UNIT - II

IIR Digital Filters

* Review of analog filter design
* Frequency transformation in the analog & Digital domains
* Design of IIR filters from Analog filters
  * Approximation of derivatives
  * Impulse invariance
  * Bilinear transformation.
* Design of Butterworth, chebyshev filters
* problems

* Realization of IIR Systems

* Structures for IIR Systems
  * Direct form - I
  * Direct form - II
  * Transposed form
  * Cascade form
  * Lattice structures
  * signal flow graphs.

* Design of IIR filters from Analog filters

There are several methods that can be used to design digital filters having an infinite duration unit sample response.

First the analog filter is designed in analog domain and then transformed into the digital domain

The system function describing an analog filter may be written as

$$H_a(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \quad -(1)$$

where $\{a_k\}$ & $\{b_k\}$ are filter coefficients

The impulse response of these filter coefficients is related to $H_a(s)$

$$H_a(s) = \int_{-\infty}^{\infty} h(t) e^{-st} \, dt \quad -(2)$$

The analog filter having rational system function $H_a(s)$ can be described by LCCDE

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad -(3)$$

where $x(t)$ is i/p & $y(t)$ is the o/p signal

The above three equivalent characterizations of an analog filter leads to alternative methods for converting the filter into the digital domain.

If the conversion technique is to be effective, It should possess the following desirable properties

1. The $j\Omega$ axis in the S-plane should map into the unit circle in the Z-plane. Thus there will be a direct relationship between the two frequency variables in the two domains

2. The LHP (Left Half plane) of the S-plane should map into the inside of the unit-circle in the Z-plane. Thus a stable analog filter will be converted to a stable digital filter.

The most widely used method for digitizing the analog filter into a digital filter

* Approximation of Derivatives

One of the simplest methods for Converting an analog filter into a digital filter is to approximate the differential equation by an equivalent difference equation.

Consider LCCDE

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Thus $$\frac{d}{dt} y(t) \Big/_{t=nT} = \frac{y(nT) - y(nT - T)}{T}$$

$$\simeq \frac{y(n) - y(n-1)}{T}$$

Where $T \rightarrow$ Sampling interval

& $y[n] \equiv y[nT]$

The analog differentiator with output $\frac{d}{dt} y(t)$ has the System fuction

$$H(s) = S.$$

while the digital System that prod. the output $\frac{y[n] - y[n-1]}{T}$ has the System fuction.

$$H(z) = \frac{1 - z^{-1}}{T}$$

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$$y(t) \longrightarrow \boxed{H(s) = s} \longrightarrow \frac{d}{dt} y(t)$$

$$y[n] \longrightarrow \boxed{H(s) = \frac{1 - z^{-1}}{T}} \longrightarrow \frac{y(n) - y(n-1)}{T}$$

By Comparing there two we get

$$\boxed{s = \frac{1 - z^{-1}}{T}}$$

The 2^{nd} derivative $\frac{d^2}{dt^2} y(t)$ is replace

by the 2^{nd} difference

$$\left.\frac{d^2 y(t)}{dt^2}\right|_{t=nT} = \frac{d}{dt}\left[\frac{d}{dt} y(t)\right]_{t=nT}$$

$$= \frac{\frac{y(nT) - y(nT - T)}{T} - \frac{y(nT - T) - y(n}{T}}{T}$$

$$= \frac{y(n) - 2y(n-1) + y(n-2)}{T^2}$$

$$\therefore \boxed{s^2 = \frac{1 - 2z^{-1} + z^{-2}}{T^2} = \left(\frac{1 - z^{-1}}{T}\right)^2}$$

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In general for the ith derivative of $y(t)$ the equivalent relationship is

$$\boxed{s^i = \left(\frac{1-z^{-1}}{T}\right)^i}$$

∴ The digital filter transfer function $H(z)$ can be obtained from analog transfer function $H_a(s)$

$$\boxed{H(z) = H_a(s) \Big/ s = \frac{1-z^{-1}}{T}}$$

we have

$$s = \frac{1-z^{-1}}{T} \Rightarrow \text{i.e.} \quad z = \frac{1}{1-sT}$$

Substitute $s = j\Omega$

$$z = \frac{1}{1-j\Omega T}$$

$$z = \frac{1}{1+\Omega^2 T^2} + j\frac{\Omega T}{1+\Omega^2 T^2}$$

$$z = x+jy \Rightarrow x^2+y^2 = x \Rightarrow \left(x-\tfrac{1}{2}\right)^2 +$$ ci

As Ω varies from $-\infty$ to ∞ the corresponding locus of points in the z-plane is a circle of radius $\frac{1}{2}$ and with center $z = \frac{1}{2}$

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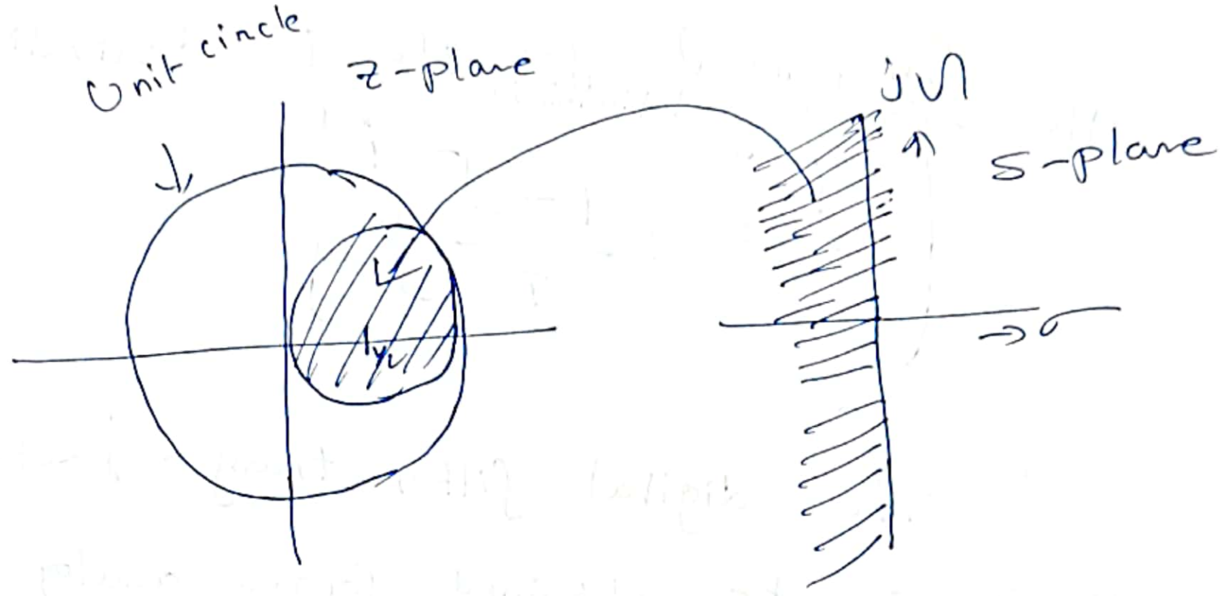


Mapping of the s-plane to z-plane

Using approximating derivative

From the above it is clear that the left half of s-plane is mapped into corresponding points inside the circle of radius 1/2 and ~~outside~~ center at $z = 0.5$ & the right half of the s-plane is mapped outside this circle.

Thus a stable analog filter gets converted into stable digital filter

This design method can be used only for transforming analog LPF & BPF which are having smaller resonant freqs

(but the $j\Omega$ axis does not map on $z = e^{j\omega}$ circle
so it restricts the filter pole location for small freq)

To overcome the limitation in mapping The forward difference equation can be substituted for the derivative instead of backward difference equation

i.e. $$\left.\frac{d}{dt}y(t)\right|_{t=nT} = \frac{y(nT+T)-y(nT)}{T}$$

$$\simeq \frac{y(n+1)-y(n)}{T}$$

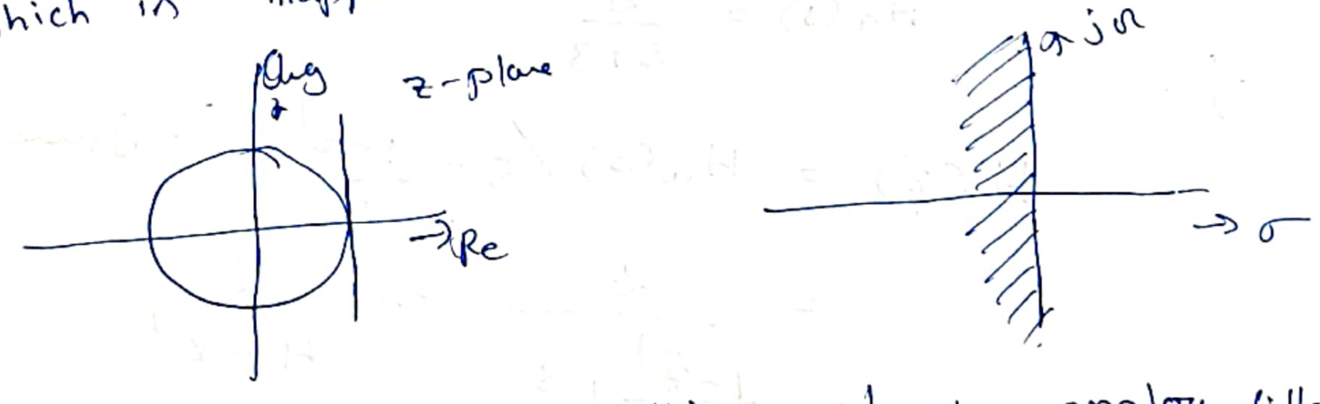
The transformation is

$$S = \frac{z-1}{T}$$

$$\Rightarrow z = 1+ST$$

when $s = j\Omega \Rightarrow z = 1+j\Omega T$

when Ω is varied from $-\infty$ to ∞. The locus of points is a straight line in z-plane which is mapped outside the circle as shown below



As a result of this stable analog filters do not always map into stable digital filters

An Nth order difference eq^n is proposed for the derivative as

$$\frac{d}{dt} y(t) \Big/_{t=nT} = \frac{1}{T} \sum_{k=1}^{N} \frac{a_k [y(nT+kT) - y(nT-kT)]}{T}$$

The transformation from S-plane to z-plane will

$$S = \frac{1}{T} \sum_{K=1}^{N} \frac{a_K (z^K - z^{-K})}{T}$$

By choosing proper values of $\{a_K\}$ the $j\Omega$ axis can be mapped on the unit circle and left half of the S-plane can be mapped into points inside the circle in the z-plane

eg: Convert the analog filter given below into a digital filter using backward difference eq^n for the derivative

$$H_a(s) = \frac{2}{s+3}$$

sol:

$$H(z) = H_a(s) \Big/ s = \frac{1-z^{-1}}{T}$$; Assume $T = 1$-sec

$$= \frac{2}{\frac{1-z^{-1}}{T} + 3} = \frac{2}{4 - z^{-1}}$$

eg: $H_a(s) = \frac{4}{s^2+9}$ & $H_a(s) = \frac{3}{(s+0.5)^2+16}$

* Impulse Invariant Transformation

The desired impulse response is obtained by uniformly sampling the impulse response of equivalent analog filter

For impulse invariant transformation

$$h[n] = h_a(t)\Big|_{t=nT} = h_a(nT)$$

Consider TF

$$H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s-P_i} \quad \text{--- (1)}$$

$$h_a(t) = L^{-1}\{H_a(s)\}$$

$$= \sum_{i=1}^{N} A_i e^{P_i t} u_a(t)$$

$$h[n] = h_a(t)\Big|_{t=nT}$$

$$= \sum_{i=1}^{N} A_i e^{P_i nT} u_a(nT)$$

wkt

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{i=1}^{N} A_i e^{P_i nT} u_a[nT]\right] z^{-n}$$

$$= \sum_{i=1}^{N} A_i \sum_{n=0}^{\infty} \left(e^{P_i T} z^{-1}\right)^n$$

$$H(z) = \sum_{i=1}^{N} \frac{A_i}{1 - e^{P_i T} z^{-1}} \quad \text{—— (2)}$$

Comparing eq^n (1) & (2)

$$\frac{1}{s - P_i} \longrightarrow \frac{1}{1 - e^{P_i T} z^{-1}}$$

analog pole at $s = P_i$ is mapped into a digital pole at $z = e^{P_i T}$

$\therefore$ relation is $\boxed{Z = e^{ST}}$

$$r e^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$|z| = r = e^{\sigma T}$$

$$\angle z = \omega = \Omega T$$

$\therefore$ The relationship b/w analog freq Ω and digital freq ω is

$$\omega = \Omega T \quad \text{(or)} \quad \Omega = \frac{\omega}{T}$$

If $\sigma < 0$ then $0 < r < 1$

i.e left half of the s-plane is mapped within the unit circle in the z-plane

If $\sigma > 0$ then $r > 1$

i.e. the right half of the s-plane is mapped outside the unit circle in the z-plane

If $\sigma = 0$ then $r = 1$

i.e. the imaginary axis is mapped on to the unit circle. thus

A stable analog filter is converted into a stable digital filter

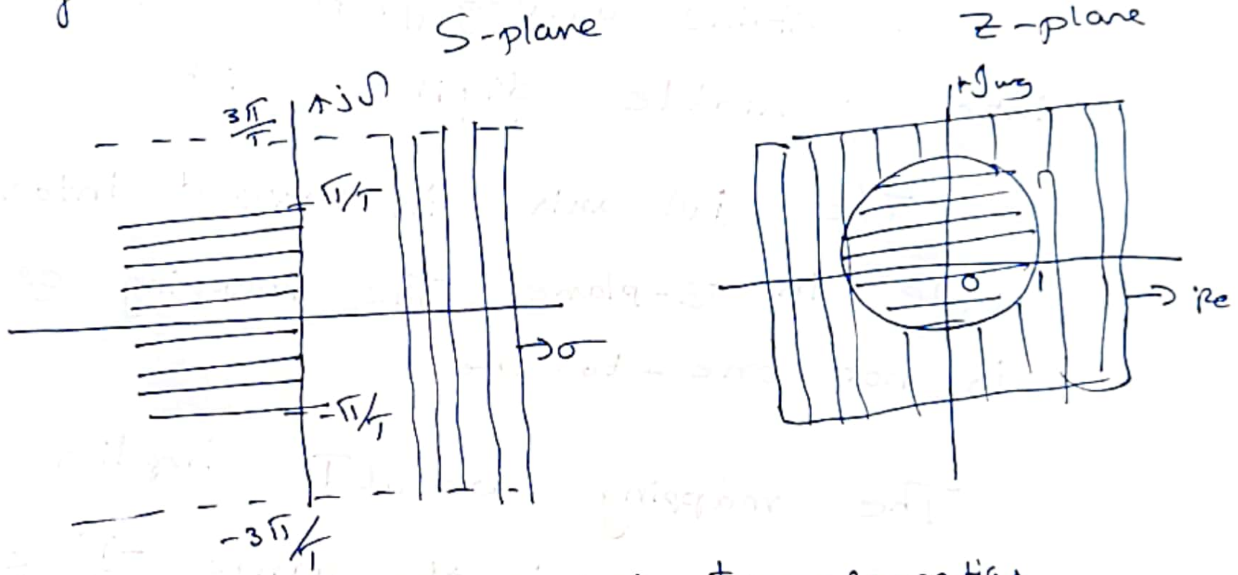
The $j\Omega$ axis is mapped into the unit circle in z-plane. The mapping of $j\Omega$ axis is not one-to-one.

The mapping $\omega = \Omega T$ implies that for values of s in the range $-\frac{\pi}{T} \leq \Omega \leq \frac{\pi}{T}$ into the corresponding values of $-\pi \leq \omega \leq \pi$ into the entire z-plane. Similarly the values of s in the range $\frac{\pi}{T} \leq \Omega \leq \frac{3\pi}{T}$ also map into the interval $-\pi \leq \omega \leq \pi$.

In general Any frequency in the interval $\frac{(2k-1)\pi}{T} \leq \Omega \leq \frac{(2k+1)\pi}{T}$ when k is an integer will also map into the interval $-\pi \leq \omega \leq \pi$ in the z-plane.

Thus the mapping from analog frequency Ω to the digital frequency ω by impulse invariance transformation is many-to-one. which simply reflects the effects of ~~sampling~~ aliasing due to sampling.

The following figure illustrates the mapping from the s-plane to z-plane



Useful impulse invariant transformation

* $$\frac{1}{(s+p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \left[\frac{1}{1-e^{-sT} z^{-1}} \right]_{s=p_i}$$

* $$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{aT} z^{-1} \cos bT}{1-2e^{-aT} z^{-1} \cos bT + e^{-2aT} z^{-2}}$$

* $$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT} z^{-1} \sin bT}{1-2e^{-aT} z^{-1} \cos bT + e^{-2aT} z^{-2}}$$

* For the analog T.F $H_a(s) = \frac{2}{(s+1)(s+3)}$

determine $H(z)$ if a) $T = 1$ sec

b) $T = 0.5$ sec Using IIT

Given

Sol: $H_a(s) = \frac{2}{(s+1)(s+3)}$

$$H_a(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

$$= \frac{1}{s+1} - \frac{1}{s+3}$$

$$H(z) = \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-3T}z^{-1}}$$

a) $T = 1$ sec

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} - \frac{1}{1-e^{-3}z^{-1}}$$

$$= \frac{1}{1-0.3678z^{-1}} - \frac{1}{1-0.0497z^{-1}}$$

$$= \frac{0.3181z^{-1}}{1-0.4175z^{-1}+0.0182z^{-2}}$$

b) $T = 0.5$ sec

$$H(z) = \frac{1}{1-e^{-0.5}z^{-1}} - \frac{1}{1-e^{-3\times0.5}z^{-1}}$$

$$= \frac{1}{1-0.606z^{-1}} - \frac{1}{1-0.223z^{-1}}$$

$$= \frac{0.383z^{-1}}{1-0.829z^{-1}+0.135z^{-2}}$$

* Convert the analog filter with T.F

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$$ into a digital filter using IIT

sol: $a = 0.1$ & $b = 3$

$$H(z) = \frac{1 - e^{-0.1T} z^{-1} \cos 3T}{1 - 2e^{-0.1T} z^{-1} \cos 3T + e^{-2(0.1)T} z^{-2}}$$

$$= \frac{1 + 0.8956\, z^{-1}}{1 + 1.7913\, z^{-1} + 0.8187\, z^{-2}}$$

* Convert the analog filter with system functi

into digital filter using IIT

$$H_a(s) = \frac{2}{(s+0.4)^2+4}$$

sol: $$H_a(s) = \frac{2}{(s+0.4)^2+2^2}$$

$a = 0.4$ & $b = 2$

$$H(z) = \frac{e^{-0.4T} z^{-1} \sin 2T}{1 - 2e^{-0.4T} z^{-1} \cos 2T + e^{-0.8T} z^{-2}}$$

$$= \frac{0.909\, z^{-1}}{1 + 0.5578\, z^{-1} + 0.449\, z^{-2}}$$

* The s/m fn of anahalog filter is $H_a(s) = \frac{2}{s(\ldots}$

find $H(z)$ using IIT for a sampling freq o

4 samples per sec

sol $T = \frac{1}{f_s} = \frac{1}{4} = 0.25\,\text{sec}$

$\because H_a(s) = \frac{2}{s(s+2)} = \frac{A}{s} + \ldots$

$= \frac{1}{s} - \frac{1}{s+2}$

$H(z) = \frac{1}{\ldots} - \frac{1}{\ldots} = \ldots$

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* Determine $H(z)$ using IIT for the analog s/m function $H_a(s) = \frac{1}{(s+1)(s^2+s+2)}$

Sol :

$$H_a(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+2}$$

$$= \frac{0.5}{s+1} - \frac{0.5s}{s^2+s+2}$$

$$= \frac{0.5}{s+1} - 0.5 \times \frac{s}{(s+0.5)^2+(1.3228)^2}$$

$$= \frac{0.5}{s+1} - 0.5\left[\frac{s+0.5}{(s+0.5)^2+(1.3228)^2} - \frac{0.5}{(s+0.5)^2+(1.32\ldots$$

$$= \frac{0.5}{s+1} - \frac{0.5(s+0.5)}{(s+0.5)^2+(1.3228)^2} + \frac{0.25 \times 1.3228}{(1.3228)((s+0.5)^2+(\ldots$$

$$H(z) = \frac{0.5}{1-e^{-T}z^{-1}} - 0.5\left[\frac{1-e^{-0.5T}z^{-1}\cos 1.3228T}{1-2e^{-0.5T}z^{-1}\cos 1.3228T}\right] + \frac{0.188\ldots \times e^{-0.5T}z^{-1}\ldots}{1-2e^{-0.5T}z^{-1}\ldots}$$

$$= \frac{0.5}{1-0.3678z^{-1}} - \frac{0.5(1-0.1487z^{-1})}{(1-0.2977z^{-1}+0.3678z^{-2})} + \frac{0.188\ldots \times 0.5\ldots}{(1-0.2977z^{-1}+0.\ldots}$$

* Bilinear Transformation

IIR filter design using approximation of derivatives & Impulse Invariant Transformation methods are appropriate only for the design of LPF & BPF whose resonant freq's are small. These techniques are not suitable for HPF & BSF.

These limitations can be overcome by using Bilinear transformation. This transformation is one-to-one mapping from s-domain to the z-domain

Let the system function of the analog filter be

$$H_a(s) = \frac{b}{s+a} = \frac{Y(s)}{X(s)} \quad \text{--- (1)}$$

$$s\,Y(s) + a\,Y(s) = b\,X(s)$$

apply ILT

$$\frac{d}{dt} y(t) + a y(t) = b x(t)$$

$$\int_{nT-T}^{nT} \frac{d}{dt} y(t)\, dt + a \int_{nT-T}^{nT} y(t)\, dt = b \int_{nT-T}^{nT} x(t)\, dt$$

Approximate the integral by the trapezoidal formulae

Trapezoid rule for numeric integration in

i.e. $\displaystyle\int_{nT-T}^{nT} a(t)\,dt = \frac{T}{2}\left[a(nT) + a(nT-T)\right]$

$$\therefore \quad y(nT) - y(nT-T) + \frac{aT}{2}\left[y(nT) + y(nT-T)\right]$$
$$= \frac{bT}{2}\left[x(nT) + x(nT-T)\right]$$

$$Y(z)\left[1 - z^{-1}\right] + \frac{aT}{2}\left[1 + z^{-1}\right]Y(z) = \frac{bT}{2}\left[1 + z^{-1}\right]X(z)$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{bT}{2}(1 + z^{-1})}{(1 - z^{-1}) + \frac{aT}{2}(1 + z^{-1})}$$

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + a} \quad \text{—} (2)$$

Comparing eq^n (1) & (2)

$$\boxed{S = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)}$$

or

$$S = \frac{2}{T}\left(\frac{z - 1}{z + 1}\right)$$

$z = re^{j\omega}$ & $S = \sigma + j\Omega$

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$$S = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$$

$$= \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)\left(\frac{re^{-j\omega}+1}{re^{-j\omega}+1}\right)$$

$$= \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega} + j\,\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right)$$

$$\because \quad S = \sigma + j\Omega$$

$$\boxed{\sigma = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega}\right)}$$

If $r < 1$ then $\sigma < 0$

i.e. Left half of the s-plane is mapped inside the unit circle in the z-plane

If $r > 1$ then $\sigma > 0$

i.e. Right half of the s-plane is mapped outside the unit circle in the z-plane

If $r = 1$ then $\sigma = 0$

i.e. the imaginary axis in the s-pla[ne] is mapped onto the unit circle in the z-[plane]

Thus a stable analog filter gets converted into stable digital filter when $r=1$

$$\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right)\Big/_{r=1}$$

$$= \frac{2}{T}\left(\frac{2\sin\omega}{1+1+2\cos\omega}\right)$$

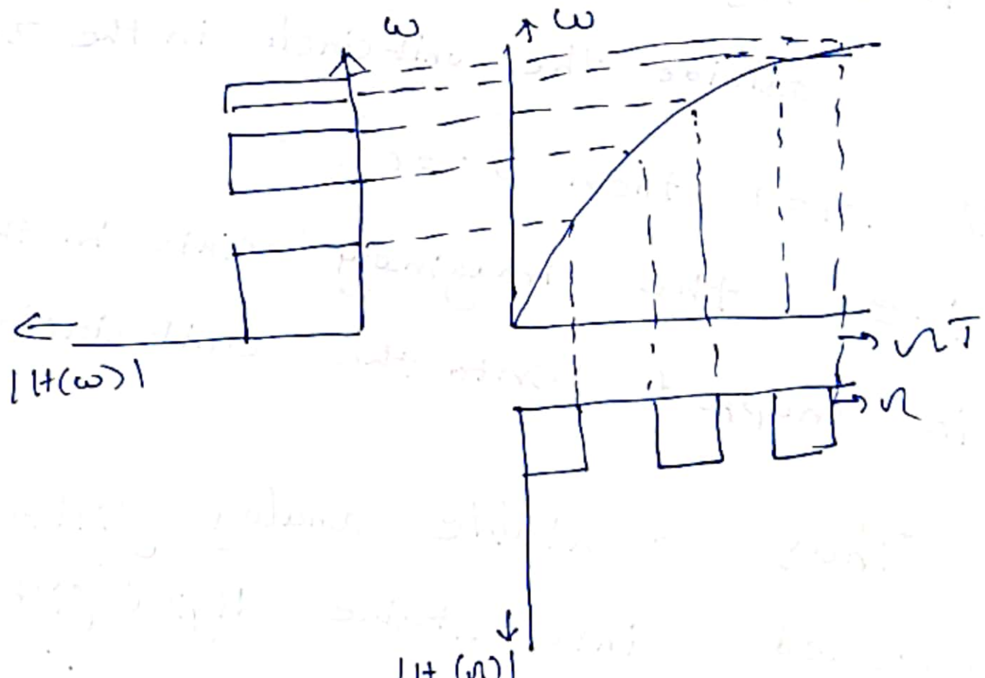
$$= \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$$

$$= \frac{2}{T}\cdot\frac{2\sin\omega/2\,\cos\omega/2}{2\cos^2\omega/2}$$

$$\boxed{\Omega = \frac{2}{T}\tan\omega/2}$$

(or)

$$\boxed{\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)}$$



The mapping is nonlinear and the lower freq's in the analog domain are expanded in digital domain where as the higher freq's are compressed. This is due to the nonlinearity of arc tangent function. "Freq wrapping effect"

* Convert the following analog filter with Transfer function $H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$ into a digital IIR filter by using bilinear transformation. The digital IIR filter is having a resonant freq $\omega_r = \pi/2$

Sol: $\Omega_c = 3$

$$\because \Omega_c = \frac{2}{T} \tan \frac{\omega_r}{2}$$

$$\Rightarrow T = \frac{2}{\Omega_c} \tan \frac{\omega_r}{2}$$

$$= \frac{2}{3} \tan \frac{\pi/2}{2} = 0.666 \text{ sec}$$

$$\therefore H(z) = H_a(s) \Big/ s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{s+0.1}{(s+0.1)^2+9} \Big/ s = \frac{2}{\frac{2}{3}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

(or)

$$s = 3\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{3.1 + 0.2z^{-1} - 2.9z^{-2}}{18.61 + 0.02z^{-1} + 17.41z^{-2}}$$

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* Convert the analog filter with system function $H_a(s) = \frac{s+0.5}{(s+0.5)^2+16}$ into a digital IIR filter should have a resonant freq at $\omega_r = \pi/2$

Sol: $\Omega_c = 4$

$$\Omega_c = \frac{2}{T} \tan\frac{\omega_r}{2}$$

$$T = \frac{2}{\Omega_c} \tan\frac{\omega_r}{2}$$

$$= \frac{2}{4} \tan\frac{\pi/2}{2} = 0.5 \text{ sec}$$

$$H(z) = H_a(s) \Big/_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{s+0.5}{(s+0.5)^2+16} \Big/_{s = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{4.5 + z^{-1} - 3.5 z^{-2}}{36.25 + 0.5 z^{-1} + 28.25 z^{-2}}$$

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* Apply the bilinear transformation to

$$H_a(s) = \frac{4}{(s+3)(s+4)}$$ with $T = 0.5$ sec

and find $H(z)$

Sol: $H(z) = H_a(s)\Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$

$$= \frac{4}{(s+3)(s+4)}\Bigg/_{s = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{1}{2}\frac{(1+z^{-1})^2}{(7-z^{-1})}$$ //

* Obtain $H(z)$ from $H_a(s)$ when $T = 1$ sec

and $H_a(s) = \frac{3s}{s^2+0.5s+2}$ using

bilinear transformation

Given $T = 1$ sec

$$H(z) = H_a(s)\Big/_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{3s}{s^2+0.5s+2}\Bigg/_{s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{6+6z^{-1}}{7-4z^{-1}+5z^{-2}}$$

* Using bilinear transformation obtain H(z) from $H_a(s)$ when T = 1 sec &

$$H_a(s) = \frac{s^3}{(s+1)(s^2+2s+2)}$$

Sol : $H(z) = H_a(s) \Big/ s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

$$= \frac{8(1-z^{-1})^3}{(3-z^{-1})(10-4z^{-1}+2z^{-2})}$$

* A digital filter with 3 dB bandwidth of 0.4π is to be designed from the analog filter whose s/m response is

$$H(s) = \frac{\Omega_c}{s+\Omega_c}$$

use the bilinear transformation and obtain H(z)

Sol : $\because \Omega_c = \frac{2}{T}\tan\frac{\omega_n}{2}$

$$= \frac{2}{T}\tan\frac{0.4\pi}{2} = \frac{1.453}{T}$$

$$H(z) = H_a(s) \Big/ s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

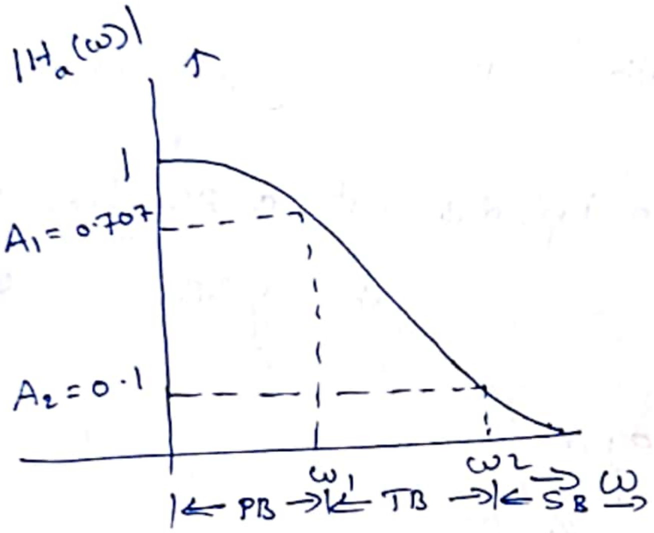
$$= \frac{\Omega_c}{s+\Omega_c}$$

$$= \frac{1.453/T}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+\frac{1.453}{T}} = \frac{1+z^{-1}}{3.376}$$

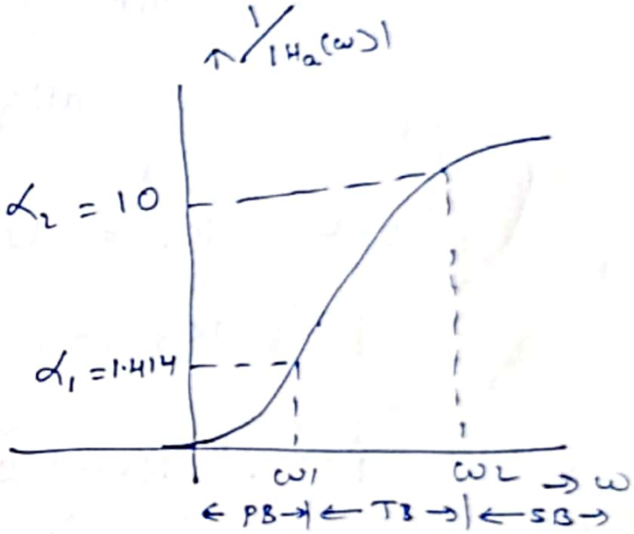
* Analog filter design

Specifications of the LOW-PASS FILTER (LPF)

The magnitude response of LPF in terms of gain & attenuation are shown in below



Gain vs ω



Attenuation vs ω

Let ω_1 = passband freq in rad/sec

ω_2 = stopband "

Let the gain at the passband freq ω_1 be A_1

and gain " stopband " ω_2 be A_2

i.e. $A_1 = |H(\omega)|_{\omega=\omega_1}$

$A_2 = |H(\omega)|_{\omega=\omega_2}$

The maximum value of normalised gain is Unity. So A_1 & A_2 are less than 1 & α_1 and α_2 are > 1

From the above figure

A_1 is assumed as $1/\sqrt{2}$ & A_2 is assumed as 0.1

Hence $\alpha_1 = \sqrt{2} = 1.414$ & $\alpha_2 = 1/0.1 = 10$

Another popular unit that is used for filter specification is dB.

When the gain expressed in dB, it will be -ve dB

" attenuation " , " +ve dB

Let K_1 = Gain in dB at a PB freq ω_1

K_2 = " at a SB " ω_2

i.e. $K_1 = 20\log_{10} A_1$

$\Rightarrow A_1 = 10^{\frac{K_1}{20}}$

$\|^{ly}$ $K_2 = 20\log_{10} A_2$

$\Rightarrow A_2 = 10^{\frac{K_2}{20}}$

When expressed in dB the gain & attenuation will have only change in sign

because $\log \alpha = \log\left(\frac{1}{A}\right) = -\log A$

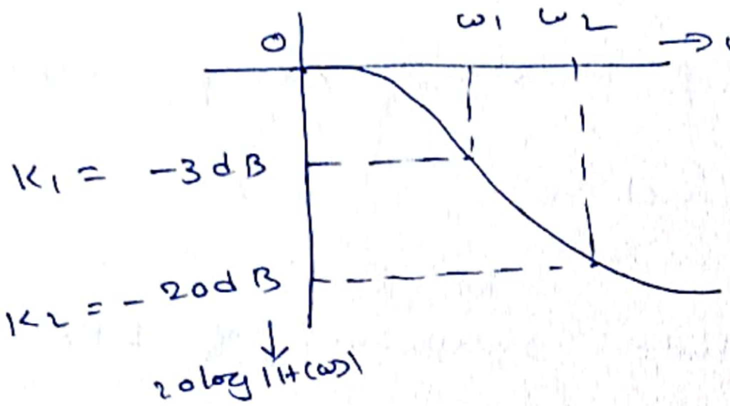
$\therefore$ when dB is +ve $\rightarrow$ Attenuation

" " -ve $\rightarrow$ Gain

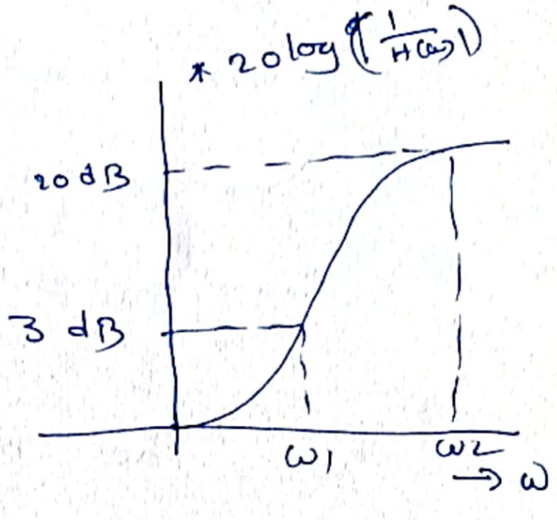
when $A_1 = 0.707 \Rightarrow K_1 = 20\log_{10} 0.707 = -3\,dB$

$A_2 = 0.1 \Rightarrow K_2 = 20\log_{10} 0.1 = -20\,dB$

The magnitude response of LPF in terms of dB shown in below



dB-Gain vs ω



dB - Attenuation vs ω

Sometimes the specifications are given in terms of passband ripple δ_p and stopband ripple δ_s

In this case, the dB-gain & attenuation can be estimated as follows

$$K_1 = 20\log_{10}(1 - \delta_p)$$

$$K_2 = 20\log_{10}\delta_s$$

$$\alpha_1 = -20\log(1-\delta_p)$$

$$\alpha_2 = -20\log\delta_s$$

If the ripples are specified in dB, then the minimum passband ripple is equal to K_1 & the -ve of maximum passband attenuation is equal to K_1

* Design of LP Digital Butterworth Filter

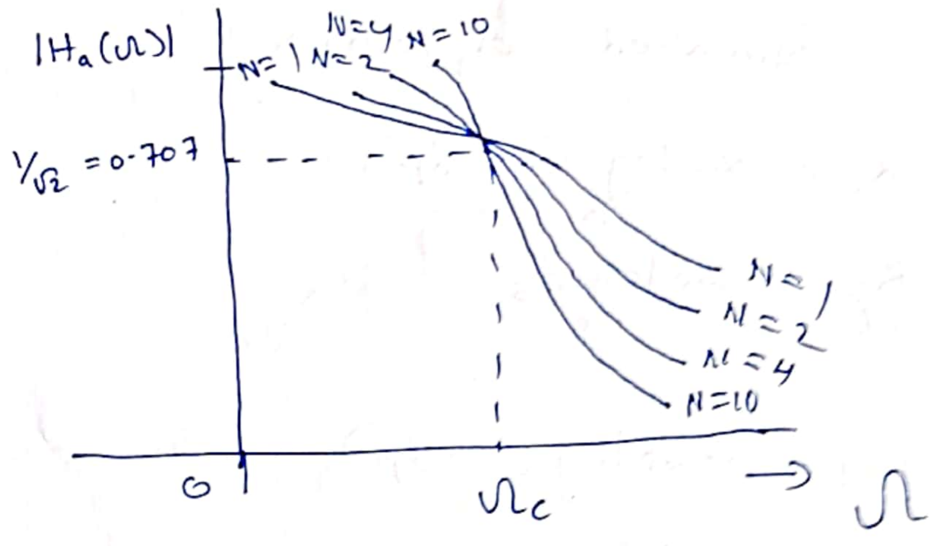
The analog Butterworth filter is designed by approximating the ideal frequency response using an error function.

The magnitude response of LPF obtained by this approximation is given by

$$\boxed{|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}}$$

where Ω_c = 3 dB cutoff frequency

and N = Order of the filter



Magnitude response of Butterworth LPF for various values of N

The magnitude response approaches the ideal response as the value of N increases.

The phase response of the Butterworth filter becomes more nonlinear with increasing N.

∵ The freq response of the filter depends on its order N. The order N has to be estimated to satisfy the given specification.

Usually the specifications of the filter are given in terms of gain A or attenuation α at a PB & SB freq as given below

$$A_1 \le |H(\omega)| \le 1 \quad , \quad 0 \le \omega \le \omega_1$$

$$|H(\omega)| \le A_2 \quad , \quad \omega_2 \le \omega \le \pi$$

Let Ω_1 & Ω_2 be the analog edge freq's corresponding to digital freqs ω_1 & ω_2

The values Ω_1 & Ω_2 are obtained using the bilinear transformation or impulse invariant transformation

$$A_1^2 \le \frac{1}{1+\left(\frac{\Omega_1}{\Omega_c}\right)^{2N}} \le 1 \quad \text{—} (1)$$

&

$$\frac{1}{1+\left(\frac{\Omega_2}{\Omega_c}\right)^{2N}} \le A_2^2 \quad \text{—} (2)$$

→ The above two eq^{ns} can be written as

$$① \Rightarrow \quad \left(\frac{\Omega_1}{\Omega_c}\right)^{2N} \leq \frac{1}{A_1^2} - 1 \quad —— ③$$

$$② \Rightarrow \quad \& \quad \left(\frac{\Omega_2}{\Omega_c}\right)^{2N} \geq \frac{1}{A_2^2} - 1 \quad —— ④$$

Assuming equality

$$\frac{③}{④} \Rightarrow \left(\frac{\Omega_1}{\Omega_2}\right)^{2N} = \frac{\frac{1}{A_1^2} - 1}{\frac{1}{A_2^2} - 1}$$

$$\Rightarrow \boxed{N = \frac{1}{2} \frac{\log\left\{\left(\frac{1}{A_2^2} - 1\right) / \left(\frac{1}{A_1^2} - 1\right)\right\}}{\log\left(\frac{\Omega_2}{\Omega_1}\right)}} \quad —— ⑤$$

If N is not an integer, the value of N is chosen to be the next nearest integer

& we can get

$$\boxed{\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}} \quad —— ⑥$$

when parameters A_1 & A_2 are given in dB.

$$A_1 dB = -20 \log_{10} A_1 \Rightarrow \log_{10} A_1 = -\frac{A_1 dB}{20}$$

$$\Rightarrow A_1 = 10^{-\frac{A_1 dB}{20}}$$

$$\boxed{\cancel{N = \frac{1}{2} \log_{10} \left[\frac{10^{0.1 A_2 dB} - 1}{10^{0.1 A_1 dB} - 1}\right]}}$$

$$\therefore \boxed{N = \frac{1}{2} \frac{\log \left[\frac{10^{0.1 A_2 dB} - 1}{10^{0.1 A_1 dB} - 1} \right]}{\log \left(\frac{\Omega_2}{\Omega_1} \right)}}$$

and

$$\boxed{\Omega_c = \frac{\Omega_1}{\left(10^{0.1 A_1 dB} - 1\right)^{1/2N}}}$$

(or)

$$\boxed{\Omega_c = \frac{\Omega_2}{\left(10^{0.1 A_2 dB} - 1\right)^{1/2N}}}$$

Infact

$$\Omega_c = \frac{1}{2} \left[\frac{\Omega_1}{\left(10^{0.1 A_1 dB} - 1\right)^{1/2N}} + \frac{\Omega_2}{\left(10^{0.1 A_2 dB} - 1\right)^{1/2N}} \right]$$

The unnormalized transfer function of the Butterworth filter is usually written in factored form as

$$H_a(s) = \prod_{K=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_K \Omega_c s + \Omega_c^2}$$ when N is even

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{K=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_K \Omega_c s + \Omega_c^2}$$ when N is odd

Where $b_K = 2 \sin \left[\frac{(2K-1)\pi}{2N} \right]$

If S/Ω_c is replaced by S_n then the normalised Butterworth filter transfer function is given by

$$H_a(s) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{S_n^2 + b_k S_n + 1} \quad \text{when N is even}$$

$$H_a(s) = \frac{1}{S_n + 1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{S_n^2 + b_k S_n + 1} \quad \text{when N is odd}$$

where $b_k = 2 \sin \left[\frac{(2k-1)\pi}{2N} \right]$

poles of the normalised Butterworth filter

The Butterworth LPF has a magnitude squared response given by

$$|H_a(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

w.k.t The freq response $H_a(\Omega)$ of an analog filter is obtained by substituting $s = j\Omega$ in the analog T.F $H_a(s)$.

$$H_a(s)\, H_a(-s) = \frac{1}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}} = \frac{1}{1 + \left(\frac{s^2}{j^2\Omega_c^2}\right)^{N}}$$

replace s/Ω_c by S_n i.e.e ($\Omega_c = 1$ rad/sec)

∴ Normalised T.F

$$H_a(s_n)\, H_a(-s_n) = \frac{1}{1+(-s_n^2)^N}$$

∴ The T.F of the above eq^n will have 2N poles which are given by the roots of denominator polynomial.

The poles of the T.F symmetrically lie on a unit circle in s-plane with angular spacing of π/N.

i.e. $1+(-s_n^2)^N = 0$

For N is odd $\Rightarrow$ $s^{2N} = 1 = e^{j2\pi k}$

$$S_k = e^{\frac{j\pi k}{N}} \quad ; k = 1, 2, \text{---} 2N$$

For N is even $\Rightarrow$ $s^{2N} = -1 = e^{j(2k-1)\pi}$

$$S_k = e^{j(2k-1)\pi/2N} \quad \text{for } k = 1, 2, \text{--} 2N$$

Angle between any two poles $\theta = 360^\circ/2N$

If the order of the filter N is even then location of the first pole is at $\theta/2$

The location of the subsequent poles are respectively at (counter clockwise direction)

$$\left(\frac{\theta}{2}+\theta\right), \left(\frac{\theta}{2}+2\theta\right), \text{---}, \left(360-\frac{\theta}{2}\right)$$

If N is odd $\rightarrow$ first pole is on X-axis, Subsequent poles $0, 2\theta, \text{--} 360-\theta$

Properties of Butterworth filters

* The Butterworth filters are all pole designs
* The filter order N Completely specifies the filter
* The magnitude response approaches the ideal response as the value of N increases.
* The magnitude is maximally flat at the origin
* The magnitude is monotonically decreasing function of Ω
* At the cutoff freq Ω_c, the magnitude of normalized Butterworth filter is $1/\sqrt{2}$. Hence the dB magnitude at the cutoff freq will be 3 dB less than the maximum value

* Design procedure for LP Digital Butterworth IIR filter

Let A_1 = Gain at a PB freq ω_1

A_2 = " SB " ω_2

Ω_1 = Analog freq corresponding to ω_1

Ω_2 = " " " ω_2

Step 1 : Choose the type of transformation

i.e. either bilinear (or) IIT

Step 2 : Calculate the ratio of analog edge freqs Ω_2/Ω_1

→ For bilinear transformation

$$\Omega_1 = \frac{2}{T} \tan\frac{\omega_1}{2}$$

$$\Omega_2 = \frac{2}{T} \tan\frac{\omega_2}{2}$$

$$\therefore \frac{\Omega_2}{\Omega_1} = \frac{\tan \omega_2/2}{\tan \omega_1/2}$$

→ For IIT

$$\Omega_1 = \frac{\omega_1}{T}$$

$$\Omega_2 = \frac{\omega_2}{T}$$

$$\therefore \frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1}$$

Step 3 : Decide the order of the filter

$$N \geq \frac{1}{2} \frac{\log\left\{\left(\frac{1}{A_2^2}-1\right)/\left(\frac{1}{A_1^2}-1\right)\right\}}{\log \frac{\Omega_2}{\Omega_1}}$$

Choose N such that it is an integer just greater than

Step 4   Calculate analog Cutoff freq

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$$

For BT : $\Omega_c = \dfrac{\frac{2}{T} \tan \omega_1/2}{\left(\frac{1}{A_1^2} - 1\right)^{1/2N}}$

For IIT : $\Omega_c = \dfrac{\omega_1/T}{\left(\frac{1}{A_1^2} - 1\right)^{1/2N}}$

Step 5 : Determine the T.F of the analog filter

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$ When N ine
For unity dc g filt

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$ When N i

where $b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$

For normalised case $\Omega_c = 1$ rad/sec

Step 6 : Using the chosen transformation transform the analog filter T.F to digital filter T.F H(Z)

Step 7 : Realize the digital filter T.F H(Z) by a suitable structure

* Design a Butterworth digital filter using BT. The specifications of the desired LPF are

$$0.9 \le |H(\omega)| \le 1 \quad ; \quad 0 \le \omega \le \frac{\pi}{2}$$

$$|H(\omega)| \le 0.2 \quad ; \quad \frac{3\pi}{4} \le \omega \le \pi$$

with $T = 1$ sec.

Sol):- From Given specifications

$$A_1 = 0.9 \quad \& \quad \omega_1 = \frac{\pi}{2}$$

$$A_2 = 0.2 \quad \& \quad \omega_2 = \frac{3\pi}{4} \quad \& \quad T = 1\text{sec}$$

1. Using BT $\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 2\tan\left[\frac{3\pi/4}{2}\right] = 4.828$

$$\Omega_1 = \frac{2}{T}\tan\omega_1/2 = 2\tan\left[\frac{\pi/2}{2}\right] = 2$$

2. $$\frac{\Omega_2}{\Omega_1} = \frac{4.828}{2} = 2.414$$

3. $$N \ge \frac{1}{2}\frac{\log_{10}\left\{\left(\frac{1}{A_2^2}-1\right)/\left(\frac{1}{A_1^2}-1\right)\right\}}{\log_{10}\left(\Omega_2/\Omega_1\right)}$$

$$\ge \frac{1}{2}\frac{\log_{10}\left\{\left(\frac{1}{(0.2)^2}-1\right)/\left(\frac{1}{(0.9)^2}-1\right)\right\}}{\log_{10} 2.414}$$

$$\ge 2.626$$

Choose $N = 3$

4. $$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}} = \frac{2}{\left[\frac{1}{(0.9)^2}-1\right]^{1/2\times3}} = 2.5467$$

For N odd, T.F

5. $$H_a(s) = \frac{\Omega_c}{s+\Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

where $b_k = 2 \sin\left(\frac{(2k-1)\pi}{2N}\right)$

$$\therefore H_a(s) = \frac{\Omega_c}{s+\Omega_c} \cdot \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2}$$

when $b_1 = 2 \sin\left(\frac{(2\times 1-1)\pi}{2\times 3}\right) = 1$

$$H_a(s) = \left(\frac{2.5467}{s+2.5467}\right)\left(\frac{(2.5467)^2}{s^2 + 2.5467\, s + (2.5467)^2}\right)$$

6. Convert $H_a(s)$ to $H(z)$ using BT

$$H(z) = H_a(s) \Big/_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{0.2332\,(1+z^{-1})^3}{1+0.439\,z^{-1} + 0.3845\,z^{-2} + 0.0416\,z^{-3}}$$

* Design a digital Butterworth filter satisfying the following constraints

$$0.8 \le |H(\omega)| \le 1 \quad ; \quad 0 \le \omega \le 0.2\pi$$

$$|H(\omega)| \le 0.2 \quad ; \quad 0.32\pi \le \omega \le \pi$$

with $T = 1s$. Apply IIT

Sol): Given specifications

$A_1 = 0.8$, $\omega_1 = 0.2\pi$

$A_2 = 0.2$; $\omega_2 = 0.32\pi$ & $T = 1s$

* Here IIT method

$$\Omega_2 = \frac{\omega_2}{T} = \frac{0.32\pi}{1} = 0.32\pi$$

$$\Omega_1 = \frac{\omega_1}{T} = \frac{0.2\pi}{1} = 0.2\pi$$

$$\frac{\Omega_2}{\Omega_1} = \frac{0.32\pi}{0.2\pi} = 1.6$$

$$N \ge \frac{1}{2} \frac{\log_{10}\left\{\left(\frac{1}{A_2^2}-1\right) / \left(\frac{1}{A_1^2}-1\right)\right\}}{\log_{10}\left(\frac{\Omega_2}{\Omega_1}\right)}$$

$$\ge 3.9931$$

$\therefore$ $N = 4$

$$\Omega_c = \frac{\Omega_1}{\left(\frac{1}{A_1^2}-1\right)^{1/2N}} = \frac{0.2\pi}{\left(\frac{1}{0.8^2}-1\right)^{1/2\times4}} = 0.675 \text{ rad/s}$$

Scanned with CamScanner

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

where $b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$

Here $N = 4 \quad \therefore \quad k = 1, 2$

for $k = 1 \Rightarrow b_1 = 0.765$

$k = 2 \Rightarrow b_2 = 1.848$

$$H_a(s) = \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2} \times \frac{\Omega_c^2}{s^2 + b_2 \Omega_c s + \Omega_c^2}$$

$$= \frac{(0.675)^2}{s^2 + (0.765 \times 0.675)s + (0.675)^2} \times \frac{(0.675)^2}{s^2 + (1.848 \times 0.675)s + (0.675)^2}$$

$$= \frac{0.2076}{(s^2 + 0.516s + 0.456)(s^2 + 1.247s + 0.456)}$$

$$H_a(s) = \frac{As + B}{s^2 + 0.516s + 0.456} + \frac{Cs + D}{s^2 + 1.247s + 0.456}$$

$$\therefore = \frac{-0.622s - 0.321}{\underbrace{s^2 + 0.516s + 0.456}} + \frac{0.622s + 0.776}{s^2 + 1.247s + 0.456}$$

Find roots then use IIT method for H(z)

* Design a LP Butterworth digital filter to give response of 3dB or less for freqs upto 2KHz and an attenuation of 20dB or more beyond 4KHz. Use the BT method and obtain H(z) of the desired filter

Sol: Attenuation at PB freq $\omega_1 = 3dB$

$\therefore$ Gain at PB edge freq ω_1 is $K_1 = -3dB$

$$A_1 = 10^{K_1/20} = 10^{-3/20} = 0.707$$

Attenuation at SB freq $\omega_2 = 20dB$

$\therefore$ Gain at SB edge freq ω_2 is $K_2 = -20dB$

$$A_2 = 10^{K_2/20} = 10^{-20/20} = 0.1$$

PB edge freq $f_1 = 2KHz$

SB edge freq $f_2 = 4KHz$

Let the sampling freq be 10000Hz

Normalised $\omega_1 = 2\pi f_1/f_s = 0.4\pi$

Normalised $\omega_2 = 2\pi f_2/f_s = 0.8\pi$

Analog edge freqs using BT

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = 14530.8 \text{ rad/sec}$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 61553.6 \text{ rad/sec}$$

$$\frac{\Omega_2}{\Omega_1} = 4.236$$

$$N \geq \frac{1}{2} \frac{\log_{10}\left\{\left(\frac{1}{A_2^2}-1\right) / \left(\frac{1}{A_1^2}-1\right)\right\}}{\log_{10}\left(\Omega_2/\Omega_1\right)} \geq 1.59$$

$\therefore N = 2$

$$\Omega_c = \frac{\Omega_1}{\left(\frac{1}{A_1^2}-1\right)^{1/2N}} = 1.4530$$

Unnomalised $\Omega_c = f_s \times 1.4530 = 14530 \text{ rad/s}$

T.F $$H_a(s) = \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2}$$

$b_1 = 1.414$

$$H_a(s) = \frac{(14530)^2}{s^2 + 1.444 \times 14530 s + (14530)^2}$$

$=$

BT

$$H(z) = H_a(s) \Big/ s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= H_a(s) \Big/ s = 20000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{0.528}{2.5552 - 0.946 z^{-1} + 0.5008 z^{-2}}$$

* Design a LP Butterworth filter using BT method for satisfying the following constraints

PB : 0 - 400 Hz SB : 2.1 - 4 KHz

PB ripple : 2 dB SB ~~ripple~~ attenuation : 20 dB

Sampling freq : 10 KHz

Sol: Given $\alpha_1 = 2$ dB

$K_1 = -2$ dB

$\therefore A_1 = 10^{K_1/20} = 0.794$

$\alpha_2 = 20$ dB

$K_2 = -20$ dB

$\therefore A_2 = 10^{K_2/20} = 0.1$

PB edge freq $f_1 = 400$ Hz & $f_s = 10$ KHz

SB " $f_2 = 2.1$ KHz

Normalizing $\omega_1 = 2\pi f_1 / f_s = 0.25$ rad & $\omega_2 = 2\pi f_2 / f_s = 1.319$ rad

Analog filter edge freq's $\Omega_1 = \frac{2}{T} \tan \frac{\omega_1}{2} = 2 \times 10000 \times \tan \frac{0.25}{2} = 2513.102$ rad/sec

$\Omega_2 = \frac{2}{T} \tan \frac{\omega_2}{2} = 15,506.08$ rad/sec

$\frac{\Omega_2}{\Omega_1} = 6.1703$

$$N \geq \frac{1}{2} \frac{\log\left\{\left(\frac{1}{A_2^2}-1\right)\Big/\left(\frac{1}{A_1^2}-1\right)\right\}}{\log\left(\Omega_2/\Omega_1\right)} \geq 1.409 \simeq 2$$

(or) ~~$N \geq \frac{1}{2}\log$~~

$$\Omega_c = \frac{\Omega_2}{\left(\frac{1}{A_2^2}-1\right)^{1/2N}}$$

$= 4915.788$ rad/sec

TF

$$H_a(s) = \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2}$$

where $b_1 = 2\sin\left(\frac{(2\times1-1)\pi}{2\times2}\right) = 1.414$

$$H_a(s) = \frac{(4915.788)^2}{s^2 + 1.414 \times 4915.788 s + (4915.788)^2}$$

$$\therefore H(z) = H_a(s)\Big/_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{0.042 + 0.085 z^{-1} + 0.042 z^{-2}}{1 - 1.335 z^{-1} + 0.506 z^{-2}}$$

The poles are

$$s_k = \pm \Omega_c \left(e^{j\frac{(2k+N+1)\pi}{2N}} \right); k = 0, 1 \ldots N$$

$P_1 = -3475.6 + j\,3475.6$

$P_2 = -3475.6 - j\,3475.6$

* A digital LPF is required to meet the following specifications

PB attenuation $\leq 1dB$ PB edge $= 4KHz$

SB " $\geq 40dB$ SB " $= 8KHz$

Sampling rate $= 24KHz$

Design a Butterworth filter using B.T

Sol: Given $\alpha_1 = 1\,dB \Rightarrow K_1 = -1\,dB \Rightarrow A_1 = 10^{K_1/20} = 0.891$

$\alpha_2 = 40\,dB \Rightarrow K_2 = -40\,dB \Rightarrow A_2 = 10^{K_2/20} = 0.01$

$\because f_s = 24KHz$, $f_1 = 4KHz$ & $f_2 = 8KHz$

Normalized angular freq^n

$$\omega_1 = 2\pi f_1/f_s = 2\pi \times \frac{4000}{24000} = 1.047\, rad/s$$

$$\omega_2 = 2\pi f_2/f_s = 2\pi \times \frac{8000}{24000} = 2.094\, rad/s$$

$$\Omega_1 = \frac{2}{T} \tan\frac{\omega_1}{2} = 27706.49\, rad/s$$

$$\Omega_2 = \frac{2}{T} \tan\frac{\omega_2}{2} = 83100.52 \;\; ''$$

$$\therefore \frac{\Omega_2}{\Omega_1} = \frac{83000.52}{27706.49} = 2.9957$$

order of the filter

$$N \geq \frac{1}{2} \frac{\log_{10}\left\{\left(\frac{1}{A_2^2}-1\right) / \left(\frac{1}{A_1^2}-1\right)\right\}}{\log_{10}\left(\Omega_2/\Omega_1\right)}$$

$$\geq 4.8 = 5$$

Cutoff freq

$$\Omega_c = \frac{\Omega_1}{\left(\frac{1}{A_1^2} - 1\right)^{\frac{1}{2N}}} = 31,715 \text{ rad/sec}$$

Analog filter T.F

$$H_a(s) = \left(\frac{\Omega_c}{s+\Omega_c}\right)\left(\frac{\Omega_c^2}{s^2+b_1\Omega_c s+\Omega_c^2}\right)\left(\frac{\Omega_c^2}{s^2+b_2\Omega_c s+\Omega_c^2}\right)$$

$$b_1 = 2\sin\left(\frac{(2\times1-1)\pi}{2N}\right) = 0.618$$

$$b_2 = 2\sin\left(\frac{(2\times2-1)\pi}{2N}\right) = 1.618$$

$$\therefore H_a(s) = \left(\frac{31715}{s+31715}\right)\left(\frac{(31715)^2}{s^2+0.618\times31715s+(31715)^2}\right)$$

$$\cdot\left(\frac{(31715)^2}{s^2+1.618\times31715s+(31715)^2}\right)$$

$$H(z) = H_a(s)\Big/_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= H_a(s)\Big/_{s=2\times24k\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

* Design a digital IIR LPF with passband edge at 1000 Hz and stop band edge at 1500 Hz for a sampling frequency of 5000 Hz. The filter is to have a passband ripple of 0.5 dB & a SB ripple below 30 dB. Design Butterworth filter using the BT.

Sol: Given

$$f_s = 5000 \text{ Hz}$$

$$f_1 = 1000 \text{ Hz} \Rightarrow \omega_1 = \frac{2\pi f_1}{f_s} = 0.4\pi$$

$$f_2 = 1500 \text{ Hz} \Rightarrow \omega_2 = \frac{2\pi f_2}{f_s} = 0.6\pi$$

$$\alpha_1 = 0.5 \text{ dB} \Rightarrow K_1 = -0.5 \text{dB} \Rightarrow A_1 = 10^{K_1/20} = 0.94$$

$$\alpha_2 = 30 \text{ dB} \Rightarrow K_2 = -30 \text{dB} \Rightarrow A_2 = 10^{K_2/20} = 0.031$$

$$\Omega_1 = \frac{2}{T} \tan \frac{\omega_1}{2} = 7265.425 \text{ rad/sec}$$

$$\Omega_2 = \frac{2}{T} \tan \frac{\omega_2}{2} = 13763.819 \text{ rad/sec}$$

$$\frac{\Omega_2}{\Omega_1} = 1.8944$$

order of the filt

$$\therefore N \geq \frac{1}{2} \frac{\log_{10} \left\{ \left(\frac{1}{A_2^2} - 1\right) \Big/ \left(\frac{1}{A_1^2} - 1\right) \right\}}{\log_{10} \left(\Omega_2/\Omega_1\right)}$$

$$\geq 7.35$$

$$\simeq 8$$

Cutoff freq

$$\Omega_c = \frac{\Omega_1}{\left(\frac{1}{A_1^2} - 1\right)^{\frac{1}{2N}}} = 8292 \text{ rad/s}$$

S/m function

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

where $b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$

$b_1 = 0.390$

$b_2 = 1.111$

$b_3 = 1.662$

$b_4 = 1.961$

$$H_a(s) = \left(\frac{(8292)^2}{s^2 + 0.390 \times 8292 s + (8292)^2}\right)\left[\frac{(8292)^2}{s^2 + 1.111 \, 8292 s + (8292)^2}\right]$$

$$\left(\frac{(8292)^2}{s^2 + 1.662 \times 8292 s + (8292)^2}\right)\left[\frac{(8292)^2}{s^2 + 1.961 \times 8292 s + (8292)^2}\right]$$

$$H(z) = H_a(s) \Big/ s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$= H_a(s) \Big/ = 2 \times 5k \left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

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* Find the filter order for the following specificat

$$\sqrt{0.5} \leq |H(\omega)| \leq 1 \qquad 0 \leq \omega \leq \frac{\pi}{2}$$

$$|H(\omega)| \leq 0.2 \qquad \frac{3\pi}{4} \leq \omega \leq \pi$$

with $T = 1$ Sec. Use the IIT

Sol: Given $A_1 = \sqrt{0.5}$

$A_2 = 0.2$

$\omega_1 = \pi/2$

$\omega_2 = 3\pi/4$ & $T = 1$ Sec

$$\frac{\Omega_2}{\Omega_1} = \frac{\omega_2/T}{\omega_1/T} = 1.5$$

$\therefore$ order the filter

$$N \geq \frac{1}{2} \frac{\log_{10}\left[\left(\frac{1}{A_2^2} - 1\right) / \left(\frac{1}{A_1^2} - 1\right)\right]}{\log_{10}\left(\Omega_2/\Omega_1\right)}$$

$$\geq 3.919$$

$$= 4$$

* Determine the order and the poles of a LP Butterworth filter that has −3 dB bandwidth of 500 Hz and an attenuation 40 dB at 1000 Hz

Sol: Given

PB edge freq $f_1 = 500\,Hz \Rightarrow$ ~~...~~

Gain at PB edge $K_1 = -3\,dB$

SB edge freq $f_2 = 1000\,Hz$

Gain at SB edge $K_2 = -40\,dB$

Normalized freq's are

$$\omega_1 = \frac{2\pi f_1}{f_s} = 0.5\pi$$

$$\omega_2 = \frac{2\pi f_2}{f_s} = \pi$$

For IIT

$$\frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1} = 2$$

$$\therefore N \geq \frac{1}{2} \frac{\log_{10}\left\{\left(\frac{1}{A_2^2}-1\right)/\left(\frac{1}{A_1^2}-1\right)\right\}}{\log\left(\frac{\Omega_2}{\Omega_1}\right)}$$

$$\geq 6.64 = 7$$

The poles are at

$$\Omega_c = \frac{\Omega_1}{\left(\frac{1}{A_1^2}-1\right)^{\frac{1}{2N}}}$$

$$S_K = \Omega_c\, e^{j\left[\frac{\pi}{2} + (2K+1)\frac{\pi}{2N}\right]}$$

$$= 1000\pi\, e^{j\left[\frac{\pi}{2} + (2K+1)\frac{\pi}{14}\right]}$$

; $K = 0,1,2,3,4,5,6 < N$

* Design of LP Chebyshev Filter

There are two types of Chebyshev approximations.

In Type-I approximation, the error function is selected such that the magnitude response is equiripple in the PB and monotonic in the SB.

In Type-II approximation, the error function is selected such that the magnitude function is monotonic in the PB and equiripple in the SB.

The Type-II magnitude response is also called inverse Chebyshev response.

The magnitude response of N^{th} order Type-I chebysheve LPF is given by

$$\boxed{|H_a(\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_c}\right)}}$$

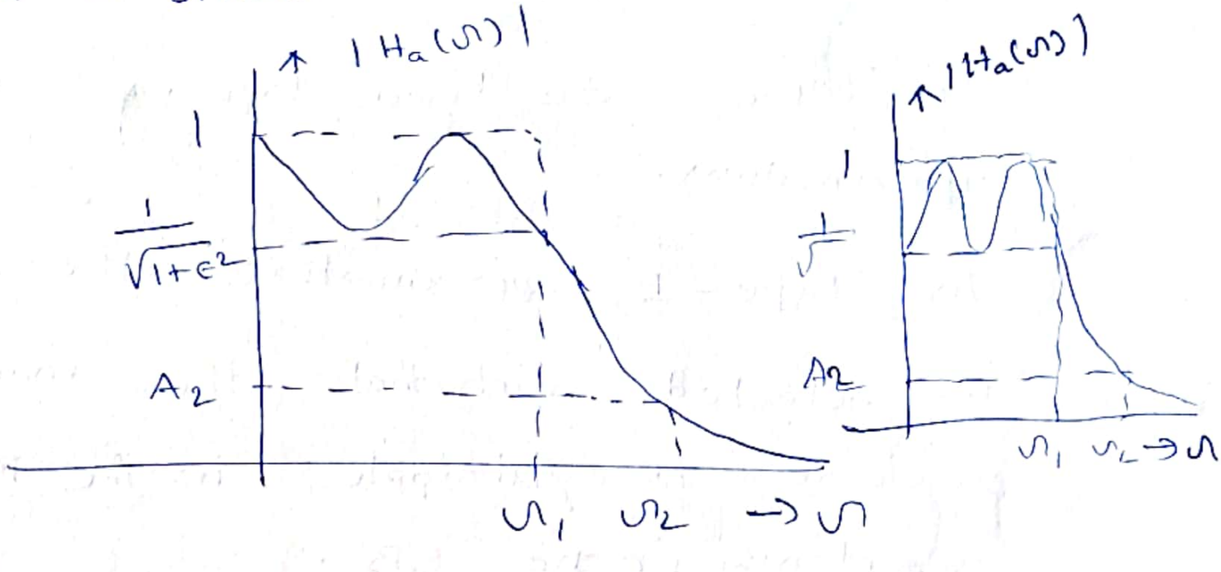
$$C_N(x) = \begin{cases} \cos(N\cos^{-1}x); & |x|\le 1 \text{ (PB)} \\ \cosh(N\cosh^{-1}x); & |x|\ge 1 \text{ (SB)} \end{cases}$$

where ϵ is attenuation Constant given by

$$\epsilon = \left(\frac{1}{A_1^2} - 1\right)^{1/2}$$

& $C_N\left(\frac{\Omega}{\Omega_c}\right)$ is the Chebysheve polynomial of the first kind of

The magnitude response of Type-I Chebyshev filter is shown below



$\because$ $A_1 \leq |H(\omega)| \leq 1$; $0 \leq \omega \leq \omega_1$

$|H(\omega)| \leq A_2$; $\omega_2 \leq \omega \leq \pi$

we have

$$A_1^2 \leq \frac{1}{1+\epsilon^2 C_N^2\left(\frac{\Omega_1}{\Omega_c}\right)} \leq 1$$

$$\frac{1}{1+\epsilon^2 C_N^2\left(\frac{\Omega_2}{\Omega_c}\right)} \leq A_2^2$$

Assuming $\Omega_c = \Omega_1 \Rightarrow C_N\left(\frac{\Omega_1}{\Omega_c}\right) \Rightarrow C_N(1) = 1$

$\therefore$ From the inequality involving A_1^2

$$A_1^2 \leq \frac{1}{1+\epsilon^2}$$

$$\Rightarrow \boxed{\epsilon = \left(\frac{1}{A_1^2} - 1\right)^{1/2}}$$

The order of the filter can be determined from the inequality for A_2^2

Assuming $\Omega_c = \Omega_1$

$$C_N(\Omega_2/\Omega_1) \geq \frac{1}{\epsilon}\left(\frac{1}{A_2^2}-1\right)^{\frac{1}{2}}$$

$\because \Omega_2 > \Omega_1$

$$\cosh[N \cosh^{-1}(\Omega_2/\Omega_1)] \geq \frac{1}{\epsilon}\left(\frac{1}{A_2^2}-1\right)^{\frac{1}{2}}$$

$$\Rightarrow \boxed{N \geq \frac{\cosh^{-1}\left\{\frac{1}{\epsilon}\left(\frac{1}{A_2^2}-1\right)^{\frac{1}{2}}\right\}}{\cosh^{-1}(\Omega_2/\Omega_1)}}$$

Choose N to be the next nearest integer.

The T.F of chebyshev filters

$$H_a(s) = \prod_{K=1}^{N/2} \frac{B_K \Omega_c^2}{s^2 + b_K \Omega_c s + c_K \Omega_c^2} \quad \text{when N is even}$$

$$H_a(s) = \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \prod_{K=1}^{\frac{N-1}{2}} \frac{B_K \Omega_c^2}{s^2 + b_K \Omega_c s + c_K \Omega_c^2} \quad \text{when N is odd}$$

when $b_K = 2 y_N \sin\left(\frac{(2K-1)\pi}{2N}\right)$

$c_K = y_N^2 + \cos^2\left(\frac{(2K-1)\pi}{2N}\right)$

$c_0 = y_N$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2}+\frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2}+\frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right\}$$

For even value of N & unity dc gain filter the parameter B_K are evaluated

$$H_a(s)\Big/_{s=0} = \frac{1}{(1+\epsilon^2)^{1/2}}$$

For odd value of N & unity dc gain filter the parameter B_K are evaluated

$$H_a(s)\Big/_{s=0} = 1$$

Poles of a normalised Chebyshev filter

The T.F from magnitude square function

$$H_a(s)\,H_a(-s) = \frac{1}{1+\epsilon^2 C_N^2\left(\frac{s/j}{\Omega_c}\right)} \quad (\because s = j\Omega)$$

$$H_a(s_n)\,H_a(-s_n) = \frac{1}{1+\epsilon^2 C_N^2(-js_n)}$$

$\therefore$ poles are $1+\epsilon^2 C_N^2(-js_n) = 0$

The solution to the above eq^n gives 2N poles

$$s_n = -\sin x \sinh y + j \cos x \cosh y = \sigma_n + j\Omega_n$$

where $n = 1, 2, --(N+1)/2$ for N odd

$n = 1, 2, ---N/2$ for N even

For N even poles complex
For N odd one in real all are complex

& $x = \frac{(2n-1)\pi}{2N}$; $n = 1, 2, --N$ & $y = \pm\frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right)$

The unnormalised poles from Normalised $\boxed{s_n' = s_n \Omega_c}$ $n = 1, 2, --N$

Design procedure for LP Digital Chebyshev IIR filter

1. Choose the type of transformation (BT/IIT)
2. Calculate the attenuation constant

$$\epsilon = \left(\frac{1}{A_1^2} - 1\right)^{1/2}$$

3. Calculate the ratio of analog edge freqs Ω_2/Ω_1

i.e. $$\frac{\Omega_2}{\Omega_1} = \frac{\frac{2}{T}\tan\omega_2/2}{\frac{2}{T}\tan\omega_1/2} = \frac{\tan\omega_2/2}{\tan\omega_1/2} \text{ for BT}$$

$$\frac{\Omega_2}{\Omega_1} = \frac{\omega_2/T}{\omega_1/T} = \frac{\omega_2}{\omega_1} \text{ for IIT}$$

4. Order of the filter N given by

$$N \geq \frac{\cosh^{-1}\left\{\frac{1}{\epsilon}\left(\frac{1}{A_2^2} - 1\right)\right\}}{\cosh^{-1}\left(\Omega_2/\Omega_1\right)}$$

5. Calculate analog cutoff freq

$$\Omega_c = \frac{\Omega_1}{\left(\frac{1}{A_1^2} - 1\right)^{1/2N}}$$

6. Analog filter T.F

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2} \quad \text{for N even}$$

$$H_a(s) = \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2} \quad \text{for N odd}$$

where $b_k = 2 y_N \sin\left(\frac{(2k-1)\pi}{2N}\right)$

$$c_k = y_N^2 + \cos^2 \frac{(2k-1)\pi}{2N}$$

$$c_0 = y_N$$

$$y_N = \frac{1}{2}\left\{ \left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2} + \frac{1}{\epsilon}\right]^{-\frac{1}{N}} \right\}$$

For even valey N

$$H_a(s)\big|_{s=0} = \frac{1}{(1+\epsilon^2)^{1/2}}$$

For odd valey N

$$\sum_{k=0}^{\frac{N-1}{2}} \frac{B_k}{C_k} = 1$$

(It is taken $B_0 = B_1 = B_2 = \cdots = B_k$)

7. Convert $H_a(s)$ to $H(z)$ by chosen Trayomation

* Design a chebyshev IIR digital LPF to satisfy the Constraints

$$0.707 \le |H(\omega)| \le 1 \quad ; 0 \le \omega \le 0.2\pi$$

$$|H(\omega)| \le 0.1 \quad ; 0.5\pi \le \omega \le \pi$$

Using bilinear transformation & assuming $T = 1$ Sec

Sol: Given $A_1 = 0.707$

$A_2 = 0.1$

$\omega_1 = 0.2\pi$

$\omega_2 = 0.5\pi$ & $T = 1$ sec

→ Attenuation constant

$$\epsilon = \left(\frac{1}{A_1^2} - 1\right)^{1/2} = \left(\frac{1}{(0.707)^2} - 1\right)^{1/2} = 1$$

→ $$\frac{\Omega_2}{\Omega_1} = \frac{2/T \tan \frac{\omega_2}{2}}{2/T \tan \omega_1/2} = 3.0779$$

→ $$N \ge \frac{\cosh^{-1}\left[\frac{1}{\epsilon}\left(\frac{1}{A_2^2} - 1\right)^{1/2}\right]}{\cosh^{-1}(\Omega_2/\Omega_1)}$$

$$\ge 1.669$$

$$\simeq 2$$

Analog Cutoff freq

$$\Omega_c = \frac{\Omega_1}{\left(\frac{1}{A_1^2} - 1\right)^{\frac{1}{2N}}} = 0.6498$$

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T.F

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + C_k \Omega_c^2} = \frac{B_1 \Omega_c^2}{s^2 + b_1 \Omega_c s + C_1 \Omega_c^2}$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2} + \frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right\}$$

$$= 0.455$$

$$b_k = 2y_N \sin\frac{(2k-1)\pi}{2N}$$

$$b_1 = 2 \times 0.455 \sin\left(\frac{(2\times1-1)\pi}{2\times2}\right) = 0.6435$$

$$C_1 = y_N^2 + \cos^2\frac{(2k-1)\pi}{2N}$$

$$= (0.455)^2 + \cos^2\left(\frac{(2\times1-1)\pi}{2\times2}\right)$$

$$= 0.707$$

N in even

$$\prod_{k=1}^{N/2} \frac{B_k}{C_k} = \frac{1}{(1+\epsilon^2)^{1/2}} = 0.707$$

$$\Rightarrow B_1 = C_1 \times 0.707 = 0.5$$

$$H_a(s) = \frac{0.5 \times (0.6498)^2}{s^2 + (0.6435)(0.6498)s + 0.707(0.6498)^2} = \frac{0.2111}{s^2 + 0.4181s + 0.2985}$$

$$\therefore H(z) = H_a(s) \Big/ s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{0.2111}{s^2 + 0.4181s + 0.2985} \Big/ s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

* Determine the system function $H(z)$ of the lowest order chebyshev IIR digital filter with the following specifications

3 dB ripple in PB $0 \le \omega \le 0.2\pi$

25 dB attenuation in SB $0.45\pi \le \omega \le \pi$

Sol: Given

$\alpha_1 = 3\,dB \Rightarrow K_1 = -3\,dB \Rightarrow A_1 = 10^{K_1/20} = 0.707$

$\alpha_2 = 25\,dB \Rightarrow K_1 = -25\,dB \Rightarrow A_2 = 10^{K_2/20} = 0.0562$

$\omega_1 = 0.2\pi$ & $\omega_2 = 0.45\pi$

Let $T=1$ & BT is used

Attenuation constant

$$\rightarrow \epsilon = \left(\frac{1}{A_1^2} - 1\right)^{1/2} = 1$$

$$\rightarrow \frac{\Omega_2}{\Omega_1} = \frac{2/T \tan \omega_2/2}{2/T \tan \omega_1/2} = 2.628$$

$$\rightarrow N \ge \frac{\cosh^{-1}\left\{\frac{1}{\epsilon}\left(\frac{1}{A_2^2} - 1\right)^{\frac{1}{2}}\right\}}{\cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)}$$

$$\ge 2.20 \simeq 3$$

$$\rightarrow \Omega_c = \frac{\Omega_1}{\left(\frac{1}{A_1^2} - 1\right)^{1/2N}} = 1.708$$

T.F $(N=3)$

$$\rightarrow H_a(s) = \frac{B_0 \Omega_c}{S + C_0 \Omega_c} \cdot \frac{B_1 \Omega_c}{s^2 + b_1 \Omega_c s + c_1 \Omega_c^2}$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2} + \frac{1}{\epsilon}\right]^{\frac{-1}{N}}\right\}$$

$$= 0.5959$$

$$C_0 = y_N = 0.5959$$

$$b_1 = 2y_N \sin\left[\frac{(2\times1-1)\pi}{2\times3}\right] = 0.5959$$

$$c_1 = y_N^2 + \cos^2\left(\frac{(2\times1-1)\pi}{2\times3}\right) = (0.5959)^2 + \cos^2 \pi/6 = 1.105$$

For N odd

$$\prod_{k=0}^{\frac{N-1}{2}} \frac{B_k}{C_k} = 1$$

$$B_0 = C_0 = 0.5959 \quad B_1 = C_1 = 1.105$$

$$H_a(s) = \left(\frac{0.5959}{s + 0.5959\times1.708}\right)\left(\frac{1.105\ (1.708)^2}{s^2 + 0.599\times1.708\, s + 1.105(1.708)^2}\right)$$

Using Bilinear Transformation

$$H(z) = H_a(s)\Big/_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= H_a(s)\Big/_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{3.25(1+z^{-1})^3}{7.423 - 1.554z^{-1} + 7.023z^{-2}}$$

* The specifications of the desired LPF is

$$0.9 \le |H(\omega)| \le 1.0 \; ; \; 0 \le \omega \le 0.3\pi$$

$$|H(\omega)| \le 0.15 \; ; \; 0.5\pi \le \omega \le \pi$$

Design a chebyshev digital filter using BT

sol: Given

$$A_1 = 0.9$$

$$A_2 = 0.15$$

$$\omega_1 = 0.3\pi$$

$$\omega_2 = 0.5\pi$$

$$\rightarrow \quad \epsilon = \left(\frac{1}{A_1^2} - 1\right)^{1/2} = 0.484$$

$$\rightarrow \quad \frac{\Omega_2}{\Omega_1} = \frac{\frac{2}{T}\tan\frac{\omega_2}{2}}{\frac{2}{T}\tan\frac{\omega_1}{2}} = 1.962$$

$$\rightarrow \quad N \ge \frac{\cosh^{-1}\left\{\left(\frac{1}{\epsilon}\left(\frac{1}{A_2^2}-1\right)^{1/2}\right)\right\}}{\cosh^{-1}(\Omega_2/\Omega_1)}$$

$$\ge 2.55 \simeq 3$$

$$\rightarrow \quad \Omega_c = \frac{\Omega_1}{\left(\frac{1}{A_1^2} - 1\right)^{\frac{1}{2N}}} = 1.13 \text{ rad/sec}$$

$\rightarrow$ For $N = 3$

$$H_a(s) = \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \cdot \frac{B_1 \Omega_c^2}{s^2 + b_{1}\Omega_c s + c_1 \Omega_c^2}$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2} + \frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right\}$$

$$\simeq 0.511$$

$$C_0 = y_N = 0.511$$

$$C_K = y_N^2 + \cos^2\left(\frac{(2K-1)\pi}{2N}\right)$$

when $K=1 \Rightarrow c_1 = y_N^2 + \cos^2\left(\frac{\pi}{6}\right) = (0.511)^2 + 0.75 = 1.011$

$$b_K = 2y_N \sin\left(\frac{(2K-1)\pi}{2N}\right)$$

when $K=1 \Rightarrow b_1 = 2 \times 0.511 \left(\frac{1}{2}\right) = 0.511$

~~where $s=0 \Rightarrow H_a(0) =$~~ $\cancel{B_0 B_1}$

$$H_a(s) = \left(\frac{B_0(1.13)}{s+0.511 \times 1.13}\right)\left(\frac{B_1(1.13)^2}{s^2+0.511 \times 1.13 s + 1.011(1.13)^2}\right)$$

when $s=0$

$$H_a(0) = \frac{B_0 B_1 (1.442)}{(0.511)(1.13)(1.011)(1.13)^2} = 1.935 B_0 B_1$$

when N is odd

let $H_a(0) = 1 \Rightarrow 1.935 B_0 B_1 = 1$

let $B_0 = B_1 \Rightarrow B_0^2 = \frac{1}{1.935} = 0.516 \Rightarrow B_0 = 0.718$

$$B_0 = B_1 = 0.718$$

$$H_a(s) = \frac{0.744}{(s+0.577)(s^2+0.577 s + 1.29)}$$

$$\therefore H(z) = H_a(s) \Big/ s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{0.744(1+z^{-1})^3}{(2.577-1.423z^{-1})(6.83-5.42z^{-1}+3.78)}$$

* Determine the system function of the lowest order Chebyshev digital filter that meets the following specifications.

2 dB ripple in the PB $0 \le \omega \le 0.25\pi$

Atleast 50 dB attenuation in SB $0.4\pi \le \omega \le \pi$

Sol: Given

Ripple in PB = 2 dB $\Rightarrow K_1 = -2$ dB

$\Rightarrow A_1 = 10^{K_1/20} = 0.794$

Attenuation in SB = 50 dB $\Rightarrow K_2 = -50$ dB

$\Rightarrow A_2 = 10^{K_2/20} = 0.0031$

$\omega_1 = 0.25\pi$

$\omega_2 = 0.4\pi$

$$\rightarrow \quad \epsilon = \left(\frac{1}{A_1^2} - 1\right)^{1/2} = 0.765$$

$$\rightarrow \quad \frac{\Omega_2}{\Omega_1} = \frac{\frac{2}{T}\tan\frac{\omega_2}{2}}{\frac{2}{T}\tan\frac{\omega_1}{2}} = 1.754$$

$$\rightarrow \quad N \ge \frac{\cosh^{-1}\left[\frac{1}{\epsilon}\left(\frac{1}{A_2^2} - 1\right)^{1/2}\right]}{\cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)}$$

$$\ge 5.786$$

$$\simeq 6$$

$$\rightarrow \quad \Omega_c = \frac{\Omega_1}{\left(\frac{1}{A_1^2} - 1\right)^{\frac{1}{2N}}} = 0.866 \text{ rad/s}$$

$$\rightarrow \quad H_a(s) = \left(\frac{B_1 \Omega_c^2}{s^2 + b_1 \Omega_c s + c_1 \Omega_c^2}\right)\left(\frac{B_2 \Omega_c^2}{s^2 + b_2 \Omega_c s + c_2 \Omega_c^2}\right)\left(\frac{B_3 \Omega_c^2}{s^2 + b_3 \Omega_c s + c_3 \Omega_c^2}\right)$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2} + \frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right\}$$

$$= 0.1.183$$

$$C_0 = y_N = 0.183$$

$$C_k = y_N^2 + \cos^2\left(\frac{(2k-1)\pi}{2N}\right) = 0.9664$$

$$b_1 = 2 y_N \sin\left(\frac{(2\times1-1)\pi}{2\times6}\right) = 0.094$$

$$C_2 = y_N^2 + \cos^2\left(\frac{(2\times2-1)\pi}{2\times6}\right) = 0.5334$$

$$b_2 = 2 y_N \sin\left(\frac{(2\times2-1)\pi}{2\times6}\right) = 0.258$$

$$c_3 = y_N^2 + \cos^2\left(\frac{(2\times3-1)\pi}{2\times6}\right) = 0.1$$

$$b_3 = 2 y_N \sin\left(\frac{(2\times3-1)\pi}{2\times6}\right) = 0.353$$

let $B_1 = B_2 = B_3$ & $H_a(0) = 1$

$$\frac{B_1 B_2 B_3 \Omega_c^6}{C_1 C_2 C_3 \Omega_c^6} = 1$$

$$B_1 = B_2 = B_3 = (C_1 C_2 C_3)^{1/3} = 0.371$$

$$\therefore H_a(s) = \left(\frac{0.371 \times (0.866)^2}{s^2 + 0.094 \times 0.866s + 0.966 \times (0.866)^2}\right) \times$$

$$\left(\frac{0.371 \times (0.866)^2}{s^2 + 0.258 \times 0.866s + 0.333 \times (0.866)^2}\right) \times$$

$$\left(\frac{0.371 \times (0.866)^2}{s^2 + 0.353 \times 0.866s + 0.1 \times (0.866)^2}\right)$$

$$H_a(s) = \frac{0.278 \times 0.278 \times 0.278}{(s^2 + 0.018s + 0.724)(s^2 + 0.223s + 0.399)(s^2 + 0.303s + 0.07\ldots}$$

BT

$$\therefore H(z) = H_a(s) \Big/ _{s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)}$$

$$= \frac{0.278\,(1 + z^{-1})^2}{4.684 - 7.852z^{-1} + 3.464z^{-2}}$$

* Find the Chebyshev filter order for the following specifications

$$\sqrt{0.6} \le |H(\omega)| \le 1 \qquad 0 \le \omega \le \pi/2$$

$$|H(\omega)| \le 0.25 \qquad 3\pi/2 \le \omega \le \pi$$

with $T = 1$ sec use the IIT.

Sol:. Given

$A_1 = 0.774$

$A_2 = 0.25$

$\omega_1 = \pi/2$

$\omega_2 = 3\pi/2$ & $T = 1$ sec, IIT

→ $\epsilon = \left(\frac{1}{A_1^2} - 1\right)^{1/2} = 0.818$

IIT Method

→ $\frac{\Omega_2}{\Omega_1} = \frac{\omega_2/T}{\omega_1/T} = \frac{\omega_2}{\omega_1} = 3$

→ $$N \ge \frac{\cosh^{-1}\left[\frac{1}{\epsilon}\left(\frac{1}{A_2^2} - 1\right)^{1/2}\right]}{\cosh^{-1}\left(\Omega_2/\Omega_1\right)}$$

≥ 1.268

$\simeq 2$

* Find the order for the following Specifications

$$\sqrt{0.5} \le |H(\omega)| \le 1 \qquad 0 \le \omega \le \frac{\pi}{2}$$ with $T = 1$ sec

$$|H(\omega)| \le 0.2 \qquad \frac{3\pi}{4} \le \omega \le \pi$$ using IIT

Sol : Given

$A_1 = 0.707$

$A_2 = 0.2$

$\omega_1 = \pi/2$

$\omega_2 = 3\pi/4$ & $T = 1$ sec

$$\rightarrow \quad \epsilon = \left(\frac{1}{A_1^2} - 1\right)^{1/2} = 1$$

$$\rightarrow \quad \frac{\Omega_2}{\Omega_1} = \frac{\omega_2/T}{\omega_1/T} = \omega_2/\omega_1 = \frac{3}{2} = 1.5$$

$$\rightarrow \quad N \ge \frac{\cosh^{-1}\left[\frac{1}{\epsilon}\left(\frac{1}{A_1^2} - 1\right)^{1/2}\right]}{\cosh^{-1}\left(\Omega_2/\Omega_1\right)}$$

$$\ge 2.36$$

$$\simeq 3$$

* Determine the lowest order of Chebyshev filter that meets the following specifications

1 dB ripple in the PB $0 \leq |\omega| \leq 0.3\pi$

Atleast 60 dB attenuation in the SB ; $0.35\pi \leq |\omega| \leq \pi$

use the B.T

Sol:

$\alpha_1 = 1\,dB \Rightarrow K_1 = -1\,dB \Rightarrow A_1 = 10^{K_1/20} = 0.891$

$\alpha_2 = 60\,dB \Rightarrow K_2 = -60\,dB \Rightarrow A_2 = 10^{K_2/20} = 0.001$

$\omega_1 = 0.3\pi$

$\omega_2 = 0.35\pi$ & $T = 1\,sec$

$$\rightarrow \epsilon = \left(\frac{1}{A_1^2} - 1\right)^{1/2} = 0.509$$

$$\rightarrow \frac{\Omega_2}{\Omega_1} = \frac{\frac{2}{T}\tan\frac{\omega_2}{2}}{\frac{2}{T}\tan\omega_1/2} = 1.2$$

$$\rightarrow N \geq \frac{\cosh^{-1}\left[\frac{1}{\epsilon}\left(\frac{1}{A_2^2} - 1\right)^{1/2}\right]}{\cosh^{-1}\left(\frac{\Omega_2}{\Omega_1}\right)}$$

$$\geq 13.338$$

$$\approx 14$$

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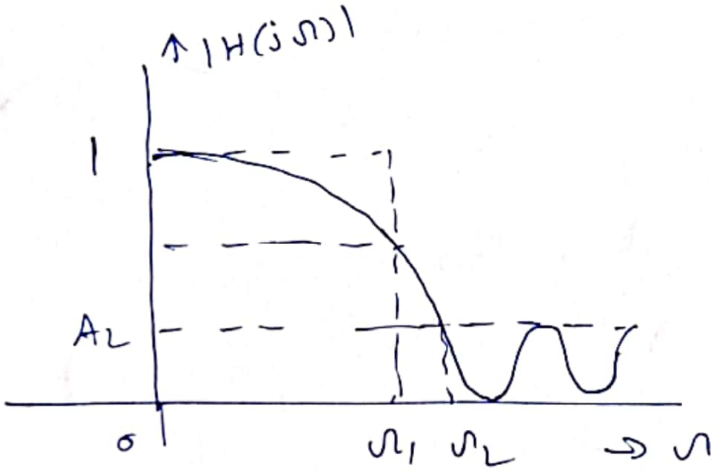
* Inverse Chebyshev Filters

They are also called Type-II chebyshev filters

The magnitute response of the inverse Chebyshev filter is given by

$$|H(j\Omega)| = \frac{\epsilon\, C_N(\Omega_2/\Omega)}{[1+\epsilon^2 C_N^2(\Omega_2/\Omega)]^{1/2}}$$



The parameters of the inverse chebyshev filter are obtained by considering the LPF with the desired specifications

$$0.707 \le |H(\Omega)| \le 1 \qquad 0 \le \Omega \le \Omega_c$$

$$|H(\Omega)| \le A_2 \qquad \Omega \le \Omega_a$$

the attenuation constant

$$\epsilon = \frac{A_1}{(1-A_2^2)^{1/2}}$$

The order of the filter is given by

$$N \geq \frac{\cosh^{-1}(1/\epsilon)}{\cosh^{-1}(\Omega_2/\Omega_c)}$$

$$= \frac{\cosh^{-1}\left(\frac{1}{A_2^2}-1\right)^{1/2}}{\cosh^{-1}(\Omega_2/\Omega_c)}$$

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* Frequency Transformation in Analog Domain

The LPF can be considered as a prototype filter and its system $H_p(s)$ can be determined HPF, BPF & BSF are designed by designing a LPF and then transforming LPF transfer function into required filter function by frequency transformation

1. Analog Frequency Transformation:

Given normalized LPF, ($\Omega_c = 1$ rad/sec)

Given Unnormalized LP

Type	Transformation
Lowpass	$s \rightarrow \Omega_c \frac{s}{\Omega_c^*}$
Highpass	$s \rightarrow \Omega_c \frac{\Omega_c^*}{s}$
Bandpass	$s \rightarrow \Omega_c \frac{(s^2 + \Omega_1\Omega_2)}{s(\Omega_2 - \Omega_1)}$
Bandstop	$s \rightarrow \Omega_c \frac{s(\Omega_2 - \Omega_1)}{s^2 + \Omega_1\Omega_2}$

where $\Omega_0 = \sqrt{\Omega_1\Omega_2}$

$$\text{Quality factor} = \frac{\Omega_0}{\Omega_2 - \Omega_1}$$

$\Omega_c \rightarrow$ cutoff freq of ~~new~~ LP filter

$\Omega_c^* \rightarrow$ " " newfilter

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Digital Frequency transformation

prototype LPF with cutoff freq ω_c

$$z^{-1} \rightarrow g(z^{-1}) = \pm \prod_{k=1}^{n} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}}$$

Type	Transformation	Parameters
Lowpass	$z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \frac{\sin[(\omega_c - \omega_c^*)/2]}{\sin[(\omega_c + \omega_c^*)/2]}$
Highpass	$z^{-1} \rightarrow -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = \frac{-\cos[(\omega_c - \omega_c^*)/2]}{\cos[(\omega_c + \omega_c^*)/2]}$
Bandpass	$z^{-1} \rightarrow -\frac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}$	$\alpha_1 = -\frac{2\alpha K}{(K+1)}$, $\alpha_2 = (K-1)/(K+1)$, $\alpha = \frac{\cos(\omega_2 + \omega_1)/2}{\cos(\omega_2 - \omega_1)/2}$, $K = \cot\left(\frac{\omega_2 - \omega_1}{2}\right)\tan\left(\frac{\omega_c}{2}\right)$
Bandstop	$z^{-1} \rightarrow \frac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}$	$\alpha = \frac{\cos(\omega_2 + \omega_1)/2}{\cos(\omega_2 - \omega_1)/2}$, $K = \tan\left(\frac{\omega_2 - \omega_1}{2}\right)\tan\frac{\omega_c}{2}$, $\alpha_1 = \frac{-2\alpha K}{(K+1)}$, $\alpha_2 = \frac{1-K}{1+K}$

* A prototyp LPF has the s/m fn

$$H_p(s) = \frac{1}{s^2+3s+2}$$ obtain a BPF

with $\Omega_0 = 3$ rad/s and $Q = 12$

Sol.

LP $\rightarrow$ BP

$$s \rightarrow \Omega_c \frac{s^2+\Omega_1\Omega_2}{s(\Omega_2-\Omega_1)} = \Omega_c \frac{s^2+\Omega_0^2}{s(\Omega_0/Q)}$$

$$s \rightarrow \Omega_c \frac{s^2+3^2}{s\left(\frac{3}{12}\right)} = 4\Omega_c\left(\frac{s^2+9}{s}\right)$$

$$\therefore H(s) = H_p(s) \Big/ s = 4\Omega_c\left(\frac{s^2+9}{s}\right)$$

$$= \frac{1}{16}\,\frac{s^2}{\Omega_c^2 s^4 + 0.75\,\Omega_c s^3 + (18\Omega_c^2 + 0.125)s^2 + 6.75\,\Omega_c s + 81\Omega_c^2}$$

* Transform the prototype LPF with s/m function $H_p(s) = \dfrac{\Omega_c}{s+2\Omega_c}$ into a HPF with cutoff freq Ω_c^*

Sol

$$s \rightarrow \Omega_c \frac{\Omega_c^*}{s}$$

$$H_{hp}(s) = H_p(s) \Big/ s = \Omega_c \frac{\Omega_c^*}{s} = \frac{s}{2s+\Omega_c^*}$$

IIT is not suitable for HP or BSF where resonant freq's are higher. In such a case LP prototype filter is converted into HPF using the analog freq transformation and transformed later to a digital filter using IIT. This will result in aliasing problem.

If the same prototype filter is first transformed into digital domain using IIT and later converted into HPF using digital frequency transformation then it will not have any aliasing problem.

Whenever the BT is used it is of no significance whether analog frequency transformation or digital freq transformation used

* Realization of IIR Systems

Consider the N^{th} order LCCDE

$$\sum_{K=0}^{N} a_K\, y(n-K) = \sum_{K=0}^{N} b_K\, x(n-K) \quad \text{--- (1)}$$

taking the Z.T on both sides

$$\sum_{K=0}^{N} a_K\, z^{-K}\, Y(z) = \sum_{K=0}^{N} b_K\, z^{-K}\, X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{K=0}^{N} b_K\, z^{-K}}{\sum_{K=0}^{N} a_K\, z^{-K}}$$

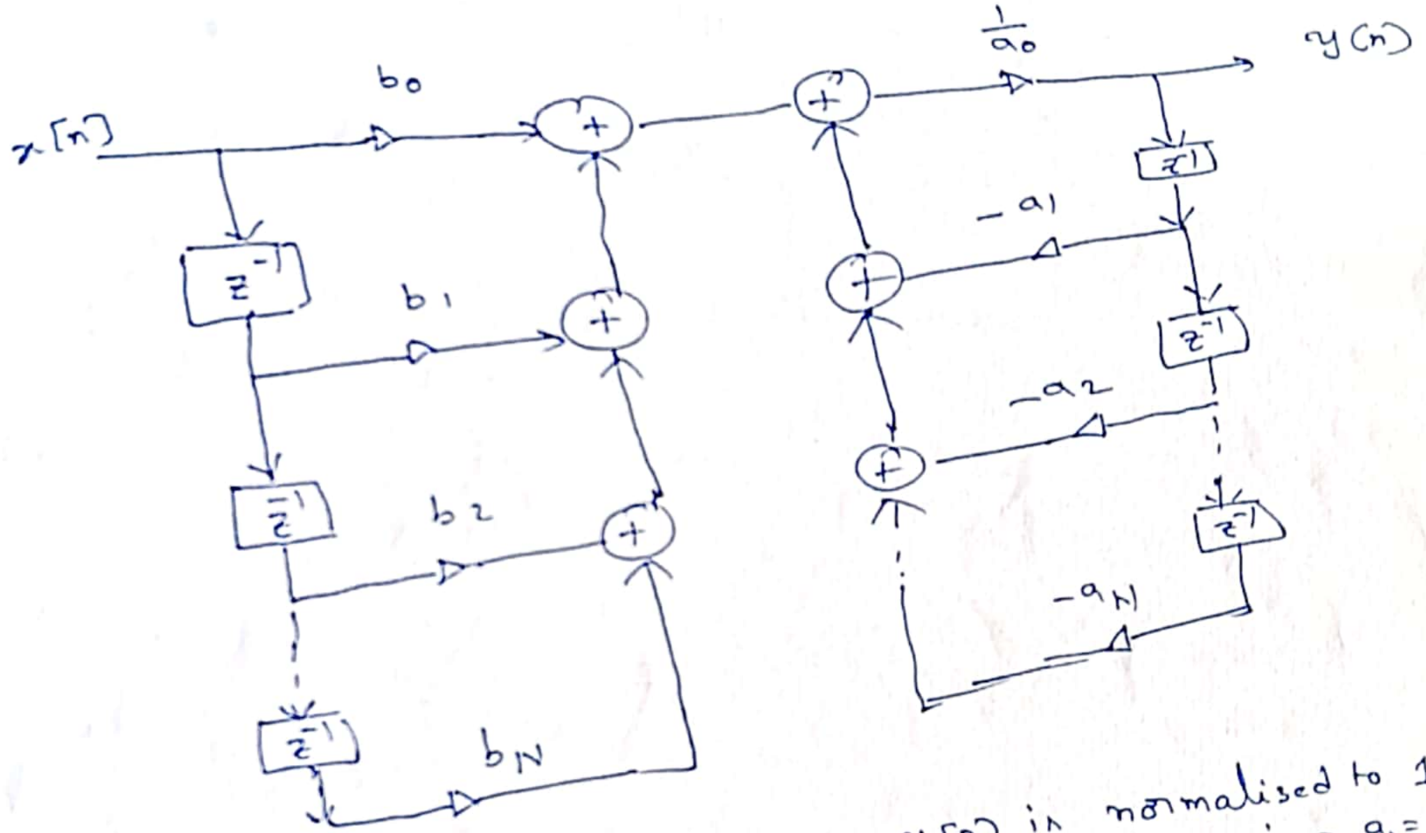
Expand the eq^n (1)

$$a_0\, y(n) + \sum_{K=1}^{N} a_K\, y(n-K) = \sum_{K=0}^{N} b_K\, x(n-K)$$

$$y[n] = \frac{1}{a_0}\left[\sum_{K=0}^{N} b_K\, x(n-K) - \sum_{K=1}^{N} a_K\, y(n-K)\right]$$

The above eq^n is realized as

$$y[n] = \frac{1}{a_0}\left[\sum_{k=0}^{N} b_k\, x(n-k) - \sum_{k=1}^{N} a_k\, y(n-k)\right]$$



Generally the coefficient y[n] is normalised to 1 i.e $a_0 = 1$

* Structures for IIR Systems

Causal IIR systems are characterised by the constant difference eq^n

$$y[n] = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k) \quad \text{--- (1)}$$

apply Z.T

$$Y(z)\left[1+\sum_{k=1}^{N} a_k z^{-k}\right] = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} \quad \text{--- (2)}$$

The above eq^n is std form of system T.F

Direct form - I Realization

The digital system structure determined directly from either eq^n (1) or eq^n (2) is called direct form - I

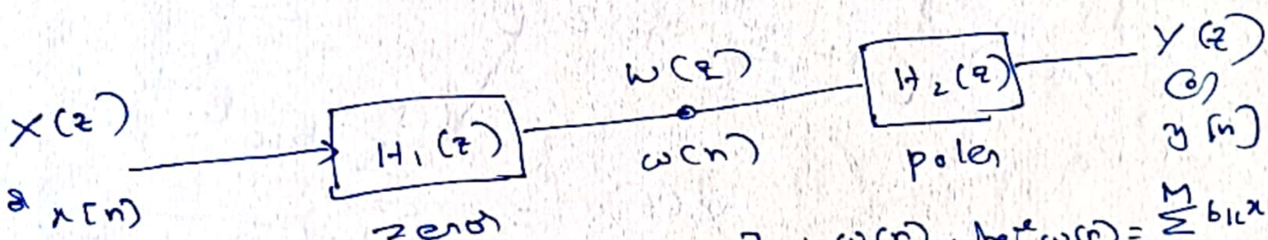
In this case the sh fn is divided into two parts connected in cascade, the 1st part containing only zeros followed by the part containing only poles. An intermediate sequence $w(n)$ is introduced.

i.e $$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

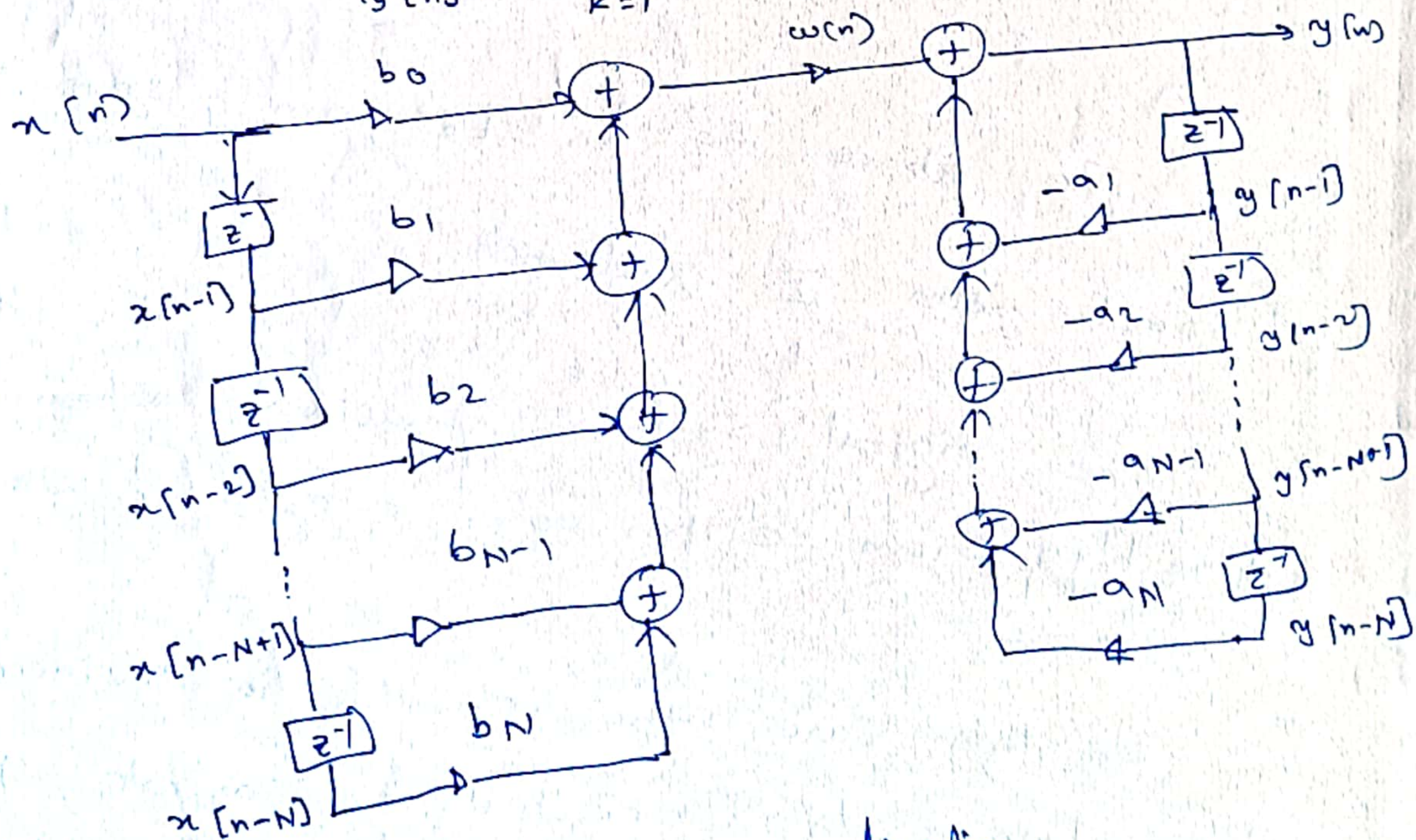
$$H(z) = \underbrace{\left(\sum_{k=0}^{M} b_k z^{-k}\right)}_{\text{zeros}} \underbrace{\left(\frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}\right)}_{\text{poles}}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \cdot \frac{Y(z)}{W(z)}$$



$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + w(n) \quad \text{where } w(n) = \sum_{k=0}^{M} b_k x(n-k)$$



Direct form - I realisation required $M+N$ storage elements.

* Obtain the direct form - I realization for a 3rd order IIR T.F

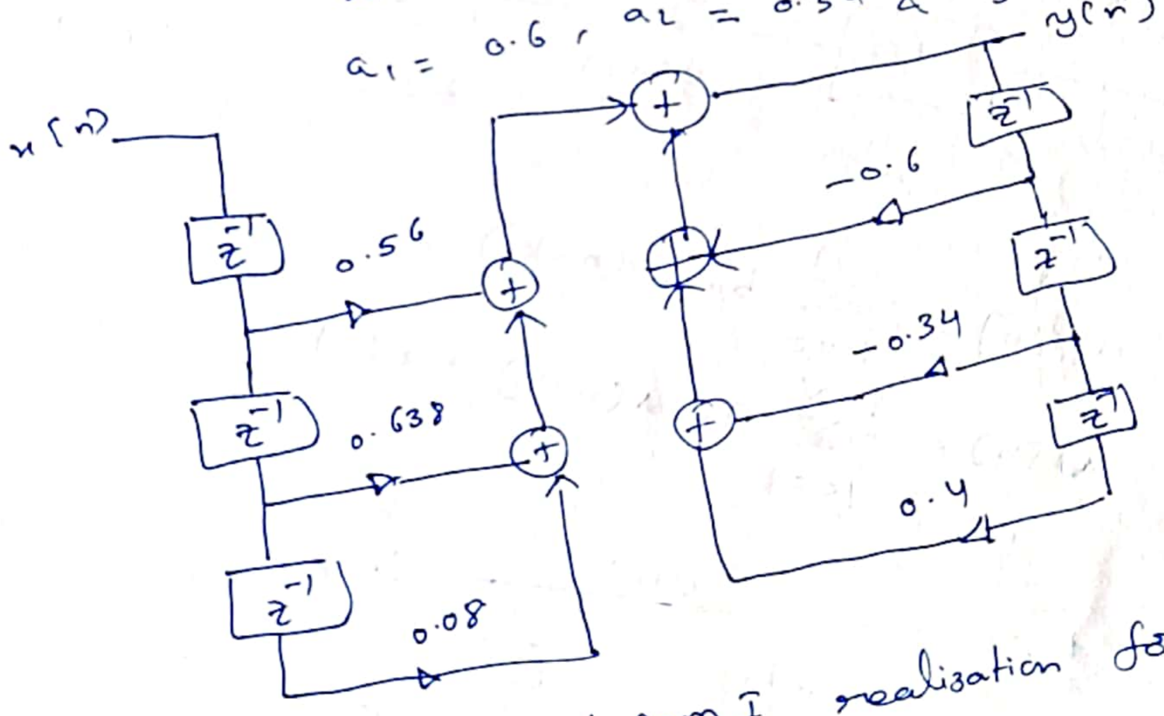
$$H(z) = \frac{0.28z^2 + 0.319z + 0.04}{0.5z^3 + 0.3z^2 + 0.17z - 0.2}$$

$$H(z) = \frac{0.56z^{-1} + 0.638z^{-2} + 0.08z^{-3}}{1 + 0.6z^{-1} + 0.34z^{-2} - 0.4z^{-3}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

$b_0 = 0$, $b_1 = 0.638$, $b_2 = 0.008$

$a_1 = 0.6$, $a_2 = 0.34$ & $a_3 = 0.4$



* Obtain the direct form I realization for the system described by the following Difference eq^n

i) $y[n] = 0.5y[n-1] - 0.25y[n-2] + x[n] + 0.4x[n-1]$

ii) $y[n] = 2y[n-1] + 3y[n-2] + x[n] + 2x(n-1) + 3x[n-2]$

iii) $y[n] = 0.5y[n-1] + 0.06y(n-2) + 0.3x[n] + 0.5x[n-1]$

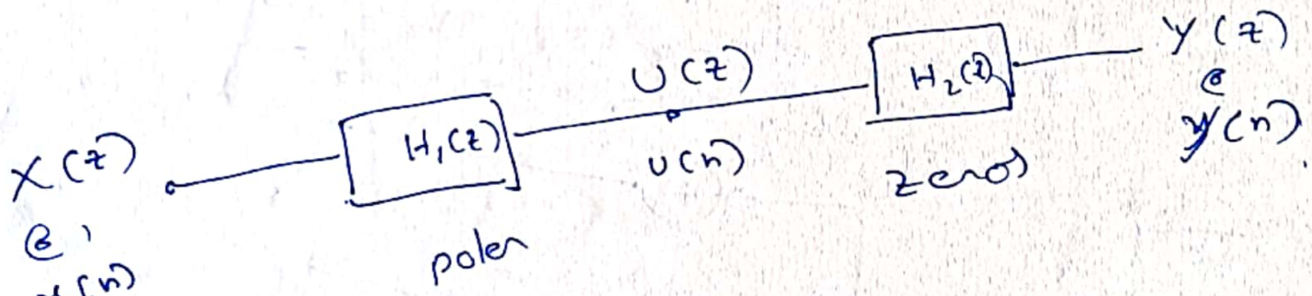
Direct form - II Realization

In this case the poles of $H(z)$ are realised first and the zeros second. ($\because$ considering linear system)

$$H(z) = H_1(z)\, H_2(z)$$

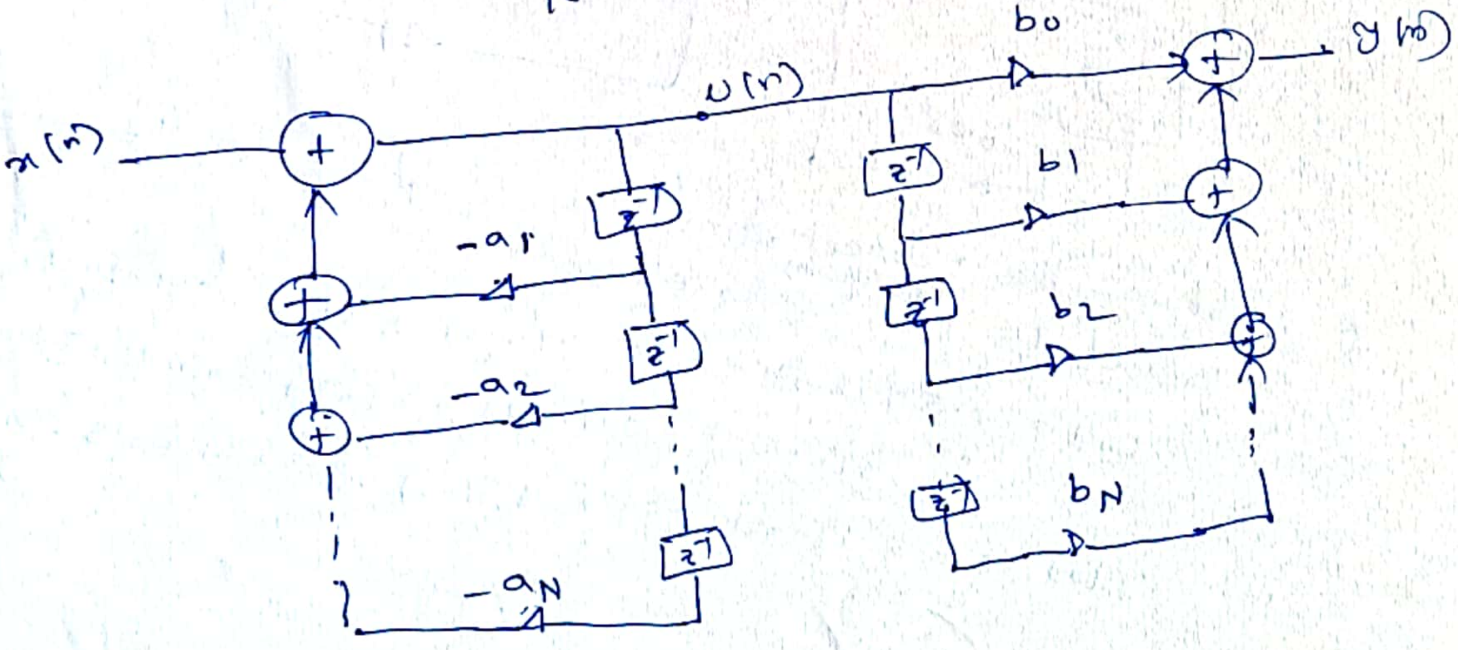
where
$$H_1(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

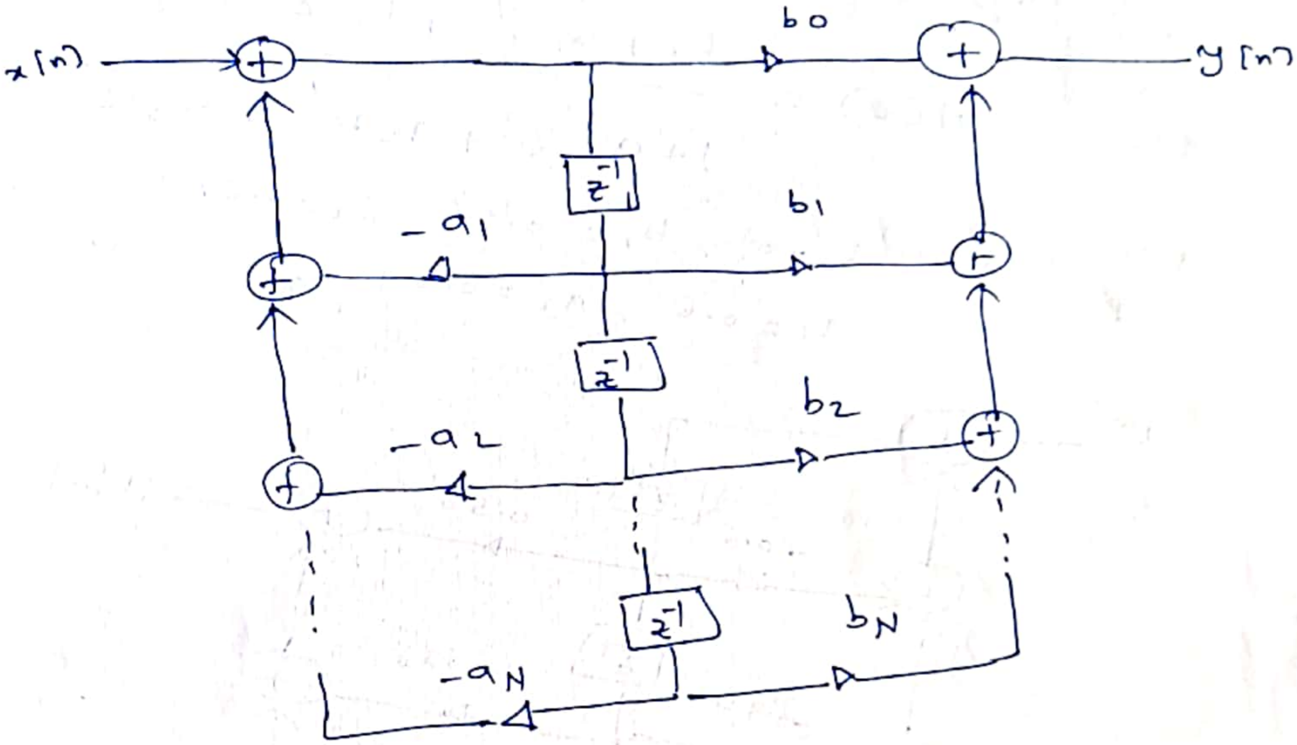
$$H_2(z) = \sum_{k=0}^{M} b_k z^{-k}$$



$$y[n] = \sum_{k=0}^{M} b_k\, u(n-k)$$

$$u(n) = -\sum_{k=1}^{N} a_k\, u(n-k) + x(n)$$





Direct form - II requires only the larger of M or N storage elements.

It uses minimum no. of storage elements and hence said to Canonic structure.

* Determine the direct form II realization for a 3rd order T.F

$$H(z) = \frac{0.28z^2 + 0.319z + 0.04}{0.5z^3 + 0.3z^2 + 0.17z - 0.2}$$

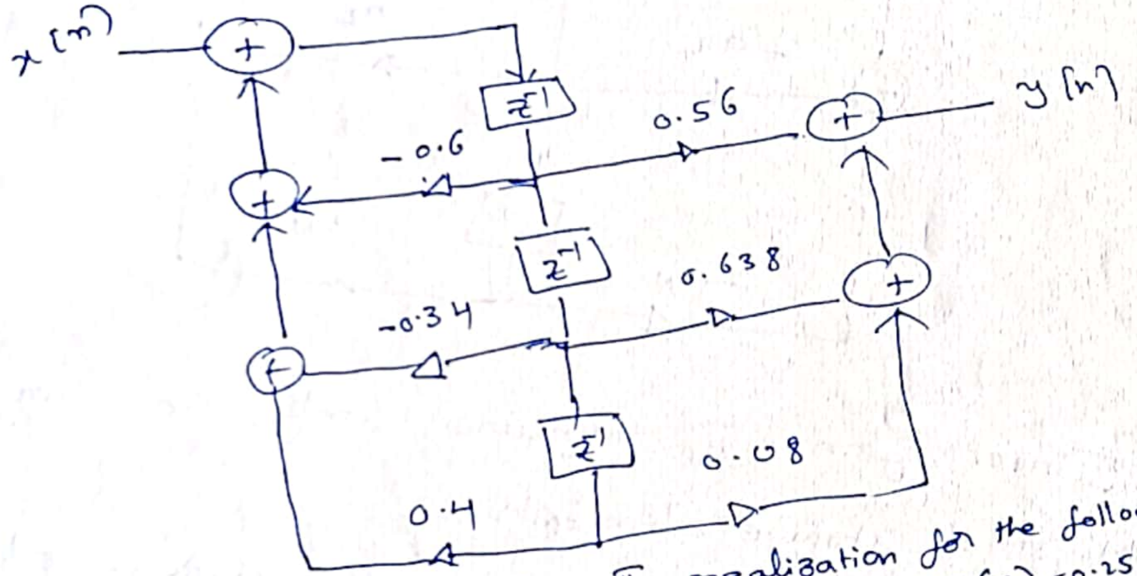
$$H(z) = \frac{0.56z^{-1} + 0.638z^{-2} + 0.08z^{-3}}{1 + 0.6z^{-1} + 0.34z^{-2} - 0.4z^{-3}}$$

Compare with

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

$b_0 = 0$, $b_1 = 0.56$, $b_2 = 0.638$

$a_1 = 0.6$, $a_2 = 0.34$, $a_3 = -0.4$



* Determine the Direct form II realization for the following s/m

i) $y[n] = -0.1y(n-1) + 0.72\,y(n-2) + 0.7x(n) - 0.252x(n-2)$

ii) $y(n) + y(n-1) - 4y(n-3) = x(n) + 3x(n$

iii) $y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{2}x(n$

Remark

→ Both are lack of H/w flexibility

→ Due to finite precision arithmetic coefficient to quatisation the sensitivity of the effects increases with the order of the filter. This sensitivity may change the coefficient values and hence the freq response, there by causing the filter to become unstable

Remedy

→ To over come these effects Cascade & parallel realizations Can be implemented

Cascade form Realization

In this Realization the T.F $H(z)$ is broken into a product of T.Fs.

$$x(n) \rightarrow \boxed{H_1(z)} \rightarrow \boxed{H_2(z)} \rightarrow \cdots \rightarrow \boxed{H_K(z)} \rightarrow y(n)$$

i.e. $H(z) = H_1(z)\, H_2(z) \cdots H_K(z)$

$$\boxed{H(z) = \prod_{k=1}^{K} H_k(z)}$$

By factorizing the NR & DR polynomials of the T.F

$$H(z) = \frac{G \cdot \prod_{k=1}^{M_1} (1 - g_k z^{-1}) \prod_{k=1}^{M_2} (1 - h_k z^{-1})(1 - h_k^* z^{-1})}{\prod_{k=1}^{N_1} (1 - c_k z^{-1}) \prod_{k=1}^{N_2} (1 - d_k z^{-1})(1 - d_k^* z^{-1})}$$

Where $M = M_1 + 2M_2$

& $N = N_1 + 2N_2$

& $G \rightarrow$ gainfactor.

In the above eq^n the 1^{st} order factors represents real zeros at g_k & real poles at c_k and 2^{nd} order factors represent Complex Conjugate zeros at h_k & h_k^* & Complex Conjugate poles at d_k & d_k^*

i.e. The NR & DR polynomials of the T.F $H(z)$ are factored into a product of 1^{st} & 2^{nd} order polynomial.

The advatage of the Cascade form is that the overall T.F of the filter can be determined and also it has the same zeros & poles as the individual components since T.F has the product of the components

* Obtain a Cascade form realization of the System characterised by the T.F

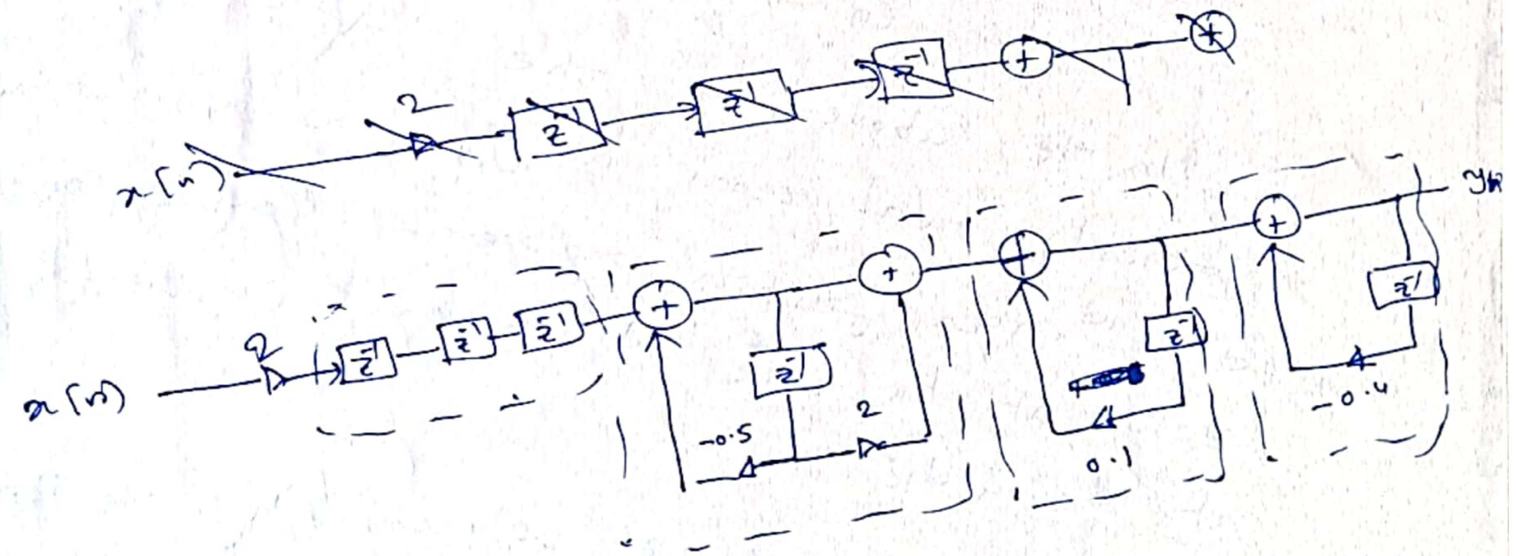
$$H(z) = \frac{2(z+2)}{z(z+0.5)(z-0.1)(z+0.4)}$$

Sol

$$H(z) = \frac{2z^{-3}(1+2z^{-1})}{(1+0.5z^{-1})(1-0.1z^{-1})(1+0.4z^{-1})}$$

$$H(z) = 2z^{-3} \cdot \frac{(1+2z^{-1})}{(1+0.5z^{-1})} \cdot \frac{1}{(1-0.1z^{-1})} \cdot \frac{1}{(1+0.4z^{-1})}$$

$$= 2 \; H_1(z) \quad H_2(z) \quad H_3(z) \quad H_4(z)$$



* obtain Cascade form realization of the following System T.F

i) $$H(z) = \frac{1+\frac{1}{2}z^{-1}}{(1-z^{-1}+\frac{1}{4}z^{-2})(1-z^{-1}+\frac{1}{2}z^{-2})}$$

ii) $$H(z) = \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}} \cdot \frac{1-z^{-1}+\frac{1}{4}z^{-2}}{1+z^{-1}+\frac{1}{4}z^{-2}} \cdot \frac{1+z^{-1}}{1+z^{-1}+\frac{1}{2}z^{-2}}$$

* Realise the system with difference eq^n

$$y[n] = \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + x[n] + \frac{1}{3}x[n-1]$$

Sol. Apply Z.T for the above eq^n

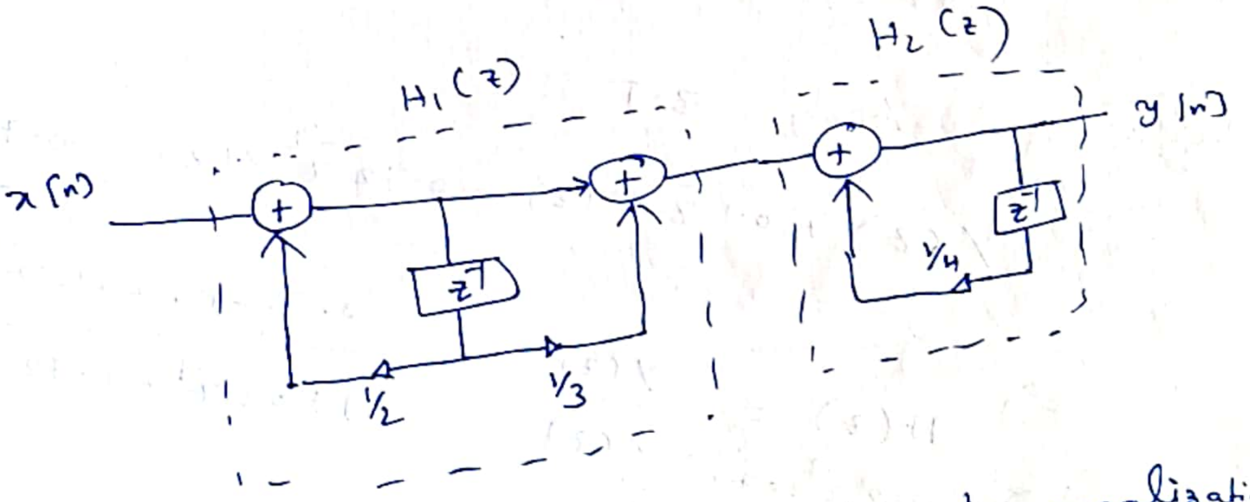
$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

$$H(z) = \frac{1+\frac{1}{3}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

$$H(z) = H_1(z)\,H_2(z)$$

$$H(z) = \left(\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$



* Using 1st order section, Obtain a Cascade realization

i) $H(z) = \frac{(1+\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-\frac{1}{8}z^{-1})}$

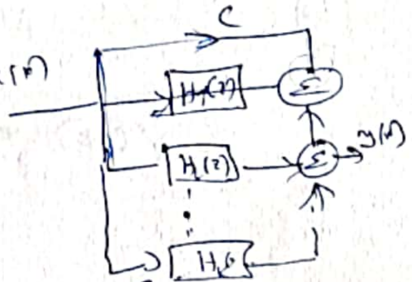
ii) $H(z) = \frac{1+\frac{1}{4}z^{-1}+\frac{1}{2}z^{-2}}{(1+\frac{1}{2}z^{-1})(1+z^{-1}+\frac{1}{2}z^{-2})(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2})}$

iii) $H(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$

$= \frac{(1+z^{-1})(1+}{(1-\frac{1}{2}z^{-1})(1-}$

* Parallel form Realization

A parallel form realization of an IIR system can be obtained by performing a partial fraction expansion of $H(z)$



$$\boxed{H(z) = C + \sum_{K=1}^{N} \frac{C_K}{1 - P_K z^{-1}}}$$

$$H(z) = C + \frac{C_1}{1 - P_1 z^{-1}} + \frac{C_2}{1 - P_2 z^{-1}} + \cdots + \frac{C_N}{1 - P_N z^{-1}}$$

$$H(z) = C + H_1(z) + H_2(z) + \cdots + H_N(z)$$

$$\left(H(z) = \sum_{K=0}^{M-N} C_K z^{-K} + \frac{b_{00}}{1 + a_{01} z^{-1}} + \sum_{K=1}^{N} \frac{b_{K0} + b_{K1} z^{-1}}{1 + a_{K1} z^{-1} + a_{K2} z^{-2}} \right)$$

* Realize the system given by the difference eq^n $y(n) = -0.1 y(n-1) + 0.72 y(n-2) + 0.7 x(n) - 0.252 x(n-2)$ in parallel form

Sol: Apply Z.T

$$Y(z) + 0.1 z^{-1} Y(z) - 0.72 z^{-2} Y(z) = 0.7 X(z) - 0.252 z^{-2} X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{0.7 - 0.252 z^{-2}}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

$$-0.72 z^{-2} + 0.1 z^{-1} + 1 \overline{) -0.252 z^{-2} + 0.7 } \left(0.35 \right.$$

$$-0.252 z^{-2} \pm 0.035 z^{-1} \pm 0.35$$

$$\cancel{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

$$0.35 - 0.035 z^{-1}$$

$$0.7 - 0.252 z^{-2} \; (0.7$$

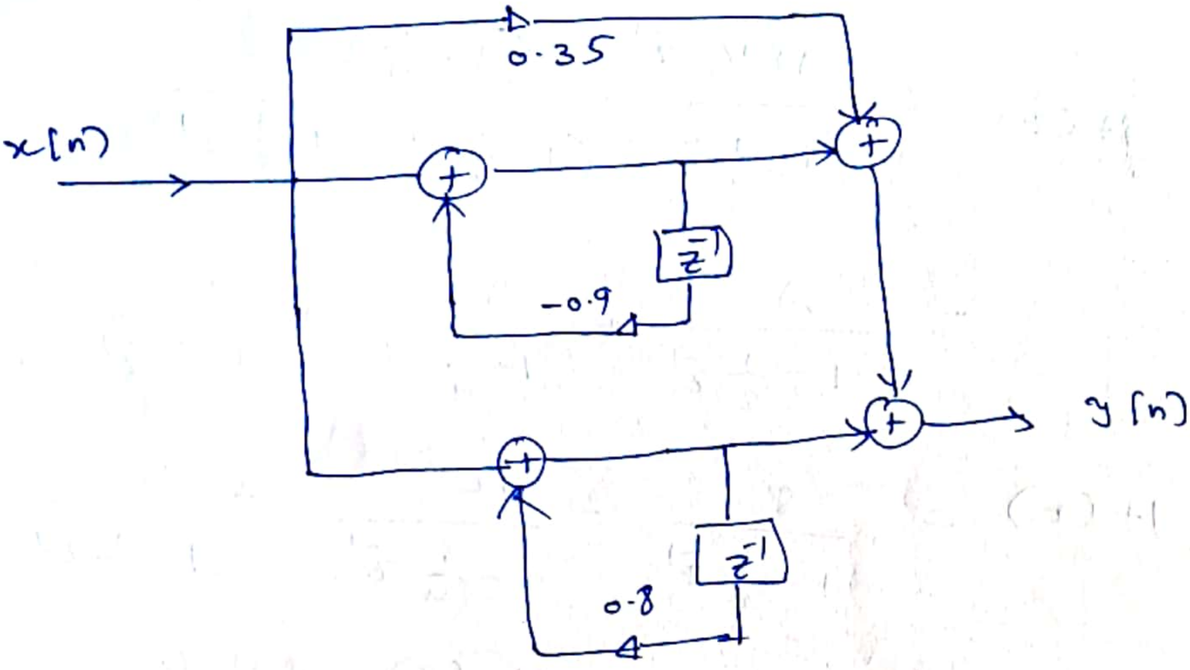
$$0.7 + 0.07 z^{-1} - 0.504 z^{-2}$$

$$- \quad - \quad +$$

$$-0.07 z^{-1} + 0.252 z^{-2}$$

$$H(z) = 0.35 + \frac{0.35 - 0.035 z^{-1}}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

$$= 0.35 + \frac{0.206}{1 + 0.9 z^{-1}} + \frac{0.144}{1 - 0.8 z^{-1}}$$



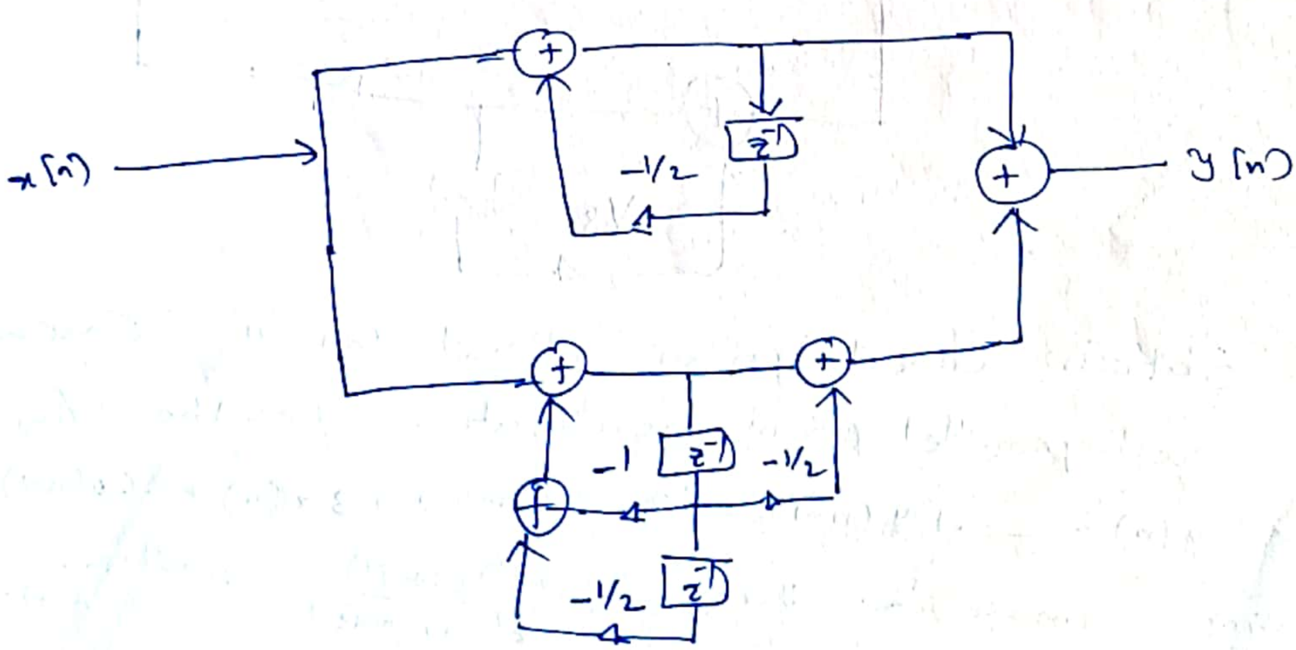
Realize in parallel form

* i) $H(z) = \dfrac{2 + z^{-1} + \frac{1}{4} z^{-2}}{(1+\frac{1}{2}z^{-1})(1+z^{-1}+\frac{1}{2}z^{-2})}$

ii) $H(z) = \dfrac{1+z^{-1}+z^{-2}}{(1+\frac{1}{2}z^{-1})(1+\frac{1}{6}z^{-1}}$

$$= \frac{A}{1+\frac{1}{2}z^{-1}} + \frac{B + Cz^{-1}}{1+z^{-1}+\frac{1}{2}z^{-2}}$$

$$= \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{1-\frac{1}{2}z^{-1}}{1+z^{-1}+\frac{1}{2}z^{-2}}$$



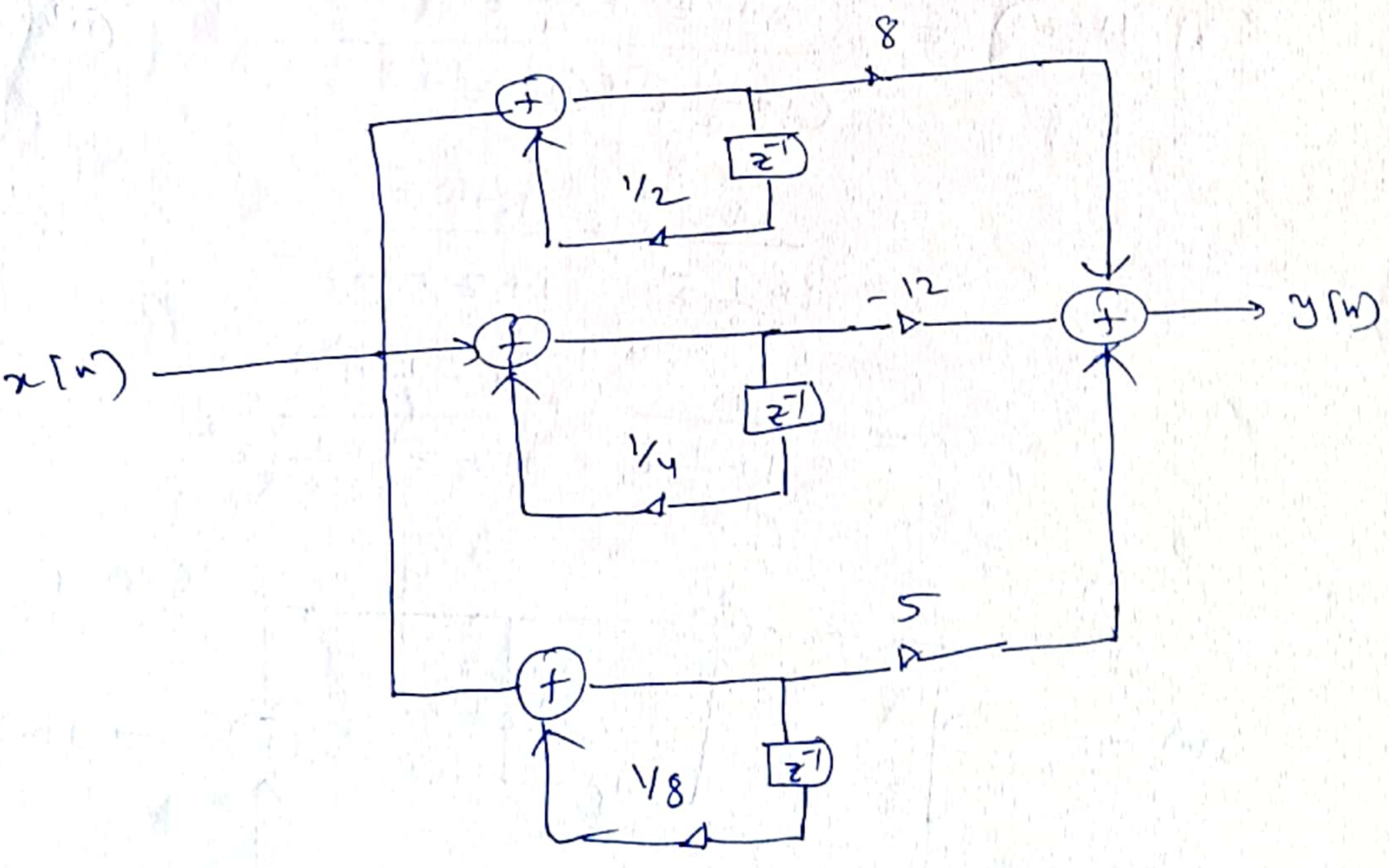
* Realise the following S/m in parallel form

$$H(z) = \frac{(1+1/2\, z^{-1})\,(1+\frac{1}{4} z^{-1})}{(1-\frac{1}{2} z^{-1})\,(1-\frac{1}{4} z^{-1})\,(1-\frac{1}{8} z^{-1})}$$

$$= \frac{A}{1-\frac{1}{2} z^{-1}} + \frac{B}{1-\frac{1}{4} z^{-1}} + \frac{C}{1-\frac{1}{8} z^{-1}}$$

$$H(z) = \frac{8}{1-\frac{1}{2} z^{-1}} - \frac{12}{1-\frac{1}{4} z^{-1}} + \frac{5}{1-\frac{1}{8} z^{-1}}$$

$$H(z) = H_1(z) + H_2(z) + H_3(z)$$



* obtain direct form I, direct form II, Cascade form and parallel form realization for the S/m

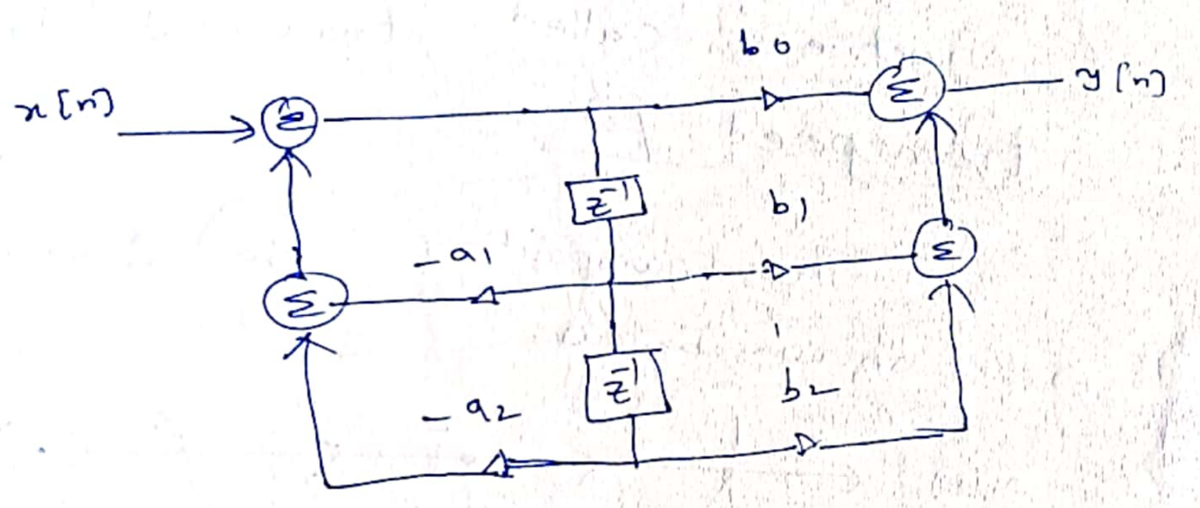
$$y(n) = -0.1\, y(n-1) + 0.2\, y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$$

Hint Cascade form: $H(z) = \dfrac{(3+0.6z^{-1})(1+z^{-1})}{(1+0.5z^{-1})(1-0.4z^{-1})}$ parallel form: $H(z) = -3 + \dfrac{7}{1-0.4z^{-1}} - \dfrac{1}{1+0.5z^{-1}}$

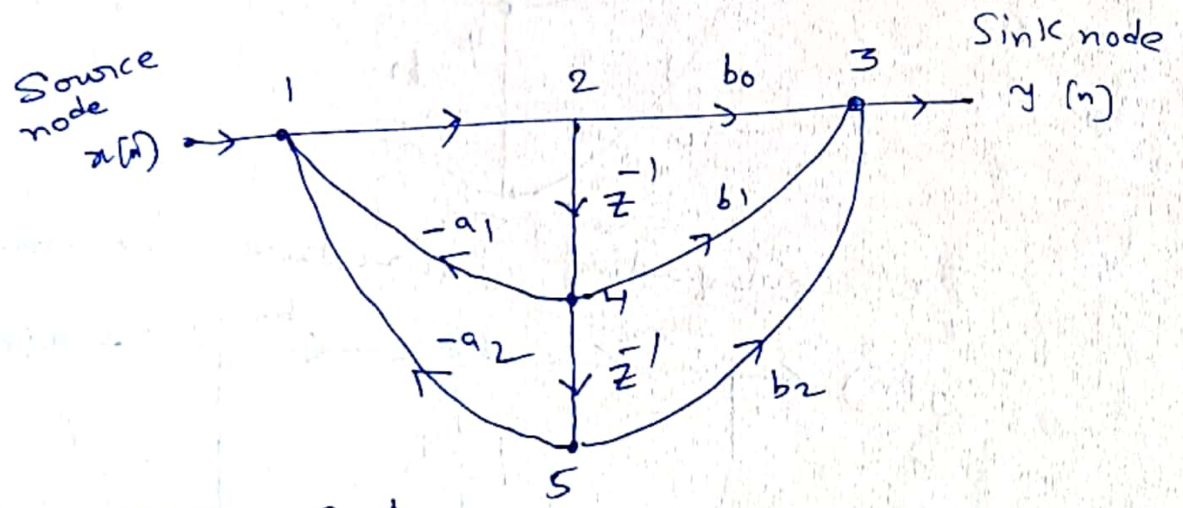
* Signal flow Graph (SFG)

A SFG is a graphical representation to a block diagram structure. The basic elements of SFG are branches & nodes. The SFG is basically a set of branches that connected at nodes.

Let us consider 2nd order filter structure shown below



and its SFG is shown below

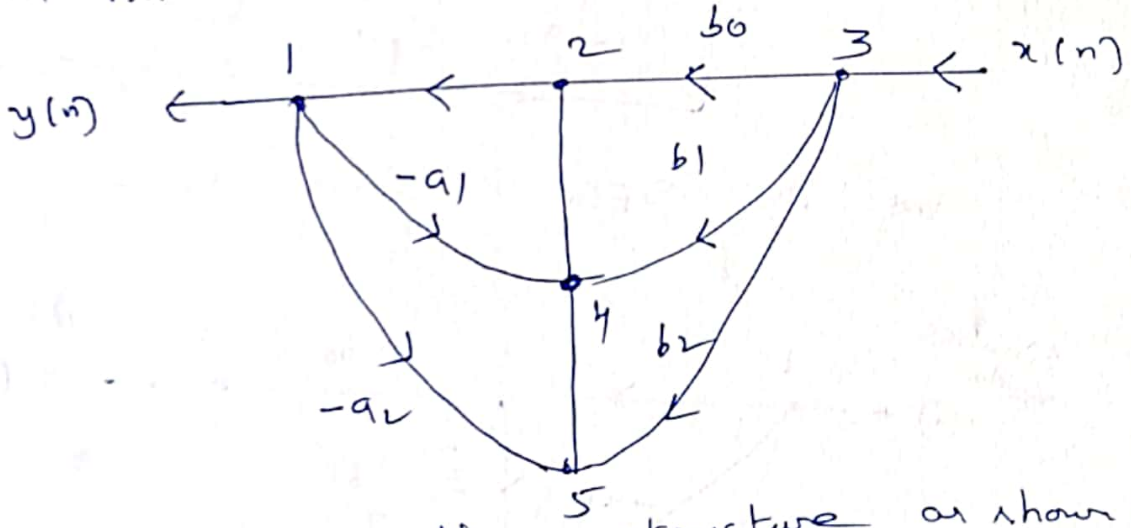


The above SFG contains 5 nodes out of which two (2) nodes are summing (1 & 3) nodes while other three (3) nodes represent branching (2, 4, 5) nodes.

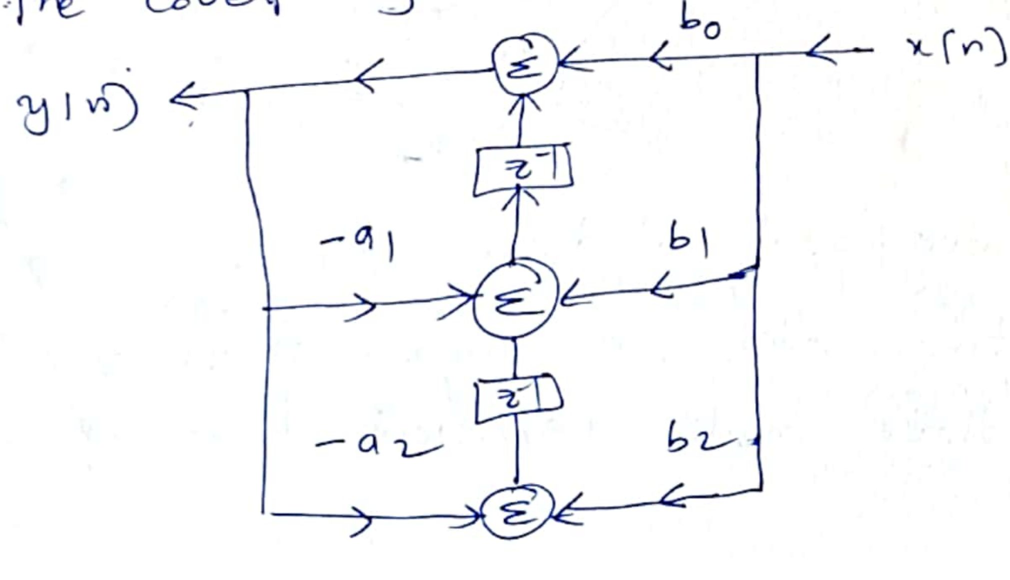
One technique that is useful in deriving new system structure for IIR systems from the transposition (or) flow-graph reversal theorem.

This theorem simply states that if we reverse the direction of all branch transmittances and interchange the I/P & O/P in the flow graph & (summing as branching & vice versa) and the s/m remains unchanged. The resulting structure is called transposed structure (or) transposed form.

The transposition of the above SFG is shown below



The corresponding structure as shown below



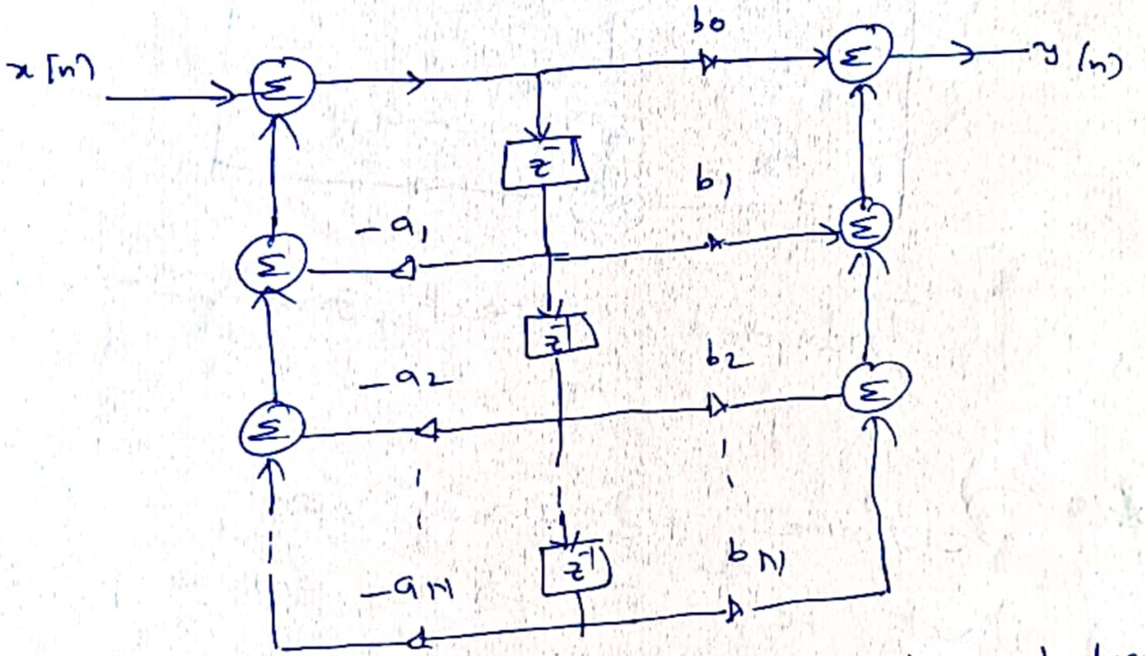
* Transposed form

→ Reverse all the signal flow direction.

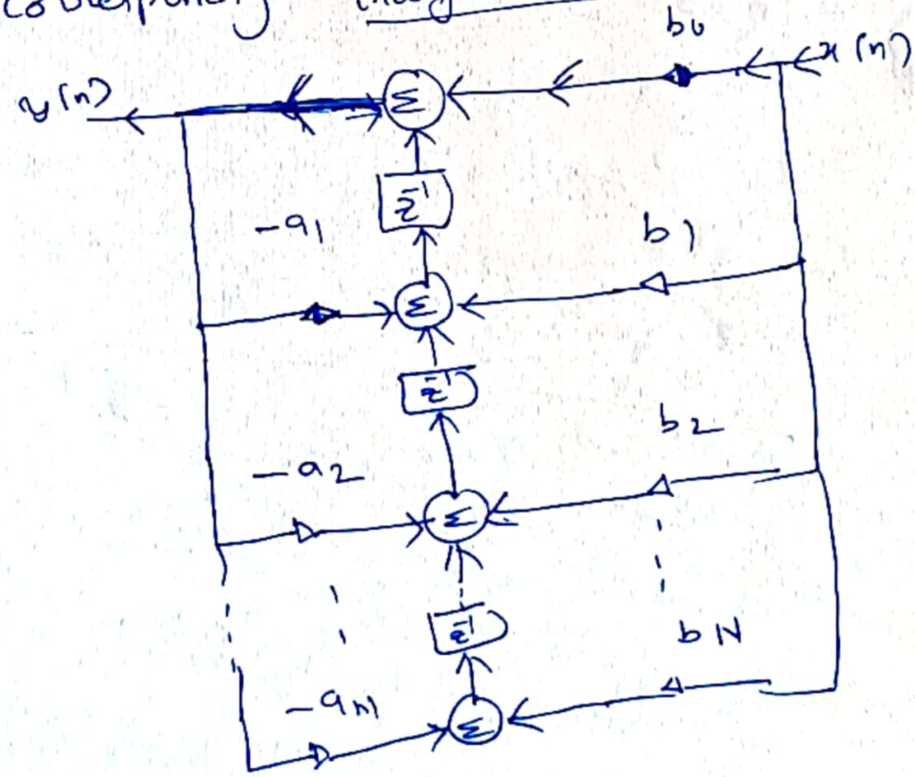
→ change nodes into summing & summing into nodes.

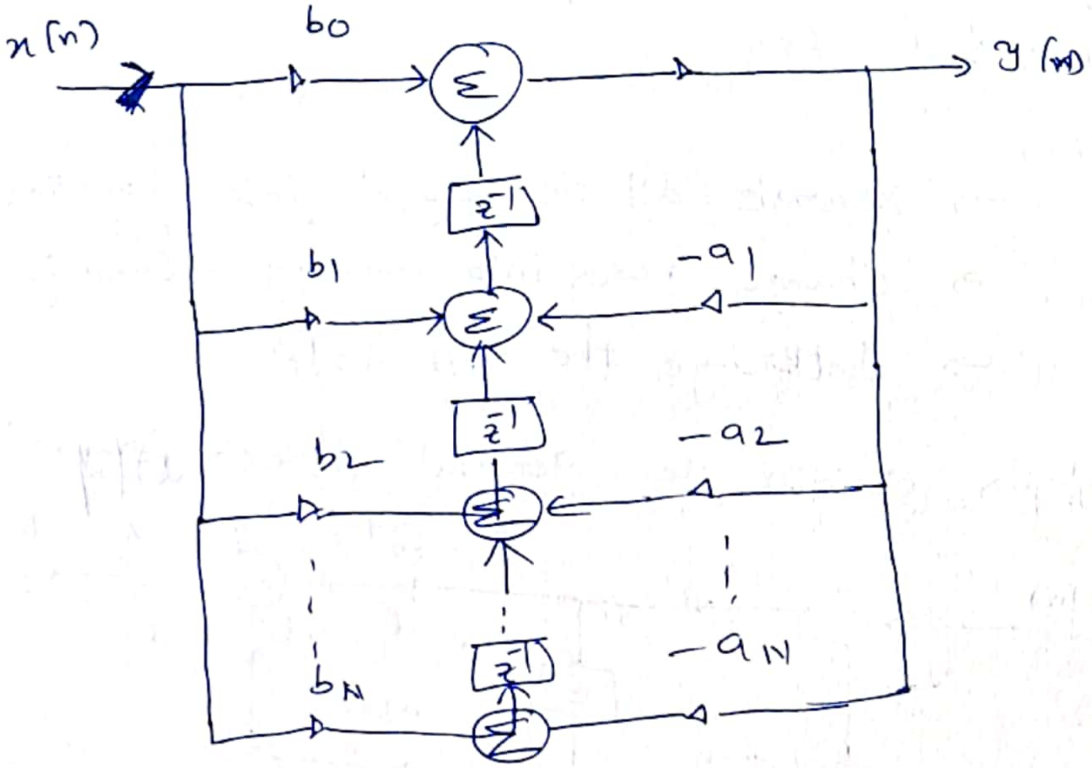
→ Interchange the I/P & O/P

Let us consider the general direct form-II structure.



The corresponding transposed form shown below





Transposed direct form - II realization

* Determine the direct form - II & Transposed direct form - II for the following given sln

$$y[n] = \frac{1}{2} y[n-1] - \frac{1}{4} y[n-2] + x[n] + x[n-1]$$

Sol:

Apply Z.T

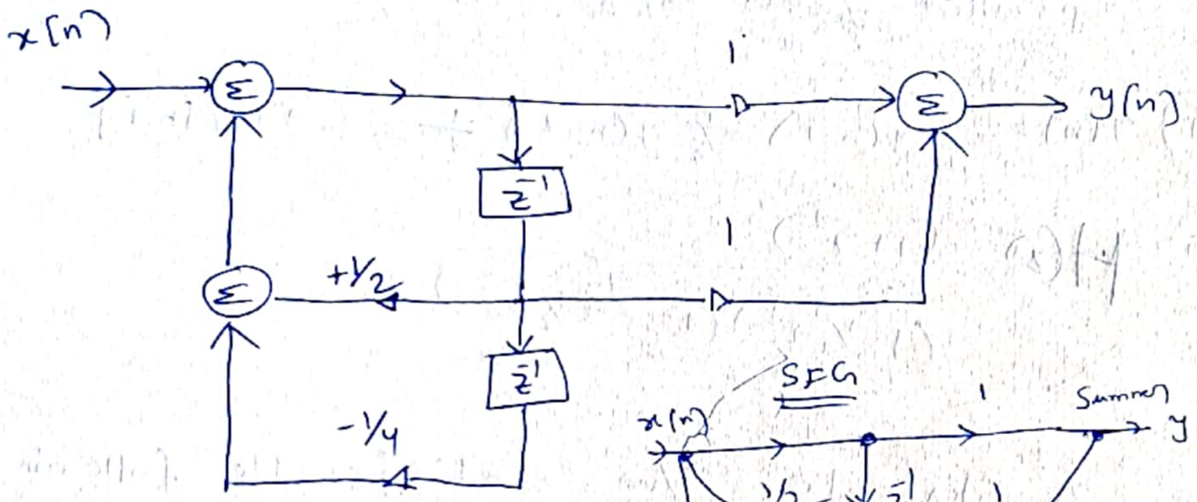
$$Y(z) = \frac{1}{2} z^{-1} Y(z) - \frac{1}{4} z^{-2} Y(z) + X(z) + z^{-1} X(z)$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2}}}$$

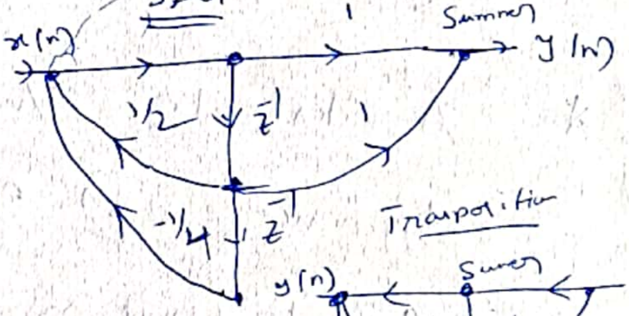
where $b_0 = 1$, $b_1 = 1$

$a_1 = -\frac{1}{2}$ & $a_2 = \frac{1}{4}$

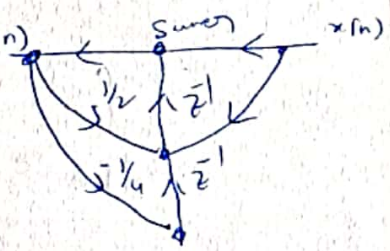
direct form - II



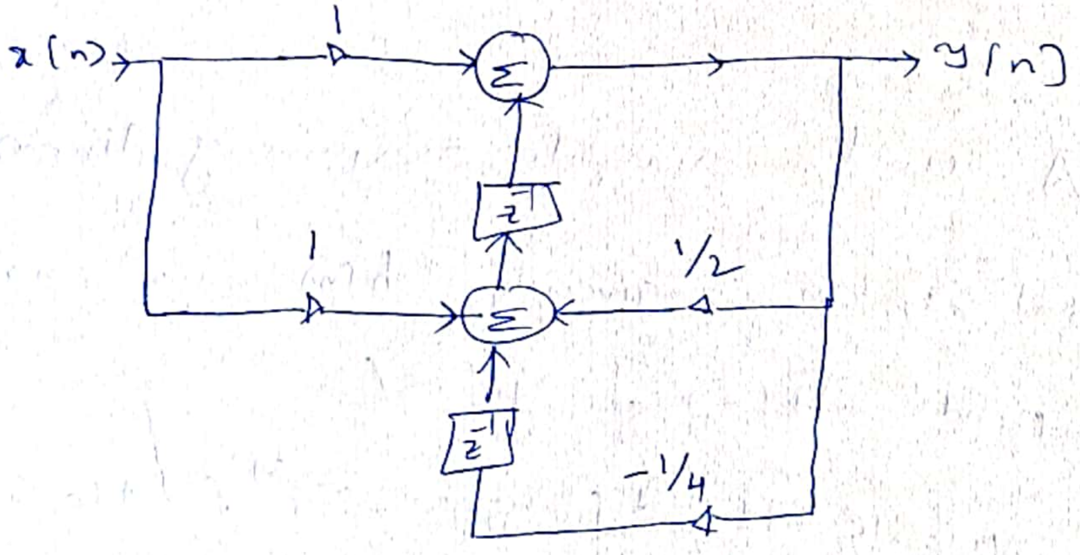
SFG



Transposition



Transposed form



* Obtain the transposed direct form - II for the following s/m

$$y[n] = \frac{1}{2} y[n-1] + \frac{1}{4} y[n-2] + x[n] + \frac{1}{2} x[n-1]$$

* obtain the direct form, Cascade form & parallel form realization of

i) $y[n] = -\frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x[n] + x[n-1]$

ii) $H(z) = \frac{(1+z^{-1})^3}{(1-\frac{1}{4}z^{-1})(1-z^{-1}+\frac{1}{2}z^{-2})}$

* A discrete s/m satisfies the following difference eqn $y[n] - 7y[n-1] + 10y(n-2) = x(n) - 2x(n-1)$ find its unit sample response & realise the system with minimum no of delay units

* A unit sample response of linear digital system is given by $h[n] = na^n ; n \geq 0 ; |a| < 1$

1) Find $H(z)$ of the s/m
2) write the difference eqn for the s/m
3) Sketch the realization diagram of the s/m with min no of delay units.

Scanned with CamScanner

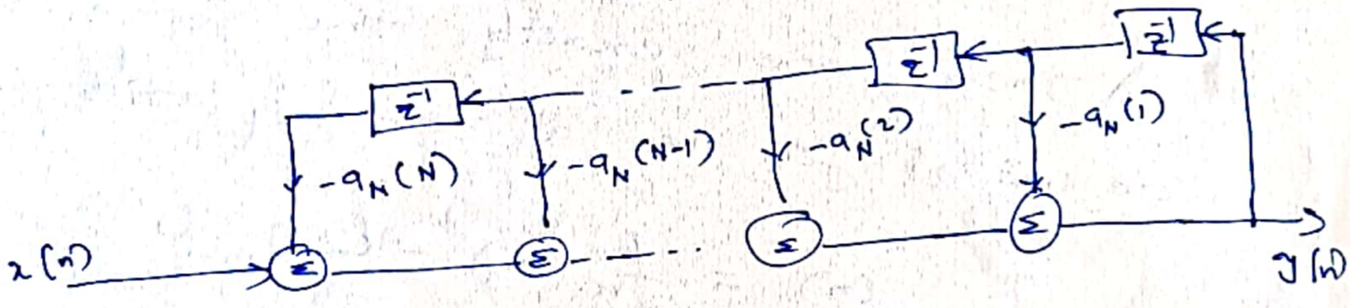
Lattice structure for IIR Systems

Consider an all pole s/m with system function

$$H(z) = \frac{1}{1 + \sum_{K=1}^{N} a_N(K) z^{-K}} = \frac{1}{A_N(z)}$$

The difference eq^n for this s/m

$$y[n] = x[n] - \sum_{K=1}^{N} a_N(K)\, y(n-K) \quad (1)$$



If we interchange the role of I/P & O/P then

$$x[n] = y[n] - \sum_{K=1}^{N} a_N(K)\, x(n-K)$$

$$\Rightarrow y[n] = x[n] + \sum_{K=1}^{N} a_N(K)\, x(n-K) \quad (2)$$

The above eq^n describes an FIR s/m having the s/m function $H(z) = A_N(z)$

The s/m described in eq^n (1) represents an IIR s/m with s/m function $H(z) = \frac{1}{A_N(z)}$

Thus one s/m can be obtained from the other simply by interchanging roles of the i/p & o/p.

Consider a all-zero lattice filter & then redefine the input as

$x[n] = f_N(n)$ & o/p as

$y[n] = f_0[n]$

these are exactly the opposite definitions for the all zero lattice filter

$$f_{m-1}(n) = f_m(n) - K_m g_{m-1}(n-1) \; ; \; m = N, N-1, \ldots 1$$

$g_m(n)$ remains unchanged

The result of these changes is the set of equations

$$f_N(n) = x[n]$$

$$f_{m-1}(n) = f_m(n) - K_m g_{m-1}(n-1) \; ; \; m = N, N-1, \ldots 1$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1) \; ; \; m = N, N-1, \ldots 1$$

$$y[n] = f_0[n] = g_0(n)$$

which corresponds to the structure shown below

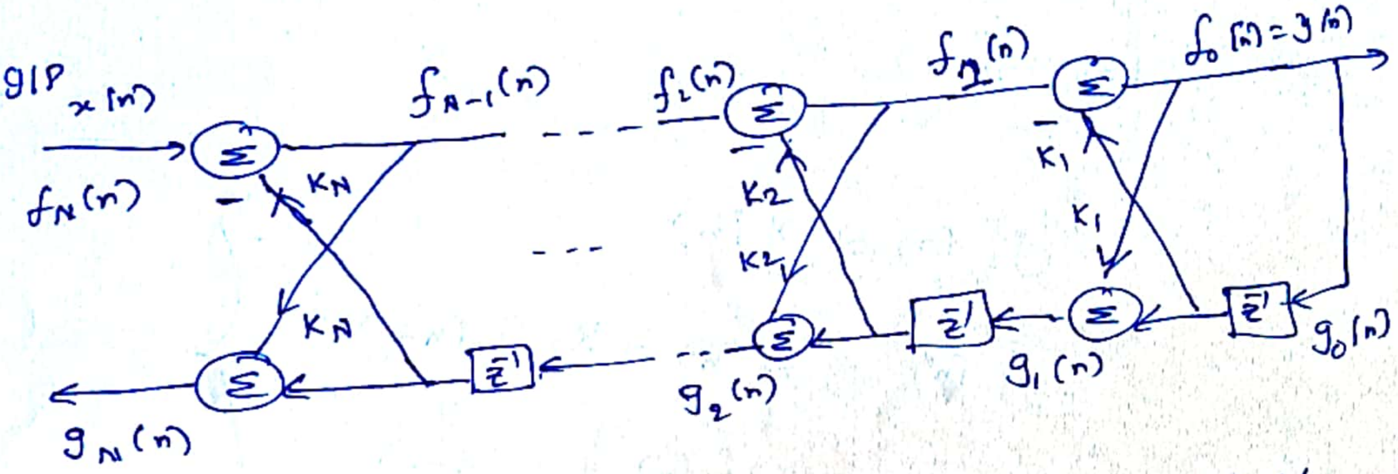


fig: Lattice Structure for all pole IIR s/m

Let us consider $N=1$

$$\left.\begin{aligned} x[n] &= f_1(n) \\ f_0(n) &= f_1(n) - k_1 g_0(n-1) \\ g_1(n) &= k_1 f_0(n) + g_0(n-1) \\ y(n) &= f_0(n) \end{aligned}\right\} \quad (3)$$

$$= x(n) - k_1 y(n-1)$$

$$g_1(n) = k_1 y(n) + y(n-1) \quad (4)$$

eq^n (3) represent 1st order all pole IIR s/m

while eq^n (4) represent 1st order FIR s/m

Comparing with

$$y(n) = x(n) - a_1(1)\, y(n-1)$$

$$\boxed{k_1 = a_1(1)}$$

Let $N = 2$

$$\left.\begin{aligned} f_2(n) &= x(n) \\ f_1(n) &= f_2(n) - K_2\, g_1(n-1) \\ g_2(n) &= K_2 f_1(n) + g_1(n-1) \\ f_0(n) &= f_1(n) - K_1 g_0(n-1) \\ g_1(n) &= K_1 f_0(n) + g_0(n-1) \\ y(n) &= f_0(n) = g_0(n) \end{aligned}\right\} \quad — (5)$$

After some simple substitution & manipulation

$$y(n) = -K_1(1+K_2)\, y(n-1) - K_2\, y(n-2) + x(n) \quad — (6)$$

$$g_2(n) = K_2\, y(n) + K_1(1+K_2)\, y(n-1) + y(n-2) \quad — (7)$$

The above eq^n (6) represents two-pole IIR S/m

eq^n (7) represents i/p - o/p eq^n for two-zero FIR S/m

∴ by Comparing with

$$y(n) = x(n) - a_2(1)\, y(n-1) - a_2(2)\, y(n-2)$$

$$\boxed{a_2(0) = 1, \quad a_2(1) = K_1(1+K_2) \;\&\; a_2(2) = K_2}$$

* Conversion from lattice structure to direct form

Consider —

for $N = 3$

$$y[n] = x[n] - a_3(1)\,y(n-1) - a_3(2)\,y(n-2) - a_3(3)\,y(n-3)$$

from the lattice structure

$$y[n] = f_0[n] = g_0[n]$$

$$x[n] = f_3[n] = f_2[n] + K_3\, g_2(n-1)$$

$$g_3(n) = K_3 f_2(n) + g_2(n-1)$$

sub $m = 2$

$$f_2[n] = f_1[n] + K_2\, g_1(n-1)$$

$$g_2(n-1) = K_2 f_1(n-1) + g_1(n-2)$$

$$x[n] = \underbrace{f_2[n]} + K_3\, g_2(n-1)$$

$$x[n] = f_1[n] + K_2\, g_1(n-1) + K_3\,[K_2 f_1(n-1) + g_1(n-2)]$$

$$= f_0(n) + K_1 g_0(n-1) + K_2[K_1 f_0(n-1) + g_0(n-2)]$$
$$+ K_3 K_2\,[f_0(n-1) + K_1 g_0(n-2)]$$
$$+ K_3\,[K_1 f_0(n-2) + g_0(n-3)]$$

$$= y(n) + [K_1(1+K_2) + K_2 K_3]\, y(n-1)$$
$$+ [K_2 + K_1 K_3(1+K_2)]\, y(n-2) + K_3\, y(n-3)$$

$$= y(n) + [a_2(1) + a_2(2)\, a_3(3)]\, y(n-1)$$
$$+ [a_2(2) + a_2(1)\, a_3(3)]\, y(n-2)$$
$$+ a_3(3)\, y(n-3)$$

$$x(n) = y(n) + a_3(1)\, y(n-1) + a_3(2)\, y(n-2) + a_3(3)\, y(n-3)$$

$$a_3(0) = 1$$
$$a_3(1) = a_2(1) + a_2(2)\, a_3(3)$$
$$a_3(2) = a_3(2) + a_2(1)\, a_3(3)$$
$$K_3 = a_3(3)$$

In general

$$\boxed{\begin{aligned} a_m(0) &= 1\,; \\ a_m(k) &= a_{m-1}(k) + a_m(m)\, a_{m-1}(m-k) \\ a_m(m) &= K_m \end{aligned}}$$

These are used to convert lattice structure to direct form

* Conversion from direct form to lattice structure

Consider

For 3-stage IIR Sm

$$x[n] = f_3[n] = f_2[n] + k_3 g_2[n-1] \quad (1)$$

$$g_3[n] = k_3 f_2[n] + g_2[n-1]$$

from which

$$g_2[n-1] = g_3[n] - k_3 f_2[n]$$

$$(1) \Rightarrow f_3[n] = f_2[n] + k_3 [g_3[n] - k_3 f_2[n]]$$

$$\Rightarrow f_2[n] = \frac{f_3[n] - k_3 g_3[n]}{1 - k_3^2}$$

$$= \frac{y[n] + a_3(1) y(n-1) + a_3(2) y(n-2) + a_3(3) y(n-3) - k_3 a_3(3) y(n) - k_3 a_3(2) y(n-2) - k_3 a_3(2) y(n-1) - a_3(3) y(n-3)}{1 - k_3^2}$$

$$= y[n] + \frac{a_3(1) - a_3(3) a_3(2)}{1 - a_3^2(3)} y(n-1) + \frac{a_3(2) - a_3(2) a_3(3)}{1 - a_3^2(3)} y(n-2)$$

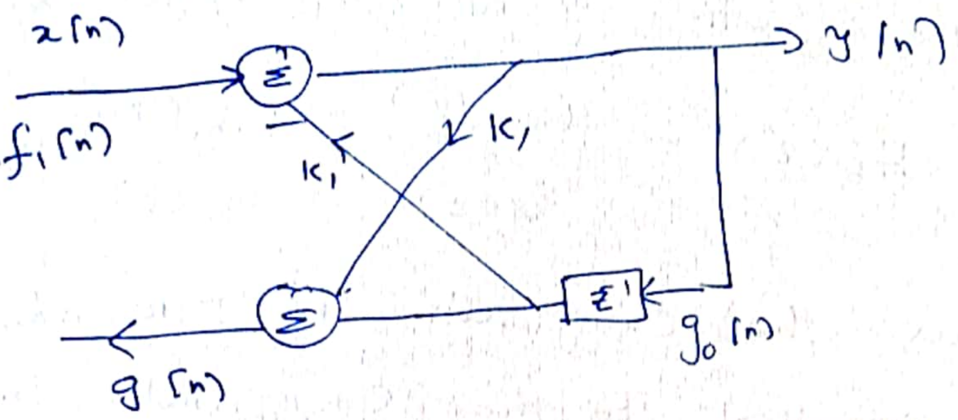


fig: Single pole lattice S/m

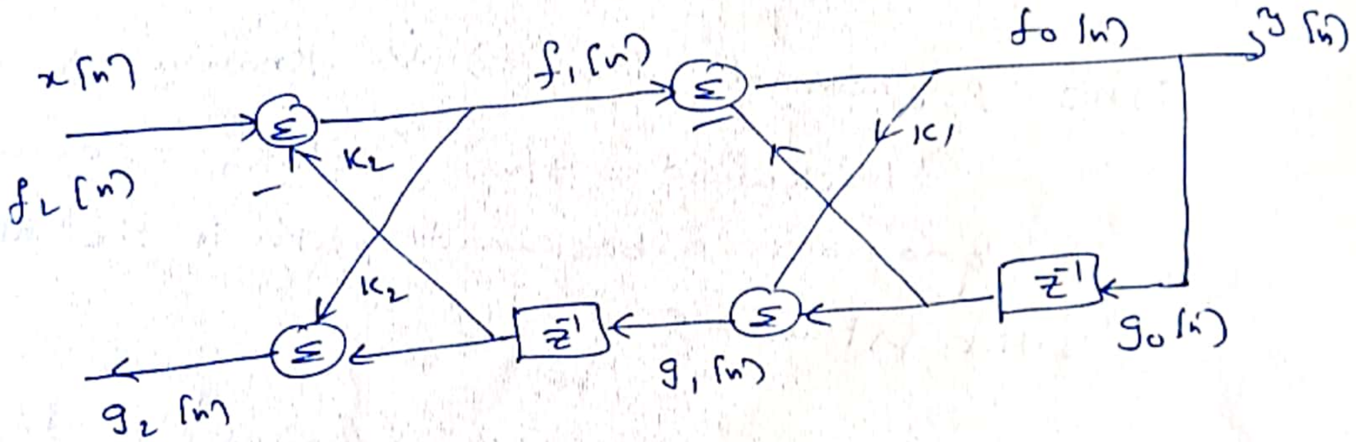


fig: Two pole lattice S/m

The coefficient for the FIR S/m are identical to those in the IIR S/m except that they occur in reverse order

for any N, the S/m function for all-pole IIR S/m

$$H_a(z) = \frac{Y(z)}{X(z)} = \frac{F_0(z)}{F_m(z)} = \frac{1}{A_m(z)}$$

The s/m function of all zero (FIR) s/m is

$$H_b(z) = \frac{G_m(z)}{Y(z)} = \frac{G_m(z)}{G_0(z)} = B_m(z) = z^{-m} A_m(z^{-1})$$

thus the coefficients in FIR S/m $H_b(z)$ are identical to the coefficients in $A_m(z)$ except that they occur in reverse order.

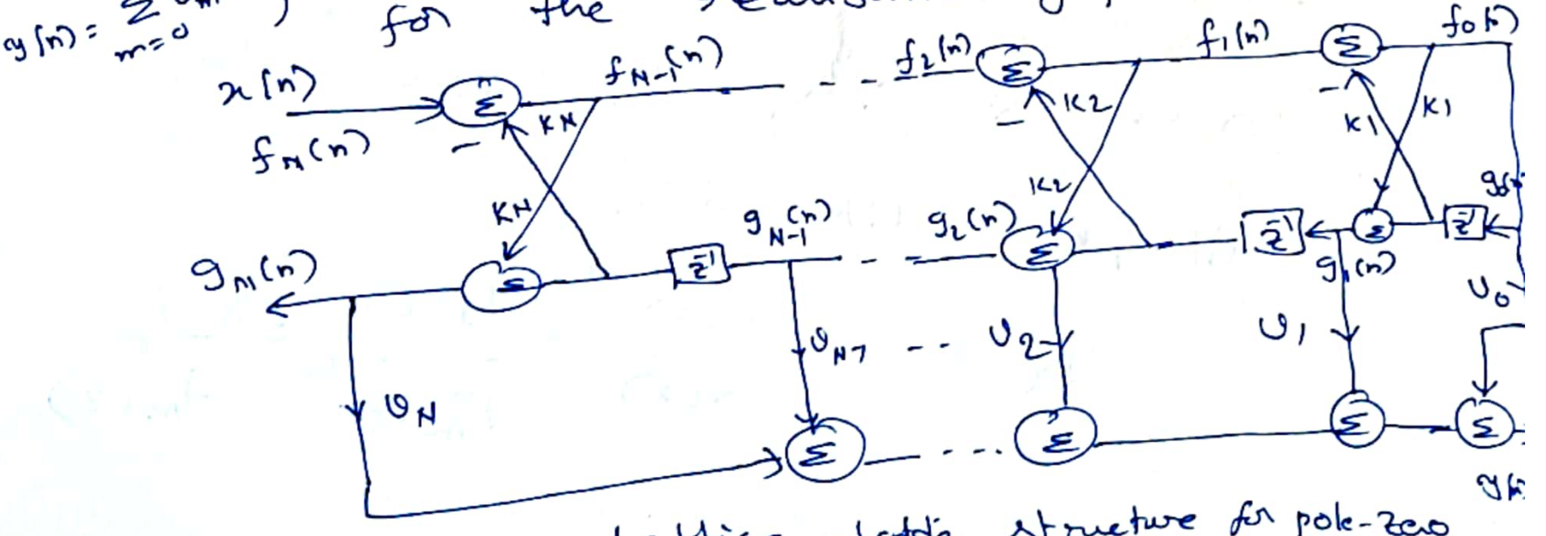
The polynomial $B_m(z)$ is known as backward s/m function because it provides a backward path in the all-pole lattice structure

Consider an IIR S/m with S/m function

$$H(z) = \frac{\sum_{k=0}^{M} C_M(k) z^{-k}}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}} = \frac{C_M(z)}{A_N(z)}$$

$y(n) = \sum_{m=0}^{M} v_m g_m(n)$

If $M = N$ the lattice - ladder structure for the realisation of pole-zero S/m



Lattice - ladder structure for pole-zero s/m

Where $\{\upsilon_m\}$ are the parameters that determine the zeros of the s/m

$$H(z) = \sum_{m=0}^{M} \upsilon_m \frac{G_m(z)}{X(z)}$$

$\because$ $X(z) = F_N(z)$ & $F_0(z) = G_0(z)$

$$H(z) = \sum_{m=0}^{M} \upsilon_m \frac{G_m(z)}{G_0(z)} \cdot \frac{F_0(z)}{F_N(z)}$$

$$= \sum_{m=0}^{M} \upsilon_m \frac{G_m(z)}{G_0(z)} \cdot \frac{1}{\frac{F_N(z)}{F_0(z)}}$$

$$= \sum_{m=0}^{M} \upsilon_m \frac{B_m(z)}{A_N(z)}$$

$$= \frac{\sum_{m=0}^{M} \upsilon_m B_m(z)}{A_N(z)}$$

Comparing the above eq^n

$$H(z) = \frac{\sum_{K=0}^{M} C_M(K) z^{-K}}{1 + \sum_{K=1}^{N} a_N(K) z^{-K}} = \frac{C_M(z)}{A_N(z)}$$

we get $C_M(z) = \sum_{m=0}^{M} \upsilon_m B_m(z)$

The ladder parameters are determined from the above eq^n which can be expressed as

$$C_m(z) = \sum_{k=0}^{m-1} \vartheta_k B_k(z) + \vartheta_m B_m(z)$$

or) $$C_m(z) = C_{m-1}(z) + \vartheta_m B_m(z)$$

lattice ladder filter structures requires minimum amount of memory.

Used in many practical applications including speech processing systems, adaptive filtering & geophysical signal processing

$\star$ Realise the sln using lattice structure

$$H(z) = \frac{1}{1 + \frac{2}{5} z^{-1} + \frac{3}{4} z^{-2} + \frac{1}{8} z^{-3}}$$

$\therefore$ $m = 3$; $K_0 = a_3(0) = 1$

$a_3(1) = 2/5$

$a_3(2) = \frac{3}{4}$

$K_3 = a_3(3) = \frac{1}{3}$

$m=3$

$\underline{k=1}$

$$a_2(1) = \frac{a_3(1) - k_3 a_3(2)}{1 - k_3^2}$$

$$= \frac{\frac{2}{5} - \frac{1}{3} \cdot \frac{3}{4}}{1 - \frac{1}{9}} = 0.16875$$

$\underline{k=2}$

$$a_2(2) = \frac{a_3(2) - k_3 a_3(2)}{1 - k_3^2}$$

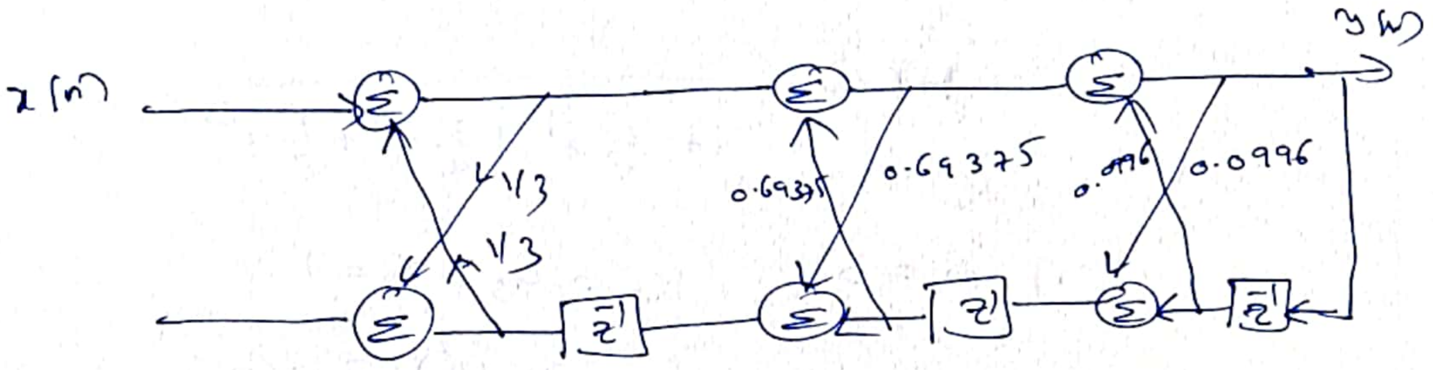
$$K_2 = a_2(2) = \frac{\frac{3}{4} - \frac{1}{2} \cdot \frac{2}{3}}{1 - 1/9} = 0.69375$$

$m=2$

$\underline{k=1}$

$$a_1(1) = \frac{a_2(1) - k_2 a_2(1)}{1 - k_2^2}$$

$K_1 = a_1(1) = 0.0996$


x(n)
y(n)
1/3
1/3
0.69375
0.69375
0.0996
0.0996
z^{-1}
z^{-1}
z^{-1}

* Sketch the lattice - ladder realisation for the s/m

$$H(z) = \frac{1-0.8z^{-1}+0.15z^{-2}}{1+0.1z^{-1}-0.72z^{-2}} = \frac{B(z)}{A(z)}$$

Sol: For all pole s/m $\frac{1}{A(z)}$, we have

$K_2 = -0.72$

$K_1(1+K_2) = 0.1$

$\Rightarrow K_1 = 0.357$

For all zero s/m $C_2(z) = 1-0.8z^{-1}+0.15z^{-2}$

$A_2(z) = 1-0.8z^{-1}+0.15z^{-2}$

$B_2(z) = 0.15-0.8z^{-1}+z^{-2}$

$K_2 = 0.15$

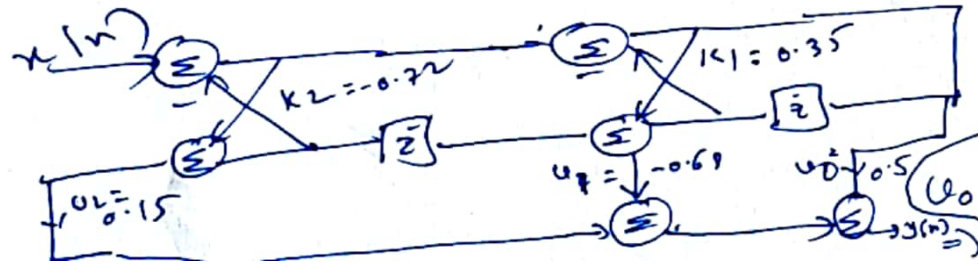
$$A_1(z) = \frac{A_2(z)-K_2B_2(z)}{1-K_2^2} = 1-0.696z^{-1}$$

$B_1(z) = -0.696+z^{-1}$ $\therefore K_1 = 0.696$

$A_0(z) = B_0(z) = 1$

$$C_2(z) = \sum_{m=0}^{2} v_m B_m(z) = v_0 + v_1 B_1(z) + v_2 B_2(z)$$

$= 1-0.8z^{-1}+0.15z^{-2}$



$v_2 = 0.15$

$v_1 - 0.18v_2 = -0.8$

$v_0 - 0.696v_1 + 0.15v_2 = 1$

$\Rightarrow v_1 = -0.68,\ v_0 = 0.5$

* Realise the s/m using lattice structure

$$H(z) = \frac{1+2z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{4}z^{-2}}$$

Sol..

$a_2(0) = 1$

$a_2(1) = \frac{3}{4}$

$a_2(2) = K_2 = \frac{1}{4}$

$m=2$
$k=1$

$$a_1(1) = \frac{a_2(1) - K_2 a_2(1)}{1-K_2^2}$$

$$= \frac{\frac{3}{4} - \frac{1}{4}\cdot\frac{3}{4}}{1-(1/4)^2}$$

$K_1 = a_1(1) = 0.6$