UNIT-5 Vector Integration

In this chapter we shall define line, surface and volume integrals which occur frequently in connection with physical and Engineering problems.

The concept of Line integral is a natural generalization of the concept of definite integral $\int_a^b f(x)\,dx$, $f(x) \in [a,b]$.

The concept of double integral (or) surface integral is a generalization of the concept of double integral. It is defined in two dimensional xy-plane (or) yz-plane (or) xz plane.

The concept of volume integral is defined in three-dimensional which is a function of x, y, z.

Line integrals:

Any integral which is to be evaluated along the line (or) curve is called the line integral. This line integral we use to measure the length of the curve along the path.

Circulation:

If $\bar{v}$ represents the velocity of a fluid particle and 'C' is a closed curve then the line integral $\oint_C \bar{v} \cdot d\bar{r}$ is called the circulation of $\bar{v}$ around the curve 'C'.

If $\int_C \bar{v} \cdot d\bar{r} = 0$ then $\bar{v}$ is called conservative and no work is done and the energy is conserved.

If the circulation of $\bar{v}$ round every closed curve in a region 'D' vanishes then $\bar{v}$ is said to be irrotational in 'D'.

Work done by a Force :

If $\bar{F}$ represents the force vector acting on a particle moving along an arc AB, then the work done during a small displacement $\overline{\delta r}$ is $\bar{F}\cdot\overline{\delta r}$. Hence the total work done is $\bar{F}$ during displacement from A to B is given by the line integral $\int_A^B \bar{F}\cdot\overline{dr}$

If the force $\bar{F}$ is conservative, $\bar{F} = \nabla\phi$ then the work done is independent of the path and vice-versa. In this case $\operatorname{curl}\bar{F} = \operatorname{curl}(\operatorname{grad}\phi) = \bar{0}$ and ϕ is scalar potential.

Note: ① $\bar{F}$ is conservative force field if $\nabla\times\bar{F} = \bar{0}$

② A conservative force field is also irrotational $(\because \nabla\times\bar{F} = \bar{0})$

Problems:

Find the work done by the force $\bar{F} = (3x^2-6yz)\bar{i}+(2y+3xz)\bar{j}+(1-4xyz^2)\bar{k}$ in moving particle from the point $(0,0,0)$ to $(1,1,1)$ along the curve $c: x=t,\ y=t^2,\ z=t^3$.

Soln: We know that work done $= \int_C \bar{F}\cdot\overline{dr} \rightarrow ①$

c: $O(0,0,0)$ to $A(1,1,1)$

Here $\bar{F} = (3x^2-6yz)\bar{i}+(2y+3xz)\bar{j}+(1-4xyz^2)\bar{k}$

$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} \Rightarrow \boxed{\overline{dr} = dx\bar{i}+dy\bar{j}+dz\bar{k}}$

$$\bar{F}\cdot\overline{dr} = \left[(3x^2-6yz)\bar{i}+(2y+3xz)\bar{j}+(1-4xyz^2)\bar{k}\right]\cdot(dx\bar{i}+dy\bar{j}+dz\bar{k})$$

$$= (3x^2-6yz)\,dx + (2y+3xz)\,dy + (1-4xyz^2)\,dz$$

$x=t$	$y=t^2$	$z=t^3$
$dx=dt$	$dy=2t\,dt$	$dz=3t^2dt$

$$\therefore \int_{OA}\bar{F}\cdot\overline{dr} = \int_{t=0}^{1}(3t^2-6t^5)\,dt + (2t^2+3t^4)\,2t\,dt + (1-4t^9)\,3t^2dt$$

$$\int_{OA} \bar{F}\cdot\bar{dr} = \int_{t=0}^{1} (3t^2 - \cancel{6t^5} + 4t^3 + \cancel{6t^5} + 3t^2 - 12t^{11})\, dt$$

$$= \int_{t=0}^{1} (-12t^{11} + 4t^3 + 6t^2)\,dt$$

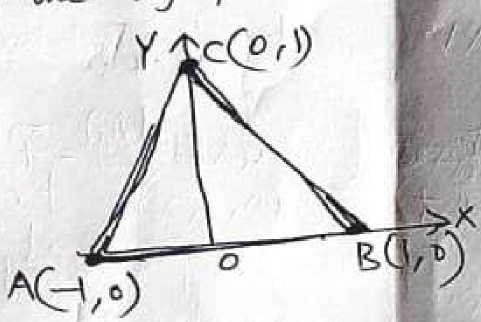
$$= \left[-12\left(\frac{t^{12}}{12}\right) + 4\left(\frac{t^4}{4}\right) + 6\left(\frac{t^3}{3}\right)\right]_{t=0}^{1}$$

$$= \left(-t^{12} + t^4 + 2t^3\right)_{t=0}^{1}$$

$$= (-1+1+2) - (0+0+0)$$

$$= 2$$

(2) Compute the line integral $\int (y^2 dx - x^2 dy)$ round the Triangle whose vertices are $(1,0), (0,1), (-1,0)$ in the xy-plane.



Soln: Equation of AB = x-axis = $y=0$

Equation of BC = $x+y=1$

equation of CA = $y-x=1$

$$\therefore \int_C (y^2dx - x^2dy) = \int_{AB} + \int_{BC} + \int_{CA} \rightarrow (1)$$

along the line AB: $y=0 \Rightarrow dy=0$, $\int_{AB}(y^2dx - x^2dy) = 0$

along the line BC: $x+y=1$

$y = 1-x$, $dy = -dx$

$$\int_{BC}(y^2dx - x^2dy) = \int_{x=1}^{0} (1-x)^2dx - x^2(-dx) = \left[\frac{(1-x)^3}{3\times(-1)} + \frac{x^3}{3}\right]_{x=1}^{0} = -\frac{1}{3} - \frac{1}{3} = \frac{-2}{3}$$

along the line CA: $y-x=1$

$y = x+1$, $dy = dx$

$$\int_{CA}(y^2dx - x^2dy) = \int_{x=0}^{-1} (x+1)^2dx - x^2dx = \left[\frac{(x+1)^3}{3} - \frac{x^3}{3}\right]_{x=0}^{-1} = \left(0+\frac{1}{3}\right) - \left(\frac{1}{3}+0\right) = 0$$

from (1): Required line integral $= \int_{AB} + \int_{BC} + \int_{CA} = 0 - \frac{2}{3} + 0 = \frac{-2}{3}$

(3) Find workdone in moving particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Soln: Given $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$

the straight line from $O(\overset{x_1,y_1,z_1}{0,0,0})$ to $A(\overset{x_2,y_2,z_2}{2,1,3})$

Equation of OA is $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1} = t$ (say)

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$$

$x = 2t,\ y = t,\ z = 3t$ parametric eqns.

when $x=0, y=0, z=0 \Rightarrow t = 0$

$x=2, y=1, z=3 \Rightarrow t = 1$

$\therefore$ workdone by force in moving particle

$$\int_C \bar{F}\cdot\bar{dr} = \int_{OA} (3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k})\cdot(dx\bar{i} + dy\bar{j} + dz\bar{k})$$

$$= \int_{(0,0,0)}^{(2,1,3)} 3x^2\,dx + (2xz-y)\,dy + z\,dz$$

$\bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k} = 1$

$\bar{i}\cdot\bar{j} = \bar{j}\cdot\bar{k} = \bar{k}\cdot\bar{i} = 0$

Now $x = 2t$, $dx = 2dt$ | $y = t$, $dy = dt$ | $z = 3t$, $dz = 3dt$

$$\Rightarrow \int_{t=0}^{1} 3(2t)^2(2dt) + (2(2t)(3t) - t)dy + 3t(3dt)$$

$$= \int_{t=0}^{1} (24t^2 + 12t^2 - t + 9t)\,dt$$

$$= \int_{t=0}^{1} (36t^2 + 8t)\,dt$$

$$= \left[36\left(\frac{t^3}{3}\right) + 8\left(\frac{t^2}{2}\right)\right]_{t=0}^{1}$$

$$= \left(\frac{36}{3} + \frac{8}{2}\right) - (0+0)$$

$$= 12 + 4$$

$$= 16 \text{ units.}$$

Example 21 : If $\overline{F}=(5xy-6x^2)i+(2y-4x)j$, Evaluate $\int_C \overline{F}\cdot d\overline{R}$, where C is the curve in the xy - plane $y=x^3$ from $(1,1)$ to $(2,8)$. **[JNTU (A) June 2010 (Set No. 2)]**

Solution : We are given $\overline{F}=(5xy-6x^2)\overline{i}+(2y-4x)\overline{j}$

Since the particle moves in the xy-plane ($z=0$), we take $\overline{R}=x\,\overline{i}+y\,\overline{j}$.

Then $\int_c \overline{F}\cdot d\overline{R}=\int_c[(5xy-6x^2)\overline{i}+(2y-4x)\overline{j}]\cdot[dx\,\overline{i}+dy\,\overline{j}]$ where c is the parabola $y=x^3$

$$=\int_c (5xy-6x^2)dx+(2y-4x)dy \quad \ldots (1)$$

Substituting $y=x^3$, where x goes from 1 to 2, (1) becomes

$$\int_c \overline{F}\cdot d\overline{R}=\int_{x=1}^{2}[5x(x^3)-6x^2]dx+(2x^3-4x)3x^2dx$$

$$=\int_1^2(5x^4-6x^2+6x^5-12x^3)dx=\int_1^2(6x^5+5x^4-12x^3-6x^2)dx$$

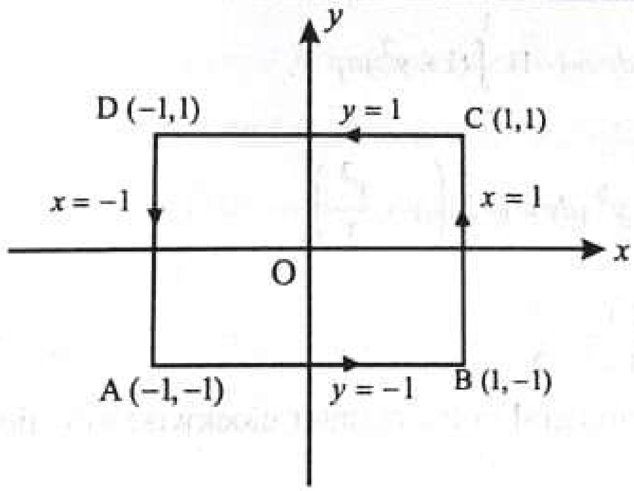
$$=\left(6\cdot\frac{x^6}{6}+5\cdot\frac{x^5}{5}-12\cdot\frac{x^4}{4}-6\cdot\frac{x^3}{3}\right)_1^2=[x^6+x^5-3x^4-2x^3]_1^2$$

$$=[2^6+2^5-3\cdot(2^4)-2(2^3)]-[1+1-3-2]$$

$$=8(8+4-6-2)-(-3)\quad=32+3=35.$$

Example 22 : Evaluate the line integral $\int_c[(x^2+xy)dx+(x^2+y^2)dy]$ where c is the square formed by the lines $x=\pm1$ and $y=\pm1$. **[JNTU (A) June 2010, 2011 (Set No. 1)]**

Solution :



Here $\int_C \overline{F}\cdot d\overline{r}=\int_C[(x^2+xy)dx+(x^2+y^2)dy]$

In the counter clockwise direction

$$\int_C \overline{F}\cdot d\overline{r} = \int_{AB} \overline{F}\cdot d\overline{r} + \int_{BC} \overline{F}\cdot d\overline{r} + \int_{CD} \overline{F}\cdot d\overline{r} + \int_{DA} \overline{F}\cdot d\overline{r} \quad \text{... (1)}$$

Along AB :

Here $y=-1$. $\therefore\ dy=0$

$$\therefore \int_{AB} \overline{F}\cdot d\overline{r} = \int_{-1}^{1}(x^2-x)dx = \int_{-1}^{1}x^2dx - \int_{-1}^{1}x\,dx$$

$$= 2\int_0^1 x^2dx - 0 = 2\left(\frac{x^3}{3}\right)_0^1 = \frac{2}{3} \quad \text{... (2)}$$

Along BC :

Here $x=1$. $\therefore\ dx=0$.

$$\therefore \int_{BC} \overline{F}\cdot d\overline{r} = \int_{-1}^{1}(1+y^2)dy = 2\int_0^1(1+y^2)dy$$

$$= 2\left(y+\frac{y^3}{3}\right)_0^1 = 2\left(1+\frac{1}{3}\right) = \frac{8}{3} \quad \text{... (3)}$$

Along CD :

Here $y=1$. $\therefore\ dy=0$.

$$\therefore \int_{CD} \overline{F}\cdot d\overline{r} = \int_{1}^{-1}(x^2+x)dx = (-1)\int_{-1}^{1}(x^2+x)dx$$

$$= (-1)\left[2\int_0^1 x^2dx + 0\right] = -\frac{2}{3} \quad \text{... (4)}$$

Along DA :

Here $x=-1$. $\therefore\ dx=0$.

$$\therefore \int_{DA} \overline{F}\cdot d\overline{r} = \int_{1}^{-1}(1+y^2)dy = (-1)\int_{-1}^{1}(1+y^2)dy$$

$$= (-2)\int_0^1(1+y^2)dy = (-2)\left(y+\frac{y^3}{3}\right)_0^1$$

$$= (-2)\left(1+\frac{1}{3}\right) = -\frac{8}{3} \quad \text{... (5)}$$

Hence the required line integral in the counter clockwise direction is

$\int_C \overline{F}\cdot d\overline{r} = \frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3} = 0$, using (1).

$B(1,0)$ ———————— $C(0,1)$

x_1, y_1 $\quad\quad$ x_2, y_2

eqn of Line joining b/w two points

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{1-0}{0-1}(x-1)$$

$$y = -(x-1)$$

$$y = -x + 1$$

$$\boxed{y + x = 1}$$

Surface Integrals

Any integral evaluated over a surface is called a surface integral. It is denoted by $\int_S \bar{F}\cdot\bar{n}\, dS$.

Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

where F_1, F_2, F_3 are functions of x, y, and z. S is the surface over which the integral is evaluated and $\bar{n}$ is the outward drawn unit normal vector to the surface S.

We have to project the surface S on the xy-plane (or) yz-plane (or) zx-plane.

1) If R_1 is the projection on the xy-plane, then the outward drawn unit normal $\bar{n}$ is along z-axis ie; $\bar{k}$.

$$\therefore \int_S \bar{F}\cdot n\, dS = \iint_{R_1} \frac{\bar{F}\cdot\bar{n}\, dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

2) If R_2 is the projection on yz-plane, then $\bar{n} = \bar{i}$.

$$\therefore \int_S \bar{F}\cdot\bar{n}\, dS = \iint_{R_2} \frac{\bar{F}\cdot\bar{n}\, dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

3) If R_3 is the projection on zx-plane, then $\bar{n} = \bar{j}$.

$$\therefore \int_S \bar{F}\cdot\bar{n}\, dS = \iint_{R_3} \frac{\bar{F}\cdot\bar{n}\, dz\,dx}{|\bar{n}\cdot\bar{j}|}$$

In Cartesian co-ordinates, it can also be represented by

$$\int_S \bar{F}\cdot\bar{n}\,dS = \iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy.$$

Problems:

1) Evaluate $\int_S \bar{F}\cdot\bar{n}\,dS$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is part of the surface of the plane $2x+3y+6z=12$ located in first octant.

Sol: Surface is denoted by $\phi = 2x+3y+6z-12$.

Normal to the surface is $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Unit normal $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}$

Let R be projection of S on xy-plane.

$$\therefore \int_S \bar{F}\cdot\bar{n}\,dS = \iint_R \frac{\bar{F}\cdot\bar{n}\,dx\,dy}{|\bar{n}\cdot\bar{k}|} \quad \text{——} (1)$$

Now, $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$

$$\bar{F}\cdot\bar{n} = \frac{1}{7}(36z - 36 + 18y) = \frac{6}{7}(6z - 6 + 3y)$$

$$\bar{n}\cdot\bar{k} = \left(\frac{2\bar{i}}{7} + \frac{3\bar{j}}{7} + \frac{6\bar{k}}{7}\right)\cdot\bar{k} = \frac{6}{7}$$

$$\therefore |\bar{n}\cdot\bar{k}| = \frac{6}{7}$$

How to find the limits. Since integration is on xy-plane, $z=0$ and we have to find limits for x and y.

Surface is $2x+3y+6z=12$

$z=0 \Rightarrow 2x+3y=12.$

$\Rightarrow 3y = 12-2x$

$\Rightarrow y = \frac{12-2x}{3}$

Putting $y=0$, $2x=12 \Rightarrow x=6$

Since we take first octant,

$x \to 0$ to 6

$y \to 0$ to $\frac{12-2x}{3}$.

$$\therefore \int_S \bar{F}\cdot\bar{n}\,dS = \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} \frac{\cancel{6/7}(6z-6+3y)}{\cancel{6/7}}\,dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} [(12-2x-3y)-6+3y]\,dx\,dy$$

$[\because 6z = 12-2x-3y]$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6-2x)\,dx\,dy$$

$$= 2\int_{x=0}^{6}\left[(3-x)\,dy\right]dx$$

$$= 2\int_{x=0}^{6} (3-x)\,y\Big|_0^{\frac{12-2x}{3}}\,dx$$

$$= 2\int_{x=0}^{6} (3-x)\left(\frac{12-2x}{3}\right)dx.$$

$$= \frac{4}{3}\int_{x=0}^{6} (3-x)(6-x)\,dx = \frac{4}{3}\int_{x=0}^{6}(18-9x+x^2)\,dx$$

$$= \frac{4}{3}\left(18x - \frac{9x^2}{2} + \frac{x^3}{3}\right)_0^6 = \underline{24}$$

2) Evaluate $\int \bar{F}\cdot\bar{n}\,dS$ where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$ and S is the surface $x^2+y^2=16$ included in the first octant b/w $z=0$ and $z=5$.

Soln: [Since limits of z are directly given, we can project the surface to zx plane (or) yz plane].

Surface $\phi = x^2+y^2-16$

Normal to surface $= \nabla\phi = 2x\bar{i} + 2y\bar{j}$

$\bar{n}$ = unit normal to surface, $\cancel{\frac{2x\bar{i}+2y\bar{j}}{\sqrt{4+4}}}$

$$= \frac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}} = \frac{\cancel{2}(x\bar{i}+y\bar{j})}{\cancel{2}\sqrt{x^2+y^2}}$$

$$= \frac{x\bar{i}+y\bar{j}}{4} \qquad \left[\because \phi \text{ is } x^2+y^2=16 \quad \therefore \sqrt{x^2+y^2}=4\right]$$

$\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$

$\bar{F}\cdot\bar{n} = \dfrac{xz + xy}{4}$

Let R the projection of S on yz plane.

$\therefore |\bar{n}\cdot\bar{i}| = \left|\left(\frac{x\bar{i}}{4} + \frac{y}{4}\bar{j}\right)\cdot\bar{i}\right| = \frac{x}{4}$

On yz-plane, $x=0$ $\therefore y^2=16 \Rightarrow y=\pm 4$

In first octant $y \to 0$ to 4.

Given $z \to 0$ to 5.

$$\therefore \int_S \bar{F}\cdot\bar{n}\,dS = \int_{y=0}^{4}\int_{z=0}^{5} \frac{\cancel{x}(z+y)}{\cancel{4}} \frac{dy\,dz}{\cancel{x/4}}$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} (z+y)\,dy\,dz$$

$$= \int_{y=0}^{4} \left[\int_{z=0}^{5} (y+z)\, dz \right] dy$$

$$= \int_{y=0}^{4} \left(yz + \frac{z^2}{2} \right)_0^5 dy$$

$$= \int_{y=0}^{4} \left(5y + \frac{25}{2} \right) dy = \left(\frac{5y^2}{2} + \frac{25}{2} y \right)_0^4$$

$$= \left(\frac{80}{2} + \frac{100}{2} \right) = \underline{\underline{90}}.$$

<u>HW.</u> 1) Evaluate $\int_S \bar{F}\cdot\bar{n}\, dS$ where $\bar{F} = 12x^2y\,\bar{i} - 3yz\,\bar{j} + 2z\,\bar{k}$ and S is part of plane $x+y+z=1$ in first octant.

Ans: $\frac{-55}{24}$

~~2) 2~~

2) Evaluate $\int_S \bar{F}\cdot\bar{n}\, dS$ if $\bar{F} = yz\,\bar{i} + 2y^2\,\bar{j} + xz^2\, k$ and S is surface of cylinder $x^2+y^2=9$ in the first octant b/w $z=0$ and $z=2$.

Ans: 78.

~~Volume Integral~~.

Flux of a vector field

$$\text{Flux} = \iint \bar{V}\cdot\bar{n}\, dS.$$

1) Compute flux of water through cycloid $y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$ if $\bar{V} = 3z^2\,\bar{i} + 6\,\bar{j} + 6xz\,\bar{k}$.

[Here take $|\bar{n}\cdot\bar{j}| = 1$].

$$\therefore \text{Flux} = \int_{x=0}^{2} \int_{z=0}^{3} (6xz^2 - 6)\, dx\, dz = \underline{\underline{72}}.$$

Volume Integrals

Let V be the volume bounded by a surface. Let $\bar{F}$ be defined over V. Then the volume integral is given by $\int_V \bar{F}\,dV$.

If $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$, where F_1, F_2, F_3 are fns of x, y, z

$$\int_V \bar{F}\,dV = \iiint_{x\,y\,z} (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\,dx\,dy\,dz$$

$$= \bar{i}\iiint F_1\,dx\,dy\,dz + \bar{j}\iiint F_2\,dx\,dy\,dz + \bar{k}\iiint F_3\,dx\,dy\,dz.$$

Example 1 : If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$ evaluate $\int_V \bar{F}dv$ where V is the region bounded by the surfaces $x=0, x=2, y=0, y=6, z=x^2, z=4$.

Solution : Given $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$.

$\therefore$ The volume integral is

$$\int_V \bar{F}dv = \iiint (2xz\bar{i} - x\bar{j} + y^2\bar{k})dx\,dy\,dz$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xz\,dx\,dy\,dz - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2\,dx\,dy\,dz$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6} [xz^2]_{x^2}^{4}\,dx\,dy - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6} (xz)_{x^2}^{4}\,dx\,dy + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6} y^2(z)_{x^2}^{4}\,dx\,dy$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6} x(16-x^4)dx\,dy - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6} x(4-x^2)dx\,dy$$

$$+\bar{k}\int_{x=0}^{2}\int_{y=0}^{2} y^2(4-x^2)dx\,dy$$

Scanned with Cam

$$= \bar{i} \int_{x=0}^{2} (16x - x^5)(y)_0^6 dx - \bar{j} \int_{x=0}^{2} (4x - x^3)(y)_0^6 dx$$

$$+ \bar{k} \int_{x=0}^{2} (4 - x^2)\left(\frac{y^3}{3}\right)_0^6 dx$$

$$= \bar{i}\left(8x^2 - \frac{x^6}{6}\right)_0^2 (6) - \bar{j}\left(2x^2 - \frac{x^4}{4}\right)_0^2 (6) + \bar{k}\left(4x - \frac{x^3}{3}\right)_0^2 \left(\frac{216}{3}\right)$$

$$= 128\bar{i} - 24\bar{j} + 384\bar{k}$$

Example 2 : If $\bar{F} = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$ then evaluate (*i*) $\int_V \nabla.\bar{F}\, dv$ and (*ii*) $\int_V \nabla \times \bar{F} dv$ where V is the closed region bounded by $x = 0, y = 0, z = 0, 2x + 2y + z = 4$. [JNTU (K) May 2011S (Set No. 1)]

Solution : (*i*) $\nabla \cdot \bar{F} = \bar{i} \cdot \frac{\partial \bar{F}}{\partial x} + \bar{j} \cdot \frac{\partial \bar{F}}{\partial y} + \bar{k} \cdot \frac{\partial \bar{F}}{\partial z} = 4x - 2x = 2x.$

The limits are : $z = 0$ to $z = 4 - 2x - 2y$, $y = 0$ to $\frac{4-2x}{2}$ (*i.e.*) $2 - x$ and $x = 0$ to $\frac{4}{2}$ (*i.e.*) 2

$$\therefore \int_V \nabla.\bar{F} dv = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2x\, dx\, dy\, dz = \int_{x=0}^{2} \int_{y=0}^{2-x} (2x)(z)_0^{4-2x-2y} dx\, dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} 2x(4 - 2x - 2y) dx\, dy = 4\int_{x=0}^{2} \int_{y=0}^{2-x} (2x - x^2 - xy) dx\, dy$$

$$= 4\int_0^2 \left(2xy - x^2 y - \frac{xy^2}{2}\right)_0^{2-x} dx = 4\int_0^2 \left[(2x - x^2)(2 - x) - \frac{x}{2}(2 - x)^2\right] dx$$

$$= 4\int_0^2 \frac{2-x}{2}\left[(4x - 2x^2) - (2x - x^2)\right] dx = 2\int_0^2 (x^3 - 4x^2 + 4x)\, dx$$

$$= 2\left[\frac{x^4}{4} - 4.\frac{x^3}{3} + 4.\frac{x^2}{2}\right]_0^2 = \left[\frac{x^4}{2} - \frac{8x^3}{2} + 4x^2\right]_0^2 = -8$$

$$(ii) \quad \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix} = \bar{j} - 2y\bar{k}$$

$$\therefore \int_V \nabla \times \bar{F} dv = \iiint_V (\bar{j} - 2y\bar{k}) dx\, dy\, dz = \int_{x=0}^{2} \int_{y=0}^{2-x} (\bar{j} - 2y\bar{k})(z)_0^{4-2x-2y} dx\, dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} (\bar{j} - 2y\bar{k})(4 - 2x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \left\{\bar{j}\,[(4-2x) - 2y] - \bar{k}\left[(4-2x)\cdot 2y - 4y^2\right]\right\} dx\,dy$$

$$= \int_{x=0}^{2} \bar{j}\left[(4-2x)y - y^2\right]_0^{2-x} dx - \bar{k} \int_{x=0}^{2} \left[(4-2x)y^2 - \frac{4y^3}{3}\right]_0^{2-x} dx$$

$$= \bar{j}\int_0^2 (2-x)^2\,dx - \bar{k}\int_0^2 \frac{2}{3}(2-x)^3\,dx$$

$$= \bar{j}\left[\frac{(2-x)^3}{-3}\right]_0^2 - \frac{2\bar{k}}{3}\left[\frac{(2-x)^4}{-4}\right]_0^2 = \frac{8}{3}(\bar{j} - \bar{k})$$

Vector Integral Theorems.

Green's Theorem in a plane.

(Transformation b/w line integral and double integral)

Stmt: If R is a closed region in xy-plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where C is traversed in the positive (anti-clockwise) direction.

Note: Area of region R bounded by simple closed curve C.

Let $N = x$, $M = -y$.

Then $\frac{\partial N}{\partial x} = 1$, $\frac{\partial M}{\partial y} = -1$. Applying Green's theorem,

$$\therefore \oint_C -y\,dx + x\,dy = \iint_R (1+1)\,dx\,dy = 2\iint_R dx\,dy$$

$$\therefore \iint_R dx\,dy = \frac{1}{2}\oint_C x\,dy - y\,dx$$

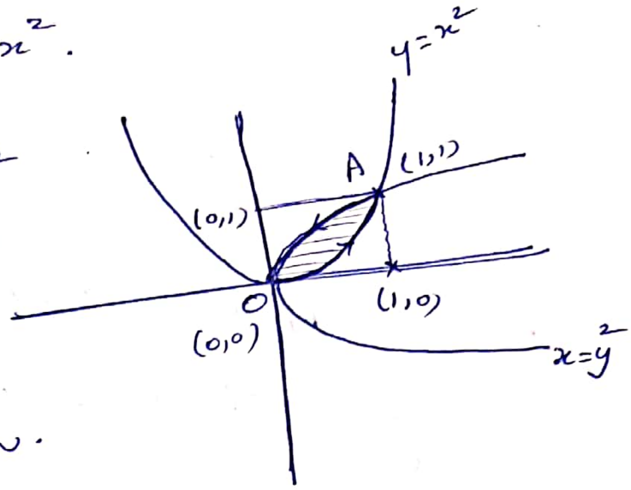
$\iint_R dx\,dy$ represents area of region R.

$$A = \text{Area of region } R = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

$$\boxed{A = \frac{1}{2}\oint_C x\,dy - y\,dx}$$

Problems:

1) Verify Green's theorem in plane for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region bounded by $y = \sqrt{x}$ and $y = x^2$.



Sol: $y = \sqrt{x} \Rightarrow y^2 = x$ (or) $x = y^2$

The region bounded by the 2 parabolas $x = y^2$ and $y = x^2$ is the shaded region.

The points of intersection of the 2 parabolas are O(0,0,0) and A. Solving $x = y^2$ and $y = x^2$, we get A(1, 1).

Let $M = 3x^2 - 8y^2$ $\qquad N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

$\therefore$ By Green's theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R (-6y + 16y)\,dx\,dy$$

$$= 10 \iint_R y\,dx\,dy$$

$$= 10 \int_{x=0}^{1} \left[\int_{y=x^2}^{\sqrt{x}} y\,dy\right] dx$$

$$= 10 \int_{x=0}^{1} \left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$= 10 \int_{x=0}^{1} \left[\frac{(\sqrt{x})^2}{2} - \frac{(x^2)^2}{2} \right] dx$$

$$= 5 \int_{x=0}^{1} (x - x^4)\, dx = 5\left(\frac{x^2}{2} - \frac{x^5}{5} \right)_0^1$$

$$= 5\left(\frac{1}{2} - \frac{1}{5} \right) = 5\left(\frac{5-2}{10} \right)$$

$$= \underline{\underline{\frac{3}{2}}}$$

<u>Verification</u>

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AO} \bar{F}\cdot d\bar{r} = I_1 + I_2.$$

<u>Along I_1</u> $\quad y = x^2$

$dy = 2x\,dx \quad , x \to 0$ to 1

$$\therefore \int_C (3x^2 - 8y^2)dx + (4y - 6xy)\, dy$$

$$= \int_{x=0}^{1} [3x^2 - 8(x^2)^2]\, dx + [4x^2 - 6x(x^2)]\, 2x\,dx$$

$$= \int_{x=0}^{1} (3x^2 - 8x^4 + 8x^3 - 12x^4)\, dx$$

$$= \int_{x=0}^{1} (3x^2 + 8x^3 - 20x^4)\, dx$$

$$= \frac{3x^3}{3} + \frac{8x^4}{4} - \frac{20x^5}{5} \Big)_0^1$$

$$= x^3 + 2x^4 - 4x^5 \Big)_0^1$$

$$= 1 + 2 - 4 = \underline{-1}.$$

<u>Along I_2.</u> $\quad x = y^2$

$dx = 2y\,dy. \quad y \to 1$ to 0.

$$\therefore \int_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy =$$

$$= \int_{y=1}^{0} [3(y^4) - 8y^2]\, 2y\, dy + [4y - 6(y^2)y]\, dy$$

$$= \int_{y=1}^{0} (6y^5 - 16y^3 + 4y - 6y^3)\, dy$$

$$= \int_{y=1}^{0} (4y - 22y^3 + 6y^5)\, dy = \left.\frac{4y^2}{2} - \frac{22y^4}{4} + \frac{6y^6}{6}\right)_1^0$$

$$= \left. 2y^2 - \frac{11}{2}y^4 + y^6 \right)_1^0$$

$$= 0 - \left(2 - \frac{11}{2} + 1\right) = -\left(3 - \frac{11}{2}\right)$$

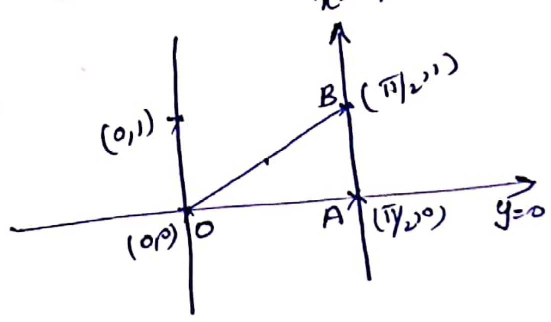
$$= -\left(\frac{6-11}{2}\right) = \frac{5}{2}$$

$$\therefore \int_C M\,dx + N\,dy = I_1 + I_2 = -1 + \frac{5}{2} = \frac{3}{2}$$

$\therefore$ LHS = RHS.

Hence Green's theorem is verified.

2. Evaluate by Green's theorem $\oint_C (y - \sin x)\,dx + \cos x\,dy$ where C is the triangles enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$, $\pi y = 2x$.



Sol: $M = y - \sin x$, $N = \cos x$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x.$$

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\int_C (y - \sin x)\,dx + \cos x\,dy = -\iint_R (1 + \sin x)\,dx\,dy.$$

To find x limits, the x co-ordinates on x-axis in OAB. i.e; $x \to 0$ to $\pi/2$.

To find y limits, $y \to 0$ to $y = \frac{2x}{\pi}$.

$$\therefore \iint_R (1+\sin x)\,dx\,dy = \int_{x=0}^{\pi/2}\left[\int_{y=0}^{\frac{2x}{\pi}} (1+\sin x)\,dy\right]dx$$

$$= \int_{x=0}^{\pi/2} (y)_0^{\frac{2x}{\pi}}(1+\sin x)\,dx$$

$$= \frac{2}{\pi}\int_{x=0}^{\pi/2} (x + x\sin x)\,dx$$

$$= \frac{-2}{\pi}\left[\frac{x^2}{2} + x(-\cos x) + 1(-\sin x)\right]_0^{\pi/2}$$

$$= \frac{-2}{\pi}\left[\frac{x^2}{2} - x\cos x + \sin x\right]_0^{\pi/2}$$

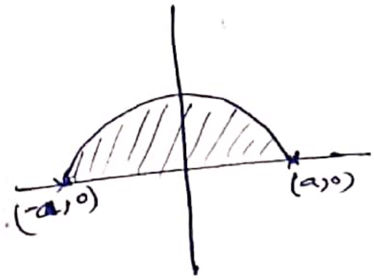
$$= \frac{-2}{\pi}\left[\left(\frac{\pi^2}{8} - 0 + 1\right) - (0 - 0 - 0)\right]$$

$$= \frac{-2}{\pi}\left(\frac{\pi^2}{8} + 1\right) = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

3) Apply Green's theorem to evaluate $\int_C (2x^2 - y^2)\,dx + (x^2+y^2)\,dy$ where C is the boundary of the area enclosed by x-axis and upper half of the circle $x^2+y^2=a^2$.

Sol: Let $M = 2x^2 - y^2$ $\quad N = x^2+y^2$

$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2x$



$\therefore$ By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

⑥

$\therefore \int_C (2x^2-y^2)dx + (x^2+y^2)dy = \iint_R (2x+2y)\,dx\,dy$

[Changing into polar co-ordinates,
$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy \to r\,dr\,d\theta.$
$r \to 0$ to a, $\theta \to 0$ to π]

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi} 2(r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi}\left[\int_{r=0}^{a} r^2(\cos\theta+\sin\theta)\,dr\right]d\theta$$

$$= 2\int_{\theta=0}^{\pi}\left(\frac{r^3}{3}\right)_0^a(\cos\theta+\sin\theta)\,d\theta$$

$$= \frac{2a^3}{3}\left[\sin\theta - \cos\theta\right]_0^{\pi}$$

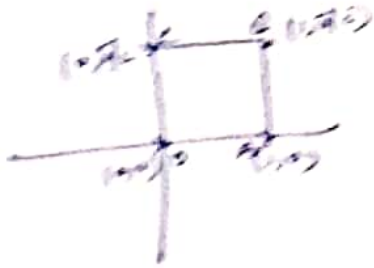
$$= \frac{2a^3}{3}\left[(\sin\pi - \cos\pi) - (\sin 0 - \cos 0)\right]$$

$$= \frac{2a^3}{3}(1+1) = \frac{4a^3}{3} \quad \left\{\begin{array}{l}\sin\pi = 0\\ \cos\pi = -1\\ \cos 0 = 1\end{array}\right\}$$

4) Find the circulation of $\bar{F}$ round the curve C where $\bar{F} = (e^x\sin y)\bar{i} + (e^x\cos y)\bar{j}$ and C is rectangle with vertices $(0,0), (1,0), (1,\pi/2), (0,\pi/2)$.

Sol: $M = e^x\sin y$, $N = e^x\cos y$

$\frac{\partial M}{\partial y} = e^x\cos y$, $\frac{\partial N}{\partial x} = e^x\cos y$



Circulation $= \int_C \bar{F}\cdot d\bar{r}$

$= \int_C e^x\sin y\,dx + e^x\cos y\,dy$

$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$ by Green's theorem

$= \iint (e^x\cos y - e^x\cos y)\,dx\,dy = \underline{0}$

5) A vector field is given by $\bar{F} = \sin y\,\bar{i} + x(1+\cos y)\bar{j}$. Evaluate the line integral around the circular path $x^2+y^2=a^2,\ z=0$ 1) directly (2) using Green's theorem.

Soln: 1) Using line integral:

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \sin y\,dx + x(1+\cos y)\,dy$$

$$= \int_C \sin y\,dx + x\cos y\,dy + x\,dy \qquad \begin{cases} x = a\cos\theta \\ dx = -a\sin\theta\,d\theta \\ y = a\sin\theta \\ dy = a\cos\theta\,d\theta \end{cases}$$

$$= \int_C d(x\sin y) + x\,dy$$

$$= \int_0^{2\pi} d\left[a\cos\theta \sin(a\sin\theta)\right] + \left[a\cos\theta \cdot a\cos\theta\right]d\theta$$

$$= a\cos\theta\sin(a\sin\theta)\Big]_0^{2\pi} + \int_0^{2\pi} a^2\cos^2\theta\,d\theta$$

$$= a\cos\theta\sin(a\sin\theta)\Big]_0^{2\pi} + 4a^2\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= \left[a\cos 2\pi\sin(a\sin 2\pi) - a\cos 0\sin(a\sin 0)\right] + 4a^2\,\frac{1}{2}\cdot\frac{\pi}{2}.$$

$$= 0 + \frac{4a^2\pi}{4} = \underline{\pi a^2}.$$

2) By Green's Theorem.

$M = \sin y \quad , \quad N = x(1+\cos y)$

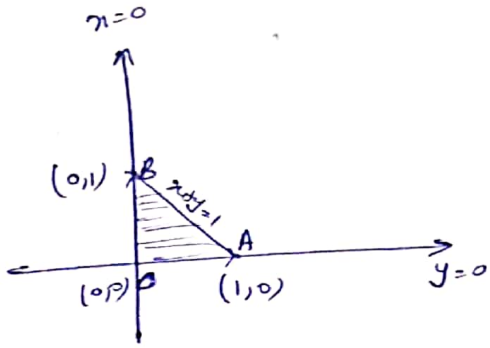
$\frac{\partial M}{\partial y} = \cos y \quad , \quad \frac{\partial N}{\partial x} = 1+\cos y$

$$\therefore \int_C \sin y\,dx + x(1+\cos y)\,dy = \iint_R \left[(1+\cos y) - \cos y\right]dx\,dy$$

$$= \iint_R dx\,dy = \text{Area of circle with radius 'a'} = \underline{\pi a^2}$$

6) Verify Green's theorem for $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the region bounded by $x=0,\ y=0,\ x+y=1$.

Soln:



$M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$, $\frac{\partial N}{\partial x} = -6y$

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$x+y=1$

$y=1-x$

$$= \iint_R (-6y + 16y)\,dx\,dy$$

$$= 10 \iint_R y\,dx\,dy = 10 \int_{x=0}^{1} \left[\int_{y=0}^{1-x} y\,dy \right] dx$$

$$= 10 \int_{x=0}^{1} \left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$= \frac{10}{2} \int (1-x)^2\,dx = 5 \left.\frac{(1-x)^3}{-3}\right]_0^1$$

$$= -\frac{5}{3}(0-1) = \frac{5}{3}$$

Verification

$$\int_C F\cdot d\bar{r} = \left(\int_{OA} + \int_{AB} + \int_{BO}\right) F\cdot d\bar{r}.$$

Along OA. (0,0) to (1,0)

$x \to 0$ to 1, $y=0$, $dy=0$.

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_{x=0}^{1} 3x^2\,dx = \left.\frac{\cancel{3}x^3}{\cancel{3}}\right)_0^1 = 1$$

Along AB. $x+y=1 \Rightarrow x = 1-y$.
$dx = -dy$

$$\therefore \int_{AB} \bar{F}\cdot d\bar{r} = \int_{y=0}^{1} [3(1-y)^2 - 8y^2](-dy) + [4y - 6(1-y)y]\,dy$$

$$= \int_{y=0}^{1} (3+3y^2-6y-8y^2)(-dy) + (4y-6y+6y^2)\,dy$$

$$= \int_{y=0}^{1} (5y^2+6y-3-2y+6y^2)\,dy$$

$$= \int_{y=0}^{1} (11y^2+4y-3)\,dy = \left(\frac{11y^3}{3} + \frac{4y^2}{2} - 3y\right)_0^1$$

$$= \frac{11}{3} + 2 - 3 = \frac{11}{3} - 1 = \underline{\underline{\frac{8}{3}}}$$

Along BO. $(0,1)$ to $(0,0)$

$x=0 \Rightarrow dx = 0 \quad y \to 1$ to 0.

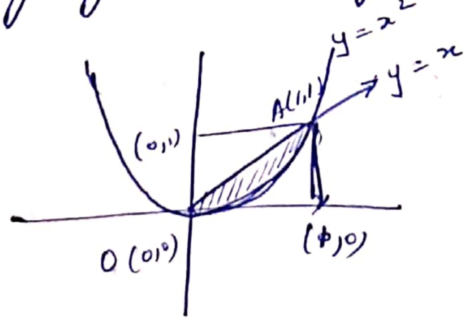
$$\int_{BO} \bar{F}\cdot d\bar{r} = \int_{y=1}^{0} 4y\,dy = 4\frac{y^2}{2}\Big)_1^0 = 2y^2\Big)_1^0 = 0-2 = \underline{\underline{-2}}.$$

$$\therefore \text{Total} = \int_C \bar{F}\cdot d\bar{r} = \cancel{\tfrac{1}{3}} + \frac{8}{3} - 2 = \cancel{3-2}\ \frac{8}{3} - 1 = \underline{\underline{\frac{5}{3}}}$$

$\therefore$ LHS = RHS = $5/3$

$\therefore$ Green's theorem is verified.

7) Verify Green's Theorem for $\int_C (xy+y^2)\,dx + x^2\,dy$ where C is bounded by $y=x$ and $y=x^2$.



$x \to 0$ to 1
$y \to x^2$ to x.

$M = xy + y^2$, $N = x^2$

$\frac{\partial M}{\partial y} = x + 2y$ $\quad \frac{\partial N}{\partial x} = 2x.$

By Green's theorem, $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$

$$= \iint_R [2x - (x + 2y)]\,dx\,dy$$

$$= \iint_R (x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1} \left[\int_{y=x^2}^{x} (x - 2y)\,dy \right] dx$$

$$= \int_{x=0}^{1} \left(xy - \frac{2y^2}{2} \right)_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left(xy - y^2 \right)_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} (x^2 - x^2) - (x^3 - x^4)\,dx.$$

$$= \int_{x=0}^{1} (x^4 - x^3)\,dx \quad \left(\frac{x^5}{5} - \frac{x^4}{4} \right)_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{4 - 5}{20} = \frac{-1}{20}$$

Verification

$\int_C \bar{F} \cdot d\bar{r} = \int_{OA} \bar{F} \cdot d\bar{r} + \int_{AO} \bar{F} \cdot d\bar{r} = I_1 + I_2.$

$I_1 = \int_{OA} \bar{F} \cdot d\bar{r}$ and $I_2 = \int_{AO} \bar{F} \cdot d\bar{r}.$

Along OA. $y = x^2$ $\quad x \to 0$ to 1

$dy = 2x\,dx$.

$\therefore \int_C (xy+y^2)\,dx + x^2 dy = \int_x [(x^3 + x^4) + x^2 \, 2x]\,dx$

$= \int_{x=0}^{1} (3x^3 + x^4)\,dx = \left.\frac{3x^4}{4} + \frac{x^5}{5}\right)_0^1$

$= \frac{3}{4} + \frac{1}{5} = \frac{15+4}{20}$

$= \frac{19}{20}$.

Along AO $\quad y = x \quad x \to 0$ to 1

$dy = dx$

$\int_C (xy+y^2)\,dx + x^2 dy = \int_{x=1}^{0} (x^2 + x^2 + x^2)\,dx$

$= \int_1^0 3x^2\,dx = \left.\frac{\cancel{3}x^3}{\cancel{3}}\right)_1^0$

$= 0 - 1 = -1$.

$\therefore \int_C \bar{F}\cdot d\bar{r} = \frac{19}{20} + (-1) = \frac{-1}{20}$.

$\therefore$ LHS = RHS.

$\therefore$ Green's theorem is verified.

HW

1). Evaluate $\int_C (x^2 + xy)\,dx + (x^2+y^2)\,dy$ where C is the boundary of the region bounded by the lines $x=0, x=1, y=0, y=1$. Ans: $\frac{1}{2}$.

2) Verify Green's theorem for $\int_C (x^2 - xy^3)\,dx + ~~(y^2 dx~~ + (y^2 - 2xy)\,dy$

12

where C is a square with vertices $(0,0)$, $(2,0)$ $(2,2)$, $(0,2)$. Ans: 8

3) Evaluate using Green's theorem $\int_C (2xy - x^2)dx + (x^2+y^2)dy$ where C is the closed curve in xy-plane bounded by the curves $y = x^2$ and ~~$x^2 = y$~~ $y^2 = x$.

Ans: 0

4) Find area bounded by 1) $x = a\cos\theta, y = b\sin\theta$ (or) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow$ Ans: πab

2) $x = a\cos\theta, y = a\sin\theta$ (or) $x^2+y^2 = a^2 \rightarrow$ Ans: πa^2.

Area.

1) Find the area bounded by the curve $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$, $a > 0$, $0 < \theta < 2\pi$ using Green's theorem.

$$\text{Area of curve} = \frac{1}{2}\int_C x\,dy - y\,dx.$$

$$= \frac{1}{2}\int_0^{2\pi} a(\theta - \sin\theta)\,a(\cancel{1-\cos\theta}\ \sin\theta) - a(1-\cos\theta)\,a(1-\cos\theta)$$

$x = a(\theta - \sin\theta)$

$dx = a(1-\cos\theta)$

$y = a(1-\cos\theta)$

$dy = a(0+\sin\theta) = a\sin\theta$

$$= \frac{a^2}{2}\int_0^{2\pi}[(\theta\sin\theta - \sin^2\theta) - (1-\cos\theta)^2]\,d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}\left[\theta\sin\theta - \left(\frac{1-\cos 2\theta}{2}\right) - (1+\cos^2\theta - 2\cos\theta)\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}\left[\theta\sin\theta - \frac{1}{2} + \frac{\cos 2\theta}{2} - 1 - \left(\frac{1+\cos 2\theta}{2}\right) + 2\cos\theta\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}(\theta\sin\theta - 2 + 2\cos\theta)\,d\theta = \frac{a^2}{2}\left(-\theta\cos\theta + \sin\theta - 2\theta + 2\sin\theta\right)_0^{2\pi}$$

$$= \frac{a^2}{2}\{[-2\pi\cos 2\pi - 2(2\pi)] - [0]\}$$

$$= \frac{a^2}{2}(-2\pi - 4\pi) = \frac{a^2}{2}(-6\pi)$$

$$= -3\pi a^2$$

$\therefore$ Area $= 3\pi a^2$ sq. units.

Scanned with Cam

Stoke's Theorem.

(Transformation b/w Line integral and Surface integral).

Stmt: Let S be an open surface bounded by a closed non-intersecting curve. If $\bar{F}$ is any differentiable vector point function, then

$$\oint_C \bar{F}.d\bar{r} = \int_S \text{curl } \bar{F}.\bar{n}\, dS$$

where C is traversed in the positive direction and $\bar{n}$ is the outward drawn normal at any point of the surface.

$$\int_C \bar{F}.d\bar{r} = \int_S (\nabla\times\bar{F}).\bar{n}\, dS.$$

Deduction of Green's Theorem from Stoke's Theorem.

Let the surface S lie on xy-plane. Then $z=0$. Let $\bar{F} = F_1\bar{i} + F_2\bar{j}$ ~~$F_3\bar{k}$~~ and and $\bar{n} = \bar{k}$.

$$\bar{r} = x\bar{i} + y\bar{j}.$$

Then $\bar{F}.d\bar{r} = F_1\,dx + F_2\,dy.$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \left(0-\frac{\partial F_2}{\partial z}\right)\bar{i} - \bar{j}\left(0-\frac{\partial F_1}{\partial z}\right) + \bar{k}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$\therefore (\nabla\times\bar{F}).\bar{n} = (\nabla\times\bar{F}).\bar{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

In xy-plane, $dS = \dfrac{dx\,dy}{|\bar{n}.\bar{k}|} = \dfrac{dx\,dy}{|\bar{k}.\bar{k}|} = dx\,dy$

$$\therefore \int_C \bar{F}.d\bar{r} = \int_S (\nabla\times\bar{F}).\bar{n}\, dS \Rightarrow \int_C F_1\,dx + F_2\,dy = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

which is Green's theorem in a plane.

1) Verify Stoke's theorem for $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ where S is the circular disc, $x^2+y^2 \le 1$, $z=0$.

Sol. $\bar{F} = -y^3\bar{i} + x^2\bar{j}$. The circle is $x^2+y^2=1$.

$\therefore x = \cos\theta, \; y = \sin\theta$

$dx = -\sin\theta\, d\theta, \; dy = \cos\theta\, d\theta$.

$\therefore$ By Stoke's theorem, $\int_C \bar{F}\cdot d\bar{r} = \int_S (\nabla\times\bar{F})\cdot\bar{n}\, dS$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C -y^3 dx + x^3 dy$$

$$= \int_{\theta=0}^{2\pi} -\sin^3\theta(-\sin\theta\, d\theta) + \cos^3\theta(\cos\theta)\, d\theta$$

$$= \int_{\theta=0}^{2\pi} (\cancel{-}\cos^4\theta + \sin^4\theta)\, d\theta$$

$$= 4\int_0^{\pi/2} \cos^4\theta\, d\theta + 4\int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= 4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{3\pi}{2}$$

Now, $$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2+3y^2)$$

On xy-plane, $dS = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \dfrac{dx\,dy}{|\bar{k}\cdot\bar{k}|} = dx\,dy$.

$\bar{n} = \bar{k}$

$\therefore \iint_S (\nabla\times\bar{F})\cdot\bar{k}\,dS = \iint (3x^2+3y^2)\,dx\,dy = 3\iint (x^2+y^2)\,dx\,dy$

Put $x = r\cos\theta, \; y = r\sin\theta, \; dx\,dy = r\,dr\,d\theta$

$$\therefore \iint_S (\nabla\times\bar{F})\cdot\bar{n}\,dS = 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cdot r\,dr\,d\theta = 3\int_0^{2\pi}\left(\frac{r^4}{4}\right)_0^1 d\theta$$

$$= \frac{3}{4}\int_{\theta=0}^{2\pi} d\theta = \frac{3}{4}(\theta)_0^{2\pi} = \frac{3}{4}\times 2\pi = \frac{3\pi}{2}$$

$\therefore$ LHS = RHS

Hence Stoke's theorem is verified.

2) Verify Stokes theorem for $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane.

The boundary of C is $x^2+y^2=1, z=0$.

$\therefore x = \cos\theta,\ y = \sin\theta,\ \theta \to 0$ to 2π.

$dx = -\sin\theta\, d\theta,\ dy = \cos\theta\, d\theta$.

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_C F_1 dx + F_2 dy + F_3 dz = \int_C (2x-y)dx - yz^2 dy - y^2 z dz$$

$$= \int_C (2x-y)dx \qquad [\because z=0, dz=0].$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\, d\theta)$$

$$= \int_{\theta=0}^{2\pi} (-2\cos\theta\sin\theta + \sin^2\theta)\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \sin^2\theta\, d\theta - \int_{\theta=0}^{2\pi} \sin 2\theta\, d\theta$$

$$= 4\int_0^{\pi/2} \sin^2\theta\, d\theta - \left(-\frac{\cos 2\theta}{2}\right)_0^{2\pi}$$

$$= 4\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{2}(\cos 4\pi - \cos 0)$$

$$= \pi + 0 = \pi$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & yz^2 & -y^2z \end{vmatrix} = \bar{i}(-2yz+2yz) - \bar{j}(0-0) + \bar{k}(0+1) = \bar{k}.$$

$\therefore \bar{n} = \bar{k}$. on xy-plane. $dS = dxdy$

$$\therefore \int_S (\nabla \times \bar{F}) \cdot \bar{n}\, dS = \iint_R \bar{k} \cdot \bar{k}\, dxdy$$

$$= \iint_R dxdy$$

$$= \{ \text{Area of region } R$$

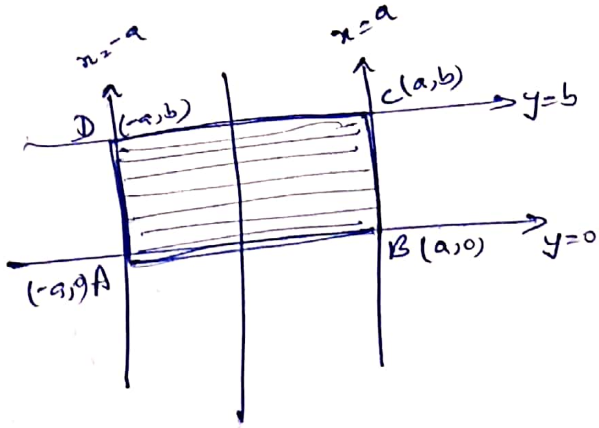
$$= \text{Area of circle } x^2+y^2=1$$

$$= \pi(1)^2 = \underline{\pi}$$

$\therefore$ LHS = RHS

$\therefore$ Stoke's theorem is verified.

3) Verify Stoke's theorem for $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ taken round the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$.



Sol: The rectangle is ABCD with vertices $(-a,0)$, $(a,0)$, (a,b) and $(-a,b)$.

$$\int_C \bar{F} \cdot d\bar{r} = \int_C F_1 dx + F_2 dy$$

$$= \int_C (x^2+y^2)dx - 2xy\,dy$$

$$= \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} \bar{F} \cdot d\bar{r}$$

1) Along AB, $x \to -a$ to a, $y = 0$, $dy = 0$.

$$\therefore \int_{AB} \bar{F} \cdot d\bar{r} = \int_{x=-a}^{a} x^2\, dx = \left(\frac{x^3}{3}\right)_{-a}^{a} = \frac{a^3}{3} + \frac{+a^3}{3} = \frac{2a^3}{3}$$

2) Along BC $x = a$, $y \to 0$ to b, $dx = 0$.

$$\int_{BC} \bar{F} \cdot d\bar{r} = \int_{y=0}^{b} -2ay\,dy = -2a\left(\frac{y^2}{2}\right)_0^b = -ab^2$$

3) Along CD $y = b$, $dy = 0$, $x \to a$ to $-a$.

$$\int_{CD} \bar{F} \cdot d\bar{r} = \int_{x=a}^{-a} (x^2 + b^2)\,dx = \left(\frac{x^3}{3} + b^2 x\right)_a^{-a}$$

$$= \left(-\frac{a^3}{3} - ab^2\right) - \left(\frac{a^3}{3} + ab^2\right)$$

$$= -\frac{2a^3}{3} - 2ab^2$$

4) Along DA. $x = -a$, $dx = 0$, $y \to b$ to 0.

$$\int_{DA} \bar{F} \cdot d\bar{r} = \int_{y=b}^{0} 2ay\,dy = 2a\left(\frac{y^2}{2}\right)_b^0 = a(0 - b^2)$$
$$= -ab^2.$$

$$\therefore \int_{C} \bar{F} \cdot d\bar{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2$$
$$= \underline{\underline{-4ab^2}}$$

Consider $\int_S \text{curl}\,\bar{F} \cdot \bar{n}\,dS$

Vector perpendicular to xy-plane is $\bar{n} = \bar{k}$.

$$\text{Curl}\,\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = -4y\bar{k}$$

$\bar{n} = \bar{k}$ and $dS = dx\,dy$.

$$\int_C (\text{curl}\,\bar{F}\cdot\bar{n})\,ds = \int_{x=-a}^{a}\int_{y=0}^{b} (-4y\,\bar{k}\cdot\bar{k})\,dx\,dy$$

$$= \int_{x=-a}^{a}\left[\int_{y=0}^{b} -4y\,dy\right]dx$$

$$= \int_{x=-a}^{a}\left(-4\frac{y^2}{2}\right)_0^b dx$$

$$= \int_{x=-a}^{a} -2b^2\,dx = -2b^2(x)_{-a}^{a}$$

$$= -2b^2(a+a)$$

$$= -2b^2\times 2a = \underline{-4ab^2}$$

$\therefore$ LHS = RHS

Hence Stoke's theorem is verified.

4) Verify Stoke's theorem for $\bar{F} = y^2\bar{i} + y\bar{j} - zx\bar{k}$ and S is the upper half of the sphere $x^2+y^2+z^2=a^2$, $z\geq 0$.

Sol: C is the circle $x^2+y^2=a^2$. On C, $z=0$.

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_C y^2dx + y\,dy$$

Since C is a circle, put $x = a\cos\theta$

$dx = -a\sin\theta\,d\theta$

$y = a\sin\theta$, $dy = a\cos\theta\,d\theta$, $\theta \to 0$ to 2π.

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_{\theta=0}^{2\pi} (a^2\sin^2\theta)(-a\sin\theta)d\theta + a\sin\theta\, a\cos\theta\, d\theta$$

$$= \int\left(-a^3\sin^3\theta + a^2\frac{\sin 2\theta}{2}\right)d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[-\frac{a^3}{4}(3\sin\theta - \sin 3\theta) + \frac{a^2}{2}\sin 2\theta\right]d\theta$$

(7)

$$= \left[-\frac{a^3}{4}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right) + \frac{a^2}{2}\left(-\frac{\cos 2\theta}{2}\right)\right]_0^{2\pi}$$

$$= -\frac{a^3}{4}\left[\left(-3+\frac{1}{3}\right)-\left(-3+\frac{1}{3}\right)\right] - \frac{a^2}{4}\left[\cos 4\pi - \cos 0\right]$$

$$= \underline{\underline{0}}$$

$$\text{Curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & y & -zx \end{vmatrix}$$

$$= \bar{i}(0-0) - \bar{j}(-z-0) + \bar{k}(0-2y)$$

$$= z\bar{j} - 2y\bar{k}$$

Surface $\phi = x^2+y^2+z^2-a^2$

Normal to the surface $= \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$$= i\,2x + \bar{j}\,2y + \bar{k}\,2z$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\bar{i}+y\bar{j}+z\bar{k}}{a}$$

$$\therefore \text{Curl } \bar{F}\cdot\bar{n} = (z\bar{j}-2y\bar{k})\cdot\left(\frac{x}{a}\bar{i}+\frac{y}{a}\bar{j}+\frac{z}{a}\bar{k}\right)$$

$$= \frac{zy}{a} - \frac{2yz}{a} = \frac{-yz}{a}.$$

$\int_S \cancel{\text{curl } \bar{F}\cdot\bar{n}\, dS =}$

Changing into spherical co-ordinates,

$x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$

$$\int_S (\nabla \times \bar{F}) \cdot \bar{n}\, dS = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} -a \sin\theta \sin\phi \cdot a\cos\theta \sin\theta \, d\theta\, d\phi$$

$$= \int_\theta \int_\phi -a^2 \sin^2\theta \cos\theta \sin\phi \, d\theta\, d\phi$$

$$= -a^2 \int_\theta \int_\phi \left(\sin^2\theta \cos\theta \,(\sin\phi \, d\phi)\right) d\theta$$

$$= -a^2 \int_\theta \sin^2\theta \cos\theta \left(-\cos\phi\right)_0^{2\pi} d\theta$$

$$= +a^2 \int_\theta \sin^2\theta \cos\theta \left(\cos 2\pi - \cos 0\right) d\theta$$

$$= 0$$

$\therefore$ LHS = RHS

Hence Stoke's theorem is verified.

5) Apply Stoke's theorem to evaluate $\int_C y\,dx + z\,dy + x\,dz$ where C is the curve of intersection of sphere $x^2+y^2+z^2=a^2$ and $x+z=a$

Sol: Equation of plane is $x+z=a$

Dividing by a, $\frac{x}{a}+\frac{z}{a}=1$

The points $A(a,0,0)$ and $B(0,0,a)$ satisfies the eqn. of the plane. These points also ~~The intersection~~ satisfy the eqn of sphere.

$\therefore$ The intersection of sphere and plane is a circle with diameter AB.

$$AB = \sqrt{(a-0)^2+(0-0)^2+(0-a)^2} \quad \text{(distance formula)}$$

$$= \sqrt{a^2+a^2} = \sqrt{2a^2} = \sqrt{2}\,a$$

$\therefore$ Radius of circle of intersection $= \frac{\sqrt{2}a}{2} = \frac{a}{\sqrt{2}}$

Given $\bar{F}\cdot d\bar{r} = y\,dx + z\,dy + x\,dz$

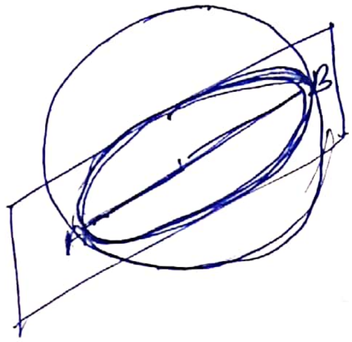
$$\therefore \bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \bar{i}(0-1) + \bar{j}(1-0) + \bar{k}(0-1)$$

$$= -(\bar{i}+\bar{j}+\bar{k})$$

Let $\bar{n}$ be the unit normal to Surface ϕ.

$$\therefore \bar{n} = \frac{\nabla\phi}{|\nabla\phi|} \quad \text{where } \phi = x+z-a$$



$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i} + \bar{k}$$

$$\therefore \bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\bar{i}+\bar{k}}{\sqrt{1^2+1^2}} = \frac{\bar{i}+\bar{k}}{\sqrt{2}}$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,dS$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C y\,dx + z\,dy + x\,dz$$

$$= -\int_S (\bar{i}+\bar{j}+\bar{k})\cdot\left(\frac{\bar{i}+\bar{k}}{\sqrt{2}}\right) dS$$

$$= -\int_S \left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) dS = -\frac{2}{\sqrt{2}}\int_S dS$$

$$= \cancel{\sqrt{2}S} = \cancel{-\sqrt{2}\left(\frac{\pi a^2}{2}\right)} = \cancel{\frac{\pi a^2}{\sqrt{2}}}.$$

$$= -\sqrt{2}\,(\text{Area of circle})$$

$$= -\sqrt{2}\,\pi\left(\frac{a}{\sqrt{2}}\right)^2 \qquad \left[\because \text{Radius} = \frac{a}{\sqrt{2}}\right]$$

$$= -\sqrt{2}\,\pi\frac{a^2}{2} = \frac{-\pi a^2}{\sqrt{2}}$$

Gauss' Divergence Theorem.

Stmt: Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V \operatorname{div} \bar{F}\, dV = \int_S \bar{F}\cdot\bar{n}\, dS$$

where $\bar{n}$ is the outward drawn normal vector at any point of S.

Note: This theorem is a transformation between surface integral and volume integral.

Cartesian Form:

Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

and $\bar{n} = \cos\alpha\,\bar{i} + \cos\beta\,\bar{j} + \cos\gamma\,\bar{k}$ where $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of $\bar{n}$.

$$\therefore\ \bar{F}\cdot\bar{n} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$$

$$\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

$$\therefore \iiint_{x\,y\,z} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

$$= \int_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\alpha)\, dS$$

$$= \iint F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy.$$

1) Compute $\int(ax^2+by^2+cz^2)\,dS$ over the surface of the sphere $x^2+y^2+z^2=1$.

Sol: By divergence theorem, $\int_S \bar{F}\cdot\bar{n}\,dS = \int_V \nabla\cdot\bar{F}\,dV$

Given $\int (ax^2 + by^2 + cz^2) ds$

$\Longrightarrow \bar{F} \cdot \bar{n} = ax^2 + by^2 + cz^2$

Given Surface S is $x^2+y^2+z^2=1$. (we call it ϕ)

Let $\phi = x^2+y^2+z^2-1$

$\nabla\phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

$$\therefore \bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$= x\bar{i}+y\bar{j}+z\bar{k} \quad [\because x^2+y^2+z^2=1].$$

$$\therefore \bar{F}\cdot(x\bar{i}+y\bar{j}+z\bar{k}) = ax^2+by^2+cz^2$$
$$= (ax\bar{i}+by\bar{j}+cz\bar{k})\cdot(x\bar{i}+y\bar{j}+z\bar{k})$$

$$\therefore \bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}.$$

$$\text{Div } \bar{F} = \nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$
$$= a+b+c.$$

$$\therefore \int (ax^2+by^2+cz^2)\,dS = \int_V \text{div } \bar{F}\, dV$$
$$= \int_V (a+b+c)\, dV$$
$$= (a+b+c)\int_V dV \quad \left[\because \int_V dV \text{ is volume of sphere} = \frac{4\pi r^3}{3},\ r=1\right]$$
$$= (a+b+c)\frac{4\pi}{3}$$
$$= \frac{4\pi}{3}(a+b+c)$$

2. By transforming into triple integral, evaluate $\iint x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ where S is the closed surface consisting of the cylinder $x^2+y^2=a^2$ and $z=0, z=b$.

Sol: We know that in cartesian form, by Gauss' Divergence thm,

$$\iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV$$

$\therefore F_1 = x^3,\quad F_2 = x^2 y,\quad F_3 = x^2 z.$

$$\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \cancel{6x +}\ 3x^2 + x^2 + x^2$$
$$= 5x^2.$$

$$\therefore \iint_S x^3\,dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy = \iiint 5x^2\,dx\,dy\,dz$$

To find limits of x, y, z.

z limits $\rightarrow z \rightarrow 0$ to b (given)

$x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2$
$\Rightarrow y = \pm\sqrt{a^2 - x^2}$
$\Rightarrow y \rightarrow -\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

Put $y = 0$, $\quad x^2 = a^2 \Rightarrow x = \pm a$
$\Rightarrow x \rightarrow -a$ to a.

$$\therefore \iiint_{xyz} 5x^2\,dx\,dy\,dz = \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=0}^{b} 5x^2\,dx\,dy\,dz$$

$$= 5 \times 2 \int_{x=0}^{a} 2 \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{b} x^2\,dx\,dy\,dz$$

$$= 20 \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} (x^2 z)_0^b\,dx\,dy$$

$$= 20b \int_{x=0}^{a}\left[\int_{y=0}^{\sqrt{a^2-x^2}} x^2\,dy\right] dx$$

$$= 20b \int_{x=0}^{a} (x^2 y)_0^{\sqrt{a^2-x^2}}\,dx.$$

$$= 20b \int_{x=0}^{a} x^2 \sqrt{a^2 - x^2}\, dx .$$

[Put $x = a\sin\theta$; when $x = 0$, $a\sin\theta = 0 \Rightarrow \theta = 0$

$\therefore dx = a\cos\theta\, d\theta$; when $x = a$, $a\sin\theta = a \Rightarrow \sin\theta = 1$

$\Rightarrow \theta = \pi/2$]

$$= 20b \int_{\theta=0}^{\pi/2} a^2 \sin^2\theta \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta$$

$$= 20b \int_{\theta=0}^{\pi/2} a^2\sin^2\theta \cdot a\cos\theta \cdot a\cos\theta\, d\theta$$

$$= 20ba^4 \int_{\theta=0}^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta .$$

$$= 5\cancel{20}\, a^4 b \int_{\theta=0}^{\pi/2} 4\sin^2\theta\cos^2\theta\, d\theta$$

$$= 5a^4 b \int_{\theta=0}^{\pi/2} (2\sin\theta\cos\theta)^2\, d\theta$$

$$= 5a^4 b \int_{0}^{\pi/2} (\sin 2\theta)^2\, d\theta$$

$$= 5a^4 b \int_{0}^{\pi/2} \frac{1 - \cos 4\theta}{2}\, d\theta$$

$$= 5a^4 b \left[\frac{1}{2}\theta - \frac{\sin 4\theta}{8} \right]_0^{\pi/2}$$

$$= 5a^4 b \left[\frac{1}{2} \cdot \frac{\pi}{2} \right] = \underline{\underline{\frac{5}{4}\pi a^4 b}}$$

3) Use divergence theorem to evaluate $\iint_S \bar{F}\cdot\bar{n}\,dS$ (5)

when $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$.

$$\iint_S \bar{F}\cdot\bar{n}\,d\bar{S} = \int_V \nabla\cdot\bar{F}\,dV$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$$

$$= 3x^2 + 3y^2 + 3z^2.$$

$$\therefore \int_V 3(x^2+y^2+z^2)\,dx\,dy\,dz$$

Changing into spherical co-ordinates,

$$x^2+y^2+z^2 = r^2,$$

$$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$\therefore \iint_S \bar{F}\cdot\bar{n}\,dS = 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2\cdot r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^4\sin\theta\,dr\,d\theta\,d\phi$$

$$= 3\int_{\theta}\int_{\phi}\left(\frac{r^5}{5}\right)_0^a \sin\theta\,d\theta\,d\phi$$

$$= \frac{3a^5}{5}\int_{\phi}(-\cos\theta)_0^{\pi}\,d\phi$$

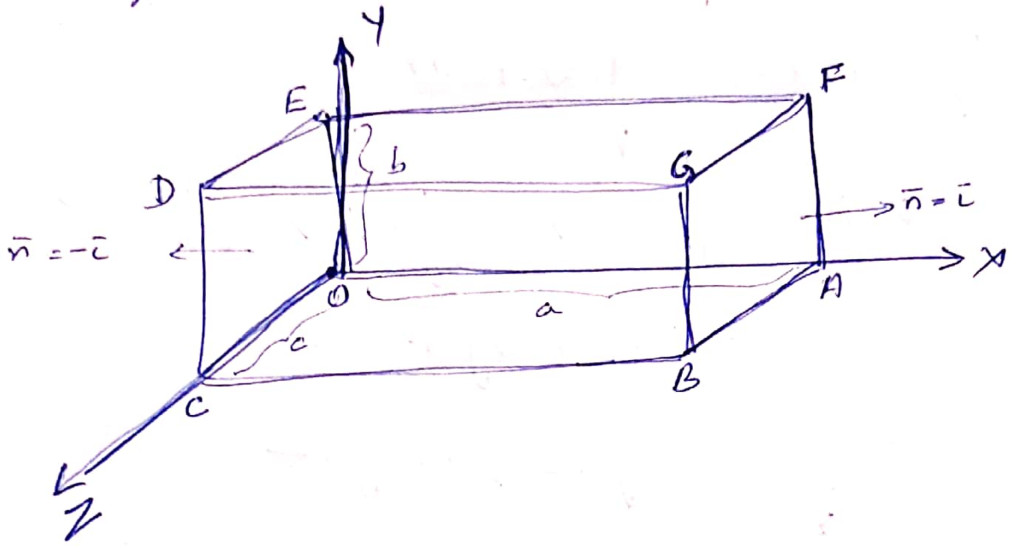
$$= \frac{3a^5}{5}\int_{\phi} 2\,d\phi = \frac{2a^5}{5}\int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{6a^5}{5}(\phi)_0^{2\pi} = \frac{2a^5}{5}.2\pi$$

$$= \frac{12\pi a^5}{5}$$

4) Verify Gauss' Divergence theorem for

$\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ taken over a rectangular parallelopiped. $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$



Soln: By divergence theorem,

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div} \bar{F}\,dV$$

$$R.H.S = \int_V \nabla\cdot\bar{F}\,dV$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2x + 2y + 2z = 2(x+y+z)$$

$$\therefore RHS = \int_V 2(x+y+z)\,dV = 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_x\int_y \left(xz + yz + \frac{z^2}{2}\right)_0^c dx\,dy$$

$$= 2\int_x\int_{y=0}^{b}\left(cx + cy + \frac{c^2}{2}\right)dy\,dx$$

$$= 2\int_x \left(cxy + \frac{cy^2}{2} + \frac{c^2}{2}y\right)_0^b dx$$

$$= 2\int_x \left(cxb + \frac{cb^2}{2} + \frac{c^2 b}{2}\right)dx$$

$$= 2\left(bc\frac{x^2}{2} + \frac{b^2c}{2}x + \frac{bc^2}{2}x\right)_0^a$$

$$= 2\left(\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right)$$

$$= \underline{abc(a+b+c)}$$

__Verification__

To evaluate $\int_S \bar{F}\cdot\bar{n}\,dS$ where S has six faces.

1) On the face ABGF (parallel to YZ plane)

$x = a,\ y \to 0 \text{ to } b,\ z \to 0 \text{ to } c,\ \bar{n} = \bar{i},\ dS = dy\,dz$

$$\therefore \int_{S_1} \bar{F}\cdot\bar{n}\,dS = \int_{y=0}^{b}\int_{z=0}^{c} (x^2 - yz)\,dy\,dz$$

$$= \int_{y=0}^{b} \left(x^2 z - y\frac{z^2}{2}\right)_0^c = \int_y \left(x^2c - y\frac{c^2}{2}\right)dy$$

$$= \int_{y=0}^{b} \left(a^2c - y\frac{c^2}{2}\right)dy \qquad \cdot (x=a)$$

$$= \left(a^2cy - \frac{y^2}{2}\frac{c^2}{2}\right)_0^b = a^2bc - \frac{b^2c^2}{4}$$

2) On the face OCDE (on the yz-plane)

$x = 0,\ y \to 0 \text{ to } b,\ z \to 0 \text{ to } c,\ \bar{n} = -\bar{i},\ dS = dy\,dz$

$$\int_{S_2} \bar{F}\cdot\bar{n}\,dS = \int_{y=0}^{b}\int_{z=0}^{c} (0 + yz)\,dy\,dz = \int_y \left(+y\frac{z^2}{2}\right)_0^c dy$$

$$= \int_{y=0}^{b} y\frac{c^2}{2}\,dy = \left.\frac{c^2}{2}\frac{y^2}{2}\right)_0^b = \frac{b^2c^2}{4}$$

3) On the face DGFE (parallel to zx plane)

$x \to 0 \text{ to } a,\ y = b,\ z \to 0 \text{ to } c,\ n = \bar{j},\ dS = dz\,dx.$

$$\int_{S_3} \bar{F}\cdot\bar{n}\,dS = \int_{x=0}^{a}\int_{z=0}^{c} (y^2 - zx)\,dx\,dz = \int_x \left(y^2z - \frac{z^2}{2}x\right)_0^c$$

$$= \int_x \left(b^2c - \frac{c^2x}{2}\right)dx$$

$$= \left(b^2cx - \frac{c^2x^2}{4}\right)_0^a = ab^2c - \frac{a^2c^2}{4}$$

4) On the face OABC (on the zx plane)

$y=0$, $x \to 0$ to a, $z \to 0$ to c, $\bar{n} = -\bar{j}$, $dS = dz\,dx$.

$$\therefore \int_{S_4} \bar{F}\cdot\bar{n}\,dS = \int_{x=0}^{a}\int_{z=0}^{c} (zx)\,dx\,dz = \int_x \left(\frac{z^2}{2}x\right)_0^c dx$$

$$= \int_{x=0}^{a} \frac{c^2 x}{2}\,dx = \frac{c^2}{2}\frac{x^2}{2}\Big)_0^a = \frac{a^2c^2}{4}$$

5) On the face CBGD (parallel to XY plane)

$z=c$, $x \to 0$ to a, $y \to 0$ to b, $n = \bar{k}$, $dS = dx\,dy$

$$\int_{S_5} \bar{F}\cdot\bar{n}\,dS = \int_x\int_y (z^2 - xy)\,dx\,dy$$

$$= \int_x\int_y (c^2 - xy)\,dx\,dy$$

$$= \int_x \left(c^2y - \frac{xy^2}{2}\right)_0^b dx = \int_x \left(bc^2 - \cancel{xy}\,\frac{xb^2}{2}\right)dx$$

$$= \left(bc^2x - \frac{b^2}{2}\frac{x^2}{2}\right)_0^a = abc^2 - \frac{a^2b^2}{4}.$$

6) On the face OAFE (on the xy-plane).

$z=0$, $x \to 0$ to a, $y \to 0$ to b, $n = -\bar{k}$, $dS = dx\,dy$.

$$\int_{S_6} \bar{F}\cdot\bar{n}\,dS = \int_{x=0}^{a}\int_{y=0}^{b} xy\,dx\,dy = \int_x \left(\frac{xy^2}{2}\right)_0^b dx$$

$$= \int_x \frac{b^2}{2}x\,dx$$

$$= \frac{b^2}{2}\frac{x^2}{2}\Big)_0^a = \frac{a^2b^2}{4}.$$

$$\therefore \int_S \bar{F}\cdot\bar{n}\,dS = a^2bc - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + ab^2c - \cancel{\frac{a^2c^2}{4}} + \cancel{\frac{a^2c^2}{4}}$$

$$+ abc^2 - \cancel{\frac{a^2c^2}{4}} + \cancel{\frac{a^2c^2}{4}}$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c).$$

$\therefore$ LHS = RHS. $\therefore$ Gauss' thm is verified.