GATE-CE.26

EE23BTECH11046 - Poluri Hemanth*

Question: The solution of second-order differential equation
$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ with boundary conditions $y(0) = 1$ and $y(1) = 3$.
(GATE 2021 CE.26)
Solution:

Symbol	Values	Description
$Y(s)$	-	y in s domain
$y(x)$	-	y in x domain
$y(0)$	1	y at $x = 0$
$y(1)$	3	$y(x)$ at $x = 1$
$u(x)$	$= \begin{cases} 1 & \text{if } x > 0 \\ 0 & o.w \end{cases}$	unit step function

TABLE I
Parameters

Applying Laplace transform

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) \tag{1}$$

$$Y(s)(s^2 + 2s + 1) = s - 2 - y'(0) \tag{2}$$

$$\implies Y(s) = \frac{s - 2 - y'(0)}{s^2 + 2s + 1} \tag{3}$$

$$= \frac{1}{s+1} - \frac{2 + y'(0)}{(s+1)^2} \tag{4}$$

For inversion of $Y(s)$ in partial fractions-

$$\frac{b}{(s+a)^n} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} \frac{b}{(n-1)!} \cdot x^{n-1}e^{-ax} \cdot u(x) \tag{5}$$

Applying Laplace inverse-

From (4),(5)

$$y(x) = \frac{1}{0!}e^{-x} \cdot u(x) - \frac{3 + y'(0)}{1!}x \cdot e^{-x} \cdot u(x) \tag{6}$$

$$= (1 - (3 + y'(0))x)e^{-x}u(x) \tag{7}$$

From (7),

$$y(1) = (1 - 3 - y'(0))e^{-1} \tag{8}$$

$$3 = (1 - 3 - y'(0))e^{-1} \tag{9}$$

$$\implies y'(0) = -(2 + 3e) \tag{10}$$

From(10),(7)

$$y(x) = (e^x + (3e - 1)xe^{-x})u(x) \tag{11}$$

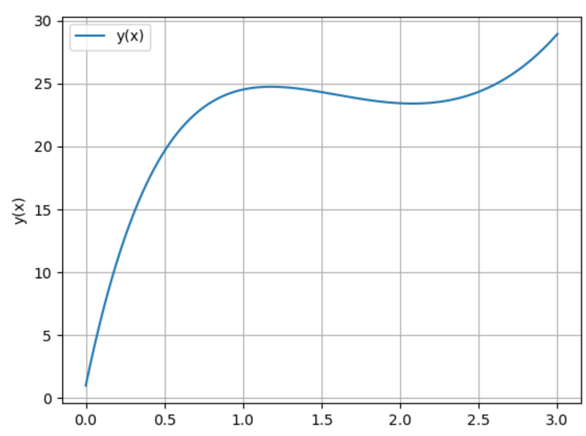

Fig. 1. Plot of y(x)