Unit-3 Multi variable calculas :-

Partial differentiation and its application :~

Higher order partial derivatives :~

In general the 1^{st} order partial derivati[ves] $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also function's of x and y and they can be deferentiated repeatedly to get higher order partial derivatives.

i.e $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, $\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2}$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3}$, $\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial y^3}$, $\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) =$

Note :~ $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$, $\frac{\partial^2 f}{\partial x^2} = f_{xx}$,

$\frac{\partial^2 f}{\partial y^2} = f_{yy}$, $\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$, $\frac{\partial^2 f}{\partial y \partial x} = f_{yx}$, $\frac{\partial^2 f}{\partial x \cdot}$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = f_{yx}}$$

Problems :~

1) find 1^{st} and 2^{nd} order partial derivatives of $ax^2+2hxy+$ and verify $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

A) Let $f(x,y) = ax^2+2hxy+by^2$

first order derivatives :~

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[ax^2+2hxy+by^2\right]$$

$= a(2x) + 2hy(1) + 0$

$= 2ax + 2hy$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[ax^2 + 2hxy + by^2]$

$= 0 + 2hx(1) + b(2y)$

$= 2by + 2hx$

Second order partial derivatives :-

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]$

$= \frac{\partial}{\partial x}[2ax + 2hy]$

$= 2a(1) + 0$

$= 2a$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]$

$= \frac{\partial}{\partial y}[2by + 2hx]$

$= 2b(1) + 0$

$= 2b$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$

$= \frac{\partial}{\partial x}[2by + 2hx]$

$= 0 + 2h(1)$

$= 2h$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$

$= \frac{\partial}{\partial y}[2ax + 2hy]$

$= 0 + 2h(1)$

$= 2h$

Here $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2h$

2) Find 1^{st} & 2^{nd} order partial derivatives of $x^3 + y^3 - 3axy$ and verify $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

A) Let $f(x, y) = x^3 + y^3 - 3axy$

first order partial derivative's

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^3 + y^3 - 3axy]$

$$= 3x^2 + 0 - 3ay(1)$$
$$= 3x^2 - 3ay.$$
$$\frac{df}{dy} = \frac{d}{dy}[x^3+y^3-3axy]$$
$$= 0 + 3y^2 - 3ax(1)$$
$$= 3y^2 - 3ax$$

2nd order partial derivatives:-

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left[\frac{df}{dx}\right] = \frac{d}{dx}[3x^2-3ay] = 3(2x)-0 = 6x$$

$$\frac{d^2f}{dxdy} = \frac{d}{dx}\left[\frac{df}{dy}\right] = \frac{d}{dx}[3y^2-3ax] = 0-3a(1) = -3a$$

$$\frac{d^2f}{dy^2} = \frac{d}{dy}\left[\frac{df}{dy}\right] = \frac{d}{dy}[3y^2-3ax] = 3(2y)-0 = 6y$$

$$\frac{d^2f}{dydx} = \frac{d}{dy}\left[\frac{df}{dx}\right] = \frac{d}{dy}[3x^2-3ay] = 0-3a(1) = -3a$$

Here,

$$\therefore \frac{d^2f}{dxdy} = \frac{d^2f}{dydx} = -3a$$

3) If $w=(y-z)(z-x)(x-y)$, find the value of $\frac{dw}{dx}+\frac{dw}{dy}+\frac{dw}{dz}$

Sol) Given $w = (y-z)(z-x)(x-y)$

$$= [zy-xy-z^2+zx](x-y)$$
$$= [-z^2+zx+zy-xy][x-y]$$
$$= -z^2x+zx^2+\cancel{zxy}-x^2y+z^2y-\cancel{xyz}-y^2z+xy$$

$$= xy^2 + yz^2 - x^2y - xz^2 + x^2z - y^2z$$

$$\frac{dw}{dx} = \frac{d}{dx}\left[xy^2 + yz^2 - x^2y - xz^2 + x^2z - y^2z\right]$$

$$= y^2(1) + 0 - 2xy - z^2(1) + z(2x) - 0]$$

$$= y^2 - z^2 - 2xy + 2xz$$

$$\frac{dw}{dy} = \frac{d}{dy}\left[xy^2 + yz^2 - xz^2 + x^2z - y^2z\right]$$

Similarly $\frac{dw}{dy} = z^2 - x^2 - 2yz + 2yx$

$$\frac{dw}{dz} = x^2 - y^2 - 2zx + 2zy$$

Now $\frac{dw}{dx} + \frac{dw}{dy} + \frac{dw}{dz}$

$$\Rightarrow \cancel{y^2} - \cancel{z^2} - 2\cancel{xy} + 2\cancel{xz} + \cancel{z^2} - \cancel{x^2} - 2\cancel{yz} + 2\cancel{yx} + \cancel{x^2} - \cancel{y^2} - 2\cancel{zx} + 2\cancel{zy}$$

$$\Rightarrow 0 \,//$$

$$\therefore \frac{dw}{dx} + \frac{dw}{dy} + \frac{dw}{dz} = 0 \,//$$

4) If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, $x^2+y^2+z^2 \neq 0$ then prove that

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}$$

A) Given $u = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$

Now $\frac{du}{dx} = \frac{d}{dx}(x^2+y^2+z^2)^{-1/2}$

$$= \frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2-1}(\cancel{2}x+0+0)$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{d^2u}{d^2x^2} = \frac{d}{dx}\left[\frac{du}{dx}\right]$$

$\Rightarrow -\frac{d}{dx}\left[x(x^2+y^2+z^2)^{-3/2}\right]$

$\Rightarrow -\left[x\,\frac{-3}{2}(x^2+y^2+z^2)^{-3/2-1}(2x+0+0)+(x^2+y^2+z^2)^{-3/2}\cdot(1)\right]$

$\Rightarrow -\left[-3x^2[x^2+y^2+z^2]^{-5/2}+(x^2+y^2+z^2)^{-3/2}\right]$

$\Rightarrow -\left[(x^2+y^2+z^2)^{-5/2}\left[-3x^2+(x^2+y^2+z^2)^{1}\right]\right]$

$\Rightarrow -\left[(x^2+y^2+z^2)^{-5/2}(-2x^2+y^2+z^2)\right]$

$\Rightarrow \frac{d^2u}{dx^2}=(2x^2-y^2-z^2)(x^2+y^2+z^2)^{-5/2}$

Similarly

$$\frac{d^2u}{dy^2}=(2y^2-z^2-x^2)(x^2+y^2+z^2)^{-5/2}$$

$$\frac{d^2u}{dz^2}=(2z^2-x^2-y^2)(x^2+y^2+z^2)^{-5/2}$$

Now $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}$

$\Rightarrow (x^2+y^2+z^2)^{-5/2}\left[\cancel{2x^2}-\cancel{y^2}-\cancel{z^2}+\cancel{2y^2}-\cancel{z^2}-\cancel{x^2}+2z^2-\cancel{x^2}-\cancel{y^2}\right]$

$\Rightarrow (x^2+y^2+z^2)^{-5/2}[0]$

$\Rightarrow 0$

$\therefore \frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0$ //

5) If $u=\log(x^3+y^3+z^3-3xyz)$ then prove that

$$\left(\frac{d}{dx}+\frac{d}{dy}+\frac{d}{dz}\right)^2 u=\frac{-9}{(x+y+z)^2}.$$

A) Given $u=\log(x^3+y^3+z^3-3xyz)$

$$\frac{du}{dx} \Rightarrow \frac{d}{dx} \log(x^3+y^3+z^3-3xyz)$$

$$\Rightarrow \frac{1}{x^3+y^3+z^3-3xyz}\left[3x^2+0+0-3yz(1)\right]$$

$$\Rightarrow \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

Similarly

$$\frac{du}{dy} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{du}{dz} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

Now $\frac{du}{dx}+\frac{du}{dy}+\frac{du}{dz}$

$$\Rightarrow \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} + \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} + \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\Rightarrow \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\Rightarrow \frac{3x^2+3y^2+3z^2-3\left[xy+yz+zx\right]}{x^3+y^3+z^3-3xyz}$$

$$\Rightarrow \frac{3\left[x^2+y^2+z^2-xy-yz-zx\right]}{x^3+y^3+z^3-3xyz}$$

$$\Rightarrow \frac{3\,\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{(x+y+z)\,\cancel{\left[x^2+y^2+z^2-xy-yz-zx\right]}}$$

$$\Rightarrow \frac{3}{(x+y+z)}$$

$$\left(\frac{d}{dx}+\frac{d}{dy}+\frac{d}{dz}\right)^2\left[u\right] = \left[\frac{d}{dx}+\frac{d}{dy}+\frac{d}{dz}\right]\left[\frac{d}{dx}+\frac{d}{dy}+\frac{d}{dz}\right]u$$

$$= \left[\frac{d}{dx}+\frac{d}{dy}+\frac{d}{dz}\right]\left[\frac{du}{dx}+\frac{du}{dy}+\frac{du}{dy}\right]$$

$$\Rightarrow \left(\frac{d}{dx}+\frac{d}{dy}+\frac{d}{dz}\right)\left[\frac{3}{x+y+z}\right]$$

$$\Rightarrow \frac{d}{dx}\left[\frac{3}{x+y+z}\right]+\frac{d}{dy}\left[\frac{3}{x+y+z}\right]+\frac{d}{dz}\left[\frac{3}{x+y+z}\right]$$

$$\Rightarrow \frac{d}{dx}\left[\frac{3}{x+y+z}\right]$$

$$\Rightarrow \frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}$$

$$\Rightarrow \frac{-9}{(x+y+z)^2}$$

$$\therefore \left[\frac{d}{dx}+\frac{d}{dy}+\frac{d}{dz}\right]^2 u = \frac{-9}{(x+y+z)^2}$$

6) verify $\dfrac{d^2u}{dx\,dy} = \dfrac{d^2u}{dy\,dx}$ $\quad u = \tan^{-1}\left(\dfrac{x}{y}\right)$

Sol) Given $u = \tan^{-1}\left[\dfrac{x}{y}\right]$

$$\frac{du}{dx} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= \frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}(1)$$

$$= \frac{\cancel{y^2}}{x^2+y^2}\times\frac{1}{\cancel{y}}$$

$$= \frac{y}{x^2+y^2}$$

$$\frac{du}{dy} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{d}{dy}\left[\frac{x}{y}\right]$$

$$= \frac{1}{1+\frac{x^2}{y^2}}\times x\times\left[\frac{-1}{y^2}\right]$$

$$\Rightarrow \frac{y^2}{x^2+y^2}\times\frac{-}{y^2}$$

$$\Rightarrow \frac{-x}{x^2+y^2}$$

$$\frac{d^2u}{dx\,dy} = \frac{d}{dx}$$

$$\frac{d^2u}{dy\,dx} =$$

7) verify

$$\Rightarrow \frac{y^2}{x^2+y^2} \times \frac{-x}{y^2}$$

$$\Rightarrow \frac{-x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{-x}{x^2+y^2}\right]$$

$$= \frac{(x^2+y^2)(-1) - (-x)(2x+0)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2}\right]$$

$$= \frac{(x^2+y^2)(1) - y.(0+2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} \,\text{"}$$

7) verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for $u = \sin^{-1}\frac{x}{y}$

8) If $u = \tan^{-1}\left[\frac{2xy}{x^2-y^2}\right]$, $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$

A) Let $u = \tan^{-1}\left[\frac{2xy}{x^2-y^2}\right]$

$$\frac{du}{dx} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2} \times 2y \frac{d}{dx}\left[\frac{x}{x^2-y^2}\right]$$

$$= \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}} \times 2y \left[\frac{(x^2-y^2)(1) - x(2x-0)}{(x^2-y^2)^2}\right]$$

$$= \frac{\cancel{(x^2-y^2)^2}}{(x^2-y^2)^2+4x^2y^2} * 2y \left[\frac{x^2-y^2-2x^2}{\cancel{(x^2-y^2)^2}}\right]$$

$$= \frac{2y(-x^2-y^2)}{(x^2-y^2)^2+4x^2y^2}$$

$$= \frac{2y(-x^2-y^2)}{(x^2)^2+(y^2)^2-2x^2y^2+4x^2y^2}$$

$$= \frac{2y\,\cancel{(-x^2-y^2)}}{(x^2+y^2)^{\cancel{2}}}$$

$$= \frac{-2y}{(x^2+y^2)}$$

NOW $$\frac{d^2u}{dx^2} = \frac{du}{dx}\left[\frac{d}{dx}\right]$$

$$= \frac{d}{dx}\left[\frac{-2y}{(x^2+y^2)}\right]$$

$$= -2y \frac{d}{dx}\left[\frac{1}{x^2+y^2}\right]$$

$= -2y \left[\frac{-\ldots}{(x\ldots}\right.$

$= -2y \left[\ldots\right.$

$= \frac{+4\ldots}{(x^2+\ldots}$

$\frac{du}{dy} = \ldots$

Now

$$= -2y\left[\frac{-1}{(x^2+y^2)^2}\times(2x+0)\right]$$

$$= -2y\left[\frac{-2x}{(x^2+y^2)^2}\right]$$

$$= \frac{+4xy}{(x^2+y^2)^2}$$

$$\frac{du}{dy} = \frac{d}{dy}\left[\tan^{-1}\frac{2xy}{x^2-y^2}\right]$$

$$= \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\times\frac{d}{dy}\left[\frac{2xy}{x^2-y^2}\right]$$

$$= \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}}\times 2x\,\frac{d}{dy}\left[\frac{y}{x^2-y^2}\right]$$

$$= \frac{(x^2-y^2)^2}{(x^2-y^2)^2+4x^2y^2}\times\frac{(2x)\times[x^2-y^2(1)-y(0-2y)]}{(x^2-y^2)^2}$$

$$= \frac{2x[x^2-y^2+2y^2]}{(x^2-y^2)^2+4x^2y^2}$$

$$= \frac{2x[x^2-y^2+2y^2]}{(x^2)^2+(y^2)^2-2x^2y^2+4x^2y^2}$$

$$= \frac{2x[x^2+y^2]}{x^2+y^2+2x^2y^2}$$

$$= \frac{2x\,(\cancel{x^2+y^2})}{(x^2+y^2)^{\cancel{2}}}$$

$$= \frac{2x}{(x^2+y^2)}$$

Now $\dfrac{d^2u}{dy^2} = \dfrac{d}{dy}\left[\dfrac{du}{dy}\right]$

$$= \frac{d}{dy}\left[\frac{2x}{x^2+y^2}\right]$$

$$= 2x\frac{d}{dy}\left[\frac{1}{x^2+y^2}\right]$$

$$= 2x\left[\frac{-1}{(x^2+y^2)^2}\right](0+2y)$$

$$= 2x\left[\frac{-1}{(x^2+y^2)^2}\right]2y$$

$$= \frac{-4xy}{(x^2+y^2)^2}$$

$$\therefore \frac{d^2u}{dx^2}+\frac{d^2u}{dy^2} = \frac{4xy}{(x^2+y^2)^2} - \frac{4xy}{(x^2+y^2)^2} = 0$$

Homogeneous functions :-

Defination:- A function $f(x,y)$ is said be homogeneous function of degree 'n', if the degree of each term in $f(x,y)$ is n, where n is a real number.

(or)

A function $f(x,y)$ is said to be homogeneous function of degree n in x, y if $f(kx,ky) = k^n f(x,y)$ $(n \in R)$

Example - 1 :-

1) $f(x,y) = x^2+y^2$

A) Let $f(x,y) = x^2+y^2$

$$f(kx,ky) = (kx)^2+(ky)^2$$

$$= k^2x^2+k^2y^2$$

$$= k^2(x^2+y^2)$$

$f(Kx, Ky) = K^2 f(x, y)$

$\therefore f(x, y)$ is homogeneous function of degree 2

Example 2:-

2) $f(x, y) = \frac{3x}{y} + \log\left[\frac{y}{x}\right]$

A) $f(Kx, Ky) = \frac{3Kx}{Ky} + \log\left(\frac{Ky}{Kx}\right)$

$= \frac{3x}{y} + \log\left(\frac{y}{x}\right)$

$= K^0\left[\frac{3x}{y} + \log\left(\frac{y}{x}\right)\right]$

$f(Kx, Ky) = K^0 f(x, y)$

$\therefore f(x, y)$ is homogeneous function of degree 0.

Example 3:-

3) $f(x, y) = \log y + 2\log x$

A) Let $f(Kx, Ky) = \log(Ky) + 2\log(Kx)$

$= \log K + \log y + 2\log K + 2\log x$

$= 3\log K + \log y + 2\log x$

$f(Kx, Ky) = 3\log K + f(x, y)$

$\therefore f(x, y)$ is non homogeneous.

Example 4:-

4) $f(x, y) = x^{\frac{1}{3}} y^{\frac{3}{4}} \tan^{-1}\left[\frac{y}{x}\right]$

A) Let $f(Kx, Ky) = Kx^{\frac{1}{3}} Ky^{\frac{3}{4}} \tan^{-1}\left[\frac{Ky}{Kx}\right]$

$= K^{\frac{1}{3}} x^{\frac{1}{3}} \cdot K^{\frac{3}{4}} \cdot y^{\frac{3}{4}} \tan^{-1}\left(\frac{y}{x}\right)$

$= K^{\frac{1}{3}+\frac{3}{4}} \cdot x^{\frac{1}{3}} y^{\frac{3}{4}} \tan^{-1}\left(\frac{y}{x}\right)$

$\frac{1}{3}+\frac{3}{4}$

$\frac{6+9}{12} = \frac{15}{12}$

$f(Kx, Ky) = K^{\frac{13}{12}} f(x, y)$

$\therefore f(x, y)$ is homogeneous function of degree $\frac{13}{12}$.

Example 5 :-

5) Let $u = \sin^{-1}\left[\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right]$

A) $$u(kx, ky) = \sin^{-1}\left[\frac{\sqrt{kx}-\sqrt{ky}}{\sqrt{kx}+\sqrt{ky}}\right]$$

$$= \sin^{-1}\left(\frac{\sqrt{k}\,[\sqrt{x}-\sqrt{y}]}{\sqrt{k}\,[\sqrt{x}+\sqrt{y}]}\right)$$

$$= \sin^{-1}\left[\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right]$$

$$u(kx, ky) = k^{0}\, u(x, y)$$

$\therefore u(x, y)$ is homogeneous function of degree 0.

Euler's theorem on homogeneous function :-

Statement :- If $z = f(x, y)$ is a homogeneous function of degree n then $x\dfrac{dz}{dx} + y\dfrac{dz}{dy} = nz$

Example 1 :-

1) $$f(x, y) = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

A) $$f(kx, ky) = \frac{\sqrt{kx}-\sqrt{ky}}{\sqrt{kx}+\sqrt{ky}}$$

$$= \frac{\cancel{\sqrt{k}}\,(\sqrt{x}-\sqrt{y})}{\cancel{\sqrt{k}}\,(\sqrt{x}+\sqrt{y})}$$

$$= k^{0}\,\frac{(\sqrt{x}-\sqrt{y})}{\sqrt{x}+\sqrt{y}}$$

$$f(kx, ky) = k^{0}\, f(x, y)$$

$\therefore f(x, y)$ is homogeneous function of degree is 0

By Euler's theorem

$$x \cdot \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f$$

$$\therefore x \cdot \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 0$$

Example 2 :-

2) $u = \tan^{-1}\left[\frac{y}{x}\right]$

A) $u(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$

$$u(kx, ky) = \tan^{-1}\left(\frac{ky}{kx}\right)$$

$$= \tan^{-1}\left(\frac{y}{x}\right)$$

$$u(kx, ky) = k^0 \tan^{-1}\left(\frac{y}{x}\right)$$

$\therefore f(x, y)$ is homogeneous function of degree 0.

By Euler's theorem

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n f$$

$$\therefore x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 0$$

3) verify Euler's theorem for $z = ax^2 + 2hxy + by^2$

Sol) Given function $z = ax^2 + 2hxy + by^2$

Given function is homogeneous of degree $n = 2$

Now we need to show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$,

LHS. Let $z = ax^2 + 2hxy + by^2$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[ax^2 + 2hxy + by^2]$$

$$= a(2x) + 2hy(1) + 0$$

$$= 2ax + 2hy$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[ax^2 + 2hxy + by^2]$$

$$= 0 + 2hx(1) + b(2y)$$

$$= 2by + 2hx$$

Now $x \frac{dz}{dx} + y \cdot \frac{dz}{dy}$

$\Rightarrow x[2ax + 2hy] + y[2by + 2hx]$

$\Rightarrow 2ax^2 + 2hxy + 2by^2 + 2hxy$

$\Rightarrow 2ax^2 + 2by^2 + 4hxy$

$= 2[ax^2 + 2hxy + by^2]$

$= 2[z] = 2z.$

$\therefore$ LHS = RHS

Euler's theorem is verified.

5) Verify Euler's theorem for the function
$xy + yz + zx$

A) Given function $u = xy + yz + zx$

Given function is homogeneous of degree 2 $\therefore$

$$x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} + z \cdot \frac{du}{dz} = 2u$$

LHS $\Rightarrow$ $\frac{du}{dx} = \frac{d}{dx}[xy + yz + zx]$

$= y(1) + 0 + z(1)$

$= y + z$

$\frac{du}{dy} = \frac{d}{dy}[xy + yz + zx]$

$= x(1) + z(1) + 0$

$= x + z$

$\frac{du}{dz} = \frac{d}{dz}[xy + yz + zx]$

$= 0 + y(1) + x(1)$

$= x + y$

$x\left[\frac{du}{dx}\right] = x[y + z] = xy + zx$

$y\left[\frac{du}{dy}\right] = y[x + z] = xy + yz$

$z \cdot \left[\frac{du}{dz}\right] = z[x+y] = zx + yz$

$x\frac{du}{dx} + y\cdot\frac{du}{dy} + z\frac{du}{dz} = xy+zx+xy+yz+zx+yz$

$= 2xy + 2zx + 2yz$

$= 2[xy+yz+zx]$

$= 2u$

$\therefore$ LHS = RHS

Euler's theorem is verified.

6) If $u = \log\left[\frac{x^2+y^2}{x+y}\right]$ then prove that $x\frac{du}{dx} + y\cdot\frac{du}{dy} = 1$

Sol) Given $u = \log_e\left[\frac{x^2+y^2}{x+y}\right]$

$u(kx, ky) = \log\left[\frac{k^2x^2+k^2y^2}{kx+ky}\right]$

$= \log\left[\frac{k^2(x^2+y^2)}{k(x+y)}\right]$

$= \log\left[\frac{k(x^2+y^2)}{x+y}\right]$

$\therefore u(x,y)$ is non homogeneous.

So convert the given function into homogeneous as Below.
$e^u = \frac{x^2+y^2}{x+y} = f(x,y)$

Here $f(x,y)$ is homogeneous of degree $n=1$

from Euler's theorem

$x\cdot\frac{df}{dx} + y\cdot\frac{df}{dy} = nf = 1f = f$

$x\cdot\frac{de^u}{dx} + y\cdot\frac{de^u}{dy} = e^u$

$x\cdot e^u\frac{du}{dx} + y\cdot e^u\cdot\frac{du}{dy} = e^u$

$\cancel{e^u}\left[x\cdot\frac{du}{dx} + y\cdot\frac{du}{dy}\right] = \cancel{e^u}$

$$\boxed{x\cdot\frac{du}{dx} + y\cdot\frac{du}{dy} = 1}.$$

7) If $u = \log\left[\frac{x^4+y^4}{x+y}\right]$ then show that $x\cdot\frac{du}{dx} + y\cdot\frac{du}{dy} = 3$

8) If $U = \sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]$ then prove that $x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} = \operatorname{Tan} u$.

Sol) Let $U = \sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]$

Given function is non-homogeneous. So convert it into homogeneous as follows.

Let $U = \sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]$

$$\sin u = \left[\frac{x^2+y^2}{x+y}\right] = f \text{ (say)}$$

Here $f(x, y)$ is homogeneous of degree $n = 1$

By Euler's theorem

$$\Rightarrow x \cdot \frac{df}{dx} + y \cdot \frac{df}{dy} = nf = f$$

$$\Rightarrow x \cdot \frac{d(\sin u)}{dx} + y \cdot \frac{d}{dy}(\sin u) = \sin u$$

$$\Rightarrow x \cos u \cdot \frac{du}{dx} + y \cdot \cos u \cdot \frac{du}{dy} = \sin u$$

$$\Rightarrow \cos u \left[x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy}\right] = \sin u$$

$$\Rightarrow \boxed{x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} = \operatorname{Tan} u}$$

9) If $U = \operatorname{Tan}^{-1}\left[\frac{x^3+y^3}{x+y}\right]$ then prove that $x \cdot \frac{du}{dx} + y \cdot \frac{du}{dy} = \sin 2u$

Sol) Given $U = \operatorname{Tan}^{-1}\left[\frac{x^3+y^3}{x+y}\right]$

Given function is non homogeneous and then convert it into homogeneous as follows.

Let $U = \operatorname{Tan}^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

$$\operatorname{Tan} u = \left(\frac{x^3+y^3}{x+y}\right) = f(x, y) \text{ Say}$$

Here $f(x,y)$ is homogeneous of degree $n=2$.

By Euler's theorem

$$\Rightarrow x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=nf=2f$$

$$\Rightarrow x\cdot\frac{\partial}{\partial x}[\operatorname{Tan}u]+y\cdot\frac{\partial}{\partial y}[\operatorname{Tan}u]=2\operatorname{Tan}u$$

$$\Rightarrow x\cdot\sec^2u\,\frac{\partial u}{\partial x}+y\cdot\sec^2u\,\frac{\partial u}{\partial y}=2\operatorname{Tan}u$$

$$\Rightarrow \sec^2u\left[x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}\right]=2\operatorname{Tan}u$$

$$\Rightarrow x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=\frac{2\operatorname{Tan}u}{\sec^2u}$$

$$\Rightarrow x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=2\,\frac{\sin u}{\cos u}\times\cos^2u$$

$$\Rightarrow x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=2\sin u\cos u \qquad \sin 2\theta=2\sin\theta\cos\theta$$

$$\Rightarrow \boxed{x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=\sin 2u}$$

10) If $u=\sin^{-1}\frac{x}{y}+\operatorname{Tan}^{-1}\frac{y}{x}$ then show that $x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=0$

Sol) Given $u=\sin^{-1}\frac{x}{y}+\operatorname{Tan}^{-1}\frac{y}{x}$

Here
$$u(kx,ky)=\sin^{-1}\left(\frac{kx}{ky}\right)+\operatorname{Tan}^{-1}\left(\frac{ky}{kx}\right)$$
$$=\sin^{-1}\left(\frac{x}{y}\right)+\operatorname{Tan}^{-1}\left(\frac{y}{x}\right)$$
$$=k^0\left[\sin^{-1}\left(\frac{x}{y}\right)+\operatorname{Tan}^{-1}\left(\frac{y}{x}\right)\right]$$
$$=k^0\,u(x,y)$$

Here $u(x,y)$ is homogeneous of degree $n=0$

From Euler's theorem

$$\Rightarrow x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=nu$$

$$\Rightarrow \boxed{x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=0}$$

The chain rule of partial differentiation :-

i) If $z = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$ then z is called composite function of a variable of t.

ii) If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \phi(u, v)$ then z is called composite function of two variables u, v.

Note :- Let $z = f(u, v)$ where $u = \phi(x, y)$ $v = \psi(x, y)$

then $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$

Similarly., $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$

Total differential co-efficient :-

* Let $z = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$ then the derivative of z with respect to t.
* i.e $\frac{dz}{dt}$ is called the total differential co-efficient or total derivative of z.

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}}$$

Problems :-

1) If $u = x^2 + y^2$, $x = at^2$, $y = 2at$ then find $\frac{du}{dt}$?

Sol) Given $u = x^2 + y^2$, $x = at^2$, $y = 2at$

Now, $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \rightarrow ①$

Let $u = x^2 + y^2$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2)$
$= 2x$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2)$
$= 2y$

$$\text{let } x = at^2 \qquad\qquad \text{Let } y = 2at$$

$$\frac{dx}{dt} = \frac{d}{dt}[at^2] \qquad\qquad \frac{dy}{dt} = \frac{d}{dt}[2at]$$

$$= a(2t) \qquad\qquad = 2a(1)$$

$$= 2at \qquad\qquad = 2a$$

from ① $\frac{du}{dt} = 2x[2at] + 2y[2a]$

$$= 4xat + 4ay$$

$$= 4[at^2]at + 4a[2at]$$

$$= 4a^2t^3 + 8a^2t$$

$$= 4a^2t[t^2+2]$$

2) If $u = x^2+y^2+z^2$, $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$ then find $\frac{du}{dt}$?

Sol) Given $u = x^2+y^2+z^2$, $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$

Now $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} \longrightarrow ①$

Let $u = x^2+y^2+z^2$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2) = 2x+0+0 = 2x$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x^2+y^2+z^2] = 0+2y+0 = 2y$

$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(x^2+y^2+z^2) = 0+0+2z = 2z$

Let $x = e^t$

$\frac{dx}{dt} = \frac{d}{dt}(e^t) = e^t$

Let $y = e^t \sin t$

$\frac{dy}{dt} = \frac{d}{dt}(e^t \sin t) = e^t \cos t + \sin t \, e^t$

Let $z = e^t \cos t$

$\frac{dz}{dt} = \frac{d}{dt}(e^t \cos t) = e^t(-\sin t) + \cos t \, e^t$

from ① $\frac{du}{dt} = 2x(e^t) + 2y(e^t\cos t + \sin t\, e^t] + 2z[e^t[-\sin t] + \cos t\, e^t]$

$$\frac{du}{dt} = 2[e^t][e^t] + 2[e^t \sin t][e^t \cos t + \sin t\, e^t] + 2e^t\cos t[-e^t \sin t + \cos t\, e^t]$$

$$= 2e^{2t} + 2e^{2t}\cancel{\sin t}\cos t + 2e^{2t}\sin^2 t - 2e^{2t}\cancel{\sin t}\cos t + 2e^{2t}\cos^2 t$$

$$= 2e^{2t} + 2e^{2t}\sin^2 t + 2e^{2t}\cos^2 t$$

$2e^{2t} + 2e^{2t}[\sin^2 t + \cos^2 t]$

$2e^{2t} + 2e^{2t}[1]$

$= 4e^{2t}$ //

3] If $u = x^2y^3$, $x = \log t$, $y = e^t$ then find $\frac{du}{dt}$?

4] If $u = f(2x-3y, 3y-4z, 4z-2x)$ prove that

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$$

Sol) Given $u = f[2x-3y, 3y-4z, 4z-2x]$

$u = f[r, s, t]$

Where $r = 2x-3y$, $s = 3y-4z$, $t = 4z-2x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(2) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-2)$$

$$\frac{\partial u}{\partial x} = 2\cdot\frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial x} = 2\left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\right)$$

$$\frac{1}{2}\cdot\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(-3) + \frac{\partial u}{\partial s}(3) + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial y} = 3\frac{\partial u}{\partial s} - \frac{3\partial u}{\partial r}$$

$$\frac{1}{3}\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} [0] + \frac{\partial u}{\partial s} (-4) + \frac{\partial u}{\partial t} [4]$$

$$\frac{\partial u}{\partial z} = 4\frac{\partial u}{\partial t} - 4\frac{\partial u}{\partial s}$$

$$\frac{1}{4}\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s} \rightarrow (3)$$

$$(1)+(2)+(3) \Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}} + \cancel{\frac{\partial u}{\partial s}} - \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial t}} - \cancel{\frac{\partial u}{\partial s}}$$

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$$

Hence proved.

5) If $u = f(y-z, z-x, x-y)$ then show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Sol] Given $u = f(y-z, z-x, x-y)$ $u = f(r, s, t)$

Let $r = y-z$, $s = z-x$, $t = x-y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x} \longrightarrow (1)$$

Now $\frac{\partial r}{\partial x} = 0-0 = 0$ $\quad \frac{\partial s}{\partial x} = 0-1 = -1$ $\quad \frac{\partial t}{\partial x} = 1-0 = 1$

$\frac{\partial r}{\partial y} = 1-0 = 1$ $\quad \frac{\partial s}{\partial y} = 0-0 = 0$ $\quad \frac{\partial t}{\partial y} = 0-1 = -1$

$\frac{\partial r}{\partial z} = 0-1 = -1$ $\quad \frac{\partial s}{\partial z} = 1-0 = 1$ $\quad \frac{\partial t}{\partial z} = 0-0 = 0$

From eq (1)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} [0] + \frac{\partial u}{\partial s} [-1] + \frac{\partial u}{\partial t} [1]$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \longrightarrow (2)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} \rightarrow (3)$$

$$= \frac{\partial u}{\partial r} [1] + \frac{\partial u}{\partial s} [0] + \frac{\partial u}{\partial t} [-1]$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \longrightarrow (4)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\,\frac{\partial t}{\partial z} \longrightarrow (5)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}[-1] + \frac{\partial u}{\partial s}[1] + \frac{\partial u}{\partial t}[0]$$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \longrightarrow (6)$$

add (2) + (4) + (6)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -\cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}} + \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}} - \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

6) If $u = f\left[\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right]$ then prove that

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z} = 0$$

Sol) Given $u = f\left[\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right]$

$$u = f[r, s, t]$$

where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} \longrightarrow (1)$$

Now
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\,\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z} \longrightarrow (3)$$

Now, $\frac{\partial r}{\partial x} = \frac{1}{y}(1) = \frac{1}{y}$ $\qquad$ $\frac{\partial s}{\partial y} = \frac{1}{z}(1) = \frac{1}{z}$

$\frac{\partial r}{\partial y} = x\left[\frac{-1}{y^2}\right] = \frac{-x}{y^2}$ $\qquad$ $\frac{\partial s}{\partial x} = 0$

$\frac{\partial r}{\partial z} = 0$ $\qquad$ $\frac{\partial s}{\partial z} = y\left[\frac{-1}{z^2}\right] = \frac{-y}{z^2}$

$$\frac{\partial t}{\partial x} = z\left[\frac{-1}{x^2}\right] = \frac{-z}{x^2}$$

$$\frac{\partial t}{\partial y} = 0$$

$$\frac{\partial t}{\partial z} = \frac{1}{x}[1] = \frac{1}{x}$$

from (1) =

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left[\frac{1}{y}\right] + \frac{\partial u}{\partial s}[0] + \frac{\partial u}{\partial t}\left[\frac{-z}{x^2}\right]$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}\cdot\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t}$$

multiply by 'x' on B.S

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\cdot\frac{\partial u}{\partial t} \longrightarrow (4)$$

from eq (2)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left[\frac{-x}{y^2}\right] + \frac{\partial u}{\partial s}\left[\frac{1}{z}\right] + \frac{\partial u}{\partial t}[0]$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}$$

Multiply by y on B.S.

$$y\frac{\partial u}{\partial y} = \frac{-x}{y}\left(\frac{\partial u}{\partial r}\right) + \frac{y}{z}\left[\frac{\partial u}{\partial s}\right] \longrightarrow (5)$$

from eq (3)

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}[0] + \frac{\partial u}{\partial s}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial t}\left[\frac{1}{x}\right]$$

$$\frac{\partial u}{\partial z} = \frac{-y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\cdot\frac{\partial u}{\partial t}$$

Multiply z on B.S

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t} \longrightarrow (6)$$

(4) + (5) + (6)

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z} = \cancel{\frac{x}{y}\frac{\partial u}{\partial r}} - \cancel{\frac{z}{x}\frac{\partial u}{\partial t}} - \cancel{\frac{x}{y}\frac{\partial u}{\partial r}} + \cancel{\frac{y}{z}\frac{\partial u}{\partial s}}$$

$$-\cancel{\frac{y}{z}\frac{\partial u}{\partial s}} + \cancel{\frac{z}{x}\frac{\partial u}{\partial t}}$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = 0$$

*** **Jacobian :-**

LAQ's
2 Marks

Defination :- Let $u = u[x, y]$ $v = v[x, y]$ then these two simultaneous relation constitute a transformation from (x, y) to (u, v).

The determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ or $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ is called the Jacobian of u, v with respect to x, y. The determinant value is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ (or) $J\left[\frac{u, v}{x, y}\right]$ then $\frac{\partial(u,v)}{\partial(x,y)}$ (or) $J\left(\frac{u,v}{x,y}\right) =$

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Note :- $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ (or) $J\left[\frac{u, v, w}{x, y, z}\right]$

$$\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Note :- If $J = \frac{\partial(u,v)}{\partial(x,y)}$, $J^1 = \frac{\partial(x,y)}{\partial(u,v)}$ then

$$\boxed{J \cdot J^1 = 1}$$

1) If $x = u($ …

$\frac{\partial(x,y)}{\partial(u,v)} = 1$ …

Sol) Given x …

$y =$ …

Now $\frac{\partial x}{\partial u}$ …

$\frac{\partial(x,y)}{\partial(u,v)}$ …

2) If $u =$ …

then …

Sol) Given

$\frac{\partial u}{\partial x}$ …

$\frac{\partial u}{\partial y}$ …

$\frac{\partial u}{\partial z}$ …

Problems :-

1) If $x = u(1+v)$, $y = v(1+u)$ then prove that
$$\frac{\partial(x,y)}{\partial(u,v)} = 1+u+v$$

Sol) Given $x = u(1+v) = u+uv$
$y = v(1+u) = v+vu$

Now
$$\frac{\partial x}{\partial u} = 1+(1)v = 1+v$$
$$\frac{\partial x}{\partial v} = 0+u(1) = u$$
$$\frac{\partial y}{\partial u} = 0+v(1) = v$$
$$\frac{\partial y}{\partial v} = 1+u(1) = 1+u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$
$$= \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$
$$= 1+v(1+u) - uv$$
$$= 1+u+v+\cancel{uv} - \cancel{uv}$$
$$\frac{\partial(x,y)}{\partial(u,v)} = 1+u+v \,//$$

2) If $u = x^2-2y$, $v = x+y+z$, $w = x-2y+3z$,
then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol) Given $u = x^2-2y$
$$\frac{\partial u}{\partial x} = 2x-0 = 2x$$
$$\frac{\partial u}{\partial y} = 0-2(1) = -2$$
$$\frac{\partial u}{\partial z} = 0-0 = 0$$

Let $v = x+y+z$
$$\frac{\partial v}{\partial x} = 1+0+0 = 1$$
$$\frac{\partial v}{\partial y} = 0+1+0 = 1$$
$$\frac{\partial v}{\partial z} = 0+0+1 = 1$$

Let $w = x-2y+3z$
$$\frac{\partial w}{\partial x} = 1-0+0 = 1$$
$$\frac{\partial w}{\partial y} = 0-2(1)+0 = -2$$
$$\frac{\partial w}{\partial z} = 0-0+3(1) = 3$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} \overset{+}{2x} & \overset{-}{-2} & \overset{+}{0} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 2x(3+2) + 2(3-1) + 0$$

$$= 2x(5) + 2(2)$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 10x + 4 \;//$$

$$= \cancel{10x=1+4}$$

3] find $J\left[\frac{u,v}{x,y}\right]$ of the following.

i) $u = e^x$, $v = e^y$

ii) $u = \frac{y^2}{x}$, $v = \frac{x^2}{y}$

iii) $u = x\cos y$, $v = y\sin x$

iv) $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$

Sol) i] Let $u = e^x$, $v = e^y$

$u_x = e^x$, $u_y = 0$

$v_x = e^0 = 0$, $v_y = e^y$

$$\therefore J\left[\frac{u,v}{x,y}\right] = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} e^x & 0 \\ 0 & e^y \end{vmatrix}$$

$$= e^x e^y - 0$$

$$= e^{x+y} \;//$$

ii) Let $u = \frac{y^2}{x}$ $v = \frac{x^2}{y}$

$u_x = \frac{-y^2}{x^2}$, $u_y = \frac{2y}{1}\left[\frac{1}{x}\right] = \frac{2y}{x}$

$v_x = \frac{2x}{y}$, $v_y = \frac{-x^2}{y^2}$

$$\therefore J\left[\frac{u,v}{x,y}\right] = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & \frac{-x^2}{y^2} \end{vmatrix}$$

$$= \frac{-\cancel{y^2}}{\cancel{x^2}} \times \frac{-\cancel{x^2}}{\cancel{y^2}} - \frac{2\cancel{y}}{\cancel{x}} \times \frac{2\cancel{x}}{\cancel{y}}$$

$$= 1 - 4 = -3$$

iii) Let $u = x\cos y$, $v = y\sin x$

$u_x = (1)\cos y = \cos y$, $u_y = x[-\sin y] = -x\sin y$

$v_x = y(\cos x) = y\cos x$, $v_y = 1[\sin x] = \sin x$

$$\therefore J\left[\frac{u,v}{x,y}\right] = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \cos y & -x\sin y \\ y\cos x & \sin x \end{vmatrix}$$

$$= \cos y(\sin x) + xy\cos x\sin y$$

$$= \sin x\cos y + xy\cos x\sin y$$

iv) Let $u = \frac{2x-y}{2} = x - \frac{y}{2}$ $v = \frac{y}{2}$

$u_x = 1 - 0 = 1$ $u_y = 0 - \frac{1}{2} = \frac{-1}{2}$

$v_x = 0 = 0$ $v_y = \frac{1}{2}$

$$\therefore J\left[\frac{u,v}{x,y}\right] = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$\begin{vmatrix} 1 & \frac{-1}{2} \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2} - 0 = \frac{1}{2}$$ //

4) If $x = u(1-v)$, $y = uv$ then prove that $JJ' = 1$

Sol) Let $x = u - uv$ Let $y = uv$

$x_u = 1-(1)v = 1-v$ $y_u = (1)v = v$

$x_v = 0-(1)u = -u$ $y_v = u(1) = u$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= (1-v)u + u(v)$$

$$= u - \cancel{uv} + \cancel{uv}$$

$$J = u$$

~~Let $x = u(1+y) \Rightarrow u = \frac{x}{1+y}$~~

Let $x = u(1-v)$ Let $y = uv$

$x = u - uv$ $v = \frac{y}{u}$

$x = u - y$ $\boxed{v = \frac{y}{x+y}}$

$\boxed{u = x+y}$

$\therefore u_x = 1+0 = 1$

$\therefore u_y = 0+1 = 1$

$$v_x = y\left[\frac{-1}{(x+y)^2} \times (1+0)\right]$$

$$\therefore v_x = \frac{-y}{(x+y)^2}$$

$$v_y = \frac{x+y\cdot(1) - y\cdot(0+1)}{(x+y)^2}$$

$$v_y = \frac{x+\cancel{y}-y}{(x+y)^2}$$

$$\therefore v_y = \frac{x}{(x+y)^2}$$

$J' = \begin{vmatrix} u_x \\ v_x \end{vmatrix}$

$= \begin{vmatrix} 1 \\ \frac{-y}{(x+y)^2} \end{vmatrix}$

$= \frac{x}{(x+y)}$

$= \frac{x}{(x}$

$=$

$\therefore J'$

$\therefore J \cdot J' =$

5) If $y_1 =$

Show tha

$u = y$

Sol) Let y_1

$\frac{\partial y_1}{\partial x_1} =$

$\frac{\partial y_1}{\partial x_2} =$

$\frac{\partial y_1}{\partial x_3}$

Le

$\frac{\partial y_3}{\partial x_1}$

$\frac{\partial y_3}{\partial x_2}$

$$J' = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{x+y}{(x+y)^2}$$

$$= \frac{1}{x+y} = \frac{1}{u}$$

$$\therefore J' = \frac{1}{u}$$

$$\therefore J \cdot J' = u \cdot \frac{1}{u} = 1$$

5) If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_1 x_3}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$ then

*** Show that $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4$

(or)

$u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$, S.T $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

Sol) Let $y_1 = \frac{x_2 x_3}{x_1}$

$$\frac{\partial y_1}{\partial x_1} = x_2 x_3 \left(\frac{-1}{x_1^2}\right) = \frac{-x_2 x_3}{x_1^2}$$

$$\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}[1] = \frac{x_3}{x_1}$$

$$\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}[1] = \frac{x_2}{x_1}$$

Let $y_2 = \frac{x_1 x_3}{x_2}$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}[1] = \frac{x_3}{x_2}$$

$$\frac{\partial y_2}{\partial x_2} = \frac{x_1 x_3}{1}\left(\frac{-1}{x_2^2}\right) = \frac{-x_1 x_3}{x_2^2}$$

$$\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}(1) = \frac{x_1}{x_2}$$

Let $y_3 = \frac{x_1 x_2}{x_3}$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}(1) = \frac{x_2}{x_3}$$

$$\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}(1) = \frac{x_1}{x_3}$$

$$\frac{\partial y_3}{\partial x_3} = \frac{x_1 x_2}{1} = x_1 x_2\left[\frac{-1}{x_3^2}\right] = \frac{-x_1 x_2}{+x_3^2}$$

$$\therefore \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_1 x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{-x_2 x_3}{x_1^2}\left[\frac{-x_1 \cancel{x_3}}{x_2^2}\left(\frac{-x_1 \cancel{x_2}}{x_3^2}\right) - \frac{x_1}{x_2}\left(\frac{x_1}{x_3}\right)\right] - \frac{x_3}{x_1}\left[\frac{\cancel{x_3}}{\cancel{x_2}}\left(\frac{-x_1 \cancel{x_2}}{x_3^2}\right) - \frac{x_1}{\cancel{x_2}}\left(\frac{\cancel{x_2}}{x_3}\right)\right] + \frac{x_2}{x_1}\left[\frac{\cancel{x_3}}{x_2}\left(\frac{x_1}{\cancel{x_3}}\right) + \frac{x_1 \cancel{x_3}}{x_2^2}\left(\frac{\cancel{x_2}}{\cancel{x_3}}\right)\right] = 0$$

$$= \frac{-x_2 x_3}{x_1^2}\left[\frac{x_1^2}{x_2 x_3} - \frac{x_1^2}{x_2 x_3}\right] - \frac{x_3}{x_1}\left(\frac{-x_1}{x_3} - \frac{x_1}{x_3}\right) + \frac{x_2}{x_1}\left(\frac{x_1}{x_2} + \frac{x_1}{x_2}\right)$$

$$= \frac{-x_2 x_3}{x_1^2}[0] + \frac{\cancel{x_3}}{\cancel{x_1}}\left(\frac{2\cancel{x_1}}{\cancel{x_3}}\right) + \frac{\cancel{x_2}}{\cancel{x_1}}\left[\frac{2\cancel{x_1}}{\cancel{x_2}}\right]$$

$$= 2 + 2$$

$$= 4 \;//$$

6) If $u = x+y+z$, $uv = y+z$, $uvw = z$ then show that $\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$

Sol) Given $u = x+y+z \longrightarrow ①$

$uv = y+z \longrightarrow ②$

$uvw = z \longrightarrow ③$

from ① => $u = x+y+z$

$x = u - y - \ldots$

$x = u - (\ldots$

$\boxed{x = u - \ldots}$

from ③ => $\boxed{z = u \ldots}$

Let $x = u - uv$

$\frac{\partial x}{\partial u} = 1 - v(1) = 1 - v$

$\frac{\partial x}{\partial v} = 0 - u(1) = -u$

$\frac{\partial x}{\partial w} = 0 - 0 = 0$

$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} \ldots \\ \frac{\partial y}{\partial u} \ldots \\ \frac{\partial z}{\partial u} \ldots \end{vmatrix}$

$= \begin{vmatrix} 1- \ldots \\ v - v \ldots \\ vw \ldots \end{vmatrix}$

$= 1 - v[\ldots$

$= 1 - v[u \ldots$

$= u^2 v - u^2 \ldots$

$= u^2 v \;//$

7) If $x = r\cos\theta$ ...

$\frac{\partial(x, y)}{\partial(r, \theta)} \cdot \frac{\partial(r, \ldots}{\partial(x \ldots}$

from ① $\Rightarrow$ $u = x+y+z$

$x = u - y - z$

$x = u - [y+z]$

$\boxed{x = u - uv}$

from ② $\Rightarrow$ $uv = y+z$

$y = uv - z$

$\boxed{y = uv - uvw}$

from ③ $\Rightarrow$ $\boxed{z = uvw}$

Let $x = u - uv$

$\frac{\partial x}{\partial u} = 1 - v(1) = 1 - v$

$\frac{\partial x}{\partial v} = 0 - u(1) = -u$

$\frac{\partial x}{\partial w} = 0 - 0 = 0$

Let $y = uv - uvw$

$\frac{\partial y}{\partial u} = (1)v - (1)vw = v - vw$

$\frac{\partial y}{\partial v} = u(1) - uw(1) = u - uw$

$\frac{\partial y}{\partial w} = 0 - uv(1) = -uv$

Let $z = uvw$

$\frac{\partial z}{\partial u} = (1)vw = vw$

$\frac{\partial z}{\partial v} = (1)uw = uw$

$\frac{\partial z}{\partial w} = (1)uv = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= 1-v[u-uw(uv)+uv(wu)] + u[v-vw(uv)+uv(vw)]$$

$$= 1-v[u^2v - u^2vw + u^2vw] + u[uv^2 - uv^2w + uv^2w]$$

$$= u^2v - u^2\cancel{v}w + u^2\cancel{v}w - u^2\cancel{v}^2 - \cancel{u}^2v^2w + u^2\cancel{v}w + \cancel{u}^2v^2 - u\cancel{v}^2w + u^2\cancel{v}^2w$$

$$= u^2v \quad //$$

7) If $x = r\cos\theta$, $y = r\sin\theta$ then show that

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

Sol) Let $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = (1)\cos\theta = \cos\theta \qquad \frac{\partial y}{\partial r} = (1)\sin\theta = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta .$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= \cos\theta\, r\cos\theta + r\sin\theta\sin\theta$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r[\sin^2\theta + \cos^2\theta]$$

$$J = r$$

$\frac{\partial(r,\theta)}{\partial(x,y)}$ where $r = \frac{x}{\cos\theta}$, $r = \frac{y}{\sin\theta}$

$$\frac{\partial r}{\partial x} = \frac{1}{\cos\theta}$$

$$\frac{\partial r}{\partial y} = \frac{1}{\sin\theta}$$

[Or]

Let $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$$= r^2[\cos^2\theta + \sin^2\theta]$$

$$= r^2[1]$$

$$x^2 + y^2 = r^2$$

Let $\frac{y}{x} = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}$

$$\frac{y}{x} = \tan\theta$$

$$\boxed{\theta = \tan^{-1}\frac{y}{x}}$$

Let $r^2 = x^2+y^2 = \quad r = \sqrt{x^2+y^2}$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \times \cancel{2}x$$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{x}{\sqrt{r^2}}$$

$$= \frac{x}{r}$$

$$= \frac{\cancel{r}\cos\theta}{\cancel{r}}$$

$$\boxed{\frac{\partial r}{\partial x} = \cos\theta}$$

$$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \times \cancel{2}y$$

$$= \frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{y}{\sqrt{r^2}}$$

$$= \frac{y}{r}$$

$$= \frac{\cancel{r}\sin\theta}{\cancel{r}}$$

$$\boxed{\frac{\partial r}{\partial y} = \sin\theta}$$

Let $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times y \times \frac{-1}{x^2}$$

$$= \frac{\cancel{x^2}}{x^2+y^2} \times \frac{-y}{\cancel{x^2}}$$

$$= \frac{-y}{x^2+y^2}$$

$$= \frac{-\cancel{r}\sin\theta}{r^{\cancel{2}}}$$

$$\boxed{\frac{\partial\theta}{\partial x} = \frac{-\sin\theta}{r}}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x}$$

$$= \frac{x^{\cancel{2}}}{x^2+y^2} \times \frac{1}{\cancel{x}}$$

$$= \frac{x}{x^2+y^2}$$

$$= \frac{\cancel{r}\cos\theta}{r^{\cancel{2}}}$$

$$\boxed{\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix}$$

$$= \frac{\cos^2\theta}{r} + \frac{\sin^2\theta}{r}$$

$$= \frac{1}{r}[\sin^2\theta + \cos^2\theta]$$

$\Rightarrow \frac{1}{r}(1)$ -

$J' \Rightarrow \frac{1}{r}$

$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$ //

8] If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ then show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

Sol) Let $u = x^2 - y^2$, $v = 2xy$

$u = r^2\cos^2\theta - r^2\sin^2\theta$

$u = r^2[\cancel{\sin^2\theta} - \cancel{\cos^2\theta}]$

$u = r^2[\cos^2\theta - \sin^2\theta]$

$\boxed{u = r^2[\cos 2\theta]}$

$v = 2r\cos\theta \, r\sin\theta$

$v = 2r^2 \sin\theta\cos\theta$

$v = r^2 \, 2\sin\theta\cos\theta$

$\boxed{v = r^2 \sin 2\theta}$

Let $u = r^2\cos 2\theta$ $\qquad$ $v = r^2 \sin 2\theta$

$\frac{\partial u}{\partial r} = 2r\cos 2\theta$ // $\qquad$ $\frac{\partial v}{\partial r} = 2r\sin 2\theta$ //

$\frac{\partial u}{\partial \theta} = r^2(-\sin 2\theta)(2)$ $\qquad$ $\frac{\partial v}{\partial \theta} = r^2[\cos 2\theta]2$

$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta$ // $\qquad$ $\frac{\partial v}{\partial \theta} = 2r^2\cos 2\theta$ //

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix}$$

$$= 2r\cos 2\theta[2r^2\cos 2\theta] + 2r^2\sin 2\theta[2r\sin 2\theta]$$

$$= 4r^3\cos^2 2\theta + 4r^3 \sin^2 2\theta$$

$= 4r^3 [\sin^2\theta + \cos^2\theta]$

$= 4r^3 [1]$

$= 4r^3$

9) If $U = \frac{x+y}{1-xy}$, $V = \tan^{-1}x + \tan^{-1}y$ then find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol) Let $U = \frac{x+y}{1-xy}$

$$\frac{\partial U}{\partial x} = \frac{(1-xy)\cdot(1) - (x+y)(0-y)}{(1-xy)^2}$$

$$= \frac{(1-xy) + (xy+y^2)}{(1-xy)^2}$$

$$= \frac{1 - \cancel{xy} + \cancel{xy} + y^2}{(1-xy)^2}$$

$$= \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial U}{\partial y} = \frac{(1-xy)(1) - x+y(0-x)}{(1-xy)^2}$$

$$= \frac{(1-xy) + (x^2+xy)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2}$$

$$= \frac{1+x^2}{(1-xy)^2}$$

Let $V = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} + 0 \qquad \frac{\partial v}{\partial y} = 0 + \frac{1}{1+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} \qquad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$= 0$$

Functional dependence :-

If the functions u and v of the independent variables x and y are functionally dependent then the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ vanishes. $\left[\frac{\partial(u,v)}{\partial(x,y)} = 0\right]$

Problems :-

1) If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$. then prove that u and v are functionaly dependent and find the relation between them.

Sol) Let $u = \frac{x+y}{1-xy}$

$$\frac{\partial u}{\partial x} = \frac{[1-xy][1] - (x+y)(0-y)}{(1-xy)^2}$$

$$= \frac{[1-xy] + xy - y^2}{[1-xy]^2}$$

$$= \frac{1 - \cancel{xy} + \cancel{xy} - y^2}{(1-xy)^2}$$

$$= \frac{1-y^2}{(1-xy)^2}$$

le

$$\frac{\partial u}{\partial y} = \frac{(1-xy)(1) - (x+y)(0-x)}{(1-xy)^2}$$

$$= \frac{1 - \cancel{xy} + x^2 - \cancel{xy}}{(1-xy)^2}$$

$$= \frac{1+x^2}{(1-xy)^2}$$

Let $v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} \qquad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1-y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2}$$

$$= 0$$

$\therefore$ u, v ar

Relation:- Let

2) Show that

$v = x^2 + y^2 +$

related. f

Sol]

$\frac{\partial u}{\partial y}$

$\frac{\partial u}{\partial z}$

Let

$\frac{\partial v}{\partial x}$

Le

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$= 0 \,//$$

$\therefore$ u, v are functionally dependent.

Relation:- Let $v = \tan^{-1}x + \tan^{-1}y$

$$v = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\boxed{v = \tan^{-1} u}$$

(or)

$$\boxed{u = \tan v}$$

2) Show that the functions $u = xy+yz+zx$, $v = x^2+y^2+z^2$, $w = x+y+z$ are functionally related. find the relation b/w them?

Sol) Let $U = xy+yz+zx$

$$\frac{dU}{dx} = (1)y + 0 + z[1] = y+z$$

$$\frac{dU}{dy} = x(1) + (1)(z) + 0 = x+z$$

$$\frac{dU}{dz} = 0 + y[1] + (1)[x] = y+x$$

Let $V = x^2+y^2+z^2$

$$\frac{dV}{dx} = 2x \qquad \frac{dV}{dy} = 2y \qquad \frac{dV}{dz} = 2z$$

Let $W = x+y+z$

$$\frac{dW}{dx} = 1 \qquad \frac{dW}{dy} = 1 \qquad \frac{dW}{dz} = 1$$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= y+z\,(2y-2z) - x+z\,[2x-2z] + y+x\,[2x-2y]$$

$$= \cancel{2y^2} + \cancel{2yz} - \cancel{2yz} - \cancel{2z^2} - \cancel{2x^2} - \cancel{2zx} + \cancel{2zx} + \cancel{2z^2} + \cancel{2xy} + \cancel{2x^2} - \cancel{2y^2} - \cancel{2xy}$$

$$= 0 \,//$$

$\therefore$ u, v, w are functionally dependent.

Relation :- Let $w = x + y + z$

$$w^2 = (x+y+z)^2$$

$$w^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$$w^2 = x^2+y^2+z^2 + 2[xy + yz + zx]$$

$$\boxed{w^2 = v + 2u}$$

3) verify $u = 2x - y + 3z$, $v = 2x - y - z$, $w = 2x - y + z$ are functionally dependent & if so find the relation b/w them.

A) Let $u = 2x - y + 3z$ Let $v = 2x - y - z$

$$\frac{du}{dx} = 2(1) - 0 + 0 = 2 \qquad \frac{dv}{dx} = 2(1) - 0 - 0 = 2$$

$$\frac{du}{dy} = 0 - 1 + 0 = -1 \qquad \frac{dv}{dy} = 0 - 1 - 0 = -1$$

$$\frac{du}{dz} = 2(0) - 0 + 3(1) = 3 \qquad \frac{dv}{dz} = 0 - 0 - 1 = -1$$

Let $w = 2x - y + z$

$$\frac{dw}{dx} = 2(1) - 0 + 0 = 2$$

$$\frac{dw}{dy} = 2(0) - 1 + 0 = -1$$

$$\frac{dw}{dz} = 2(0) - 0 + 1 = 1$$

then
$$\frac{d(u,v)}{d(x,y)} = \begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= 2(-1-1) + 1(2+2) + 3(-2+2)$$

$$= -4 + 4$$

$$= 0$$

$\therefore$ u, v, w

Relation :-

4) i) $u = e^x s$

ii) $u = \frac{x}{y}$

Sol) i) Let $u =$

$\frac{du}{dx} = e$

$\frac{du}{dy} = e$

$\therefore$ u,

no re

ii) Le

$\frac{du}{dx}$

$\frac{du}{dy}$

$\therefore u, v, w$ are functionally dependent.

Relation:- Let $u+v$

$\Rightarrow 2x-y+3z+2x-y-z$

$\Rightarrow 4x-2y+2z$

$\Rightarrow 2[2x-y+z]$

$\boxed{u+v \Rightarrow 2[w]}$

4) i) $u=e^x \sin y$, $v=e^x \cos y$

ii) $u=\frac{x}{y}$, $v=\frac{x+y}{x-y}$

Sol) i) Let $u=e^x \sin y$ Let $v=e^x \cos y$

$\frac{du}{dx}=e^x \sin y$ $\frac{dv}{dx}=e^x \cos y$

$\frac{du}{dy}=e^x \cos y$ $\frac{dv}{dy}=e^x(-\sin y)=-e^x \sin y$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix}$$

$$=e^x(\sin y)[-e^x \sin y]-e^x \cos y[e^x \cos y]$$

$$=-e^{x^2}\sin^2 y-e^{x^2}\cos^2 y$$

$$=-e^{x^2}[\sin^2 y-\cos^2 y]$$

$$=-e^{x^2}[\cos 2y]$$

$\therefore u, v$ are not functionally dependent and having no relation.

ii) Let $u=\frac{x}{y}$

$\frac{du}{dx}=\frac{1}{y}(1)=\frac{1}{y}$

$\frac{du}{dy}=x\left[\frac{-1}{y^2}\right]=\frac{-x}{y^2}$

Let $v=\frac{x+y}{x-y}$

$\frac{dv}{dx}=\frac{(x-y)(1)-x+y(1)}{(x-y)^2}$

$=\frac{\cancel{x}-y-\cancel{x}-y}{(x-y)^2}$

$=\frac{-2y}{(x-y)^2}$

$$\frac{dv}{dy} = \frac{(x-y)(1)+(x+y)(-1)}{(x-y)^2}$$

$$= \frac{x-y+x+\cancel{y}}{(x-y)^2}$$

$$= \frac{+2x}{(x-y)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{y} & \frac{-x}{y^2} \\ \frac{-2y}{(x-y)^2} & \frac{+2x}{(x-y)^2} \end{vmatrix}$$

$$= \frac{1}{y}\left(\frac{+2x}{(x-y)^2}\right) - \frac{x}{y^2}\left(\frac{2y}{(x-y)^2}\right)$$

$$= \frac{2x}{y(x-y)^2} - \frac{2xy}{y^2(x-y)^2}$$

$$= \frac{2x}{y(x^2+y^2-2xy)} - \frac{2xy}{y^2[x^2+y^2-2xy]}$$

$$= \boxed{\frac{2x}{x^2y+y^3-2xy^2} - \frac{2xy}{x^2y^2+y^4-2xy^4}} \times$$

$$= \frac{2x[1-y]}{y[x-y)^2}$$

$$= \frac{2x}{y(x-y)^2} - \frac{2x}{y(x-y)^2}$$

$$= 0$$

$\therefore$ u, v are functionally dependent

Relation :- $u = \frac{x}{y}$ $v = \frac{x+y}{x-y}$

$$\Rightarrow \frac{\cancel{y}\left(\frac{x}{y}+1\right)}{\cancel{y}\left[\frac{x}{y}-1\right]}$$

$$\Rightarrow \frac{u+1}{u-1}$$

Maximum
variables
working
values of
Le
x and y
step 1 :
solve th
Let (a,
values.
Step 2 :
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Step 3 :
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Maximum and minimum of functions of two variables :-

Working rule to find the maximum (or) minimum values of f(x,y)

Let $f(x,y)$ be a function of two variables x and y

Step 1 :- find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate each to zero. Solve these equations for x and y.

Let (a_1,b_1), (a_2,b_2) ...... be the pairs of values.

Step 2 :- find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ for each pair of values obtained in step 1.

Step 3 :- i) If $rt - s^2 > 0$ and $r < 0$ at (a_1,b_1) then (a_1,b_1) is a point of maximum and $f(a_1,b_1)$ is maximum value.

ii) If $rt - s^2 > 0$ and $r > 0$ at (a_1,b_1) then (a_1,b_1) is a point of maximum and $f(a_1,b_1)$ is minimum value.

iii) If $rt - s^2 < 0$ at (a_1,b_1) then $f(a_1,b_1)$ is not an extreme value. i.e, there is neither a maximum nor a minimum at (a_1,b_1). In this case (a_1,b_1) is a saddle point

iv) If $rt - s^2 = 0$ at (a_1,b_1) then no conclusion can be drawn about maximum or minimum and needs further investigation

similarly exami in the other pairs of values (a_2,b_2), (a_3,b_3) .... one by one

Problems :

1) If $f(x,y) = xy + (x-y)$ then find the stationary points ?

Sol) Given $f(x,y) = xy + (x-y)$

Now $\frac{df}{dx} = (1)y + (1-0) = y+1$

$\frac{df}{dy} = x(1) + (0-1) = x-1$

Let $\frac{df}{dx} = 0 \Rightarrow y+1 = 0$

$\boxed{y = -1}$

$\frac{df}{dy} = 0 \Rightarrow x-1 = 0$

$\boxed{x = 1}$

$\therefore (1,-1)$ are the stationary points.

2) Find the maximum and minimum values of x^3+y^3-3axy.

Sol) Let $f(x,y) = x^3+y^3-3axy$

$\frac{df}{dx} = 3x^2+0-3ay(1) = 3x^2-3ay$

$\frac{df}{dy} = 0+3y^2-3ax(1) = 3y^2-3ax$

Let $\frac{df}{dx} = 0 \Rightarrow 3x^2-3ay = 0$

$\cancel{3}x^2 = \cancel{3}ay$

$\boxed{y = \frac{x^2}{a}}$

$\frac{df}{dy} = 0 \Rightarrow 3y^2-3ax = 0$

$\cancel{3}y^2 = \cancel{3}ax$

$\boxed{x = \frac{y^2}{a}}$

from
Let

$\therefore$ Stat

Now, r

$s = \frac{d^2 f}{dx}$

At (a,
Now rt

Let r=
Here rt
Minimum

from ① & ②

Let $y = \frac{x^2}{a}$ Let $x = \frac{y^2}{a}$

$y = \frac{\left(\frac{y^2}{a}\right)^2}{a}$ $x = \frac{\left(\frac{x^2}{a}\right)^2}{a}$

$y = \frac{y^4}{a^3}$ $x = \frac{x^4}{a^3}$

$1 = \frac{y^3}{a^3}$ $1 = \frac{x^3}{a^3}$

$y^3 = a^3$ $x^3 = a^3$

$\boxed{y = a}$ $\boxed{x = a}$

$\therefore$ stationary point is (a, a)

Now, $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]$

$= \frac{\partial}{\partial x}\left[3x^2 - 3ay\right)$

$= 3(2x) - 0$

$= 6x$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$

$= \frac{\partial}{\partial x}\left[3y^2 - 3ax\right]$

$= 0 - 3a$

$= -3a$

$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]$

$= \frac{\partial}{\partial y}\left[3y^2 - 3ax\right]$

$= 3[2y] - 0$

$= 6y$

At $[a, a]$

Now $rt - s^2 = (6x)[6y] - (-3a]^2$

$= 36xy - 9a^2$

$= 36a^2 - 9a^2$

$rt - s^2 = 27a^2 > 0$

Let $r = 6x = 6a > 0$

Here $rt - s^2 > 0$, $r > 0$ then (a, a) is point of minimum.

Minimum value $= f(a, a) = a^3 + a^3 - 3a(a)(a)$

$= 2a^3 - 3a^3$

$= -a^3$ //

3) Find the maximum and minimum values of
$f(x,y) = x^3+3xy^2-3x^2-3y^2+4$

Sol) Given $f(x,y) = x^3+3xy^2-3x^2-3y^2+4$

Now $\frac{df}{dx} = 3x^2+3y^2(1)-3(2x)-0+0$
$= 3x^2+3y^2-6x$

$\frac{df}{dy} = 0+3x(2y)-0-3[2y]+0$
$= 6xy-6y$

Let $\frac{df}{dx} = 0 \Rightarrow 3x^2+3y^2-6x = 0$
$\Rightarrow x^2+y^2-2x = 0$

$\frac{df}{dy} = 0 \Rightarrow 6xy-6y = 0$
$6y[x-1] = 0$
$6y = 0$ (or) $x-1 = 0$
$y = 0$ $\quad x = 1$

If $x=1$ then, $1^2+y^2-2(1) = 0$
$y^2-1 = 0$
$y^2 = 1$
$y = \pm 1$

Here $x=1$ then $y=1 \Rightarrow [1,1]$
$x=1$ then $y=-1 \Rightarrow [1,-1]$

If $y=0$ then $\Rightarrow x^2+0-2x = 0$
$\Rightarrow x^2-2x = 0$
$\Rightarrow x[x-2] = 0$
$x = 0$ (or) $x = 2$

Here $y=0$ then $x=0 \Rightarrow [0,0]$
$y=0$ then $x=2 \Rightarrow [2,0]$

$\therefore$ stationary points are $[0,0]$, $[2,0]$ $[1,1]$ $[1,-1]$

Now $r = \frac{d^2f}{dx^2} = \frac{d}{dx}\left[\frac{df}{dx}\right] \Rightarrow \frac{d}{dx}\left[3x^2+3y^2-6x\right]$

$\Rightarrow 3(2x)+0-6(1)$

$= 6x-6$

$s = \frac{d^2f}{dxdy} = \frac{d}{dx}\left[\frac{df}{dy}\right] = \frac{d}{dx}\left[6xy-6y\right]$

$= 6y[1]-0$

$= 6y$

$t = \frac{d^2f}{dy^2} = \frac{d}{dy}\left[\frac{df}{dy}\right] = \frac{d}{dy}\left[6xy-6y\right]$

$= 6x[1]-6(1)$

$= 6x-6$

At $[0,0]$:-

$rt-s^2 = 6x-6[6x-6]-(6y)^2$

$= 0-6(0-6)-0$

$= 36 > 0$

Let $r = 6x-6 \Rightarrow 6(0)-6 = -6 < 0$

$\therefore (0,0)$ is a point of maximum

$\therefore$ maximum value $= f(0,0)$

$= x^3+3xy^2-3x^2-3y^2+4$

$= 0+0+0-0+4$

$= 4$ //

At $[2,0]$:-

$rt-s^2 = (6x-6)(6x-6)-(6y)^2$

$= 6(2)-6\,(6(2)-6)-(6(0))^2$

$= 12-6[12-6]-0$

$= 6\cdot 6$

$= 36 > 0$

Let $r = 6x-6 \Rightarrow 6[2]-6 = 12-6 = 6 > 0$

$\therefore [2,0]$ is a point of minimam

$\therefore$ Minimum Value $= f(2,0)$

$= (2)^3+3(2)(0)^2-3(2)^2-3(0)^2+4$

$= 8-12+4 = 0$ //

At [1,1] :-

$rt - s^2 = (6x-6)(6x-6) - (6y)^2$

$= 6(1)-6\,[6(1)-6] - 6(1)^2$

$= 6-6\,[6-6] - 36$

$= -36 < 0$

$r = 6x-6 \Rightarrow 6(1)-6 = 0 \rightarrow$ No Need to find to step 3

∴ [1,1] is a Saddle point

i.e, At [1,1] the given function $f(x,y)$ has neither maximum nor minimum.

At [1,-1] :-

$rt - s^2 = (6x-6)(6x-6) - (6y)^2$

$= [6(1)-6][6(1)-6] - [6[-1]]^2$

$= (6-6)(6-6) - 36$

$= -36 < 0$

∴ [1,-1] is a saddle point having neither maximum nor minimum.

4) Examine the function $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$ for extreme values.

Sol) Given function $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$

Now $\frac{df}{dx} = 4x^3+0-2(2x)+4y(1)-0$

$= 4x^3-4x+4y$

$\frac{df}{dy} = 0+4y^3-0+4x(1)-2(2y)$

$= 4y^3+4x-4y$

Let $\frac{df}{dx} = 0$ $\qquad$ $\frac{df}{dy} = 0$

$4x^3-4x+4y = 0$ $\qquad$ $4y^3+4x-4y = 0$

$= 4[x^3-x+y] = 0$

eq

$\Rightarrow x^3+y^3 =$

$\Rightarrow (x+y)$

then

put

If $y=0$

If $y=\sqrt{2}$

If $y=-\sqrt{2}$

Stationary

$r = \frac{d^2f}{dx^2}$

$t = \frac{d^2f}{dy^2}$

$s = \frac{d^2f}{dx}$

$= 4[x^3 - x + y] = 0$ $\qquad\qquad$ $4[y^3 + x - y] = 0$

$= x^3 - x + y = 0 \longrightarrow ①$ $\qquad\qquad$ $y^3 + x - y = 0 \longrightarrow ②$

eq ① + eq ②

$\Rightarrow x^3 - \cancel{x} + \cancel{y} + y^3 + \cancel{x} - \cancel{y} = 0$

$\Rightarrow x^3 + y^3 = 0$

$\Rightarrow (x+y)(x^2 - xy + y^2) = 0$

then $x + y = 0$ (or) $x^2 - xy + y^2 = 0$

$\boxed{x = -y}$

from ① put $x = -y$ then,

$\Rightarrow x^3 - (-y) + y = 0$

$\Rightarrow x^3 + 2y = 0$

$\Rightarrow -y^3 + 2y = 0$

$\Rightarrow -y[y^2 - 2] = 0$

$\Rightarrow y = 0$ (or) $y^2 = 2$

$y = \pm\sqrt{2}$

If $y = 0$ then $x = 0$ then $(0,0)$

If $y = \sqrt{2}$ then $x = -\sqrt{2}$ then $[-\sqrt{2}, \sqrt{2}]$

If $y = -\sqrt{2}$ then $x = \sqrt{2}$ then $[\sqrt{2}, -\sqrt{2}]$

Stationary points are $(0,0)$, $(-\sqrt{2}, \sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$

$$r = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) \Rightarrow 4(3x^2) - 4(1) + 0$$

$$\Rightarrow 12x^2 - 4$$

$$t = \frac{d^2f}{dy^2} = \frac{d}{dy}\left(\frac{df}{dy}\right) = 4(3y^2) + 0 - 4$$

$$= 12y^2 - 4$$

$$s = \frac{d^2f}{dx\,dy} = \frac{d}{dx}\left[\frac{df}{dy}\right]$$

$$= 4(1) - 4(0)$$

$$= 4$$

At $[0,0]$

$$rt - s^2 = (12x^2-4)(12y^2-4) - 4^2$$
$$= (0-4)(0-4) - 16$$
$$= 16 - 16$$
$$= 0$$

At $(0,0)$ no conclusion can be drawn needs for the investigation.

At $[-\sqrt{2}, \sqrt{2}]$

$$rt - s^2 = (12x^2-4)(12y^2-4) - 4^2$$
$$= (12(-\sqrt{2})^2-4)(12(\sqrt{2})^2-4) - 16$$
$$= (24-4)(24-4) - 16$$
$$= (20)(20) - 16$$
$$= 400 - 16$$
$$= 384 > 0$$

Let $r = 12x^2 - 4 \Rightarrow 12(-\sqrt{2})^2 - 4 = 20 > 0$

At $[-\sqrt{2}, \sqrt{2}]$ is a point of minimum

$\therefore$ minimum value is $f(-\sqrt{2}, \sqrt{2}) =$

$$\Rightarrow (-\sqrt{2})^4 + (\sqrt{2})^4 - 2(\sqrt{2})^2 + 4(-\sqrt{2})(\sqrt{2}) - 2(\sqrt{2})^2$$
$$\Rightarrow \cancel{4} + \cancel{4} - \cancel{4} - 8 - \cancel{4}$$
$$= -8$$

At $[\sqrt{2}, -\sqrt{2}]$

Similarly $[\sqrt{2}, -\sqrt{2}]$ is a point of minimum and minimum value is -8.

5) Find the maximum values of the function
*** $f(x,y) = x^3 y^2 (1-x-y)$

Sol) Given $f(x,y) = x^3y^2(1-x-y)$

$$= x^3y^2 - x^4y^2 - x^3y^3$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$\frac{\partial f}{\partial y} = x^3 2y - x^4 2y - x^3 3y^2$$

$$= 2x^3y - 2x^4y - 3x^3y^2$$

Let $\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$$\Rightarrow x^2y^2[3-4x-3y] = 0$$

$x^2y^2 = 0$ [or] $3-4x-3y = 0$

$x=0,\ y=0$ or $3-4x-3y=0 \longrightarrow ①$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2x^3y - 2x^4y - 3x^3y^2$

$$\Rightarrow x^3y(2-2x-3y)$$

$x=0$ or $y=0$ (or) $2-2x-3y=0 \longrightarrow ②$

Solve ① & ②

from ① & ② if $x=0$ then $y=0 \Rightarrow (0,0)$

from ②, put $x=0$ then $2-0-3y=0$

$$2-3y=0$$
$$3y=2$$
$$y=\frac{2}{3} \Rightarrow \left(0,\frac{2}{3}\right)$$

from ②, put $y=0$ then $2-2x-0=0$

$$2-2x=0$$
$$\cancel{2}x=\cancel{2}$$
$$x=1 \Rightarrow (1,0)$$

from ①, put $x=0$ then $3-0-3y=0$

$$3-3y=0$$
$$\cancel{3}y=\cancel{3}$$
$$y=1 \Rightarrow (0,1)$$

from ①, put $y=0$ then $3-4x-0=0$

$$3-4x=0$$
$$4x=3$$
$$x=\frac{3}{4} \Rightarrow \left(\frac{3}{4},0\right)$$

Solve ① & ②

$$3 - 4x - 3y = 0$$
$$2 - 2x - 3y = 0$$
$$1 - 2x = 0$$
$$2x = 1$$
$$x = \frac{1}{2}$$

Put $x = \frac{1}{2} \Rightarrow 3 - 4\left[\frac{1}{2}\right]^2 - 3y = 0$

$$\Rightarrow 3 - 2 - 3y = 0$$
$$\Rightarrow -3y + 1 = 0$$
$$\Rightarrow 3y = 1$$
$$y = \frac{1}{3} \Rightarrow \left(\frac{1}{2}, \frac{1}{3}\right)$$

$\therefore$ stationary points are $(0,0), (0, \frac{2}{3}), (1,0), (0,1), (3/4, 0), \left(\frac{1}{2}, \frac{1}{3}\right)$

~~At (0,0)~~

~~$rt - s^2$~~

$$r = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left[\frac{df}{dx}\right]$$
$$= 3(2x)y^2 - 4(3x^2)y^2 - 3(2x)y^3$$
$$= 6xy^2 - 12x^2y^2 - 6xy^3$$

$$s = \frac{d^2 f}{dx\,dy} = \frac{d}{dx}\left[\frac{df}{dy}\right]$$
$$= 2(3x^2)y - 2(4x^3)y - 3(3x^2)y^2$$
$$= 6x^2y - 8x^3y - 9x^2y^2$$

$$t = \frac{d^2 f}{dy^2} = \frac{d}{dy}\left[\frac{df}{dy}\right]$$
$$= 2x^3(1) - 2x^4(1) - 3x^3(2y)$$
$$= 2x^3 - 2x^4 - 6x^3y$$

At $(0,0), (0, 2/3), (1,0), (0,1), (3/4, 0)$

~~$rt - s^2$~~ we have $rt - s^2 = 0$

So no conclusion can be drawn needs further investigation.

At $\left(\frac{1}{2}, \frac{1}{3}\right)$

$rt - s^2 = \frac{1}{9.6}$

$r = \frac{-1}{9}$

$(1/2, 1/3)$ is

So maximum

Sol) Given

$\frac{df}{dx} =$

from ①

cos

If

At $\left(\frac{1}{2}, \frac{1}{3}\right)$

$rt-s^2 = \frac{1}{9.64} > 0$

$r = \frac{-1}{9} < 0$

$(1/2, 1/3)$ is a point of maximum

So maximum value is $f\left[\frac{1}{2}, \frac{1}{3}\right] = \left(\frac{1}{2}\right)^3\left[\frac{1}{3}\right]^2\left\{1-\frac{1}{2}-\frac{1}{3}\right]$

$= \frac{1}{432}$ //

6) Examine for minimum and maximum values of $\sin x+\sin y+\sin(x+y)$ (or) $\sin x \sin y \sin(x+y)$, $0<x<\pi$, $0<y<\pi$.

Sol) Given $f(x,y) = \sin x+\sin y+\sin(x+y)$

$\frac{df}{dx} = \cos x + \cos(x+y)$

$\frac{df}{dy} = \cos y + \cos(x+y)$

Let $\frac{\partial f}{dx} = 0 \Rightarrow \cos x + \cos(x+y) = 0 \longrightarrow ①$

$\frac{df}{dy} = 0 \Rightarrow \cos y + \cos(x+y) = 0 \longrightarrow ②$

solve ① & ②

$$\begin{array}{r} \cos x + \cancel{\cos(x+y)} = 0 \\ \cos y + \cancel{\cos(x+y)} = 0 \\ (-) \quad (-) \\ \hline \cos x - \cos y = 0 \end{array}$$

$\cos x = \cos y$

$\boxed{x = y}$

from ①, put $x=y$ $\quad \cos A+\cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\cos y + \cos(y+y) = 0$

$\cos y + \cos 2y = 0$

$2\cos\left(\frac{3y}{2}\right)\cos\left(\frac{-y}{2}\right) = 0 \quad \cos(-\theta) = \cos\theta$

$\cos\left(\frac{3y}{2}\right) = 0$ (or) $\cos\left(\frac{+y}{2}\right) = 0$

$\frac{3y}{2} = \pm\frac{\pi}{2}$ (or) $\frac{y}{2} = \pm\frac{\pi}{2}$

$y = \pm\frac{\pi}{3}$ (or) $y = \pm\pi$

If $y = \pm\frac{\pi}{3}$ then $x = \pm\frac{\pi}{3} \Rightarrow \left[\pm\pi/3, \pm\pi/3\right]$

If $y=\pi$ then $x=\pi \Rightarrow (\pm\pi, \pm\pi)$

∴ Stationary points are $\left[\pm\pi/3, \pm\pi/3\right]$, $\left[\pm\pi, \pm\pi\right]$

$$r = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$
$$= \frac{d}{dx}\left(\cos x + \cos(x+y)\right)$$
$$= -\sin x - \sin(x+y)$$

$$s = \frac{d^2f}{dx\,dy} = \frac{d}{dx}\left(\frac{df}{dy}\right)$$
$$= \frac{d}{dx}\left(\cos y + \cos(x+y)\right)$$
$$= 0 - \sin(x+y)$$
$$= -\sin(x+y)$$

$$t = \frac{d^2f}{dy^2} = \frac{d}{dy}\left(\frac{df}{dy}\right)$$
$$= \frac{d}{dy}\left(\cos y + \cos(x+y)\right)$$
$$= -\sin y - \sin(x+y)$$

rt - At $\left(\pm\pi/3, \pm\pi/3\right)$

$$rt - s^2 = -\sin x - \sin(x+y)\left[-\sin y - \sin(x+y)\right] - \left[-\sin(x+y)\right]^2$$

$$\Rightarrow \left\{\sin\frac{\pi}{3} + \sin\frac{2\pi}{3}\right\}\left[\sin\frac{\pi}{3} + \sin\frac{2\pi}{3}\right] - \left[\sin\frac{2\pi}{3}\right]^2$$

$$\Rightarrow \left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right]\left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow 2\left(\frac{\sqrt{3}}{2}\right) 2\left(\frac{\sqrt{3}}{2}\right) - \frac{3}{4}$$

$$\Rightarrow 3 - \frac{3}{4}$$

$$\Rightarrow \frac{9}{4} > 0$$

and at $\left[-\frac{\pi}{3}, -\frac{\pi}{3}\right]$ is a point of minimium

minimum value of $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

$$= -\frac{3\sqrt{3}}{2} \ngtr 0$$

Maximum Value of $f\left(\pm\pi/3, \pm\pi/3\right) = \sin\pi/3 + \sin\pi/3 + \sin 2\pi/3$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{2}$$

At (π, π)

$rt - s^2 = -\sin x - \sin x$ ...

$= (-\sin\pi - \sin$ ...

$= (0)(0)-$ ...

$= 0$,

∴ $(\pm\pi, \pm\pi)$, no

Lagrange's method

working rule :- Le...

$\phi(x, y, z) = 0$ is

Step1 :- Form lag...

$F(x, y, z) = f$...

where λ is calle...

determined

Step2 :- Find ...

Step3 :- Solve

These (x, y, z)

Prob...

1) find the m...

that $xyz =$ ...

Sol) Given ., Let

Let $\phi(x, y, z$ ...

form lagrang...

$F = (x^2 + y^2$ ...

Let $\frac{dF}{dx} =$ ...

At (π, π)

$$rt - s^2 = -\sin x - \sin(x+y)\,[-\sin y - \sin(x+y)] - [-\sin(x+y)]^2$$
$$= [-\sin\pi - \sin(2\pi)]\,[-\sin\pi - \sin 2\pi] - [-\sin(2\pi)]^2$$
$$= (0)(0) - (0)$$
$$= 0,$$

$\therefore$ $(\pm\pi, \pm\pi)$, no conclusion can be drawn.

Lagrange's method of undetermined multiplayers:-

working rule:- Let $f(x,y,z)$ be a function and $\phi(x,y,z) = 0$ is a condition.

Step1:- Form lagrangion function.

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z) \longrightarrow (1)$$

where λ is called Lagrangion multiplier to be determined

Step2:- Find $\frac{dF}{dx} = 0 \Rightarrow \frac{df}{dx} + \lambda \cdot \frac{d\phi}{dx} \longrightarrow (2)$

$$\frac{dF}{dy} = 0 \Rightarrow \frac{df}{dy} + \lambda \cdot \frac{d\phi}{dy} \longrightarrow (3)$$

$$\frac{dF}{dz} = 0 \Rightarrow \frac{df}{dz} + \lambda \cdot \frac{df}{dz} \longrightarrow (4)$$

Step 3:- Solve (1), (2), (3), (4) equations for (x,y,z).

These (x,y,z) gives Stationary points.

Problems:-

1) find the minimum value of $x^2+y^2+z^2$, given that $xyz = a^3$

Sol) Given, Let $f(x,y,z) = x^2+y^2+z^2$

Let $\phi(x,y,z) \equiv xyz - a^3$

form lagrangion function $F[x,y,z] = f(x,y,z) + \lambda\phi(x,y,z)$

$$F = (x^2+y^2+z^2) + \lambda(xyz - a^3)$$

Let $\frac{dF}{dx} = 0 \Rightarrow (2x+0+0) + \lambda(yz(1) - 0) = 0$

$$\Rightarrow 2x + \lambda yz = 0$$
$$\Rightarrow 2x = -\lambda yz$$
$$\Rightarrow \frac{x}{yz} = \frac{-\lambda}{2} \longrightarrow (1)$$

Similarly, $\frac{dF}{dy} = 0 \Rightarrow \frac{y}{xz} = \frac{-\lambda}{2} \longrightarrow (2)$

$\frac{dF}{dz} = 0 \Rightarrow \frac{z}{xy} = \frac{-\lambda}{2} \longrightarrow (3)$

from (1) & (2) & (3), we get

$$\frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy} = \frac{-\lambda}{2}$$

Let $\frac{x}{y\cancel{z}} = \frac{y}{x\cancel{z}} \Rightarrow x^2 = y^2$

Let $\frac{y}{\cancel{x}z} = \frac{z}{\cancel{x}y} \Rightarrow y^2 = z^2$

Then we can write $x^2 = y^2 = z^2 \Rightarrow x = y = z$ (x, y, z)

Let $xyz = a^3$

$x x x = a^3$

$x^3 = a^3$

$x = a$

if $x = a$ then $y = a$ & $z = a$

$\therefore$ Stationary point is $(x, y, z) = (a, a, a)$

Minimum Value is $f(a,a,a) = a^2 + a^2 + a^2$

$= 3a^2$

2) Find the minimum value of $x^2 + y^2 + z^2$ given that $x + y + z = 3a$

Sol) Let $f(x, y, z) = x^2 + y^2 + z^2$

Let $\phi(x, y, z) = x + y + z = 3a$

form Lagrangian funtion.

$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$

$F = (x^2 + y^2 + z^2) + \lambda(x + y + z - 3a)$

Let $\frac{dF}{dx} = 0 \Rightarrow (2x + 0 + 0) + \lambda(1 + 0 + 0 - 0)$

$\Rightarrow 2x + \lambda = 0$

$\Rightarrow 2x = -\lambda$

$x = \frac{-\lambda}{2} \longrightarrow (1)$

Similarly., $\frac{dF}{dy}=0 \Rightarrow y=\frac{-\lambda}{2}$ ⟶ (2)

$\frac{dF}{dz}=0 \Rightarrow z=\frac{-\lambda}{2}$ ⟶ (3)

from (1) & (2) & (3) we get.

$x=y=z=\frac{-\lambda}{2}$

$x=y=z$

Then we can stationary points is

let $x+y+z=3a$

$x+x+x=3a$

$3x=3a$

$x=a$

If $x=a$ then $y=a$ & $z=a$

$\therefore$ stationary point is $(x,y,z)=(a,a,a)$

Minimum value of $f(a,a,a)=a^2+a^2+a^2$

$=3a^2$ //

3) Find the volume of the Largest rectangular
*** Parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol) Let $2x, 2y, 2z$ be the length, breadth, height of the largest rectangular parallelopiped.

$\therefore$ volume $= l\times b\times h$

$= 2x\,2y\,2z$

$V = 8xyz$ [say function]

Given ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Let $\phi \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

form the Lagrangeon method function.

$F(x,y,z)=V(x,y,z)+\lambda\,\phi(x,y,z)$

$= 8xyz+\lambda\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right]$

Let $\frac{\partial F}{\partial x} = 0 \Rightarrow 8(1)yz + \lambda\left[\frac{2x}{a^2} + 0 + 0 - 0\right] = 0$

$$\Rightarrow 8yz + \lambda \cdot \frac{2x}{a^2} = 0$$

$$\Rightarrow {}^{4}\cancel{8}yz = \frac{-\lambda \cancel{2}x}{a^2}$$

$$\Rightarrow \frac{a^2 yz}{x} = \frac{-\lambda}{4} \longrightarrow (1)$$

Similarly $\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{b^2 xz}{y} = \frac{-\lambda}{4} \longrightarrow (2)$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{c^2 xy}{z} = \frac{-\lambda}{4} \longrightarrow (3)$$

from (1); (2), (3) we get

$$\frac{a^2 yz}{x} = \frac{b^2 xz}{y} = \frac{c^2 xy}{z} = \frac{-\lambda}{4}$$

Let $\frac{a^2 y\cancel{z}}{x} = \frac{b^2 x\cancel{z}}{y}$

$$a^2 y^2 = b^2 x^2$$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} \longrightarrow (4)$$

$$\frac{b^2 \cancel{x} z}{y} = \frac{c^2 \cancel{x} y}{z}$$

$$b^2 z^2 = c^2 y^2$$

$$\frac{y^2}{b^2} = \frac{z^2}{c^2} \longrightarrow (5)$$

from (4) & (5) we have,

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Given ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$3\frac{x^2}{a^2} = 1$$

$$x^2 = \frac{a^2}{3}$$

$$x = \pm \frac{a}{\sqrt{3}}$$

consider $x = \frac{a}{\sqrt{3}}$ (only positive)

then., $y = \frac{b}{\sqrt{3}}$

$z = \frac{c}{\sqrt{3}}$

$\therefore$ Volume $= 8xyz$

$$= 8 \frac{a}{\sqrt{3}} \frac{b}{\sqrt{3}} \frac{c}{\sqrt{3}}$$

$$= \frac{8abc}{3\sqrt{3}} \text{ c.units}$$

4) Find the point on the plane $3x+2y+z-12=0$
*** which is nearest to the origin?

Sol) Let $P(x,y,z)$ be a point on the given plane.

Let $OP = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ here $O = (0,0,0)$ (x_1, y_1, z_1), $P = (x, y, z)$ (x_2, y_2, z_2)

$$OP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$OP = \sqrt{x^2+y^2+z^2}$$

$$OP^2 = x^2+y^2+z^2$$

Now Let $f(x,y,z) = x^2+y^2+z^2$

Given plane $3x+2y+z-12=0$

Let $\phi = 3x+2y+z-12$

Form Lagrangeon function

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$= x^2+y^2+z^2 + \lambda(3x+2y+z-12)$$

Let $\frac{dF}{dx} = 0 \Rightarrow 2x+0+0+\lambda(3+0+0-0)$

$\Rightarrow 2x+3\lambda$

$\Rightarrow 2x = -3\lambda$

$\Rightarrow \boxed{x = \frac{-3\lambda}{2}}$

$\frac{dF}{dy}=0 \Rightarrow 0+2y+0+\lambda[0+2(1)+0-0]$

$\Rightarrow 2y+2\lambda=0$

$\Rightarrow \cancel{2}y=-\cancel{2}\lambda$

$\boxed{y=-\lambda}$

$\frac{dF}{dz}=0 \Rightarrow 0+0+2z+\lambda[0+0+1-0]$

$\Rightarrow 2z+\lambda$

$\Rightarrow 2z=-\lambda$

$\boxed{z=\frac{-\lambda}{2}}$

Given plane,

Let $3x+2y+z=12$

$3\left[\frac{-3\lambda}{2}\right]+2[-\lambda]+1\left[\frac{-\lambda}{2}\right]=12$

$\frac{-9\lambda}{2}-2\lambda-\frac{\lambda}{2}=12$

$-9\lambda-4\lambda-\lambda=24$

$-14\lambda=24$

$-\lambda=\frac{\cancel{24}^{12}}{\cancel{14}_{7}}$

$\boxed{\lambda=\frac{-12}{7}}$

Now $x=\frac{-3\lambda}{2}=\frac{+3}{\cancel{2}}\left[\frac{+\cancel{12}^{6}}{7}\right]=\frac{18}{7}$

$y=-\lambda=-\left[-\frac{12}{7}\right]=\frac{12}{7}$

$z=\frac{-\lambda}{2}=-\frac{1}{\cancel{2}}\left[\frac{-\cancel{12}^{6}}{7}\right]=\frac{6}{7}$,,

Hence $\left[\frac{18}{7},\frac{12}{7},\frac{6}{7}\right]$ is the point on the given plane which is nearest to the origin

Minimum distance of $OP=\sqrt{x^2+y^2+z^2}$

$=\sqrt{\left(\frac{18}{7}\right)^2+\left(\frac{12}{7}\right)^2+\left(\frac{6}{7}\right)^2}$

$= \sqrt{\frac{504}{7\times 7}}$

$= \sqrt{\frac{72}{7}}$ //

5) find the point on the plane $x+2y+3z=4$ that is closest to the origin.

Sol) Given $P(x,y,z)$ be a point on the given point

Let $OP = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$= \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$

$OP = \sqrt{x^2+y^2+z^2}$

$OP^2 = x^2+y^2+z^2$

Now Let $f(x,y,z) = x^2+y^2+z^2$

Given plane $x+2y+3z=4$

$x+2y+3z-4=0$

Let $\phi = x+2y+3z-4=0$

from Lagrangeous theorem function

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$= x^2+y^2+z^2 + \lambda[x+2y+3z-4]$

Let $\frac{dF}{dx} = 0 \Rightarrow 2x+0+0 + \lambda[1+0+0-0]$

$\Rightarrow 2x+\lambda$

$\Rightarrow 2x = -\lambda$

$x = \frac{-\lambda}{2}$

$\frac{dF}{dy} = 0 \Rightarrow 0+2y+0+\lambda[0+2+0-0]$

$\Rightarrow 2y+2\lambda$

$\Rightarrow \cancel{2}y = -\cancel{2}\lambda$

$\boxed{y = -\lambda}$

$\frac{dF}{dz} = 0 \Rightarrow 0+0+2z+\lambda[0+0+3-0]$

$\Rightarrow 2z+\lambda = 0$

$\Rightarrow 2z = -\lambda$

$\boxed{z = \frac{-3\lambda}{2}}$

Let $x+2y+3z=4$

$$\frac{-\lambda}{2}+2(-\lambda)+3\left(\frac{-3\lambda}{2}\right)=4$$

$$-\lambda+4(-\lambda)+9(-\lambda)=8$$

$$-\lambda-4\lambda-9\lambda=8$$

$$-\lambda-13\lambda=8$$

$$-8\lambda=18$$

$$\boxed{\lambda \neq +1}$$

$$-14\lambda=8$$

$$\lambda=\frac{-8}{14} \qquad \lambda=\frac{-4}{7}$$

Now $x=\frac{-\lambda}{2}=\frac{-4}{7}\times\frac{1}{2}=\frac{-4}{14}=\frac{+2}{7}=\frac{2}{7}$

$$y=-\lambda=\frac{4}{7}$$

$$z=\frac{-3\lambda}{2}=\frac{-3\left[\frac{-4}{7}\right]}{2}=\frac{12}{7}\times\frac{1}{2}=\frac{12}{14}=\frac{6}{7}$$

Hence $\left[\frac{2}{7},\frac{-4}{7},\frac{6}{7}\right]$ is the nearest point on the plane which is nearest to origin.

$$OP \neq \sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{-4}{7}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{4}{49}+\frac{16}{49}+\frac{36}{49}}=\sqrt{\frac{56}{49}}$$

6) Find the points on the sphere $x^2+y^2+z^2=4$ that are closest and farthest from the point $(3,1,-1)$

Sol) Let $P(x,y,z)$ be any point on the sphere and $A(3,1,-1)$ be the given point

Now $AP=\sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$

$$AP^2=(x-3)^2+(y-1)^2+(z+1)^2$$

Let $f(x,y,z)=(x-3)^2+(y-1)^2+(z+1)^2$

and $\phi(x,y,z)=x^2+y^2+z^2-4$

form Lagrangeous function.

$$F=f(x,y,z)+\lambda\phi(x,y,z)$$

$F=(x-3)^2+(y-1)^2+(z+$

Let $\frac{\partial F}{\partial x}=0 \Rightarrow 2(x$

Let $\frac{\partial F}{\partial y}=0 \Rightarrow$

Let $\frac{\partial F}{\partial z}=$

Sub $x,y,$

Given pla

Let x^2

$$F = (x-3)^2 + (y-1)^2 + (z+1)^2 + \lambda(x^2+y^2+z^2-4)$$

Let $\frac{\partial F}{\partial x} = 0 \Rightarrow 2(x-3) + 0 + 0 + \lambda(2x+0+0-0)$

$\Rightarrow 2x - 6 + 2\lambda x = 0$

$\Rightarrow 2x[1+\lambda] - 6\} = 0$

$\Rightarrow 2x[1+\lambda] = 6$

$\Rightarrow x = \frac{\cancel{6}^{3}}{(1+\lambda)\cancel{2}} = \frac{3}{1+\lambda}$

Let $\frac{\partial F}{\partial y} = 0 \Rightarrow 0 + 2(y-1) + 0 + \lambda(0+2y+0-0) = 0$

$\Rightarrow 2y - 2 + 2\lambda y = 0$

$\Rightarrow 2y(1+\lambda) - 2 = 0$

$\Rightarrow 2y(1+\lambda) = 2$

$\Rightarrow y = \frac{\cancel{2}}{(1+\lambda)\cancel{2}} = \frac{1}{1+\lambda}$

Let $\frac{\partial F}{\partial z} = 0 \Rightarrow 0 + 0 + 2(z+1) + \lambda(0+0+2z-0)$

$\Rightarrow 2z + 2 + 2\lambda z = 0$

$\Rightarrow 2z(1+\lambda) + 2 = 0$

$\Rightarrow 2z(1+\lambda) = -2$

$\Rightarrow z = \frac{-\cancel{2}}{(1+\lambda)\cancel{2}} = \frac{-1}{1+\lambda}$

Sub x, y, z

Given plane,

Let $x^2 + y^2 + z^2 = 4$

$$\Rightarrow \left[\frac{3}{1+\lambda}\right]^2 + \left[\frac{1}{1+\lambda}\right]^2 + \left[\frac{-1}{1+\lambda}\right]^2 = 4$$

$$\Rightarrow \frac{9}{(1+\lambda)^2} + \frac{1}{(1+\lambda)^2} + \frac{1}{(1+\lambda)^2} = 4$$

$$\Rightarrow \frac{11}{(1+\lambda)^2} = 4$$

$$\Rightarrow (1+\lambda)^2 = \frac{11}{4}$$

$$\Rightarrow 1+\lambda = \pm\sqrt{11/4} = \pm\frac{\sqrt{11}}{2}$$

$$\therefore\ x = \frac{3}{\pm\frac{\sqrt{11}}{2}} = \pm\frac{6}{\sqrt{11}}$$

$$y = \frac{1}{\frac{\sqrt{11}}{2}} = \pm\frac{2}{\sqrt{11}}$$

$$z = \frac{-1}{\pm\sqrt{\frac{11}{2}}} = \pm\frac{-2}{\sqrt{11}}$$

Now $x =$

Hence $\left[\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right]$ is the fartest point and $\left[\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right]$ is the closest point $[3, 1, -1]$