UNIT-IV Probability Distributions

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Random Variables

Suppose that to each point of a sample space we assign a number. We then have a *function* defined on the sample space. This function is called a *random variable* (or *stochastic variable*) or more precisely a *random function* (*stochastic function*). It is usually denoted by a capital letter such as X or Y. In general, a random variable has some specified physical, geometrical, or other significance.

EXAMPLE 2.1 Suppose that a coin is tossed twice so that the sample space is $S = \{HH, HT, TH, TT\}$. Let X represent the number of heads that can come up. With each sample point we can associate a number for X as shown in Table 2-1. Thus, for example, in the case of HH (i.e., 2 heads), $X = 2$ while for TH (1 head), $X = 1$. It follows that X is a random variable.

Table 2-1

Sample Point	HH	HT	TH	TT
X	2	1	1	0

It should be noted that many other random variables could also be defined on this sample space, for example, the square of the number of heads or the number of heads minus the number of tails.

1 **Discrete Random Variable.** If a random variable takes at most a countable number of values, it is called a discrete random variable. *In other words, a real valued function defined on a discrete sample space is called a discrete random variable.*

1.1 **Probability Mass Function** (*and probability distribution of a discrete random variable*).

Suppose X is a one-dimensional discrete random variable taking at most a countably infinite number of values $x_1, x_2, \ldots$ With each possible outcome x_i, we associate a number $p_i = P(X = x_i) = p(x_i)$, called the probability of x_i. The numbers $p(x_i)$; $i = 1, 2, \ldots$ must satisfy the following conditions :

$$(i) \quad p(x_i) \geq 0 \ \forall \ i, \quad (ii) \sum_{i=1}^{\infty} p(x_i) = 1$$

This function p is called the probability mass function of the random variable X and the set $\{x_i, p(x_i)\}$ is called the probability distribution (p.d.) of the r.v. X.

PROBLEM

A random variable X has the following probability distribution :

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k, (ii) Evaluate $P(X < 6)$, $P(X \geq 6)$, and $P(0 < X < 5)$, (iii) If $P(X \leq c) > \frac{1}{2}$, find the minimum value of c, and (iv) Determine the distribution function of X. **[Madurai Univ. B.Sc., Oct. 1988]**

Solution. Since $\sum_{x=0}^{7} p(x) = 1$, we have

$$\Rightarrow \quad k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow \quad 10k^2 + 9k - 1 = 0$$

$$\Rightarrow \quad (10k - 1)(k + 1) = 0 \Rightarrow k = 1/10$$

[$\because k = -1$, is rejected, since probability canot be negative.]

(ii) $P(X < 6) = P(X = 0) + P(X = 1) + \ldots + P(X = 5)$

$$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = 1 - P(X < 6) = \frac{19}{100}$$

$$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 8k = 4/5$$

(*iii*) $P(X \le c) > \frac{1}{2}$. By trial, we get $c = 4$.

(*iv*)

X	$F_X(x) = P(X \le x)$
0	0
1	$k = 1/10$
2	$3k = 3/10$
3	$5k = 5/10$
4	$8k = 4/5$
5	$8k + k^2 = 81/100$
6	$8k + 3k^2 = 83/100$
7	$9k + 10k^2 = 1$

Discrete Probability Distributions

Let X be a discrete random variable, and suppose that the possible values that it can assume are given by $x_1, x_2, x_3, \ldots$, arranged in some order. Suppose also that these values are assumed with probabilities given by

$$P(X = x_k) = f(x_k) \qquad k = 1, 2, \ldots \tag{1}$$

It is convenient to introduce the *probability function*, also referred to as *probability distribution*, given by

$$P(X = x) = f(x) \tag{2}$$

For $x = x_k$, this reduces to (1) while for other values of x, $f(x) = 0$.

In general, $f(x)$ is a probability function if

1. $f(x) \geq 0$
2. $\sum_x f(x) = 1$

where the sum in 2 is taken over all possible values of x.

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P-1 Find the probability function corresponding to the random variable X of Example 2.1. Assuming that the coin is fair, we have

$$P(HH) = \frac{1}{4} \qquad P(HT) = \frac{1}{4} \qquad P(TH) = \frac{1}{4} \qquad P(TT) = \frac{1}{4}$$

Then

$$P(X = 0) = P(TT) = \frac{1}{4}$$

$$P(X = 1) = P(HT \cup TH) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X = 2) = P(HH) = \frac{1}{4}$$

The probability function is thus given by Table 2-2.

Table 2-2

x	0	1	2
$f(x)$	1/4	1/2	1/4

Distribution Functions for Random Variables

The *cumulative distribution function*, or briefly the *distribution function*, for a random variable X is defined by

$$F(x) = P(X \leq x) \tag{3}$$

where x is any real number, i.e., $-\infty < x < \infty$.

The distribution function $F(x)$ has the following properties:

1. $F(x)$ is nondecreasing [i.e., $F(x) \leq F(y)$ if $x \leq y$].
2. $\lim_{x \to -\infty} F(x) = 0$; $\lim_{x \to \infty} F(x) = 1$.
3. $F(x)$ is continuous from the right [i.e., $\lim_{h \to 0^+} F(x + h) = F(x)$ for all x].

Continuous Random Variables

A nondiscrete random variable X is said to be *absolutely continuous*, or simply *continuous*, if its distribution function may be represented as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du \qquad (-\infty < x < \infty) \tag{1}$$

where the function $f(x)$ has the properties

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

It follows from the above that if X is a continuous random variable, then the probability that X takes on any one particular value is zero, whereas the *interval probability* that X lies *between two different values*, say, a and b, is given by

$$P(a < X < b) = \int_{a}^{b} f(x)\,dx \tag{2}$$

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P-2 (a) Find the constant c such that the function

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is a density function, and (b) compute $P(1 < X < 2)$.

(a) Since $f(x)$ satisfies Property 1 if $c \geq 0$, it must satisfy Property 2 in order to be a density function. Now

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{3} cx^2\,dx = \left.\frac{cx^3}{3}\right|_{0}^{3} = 9c$$

and since this must equal 1, we have $c = 1/9$.

(b)
$$P(1 < X < 2) = \int_{1}^{2} \frac{1}{9}x^2\,dx = \left.\frac{x^3}{27}\right|_{1}^{2} = \frac{8}{27} - \frac{1}{27} = \frac{7}{27}$$

In case $f(x)$ is continuous, which we shall assume unless otherwise stated, the probability that X is equal to any particular value is zero. In such case we can replace either or both of the signs $<$ in (8) by $\leq$. Thus, in Example 2.5,

$$P(1 \leq X \leq 2) = P(1 \leq X < 2) = P(1 < X \leq 2) = P(1 < X < 2) = \frac{7}{27}$$

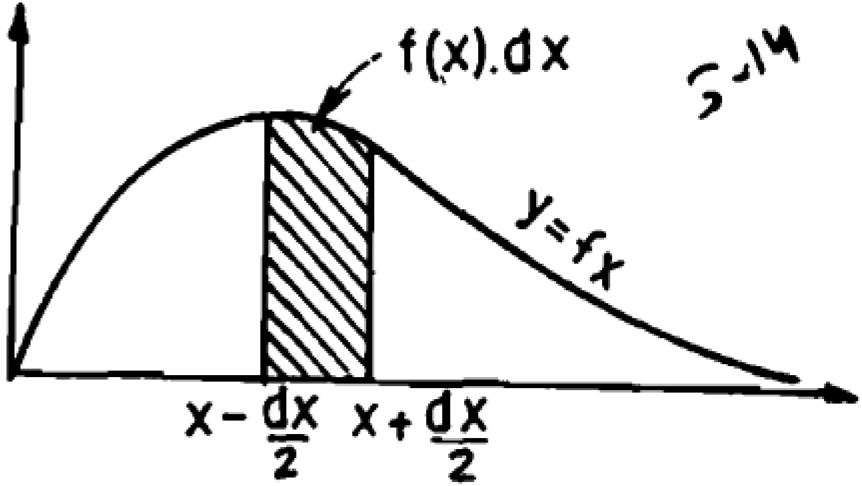
2 **Continuous Random Variable.** A random variable X is said to be continuous if it can take all possible values between certain limits. *In other words, a random variable is said to be continuous when its different values cannot be put in 1-1 correspondence with a set of positive integers.*

A continuous random variable is a random variable that (at least conceptually) can be measured to any desired degree of accuracy. Examples of continuous random variables are age, height, weight etc.

2.1 **Probability Density Function** *(Concept and Definition)*. Consider the small interval $(x, x + dx)$ of length dx round the point x. Let $f(x)$ be any continuous function of x so that $f(x)\,dx$ represents the probability that X falls in the infinitesimal interval $(x, x + dx)$. Symbolically

$$P(x \le X \le x + dx) = f_X(x)\,dx \qquad 1$$

In the figure, $f(x)\,dx$ represents the area bounded by the curve $y = f(x)$, x–axis and the ordinates at the points x and $x + dx$. The function $f_X(x)$ so defined is known as *probability density function or simply density function of random variable X and is usually abbreviated as p.d.f.* The expression, $f(x)\,dx$, usually written as $dF(x)$, is known as the *probability differential* and the curve $y = f(x)$ is known as the *probability density curve* or simply *probability curve*.



DISCRETE DISTRIBUTION (5 TYPES)

BINOMIAL DISTRIBUTION

Definition. *A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by*

$$P(X=x)=p(x)=\begin{cases}\binom{n}{x}p^x q^{n-x}\ ; x=0,1,2,\ldots,n\ ; q=1-p \\ 0,\ otherwise\end{cases} \quad (1)$$

The two independent constants n and p in the distribution are known as the *parameters* of the distribution. 'n' is also, sometimes, known as the degree of the binomial distribution.

Binomial distribution is a discrete distribution as X can take only the integral values, *viz.*, 0, 1, 2,..., n. Any variable which follows binomial distribution is known as *binomial variate.*

We shall use the notation $X \sim B(n, p)$ to denote that the random variable X follows binomial distribution with parameters n and p.

The probability $p(x)$ in (1) is also sometimes denoted by $b(x, n, p)$.

PROBLEM

Ten coins are thrown simultaneously. Find the probability of getting at least seven heads.

Solution. p = Probability of getting a head $= \frac{1}{2}$

q = Probability of not getting a head $= \frac{1}{2}$

The probability of getting x heads in a random throw of 10 coins is

$$p(x) = \binom{10}{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{10-x} = \binom{10}{x}\left(\frac{1}{2}\right)^{10} \; ; x = 0, , 2, ..., 10$$

$\therefore$ Probability of getting at least seven heads is given by

$$P(X \geq 7) = p(7) + p(8) + p(9) + p(10)$$

$$= \left(\frac{1}{2}\right)^{10}\left\{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right\}$$

$$= \frac{120 + 45 + 10 + 1}{1024} = \frac{176}{1024}.$$

PROBLEM A and B play a game in which their chances of winning are in the ratio 3 : 2. Find A's chance of winning at least three games out of the five games played.

Solution. Let p be the probability that 'A' wins the game. Then we are given $p = 3/5 \Rightarrow q = 1 - p = 2/5$.

Hence, by binomial probability law, the probability that out of 5 games played, A wins 'r' games is given by :

$$P(X = r) = p(r) = \binom{5}{r} \cdot (3/5)^r \, (2/5)^{5-r} \; ; \; r = 0, 1, 2, \ldots, 5$$

The required probability that 'A' wins at least three games is given by :

$$P(X \geq 3) = \sum_{r=3}^{5} \binom{5}{r} \frac{3^r \cdot 2^{5-r}}{5^5}$$

$$= \frac{3^3}{5^5}\left[\binom{5}{3} 2^2 + \binom{5}{4} . 3 \times 2 + 1.3^2 \times 1\right] = \frac{27 \times (40 + 30 + 9)}{3125} = 0{\cdot}68$$

POISSON DISTRIBUTION

Definition. *A random variable X is said to follow a Poisson distribution if it assumes only non-negative values and its probability mass function is given by*

$$p(x, \lambda) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x\,!}; \; x = 0, 1, 2, \ldots; \lambda > 0$$

$$= 0, \text{ otherwise} \qquad \ldots(7\cdot 14)$$

Here λ is known as the parameter of the distribution.

We shall use the notation $X \sim P(\lambda)$ to denote that X is a Poisson variate with paramter λ.

Remarks 1. It should be noted that

$$\sum_{x=0}^{\infty} P(X = x) = e^{-\lambda} \sum_{x=0}^{\infty} \lambda^x / x\,! = e^{-\lambda} e^{\lambda} = 1$$

2. The corresponding distribution function is:

$$F(x) = P(X \leq x) = \sum_{r=0}^{x} p(r) = e^{-\lambda} \sum_{r=0}^{x} \lambda^r / r\,!; \; x = 0, 1, 2, \ldots.$$

PROBLEM

A car hire firm has two cars which it fires out day by day. The number of demands for a car on each day is distributed as Poisson variate with mean 1·5., Calculate the proportion of days on which (i) neither car is used, and (ii) some demand is refused.

Solution. The proportion of days on which there are x demands for a car

$$= P\{\text{of } x \text{ demands in a day}\}$$

$$= \frac{e^{-1\cdot 5}(1\cdot 5)^x}{x!},$$

since the number of demands for a car on any day is a Poisson variate with mean 1·5. Thus

$$P(X = x) = \frac{e^{-1\cdot 5}(1\cdot 5)^x}{x!}; \quad x = 0, 1, 2, \ldots$$

(*i*) Proportion of days on which neither car is used is given by

$$P(X = 0) = e^{-1\cdot 5}$$

$$= \left[1 - 1\cdot 5 + \frac{(1\cdot 5)^2}{2!} - \frac{(1\cdot 5)^3}{3!} + \frac{(1\cdot 5)^4}{4!} - \ldots\right]$$

$$= 0\cdot 2231$$

(*ii*) Proportion of days on which some demand is refused is

$$P(X > 2) = 1 - P(X \leq 2)$$
$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - e^{-1\cdot 5}\left[1 + 1\cdot 5 + \frac{(1\cdot 5)^2}{2!}\right]$$
$$= 1 - 0\cdot 2231 \times 3\cdot 625 = 0\cdot 19126$$

PROBLEM

A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantees that not more than 10 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality ?

Solution. We are given $n = 100$.

Let p = Probability of a defective pin = 5% = 0·05

$\therefore$ λ = Mean number of defective pins in a box of 100

$= np = 100 \times 0{\cdot}05 = 5$

Since 'p' is small, we may use Poisson distribution.

Probability of x defective pins in a box of 100 is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-5}5^x}{x!};\ x = 0, 1, 2, \ldots$$

Probability that a box will fail to meet the guaranteed quality is

$$P(X > 10) = 1 - P(X \le 10) = 1 - \sum_{x=0}^{10} \frac{e^{-5}5^x}{x!} = 1 - e^{-5} \sum_{x=0}^{10} \frac{5^x}{x!}$$

PROBLEM *Six coins are tossed 6,400 times. Using the Poisson distribution, find the approximate probability of getting six heads r times.*

Solution. The probability of obtaining six heads in one throw of six coins (a single trial), is $p = (½)^6$, assuming that head and tail are equally probable.

$$\therefore \qquad \lambda = np = 6400 \times (½)^6 = 100.$$

Hence, using Poisson probability law, the required probability of getting 6 heads r times is given by :

$$P(X = r) = \frac{e^{-\lambda} \cdot \lambda^r}{r\,!} = \frac{e^{-100} \cdot (100)^r}{r\,!}\ ; \ r = 0, 1, 2, \ldots$$

CONTINUOUS DISTRIBUTION (NORMAL DISTIRBUTION)

Definition. *A random variable X is said to ahve a normal distribution with parameters μ (called "mean") and σ^2 (called "variance") if its density function is given by the probability law :*

$$f(x\,;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left\{\frac{x-\mu}{\sigma}\right\}^2\right]$$

or $$f(x\,;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$$

$$-\infty<x<\infty,\ -\infty<\mu<\infty,\ \sigma>0 \quad \ldots 1$$

Remarks. 1. A random variable X with mean μ and variance σ^2 and following the normal law (8·3) is expressed by $X \sim N(\mu, \sigma^2)$

2. If $X \sim N(\mu, \sigma^2)$, then $Z=\dfrac{X-\mu}{\sigma}$, is a standard normal variate with

$$E(Z)=0 \text{ and } \mathrm{Var}(Z)=1$$

and we write $Z \sim N(0,1)$.

Chief Characteristics of the Normal Distribution and Normal Probability Curve. The normal probability curve with mean μ and standard deviation σ is given by the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

and has the following properties :

(*i*) The curve is bell shaped and symmetrical about the line $x = \mu$.

(*ii*) Mean, median and mode of the distribution coincide.

(*iii*) As x increases numerically, $f(x)$ decreases rapidly, the maximum probability occurring at the point $x = \mu$, and given by $[p(x)]_{max} = \dfrac{1}{\sigma\sqrt{2\pi}}$.

(*iv*) $\beta_1 = 0$ and $\beta_2 = 3$.

(*v*) $\mu_{2r+1} = 0, (r = 0, 1, 2, \ldots)$,
and $\mu_{2r} = 1.3.5 \ldots (2r-1)\sigma^{2r}, (r = 0, 1, 2, \ldots)$.

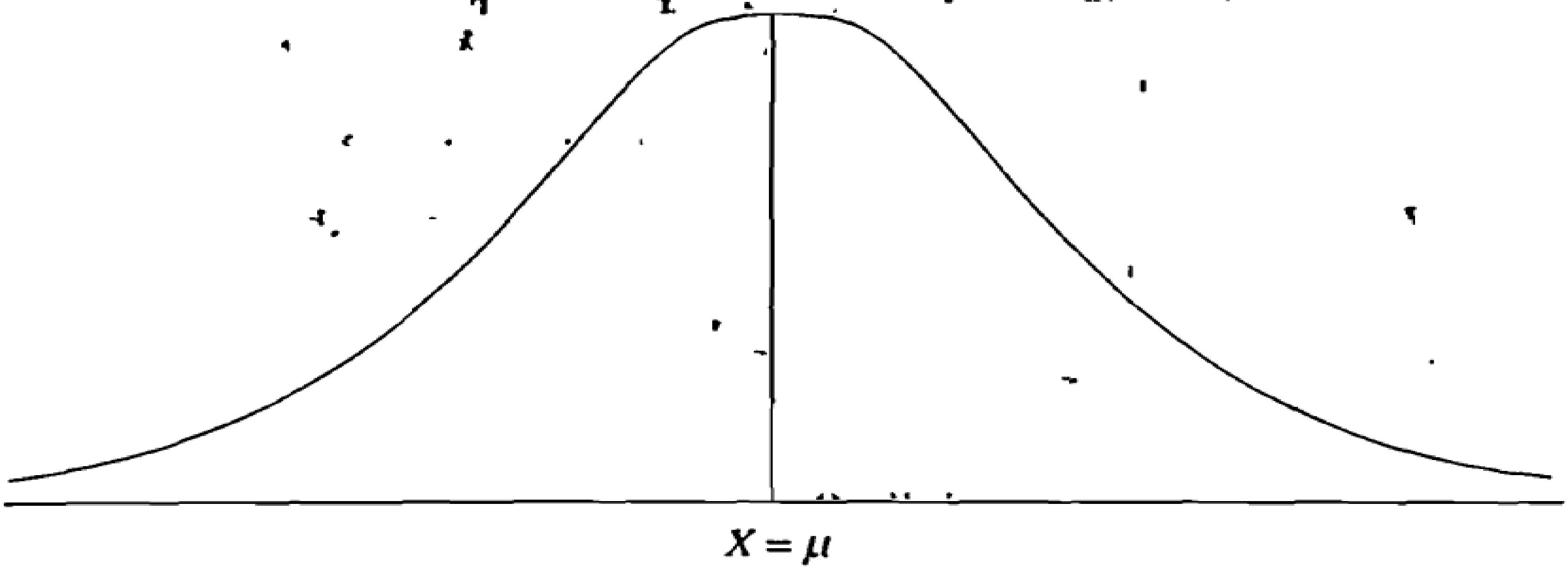
(*vi*) Since $f(x)$ being the probability, can never be negative, no portion of the curve lies below the x-axis.

(*vii*) Linear combination of independent normal variates is also a normal variate.

(*viii*) x-axis is an asymptote to the curve.

(*ix*) The points of inflexion of the curve are given by

$$\left[x = \mu \pm \sigma, f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}\right]$$



(*Normal Probability Cuve*)

PROBLEM *For a certain normal distribution, the first moment about 10 is 40 and the fourth moment about 50 is 48. What is the arithmetic mean and standard deviation of the distribution ?*

Solution. We know that if μ_1' is the first moment about the point $X = A$, then arithmetic mean is given by:

$$\text{Mean} = A + \mu_1'$$

We are given

μ_1' (about the point $X = 10$) $= 40$ $\Rightarrow$ Mean $= 10 + 40 = 50$

Also we are given

μ_4' (about the point $X = 50$) $= 48$, *i.e.*, $\mu_4 = 48$ $(\because \text{Mean} = 50)$

But for a normal distribution with standard deviation σ,

$$\mu_4 = 3\sigma^4 \Rightarrow 3\sigma^4 = 48 \quad i.e., \quad \sigma = 2$$

PROBLEM *X is normally distributed and the mean of X is 12 and S.D. is 4. (a) Find out the probability of the following :*

(i) $X \geq 20$, (ii) $X \leq 20$, and (iii) $0 \leq X \leq 12$

(b) *Find* x', *when* $P(X > x') = 0{\cdot}24$.

(c) *Find* x_0' *and* x_1', *when* $P(x_0' < X < x_1') = 0{\cdot}50$ *and* $P(X > x_1') = 0{\cdot}25$

Solution. (a) We have $\mu = 12, \sigma = 4$, *i.e.*, $X \sim N(12, 16)$.

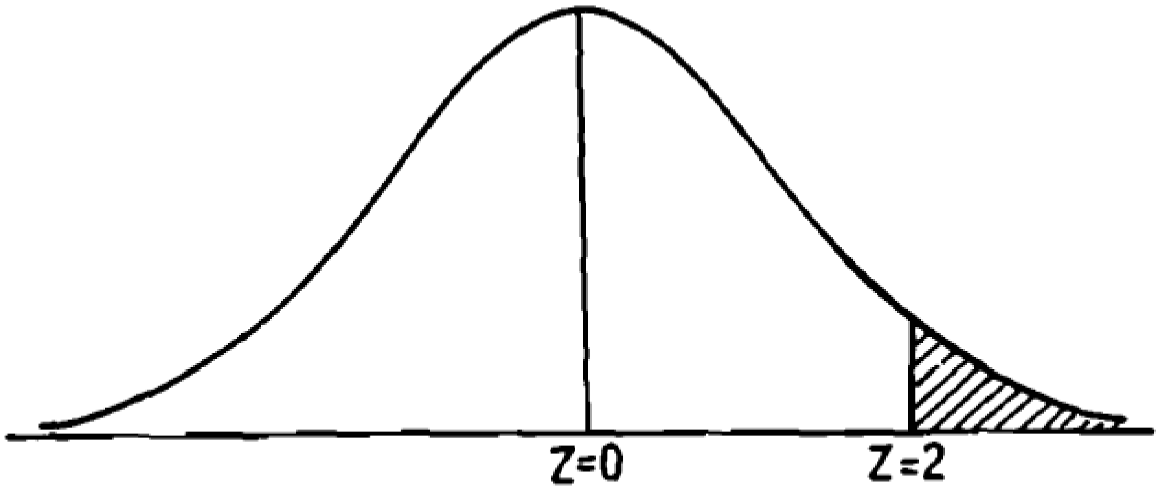
(i) $P(X \geq 20) = ?$

When $X = 20$, $Z = \dfrac{20 - 12}{4} = 2$

$\therefore \quad P(X \geq 20) = P(Z \geq 2) = 0{\cdot}5 - P(0 \leq Z \leq 2) = 0{\cdot}5 - 0{\cdot}4772 = 0{\cdot}0228$

(ii) $P(X \leq 20) = 1 - P(X \geq 20)$ ($\because$ Total probability = 1)

$= 1 - 0{\cdot}0228 = 0{\cdot}9772$



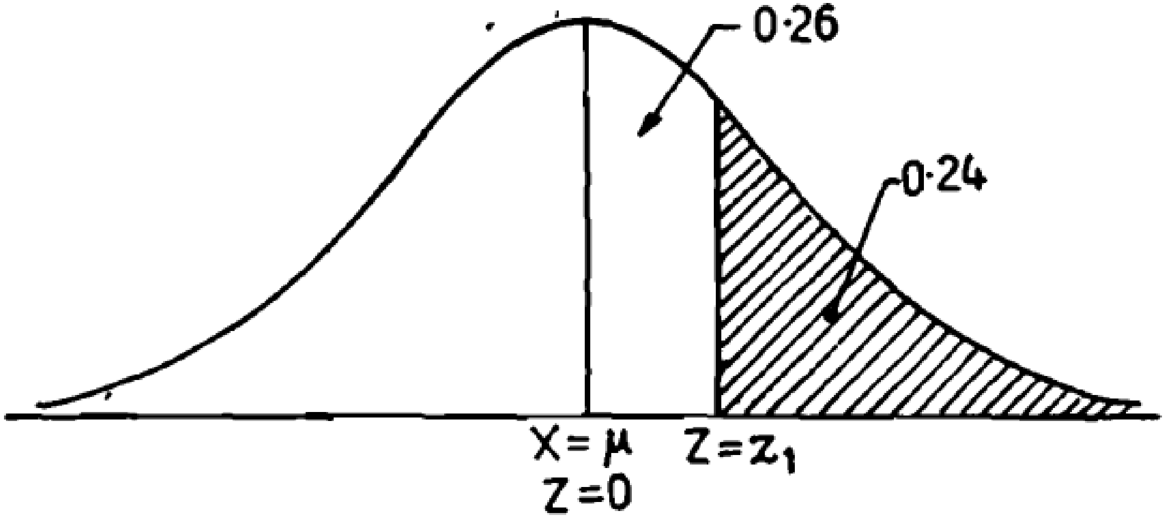
(*iii*) $P(0 \leq X \leq 12) = P(-3 \leq Z \leq 0)$ $\qquad \left(Z = \dfrac{X-12}{4} \right)$

$= P(0 \leq Z \leq 3) = 0{\cdot}49865$ (From symmetry)

(b) When $X = x'$, $Z = \dfrac{x' - 12}{4} = z_1$ (say)

then, we are given

$$P(X > x') = 0.24 \quad \Rightarrow \quad P(Z > z_1) = 0.24, \quad i.e., \quad P(0 < Z < z_1) = 0.26$$



$\therefore$ From normal tables,

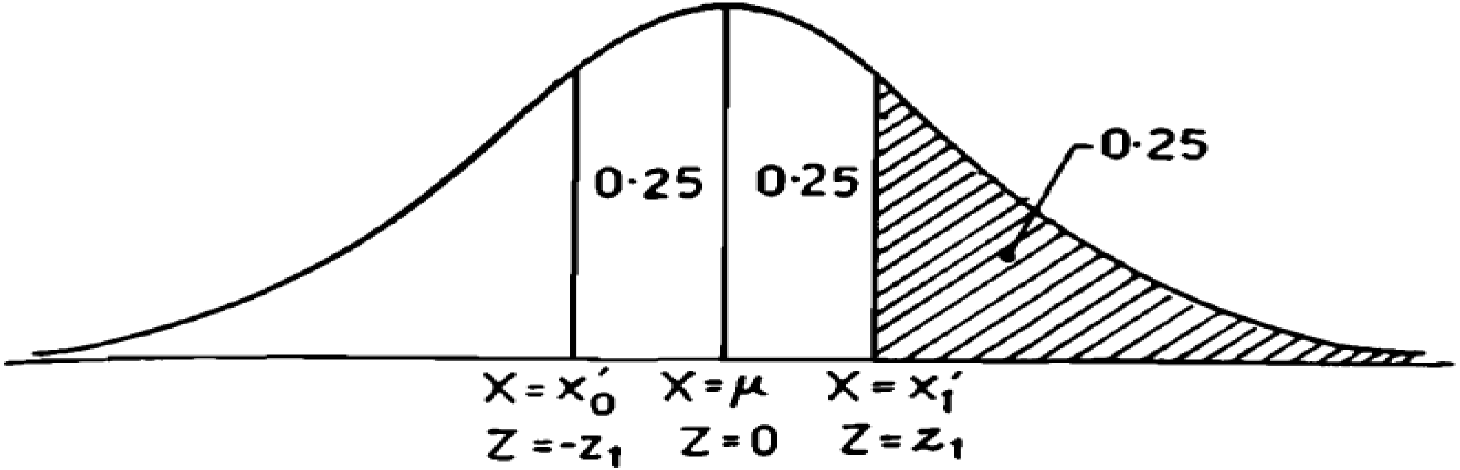
$$z_1 = 0.71 \text{ (approx.)}$$

Hence $\quad \dfrac{x_1' - 12}{4} = 0.71 \quad \Rightarrow \quad x_1' = 12 + 4 \times 0.71 = 14.84$

(c) We are given

$P(x_0' < X \leq x_1') = 0{\cdot}50$ and $P(X > x_1') = 0{\cdot}25$ ...(*)

From (*), obviously the points x_0' and x_1' are located as shown in the figure.



When $X = x_1'$, $Z = \dfrac{x_1' - 12}{4} = z_1$ (say)

and when $X = x_0'$, $Z = \dfrac{x_0' - 12}{4} = -z_1$ (It is obvious from the figure)

We have

$$P(Z > z_1) = 0.25 \quad \Rightarrow \quad P(0 < Z < z_1) = 0.25$$

$$\therefore \quad z_1 = 0.67 \qquad \text{(From tables)}$$

Hence

$$\frac{x_1' - 12}{4} = 0.67 \quad \Rightarrow \quad x_1' = 12 + 4 \times 0.67 = 14.68$$

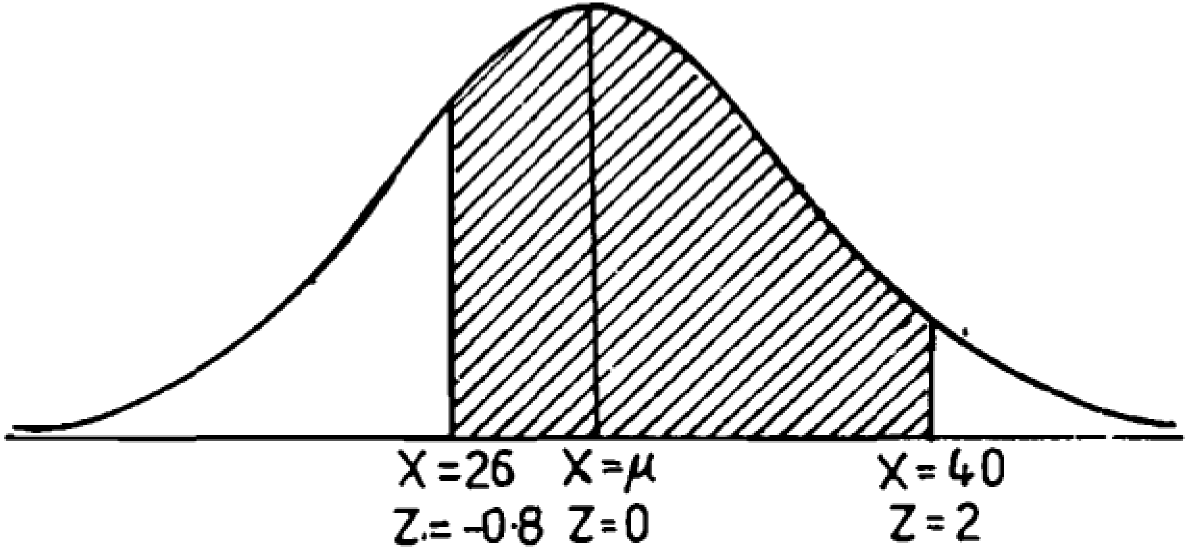
and

$$\frac{x_0' - 12}{4} = -0.67 \quad \Rightarrow \quad x_0' = 12 - 4 \times 0.67 = 9.32$$

PROBLEM *X is a normal variate with mean 30 and S.D. 5. Find the probabilities that*

(*i*) $26 \leq X \leq 40$, (*ii*) $X \geq 45$, and (*iii*) $|X - 30| > 5$.

Solution. Here $\mu = 30$ and $\sigma = 5$.

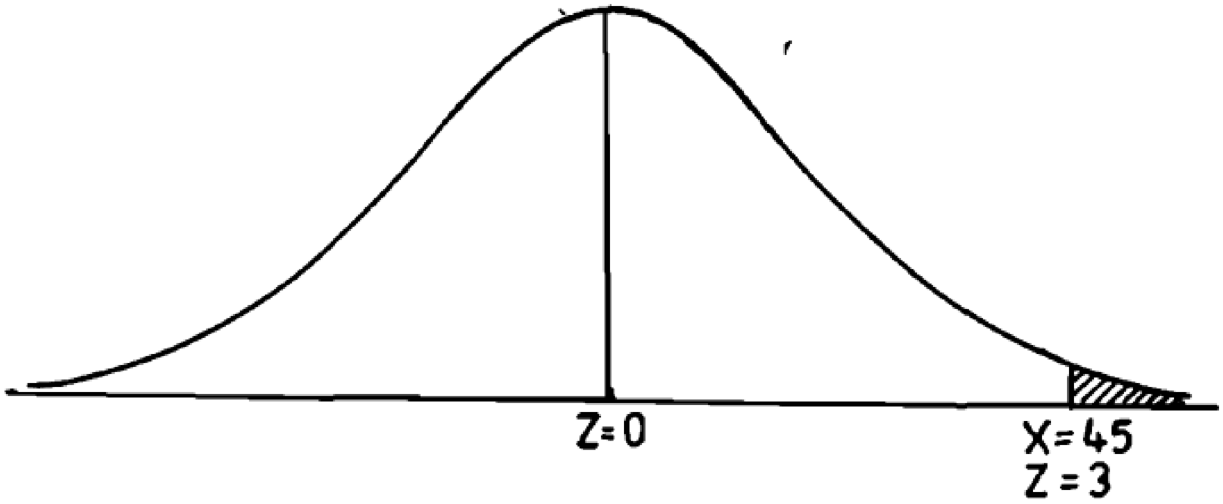


(*i*) When $X = 26$, $Z = \frac{X - \mu}{\sigma} = \frac{26 - 30}{5} = -0{\cdot}8$

and when $$X = 40,\ Z = \frac{40 - 30}{5} = 2$$

$$\begin{aligned} \therefore \quad P(26 \le X \le 40) &= P(-0{\cdot}8 \le Z \le 2) \\ &= P(-0{\cdot}8 \le Z \le 0) + P(0 \le Z \le 2) \\ &= P(-0{\cdot}8 \le Z \le 0) + 0{\cdot}4772 && \text{(From tables)} \\ &= P(0 \le Z \le 0{\cdot}8) + 0{\cdot}4772 && \text{(From symmetry)} \\ &= 0{\cdot}2881 + 0{\cdot}4772 = 0{\cdot}7653 \end{aligned}$$

$$P(X \ge 45) = ?$$



When $X = 45$, $\quad Z = \dfrac{45 - 30}{5} = 3$

$$\therefore \quad P(X \geq 45) = P(Z \geq 3) = 0{\cdot}5 - P(0 \leq Z \leq 3)$$
$$= 0{\cdot}5 - 0{\cdot}49865 = 0{\cdot}00135$$

(*iii*) $$P(|X - 30| \leq 5) = P(25 \leq X \leq 35) = P(-1 \leq Z \leq 1)$$
$$= 2\,P(0 \leq Z \leq 1) = 2 \times 0{\cdot}3413 = 0{\cdot}6826$$

$$\therefore \quad P(|X - 30| > 5) = 1 - P(|X - 30| \leq 5)$$
$$= 1 - 0{\cdot}6826 = 0{\cdot}3174$$

PROBLEM

The mean yield for one-acre plot is 662 kilos with a s.d. 32 kilos. Assuming normal distribution, how many one-acre plots in a batch of 1,000 plots would you expect to have yield (i) over 700 kilos, (ii) below 650 kilos, and (iii) what is the lowest yield of the best 100 plots?

Solution. If the r.v. X denotes the yield (in kilos) for one-acre plot, then we are given that $X \sim N(\mu, \sigma^2)$, where $\mu = 662$ and $\sigma = 32$.

(*i*) The probability that a plot has a yield over 700 kilos is given by

$$P(X > 700) = P(Z > 1 \cdot 19) \; ; \qquad Z = \frac{X - 662}{32}$$

$$= 0 \cdot 5 - P(0 \le Z \le 1 \cdot 19)$$

$$= 0 \cdot 5 - 0 \cdot 3830$$

$$= 0 \cdot 1170$$

Hence in a batch of 1.000 plots, the expected number of plots with yield over 700 kilos is $1{,}000 \times 0 \cdot 117 = 117$.

(*ii*) Required number of plots with yield below 650 kilos is given by

$$1000 \times P(X < 650) = 1000 \times P(Z < -0{\cdot}38) \qquad \left[Z = \frac{650 - 662}{32} \right]$$

$$= 1000 \times P(Z > 0{\cdot}38) \qquad \text{(By symmetry)}$$

$$= 1000 \times [0{\cdot}5 - P(0 \le Z \le 0{\cdot}38)]$$

$$= 1000 \times [0{\cdot}5 - 0{\cdot}1480] = 1000 \times 0{\cdot}352$$

$$= 352$$

(*iii*) The lowest yield, say, x_1 of the best 100 plots is given by

$$P(X > x_1) = \frac{100}{1000} = 0{\cdot}1$$

When $\quad X = x_1, \; Z = \dfrac{x_1 - \mu}{\sigma} = \dfrac{x_1 - 662}{32} = z_1 \text{ (say)} \qquad \ldots(*)$

such that $\quad P(Z > z_1) = 0{\cdot}1 \quad \Rightarrow \quad P(0 \le Z \le z_1) = 0{\cdot}4$

$\Rightarrow \quad z_1 = 1{\cdot}28$ (approx.) $\quad$ [From Normal Probability Tables]

Substituting in (*), we get

$$x_1 = 662 + 32\,z = 662 + 32 \times 1{\cdot}28$$

$$= 662 + 40{\cdot}96 = 702{\cdot}96$$

Hence the best 100 plots have yield over 702·96 kilos.

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997