Stochastic process & Markov chains

stochastic (or) random process :

* A stochastic or random process is defined as a family of random variables. $\{x(t_n) ; n = 1, 2, 3 ----\}$.
* The Random variable $x(t)$ stands for the observation at time 't'. The No. of States 'n' may be finite or infinite depending on time range.
* for example, Let us consider the poisson distribution, $p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ where $n = 1, 2, 3 ----$

This distribution represent a stochastic process with infinite No. of State. In this example, the RV 'n' denotes the No of occurences between the interval 'o' and 't'. Thus, the states of the system at any time or given by $n = 0, 1, 2, 3 -----$

* Markov process :-

Markov process is a stochastic process which has the property that the probability of transition from a given state to from any future states. depends only on the present state and not on the manner in which it was reached.

for example, suppose a car rental company is running agencies in different cities. A car sent to one city may return to any city where the companies agency is available

* If this situation is considered are the Markov process the different rental cities would be in the
" "

*A p... transition probability p_{ij} would be the probability that a car rented to city 'i' would return to city 'i'. where j may be equal 'i'. The Mathematical structure of this problem is to determine expected long fraction of cart at each city and mean No of steps a car would make starting from city 'i' before returning the location.

Definition :-

The Stochastic or random process is called Markov process if the occurence of the future state depends on the immediately preceding state and only on it. if $t_1 < t_2 < --- < t_n$ represents the points in time scale then the family of RV $\{x(t_n)\}$ is said to be Markov process provides its holds the Markovian property

$$P\{x(t_n) = x_n \mid x(t_{n-1}) = x_{n-1} --- x(t_0) = x_0\}$$

$$= P\{x(t_n) = x_n \mid x(t_{n-1}) = x_{n-1}\}$$

for all $x(t_0), x(t_1), x(t_2) ---$

*Markov process is a sequence of 'n' expeiments in which each experiment 'n' possible outcomes. $x_1, x_2 - x_n$ for individual outcome is called a state and the probability of the probability depends only on the outcome of the preceding experiment. ~~the~~

Characteristic of Markov process :

Markov analysis is based on the following characteristics

1. The states are both collectively exhaustive and

of them absorbing ... is ...

absorbing. if a customer would never switch to a particular brand.

3. The transition probabilities are stationary.

4. The probability of moving from one state to another depends only for the immediately preceding state.

5. The transistion probabilities of moving to alternative state in the next time period. Given a state in the current, time period must sum to unity.

6. The process has set of initial probabilities that may be either given or determinant.

uses :-

Markov process is widely used;

1. In examining and predicting the behaviour of consumers. in terms of their brand loyality and their switching patterns to another brands.

2. In the study of equipment maintence and failure problems.

3. In analyzing accounts receivable that will ultimately become bad debts.

4. To study the stock market price movements

Transistion probability :- (2M) Manu

* The probability of mouing from one state to another or remaining in the same state during a single time period is called the transistion probability

* Mathematically the probability

$$P_{x_n, x_{n-1}} = P(x(t_n) = x_n / x(t_{n-1}) = x_{n-1})$$ is

the conditional probability of the system which is now in the state 'x_n' at time 't_n' provided that it was previously in state x_{n-1} at time 't_{n-1}' s.

* Sometimes this probability is known as one-step transistion probability, because it describes the system during the time interval (t_{n-1}, t_n)

Transistion probability matrix (def) 2M

The transistion probabilities can be arranged in a matrix from such a matrix is called one-step transistion probability matrix denoted by 'p' and is defined as $p = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1m} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{m1} & p_{m2} & p_{m3} & \cdots & p_{mm} \end{bmatrix}$

The matrix 'p' is a square matrix whose each element is non-negative and sum of elements of each row is unity i.e; $\sum_{j=1}^{m} p_{ij} = 1$

for $i = 1, 2, \cdots m$

Construction of Transistion matrix :- The following procedure is Adapted for constructing a state transistion matrix -

Step-1 :- ~~find~~ first compute retention probability (staying) which can be obtained by dividing the no. of customers retaind by the no. of customers originally served

$$\boxed{\text{Number retained} = \text{Original number served} - \text{Number lost}}$$

Step 2 :- Calculate the probabilities associated with customers gains or losses. for customer gain with probabilities, divide the number of customers each entry has gained by the number of customers served by the source of the gain.

rows – losses
column – gain

for customer loss probabilities divide the number of customers in each entity has lost by the original no. of customers of it served

Step 3 :- results of Step-I and Step-II are used to construct the state transistion matrix. The retention probabilities are listed along the main diagonal. The remaining entries are obtained as follows-

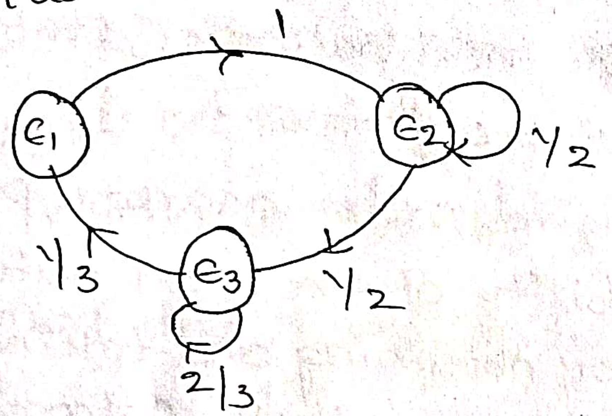
i. if loss probabilities are used, insert them in the appropriate row cells of the state transistion matrix.

ii. Gain probabilities are used, insert them in the appropriate column cells of the state transistion matrix.

In both the cases, state transistion matrixes thus obtained will be the same.

Diagramatic representation of transistion probabilities
The transistion probabilities can be represented by two types of diagrams i.e; transistion diagram:- (venndiagram), transistion diagram show the transistion probabilities or shifts that can occur in any particular situation.

Eg :-



The arrows each state fradicates the possible states to which a process can move from given state.
The matrix of transistion probabilities which correspond to the above diagram is.

$$
P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array}
\begin{array}{c}
\begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\
\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \end{bmatrix}
\end{array}
$$

ii. probability tree diagram :- As the name implies, this diagram emphasizes the probabilities and their movement from 1 step to another, alongwith all possible branches (or) paths that may connect the outcomes over a period of time.

first and higher order Markov process :- [2M]

Markov process is based on the following
— First order Markov process is based on the

i. The set of possible outcomes is finite.

ii. The probability of the next outcome (state) depends only on the immediately preceeding outcome.

iii. The transistion probabilities are constant over time.

The second order Markov process assumes that the probability of the next outcome (state) may depend on the two previous outcomes!

Similarlly, a third order markov process assumes that the probability of the next outcome (state) can be calculated by obtaining and taking account of the outcomes of the past three outcomes.

n-step transition probabilities :-

Suppose the system which occupies state E_i at time $t=0$, then we may be interested in finding out the probability that the system moves to state E_j at time $t=n$.

These time periods are sometimes referred to as number of steps

If the n-steps transition probability is denoted by $P_{ij}^{(n)}$ then these transition probabilities can be represented in matrix form as given below.

$$P^{(n)} = \begin{array}{c} \\ E_1 \\ E_2 \\ \vdots \\ \vdots \\ E_m \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \cdots & E_n \end{array} \\ \begin{bmatrix} P_{11}^{(n)} & P_{12}^{(n)} & \cdots & P_{1m}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} & \cdots & P_{2m}^{(n)} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ P_{n1}^{(n)} & P_{n2}^{(n)} & \cdots & P_{nm}^{(n)} \end{bmatrix} \end{array}$$

(n)

which occupies state E_2 will move to state E_1 after n-steps.

Let $p_{ij}(n)$ be the probability that the system occupies state E_i will move to state E_j in one transition. It should be noted that the transition probability p_{ij} is independent of time whereas the absolute probability $p_i(n)$ depends on time.

If the number of possible states be m, then

$\sum_{i=1}^{m} p_i(n) = 1$ and $\sum_{j=1}^{n} p_{ij} = 1$, for all i.

If all the state probabilities are known at time $t = n$, then the state probabilities at time $t = n+1$ can be determined by the equation,

$$P_j(n+1) = \sum_{i=1}^{m} P_i(n) p_{ij} \; ; n = 0, 1, 2 \ldots$$

Rewriting the equations for each states be m, time $t = n+1$;

$$p_1(n+1) = p_1(n) p_{11} + p_2(n) p_{21} + \cdots + p_m(n) p_{m1}$$

$$p_2(n+2) = p_1(n) p_{12} + p_2(n) p_{22} + \cdots + p_m(n) p_{m2}$$

$$\ldots$$

$$\ldots$$

$$p_n(n+1) = p_1(n) p_{1m} + p_2(n) p_{2m} + \cdots + p_m(n) p_{mn}$$

This system of equations can be written in matrix form as $\boxed{R(n+1) = R(n) P} \rightarrow$ (1) (or) $\boxed{\pi(n+1) = \pi(n) P}$ $\boxed{a_{n+1} = a_n P}$

where $R(n+1)$ is the row vector of state probabilities at time $t = n+1$, $R(n)$ is the row vector of state probabilities at time $t = n$ and P is the transition probability after | Note :- If the state probabilities at time $t=0$ are known, then we can find the state probabilities at any time by solving the matrix equation ①.

occupied | Markov chain :

1. If | Let $p_j^{(0)}$ $(j=0,1,2,---)$ be the absolute probability such

p_{ij} is | that the system be in state E_i at time t_0, where

bability | E_j $(j=0,1,2---)$ denote the exhaustive and mutually exclusive outcomes of a system at any time.

n | Also it is assumed that the system is Markovian

we now define $p_{ij} = P\{x(t_n)=j \mid x(t_{n-1})=i\}$ as the one-step transistion probability of going from

time | state i to at time t_{n-1} to state j at time t_n.

i+1 can | It is also assumed here that these probabilities from state E_i to state E_j $(i=0,1,2,---j=0,1,2----)$ are expressed in the matrix form as below.

time

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & --- \\ p_{10} & p_{11} & p_{12} & --- \\ p_{20} & p_{21} & p_{22} & --- \\ - & - & - & - \\ - & - & - & - \end{bmatrix}$$

p_{m1}

p_{m2} | This matrix P is known as stochastic matrix or homogenous matrix.

The probabilities p_{ij} must satisfy the boundary conditions

p_{mn} | $\sum p_{ij} = 1$, for all i, $p_{ij} \geq 0$, for all i and j

trix | Definition : Markov chain :-

The transistion matrix p as defined above together

ilities at | with the initial probabilities $\{p_j^{(0)}\}$ associated with

obabilities | the states E_j $(j=0,1,2---)$ completely define a

ability | Markov chain.

i. Ergodic Markov chain ii. Regular Markov chain

An Ergodic markov chain is defined as a chain having the property that it is possible to pass from one state to another in a finite number of steps, regardless of present state.

A regular Markov chain is defined as a chain having a transistion matrix p such that for some power of p it has only non-zero positive probability values.

Note i. A special type of ergodic markov chain is the regular markov chain.

ii. All the regular chains must be ergodic chains.

iii. To check or ergodic chain is regular, continue squaring the transistion matrix p untill all zeros are removed.

prblms: Manu 10M

1. Two manufacturers A and B are competing with each other in a restricted market over the years, A's' customers have exhibited a high degree of loyality as measured by the fact that customers using A's' product from 80% of time. Also formen customers purchasing the product from B have switched back to A's' 60% of time.

a. Construct and interpret the state transition matrix in terms of i. retention and loss ii. retention and gain

b. calculate the probability of customer purchasing A's product at the end of the second period?

Sol :-

a. The transistion probabilities can be arranged in a present P = purchase (n=0)

$$\begin{array}{cc} A \\ B \end{array}\begin{bmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{bmatrix}$$

retention & gain

retention losses

Clearly the probability of customers purchase at the next step (n=1) depends upon the product which a customer is having at present (n=0).

∴ Each probability in the above matrix must be conditional probability for passing from one state to another.

The conditional probabilities in the above matrix can be stated as.

i. $P(B_0|A_1) = P_{21} = 0.60$

This indicates that the customer now using B's product at n=0 (present purchase) will purchase A's product at n=1 (next purchase) is 0.60

This means loss to B's product.

i. $P(A_0|A_1) = P_{11} = 0.80$

This indicates that the probability that the customers now using A's product at n=0 (present purchase) will again purchase A's product at n=1 (next purchase) is 0.80. This means retention to A's product.

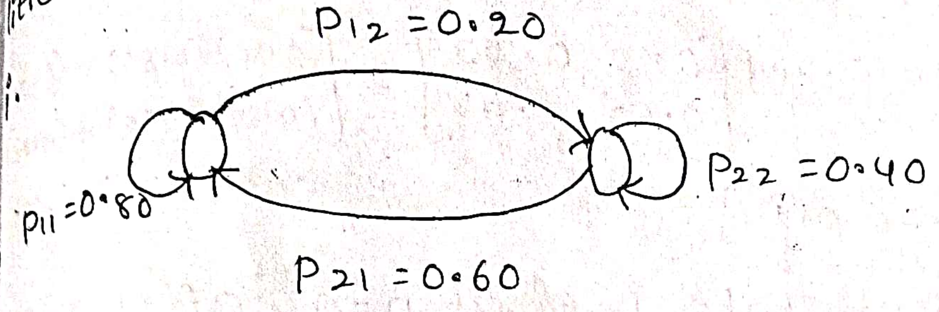
iii. $P(A_0|B_1) = P_{12} = 0.20$

This indicates that the probability that the customer now using A's product at n=0 will purchase B's product at n=1 is 0.20. This means loss to A's product.

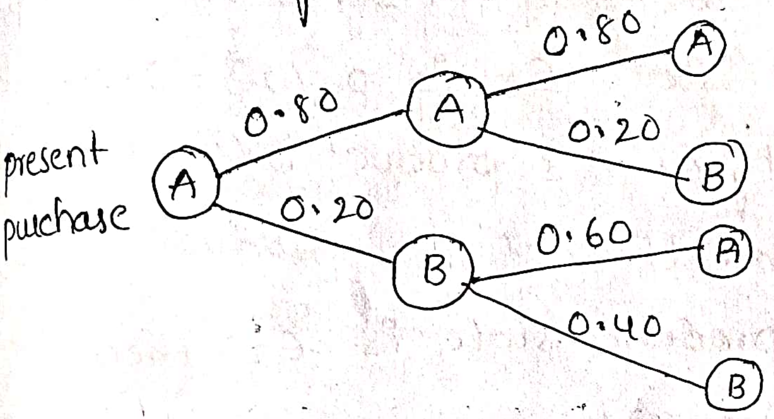
iv. $P(B_0|B_1) = P_{22} = 0.40$

This indicates that the probability that the customer now using B's product n=0 will purchase B's product at n=1 is 0.40. This means retention to B's product.

i. Diagramatic representation of transition probabilities:

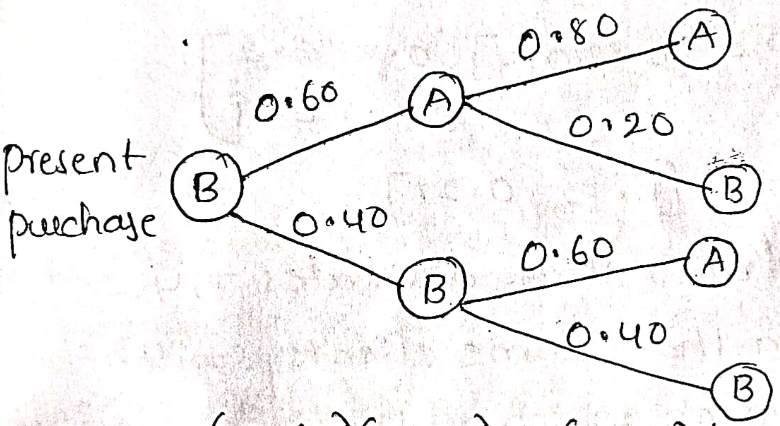


ii. Tree diagram



$p_{11} = (0.80)(0.80) + (0.20)(0.60) = 0.76$

$p_{12} = (0.80)(0.20) + (0.20)(0.40) = 0.24$



$p_{22} = (0.60)(0.20) + (0.40)(0.40) = 0.28$

$p_{21} = (0.60)(0.80) + (0.40)(0.60) = 0.72$

Calculations of probability:

We have that $R(n+1) = R(n)\,P$

At $n = 0$, we get $R(1) = R(0)\,P$ $\quad \left\{ \because p_1(0) = 1,\ p_2(0) = 0 \Rightarrow R(0) = [1\ \ 0] \right.$

$$\Rightarrow R(1) = [1\ \ 0] \begin{bmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{bmatrix} = [0.80\ \ 0.20]$$

probabi- and the probability of the next state E_2 is $P_2(1) = 0.2$

At $n=1$, we get $R(2) = R(1)\,P$

$$R(2) = [0.80 \quad 0.20] \begin{bmatrix} 0.80 & 0.20 \\ 0.60 & 0.40 \end{bmatrix} = \begin{bmatrix} (0.80)(0.80) + (0.20)(0.60) \\ (0.80)(0.20) + (0.20)(0.40) \end{bmatrix}$$

$$= [0.76 \quad 0.24]$$

If the present state is E_1 at $n=0$, after 2-steps, the probability of being in the state ~~of~~ E_1 is $P_1(2) = 0.76$ and in the state E_2 is $P_2(2) = 0.24$.

i.e; The probability of A's product after the end of 2-steps is 76% and that of B's product is 24%

Similarly if the present state is E_2 then.

where $R(1) = [P_1(1) \quad P_2(1)]$

$$R(1) = [0 \quad 1] \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = [0.6 \quad 0.4]$$

and $R(2) = R(1)\,P = [0.6 \quad 0.4] \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$

$$= [0.72 \quad 0.28]$$

2. In a small town with three advocates x, y and z each advocate known that some clients switch back and forth, depending on which advocate is available at the time the client needs one.

There are no new clients in the current legal market; however none of the old clients are leaving the area.

$P_2(0) = 0$ During a stack period, the three advocates collected data

$= [1 \quad 0]$ which identical the number of clients each advocate had seen during the preceding year.

Tables given below summarize the results of this study & the number in which clients were gained or lost respectively

Data summary - client activity

Advocate	clients as of Jan 1, 2005	Change during the year		clients as of Jan 1, 2006
		Gain	loss	
X	400	75	50	425
Y	500	50	150	400
Z	500	100	25	573

Gain - loss summery

Advocate	clients as of Jan 1, 2005	Gains			Losses			Clients as of Jan 1, 2006
		from X	Y	Z	TO X	Y	Z	
X	400	0	50	25	0	50	0	425
Y	500	50	0	0	50	0	100	400
Z	500	0	100	0	25	0	0	575

Sol :- Calculation of probabilities :-

i. Retention probability :

Retention probabilities are calculated by dividing the number of customers that were retained by the number of clients that were originally served.

No. of clients retained = The original number of clients minus the number of customers that were lost.

Retention probabilities

Advocate	clients as of Jan 1, 2005	Number lost		Number retained	probability of retention
X	400	50	$\frac{400-50}{400}$ =	350	350/400 = 0.87
Y	500	150	$\frac{500-150}{500}$ =	350	350/500 = 0.70

ii. Gain & loss probabilities :

The probabilities associated with client gain (loss) are calculated by dividing the number of clients gained (losses) by the number of clients served at the source of gain (loss).

Gain probability.

original number served	Advocates	probability of Gain		
		from x	from y	from z
400	x	P_{xx} 0/400 = 0	P_{yx} 50/500 = 0.1	25/500 = 0.05 P_{zx}
500	y	P_{xy} 50/400 = 0.125	0/500 = 0 P_{yy}	0/500 = 0 P_{zy}
500	z	P_{xz} 0/400 = 0	100/500 = 0.2 P_{yz}	0/500 = 0 P_{zz}

loss probability

original number served	Advocates	probability of loss		
		TO x	TO y	TO z
400	x	0/400 = 0	50/400 = 0.125	0/400 = 0
500	y	50/500 = 0.1	0/500 = 0	100/500 = 0.2
500	z	25/500 = 0.05	0/500 = 0	0/500 = 0

iii. Transistion matrix :-

list the retention probabilities along the main diagonal.

If loss probabilities are used, insert them in appropriate row cells.

If gain probabilities are used, insert them in appropriate

as of , 2006

clients as of Jan 1, 2006.

425
400
575

dividing by the

clients t.

probability of retention

400 = 0.875
500 = 0.700

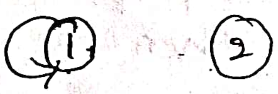
Transistion Matrix

$$P = \begin{array}{c} \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 0.875 & 0.125 & 0 \\ 0.1 & 0.7 & 0.2 \\ 0.05 & 0 & 0.95 \end{bmatrix} \end{array}$$

Classification of States :

The states of Markov chain can be classified based on the transition probability p_{ij}.

① A state i is said to be absorbing if it return to itself with certainity in one transition i.e; $p_{ij}=1$

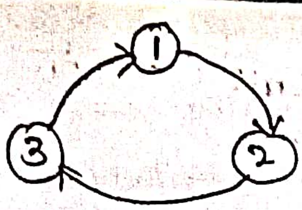


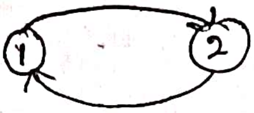
② A state is said to be transi~~ent~~ent if it can reach another state but cannot itself be reached back from another state.

Mathematically, it will happen, if $\lim_{n\to\infty} P_{ij}^{(n)}=0$, for all i

③ A state is said to be a recurrent state if, the probability of being revised from other state is one. This can happen if and only if the state is not transient

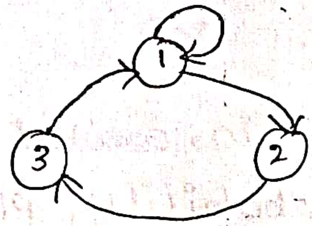
④ A state j is said to be periodic with period $t>1$, if a return is possible only in t, $2t$, $3t$ steps This returns that $P_{ii}^{(n)}=0$ whenever 'n' is not divisible





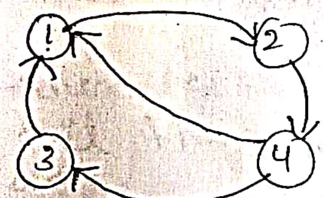
If the chain is not periodic, then it is said to be "a periodic"

Ex :-



(5) A Markov chain is irreducible if all states belong one class (All the states communicate with each other) (or) If a chain does not contain any other proper closed subset other than the state space, then the chain is called irreducible, otherwise it is called reducible.

Ex :-



irredeucible.

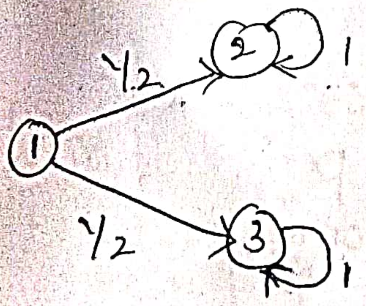
In irreducible markov chain every state can be reached from every other state.

Examples :-

(1)

It is absorbing, aperiodic and transient

(2)



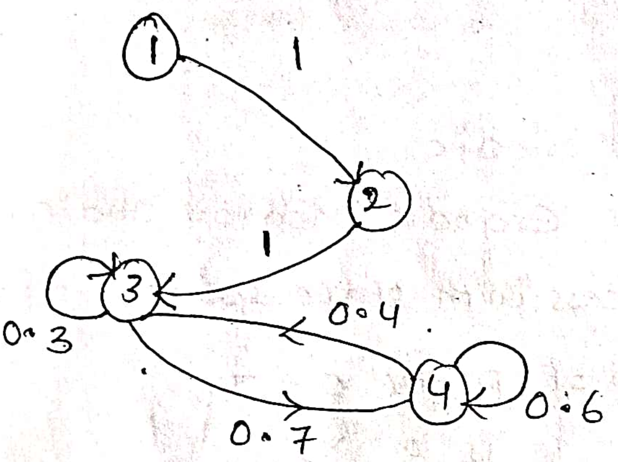
It is absorbing, not irreducible (not transitive)

UNIT III

1. Classify the states of the following Markov chain.

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0.3 & 0.7 \\ 4 & 0 & 0 & 0.4 & 0.6 \end{array}$$

Sol :- The Markov chain is given below



States 1 & 2 are transient

State 3 & 4 are absorbing and constitute a closed set i.e; irreducible

The chain is aperiodic.

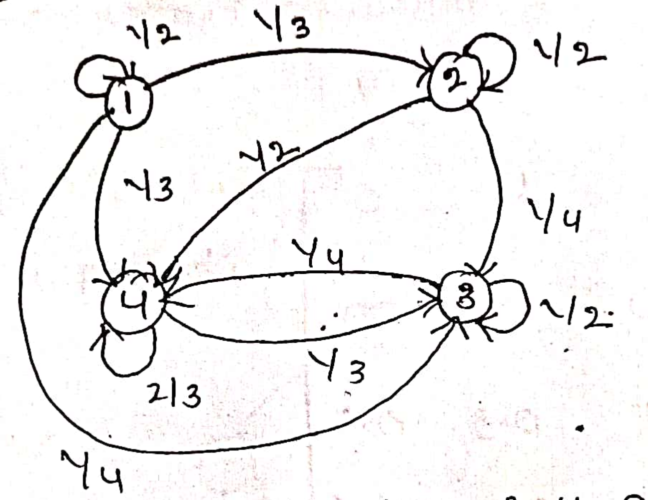
<u>Defn</u> :- A closed Markov chain is said to <u>ergodic</u> if all of its states are <u>recurrent</u> and <u>aperiodic</u>.

2. Determine if the following transition matrix is ergodic Markov chain

future state

$$\begin{array}{cccccc} & & 1 & 2 & 3 & 4 \\ \text{present} & 1 & \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{3} \end{array}$$

Sol :- The Markov chain is given below



Here the states 1, 2, 3, 4 are absorbing

It is possible to go from one state to another state directly or indirectly.

∴ It is irreducible.

Also the chain is ~~not~~ a periodic

Hence the given chain is Ergodic Markov chain.

3. Consider a Markov process with state space $S = \{0, 1, 2\}$ and transition matrix

$$P = \begin{bmatrix} p & q & 0 \\ 1/2 & 0 & 1/2 \\ p-1/2 & 7/10 & 1/5 \end{bmatrix}$$

a. What can you say about the values of p and q

b. Calculate the transition probabilities $p_{ij}^{(3)}$

c. Draw the transition graph for the process represented by P.

Sol :- Since $\sum p_{ij} = 1$, we have

$$p - 1/2 + 7/10 + 1/5 = 1$$

$$p = 1 - \frac{1}{5} - \frac{7}{10} + \frac{1}{2} = \frac{10-2+5-7}{10} = \frac{6}{10} = \frac{3}{5}$$

$$q = 1 - \frac{3}{5} = \frac{5-3}{5} = \frac{2}{5}$$

$$p = \begin{bmatrix} 3/5 & 2/5 & 0 \\ 1/2 & 0 & 1/2 \\ 1/10 & 7/10 & 1/5 \end{bmatrix}$$

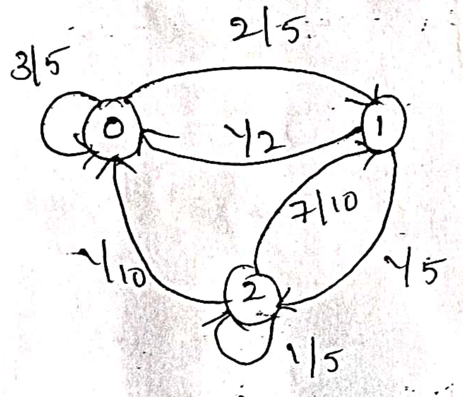
$$\left[\frac{3}{5} - \frac{1}{2} = \frac{6-5}{10} = \frac{1}{10} \right]$$

b. Transition probabilities of $p_{ij}^{(3)}$ are the elements of $(p^3)_{ij}$

$$P_3 = \begin{array}{c|c} & \begin{matrix} 0 & \quad 1 & \quad 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 3/5 & 2/5 & 0 \\ 1/2 & 0 & 1/2 \\ 1/10 & 7/10 & 1/5 \end{bmatrix}^3 \end{array} = \frac{1}{(10)^3} \begin{bmatrix} 6 & 4 & 0 \\ 5 & 0 & 5 \\ 1 & 7 & 2 \end{bmatrix}^3$$

$$= \begin{bmatrix} 0.467 & 0.364 & 0.16 \\ 0.495 & 0.21 & 0.295 \\ 0.387 & 0.445 & 0.168 \end{bmatrix} \qquad \left[p^3 = p \times p \times p \right]$$

c. Transition graph :-



4. Consider a homogenous Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \\ 3/4 & 1/4 & 0 \end{bmatrix}$$ calculate the 3-step transition matrix.

Sol :- Given $P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \\ 3/4 & 1/4 & 0 \end{bmatrix}$

we have that the n-step transition probability matrix

$\therefore$ 3-step probability transition matrix $P_{ij}^{(3)} = P^3$

i.e; $P_{ij}^{(3)} = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \\ 3/4 & 1/4 & 0 \end{bmatrix}^3$

$$P^3 = \left(\frac{1}{4}\right)^3 \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 0 \end{bmatrix}^3 = \frac{1}{64}\begin{bmatrix} 29 & 21 & 14 \\ 26 & 18 & 20 \\ 32 & 15 & 17 \end{bmatrix}.$$

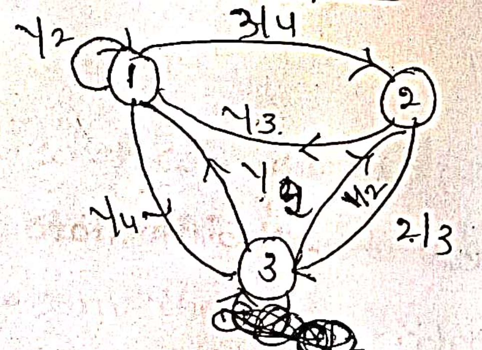
5. Consider the Markov chain with 3 states $S = \{1, 2, 3\}$ that has the following transition matrix $P = \begin{bmatrix} 1/2 & 3/4 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

a. Draw the state transition diagram for this chain.

b. If we know $P(x_1 = 1) = P(x_1 = 2) = 1/4$, find $P(x_1 = 3, x_2 = 2, x_3 = 1)$

Sol:- Given $P = \begin{bmatrix} 1/2 & 3/4 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

Transition diagram.



(b) Given $P(x_1 = 1) = P(x_1 = 2) = 1/4$

$\therefore P(x_1 = 3) = 1 - P(x_1 = 1) - P(x_1 = 2)$

$= 1 - 1/4 - 1/4 = 1 - \frac{1}{2} = \frac{1}{2}$

We can write $P(x_1 = 3, x_2 = 2, x_3 = 1) = P(x_1 = 3)\, P_{32}\, P_{21}$

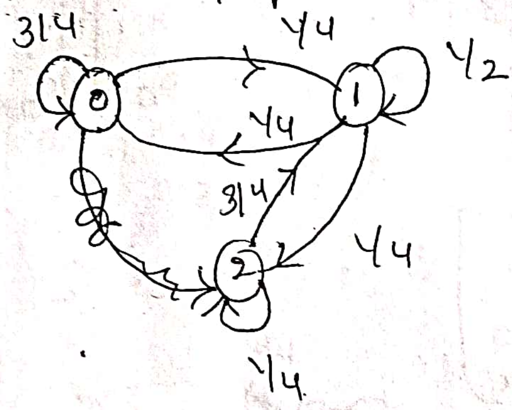
$= 1/2 \cdot 1/2 \cdot 1/3 = \frac{1}{12}.$

6. The transition probability matrix of a Markov chain with states 0, 1, 2 is $P = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$ and the initial state distribution of the chain is $P(x_0 = i) = 1/3$, $i = 0, 1, 2$ obtain i. $P(x_2 = 2)$ ii. $P(x_2 = 2, x_1 = 1, x_0 = 2)$

$\boxed{\text{Manu 10M}}$

Sol :- Given $P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 3/4 & 1/4 & 0 \\ 1 & 1/4 & 1/2 & 1/4 \\ 2 & 0 & 3/4 & 1/4 \end{array}$

Transition diagram



Given $P(x_0 = i) = 1/3$, $i = 0, 1, 2$

$\therefore P(x_0 = 0) = P(x_0 = 1) = P(x_0 = 2) = 1/3$

$P(x_1 = 0) = P(x_1 = 1) = P(x_1 = 2) = 1/3$

$P(x_2 = 0) = P(x_2 = 1) = P(x_2 = 2) = 1/3$

i. $P(x_2 = 2) = 1 - P(x_2 = 0) - P(x_1 = 1)$

$= 1 - \frac{1}{3} - \frac{1}{3}$

$= \frac{1}{3}$

ii. $P(x_2 = 2, x_1 = 1, x_0 = 2) = P(x_2 = 2)\, P_{21} \cdot P_{12}$

chain wit

al state

$x_0 = j0 = 1/3$

$, x_d = 2)$

7. Test the following transition matrix to see if the Markov chain is regular and ergodic where x is some positive p_{ij} value.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & x & x & 0 \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ 0 & x & x & 0 \end{bmatrix} \end{array}$$

Sol :- we compute

$$P^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & x & x & 0 \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ 0 & x & x & 0 \end{bmatrix} \end{array} \begin{bmatrix} 0 & x & x & 0 \\ x & 0 & 0 & x \\ x & 0 & 0 & x \\ 0 & x & x & 0 \end{bmatrix}$$

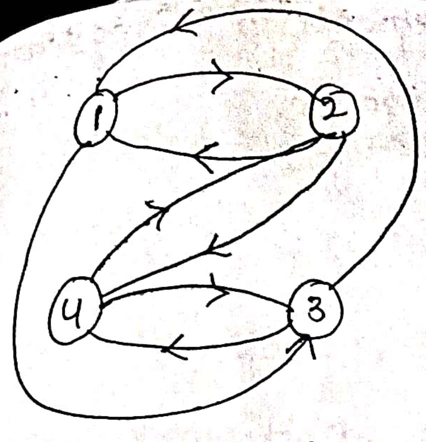
$$\Rightarrow P^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix} \end{array}$$

Similarly $P^4 = \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix}$ $P^8 = \begin{bmatrix} x & 0 & 0 & x \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ x & 0 & 0 & x \end{bmatrix}$

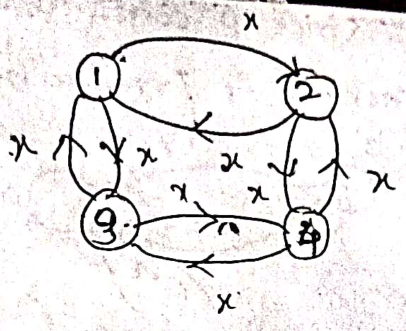
From this we observe p raised to an even number power gives the result as above, while p raised to an odd numbers power will give the original matrix.

Since all the elements are not non-zero positive elements, the given matrix is not regular.

But, it is ergodic :- since it is possible to go from one state to another.



(or)

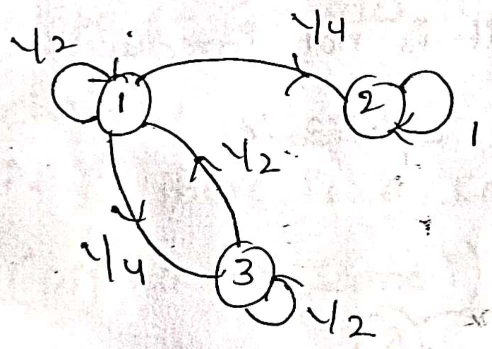


8. Which of the following stochastic matrix is regular and ergodic Markov chain.

i. $P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$ ii. $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

iii. $\begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}$

Sol :- Since $P_{22} = 1$, it is not regular



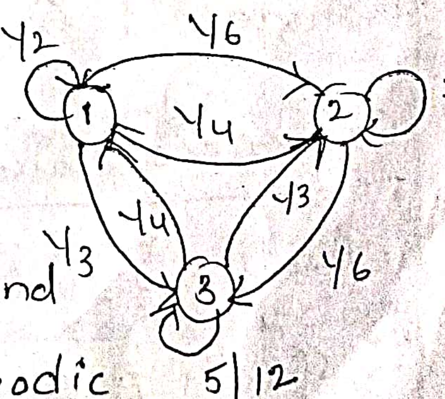
It is not irreducible state

1, 2 are not recurrent

No transition from state 2 to state 1, and from state 2 to state 3

It is not ergodic.

ii. Since $P^2 = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/13 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$ containing all +ve elements!

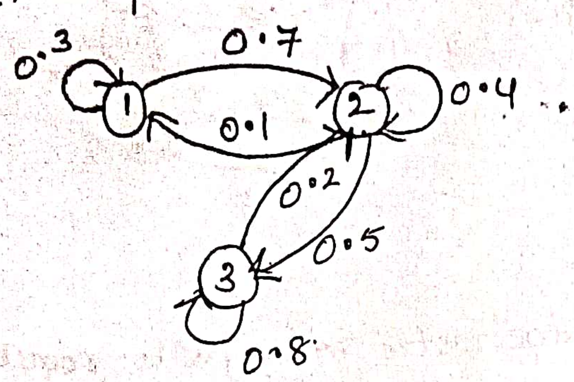


P is a regular Markov chain

Since all states are recurrent and P is not periodic, it is an ergodic chain.

iii. Since P contains all the elements, P is regular

since P is irreducible, it is ergodic



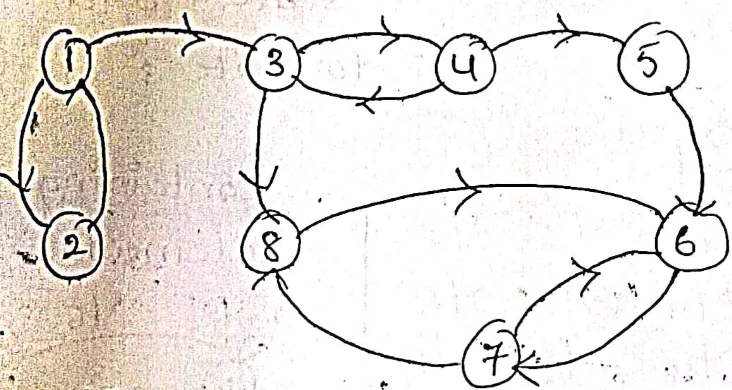
<u>Examples :-</u>

i. $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ It is irreducible but not aperiod

ii. $P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ It not irreducible and is aperiodic

iii. $P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$ It is not irreducible and is aperiodic.

iv.



<u>Equivalence classes</u> :-

Class 1 = { state 1, state 2 }

∴ State 1 & state 2 communicate each other but not

Class 2 = {state 3, state 4}

class 3 = {state 5}

class 4 = {state 6, state 7, state 8}

Class 1 = {state 1, state 2} is aperiodic since it has a self transition, $p_{22} > 0$

Class 2 = {state 3, state 4} is periodic with period 2.

Class 4 is aperiodic

Because, we can go from state 6 to state 6 in two steps (6-7-6) and in three steps (6-7-8-6)

Since gcd (2, 3) = 1, we conclude that state 6 and class 4 is aperiodic.

Note :- consider a ~~peri~~ finite irreducible Markov chain.

i. If there is a self transition in the chain ($p_{ij} > 0$ for same i) then the chain is aperiodic.

ii. Suppose, you can go from state i to state j in l-steps i.e; $p_{ij}^{(l)} > 0$, also in m steps ($p_{ij}^{(m)} > 0$) If gcd (l, m) = 1, then the state i is aperiodic.

Steady state (Equilibrium) condition : (2M Manu)

The determination of "steady state" conditions is a regular ergodic Markov chain can be accomplished most readily by computing p^n for larger values of large term probability that the system will be in the state (or) any number of transitions applied to the systems has no impact on the state vector i.e the current behaviour of the system will continue into the future.

Mathematically,

$$\lim_{n\to\infty} R(n+1) = \lim_{n\to\infty} R(n)\,p \quad (\text{or}) \quad R = Rp \to (1)$$

Therefore, as $n\to\infty$, $R(n)$ becomes constant (i.e; independent of time) and then the system is said have reached to a steady state equilibrium.

Note :- As $n\to\infty$, the state probabilities at $(n+1)^{th}$ period are very close to those that n^{th} period.

problms

1. Consider a Markov chain with only two states $S\{0,1\}$ and transition matrix $p = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$ determine the stationary distribution of p.

Sol :- Let $R = (R_0, R_1)$ be the stationary distributions

Given $p = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$

We have $R = Rp$ (or) $Rp = R$, where $R_0 + R_1 = 1$

i.e; $[R_0 \;\; R_1]\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = [R_0 \;\; R_1]$

$$\frac{1}{2}R_0 + \frac{1}{3}R_1 = R_0 \Rightarrow -\frac{R_0}{2} + \frac{R_1}{3} = 0 \to (1)$$

$$\frac{1}{2}R_0 + \frac{2}{3}R_1 = R_1 \Rightarrow \frac{R_0}{2} - \frac{R_1}{3} = 0 \to (2)$$

We have $R_0 + R_1 = 1 \Rightarrow R_1 = 1 - R_0$

$$① \Rightarrow -\frac{R_0}{2} + \frac{1-R_0}{3} = 0 \Rightarrow \frac{R_0}{2} - \frac{R_0}{3} = -1/3$$

$$\Rightarrow \left(-1/2 - 1/3\right) R_0 = -1/3 \Rightarrow \left[\frac{-2-2}{6}\right] R_0 = -1/3$$

$$\Rightarrow -\frac{5R_0}{6} = -1/3 \Rightarrow \boxed{R_0 = \frac{2}{5}}$$

$$R_1 = 1 - \frac{2}{5} = \frac{3}{5} \Rightarrow \boxed{R_1 = \frac{3}{5}}$$

2. Consider the matrix of transition probabilities of a product available in the matrix in two brands A & B, $P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$ determine the Market share of each brand in equilibrium position

Sol:- Given $P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$ and $R = [A, B]$

At equilibrium condition, we have $RP = R$

Where $A + B = 1 \rightarrow ①$

$$[A\ B] \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix} = [A\ B]$$

$$(0.9)A + (0.3)B = A \rightarrow ①$$

$$(0.1)A + (0.7)B = B \rightarrow ②$$

from ①

$$B = 1 - A$$

$$① \Rightarrow (0.9)A + (0.3)(1-A) = A$$

$$\Rightarrow (0.9)A + 0.3 - (0.3)A - A = 0$$

$\Rightarrow -(0.4)A = -0.3$

$A = \frac{0.3}{0.4} = \frac{3}{4}$

Hence $B = 1 - \frac{3}{4} = \frac{1}{4}$

The market share of brand A is 75% and the brand B is 25%

3. A manufacturing company has a certain piece of equipment that is inspected at the end of each day and classified as 'just overhauled', 'good', 'fair', or 'inoperative'. if the item is inoperative, it is overhauled a procedure that takes one day. Let us denote the four classifications as states 1, 2, 3 and 4 respectively. Assume that the working condition of the equipment follows a Markov processes with the following transition matrix.

$$\begin{array}{cc} & \text{Tomorrow} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{Today} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

It is costs Rs 125 to overhaul a machine (including lost time) on the average, and Rs 75 in production is lost if a machine is found inoperative. using steady State probabilities, compute the expected per day cost of maintance.

Sol:- Since the given matrix 'p' is an ergodic regular

Let the steady state probabilities p_1, p_2, p_3 and p_4 represent the proportion of times that the machine will be in states 1, 2, 3 and 4 respectively.

From the steady state equations, we have

$$R = RP$$

i.e; $(p_1\ p_2\ p_3\ p_4) = (p_1\ p_2\ p_3\ p_4)\begin{bmatrix} 0 & 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow p_1 = p_4 \rightarrow (1)$

$p_2 = \frac{3}{4} p_1 + \frac{1}{2} p_2 \rightarrow (2)$

$p_3 = \frac{1}{4} p_1 + \frac{1}{2} p_2 + \frac{1}{2} p_3 \rightarrow (3)$

$p_4 = \frac{1}{2} p_3 \rightarrow (4)$

and $p_1 + p_2 + p_3 + p_4 = 1 \rightarrow (5)$

Solving these simultaneous equations we get

$p_1 = \frac{2}{11}$, $p_2 = \frac{3}{11}$, $p_3 = \frac{4}{11}$ & $p_4 = \frac{2}{11}$

On an average, 2 out of every 11 day's the machine will be overhauled, 3 out of every 11 days it will be in good condition, 4 out of every 11 days it will be in fair condition and 2 out of every 11 days it will be found inoperative at the end of the days.

Hence the expected (average) cost per day of maintenance will be given by

$$\left(2/11\right)(125) + \left(\frac{3}{11}\right)(75) = Rs\ 3.36$$

Markov Analysis :-

ii. The rate of gains and loses for the current market
iii. The projected market share of each item in the system in the future.
iii. The equilibrium state fact could exist.

The best example of Markov analysis is 'Brand switching Model's, which emphasis on the time behaviour of customers who make repeated purchase of a product class, but from time to time may switch over from one brand to another. Here the state is generally the customer's prefrelence for a particular brand.

Brand switching Example :-

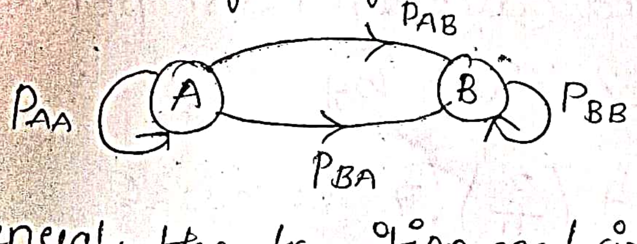
Let us consider a consumer sample distributed over two brands A and B, the samples being the representative of the entire group the standpoint of their brand loyally and their switching patters.

The behaviour of the large groups can be better described in probabilistic terms. This probabilistic description can be represented by transition matrix as explained by the following diagram.

Transition matrix :

$$\text{from} \quad \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} P_{AA} & P_{AB} \\ P_{BA} & P_{BB} \end{bmatrix} \end{array}$$

Brand switching diagram :



In general, the transition matrix for n brands $A_1, A_2, \ldots A_n$ is given by

$$P = \begin{array}{c|c} & \begin{array}{cccccc} A_1 & A_2 & \cdots & A_j & \cdots & A_n \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_i \\ \vdots \\ A_n \end{array} & \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{ij} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i1} & p_{i2} & \vdots & p_{ij} & \vdots & p_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \vdots & p_{nj} & \vdots & p_{nn} \end{bmatrix} \end{array}$$

Where $\sum_{j=1}^{n} p_{ij} = 1, \; i = 1, 2, \ldots n$

Markovian Algorithm

The following algorithm gives a procedure for solving the application problems.

Step-1 :- from the given data, first determine the probabilities associated with retentions, gain and losses.

Step-2 :- Develop the appropriate state-transition matrix.

Step-3 :- With the help of the transition matrix obtained in step-2, determine the expected future market shares for the next period.

The procedure is as follows.

(Market shares beginning of period 1) × (state-transition matrix) = Expected market shares beginning of period 2

Step-4 :- If more than one expected market share calculation is needed, then apply the following

Step-2, find the equilibrium conditions for the current problem.

Problems

1. Suppose there are two market products of brand A and B respectively. Let each of these two brands have exactly 50 of the total market in same period and let the market be of a fixed size. The transition matrix is given as follows

$$\text{from}\quad \begin{array}{c|cc} & & \text{To} \\ & A & B \\ \hline A & 0.9 & 0.1 \\ B & 0.5 & 0.5 \end{array}$$

If the initial market share breakdown is 50% for each brand then determine their market shares in the steady state

Sol:- Let $P = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$ be the given transition matrix

It is given that the initial market shares at the beginning of period 1 are 50% each. i.e; $R = (0.5\ 0.5)$

Market shares of brands A & B in second period (or) after one steady period is given by

(Market shares in the beginning of period 1) & (state transition matrix) = (expected market shares in the beginning of period 2)

$$\text{i.e; } (0.5\ \ 0.5) \times \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = (0.5 \times 0.9 + 0.5 \times 0.5 + 0.5 \times 0.1 + 0.5 \times 0.5)$$

$$= (0.7\ \ 0.3) = (70\%\ \ 30\%)$$

After two steady periods

$$(0.7\ \ 0.3)\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = (0.7 \times 0.9 + 0.3 \times 0.5 + 0.7 \times 0.1 + 0.3 \times 0.5)$$

$$= (0.78\ \ 0.22)$$

Similarly $(0.78\ \ 0.22)\begin{bmatrix}0.9 & 0.1\\0.5 & 0.5\end{bmatrix} = (81.2\ \ 18.8)$

$(81.2\ \ 18.8)\begin{bmatrix}0.9 & 0.1\\0.5 & 0.5\end{bmatrix} = (82.48\ \ 17.52)$

$(82.18\ \ 17.52)\begin{bmatrix}0.9 & 0.1\\0.5 & 0.5\end{bmatrix} = (82.992\ \ 17.008)$

$(82.992\ \ 17.008)\begin{bmatrix}0.9 & 0.1\\0.5 & 0.5\end{bmatrix} = (83\ \ 17)$.

After 6 time periods the resulting market shares are 83% and 17% respectively.

∴ The equilibrium position of market shares of A and B will be 5/6 and 1/6 of the total Market respectively.

2. A Market research team has conducted a survey of customer buying habits with respect to three brands of talcum powder in an area. It estimates at present 20% of the customers buy brand A, 50% of the customers buy brand B and 30% of the customers buy brand C. In addition, the team has analysed its survey and has determined the following brand switching matrix.

Brand Next bought / Brand just Bought	A	B	C
A	0.6	0.3	0.1
B	0.4	0.5	0.1
C	0.2	0.1	0.7

Determine the expected distribution of consumers two time periods later.

Sol :- Given transition matrix is

$$\begin{array}{cccc} & A & B & C \\ A & 0.6 & 0.3 & 0.1 \\ B & 0.4 & 0.5 & 0.1 \\ C & 0.2 & 0.1 & 0.7 \end{array}$$

Given that the initial buying of the three brands A, B & C are (20% 50% 30%)

i.e (0.2, 0.5, 0.3)

After one steady period;

[expected distribution in present period] × [Transition matrix] = [expected distribution in next period]

i.e $$(0.2 \quad 0.5 \quad 0.3)\begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} = (0.38 \quad 0.34 \quad 0.28)$$

After the steady periods;

$$(0.38 \quad 0.34 \quad 0.28)\begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} = (0.42 \quad 0.312 \quad 0.268)$$

Hence after two periods.

expected distribution of customers for three brands A, B & C are 0.42, 0.312, 0.268 respectively

(or) 42% 31.2% & 26.8% respectively

3. Suppose there are three dairies in a town, say A, B and C. They supply all the milk consumed in the town. It is known by all the dairies that consumers switched from dairy to dairy overtime because of advertising dissatisfaction with service and other reasons. All these dairies maintain records of the number of their customers and the dairy from which may

June.

Dairy	June (1) customer	Gain from			Losses To			July 1 customer
		A	B	C	A	B	C	
A	200	0	35	25	0	20	20	220
B	500	20	0	20	35	0	15	490
C	300	20	15	0	25	20	0	290

We assume that the matrix of transition probabilities remain fairy stable and that the July market shares are

$A = 22\%$ $B = 49\%$ & $C = 29\%$.

Managers of these dairies are willing to know:

i. market share of their dairies on 1st August & 1st sep

ii. their market shares in steady state.

Sol:- We have that the transition probability matrix is

$$P = \begin{bmatrix} P_{AA} & P_{AB} & P_{AC} \\ P_{BA} & P_{BB} & P_{BC} \\ P_{CA} & P_{CB} & P_{CC} \end{bmatrix}$$

i.e., The propability transition matrix is

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} \frac{160}{200} = 0.8 & \frac{20}{200} = 0.1 & \frac{20}{200} = 0.1 \\ 35/500 = 0.07 & \frac{450}{500} = 0.9 & \frac{15}{500} = 0.03 \\ 25/30 = 0.083 & \frac{20}{300} = 0.067 & \frac{255}{300} = 0.85 \end{bmatrix} \end{array}$$

Retention probabilities:

for brand A, $P_{AA} = \dfrac{\text{Number of retained}}{\text{Number of customer origanally served}}$

Number retained = (Original no - number lost)

$$P_{AA} = \frac{160}{200} = 0.8$$

$$\text{lly } P_{BB} = \frac{450}{500} = 0.9, \quad P_{CC} = \frac{255}{300} = 0.85$$

Gain probabilities:

from B to A, $P_{BA} = \dfrac{\text{No. of Customers gained from B}}{\text{Total no. of customers served for the source of gain (B)}}$

$$P_{BA} = \frac{35}{500} = 0.07$$

from c to A;

$$P_{CA} = \frac{\text{No. of customer gained from c}}{\text{Total no. of customer of c}} = \frac{25}{300} = 0.083$$

from A to B; $P_{AB} = \dfrac{20}{200} = 0.1$

from c to B; $P_{CB} = \dfrac{20}{300} = 0.067$

from A to c; $P_{AC} = \dfrac{20}{200} = 0.1$

from B to c; $P_{BC} = \dfrac{15}{500} = 0.03$

The transition matrix is,

$$\begin{bmatrix} 0.80 & 0.1 & 0.1 \\ 0.07 & 0.9 & 0.03 \\ 0.083 & 0.067 & 0.85 \end{bmatrix}$$

The market shares (on july 1st) are of the three dairi[es] A, B & C are 22%, 49% and 29% respectively.

After one steady period (Market share on 1st August:)

$$(0.22 \quad 0.49 \quad 0.29)\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.07 & 0.9 & 0.03 \\ 0.083 & 0.067 & 0.85 \end{bmatrix} = (0.234 \quad 0.483 \quad 0.283)$$

After two steady period (i.e; Market shares on 1st sep):

$$(0.234 \quad 0.483 \quad 0.283)\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.07 & 0.9 & 0.03 \\ 0.083 & 0.067 & 0.85 \end{bmatrix} = (0.245 \quad 0.477 \quad 0.278)$$

ii. The steady state market shares are given by

$$[x \quad y \quad z]\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.07 & 0.9 & 0.03 \\ 0.083 & 0.067 & 0.85 \end{bmatrix} = [x \quad y \quad z]$$

$$\Rightarrow (0.8)x + (0.07)y + (0.083)z = x \rightarrow (1)$$
$$(0.1)x + (0.9)y + (0.067)z = y \rightarrow (2)$$
$$(0.1)x + (0.03)y + (0.85)z = z \rightarrow (3)$$

and $x+y+z=1$

Solving (1) (2) (3) & (4) we get

$x = 0.273 \quad y = 0.454 \quad z = 0.273$

4. The number of units of an item that are withdrawn from inventory on a day-to-day basis is a Markov chain process in which requirements for tomorrow depend on today's requirements. A one day transition matrix is given below:

Number of units withdraw from inventory

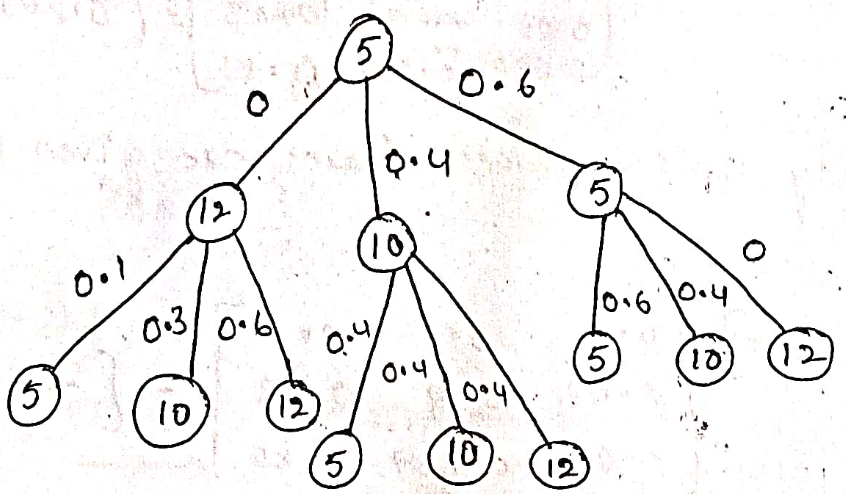
$$\begin{array}{cc} & \text{Tomorrow} \\ & \begin{array}{ccc} 5 & 10 & 12 \end{array} \\ \text{Today} \begin{array}{c} 5 \\ 10 \end{array} & \begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{array}$$

i. Construct a tree diagram showing inventory requirements two consecutive days

ii. Develop a two-day transition matrix

iii. comment on "how a two day transition matrix might be helpful to a manager who is responsible for inventory management".

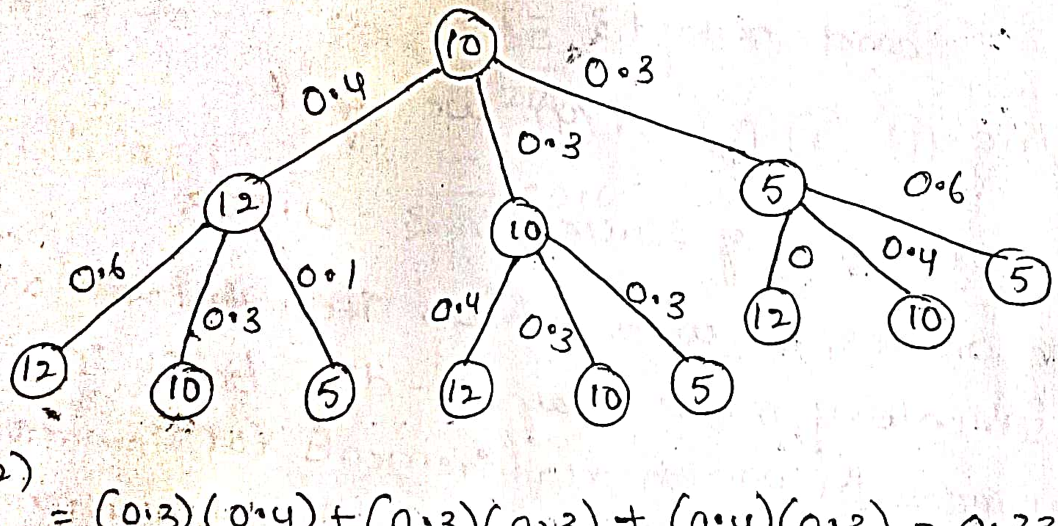
Sol :-



$$P_{11}^{(2)} = (0.6)(0.6) + (0.4)(0.3) + (0)(0.1) = 0.48$$

$$P_{12}^{(2)} = (0.6)(0.4) + (0.4)(0.3) + (0)(0.3) = 0.36$$

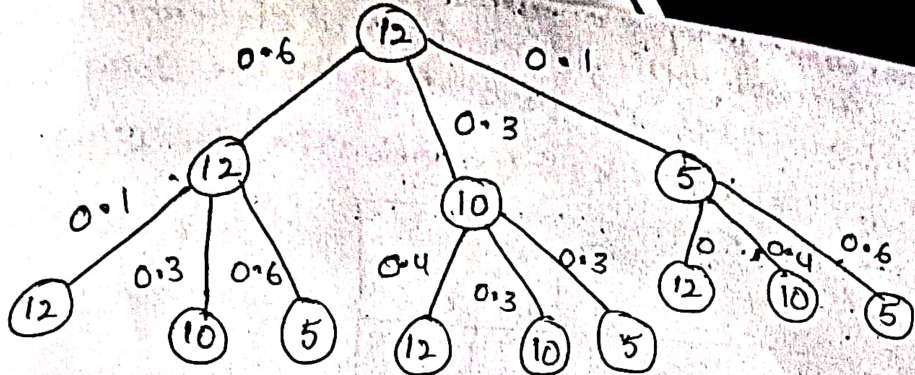
$$P_{13}^{(2)} = (0.6)(0) + (0.4)(0.4) + (0)(0.6) = 0.16$$



$$P_{22}^{(2)} = (0.3)(0.4) + (0.3)(0.3) + (0.4)(0.3) = 0.33$$

$$P_{21}^{(2)} = (0.3)(0.6) + (0.3)(0.3) + (0.4)(0.1) = 0.31$$

$$P_{23}^{(2)} = (0.3)(0) + (0.3)(0.4) + (0.4)(0.6) = 0.36$$



$$p_{31}^{(2)} = (0.1)(0.6) + (0.3)(0.3) + (0.6)(0.1) = 0.21$$

$$p_{32}^{(2)} = (0.1)(0.4) + (0.3)(0.3) + (0.6)(0.3) = 0.31$$

$$p_{33}^{(2)} = (0.1)(0) + (0.3)(0.4) + (0.6)(0.6) = 0.48$$

ii. If the transition matrix is p, then the two-day transition matrix is given by

$$p^2 = \begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$$

$$= \begin{array}{c} \\ 5 \\ 10 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 10 & 12 \end{array} \\ \begin{bmatrix} 0.48 & 0.36 & 0.16 \\ 0.31 & 0.33 & 0.36 \\ 0.21 & 0.31 & 0.48 \end{bmatrix} \end{array}$$

iii. The two day transition matrix can be used for guiding ordering decisions.

for ex : if today the manager experiences a demand for five units, then two days later the probability of requiring 5 units is 0.48, 10 units is 0.36 and 12 units is 0.16.

5. A housewife buys three kind of cereals: A, B & C she never buys the some cereal on successive weeks if she buys cereal A then the next week she buys cereal B, However, if

times as likely to buy A as the other brand. Obtain the transition probability matrix and determine how often she would buy each of the cereals in the long run.

Sol :- We present the information in the following table.

Cereals	Cereals		
	A	B	C
A	0	1	0
B	0·75	0	0·25
C	0·75	0·25	0

The transition matrix is given by

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0.75 & 0 & 0.25 \\ C & 0.75 & 0.25 & 0 \end{array}$$

00 =
1
50 + 25
75

If x, y and z represent the steady state probabilities of states A, B and C respectively, then we have

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

Where $x+y+z=1$ and $x \geq 0,\ y \geq 0,\ z \geq 0$

The equations are

$$0+(0.75)y+(0.75)z = x$$

$$x+0+(0.25)z = y$$

$$0+(0.25)y+0 = z$$

(or)

$$x-0.75y-0.75z=0 \rightarrow (1)$$

$$x-y+0.25z=0 \rightarrow (2)$$

day
d for
demand
bility of
and 12.
, B & c she
f she buys
However, if

Solving these equations, we get:

$x = 0.428,\ y = 0.457,\ z = 0.114$

Hence, the long run, the house wife would buy the cereal A, B and C with a frequency of 42.8%, 45.7% and 11.4% respectively.

6. The price of an equity share of a company may increase, decrease or remain constant on any given day. It is assumed that the change in price any day affects the change on the following day as described by the following transition matrix:

Change Tomorrow

$$\begin{array}{ccc} & \text{Increase} & \text{decrease} & \text{unchanged} \\ \text{Change Today} \quad \text{Increase} & & & \\ \text{decrease} & & & \\ \text{unchanged} & & & \end{array}$$

$$\begin{array}{r} \text{Increase} \\ \text{decrease} \\ \text{unchanged} \end{array}\begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.7 & 0.1 & 0.2 \\ 0.4 & 0.5 & 0.1 \end{bmatrix}$$

i. if the price of share increased today, what are chances that it will increase, decrease or unchanged tomorrow?

ii. If the prices of the share decreased today, what are the chances that it will increase tomorrow?

iii. If the price of the share remained unchanged today, what are the chances that it will increase, decrease or remain unchanged day after tomorrow?

Sol :-

i. Given that the probability of increase in price of the share today is 1.

∴ The probability of decreasing or unchanged is zero

After one steady period,

$$\begin{bmatrix}\text{Today's}\\\text{probabilities}\end{bmatrix} \times \begin{bmatrix}\text{Transition}\\\text{matrix}\end{bmatrix} = \begin{bmatrix}\text{expected probabilities for}\\\text{tomorrow}\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.2 & 0.3\\0.7 & 0.1 & 0.2\\0.4 & 0.5 & 0.1\end{bmatrix} = \begin{bmatrix}0.5 & 0.2 & 0.3\end{bmatrix}$$

This gives that the price will increase, decrease or unchanged tomorrow by 50%, 20% and 30% respectively.

ii. If the probability of decrease in price of a share is 1, then the probability of the other two events is zero.

∴ probabilities for today are (0 1 0)

After one steady period,

$$\begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.2 & 0.3\\0.7 & 0.1 & 0.2\\0.4 & 0.5 & 0.1\end{bmatrix} = \begin{bmatrix}0.7 & 0.1 & 0.2\end{bmatrix}$$

This indicates that the elements of price increase tomorrow is 70%.

iii. Let the probability of prices remain unchanged today be 1 and the probabilities of the other two events be zero.

then the probabilities for today are (0 0 1)

After one steady period

$$\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{bmatrix}0.5 & 0.2 & 0.3\\0.7 & 0.1 & 0.2\\0.4 & 0.5 & 0.1\end{bmatrix} = \begin{bmatrix}0.4 & 0.5 & 0.1\end{bmatrix}$$

After two steady periods;

8. on january 1 (this year), Bakery A had 40% of its local market share while the other two bakeries B and C had 40% and 20% respectively of the market share. Based upon a study by a marking research time, the following facts were compiled, Bakery A retains 90% of its customers while gaining 5% competitor B's customers and 10% of C's customer. Bakery B retains 85% of its customers while gaining 5% of A's customers and 7% of C's customers, Bakery C retains 83% of its customers and gains 5% of A's customers and 10% of B's customers. what will each firm's share be on january 1 (next year) and what will each firm's market share be at equilibrium?

Sol:- from the given data, we can form the state-transition matrix as given below.

$$\text{Bakery}\quad \begin{array}{c|ccc} & A & B & C \\ \hline A & 0.90 & 0.05 & 0.05 \\ B & 0.05 & 0.85 & 0.10 \\ C & 0.10 & 0.07 & 0.83 \end{array}\Big\downarrow \text{ Retention and Gain}$$

Bakery (columns)

$\longrightarrow$ retention and loss

Given that the initial (i.e; on january 1, present year) market shares of three Bakeries A, B & C are 40%, 40% and 20% respectively.

The expected market shares for A, B & C on january 1, next year are determined as follows.

$$\begin{bmatrix}\text{Market shares} \\ \text{on Jan 1} \\ \text{Present year}\end{bmatrix} \times \begin{bmatrix}\text{state-transition} \\ \text{matrix}\end{bmatrix} = \begin{bmatrix}\text{expected market} \\ \text{Shares on Jan 1,} \\ \text{next year}\end{bmatrix}$$

$$\begin{bmatrix}0.40 & 0.40 & 0.20\end{bmatrix}\begin{bmatrix}0.90 & 0.05 & 0.05 \\ 0.05 & 0.85 & 0.10 \\ 0.10 & 0.07 & 0.85\end{bmatrix} = \begin{bmatrix}0.400 & 0.374 & 0.226\end{bmatrix}$$

Hence the market shares of Bakeries A, B and C on January 1 next year will be 40% 37.4% and 22.6% respectively.

If x, y, z be the market shares at equilibrium state, then we have

$$[x \ y \ z]\begin{bmatrix} 0.90 & 0.05 & 0.05 \\ 0.05 & 0.85 & 0.10 \\ 0.10 & 0.07 & 0.83 \end{bmatrix} = [x \ y \ z]$$

and $x+y+z=1$ where $x, y, z \geq 0$

The equations are

$$(0.90)x + (0.05)y + (0.10)z = x$$
$$(0.05)x + (0.85)y + (0.07)z = y$$
$$(0.05)x + (0.10)y + (0.83)z = z$$

The eqn's can be written as

$$-0.10x + 0.05y + (0.10)z = 0 \rightarrow (1)$$
$$0.05x - 0.15y + 0.07z = 0 \rightarrow (2)$$
$$0.05x + 0.10y - 0.17z = 0 \rightarrow (3) \ \& \ x+y+z=1 \rightarrow (4)$$

Solving, we get $x = 0.43$, $y = 0.28$, $z = 0.29$

The market shares at equilibrium are

for Bakery A : 43% of the total market

for Bakery B = 28% of the total market

for Bakery C = 29% of the total market

Some important problems : [Markov chain]

Notation : $P(x_n)$; i means the probability of the state 'i' after 'n' time periods.

Ex :- $P(x_3) = 1$ means probability of the state 1 after '3' time periods (or) 3 states.

Note :- formula to calculate the probability of states:

$\pi_n = \pi_0 P^n$ (or) $\pi_{n+1} = \pi_n P$

In other notation ; $a_n = a_0 P^n$ (or) $a_{n+1} = \pi_n P$

Here a_0 (or) π_0 is the initial probability and P is a state transition matrix [or transition probability matrix (TPM)]

1. A man either uses his car or takes a bus or a train to work each day. The transition probability matrix of the markov chain with these three states 1 (car), 2 (bus) and 3 (train) is

future (tomorrow)

$$\text{present (today)} \quad P = \begin{array}{c|ccc} & C & B & T \\ \hline C & 0.1 & 0.5 & 0.4 \\ B & 0.6 & 0.2 & 0.2 \\ T & 0.3 & 0.4 & 0.3 \end{array}$$

and the initial probability is $(0.7, 0.2, 0.1)$ calculate $P(X_2 = 3)$

Sol :- Given TPM is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and initial probability is $a_0 =$

$[0.7, 0.2 \ 0.1]$

probability after 2 time periods is given by $a_0 P^2 = a_2$

i.e; $\begin{bmatrix} 0.7 & 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} = \begin{bmatrix} \text{probability after} \\ \text{2 periods} \end{bmatrix}$

Now $P^2 = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$

$\therefore [0.7 \quad 0.2 \quad 0.1] \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix} = [0.385 \quad 0.336 \quad 0.279]$

Hence the probability of state '3' after 2 time periods is $p(x_2 = 3) = 0.279$ (or) 27.9 %

2. Three boys A, B & C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is as likely to throw the ball to B as to A. If the initial probability distribution of three states A, B and C is (0.3, 0.4, 0.3) respectively. find i. Transition matrix ii. $p(x_2 = B)$ iii. $p(x_3 = B, x_2 = C, x_1 = B, x_0 = A)$

Sol i. using the given information, we can write the transition probability matrix as

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/2 & 1/2 & 0 \end{array}$$

The initial probabilities are $q_0 = [0.3 \quad 0.4 \quad 0.3]$

ii. To find $p(x_2 = B)$:-

i.e., The probability of B after 2 time periods.

we have $q_n = q_0 P^n$

$\therefore q_2 = q_0 P^2$

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$\therefore$ probability after '2' states (or) '2' time periods is

$$\begin{bmatrix} 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} = \begin{array}{ccc} \begin{bmatrix} 0.20 & 0.35 & 0.45 \end{bmatrix} \\ A \quad\quad B \quad\quad C \end{array}$$

$\therefore p(x_2 = B) = 0.35$

iii. $p(x_3 = B, x_2 = C, x_1 = B, x_0 = A)$:

$$q_0 \dashrightarrow A \xrightarrow{(1)} B \xrightarrow{(1)} C \xrightarrow{(1)} B$$

$$p(x_3 = B, x_2 = C, x_1 = B, x_0 = A) = q_0(A)\, P_{AB}\, P_{BC}\, P_{CB}$$

$$= 0.3 \times 1 \times 1 \times \frac{1}{2}$$

$$= 0.15$$

[where $q_0(A)$ is the initial probability of A and we can find the values P_{AB}, P_{BC} / P_{CB} from the TPM]

iv. steady-state probability (or) equilibrium (or) limiting probabilities of boys:

The steady state probability can be given by

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

The equations are $1/2\, z = x \rightarrow (1)$

$x + z/2 = y \rightarrow (2)$

$y = z \rightarrow (3)$ & $x + y + z = 1 \rightarrow (4)$

Solving, we get $x = 0.20$, $y = 0.40$, $z = 0.40$

3. The TPM of the Markov chain with three states 1, 2, 3 is $P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$ and the initial probability is $(0.5\ 0.3\ 0.2)$ calculate $P(x_3 = 3, x_1 =$

Sol :- $P(x_3 = 3, x_1 = 1)$

$$\xrightarrow{q_0} 1 \xrightarrow{(2)} 3$$

$\Rightarrow q_0(1) \cdot P_{13}^{(2)} = 0.5 \times P_{13}^{(2)}$

To find $P_{13}^{(2)}$: probability of moving state 1 to state 3 in 2 state periods is given by P^2.

$$\therefore P^2 = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.27 & 0.39 & 0.34 \\ 0.20 & 0.48 & 0.32 \\ 0.23 & 0.39 & 0.38 \end{bmatrix}$$

$\therefore P_{13}^{(2)} = 0.34$

Hence $P(x_3 = 3, x_1 = 1) = (0.5) \times (0.34) = 0.17$

4. The TPM of the Markov chain with three states 0, 1, 2 is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & --- \\ --- & 0.6 & 0.3 \\ 0.4 & --- & 0.3 \end{bmatrix} \end{array}$$

and the initial probability is $(0.5, 0.3, 0$

i. fill the blanks in TPM

ii. $P(x_3 = 2, x_2 = 1, x_1 = 0, x_0 = 2)$ iii. $P(x_3 = 2, x_1 = 0, x_0 = 2)$

iv. $P(x_2 = 2)$

Sol :- since $\Sigma P_{ij} = 1$, we write TPM 'P' as

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} \end{array}$$

and $q_0 = [0.5\ \ 0.3\ \ 0.2]$

ii. $P(x_3 = 2, x_2 = 1, x_1 = 0, x_0 = 2)$:

$$q_0 \dashrightarrow 2 \xrightarrow{(1)} 0 \xrightarrow{(1)} 1 \xrightarrow{(1)} 2$$

3. The TPM of the Markov chain with three states 1, 2, 3 is $P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$ and the initial probability is $(0.5\ 0.3\ 0.2)$ calculate $P(x_3 = 3, x_1 =$

Sol :- $P(x_3 = 3, x_1 = 1)$

$$q_0 \dashrightarrow 1 \xrightarrow{(2)} 3$$

$$\Rightarrow q_0(1) \cdot P_{13}^{(2)} = 0.5 \times P_{13}^{(2)}$$

To find $P_{13}^{(2)}$: probability of moving state 1 to state 3 in 2 state periods is given by P^2.

$$\therefore P^2 = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.27 & 0.39 & 0.34 \\ 0.20 & 0.48 & 0.32 \\ 0.23 & 0.39 & 0.38 \end{bmatrix}$$

$$\therefore P_{13}^{(2)} = 0.34$$

Hence $P(x_3 = 3, x_1 = 1) = (0.5) \times (0.34) = 0.17$

4. The TPM of the Markov chain with three states 0, 1, 2 is

$$P = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0.2 & 0.3 & --- \\ --- & 0.6 & 0.3 \\ 0.4 & --- & 0.3 \end{bmatrix} \end{array}$$

and the initial probability is $(0.5, 0.3, 0$ i. fill the blanks in TPM

ii. $P(x_3 = 2, x_2 = 1, x_1 = 0, x_0 = 2)$ iii. $P(x_3 = 2, x_1 = 0, x_0 = 2)$

iv. $P(x_2 = 2)$

Sol :- Since $\Sigma P_{ij} = 1$, we write TPM 'P' as

$$P = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} \end{array}$$

and $q_0 = [0.5\ \ 0.3\ \ 0.2]$

ii. $P(x_3 = 2, x_2 = 1, x_1 = 0, x_0 = 2)$:

$$q_0 \dashrightarrow 2 \xrightarrow{(1)} 0 \xrightarrow{(1)} 1 \xrightarrow{(1)} 2$$

$= q_0(2) \cdot P_{20} \cdot P_{01} \cdot P_{12}$

$= 0.2 \times 0.4 \times 0.3 \times 0.3 = 0.0072$

iii. $P(x_3 = 2, x_1 = 0, x_0 = 2)$

$\xrightarrow{q_0} 2 \xrightarrow{(1)} 0 \xrightarrow{(2)} 2$

$= q_0(2) \cdot P_{20} \, P_{02}^{(2)}$

$= 0.2 \times 0.4 \times P_{02}^{(2)}$

To find $P_{02}^{(2)}$: (probability from state 0 to 2 after 2 time period)

$$P^2 = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.27 & 0.39 & 0.34 \\ 0.20 & 0.48 & 0.32 \\ 0.23 & 0.39 & 0.38 \end{bmatrix}$$

$\therefore P_{02}^{(2)} = 0.34$

$\therefore P(x_3=2, x_1=0, x_0=2) = 0.2 \times 0.4 \times 0.34$

$= 0.0272$

iv. $P(x_2 = 2)$

i.e; The probability of '2' after '2' time periods

$\Rightarrow q_2 = q_0 P^2$

required probability is

$$\begin{bmatrix} 0.5 & 0.3 & 0.2 \end{bmatrix} \begin{bmatrix} 0.27 & 0.39 & 0.34 \\ 0.20 & 0.45 & 0.32 \\ 0.23 & 0.39 & 0.38 \end{bmatrix} = \begin{bmatrix} 0.241 & 0.408 & 0.342 \end{bmatrix}$$

5.

5. A professor tried not to be late for class too often, If he is late one day, he is 90% sure to be on time next day. If he is on time then the next time there is a 30% chance of his being late. In the long run, how often is he late for class?

Sol:- The required TPM is $p = \begin{array}{c|cc} & L & O \\ \hline L & 0.1 & 0.9 \\ O & 0.3 & 0.7 \end{array}$ | L = late, O = on time

Let $q = [x \ y]$ with $x + y = 1$ be the stationary distribution such that $qp = q$

i.e; $[x \ y]\begin{bmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{bmatrix} = [x \ y]$

$\Rightarrow (0.1)x + (0.3)y = x \Rightarrow -0.9x + 0.3y = 0 \rightarrow (1)$

$(0.9)x + (0.7)y = y \Rightarrow 0.9x - 0.3y = 0 \rightarrow (2)$

and $x + y = 1 \rightarrow (3)$

solving (1) & (3) we get

$x = 0.25 \quad y = 0.75$

Hence in the long run, there are 25% chances that he comes late for classes.

6. A man either drives his car or takes a train to work each day. suppose he never takes the train two days in a row, but he drives to work, then the next day he is just likely to drive again as he is to take the train. find transition matrix and also find the probability that he changes from going by train to driving exactly in four days?

Sol:- The required TPM is $p = \begin{array}{c|cc} & T & C \\ \hline T & 0 & 1 \\ C & 1/2 & 1/2 \end{array}$

because he will not travel by train for two days in a row.

The probability that he changes from going by train to driving exactly in four days is

$p(x_4 = c \mid x_0 = T) = P_{TC}^{(4)}$

We have to compute p^4 and find the required probability P_{TC}'

Now $p^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$

$p^4 = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix}$

$$p^4 = \begin{array}{cc} & \begin{array}{cc} T & \quad C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix} \end{array}$$

$\therefore$ The required probability $P_{TC}^{(4)}$ is $5/8$

7. In a certain market there are three brands of lipsticks A, B and C. Given that a lady lost purchased lipstic of brand A, there is 70% chances that she would continue with brand A, 20% and 10% chances that she would shift to brands B and C respectively. Given that a lady last purchased lipstic of brand B, there is 50% chances that she would shift to brand A and 10% chances to brand C. Given that a lady last purchase lipstic brand C, there is 60%, 20% chance that she would shift to brand A and B respectively. The present market share of the three brands A, B & C is 60%, 30% & 10% respectively. using this information find i. find transition probability matrix ii. probability that a customer who is currently a brand A purchases will purchase brand B after two time periods. iii. $p(x_2 = C \mid x_0 = A)$ iv. $p(x_2 = C \mid x_0 = C)$. v. probability that brand B purchase will purchase brand A three periods from now. vi. probability that a customer will purchase brand A, B, C three periods from now. vii. The market share of brands A, B, & C in the steady state (or) long run (or) equilibrium.

Sol :- i. The transition probability matrix is

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}$$

Given initial probabilities are $q_0 = [0.6 \quad 0.3 \quad 0.1]$

ii. required probability is $p(x_2 = B / x_0 = A) = P_{AB}^{(2)}$

Now
$$P^2 = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}$$

$$= \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.65 & 0.24 & 0.11 \\ 0.61 & 0.28 & 0.11 \\ 0.64 & 0.24 & 0.12 \end{bmatrix} \end{array}$$

$\therefore P_{AB}^{(2)} = 0.24$

iii. probability that a customer currently purchasing brand A will purchase brand C after two time periods

i.e; $p(x_2 = c / x_0 = A) = P_{AC}^{(2)} = 0.11$

iv. probability that brand c will retain its customers after two time periods.

i.e; $p(x_2 = c / x_0 = c) = P_{cc}^{(2)} = 0.12$

v. probability that brand B purchases will purchase brand A three periods from now

$$p(x_3 = A / x_0 = B) = P_{BA}^{(3)}$$

Now
$$P^3 = P^2 \cdot P = \begin{bmatrix} 0.65 & 0.24 & 0.11 \\ 0.61 & 0.28 & 0.11 \\ 0.64 & 0.24 & 0.12 \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}$$

$$p^3 = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.64 & 0.25 & 0.11 \\ 0.63 & 0.26 & 0.11 \\ 0.64 & 0.25 & 0.11 \end{bmatrix} \end{array}$$

$\therefore P_{BA}^{(3)} = 0.63$

vi. probability that a customer will purchase brands A, B, C three periods from now:

required probability $= p(x_3 = A),\ p(x_3 = B),\ p(x_3 = C) \Rightarrow$

$$[0.6\ \ 0.3\ \ 0.1]\begin{bmatrix} 0.64 & 0.25 & 0.11 \\ 0.63 & 0.26 & 0.11 \\ 0.64 & 0.25 & 0.11 \end{bmatrix} = [0.637\ \ 0.253\ \ 0.11]$$

The market shares after 3 periods are

brand A = 63.7%

brand B = 25.3%

brand C = 11%

vii. Let $q = [x\ y\ z]$ be the stationary distribution such that $qP = q$ where $x + y + z = 1$

i.e; $$[x\ \ y\ \ z]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.2 \end{bmatrix} = [x\ \ y\ \ z]$$

$\Rightarrow (0.7)x + (0.5)y + (0.6)z = x$

$(0.2)x + (0.4)y + (0.2)z = y$

$(0.1)x + (0.1)y + (0.2)z = z$

$\Rightarrow -(0.3)x + (0.5)y + (0.6)z = 0 \rightarrow (1)$

$(0.2)x - (0.6)y + (0.2)z = 0 \rightarrow (2)$

$(0.1)x + (0.1)y - (0.8)z = 0 \rightarrow (3)$

& $x + y + z = 1 \rightarrow (4)$

Solving ①, ②, ③ & ④ we get

$x = 0.6389$, $y = 0.2500$, $z = 0.1111$

Hence in the steady state, the market share of the brand A, B and C will be 63.89% 25% & 11.11% respectively

H.W

⑧. 80% of students who do maths work during one study period will do the some works at the next study period and 30% of students who do english work during one study period will do the english work in the next study period. Initially there were 60 students do maths and 40 students do english work. Calculate i. The transition probability matrix. ii. The number of students who do maths work & english work for the next consequent 2 study periods.

Sol :- i. State - Transition matrix is $\begin{bmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{bmatrix}$

Initial probabilities are (60 - 40)

After one steady period

$$\begin{bmatrix} \text{expected distribution} \\ \text{in present study period} \end{bmatrix} \times \begin{bmatrix} \text{Transition} \\ \text{matrix} \end{bmatrix} = \begin{bmatrix} \text{expected distribution} \\ \text{in next study period} \end{bmatrix}$$

$$\begin{bmatrix} 60 & 40 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{bmatrix} = (76 \quad 24)$$

After the steady period

$$(76 \quad 24) \begin{bmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{bmatrix} = (77.6 \quad 22.4)$$

Hence after two steady period.

The number of students who do maths & english works are 77.6% and 22.4%

Q. Consider a bike share problem with only 3 stations A, B & C. Suppose that all bikes must be returned to the station at the end of the day, so that all the bikes are at some station each day, the distribution of bikes at each station changes as the bike get returned to diffeient stations from where they are borrowed.

Of the bikes borrowed from station A, 30% are return to station A, 50% end up at station B and 20% end up at station C.

Of the bikes borrowed from station B, 10% end up at station A, 60% ~~end up~~ have been returned.

Of the bikes borrowed from station C, 10% end up at station A, 10% end up at station B and 80% are returned to station C.

i. Express this information as a transition probability matrix & determine the probabilities of bike being at a particular station after 2 days.

ii. Suppose when we start observing the bike share program, 30% of the bikes at station A, 45% of bikes are at station B and 25% are at station C, determine the distribution of bikes at the end of the next day & after 2 days.

Sol:- Transition matrix

i.

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0.3 & 0.5 & 0.2 \\ B & 0.1 & 0.6 & 0.3 \\ C & 0.1 & 0.1 & 0.8 \end{array}$$

Transition propability matrix after 2 days

$$P^2 = \begin{bmatrix} 0.16 & 0.47 & 0.37 \\ 0.12 & 0.44 & 0.44 \\ 0.12 & 0.19 & 0.69 \end{bmatrix}$$

ii. Initial probabilities are $\begin{pmatrix} 0.3 & 0.45 & 0.25 \end{pmatrix}$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ A $\quad$ B $\quad$ C

$$\begin{pmatrix} 0.3 & 0.45 & 0.25 \end{pmatrix}\begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} = \begin{pmatrix} 0.160, & 0.445 & 0.395 \end{pmatrix}$$

$\therefore$ after 1 day 16% of bikes at station A, 44.5% at B, 39.5% at C

After 2 days

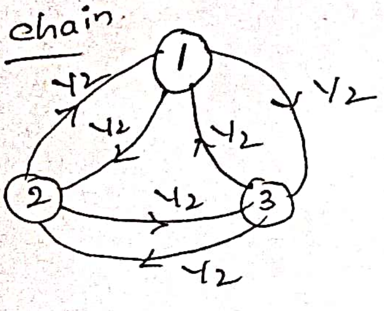
$$\begin{pmatrix} 0.160 & 0.445 & 0.395 \end{pmatrix}\begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} = \begin{pmatrix} 0.1320, & 0.3865 & 0.481 \end{pmatrix}$$

after 2 days 13.20% of bikes at station A, 38.65% at B 0.4815% at C

2. Three childrens denoted by 1, 2, 3 arranged in a circle to play a game of throughing a ball to one another at each stage the child having the ball is equally likely to throw it into any one of the other 2 children. Suppose that x_0 denote the child who had the ball initially and x_n $(n \geq 1)$ denote the child who had the ball after n throws. Calculate determine the transition propability matrix P. Calculate $Pr\{x_2 = 3 / x_0 = 2\}$, $Pr\{x_2 = 1 / x_0 = 1\}$, $Pr\{x_2 = 2 / x_0 = 3\}$, and the probability that the child who had originally the ball will have it after 2 throws

Sol:- Transistion probability matrix
$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \end{array}$$

Markov chain



$$i.\ Pr\{\cancel{x_2}=1 \mid x_0=1\} = \frac{P(x_1=1, x_0=1)}{P(x_0=1)} = 0$$

Since the throws are equally likely at all stages, we have

$$P(x_0=1)=P(x_0=2)=P(x_0=3) = 1/3 = 0.33$$

$$P(x_1=1)=P(x_1=2)=P(x_1=3) = 1/3$$

$$P(x_2=1)=P(x_2=2)=P(x_2=3) = 1/3$$

$$\therefore P(x_0=i), i) = 1/3,\ i=1,2,3$$

$$i.\ Pr\{x_2=1 \mid x_0=1\} = \frac{P(x_2=1, x_0=1)}{P(x_0=1)} = \frac{0}{1/3} = 0$$

$$ii.\ Pr\{x_2=2 \mid x_0=3\} = \frac{P(x_2=2, x_0=3)}{P(x_0=3)} = \frac{0.5}{\frac{1}{3}} = 1.5$$

$$iii.\ Pr\{x_2=3 \mid x_0=2\} = \frac{P(x_2=3, x_0=2)}{P(x_0=2)} = \frac{0.5}{\frac{1}{3}} = 1.5$$

probability that the child who will had the ball ~~after~~ 2 throws is given by the matrix P^2

$$\begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

3. The TPM of a Markov process $\{x_n\}$ has 3 states 1, 2 & 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ with initial distribution 0.7, 0.2 & 0.1 respectively find i. $P\{x_3=2, x_2=3, x_1=3, x_0=2\}$

ii. $P\{x_3=1 \mid x_2=3, x_0=2\}$ $[P(x_3=1) \mid P(x_2=3)]$

iii. $P\{x_2=3\}$ probability of state 3 after 2 time period.

Sol :- Given Transition probability matrix P:

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

and $q_0 = \begin{array}{c} [0.7 \quad 0.2 \quad 0.1] \\ \ \ 1 \qquad\ 2 \qquad\ 3 \end{array}$

i. $P\{x_3 = 2, x_2 = 3, x_1 = 3, x_0 = 2\}$:

$$q_0 \dashrightarrow 2 \xrightarrow{(1)} 3 \xrightarrow{(1)} 3 \xrightarrow{(1)} 2$$

$= q_0(2) P_{23} P_{33} P_{32}$

$= 0.2 \times 0.2 \times 0.3 \times 0.4 = 4.8 \times 10^{-3}$

ii. $P\{x_3 = 1 \mid x_2 = 3, x_0 = 2\} = P(x_3 = 1 \mid x_2 = 3)$

$q_0 \rightarrow 3 \xrightarrow{(2)} 1$

$\dfrac{P(x_3 = 1, x_2 = 3)}{P(x_2 = 3)} = \dfrac{0.4}{0.2}$ $= q_0(3) P_{31}$

$= 0.1 \times 0.3 = 0.03$

iii. $P(x_2 = 3)$

$$[0.7 \quad 0.2 \quad 0.1] \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} = (0.22 \quad 0.43 \quad 0.35)$$

After 2 time period

$$(0.22 \quad 0.43 \quad 0.35) \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} = (0.385 \quad 0.336 \quad 0.2$$