COURSE MATERIAL	
SUBJECT	NUMERICAL METHODS AND PROBABILITY STATISTICS (20A54402)
UNIT	2
COURSE	B.TECH
SEMESTER	2 - 2
DEPARTMENT	HUMANITIES & SCIENCE
PREPARED BY (Faculty Name/s)	Department of Mathematics

8. LECTURE NOTES

Interpolation

Introduction:-

If we consider the statement $y = f(x) x_0 \le x \le x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0 \le x \le x_n$. If the function $f(x)$ is single valued and continuous and is known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, ...x_n$ can be calculated. The problem now is if we are given the set of tabular values

$$\begin{array}{llll} x : x_0 & x_1 & x_2 .... x_n \\ y : y_0 & y_1 & y_2 .... y_n \end{array}$$

Satisfying the relation $y = f(x)$ and the explicit definition of $f(x)$ is not known, then it is possible to find a simple function say $f(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process to finding $\phi(x)$ is called interpolation. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation

Errors in Polynomial Interpolation:-

Suppose the function $y(x)$ which is defined at the points $(x_i, y_i) i = 0,1,2,3 ----n$ is continuous and differentiable $(n+1)$ times let $\phi_n(x)$ be polynomial of degree not exceeding n such that $\phi_n(x_i) = y_i, i = 1,2 ---n \rightarrow (1)$ be the approximation of $y(x)$ using this $\phi_n(x_i)$ for other value of x, not defined by (1) the error is to be determined

Since $y(x) - \phi_n(x) = 0 \;\; for \;\; x - x_0, x_1, ...x_n$ we put

$$y(x) - \phi_n(x) = L\pi_{n+1}(x)$$

Where $\pi_{n+1}(x)=(x-x_0).........(x-x_n)\rightarrow(3)$ and L to be determined such that the equation (2) holds for any intermediate value of x such as $x=x^1, x_0<x^1<x_n$

Clearly $L=\dfrac{y(x^1)-\phi_n(x^1)}{\pi_{n+1}(x^1)}\rightarrow(4)$

We construct a function $F(x)$ such that $F(x)=F(x_n)=F(x^1)$. Then $F(x)$ vanishes $(n+2)$ times in the interval $[x_0,x_n]$. Then by repeated application of Rolle's theorem. $F^1(x)$ Must be zero $(n+1)$ times, $F^{11}(x)$ must be zero n times…….. in the interval $[x_0,x_n]$. Also $F^{n+1}(x)=0$ once in this interval. Suppose this point is $x=\varepsilon$, $x_0<\varepsilon<x_n$ differentiate (5) $(n+1)$ times with respect to x and putting $x=\varepsilon$, we get

$y^{n+1}(\varepsilon)-L(n+1)!=0$ Which implies that $L=\dfrac{y^{n+1}(\varepsilon)}{(n+1)!}$

Comparing (4) and (6) , we get

$$y(x^1)-\phi_n(x^1)=\frac{y^{n+1}(\varepsilon)}{(n+1)!}\pi_{n+1}(x^1)$$

This can be written as $y(x)-\phi_n(x)=\dfrac{\pi_{n+1}(x)}{(n+1)!}y^{n+1}(\varepsilon)$

This given the required expression $x_0<\varepsilon<x_n$ for error

2.1. Finite Differences:-

1. **Introduction:-**

In this chapter, we introduce what are called the forward, backward and central differences of a function $y=f(x)$. These differences and three standard examples of finite differences and play a fundamental role in the study of differential calculus, which is an essential part of numerical applied mathematics

2. **Forward Differences:-**

Consider a function $y=f\left(x\right)$ of an independent variable x. let $y_0, y_1, y_2, ...y_r$ be the values of y corresponding to the values $x_0, x_1, x_2....x_r$ of x respectively. Then the differences y_1-y_0, y_2-y_1----- are called the first forward differences of y, and we denote them by $\Delta y_0, \Delta y_1, ...$ that is

$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2.........$$

In general $\Delta y_r = y_{r+1} - y_r \therefore r = 0,1,2-----$

Here, the symbol Δ is called the forward difference operator

The first forward differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1......$ that is

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

In general $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad r = 0,1,2.......$ similarly, the n^th^ forward differences are defined by the formula.

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \quad r = 0,1,2.......$$

While using this formula for $n=1$, use the notation $\Delta^0 y_r = y_r$ and we have $\Delta^n y_r = 0 \forall n = 1,2...... \ and \ r = 0,2,.........$ the symbol Δ^n is referred as the n^th^ forward difference operator.

3. **Forward Difference Table:-**

The forward differences are usually arranged in tabular columns as shown in the following table called a forward difference table

Values of x	Values of y	First differences	Second differences	Third differences	Fourth differences
x_o	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
X4	y_4	$= y_4 - y_3$			

Example -finite forward difference table for $y = x^3$

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

4. Backward Differences:- As mentioned earlier, let $y_0, y_1 ...... y_r ......$ be the values of a function $y = f(x)$ corresponding to the values $x_0, x_1, x_2 .... x_r ......$ of x respectively. Then, $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, ...$ are called the first backward differences

In general $\nabla y_r = y_r - y_{r-1},\ r = 1,2,3.........\rightarrow(1)$

The symbol ∇ is called the backward difference operator, like the operator Δ, this operator is also a linear operator

Comparing expression (1) above with the expression (1) of section we immediately note that $\nabla y_r = \nabla y_{r-1}, r = 0,1,2.......\rightarrow(2)$

The first backward differences of the first background differences are called second differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3 ---\nabla^2{}_r ----$

i.e.,.. $\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2$ ....

In general $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, r = 2,3.....\rightarrow(3)$ similarly, the nth backward differences are defined by the formula $\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, r = n, n+1.....\rightarrow(4)$ While using this formula, for n = 1 we employ the notation $\nabla^0 y_r = y_r$

If $y = f(x)$ is a constant function, then y = c is a constant, for all x, and we get $\nabla^n y_r = 0 \forall n$ the symbol ∇^n is referred to as the nth backward difference operator

5. Backward Difference Table:-

X	Y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		∇y_1 =y1-y0		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2 **=y2-y1**		$\nabla^3 y3$
x_2	y_2		$\nabla^2 y_3$	
		∇y_3 =y3-y2		
x_3	y_3			

6. Central Differences:-

With $y_0, y_1, y_2....y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2.....x_r....\ of\ x$, we define the first central differences

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2} ----$ as follows

$$\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \delta y_{5/2} = y_3 - y_2 ----$$

$$\delta y_{r-1/2} = y_r - y_{r-1} \rightarrow (1)$$

The symbol δ is called the central differences operator. This operator is a linear operator comparing expressions (1) above with expressions earlier used on forward and backward differences we get

$$\delta y_{1/2} = \Delta y_0 = \nabla y_1, \delta y_{3/2} = \Delta y_1 = \nabla y_2.....$$

In general $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}, n = 0,1,2...... \rightarrow (2)$

The first central differences of the first central differences are called the second central differences and are denoted by $\delta^2 y_1, \delta^2 y_2...$

Thus $\delta^2 y_1 = \delta_{3/2} - \delta y_{1/2}, \delta^2 y_2 = \delta_{5/2} - \delta_{3/2}.......$

$$\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2} \rightarrow (3)$$

Higher order central differences are similarly defined. In general the n[th] central differences are given by

i) for odd $n : \delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}, r = 1,2.... \rightarrow (4)$

ii) for even $n : \delta^n y_r = \delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2}, r = 1,2.... \rightarrow (5)$

while employing for formula (4) for $n = 1$, we use the notation $\delta^0 y_r = y_r$

If y is a constant function, that is if $y = c$ a constant, then $\delta^n y_r = 0\ \ for\ \ all\ \ n \geq 1$

7. Central Difference Table

x_0	y_0	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{2/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Example: Given $f(-2)=12, f(-1)=16, f(0)=15, f(1)=18, f(2)=20$ from the central difference table and write down the values of $\delta y_{3/2}, \delta^2 y_0$ and $\delta^3 y_{7/2}$ by taking $x_0 = 0$

Sol. The central difference table is

x	$y = f(x)$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
-2	12				
		4			
-1	16		-5		
		-1		9	
0	15		4		-14
		3		-5	
1	18		-1		

		2			
2	20				

5. Symbolic Relations and Separation of symbols:

We will define more operators and symbols in addition to Δ, ∇ and δ already defined and establish difference formulae by symbolic methods

Averaging Operator-Definition:- The averaging operator μ is defined by the equation

$$\mu y_r = \frac{1}{2}\left[y_{r+1/2} + y_{r-1/2}\right]$$

Shift Operator-Definition:- The shift operator E is defined by the equation $Ey_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives $E^2 y_r = E\left(Ey_r\right) = E\left(y_{r+1}\right) = y_{r+2}$

Generalizing $E^n y^r = y_{r+n}$

Relationship Between Δ *and* E

We have
$$\begin{aligned} \Delta y_0 &= y_1 - y_0 \\ &= Ey_0 - y_0 = \left(E-1\right)y_0 \\ \Rightarrow \Delta &= E - y\left(or\right)E = 1 + \Delta \end{aligned}$$

$$\begin{aligned} \Delta y_0 &= y_1 - y_0 \\ &= Ey_0 - y_0 = \left(E-1\right)y_0 \\ \Rightarrow \Delta &= E - y\left(or\right)E = 1 + \Delta \end{aligned}$$

Some more relations

$$\begin{aligned} \Delta^3 y_0 &= \left(E-1\right)^3 y_0 = \left(E^3 - 3E^2 + 3E - 1\right)y_0 \\ &= y_3 - 3y_2 + 3y_1 - y_0 \end{aligned}$$

Inverse Shift Operator-Definition

Inverse operator E^{-1} is defined as $E^{-1}y_r = y_{r-1}$

In general $E^{-n}y_n = y_{r-n}$

We can easily establish the following relations

i) $\nabla \equiv 1 - E^{-1}$

ii) $\delta \equiv E^{1/2} - E^{-1/2}$

iii) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

iv) $\Delta = \nabla E = E^{1/2}$

v) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

Differential Operator-Definition The operator D is defined as $Dy(x) = \frac{\partial}{\partial x}\left[y(x)\right]$

Relation between the Operators D and E

Using Taylor's series we have, $y(x+h) = y(x) + hy^1(x) + \frac{h^2}{2!}y^{11}(x) + \frac{h^3}{3!}y^{111}(x) + ----$

This can be written in symbolic form

$$Ey_x = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ----\right]y_x = e^{hD}.y_x$$

We obtain in the relation $E = e^{hD} \rightarrow (3)$

❖ If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is constant

Proof:

Let $f(x) = a_0x^n + a_1x^{n-1} + ------ + a_{n-1}x + a_n$ where $a_0, a_1, a_2 .... a_n$ are constants and $a_0 \neq 0$. If h is the step- length, we know the formula for the first forward difference

$$\Delta f(x) = f(x+h) - f(x) = \left[a_0(x+h)^n + a_1(x+h)^{n-1} + ----- + a_{n-1}(x+h) + a_n\right]$$
$$-\left[a_0x^n + a_1x^{n-1} + ----- + a_{n-1}x + a_n\right]$$

$$= a_0\left[\left\{x^n + n.x^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}.h^2 + ---\right\} - x^n\right] +$$
$$a_1\left[\left\{x^{n-1} + (n-1)x^{n-2}.h + \frac{(n-1)(n-2)}{2!}x^{n-3}.h^2 + ---\right\} - x^{n-1}\right] +$$
$$------ + a_{n-1}h$$
$$= a_0 nhx^{n-1} + b_2x^{n-2} + b_3x^{n-3} + ---- + b_{n-3}x + b_{n-2}$$

Where $b_2, b_3, ...b_{n-2}$ are constants. Here this polynomial is of degree $(n-1)$, thus, the first difference of a polynomial of n^th degree is a polynomial of degree $(n-1)$

Now

$$\Delta^2 f(x) = \Delta[\Delta f(x)]$$
$$= \Delta\left[a_0 nh.x^{n-1} + b_2x^{n-2} + b_3x^{n-3} + ---- + b_{n-1}x + b_{n-2}\right]$$
$$= a_0 nh\left[(x+h)^{n-1} - x^{n-1}\right] + b_2\left[(x+h)^{n-2} - x^{n-2}\right] + --- + b_{n-1}\left[(x+h) - x\right]$$
$$= a_0 n^{(n-1)}h^2x^{n-2} + c_3x^{n-3} + ------ + c_{n-4}x + c_{n-3}$$

Where $c_3....c_{n-3}$ are constants. This polynomial is of degree $(n-2)$

Thus, the second difference of a polynomial of degree n is a polynomial of degree $(n-2)$ continuing like this we get $\Delta^n f(x) = a_0 n(n-1)(n-2)-----2.1.h^n = a_0 h^n (n!)$

$\therefore$ which is constant

Note:-

1. As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1} f(x) = 0, \Delta^{n+2} f(x) = 0, ...$
2. The converse of above result is also true that is, if $\Delta^n f(x)$ is tabulated at equal spaced intervals and is a constant, then the function $f(x)$ is a polynomial of degree n

Example:-

1. Form the forward difference table and write down the values of $\Delta f(10)$, $\Delta^2 f(10), \Delta^3 f(15)$ *and* $\Delta^4 y(15)$

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

Sol.

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	**19.97f(10)**					
		1.54$\Delta f(10)$				
15	**21.51f(15)**		**-0.58**$\Delta^2 f(10)$			
		0.96		0.67		
20	**22.47f(20)**		0.09		- 0.68	
		1.05		**- 0.01**$\Delta^3 f(15)$		0.72
25	**23.52f(25)**		0.08		**0.04** $\Delta^4 f(15)$	
		1.13		0.03		
30	**24.65f(30)**		0.11			
		1.24				
35	**25.89f(35)**					

We note that the values of x are equally spaced with step- length h = 5

Note: - $\therefore x_0 = 10, x_1 = 15 ----x_5 = 35$ and

$$y_0 = f(x_0) = 19.97$$
$$y_1 = f(x_1) = 21.51$$

$$y_5 = f(x_5) = 25.89$$

$$y_5 = f(x_5) = 25.89$$

From table

$$\Delta f(10) = \Delta y_0 = 1.54$$
$$\Delta^2 f(10) = \Delta^2 y_0 = -0.58$$
$$\Delta^3 f(15) = \Delta^3 y_1 = -0.01$$
$$\Delta^4 f(15) = \Delta^3 y_1 = 0.04$$

2. **Evaluate**

$(i)\Delta \cos x$
$(ii)\Delta^2 \sin(px+q)$
$(iii)\Delta^n e^{ax+b}$

Sol. Let h be the interval of differencing

$$(i)\Delta \cos x = \cos(x+h) - \cos x$$
$$= -2\sin\left(x+\frac{h}{2}\right)\sin\frac{\mathrm{h}}{2}$$
$$(ii)\Delta \sin(px+q) = \sin\left[p(x+h)+q\right] - \sin(px+q)$$ $\Delta f(x) = f(x+h) - f(x)$,for ward formula
$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}$$
$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$\nabla f(x) = f(x) - f(x-h)$ back ward formula

$$\Delta^2 \sin(px+q) = 2\sin\frac{ph}{2}\Delta\left[\sin(px+q)+\frac{1}{2}(\pi+ph)\right]$$

$$= \left[2\sin\frac{ph}{2}\right]^2 \sin\left[px+q+\frac{1}{2}(\pi+ph)\right]$$

$$(iii)\,\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$$
$$= e^{(ax+b)}\left(e^{ah-1}\right)$$
$$\Delta^2 e^{ax+b} = \Delta\left[\Delta\left(e^{ax+b}\right)\right] - \Delta\left[\left(e^{ah}-1\right)\left(e^{ax+b}\right)\right]$$
$$= \left(e^{ah}-1\right)^2 \Delta\left(e^{ax+h}\right)$$
$$= \left(e^{ah}-1\right)^2 e^{ax+b}$$

Proceeding on, we get $\Delta^n\left(e^{ax+b}\right) = \left(e^{ah}-1\right)^n e^{ax+b}$

3. Using the method of separation of symbols show that

$$\Delta^n \mu_{x-n} = \mu_{x-n} - n\mu_{x-1} + \frac{n(n-1)}{2}\mu_{x-2} + ----+(-1)^n \mu_{x-n}$$

Sol. To prove this result, we start with the right hand side. Thus

$$\mu x - n\mu x - 1 + \frac{n(n-1)}{2}\mu x - 2 + ------+(-1)^n \mu x - n$$
$$= \mu x - nE^{-1}\mu x + \frac{n(n-1)}{2}E^{-2}\mu x + ------+(-1)^n E^{-n}\mu x$$
$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2}E^{-2} + ------+(-1)^n E^{-n}\right]\mu x = \left(1-E^{-1}\right)^n \mu x$$
$$= \left(1-\frac{1}{E}\right)^n \mu n = \frac{(E-1)^n}{E}\mu n$$
$$= \frac{\Delta^n}{E^n}\mu x = \Delta^n E^{-n}\mu x$$

$= \Delta^n \mu_{x-n}$ This is left hand side

4. **Find the missing term in the following data**

X	0	1	2	3	4
Y	1	3	9	-	81

Why this value is not equal to 3^3. Explain

Sol. Consider $\Delta^4 y_0 = 0$

x	Y	Δ	Δ2	Δ3	Δ4	
0	1	2	4	x-19	124-4x=0	
1	3	6	x-15	105-3x		
2	9	X-9	90-2x			
3	**X**	81-x				
4	81					

$\Rightarrow 4y_0 - 4y_3 + 5y_2 - 4y_1 + y_0 = 0$ 124-4x=0— **x=124/4=31**

Substitute given values we get

$81 - 4y_3 + 54 - 12 + 1 = 0 \Rightarrow y_3 = 31$

From the given data we can conclude that the given function is $y = 3^x$. To find y_3, we have to assume that y is a polynomial function, which is not so. Thus we are not getting $y = 3^3 = 27$

2.2. a. Newton's Forward Interpolation Formula:-

Let $y = f(x)$ be a polynomial of degree n and taken in the following form

$$y = f(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2) + --- \\ + b_n(x-x_0)(x-x_1) ----(x-x_{n-1}) \to (1)$$

This polynomial passes through all the points $[xi; yi]$ for i = 0 to n. there fore, we can obtain the y_i's by substituting the corresponding x_i's as

$$at\ \ x = x_0, y_0 = b_0$$
$$at\ \ x = x_1, y_1 = b_0 + b_1(x_1 - x_0)$$
$$at\ \ x = x_2, y_2 = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \to (1)$$

Let 'h' be the length of interval such that x_i's represent

$x_0, x_0+h, x_0+2h, x_0+3h----x_0+xh$

This implies $x_1 - x_0 = h, x_2 - x_0 - 2h, x_3 - x_0 = 3h ----x_n - x_0 = nh \rightarrow (2)$

From (1) and (2), we get

$$
\begin{aligned}
&y_0 = b_0 \\
&y_1 = b_0 + b_1 h \\
&y_2 = b_0 + b_1 2h + b_2 (2h) h \\
&y_3 = b_0 + b_1 3h + b_2 (3h)(2h) + b_3 (3h)(2h) h \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&y_n = b_0 + b_1 (nh) + b_2 (nh)(n-1)h + ---+ b_n (nh)\left[(n-1)h\right]\left[(n-2)h\right] \rightarrow (3)
\end{aligned}
$$

Solving the above equations for $b_0, b_{11}, b_2 \ldots\ldots b_n$, we get $b_0 = y_0$

$$
b_1 = \frac{y_1 - b_0}{h} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}
$$

$$
b_2 = \frac{y_2 - b_0 - b_1 2h}{2h^2} = y_2 - y_0 - \frac{(y_1 - y_0)}{h} 2h
$$

$$
= \frac{y_2 - y_0 - 2y_1 - 2y_0}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}
$$

$$
\therefore b_2 = \frac{\Delta^2 y_0}{2!h^2}
$$

Similarly, we can see that

$$
b_3 = \frac{\Delta^3 y_0}{3!h^3}, b_4 = \frac{\Delta^4 y_0}{4!h^4} -----b_n = \frac{\Delta^n y_0}{n!h^n}
$$

$$
\therefore y = f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1)
$$

$$
+ \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + ----+
$$

$$
+ \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1) ---(x - x_{n-1}) \rightarrow (3)
$$

If we use the relationship $x = x_0 + ph \Rightarrow x - x_0 = ph,\ where\ p = 0,1,2,\ldots\ldots n$

Then

$$x-x_1=x-(x_0+h)=(x-x_0)-h$$
$$=ph-h=(p-1)h$$
$$x-x_2=x-(x_1+h)=(x-x_1)-h$$
$$=(p-1)h-h=(p-2)h$$
....
$$x-x_i=(p-i)h$$
....
$$x-x_{n-1}=[p-(n-1)]h$$

Equation (3) becomes for $p=\dfrac{x-x_0}{h}$

$$y=f(x)=f(x_0+ph)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+----+$$
$$\frac{p(p-1)(p-2)----(p-(n-1))}{n!}\Delta^n y_0\rightarrow(4)$$

2.2. b. Newton's Backward Interpolation Formula:-

If we consider

$$y_n(x)=a_0+a_1(x-x_n)+a_2(x-x_n)(x-x_{n-1})+a_3(x-x_n)(x-x_{n-1})(x-x_{n-2})+----(x-x_i)$$

and impose the condition that y and $y_n(x)$ should agree at the tabulated

points $x_n, x_n-1, ...x_2, x_1, x_0$

We obtain

$$y_n(x)=y_n+p\nabla y_n+\frac{p(p+1)}{2i}\nabla^2 y_n+---$$
$$\frac{p(p+1)----[p+(n-1)]}{n!}\nabla^n y_n+----\rightarrow(6)$$

Where $p=\dfrac{x-x_n}{h}$

This uses tabular values of the left of y_n . Thus this formula is useful formula is useful for interpolation near the end of the table values

Formula for Error in Polynomial Interpolation:-

If $y = f(x)$ is the exact curve and $y = \phi_n(x)$ is the interpolating curve, then the error in polynomial interpolation is given by

$$Error = f(x) - \phi_n(x) = \frac{(x-x_0)(x-x_1)----(x-x_n)}{(n+1)!} f^{n+1}(\varepsilon) \rightarrow (7)$$

for any x, where $x_0 < x < x_n$ and $x_0 < \varepsilon < x_n$

The error in Newton's forward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p-1)(p-2).......(p-n)}{(n+1)!} \Delta^{n+1} f(\varepsilon)$$

Where $p = \frac{x - x_0}{h}$

The error in Newton's backward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p+1)(p+2).......(p+n)}{(n+1)!} h^{n+1} y^{n+1} f(\varepsilon) \text{ Where } p = \frac{x - x_n}{h}$$

Examples:-

1. Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula

Percentage of lead(p)-X	50	60	70	80
Temperature $(Q^\circ c)$-Y	205	225	248	274

Sol. The difference table is

X	Y	Δ	Δ^2	Δ^3
50=X_0	**205=Y_0**			
		20=$\Delta Y0$		
60	225		**3=$\Delta^2 Y_0$**	
		23		**0=$\Delta^3 Y_0$**
70	248		3	
		26		
80	274			

Let temperature = $f(x)$,X=54

$$x_0 + ph = 24, x_0 = 50, h = 10$$
$$50 + p(10) = 54 \ (or)\ p = 0.4$$

By Newton's forward interpolation formula

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{n!}\Delta^3 y_0 + ---$$
$$f(54) = 205 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(3) + \frac{(0.4)(0.4-1)(0.4-2)}{3!}(0)$$
$$= 205 + 8 - 0.36$$
$$= 212.64$$

Melting point = 212.64

2. Using Newton's forward interpolation formula, and the given table of values

X	1.1	1.3	1.5	1.7	1.9
$f(x)$	0.21	0.69	1.25	1.89	2.61

Obtain the value of $f(x)$ *when* $x = 1.4$

Sol.

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4
1.1	**0.21**				
		0.48			
1.3	0.69		**0.08**		
		0.56		0	
1.5	1.25		0.08		0
		0.64		0	
1.7	1.89		0.08		
		0.72			
1.9	2.61				

If we take $x_0 = 1.3$ *then* $y_0 = 0.69$,

$$\Delta y_0 = 0.56, \Delta^2 y_0 = 0.08, \Delta^3 y_0 = 0, L = 0.2, x = 1.3$$
$$x_0 + ph = 1.4(or)1.3 + p(0.2) = 1.4, p = \frac{1}{2}$$

Using Newton's interpolation formula

$$f(1.4) = 0.69 + \frac{1}{2} \times 0.56 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!} \times 0.08$$
$$= 0.69 + 0.28 - 0.01 = 0.96$$

3. The population of a town in the decimal census was given below. Estimate the population for the 1895

Year x	1891	1901	1911	1921	1931
Population of y	46	66	81	93	101

Sol. Putting $L=10, x_0=1891, x=1895$ in the formula $x=x_0+ph$ we obtain $p=2/5=0.4$

X	Y	Δ	Δ^2	Δ^3	Δ^4
1891=x0	46=y0				
		20$\Delta y0$			
1901	66		**-5$\Delta^2 Y0$**		
		15		**2**	
1911	81		-3		**-3**
		12		-1	
1921	93		-4		
		8			
1931	101				

$$y(1895)=46+(0.4)(20)+\frac{(0.4)(0.4-1)}{6}-(-5)$$
$$+\frac{(0.4-1)0.4(0.4-2)}{6}(2)$$
$$+\frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}$$
$$=54.45\ thousands$$

2.3. Gauss's Interpolation Formula:- We take x_0 as one of the specified of x that lies around the middle of the difference table and denote x_0-rh by $x-r$ and the corresponding value of y by $y-r$. Then the middle part of the forward difference table will appear as shown in the next page

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-4}	y_{-4}					
x_{-3}	y_{-3}	Δy_{-4}				
x_{-2}	y_{-2}	Δy_{-3}	$\Delta^2 y_{-4}$			
x_{-1}	y_{-1}	Δy_{-2}	$\Delta^2 y_{-3}$	$\Delta^3 y_{-4}$		
x_0	y_0	Δy_{-1}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-4}$	
x_1	y_1	Δy_0	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-4}$
x_2	y_2	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-3}$
x_3	y_3	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_{-1}$	$\Delta^5 y_{-2}$
x_4	y_4	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_0$	$\Delta^5 y_{-1}$

$$
\begin{aligned}
&\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1} \\
&\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \\
&\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \\
&\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} -----(1) \;\; \textit{and} \\
&\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2} \\
&\Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2} \\
&\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2} \\
&\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} -----(2)
\end{aligned}
$$

By using the expressions (1) and (2), we now obtain two versions of the following Newton's forward interpolation formula

$$
y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + ----] \quad \text{-----------------3}
$$

Here y_p is the value of y at $x = x_p = x_0 + ph$,**P= (x-x₀)/h**

2.3.a. Gauss Forward Interpolation Formula:-

Substituting for $\Delta^2 y_0, \Delta^3 y_0, ...$ from (1)in the formula (3), we get

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1}$$
$$+\Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{-1} + \Delta^5 y_{-1} + ----]$$

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1}) + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$
$$+\frac{p(p+1)(p-1)(p-2)}{4!}(\Delta^4 y_{-1}) + ----]$$

Substituting $\Delta^4 y_{-1}$ *from* (2), this becomes

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+\frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + ----] \quad \text{-----------------4}$$

Note:- we observe from the difference table that

$\Delta y_0 = \delta y_{1/2}, \Delta^2 y_{-1} = \delta^2 y_0, \Delta^3 y_{-1} = \delta^3 y_{1/2}, \Delta^4 y_{-2} = \delta^4 y_0$ and so on. Accordingly the formula

(4) can be written in the notation of central differences as given below

$$y_p = [y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2}$$
$$+\frac{(p+1)(p-1)p(p-2)}{4!}\delta^4 y_0 + ----] \quad \text{-----------------5}$$

2.3.b. Gauss's Backward Interpolation formula:-

Let us substitute for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ ----- from (1) in the formula (3), thus we obtain

$$y_p = [y_0 + p\left(\Delta y_{-1} + \Delta^2 y_{-1}\right) + \frac{p(p-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right) + \frac{(p-1)p(p-2)}{3!}\left(\Delta^3 y_{-1} + \Delta^4 y_{-1}\right) + \frac{(p-1)(p-2)p(p-3)}{4!}\left(\Delta^4 y_{-1} + \Delta^5 y_{-1}\right) + ----]$$

$$= [y_0 + p\left(\Delta y_{-1}\right) + \frac{(p+1)}{2!}p\left(\Delta^2 y_{-1}\right) + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\left(\Delta^4 y_{-1}\right) + ----]$$

Substituting for $\Delta^3 y_{-1}$ and $\Delta^4 y_{-1}$ from (2) this becomes

$$y_p = [y_0 + p\left(\Delta y_{-1}\right) + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\left(\Delta^3 y_{-1} + \Delta^4 y_{-2}\right) + \frac{(p+1)p(p-1)(p-2)}{4!}\left(\Delta^4 y_{-2} + \Delta^5 y_{-2}\right) + ----]$$

2.4. Lagrange's Interpolation Formula:-

Let $x_0, x_1, x_2, \ldots.\ x_n$ be the $(n+1)$ values of x which are not necessarily equally spaced. Let $y_0, y_1, y_2 \ldots\ldots.. y_n$ be the corresponding values of $y = f(x)$ let the polynomial of degree n for the function $y = f(x)$ passing through the $(n+1)$ points $\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right) ---- \left(x_n, f(x_n)\right)$ be in the following form

$$y = f(x) = a_0(x-x_1)(x-x_2)\ldots\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\ldots\ldots\ldots(x-x_n) + a_2(x-x_0)(x-x_1)\ldots\ldots.(x-x_n) + \ldots\ldots.. + a_n(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1}) \to (1)$$

Where $a_0, a_1, a_2 \ldots.$ a^n are constants

Since the polynomial passes through $\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right)\ldots\ldots\left(x_n, f(x_n)\right)$. The constants can be determined by substituting one of the values of $x_0, x_1, \ldots..x_n$ *for* x in the above equation

Putting $x = x_0$ in (1) we get, $f(x_0) = a_0(x-x_1)(x_0-x_2)(x_0-x_n)$

$$\Rightarrow a_0 = \frac{f(x_0)}{(x-x_1)(x_0-x_2)\ldots.(x_0-x_n)}$$

Putting $x = x_1$ in (1) we get, $f(x_1) = a_1(x - x_0)(x_1 - x_2) - - - - (x_1 - x_n)$

$$\Rightarrow a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)....(x_1 - x_n)}$$

Similarly substituting $x = x_2$ in (1), we get

$$\Rightarrow a_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)......(x_2 - x_n)}$$

Continuing in this manner and putting $x = x_n$ in (1) we

get $a_n = \dfrac{f(x_n)}{(x_n - x_0)(x_n - x_1) - - - - (x_n - x_{n-1})}$

Substituting the values of $a_0, a_1, a_2....a_n$, we get

$$f(x) = \frac{(x - x_1)(x - x_2).......(x - x_n)}{(x_0 - x_1)(x_0 - x_2).........(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2).....(x - x_n)}{(x_1 - x_0)(x_1 - x_2)....(x_1 - x_n)}$$

$$f(x_1) + \frac{(x - x_0)(x - x_1)(x - x_2).....(x - x_n)}{(x_2 - x_0)(x_2 - x_1)......(x_2 - x_n)} + ...f(x_2) + \frac{(x - x_0)(x - x_1).....(x - x_{n-1})}{(x_n - x_1)(x_n - x_2).....(x_n - x_{n-1})} f(x_n)$$

Examples:-

1. Using Lagrange's formula calculate $f(3)$ from the following table

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Sol. Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_5 = 6, x_4 = 5$

$$f(x_0) = 1, f(x_1) = 14, f(x_2) = 15, f(x_3) = 5, f(x_4) = 6, f(x_5) = 19$$

From langrage's interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} f(x_0)$$
$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} f(x_2)$$

$$-------------------$$

$$-------------------$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} f(x_5)$$

Here $x = 3$ then

$$f(3) = \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \times 1 +$$
$$\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \times 14 +$$
$$\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times 15 +$$

$$\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} \times 5 +$$
$$\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times 6 +$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} \times 19$$
$$= \frac{12}{240} - \frac{18}{60} \times 14 + \frac{36}{48} \times 15 + \frac{36}{48} \times 5 - \frac{18}{60} \times 6 + \frac{12}{40} \times 19$$

$$= 0.05 - 4.2 + 11.25 + 3.75 - 1.8 + 0.95$$

$$= 10$$

f(3)=10

2. Find $f(3.5)$ using Lagrange method of 2^{nd} and 3^{rd} order degree polynomials.

x	1	2	3	4
$f(x)$	1	2	9	28

Sol: By lagrange's interpolation formula

$$f(x)=\sum_{k=0}^{n} f(x_k)\frac{(x-x_0).......(x-x_{k-1})(x-x_{k+1})(x-x_n)}{(x_k-x_0).......(x_k-x_{k-1}).......(x_k-x_n)}$$

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+$$
$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)+$$

$$\therefore f(3.5)=\frac{(3.5-2)(3.5-3)(3.5-4)}{(1-2)(1-3)(1-4)}(1)+\frac{(3.5-1)(3.5-3)(3.5-4)}{(2-1)(2-3)(2-4)}(2)+$$

$$\frac{(3.5-1)(3.5-2)(3.5-4)}{(3-1)(3-2)(3-4)}(9)+$$
$$\frac{(3.5-1)(3.5-2)(3.5-3)}{(4-1)(4-2)(4-3)}(28)+$$

- - - -

=0.0625+(-0.625)+8.4375+8.75

=16.625

$$f(x)=\frac{(x-2)(x-3)(x-4)}{-6}(1)+\frac{(x-1)(x-3)(x-4)}{2}(2)$$
$$+\frac{(x-1)(x-2)(x-4)}{(-2)}(9)+\frac{(x-1)(x-2)(x-3)}{6}(28)$$

$$=\frac{(x^2-5x+6)(x-4)}{-6}+(x^2-4x+3)(x-4)+\frac{(x^2-3x+2)}{-2}(x-4)(9)+\frac{(x^2-3x+2)}{6}(x-3)(28)$$

$$=\frac{x^3-9x^2+26x-24}{-6}+x^3-8x^2+9x-12+\frac{x^3-7x^2+14x-8}{-2}(9)+\frac{x^3-6x^2+11x-6}{6}(28)$$

$$=\frac{\left[-x^3+9x^2-26x+24+6x^3-48x^2+114x-72-27x^3+189x^2-378x+216+308x+28x^3-168x^2-168\right]}{6}$$

$$=\frac{6x^3-18x^2+18x}{6}\Rightarrow f(x)=x^3-3x^2+3x$$

$$\therefore f(3.5)=(3.5)^3-3(3.5)^2+3(3.5)=16.625$$

Gauss Formula Example:

1. Find y(25), given that $y_{20} = 24, y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$ using Gauss forward difference formula :

Solution: Given

X	20	24	28	32
Y	24	32	35	40

By Gauss Forward difference formula

$$y_p=[y_0+p(\Delta y_0)+\frac{p(p-1)}{2!}\Delta^2 y_{-1}+\frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+\frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2})+----]\rightarrow(4)$$

We take $x_0 = 24$ as origin.

$X_0 = 24$, $h = 4$, $x = 25$, $p = (x-x_0)/h$, $p = (25-24)/4 = 0.25$

Gauss Forward difference table is

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
20=x-1	24=y-1			
24=x0	**32=y0**	$\Delta y_{-1} = 8$		
28=x1	35=y1	$\Delta y_0 = 3$	$\Delta^2 y_{-1} = -5$	
32=x2	40=y2	$\Delta y_1 = 5$	$\Delta^2 y_0 = 2$	$\Delta^3 y_{-1} = 7$

By gauss forward interpolation Formula

We get y(25) = 32 +0.25(3) + $(\frac{0.25)(0.25-1)}{2})(-5)$ + $\frac{(0.25+1)(0.25)(0.25-1)}{6}(7)$ = 32 +0.75

+0 .46875 - 0.2734 = 32.945

Y(25) = 32.945.

2. Example:

Use Gauss Backward interpolation formula to find f(32) given that f(25) = 0.2707, f(30) = 0.3027, f(35) =0 .3386 , f(40) =0 .3794.

Solution: let x_0 = 35 and difference table is

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
25=x-2	0.2707=y-2			
30=x-1	0.3027=y-1	0.032		
35=x0	**0.3386=y0**	**0.0359**	**0.0039**	
40=x1	0.3794=y1	0.0408	0.0049	0.0010

From the table y_0 = 0.3386 ,h=5,P=(x-x0)/h=(32-35)/5=-0.6

Δy_{-1} = 0.0359, $\Delta^2 y_{-1}$ = 0.0049, $\Delta^3 y_{-2}$ = 0.0010, x_p = 32 p = (x_p- x_0)/h = (32-35)/5 = -0.6

By Gauss Backward difference formula

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-2})$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + ----]$$

f(32) = 0.3386 + (-0.6)(0.0359) + (-0.6)(-0.6+1)(0.0049)/2 + (-0.6)(0.36-1)(0.00010)/6 =0 .3165

2.5 Stirling's Formula:

Stirling's formula is arithmetic mean of Gauss forward interpolation and Gauss Backward Interpolation formulae

We know that from Gauss forward formula

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + ----] \rightarrow (4)$$

And from Gauss backward formula

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-2}) + \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + ----]$$

----(5)

From 4,5, we found arithmetic mean the Stirling's formula is defined as

$P=\frac{X-X0}{h}$, where $h=x_1-x_0$

$Y(x) = y_0 + P\left[\frac{\Delta Y-1+\Delta Y0}{2}\right] + \frac{P^2}{2}\Delta^2 Y_{-1} + \frac{[P(P^2-1)]}{3!}\left[\frac{\Delta^3 Y-2+\Delta^3 Y-1}{2}\right] + \frac{P^2(P^2-1)}{4!}\Delta^4 Y_{-2} + ----$

2.5 Bessel's Formula:

Y=f(x) is a function with data (x_i, y_i) with $P=\frac{X-X0}{h}$, where $h=x_1-x_0$ then Bessel's formula is defined as follows

$Y(X)= Y_0+P\Delta Y0 + \frac{P(P-1)}{2!}\left[\frac{\Delta^2 Y0+\Delta^2 Y-1}{2}\right] + \frac{(P-\frac{1}{2})P(P-1)}{3!}\Delta^3 Y_{-1} + \frac{P(P-1)(P+1)(P-2)}{4!}\left[\frac{\Delta^4 Y-2+\Delta^4 Y-1}{2}\right] + ---$

Examples:

1. Using Striling's formula, compute f(1.22) from the following data

X	1.0	1.1	**1.2**	1.3	1.4
F(x)	0.841	0.891	0.932	0.963	0.985

Sol. Chose X0=1.2 is origin and length h=0.1 and P=$\frac{X-X0}{h}=\frac{1.22-1.2}{0.1}$=0.2
Next we construct central difference table by using above data and evaluate required value by Stirling's formula

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-2}=1.0	y_{-2}=0.841				
x_{-1}=1.1	y_{-1}=0.891	Δy_{-2}=0.05			
x_0=1.2	y_0=0.932	Δy_{-1}=0.041	$\Delta^2 y_{-2}$=-0.009		
x_1=1.3	y_1=0.963	Δy_0=0.031	$\Delta^2 y_{-1}$=-0.01	$\Delta^3 y_{-2}$=-0.001	
x_2=1.4	y_2=0.985	Δy_1=0.022	$\Delta^2 y_0$=-0.009	$\Delta^3 y_{-1}$=0.001	$\Delta^4 y_{-2}$=0.02

Use Stirling's formula

$$Y(x) = y_0 + P\left[\frac{\Delta Y-1+\Delta Y0}{2}\right] + \frac{P^2}{2}\Delta^2 Y_{-1} + [P(P^2-1)]/3!\left[\frac{\Delta^3 Y-2+\Delta^3 Y-1}{2}\right] + \frac{P^2(P^2-1)}{4!}\Delta^4 Y_{-2} + ----$$

Then Y(1.22) = 0.932+0.2$\frac{(0.041+0.031)}{2}$ +$\frac{0.04}{2!}$ (-0.01)+$\frac{0.2(0.04-1)}{4!}(0.002)$ = 0.93899

2. Find f(16) by Stirling's formula from the following table

x	0	5	10	15	20	25	30
F(x)	0	0.0875	0.1763	0.2679	0.364	0.4663	0.5774

Examples:

1. Use Bessel's formula to compute f(1.95) from the following data

X	1.7	1.8	1.9	2.0	2.1	2.2	2.3
F(X)	2.979	3.144	3.283	3.391	3.463	3.997	4.491

Sol. Choose the origin at X_0 =2.0, given h=0.1 and P= $\frac{x-x0}{h}=\frac{1.95-2.0}{0.1}$ =-0.5
Next by using Bessel's formula and central difference table we can evaluate the required solution

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-3}=1.7	y_{-3}=2.979					
x_{-2}=1.8	y_{-2}=3.144	Δy_{-3}=0.165				
x_{-1}=1.9	y_{-1}=3.283	Δy_{-2}=0.139	$\Delta^2 y_{-3}$=-0.026			
x_0 **=2.0**	y_0 **=3.391**	Δy_{-1}=0.108	$\Delta^2 y_{-2}$=-0.031	$\Delta^3 y_{-3}$=-0.005		
x_1=2.1	y_1=3.463	Δy_0=0.072	$\Delta^2 y_{-1}$=-0.036	$\Delta^3 y_{-2}$=-0.005	$\Delta^4 y_{-3}$=0	
x_2=2.2	y_2=3.997	Δy_1=0.53	$\Delta^2 y_0$=0.462	$\Delta^3 y_{-1}$=0.498	$\Delta^4 y_{-2}$ =0.503	$\Delta^5 y_{-3}$=0.503
x_3=2.3	y_3=4.491	Δy_2=0.494	$\Delta^2 y_1$=-0.04	$\Delta^3 y_0$=-0.502	$\Delta^4 y_{-1}$=-1	$\Delta^5 y_{-2}$=-1.503

Bessel's formula $\Delta^6 y_{-3} = -2.006$

Y(X)= $Y_0 + P\Delta Y0 + \frac{P(P-1)}{2!}\left[\frac{\Delta^2 Y0+\Delta^2 Y-1}{2}\right] + \frac{(P-\frac{1}{2})P(P-1)}{3!}\Delta^3\ Y_{-1} + \frac{P(P-1)(P+1)(P-2)}{4!}\left[\frac{\Delta^4 Y-2+\Delta^4 Y-1}{2}\right] + ---$

Y(1.95) = 3.391+(-0.5)(0.072) + $\frac{(-0.5)(-0.5-1)(-0.036+0.462)}{2.2} + \frac{(-0.5-0.5)(-0.5)(-0.5-1)(-0.5-2)(0.503-1)}{24.2}$

Y(1.95) = 3.3629

2. Compute Y(25) by using Bessel's formula to the following table

X	20	24	28	32
Y	2854	3162	3544	3992

9. Practice Quiz

1. Newton's backward Interpolation formula is ………………. [a]

a.

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-2}) + \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + ----]$$

b. $y+y_0$

c. y_0

d. None

2. The ……………………….Interpolation formula is used to estimate y, if the x-values are unequally spaced. [c]

a. Newton formula

b. Gauss formula

c. Lagrange's formula

d. Bessel's formula

3. Averaging Operator formula [d]

a.Δ

b. ∇

c. $\cup$

d. $\mu y_r = \frac{1}{2}[y_{r+1/2} + y_{r-1/2}]$

4. The relation between Δ *and* E [c]

a. $\Delta = E$

b. $\nabla = E$

c. $\Delta = E - 1$

d. $\Delta = E + 1$

5. Find the missing term in the following data [c]

x	0	1	2	3	4
Y	1	3	9	-	81

a. 10

b. 19

c. 27

d. 0

6. The relation between ∇ and E^{-1} [a]

a. $\nabla \equiv 1 - E^{-1}$

b. $\delta \equiv E^{1/2} - E^{-1/2}$

c. $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

d. $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

7. The relation between δ *and* E [b]

a. $\nabla \equiv 1 - E^{-1}$

b. $\delta \equiv E^{1/2} - E^{-1/2}$

c. $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

d. $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

8. The relation between μ *and* E [c]

a. $\nabla \equiv 1 - E^{-1}$

b. $\delta \equiv E^{1/2} - E^{-1/2}$

c. $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$