GATE-CS.51

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Question: Consider the following recurrence:

$$f(1) = 1; \tag{1}$$
$$f(2n) = 2f(n) - 1, \text{ for } n \geq 1; \tag{2}$$
$$f(2n + 1) = 2f(n) + 1, \text{ for } n \geq 1. \tag{3}$$

Then, which of thefollowing is/are **TRUE?**
(A) $f(2^n - 1) = 2^n - 1$
(B) $f(2^n) = 1$
(C) $f(5 \cdot 2^n) = 2^{n+1} + 1$
(D) $f(2^n + 1) = 2^n + 1$
[GATE-CS.51 2022]
Solution:
(A)

$$f(2^k - 1) = 2^k - 1 \text{ (for k=1)} \tag{4}$$
$$f(1) = 1 \text{ (From (1))} \tag{5}$$
$$f(2^{k+1} - 1) = f(2(2^k - 1) + 1) \tag{6}$$

From (2),(6)

$$f(2^{k+1} - 1) = 2f(2^k - 1) + 1 \tag{7}$$
$$= 2(2^k - 1) + 1 \tag{8}$$
$$= 2^{k+1} - 1 \tag{9}$$

Hence $f(2^n - 1) = 2^n - 1$ for $n \geq 1$
So statement A is TRUE
(B)

Let $f(2^n) = 1$ for any n≥0
From (2)

$$f(2^{n+1}) = 2f(2^n) - 1 \tag{10}$$
$$= 1 \tag{11}$$

Therefore B will be proved
For $n = 0$

$$f(2) = 2f(1) - 1 \text{ From (1)}, \tag{12}$$
$$= 1 \tag{13}$$

Hence $f(2^n) = 1$ for every $n \geq 0$ value.
So statement B is TRUE.

(C)
Let, $f(5 \cdot 2^n) = 2^{n+1} + 1$ be true for any n value,

$$f(5 \cdot 2^{n+1}) = f(2(5 \cdot 2^n)) \tag{14}$$
$$\text{From (2)} \tag{15}$$
$$= 2f(5 \cdot 2^n) - 1 \tag{16}$$
$$= 2(2^{n+1} + 1) - 1 \tag{17}$$
$$= 2^{n+2} + 1 \tag{18}$$

Therefore C will be proved
For $n = 0$

$$f(5) = f(2 \cdot 2 + 1) \tag{19}$$
$$\text{From (3)}$$
$$= 2f(2) + 1 \tag{20}$$
$$= 3 \tag{21}$$

Hence $f(5.2^n) = 2^{n+1} + 1$ for $n \geq 1$
So statement C is TRUE.
(D)
From (3)

$$f(2^n + 1) = f(2 \cdot 2^{n-1}) + 1 \tag{22}$$
$$\text{From (3)}$$
$$= 2f(2^{n-1}) + 1 \tag{23}$$
From statement (B)
$$f(2^n + 1) = 3 \text{ (for n≥1)} \tag{24}$$

Hence statement D is FALSE.