CA-1

for an Elastic

The variational problem is

$$\delta\pi = \int_{\Omega^e} \delta \underline{\varepsilon}^T \underline{C}\, \underline{\varepsilon}\, d\Omega - \int \delta u_i^h b_i d\Omega$$

$$- \int_{\Gamma^{h_i}} \delta u_i^h h_i\, d\Gamma = 0$$

$$\Rightarrow \sum_e \delta d^{e^T} \Big[\underbrace{\int_{\Omega^e} \underline{B}^T \underline{C}\, \underline{B}\, d\Omega}_{\text{Stiffness matrix}} - \int_{\Omega^e} \underline{N}^T \underline{b}\, d\Omega$$

$$- \int_{\Gamma^{h_i}} \underline{N}^T h\, d\Gamma \Big] = 0$$

for an Q4 - FI :

4 node Quadrilateral Element with 4 Gaussian points for integration

$$\Rightarrow \int_{\Omega^e} \underline{B}^T \underline{C}\, \underline{B}\, d\Omega = \sum_{i=1}^{4} \underbrace{w_i \underset{8\times3}{\underline{B}^T(\xi_i, \eta_i)} \underset{3\times3}{\underline{C}} \underset{3\times8}{B(\xi_i, \eta_i)} J(\xi_i, \eta_i)}_{\text{local } 8\times8 \text{ matrix}}$$

for an Q4-RI

only one gaussian point for integration ie (0,0)

$$\Rightarrow \int_{\Omega^e} \underset{\sim}{B}^v \underset{\sim}{C} \underset{\sim}{B}\, d\Omega = 4\, J(0,0) \cdot \underset{8\times3}{\underset{\sim}{B}^T(0,0)}\ \underset{3\times3}{\underset{\sim}{C}}\ \underset{3\times8}{\underset{\sim}{B}(0,0)}$$

for Q4-SRI:

we decompose strain into volumetric and deviatoric

i.e.

$$\int_{\Omega} \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{C} \underset{\sim}{\varepsilon}\, d\Omega = \int \delta \underset{\sim}{\varepsilon}^{vT} \underset{\sim}{C} \underset{\sim}{\varepsilon}^v d\Omega + \int \delta \underset{\sim}{\varepsilon}^{dT} \underset{\sim}{C} \underset{\sim}{\varepsilon}^d d\Omega$$

$$\underset{\sim}{K}^e = \underset{\sim}{K}^{ve} + \underset{\sim}{K}^{de}$$

$$= \sum_e \delta d^{eT} \left[\int \underset{\sim}{B}^{vT} \underset{\sim}{C} \underset{\sim}{B}^v d\Omega + \int \underset{\sim}{B}^{dT} \underset{\sim}{C} \underset{\sim}{B}^d d\Omega \right]$$

we have 4 gaussian pts for deviatoric part.

and 1 gaussian pt for volumetric part.

$$\Rightarrow \int \underset{\sim}{B}^{dT} \underset{\sim}{C} \underset{\sim}{B}^d = \sum_{i=1}^{4} w_i\, \underset{8\times4}{\underset{\sim}{B}^{dT}(\xi_i,\eta_i)}\ \underset{4\times4}{\underset{\sim}{C}}\ \underset{4\times8}{\underset{\sim}{B}^d(\xi_i,\eta_i)}\, J(\xi_i,\eta_i)$$

$$\int \underset{\sim}{B}^{vT} \underset{\sim}{C} \underset{\sim}{B}^v = 4\, J(0,0) \cdot \underset{8\times4}{\underset{\sim}{B}^{vT}(0,0)}\ \underset{4\times4}{\underset{\sim}{C}}\ \underset{4\times8}{\underset{\sim}{B}^v(0,0)}$$

for RI & FI:

$$C = \begin{array}{cc} & \begin{array}{ccc} 11 & 22 & 12 \end{array} \\ & \left[\begin{array}{ccc} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{array}\right] \begin{array}{c} 11 \\ 22 \\ 12 \end{array} \end{array}$$

for SRI;

$$C = \begin{array}{cc} & \begin{array}{cccc} 11 & 22 & 33 & 12 \end{array} \\ & \left[\begin{array}{cccc} \lambda+2\mu & \lambda & \lambda & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 \\ 0 & 0 & 0 & \mu \end{array}\right] \begin{array}{c} 11 \\ 22 \\ 33 \\ 12 \end{array} \end{array}$$

$$B = \left[B_1, B_2, B_3, B_4\right]_{3\times 8}; \quad B_I = \left[\begin{array}{cc} N_{I,1} & 0 \\ 0 & N_{I,2} \\ N_{I,2} & N_{I,1} \end{array}\right]_{3\times 2}$$

where.

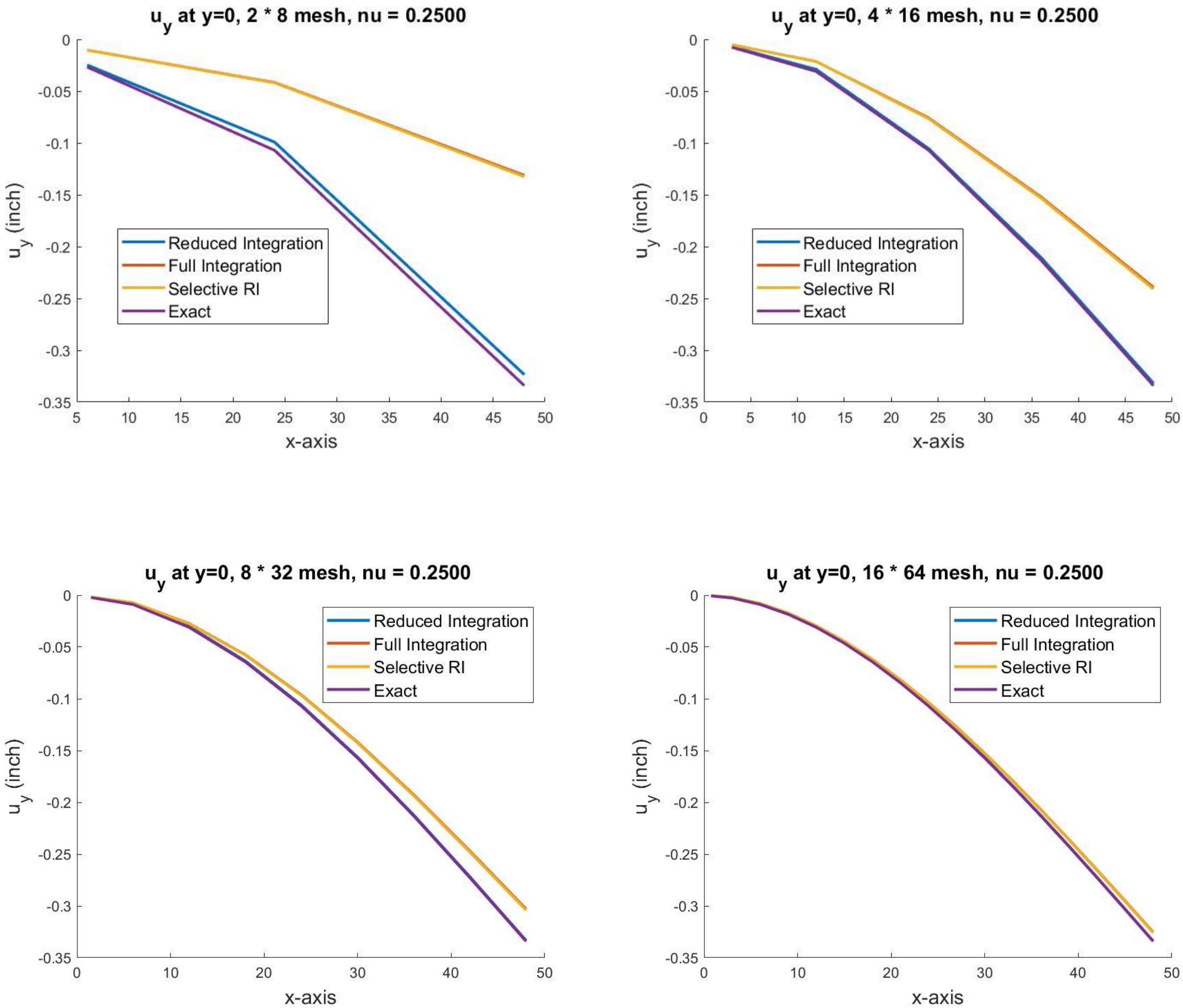
$$B^{v} = \left[B^{v}_1, B^{v}_2, B^{v}_3, B^{v}_4\right]_{3\times 8}; \quad B^{v}_I = \left[\begin{array}{cc} \frac{1}{3}N_{I,1} & \frac{1}{3}N_{I,2} \\ \frac{1}{3}N_{I,1} & \frac{1}{3}N_{I,2} \\ \frac{1}{3}N_{I,1} & \frac{1}{3}N_{I,2} \\ 0 & 0 \end{array}\right]$$

$$B^d = \left[\underset{\sim}{B}_1^d, \underset{\sim}{B}_2^d, \underset{\sim}{B}_3^d, \underset{\sim}{B}_4 \right]_{3\times 8}$$

$$B_I^d = \begin{bmatrix} \frac{2}{3} N_{I,1} & -\frac{1}{3} N_{I,2} \\ -\frac{1}{3} N_{I,1} & \frac{2}{3} N_{I,2} \\ -\frac{1}{3} N_{I,1} & -\frac{1}{3} N_{I,2} \\ N_{I,2} & N_{I,1} \end{bmatrix}$$

Computing these, we get the corresponding Element Stiffness matrices for Q4-FI, RI and SRI

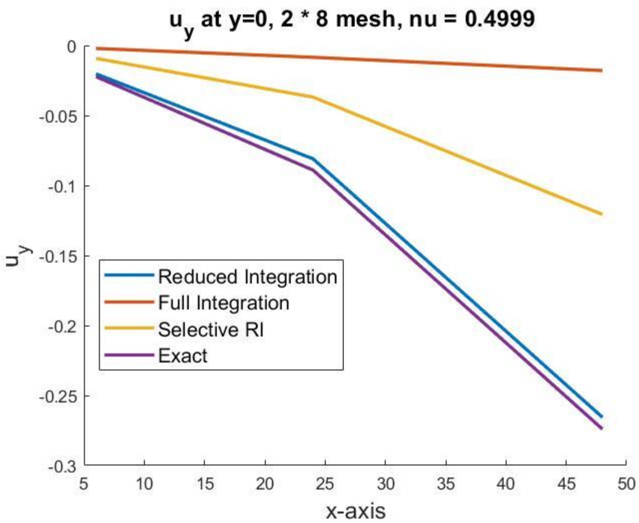
b) Deformation

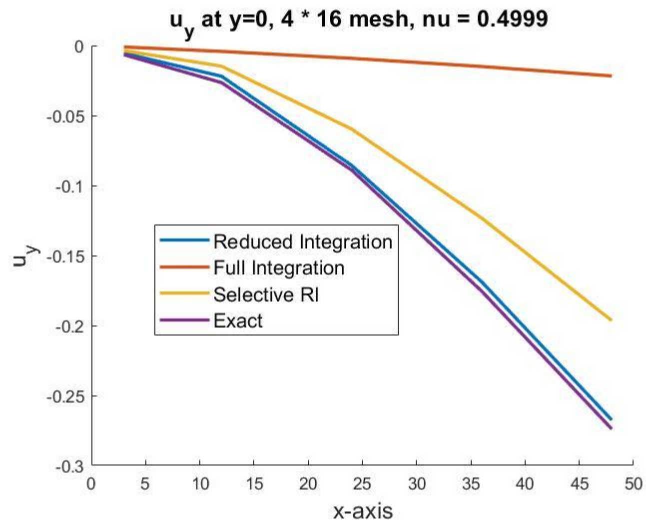


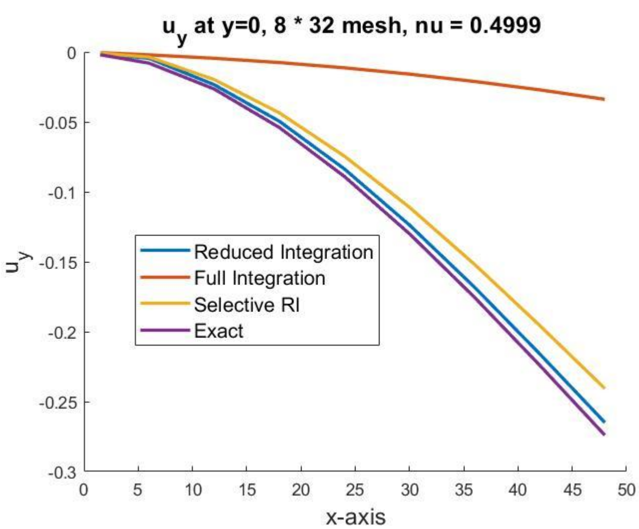
The above figure shows the variation of FEM solution for Displacement with mesh refinement and the solution obtained with different methods of integration.
We see that, for Poisson's ratio of 0.25, there is no difference for full integration and SRI solutions. With better refinement of mesh(smaller element size) the fem solutions of FI and SRI have less error and they are closer to the exact (analytical solution)

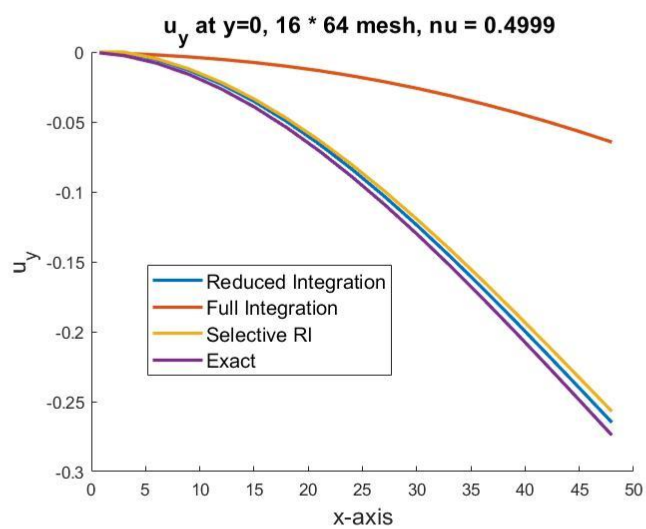
Unexpectedly, the reduced integration solution seems to have less displacement error than other methods

Deformation





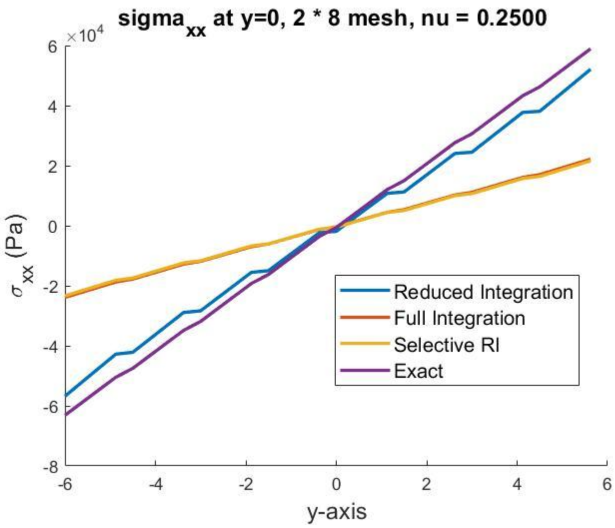


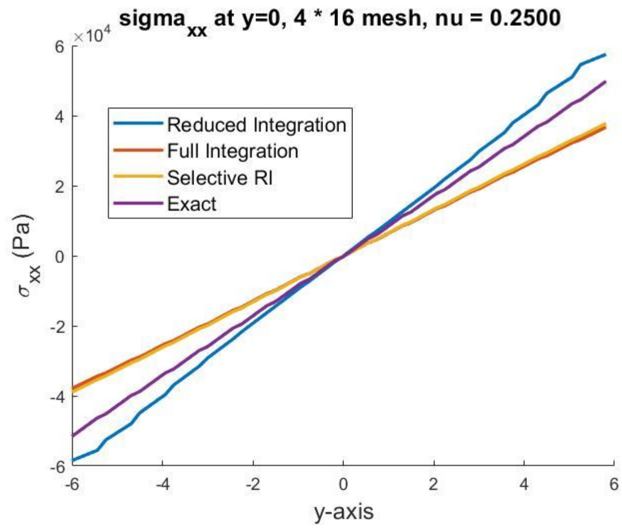


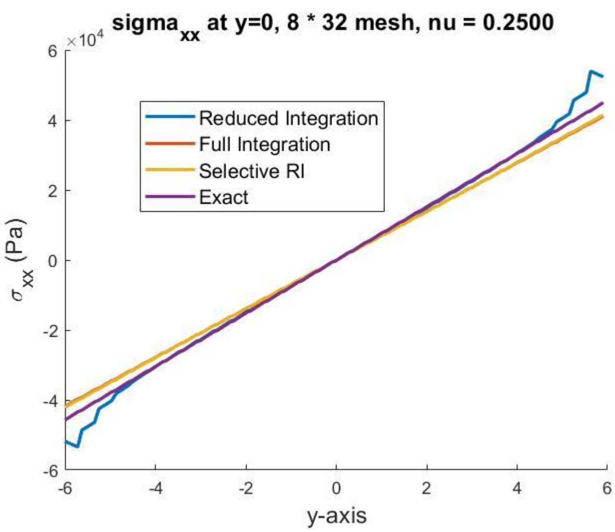
For Poisson's ratio of 0.4999, near incompressible material, the FI solution seems to be very small in comparison to other solutions(exact, RI and SRI). even though the SRI solution has higher error at coarse discretization, with finer mesh, we can see that the accuracy of the solution improved and is very coles to exact solution.

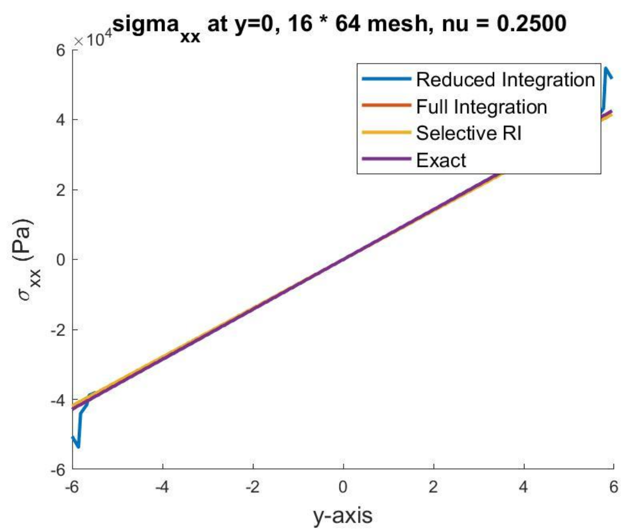
At closer inspection, the RI solution seems to be less smoother in comparison to SRI solution. It has slight oscillation in the U_y solution. This oscillation can be seen more clearly if the bar undergoes higher deformation, i.e, at greater load condition.

Normal stress









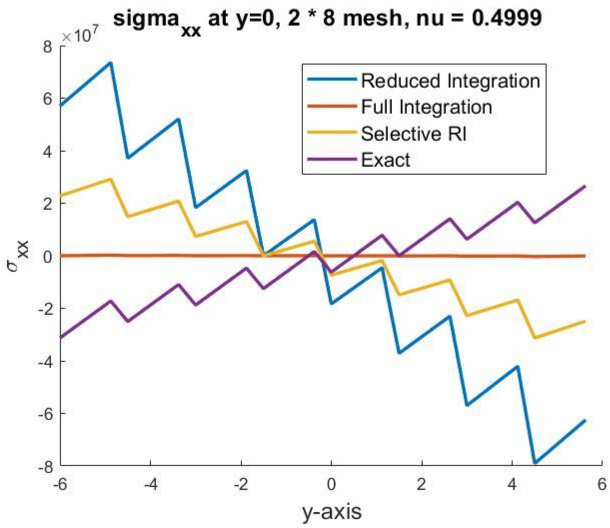
The Sigma_xx plots seems to be slightly wavy, as they are being computed at discrete points on the elements and they are not necessarily continuous in nature. The exact solution for displacement is obtained at the nodes and then the stress are being computed as per the shape functions and chosen points to report.

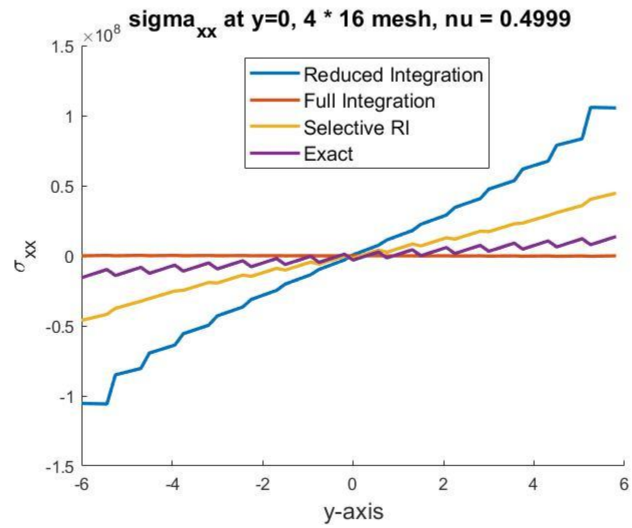
Here i used the points [(-0.5,-1),(0,-1),(0.5,-1), (1,-1),(1,-0.5),(1,0),(1,0.5)] in local coordinates, as the positions in an element where stresses are being reported.

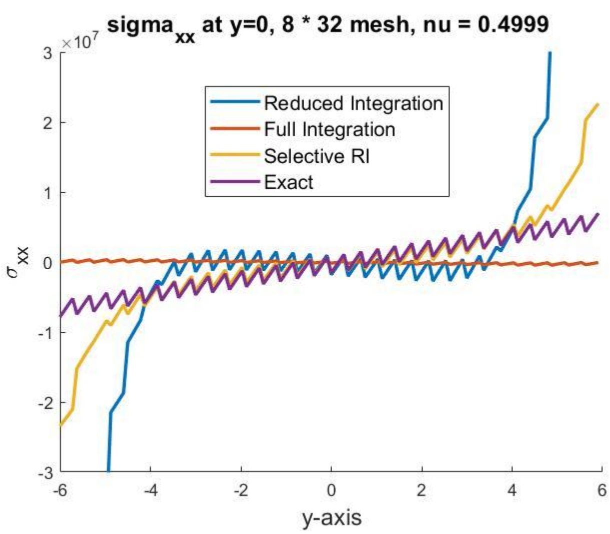
We see that the Reduced integration is closer to exact solution but the oscillation about its solution is quite big in comparison to SRI and FI solutions. The SRI and FI solutions nearly the same for Nu = 0.25 case and they improve in accuracy with finer discretization.

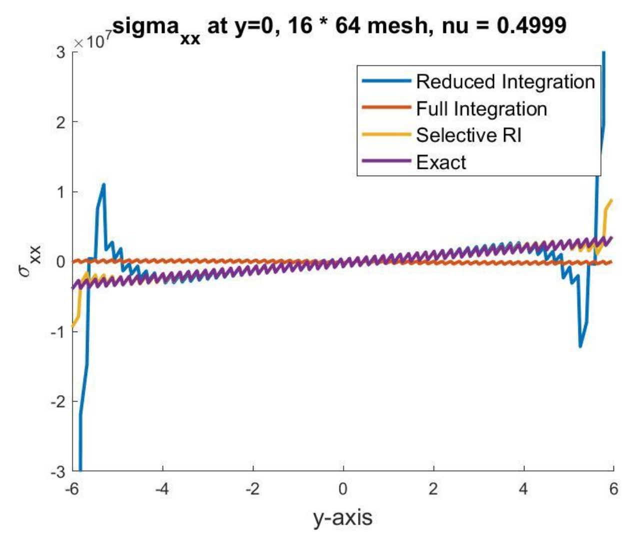
But even at finer discretization the oscillation is still observed in RI solution

Normal stress







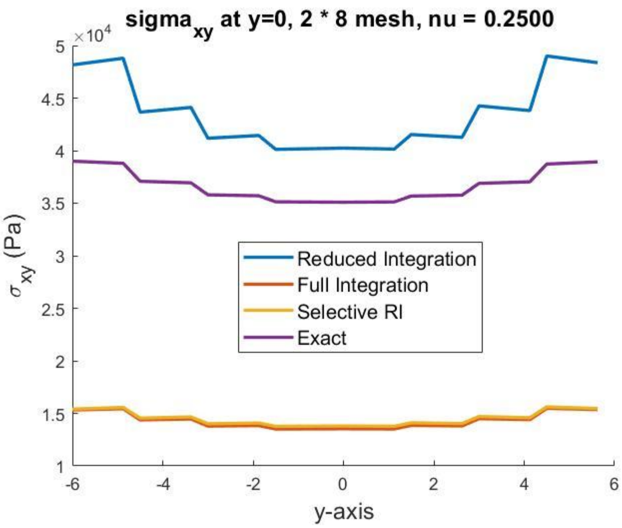


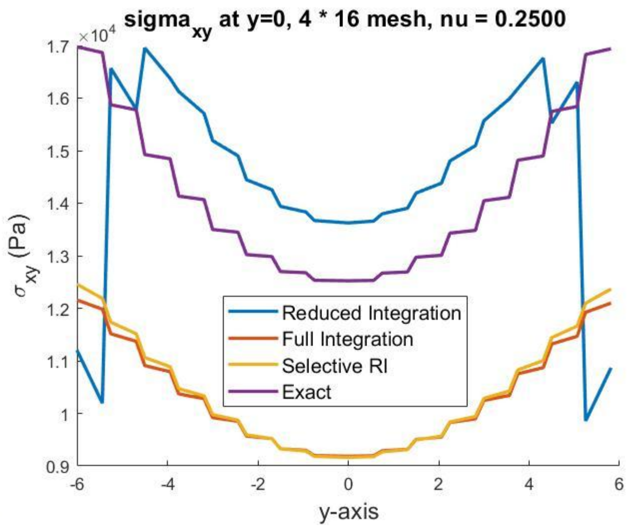
At nu = 0.4999, near incompressible case, we find that at very coarse meshing, the solution of any method is far off from the exact solution. With the refining of mesh the accuracy improves, but the stress oscillation cannot be completely removed.

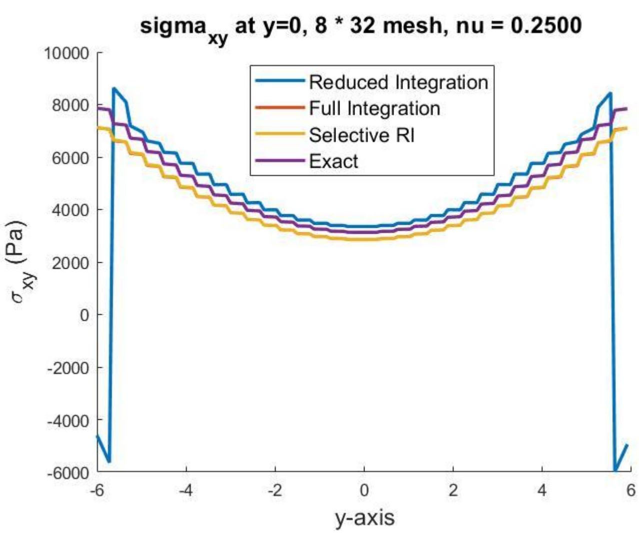
The FI method is grossly different from the real solution and this can be understood because of the volumetric locking. This solution is not improved much with meshing refinement.

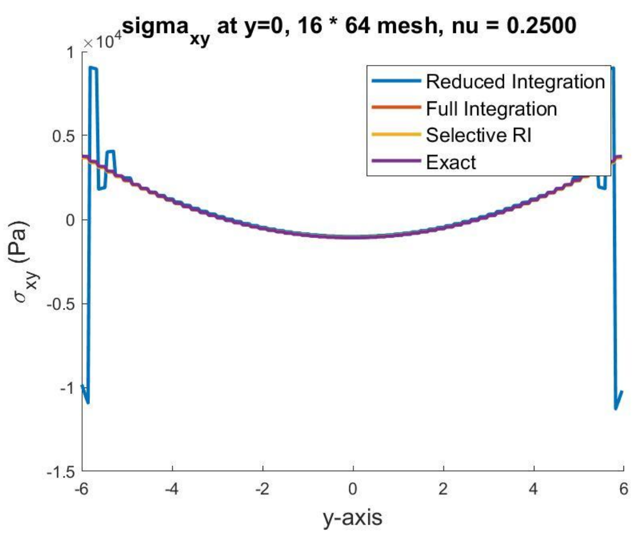
The accuracy of the RI and SRI methods improve with meshing refinement but the oscillation of solution still exist. Though this oscillation is less in SRI method in comparison to RI method, especially at finer refinement.

Shear stress





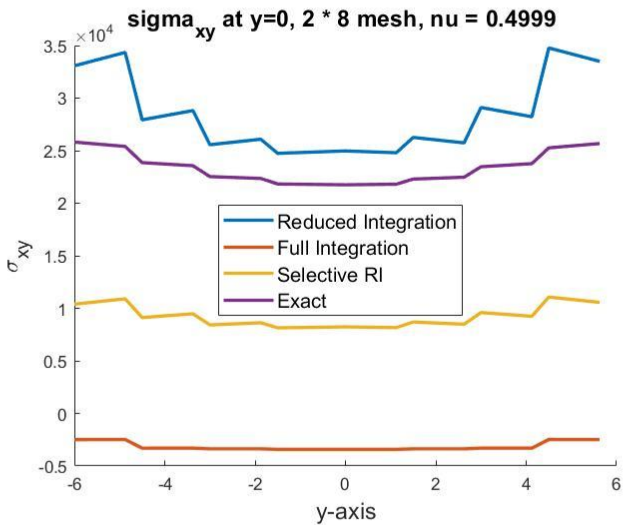


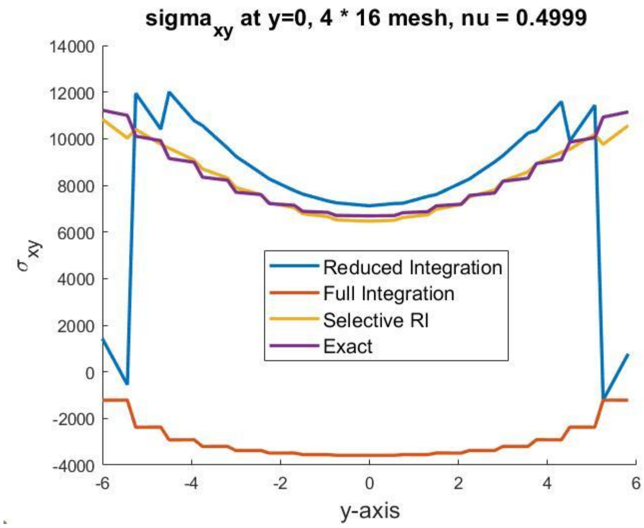


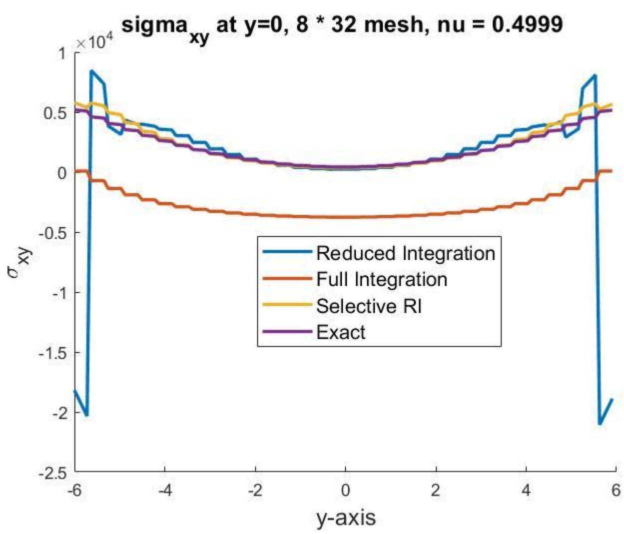
Similar to the Sigma_xx solutions, the Sigma_xy solution from RI, SRI and FI improves with mesh refinement for Nu = 0.25 case. But it can be observed that the stress oscillation of the RI solution is quite high here especially towards the periphery points.

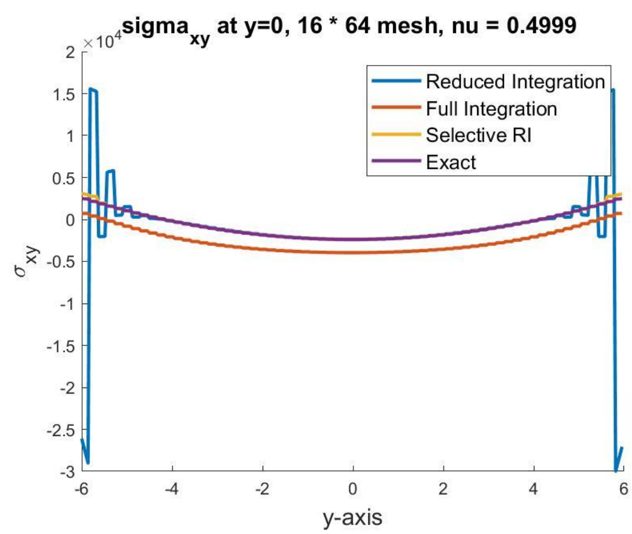
On the other hand The SRI and FI methods though have lower accuracy at coarse refinement, the accuracy improves as we have finer mesh.

Shear stress









For near incompressible case, the FI shear stress solution is vastly different than the exact solution because of the locking and also does not improve with meshing. While the SRI solution of shear stress improves with mesh refinement. Though the RI solution also improves with mesh refinement, the oscillations of the solutions still exist and cannot be completely removed with mesh refinement alone.

C) Locking occurs in near-incompressible materials when using full integration because the stiffness matrix of the element becomes overly stiff.
In such materials, the deformation is mostly incompressible, meaning any deformation results only in deviatoric part and keeping the volumetric part unchanged.
As integration points are distributed throughout the element where the deformation is mostly incompressible, the stiffness matrix of the element becomes overly stiff and as we approach Poisson's ratio of 0.5, the stiffness matrix becomes singular resulting inaccurate results.

Reduced integration reduces the number of integration points used for integration. Instead of integrating over the whole element, integration is performed only over a reduced number of Gauss points, in Q4 element at the centre(0,0) locally in an element. This softens the stiffness matrix (K) of the element thus providing with a better solution that FI method at near incompressible case.

When we come to Selective Reduced Integration(SRI) method, we separate the stiffness matrix into Volumetric and Deviatoric parts. Since the incompressibility will not affect deviatoric part, we can do the full integration(4 gaussian points for Q$ elements) for this part but for the volumetric part we do a reduced integration(only 1 gaussian point). This increases the accuracy of the solution.

In the RI method the stress solution(pressure) will be a constant in an element as we reduced the no of integration points. While the original assumption for FI is displacement Bi linear and pressure linear in an element. In the SRI method this situation slightly improves as we have the deviatoric part being linear and the volumetric part being constant. Thus have much closer semblance to the original assumption .

d) Reduced integration can cause hourglass instability because of the lower integration points. This causes some elements to become underdetermined or unstable, leading to numerical oscillations or instabilities.

We see this hourglass modes clearly in the stress solutions where the RI solutions are giving spurious modes especially near the periphery. These stress oscillations are a result of reduced integration resulting in constant pressure in an element contrary to assumed linear solution. The SRI solutions are better in minimising this hourglass modes as the deviatoric part is still linear in nature.
We also see that the SRI stress oscillations diminish with mesh refinement but no such behaviour exist for RI method.

Code at: https://github.com/hemanth297/CA1_full