GATE-CS.51

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Question: Consider the following recurrence:

$$f(1) = 1; \quad (1)$$
$$f(2n) = 2f(n) - 1, \text{ for } n \geq 1; \quad (2)$$
$$f(2n+1) = 2f(n) + 1, \text{ for } n \geq 1. \quad (3)$$

Then, which of thefollowing is/are **TRUE?**
(A) $f(2^n - 1) = 2^n - 1$
(B) $f(2^n) = 1$
(C) $f(5 \cdot 2^n) = 2^{n+1} + 1$
(D) $f(2^n + 1) = 2^n + 1$
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Solution:
(A) let $f(2^k - 1) = 2^k - 1$ for any $k \geq 1$,

$$f(2^{k+1} - 1) = f(2(2^k - 1) + 1) \quad (4)$$

From (3),

$$= 2f(2^k - 1) + 1 \quad (5)$$
$$= 2(2^k - 1) + 1 \quad (6)$$
$$= 2^{k+1} - 1 \quad (7)$$

From (1),(7)

$$f(2 - 1) = 2 - 1 \ (k = 0) \quad (8)$$
$$= 1 \quad (9)$$

Hence $f(2^n - 1) = 2^n - 1$ for $n \geq 1$
So statement A is TRUE
(B)
Let $f(2^k) = 1$ for any $k \geq 0$

$$f(2^{k+1}) = f(2 \cdot 2^k) \quad (10)$$

From (2)

$$= 2f(2^k) - 1 \quad (11)$$
$$= 2 - 1 \quad (12)$$
$$= 1 \quad (13)$$

From (11),(2)

$$f(2) = 2f(1) - 1 \ (k = 0) \quad (14)$$
$$= 2 - 1 \quad (15)$$
$$= 1 \quad (16)$$

Hence $f(2^n) = 1$ for every $n \geq 0$ value.
So statement B is TRUE.
(C)
Let,$f(5 \cdot 2^k) = 2^{k+1} + 1$ be true for any $k \geq 0$,

$$f(5 \cdot 2^{k+1}) = f(2(5 \cdot 2^k)) \quad (17)$$

From (2)

$$= 2f(5 \cdot 2^k) - 1 \quad (18)$$
$$= 2(2^{k+1} + 1) - 1 \quad (19)$$
$$= 2^{k+2} + 1 \quad (20)$$

From (20)

$$f(5) = 2^1 + 1 \ (k = -1) \quad (21)$$
$$= 3 \quad (22)$$

Proof:-

$$f(5) = f(2 \cdot 2 + 1) \quad (23)$$

From (3),(16)

$$= 2f(2) + 1 \quad (24)$$
$$= 3 \quad (25)$$

Hence $f(5.2^n) = 2^{n+1} + 1$ for $n \geq 1$
So statement C is TRUE.
(D)

$$f(2^n + 1) = f(2 \cdot 2^{n-1} + 1) \quad (26)$$

From (3),(26),(16)

$$= 2f(2^{n-1}) + 1 \quad (27)$$
$$= 2 + 1 \quad (28)$$
$$= 3 \quad (29)$$

Hence statement D is FALSE.