GATE-CS.51

EE23BTECH11046 - Poluri Hemanth*

Question: Consider the following recurrence:

$$f(1) = 1; \quad (1)$$
$$f(2n) = 2f(n) - 1, \text{ for } n \geq 1; \quad (2)$$
$$f(2n+1) = 2f(n) + 1, \text{ for } n \geq 1. \quad (3)$$

Then, which of thefollowing is/are **TRUE?**
(A) $f(2^n - 1) = 2^n - 1$
(B) $f(2^n) = 1$
(C) $f(5 \cdot 2^n) = 2^{n+1} + 1$
(D) $f(2^n + 1) = 2^n + 1$
[GATE-CS.51 2022]
Solution:
(A)

$$f(2^k - 1) = 2^k - 1 \text{ (for k=1)} \quad (4)$$
$$f(1) = 1 \text{ (From (1))} \quad (5)$$
$$f(2^{k+1} - 1) = f(2(2^k - 1) + 1) \quad (6)$$

From (2),(6)

$$f(2^{k+1} - 1) = 2f(2^k - 1) + 1 \quad (7)$$
$$= 2(2^k - 1) + 1 \quad (8)$$
$$= 2^{k+1} - 1 \quad (9)$$

Hence $f(2^n - 1) = 2^n - 1$ for $n \geq 1$
So statement A is TRUE
(B)

Let $f(2^n) = 1$ for any n≥0
From (2)

$$f(2^{n+1}) = 2f(2^n) - 1 \quad (10)$$
$$= 1 \quad (11)$$

Therefore B will be proved
For $n = 0$

$$f(2) = 2f(1) - 1 \text{ From (1),} \quad (12)$$
$$= 1 \quad (13)$$

Hence $f(2^n) = 1$ for every $n \geq 0$ value.
So statement B is TRUE.

(C)
Let,$f(5 \cdot 2^n) = 2^{n+1} + 1$ be true for any n value,

$$f(5 \cdot 2^{n+1}) = f(2(5 \cdot 2^n)) \quad (14)$$
$$\text{From (2)} \quad (15)$$
$$= 2f(5 \cdot 2^n) - 1 \quad (16)$$
$$= 2(2^{n+1} + 1) - 1 \quad (17)$$
$$= 2^{n+2} + 1 \quad (18)$$

Therefore C will be proved

$$f(5) = f(2 \cdot 2 + 1) \quad (19)$$
$$\text{From (3)}$$
$$= 2f(2) + 1 \quad (20)$$
$$= 3 \quad (21)$$

Hence $f(5.2^n + 1) = 2^{n+1} + 1$ for $n \geq 1$
So statement C is TRUE.
(D)
From (3)

$$f(2^n + 1) = f(2 \cdot 2^{n-1}) + 1 \quad (22)$$
$$\text{From (3)}$$
$$= 2f(2^{n-1}) + 1 \quad (23)$$

From statement (B)

$$f(2^n + 1) = 3 \text{ (for n≥1)} \quad (24)$$

Hence statement D is FALSE.