Strength of Materials - 2

Unit-2

Deflection of Beams

The deformation of a beam is usually expressed in terms of its deflection from its original unloaded position. The deflection is measured from the original neutral surface of the beam to the neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known as the "elastic curve" of the beam.

Slope of Beam: Slope of beam is the angle between deflected beam ~~to~~ the actual beam at same point.

Deflection of beam: Deflection of beam is defined as the vertical displacement of a point on a loaded beam. There are many methods to find out the slope and deflection at a section in a loaded beam.

Assumptions:

The following assumptions are undertaken in order to desire a differential equation of elastic curve for the loaded beam

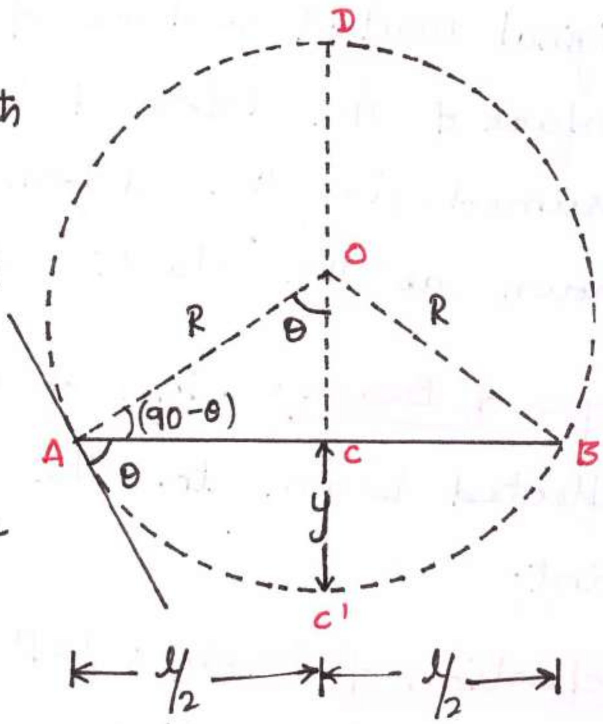
→ stress is proportional to strain ie hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.

→ The curvature is always small.

→ Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

* Deflection and Slope of a beam subjected to Uniform Bending:

A beam AB of length L is subjected to a uniform bending moment M. As the beam is subjected to a constant bending moment, hence it will bend into a circular arc.



The initial position of the beam is shown by ACB, where as the deflected position is shown by AC'B.

Let R = Radius of Curvature of deflected beam

y = deflection of beam at the centre

I = moment of inertia of beam section

E = Young's Modulus for beam material.

θ = slope of beam at the end A.

Hence $\tan\theta = \theta$ where θ is in radians.

Hence θ becomes the slope as slope is

$$\frac{dy}{dx} = \tan\theta = \theta$$

Now $AC = BC = \frac{L}{2}$

Also from the geometry of a circle, we know that

$$AC \times CB = DC \times CC'$$

$$\frac{L}{2} \times \frac{L}{2} = (2R - y) \times y$$

$$\frac{L^2}{4} = 2Ry - y^2$$

For a practical beam, the deflection y is a small quantity. Hence the square of a small quantity will be negligible. Hence neglecting y^2 in the above equation, we get

$$\frac{L^2}{4} = 2Ry$$

$$\Rightarrow \quad y = \frac{L^2}{8R}$$

But from bending equation we have

$$\frac{M}{I} = \frac{E}{R}$$

$$\therefore \quad R = \frac{EI}{M}$$

$$\Rightarrow \quad y = \frac{L^2}{8\frac{EI}{M}} = \frac{ML^2}{8EI}$$

Value of slope (θ)

From Δ^{le} AOB, we know that

$$\sin\theta = \frac{AC}{AO} = \frac{\left(\frac{L}{2}\right)}{R} = \frac{L}{2R}$$

since the angle θ is very small, hence

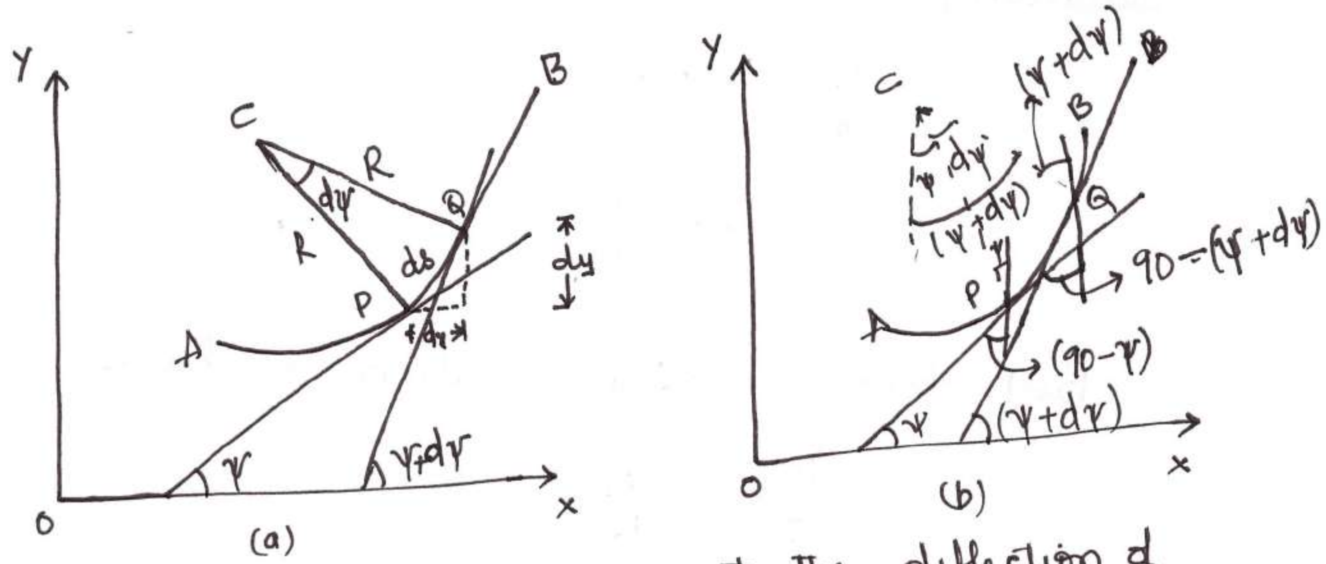
$$\sin\theta = \theta$$

$$\Rightarrow \quad \theta = \frac{L}{2R}$$

$$= \frac{L}{2 \times \frac{EI}{M}}$$

$$\boxed{\theta = \frac{ML}{2EI}}$$

* Relation between slope, deflection and radius of Curvature.



Let the curve AB represents the deflection of a beam as shown in fig (a). Consider a small portion PQ of this beam. Let the tangents at P & Q make angle ψ and $\psi + d\psi$ with x-axis.

Normal at P and Q will meet at C such that

$$PC = QC = R$$

The point C is known as centre of curvature of PQ.

Let the length of PQ is equal to ds.

from fig (b), we see that

$$\angle PCQ = d\psi$$

$$\therefore \quad PQ = ds = R \cdot d\psi$$

$$\text{or} \quad R = \frac{ds}{d\psi} \qquad \text{—— (1)}$$

But if x and y be the coordinates of P then

$$\tan\psi = \frac{dy}{dx} \qquad \text{—— (2)}$$

$$\sin\psi = \frac{dy}{ds}$$

$$\cos\psi = \frac{dx}{ds}$$

eq (1) can be written as

$$R = \frac{ds}{d\psi} = \frac{\left(\frac{ds}{dx}\right)}{\left(\frac{d\psi}{dx}\right)} = \frac{\left(\frac{1}{\cos\psi}\right)}{\left(\frac{d\psi}{dx}\right)}$$

$$R = \frac{\sec\psi}{\left(\frac{d\psi}{dx}\right)} \qquad \text{—— (3)}$$

differentiating eq (2) wrt x we get

$$\sec^2\psi \cdot \frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \quad \frac{d\psi}{dx} = \frac{\left(\frac{d^2y}{dx^2}\right)}{\sec^2\psi}$$

substituting this value of $\frac{d\psi}{dx}$ in eq (3)

$$R = \frac{\sec\psi}{\left[\frac{\left(\frac{d^2y}{dx^2}\right)}{\sec^2\psi}\right]} = \frac{\sec^3\psi}{\left(\frac{d^2y}{dx^2}\right)}$$

Taking the reciprocal both sides we get

$$\frac{1}{R} = \frac{\left(\frac{d^2y}{dx^2}\right)}{\sec^3\psi} = \frac{\left(\frac{d^2y}{dx^2}\right)}{(\sec^2\psi)^{3/2}}$$

$$= \frac{\left(\frac{d^2y}{dx^2}\right)}{(1+\tan^2\psi)^{3/2}}$$

For a practical beam, the slope $\tan\psi$ at any point is a small quantity. Hence $\tan^2\psi$ can be neglected.

$$\therefore \quad \frac{1}{R} = \frac{d^2y}{dx^2} \qquad \text{—— (4)}$$

from bending equation we have

$$\frac{M}{I} = \frac{E}{R}$$

$$\Rightarrow \quad \frac{1}{R} = \frac{M}{EI} \qquad \text{—— (5)}$$

Equating eq 4 & 5 we get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\therefore \quad M = EI \frac{d^2y}{dx^2} \qquad \text{—— (6)}$$

The above equation is called as differential equation for elastic line of beam.

differentiating eq (6) we get

$$\frac{dM}{dx} = EI \frac{d^3y}{dx^3}$$

But $\frac{dM}{dx} = F$ (shear force)

$$\therefore \quad F = EI \frac{d^3y}{dx^3} . \qquad \text{—— (7)}$$

differentiating eq (7) wrt x we get

$$\frac{dF}{dx} = EI \frac{d^4y}{dx^4}$$

But $\frac{dF}{dx} = w$ (the rate of loading)

$$\therefore \quad w = EI \frac{d^4y}{dx^4} . \qquad \text{—— (8)}$$

Hence, the relation between curvature, slope, deflection etc at a section is given by deflection = y

slope = $\frac{dy}{dx}$

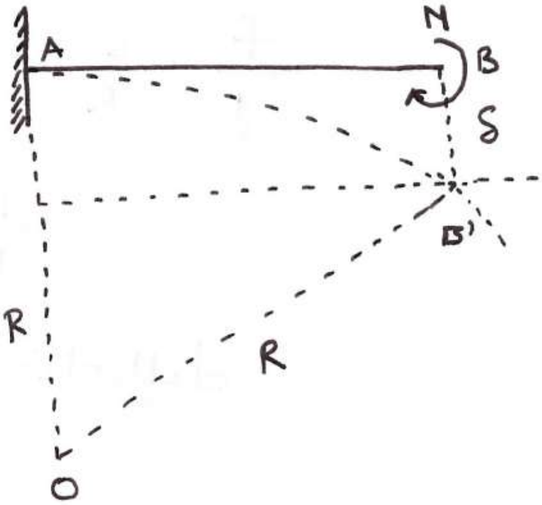
Bending moment = $EI\frac{d^2y}{dx^2}$

shear force = $EI\frac{d^3y}{dx^3}$

rate of loading = $EI\frac{d^4y}{dx^4}$.

P-1] A cantilever of length 'l' is subjected to a couple M at its free end. Find the slope and deflection of the end.

Sol:- Since the cantilever is subjected to a constant bending moment, it will bend to the shape of a circular arc of radius R such that



$$\frac{1}{R} = \frac{M}{EI}$$

Let i_b = slope at the free end B

$$i_b = \sin i_b = \frac{AB'}{R} = \frac{l}{R} = \frac{Ml}{EI}$$

Let δ = deflection

by properties of circles

$$\delta(2R - \delta) = l^2$$

$$2R\delta - \delta^2 = l^2$$

Ignoring δ^2 which is very small quantity

$$\delta = \frac{l^2}{2R} = \frac{l^2}{2} \cdot \frac{M}{EI} = \frac{Ml^2}{2EI}$$

P-2] A Cantilever of length 1.25m is subjected to a couple M_0 at the free end. The longitudinal (normal) strain at the top surface is 0.0015 and the distance of the top surface of the cantilever from the neutral layer is 90mm. Find the radius of the neutral layer and the vertical deflection at the end of the cantilever.

Sol:- $l = 1.25\,m = 1250\,mm$

Constant BM $= M_0$

$$\frac{f}{y} = \frac{E}{R} \quad \therefore \quad \frac{y}{R} = \frac{f}{E} = 0.0015$$

$$\frac{90}{R} = 0.0015 \Rightarrow R = 60000\,mm = 60\,m$$

$$\text{deflection } \delta = \frac{l^2}{2R} = \frac{1250^2}{2 \times 60000}$$

$$= \underline{13.02\,mm}$$

P-3] A 500 mm long and 5mm thick strip of steel is subjected to end couples M_0. The central deflection is found to be 6.25 mm. Find the longitudinal normal strain at the top surface of the strip.

Sol:- $l = 500\,mm$

$t = 5\,mm$

Const BM $= M_0$ $\quad \delta = 6.25\,mm$

$$\delta = \frac{l^2}{8R} \quad \therefore \quad R = \frac{l^2}{8\delta} = \frac{500^2}{8 \times 6.25}$$

$$= 5000\,mm.$$

$$\frac{f}{y} = \frac{E}{R} \Rightarrow \quad \therefore \frac{f}{E} = \frac{y}{R} = \frac{t/2}{R} = \frac{2.5}{500}$$

$$= 0.0005$$

$\therefore$ longitudinal normal strain at the top surface $= 0.0005$

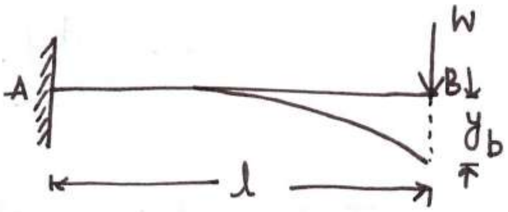
* Methods of determining slope and deflection at a section in a loaded beam:

The following are the important methods for finding the slope and deflection at a section in a loaded beam

(i) Double integration method
(ii) Moment Area Method
(iii) Macaulay's Method.

* Cantilevers subjected to Various types of load: (Double Integration Method)

* Cantilever of length l carrying a point load at free end:



Consider a cantilever beam AB of a uniform section and of length 'l' fixed at the end A and free at end B. Let a concentrated load W be applied at the free end.

Let the moment of inertia of the section of the cantilever about the neutral axis be I.

Consider any section X of the cantilever distant x from the fixed end A.

The bending moment at the section is given by

$$M = EI \frac{d^2y}{dx^2}$$

$$\Rightarrow \quad EI \frac{d^2y}{dx^2} = -W(l-x)$$

Integrating the above equation we get

$$EI\frac{dy}{dx} = -W\left(lx - \frac{x^2}{2}\right) + C_1$$

where C_1 is the constant of integration

At A, the slope being zero

we have $x = 0$, $\frac{dy}{dx} = 0$

$$\therefore \quad C_1 = 0$$

$$\Rightarrow \quad EI\frac{dy}{dx} = -W\left(lx - \frac{x^2}{2}\right) \longrightarrow (1)$$

slope Eq.

$$\therefore \quad \frac{dy}{dx} = \frac{-Wx}{2EI}(2l - x)$$

Integrating above equation we get

$$EI\,y = -W\left[\frac{lx^2}{2} - \frac{x^3}{6}\right] + C_2$$

where C_2 is a constant of integration.

At A the deflection being zero we have

$x = 0$, $y = 0$

$$\therefore \quad C_2 = 0$$

$$\therefore \quad EI\,y = -W\left[\frac{lx^2}{2} - \frac{x^3}{6}\right] \longrightarrow (2)$$

deflection eq

$$y = -\frac{Wx^2}{EI}\left[3l - x\right]$$

Hence the slope and deflection at any section can be determined by eq (1) & (2)

The slope and deflection at the free end can (7)
be determined by substituting $x = l$ in these eq's.
Let the slope and deflection at B be i_b and y_b resp.

we have

$$EI\, i_b = -W\left(l \cdot l - \frac{l^2}{2}\right)$$

$$\therefore \quad EI\, i_b = -\frac{Wl^2}{2}$$

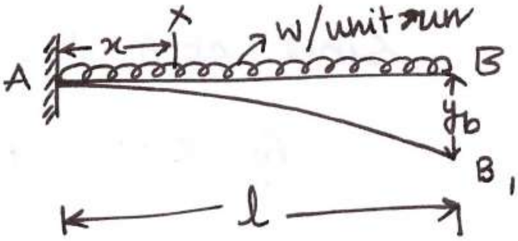
$$\therefore \quad i_b = -\frac{Wl^2}{2EI}$$

also $$EI\, y_b = -W\left[l \cdot \frac{l^2}{2} - \frac{l^3}{6}\right] = -\frac{Wl^3}{3}$$

$$\therefore \quad y_b = \frac{-Wl^3}{3EI}$$

$$\therefore \quad \boxed{\text{downward deflection of } B = \frac{Wl^3}{3EI}}$$

* Cantilever of length l carrying a uniformly distribut load W per unit run over whole length.



Consider a cantilever beam AB of length l fixed at A and free at B carrying a uniformly distributed load of W/unit run over the whole length.

The BM at any section X distant x from the fixed end is given by

$$M = EI\frac{d^2y}{dx^2} \Rightarrow EI\frac{d^2y}{dx^2} = -\frac{W}{2}(l-x)^2$$

Integrating the above equation we get

$$EI\frac{dy}{dx} = -\frac{W}{6}(l-x)^3 + C_1$$

At A the fixed end the slope being zero we have

$$x=0,\ \frac{dy}{dx}=0 \quad \therefore C_1 = \frac{-Wl^3}{6}$$

$$\therefore EI\frac{dy}{dx} = \frac{W}{6}(l-x)^3 - \frac{Wl^3}{6} \quad \text{— (1)}$$

slope eq.

$$\frac{dy}{dx} = -\frac{W}{6EI}\left[(l-x)^3 - l^3\right]$$

$$= \frac{-Wx}{6EI}(3l^2 - 3lx + x^2)$$

Integrating the above eq. we get

$$EI\,y = \frac{-W}{24}(l-x)^4 - \frac{Wl^3}{6}x + C_2$$

Since at A the deflection is zero, we get

@ $x=0,\ y=0$

$$\therefore 0 = \frac{-Wl^4}{24} + C_2 \quad \therefore C_2 = \frac{Wl^4}{24}$$

$$\therefore EIy = -\frac{W}{24}(l-x)^4 - \frac{Wl^3}{6}x + \frac{Wl^4}{24} \quad \text{— (2)}$$

$$EIy = -\frac{W}{24}\left[(l-x)^4 + 4l^3x - l^4\right]$$

$$= \frac{-wx^2}{24}[6l^2 - 4lx + x^2]$$

$$\therefore y = -\frac{wx^2}{24EI}[6l^2 - 4lx + x^2]$$

From the above eq the slope and deflection at any section can be determined.

To find the slope i_b at B, substituting $x = l$ in the slope equation

we get, $EI\, i_b = \frac{-wl^3}{6}$ $\therefore i_b = \frac{-wl^3}{6EI}$.

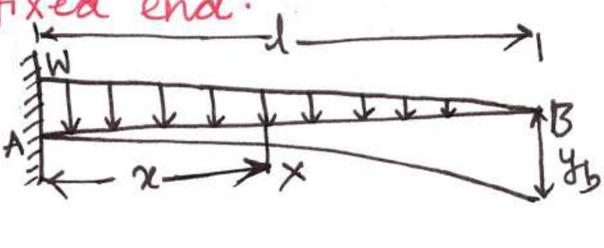
To find the deflection y_b at B, putting $x = l$ in the deflection equation, we get

$$EI\, y_b = -\frac{wl^4}{6} + \frac{wl^4}{24} = \frac{-wl^4}{8}$$

$$\therefore y_b = \frac{-wl^4}{8EI}$$

$$\therefore \boxed{\text{downward deflection @ B} = \frac{-wl^4}{8EI}}$$

* Cantilever of length l carrying a distributed load whose intensity varies uniformly from zero at the free end to w per unit run at the fixed end.



consider a cantilever beam AB of length 'l' and carrying a distributed load whose intensity varies uniformly from zero at the free end to w per unit run at the fixed end.

Consider a section x at a distance x from the fixed end A.

Intensity of loading at x

$$= \frac{(l-x)}{l} w \text{ per unit run.}$$

The BM at the section x is given by

$$EI \frac{d^2y}{dx^2} = M$$

$$\Rightarrow \quad EI \frac{d^2y}{dx^2} = -\frac{1}{2}(l-x)\frac{w}{l}(l-x) \cdot \frac{(l-x)}{3}$$

$$= \frac{-w(l-x)^3}{6l}$$

Integrating the above eq we get

$$\int EI \frac{d^2y}{dx^2} = \int -\frac{w(l-x)^3}{6l}$$

$$\rightarrow \quad EI \frac{dy}{dx} = \frac{w(l-x)^4}{24l} + C_1$$

At A the slope is zero

$\therefore x = 0$

$$\Rightarrow \frac{dy}{dx} = 0$$

$$\therefore \quad 0 = \frac{Wl^3}{24} + C_1 \quad \therefore \quad C_1 = \frac{-Wl^3}{24}$$

$$EI\frac{dy}{dx} = \frac{W(l-x)^4}{24l} - \frac{Wl^3}{24} \quad \text{— (1)}$$

slope eq.

Integrating the above eq we get

$$\int EI\frac{dy}{dx} = \int \frac{W(l-x)^4}{24l} - \int \frac{Wl^3}{24}$$

$$\Rightarrow \quad EIy = -\frac{W(l-x)^5}{120\,l} - \frac{Wl^3}{24}x + C_2$$

The deflection at A is 0

$$\therefore \quad x = 0, \quad y = 0$$

$$\Rightarrow \quad 0 = \frac{-Wl^4}{120} + C_2$$

$$\therefore \quad C_2 = \frac{Wl^4}{120}$$

$$\therefore \quad EIy = -\frac{W(l-x)^5}{120\,l} - \frac{Wl^3}{24}x + \frac{Wl^4}{120} \quad \text{— (2)}$$

deflection eq.

To find the slope at B at the free end, putting $x = l$ in the slope eq we get

$$EI\,i_b = \frac{-Wl^3}{24}$$

$$i_b = \frac{-Wl^3}{24EI}$$

To find the deflection at B putting $x = l$, in the deflection equation we get

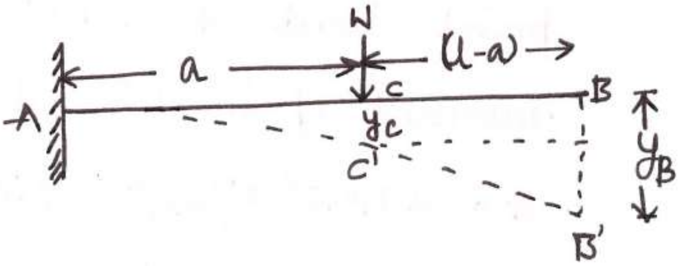
$$EI\, y_b = \frac{-Wl^4}{24} + \frac{Wl^4}{120} = \frac{-Wl^4}{120}(5-1)$$

$$= \frac{-Wl^4}{30}$$

$$\therefore y_b = \frac{-Wl^4}{30\,EI}$$

$\therefore$ downward deflection @ B = $\dfrac{-Wl^4}{30\,EI}$

* Cantilever with a point load at a distance 'a' from the fixed end: ⑩



A cantilever AB of length 'l' fixed at point A and free at point B and carrying a point load W at a distance 'a' from the fixed end A.

Let θ_c = slope at point C ie $\left(\frac{dy}{dx}\right)$ at C

y_c = deflection at point C

y_B = deflection at point B

$\therefore$ slope @ C $\theta_c = \dfrac{Wa^2}{2EI}$

deflection @ C $y_c = \dfrac{Wa^3}{3EI}$

The beam will bend only between A & C, but from C to B it will remain straight since BM b/w C and B is zero.

Since the portion CB of cantilever is straight

$\therefore$ slope at C = slope at B

$$\theta_c = \theta_B = \frac{Wa^2}{2EI}$$

$$\therefore y_B = y_c + \theta_c(l - a)$$

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(l-a).$$

P-4] A cantilever of length 3m is carrying a point load of 25 KN at the free end. If the moment of inertia of the beam = 10^8 mm^4 and $E = 2.1 \times 10^5$ N/mm^2 find slope of the cantilever at the free end and deflection at the free end.

Sol:- Given

Length = $L = 3m = 3000mm$

point load $W = 25$ KN $= 25000N$

$I = 10^8$ mm^4

$E = 2.1 \times 10^5$ N/mm^2.

(i) slope at the free end is given by

$$\theta_B = \frac{Wl^2}{2EI} = \frac{25000 \times 3000^2}{2 \times 2.1 \times 10^5 \times 10^8}$$

$$= \underline{0.0005357 \text{ rad}}.$$

(ii) deflection at the free end

$$y_B = \frac{Wl^3}{3EI} = \frac{25000 \times 3000^3}{3 \times 2.1 \times 10^5 \times 10^8}$$

$$= \underline{10.71 \text{ mm}}$$

P-5] A cantilever of length 3m is carrying a point load of 50 KN at a distance of 2m from the fixed end. If $I = 10^8$ mm^4, $E = 2 \times 10^5$ N/mm^2 find slope at the free end and deflection at free end.

Sol :- Given

$l = 3m = 3000m$

$W = 50\,kN = 50,000\,N$

distance between The load and the fixed end

$a = 2m = 2000\,mm$

$I = 10^8\,mm^4$

$E = 2\times10^5\,N/mm^2$

(i) slope at the free end

$$\theta_B = \frac{Wa^2}{2EI} = \frac{50000 \times 2000^2}{2\times 2\times 10^5 \times 10^8}$$

$$= \underline{0.005\ rad}$$

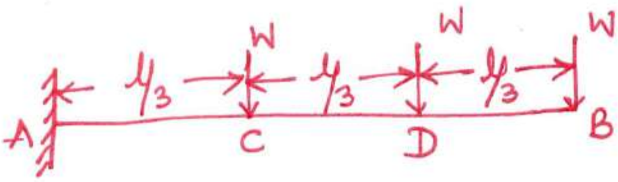
(ii) deflection at the free end

$$y_B = \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(l-a)$$

$$= \frac{50000\times 2000^3}{3\times 2\times 10^5\times 10^8} + \frac{50000 \times 2000^2}{2\times 2\times 10^5\times 10^5}(3000-2000)$$

$$= 6.67 + 5 = \underline{11.67\,mm}$$

P-6] A cantilever AB of length l carries three point loads, W each at distances $\frac{l}{3}$, $\frac{2l}{3}$ and l from the fixed end A. Determine the slope and deflection at the free end B.



Sol :-

Slope at the free end B

Slope at B due to the load W @ C $= \dfrac{W\left(\frac{l}{3}\right)^2}{2EI}$

$= \dfrac{Wl^2}{18EI}$

Slope at B due to load W at D $= \dfrac{W\left(\frac{2l}{3}\right)^2}{2EI}$

$= \dfrac{2Wl^2}{9EI}$

Slope at B due to load at B $= \dfrac{Wl^2}{2EI}$

$\therefore$ Total slope at $B = \dfrac{Wl^2}{18EI} + \dfrac{2Wl^2}{9EI} + \dfrac{Wl^2}{2EI}$

$= \dfrac{Wl^2}{18EI}(1+4+9) = \underline{\dfrac{7Wl^2}{9EI}}$

Deflection at the free end B

deflection @ B due to load W at C

$= \dfrac{W\left(\frac{l}{3}\right)^3}{3EI} + \dfrac{W\left(\frac{l}{3}\right)^2}{2EI} \cdot \dfrac{2l}{3} = \dfrac{4}{81}\dfrac{Wl^3}{EI}$

deflection @ B due to load W at D

$= \dfrac{W\left(\frac{2l}{3}\right)^3}{3EI} + \dfrac{W\left(\frac{2l}{3}\right)^2}{2EI} \cdot \dfrac{l}{3}$

$= \dfrac{14}{81}\dfrac{Wl^3}{EI}$

deflection at B due to the load W at B

$$= \frac{1}{3}\frac{Wl^3}{EI}$$

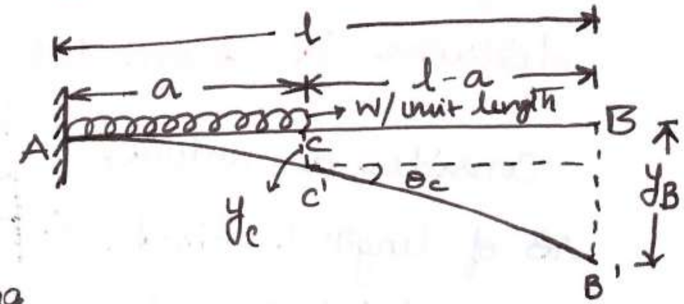
$\therefore$ Total deflection at B

$$= \frac{4}{81}\frac{Wl^3}{EI} + \frac{14}{81}\frac{Wl^3}{EI} + \frac{1}{3}\frac{Wl^3}{EI}$$

$$= \frac{Wl^3}{81EI}(4 + 14 + 27)$$

$$= \frac{5}{9}\frac{Wl^3}{EI}$$

* Deflection of a cantilever with a uniformly distributed load for a distance 'a' from the fixed end.



Consider a cantilever AB of length L fixed at the point A and free at the point B and carrying a UDL of W/unit length for a distance of 'a' from the fixed end.

The beam will bend only between A and C, but from C to B it will remain straight since BM between C and B is zero. The deflected shape of cantilever is shown by AC'B' in which portion C'B' is straight.

Let θ_c = slope at C i.e. $\left(\frac{dy}{dx}\right)$ at C

y_c = deflection at C

y_B = deflection at B

$$\therefore \quad \theta_C = \frac{wa^3}{6EI}$$

$$y_C = \frac{wa^4}{8EI}$$

Since portion $C'B'$ of the cantilever is straight, therefore slope at C = slope at B

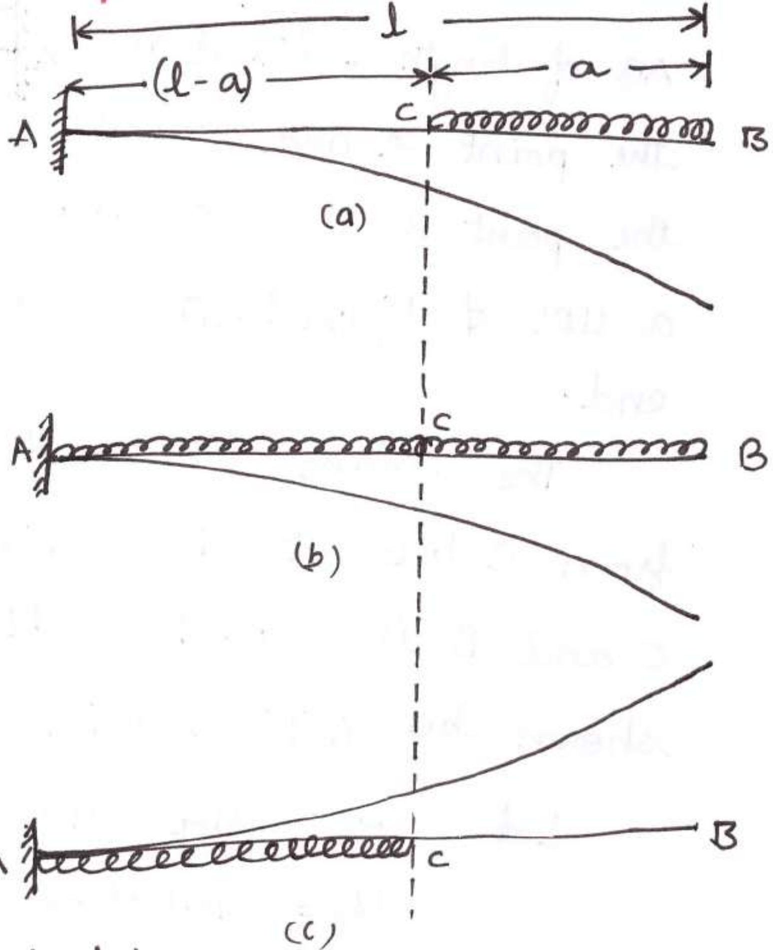
$$\text{ie} \quad \theta_C = \theta_B = \frac{wa^3}{6EI}$$

$$\therefore \quad y_B = y_C + \theta_C (l-a)$$

$$= \frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l-a).$$

* Deflection of a cantilever with a UDL for a distance 'a' from the free end.

Consider a Cantilever AB of length L fixed at the point A and free at the point B and carrying a UDL of w/unit length for a distance 'a' from the free end shown in fig (a)



The slope and deflection at the point B are determined by considering

(i) The whole cantilever AB loaded with a UDL of w/unit length as shown in fig (B)

(ii) a part of cantilever from A to C of length $(l-a)$ loaded with an upward uniformly distributed load of w/unit length as shown in fig (c).

Then slope at B = slope due to downward UDL over whole span - slope due to upward UDL from A to C.

deflection at B = deflection due to downward UDL over the whole span - deflection due to upward UDL from A to C.

(a) Now slope B due to downward UDL over the whole length $= \dfrac{wl^3}{6EI}$

(b) Slope at B or at C due to upward UDL over length $(l-a)$ $= \dfrac{w(l-a)^3}{6EI}$

$\therefore$ Net slope at B is given by

$$\theta_B = \frac{wl^3}{6EI} - \frac{w(l-a)^3}{6EI}$$

The downward deflection of point B due to downward distributed load over the whole span AB.

$$= \frac{wl^4}{8EI}$$

The upward deflection of point B due to upward udl acting on the portion AC is equal to

Upward deflection of C + slope at C × CB

$$= \frac{w(l-a)^4}{8EI} + \frac{w(l-a)^3}{6EI} \times a$$

∴ Net downward deflection of the free end B

$$y_B = \frac{wl^4}{8EI} - \left[\frac{w(l-a)^4}{8EI} + \frac{w(l-a)^3}{6EI} \times a\right]$$

P-7] Determine the slope and deflection of the free end of a cantilever of length 3m which is carrying a udl of 10 KN/m over a length of 2m from the fixed end. Take $I = 10^8\ mm^4$ and $E = 2\times10^5\ N/mm^2$.

Sol:: Given

$L = 3m = 3000\,mm$

$w = 10\ KN/m = 10\ N/mm$

length of udl from fixed end $a = 2m = 2000\ mm$.

$I = 10^8\ mm^4$

$E = 2\times10^5\ N/mm^2$

Let θ_B = slope at the free end

y_B = deflection at the free end

$$\therefore \theta_B = \frac{wa^3}{6EI} = \frac{10 \times 2000^3}{6 \times 2 \times 10^5 \times 10^8}$$

$$= \underline{\underline{0.00066 \text{ rad}}}$$

$$y_B = \frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l-a)$$

$$= \frac{10 \times 2000^4}{8 \times 2 \times 10^5 \times 10^8} + \frac{10 \times 2000^3}{6 \times 2 \times 10^5 \times 10^8}(3000-2000)$$

$$= 1 + 0.67 = \underline{\underline{1.67 \text{ mm}}}$$

P-8] A cantilever 120 mm wide and 200 mm deep is 2.5 m long. What is the UDL which the beam can carry in order to produce a deflection of 5 mm at the free end? Take E = 200 GN/m².

Sol:- Given

width $b = 120$ mm

depth $d = 200$ mm

length $l = 2.5$ m $= 2500$ mm

deflection $y_B = 5$ mm

$E = 200 \text{ GN/m}^2 = 200 \times 10^9 \text{ N/m}^2$

$= 2 \times 10^5 \text{ N/mm}^2$

Moment of Inertia $I = \frac{bd^3}{12}$

$$= \frac{120 \times 200^3}{12}$$

$$= 8 \times 10^7 \ mm^4$$

Let w = udl per m length in N

W = Total load

$= w \times L = w \times 2.5$ N.

$$y_B = \frac{Wl^3}{8EI}$$

$$5 = \frac{2.5w \times 2500^3}{8 \times 2 \times 10^5 \times 8 \times 10^7}$$

$$\therefore w = \underline{\underline{16.384 \ KN/m}}$$

P-9] A cantilever of length 3m carries a UDL of 10 KN/m over a length of 2m from free end. If $I = 10^8 \ mm^4$ and $E = 2 \times 10^5 \ N/mm^2$ find (i) slope at the free end and (ii) deflection at free end.

Sol:- Given

$l = 3m = 3000 \ mm$

udl $w = 10 \ KN/m = 10 \ N/mm$

$a = 2m = 2000 \ mm$.

$I = 10^8 \ mm^4$; $E = 2 \times 10^5 \ N/mm^2$

Let θ_B = slope at free end ie $\left(\frac{dy}{dx}\right)$ at B

y_B = deflection at free end

$$\therefore \theta_B = \frac{Wl^3}{6EI} - \frac{W(l-a)^3}{6EI}$$

$$= \frac{10 \times 3000^3}{6 \times 2 \times 10^5 \times 10^8} - \frac{10(3000-2000)^3}{6 \times 2 \times 10^5 \times 10^8}$$

$$= 0.00225 - 0.000083$$

$$= \underline{\underline{0.002167 \text{ rad}}}$$

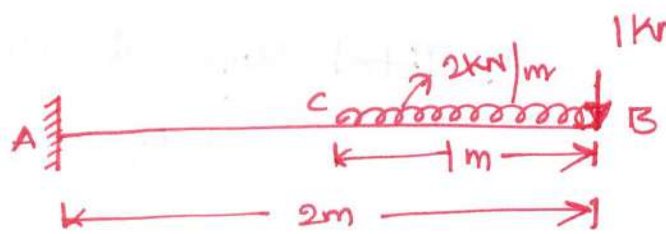
$$y_B = \frac{Wl^4}{8EI} - \left[\frac{W(l-a)^4}{8EI} + \frac{W(l-a)^3}{6EI} \times a\right]$$

$$= \frac{10 \times 3000^4}{8 \times 2 \times 10^5 \times 10^8} - \left[\frac{10(3000-2000)^4}{8 \times 2 \times 10^5 \times 10^8} + \frac{10(3000-2000)^3}{6 \times 2 \times 10^5 \times 10^8} \times 2000\right]$$

$$= 5.0625 - [0.0625 + 0.1667]$$

$$= \underline{4.8333 \text{ mm}}$$

P-10] A cantilever of length 2m carries a udl 2KN/m over a length of 1m from the free end and a point load of 1KN at the free end. Find the slope and deflection at the free end. $E = 2.1 \times 10^5$ N/mm² and $I = 6.667 \times 10^7$ mm⁴.



Sol:- Given

$l = 2m = 2000mm$

Udl $w = 2\ KN/m = \frac{2 \times 1000}{1000} = 2\ N/mm$

$a = 1m = 1000\ mm$

Point load $W = 1KN = 1000\ N$

$E = 2.1 \times 10^5\ N/mm^2$

$I = 6.667 \times 10^7\ mm^4$

(i) Slope at the free end

Let θ_1 = Slope at free end due to point load

θ_2 = Slope at free end due to udl.

$$\therefore \theta_1 = \frac{Wl^2}{2EI}$$

$$= \frac{1000 \times 2000^2}{2 \times 2.1 \times 10^5 \times 6.667 \times 10^7} = \underline{\underline{0.0001428\ rad}}$$

$$\theta_2 = \frac{wl^3}{6EI} - \frac{w(l-a)^3}{6EI}$$

$$= \frac{2 \times 2000^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \frac{2 \times (2000 - 1000)^3}{6 \times 2.1 \times 10^5 \times 6.667 \times 10^7}$$

$$= 0.0001904 - 0.000238$$

$$= \underline{\underline{0.0001666\ rad}}$$

$\therefore$ Total slope at the free end

$$= \theta_1 + \theta_2 = 0.0001428 + 0.0001666$$

$$= \underline{\underline{0.0003094\ rad}}$$

(ii) deflection at the free end

let y_1 = deflection at free end due to point load

y_2 = deflection at free end due to udl.

$$\therefore\ y_1 = \frac{Wl^3}{3EI} = \frac{1000 \times 2000^3}{3 \times 2.1 \times 10^5 \times 6.667 \times 10^7}$$

$$= \underline{\underline{0.1904\ mm}}$$

$$y_2 = \frac{wl^4}{8EI} - \left[\frac{w(l-a)^4}{8EI} + \frac{w(l-a)^3}{6EI} \times a\right]$$

$$= \frac{2 \times 2000^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} - \left[\frac{2(2000-1000)^4}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7} + \frac{2(2000-1000)^3 \times 1000}{8 \times 2.1 \times 10^5 \times 6.667 \times 10^7}\right]$$

$$= 0.2857 - [0.01785 + 0.0238]$$

$$= \underline{\underline{0.244\ mm}}$$

$\therefore$ Total deflection at free end

$$y_B = y_1 + y_2$$

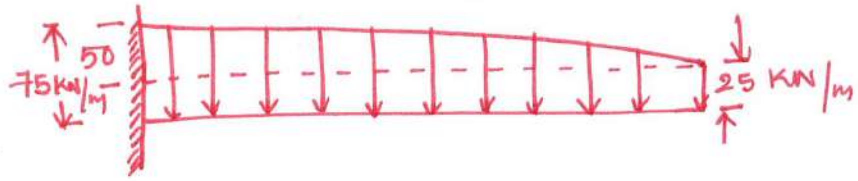
$$= 0.1904 + 0.244$$

$$= 0.4344\ mm.$$

$\therefore\ \underline{\underline{\theta_B = 0.0003094\ rad}}$ $\quad y_B = \underline{\underline{0.4344\ mm}}$

P-11] A cantilever of length 2m carries a UVL of 25 KN/m at the free end to 75 KN/m at the fixed end. If $E = 1 \times 10^5\ N/mm^2$ and $I = 10^8\ mm^4$ determine the slope and deflection of the cantilever at the free end.

Sol:- Given



$l = 2m = 2000mm$

load at free end $= 25\ KN/m = \frac{25 \times 1000}{1000} \times 25\ N/mm$

Load at fixed end $= 75\ KN/mm = 75\ N/mm$

$$E = 1 \times 10^5\ N/mm^2$$

$$I = 10^8\ mm^4$$

The load acting on the cantilever is equivalent to a udl of 25 KN/m over the entire span and a triangular of zero intensity at free end and 50 N/mm at the fixed end.

(i) slope at the free end.

Let θ_1 = slope at free end due to udl

θ_2 = slope at free end due to UVL

$$\therefore\ \theta_1 = \frac{wl^3}{6EI} = \frac{25 \times 2000^3}{6 \times 1 \times 10^5 \times 10^8} = 0.0033\ rad$$

$$\theta_2 = \frac{Wl^3}{24EI} = \frac{50 \times 2000^3}{24 \times 10^5 \times 10^8} = 0.00167\ rad$$

VASIREDDY VENKATADRI
INSTITUTE OF TECHNOLOGY

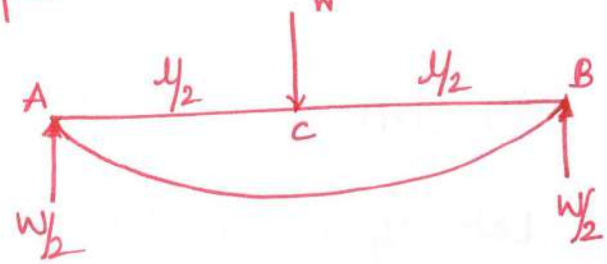
Certificate of Participation

This is to certify that Mr/Ms D. Yogeswar (Asst. professor) of ALTS, Ananthapuram bearing Regd no - has participated in Two day advanced technical training program on BIM TOOLS from March 8 to 9, 2018 organized by Department of Civil Engineering, Vasireddy Venkatadri Institute of Technology, Nambur(V), Guntur(Dt).

Coordinator | HOD | Principal

* Simply Supported Beams:

* Simply supported beam of span l carrying a point load at mid span:



Consider a simply supported beam AB of span l carrying a point load W at mid span C.

Since the load is symmetrically applied the maximum deflection will occur at mid span

Vertical reaction = $\frac{W}{2}$

Consider the left half AC of the span

The BM at any section in AC distant x from A is given by

$$EI\frac{d^2y}{dx^2} = +\frac{W}{2}x$$

Integrating above eq we get

$$\int EI\frac{d^2y}{dx^2} = \int \frac{W}{2}x$$

$$\Rightarrow \quad EI\frac{dy}{dx} = \frac{Wx^2}{4} + C_1$$

Since the maximum deflection occurs at mid span C the slope at C is zero.

ie @ $x = l/2 \rightarrow \frac{dy}{dx} = 0$.

$$\therefore \quad 0 = \frac{w}{4}\left(\frac{l}{2}\right)^2 + C_1 \rightarrow C_1 = \frac{-wl^2}{16}$$

$$\therefore \quad EI\frac{dy}{dx} = \frac{wx^2}{4} - \frac{wl^2}{16} \quad \text{—— (1)}$$

slope eq.

To find the slope at A put $x=0$

$$\therefore \quad EI\, i_a = \frac{-wl^2}{16} \Rightarrow i_a = -\frac{wl^2}{16EI}$$

Integrating slope equation we get

$$\int EI\frac{dy}{dx} = \int\frac{wx^2}{4} - \int\frac{wl^2}{16}$$

$$EI\,y = \frac{wx^3}{12} - \frac{wl^2x}{16} + C_2.$$

since at A deflection is zero

we have at $x=0$, $y=0$ $\quad C_2 = 0$

$$\therefore \quad EIy = \frac{wx^3}{12} - \frac{wl^2}{16} * x \quad \text{—— (2)}$$

deflection eq.

To find the deflection at C put $x = l/2$

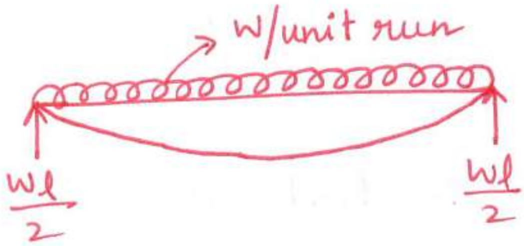
$$\Rightarrow EIy_c = \frac{w}{12}\left(\frac{l}{2}\right)^3 - \frac{wl^2}{16}\left(\frac{l}{2}\right) = -\frac{wl^3}{48}$$

$$\therefore \quad y_c = \frac{-wl^3}{48EI}$$

$\therefore$ $\boxed{\text{Downward deflection of } C = \frac{wl^3}{48EI}}$

* Simply supported beam of span l carrying a udl of w per unit run over the whole span.

Consider a simply supported beam of span l carrying a udl w per unit run over the whole span.



Vertical reaction = $\frac{wl}{2}$

The BM at any section distant x from the end A is given by

$$EI\frac{d^2y}{dx^2} = \frac{wl}{2}x - \frac{wx^2}{2}$$

On Integrating the above equation we get

$$EI\frac{dy}{dx} = \frac{wl}{4}x^2 - \frac{wx^3}{6} + C_1$$

The loading being symmetrical, the maximum deflection will occur at mid span and hence the slope at mid span equals zero.

ie @ $x = l/2$ $\quad \frac{dy}{dx} = 0$

$$\therefore 0 = \frac{wl}{4}\left(\frac{l}{2}\right)^2 - \frac{w}{6}\left(\frac{l}{2}\right)^3 + C_1$$

$$C_1 = \frac{-wl^3}{24}$$

$$EI\frac{dy}{dx} = \frac{wl}{4}x^2 - \frac{wx^3}{6} - \frac{wl^3}{24} \quad \text{--- (1)}$$

Slope equation.

$$\therefore \frac{dy}{dx} = \frac{wl}{24EI}\left[l^3 - 6lx^2 + 4x^3\right]$$

Integrating the slope eq we get

$$EIy = \frac{wl}{12}x^3 - \frac{wx^4}{24} - \frac{wl^3}{24}x + C_2$$

At A the deflection being zero, we have.

at $x=0$, $y=0$

$$\therefore C_2 = 0$$

$$\rightarrow \quad EIy = \frac{wl}{12}x^3 - \frac{wx^4}{24} - \frac{wl^3}{24}x \quad \text{--- (2)}$$

deflection eq.

$$\therefore y = -\frac{wx}{24EI}\left[l^3 - 2lx^2 + x^3\right]$$

To find the maximum deflection which occurs at mid span section C, putting $x = \frac{l}{2}$ in the above equation.

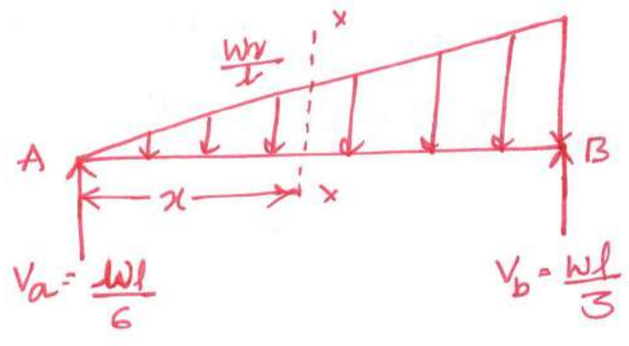
$$y_c = -\frac{w}{24EI}\frac{l}{2}\left[l^3 - 2l\cdot\frac{l^2}{4} + \frac{l^3}{8}\right]$$

$$= -\frac{5}{384}\frac{wl^4}{EI}$$

To find the slope at A put $x=0$ in slope eq

$$EIi_a = \frac{-wl^3}{24} \Rightarrow i_a = -\frac{wl^3}{24EI}.$$

* Simply supported beam of span l carries a triangular load whose intensity varies from zero at one end to 'w' at the other end.



Consider a simply supported beam carrying the triangular load. Let V_a and V_b be the reactions at the supports A & B.

Taking moments about the end A

$$V_b \times l = \frac{wl}{2}\,\frac{2}{3}l$$

$$\therefore V_b = \frac{wl}{3} \text{ and } V_a = \frac{wl}{2} - \frac{wl}{3} = \frac{wl}{6}$$

Consider any section x distant x from A

The BM at the section $x = M_x$

$$M_x = \frac{wl}{6}x - \frac{1}{2}x \cdot \frac{wx}{l} \cdot \frac{x}{3}$$

$$= \frac{wl}{6}x - \frac{wx^3}{6l}$$

The equation to the deflected shape of the beam is given by

$$EI\,\frac{d^2y}{dx^2} = M_x$$

$$EI\frac{d^2y}{dx^2} = \frac{wl}{6}x - \frac{wx^3}{6l}$$

Integrating we get $EI\frac{dy}{dx} = \frac{wlx^2}{12} - \frac{wx^4}{24l} + C_1$

[Slope eq]

Integrating again

$$EI\,y = \frac{wlx^3}{36} - \frac{wx^5}{120l} + C_1x + C_2$$

[deflection eq]

at $x=0,\ y=0 \quad \therefore C_2 = 0$

at $x=l,\ y=0 \quad \therefore 0 = \frac{wl^4}{36} - \frac{wl^4}{120} + C_1 l$

$$\therefore C_1 = \frac{7}{360}wl^3$$

$$\therefore EIy = \frac{wlx^3}{36} - \frac{wx^5}{120l} - \frac{7}{360}wl^3x$$

The deflection is maximum at a section where the slope is zero.

Equating the general ~~eqn~~ expression for slope to zero

$$\Rightarrow \frac{wlx^2}{12} - \frac{wx^4}{24l} - \frac{7}{360}wl^3 = 0$$

Let $x = nl$

$$n^2\frac{wl^3}{12} - n^4\frac{wl^3}{24} - \frac{7}{360}wl^3 = 0$$

$$15n^4 - 30n^2 + 7 = 0$$

solving as a quadratic in n^2

$$n^2 = 0.2697032$$

$$\therefore \quad n = 0.5193 \quad \therefore \quad x = 0.5193\,l$$

substituting the above value of x in deflection equation.

$$EI\,y_{max} = \frac{wl}{36}(0.5193l)^3 - \frac{w}{120l}(0.5193l)^5 - \frac{7}{360}wl^3(0.5193l)$$

$$= -0.006522\,wl^4$$

$$\therefore \quad y_{max} = -0.006522\frac{wl^4}{EI}.$$

P-12] A beam of 6m long, simply supported at its end, is carrying a point load of 50 KN at its centre. The moment of inertia of the beam is 78×10^6 mm^4. If E for the material of the beam = 2.1×10^5 N/mm^2. Calculate (i) deflection at the centre of the beam (ii) slope at the supports.

Sol: Given

$$l = 6\text{m} = 6000\text{ mm}$$

$$W = 50\text{ KN} = 50000\text{N}$$

$$I = 78 \times 10^6\text{ mm}^4$$

$$E = 2.1 \times 10^5\text{ N/mm}^2$$

Let y_c = deflection at the centre

θ_A = slope at the support.

$$\therefore \quad y_c = \frac{Wl^3}{48EI}$$

$$= \frac{50000 \times 6000^3}{48 \times 2.1 \times 10^5 \times 78 \times 10^6}$$

$$= \underline{\underline{13.736 \text{ mm}}}$$

$$\theta_A = \theta_B = \frac{-Wl^2}{16EI}$$

$$= \frac{50000 \times 6000^2}{16 \times 2.1 \times 10^5 \times 78 \times 10^6}$$

$$= \underline{\underline{0.06868 \text{ rad}}}$$

P-13] A beam 4m long, simply supported at its ends, carries a point load W at its centre. If the slope at the ends of the beam is not to exceed 1°, find the deflection at the centre of the beam.

Sol :- Given

$L = 4\text{m} = 4000 \text{ mm}$

Point load at centre = W

Slope at the ends $\theta_A = \theta_B = 1° = \frac{1 \times \pi}{180} = 0.01745$ rad

Let y_c = deflection at the centre

$$\therefore \quad \theta_A = \frac{Wl^2}{16EI}$$

$$\Rightarrow \quad 0.01745 = \frac{Wl^2}{16EI}$$

$$y_c = \frac{Wl^3}{48EI}$$

$$= \frac{Wl^2}{16EI} \times \frac{l}{3} = 0.01745 \times \frac{4000}{3}$$

$$= \underline{\underline{23.26 \text{ mm}}}$$

P-14] A beam of uniform rectangular section 200mm wide and 300 mm deep is simply supported at its ends. It carries a uniformly distributed load of 9 kN/m run over the entire span of 5 m. If the value of E for the beam material is 1×10^4 N/mm² find (i) slope at supports (ii) maximum deflection.

Sol:- Given

width $b = 200$ mm

depth $d = 300$ mm

$$I = \frac{bd^3}{12} = \frac{200 \times 300^3}{12} = 4.5 \times 10^8 \text{ mm}^4$$

$$w = 9 \text{ kN/m} = 9000 \text{ N/m}$$

$$l = 5 \text{ m} = 5000 \text{ mm}$$

$\therefore$ Total load $W = w \times L$

$= 9000 \times 5 = 45000\,N$

$E = 1 \times 10^4\ N/mm^2$

Let θ_A = slope at the support

y_c = maximum deflection.

$$\therefore \theta_A = \frac{Wl^2}{24EI}$$

$$= \frac{45000 \times 5000^2}{24 \times 1 \times 10^4 \times 4.5 \times 10^8} = \underline{\underline{0.0104\ rad}}$$

$$y_c = \frac{5}{384} \frac{Wl^3}{EI}$$

$$= \frac{5}{384} \frac{45000 \times 5000^3}{1 \times 10^4 \times 4.5 \times 10^8}$$

$$= \underline{\underline{16.27\ mm}}$$

P-15] A beam of length 5m and of uniform rectangular section is simply supported at its ends. It carries a uniformly distributed load of 9 KN/m run over the entire length. Calculate the width and depth of the beam if permissible bending stress is 7 N/mm² and central deflection is not to exceed 1cm.

$E = 1 \times 10^4\ N/mm^2$

Sol:- Given

$l = 5m = 5000\ mm$

$w = 9\ kN/m$

Total load $W = w \times l = 9 \times 5 = 45\ kN$

$= 45000\ N$

Bending stress $f = 7\ N/mm^2$

Central deflection $y_c = 1\ cm = 10\ mm$

$E = 1 \times 10^4\ N/mm^2$

Let b = width of beam

d = depth of beam.

$$MOI = \frac{bd^3}{12}$$

WKT

$$y_c = \frac{5}{384}\frac{Wl^3}{EI}$$

$$\Rightarrow \quad 10 = \frac{5}{384}\frac{45000 \times 5000^3}{1 \times 10^4 \times \frac{bd^3}{12}}$$

$$bd^3 = 878.906 \times 10^7\ mm^4 \qquad \text{--- (1)}$$

The maximum bending moment for a simply supported beam carrying a udl is given by

$$M = \frac{wl^2}{8} = \frac{Wl}{8}$$

$$= \frac{45000 \times 5}{8} \text{ Nm}$$

$$= \frac{45000 \times 5}{8} \times 1000 \text{ Nmm}$$

$$= 28125000 \text{ Nmm}$$

Now using the bending equation as

$$\frac{M}{I} = \frac{f}{y}$$

$$\frac{28125000}{\left(\frac{bd^3}{12}\right)} = \frac{7}{\left(\frac{d}{2}\right)}$$

$$\Rightarrow \quad bd^2 = 24107142.85 \text{ mm}^3 \text{ —— (2)}$$

dividing eq (1) by eq (2) we get

$$d = \frac{838.906 \times 10^7}{24107142.85} = \underline{364.58 \text{ mm}}$$

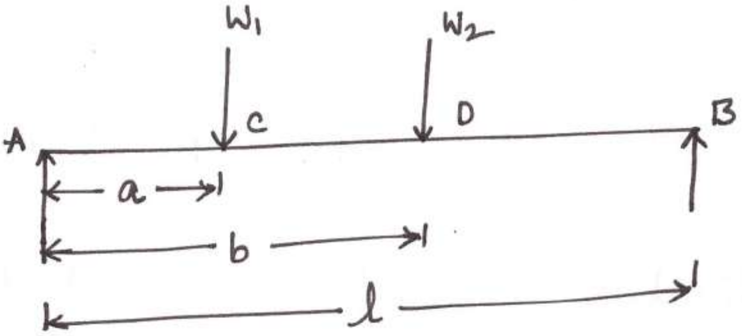
$$\Rightarrow \quad b(364.58)^2 = 24107142.85$$

$$\Rightarrow \quad b = \frac{24107142.85}{(364.58)^2} = \underline{181.36 \text{ mm}}$$

* Macaulay's Method:

Macaulay's Method is convinient for determining the deflection of a beam subjected to point loads or in general discontinuous loads. The method mainly consists in the special manner in which the bending moment at any section is expressed and in the manner in which the integration is carried.

Consider a simply supported beam loaded as shown:



The simply supported beam AB supported at A and B, having a span l and carrying the loads W_1 & W_2 at C and D at distances a and b from the end A.

Let V_a & V_b the Vertical reactions at A and B.

At any section between A and C distant x from A the bending moment is given by

$$M_x = V_a x$$

This expression for the bending moment holds good for all values of x between $x = 0$ & $x = a$

At any section between C & D and distant x from A, the bending moment is given by

$$M_a = V_a x - W_1 (x-a)$$

This expression holds good for all values of x between $x = a$ & $x = b$.

At any section between D and B and at a distant x from the end A, the bending moment is given by

$$M_x = V_a x - W_1 (x-a) - W_2 (x-b)$$

This expression holds good for all values of x between $x = b$ and $x = l$.

∴ In general at any section the bending moment is given by

$$M_x \cdot EI \frac{d^2y}{dx} = V_a x \,| - W_1 (x-a) \,| - W_2 (x-b) \qquad (1)$$

The manner in which the above expression is written should be noted. As the magnitude of x goes on increasing so that the law of loading changes additional expressions appear.

For values of x between $x = 0$ and $x = a$, only the first term of the above expression should be considered.

For x between $x = a$ & $x = b$, only first two terms of the above expression should be consider

For x between $x = b$ & $x = l$ all terms of above equation to be considered.

Integrating eq ① we get, the general expression for slope.

$$EI\frac{dy}{dx} = V_a\frac{x^2}{2} + C_1 \Big| - \frac{W_1(x-a)^2}{2} \Big| - \frac{W_2(x-b)^2}{2}$$

It is very important to note the following two points.

(a) The constant of integration C_1 should be written after the first term of the above expression.

(b) The quantity $(x-a)$ should be integrated as a whole ie as $\frac{(x-a)^2}{2}$ and not as $\frac{x^2}{2} - ax$.

|||ly $(x-b)$ as $\frac{(x-b)^2}{2}$

(c) The constant C_1 is valid for all values of x.

Integrating equation ②, we get diflection equation

$$EIy = V_a\frac{x^3}{6} + C_1x + C_2 \Big| - \frac{W_1(x-a)^3}{6} \Big| - \frac{W_2(x-b)^3}{6}$$

The constants C_1 and C_2 can be evaluated if the end conditions are known.

For instance, when the beam is simply supported the deflection is zero at A and B ie @ $x=0$ & $x=l$, $y=0$.

Putting $x=0$ & $y=0$ in the deflection eq we get $C_2=0$

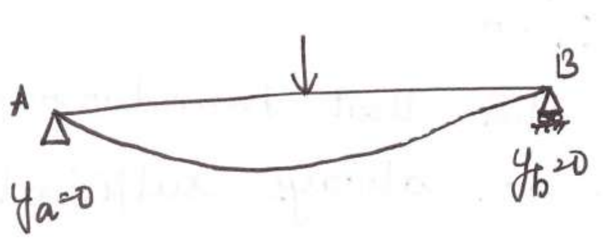
putting $x=l$ & $y=0$ in the deflection eq, the C_1 can be evaluated. Once the constants C_1 and C_2 are known, the slope and deflection at any section can be determined.

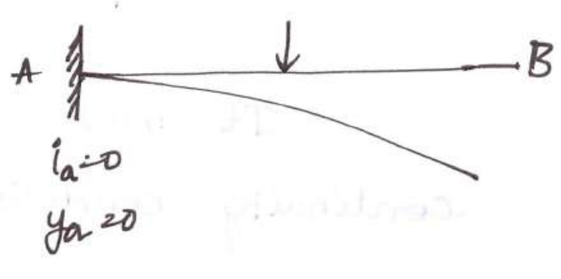
In general, the constants of integration are determined from known conditions about the slopes and deflections. The conditions fall into three categories, namely.

(i) Boundary Conditions
(ii) Continuity condition
(iii) Symmetry conditions.

(i) <u>Boundary Conditions</u>: These conditions relate to the deflections and slopes at the supports of a beam.

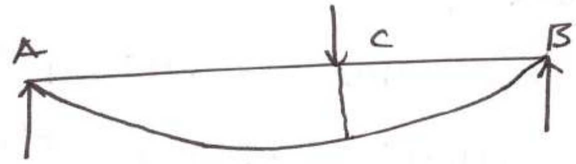
For instance, at a simple support we know that the deflection is zero, as shown in fig(a), and at a fixed support, both the deflection and the slope are zero as shown in fig (b). Each such boundary condition offers one equation for evaluating the constants of integration.





(ii) Continuity Conditions:

These conditions occurs at points where the regions of integrations meet, such as at the point C of the beam shown below.



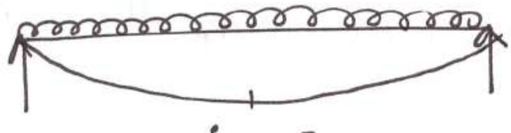
$i_{ca} = i_{cb}$

$y_{ca} = y_{cb}$

Since there is continuity of the deflected curve of the beam at C, the deflection at C determined for the left part must be equal to the deflection at C determined for the right part, similarly the slope at C determined for the left part must be equal to the slope at C determined for the right part.

(iii) Symmetry Conditions:

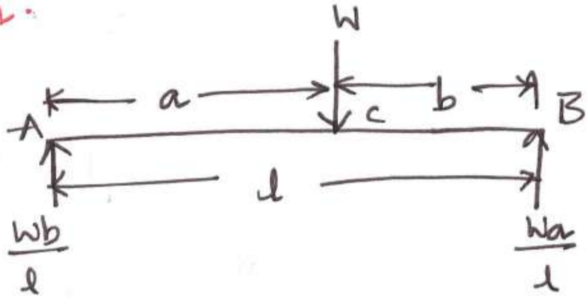
For a beam simply supported at its ends and carrying say a udl over the whole span, we know the slope at the midpoint of the deflected curve must be zero.



$i_c = 0$

It must be noted that boundary and continuity conditions are always sufficient to to determine the constants of integration. symmetry conditions provide additional equations which may be used for easier simplification. symmetry conditions provide additional equations which are not independent of others equations obtained by boundary and continuity conditions.

P-16] A beam of length l simply supported at the ends carries a point load W at a distance 'a' from the left end. Find the deflection under the load and the maximum deflection.



Sol:- Let AB be the beam of span l carrying the load W at C.

Let $AC = a$

$CB = b$

$a > b$

The vertical reactions at A and B are given by

$$V_a = \frac{Wb}{l} \qquad V_b = \frac{Wa}{l}$$

Following Macaulay's method the bending moment at any section distant x from A is given by

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{l}x \Big| - W(x-a)$$

Integrating we get $\quad EI\frac{dy}{dx} = \frac{Wbx^2}{2l} + C_1 \Big| - \frac{W(x-a)^2}{2}$ (slope eq)

Integrating again we get

$$EI\,y = \frac{Wbx^3}{6l} + C_1x + C_2 \Big| - \frac{W(x-a)^3}{6} \quad \text{- (deflection eq)}$$

At A the deflection is zero

ie At $x=0$

$y=0 \Rightarrow C_2 = 0$

At B the deflection is zero

ie At $x = l$

$y = 0$

$$\Rightarrow 0 = \frac{Wbl^2}{6} + C_1 l - \frac{W(l-a)^3}{6}$$

$$C_1 l = \frac{W(l-a)^3}{6} - \frac{Wbl^2}{6}$$

$$\Rightarrow C_1 l = \frac{Wb^3}{6} - \frac{Wbl^2}{6} = \frac{-Wb}{6}(l^2-b^2)$$

$$\therefore C_1 = \frac{-Wb}{6l}(l^2-b^2)$$

Hence the slope and deflection at any section are given by

$$EI\frac{dy}{dx} = \frac{Wbx^2}{2l} - \frac{Wb}{6l}(l^2-b^2) \Big| - \frac{W(x-a)^2}{2}$$

$$EI\,y = \frac{Wbx^3}{6l} - \frac{Wb}{6l}(l^2-b^2)x \Big| - \frac{W(x-a)^3}{6}$$

To find the deflection y_c under load, putting $x = a$ in the deflection eq we get

$$EI\,y_c = \frac{Wba^3}{6l} - \frac{Wb}{6l}(l^2-b^2)a$$

$$= \frac{-Wb}{6l}\,a(l^2-b^2-a^2)$$

But $l = a+b$.

$$EI\,y_c = -\frac{Wba}{6l}(a^2+b^2+2ab-b^2-a^2)$$

$$= -\frac{Wba}{6l}(2ab) = \frac{-Wa^2b^2}{3l}$$

$$\therefore\ y_c = \frac{-Wa^2b^2}{3EIl}.$$

To find the maximum deflection

The maximum deflection will occur on the larger segment AC. Further at the point of maximum deflection the slope is zero.

Hence equating the slope at a section in AC to zero we have

$$0 = \frac{Wbx^2}{2l} - \frac{Wb}{6l}(l^2-b^2)$$

$$x^2 = \frac{l^2-b^2}{3} \quad \therefore\ x = \sqrt{\frac{l^2-b^2}{3}} \text{ or } \sqrt{\frac{a^2+2ab}{3}}$$

The max deflection y_{max} can be determined by putting $x = \sqrt{\frac{l^2-b^2}{3}}$ in the expression for deflection

$$EI\,y_{max} = \frac{Wb}{6l}\left(\frac{l^2-b^2}{3}\right)^{3/2} - \frac{Wb}{6l}(l^2-b^2)\left(\frac{l^2-b^2}{3}\right)^{1/2}$$

$$= \frac{-Wb}{6l}(l^2-b^2)^{3/2}\left[\frac{1}{\sqrt{3}} - \frac{1}{3^{3/2}}\right]$$

$$= \frac{-Wb}{6l}(l^2-b^2)^{3/2}\left[\frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}\right]$$

$$= -\frac{Wb}{6l}(l^2-b^2)^{3/2} \cdot \frac{2\sqrt{3}}{9}$$

$$= \frac{-Wb\,(l^2-b^2)^{3/2}}{9\sqrt{3}\,l}$$

$$\therefore\ y_{max} = -\frac{Wb\,(l^2-b^2)^{3/2}}{9\sqrt{3}\,EI\,l} \qquad [a \geq b]$$

(or)

$$y_{max} = \frac{-Wb\,(a^2+2ab)^{3/2}}{9\sqrt{3}\,EI\,l}.$$

End Slopes:

To find the slope at A putting $x=0$ in slope eq.

$$EI\,i_a = \frac{-Wb}{6l}(l^2-b^2)$$

$$\therefore\ i_a = \frac{-Wb}{6EIl}(l^2-b^2) = \frac{-Wab(l+b)}{6EIl}.$$

To find the slope at B, putting $x=l$ in slope eq.

$$EI\,i_b = \frac{Wbl}{2} - \frac{Wb}{6l}(l^2-b^2) - \frac{Wb^2}{2}$$

$$EI\,i_b = \frac{Wbl}{3} + \frac{Wb^3}{6l} - \frac{Wb^2}{2} = \frac{Wb}{6l}(2l^2-3lb+b^2)$$

$$EI\, i_b = \frac{Wb}{6l}(l-b)(2l-b)$$

$$= \frac{Wab(2l-b)}{6l} = \frac{Wab(l+a)}{6l}$$

$$\therefore i_b = \frac{Wab(2l-b)}{6EIl} = \frac{Wab\,(l+a)}{6EIl}.$$

The end slopes i_a and i_b are functions of the position of the load. They reach their maximum values when the load is close to the mid point of the span.

Consider the slope $i_b = \dfrac{-Wb(l^2-b^2)}{6EIl}$.

For i_a to be maximum

$$\frac{di_a}{db} = 0$$

$$\Rightarrow \quad \frac{di_a}{db} = \frac{-W}{6EIl}(l^2-3b^2) = 0$$

$$b = \frac{l}{\sqrt{3}} = 0.57735\,l$$

$$\text{and} \quad a = 0.42265\,l$$

$$\therefore \quad i_{a_{max}} = \frac{-W}{6EI}\frac{l}{\sqrt{3}}\left(l^2 - \frac{l^2}{3}\right)$$

$$= \frac{-Wl^2\sqrt{3}}{27EI}.$$

P-17] A beam of length 6m is simply supported at its ends and carries a point load of 40KN at a distance of 4 m from the left support. Find the deflection under the load and max deflection. Also calculate the point at which maximum deflection takes place. Given MOI of beam = 7.33×10^7 mm^4 and $E = 2 \times 10^5$ N/mm^2

Sol:- Given

length $l = 6m = 6000$ mm

point load $W = 40$ KN $= 40,000$ N

distance of point load from left support

$a = 4m = 4000$ mm

$\therefore b = l - a = 6 - 4 = 2m = 2000$ mm.

Let y_c = deflection under the load

y_{max} = maximum deflection.

Using equation $y_c = \dfrac{Wa^2b^2}{3EIl}$

$$\Rightarrow y_c = \frac{40000 \times 4000^2 \times 2000^2}{3 \times 2 \times 10^5 \times 7.33 \times 10^7 \times 6000}$$

$= \underline{9.7 \text{ mm}}$

P-18] A beam of length 6m is simply supported at its ends and carries two point loads of 48 KN and 40 KN at a distance of 1m and 3m respectively from the left support. find

(i) deflection under each load

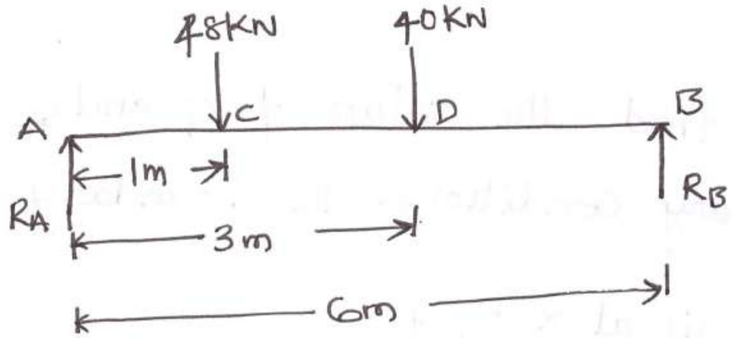
(ii) maximum deflection

(iii) the point at which maximum deflection occurs.

$E = 2\times10^5\ N/mm^2 \quad I = 85\times10^6\ mm^4.$

Sol :- Given

$I = 85\times10^6\ mm^4 \qquad E = 2\times10^5\ N/mm^2.$



Reactions R_A and R_B:

Taking moments about A

$$R_B\times6 = 48\times1 + 40\times3 = 168$$

$$\therefore R_B = \frac{168}{6} = 28\ KN$$

$$R_A + R_B = 48 + 40$$

$$\Rightarrow R_A = 60\ KN.$$

Consider the section x in the last part of the beam (ie in length DB) at a distance x from the left support A. The BM at this section is given by

$$EI \frac{d^2y}{dx^2} = R_A x \Big| - 48(x-1) \Big| - 40(x-3)$$

$$= 60x \Big| - 48(x-1) \Big| - 40(x-3)$$

integrating the above equation, we get

$$EI \frac{dy}{dx} = \frac{60x^2}{2} + C_1 \Big| - \frac{48(x-1)^2}{2} \Big| - \frac{40(x-3)^2}{2} \quad -(1)$$

integrating the above equation we get

$$EI\,y = \frac{30x^3}{3} + C_1x + C_2 \Big| - \frac{24(x-1)^3}{3} \Big| - \frac{20(x-3)^3}{3} \quad -(2)$$

To find the values of C_1 and C_2 use two boundary conditions. The boundary conditions are

(i) at $x = 0$, $y = 0$

(ii) at $x = 6$ $y = 0$.

⇒ substituting condition (i) in eq (2)

$$0 = 0 + 0 + C_2 \qquad \therefore C_2 = 0$$

⇒ substituting condition (ii) in eq (2)

$$0 = 10 \times 6^3 + C_1 \times 6 + 0 - 8(6-1)^3 - \frac{20}{3}(6-3)^3$$

$$= 2160 + 6C_1 - 1000 - 180 = 980 + 6C_1$$

$$\therefore C_1 = -163.33$$

substituting the values of C_1 and C_2 in eq (2) we get

$$EIy = 10x^3 - 163.33x \,\Big|\, -8(x-1)^3 \,\Big|\, -\frac{20}{3}(x-3)^3 \quad \text{--- (3)}$$

(i) Deflection under first load:

ie @ $x = 1$

$\therefore$ Substitute $x = 1$ in eq (3)

$$\Rightarrow \quad EIy_c = 10 \times 1^3 - 163.33 \times 1$$

$$= -153.33 = -153.33 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \quad y_c = \frac{-153 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = -9.019 \text{ mm}$$

(b) deflection under second load

@ $x = 3$

substituting $x = 3$ in eq (3)

$$\rightarrow \quad EI\, y_d = 10 \times 3^3 - 163.33 \times 3 - 8(3-1)^3$$

$$= -283.99 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \quad y_D = \frac{-283.99 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = -16.7 \text{mm}$$

(c) Maximum deflection:

The deflection is likely to be maximum at a section between C and D. For maximum deflection $\frac{dy}{dx}$ should be zero. Hence equate the eq (1) to zero

$$\Rightarrow \quad 0 = 30x^2 + 9 - 24(x-1)^2$$

$$\rightarrow \quad 30x^2 - 163.33 - 24(x^2+1-2x) = 0$$

$$\Rightarrow \quad 6x^2 + 48x - 187.33 = 0$$

on solving above equation we get

$$\underline{x = 2.87} \text{ m}$$

$\therefore$ @ $x = 2.87$

$$EI\, y_{max} = 10 \times 2.87^3 - 163.33 \times 2.87 - 8(2.87-1)^3$$

$$= 236.39 - 468.75 - 52.31$$

$$= 284.67 \times 10^{12} \text{ N mm}^3$$

$$\therefore \quad y_{max} = \frac{-284.67 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = \underline{-16.745 \text{ mm}}$$

P-19] A beam ABC of length 9m has one support of the left end and the other support at a distance of 6m from the left end. The beam carries a point load of 1KN at right and also carries a udl of 4 KN/m over a length of 3m as shown. Determine the slope and deflection at C.

$E = 2 \times 10^5 \text{ N/mm}^2$

$I = 5 \times 10^8 \text{ mm}^4$

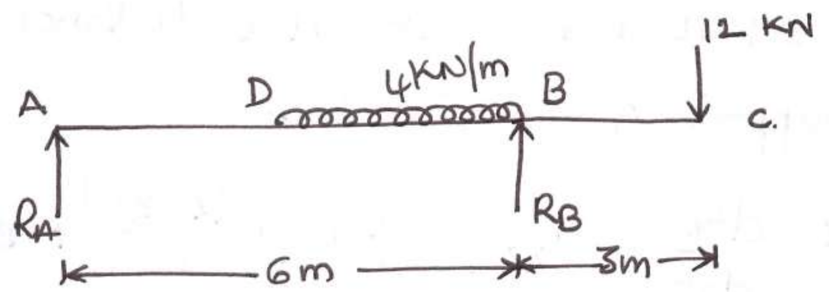
Sol :- Given

Point load $W = 12$ KN

Udl. $w = 4$ KN/m

$E = 2 \times 10^5$ N/mm²

$I = 5 \times 10^8$ N/mm²



Reactions R_A and R_B :

Taking moments about A, we get

$$R_B \times 6 = 4 \times 3 \times \left(3 + \frac{3}{2}\right) + 12 \times 9$$

$$= 162$$

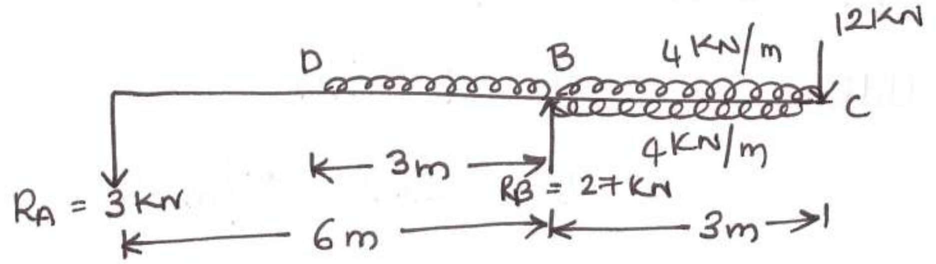
$$\therefore \quad R_B = 27 \text{ KN} \quad (\uparrow)$$

$$R_A + R_B = 12 + 12 = 24$$

$$\therefore \quad R_A = 24 - 27 = -3 \text{ KN} \; (\downarrow)$$

Negative sign shows that R_A will be acting downward

In order to obtain general expression for the bending moment at a distance x from the left end A, which will apply for all values of x, it is necessary to extend the udl upto C, compensating with an equal upward load of 4 KN/m over the span BC.



Applying Macaulay's method

The BM at any section at a distance x from the support A is given by

$$EI\frac{d^2y}{dx^2} = -R_A x \Big| -4(x-3)\frac{(x-3)}{2}\Big| + R_B(x-6)\Big|$$

$$+4(x-6)\frac{(x-6)}{2}$$

$$= -3x\Big| -2(x-3)^2\Big| + 27(x-6)\Big| + 2(x-6)^2.$$

Integrating the above equation

$$EI\frac{dy}{dx} = \frac{-3x^2}{2} + C_1\Big| -2\frac{(x-3)^3}{3}\Big| + \frac{27(x-6)^2}{2}\Big| + \frac{2(x-6)^3}{3} \quad \text{—(1)}$$

Integrating the above equation

$$EIy = -\frac{3}{2}\frac{x^3}{3} + C_1x + C_2\Big| -\frac{2}{3}\frac{(x-3)^4}{4}\Big| + \frac{27}{2}\frac{(x-6)^3}{3}$$

$$\Big| + \frac{2}{3}\frac{(x-6)^4}{4}$$

$$= -\frac{x^3}{2} + C_1x + C_2\Big| - \frac{(x-3)^4}{6}\Big| + \frac{9}{2}(x-6)^3$$

$$\Big| + \frac{1}{6}(x-6)^4 \quad \text{—(2)}$$

where C_1 and C_2 are constant of integration.

Their values are obtained from boundary conditions which are

(a) at $x = 0$ $y = 0$ and

(b) at $x = 6$ $y = 0$

(a) substituting $x = 0$ & $y = 0$ in eq 2 (upto first term)

$$\Rightarrow \quad 0 = 0 + C_1 \times 0 + C_2 =$$

$$\therefore \quad C_2 = 0$$

(b) substituting $x = 6$ & $y = 0$ in eq 2 (upto second term)

$$\rightarrow \quad 0 = \frac{-6^3}{2} + C_1 \times 6 + 0 - \frac{(6-3)^4}{c}$$

$$= -108 + 6C_1 - 13.5$$

$$\therefore \quad C_1 = \frac{121.5}{6} = 20.25$$

substituting the values of C_1 and C_2 in eq 1 & 2

$$EI\frac{dy}{dx} = -\frac{3}{2}x^2 + 20.25 \Big| - \frac{2(x-3)^2}{3} \Big| + \frac{27}{2}(x-6)^2 \Big| + \frac{2}{3}(x-6)^3 \quad -③$$

$$EI\,y = \frac{-x^3}{2} + 20.25x \Big| - \frac{1}{6}(x-3)^4 \Big| + \frac{9}{2}(x-6)^3 \Big| + \frac{1}{6}(x-6)^4 \quad -④$$

(i) Slope at point C

By substituting $x = 9m$ in eq 3 we get the slope at C. Here complete equation is to be taken as point $x = 9$ m lies in the last part of beam

$$\therefore EI\theta_c = \frac{-3}{2} \times 9^2 + 20.25 - \frac{2}{3}(9-3)^3 + \frac{27}{2}(9-6)^2 + \frac{2}{3}(9-6)^3$$

$$= -121.5 + 20.25 - 144 + 121.5 + 18$$

$$= -105.75 \text{ KNm}^2$$

$$= -105.75 \times 10^9 \text{ N/mm}^2$$

$$\therefore \theta_c = \frac{-105.75 \times 10^9}{2 \times 10^5 \times 5 \times 10^8} = \underline{-0.0010575 \text{ rad}}$$

(ii) Deflection at point C.

By substituting $x = 9$ m in complete eq (4) we get

$$EI\,y_c = -\frac{9^3}{2} + 20.25 \times 9 - \frac{1}{6}(9-3)^4 + \frac{9}{2}(9-6)^3 + \frac{1}{6}(9-6)^4$$

$$= -364.5 + 182.25 - 216 + 121.5 + 13.5$$

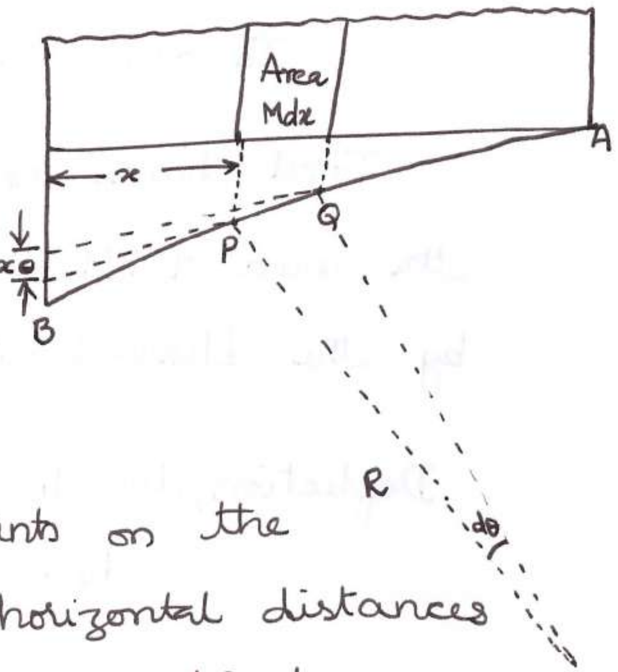
$$= -263.25 \text{ KN m}^3$$

$$= -263.25 \times 10^{12} \text{ N mm}^3$$

$$\therefore y_c = \frac{-263.25 \times 10^{12}}{2 \times 10^5 \times 5 \times 10^8}$$

$$= \underline{-2.6325 \text{ mm}}$$

* Moment Area Method - Mohr's Theorem :



Let AB represent part of the deflected form of a beam of uniform section.

Let A be a point of zero slope and zero deflection.

Let P and Q be two points on the deflection curve whose horizontal distances from B are x and $x+dx$ respectively.

Let the angle between the tangents at P and Q be $d\theta$. Obviously the angle between the normals at P and Q will also be equal to $d\theta$.

Let R be the radius of curvature of the elemental part PQ.

$$\therefore \quad d\theta = \frac{PQ}{R}$$

But $PQ = dx \quad \therefore \quad d\theta = \frac{dx}{R}$

But $\frac{1}{R} = \frac{M}{EI}$

$$\Rightarrow \quad d\theta = \frac{M}{EI}\,dx \qquad \text{——} (1)$$

Since A is point of zero slope, the total slope at B is

$$\theta = \frac{1}{EI}\sum_{x=0}^{x=BA} M\,dx .$$

$= \frac{1}{EI}$ (area of BM diagram between A and B)

In case, the slope at A is not zero, we have

Total change in slope between B and A equals the area of BM diagram between B and A divided by the flexural ridigity EI.

Deflection, due to the bending of the portion PQ

$$dy = x \, d\theta$$

substituting from eq ① $dy = \frac{M . x dx}{EI}$

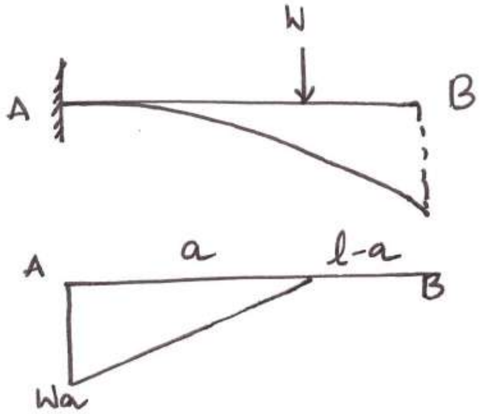
∴ Total deflection at B due to bending of all elemental portions like PQ

$$= y = \frac{1}{EI} \sum_{x=0}^{x=BA} M . x dx .$$

$= \frac{1}{EI}$ (the first moment of area of BMD b/w B & A] about B

In case the point A is not a point of zero slope, the deflection of B with respect to the tangent at A equals the first moment about B of the area of the BM diagram between B and A.

P-20] A cantilever of length l carries a point load W at a distance a from the fixed end. Find the slope and deflection at free end.



Sol:- Area of BMD between A & B

$$A = \frac{1}{2} a \cdot Wa = \frac{Wa^2}{2}$$

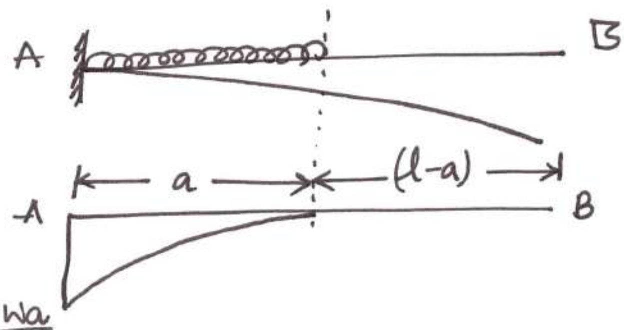
$$\therefore \text{Slope at } B = \frac{A}{EI} = \frac{Wa^2}{2EI}$$

deflection at B = $\dfrac{\text{Moment of the area of BMD between A and B about B}}{EI}$

$$\frac{A\bar{x}}{EI} = \frac{1}{EI} \cdot \frac{Wa^2}{2}\left[l - a + \frac{2a}{3}\right]$$

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(l-a).$$

P-21] A cantilever of length l carries a udl of w per unit run for a distance a from the fixed end. Find the slope and deflection at the free end.



Sol:- Area of BMD b/w A & B

$$A = \frac{1}{3} a \frac{wa^2}{2} = \frac{wa^3}{6}$$

$$\therefore \text{Slope @ } B = \frac{A}{EI} = \frac{wa^3}{6EI}$$

$\dfrac{wa}{2}$

$$\text{deflection at } B = \frac{A\bar{x}}{EI}$$

$$= \frac{1}{EI}\left[\frac{wa^3}{6}\left(l - a + \frac{3}{4}a\right)\right] \quad \frac{wa^3}{6EI}\left(l - \frac{a}{4}\right)$$