

Unit - 5

Introduction :- The future behaviour of the system is based on present input and past history of the system.

The present behaviour of the system is based on present input and past history of the system.

Past history of the system can be described by State Variables.

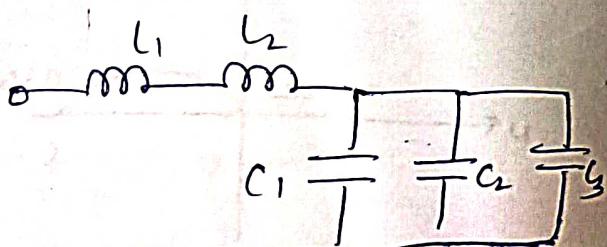
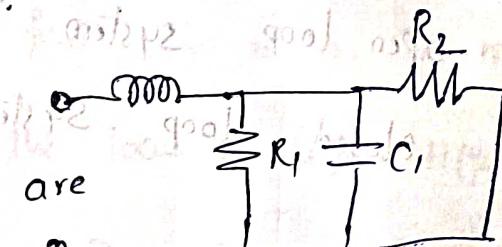
No. of State Variables:-

1) If any electrical network is given, the no. of state variables is equal to the sum of the inductors and capacitors.

2) If there are 3 storage elements then there are 3 states. If same kind of elements are connected in series or parallel then it should be treated as single component.

For example, from the above circuit, there are 5 storage elements are present but there are only two states exists.

3) If differential equation is given, the no. of state variables is equal to the order of differential equation.



Limitations of Transfer function analysis :-

- Transfer function analysis is more suitable for single input and single output systems, whereas state analysis is used for multi-input and multi-output systems.
- Transfer function (TF) analysis cannot give any idea about controllability and observability.

Initial conditions :- TF analysis is valid for only LTI systems whereas static space analysis is valid for dynamic systems i.e. systems may be linear or non-linear, time variant or time invariant.

Standard form of state variables

$$\dot{x}_{nx1} = A_{nxm} x_{nx1} + B_{nxm} u_{mx1} \quad \text{Model: } \dot{x}_{nx1} = \underbrace{A_{nxm} x_{nx1}}_{\text{state}} + \underbrace{B_{nxm} u_{mx1}}_{\text{input}} \rightarrow ①$$

$$y_{px1} = C_{pxn} x_{nx1} + D_{pxm} u_{mx1} \quad \rightarrow ②$$

Equation (1) is called as state equation

Equation (2) is called as output equation

Where A is state Matrix and x is state vector

$$\dot{x} = \frac{dx}{dt} \quad \text{Differential State Vector}$$

x - state vector

u - input vector

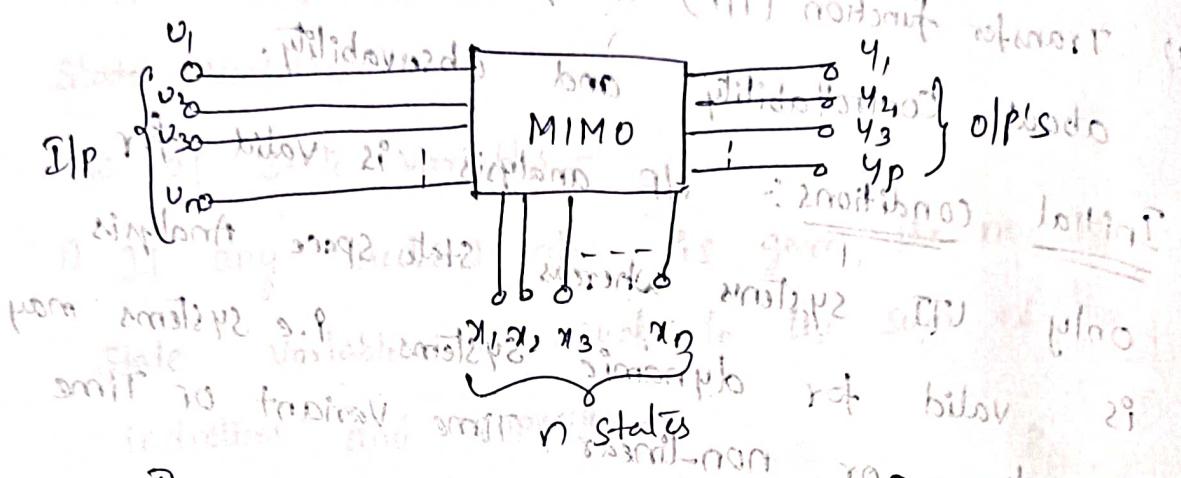
y - output vector

B - input matrix

Output matrix (for convolution)

Order of the matrix is $n \times n$ states

Consider a MIMO system with n inputs and n states and p outputs as shown in figure



Then

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

where element U_{ij} is the i th element of u_j and X_i is the i th element of X .

Note:- If the transition matrix is always zero because if the network consists of any active element

For example, here state x_1 is zero because $x_{11} = 0$

State space Models :-

1) Differential Equations

2) Transfer functions

3) Signal flow graph

4) Electrical networks

Problems on State Model to Differential Equations:-

Write the state model for the following differential equation

$$\ddot{y} + 2\dot{y} + 3y + 4y = 10 \sin t \text{ or } 2t \text{ dines}$$

Given D.E is

$$\ddot{y} + 2\dot{y} + 3y + 4y = 10 \sin t \rightarrow (1)$$

The order of given D.E is 3

$$n = 3 \quad \begin{pmatrix} 0 & 0 & 1 \\ & & \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{let } y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\dot{y} = \frac{dx_1}{dt} = x_2$$

$$\ddot{y} = \frac{d\dot{x}_1}{dt} = x_3$$

$$\dot{y} = \frac{d\dot{x}_2}{dt} = x_3$$

Sub the above values in eq ①

$$x_3 + 2x_3 + 3x_2 + 4x_1 = 10 \sin t$$

$$x_3 + 2x_3 + 3x_2 + 4x_1 = 10 \sin t$$

$$x_3 = [-4x_1 - 3x_2 - 2x_3] + 10 \sin t$$

The above equation is in the form of state

equation given by

$$\dot{x}_{nx1} = A_{nxn} x_{nx1} + B_{nxm} u_{mx1}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3 \times 1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{pmatrix}_{3 \times 3} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3 \times 1} + \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix}_{3 \times 1} u_{1 \times 1}$$

Now the output equation is given by making
terminals parallel with reference state and also

$$y = x_1$$

which is in the form of

$$y = CX + DU$$

$$\text{where } D = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$(4)_{x_1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}_{x_3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2. obtain the state equation for the differential
equation given by

$$y'' + 6y' + 5y + 3y + 2y = 5u(t)$$

Given D.E is

$$y'' + 6y' + 5y + 3y + 2y = 5u(t)$$

order of D.E is 4

$$n = 4$$

state to most cell let $y = x_1$

$$y' = \frac{dx_1}{dt} = x_2$$

$$y'' = \frac{d^2x_1}{dt^2} = x_3$$

$$\begin{aligned} 1 & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \\ 1 & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \\ 1 & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

Sub the above values in eqn ①

$$\dot{x}_4 + 6x_4 - 5x_3 - 3x_2 + 2x_1 = 3u(t)$$

$$\dot{x}_4 = (-2x_1 - 3x_2 + 5x_3 - 6x_4) + 3u(t)$$

This is the required state equation which is in the form of

$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & 5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} u$$

Now the output equation is

$$y = Cx, \text{ or } y = Cx_1$$

which is in the form of

$$y = Cu + Du$$

$$\text{where } C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In case of State Model, the transfer function :-

* To convert the State Model to the given T/F write the State Model in the form of $\dot{x} = Ax + Bu$

$$G(s) H(s) = \frac{2s+3}{s^3 + 5s^2 + 6s + 7}$$

Given

$$T/F = \frac{y(s)}{u(s)} = \frac{2s+3}{s^3 + 5s^2 + 6s + 7}$$

$$4/57 \quad \text{Given } \textcircled{1} \quad 2S+3S^2 \rightarrow (1) \text{ in standard form}$$

$$\text{and } U(S) = S^3 + 5S^2 + 6S + 2S^0 \rightarrow \text{eq } \textcircled{2}$$

$$\text{let } S^{st} = (x_1 - ex_2 + ex_3) = p \hat{x}$$

$$\text{divide } S^{st} \frac{dx_1}{dt} = x_2 \text{ by } p \hat{x} \text{ we get } \text{ eq } \textcircled{3}$$

$$S^2 \frac{dx_2}{dt} = x_3 \text{ and eq } \textcircled{3}$$

$$\text{By substituting above values in eq } \textcircled{2} \text{ we get}$$

$$U(S)_{st} = \dot{x}_3 + 5\dot{x}_3 + 6x_2 + 2x_1$$

$$\text{Equation } \dot{x}_3 = [5x_3 + 6x_2 + 2x_1] + U(S)$$

The above equation is the required state equation

which is in the form of

$$\dot{x} = Ax + Bu = p$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U$$

∴ now substitute $s = 1, s^2, s^3$ values in eq $\textcircled{2}$

By substituting $s = 1, s^2, s^3$ in $\textcircled{2}$

all coefficients of x_1, x_2, x_3 obtained

equation (1). The output equation is obtained as

$$Y(S) = [2x_2 + 3x_1] = (2)(1) (2)$$

which is in the form of

$$\frac{Y(S)}{U(S)} = \frac{(2)(1)}{(2)} = \frac{2}{2}$$

where $(\text{P}D) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$[y] = [3 \ 2 \ 1 \ 0]_{1 \times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{3 \times 1} + [0]_{1 \times 1} [0]_{1 \times 1}$$

write the state Model for the given T/F

①

$$\text{T/F} = \frac{s^3 + 6s^2 + 10}{s^5 + 10s^4 - 8s^2 + 7s + 9}$$

Given

$$\text{T/F} = \frac{Y(s)}{U(s)} = \frac{s^3 + 6s^2 + 10}{s^5 + 10s^4 - 8s^2 + 7s + 9}$$

$$Y(s) = 6s^2 + s^3 + 10s^0 \quad \text{Eq 1}$$

$$\text{and } U(s) = s^5 + 10s^4 - 8s^2 + 7s + 9 \quad \text{Eq 2}$$

The order of variables given system is 5

$$(210.0) \rightarrow (\boxed{\text{P}D=5} + \boxed{U}) = (21)$$

$$\text{Let } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \frac{dx_1}{dt} = x_2$$

$$S^2 = \frac{dx_2}{dt} = x_3$$

$$S^3 = \frac{dx_3}{dt} = x_4$$

$$S^4 = \frac{dx_4}{dt} = x_5$$

$$\text{and } S^5 = \frac{dx_5}{dt} = x_1$$

By substituting the above values in eq ② we get

$$U(s) = x_5 + 10x_5 + -8x_3 + 7x_2 + 9x_1$$

$$\dot{x}_5(s) = [-9x_1 - 7x_2 + 8x_3 - (0.75) + 0.1s]$$

This is the required state equation which is in the form of

$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -9 & -7 & 8 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5x_1 \end{bmatrix}$$

By substituting the S^3, S^2 , and S^0 values in eq(1)

The OLP equation is obtained as

$$y(s) = (x_4 + 6x_3 + 10x_1) + 0.1s$$

$$[y] = [10 \ 0 \ 6 \ 1 \ 0]_{1 \times 5} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + [0]_{1 \times 1}$$

Procedure for the conversion of signal flow graph to

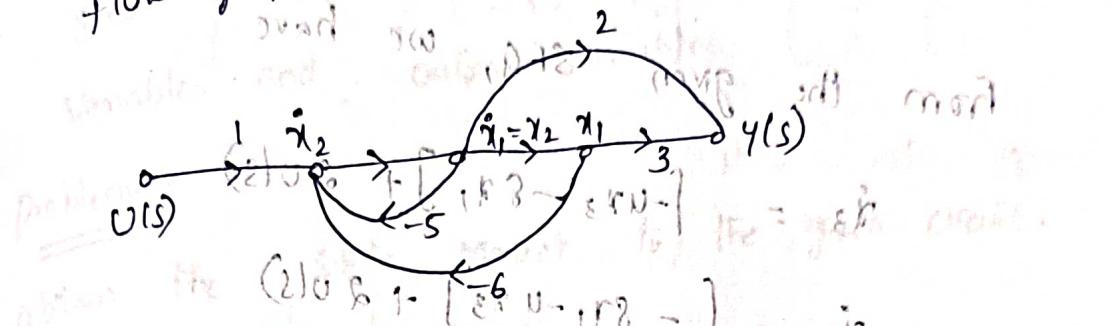
State Model :-

- ① First node should be input node
- ② Next node is the highest power of s equivalent to differential state variable
- ③ Successive nodes are integrable nodes until to get $x_1 + x_2 + x_3 + x_4 + x_5 = 0$

ii) last node should be the output node.

problem :-

obtain the state model for the following signal flow graph.



so from the given signal flow graph (SFG)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(s)$$

This is the required state equation which is

in the form of $\dot{x} = Ax + Bu$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$$

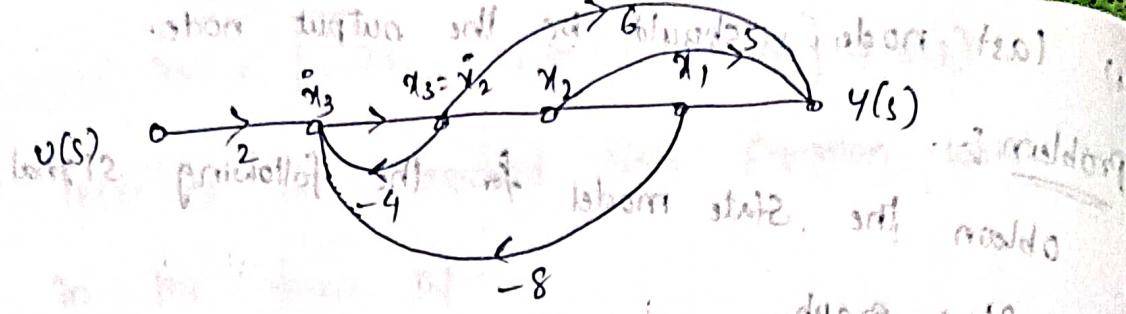
Now the output equation is

$$y(s) = 3x_1 + 2x_2$$

$$[y]_{1 \times 1} = [3 \ 2]_{1 \times 2} [x]_{2 \times 1} + [0]_{1 \times 1} [U]_{1 \times 1}$$

obtain the state Model for the following
signal flow graph. (SFG)

ratios to 1000 and 1500 for background alias (e)



From the given SFG, we have

$$\dot{x}_3 = [-u x_3 - 8 x_1] + 2 u(s)$$

$$(2) \quad \dot{x}_3 = [-8 x_1 - u x_3] + 2 u(s)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -8 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} [u]_{3 \times 1}$$

Now, the output equation is

$$y(s) = 3x_1 + 5x_2 + 6x_3 + 2u(s)$$

$$y(s) = \begin{bmatrix} 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} [u]_{3 \times 1}$$

State Model for Electrical Networks

Procedure :-

1) Select the state variables as voltage across.

parallel capacitor, current through Inductor.

2) No. of state variables equals to sum of Inductor and capacitors

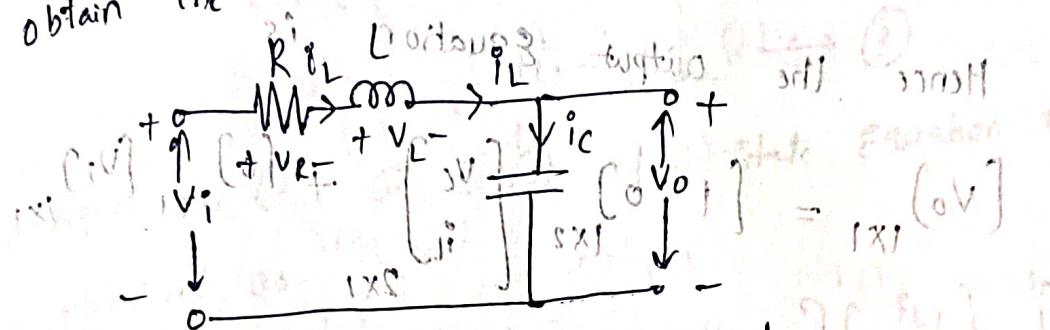
3) write independent KCL and KV at capacitor

junction. Apply KVL across inductor \mathcal{E}_L through inductors.

The resultant equation should consist of only state variables, differential state variables, input variables and output variables.

Problems:

Obtain the state Model to the given circuit.



Apply KVL to the input loop

$$-V_C + i_L R + \frac{d i_L}{dt} L + V_o = 0$$

$$i_L = \left[-\frac{1}{L} V_C - \frac{R}{L} i_L \right] + \frac{V_o}{L} \quad \text{--- (1)}$$

Here the state variables are V_C and i_L .

Hence the corresponding state vector is $\begin{bmatrix} V_C \\ i_L \end{bmatrix}$

From the given circuit, we have

$$V_o + \left(V_i - \frac{1}{L} i_L - \frac{1}{C} V_C \right) = 0$$

$$\therefore i_L = C \frac{d V_C}{d t}$$

$$\therefore V_C = \frac{1}{C} i_L \quad \text{--- (2)}$$

$$0 = C \frac{d V_C}{d t} + \frac{1}{C} i_L + V_o$$

from steps ① & ②, the state equation arising

is obtained as

$$\begin{bmatrix} \dot{v}_c \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v_i$$

As the output is taken across capacitor

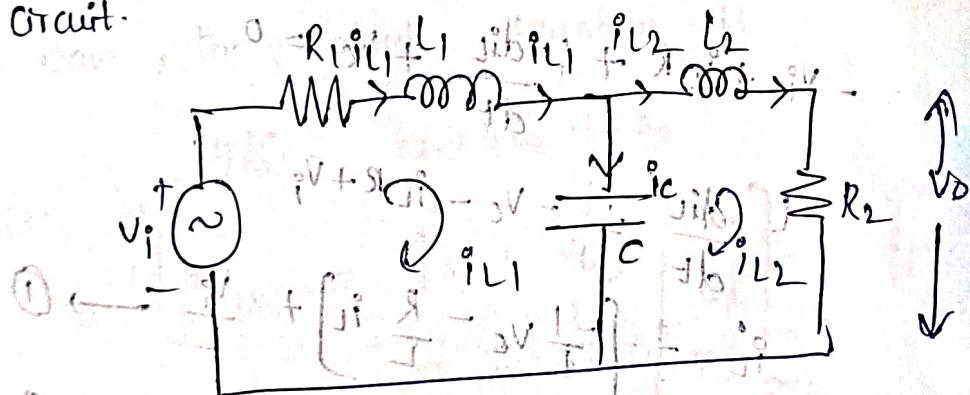
$$\therefore V_o = v_{ab} \text{ or } v_c$$

Hence the output equation is

$$\begin{bmatrix} v_o \\ i_L \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v_i$$

② Obtain the State Model for the given Electrical

Circuit.



Apply KVL to the loop

$$-v_C + i_{L1} R_1 + \frac{d i_{L1}}{dt} + v_{L2} = 0$$

$$i_{L1} = \frac{1}{L_1} v_{C1} - \frac{R_1}{L_1} i_{L1} + \frac{1}{L_1} v_i$$

$$\rightarrow ①$$

Apply KVL at loop ②

$$-v_C + L_2 \frac{d i_{L2}}{dt} + i_{L2} R_2 = 0$$

$$i_{L2} = V_{o3} - R_2 i_{L2} \rightarrow \text{doubtful answer}$$

$$i_{L2} = \frac{1}{L_2} v_c - \frac{R_2}{L_2} i_{L2} \rightarrow \text{eqn 2} \quad \text{obtained}$$

Apply KCL at node 0

$$\begin{aligned} (1) \quad & i_{L1} = i_C + i_{L2} \\ & i_{L1} = C \frac{dv_c}{dt} + i_{L2} \end{aligned}$$

$$C \frac{dv_c}{dt} = i_{L1} - i_{L2}$$

$$(2) \quad v_c = \frac{1}{C} [i_{L1} - i_{L2}] \rightarrow \text{eqn 3}$$

From eqn 0, 1, 2 & 3 The state equation is

Obtained as

$$\begin{bmatrix} i_{L1} \\ i_{L2} \\ v_c \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & 0 \\ 0 & -R_2/L_2 & 1/L_2 \\ 1/C & -1/C & 0 \end{bmatrix} \begin{bmatrix} i_{L1} \\ i_{L2} \\ v_c \end{bmatrix} + \begin{bmatrix} i_{L1} \\ 0 \\ 0 \end{bmatrix} u_i$$

~~Transfer function for State Model~~

~~$\dot{x} = Ax + Bu \rightarrow \text{eqn 1}$~~

~~$0 \cdot \text{at } \text{eqn 2} \quad y = Cx + Du \rightarrow \text{eqn 2}$~~

Since the output is taken across R_2

$$\therefore V_o = i_{L2} R_2$$

Hence the output equation is

$$[V_o] = [0 \ R_2 \ 0] \begin{bmatrix} i_{L1} \\ i_{L2} \\ v_c \end{bmatrix} + [0] [v_i]$$

Transfer function for State Model

Consider $\dot{x} = Ax + Bu \rightarrow \text{Eqn } 1$

$$\dot{x} = Ax + Bu \rightarrow 1$$

$$y = Cx + Du \rightarrow 2$$

Apply Laplace transform for equation (1)

$$sX(s) = Ax(s) + Bu(s)$$

$$1 \leftarrow X(s) [sI - A] = Bu(s)$$

$$\text{if } \text{nodeup3} \text{ matrix } X(s) = B[sI - A]^{-1} u(s) \rightarrow 3$$

Now apply Laplace transform for equation (2)

$$4 \leftarrow Y(s) = Cx(s) + Du(s) \rightarrow 4$$

$$Y(s) = C[sI - A]^{-1} u(s) + Du(s)$$

$$Y(s) = U(s) [C[sI - A]^{-1} B + D]$$

$$Y(s) = U(s) [C[sI - A]^{-1} B + D]$$

$U(s) \neq 0$ & $C[sI - A]^{-1} B + D$ equals to 0

But based on the matrix D

$$\boxed{\frac{Y(s)}{U(s)} = C[sI - A]^{-1} B}$$

Note :-

(1) The equation $|sI - A| = 0$ is known as characteristic equation

(2) The roots of above equation (C.O) are called

as eigen values. $\lambda_1 = 1$, $\lambda_2 = -2$

problems :-

Find the T/F₂ to the given state Model

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} [U]$$

$$\text{and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From the given state Model we have

$$A = \begin{bmatrix} -2 & 3 \\ 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{we have } \frac{1}{s+2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{s+2}$$

$$\frac{1}{s+2} = C (sI - A)^{-1} B \rightarrow ①$$

$$\frac{1}{s+2} = \frac{1}{\text{Adj}(sI - A)} = \frac{1}{\text{Adj}(sI - A)}$$

$$(sI - A) = (A - C) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$(sI - A) = \begin{bmatrix} s+2 & 3 \\ -4 & s+2 \end{bmatrix}$$

$$\text{Now, } \text{Adj}(sI - A) = \begin{bmatrix} s-2 & -3 \\ -4 & s+2 \end{bmatrix}$$

$$\text{And } |SI - A| = \begin{vmatrix} s+2 & 3 \\ -4 & s-2 \end{vmatrix} = (s+2)(s-2) + 12$$

from $s=12$, we get $(s+2)(s-2) + 12$

$$|SI - A| = s^2 - 4 + 12 = s^2 + 8$$

$$[(S - A)^{-1}]^T = \frac{\text{Adj}[SI - A]}{|SI - A|} = \begin{pmatrix} 1 & 4 \\ 4 & s+2 \end{pmatrix}$$

$$(SI - A)^{-1} = \frac{1}{s^2 + 8} \begin{pmatrix} s-2 & -3 \\ 4 & s+2 \end{pmatrix}$$

$$[(S - A)^{-1}]^T B = \frac{1}{s^2 + 8} \begin{pmatrix} s-2 & -3 \\ 4(s-s+2) & s+2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\text{Now, Substitute } \frac{1}{s^2 + 8} \begin{pmatrix} 3s-6-15 \\ 12+5s+10 \end{pmatrix} \quad (2 \times 1)$$

$$[(S - A)^{-1}]^T B = \frac{1}{s^2 + 8} \begin{pmatrix} 3s-21 \\ 5s+22 \end{pmatrix}$$

Now from eq ①

$$\frac{Y(s)}{U(s)} = [C \cdot (SI - A)^{-1} B + C_2]$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 8} \begin{pmatrix} 3s-21 \\ 5s+22 \end{pmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 8} \begin{pmatrix} 3s-21+5s+22 \\ 5s+22 \end{pmatrix}$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{8s+1}{s^2 + 8}}$$

* obtain the T/F to the given state Model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} s & -3 \\ 2 & s+5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

and also find the step response.

From the given state Model, we have

Sol

$$[sI - A] = \begin{bmatrix} s & -3 \\ 2 & s+5 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{and } C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

we have

$$\frac{y(s)}{u(s)} = C [sI - A]^{-1} B \rightarrow ①$$

where $[sI - A]^{-1}$ place $\text{Adj}[sI - A]$ on $sI - A$

$$\text{Now, } \text{Adj}[sI - A] = \begin{bmatrix} s+5 & 3 \\ -2 & s \end{bmatrix}$$

$$\text{and } |sI - A| = \begin{vmatrix} s & -3 \\ 2 & s+5 \end{vmatrix} = s(s+5) + 6$$

$$|sI - A| = s^2 + 5s + 6$$

$$[sI - A]^{-1} = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s+5 & 3 \\ -2 & s \end{bmatrix}$$

$$C \cdot [sI - A]^{-1} = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+5 & 3 \\ -2 & s \end{bmatrix}$$

$$C \cdot [sI - A]^{-1} = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} 0 & 2 \\ 0 & s+3 \end{bmatrix}$$

NOW

$$C \cdot [sI - A]^{-1} B = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\frac{y(s)}{u(s)} = C \cdot [sI - A]^{-1} B = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\boxed{\frac{y(s)}{u(s)} = \frac{s-2}{s^2 + 5s + 6}}$$

To find step response

Given that
Input (P.S.C Step) Signal

$$\frac{u(s)}{} = \frac{1}{s}$$

Now, we have

$$\frac{y(s)}{u(s)} = \frac{s-2}{s^2 + 5s + 6}$$

$$y(s) = \frac{s-2}{s^2 + 5s + 6} \cdot u(s)$$

$$y(s) = \frac{s-2}{s^2 + 5s + 6} \cdot \frac{1}{s}$$

$$\frac{e^{-2s} + y(s)}{2-s} = \frac{1}{s(s^2 + 5s + 6)}$$

$$\text{Consider } \left[\frac{s^2}{s(s^2 + 5s + 6)} \right] = \frac{s-2}{s(s+2)(s+3)}$$

$$\frac{S-2}{S(S+2)(S+3)} = \frac{A}{S} + \frac{B}{S+2} + \frac{C}{S+3}$$

$$S-2 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2)$$

$$\text{At } S=0$$

$$-2 = A(2)(3) \quad \text{or} \quad A = -1/3$$

$$\boxed{A = -1/3}$$

$$\text{At } S=-2$$

$$-4 = B(-2)(1) \quad \text{or} \quad B = 2$$

$$\boxed{B = 2}$$

$$\text{At } S=-3$$

$$-5 = C(-3)(-1) \quad \text{or} \quad C = -5/3$$

$$\boxed{C = -5/3}$$

$$y(s) = \frac{s-2}{s(s+2)(s+3)} = \frac{-1}{3s} + \frac{2}{s+2} - \frac{5}{3(s+3)}$$

Apply inverse Laplace transform on $B.S$

$$y(t) = \left[\frac{2}{3} t + \frac{2}{s+2} - \frac{5}{3(s+3)} \right]$$

$$y(t) = \left[\frac{2}{3} u(t) + 2 e^{-2t} u(t) - \frac{5}{3} e^{-3t} u(t) \right]$$

$$y(t) = u(t) \left[\frac{2}{3} + 2e^{-2t} + \frac{5}{3} e^{-3t} \right]$$

Solution to state equations :-

1 Laplace transform method

Consider the state equation $\dot{x} = Ax + Bu$ and O/P equation $y = Cx + Du$

$$(s+2)\dot{x} = Ax + Bu + (s+2)(s+2)x_0 = s \cdot x_0$$

$$y = Cx + Du$$

Apply LT for both Equations

$$sx(s) - x(0) = Ax(s) + Bu(s) \quad \text{--- (1)}$$

$$y(s) = Cx(s) + Du(s)$$

From Equation (1)

$$x(s)[sI - A] = Bu(s) + x(0)$$

$$x(s) = [sI - A]^{-1} Bu(s) + [sI - A]^{-1} x(0)$$

Apply Inverse Laplace transform

$$\frac{e^{-s t}}{(s+2)^2} x(t) = \mathcal{L}^{-1} \left\{ [sI - A]^{-1} Bu(s) \right\} + \mathcal{L}^{-1} \left\{ [sI - A]^{-1} x(0) \right\}$$

The equation (2) contains two parts

1st part is called "zero state Response"

2nd part is called "zero Input Response"

2. Classical Method :-

$$\boxed{\left[\frac{dx}{dt} + Ax + Bu \right] (sI - A)^{-1} U(s) = (sI - A)^{-1} P}$$

We have $\left[\frac{dx}{dt} + Ax + Bu \right] (sI - A)^{-1} U(s) = (sI - A)^{-1} P$

$$x(t) = \int_0^t e^{A(t-s)} Bu(s) ds + e^{At} x(0)$$

zSR bottom mark

$\rightarrow (3)$

From eq ② & ③ we have

$$L^{-1}\{[S\mathbf{I} - A]^{-1} x(0)\} = e^{At} x(0) \rightarrow ①$$

And $L^{-1}\{[S\mathbf{I} - A]^{-1} B u(s)\} = \int_0^t e^{A(t-s)} B u(s) ds$ $\rightarrow ③$

from equation ①, adding ② & ③ we get

$$L^{-1}\{[S\mathbf{I} - A]^{-1} x(0)\} + L^{-1}\{[S\mathbf{I} - A]^{-1} B u(s)\} = e^{At} x(0) + \int_0^t e^{A(t-s)} B u(s) ds$$

$$L^{-1}\{[S\mathbf{I} - A]^{-1}\} x(0) + L^{-1}\{[S\mathbf{I} - A]^{-1} B u(s)\} = e^{At} x(0)$$

$$L^{-1}\{[S\mathbf{I} - A]^{-1}\} x(0) + L^{-1}\{[S\mathbf{I} - A]^{-1} B u(s)\} = \phi(t) x(0)$$

$$L^{-1}\{[S\mathbf{I} - A]^{-1}\} x(0) + L^{-1}\{[S\mathbf{I} - A]^{-1} B u(s)\} = \phi(t) x(0)$$

$$L^{-1}\{[S\mathbf{I} - A]^{-1}\} x(0) + L^{-1}\{[S\mathbf{I} - A]^{-1} B u(s)\} = \phi(t) x(0)$$

where $\phi(t) = \text{State Transition matrix}$

$$\phi(t) = \text{Resolvent matrix}$$

Now from equation ⑤

$$L^{-1}\{[S\mathbf{I} - A]^{-1} B u(s)\} = \int_0^t e^{A(t-s)} B u(s) ds$$

$$L^{-1}\{[S\mathbf{I} - A]^{-1}\} x(0) + L^{-1}\{[S\mathbf{I} - A]^{-1} B u(s)\} = \phi(t) x(0)$$

$$[S\mathbf{I} - A]^{-1} = \phi(s)$$

$$L^{-1}\{\phi(s) B u(s)\} = \int_0^t e^{A(t-s)} B u(s) ds$$

$$\therefore x(t) = L^{-1}\{\phi(s) B u(s)\} + L^{-1}\{\phi(s) x(0)\}$$

$$x(t) = \phi(s) B u(s) + \phi(s) x(0)$$

$$x(t) = \phi(s) B u(s) + \phi(s) x(0)$$

Properties of STM :-

We have $\rightarrow \{c_1 x^{(A)}\} = \{c_1 x^{(A-E2)}\}$

$$\Rightarrow e^{At} = L^{-1} \{ (S - A)^{-1} \}$$

i) $\phi(0) = I$

ii) $\phi^{-1}(t) = (e^{At})^{-1} = e^{-At} = e^{(0-t)} = \phi(-t)$

$$\phi^{-1}(t) = \phi(-t)$$

iii) $\phi^k(t) = \phi(kt)$

iv) $\phi(t_1+t_2) = e^{A(t_1+t_2)} = e^{At_1 + At_2} = e^{At_1} \cdot e^{At_2}$

$$\phi(t_1+t_2) = \phi(t_1) \cdot \phi(t_2)$$

v) $\phi(t_2-t_1) = -\phi(t_2-t_1) * \phi(t_1-t_0)$

Problems on STM :-

To obtain the complete time response of the S/m

given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} x$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Given that $\{c_1 x^{(2)}\} + \{c_2 x^{(2)}\}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x ; y = [1 \ -1] x$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad \text{with } A^T = A$$

Now the required general solution is

$$x(t) = L^{-1}\{\phi(s)B\mathbf{U}(s)\} + L^{-1}\{\phi(s)x(0)\}$$

From the given state equations i.e.,

$$\begin{bmatrix} (1+s) & 1 \\ -2 & 1+s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \xrightarrow{\text{for zero input}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{① (not unique)} \quad \text{for zero input}$$

$$\text{Since } B = 0$$

① not unique

$$\begin{bmatrix} (1+s) & 1 \\ -2 & 1+s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \xrightarrow{\text{for zero input}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = (I - B)^{-1}$$

$$\text{Hence } x(t) = L^{-1}\{\phi(s)x(0)\} \rightarrow ①$$

But we know that

$$\phi(s) = (sI - A)^{-1} = \frac{\text{Adj}(sI - A)}{|sI - A|}$$

$$\text{Now } |sI - A| = \begin{vmatrix} s & 0 \\ 0 & s \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -2 & 0 \end{vmatrix} = s^2 + 2$$

$$\frac{s}{s^2 + 2} (sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 2 & s \end{bmatrix}$$

$$\text{Adj}(sI - A) = \begin{bmatrix} s & -1 \\ 2 & s \end{bmatrix} = s^2 + 2$$

$$\text{So } |sI - A| = s^2 + 2$$

$$\therefore \phi(s) = \frac{\text{Adj}(sI - A)}{|sI - A|} = \frac{s^2 + 2}{s^2 + 2} \begin{bmatrix} s & 1 \\ -2 & s \end{bmatrix}$$

$$\text{Now, } \phi(s) x(0) = 1 \xrightarrow{s^2+2} \begin{bmatrix} s & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix} = 1$$

$$\phi(s) x(0) = \frac{1}{s^2+2} \begin{bmatrix} s+1 \\ -2+s \end{bmatrix} \xrightarrow{\text{apply } L^{-1}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Apply Inversion to $b_1 s$

$$L^{-1}\{\phi(s)x(0)\} = L^{-1}\left\{\frac{1}{s^2+2} \begin{bmatrix} s+1 \\ -2+s \end{bmatrix}\right\}$$

From Equation ①

$$x(t) = L^{-1}\{\phi(s)x(0)\} = L^{-1}\left\{\frac{s+1}{s^2+2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$\text{Or } x(t) = \frac{1}{s^2+2} \begin{bmatrix} s+1 \\ s-2 \end{bmatrix}$$

$$x(t) = \frac{1}{s^2+2} \begin{bmatrix} L^{-1}\left\{\frac{s+1}{s^2+2}\right\} \\ L^{-1}\left\{\frac{s-2}{s^2+2}\right\} \end{bmatrix}$$

$$x(t) = \frac{1}{s^2+2} \left[-L^{-1}\left\{\frac{1}{s^2+2}\right\} + L^{-1}\left\{\frac{1}{s^2+2}\right\} \right]$$

$$= \frac{1}{s^2+2} \left[L^{-1}\left\{\frac{1}{s^2+2}\right\} - L^{-1}\left\{\frac{2}{s^2+2}\right\} \right]$$

$$\therefore x(t) = \begin{cases} \cos \sqrt{2}t + \frac{1}{\sqrt{2}} \sin \sqrt{2}t \\ \cos \sqrt{2}t - \sqrt{2} \sin \sqrt{2}t \end{cases}$$

To find the complete time response, we have

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \frac{d^2y}{dt^2} + y(t) = (1+2t)(1+t) x(t)$$

$$\text{i.e., } y(t) = \begin{bmatrix} r & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^t u(t), \text{ where}$$

$$y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \cos \sqrt{2}t + \frac{1}{\sqrt{2}} \sin \sqrt{2}t \\ \cos \sqrt{2}t - \frac{1}{\sqrt{2}} \sin \sqrt{2}t \end{bmatrix}$$

$$y(t) = \cos \sqrt{2}t + \frac{1}{\sqrt{2}} \sin \sqrt{2}t - \cos \sqrt{2}t + \sqrt{2} \sin \sqrt{2}t$$

$$y(t) = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sin \sqrt{2}t \end{bmatrix} = [A \cdot \alpha] \cdot i(b)$$

2. Find output (the time response for unit step input) of the system given by -

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \end{bmatrix} \text{ given initial cond.}$$

$$\begin{bmatrix} 1 & \sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix} \frac{1}{s+2} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{(s+2)}{(s+\sqrt{2})(s-\sqrt{2})} b$$

$$\text{and } y = \begin{bmatrix} 1 & \sqrt{2} \\ 2 & -\sqrt{2} \end{bmatrix} x$$

Given state equation (i.e.)

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{2} \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ (\sqrt{2})(-\sqrt{2}) & -2 \end{bmatrix} x + \begin{bmatrix} x(0) \\ 5 \end{bmatrix} u$$

$$\text{Let } \begin{bmatrix} sA^2 = \\ s - \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = B \Rightarrow \begin{bmatrix} 0 \\ s \end{bmatrix} b$$

The given system is non-homogeneous sys.

Hence the general solution is

$$x(t) = \left[-1 \right] \phi(s) B u(s) + \left[1 \right] \phi(s) x(0) \rightarrow ①$$

$$\text{But } \phi(s) = \frac{1}{s+2} \begin{bmatrix} s+1 & 1 \\ -2 & -3 \end{bmatrix}^{-1} = \frac{\text{Adj} \begin{bmatrix} s+1 & 1 \\ -2 & -3 \end{bmatrix}^{-1}}{|s+2|} \rightarrow ②$$

$$\text{Now, } SI - A = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad (3.1)$$

~~$\left[\frac{\text{det}(sI - A)}{s}, \frac{\text{det}(sI - A)}{s} \right]$~~ $\left[\begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right] = (1)I$

$$SI - A = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\text{Adj}[SI - A] = \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix} \quad (3.2)$$

$$|SI - A| = \begin{vmatrix} S+3 & 1 \\ -2 & S \end{vmatrix} = S(S+3) + 2 = S^2 + 3S + 2$$

From Equation ② $\begin{bmatrix} 1 & 0 \\ -2 & S \end{bmatrix} = X^{-1}$

$$\phi(s) = \frac{\text{Adj}[SI - A]}{|SI - A|} = \frac{1}{S^2 + 3S + 2} \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

$$\phi(s) = \frac{1}{(S+1)(S+2)} \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

$$\text{Now } \phi(s) \times (0) = \frac{1}{(S+1)(S+2)} \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\phi(s) \times (0)^2 = \frac{1}{(S+1)(S+2)} \begin{bmatrix} S+3 \\ -2 \end{bmatrix}$$

Apply ILT on both sides

$$① \left\{ (0) \times \frac{1}{(S+2)} \phi(s) \times (0) \right\} = \left[\frac{1}{(S+1)(S+2)} \begin{bmatrix} S+3 \\ -2 \end{bmatrix} \right]$$

$$② \left\{ \frac{1}{(S+2)} \left[a \phi(s) \times (0) \right] \right\} = \left[\frac{1}{(S+1)(S+2)} \begin{bmatrix} S+3 \\ -2 \end{bmatrix} \right]$$

$$\left(\begin{matrix} -1 \\ 1 \end{matrix} \right) \phi(s) \left(\begin{matrix} 1 \\ 0 \end{matrix} \right) = 1 \left(\begin{matrix} -1 \\ 1 \end{matrix} \right) \left(\begin{matrix} \frac{s+3}{(s+1)(s+2)} \\ 1 \end{matrix} \right)$$

$$\left(\begin{matrix} -1 \\ 1 \end{matrix} \right) \phi(s) \left(\begin{matrix} 1 \\ 0 \end{matrix} \right) = \left(\begin{matrix} -1 \\ 1 \end{matrix} \right) \left(\begin{matrix} \frac{-2}{(s+1)(s+2)} \\ 1 \end{matrix} \right)$$

$$\left(\begin{matrix} -1 \\ 1 \end{matrix} \right) \phi(s) \left(\begin{matrix} 1 \\ 0 \end{matrix} \right) = \left(\begin{matrix} -1 \\ 1 \end{matrix} \right) \left(\begin{matrix} \frac{2}{s+1} - \frac{1}{s+2} \\ 1 \end{matrix} \right)$$

$$\left(\begin{matrix} -1 \\ 1 \end{matrix} \right) \phi(s) \left(\begin{matrix} 1 \\ 0 \end{matrix} \right) = \left(\begin{matrix} -1 \\ 1 \end{matrix} \right) \left(\begin{matrix} \frac{-2}{s+1} + \frac{2}{s+2} \\ 1 \end{matrix} \right) \rightarrow (3)$$

NOW $\left(\begin{matrix} -1 \\ 1 \end{matrix} \right) \phi(s) B = \frac{1}{(s+1)(s+2)} \left(\begin{matrix} 2 & 1 \\ -2 & s \end{matrix} \right) \left(\begin{matrix} 0 \\ 5 \end{matrix} \right)$

$$\phi(s) \cdot B = \frac{1}{(s+1)(s+2)} \left[\begin{matrix} (s+3)0 + 1 \times 5 \\ -2 \times 0 + s \times 5 \end{matrix} \right]$$

$$\phi(s) \cdot B \cdot v(s) = \frac{1}{(s+1)(s+2)} \left[\begin{matrix} 5 \\ 5s \end{matrix} \right] v(s)$$

Given (1) that,
The input of the system is a step input

$$+ \left(\begin{matrix} CHU \\ CHU \end{matrix} \right) v(s) = \frac{1}{s} \left(\begin{matrix} 5 \\ 5s \end{matrix} \right) \cdot \frac{1}{s}$$

$$\phi(s) \cdot B \cdot v(s) = \frac{1}{(s+1)(s+2)} \left[\begin{matrix} 5 \\ 5s \end{matrix} \right] \cdot \frac{1}{s}$$

$$\phi(s) \cdot B \cdot v(s) = \frac{1}{(s+1)(s+2)} \left[\begin{matrix} 5 \\ 5s \end{matrix} \right]$$

Apply ILT on both sides

$$L^{-1} \left\{ \phi(s) B \cdot U(s) \right\} = L^{-1} \left\{ \frac{1}{s(s+1)(s+2)} \left[(s+1) \times \begin{pmatrix} 2 & 5 \\ 5 & 5 \end{pmatrix} \right] \right\}$$

$$L^{-1} \left\{ \phi(s) B \cdot U(s) \right\} = L^{-1} \left\{ \frac{5}{s(s+1)(s+2)} \right\}$$

$$\left[\left(\frac{1}{s+2} - \frac{5}{s+1} \right) I_2 + \left(\frac{5s}{s+1}(s+2) \right) \right] \left[\begin{array}{l} L^{-1} \left\{ \frac{5}{s(s+1)(s+2)} \right\} \\ L^{-1} \left\{ \frac{5}{(s+1)(s+2)} \right\} \end{array} \right]$$

(3) \leftarrow

$$\left[\begin{array}{l} (1)U^{(s+2)} - (1)U^{(s+1)} \\ (1)U^{(s+2)} + (1)U^{(s+1)} \end{array} \right] \left[\begin{array}{l} L^{-1} \left\{ \frac{5}{s(s+1)(s+2)} \right\} \\ L^{-1} \left\{ \frac{5}{(s+1)(s+2)} \right\} \end{array} \right]$$

$$L^{-1} \left\{ \phi(s) B \cdot U(s) \right\} = L^{-1} \left\{ \frac{5}{2s} - \frac{5}{s+1} + \frac{5}{2(s+2)} \right\}$$

$$\left[\begin{array}{l} \frac{5}{2s} - \frac{5}{s+1} \\ \frac{5}{2s} - \frac{5}{s+2} \end{array} \right] = 5 \cdot (2) \phi$$

$$L^{-1} \left\{ \phi(s) B \cdot U(s) \right\} = \left[\begin{array}{l} \frac{5}{2} u(t) - 5e^{-t} u(t) + \frac{5}{2} e^{-2t} u(t) \\ 5e^{-t} u(t) - 5e^{-2t} u(t) \end{array} \right]$$

→ (4)

Substitute eqn (3) & (4) in equation (1) $\boxed{\text{Ans}}$

$$x(t) = \left[\begin{array}{l} \frac{5}{2} u(t) - 5e^{-t} u(t) + \left(\frac{5}{2} \right) e^{-2t} u(t) \\ \frac{1}{2} \cdot \left(5e^{-t} \right) u(t) - 5e^{-2t} u(t) \end{array} \right] +$$

$$\left[\begin{array}{l} 2e^{-t} u(t) - e^{-2t} u(t) \\ -2e^{-t} u(t) + 2e^{-2t} u(t) \end{array} \right] = (2) \boxed{1} - (2) \boxed{2}$$

Ans

$$x(t) = \begin{cases} \frac{5}{2}u(t) - 3e^{-t}u(t) + \frac{3}{2}e^{-2t}u(t), & t \geq 0 \\ 3e^{-t}u(t) - 3e^{-2t}u(t), & t < 0 \end{cases}$$

For time response, we have initial to initial

$$\begin{aligned} y(t) &= \begin{bmatrix} -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} -2 & -3 \end{bmatrix} \left[\frac{5}{2}u(t) - 3e^{-t}u(t) + \frac{3}{2}e^{-2t}u(t) \right] \\ &\quad + \left[3e^{-t}u(t) - 3e^{-2t}u(t) \right] \\ &= -5u(t) + 6e^{-t}u(t) - 3e^{-2t}u(t) \left(-9e^{-t}u(t) + 9e^{-2t}u(t) \right) \\ &= -5u(t) - 3e^{-t}u(t) + 6e^{-2t}u(t) \\ \therefore y(t) &= \boxed{\begin{bmatrix} 6e^{-2t} & -3e^{-t} & -5 \end{bmatrix} u(t)} \end{aligned}$$

Controllability :- A system is said to be controllable if it is possible to transfer the initial state to any other state in a finite time by controlled vector.

Controllability is verified by Kalman's Test.

$$Q_C = \left\{ \begin{array}{l} \text{if } B \text{ is not } AB \\ \text{if } B^T B = A^{n-1}B \end{array} \right\}$$

where n - order of the matrix A

Conditions :-

① Rank of Q_C = Rank of A

② $|Q_C| \neq 0$

Q4 The system is (1) satisfying the above two conditions, this means said to be controllable

Problems of Controllability

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Given the system is (1) satisfying the above two conditions, this means said to be controllable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(2) $\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

∴ The order of $A^2 + A = 2 \times 2 = 4$

Now $Q_C = [B \quad AB]$ ① Preliminaries

But, $AB = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$
Preliminaries

Substitute B & AB values in equation ①

$$Q_C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Rank of Q_C :-

$$|Q_C| = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 0(-1) - 1(1) = -1 \neq 0$$

\therefore Rank of $Q_C = 2$

Rank of A :-

$$|A| = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 1(-1) - 2(1) = -3 \neq 0$$

\therefore Rank of $A = 2$

Conclusions :-

$$\textcircled{1} \quad \text{Rank of } Q_C = \text{Rank of } A$$

$$\textcircled{2} \quad |Q_C| \neq 0$$

Therefore, the given system is controllable.

a. Check the controllability of the system

$$0+2 = 0-2 = -2x_1 + u \quad \left(\begin{matrix} 0 & -2 \\ 0 & 1 \end{matrix} \right) = |A|$$

$$0+3 = 0+3 = -3x_2 + 5x_1$$

Given

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = B \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (1)$$

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$$

\therefore Order of matrix A , $n=2$

$$Q_C = [B \quad AB] = [B \quad A^T B]$$

$$\text{NON, } AB = \begin{pmatrix} -2 & 0 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \quad \text{Non-controllable}$$

$$= \begin{pmatrix} -2 \\ 5 \end{pmatrix} \quad \text{Non-controllable}$$

$$\therefore AB = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \quad \text{Non-controllable}$$

$$Q_C = \begin{pmatrix} 1 & -2 \\ 0 & 5 \end{pmatrix} \quad \text{Non-controllable}$$

Rank of Q_C :-

$$|Q_C| = \begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} = 5 + 0 = 5 \neq 0 \quad \text{Non-controllable}$$

$$0 \neq 10 \quad \text{Non-controllable}$$

$$\therefore \text{Rank of } Q_C = 2 \quad \text{Non-controllable}$$

Rank of A :-

$$|A| = \begin{vmatrix} -2 & 0 \\ 5 & -3 \end{vmatrix} = 6 - 0 = 6 \neq 0$$

$$\therefore \text{Rank of } A = 2$$

But, $\text{Rank of } Q_C = \text{Rank of } A$

$$\text{Conclusion :- } \text{Rank of } Q_C = \text{Rank of } A$$

$\Rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 5 \end{pmatrix} \neq 0$ Controllable

Hence the given system is Controllable.

Observability :-

A system is said to be observable if it is possible to determine initial states of the

system by obscuring the output for a finite time interval.

$$Q_0 = [C^T \quad A^T C^T \quad (A^T)^2 C^T \quad \dots \quad (A^T)^{n-1} C^T]$$

$$Q_0 = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

conditions :- $C^T Q_0 = 0$ if $\text{Rank } Q_0 = \text{Rank of } A^T$, i.e., controllability.

① Rank of $Q_0 = \text{Rank of } A^T$, i.e., controllability.

② $|Q_0| \neq 0$ to check observability.

problems on controllability and observability :-
check the controllability and observability for

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$\text{and } y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Given

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{and } C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Controllability :-

$$Q_0 = \begin{bmatrix} B & AB \\ \vdots & \vdots \end{bmatrix}, \quad \because n=2$$

$$\text{Now, } AB = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$\left(\begin{array}{c} \text{Controllable} \\ \text{(A)} \end{array}\right)$

$$\therefore Q_C = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0B$$

$$\text{Now, } |Q_C| = \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = 1 \cdot (-2) - (-1) \cdot 2 = 0$$

$$\therefore \text{Rank of } Q_C = 1$$

$$\text{Now, } |A|_A = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 + 0 = 2 \neq 0$$

$$\therefore \text{Rank of } A = 2$$

Conclusions :-

1) Rank of Q_C < Rank of A

$$2) |Q_C| = 0$$

Hence the given system is not controllable

Observability :-

$$Q_O = \begin{bmatrix} C \\ CA \end{bmatrix}^T = \begin{bmatrix} C \\ CA \end{bmatrix}^T = P$$

$$\text{Now } CA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$$

$$\therefore CA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore Q_O = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{Now, } |Q_O| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0B$$

\therefore Rank of $Q_0 = 1$,
have Rank of $A = 2$

we
conclusions:- $Q_0 = 1 + 1 = \begin{vmatrix} 1 & 1 \end{vmatrix} = \{1\}$ i.e.

i) Rank of $Q_0 \neq$ Rank of A

ii) $|Q_0| = 0$

Hence the given system is not observable.

* check the controllability and observability for

$$\dot{x}_1 = -2x_1 + x_2 + u \quad \text{to check for } Q_0$$

$$\dot{x}_2 = -x_2 + u$$

$$y = x_1 + x_2 \quad \text{to check for } Q_0$$

Given that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad Q_0 = \{1\}$$

$$\text{and } C = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{and } C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Controllability :-

Since $n=2$

$$Q_C = \begin{bmatrix} I & AB \\ B & A^T B \end{bmatrix} = \{1\}$$

$$\text{Now } AB = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\therefore Q_C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ Rank } Q_C = 2$$

Now, $|Q_C| = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -1 + 1 = 0$

Now, $|A| = \begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix} = 2 - 0 = 2$

Rank of $A = 2$

Conclusions :-

1) Rank of $Q_C \neq$ Rank of RA

2) $|Q_C| = 0$

Hence, the given system is not controllable

Observability :-

$$Q_O = \begin{bmatrix} I & C \\ 0 & CA \end{bmatrix} = [P] \text{ bno}$$

$$\text{Now, } CA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \triangleq E_2(0)$$

$$\therefore Q_O = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \text{ bno}$$

$$\text{Now, } |Q_O| = \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = 0 + 2 = 2 \neq 0$$

$$\left[\begin{array}{c} 1 \\ -2 \end{array} \right] = \text{Rank of } Q_O = 2 \neq 0 \text{ no}$$

We have,

$$\text{Rank of } A = 2$$

conclusions :-

1) If Rank of $Q_0 = \text{Rank of } A$ $\Rightarrow Q_0 \neq 0$ (c)

2) $|Q_0| \neq 0$

Hence the given system is observable.

* Determine the controllability and observability of

$$(A - A_1) = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

Given $A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

and $C = \begin{bmatrix} 1 & 2 \end{bmatrix}$

Order of the matrix $A = 2 \times 2$, $B = 2 \times 1$

Controllability :-

$$Q_C = \begin{pmatrix} B & AB \end{pmatrix}$$

$$\text{Now, } AB = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

$$\therefore Q_C = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\text{Now, } |Q_C| = \begin{vmatrix} 0 & 0 \\ 0 & -3 \end{vmatrix} = 0 - 0 = 0$$

∴ Rank of $Q_C = 0$ implies all zeros

$$\text{Now, } |A| = \begin{vmatrix} -1 & 0 \\ 0 & -3 \end{vmatrix} = 3 - 0 = 3 \neq 0$$

∴ Rank of $A = 2$

Conclusions:-

1) Rank of $Q_0 \neq$ Rank of A

2) $|Q_0| = 0$ Hence the given system is not controllable.

Observability

$$Q_0 = \begin{bmatrix} C \\ CA \end{bmatrix}$$

Now, $CA = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = I_2$

$$Q_0 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

Rank of $Q_0 = 4$

$$\text{Now, } |Q_0| = \begin{vmatrix} 0 & 1 \\ 1 & 2 \\ -1 & -6 \end{vmatrix} = -6 + 2 = -4 \neq 0$$

Hence, the system is observable.

Observability: Rank of $Q_0 = 2 \Rightarrow$ Controllable

and also we have

$$\text{Rank of } A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 2$$

Conclusions:-

1) Rank of $Q_0 \neq$ Rank of A

2) $|Q_0| \neq 0$

Hence, the given system is observable.

$$0 + 8 = 0 \cdot 0 \Rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$s = 4 \Rightarrow \text{to rank } \dots$$