GATE-CS.51

EE23BTECH11046 - Poluri Hemanth*

Question: Consider the following recurrence:

$$\begin{aligned} f(1) &= 1; \\ f(2n) &= 2f(n) - 1, \text{ for } n \geq 1; \\ f(2n+1) &= 2f(n) + 1, \text{ for } n \geq 1. \end{aligned}$$

Then, which of thefollowing is/are **TRUE?**
(A) $f(2^n - 1) = 2^n - 1$
(B) $f(2^n) = 1$
(C) $f(5 \cdot 2^n) = 2^{n+1} + 1$
(D) $f(2^n + 1) = 2^n + 1$
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Solution:
(A)
For k=1.

$$\begin{aligned} f(2^k - 1) &= 2^k - 1 \text{ (for k=1)} && (1) \\ &= 1 && (2) \\ f(2^{k+1} - 1) &= f(2(2^k - 1) + 1) && (3) \\ &= 2f(2^k - 1) + 1 && (4) \\ &= 2(2^k - 1) + 1 && (5) \\ &= 2^{k+1} - 1 && (6) \end{aligned}$$

Hence statement A is TRUE
(B)

Let $f(2^n) = 1$ for any random n≥1

$$\begin{aligned} f(2^{n+1}) &= 2f(2^n) - 1 && (7) \\ &= 1 && (8) \end{aligned}$$

Therefore B will be proved

$$\begin{aligned} f(2^0) &= f(1) && (9) \\ &= 1 && (10) \end{aligned}$$

Hence $f(2^n) = 1$ for every $n \geq 1$ value.
So statement B is TRUE.

(C)
Let,$f(5 \cdot 2^n) = 2^{n+1} + 1$ is true for any n value,

$$\begin{aligned} f(5 \cdot 2^{n+1}) &= f(2(5 \cdot 2^n)) && (11) \\ &= 2f(5 \cdot 2^n) + 1 && (12) \\ &= 2(2^{n+1} + 1) - 1 && (13) \\ &= 2^{n+2} + 1 && (14) \end{aligned}$$

Therefore C will be proved

$$\begin{aligned} f(5) &= f(2 \cdot 2 + 1) && (15) \\ &= 2f(2) + 1 && (16) \\ &= 3 && (17) \end{aligned}$$

Hence statement C is TRUE.
(D)
For any n≥1

$$f(2^n + 1) = 2f(2^{n-1}) + 1 \quad (18)$$

From (B)

$$f(2^n + 1) = 3 \quad (19)$$

Hence statement D is FALSE.