

GATE-CS.51

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Question: Consider the following recurrence:

$$\begin{aligned} f(1) &= 1; \\ f(2n) &= 2f(n) - 1, \text{ for } n \geq 1; \\ f(2n+1) &= 2f(n) + 1, \text{ for } n \geq 1. \end{aligned}$$

(C)

Let, $f(5 \cdot 2^n) = 2^{n+1} + 1$ is true for any n value,

$$f(5 \cdot 2^{n+1}) = f(2(5 \cdot 2^n)) \quad (11)$$

$$= 2f(5 \cdot 2^n) + 1 \quad (12)$$

$$= 2(2^{n+1} + 1) - 1 \quad (13)$$

$$= 2^{n+2} + 1 \quad (14)$$

Then, which of the following is/are **TRUE**?

(A) $f(2^n - 1) = 2^n - 1$

(B) $f(2^n) = 1$

(C) $f(5 \cdot 2^n) = 2^{n+1} + 1$

(D) $f(2^n + 1) = 2^n + 1$

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Solution:

(A)

For $k=1$.

$$f(2^k - 1) = 1 \quad (1)$$

$$= 2^k - 1 \quad (2)$$

$$f(2^{k+1} - 1) = f(2(2^k - 1) + 1) \quad (3)$$

$$= 2f(2^k - 1) + 1 \quad (4)$$

$$= 2(2^k - 1) + 1 \quad (5)$$

$$= 2^{k+1} - 1 \quad (6)$$

Therefore C will be proved

$$f(5) = f(2 \cdot 2 + 1) \quad (15)$$

$$= 2f(2) + 1 \quad (16)$$

$$= 3 \quad (17)$$

Hence statement C is TRUE.

(D)

For any $n \geq 1$

$$f(2^n + 1) = 2f(2^{n-1}) + 1 \quad (18)$$

From (B)

$$f(2^n + 1) = 3 \quad (19)$$

Hence statement D is FALSE.

Hence statement A is TRUE

(B)

Let $f(2^n) = 1$ for any random $n \geq 1$

$$f(2^{n+1}) = 2f(2^n) - 1 \quad (7)$$

$$= 1 \quad (8)$$

Therefore B will be proved

$$f(2^0) = f(1) \quad (9)$$

$$= 1 \quad (10)$$

Hence $f(2^n) = 1$ for every $n \geq 1$ value.

So statement B is TRUE.