Components of automatic control system

1. The basic components of an automatic control system are error detector, Amplifier and controller, Actuator (power Actuator), plant and sensor or a feedback system. The block diagram of an automatic control system is shown in below figure.

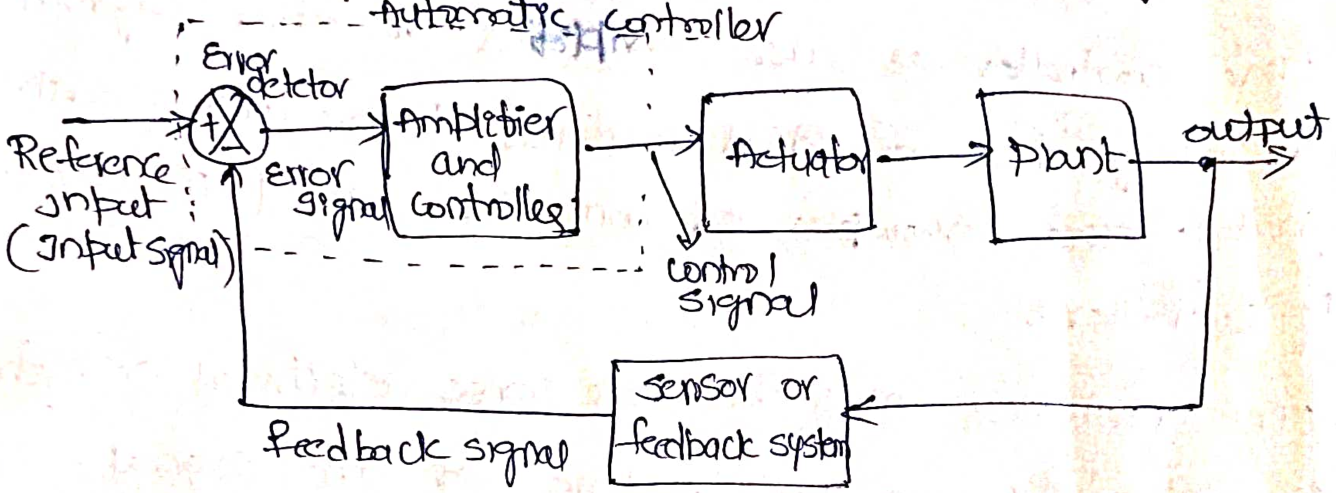


fig :- Block diagram of automatic control system

2. The plant is the open loop system whose output is controlled by closed loop system.

3. The combination of error detector, amplifier and Controller is called automatic controller. without this unit the system becomes open loop system.

4. In automatic control systems, the reference input is proportional to the desired output and the feedback signal is proportional to the current output of the system

5. The error detector compares the reference input and feedback signal and produces the error signal. As the ~~er~~ error signal is a weak signal it is further amplified by the amplifier and modifies by the controller for better control action.

6. The error signal is used to correct the output if there is a deviation from the desired value. The examples of error detector are potentiometer, LVDT.

7. The controller is also amplifies the error signal and integrates or differentiates to produce a control signal. The examples of controllers are P, PI, PD and PID controllers.

8. The controller may be electronic, electrical or hydraulic depending on the nature of the signal.

9. The actuator is a power amplifying device which amplifies the controller output and converts to the suitable form of energy for the plant.

10. This process is continues till we get the difference between reference input and feedback signal is zero as it indicates the output settles to the desired value.

11. The actuator may be pneumatic motor | value, hydraulic motor or electric motor. Examples of electric motors ~~are~~ employed ~~and~~ as actuator are DC servomotor, AC servomotor, stepper motor.

(12) The feedback system samples the output to produce a feedback signal which is proportional to current o/p.

(13) The feedback system converts the output variable to another suitable variable such as displacement, pressure or voltage.

(14) Transducers, Tachogenerators are used as a feedback systems.

Electromagnetic Field Motors:-

These motors are economical for higher power ratings, i.e., above 1KW. This type of servo motors are similar to conventional DC motors. It has the following features.

1. The number of slots and commutator segments is large to improve commutation.
2. Compoles and compensating windings are provided to eliminate sparking.
3. The diameter to length ratio is kept low to reduce inertia.
4. Oversize shafts are employed to withstand the high torque stressed...
5. Eddy currents are reduced by complete lamination of the magnetic circuit and by using low loss steel.

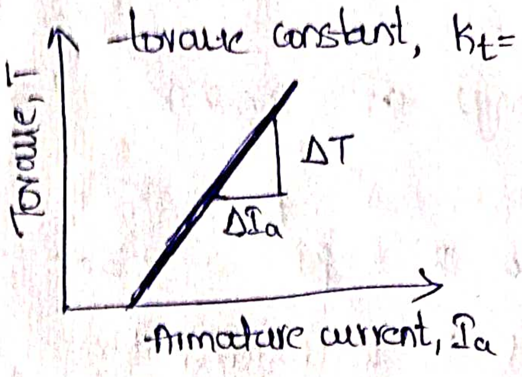
In this type of motor, the torque and speed are controlled by varying the armature current and/or the field current.

In armature controlled mode of operation, the field current is held constant and the armature current is varied to control the torque.

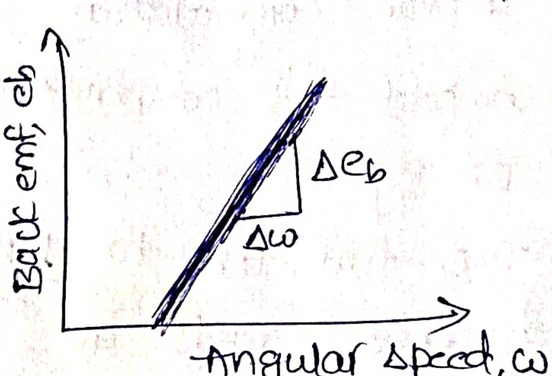
In the field controlled mode, the armature current is maintained constant and the field current is varied to control the torque.

Armature controlled DC servomotor :- It is a DC shunt Motor. The field is excited by a constant DC supply. If the field current is constant then speed is directly proportional to armature current and torque is directly proportional to armature current.

In small motors, the armature voltage is controlled by a variable resistance but in large Motors the armature voltage is controlled by thyristors in order to reduce power loss. The steady state operating characteristics of an armature controlled DC servomotors are shown in below figure.



I_a Vs T.



ω Vs e_b.

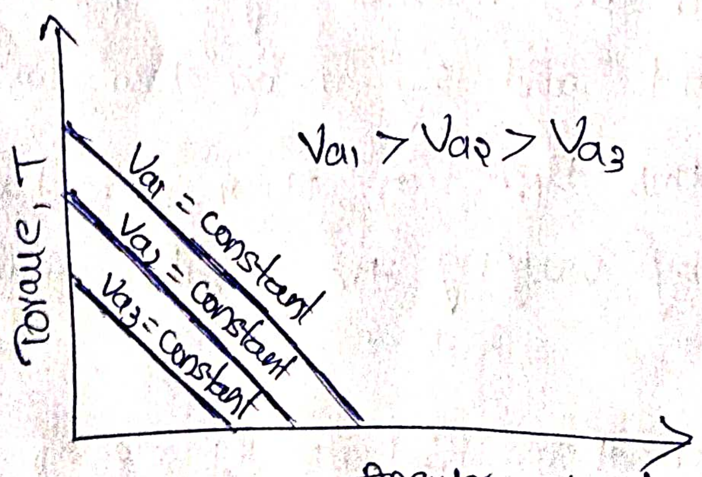
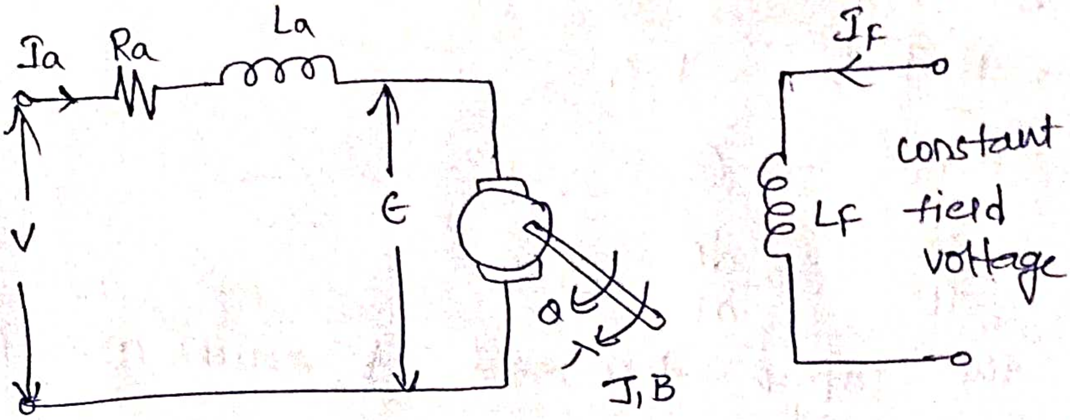


fig ω Vs T

fig: characteristics of armature controlled DC servomotor

Transfer function :-



R_a = armature Resistance

L_a = armature self inductance

i_a = armature current

i_f = field current

E = induced emf in armature

V = applied voltage

T = Torque

θ = angular displacement

J = Equivalent moment of Inertia

B = coefficient of friction.

Apply KVL to above circuit

$$V = R_a i_a + L\frac{di_a}{dt} + E \rightarrow (1)$$

when armature is rotating, the induced emf is

$$E \propto \phi\omega \Rightarrow E = K_{b1}\phi\omega$$

where ω is angular frequency.

As field current is constant replace $K_{b1}\phi$ with the one constant

$$E = K_b\omega$$

$$E = K_b\frac{d\theta}{dt} \rightarrow (2) \quad \left[\because \omega = \frac{d\theta}{dt}\right]$$

servo motor.

Torque is given

$$T \propto \phi i_a$$

$$T = \underline{K_1 \phi}\, i_a = K i_a$$

$$T = K i_a \rightarrow (3)$$

$$T = J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} \rightarrow (4)$$

Take the laplace transform of eqns (1),(2),(3),(4)

$$V(s) = R_a I_a(s) + L_a s I_a(s) + E(s) \rightarrow$$

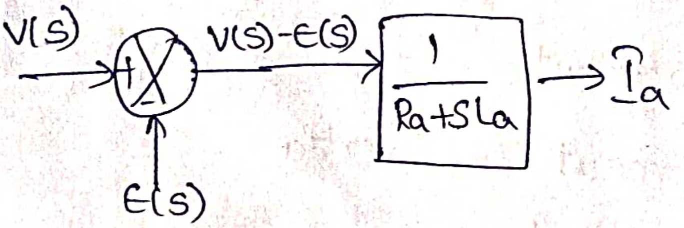
$$V(s) - E(s) = I_a(s)\left[R_a + s L_a\right] \rightarrow (a)$$

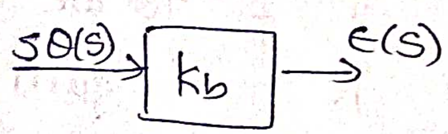
$$E(s) = K_b s \theta(s) \rightarrow (b)$$

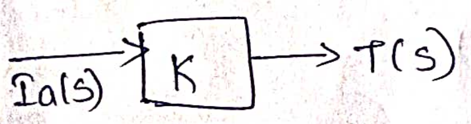
$$T(s) = K I_a(s) \rightarrow (c)$$

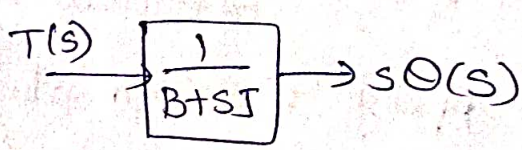
$$T(s) = s^2 J \theta(s) + B s \theta(s) \rightarrow$$

$$T(s) = s\theta(s)\left[sJ + B\right] \rightarrow (d)$$

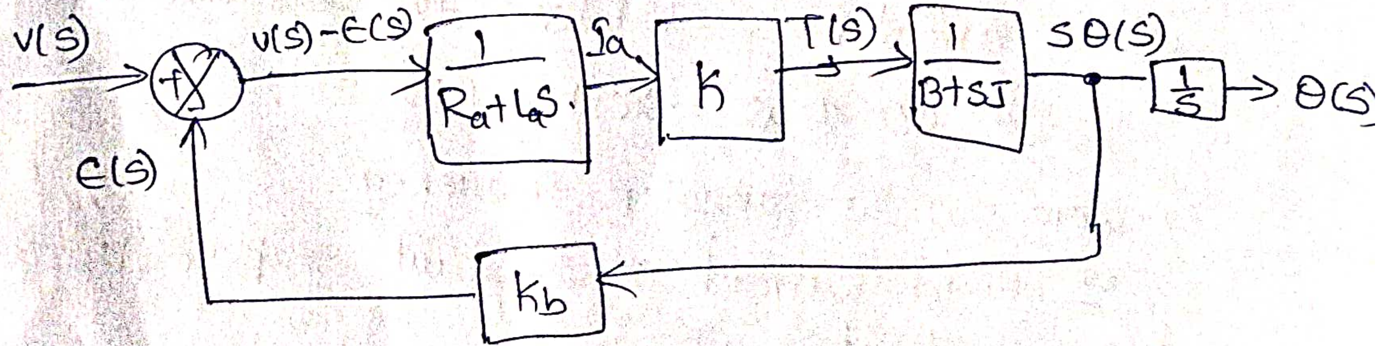


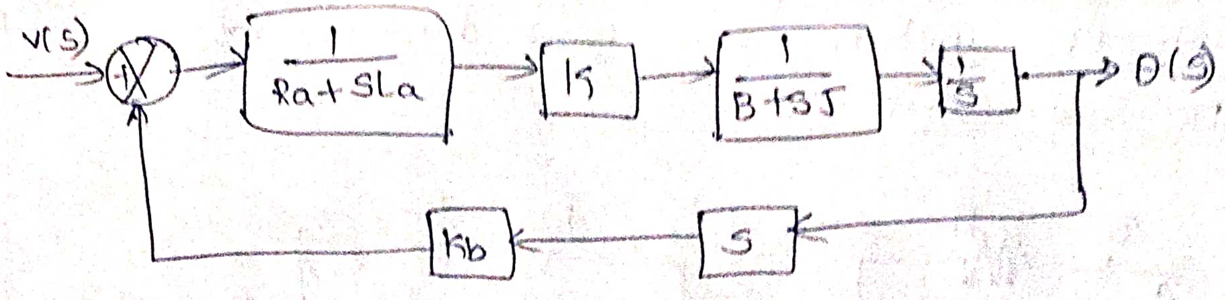


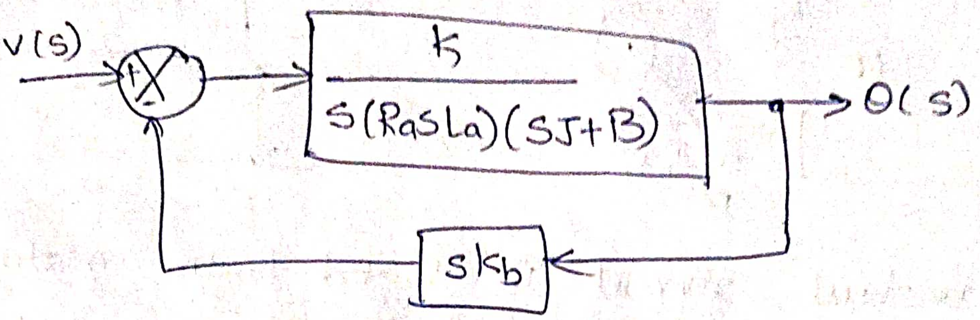




We know that V(s) is input and θ(s) is output







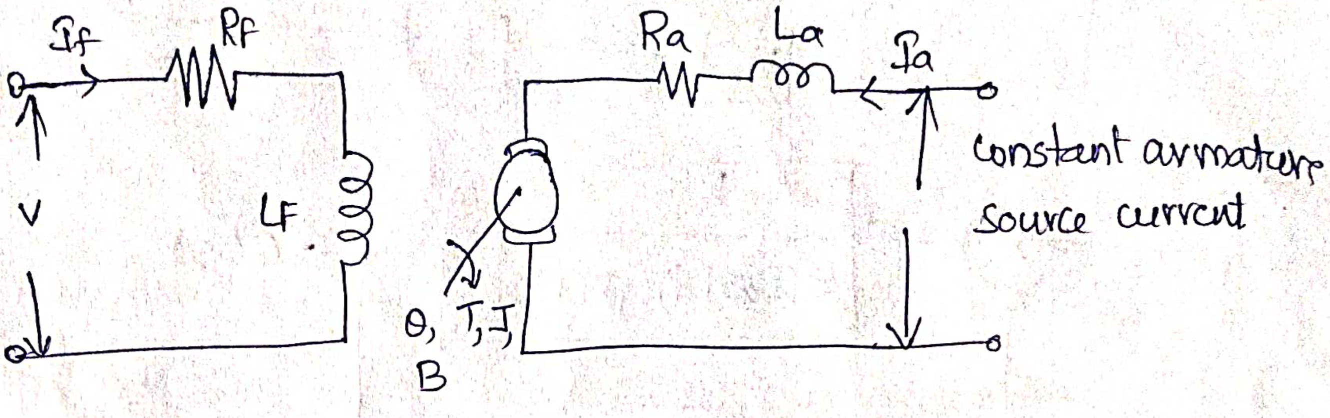
$$\frac{\theta(s)}{V(s)} = \frac{\dfrac{K}{S(RaSLa)(SJ+B)}}{1+\dfrac{K}{S(RaSLa)(SJ+B)}\,S\,K_b} = \frac{\cancel{\dfrac{K}{S(RaSLa)(SJ+B)}}}{\dfrac{S(RaSLa)(SJ+B)+SK_bK}{\cancel{S(RaSLa)(SJ+B)}}}$$

$$\boxed{\frac{\theta(s)}{V(s)} = \frac{K}{(Ra+SLa)(SJ+B)S+KK_bS}}$$

<u>Field controlled of DC Servomotor</u> :- It is a DC shunt motor. In this Motor the armature is applied or supplied with a constant current voltage.

When the armature voltage is constant the torque is directly proportional to the field flux but the field current is proportional to flux, therefore the torque of the motor is controlled by the controlling the field current.

Transfer function and Block diagram of field controlled DC servo motor :-



1. A constant current I_a is fed to the armature

2. Flux is proportional to the field current

$$\phi \propto I_f$$

$$\phi = K_f I_f \rightarrow (1)$$

3. Apply KVL to above circuit

$$V_f = R_f I_f + L_f \frac{dI_f}{dt} \rightarrow (2)$$

4. Torque is portportional to flux and armature current

$$T \propto \phi I_a$$

$$T = K' \phi I_a$$

$$= K' K_f I_f I_a$$

$$T = K K_f I_f \rightarrow (3) \quad [\because K' \Rightarrow \text{constant} \Rightarrow I_a$$

$$K = K' I_a$$

5. $$T = J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} \rightarrow (4)$$

θ = angular displacement

J = Moment of Inertia

B = coefficient of friction

Apply Laplace transform, we get

$$\phi(s) = K_f I_f(s)$$

$$V_f(s) = R_f I_f(s) + sL_f I_f(s)$$

$$= I_f(s)[R_f + sL_f]$$

$$I_f(s) = \frac{V_f(s)}{sL_f + R_f} \rightarrow (1)$$

$$T(s) = K I_f(s) K_f \rightarrow (2)$$

$$T(s) = J s^2 \theta(s) + B s \theta(s)$$

$$T(s) = \theta(s)[Js^2 + sB] \rightarrow (3)$$

Here output is angular displacement and input $V(s)$

Sub eqn (1) in eqn (2)

$$T(s) = K K_f \frac{V_f(s)}{R_f + sL_f} \rightarrow (4)$$

Sub eqn (4) in eqn (3)

$$\theta(s)[Js^2 + sB] = K K_f \frac{V_f(s)}{R_f + sL_f}$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K K_f}{s(sJ+B)(R_f + sL_f)}$$

$$= \frac{K K_f}{R_f B s\left(1 + s\frac{J}{B}\right)\left(1 + s\frac{L_f}{R_f}\right)}$$

$$\boxed{\frac{\theta(s)}{V_f(s)} = \frac{K K_f}{R_f B s(1 + s\tau_m)(1 + s\tau_f)}}$$

where $\tau_m = \frac{J}{B}$ = mechanical time constant

$\tau_f = \frac{L_f}{R_f}$ = time constant for field circuit

Ac Servomotor :- (1) The representation of Ac servomotor as a control system is shown in figure.

2. The referrence winding is excited by a constant voltage source with a frequency range 50 to 1000Hz.

3. The system can be made less susceptible to low frequency noise ~~we~~ when we operate at more than 400Hz.

4. Due to this feature, ac devices are used in aircraft and Missile control system.

5. The control windings is excited by the modulated control signal as the control signal is low frequency signal ranges from 0 to 20 Hz.

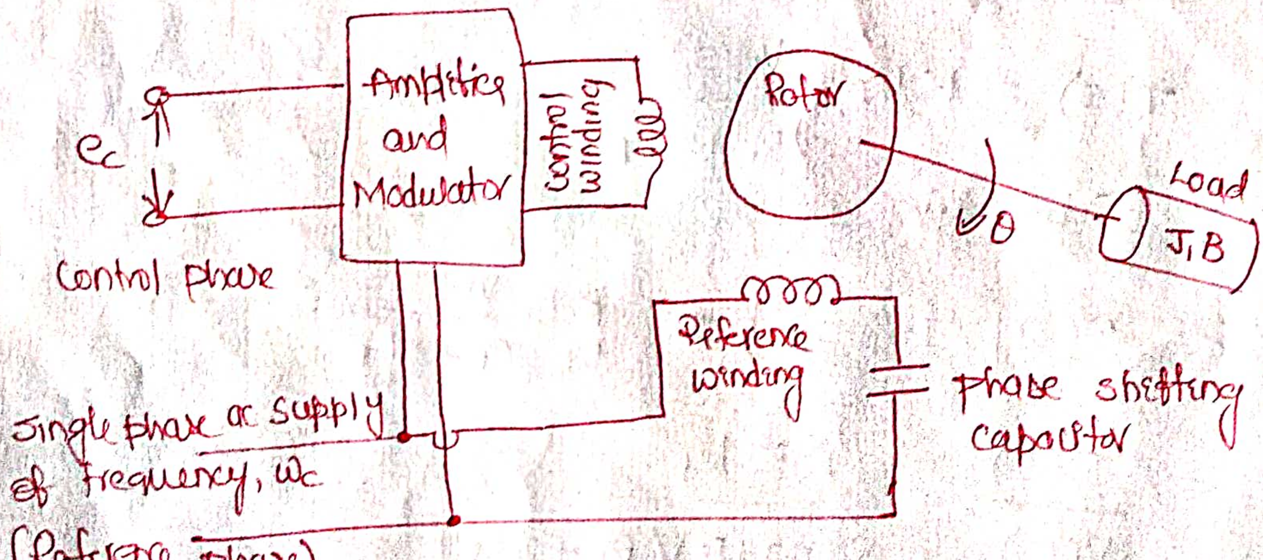


fig: symbolic representation of an ac servomotor.

6. In order to produce rotating magnetic field, the control phase voltage must and reference phase voltage must have same frequency and two voltages must be in time quadrature.

7. Hence the control signal is modulated by a carrier whose frequency is equal to the reference voltag frequency and this modulated signal is applied to control winding.

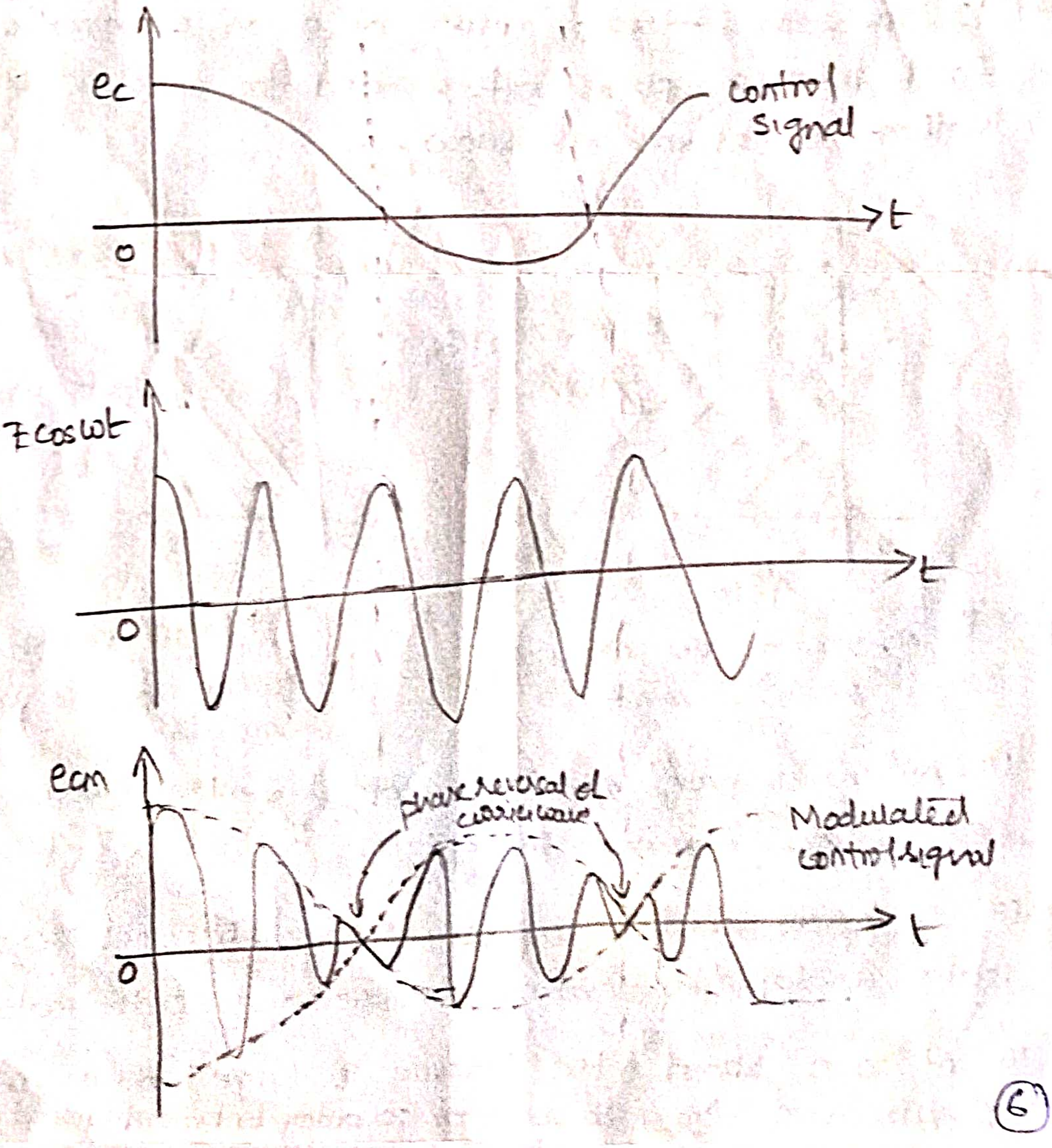
8. Here we are performing amplitud Modulation and the information is available on the envelope of the modulated Sign

9. The 90° phase difference between control signal and reference signal is obtained by the insertion of a capacitor in reference winding.

10. In modulation process control signal is modulating signal : e; e_c = control signal.

e_{car} = carrier signal = $E \cos \omega_c t$

e_{cm} = Modulated control signal.



11. In above figure, if e_c is positive then phases of e_{cm} and e_{car} are same. on ther hand if e_c is negative then there is 180° phase difference.

$$\text{i.e., } e_{cm} = |E + e_c| \cos \omega_c t \quad \text{for } e_c > 0$$

$$|E + e_c| \cos(\omega_c t + \pi) \quad \text{for } e_c < 0$$

12. This phase reversal changes the direction of rotation of the magnetic field and hence reversal in the direction of rotation of the Motor shaft.

13. The speed-torque curves of a ac servomotor for fixed reference phase voltage and for constant control voltage is shown in below.

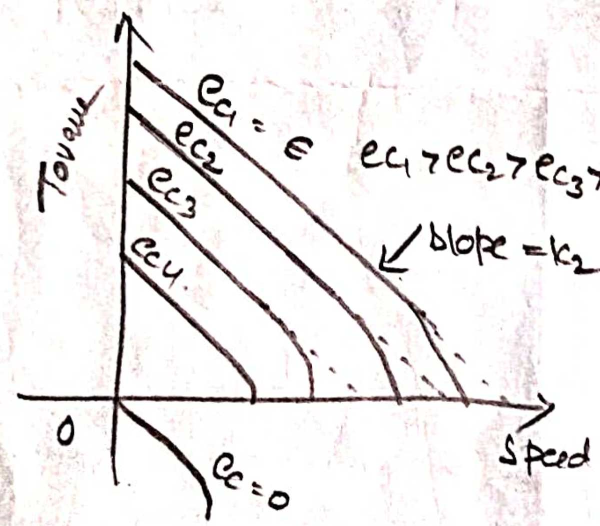


fig a) speed torque curves of an ac servomotor.

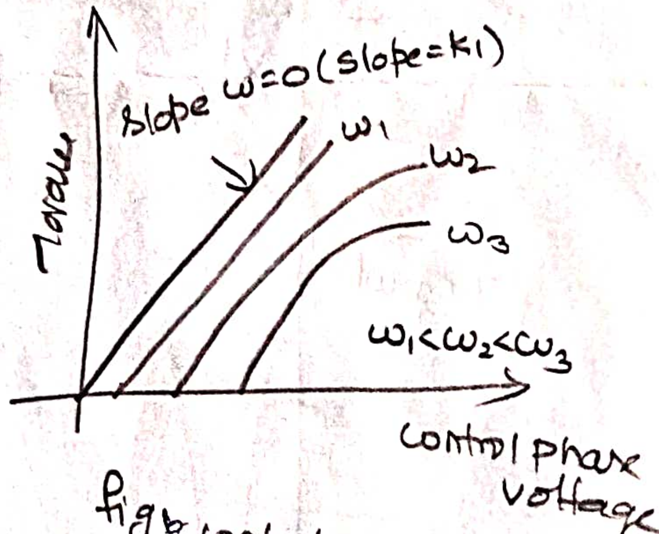


fig b control voltage Vs Torque curves of an ac servomotor.

14. All curves have negative slope but the curves for $e_c = 0$ goes through the origin.

15. So when control phase voltage becomes zero, the motor develop decelerating torque and so the motor stops

16. At zero speed, the torque is large and we need they requirement for a servomotor in order to provide rapid acceleration

(17) The speed - torque curves of ac servomotor is linear at low speed region and Non linear at high speed region.

(18) In order to derive transfer function for the motor, we will take linearizing approximations.

(19). so we are extending linear portions of speed-torque curves to high speed regions as shown by dashed lines.

(20) even with this extension, the curves are not still parallel to each other which indicates that the torque does not vary linearly with input voltage e_c for constant speed as shown in fig (b).

Transfer function of Ac servomotor:-

Let, T_m = Torque developed by servomotor

θ = Angular displacement of rotor.

ω = Angular speed = $\frac{d\theta}{dt}$.

T_l = Torque required by the load.

J = Moment of inertia of load and the rotor

B = Viscous frictional coefficient of load and the rotor.

K_1 = slope of control phase voltage Vs Torque characteristics.

K_2 = slope of speed - torque characteristic.

Torque developed by the motor, $T_m = K_1 e_c - K_2 \frac{d\theta}{dt}$

$\rightarrow (1)$

The rotating part of the Motor and the load can be modelled by equation

Load torque, $T_l = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}$ → (2)

$$J\frac{d^2\theta}{dt} + B\frac{d\theta}{dt} = K_1 e_c - k_2\frac{d\theta}{dt}$$

On taking Laplace transforms with zero initial conditions, we get

$$J s^2\theta(s) + Bs\theta(s) = K_1 E_c(s) - k_2 s\,\theta(s)$$

$$\left[Js^2 + Bs + k_2 s\right]\theta(s) = K_1 E_c(s)$$

$$\frac{\theta(s)}{E_c(s)} = \frac{K_1}{s(Js+B+k_2)} = \frac{\frac{K_1}{B+k_2}}{s\left(\frac{J}{B+k_2}s+1\right)} = \frac{K_m}{S(\tau_m s+1)}$$

where, $K_m = \frac{K_1}{B+k_2}$ = Motor gain constant

$\tau_m = \frac{J}{B+k_2}$ = Motor time constant

Potentiometer :- 1. A potentiometer is a device which can be used to convert a linear or angular displacement into a voltage.

2. A potentiometer is a variable resistance whose value varies according to angular displacement or linear displacement of the wiper contact.

3. The resistance element canbe constructed by depositing a conducting material on a plastic base.

4. The potentiometer has an input shaft to which a wiper attached.

5. The potentiometer is excited by a dc or ac voltage. The output voltage is measured at wiper contact with respect to reference.

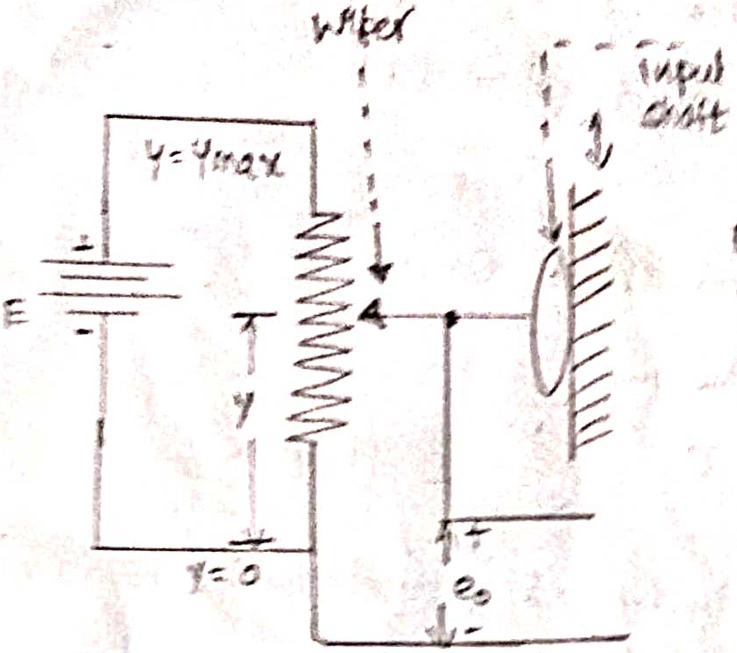


fig:- Linear displacement potentiometer

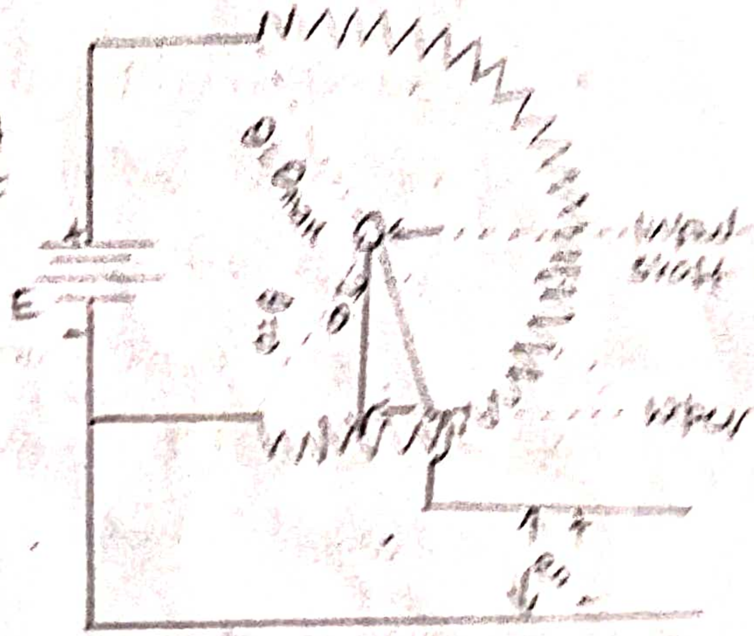


fig: angular displacement potentiometer

Linear displacement potentiometer

Y = Displacement of the wiper contact from reference ($y=0$)

Y_{max} = maximum displacement of wiper contact

E = Excitation voltage of potentiometer

e_0 = output voltage

If $y=0$, then $e_0=0$, and if $y=Y_{max}$, then $e_0=E$

for displacement y,

$$\text{output voltage}, \; e_0 = \frac{E}{Y_{max}} y = K_p y$$

where, $K_p = \frac{E}{Y_{max}}$ = sensitivity of potentiometer in volts/mm

Angular displacement potentiometer :-

θ = angular displacement of wiper arm from reference ($\theta = 0$)

θ_{max} = Maximum displacement of wiper arm

E = Excitation voltage of potentiometer

e_o = output voltage

If $\theta = 0$, then $e_o = 0$ and if $\theta = \theta_{max}$ then $e_o = E$

for displacement θ,

$$\text{output voltage, } e_o = \frac{E}{\theta_{max}}\theta = K_p\theta$$

where $K_p = \frac{E}{\theta_{max}}$ = sensitivity of potentiometer in volts/deg

Applications of potentiometer :-

1. Potentiometers can be used either to convert a mechanical motion to proportional voltage or as an error signal.

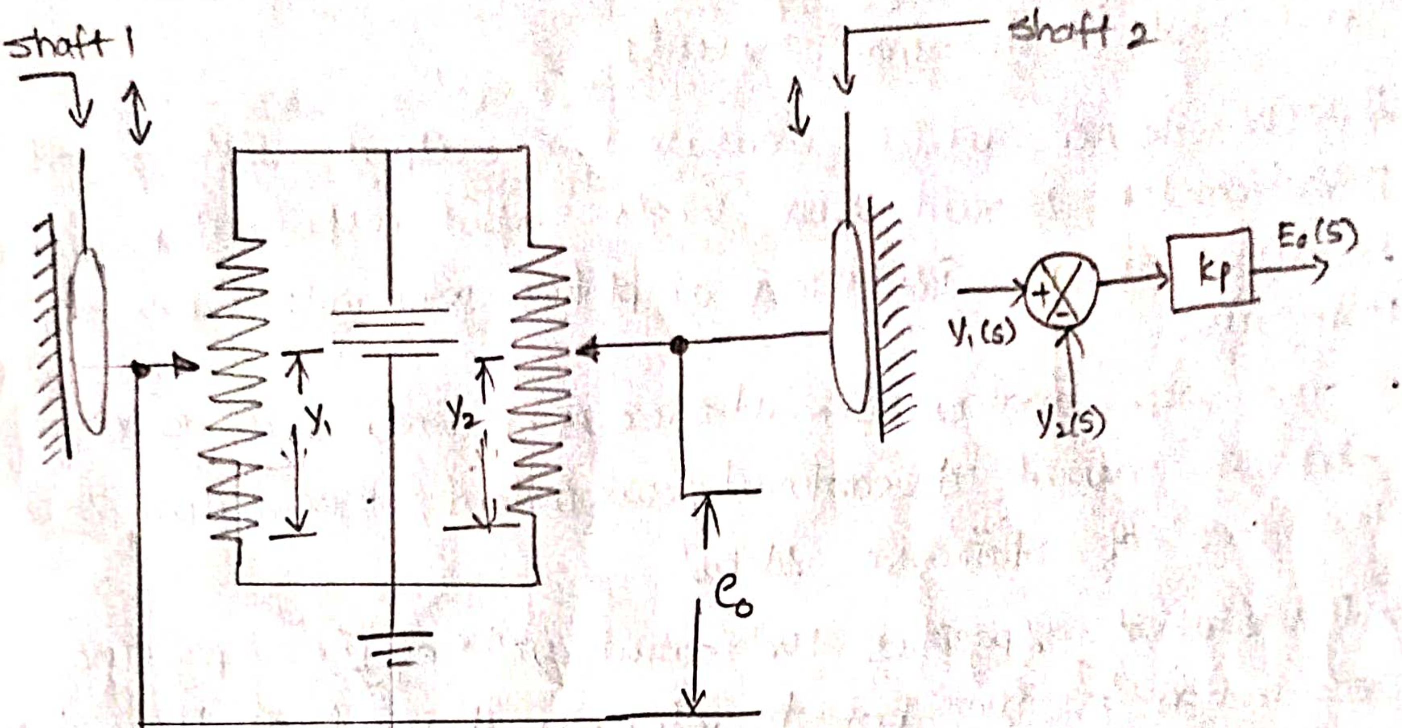
2. A potentiometer excited by dc or ac voltage is used to produce an output voltage proportional to displacement of the input shaft.

3. When potentiometers are used as error detector, two identical potentiometers are required as shown in figure.

4. Both potentiometer are excited by the same source with same voltage

5. If the wiper arm of both the potentiometers are in the same position then the voltage between two wiper arms is zero.

6. The position of one wiper arm is kept as reference input and the displacement is applied to another arm of potentiometer.



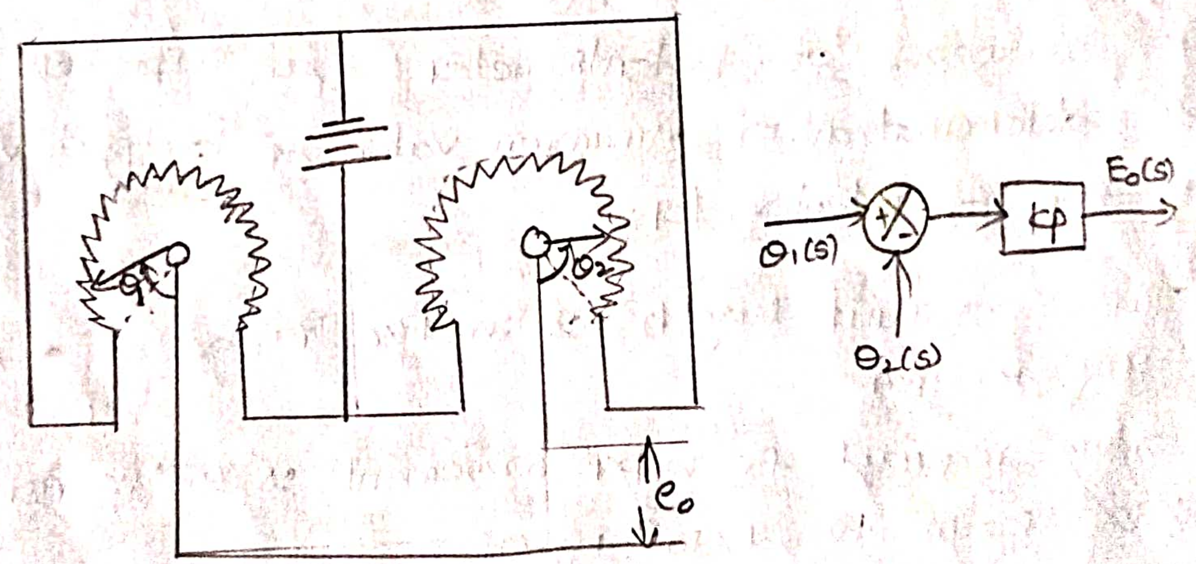


fig:- potentiometer as error detector.

7. Hence the output voltage which is measured between the two arms is proportional to the difference between the displacement of both the wiper arms.

Characteristics of potentiometer:-

1. In ideal characteristics of potentiometer, the Resistance is varied linearly with displacement. This can be achieved by having large radius, more number of turns and high resistance elements.

2. The device which Measure the output voltage of potentiometer should have high input impedance to avoid loading error. Isolation amplifier with with high input impedance.

3. The wiper makes simultaneous contacts with adjacent turns to avoid discontinuity in output. The output is in the form of staircase steps.

4. Resolution specifies the output voltage per step. The resolution of the potentiometer is defined as the ratio of number of ~~to~~ steps to ~~total~~ number of turns.

5. The resolution of potentiometer is an important factor In the determination of Minumum value of output voltage

Specification of potentiometer:-

1. Turns per unit length is in the range of 6 to 3[illegible] turns per mm.
2. Torque required for wiper movement is in the range of 1×10^{-3} kg-m to 1×10^{-2} kg-m.
3. The total resistance of the potentiometer is in the range of 5 ohms to 1-mega ohms.
4. power rating is 1 to 10 watts.
5. Heat dissipation is 1/2 watts per cm^2.

6. Excitation voltage is 11 to 90 volts

7. Voltage gradient is 0.01 to 0.05 volts per degree

Synchros:- A synchro system is a interconnection of synchro transmitter and the synchro control transformer devices and referred to as synchro pair.

The synchro pair compares the angular displacement and produces the output which is proportional to angular difference of the position of both the shafts.

This can be used in two ways

i) To control the angular position of load from a remote place / long distance.

ii) For automatic correction of changes due to disturbance in the angular position of the load.

Synchro Transmitter:-

Construction:- Synchro transmitter consists of stator and rotor. Stator is made of laminated silicon steel and slotted on inner periphery to achieve three phase winding. The stator winding is star with three coils 120° apart. The stator winding is star connected.

The rotor is of dumbbell construction with a single winding and windings are terminated on two slip rings. The single phase ac voltage is applied to rotor through slip windings.

Working Principle:- When the rotor is excited by ac voltage then current flows rotor produces magnetic field due to this field an emf is induced in stator coils.

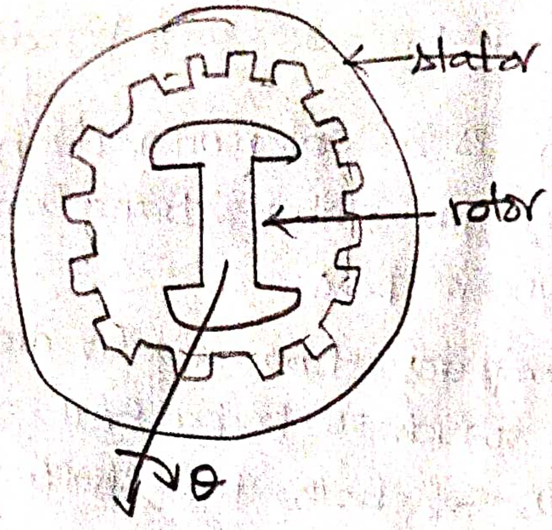


fig:- constructional features

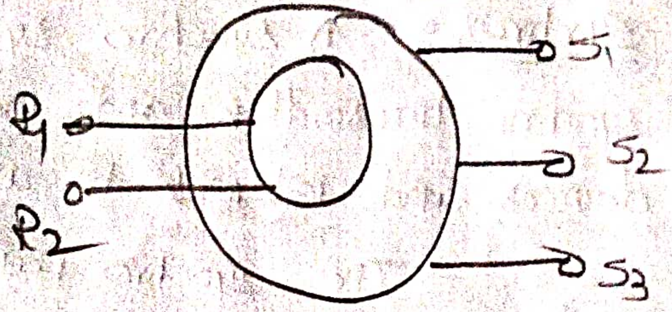


fig:- schematic symbol of a synchro transmitter

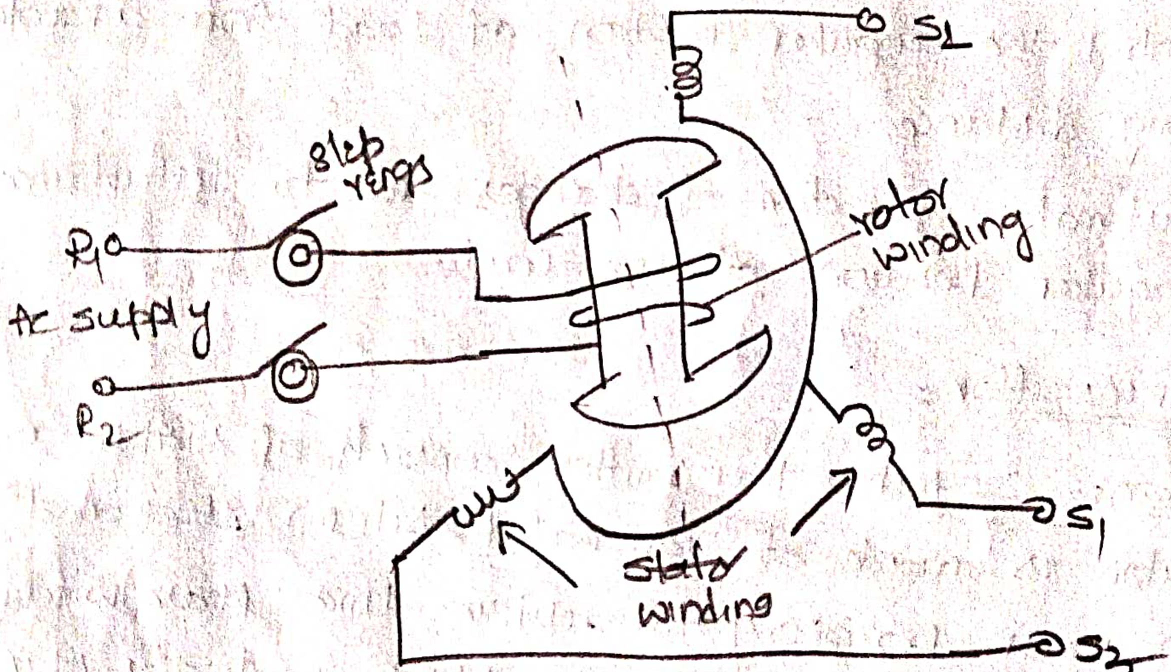


fig : electrical circuit

fig:- synchro transmitter.

Let e_r = Instantaneous value of ac voltage applied to rotor.

e_{s1}, e_{s2}, e_{s3} = Intantaneous value of emf induced in stator coils s1, s2, s3 with respect to neutral respectively.

E_r = Maximum value of rotor excitation voltage.

The voltage induced in stator coil depend on the angular position of the coil's axes with respect to rotor axes. When the rotor rotates in anti clockwise direction by angle θ, emfs are induced in stator coils. The frequency of induced emf is same as that of rotor frequency. The magnitude of induced emfs are proportional to the turns ratio and coupling coefficient. The turns ratio K_t is a constant but coupling coefficient K_c is a function of rotor angular position.

induced emf in star coil $= K_t K_c E_r \sin\omega t$.

Let e_{s2} be the reference vector.

when $\theta = 0$, the flux linkage of coil S_2 is maximum and when $\theta = 90^\circ$ the flux linkage of coil S_2 is zero. Hence the flux linkage of coil S_2 is a function of $\cos\theta$. The flux linkage of coil S_3 will be maximum after a rotation of 120° in anticlockwise direction and that of S_1 after a rotation of 240°.

Coupling coefficient, K_c for coil-S_2 $= K_1 \cos\theta$.

Coupling coefficient, K_c for coil-S_3 $= K_1 \cos(\theta - 120)$

coupling coefficient, K_c for coil-S_1 $= K_1 \cos(\theta - 240)$.

Hence emfs of stator coils with respect to neutral

$$e_{s2} = K_t K_1 \cos\theta \, E_r \sin\omega t = K E_r \cos\theta \sin\omega t \rightarrow (1)$$

$$e_{s3} = K_t K_1 \cos(\theta - 120) \, E_r \sin\omega t = K E_r \cos(\theta - 120) \sin\omega t$$

$$e_{s1} = K_t K_1 \cos(\theta - 240) \, E_r \sin\omega t = K E_r \cos(\theta - 240^\circ) \sin\omega t$$

By KVL, the coil-to-coil emf can be expressed as

$$e_{s1s2} = e_{s1} - e_{s2} = \sqrt{3}\, K E_r \sin(\theta + 240) \sin\omega t \quad \rightarrow (2)$$

$$e_{s2s3} = e_{s2} - e_{s3} = \sqrt{3}\, E_r K \sin(\theta + 120) \sin\omega t$$

$$e_{s3s1} = e_{s3} - e_{s1} = \sqrt{3}\, K E_r \sin\theta \sin\omega t.$$

When $\theta = 0$, in eqn (1) the maximum emf is induced in coil S_2 but in eqn (2) the coil-to-coil voltage e_{s3s1} is zero. This position of the rotor is defined as the electrical zero of the transmitter. The electrical zero position is used as reference for specifying the angular positions of the rotor.

Here the input to the synchro transmitter is the angular position of its rotor shaft and the output is a set of three stator coil-to-coil voltages.

Synchro control transformer :- The constructional features of synchro control transformer is similar to that of synchro transmitter, except the shape of rotor. Here the rotor is made cylindrical so that the air gap is uniform. Due to this feature, we can minimizes the changes in the rotor impedance

Working :- The generated emf of the synchro transmitter is applied as input to the stator coils of control transformer. The rotor shaft is connected to the load whose position has to be maintained constant at the desired value.

Depending on the current position of the rotor and applied emf on the stator, an emf am emf is induced in the rotor winding. Thus an emf in induced in the rotor winding. This emf is ~~used~~ used to drive a motor so that the position of the load is corrected.

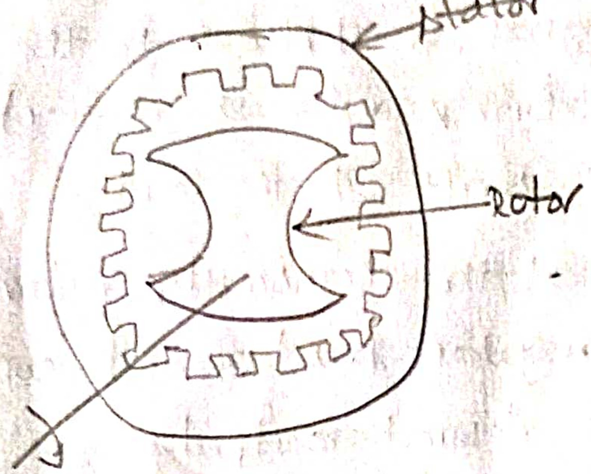


fig:- constructional feature

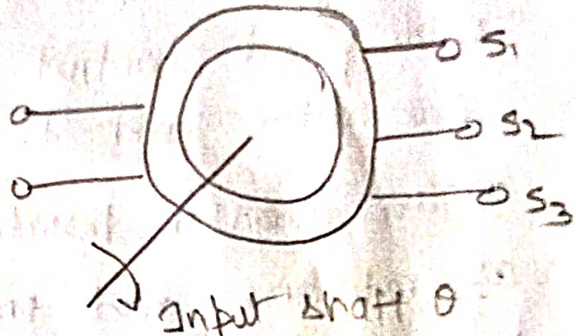


fig:- schematic symbol of a synchro control transformer

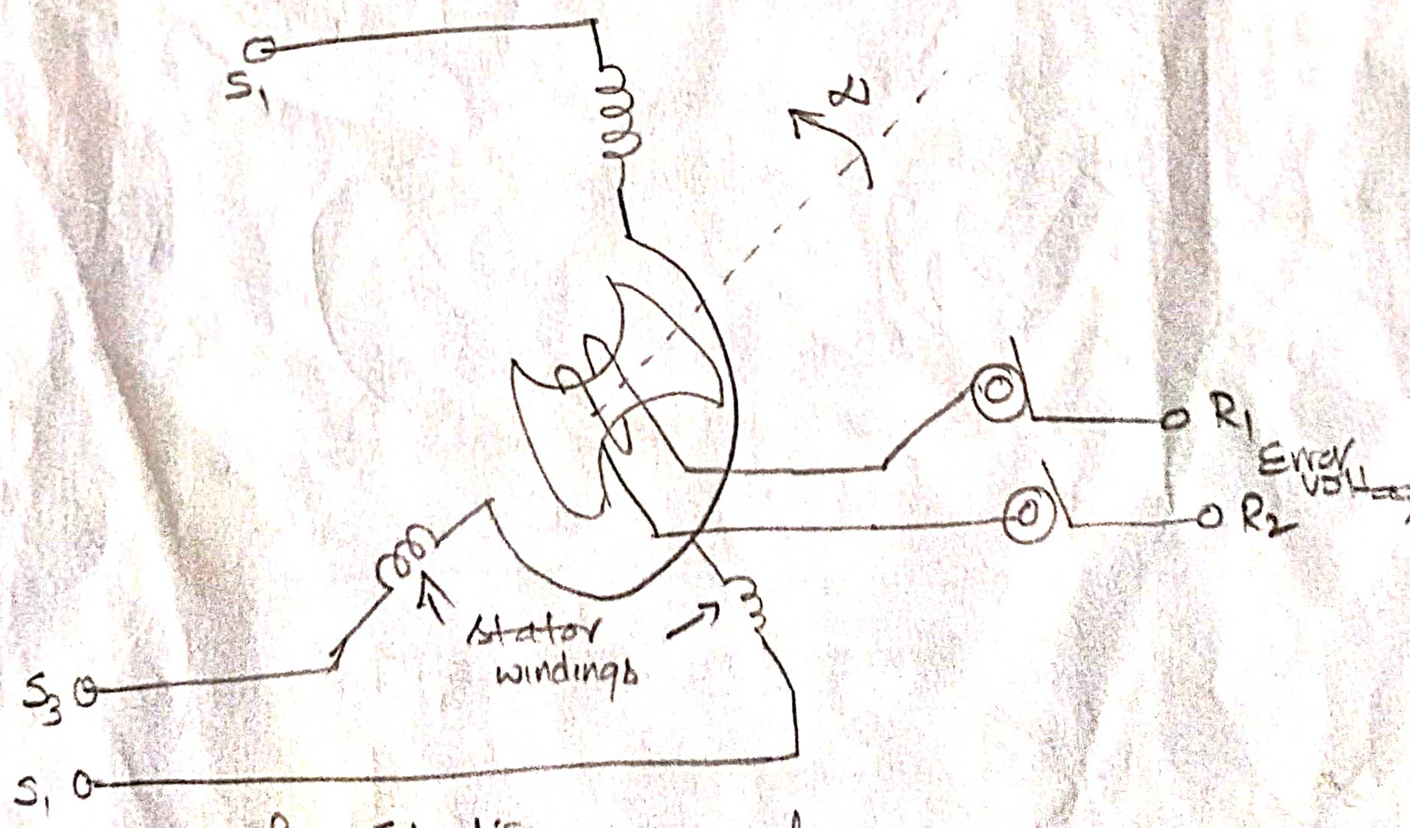


fig Electrical circuit

fig: synchro control transformer

Synchro as error detector :-

1. The synchro error detector is an interconnection of a synchro transmitter and synchro control transformer.

2. The stator leads of of transmitter are directly connected to the stator leads of the control transformer.

3. Here the input ac signal is applied to the transformer rotor and control transformer rotor is connected to a servo motor and to the shaft of the load.

4. The angular position of the transmitter rotor is used as reference and the angular position of control transformer rotor is the desired output.

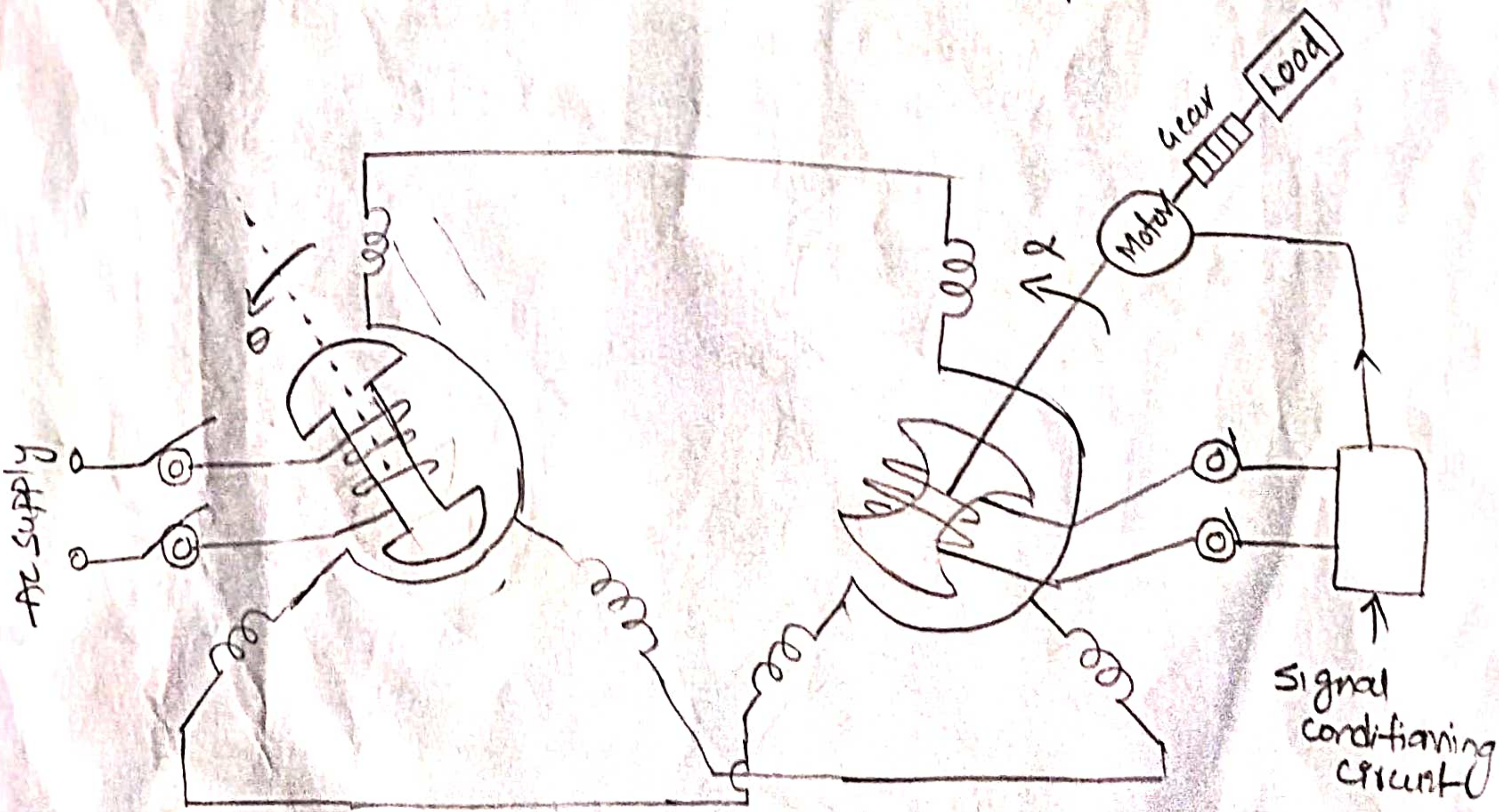


fig servo system using synchro error detector.

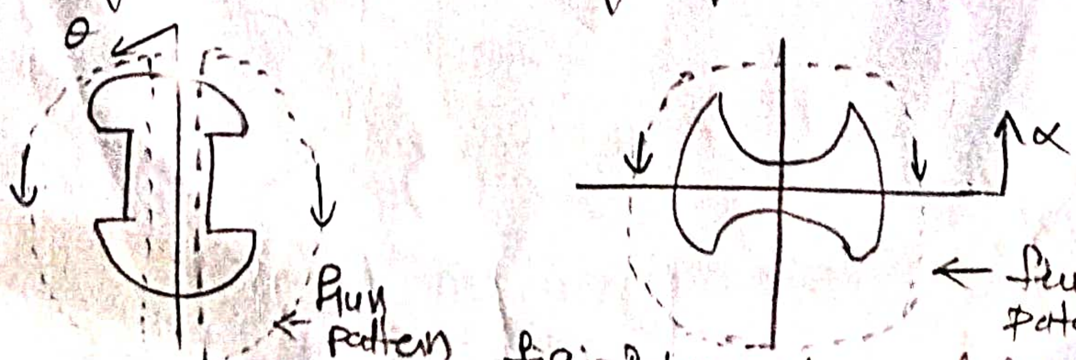


fig:- Rotor positions and flux pattern

5. when the transmitted rotor is excited by ac voltage flux is produced in rotor thereby producing emfs are induced in stator coils.

6. The induced emf available accross the rotor slip rings of control transformer is measured by a signal conditioning circuit.

7. The output of signal conditioning circuit is used to drive motor to find the position of load.

8. Initially rotor shafts of transmitter and control transformer are in alinged position.

9. In this case the transmitter rotor is in electrical zero position and control transformer rotor is in null position

10. The angular difference of both rotor axis is 90° i.e, alinged position.

11. The null position of a control transformer rotor indicates the position of rotor at which the output of rotor winding is zero.

12. ~~fro~~ when the transmitter rotor rotates through an angle θ from its electrical position then the synchro control transformer also rotates in the same direction by an angle α.

13. The angular difference between two rotor axis is equal to (90-θ+α) and the voltage is induced in in control transformer rotor is proportional to cosine of this above angle.

14. The error voltage is amplified and used to drive a servo motor.

(15) voltage accross slip rings of control transformer is

$$e_m = K' E_r \cos(90-\theta+\alpha) \sin\omega t$$

$$= K' E_r \cos(90-(\theta-\alpha)) \sin\omega t$$

$$= K' E_r \sin(\theta-\alpha) \sin\omega t$$

where K' is proportional constant.

Let $\phi(t) = \theta - \alpha$

for small values of $\phi(t)$, $\sin(\theta-\alpha) = \sin\phi(t) \cong \phi(t)$

$$\boxed{e_m = K' E_r \phi(t) \sin\omega t}$$

(16) From above equation we can say that output voltage of the synchro error detector is a modulated signal with carrier frequency.

(17) The signal conditioning circuit demodulates the voltage accross slip rings and develops a demodulated signal and amplified error voltage to drive the motor.

The demodulated error voltage, $e = K_s \phi(t)$

where K_s = sensitivity of synchro error detector in volt/degree

on taking Laplace transform to above equation we get

$$E(s) = K_s \phi(s)$$

$$\therefore \boxed{K_s = \frac{E(s)}{\phi(s)}}$$

The above equation is the transfer function of synchro error detector.

Tachometer :- 1. Tachometer used for measuring the the rotational speed and or angular velocity of the machine.

2. It works on the principle of relative motion between the magnetic field and shaft of the machine.

3. This relative motion induces emf in the conducting coil called armature which is placed between two permanent magnets.

4. The induced emf is directly proportional to the speed of the shaft.

5. Basically we are having mechanical tachometer and electrical tachometers

6. Mechanical tachometer measures speed of shaft in revolution per minute.

7. The electrical tachometer converts the angular velocity into an electrical voltage.

8. Depends on the nature of induced voltage the electrical tachometer are of two types.

i) AC Tachometer Generator.

ii) DC Tachometer Generator.

DC Tachometer Generator:-

1. It contains permanent magnet, armature commutator, brushes, variable resistor and the moving coil voltmeter

2. The machine whose speed is to be measured is coupled with the shaft of the of the DC tachometer generator.

3. When the closed conductors (armature) moves in the magnetic, an emf induced in the conductor and it depends on flux link with the conductor and the speed of the shaft.

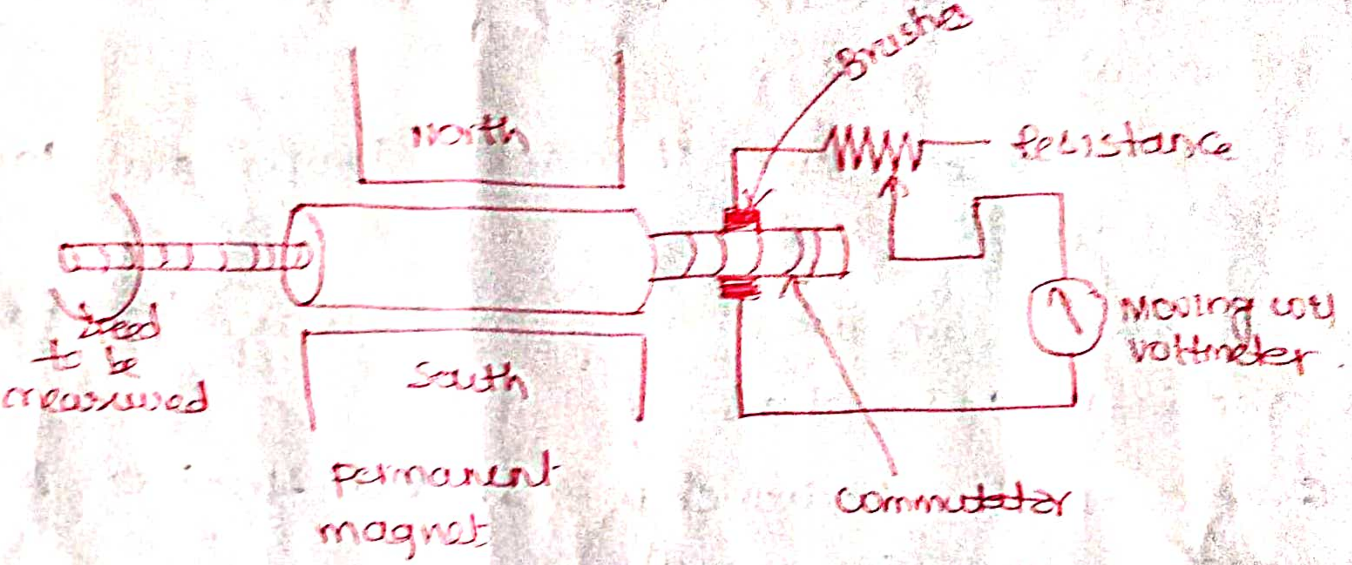


Fig:- DC Tachometer generator.

4. The commutator converts the alternating current of the armature coil to the direct current with the help of brushes.

5. The resistance is connected in series with the voltmeter for conducting heavy current of the armature.

6. The moving coil voltmeter measures the induced emf with polarity indicating the direction of motion of the shaft.

The induced emf is given as

$$E = \frac{\phi P N}{60} \times \frac{Z}{a}$$

where, E - generated voltage

ϕ - flux per pole in weber

P - Number of poles.

N - speed in revolution per minutes.

Z - Number of the conductor in armature windings.

a - Number of parallel paths in the armature windings.

$$E \propto N$$

$$E = KN$$

$$K = \text{constant} = \frac{\phi P}{60} \times \frac{Z}{a}$$

Advantages:-

1. ~~The commutator and brushes~~ The polarity of the induces voltage indicates the direction of rotation of the shaft

2. The conventional DC type voltmeter is used for measuring the induces voltage.

Disadvantages of DC generator:-

1. The commutator and brushes require the periodic maintenance.

2. The output resistance of the DC tachometer is kept high as compared to input resistance.

AC Tachometer generator :-

1. The DC tachometer generator uses the commutator and brushes which have many disadvantages.

2. We can overcome the above by having AC tachometer generator.

3. The AC tachometer has stationary armature and rotating magnetic field.

4. The rotating magnetic field produces emf in the stationary coil of stator.

5. The amplitude and frequency of the induced emf are proportional to speed of the shaft.

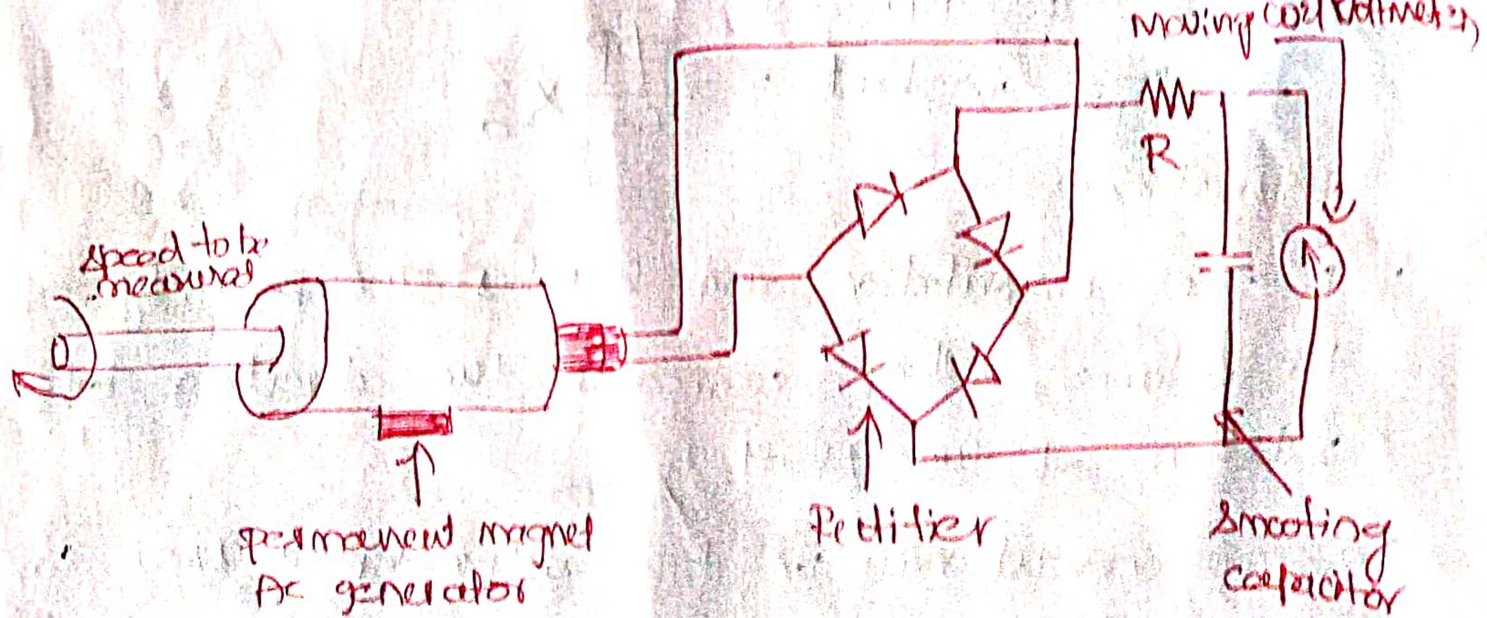


fig AC Tachometer generator

6. The above circuit is used to measure the speed of the machine (rotor) by taking the amplitude of the induced voltage.

7. The induces voltage are rectified and then passes through the capacitor filter for smoothening the ripples of rectified voltages.

AC position control system :-

① Servosystems have wide range of applications in automated industries, space crafts, missles, robots etc.

② An AC servomotor is a two phase AC Motor. It consists of a reference winding and control winding which are seperated by 90° and supplied with AC voltage phase shifted by 90°.

③ The torque produced in the Motor is adjusted controlled by adjusting the control winding voltage.

④ In AC position control system, AC servomotor is controlled in closed loop system mode by using either position or velocity feedback or the combination of both and this mechanism is called servo mechanism.

⑤ An error voltage with amplitude e corresponding to position of synchro transmitter rotor is produced.

⑥ In position feedback this error voltage is amplified and given to control winding i.e,

$$V_c = k_1 e$$

⑦ In position plus velocity feedback the error voltage and its derivative is used for feedback. i.e,

$$V_c = K_1 e + K_2 \frac{d}{dt} e$$

⑧ Torque produced by AC servomotor is directly proportional to control winding voltage.

⑨ Hence the rotor of servomotor start move towards the reference position.

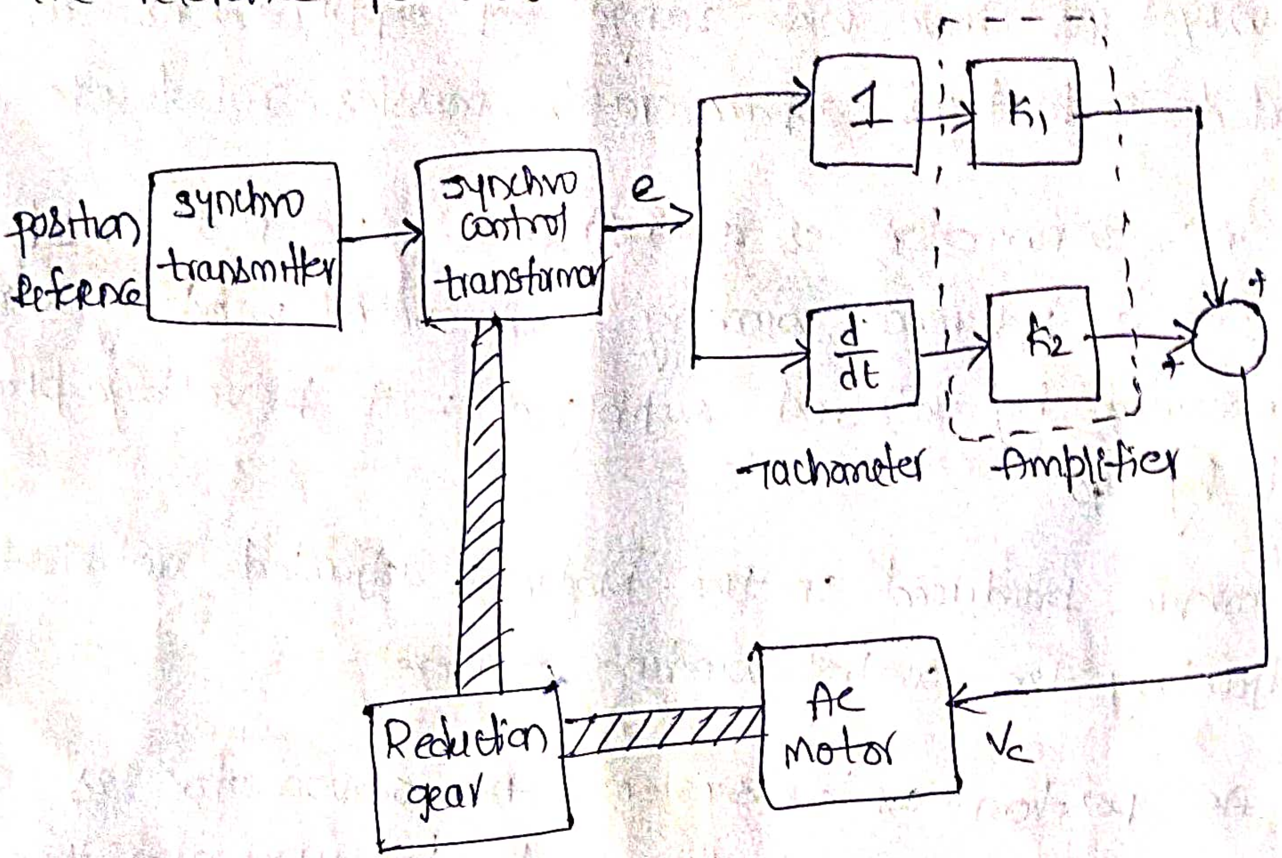


fig:- AC position control system.

Controllers :-

→ controllers will control the steady state error and transient response of the system.

ⓐ P- controller :- It is called as proportional controller. It is a device which produces output which is proportional to a input. It improves steady state error and also stability. It decreases sensitivity of the system to parameter variations. It produces constant steady state error

Transfer function = $G_c(s) = K_p$

② P-I controller :- It is known as proportional integral controller. The output of the device proportional to input and also integral of input. It increases order of system by due to this steady state error reduces. The system is less stable compared with original system.

$$\text{Transfer function} = G_c(s) = K_p + \frac{K_i}{s}$$

where, K_p = proportionality constant

K_i = integral constant

③ PD controller :- It is known as proportional derivative controller. The output of the device contains two terms, one term is proportional to input signal and another term proportional to derivative of input. It increases the damping ratio but there is a reduction in peak overshoot.

$$\text{Transfer function} = G_c(s) = K_p + K_d s$$

where K_p = proportionality constant

$K_d s$ = Derivative constant.

④ P.I.D controller :- It is known as proportional, integral derivative controller. The output of the system contains theree terms. first term is proportional to input, second term proportional integral of input and third term proportional to derivative of input. It stabilizes the gain and reduces the steady state error and also reduces the peak over shoot.

$$\text{Transfer function} = G_c(s) = K_p + \frac{K_i}{s} + K_d s$$

Effect of adding a zero to a system :-

Let us consider a second order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \quad \text{(original system)}$$

Now add a zero at $s=-z$ to the above second order closed loop system.

$$\longrightarrow \boxed{s+z} \longrightarrow \boxed{\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}} \longrightarrow$$

$$\frac{C_z(s)}{R(s)} = \frac{(s+z)\,\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

After adding a zero to the transfer function affects the gain of the system. In order to make the gain unaffected, divide the numerator by z.

$$\frac{C_z(s)}{R(s)} = \frac{(s+z)\cdot\frac{\omega_n^2}{z}}{s^2+2\xi\omega_n s+\omega_n^2} \quad \text{(Modified system)}$$

To analyse the effect of adding a zero to the system, wee have to find time domain specifications.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} + \frac{s}{z}\cdot\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}.$$

If $c(t)$ is the step response of the system $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

then step response of $\frac{s}{z}\left(\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right)$ is $\frac{1}{z}\frac{d}{dt}\left[c(t)\right]$

i.e., $c_z(t) = c(t) + \frac{1}{z}\frac{d}{dt}c(t)$. $\qquad L\{f(t)\} = F(s)$

$L\left\{\frac{d}{dt}f(t)\right\} = sF(s)$.

Where $c_z(t)$ is the response of the system with added zero and ~~c(t)~~ $c(t)$ is the response of original system.

From above equation, we can observe the following two points

1. The rise time of $C_z(t)$ is less than that of $c(t)$. Hence the peak occurs earlier in $C_z(s)$ than $C(s)$.

2. The maximum peak overshoot of $C_z(t)$ is greater than that of $c(t)$.

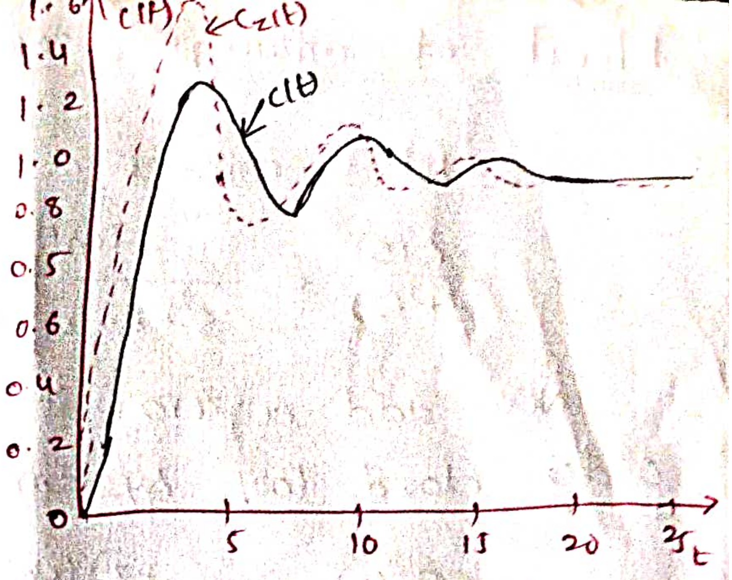


fig: Effect of adding a zero to a system.

The addition of a zero to a system decreases rise time and increases maximum peak over shoot.

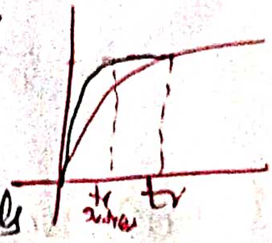
The values of rise time and peak over shoot depend on the selection of z. The smaller values of zero i.e., the zero close to origin produces large overshoot whereas the large values of zero i.e, the zero away from the origin produces negligible effect on transient response.

Therefore, selecting the zero near the origin will avoid large overshoots.

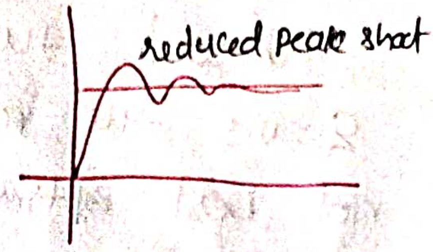
Performance indices :- when we design a control system, we are so many parameters. By handling all these parameters, it becomes difficult to design a control system. so we are taking one optimized parameter or metric a function to a design a control system, ~~parameters~~. Here we are taking one optimized parameter instead of taking ξ, M_p, t_r, t_p, t_s etc to design a control system.

So we are introducing the following errors while designing a control system.

① Integral square error (ISE) :- Integral square error is given by $ISE = \int_0^\infty e^2(t)\,dt$. It by minimizing ISE results in the reduction of rise time to limit large initial error. By minimizing ISE results in reduction of peak overshoot to limit the effect of small error lasting for a long time.



The above two requirements depends on ξ value.



Let us consider 2nd order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Best damping ratio for underdamped case ($\xi<1$) to minimize ISE. If input is $R(s) = 1/s$.

$$E(s) = R(s) - C(s).$$

$$= \frac{1}{s} - \frac{1}{s}\cdot\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$E(s) = \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}.$$

$$ISE = \int_0^\infty e^2(t)\,dt$$

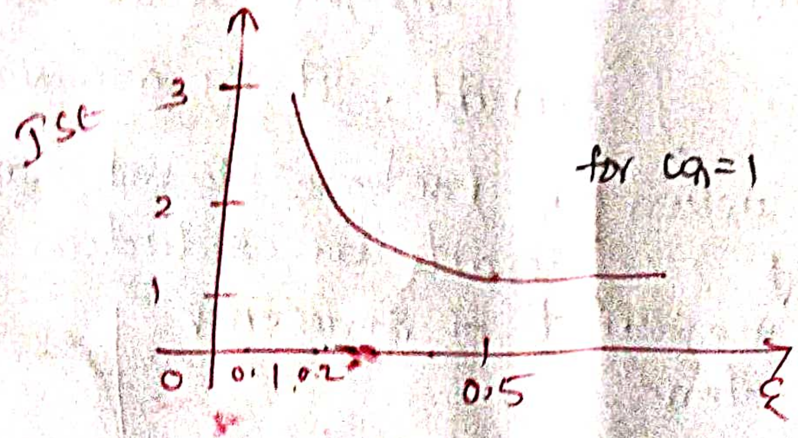
$$e(t) = r(t) - c(t)$$

$$= \cancel{1} - \cancel{1} + \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\sqrt{1-\xi^2}\,\omega_n t + \theta)$$

$$e(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\sqrt{1-\xi^2}\,\omega_n t + \theta)$$

Squaring and integrating from 0 to ∞ yields

$$ISE = \frac{1}{2\omega_n}\left(\frac{1}{2\xi} + 2\xi\right)$$



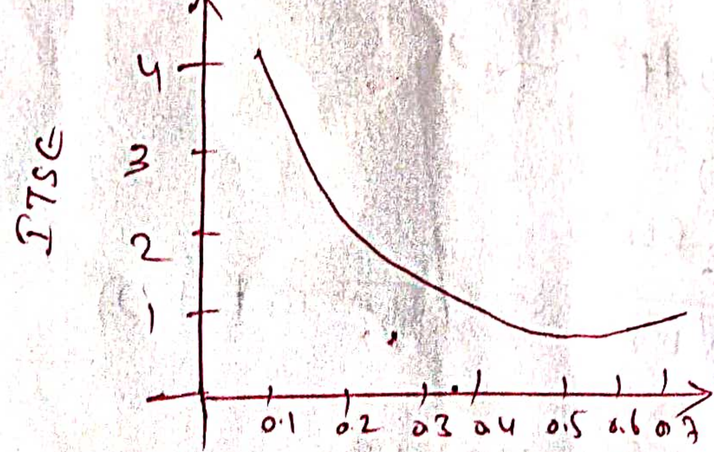
differentiate the ISE w.r.to ξ and make it equal to zero then error (ISE response) minimise and $\xi = 0.5$. After the $\xi = 0.5$ the response is constant that's why ISE is not best optimizing metric to optimize the control system so we move for ITSE.

(2) Integral time multiple square error (ITSE) :-

$$ITSE = \int_0^{\infty} t\, e^2(t)\, dt$$

solving the above using e(t) yields

$$ITSE = \frac{1}{\omega_n^2}\left(\xi^2 + \frac{1}{8\xi^2}\right)$$



when we solve $\xi = 0.7$

(3) Integral absolute Error :- (IAE) :-

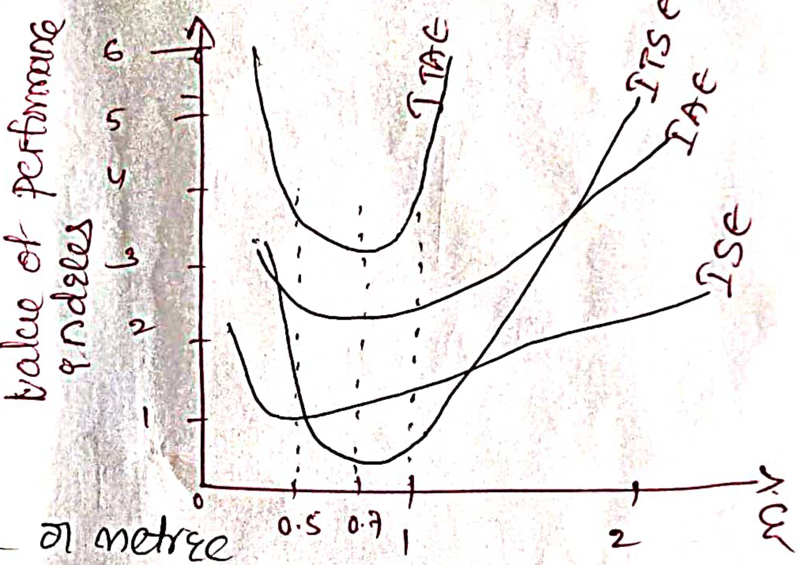
$$IAE = \int_0^\infty |e(t)|\, dt$$

(4) Integral of time multiplied absolute error (ITAE):-

$$ITAE = \int_0^\infty t\, |e(t)|\, dt$$

(a) Selecting of ISE metric is poor because it is flat and also minimum at 0.5

(b) IAE is minimized at 0.7 and gives slightly better selectivity than ISE



(c) ITAE performance index or metric is minimized at $\xi = 0.707$ and it is most sensitive among the three above indices.

(d) ITSE is somewhat less sensitive to variations.

UNIT- 2

Time Response Analysis.

* **Time Response of Control system :-** If the output of control system for an input varies with respect to time, then it is called the "time response of the control system"

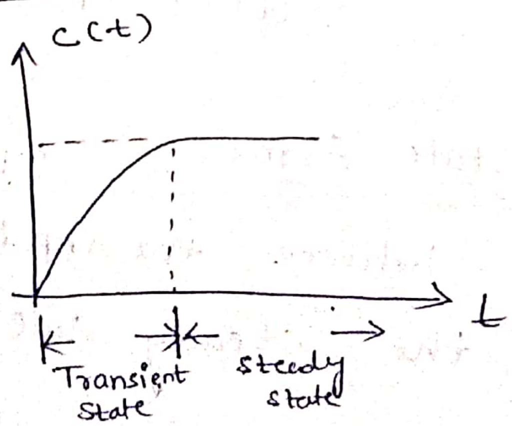
→ It is represented by c(t).

→ The response (or) output of control system is in time domain

The time Response of a control system consists of two parts

1. Transient Response
2. Steady state Response

1. Transient Response :-



→ After applying the input to the control system, output takes certain time to reach steady state.

∴ the response of the control system during the transient state is known as the transient Response.

→ Transient response means, response goes from initial state to the final state.

2. Steady state response :- The response during the steady state is called steady state response

→ In which the system output 't' approaches infinity.

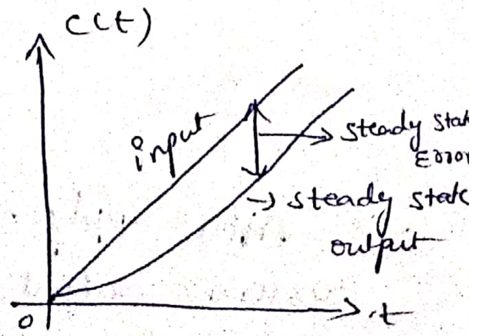
∴ thus the system response c(t) may be written as

$$\boxed{c(t) = c_{tr}(t) + c_{ss}(t)}$$

where $c_{tr}(t)$ = Transient Response

$c_{ss}(t)$ = Steady state response.

Steady State Error :- It is defined as the difference between the input and the output reached the steady state.



The steady state error will depend on the type of the input (i.e ramp, step, parabolic etc) and as well as the system type.

* **Step Response:-** The response of a system to the unit step input is called the step response

* **Impulse Response:-** The response of a system to the impulse signal input is called impulse response.
→ Impulse function is used to check the stability of a system.

* **Order of a System:-** The order of the system is given by the highest order of the differential equation representing the system.
→ If the system is represented by n^{th} order differential equation, then the system is called n^{th} order system.

Ex.. $a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \ldots\ldots$

In case of transfer function, the order of the system is given by the maximum power of 's' in the denominator

$$T(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots}{a_0 s^n + a_1 s^{n-1} + \ldots} = \frac{C(s)}{R(s)}$$

The steady state error will depend on the type of the input (i.e ramp, step, parabolic etc) and as well as the system type.

* **Step Response:-** The response of a system to the unit step input is called the step response

* **Impulse Response:-** The response of a system to the impulse signal input is called impulse response.

→ Impulse function is used to check the stability of a system.

* **Order of a System:-** The order of the system is given by the highest order of the differential equation representing the system.

→ If the system is represented by n^{th} order differential equation, then the system is called n^{th} order system.

Ex.- $a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots$

In case of transfer function, the order of the system is given by the maximum power of 's' in the denominator

$$T(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots}{a_0 s^n + a_1 s^{n-1} + \cdots} = \frac{C(s)}{R(s)}$$

where

$R(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots$

Here 'n' is the order of the system.

when $n=0$, zero order system

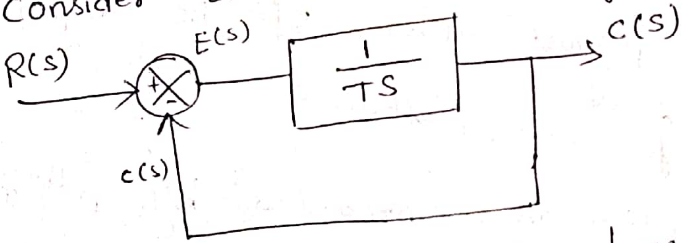
$n=1$, first order system

$n=2$, Second order system.

* **Characteristic Equation** :- The characteristic equation is nothing but setting the denominator of the closed loop transfer function to zero.

* **Time Response of first order Systems** :-

Consider the first order system



The transfer function of above block diagram is

$C(s) = E(s) \cdot \frac{1}{TS} \longrightarrow (1)$

$E(s) = R(s) - C(s) \longrightarrow (2)$ sub (2) in (1)

$C(s) = \frac{R(s) - C(s)}{TS}$

$C(s)TS = R(s) - C(s)$

$C(s)TS + C(s) = R(s)$

$(1+TS)\,C(s) = R(s)$

$$\boxed{\frac{C(s)}{R(s)} = \frac{1}{1+TS}}$$

∴ The simplified block diagram is

$$R(s) \longrightarrow \boxed{\frac{1}{1+TS}} \longrightarrow C(s)$$

∴ The input - output relation is given by

$$\boxed{\frac{C(s)}{R(s)} = \frac{1}{1+TS}}$$

Unit Step response of First order system:-

If the input is unit step then $r(t) = 1$

then laplace transform of $r(t)$ is $\boxed{R(s) = 1/s} \longrightarrow (1)$

we know the time response of first order system is

$$\frac{C(s)}{R(s)} = \frac{1}{1+TS} \Rightarrow C(s) = R(s) \cdot \frac{1}{1+TS}$$

from eq (1)

$$C(s) = \frac{1}{S} \cdot \frac{1}{1+TS} \longrightarrow (2)$$

Unit step response can be obtained by applying the inverse laplace transform to the equation (2)

$$\mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{S(1+TS)}\right]$$

Apply partial fractions

$$C(s) = \frac{1/T}{s(s+1/T)} \Rightarrow \frac{A}{S} + \frac{B}{s+1/T}$$

$$\frac{1}{T} = A(s+1/T) + BS$$

If $s = 0 \Rightarrow \frac{1}{T} = \frac{A}{T} + 0$ ∴ $\boxed{A = 1}$

If $s = -\frac{1}{T} \Rightarrow \frac{1}{T} = 0 - \frac{B}{T}$ ∴ $\boxed{B = -1}$

$$\therefore \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1/T}\right]$$

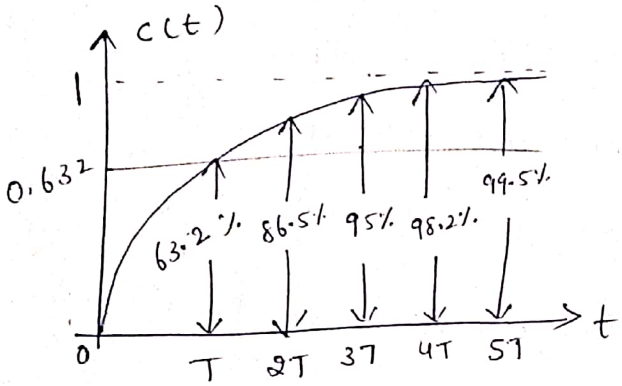
$$c(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1/T}\right]$$

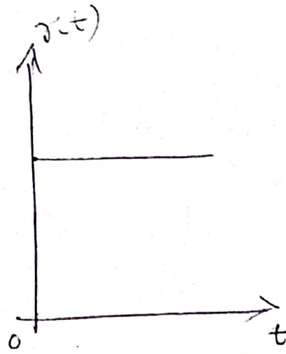
$$\boxed{c(t) = 1 - e^{-t/\tau}}$$

when $t = 0$
$c(t) = 1 - e^{0} = 0$
when $t = 2T$
$c(t) = 1 - e^{-2} = 0.865$
when $t = 3T$
$c(t) = 1 - e^{-3} = 0.95$

at $t = T$

$$c(T) = 1 - e^{-T/T} = 1 - e^{-1} = 0.632$$





unit step input

Unit step response of first order system

Unit ramp response of first order system :-

If the input is unit ramp, then $r(t) = t$

The laplace transform of ramp input is $\boxed{R(s) = \frac{1}{s^2}}$ $\rightarrow$ ①

we know the transfer function of first order system is

$$\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$$

$$\boxed{C(s) = R(s) \cdot \frac{1}{1+Ts}} \longrightarrow ②$$

sub ① in ②, $C(s) = \frac{1}{s^2} \frac{1}{(1+Ts)}$

$$C(s) = \frac{1\ (1/T)}{s^2\ (s+1/T)}$$

Apply partial fractions

$$c(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1/T}$$

$$\therefore \frac{1}{T} = As\left(s+\frac{1}{T}\right) + B\left(s+\frac{1}{T}\right) + Cs^2$$

If $s=0$

$$\frac{1}{T} = 0 + \frac{B}{T} + 0$$

$$\boxed{\therefore B = 1}$$

If $s = -1/T$

$$\frac{1}{T} = 0 + 0 + \frac{C}{T^2}$$

$$1 = \frac{C}{T}$$

$$\boxed{C = T}$$

On comparing the coefficients of s^2 $\quad A + C = 0$

$$A + T = 0$$

$$\boxed{A = -T}$$

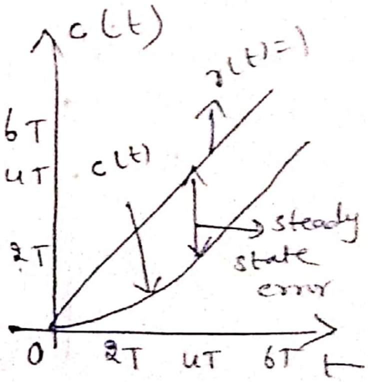
$$\therefore \quad c(s) = \frac{-T}{s} + \frac{1}{s^2} + \frac{T}{s+1/T}$$

Apply the inverse laplace transform to the above equation

$$\mathcal{L}^{-1}[c(s)] = \mathcal{L}^{-1}\left[\frac{-T}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + T\,\mathcal{L}^{-1}\left[\frac{1}{s+1/T}\right]$$

$$c(t) = -T + t + Te^{-t/T}, \quad t \geq 0$$

$$\boxed{c(t) = t + T\left(e^{-t/T} - 1\right)}$$



Steady state error

$$\boxed{e(t) = r(t) - c(t)}$$

Unit impulse response of first order system.

If the input is impulse then $\boxed{R(s) = 1}$ → (1)

we know the transfer function of first order system is

$$\frac{C(s)}{R(s)} = \frac{1}{1+TS} \rightarrow (2)$$

sub eq (1) in (2)

$$C(s) = (1) \cdot \frac{1}{1+TS}$$

$$C(s) = \frac{1}{1+TS} \Rightarrow \frac{1/T}{(S+1/T)}$$

Apply inverse laplace transform to the above equation

$$\mathcal{L}^{-1}[C(s)] = \frac{1}{T} \mathcal{L}^{-1}\left[\frac{1}{S+1/T}\right]$$

$$c(t) = \frac{1}{T} e^{-t/T}$$

∴ The time response of unit impulse is

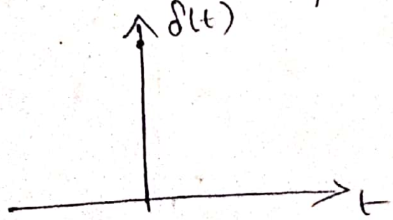
$$\boxed{c(t) = \frac{1}{T} e^{-t/T}}$$

when $t = 0$,

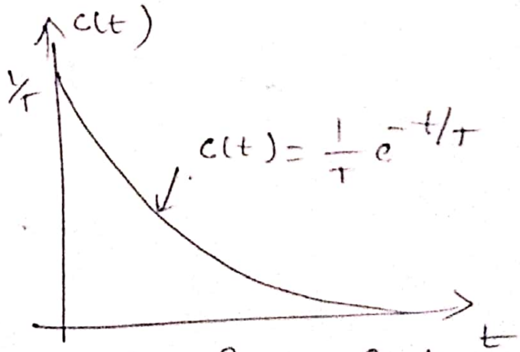
$c(t) = \frac{1}{T} e^{0} = 1/T$

when $t = 1T$

$c(t) = \frac{1}{T} e^{-T/T} = \frac{1}{T} e^{-1}$



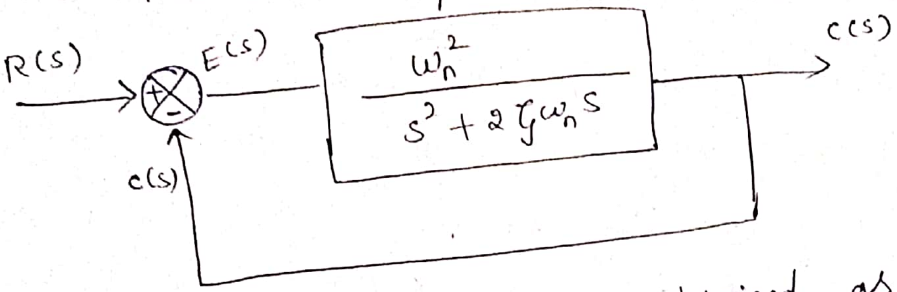
Input impulse signal



output: Impulse Response of first order system

*Second Order System :-

The closed loop second order system is given by



The transfer function can be obtained as

$$C(s) = E(s) \cdot \frac{\omega_n^2}{s^2+2\zeta\omega_n s} \longrightarrow (1)$$

$$E(s) = R(s) - C(s) \longrightarrow (2)$$

sub (2) in (1)

$$C(s) = \left[R(s) - C(s)\right] \frac{\omega_n^2}{s^2+2\zeta\omega_n s}$$

$$C(s) + C(s)\frac{\omega_n^2}{s^2+2\zeta\omega_n s} = \frac{R(s)\,\omega_n^2}{s^2+2\zeta\omega_n s}$$

$$C(s)\left[\frac{s^2+2\zeta\omega_n s+\omega_n^2}{\cancel{s^2+2\zeta\omega_n s}}\right] = \frac{R(s)\,\omega_n^2}{\cancel{s^2+2\zeta\omega_n s}}$$

$$\boxed{\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

∴ The standard form of closed loop transfer function of second order system is given by

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

where ω_n = undamped natural frequency, rad/sec

ζ = Damping ratio.

Damping Ratio :- The damping ratio is defined as the ratio of actual damping to the critical damping

→ The response of c(t) of the second order system depends on the value of damping ratio.

Depending on the damping ratio, the system can be classified into the following four cases.

Case 1 : Undamped System, $\zeta = 0$

Case 2 : Underdamped system, $0 < \zeta < 1$

Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Overdamped System, $\zeta > 1$

The characterstic equation of the second order system is given by

$$\boxed{s^2 + 2\zeta\omega_n s + \omega_n^2 = 0}$$

Roots of the above equation are given by

$$S_1, S_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$S_1, S_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

⇒ when $\zeta = 0$, $\boxed{S_1, S_2 = \pm j\omega_n}$ { Roots are purely imaginary and the system is undamped

⇒ when $\zeta = 1$, $\boxed{S_1, S_2 = -\omega_n}$; { Roots are real and equal and the system is critically damped

11

when $\zeta > 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$ { Roots are real and unequal and the system is overdamped

when $0 < \zeta < 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

$$= -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{(-1)}\sqrt{1-\zeta^2}$$

$$\boxed{s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}}$$

So, the roots are complex conjugate and the system is underdamped.

$$\boxed{\therefore\ s_1, s_2 = -\zeta\omega_n \pm j\omega_d}$$

where $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$

ω_d = damped frequency of oscillation of the system

* Response of undamped Second order system for unit step unit

for undamped System $\boxed{\zeta = 0} \longrightarrow (1)$

we know the transfer function of second order system is

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \longrightarrow (2)$$

sub (1) in (2),

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2} \longrightarrow (3)$$

when the input is unit step, $r(t) = 1$

Laplace transform of unit step is, $\boxed{R(s) = 1/s}$ → (4)

from equation (3)

$$c(s) = R(s) \cdot \frac{\omega_n^2}{s^2 + \omega_n^2}$$

from eq (4)

$$c(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + \omega_n^2}$$

Apply the partial fractions

$$c(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + \omega_n^2}$$

$$\therefore \omega_n^2 = A(s^2 + \omega_n^2) + (Bs + C)(s)$$

on coparing the coefficients of ω_n^2 $A\omega_n^2 = \omega_n^2$

$$\boxed{A = 1}$$

co-efficients of $s^2 \Rightarrow A + B = 0$

$$1 + B = 0$$

$$\boxed{B = -1}$$

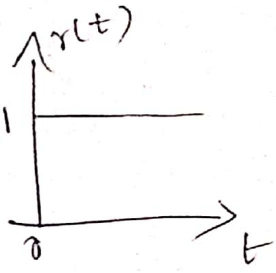
Co-efficients of $s \Rightarrow \boxed{C = 0}$

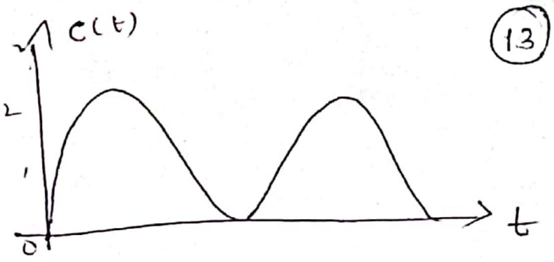
$$\therefore c(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

apply the inverse laplace transfom to the above equation

$$\mathcal{L}^{-1}[c(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2 + \omega_n^2}\right]$$

$\boxed{c(t) = 1 - \cos\omega_n t}$ this is the time domain response of second order system for the unit step input.





Unit step input

Time domain response.

Response of underdamped Second-order system for Unit step input

For underdamped system, $0 < \zeta < 1$ and the roots of the denominator are complex conjugate.

Since, $\zeta < 1$, ζ^2 is also < 1 and $1 - \zeta^2$ is always +ve

$$\therefore \; s = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)}$$

$$\boxed{s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}}$$

The damped frequency of oscillation is $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$

$$\boxed{\therefore \; s = -\zeta\omega_n \pm j\omega_d}$$

We know the ~~th~~ transfer function of second order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

when the input is unit step input, then $R(s) = 1/s$

$$\therefore \; C(s) = R(s) \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

By partial fractions

$$C(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\omega_n^2 = A(s^2 + 2\xi\omega_n s + \omega_n^2) + Bs + C(s)$$

$$\omega_n^2 = As^2 + 2\xi\omega_n s A + A\omega_n^2 + Bs^2 + Cs$$

on comparing the co-efficients

$A + B = 0$, $\quad 2\xi\omega_n + C = 0 \quad A\cancel{\omega_n^2} = \cancel{\omega_n^2}$

$\boxed{A = 1}$

$1 + B = 0$

$\boxed{B = -1}$ $\quad \boxed{C = -2\xi\omega_n}$

$$\therefore C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

adding & substracting the $\xi^2\omega_n^2$ to the denominator of second term.

$$C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2 + \xi^2\omega_n^2 - \xi^2\omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \xi^2\omega_n^2 + \omega_n^2(1 - \xi^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \xi\omega_n + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}$$

Multiply & divide by ω_d in the third term of equation

$$C(s) = \frac{1}{s} - \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{\omega_d} \frac{\omega_d}{(s + \xi\omega_n)^2 + \omega_d^2}$$

apply the inverse laplace transform to the above equation.

$$\mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right] - \frac{\zeta\omega_n}{\omega_d}\mathcal{L}^{-1}\left[\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right]$$

we know that

$$\mathcal{L}^{-1}\left[\frac{\omega}{(s+a)^2+\omega^2}\right] = e^{-at}\sin\omega t$$

$$\mathcal{L}^{-1}\left[\frac{s+a}{(s+a)^2+\omega^2}\right] = e^{-at}\cos\omega t$$

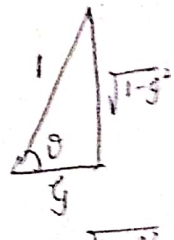
$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right]$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t\right]$$



$$\boxed{c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin(\theta+\omega_d t)\right]}$$

$\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$

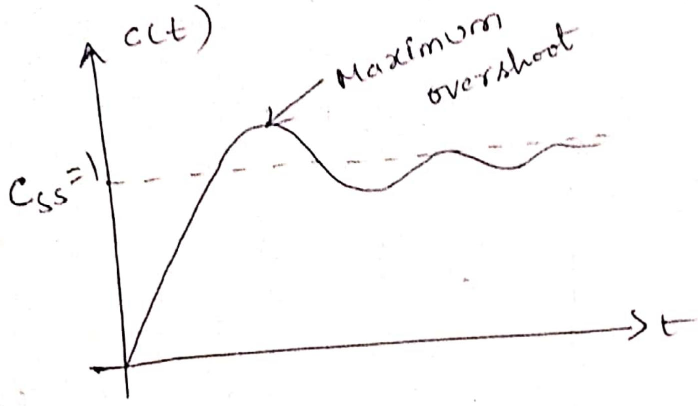
$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$

where $\boxed{\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}$

when $t = 0$, $c(0) = 0$

$t = \infty$, $c(\infty) = 1$

$c(\infty) = C_{ss}$: steady state value



Response of Critically damped Second order system for unit step unit.

The transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

for critical damping $\zeta = 1$

then

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} \Rightarrow \frac{\omega_n^2}{(s+\omega_n)^2}$$

when the input is unit step then $R(s) = 1/s$

$$\therefore C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{(s+\omega_n)^2}$$

$$\therefore C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$\omega_n^2 = A(s+\omega_n)^2 + Bs(s+\omega_n) + Cs$$

$$\omega_n^2 = As^2 + A\omega_n^2 + A\,2\omega_n s + Bs^2 + B\omega_n s + Cs.$$

on Comparing the coefficients

$A + B = 0.$

$1 + B = 0$

$\boxed{B = -1}$

$A\omega_n^2 = \omega_n^2$

$\boxed{A = 1}$

$2A\omega_n + B\omega_n + C = 0$

$2\omega_n - \omega_n + C = 0$

$\omega_n + C = 0$

$\boxed{C = -\omega_n}$

$$\therefore\ C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

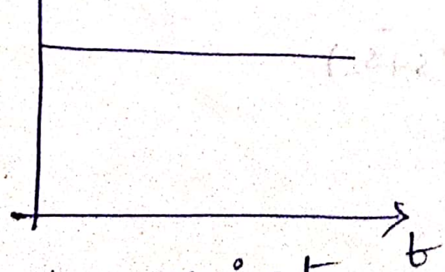
Apply the inverse laplace transform to the above equation

$$\mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+\omega_n}\right] - \omega_n \mathcal{L}^{-1}\left[\frac{1}{(s+\omega_n)^2}\right]$$

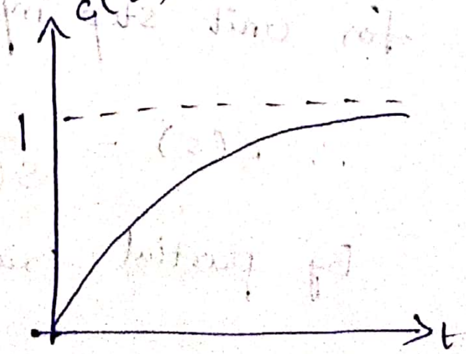
$$c(t) = 1 - e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t}$$

$$\boxed{c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)}$$

this is the time response of the ~~second~~ critically damped second order system for unit step input



unit step input



Response

Response of overdamped Second order system for unit step input

The transfer function of the second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

for overdamped system, $\zeta > 1$. so the roots of the denominator are real and distinct.

Let the roots of the denominator are s_a & s_b

$$s_a, s_b = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$
$$= -\left[\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}\right]$$

Let $s_1 = -s_a$ and $s_2 = -s_b$

$$\therefore s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$$
$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$$

The closed loop transfer function can be written in terms of s_1 and s_2 as shown below

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)}$$

for unit step input $r(t) = 1$, & $R(s) = 1/s$

$$\therefore C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

By partial fractions

$$C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

$$\cancel{C(s)}\ \omega_n^2 = A(s+s_1)(s+s_2) + B\,s(s+s_2) + C\,s(s+s_1)$$

when $s=0$

$\Rightarrow \omega_n^2 = A(s_1)(s_2) + 0 + 0$, sub s_1 & s_2

$$A = \frac{\omega_n^2}{s_1 s_2} = \frac{\omega_n^2}{[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}][\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}]}$$

$$= \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$\boxed{\therefore A = 1}$$

when $s = -s_1$

$$\omega_n^2 = A(0) + B(-s_1)(-s_1+s_2) + 0$$

$$B = \frac{\omega_n^2}{(-s_1)(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}]}$$

$$B = \frac{-\omega_n^{\cancel{2}}}{s_1\left([2\cancel{\omega_n}\sqrt{\zeta^2-1}]\right)}$$

$$\boxed{\therefore B = \frac{-\omega_n}{2\sqrt{\zeta^2-1}} \cdot \frac{1}{s_1}}$$

when $s = -s_2$

$$\omega_n^2 = 0 + 0 + C(-s_2)(-s_2+s_1)$$

$$C = \frac{\omega_n^2}{(-s_2)(-s_2+s_1)}$$

$$C = \frac{\omega_n^2}{(-s_2)\left[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]}$$

$$C = \frac{\omega_n^2}{s_2\left[2\omega_n\sqrt{\zeta^2-1}\right]}$$

$$\boxed{C = \frac{\omega_n}{\left[2\sqrt{\zeta^2-1}\right] s_2}}$$

Substitute A, B, C in c(s)

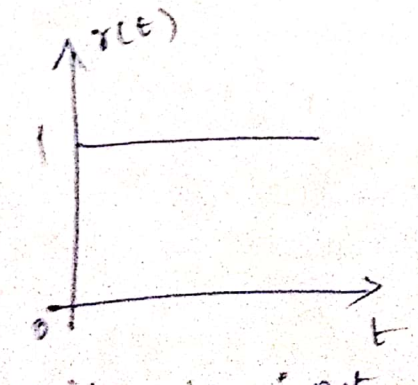
$$\therefore c(s) = \frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_1}\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_2}\frac{1}{(s+s_2)}$$

Apply the inverse laplace transform to the above equation

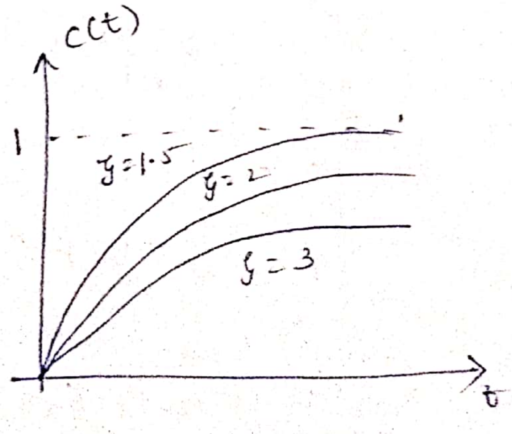
$$\mathcal{L}^{-1}[c(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_1}\mathcal{L}^{-1}\left[\frac{1}{s+s_1}\right] + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_2}\mathcal{L}^{-1}\left[\frac{1}{s+s_2}\right]$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_1}e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_2}e^{-s_2 t}$$

$$\boxed{c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right]}$$



Unit step input

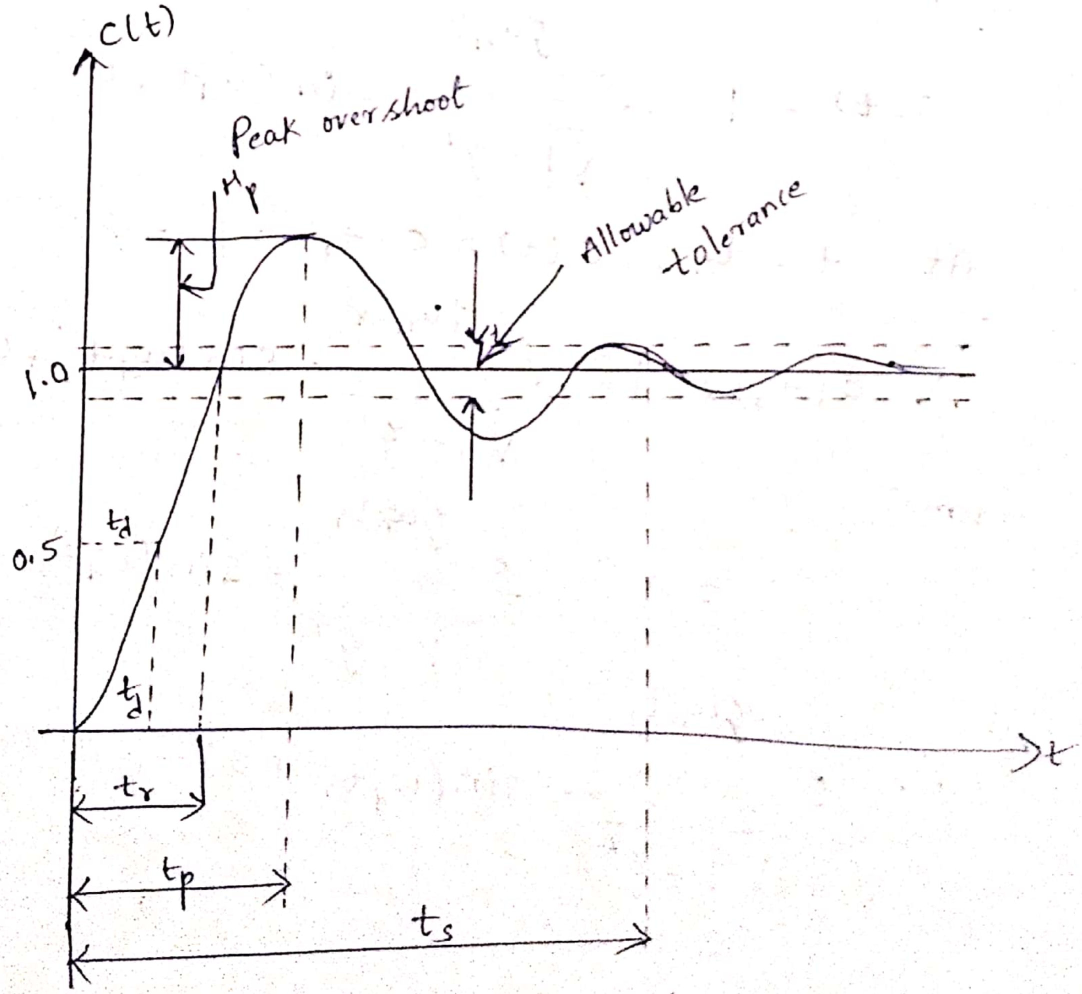


Time domain Specifications (or) Time Response Specifications

The desired performance characteristics of control system are specified in terms of time domain specifications.

Time domain Specifications are

1. Delay time t_d
2. Rise time t_r
3. Peak time t_p
4. Peak overshoot M_p
5. Settling time t_s
6. Steady State Error e_{ss}



Time domain Specifications

1. **Delay time t_d :-** It is the time required for the response to reach 50% of the final value in first attempt.

2. **Rise time t_r :-** It is the time required for the response to rise from 10% to 90% of the final value for overdamped systems and 0 to 100% of the final value for underdamped systems

Expression for the Rise time t_r :-

The unit step response of second order system for underdamped case is given by

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1 \rightarrow ①$

$$\therefore c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta)$$

from ①

$$1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta)$$

$$\therefore \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0$$

Since $e^{-\zeta\omega_n t_r} \neq 0$

so, $\sin(\omega_d t_r + \theta) = 0$

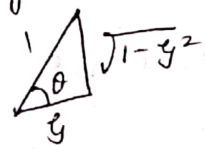
$$\therefore \omega_d t_r + \theta = \sin^{-1}(0)$$

$$\omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

$$\boxed{\therefore \text{Rise time},\ t_r = \frac{\pi - \theta}{\omega_d}} \rightarrow (2)$$

On constructing the right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get



$$\therefore \tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Here $\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$ and

Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ $\rightarrow (3)$

Substitute Equations (3) in eq (3)

$$\therefore \text{Rise time}\ \boxed{t_r = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}{\omega_n\sqrt{1-\zeta^2}}}$$

(3) **Peak time** :- It is the time required for the response to reach the peak of time response

Expression for peak time :- To find the expression for peak time, t_p, differentiate $c(t)$ with respect to t and equating to zero

i.e $\left.\frac{d\,c(t)}{dt}\right|_{t=t_p} = 0$

The unit step response of under damped Second order system is given by

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

Differentiating c(t) w.r. to t

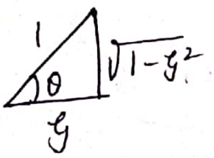
$$\frac{d}{dt} c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} (-\zeta\omega_n) \sin(\omega_d t+\theta)$$
$$+ \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right) \cos(\omega_d t+\theta)\, \omega_d$$

and we know $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \frac{d}{dt} c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\zeta\omega_n) \sin(\omega_d t+\theta) - \frac{\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_d t+\theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\zeta \sin(\omega_d t+\theta) - \sqrt{1-\zeta^2} \cos(\omega_d t+\theta)\right]$$

from



$\Rightarrow$ $\sin\theta = \sqrt{1-\zeta^2}$
$\cos\theta = \zeta$

$$\therefore \frac{d}{dt} c(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\cos\theta \sin(\omega_d t+\theta) - \sin\theta \cos(\omega_d t+\theta)\right]$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\sin(\omega_d t+\theta)\cos\theta - \cos(\omega_d t+\theta)\sin\theta\right]$$

$$\frac{dc(t)}{dt} = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\sin(\omega_d t+\theta-\theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t)$$

at $t = t_p$, $\frac{d\,c(t)}{dt} = 0$

$$\therefore \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} \sin(\omega_d t_p) = 0.$$

Since $e^{-\zeta\omega_n t_p} \neq 0$, then $\sin(\omega_d t_p) = 0$

$$\therefore \omega_d t_p = \sin^{-1}(0)$$

$$\omega_d t_p = \pi$$

$$\boxed{\therefore \text{Peak time } t_p = \frac{\pi}{\omega_d}}$$

we know, the damped frequency of oscillation

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\boxed{\therefore \text{peak time } t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

4. Peak overshoot :- It indicates the normalized difference between the time response peak and the steady output and is defined as

$$\text{Peak percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

where $c(t_p)$ = Peak response at $t = t_p$

$c(\infty)$ = Final Steady State value

The unit step response of Second order system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

At $t = \infty$, $c(t) = c(\infty) = 1 - \frac{e^{-\infty}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$

$$= 1 - 0$$

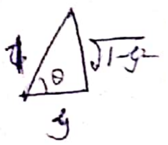
At $t = t_p$, $c(t) = c(t_p) = 1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \theta)$

we already know $t_p = \frac{\pi}{\omega_d}$ & $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\therefore c(t_p) = 1 - \frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \theta\right)$$

$$c(t_p) = 1 - \frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin(\pi + \theta)$$

$$c(t_p) = 1 + \frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin\theta$$



$\sin\theta = \sqrt{1-\xi^2}$

$$c(t_p) = 1 + \frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\cancel{\sqrt{1-\xi^2}}} (\cancel{\sqrt{1-\xi^2}})$$

$$\boxed{c(t_p) = 1 + e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}$$

$\therefore$ % Peak overshoot, $\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$

$$\% M_p = \frac{1 + e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} - 1}{1} \times 100$$

$$\boxed{\% M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100}$$

5. **Settling time (t_s) :-** It is the time required for the response to reach and stay with in a specified tolerance band (usually 2% (or) 5%) of its final value.

the response of Second order system has two components. They are

1. Decaying exponential component, $\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$
2. Sinusoidal component, $\sin(\omega_d t + \theta)$

→ Among the above two components decaying exponential term reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by the exponential component.

→ the Settling time can be found out by equating exponential component to % tolerance errors

for 2% tolerance error band, at $t = t_s$

$$\frac{e^{-\xi\omega_n t_s}}{\sqrt{1-\xi^2}} = 0.02$$

for least values of t_s,

$$e^{-\xi\omega_n t_s} = 0.02$$

$$-\xi\omega_n t_s = \ln(0.02)$$

$$-\xi\omega_n t_s = -4$$

$$\boxed{t_s = \frac{4}{\xi\omega_n}}$$

$\therefore$ Settling time, $\boxed{t_s = \frac{4}{\xi\omega_n}}$ for 2% error

for 5% tolerance error.

$$-\xi\omega_n t_s = \ln(0.05)$$

$$-\xi\omega_n t_s = -3$$

$$\boxed{t_s = \frac{+3}{\xi\omega_n}}$$

$\therefore$ Settling time, $\boxed{t_s = \frac{3}{\xi\omega_n}}$ for 5% error

6. Steady State error:- It indicates the error between the actual output and desired output as t tends to ∞ i.e

$$e_{ss} = \lim_{t\to\infty}[r(t) - c(t)]$$

Type Number of control System:-

The number of poles of the loop transfer function lying at the origin decides the type number of the system. The type number is specified for loop transfer function $G(s)H(s)$.

→ the loop transfer function can be expressed as a ratio of two polynomials in s.

$$G(s)H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)(s+z_2)(s+z_3)\cdots}{s^N(s+P_1)(s+P_2)(s+P_3)\cdots}$$

where $z_1, z_2, z_3 \cdots$ are zeros of transfer function

$P_1, P_2, P_3 \cdots$ are poles of transfer function

K = Constant

N = Number of poles at the origin.

The value of 'N' in the denominator of loop transfer function represents the type number of the system.

the System

If $N=0$, system is type-0 system

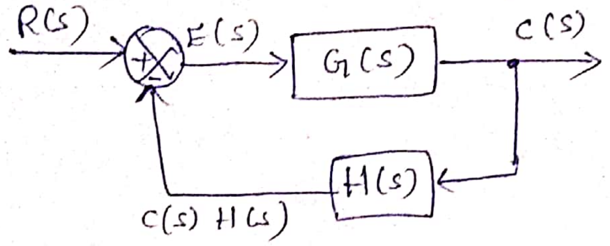
$N=1$, System is type-1 system

$N=2$, System is type-2 system and so on

Steady State Error :-

The steady state error is the value of error signal $e(t)$, when t tends to ∞. These errors arise from the nature of inputs, type of system and from non-linearity of system components.

Consider a closed loop system



the error signal $E(s) = R(s) - C(s)\,H(s). \longrightarrow ①$

The output signal $C(s) = E(s)\,G(s). \longrightarrow ②$

sub eq ② in eq ①

$$E(s) = R(s) - [E(s)\,G(s)]\,H(s)$$

$$E(s) + E(s)\,G(s)\,H(s) = R(s)$$

$$E(s)\,[1 + G(s)\,H(s)] = R(s)$$

$$\therefore E(s) = \frac{R(s)}{1 + G(s)H(s)} \longrightarrow ③$$

Let $e(t)$ = error signal in time domain

The steady state error is defined as the value of $e(t)$ when t tends to ∞

$$\therefore e_{ss} = \lim_{t\to\infty} e(t)$$

The final value theorem of laplace transform states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s).$$

So, the steady state error can be written as

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s)$$

from eq ③

$$\boxed{\therefore e_{ss} = \lim_{s\to 0} s.\frac{R(s)}{1 + G(s)H(s)}}$$

Static Error Constants

The value of steady state error depends on the type number and the input signal static error constants are 3 types.

Type-0 static error constants

1. Positional error constant $K_p = \lim_{s \to 0} G(s) H(s)$
2. Velocity error Constant $K_v = \lim_{s \to 0} s\, G(s) H(s)$
3. Acceleration error Constant $K_a = \lim_{s \to 0} s^2 G(s) H(s)$

1. Type-0 System will have a constant steady state error when the input is step signal
2. Type-1 System will have a constant steady state error when the input is ramp signal
3. Type-2 System will have a constant steady state error when the input is parabolic signal

Steady State error when the input is unit step signal

Steady state error, $e_{ss} = \lim_{s \to 0} \frac{s \cdot R(s)}{1 + G(s) H(s)}$

when the input is unit step, $R(s) = 1/s$.

$$\therefore e_{ss} = \lim_{s \to 0} \frac{\cancel{s} \cdot 1/s}{1 + G(s) H(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{1 + G(s) H(s)}$$

$$e_{ss} = \frac{1}{1 + \lim_{s \to 0} G(s) H(s)}$$

$$e_{ss} = \frac{1}{1 + K_p}$$ where K_p = positional error constant

Type - 0 System

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$= \lim_{s \to 0} K \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$K_p = K \frac{z_1 z_2 z_3 \ldots}{P_1 P_2 P_3 \ldots} = \text{Constant}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{Constant}$$

Hence in type-0 systems when the input is unit step there will be a constant steady state error.

Type-I System

$$K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} K \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{s(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$K_p = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

In systems with type number 1 and above for unit step input the value of Kp is ∞ and steady state error is zero.

Steady state error when the input is unit Ramp signal

Steady state error $e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1+G(s)H(s)}$

when the input is unit Ramp, $R(s) = \frac{1}{s^2}$

$$\therefore e_{ss} = \lim_{s \to 0} \frac{\cancel{s} \cdot 1/s^2}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{1}{s + sG(s)H(s)}$$

$$e_{ss} = \frac{1}{\lim_{s \to 0} sG(s)H(s)} = \frac{1}{K_v}$$

where $K_v = \lim_{s \to 0} sG(s)H(s)$

the constant 'K_v' is called velocity error constant

Type-0 System

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} s.K \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$= 0$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

Hence in Type-0 systems when the input is unit ramp, the steady state error is infinity

Type-1 System

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \cancel{s}K \frac{(s+z_1)(s+z_2)\ldots}{\cancel{s}(s+P_1)(s+P_2)\ldots}$$

$$K_v = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{\text{constant}} \cancel{= \infty} = \text{constant}$$

Hence in type-1 systems when the input is unit ramp there will be a constant steady state error

Type - 2 System

$$K_v = \lim_{s\to 0} s\,G(s)H(s)$$

$$= \lim_{s\to 0} sK \frac{(s+Z_1)(s+Z_2)\ldots}{s^2(s+P_1)(s+P_2)\ldots} = \infty$$

$$\boxed{K_v = \infty}$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

In systems with type number 2 and above for unit ramp unit, the value of K_v is ∞ so the steady state error is zero

Steady state error when the input is unit parabolic signal

Steady state error, $e_{ss} = \lim_{s\to 0} s \frac{R(s)}{1+G(s)R(s)}$

when the input is unit parabolic $R(s) = 1/s^3$

$$\therefore \cancel{e_{ss} = \lim_{s\to 0} s \frac{R(s)H(s)}{1+G(s)H}}$$

$$\therefore e_{ss} = \lim_{s\to 0} s \frac{1/s^3}{1+G(s)H(s)} = \lim_{s\to 0} \frac{1}{s^2+s^2G(s)H(s)}$$

$$e_{ss} = \frac{1}{\lim_{s\to 0} s^2 G(s)H(s)} = \frac{1}{K_a}$$

where $K_a = \lim_{s\to 0} s^2 G(s)H(s)$ and it is accelaration error constant

Type-0 System

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} s^2 K \frac{(s+z_1)(s+z_2)\cdots}{(s+P_1)(s+P_2)\cdots}$$

$$\boxed{K_a = 0}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity

Type-I System

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} s^{\cancel{2}} K \frac{(s+z_1)(s+z_2)\cdots}{\cancel{s}(s+P_1)(s+P_2)\cdots}$$

$$\boxed{K_a = 0}$$

~~Hence~~ $\therefore e_{ss} = \frac{1}{0} = \infty$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity

Type-2 System

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \cancel{s^2} K \frac{(s+z_1)(s+z_2)\cdots}{\cancel{s^2}(s+P_1)(s+P_2)\cdots}$$

$$\boxed{K_a = \text{Constant}}$$

$$\therefore e_{ss} = \frac{1}{\text{Constant}} = \text{Constant}$$

Hence in type-2 system for unit parabolic input, the steady state error is constant

Type-3 system

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \cancel{s^2} K \frac{(s+z_1)(s+z_2)\cdots}{s^{\cancel{3}}(s+P_1)(s+P_2)\cdots}$$

$$\boxed{K_a = \infty}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

Hence in type-3 system for unit parabolic input, the steady state error is zero and K_a is infinity

Static Error Constants for various type number of system

Error Constant	Type number of system			
	0	1	2	3
K_p	Constant	∞	∞	∞
K_v	0	Constant	∞	∞
K_a	0	0	Constant	∞

Steady State error for various types of inputs

Input Signal	Type number of Systems			
	0	1	2	3
Unit step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0