| COURSE MATERIAL | |
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| SUBJECT | NUMERICAL METHODS AND PROBABILITY THEORY (20A54402) |
| UNIT | IV |
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## 8. LECTURE NOTES

## 4.1 INTRODUCTION

Suppose a coin is tossed. The toss may result in the occurrence of 'Head' or in the occurrence of 'Tail'. Here, the chances of head and tail are equal. In other words, the probability of occurrence of head is ½ and the probability of occurrence of tail is ½ Thus, Probability is a numerical measure which indicates the chance of occurrence.

### RANDOM EXPERIMENT:

There are two types of experiments. They are

(i) Deterministic experiment and
(ii) Random experiment.

A deterministic experiment, when repeated under the same conditions, results in the same outcome. It has a unique outcome.

Random experiment is an experiment which may not result in the same outcome when repeated under the same conditions. It is an experiment which does not have a unique outcome.

For example:

1. The experiment of 'Toss of a coin' is a random experiment. It is so because when a coin is tossed the result may be 'Head' or it may be 'Tail'.

2. The experiment of 'Drawing a card randomly from a pack of playing cards' is a random experiment. Here, the result of the draw may be any one of the 52 cards.

### SAMPLE SPACE

The set of all possible outcomes of a random experiment is the Sample space. The sample space is denoted by S. The outcomes of the random experiment (elements of the sample space) are called sample points or outcomes or cases.

A sample space with finite number of outcomes is a finite sample space. A sample space with infinite number of outcomes is an infinite sample space.

Ex1. While throwing a die, the sample space is

S = {1, 2, 3, 4, 5, 6}. This is a finite sample space.

Ex2. While tossing two coins simultaneously, the sample space is

S = {HH, HT, TH, TT}. This is a finite sample space.

Ex3. Consider the toss of a coin successively until a head is obtained. Let the number of tosses be noted. Here, the sample space is

S= {1, 2, 3,4....} This is an infinite sample space

**EVENT:**

Even is a subnet of the sample space. Events are denoted by A, B, C etc.

- An event which does not contains any outcome is a null event (impossible event). It is denoted by Φ.
- An event which has only one outcome is an ELEMENTARY EVENT OR SIMPLEEVENT.
- An event which has more than outcome is a compound event.
- An event which contains all the outcomes is equal to the sample and it is called sure event or certain event.

Ex.1. While throwing a die, A= {2,4,6} is an event. It is the event that the throw results in an even number. Here, A is a compound event.

Ex.2. While tossing two coins, A={TT} is an event. It is the event that the toss results in two tails. Here, A is a simple event.

The outcomes which belong to an event are said to be favourable to that event. The event happens whenever the experiment results in a favourable outcome. Otherwise, the event does not happen

While throwing a die, the event A = {2,4,6} has three favourable outcomes, namely, 2,4 and 6. Where the throw results in 2,4 or 6, event A occurs.

**COMPLEMENT OF AN EVENT:**

Let A be an event. Then, Complement of A is the event of non-occurrence of A. It is the event constituted by the outcomes which are not favourable to A. The complement of A is denoted by A′ or $\bar{A}$ or Ac.

While throwing a die, If A = {2,4,6}, its complement is A′ = {1,3,5}. Here, A is the event that throw result in an even number. A′ is the event that throw does not result in an even number. That is, A′ is the event that throw result in an odd number.

**EQUALLY LIKELY EVENTS (Equiprobable events)**

Two or more events are equally likely if they have equal chance of occurrence. That is, equally likely events are such that none of them has greater chance of occurrence than the others.

Ex. 1. While tossing a fair coin, the outcomes 'Head' and 'Tail' are equally likely.

Ex.2. While throwing a fair die, the events A= {2,4,6}, B = {1,3, 5} &C= {1,2, 3} are equally likely.

A sample space is called an equiprobable space if the outcomes are equally likely. For instance, the sample space S = {1, 2, 3, 4, 5, 6} of throw of a fair die is equiprobable space because the six outcomes are equally likely.

**MUTUALLY EXCLUSIVE EVENTS (Disjoint events)**

Two or more events are mutually exclusive if only one of them can occur at a time. That is, the occurrence of any of these events totally excludes the occurrence of the other events. Mutually exclusive events cannot occur together.

Ex. 1. While tossing a coin, the outcomes 'Head1 and 'Tail' are mutually exclusive because when the coin is tossed once, the result cannot be Head as well as Tail.

Ex.2. While throwing a die, the events A = {2, 4, 6}, B= {3,5} and C = {1} are mutually exclusive.

If A is an event, A and A' are mutually exclusive. It should be noted that intersection of mutually exclusive events is a null event.

**EXHAUSTIVE EVENTS (Exhaustive set of events)**

A set of events is exhaustive if one" or the other of the events in the set occurs whenever the experiment is conducted.

That is, the set of events exhausts all the outcomes of the experiment

The union of exhaustive events is equal to the sample space.

Ex.1. While throwing a die, the six outcomes together are exhaustive. But here, if any one of these outcomes is left out, the remaining five outcomes are not exhaustive.

Ex.2. While throwing a die, events A = {2,4, 6}, B = {3, 6} and C = {1,5,6} together are exhaustive.

**CLASSICAL DEFINITION:**

Let a random experiment have 'n' equally likely, mutually exclusive and exhaustive outcomes. Let 'm' of these outcomes be favorable to an event A. Then, probability of A is —

$$P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}} = \frac{m}{n}$$

**Limitations of classical definition:**

This definition is applicable only when

(i) The outcomes are equally likely, mutually exclusive and exhaustive.

(ii) The number of outcomes n is finite.

## 4.2 THE AXIOMATIC APPROACH

Consider a random experiment with sample space S. Associated with this random experiment, many events can be defined. Let for every event A, a real number P(A) be assigned. Then, P(A) is the probability of event A, if the following axioms are satisfied.

Axiom 1: $P(A) \geq 0$

Axiom 2: P(S) = 1, S being the sure event or Sample space.

Axiom 3: For two mutually exclusive events A and B,

$P(A \cup B) = P(A) + P(B)$

Note that the third axiom can be generalised for any number of mutually exclusive events.

RESULT 1:

P(A) is a value between 0 and 1. That is, 0 < P(A) < 1.

Proof:

Let a random experiment have 'n' equally likely, mutually exclusive and exhaustive outcomes. Let 'm' of these outcomes be favourable to event A.

Then P(A) = = $\frac{\mathrm{m}}{\mathrm{n}}$

Here, the least possible value of m is 0. Also, the highest possible value of m is n.

And so, 0 ≤ m ≤ n.

$$\frac{0}{n} \leq \frac{m}{n} \leq \frac{n}{n}$$

$$\Rightarrow 0 \leq p(A) \leq 1$$

Thus, P(A) is a value between 0 and 1.

RESULT 2:

P(A') = 1 - P(A). That is, P(A) = 1 - P(A').

Proof:

In a random experiment with n equally likely, mutually exclusive and exhaustive outcomes, if m outcomes are favourable to event A, the remaining (n-m) outcomes are favourable to the complementary event A'. Therefore,

P(A1) = $\frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - P(A)$

Thus, P(A') = 1 - P(A). That is, P(A) = 1 - P(A').

1. Find the probability that a throw of an unbiased die results in

(i) an One (ii) an even number (iii) a multiple of 3.

Sol: The sample space is S = {1,2,3,4,5,6]. There is n = 6 equally likely, mutually exclusive and exhaustive outcomes. Let events A, Band C be—

A: throw results in an ace (number 1)

B: throw results in an even number

C: throw results in a multiple of 3

(i) Event A has one favourable outcome.

∴P[ace] = P(A)= m/n = 1/6

(ii) Event B has 3 favourable outcomes, namely, 2, 4 and 6.

∴P [even number] = P(B) = m/n = 3/6= ½

(iii) Event C has 2 favourable outcomes, namely, 3 and 6

P [multiple of 3] = P(C)= m/n = 2/6 = 1/3

2. A bag contains 3 white, 4 red and 2 green balls. One ball is selected at random from the bag. Find the probability that the selected ball is

(i) white (ii) non-white (iii) white or green.

Sol: The bag totally has 9 balls. Since the ball drawn can be any one of them, there are 9 equally likely, mutually exclusive and exhaustive outcomes. Let events A, B and C be

A: selected ball is white

B: selected ball is non-white

C: selected ball is white or green

(i) There are 3 white balls in the bag. Therefore, out of the 9 outcomes, 3 are favourable to event A.

∴P [white ball] = P(A) = 3/9 = 1/3

(ii) Event B is the complement of event A. Therefore,

∴ P (non-white ball) = P(B) = 1 - P(A) = 1 – 1/3 = 2/3

(iii) There are 3 white and 2 green balls in the bag. Therefore, out of 9 outcomes, 5 are either white or green.

∴ P [white or green ball] = P(C) = 5/9

3. One card is drawn from a well-shuffled pack of playing cards. Find the probability that the card drawn (i) is a Heart (ii) is a King (iii) belongs to red suit (iv) is a King or a Queen (v) is a King or a Heart.

Sol:

A pack of playing cards has 52 cards. There are four suits, namely, Spade, Club, Heart and Diamond (Dice). In each suit, there are thirteen denominations - Ace (1), 2, 3, 10, Jack (Knave), Queen and King.

A card selected at random may be any one of the 52 cards. Therefore, there are 52 equally likely, mutually exclusive and exhaustive outcomes. Let events A, B, C, D and E be —

A: selected card is a Heart

B: selected card is a King

C: selected card belongs to a red suit.

D: selected card is a King or a Queen

E: selected card is a King or a Heart

(i) There are 13 Hearts in a pack. Therefore, 13 outcomes are favourable to event A.

$\therefore$ P [Heart] = P(A) =13/52 = ¼

(ii) There are 4 Kings in a pack. Therefore, 4 outcomes are favourable to event B.

$\therefore$ P[King] = P(B)=4/52 =1/13

(iii) There are 13 Hearts and 13 Diamonds in a pack. Therefore, 26 outcomes are favorable to event C.

$\therefore$ P [Red card] = P(C) =26/52 = ½

(iv) There are 4 Kings and 4 Queens in a pack. Therefore, 8 outcomes are favourable to event D.

$\therefore$ P [King or Queen] = P(D) = 8/52 = 2/13

(v) There are 4 Kings and 13 Hearts in a pack. Among these, one card is Heart-King. Therefore, (4+13-1) = 16 outcomes are favourable to event E.

$\therefore$ P [King or Heart] = P(E) =16/52 = 4/13

4. A bag contains 8 tickets which are marked with the numbers 1,2, 3, .. 8. Find the probability that a ticket drawn at random from the bag is marked with (i) an even number (ii) a multiple of 3.

Sol: The selection can be any one of the eight numbers. Therefore, there are 8 equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be—

A: selected number is even.

B: selected number is a multiple of 3.

(i) Four of the selections, namely, 2, 4, 6 and 8 are favourable to event A.

$\therefore$ P [even number] = P(A) = 4/8 = ½

(ii) Two of the selections, namely, 3 and 6 are favourable to event B.

∴ P [multiple of 3] = P(B) = 2/8 = ¼.

5. A fair coin is tossed twice. Find the probability that the tosses result in

(i) two heads (ii) at least one head.

Sol: The sample space is 5 = (HH, HT, TH, TT}, There are four equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be—

A: the tosses result in 2 heads

B: the tosses result in at least one head.

(i) One outcome, HH is favourable to event A.

∴ P [two heads] = P(A) = ¼

(ii) 3 outcomes HH, HT and TH are favourable to event B.

∴ P [at least one head] = P(B) = ¾

6. Two fair dice are rolled. Find the probability that (i) both the dice show number 6 (ii) the sum of numbers obtained is 7 or 10 (iii) the sum of the numbers obtained is less than 11 (iv) the sum is divisible by 3.

Sol: The sample space is S= {(1,1), (1, 2), (1,3)…… (1,6)

(2,1), (2, 2), (2,3) …. (2,6)

……………………………..

(6,1), {6, 2), (6, 3)…. (6,6)}

There are 6x6 = 36 equally likely, mutually exclusive and exhaustive outcomes.

Let events A, B, C and D be —

A: both the dice show number 6

B: sum of the numbers obtained is 7 or 10

C: sum of the numbers obtained is less than 11.

D: sum of the numbers obtained is divisible by 3.

(i) One outcome, namely, (6, 6) is favourable to event A.

∴P [6 on both the dice] = P(A) = 1/36

(ii) Nine outcomes, namely, (6,1), (5,2), (4, 3), (3,4), (2,5), (1,6), (6, 4), (5, 5) and (4,6) are favourable to event B.

∴P [sum is 7 or 10] = 9/36 = ¼

(iii) The complement of event C is— C': sum is 11 or 12.

Event C' has three favourable outcomes, namely, (6,5), (5, 6) and (6, 6).

P [sum is less than 11] = 1 – P [sum is 11 or 12]

= 1-3/36= 1-1/12 = 11/12

The sum is divisible by 3 if it is 3, 6, 9 or 12. Therefore, the outcomes favourable to event D are (2, 1), (1, 2), (5,1), (4,2), (3,3), (2, 4), (1,5), (6, 3), (5, 4), (4, 5), (3, 6) and (6, 6). Thus, 12 outcomes are favourable.

P [sum is divisible by 3] = 12/36 = 1/3.

**7.** A box has 5 white, 4 red and 3 green balls. Two balls are drawn at random from the box. Find the probability that they are (i) of the same colour (ii) of different colours.

Sol: The box totally has 12 balls. A random draw of two balls has $12C_2$ equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be—

A: the balls drawn are of the same colour

B: the balls drawn are of different colours.

(i). Events happens when the drawn balls are both white or both red or both green. Out of $12C_2$ selections, $5C_2$ selections are both white,$4C_2$ selections are both red And $3C_2$ selections are both green. Thus, $5C_2$+$4C_2$ + $3C_2$ outcomes are Favourable to event A.

P [balls of same colour] = $\frac{5C_2+{}^4C_2+{}^3C_2}{12C_2}$

$$= \frac{10+6+3}{66} = \frac{19}{66} = 0.2879$$

(ii). Event B is the complement of event A. Therefore,

P [balls of different colours] = 1 - P[same colour]

= 1- P(A)

= 1 - 19/66

= 47/66

**8.** Two cards are drawn at random from a pack of cards. Find the probability that (i) both are Spades (ii) both are Kings (iii) one is Spade and the other is a Heart (v) the cards belong to the same suit (v) the cards belong to different suits.

Sol:

A random draw of 2 cards from a pack of 52 cards has $52C_2$ equally likely, mutually exclusive and exhaustive outcomes. Let events A, B, C, D and E be—

A: both the cards drawn are Spades

B: both the cards drawn are Kings.

C: the cards drawn are one Spade and one Heart.

D: the cards belong to the same suit.

E: the cards belong to different suits.

(i) Since there are 13 Spades in a pack, event A has $13C_2$ favourable outcomes.

Therefore, P [both spades] = $\frac{13C_2}{52C_2} = \frac{13\times6}{26\times51} = \frac{1}{17}$

(ii) Since there are 4 Kings in a pack, event B has $4C_2$ favourable outcomes.

Therefore, P [both Kings] = $\frac{4C_2}{52C_2} = \frac{2\times6}{26\times51} = \frac{1}{221}$

(iii) Here, one card should be a Spade and the other should be a Heart.

From 13 Spades, one Spade can be had in $13C_1$ ways. From 13 Hearts, one Heart can be had in $13C_1$ ways. Thus, $13C_1$ X $13C_1$ outcomes are favourable to event C.

Therefore, P [a Spade and a Heart] = $\frac{13C_1\times{}^{13}C_1}{52C_2}$

$= \frac{13\times13}{26\times51} = \frac{13}{102}$

(iv) Here, the cards should be 2 Spades or 2 Clubs or 2 Hearts or 2 Diamonds. There are 13 cards of each suit. In each case, a selection of two cards can be made in $13C_2$ ways. Thus, totally the number of favourable cases is $13C_2 + 13C_2 + 13C_2 + 13C_2$.

P [cards of same suit] = $\frac{13C_2+{}^{13}C_2+{}^{13}C_2+{}^{13}C_2}{52C_2}$

$= \frac{4\times78}{26\times51} = \frac{4}{17}$

(v) Events E is the complement of event D. Therefore,

P [cards of different suits] = 1 – P [cards of same suit]

= 1 – 4/17 = 13/17

**9.** A bag has 9 tickets marked with numbers 1, 2, 3,……9. Two tickets are drawn at random from the bag. Find the probability that both the numbers drawn are (i) even (ii) odd.

Sol:

There are $9C_2$ equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be

A: both the selected numbers are even.

B: both the selected numbers are odd.

(i) Out of 9 numbers, 4 numbers, namely, 2,4,6 and 8 are even. Therefore, $4C_2$ selections will have two even numbers. Therefore,

P [both even] = P(A) = $\frac{4C_2}{9C_2} = 6/36 = 1/6$

(ii) Out of 9 numbers, 5 numbers, namely, 1,3,5,7 and 9 are odd. Therefore, $5C_2$ selections will have two odd numbers. Therefore,

P [both odd] = P(B) = $\frac{5C_2}{9C_2} = 10/36 = 5/18$

**10.** A bag contains 3 red, 4 green and 3 yellow marbles. Three marbles are randomly drawn from the bag. What is the probability that they are of (i) the same colour (ii) different colours (one of each colour)?

Sol: There are $10C_3$ equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be

A: Selected marble are of the same colour.

B: Selected marbles are of different colours

(i)The marbles drawn should be 3 red or 4 green or 3 yellow.

Therefore, $3C_3 + 4C_3 + 3C_3$ outcomes are favourable to events A, Therefore,

P [marbles of the same colour] = $\frac{3C_3+{}^4C_3+{}^3C_3}{10C_3} = \frac{1+4+1}{120} = \frac{1}{20}$

(ii) The marbles should be one of each colour. Therefore, $3C_1$ x$3C_1$ x $3C_1$ outcomes are favourable. Therefore,

P [marbles of different colours] = $\frac{3C_1+{}^4C_1+{}^3C_1}{10C_3} = \frac{3}{10}$

## 4.3 ADDITION THEOREM ON PROBABILITY

**Statement:** For any two events A and B, P $(A \cup B)$ = P (A) + P (B) – P $(A \cap B)$

Proof:

For events A and B, $B = (A \cap B) \cup (A^1 \cap B)$

Here, A∩B and A`∩B are mutually exclusive. Therefore, by axiom 3,

$P(B) = P(A \cap B) + P(A^1 \cap B)$

$P(A^1 \cap B) = P(B) - P(A \cap B)$

Also, $A \cup B = A \cup (A^1 \cap B)$

Here, A∩B and A`∩B are mutually exclusive therefore,

$P(A \cup B) = P(A) + P(A^1 \cap B)$

By above results

P $(A \cup B)$ = P (A) + P (B) – P $(A \cap B)$.

**1.** Three unbiased dice are thrown once. Find the probability that all the three dice show the number 6.

**Sol:**

When 3 dice are thrown, there are 6x6x6 = 216 equally, mutually exclusive and exhaustive outcomes. of these 216 outcomes, 1 outcome, namely, (6, 6, 6) is favourable. Therefore, probability of all the three dice showing the number 6 is

P [all the three result in the number 6] =1/216

**2**. A fair coin is tossed five times. Find the probability of obtaining

(i) head in all the tosses, (ii) head in at least one of the tosses.

**Sol:** There are 32 equally likely, mutually exclusive and exhaustive outcomes.

(i) Out of them, one outcome is HHHHH and another outcome is TTTTT. Therefore,

P [head in ail tosses] = 1/32

(ii) P [at least one head] = 1 – P [tail in all tosses]

= 1-1/32 = 31/32

**Note:** Whenever probability of the event "at least one" has to be found, it is easier to find it by using the probability of the complementary event as follows.

P [at least one] = P[none]

**3.** In a college, there are five lecturers. Among them, three are doctorates. If a committee consisting three lecturers is formed, what is the probability that at least two of them are doctorates?

**Sol:**

From the five lecturers, three lecturers can be selected in $5C_3$ ways. Thus, there are $5C_3$ equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be —

A: Two of the selected lecturers are doctorates.

B: All the three selected lecturers are doctorates.

Then, events have $3C_2$ x $2C_1$ favourable outcomes. And, event B has $3C_3$ favourable outcomes. Here, events A and B are mutually exclusive.

.'. P [at least two doctrates] = P [two or three doctrates]= $P(A \cup B)$

$$= P(A) + P(B) = \frac{3C_2 \times {}^2C_1}{5C_3} + \frac{3C_3}{5C_3}$$

$$= \frac{3\times2}{10} + \frac{1}{10} = \frac{7}{10} = 0.7$$

**4.** What is the probability that there will be 53 Sundays in a randomly selected

(i) leap year (ii) non-leap year?

**Sol:**

(i) A leap year has 366 days, out of them, 7*52 = 364 days make 52 complete weeks. The remaining two days may occur in any of the following pattern ---

(Sunday, Monday), (Monday, Tuesday), (Tuesday, Wednesday), (Wednesday, Thursday), (Thursday, Friday), (Friday, Saturday) and (Saturday, Sunday).

Out of these 7 cases which are equally likely, mutually exclusive and exhaustive, 2 cases namely (Sunday, Monday) and (Saturday, Sunday) have Sunday. Therefore,

P [leap year has 53 Sundays] =2/7

(ii) A non- leap year has 365 days. Out of them, 364 days make 52 complete weeks. The remaining one day may be Sunday, Monday, ---- Saturday. Out of these 7 possibilities, only one is Sunday. Therefore,

P [non-leap year has 53 Sundays] =1/7

**5.** If from a pack of cards, a single card is drawn. What is the probability that it is either a spade or a king?

**Sol:** P (A) = P (a spade card) =12/52=1/4

P (B) = P (a king card) =4/52

P (either a spade or a king card) = P (A or B)

$= P(A) + P(B) - P(A \cap B)$

= (12/52) +(1/52) -(12/ (52*52))

= 4/13

**6.** A person is known to hit the target in 3 out of 4 shots, whereas another person is known to hit the target in 2 out of 3 shots. Find the probability of the targets being hit at all when they both persons try.

**Sol:** The prob. that the first person hit the target = P (A) = 3/4

The prob. that the second person hit the target = P (B) = 2/3

The two events are not mutually exclusive, since both persons hit the same target.

$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$

=3/4+2/3-(6/12)

=11/12

## 4.4 CONDITIONAL PROBABILITY

Let A and B be two events. Then, conditional probability of given A is the probability of happening of B when it is known that A has already happened. On the other hand, the probability of happening of B when nothing is known about happening of A is called unconditional probability of B.

The conditional probability of B given A is denoted by P(B\A). The unconditional probability is P{B).

Let P(A) > 0. Then, conditional probability of event B given A is defined as-----

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

If P(A) = 0, the conditional probability P(B\A) is not defined.

If A and B are independent events, occurrence of B will be independent of occurrence of A. Therefore, the conditional and unconditional probabilities are equal. That is, P(B\A) = P(B).

$$P(B) = \frac{P(A \cap B)}{P(A)}$$ That is, P (A ∩B) = P(A). P(B)

INDEPENDENT & DEPENDENT EVENTS:

Two events are said to be independent when the actual happening of one does not influence in any way the happening of the other. Events which are not independent are called dependent events.

If two events are independent, the occurrence or non-occurrence of one does not depend on the occurrence or non-occurrence of the other.

**Note:** Two events A and B are independent if and only if P(A∩B) = P(A). P(B)

## 4.5 MULTIPLICATION LAW OF PROBABILITY

If A & B are two independent events, then

P (A ∩ $B$) = P (Both A & B will happen)
= P (A) × P (B)

**1.** A bag contains 8 white and 10 black balls. Two balls are drawn in succession. What is the prob. that first is white and second is black?

Sol: Total no. of balls = 8 + 10 = 18

P (drawing one white ball from8 balls) = 8/18

P (drawing one black ball from 10 balls) = 10/18

P (drawing first white & second black) =10*8/ (18*18)

= 80/324

**2.** Two persons A & B appear in an interview for 2 vacancies for the same post. The probability of A's selection is1/7 and that of B's selection is1/8. What is the probability that, i) both of them will be selected?

ii) none of them will be selected.

**Sol:** P (A selected) =1/7

P (B selected) =1/8

P (A will not be selected) = 1-(1/7) = 6/7

P (B will not be selected) = 1-(1/8) =7/8

P (Both of them will be selected) = P (A) × P (B)

=1/ (7*8)

=1/56

**3**. A card is drawn at random from a pack of cards.

(i) What is the probability that it is a heart?

(ii) If it is known that the card drawn is red, what is the probability that it is a heart?

**Sol:** There are 52 equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be —

A: card drawn is red.

B: card drawn is heart.

There are 26 red cards and 13 hearts in a pack of cards. Therefore, event A has 26 favourable outcomes and event B has 13 favourable outcomes. Event A∩B has 13 favourable outcomes because when any of the 13 hearts is drawn A∩B happens.

Therefore, P(A) = 26/52. P(B) = 13/52 and P(A∩B) =13/52 (i)The unconditional probability of drawing a heart is ---

P(B) = 13/52 = ¼

(ii) The conditional probability of drawing a heart given that it is red card is-----

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{13/52}{26/52} = 1/2$$

**4.** A fair coin is tossed thrice. What is the probability that all the three tosses result in heads?

**Sol:**

Let events A, B, and C be------

A: the first toss results in head

B: the second toss results in head.

C: the third toss results in head.

Then, P(A) = P(B) =P(C) = ½

Since A, B, and C are results of three different tosses, they are independent.

Therefore, probability that all the three tosses result in head is ---

P [ 3 heads] = P(A∩B∩C) = P(A). P(B). P(C)

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

**5.** Two fair dice are rolled. If the sum of the numbers obtained is 4, find the probability that the numbers obtained on both the dice are even-

**Sol:** Let events A and B be —

A: the sum of the numbers is 4

B: the numbers on both the dice are even

Here, we have to find -----

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Event A has 3 favourable outcomes, namely, (1,3, (2,2) and (3,1)

$$P(B|A) = \frac{P(A \cap B)}{B(A)}$$

∴ P [Sum 4] = P(A) = 3/36

Event (A∩B) has 1 favourable outcome, namely, (2,2).

∴P [Sum 4 and number even] = P(A∩B) = 1/36

Thus, P [Number even given Sum 4] = $\frac{1/36}{3/36} = 1/3$

**6.** A box has 1 red and 3 white balls. Balls are drawn one after one from the box. Find the probability that the two balls drawn would be red if

a. the ball drawn first is returned to the box before the second draw is made. (Draw with replacement).

b. the ball drawn first is not returned before the second draw is made. (Draw without replacement).

**Sol:** Let A: the first ball drawn is red

B: the second ball drawn is red.

Draw with replacement:

Here, P(A) =1/4 Also, since the first ball- is returned before the second draw is made, P(B|A) =1/4

∴P [Two balls are red] = P(A∩B)

= P(A). P (B|A)

= 1/4 * 1/4 =1/16

Draw without replacement:

Here, Since the first ball drawn is not returned before the second draw is made, P(B|A) = 0/4

.'. P [Two balls are red} = P(A∩B)

= P(A). P(B|A)

= ¼ * 0/4 = 0

**7.** The probability that a contractor will get a plumbing contract is 2/3 and probability that he will not get an electrical contract is 5/9. If the probability of getting at least one of these contracts is 4/5, what is the probability that he will get both?

**Sol:** Let A: contractor gets plumbing contract

B: contractor gets electrical contract

Then, P(A) = 2/3, P(B`) = 5/9 and P(A∪B) = 4/5

Therefore, P(B) = 1-P(B`) = 4/9

By addition theorem we have,

P(A∪B) = P(A) +P(B) – P(A∩B)

That is, P(A∩B) = P(A) +P(B) – P(A∪B)

Therefore,

P [he gets both plumbing and electrical contract] = P(A∩B)

= P(A) +P(B) – P(A∪B)

$= 2/3 + 4/9 - 4/5 = 14/45$

**8.** A can solve 90 percent of the problems given in a book and B can solve 70 percent. What is the probability that at least one of them will solve a problem selected at random?

**Sol:** event A: student A solve the problem

event B: student B solve the problem.

P (at least one solves the problem) = 1-P (none solve the problem)

$$= 1 - P(\overline{A} \cap \overline{B})$$

$$= 1 - P(\overline{A}).P(\overline{B})$$

$$= 1 - (0.10)(0.30)$$

$$= 0.97$$

The probability that a trainee will remain with a company 0.6, The probability that an employee earns more ten Rs.10,000 per year 0.5. The probability an employee is trainee who remained with the company or who earn more then Rs.10,000 per year is 0.7. What is the probability earn more than Rs.10,000 per year given that he is a trainee who stayed with the company?

**Sol:** Event A: A trainee will remain with the company

Event B: A trainee earns more than Rs. 10,000.

Given P(A) = 0.6 P(B) = 0.5 P(A∪B) = 0.7

We need to find

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A) + P(B) - P(A \cup B)}{P(A)} = \frac{0.4}{0.6} = 0.67$$

**9.** Suppose that one of the three men, a politician a bureaucrat and an educationist will be appointed as VC of the university. The probabilities of their appointment are respectively 0.3,0.25, and 0.45. The probability that these people will promote research activities if there are appointed is 0.4,0.7 and 0.8 respectively. What is the probability that research will be promoted by the new VC

**Sol:**

event A: Politician appointed as VC
event B: bureaucrat appointed as VC
event C: Educationist appointed as VC
event D: promotion of research activities

$$= P(A \cap P) + P(B \cap D) + P(C \cap D).$$
$$= P(D|A).P(A) + P(D|B).P(B) + P(D|C).P(C)$$
$$= (0.3)(0.4) + (0.25)(0.7) + (0.45)(0.8) = 0.655$$

**10.** A box contains 4 green and 6 white bolls another box contains 7 green and 8 white bolls. Two bolls are transferred from box 1 to box 2 and then a boll is drawn from box 2. What is the probability that it is white?

**Sol:**

event A: transferred balls are green

event B: transferred balls are white

event C: Among transferred balls one green & 1 white

event D: selection of a white ball from box 2.

$$= P(A \cap D) + P(B \cap D) + P(C \cap D)$$
$$= P(D|A).P(A) + P(D|B).P(B) + P(D|C).P(C)$$
$$= \frac{4C_2}{10C_2}.\frac{8}{17} + \frac{6C_2}{10C_2}.\frac{10}{17} + \frac{4C_1 \times^6 C_1}{10C_2} \times \frac{9}{17}$$
$$= 0.5412$$

**11.** Probabilities of Husband's and wife's selection to a post are 1/5 and 1/7 respectively, what is the probability that.

(i) Both of them will be selected.

(ii) Exactly one of them will be selected

(iii) None of them will be selected

**Sol:**

event A: selection of Husband P(A) = $\frac{1}{5}$

event B: selection of Husband P(B) = $\frac{1}{7}$

(i) P (both of them will be selected) = P(A∩B) =P(A). P(B)

$$=\frac{1}{5}\times\frac{1}{7}=\frac{1}{35}$$

(ii) P (exactly one of them will be selected) = $P(\overline{A}\cap B)+P(\overline{B}\cap A)$

$$=P(\overline{A}).P(B)+P(\overline{B}).P(A)$$

$$=\frac{4}{5}\times\frac{1}{7}+\frac{6}{7}\times\frac{1}{5}=\frac{10}{35}.$$

(iii) P (none of them will be selected) =$P(\overline{A}\cap\overline{B})=P(\overline{A}).P(\overline{B})$

$$=\frac{4}{5}\times\frac{6}{7}=\frac{24}{35}$$

## 4.6 BAYE'S THEOREM

**Statement:**

If $\mathrm{E_1,E_2,E_3,\ldots.E_n}$ are 'n' mutually exclusive events with $P(E_i)\neq 0$ for each i in the sample space S and for any arbitrary event A which is a subset of $\cup_{i=1}^{n}E_i$ with $P(A)>0$ then $\mathrm{P}\left({E_i}/_{\mathrm{A}}\right)=\frac{\mathrm{P}(E_i)(\mathrm{P}\left({\mathrm{A}}/_{E_i}\right),}{\sum_{i=1}^{n}\mathrm{P(E_i)P}\left({\mathrm{A}}/_{E_i}\right)}$ for i=1 to n.

Proof:

Given $\mathrm{E_1,E_2,E_3,\ldots.E_n}$ are n mutually exclusive

$\mathrm{i.e.,E_i\cap E_j=\phi\ for\ i\neq j}$

Also $\mathrm{E_1,E_2,E_3,\ldots.E_n}$ are exhaustive events

$\mathrm{i.e.,E_1\cup E_2\cup E_3\cup\ldots\cup E_n=S}$

$$\mathrm{A=A\cap S}$$

$$\mathrm{=A\cap(E_1\cup E_2\cup E_3\cup\ldots\cup E_n)}$$

$$\mathrm{=(A\cap E_1)\cup(A\cap E_2)\cup(A\cap E_3)\cup\ldots\cup(A\cap E_n)}$$

$$P(A) = P\big((A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots \cup (A \cap E_n)\big)$$

Since $(A \cap E_1), (A \cap E_2), (A \cap E_3), \ldots (A \cap E_n)$are n mutually exclusive even

By Axiom-III

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + \ldots + P(A \cap E_n)$$

$$P\left({}^{E_i}/_{A}\right) = \frac{P(E_i \cap A)}{P(A)}$$

$$P\left({}^{E_i}/_{A}\right) = \frac{P(E_i)P\left({}^{A}/_{E_i}\right)}{P(A)}$$

$$P\left({}^{E_i}/_{A}\right) = \frac{P(E_i)P\left({}^{A}/_{E_i}\right),}{P(A\cap E_1)+P(A\cap E_2)+P(A\cap E_3)+.+P(A\cap E_n).}$$

$$P\left({}^{E_i}/_{A}\right) = \frac{P(E_i)P\left({}^{A}/_{E_i}\right)}{\sum P(A\cap E_i)}$$

$$P\left({}^{E_i}/_{A}\right) = \frac{P(E_i)(P\left({}^{A}/_{E_i}\right),}{\sum_{i=1}^{n} P(E_i)P\left({}^{A}/_{E_i}\right)}$$ (By the definition of Conditional Probability)

PROBLEMS:

**1.** In a certain college, 25% of boys and 10% of girls are studying mathematics. The girls constitute 60% of the students. (a) What is the probability that mathematics is being studied? (b) If a student is selected at random and is found to be studying mathematics, Find the probability that the student is a girl? © a boy?

Sol: Given P(Boy) = P(B) =$\frac{40}{100} = \frac{2}{5}$

P(Girl) = P(G) =$\frac{60}{100} = \frac{3}{5}$

Probability that mathematics is studied given that the student is a boy

$$P\left({}^{M}/_{B}\right) = \frac{25}{100} = \frac{1}{4}$$

Probability that mathematics is studied given that the student is a girl

$$P\left({}^{M}/_{G}\right) = \frac{10}{100} = \frac{1}{10}$$

Probability that the student studied Mathematics $P(M) = P(G)P\left({}^{M}/_{G}\right) + P(B)P\left({}^{M}/_{B}\right)$

$$= \frac{3}{5} \times \frac{1}{10} + \frac{2}{5} \times \frac{1}{4} = \frac{5}{25}$$

By Bayes theorem, probability of mathematics student is a girl

$$P\left({}^{G}/_{M}\right) = \frac{P(G)P\left({}^{M}/_{G}\right)}{P(G)P\left({}^{M}/_{G}\right)+P(B)P\left({}^{M}/_{B}\right)} = \frac{\frac{3}{5}\times\frac{1}{10}}{\frac{5}{25}} = \frac{3}{10}$$

By Bayes theorem, probability of mathematics student is a boy

$$P\left({}^{B}/_{M}\right) = \frac{P(B)P\left({}^{M}/_{B}\right)}{P(G)P\left({}^{M}/_{G}\right)+P(B)P\left({}^{M}/_{B}\right)} = \frac{\frac{2}{5}\times\frac{1}{4}}{\frac{5}{25}} = \frac{1}{2}$$

**2.** The chance that doctor A will diagnose a disease x correctly is 60%. The chance that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of doctor A, who had disease x died. What is the probability that his disease was diagnosed correctly?

Sol: Let $E_1$be the event that "disease x is diagnosed correctly by doctor A" and

$E_2$be the event that "a patient of doctor A who disease x died"

Then $P(E_1) = \frac{60}{100} = 0.6$, $P\left({}^{E_2}/_{E_1}\right) = \frac{40}{100} = 0.4$

Which implies that $P(\overline{E_1}) = 1 - 0.6 = 0.4$ and $P\left({}^{E_2}/_{\overline{E_1}}\right) = \frac{70}{100} = 0.7$

By Bayes theorem

$$P\left({}^{E_1}/_{E_2}\right) = \frac{P(E_1)P\left({}^{E_2}/_{E_1}\right)}{P(E_1)P\left({}^{E_2}/_{E_1}\right) + P(\overline{E}_1)P\left({}^{E_2}/_{\overline{E}_1}\right)}$$

$$= \frac{0.6\times0.4}{0.6\times0.4+0.4\times0.7} = \frac{6}{13}$$

**3.** Of the three men, the chances that a politician, a business man or an academician will be appointed as a vice-chancellor (V.C) of a university are 0.5, 0.3, 0.2 respectively. Probability that research is promoted by these persons if they are appointed as VC are 0.3, 0.7, 0.8 respectively. Determine the probability that research is promoted. If research is promoted, what is the probability that VC is an academician?

Sol: Let A, B, C be the events that a politician, businessmen or an academician will

be appointed as VC of the three men.

Then P(A)=0.5, P(B)=0.3, P(C)=0.2

The probabilities that research is promoted if they are appointed as VC s are

$$P(R/A) = 0.3,\ P(R/B) = 0.7,\ P(R/C) = 0.8$$

The Probability that the research is promoted

$$=P(A)P(R/A) + P(B)P(R/B) + P(C)P(R/C)$$

$$= (0.5)(0.3) + (0.3)(0.7) + (0.2)(0.8)$$

=0.52

The probability that research is promoted when the VC is an academician

$$P(C/R) = \frac{P(C)P(R/C)}{P(C)P(R/C)+P(B)P(R/B)+P(A)P(R/A)}$$

$$= \frac{0.16}{0.15+0.21+0.16} = \frac{4}{13} = 0.3077$$

**4.** A businessman goes to hotels X, Y, Z, 20%, 50%, 30% of the time respectively. It is known that 5%, 4%, 8% of the rooms in X, Y, Z hotels have faulty plumbings. What is the probability that business man's room having faulty plumbing is assigned to hotel Z?

**Sol:** Let the probabilities of business man going to hotels X, Y, Z be receptively P(X), P(Y), P(Z).

Then $P(X) = \frac{20}{100} = 0.2,\ P(Y) = \frac{50}{100} = 0.5,\quad P(Z) = \frac{30}{100} = 0.3$

Let E be the event that the hotel room has faulty plumbing. Then the probabilities that hotels X, Y, Z have faculty plumbing are

$$P(E/X) = \frac{5}{100} = 0.05,\quad P(E/Y) = \frac{4}{100} = 0.04,\quad P(E/Z) = \frac{8}{100} = 0.08,$$

The probability that the business man's room having faulty plumbing is assigned to hotel z

$$P(Z/E) = \frac{P(Z)P(E/Z)}{P(Z)P(E/Z)+P(Y)P(E/Y)+P(X)P(E/X)}$$

$$= \frac{0.3\times0.08}{0.3\times0.08+0.5\times0.04+0.2\times0.05} = \frac{4}{9}$$

**5.** There are two boxes, in box I, 11 cards are there numbered 1 to 11 and in box II, 5 cards are there numbered 1 to 5. A box is chosen and a card is drawn. If the card shows an even number, then another card is drawn from the same box. If card shows an odd number another card is drawn from the other box. Find the probability that they are from box I?

**Sol:**

Number of cards in box I = 11

Number of cards with even number = 5

Number of cards with odd number = 6

Number of cards in box II = 5

Number of cards with even number = 2

Number of cards with odd number = 3

The probability of choosing any one box=$\frac{1}{2}$

Let E be the event that both the cards are even.

For this box is choosen and a card is picked, if the first card is even then the second card is also picked from the same box and the card is also even.

Let $E_1$be the event that both the cards are from box I

$$P(E_1) = \frac{1}{2}\left(\frac{5}{11}\right)\left(\frac{4}{10}\right) = \frac{1}{11}$$

Let $E_2$ be the event that both the cards are from box II. Then

$$P(E_2) = \frac{1}{2}\left(\frac{2}{5}\right)\left(\frac{1}{4}\right) = \frac{1}{20}$$

$$\therefore P(E) = P(E_1) + P(E_2) = \frac{1}{11} + \frac{1}{20} = \frac{31}{220}$$

Let E be the event that both the cards are odd

Then a box is choosen, first card is odd and second card is picked from another box and that is also odd

Let $E_1$ be the event that the first card odd from box I and second card is odd from box II.

$$P(E_1) = \frac{1}{2}\left(\frac{6}{11}\right)\left(\frac{3}{5}\right) = \frac{9}{55}$$

Let $E_2$ be the event that first card is odd from box II and second card is odd from box I

Then $P(E_2) = \frac{1}{2}\left(\frac{3}{5}\right)\left(\frac{6}{11}\right) = \frac{9}{55}$

$$\therefore P(E) = P(E_1) + P(E_2) = \frac{9}{55} + \frac{9}{55} = \frac{18}{55}$$

The probability that both cards from box I are even $= \frac{1}{2}\left(\frac{5}{11}\right)\left(\frac{4}{10}\right) = \frac{1}{11}$

The probability that both cards from box II are even $= \frac{1}{2}\left(\frac{2}{5}\right)\left(\frac{1}{4}\right) = \frac{1}{20}$

The Probability that if both cards are even then they are from box I

$$\frac{\frac{1}{2}\left(\frac{5}{11}\right)\left(\frac{4}{10}\right)}{\frac{1}{2}\left(\frac{5}{11}\right)\left(\frac{4}{10}\right)+\frac{1}{2}\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)} = \frac{\frac{1}{11}}{\frac{1}{11}+\frac{1}{20}} = \frac{20}{31}$$

**6.** Two bolts are drawn from a box containing 4 good and 6 bad bolts. Find the probability that the second bolt is good if the first one is found to be bad.

Sol: Since in the problem it is given that "probability that the second bolt is good, if the first one is bad", we have to understand that the two bolts are drawn in succession (Without replacement).

Probability that the first bolt is found to be bad $= \frac{6}{10}$

Probability that the bolt drawn now, is found to be good $= \frac{4}{9}$

P (first is bad and second is good)$= \frac{6}{10} \times \frac{4}{9} = \frac{4}{15}$

**7.** In a factory, machine A produce 40% of the output and machine B produces 60%. On the average, 9 items in 1000 produced by A are defective and 1 item in 250 produced by B is defective. An item drawn at random from a day's output is defective. What is the probability that it was produced by A or B?

Sol:

Output produced by A=40%

$\therefore P(A) = 0.4$

Output produced by B=60%

$\therefore P(B) = 0.6$

$P(D/A)$ = Probability that items produced by A are defective $= \frac{9}{1000} = 0.009$

Similarly

$P(D/B)$ = Probability that items produced by B are defective $= \frac{1}{250} = 0.004$

$P(A/D)$ = Probability that the bolt is produced by A given that it is defective

$$= \frac{P(A)\times P(D/A)}{P(A)\times P(D/A)+P(B)\times P(D/B)}$$

$$= \frac{0.4\times 0.009}{0.4\times 0.009+0.6\times 0.004} = \frac{0.0036}{0.006} = 0.6$$

$P(B/D)$ = Probability that the bolt is produced by B given that it is defective

$$= \frac{P(B)\times P(D/B)}{P(A)\times P(D/A)+P(B)\times P(D/B)}$$

$$= \frac{0.6\times 0.004}{0.4\times 0.009+0.6\times 0.004} = \frac{0.0024}{0.006} = 0.4$$

Probability that it was produced by A or B = $P(A/D)+P(B/D)$ = 0.6+0.4=1

1

## 4.7 RANDOM VARIABLE

Suppose two fair coins are tossed. Here, the sample space is 5 = {TT, TH, HT, HH} Suppose each of the four sample points in this sample space, a number is assigned as follows.

| Sample point | TT | TH | HT | HH |
|---|---|---|---|---|
| Number | 0 | 1 | 1 | 2 |

Here, the assigned numbers indicate the number of heads obtained in each case. Let 'the number of heads' be denoted by X. Then, X is a function on the sample space. It takes the values 0,1 and 2 with probabilities —

$P[X=0] = P$ [no head] = ¼

$P[X=1] = P$ [one head] = ½

$P[X=2] = P$ [two head] = ¼

Here, X is called Random variable or Variate.

Random variable is a function which assigns a real number to every sample point in the sample space. The set of such real values is the range of the random variable.

There are two types of random variable, namely, Discrete random variable and Continuous random variable.

A Variable X which takes values $x_1, x_2, ....x_n$ with probabilities $p_1,p_2,....p_n$ is a Discrete random variable. Here, the value $x_1, x_2, ....x_n$ from the range of the random variable.

A random variable whose range is uncountable infinite is a Continuous random variable.

Ex1. Let X denote the number of heads obtained while tossing two fair coins. Then, X is a random variable which takes the values 0,1 and 2 wit respective probabilities ¼, ½ and ¼. Here, X is a discrete random variable.

Ex. 2. Let X denote the number obtained while throwing a fair die. Then, X is a discrete random variable taking values 1, 2, 3, 4, 5 and 6 with probability 1/6 each.

Ex. 3. Let X denote the weight of apples. Then, X is a continuous random variable.

Generally, random variables are denoted by X, Y, Z, etc. If X is a random variable, the values taken by X are denoted by x (small letter).

**PROBABILITY DISTRIBUTION**

A systematic presentation of the values taken by a random variable and the corresponding probabilities is called probability distribution of the random variable.

**PROBABILITY MASS FUNCTION**

Let X be a discrete random variable. And let p(x) be a function such that p(x) = P[X=x]. Then, p(x) is the probability mass function of X if it satisfies following conditions.

(i) $p(x) \geq 0$ for all x

(ii) $\sum p(x) = 1$

A similar function is defined for a continuous random variable X. It is called **probability density function** (p.d.f.). It is denoted by f(x).

**MATHEMATICAL EXPECTATION:**

Let X be a discrete random variable with probability mass function p(x). Then, mathematical expectation of X is $E(X) = \sum x.\ p(x)$

**Mathematical expectation of a function h(x) of X:**

Let X be a discrete random variable with probability mass function p(x). Then, mathematical expectation of any function h(X) of X is $E[h(X)] = \sum h(x).\ p(x)$

**1.** Two fair coins are tossed once. Find the mathematical expectation of the number of heads obtained.

Sol: Let X denote the number of heads obtained. Then, X is a random variable which takes the values 0, 1 and 2 with respective probabilities ¼, ½ and ¼ and That is,

| x | 0 | 1 | 2 |
|---|---|---|---|
| p(x) | ¼ | ½ | ¼ |

The mathematical expectation of the number of head is

$$E(X) = \sum x.p(x) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + +2 \times \frac{1}{4} = 1$$

**Note:**

1. For a random variable X, the Arithmetic Mean is E(X).
2. For a random variable X, the Variance is

$Var(X) = E[X-E(X)]^2$

$= E(X^2) - [E(X)]^2$

The Standard Deviation is the square – root of the variance.

**2.** A bag has 3 white and 4 red balls. Two balls are randomly drawn from the bag. Find the expected number of white balls in the draw.

Sol: Let X denote the number of white balls obtained in the draw. Then, X is a random variable which takes the values 0, 1 and 2 with respective probabilities –

P (x=0) = P [both red] = $\frac{4C_2}{7C_2} = \frac{2}{7}$

P (x=1) = P [one white & one red] = $\frac{3C_2 \times {}^4C_1}{7C_2} = \frac{4}{7}$

P (x=2) = P [both white] = $\frac{3C_2}{7C_2} = \frac{1}{7}$

The probability distribution of X is –

| x | 0 | 1 | 2 |
|---|---|---|---|
| p(x) | 2/7 | 4/7 | 1/7 |

$$E(X) = \sum x.p(x) = 0 \times \frac{2}{7} + 1 \times \frac{4}{7} + 2 \times \frac{1}{7} = \frac{6}{7}$$

**3**. A Random Variable 'X' has the following values of 'X'

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| P(x) | 0 | k | 2k | 2k | 3k | $K^2$ | $2K^2$ | $K+7K^2$ |

(i) Find 'K'
(ii) Find mean
(iii) Find Variance

Sol:
(i) If X is a random variable, then $\sum P(x_i) = 1$

$$\Rightarrow P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5) + P(x=6) + P(x=7) = 1$$

$$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow 10k^2 + 9k - 1 = 0$$

$$\Rightarrow k = \frac{1}{10}, -1$$

$\therefore k = \frac{1}{10}$ (Since probability is positive)

(ii) $E(X) = \sum x.p(x)$

$$= 0 \times 0 + 1 \times \mathrm{k} + 2 \times 2\mathrm{k} + 3 \times 2\mathrm{k} + 4 \times 3\mathrm{k} + 5 \times k^2 + 6 \times 2k^2 + 7 \times (7k^2 + k)$$

$$= 30\mathrm{k} + 66k^2$$

$$= \frac{30}{10} + \frac{66}{100} = 3.66$$

(iii) Variance = $E(X^2)$- $[E(X)]^2$

$$= \sum x^2.p(x) - (3.66)^2$$

$$= 0 \times 0 + 1 \times \mathrm{k} + 4 \times 2\mathrm{k} + 9 \times 2\mathrm{k} + 16 \times 3\mathrm{k} + 25 \times k^2 + 36 \times 2k^2 + 49 \times (7k^2 + k) - (3.66)^2$$

$$= 124\mathrm{k} + 440k^2 - (3.66)^2 = 3.404$$

**4.** Two dice are thrown, let X assign to each of (a, b) in S the maximum of its numbers i.e., X (a, b) = max (a, b). Find the probability distribution and also find the mean and variance of the distribution.

Sol: If we throw two dice maximum number could be 1, 2, 3, 4, 5, 6,

$\therefore X = \{1, 2, 3, 4, 5, 6\}$

If we throw two dice then the sample has 36 events

$P(x=1)=\frac{1}{36}$ , $P(x=2)=\frac{3}{36}$, $P(x=3)=\frac{5}{36}$, $P(x=4)=\frac{7}{36}$,

$P(x=5)=\frac{9}{36}$, $P(x=6)=\frac{11}{36}$

The probability distribution of X is –

| x | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| p(x) | 1/36 | 3/36 | 5/36 | 7/36 | 9/36 | 11/36 |

Mean= $E(X)=\sum x.p(x)=1\times\frac{1}{36}+2\times\frac{3}{36}+3\times\frac{5}{36}+4\times\frac{7}{36}+5\times\frac{9}{36}+6\times\frac{11}{36}=\frac{161}{36}=4.47$

Variance = $E(X^2)$- $[E(X)]^2$ = $=\sum x^2.p(x)-(4.47)^2=1.99$

**5.** The probability density f(x) of a continuous random variable is given by $f(x)=ce^{-|x|},-\infty<x<\infty$ .Show that $c=\frac{1}{2}$ and also find the mean and variance of the distribution.

Sol: We know that $\int_{-\infty}^{\infty}f(x)dx=1$

$$\Rightarrow \int_{-\infty}^{\infty}ce^{-|x|}\,dx=1$$

$$\Rightarrow 2\int_{0}^{\infty}ce^{-|x|}\,dx=1$$ since the function is Even.

$$\Rightarrow 2\int_{0}^{\infty}ce^{-x}\,dx=1$$

$$\Rightarrow 2c[\frac{e^{-x}}{-1}]_0^{\infty}=1$$

$$\Rightarrow -2c[e^{-\infty}-e^{-0}]=1$$

$$\Rightarrow -2c[0-1]=1$$

$$\Rightarrow -2c[0-1]=1$$

$$\Rightarrow 2c=1$$

$$\therefore c=\frac{1}{2}$$

Mean=$\int_{-\infty}^{\infty}xf(x)dx$

$$=\int_{-\infty}^{\infty}x\,ce^{-|x|}dx$$

$$=c\int_{-\infty}^{\infty}x\,e^{-|x|}dx=0$$ since $x\,e^{-|x|}$ is odd function

Variance= $E(X^2)$- $[E(X)]^2$

$$=\int_{-\infty}^{\infty}x^2f(x)\,dx-0$$

$$=\int_{-\infty}^{\infty}x^2ce^{-|x|}\,dx$$

$= 2\int_0^\infty x^2 c e^{-x}\, dx$ Since $x^2\, e^{-|x|}$ is even function

$= 2.\frac{1}{2}\int_0^\infty x^2 e^{-x}\, dx$

$= [-x^2e^{-x} - 2xe^{-x} - 2e^{-x}]_0^\infty$

$= [0-(-2)] = 2$

**6.** A continuous random variable has the probability density function

$f(x) = \begin{cases} kxe^{-\lambda x}, for x \geq 0, \lambda > 0 \\ 0, otherwise \end{cases}$ Determine (i) k (ii) Mean (iii) variance

**Sol:** (i) We know that $\int_{-\infty}^\infty f(x)dx = 1$

$\Rightarrow \int_0^\infty kxe^{-\lambda x}\, dx = 1$

$\Rightarrow k[x\frac{e^{-\lambda x}}{-\lambda} - 1.\frac{e^{-\lambda x}}{\lambda^2}]_0^\infty = 1$

$\Rightarrow k\left[0-\left(-\frac{1}{\lambda^2}\right)\right] = 1$

$\Rightarrow k = \lambda^2$

(ii) Mean=$\int_{-\infty}^\infty xf(x)dx$

$= \int_0^\infty xkxe^{-\lambda x}\, dx$

$= \int_0^\infty kx^2e^{-\lambda x}\, dx$

$= k[x^2\frac{e^{-\lambda x}}{-\lambda} - 2x.\frac{e^{-\lambda x}}{\lambda^2} - 2\frac{e^{-\lambda x}}{\lambda^3}]_0^\infty$

$= k\left[0-\left(-\frac{2}{\lambda^3}\right)\right]$

$= \lambda^2.\frac{2}{\lambda^3} = \frac{2}{\lambda}$

(iii) Variance= $E(X^2)$- $[E(X)]^2$

$= \int_{-\infty}^\infty x^2 f(x)\, dx - (\frac{2}{\lambda})^2$

$= \int_0^\infty kx^3e^{-\lambda x}\, dx - \frac{4}{\lambda^2}$

$= k[x^3\frac{e^{-\lambda x}}{-\lambda} - 3x^2.\frac{e^{-\lambda x}}{\lambda^2} - 6x.\frac{e^{-\lambda x}}{\lambda^3} - 6.\frac{e^{-\lambda x}}{\lambda^4}]_0^\infty - \frac{4}{\lambda^2}$

$= k\left[0-\left(-\frac{6}{\lambda^4}\right)\right] - \frac{4}{\lambda^2} = \lambda^2.\frac{6}{\lambda^4} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}$

## 9. Practice Quiz

1. The chance that a leaf year contains 53 Fridays [ B ]

   A) $\frac{1}{7}$

   B) $\frac{2}{7}$

   C) $\frac{1}{365}$

   D) $\frac{2}{365}$

2. If a card is drawn from a well shuffled packet 53 cards, then the probability that it is spade (or) queen is [ B ]

   A) $\frac{17}{52}$
   B) $\frac{4}{13}$
   C) $\frac{13}{52}+\frac{4}{51}$
   D) None

3. If a coin is tossed twice the probability of getting at least one head is [ C ]

   A) $\frac{1}{2}$

   B) $\frac{1}{4}$

   C) $\frac{3}{4}$

   D) None

4. A bag contains 3 red balls, 4 white balls and 7 black balls. The probability of drawing a red or a black ball is [ B ]

   A) $\frac{2}{7}$
   B) B) $\frac{5}{7}$
   C) C) $\frac{3}{7}$
   D) D) $\frac{4}{7}$

5. $P(\bar{A}\cap\bar{B})=$………. Where A, B are independents event [ C ]

   A) $P(\bar{A})+P(\bar{B})$
   B) $P(\bar{A})-P(\bar{B})$
   C) $P(\bar{A}).P(\bar{B})$
   D) None

6. The probability of drawing a king from pack of cards is [ A ]

A) $\frac{1}{13}$

B) $\frac{1}{26}$

C) $\frac{1}{52}$

D) None

7. A, B, C are three mutually exclusive events then $P(A \cup B \cup C) =$ [ A ]

A) $P(A) + P(B) + P(C)$

B) 0

C) $P(A)P(B)P(C)$

D) None

8. $P(A \cap B) = \frac{1}{6}, P(A) = \frac{1}{2}$ then $P\left(\frac{B}{A}\right) =$ [ A ]

A) $\frac{1}{3}$
B) B) $\frac{2}{3}$
C) C) 1
D) D) None

9. The total no. of sample events when two dice are thrown simultaneously [ C ]

A) 20
B) 25
C) 36
D) None

10. $P(A) = P(B)$ and $P(A) = 2P(C)$ then $P(A) =$ [ B ]

A) $\frac{1}{5}$

B) $\frac{2}{5}$

C) $\frac{3}{5}$

D) 1

11. Tossing of a coin is an example for …………….. variable [ A ]

A) Discrete

B) Continuous

C) both A & B