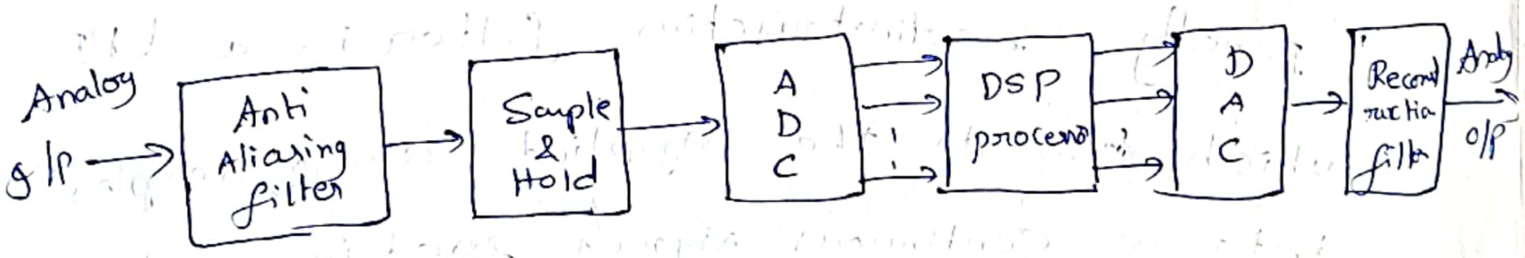
# UNIT - I

* Discrete Fourier Transform:
  * Discrete Fourier Series (DFS)
  * properties of Discrete Fourier Series
  * Discrete Fourier Transform (DFT)
  * The DFT as a linear transformation
  * Relationship of the DFT to other transforms
  * properties of DFT
* Fast Fourier Transforms
  * Efficient Computation of DFT algorithms
  * Radix 2 - Decimation - in - Time & Decimation - in - Frequency Algorithms.
  * Inverse FFT
* Illustrative problems.

# Introduction to DSP

The signal itself carries some kind of information available for observation.
processing means operating in some fashion on a signal i.e modification or extraction of some useful information.
The word digital shall mean that the processing is done with a digital computer or special purpose digital Hardware.

A System which carries out mathematical operations on sequences of samples related to a signal is referred as a DSP System.
The block diagram of basic DSP system is shown below




Anti aliasing filter is used to band limit the applied input signal to the specified frequency band.

Sample & Hold ckt provides the input signal samples at the input of ADC based on the sampling rate specified by the System design requirements.

ADC converts the analog input samples to the required digital input Sample

DSP processor
It is the heart of the System which performs the required operations on the digital signals & the processed signals are applied to the DAC

DAC converts the processed digital signals into the equivalent analog signals

Finally reconstruction filter is a LPF which converts the applied analog samples into a continuous signals used in real time applications.

The fundamental period of discrete time signal is always an integer.

Sum of periodic signals in discrete time is always periodic.

* Determine the periodicity of the following signals. Hence determine the fundamental period

i) $x[n] = \cos\left(\frac{2\pi}{5}\right) n \, u[n]$

ii) $x[n] = \cos\left(\frac{2\pi}{5} n + 30^\circ\right)$

iii) $x[n] = \sin 3n$

iv) $x[n] = e^{j(\pi/6\, n - 30^\circ)}$

v) $x[n] = \sin\left(\frac{2\pi}{5} n\right) + \cos\left(\pi/7\, n\right)$

vi) $x[n] = \sin\left(\frac{\pi}{5} n - 20^\circ\right) + \sin\left(3n + 27^\circ\right)$

vii) $x[n] = \sin\left(\pi/3\, n\right) \cos\left(\pi/7\, n\right)$

viii) $x(n) = \cos\left(\pi/7\, n + 30^\circ\right) + 10 \sin\left(3\pi/5\, n\right) - 4\cos\left(2\pi/3\, n\right)$

## * Periodicity of discrete time Signals

Discrete time signal is said to be periodic if $x[n] = x[n+N] \; \forall n$

$$= x[n+mN] \; \forall n \quad \text{--- (1)}$$

where m is an integer

The smallest value of N that satisfy the above condition is known as fundamental period (or) periodicity of $x[n]$

let $x[n] = A\cos(\omega_0 n + \theta)$ — (2)

then $x[n+N] = A\cos(\omega_0(n+N)+\theta)$

$$= A\cos(\omega_0 n + \theta + \omega_0 N) \quad \text{--- (3)}$$

eq$^n$ (3) reduces to eq$^n$ (2) if and only if

$$\omega_0 N = 2\pi K$$

$$\boxed{N = \frac{2\pi}{\omega_0} K} \quad \text{--- (4)}$$

If it is possible to convert the above eq$^n$ (4) into an integer then it is said to be periodic signal otherwise it is said to be Nonperiodic signal

i) $x[n] = \cos\left(\frac{2\pi}{5} n\right) u[n]$

$\because u[n] = \begin{cases} 1 ; n \geq 0 \\ 0 : n < 0 \end{cases}$

$\therefore$ Non periodic signal

ii) $x[n] = \cos\left(\frac{2\pi}{5} n + 30\right)$

$\omega_0 = \frac{2\pi}{5}$

$N = \frac{2\pi}{\omega_0} K = \frac{2\pi}{2\pi/5} K = 5K$

$= 5, 10, 15 \ldots$

Smallest value of $\boxed{N = 5}$ 1st harmonic of the F.P

It is an integer $\therefore$ It is periodic signal

iii) $x[n] = \sin 3n$

$\omega_0 = 3$

$N = \frac{2\pi}{\omega_0} K$

$N = \frac{2\pi}{3} K$

As there is no value of K which converts N into an integer $\therefore$ It is non periodic signal

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iv) $x[n] = e^{j(\pi/6 n - 30^\circ)}$

$\omega_0 = \pi/6$

$N = \frac{2\pi}{\pi/6} K$

$N = 12K \Rightarrow \boxed{N = 12}$ Integer

$\therefore$ It is periodic signal with F.P $N=12$

v) $x[n] = \sin(2\pi/5\, n) + \cos(\pi/7\, n)$

$N_1 = \frac{2\pi}{2\pi/5} K$ $\quad$ $N_2 = \frac{2\pi}{\pi/7} K$

$\boxed{N_1 = 5}$ $\quad$ $\boxed{N_2 = 14}$

$\therefore$ It is periodic signal

with Fundamental period

$$N = \frac{N_1 \cdot N_2}{GCM(N_1, N_2)}$$

$$N = \frac{5 \times 14}{GCM(5,14)} = \frac{70}{1}$$

$\boxed{N = 70 \text{ Samples}}$

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## * Discrete Fourier Series (DFS)

The Fourier Series representation for $x[n]$ Consists of N harmonically related exponential functions and it is expressed as

$$x[n] = \sum_{k=0}^{N-1} C_k \, e^{\frac{j2\pi nk}{N}} \; ; \; n = 0, 1, \ldots N-1 \quad \text{--- (1)}$$

$$\sum_{n=0}^{N-1} e^{\frac{j2\pi nk}{N}} = \begin{cases} N \; ; & k = 0, \pm N, \pm 2N \\ 0 \; ; & \text{otherwise} \end{cases}$$

where $\{c_k\}$ are the coefficients in the series representation

Multiply eq$^n$ (1) on both sides by $e^{\frac{-j2\pi nl}{N}}$ and summing the product from $n=0$ to $N-1$

$$\sum_{n=0}^{N-1} x[n] \, e^{\frac{-j2\pi nl}{N}} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} C_k \, e^{\frac{j2\pi (k-l)n}{N}}$$

$$\because \sum_{n=0}^{N-1} e^{\frac{j2\pi (k-l)n}{N}} = \begin{cases} N \; ; & k-l = 0, \pm N, \ldots \\ 0 \; ; & \text{otherwise} \end{cases}$$

$$\therefore \sum_{n=0}^{N-1} x[n] \, e^{\frac{-j2\pi nl}{N}} = N C_l$$

$$\Rightarrow C_l = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \, e^{\frac{-j2\pi nl}{N}} \quad \text{--- (2)}$$

Synthesis equation ① ⇒ $\boxed{x[n] = \sum_{k=0}^{N-1} C_k \, e^{\frac{j2\pi nk}{N}} \; ; n=0,1,\ldots N-1}$

Analysis equation ② ⇒ $\boxed{C_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \, e^{\frac{-j2\pi nk}{N}} \; ; k=0,1,\ldots N-1}$

The above two equations form a DFS pair

$$\therefore \boxed{x[n] \xleftrightarrow{DFS} C_k}$$

$\{C_k\}$ is a periodic sequence with fundamental period N.

$$C_{k+N} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \, e^{\frac{-j2\pi(k+N)n}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \, e^{\frac{-j2\pi nk}{N}}$$

$$= C_k$$

$$\therefore \boxed{C_{k+N} = C_k}$$

Thus the spectrum of a signal $x[n]$ is periodic with period N

* Find the DFS of $x[n] = \{1, 1, 0, 0\}$ with $N=4$ and draw the spectrum / Determine Spectra of the signal

Sol :

we know that

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \, e^{\frac{-j2\pi nk}{N}} \quad ; k = 0, 1, \dots N-1$$

$$C_k = \frac{1}{4} \sum_{n=0}^{3} x[n] \, e^{\frac{-j2\pi nk}{4}} \quad ; k = 0, 1, 2, 3$$

$$= \frac{1}{4} \left[ 1 + 1 \, e^{\frac{-j\pi k}{2}} + 0 + 0 \right]$$

$$C_k = \frac{1}{4} \left[ 1 + e^{\frac{-j\pi k}{2}} \right]$$

For $k=0$

$$C_0 = \frac{1}{4} [1+1] = \frac{1}{2}$$

$k=1$

$$C_1 = \frac{1}{4} [1 - j]$$

$k=2$

$$C_2 = 0$$

$$C_3 = \frac{1}{4} [1 + j]$$

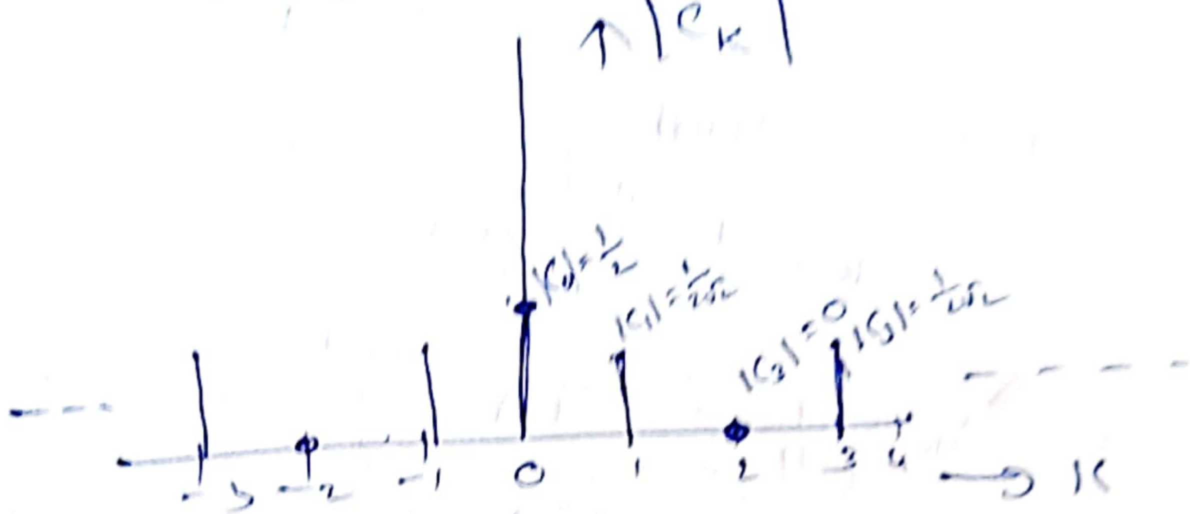
For $C_0 \Rightarrow |C_0| = \frac{1}{2}$ & $\angle C_0 = 0$

For $C_1 \Rightarrow |C_1| = \frac{\sqrt{2}}{4}$ & $\angle C_1 = \frac{-\pi}{4}$

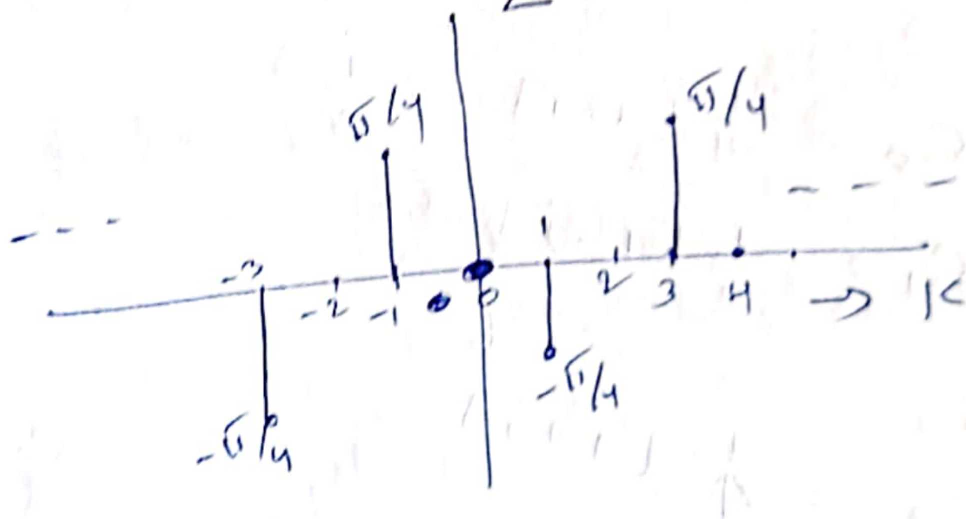
For $C_2 \Rightarrow |C_2| = 0$ & $\angle C_2 = \text{Undefined}$

For $C_3 \Rightarrow |C_3| = \frac{\sqrt{2}}{4}$ & $\angle C_3 = \frac{\pi}{4}$

Magnitude Spectrum:




Phase Spectrum:




* Determine Spectra of the signal

$x(n) = \cos \frac{\pi}{3} n$

where $\omega_0 = \pi/3$

$N = \frac{2\pi}{\omega_0} k = \frac{2\pi}{\pi/3} k = 6k = 6$ ∴ It is integer

It is periodic signal with Fundamental period $N=6$

We know that

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi nk}{N}} \;;\; k=0,1,\ldots N-1$$

$$= \frac{1}{6}\sum_{n=0}^{5} \cos\frac{\pi}{3}n \; e^{\frac{-j2\pi nk}{6}} \;;\; k=0,1,2,3,4,5$$

$$= \frac{1}{6}\Big[1+0.5\,e^{-j\pi/3\,k} - 0.5\,e^{-j\frac{2\pi}{3}k} - e^{-j\pi k} - 0.5\,e^{-j4\pi/3\,k} + 0.5\,e^{-j5\pi/3\,k}\Big]$$

For $k=0$

$$C_0 = \frac{1}{6}\big[1+0.5-0.5-1-0.5+1\big] = 0$$

$k=1$

$$C_1 = \frac{1}{6}\Big[1+0.5\,e^{-j\pi/3} - 0.5\,e^{-j2\pi/3} - e^{-j\pi} - 0.5\,e^{-j4\pi/3} + 0.5\,e^{-j5\pi/3}\Big]$$

$$= 1/2$$

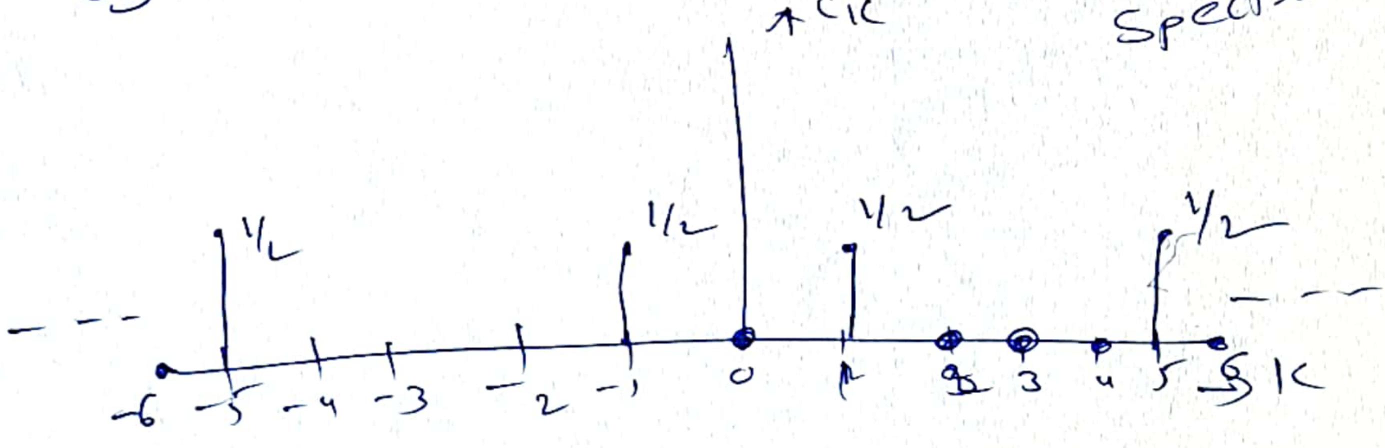
$C_2 = 0$

$C_3 = 0$

$C_4 = 0$

$C_5 = 1/2$

Amplitude Spectrum



* Determine the Spectra of the signal,

$$x[n] = \cos\sqrt{2}\pi n$$

$$\omega_0 = \sqrt{2}\pi$$

$$N = \frac{2\pi}{\omega_0} k$$

$$N = \frac{2\pi}{\sqrt{2}\pi} k = \sqrt{2} k$$

It is not an integer

So the signal is not periodic

This signal cannot be expanded in Fourier series. But the signal does ~~not~~ possess a spectrum

Its spectral content consists of the single frequency component at $\omega_0 = \sqrt{2}\pi$

* Determine Fourier Series representation of following discrete time signals

i) $x[n] = 2\cos\sqrt{3}\pi n$

ii) $x[n] = 4\cos \frac{\pi}{2} n$

iii) $x[n] = 3e^{j5\pi/2 n}$

iv) $x[n] = \{ ---1, 2, -1, \underset{\uparrow}{1}, 2, -1, 1, 2, -1, --- \}$

v) $x[n] = \{ ---1, -1, -\frac{1}{2}, 0, 0, \frac{1}{2}, \underset{\uparrow}{1}, -\frac{1}{2}, 0, 0, \frac{1}{2}, 1, -\frac{1}{2}, 0, 0, --- \}$

i) $x[n] = 2\cos\sqrt{3}\pi n$

$\omega_0 = \sqrt{3}\pi$

$N = \frac{2\pi}{\omega_0} K$

$N = \frac{2\pi}{\sqrt{3}\pi} K$

$N =$

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## properties of discrete fourier series:-

### ① Linearity property:-

statement:- If $x_1(n) \xleftrightarrow{DFS} c_{1k}$

and $x_2[n] \xleftrightarrow{DFS} c_{2k}$

then $a x_1[n] + b x_2[n] \xleftrightarrow{DFS} a c_{1k} + b c_{2k}$

Proof:- By the definition of DFS we have

$$DFS[x[n]] = c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi nk}{N}} \quad ; k = 0,1,\dots N-1$$

$$DFS[a x_1[n] + b x_2[n]] = \frac{1}{N} \sum_{n=\langle N \rangle} [a x_1[n] + b x_2[n]] e^{\frac{-j2\pi nk}{N}}$$

$$= a \cdot \frac{1}{N} \sum_{n=\langle N \rangle} x_1[n] e^{\frac{-j2\pi nk}{N}} + b \cdot \frac{1}{N} \sum_{n=\langle N \rangle} x_2[n] e^{\frac{-j2\pi nk}{N}}$$

$$= a c_{1k} + b c_{2k}$$

② Time shifting property:-

Statement:-

$$x[n] \xleftrightarrow{DFS.} c_k$$

$$x[n \pm n_0] \xleftrightarrow{DFS} e^{\pm jk\frac{2\pi}{N} n_0} c_k$$

proof:- By the definition of DFS we have

$$DFS\{x[n]\} = c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi n.k}{N}} \; ; \; k=0,1,--N-$$

$$DFS[x[n-n_0]] = \frac{1}{N} \sum_{n=0}^{N-1} x[n-n_0] e^{\frac{-j2\pi nk}{N}}$$

let $n-n_0 = r$

$$= \frac{1}{N} \sum_{r=<N>} x[r] e^{\frac{-j2\pi (r+n_0)k}{N}}$$

$$= e^{\frac{-j2\pi n_0 k}{N}} \frac{1}{N} \sum_{r=<N>} x[r] e^{\frac{-j2\pi rk}{N}}$$

$$= \boxed{e^{\frac{-j2\pi n_0 k}{N}} \cdot c_k}$$

③ Frequency shifting property:-

statement:- If $x[n] \xleftrightarrow{DFS} c_k$

then $e^{\pm jk\frac{2\pi}{N} n_0} \xleftrightarrow{DFS} c_{k \pm k_0}$

Proof:- By the definition of DFS we have,

$$DFS(x[n]) = c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi}{N} nk} \rightarrow ①$$

$$DFS\{e^{jk\frac{2\pi}{N} n_0} x(n)\} = \frac{1}{N} \sum_{n=<N>} [e^{jk\frac{2\pi}{N} n_0} x(n)] e^{\frac{-j2\pi}{N} n_0 k}$$

$$= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}(k-k_0)n} \rightarrow ②$$

By compare eq ① & ②

$$= c_{k-k_0}$$

$$e^{+jk_0\frac{2\pi}{N} n_0} \xleftrightarrow{DFS} c_{k-k_0}$$

④ conjugate property:-

statement:- If $x[n] \xleftrightarrow{DFS} c_k$

then $x^*[n] \xleftrightarrow{DFS} c^*_{-k}$

Proof :- By the definition of DFS we have

$$DFS[x(n)] = c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi}{N}nk} \quad -① $$

Apply conjugate on both sides

$$c_k^* = \left[\frac{1}{N}\sum_{n=<N>} x[n] e^{\frac{-j2\pi nk}{N}}\right]^*$$

$$c_k^* = \frac{1}{N}\sum_{n=<N>} x^*[n] e^{\frac{j2\pi nk}{N}}$$

Replace k by -k

$$c_{-k}^* = \frac{1}{N}\sum_{n=<N>} x^*[n] e^{\frac{-j2\pi nk}{N}}$$

$$= DFS(x^*[n])$$

$$x^*[n] \xleftrightarrow{DFS} c_{-k}^*$$

⑤ Time reversal property:-

statement:- If $x[n] \xleftrightarrow{DFS} c_k$

then $x[-n] \xleftrightarrow{DFS} c_{-k}$

By definition of DFS we have

$$DFS\{x[n]\} = c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi nk}{N}} \quad ; k=0,1,\ldots N-1 \quad \rightarrow ①$$

$$DFS(x[-n]) = \frac{1}{N}\sum_{n=0}^{N-1} x[-n] e^{\frac{-j2\pi nk}{N}}$$

$$= \frac{1}{N}\sum_{r<N>} x[r] e^{\frac{j2\pi rk}{N}}$$

$$= \frac{1}{N} x[r] e^{\frac{-j2\pi r(-k)}{N}} \rightarrow ②$$

$$= c_{-k}$$

compare ① & ②

$$x[-n] \xleftrightarrow{DFS} c_{-k}$$

⑥ Time scaling property:-

Statement:- If $x[n] \xleftrightarrow{DFS} c_k$

then $x_m[n] = \begin{cases} x(n/m) ; & \text{if } n \text{ is mul's of } m. \\ 0 ; & \text{otherwise} \end{cases} \xleftrightarrow{DFS} \frac{1}{m} c_k$

Proof :- By the definition of DFS,

$$DFS \; x[n] \; C_k = \frac{1}{N} \sum_{n=\langle N \rangle} e^{\frac{-j2\pi nk}{N}} \; ; \; 0 \le k \le N-1$$

$$DFS\{x(n/m)\} = \frac{1}{N} \sum_{n=\langle N \rangle} x(n/m) e^{\frac{-j2\pi nk}{N}}$$

If $x[n]$ is periodic with period $N$

$x\left[\frac{n}{m}\right]$ is also periodic with $mN$.

$$\left\{x\left[\frac{n}{m}\right]\right\} = \frac{1}{mN} \sum_{n=\langle mN \rangle} x\left[\frac{n}{m}\right] e^{\frac{-j2\pi nk}{mN}}$$

$$n/m = r \Rightarrow \frac{1}{mN} \sum_{r=\langle N \rangle} x[r] e^{\frac{-j2\pi rk}{N}}$$

$$= \frac{1}{m} C_k$$

7) Convolution in time domain property / periodic conversion property:-

Statement:- If $x_1[n] \xleftrightarrow{DFS} C_{1k}$

and $x_2[n] \xleftrightarrow{DFS} C_{2k}$

then $\sum_{r=\langle N \rangle} x_1[r] x_2[n-r] \xleftrightarrow{DFS} N C_{1k} C_{2k}$

proof:- By the definition of DFS.

$$DFSC\{x(n)\} \; C_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{\frac{-j2\pi nk}{N}} \; ; \; 0 \le k \le N-1.$$

$$DFS\left\{\sum_{r=\langle N \rangle} x_1(r) x(n)\right\} = \frac{1}{N} \sum_{n=\langle N \rangle} \left(\sum_{r=\langle N \rangle} x_1(r) x_2(n-r)\right) e^{\frac{-j2\pi nk}{N}}$$

interchange the order of summation

$$= \sum_{r=\langle N \rangle} x_1[r] \frac{1}{N} \sum_{n=\langle N \rangle} x_2[n-r] e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{r=\langle N \rangle} x_1[r] \left[e^{\frac{-j2\pi kr}{N}} \cdot C_{2k}\right]$$

$$= C_{2k} \cdot N \times \underbrace{\frac{1}{N} \sum_{r=\langle N \rangle} x_1[r] e^{\frac{-j2\pi rk}{N}}}_{\therefore C_{1k}}$$

$$= N C_{1k} C_{2k}$$

8) convolution in frequency domain property / multiplication property

statement :- If $x_1[n] \xleftrightarrow{DFS} c_{1k}$

and $x_2[n] \xleftrightarrow{DFS} c_{2k}$

then $x_1[n]x_2[n] \xleftrightarrow{DFS} \sum_{l=\langle N \rangle} c_{1l}\, c_2(k-l)$

$(c_{1k} * c_{2k})$

proof :-

By the definition of DFS

$$DFS\{x[n]\}\, c_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{\frac{-j2\pi nk}{N}} \quad ; \; 0 \le k \le N-1$$

$$DFS\{x_1[n]x_2[n]\} = \frac{1}{N}\sum_{n=\langle N\rangle} x_1[n]x_2[n]\, e^{\frac{-j2\pi nk}{N}}$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle}\left[\sum_{l=\langle N\rangle} c_{1l}\, e^{\frac{j2\pi nl}{N}}\right] x_2[n]\, e^{\frac{-j2\pi nk}{N}}$$

Interchanging the order of summation

$$= \sum_{l=\langle N\rangle} c_{1l}\underbrace{\left[\frac{1}{N}\sum_{n=\langle N\rangle} x_2[n]\, e^{\frac{-j2\pi(k-l)}{N}n}\right]}_{c_2[k-l]}$$

$$= \sum_{l=\langle N\rangle} c_{1l}\, c_2(k-l)$$

$$= c_{1k} * c_{2k}$$

9) First difference property :-

statement :- If $x[n] \xleftrightarrow{DFS} c_k$

then $x[n] - x[n-1] \xleftrightarrow{DFS} (1 - e^{\frac{-j2\pi}{N}k})c_k$

proof :- By the definition of DFS

$$DFS\{x(n)\}\, c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi nk}{N}} \quad ; \; 0 \le k \le N-1$$

$$x[n] \xleftrightarrow{DFS} c_k$$

$$x[n-1] \xleftrightarrow{DFS} e^{\frac{-j2\pi}{N}k} \cdot c_k$$

$$x[n] - x[n-1] \xleftrightarrow{DFS} c_k - e^{\frac{-j2\pi}{N}k} \cdot c_k$$

$$x[n] - x[n-1] \xleftrightarrow{DFS} \left[1 - e^{\frac{-j2\pi k}{N}}\right] c_k$$

10) Running sum property:-

Statement:- If $x[n] \xleftrightarrow{DFS} c_k$

then $\sum_{r=-\infty}^{n} x[r] \xleftrightarrow{DFS} \left[\frac{1}{1-e^{-j\frac{2\pi k}{N}}}\right] c_k$

(finite valued and periodic)

(if $c_0 = 0$)

Proof:- By the definition of DFS

$$DFS\{x[n]\}\ c_k = \frac{1}{N}\sum_{n=<N>} x[n]\, e^{\frac{-j2\pi nk}{N}} ; \quad 0 \le k \le N-1$$

consider, $\sum_{r=-\infty}^{n} x[r] = \sum_{r=-\infty}^{\infty} x[r]\underbrace{u[n-r]}_{1}$

$$u(n-r) = \begin{cases} 1; & n \ge r \\ 0; & n < r \end{cases}$$

$$= x[n] * u[n]$$

$$\therefore DFS\left\{\sum_{n=-\infty}^{n} x[r]\right\} = DFS\{x(n) * u(n)\}$$

$$= c_k \times \left(\frac{1}{1-e^{-j\frac{2\pi nk}{N}}}\right)$$

11) conjugate symmetry for real signals

Statement:- If $x[n]$ is real $\xrightarrow{DFS}$
$$\begin{cases} c_k = c_{-k}^* \\ Re\{c_k\} = Re\{c_{-k}\} \\ Img\{c_k\} = -Img\{c_{-k}\} \\ |c_k| = |c_{-k}| \\ \angle c_k = -\angle c_{-k} \end{cases}$$

12) Real & even and real & odd sil's:-

Statement:-

If $x[n]$ is real & even $\xrightarrow{DFS}$ $c_k$ is real & even

and If $x[n]$ is real & odd $\xrightarrow{DFS}$ $c_k$ is purely imaginary & odd

13) Even-odd decomposition for real sil's:-

Statement:-

$$Even\{x(n)\} \longrightarrow Re\{c_k\}$$

$$odd\{x(n)\} \longrightarrow j\,Img\{c_k\}$$

14) Parseval's relation for periodic signals:-

Statement: If $x(n)$ is periodic with period $N$ then, the avg Power, $P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |c_k|^2$

Proof:-

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n) x^*(n)$$

$$\because x(n) = \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi nk}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left( \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi nk}{N}} \right) x^*(n)$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n) \left( \sum_{k=0}^{N-1} c_k^* e^{\frac{-j2\pi nk}{N}} \right)$$

Interchanging the order of summation

$$P = \sum_{k=0}^{N-1} c_k^* \underbrace{\frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}}}_{c_k}$$

$$P = \sum_{k=0}^{N-1} c_k c_k^* \Rightarrow \sum_{k=0}^{N-1} |c_k|^2$$

Power remains same.

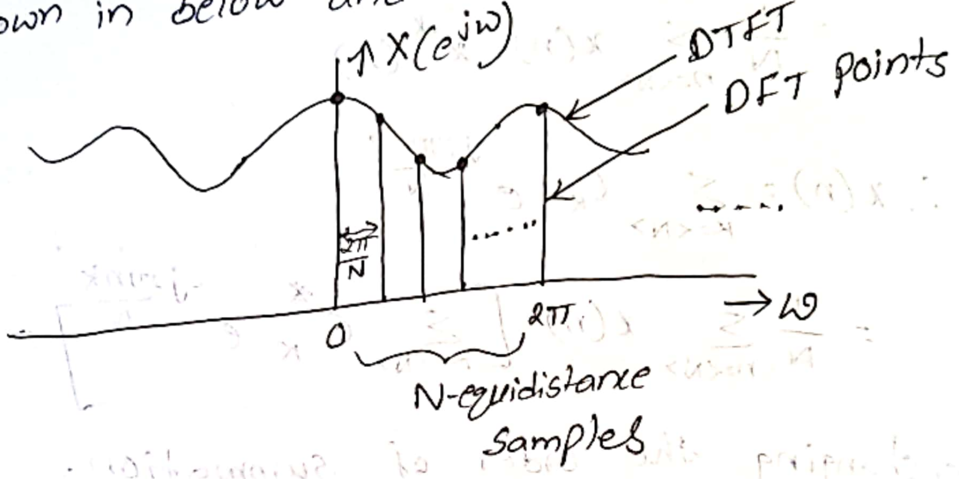
# Discrete Fourier Transform (DFT)

consider a sequence of length n, then the

DTFT of $x(n) = X(e^{jw}) = \sum_{n=-\alpha}^{\infty} x(n)e^{-jwn} \longrightarrow ①$

P & conti ... NP & DT

Let us consider the spectrum of the DTFT is shown in below and it is periodic with




sampling eq ① at $w = \frac{2\pi}{N} \cdot K$

$$① \Rightarrow X\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{n=-\alpha}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn} \longrightarrow ②$$

$$X\left(e^{j\frac{2\pi}{N}k}\right) = \dots + \sum_{n=-N}^{-1} x(n)e^{\frac{-j2\pi nk}{N}} + \sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi nk}{N}} + \sum_{n=N}^{2N-1} x(n \longrightarrow ③$$

$$X\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{l=-\alpha}^{\alpha} \sum_{n=lN}^{lN+N-1} x(n)\, e^{\frac{-j2\pi nk}{N}} \longrightarrow ④$$

$$X\left(e^{j\frac{2\pi}{N}K}\right) = \sum_{n=0}^{N-1} \sum_{l=-\alpha}^{\alpha} x(n)\, e^{\frac{-j2\pi nK}{N}}$$

replace $n = n - lN \Rightarrow lN = 0$

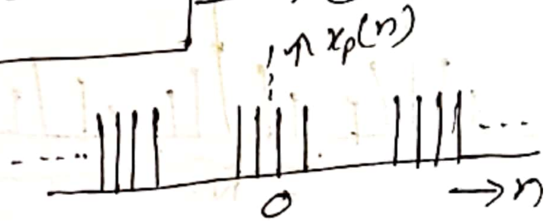
$$X\left(e^{\frac{j2\pi}{N}k}\right) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x[n-lN] e^{-j\frac{2\pi nk}{N}} \longrightarrow (5)$$

In the above eq (5)

$\sum_{l=-\infty}^{\infty} x[n-lN] \longrightarrow$ periodic sequencies

& denote by $x_p(n)$

$$(5) \Rightarrow \boxed{X\left(e^{\frac{j2\pi}{N}k}\right) = \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi nk}{N}}} \longrightarrow (6)$$




DFS

$$x_p(n) = \sum_{K=0}^{N-1} C_K e^{j\frac{2\pi nk}{N}} \quad ; \; n = 0, 1, \cdots N-1 \longrightarrow (7)$$

$$C_K = \frac{1}{N} \sum_{K=0}^{N-1} x_p(n) e^{-j\frac{2\pi nk}{N}} \quad ; \; k = 0, 1, \cdots N-1 \longrightarrow (8)$$

sub eq (6) in eq (8)

$$(8) \Rightarrow C_K = \frac{1}{N} X\left(e^{j\frac{2\pi}{N}k}\right) \longrightarrow (9)$$

sub eq (9) into eq (7)

$$(7) \Rightarrow x_p(n) = \sum_{K=0}^{N-1} \left[\frac{1}{N} X\left(e^{j\frac{2\pi}{N}k}\right)\right] e^{j\frac{2\pi nk}{N}}$$

$$\boxed{x_p(n) = \frac{1}{N} \sum_{K=0}^{N-1} X\left(e^{j\frac{2\pi}{N}k}\right) e^{j\frac{2\pi nk}{N}}} \longrightarrow (10)$$

DFT

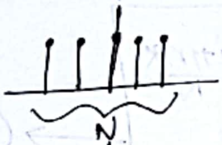
$$⑥ \Rightarrow \boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}} \quad ; k=0,1,\dots N-1} \rightarrow (11)$$

IDFT

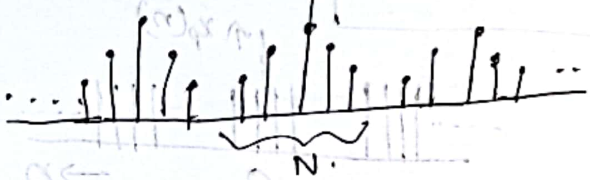
$$(12) \Rightarrow \boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi nk}{N}} \quad n=0,1,\dots N-1} \rightarrow (12)$$

The above equations (11),(12) represents DFT pair

$$\therefore\ x(n) \xleftrightarrow{DFT} X(k).$$



Non-periodic $x(n) = \begin{cases} x_p(n); & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$



periodic

## Problem:-

* compute the 4-point DFT of $x(n) = \{0,1,2,3\}$

A. we know that

$$DFT(x(n)) = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}} \quad ; k=0,1,2,\dots$$

$$IDFT(X(k)) = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi nk}{N}} \quad ; n=0,1,\dots$$

Given $N=4$

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi nk}{4}} \quad ; k=0,1,2,3.$$

$$x(k) = x(0)\, e^{\frac{-j2\pi(0)k}{4}} + x(1)\, e^{\frac{-j\pi k}{2}} + x(2)\, e^{j\pi k} + x(3\dots$$

$K=0 \Rightarrow X(0) = x(0) + x(1) + x(2) + x(3)$

$K=1 \Rightarrow X(1) = x(0) + x(1) e^{-j\frac{\pi}{2}} + x(2) e^{j\pi} + x(3) e^{-j\frac{3\pi}{2}}$

$K=2 \Rightarrow X(2) = x(0) + x(1) e^{-j\pi} + x(2) e^{j2\pi} + x(3) e^{-j3\pi}$

$K=3 \Rightarrow X(3) = x(0) + x(1) e^{-j\frac{3\pi}{2}} + x(2) e^{j3\pi} + x(3) e^{-j\frac{9\pi}{2}}$

$X(0) = 0+1+2+3 = 6$

$X(1) = 0 + 1.\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right)$

$+ 2\left(\cos\pi + j\sin\pi\right) + 3\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right)$

$= 0 + 1(0-j) + 2(-1+0) + 3\left(0 + j(1)\right)$

$= 0 - j - 2 + 3j$

$-2 + 2j = -2(1+j)$

$X(2) = 0 + 1.\left(\cos\pi - j\sin\pi\right) + 2\left(\cos 2\pi + j\sin 2\pi\right)$

$+ 3\left(\cos 3\pi - j\sin 3\pi\right)$

$= 0 + 1\left(-1 - j(0)\right) + 2\left(1 + j(0)\right) + 3\left(-1 - j(0)\right)$

$= 0 - 1 + 2 - 3 - 0$

$= -2$

$X(3) = 0 + 1.\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) + 2\left(\cos 3\pi + j\sin 3\pi\right)$

$+ 3.\left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$

$= 0 + 1\left(0 - j(-1)\right) + 2\left(-1 + j(0)\right) + 3\left(0 - j(1)\right)$

$= 0 + j - 2 - 3j = -2 - 2j$

$$x(n) = \{0,1,2,3\} \xleftrightarrow{DFT} X(k) = \{\underset{X(0)}{6}, \underset{X(1)}{-2+j2}, \underset{X(2)}{-2}, \underset{X(3)}{-2-j2}\}$$

2. Find IDFT of $X(k) = \{6, -2+j2, -2, -2-j2\}$

sol:-

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi nk}{N}} \quad ; n=0,1,\dots N-1$$

$$= \frac{1}{4}\sum_{k=0}^{3} X(k)\, e^{\frac{j2\pi nk}{4}} \quad ; n=0,1,2,3$$

$$= \frac{1}{4}\left[X(0)\cdot 1 + X(1)\cdot e^{j\frac{\pi}{2}n} + X(2)\,e^{j\pi n} + X(3)\,e^{j\frac{3\pi}{2}n}\right]$$

$$n=0 \Rightarrow x(0) = \frac{1}{4}\left[\cancel{x(0)}\, 6 + (-2+j2) - 2 + (-2-j2)\right]$$

$$= \frac{1}{4}\left[6 - 2 + \cancel{j2} - 2 - 2 - \cancel{j2}\right]$$

$$= 0$$

$$n=1 \Rightarrow x(1) = \frac{1}{4}\left[6 + (-2+j2)\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right) + (-2)\left(\cos\pi + j\right.\right.$$

$$\left. + (-2-j2)\left(\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2}\right)\right]$$

$$= \frac{1}{4}\left[6 + (-2+j2)(0+j) + (-2)(-1+0) + (-2-j2)(0-j)\right]$$

$$= \frac{1}{4}\left[6 - \cancel{2j} - 2 + \cancel{2} + \cancel{2j} - \cancel{2}\right]$$

$$= \frac{4}{4} = 1$$

$$n=2 \Rightarrow x(2) = \frac{1}{4}\Big[6 + (-2+j2)(\cos\pi + j\sin\pi) + (-2)(\cos 2\pi + j\sin 2\pi)$$
$$+ (-2-j2)(\cos 3\pi + j\sin 3\pi)\Big]$$

$$= \frac{1}{4}\Big[6 + (-2+j2)(-1+0) + (-2)(1+0)$$
$$+ (-2-j2)(-1+0)\Big]$$

$$= \frac{1}{4}\Big[6 + 2 - \cancel{2j} - \cancel{2} + \cancel{2} + \cancel{2j}\Big]$$

$$= \frac{8}{4} = 2.$$

$$n=3 \Rightarrow x(3) = \frac{1}{4}\Big[6 + (-2+j2)\Big(\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2}\Big) + (-2)(\cos 3\pi + j\sin 3\pi)$$
$$+ (-2-j2)\Big(\cos\frac{9\pi}{2} + j\sin\frac{9\pi}{2}\Big)\Big]$$

$$= \frac{1}{4}\Big[6 + (-2+j2)(0-j) + (-2)(-1+0) +$$
$$(-2-j2)(0+j)\Big]$$

$$= \frac{1}{4}\Big[6 + \cancel{2j} + 2 + 2 - \cancel{2j} + 2\Big]$$

$$= \frac{12}{4}$$

$$= 3$$

$$\therefore X(K) = \{6, -2+j2, -2, -2-j2\} \xleftrightarrow{IDFT} x(n) = \{0, 1, 2$$

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3. **EX:-** Find 4-point IDFT of $x(n) = \{10, -2+2j, -2, -2-2j$

using matrix method. twiddle factor.

**sol:-** $[x]_{N\times1} = \frac{1}{N}[W^*]_{N\times N}[X]_{N\times1}$ ; Given $N=4$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}_{4\times1} = \frac{1}{4}\begin{bmatrix} W_4^{*0} & W_4^{*0} & W_4^{*0} & W_4^{*0} \\ W_4^{*0} & W_4^{*1} & W_4^{*2} & W_4^{*3} \\ W_4^{*0} & W_4^{*2} & W_4^{*4} & W_4^{*6} \\ W_4^{*0} & W_4^{*3} & W_4^{*6} & W_4^{*9} \end{bmatrix}\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

$W_4^{*0} = e^{(j2\pi)\frac{0}{4}} = 1$ $\qquad W_4^{*4} = e^{(j2\pi)\frac{4}{4}} = \cos 2\pi + j\sin 2\pi = 1$

$W_4^{*1} = e^{(j2\pi)\frac{1}{4}} = j$ $\qquad W_4^{*6} = e^{(j2\pi)\frac{6}{4}} = \cos 3\pi + j\sin 3\pi = -$

$W_4^{*2} = e^{(j2\pi)\frac{2}{4}} = -1$ $\qquad W_4^{*9} = e^{(j2\pi)\frac{9}{4}} = \cos\frac{9\pi}{2} + j\sin\frac{9\pi}{2} =$

$W_4^{*3} = e^{j(2\pi)\frac{3}{4}} = -j$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}_{4\times1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 10-2+\cancel{2j}-2-2-\cancel{2j} \\ 10-\cancel{2j}-\cancel{2}+\cancel{2}+\cancel{2j}-2 \\ 10+2-\cancel{2j}-\cancel{2}+\cancel{2}+\cancel{2j} \\ 10+\cancel{2j}+2+2-\cancel{2j}+2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 12 \\ 16 \end{bmatrix} =$$

$\therefore X(n) = \{10, -2+2j, -2, -2-2j\} \xleftrightarrow{IDFT} x(n) = \{\underset{\uparrow}{1}, 2, 3,$

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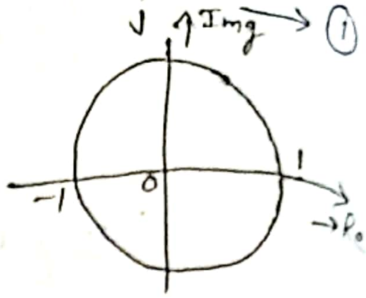
# DFT as a linear Transformation :-

$$DFT\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nK}{N}} \quad ; K = 0, 1, 2, \ldots, N-1. \rightarrow ①$$

Twiddle (or) phase factor $\boxed{W = e^{-j2\pi}}$




eq ① with twiddle factor

$$\Rightarrow X(K) = \sum_{n=0}^{N-1} x(n) \; W_N^{nK} \quad ; 0 \leq K \leq N-1 \longrightarrow ②$$

$$X(K) = x(0) W_N^{0} + x(1) \cdot W_N^{K} + x(2) W_N^{2K} + \ldots\ldots + x(N-1) W_N^{(N-1)K}$$

In matrix form

$$\underset{N \times 1}{\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}} = \underset{N \times N}{\begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(n-1) \end{bmatrix}$$

$$\boxed{\therefore [X]_{N \times 1} = [W]_{N \times N} \; [x]_{N \times 1}}$$

$$IDFT[X(K)] = x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi nK}{N}} \quad ; n = 0, 1, \ldots N-1 \rightarrow ③$$

$$③ \Rightarrow x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \; W_N^{-nK} \quad , 0 \leq n \leq N-1 \longrightarrow ④$$

$$X_{N\times 1} = W_{N\times N} \cdot x_{N\times 1}$$

MUL by $W^{-1}_{N\times N}$

$$\boxed{x_{N\times 1} = W^{-1}_{N\times N} \cdot X_{N\times 1}} \longrightarrow (5)$$

$$(4) \Rightarrow [x]_{N\times 1} = \frac{1}{N}[W^*]_{N\times N}[X]_{N\times 1} \longrightarrow (6).$$

Relation b/w (5) & (6)

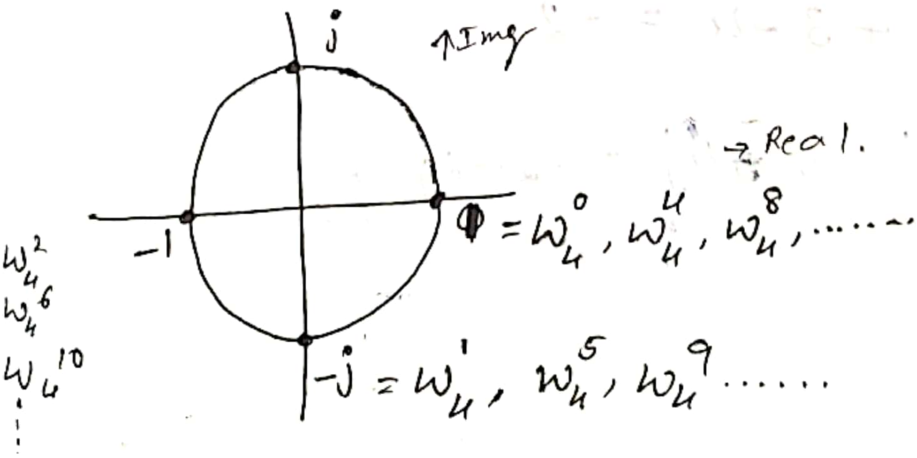
$$\boxed{\because W^{-1} = \frac{1}{N} W^*}$$

__Example__:- Find 4-point DFT of $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$ Using matrix method. twiddle factor.

__sol__:- $[X]_{N\times 1} = [W]_{N\times N} \cdot [x]_{N\times 1}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}_{4\times 1} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}_{4\times 4} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}_{4\times 1}$$

$W_4^3, W_4^7, W_4^{11} \ldots\ldots$



* compute the N-~~end~~ point DFT

(i) $\delta[n]$

(ii) $\delta[n-n_0]$

(iii) $x(n) = a^n$ ; $0 \le n \le N-1$

sol:- we know that the DFT of $x(n)$

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}} \quad ; \; k = 0, 1, \dots N-1$$

(i) $$DFT\{\delta(n)\} = \sum_{n=0}^{N-1} \delta(n) e^{\frac{-j2\pi nk}{N}}$$

$$\delta(n) = \begin{cases} 1 & ; n = 0 \\ 0 & ; n \neq 0 \end{cases}$$

$$= e^{\frac{-j2\pi nk}{N}} \Big/ _{n=0}$$

$$= 1.$$

$$\boxed{\delta(n) \xleftrightarrow{DFT} 1 \quad \forall k}$$

(ii) $$DFT\{\delta(n-n_0)\} = \sum_{n=0}^{N-1} \delta(n-n_0) e^{\frac{-j2\pi nk}{N}}$$

$n - n_0 = 0$

$n = n_0$

$n - n_0 = N$

$$\delta(n-n_0) = \begin{cases} 1 & ; n - n_0 = 0 \\ & \quad n = n_0 \\ 0 & ; n - n_0 \neq 0 \\ & \quad n \neq n_0 \end{cases}$$

$$= e^{\frac{-j2\pi nk}{N}} \Big/ _{n=n_0}$$

$W_4^0 = e^{\;\;} = 1$

$W_4^1 = e^{(-j2\pi)\frac{1}{4}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$

$W_4^2 = e^{(-j2\pi)\frac{2}{4}} = \cos\pi - j\sin\pi = -1$

$W_4^3 = e^{(-j2\pi)\frac{3}{4}} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = j$

$W_4^4 = e^{(-j2\pi)\frac{4}{4}} = \cos 2\pi - j\sin 2\pi = 1$

$W_4^6 = e^{(-j2\pi)\frac{6}{4}} = \cos 3\pi - j\sin 3\pi = -1$

$W_4^9 = e^{(-j2\pi)\frac{9}{4}} = \cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2} = -j$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}_{4\times1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}_{4\times4} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}_{4\times1}$$

$X(0) = 1 + 2 + 3 + 4 = 10$

$X(1) = 1 - 2j - 3 + 4j = -2 + 2j$

$X(2) = 1 - 2 + 3 - 4 = -2$

$X(3) = 1 + 2j - 3 - 4j = -2 - 2j$ //

$$= e^{\frac{-j2\pi n_0 k}{N}}$$

$$\boxed{\therefore \delta(n-n_0) = e^{\frac{-j2\pi n_0 k}{N}}}$$

(iii) $DFT\{a^n\} = \sum_{n=0}^{N-1} a^n e^{\frac{-j2\pi nk}{N}}$

$$= \sum_{n=0}^{N-1} \left[a e^{\frac{-j2\pi k}{N}}\right]^n$$

$$\boxed{\therefore \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}}$$

$$= \frac{1-\left(a e^{\frac{-j2\pi k}{N}}\right)^N}{1-a e^{\frac{-j2\pi k}{N}}}$$

$$= \frac{1-a^N}{1-a e^{\frac{-j2\pi k}{N}}}$$

4.

$$\boxed{\therefore a^n \xleftrightarrow{DFT} \frac{1-a^N}{1-a e^{\frac{-j2\pi k}{N}}}}$$

2. Find the DFT of the following signal.

$$x(n) = \begin{cases} 1 & ; 0 \le n \le 2 \\ 0 & ; \text{otherwise} \end{cases} \quad \text{for } N = 4.$$

Sol:- Given $N = 4$

$$x(n) = \{\underset{\uparrow}{1}, 1, 1, 0\}$$

$$DFT[x(n)] = X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nK}{N}} ; \quad K = 0, 1, \dots, N-1$$

$$= \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi nK}{N}}, \quad K = 0, 1, 2, 3$$

$$= x(0) e^{(0)} + x(1) e^{-j\frac{\pi K}{2}} + x(2) e^{-j\pi K} + x(3) e^{-j\frac{3\pi}{2}K}.$$

$$= x(0)1 + 1 \cdot \left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) + 1\left(\cos\pi - j\sin\pi\right) + 0\left(\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2}\right)$$

$$= 1 + 1(0 - j) + 1(-1 - 0) + 1(-1 - 0) + 0($$

$$= \cancel{1} - j - \cancel{1} - 1$$

$$X(0) = 1 + 1 + 1 + 0 = 3$$

$$X(1) = 1 + 1 \cdot (0 - j) + 1(-1 - 0) + 1(-1 - 0) + 0($$

$$= \cancel{1} - j - \cancel{1} - 1$$

$$X(2) = 1 + 1 \cdot e^{-j\frac{4\pi K}{4}} + 1 \cdot e^{-j}$$

$$= 1$$

$$X(3) = x(0) + x(1) e^{\frac{-3\pi j}{2}} + x(2) e^{-3j\pi}$$

$$= \cancel{1} + j + \cancel{(-1)}$$

$$= j$$

$$|X(K)| = \{3, 1, 1, 1\}$$

$\tan^{-1}$

$$\angle X(K) = \left\{0, -\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$$

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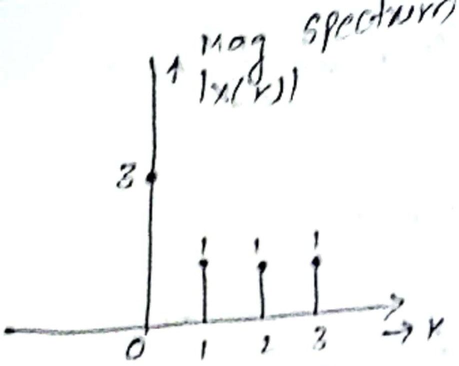


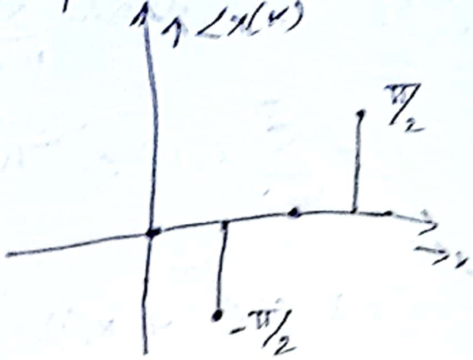


3. Find the 4-point DFT of $x(n) = \{1,0,0,1\}$ using Linear Transformation.

Sol:- $[X]_{N\times1} = [W]_{N\times N}\,[x]_{N\times1}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}_{4\times1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$X(0) = 1+0+0+1 = 2$

$X(1) = 1+0+0+j = 1+j$

$X(2) = 1+0+0-1 = 0$

$X(3) = 1+0+0-j = 1-j$

$$x(n) = \{1,0,0,1\} \xleftrightarrow{DFT} X(k) = \{2,\ 1+j,\ 0,\ 1-j\}$$

## * Relationship of the DFT to other Transforms

### 1. Relationship to the Fourier Series Coefficients of a periodic sequence:

A periodic sequence $\{x_p[n]\}$ with fundamental period N can be represented in a Fourier Series of the form

$$x_p[n] = \sum_{k=0}^{N-1} c_K \, e^{\frac{j 2\pi nk}{N}} \quad -①$$

where Fourier Series coefficients are given by the expression

$$c_K = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n] \, e^{\frac{-j 2\pi nk}{N}} \quad -②$$

we know that

$$DFT\{x[n]\} = X(K) = \sum_{n=0}^{N-1} x[n] \, e^{\frac{-j 2\pi nk}{N}} \quad -③$$

$$x[n] = x_p[n] \; ; \; 0 \le n \le N-1$$

By eqⁿ ② & ③ we get

$$c_K = \frac{1}{N} X(K)$$

$$\boxed{X(K) = N \, c_K}$$

* Relationship to the Fourier Transform of an aperiodic sequence:

We know that DTFT of $x[n]$ over one period

$$DTFT\{x[n]\} = X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n} \Big/_{\omega = \frac{2\pi}{N}k} \quad \text{—} (1)$$

&

$$X(k) = DFT\{x[n]\} = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi nk}{N}} \quad \text{—} (2)$$

$$\boxed{X(k) = X(e^{j\omega}) \Big/_{\omega = \frac{2\pi}{N}k}}$$

DFT is Sampled version of DTFT

* Relationship to the Z-transform

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{N-1} x[n]\, z^{-n}$$

$$\boxed{X(k) = X(z) \Big/_{z = e^{j\frac{2\pi}{N}k}}}$$

$$X(z) = \sum_{n=0}^{N-1} \left[ \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi nk}{N}} \right] z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left( e^{\frac{j2\pi k}{N}} z^{-1} \right)^{n}$$

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{\frac{j2\pi k}{N}} z^{-1}}$$

* Relationship to the Fourier Series Coefficients of a CT signal.

We know that Fourier series of periodic signal

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \text{--- (1)}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t f_0} \quad \text{--- (2)}$$

If we sample $x_a(t)$ at a uniform rate

$$f_s = \frac{N}{T_0} = N f_0 \quad \text{--- (3)}$$

$$\therefore x[n] \equiv x[nT_s] = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 [nT_s]}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{\frac{j2\pi nk}{N}}$$

$$= \sum_{k=0}^{N-1} \left[ \sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{\frac{j2\pi nk}{N}} \quad \text{--- (4)}$$

Using IDFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi nk}{N}} \quad \text{--- (5)}$$

Compare eq (4) & eq 5

$$\boxed{X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} = N \tilde{c}_k}$$

## Properties of DFT

1. Linearity property

Stmt: If $x_1[n] \xleftrightarrow{DFT} X_1(k)$

and $x_2[n] \xleftrightarrow{DFT} X_2(k)$

then $a x_1[n] + b x_2[n] \xleftrightarrow{DFT} a X_1(k) + b X_2(k)$

Proof: By the definition of DFT of $x(n)$

$$DFT\{x[n]\} = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi}{N}kn}, \quad 0 \le k \le N-1$$

$$\therefore DFT[a x_1[n] + b x_2[n]] = \sum_{n=0}^{N-1} [a x_1[n] + b x_2[n]] e^{\frac{-j2\pi}{N}kn}$$

$$= a \sum_{n=0}^{N-1} x_1[n] e^{\frac{-j2\pi}{N}kn} + b \sum_{n=0}^{N-1} x_2[n] e^{\frac{-j2\pi}{N}kn}$$

$$= a X_1(k) + b X_2(k)$$

$$\therefore \boxed{a x_1[n] + b x_2[n] \xleftrightarrow{DFT} a X_1(k) + b X_2(k)}$$

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2. periodicity property

If $x[n] \xleftrightarrow{DFT} X(K)$

~~stmt~~ $x[\cancel{n+N}] \xleftrightarrow{DFT}$

then $x[n+N] = x(n) \quad \forall n$

and $X[K+N] = X(K) \quad \forall K$

proof: By the definition IDFT of $X(K)$

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi}{N} nK} \quad ; \quad 0 \le n \le N-1$$

$$x[n+N] = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \, e^{j\frac{2\pi}{N} K(n+N)}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) \, e^{j\frac{2\pi}{N} Kn} \cdot \underbrace{e^{j\frac{2\pi}{N} KN}}_{1}$$

$$\boxed{x[n+N] = x[n]} \; //$$

$$X(K) = \sum_{n=0}^{N-1} x(n) \, e^{-j\frac{2\pi}{N} nK} \quad ; 0 \le K \le N-1$$

$$X(K+N) = \sum_{n=0}^{N-1} x[n] \, e^{-j\frac{2\pi}{N} n(K+N)}$$

$$= \sum_{n=0}^{N-1} x[n] \, e^{-j\frac{2\pi}{N} nK} \cdot \underbrace{e^{-j\frac{2\pi}{N} nN}}_{1}$$

$$\boxed{X(K+N) = X(K)} \; //$$

3) Circular shift of a sequence

Stmt: If $x[n] \xleftrightarrow{DFT} X(K)$

then $x((n-n_0))_N \xleftrightarrow{DFT} e^{-j\frac{2\pi}{N}kn_0} \cdot X(K)$

proof: By the definition of DFT of $x[n]$

we've

$$DFT\{x[n]\} = X(K) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}}$$

$$DFT[x((n-n_0))_N] = \sum_{n=0}^{N-1} [x((n-n_0))_N] e^{-j\frac{2\pi nk}{N}}$$

$$= \underbrace{\sum_{n=0}^{n_0-1} x((n-n_0))_N e^{-j\frac{2\pi nk}{N}}} + \underbrace{\sum_{n=n_0}^{N-1} x((n-n_0))_N e^{-j\frac{2\pi nk}{N}}} \quad \text{---} (1)$$

Consider (1) $\Rightarrow$ $\sum_{n=0}^{n_0-1} x((n-n_0))_N e^{-j\frac{2\pi nk}{N}} = \sum_{n=0}^{n_0-1} x(N-n_0+n) e^{-j\frac{2\pi nk}{N}}$

Let $N-n_0+n = l$

$$= \sum_{l=N-n_0}^{N-1} x(l) e^{-j\frac{2\pi k}{N}(l-N+n_0)}$$

$$= \sum_{l=N-n_0}^{N-1} x(l) e^{-j\frac{2\pi}{N}k(l+n_0)} \quad \text{---} (2)$$

Similarly

Consider

$$④ \Rightarrow \sum_{n=n_0}^{N-1} x((n-n_0))_N \, e^{-j2\pi nk/N} = \sum_{n=n_0}^{N-1} x(N+n-n_0) \, e^{-j\frac{2\pi nk}{N}}$$

Let $l = N+n-n_0$

$$= \sum_{l=N}^{2N-1-n_0} x(l) \, e^{-j\frac{2\pi}{N}k(l-N+n_0)}$$

$$= \sum_{l=0}^{N-1-n_0} x(l) \, e^{-j\frac{2\pi}{N}k(l+n_0)} \quad \text{--- } ③$$

Substitute $eq^n$ ② & ③ in ①

$$① \Rightarrow DFT[x((n-n_0))_N] = \sum_{l=N-n_0}^{N-1} x(l) \, e^{-j\frac{2\pi}{N}k(l+n_0)} + \sum_{l=0}^{N-1-n_0} x(l) \, e^{-j\frac{2\pi}{N}k(l+n_0)}$$

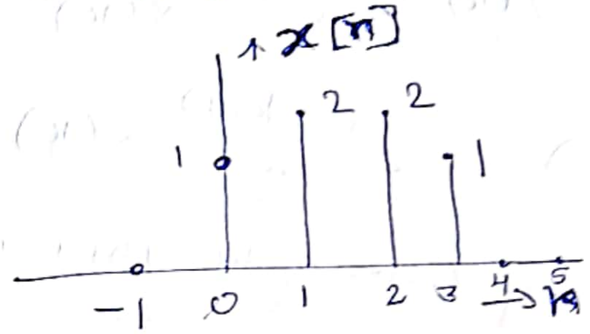
$$= \sum_{l=0}^{N-1} x(l) \, e^{-j\frac{2\pi}{N}k(l+n_0)}$$

$$= e^{-j\frac{2\pi}{N}kn_0} \cdot \underbrace{\sum_{l=0}^{N-1} x(l) \, e^{-j\frac{2\pi}{N}kl}}$$

$$= e^{-j\frac{2\pi}{N}kn_0} \cdot X(k) \; //$$

* Consider the finite length sequence $x[n]$ is shown below. The five point DFT of $x[n]$ is denoted by $X(k)$. plot the sequence whose DFT is

$$Y(k) = e^{\frac{-j4\pi k}{5}} X(k)$$



$$\underbrace{Y(k) = e^{\frac{-j4\pi k}{5}} X(k)}$$

$$Y(k) = e^{\frac{-j2\pi k(2)}{5}} X(k)$$

apply IDFT

Sol):

$\therefore\ y[n] = x((n-2))_5$

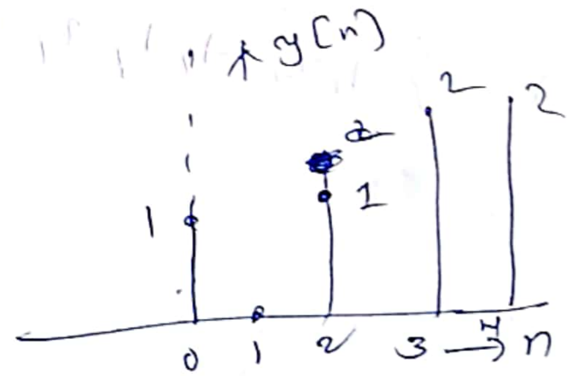
$y(0) = x((0-2))_5 = x((-2))_5 = x(-2 \text{ modulo } 5)$
$= x(3) = 1$

$y(1) = x((1-2))_5 = x((-1))_5 = x(4) = 0$

$y(2) = x((2-2))_5 = x((0))_5 = x(0) = 1$

$y(3) = x((3-2))_5 = x((1))_5 = x(1) = 2$

$y(4) = x((4-2))_5 = x((2))_5 = x(2) = 2$



* If the DFT of the sequence $x[n] = \{1, 2, -1, 1, 2, -1\}$ is $X(k)$. Find the sequence $y(n)$ whose DFT $Y(k) = e^{-j\pi k} X(k)$

Sol:

$$Y(k) = e^{-j\frac{2\pi k(3)}{6}} X(k)$$

the corresponding IDFT is

$$y[n] = x((n-3))_6$$

$$y(0) = x((0-3))_6 = x((-3))_6 = x(3) = 1$$

$$y(1) = x((1-3))_6 = x((-2))_6 = x(4) = 2$$

$$y(2) = x((2-3))_6 = x((-1))_6 = x(5) = -1$$

$$y(3) = x((3-3))_6 = x((0))_6 = x(0) = 1$$

$$y(4) = x((4-3))_6 = x((1))_6 = x(1) = 2$$

$$y(5) = x((5-3))_6 = x((2))_6 = x(2) = 1$$

$$y[n] = \{1, 2, -1, 1, 2, 1\}$$

4) Time reversal of a sequence

Stmt) If $x[n] \xleftrightarrow[N]{DFT} X(k)$

then $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$

proof: By the definition of DFT of $x[n]$ we've

$$DFT\{x[n]\} = X(k) = \sum_{n=0}^{N-1} x[n] \, e^{-j\frac{2\pi nk}{N}}$$

$$DFT[x((-n))_N] = \sum_{n=0}^{N-1} x((-n))_N \, e^{-j\frac{2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} x(N-n) \, e^{-j\frac{2\pi nk}{N}}$$

let $N-n = l$

$$= \sum_{l=N}^{1} x(l) \, e^{-j\frac{2\pi}{N}k(N-l)}$$

$$= \sum_{l=0}^{N-1} x(l) \, e^{j\frac{2\pi lk}{N}}$$

$$= \sum_{l=0}^{N-1} x(l) \, e^{-j\frac{2\pi}{N}(N-k)l}$$

$$= X(N-k)$$

$$= X((-k))_N$$

//

5) Circular frequency shift

Stmt: If $x[n] \xleftrightarrow[N]{DFT} X(k)$

then $x[n]\, e^{\frac{j2\pi l n}{N}} \xleftrightarrow[N]{DFT} X((k-l))_N$

proof: By the definition of DFT of $x[n]$, we've

$$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x[n]\, e^{\frac{-j2\pi nk}{N}}$$

$$DFT\left\{x(n)\, e^{\frac{j2\pi}{N}ln}\right\} = \sum_{n=0}^{N-1}\left\{x[n]\, e^{\frac{j2\pi}{N}ln}\right\} e^{\frac{-j2\pi k}{N}n}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}(k-l)n}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}(N+k-l)n}$$

$$= X(N+k-l)$$

$$= X((k-l))_N$$

---

$x_R[n] \longrightarrow X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$

$jx_I[n] \longrightarrow X_{cod}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$

$x_{ce}[n] = \frac{1}{2}[x[n] + x^*(N-n)] \longrightarrow X_R(k)$

$x_{cod}[n] = \frac{1}{2}[x[n] - x^*(N-n)] \longrightarrow jX_I(k)$

6) a) Complex Conjugate property

stmt: If $x[n] \xleftrightarrow[N]{DFT} X(k)$

then $x^*[n] \xleftrightarrow[N]{DFT} X^*(N-k) = X^*((-k))_N$

proof: By the definition of DFT of $x(n)$, we've

$$DFT[x[n]] = X(k) = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi nk}{N}}$$

$$DFT[x^*[n]] = \sum_{n=0}^{N-1} x^*[n]\, e^{-j\frac{2\pi nk}{N}}$$

$$= \left[\sum_{n=0}^{N-1} x[n]\, e^{j\frac{2\pi nk}{N}}\right]^*$$

$$= \underbrace{\left[\sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}(N-k)n}\right]^*}$$

$$= X^*(N-k)$$

$$= X^*((-k))_N$$

- Symmetry properties

If $x[n]$ is real then $X(k) = X^*(N-k)$

$X_R(k) = X_R(N-k)$

$X_I(k) = -X_I(N-k)$

$|X(k)| = |X(N-k)|$

$\angle X(k) = -\angle X(N-k)$

$x_e[n] = \frac{1}{2}[x[n] + x[N-n]] \longrightarrow X_R(k)$

$x_{od}[n] = \frac{1}{2}[x[n] - x[N-n]] \longrightarrow jX_I(k)$

5) b)

Stmt If $x[n] \xrightarrow[N]{DFT} X(K)$

Then $x^*(N-n) = x^*((-n))_N \xleftrightarrow{DFT} X^*(K)$

proof: By the definition DFT of $x[n]$, we've

$$DFT[x[n]] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$DFT[x^*(N-n)] = \sum_{n=0}^{N-1} x^*(N-n) e^{-j\frac{2\pi k}{N}n}$$

let $l = N-n$

$$= \sum_{l=N}^{1} x^*(l) e^{-j\frac{2\pi}{N}k(N-l)}$$

$$= \sum_{l=0}^{N-1} x^*(l) e^{j\frac{2\pi}{N}kl}$$

$$= \left[\underbrace{\sum_{l=0}^{N-1} x(l) e^{-j\frac{2\pi}{N}kl}}\right]^*$$

$$= X^*(K)$$

Note: If $x[n]$ is real

$x^*[n] = x[n] \xleftrightarrow[N]{DFT} X^*(N-K)$

i.e $\boxed{X(K) = X^*(N-K)}$

7) Circular Convolution property

Stmt: If $x_1[n] \xleftrightarrow[N]{DFT} X_1(k)$

and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

then $x_1[n] \circledast x_2[n] \xleftrightarrow[N]{DFT} X_1(k)\, X_2(k)$

proof: we know that

$$X_1(k) = \sum_{m=0}^{N-1} x_1(m)\, e^{-j\frac{2\pi}{N}km}$$

and $$X_2(k) = \sum_{l=0}^{N-1} x_2(l)\, e^{-j\frac{2\pi}{N}kl}$$

Let $$X_3(k) = X_1(k)\, X_2(k)$$

$$= \sum_{m=0}^{N-1} x_1(m)\, e^{-j\frac{2\pi}{N}km} \cdot \sum_{l=0}^{N-1} x_2(l)\, e^{-j\frac{2\pi}{N}kl}$$

The IDFT of $X_3(k)$ is given by

$$x_3[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k)\, e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[ X_1(k)\, X_2(k) \right] e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[ \sum_{m=0}^{N-1} x_1[m]\, e^{-j\frac{2\pi}{N}km} \cdot \sum_{l=0}^{N-1} x_2(l)\, e^{-j\frac{2\pi}{N}kl} \right] e^{\frac{j2\pi nk}{N}}$$

Interchanging the order of Summation

$$x_3[n] = \frac{1}{N} \sum_{m=0}^{N-1} x_1[m] \sum_{l=0}^{N-1} x_2(l) \underbrace{\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m-l)}}$$

$= N$ for $l = \underset{\text{modulo } N}{n-m}$

$= 0$ for $l \neq n-m$

$$\therefore x_3[n] = \frac{1}{N} \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N \cdot N$$

$$x[n] = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N \quad ; \; m = 0, 1, \ldots N-1$$

$$= x_1[n] \circledast x_2[n] \,//$$

8) Circular Correlation property

Stmt: If $x[n] \xleftrightarrow[N]{DFT} X(K)$

and $y[n] \xleftrightarrow[N]{DFT} Y(K)$

then $r_{xy}(l) \xleftrightarrow[N]{DFT} X(K)\, Y^*(K)$

proof: Consider

Correlat $(x(n) \& y(n))$ $\quad r_{xy}(l) = \sum_{n=0}^{N-1} x[n]\, y^*((n-l))_N$

$= x(l) \circledast y^*(-l)$ = Circular Convg $[x(n) \& y^*(n)]$

$$\therefore \text{DFT}[r_{xy}(l)] = \text{DFT}[x(n) \circledast y^*(-n)]$$

$$= X(K)\, Y^*(K) \,//$$

9) Multiplication of two sequences

Stmt: If $x_1[n] \xleftrightarrow[N]{DFT} X_1(K)$

and $x_2[n] \xleftrightarrow[N]{DFT} X_2(K)$

then $x_1(n)\, x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} X_1(K) \; (N) \; X_2(K)$.

proof: By the definition of DFT & IDFT:

$$IDFT[X(K)] = x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{+j\frac{2\pi nK}{N}}$$

$$DFT[x[n]] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi nK}{N}}$$

$$DFT[x_1[n]\, x_2[n]] = \sum_{n=0}^{N-1} \underbrace{x_1[n]}\, x_2[n]\, e^{-j\frac{2\pi nK}{N}}$$

$$= \sum_{n=0}^{N-1} \left[ \frac{1}{N} \sum_{l=0}^{N-1} X_1(l)\, e^{j\frac{2\pi}{N} nl} \right] x_2[n]\, e^{-j\frac{2\pi nK}{N}}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \underbrace{\sum_{n=0}^{N-1} x_2[n]\, e^{-j\frac{2\pi}{N}(K-l)n}}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_1(l)\, X_2(K-l)$$

$$= \frac{1}{N} X_1(K) \; (N) \; X_2(K)$$

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10) Parseval's Theorem

Stmt. If $x[n] \xleftrightarrow[N]{DFT} X(K)$

and $y[n] \xleftrightarrow[N]{DFT} Y(K)$

when $x[n]$ & $y[n]$ are complex valued sequences

then $$\sum_{n=0}^{N-1} x[n]\, y^*[n] = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, Y^*(K)$$

proof: Consider $$\sum_{n=0}^{N-1} x[n]\, y^*[n]$$

$$= \sum_{n=0}^{N-1} x[n] \left[ \frac{1}{N} \sum_{K=0}^{N-1} Y(K)\, e^{\frac{j2\pi nK}{N}} \right]^*$$

Interchanging the order of summation

$$= \frac{1}{N} \sum_{K=0}^{N-1} Y^*(K) \underbrace{\left[ \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi nK}{N}} \right]}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, Y^*(K)$$

* Fast Fourier Transforms

Efficient Computation of the DFT : algorithms

To compute all N values of the DFT requires $N^2$ complex multiplications & $N^2 - N$ complex additions

So, Direct computation of the DFT is basically inefficient.

∴ An efficient algorithm for DFT computation is the FFT algorithm because FFT algorithms exploit the below two properties of twiddle factor / phase factor

1. Symmetry property

$$W_N^{K+\frac{N}{2}} = \left(e^{-j\frac{2\pi}{N}}\right)^{K+\frac{N}{2}} = e^{-j\frac{2\pi}{N}K} \cdot e^{-j\pi}$$

$$= -\left(e^{-j\frac{2\pi}{N}}\right)^K = -W_N^K$$

$$\boxed{W_N^{K+\frac{N}{2}} = -W_N^K}$$

2. Periodicity property

$$W_N^{K+N} = \left(e^{-j\frac{2\pi}{N}}\right)^{K+N} = \left(e^{-j\frac{2\pi}{N}}\right)^K \cdot e^{-j2\pi} = W_N^K$$

$$= \left(e^{-j\frac{2\pi}{N}}\right)^K$$

$$\boxed{W_N^{K+N} = W_N^K}$$

So the above two properties reduces the no. of Complex multiplication required to perform DFT from $N^2 \xrightarrow{\text{reduces to}} \frac{N}{2} \log_2 N$

& Complex additions from $N^2 - N \xrightarrow{\text{reduced}} N \log_2 N$

* Radix-2 FFT Algorithm : $\boxed{N = r^L}$ $r \rightarrow$ radix of the FFT ($r=2$)

* Decimation in Time Algorithm (DIT-FFT)

The no. of output points $N$ can be expressed as a power of 2 i.e. $N = 2^L$ where $L$ is an integer

Let $x[n]$ is an N-pt sequence, where N is power of 2

Decimate or break this sequence into two sequences of length $\frac{N}{2}$

i.e. $x_e[n] = x(2n)$ ; $n = 0, 1, --, \frac{N}{2} - 1$

$x_o[n] = x(2n+1)$ ; $n = 0, 1, --, \frac{N}{2} - 1$

The N-pt DFT of $x[n]$ can be written as

$$X(K) = \sum_{n=0}^{N-1} x[n] W_N^{nK} \quad ; 0 \leq K \leq N-1 \qquad (1)$$

Now decimate

$$X(K) = \sum_{\substack{n=0 \\ (\text{even})}}^{N-1} x[n] W_N^{nK} + \sum_{n=0 (\text{odd})}^{N-1} x[n] W_N^{nK}$$

Substitute $n = 2r$ for n 'even' & $n = 2r+1$ for n 'odd'

$$X(K) = \sum_{r=0}^{\frac{N}{2}-1} x(2r)\, W_N^{2rK} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\, W_N^{(2r+1)K}$$

$$W_N^2 = \left(e^{-j\frac{2\pi}{N}}\right)^2 = e^{-j\frac{2\pi}{N/2}} = W_{N/2}$$

$$X(K) = \sum_{r=0}^{\frac{N}{2}-1} x(2r)\, W_{N/2}^{rK} + W_N^K \sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\, W_{N/2}^{rK}$$

$$\boxed{X(K) = G(K) + W_N^K\, H(K)}\ ;\ K = 0, 1, \ldots N-1 \quad —(2)$$

where $G(K)$ & $H(K)$ are the $\frac{N}{2}$ pt DFTs of even & odd numbered sequences

& $G(K)$ & $H(K)$ are periodic with period $\frac{N}{2}$

i.e $G(K+\frac{N}{2}) = G(K)$

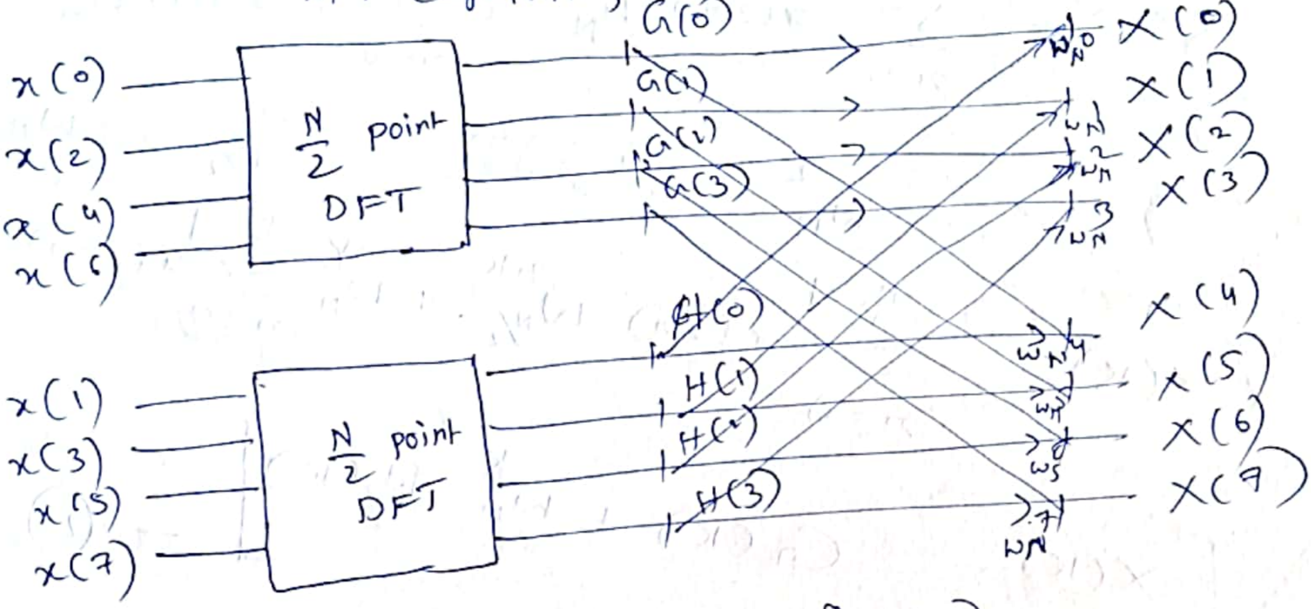
& $H(K+\frac{N}{2}) = H(K)$

& $W_N^{K+\frac{N}{2}} = -W_N^K$

(2) $\Rightarrow$ $X(K) = \begin{cases} G(K) + W_N^K\, H(K) ; & 0 \le K \le \frac{N}{2}-1 \\ G(K+\frac{N}{2}) + W_N^{K+\frac{N}{2}}\, H(K+\frac{N}{2}) ; & \frac{N}{2} \le K \le N \end{cases}$

$$\therefore \boxed{X(K) = \begin{cases} G(K) + W_N^K\, H(K) ; & 0 \le K \le \frac{N}{2}-1 \\ G(K) - W_N^K\, H(K) ; & \frac{N}{2} \le K \le N-1 \end{cases}} \quad —(3)$$

DIT decomposition of an 8 pt DFT into two 4-pt DFT computations




$$X(0) = G(0) + W_8^0 H(0)$$

$$X(1) = G(1) + W_8^1 H(1)$$

$$X(2) = G(2) + W_8^2 H(2)$$

$$X(3) = G(3) + W_8^3 H(3)$$

$$X(4) = G(0+4) + W_8^4 G(0+4)$$

$$X(4) = G(0) + W_8^4 H(0)$$

$$X(5) = G(1) + W_8^5 H(1)$$

$$X(6) = G(2) + W_8^6 H(2)$$

$$X(7) = G(3) + W_8^7 H(3)$$

The above process may be continued by expressing each of the two $\frac{N}{2}$-pt DFT's $G(k)$ & $H(k)$ as a combination two $\frac{N}{4}$ pt DFTS

$$G(K) = \sum_{n=0}^{\frac{N}{2}-1} g(n) W_{N/2}^{nK}$$

$$= \sum_{l=0}^{\frac{N}{4}-1} g(2l) W_{N/2}^{2lK} + \sum_{l=0}^{\frac{N}{4}-1} g(2l+1) W_{N/2}^{(2l+1)K}$$

$$= \sum_{l=0}^{\frac{N}{4}-1} g(2l) W_{N/4}^{lK} + W_{N/2}^{K} \sum_{l=0}^{\frac{N}{4}-1} g(2l+1) W_{N/4}^{lK}$$

$$\boxed{G(K) = A(K) + W_{N/2}^{K} B(K)} \quad (4)$$

where $A(K)$ & $B(K)$ are $\frac{N}{4}$ pt DFT of even & odd seque periodic with $\frac{N}{4}$ period $lK$

$$H(K) = \sum_{l=0}^{\frac{N}{4}-1} h(2l) W_{N/4}^{lK} + W_{N/2}^{K} \sum_{l=0}^{\frac{N}{4}-1} h(2l+1) W_{N/4}^{lK}$$

$$\boxed{H(K) = C(K) + W_{N/2}^{K} D(K)} \quad (5)$$

for $K=0$

$$G(0) = A(0) + W_{N/2}^{0} B(0) = A(0) + W_{N}^{0} B(0)$$

$K=1$

$$G(1) = A(1) + W_{N/2}^{1} B(1) = A(1) + W_{N}^{2} B(1)$$

$$G(2) = A(2) + W_{N/2}^{2} B(2) = A(0) + W_{N}^{4} B(0)$$

$$G(3) = A(3) + W_{N/2}^{3} B(3) = A(1) + W_{N}^{6} B(1)$$

$$H(0) = C(0) + W_{N}^{0} D(0)$$

$$H(1) = C(1) + W_{N}^{2} D(1)$$

$$H(2) = C(0) + W_{N}^{4} D(0)$$

$$H(3) = C(1) + W_{N}^{6} D(1)$$

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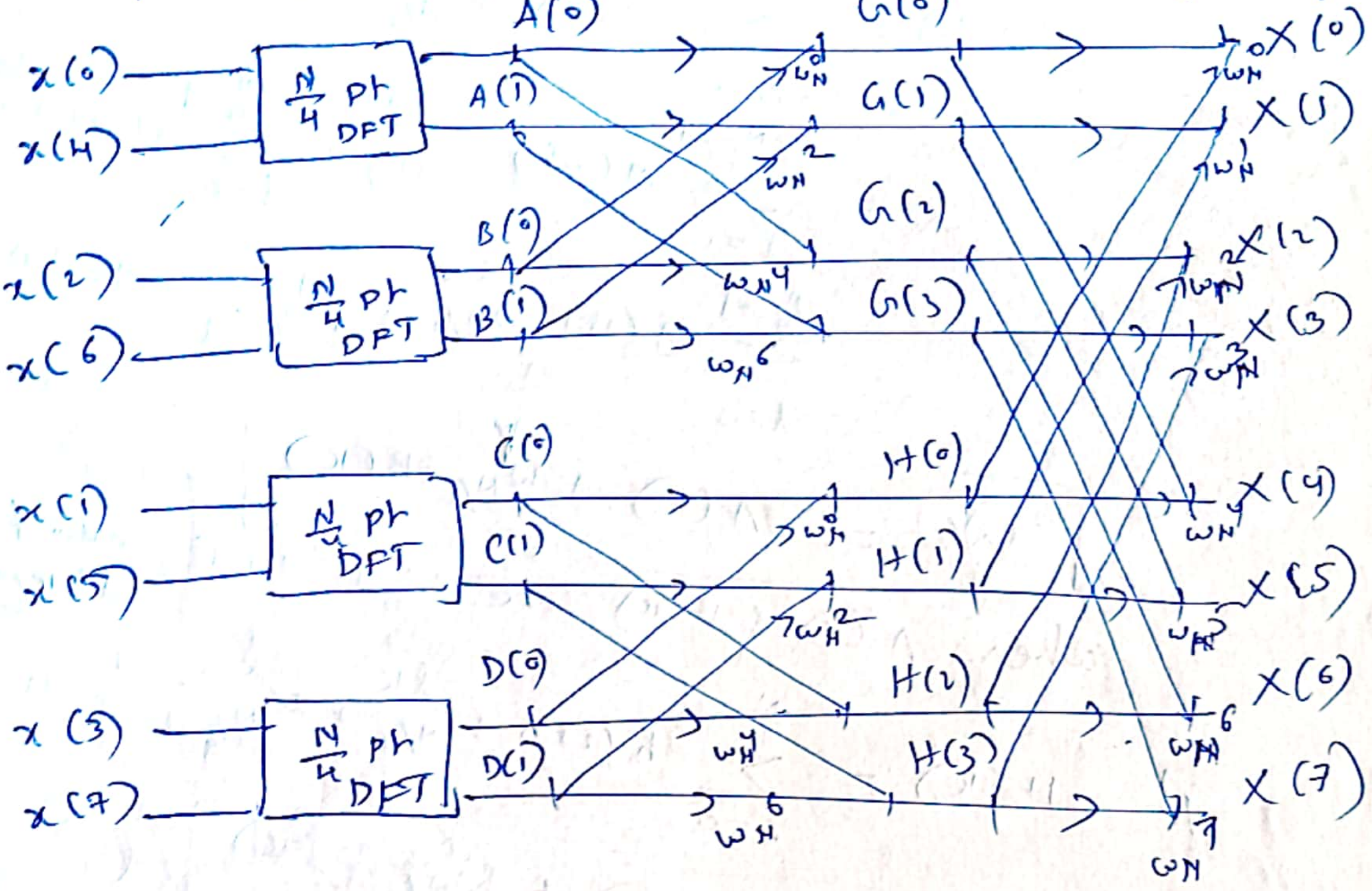
The flow graph of the 2nd stage DIT-FFT algorithm for N=8

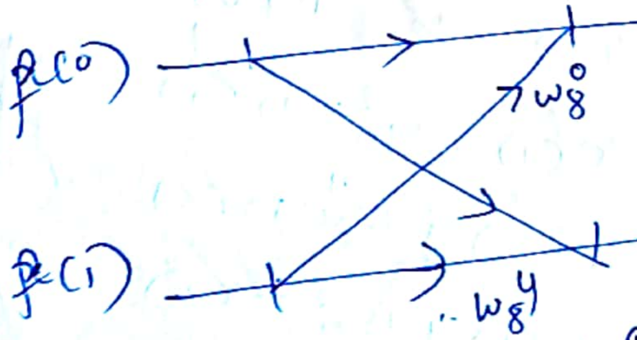


The above process continue until we are left with 2 pt DFTs (or) $\log_2 N$ stages

N=8

Two pt DFT $F(K)$ can be evaluated

$$F(0) = f(0) + W_8^0 f(1) = f(0) + W_8^0 f(1)$$
$$F(1) = f(0) + W_8^4 f(1) = f(0) - W_8^0 f(1)$$



$$F(0) = f(0) + W_N^0 f(1)$$

$$F(1) = f(0) - W_N^0 f(1)$$

In general

For N = 8 DIT-FFT algorithm Consists of 3 stages

| Index | binary rep | bit reversed binary | bit rev index |
| --- | --- | --- | --- |
| 0 | 000 | 000 | 0 |
| 1 | 001 | 100 | 4 |
| 2 | 010 | 010 | 2 |
| 3 | 011 | 110 | 6 |
| 4 | 100 | 001 | 1 |
| 5 | 101 | 101 | 5 |
| 6 | 110 | 011 | 3 |
| 7 | 111 | 111 | 7 |

∴ reduced 8-pt DIT-FFT flow-graph shown below

No. of stages $\log_2 N = \log_2 8 = 3$ stages.

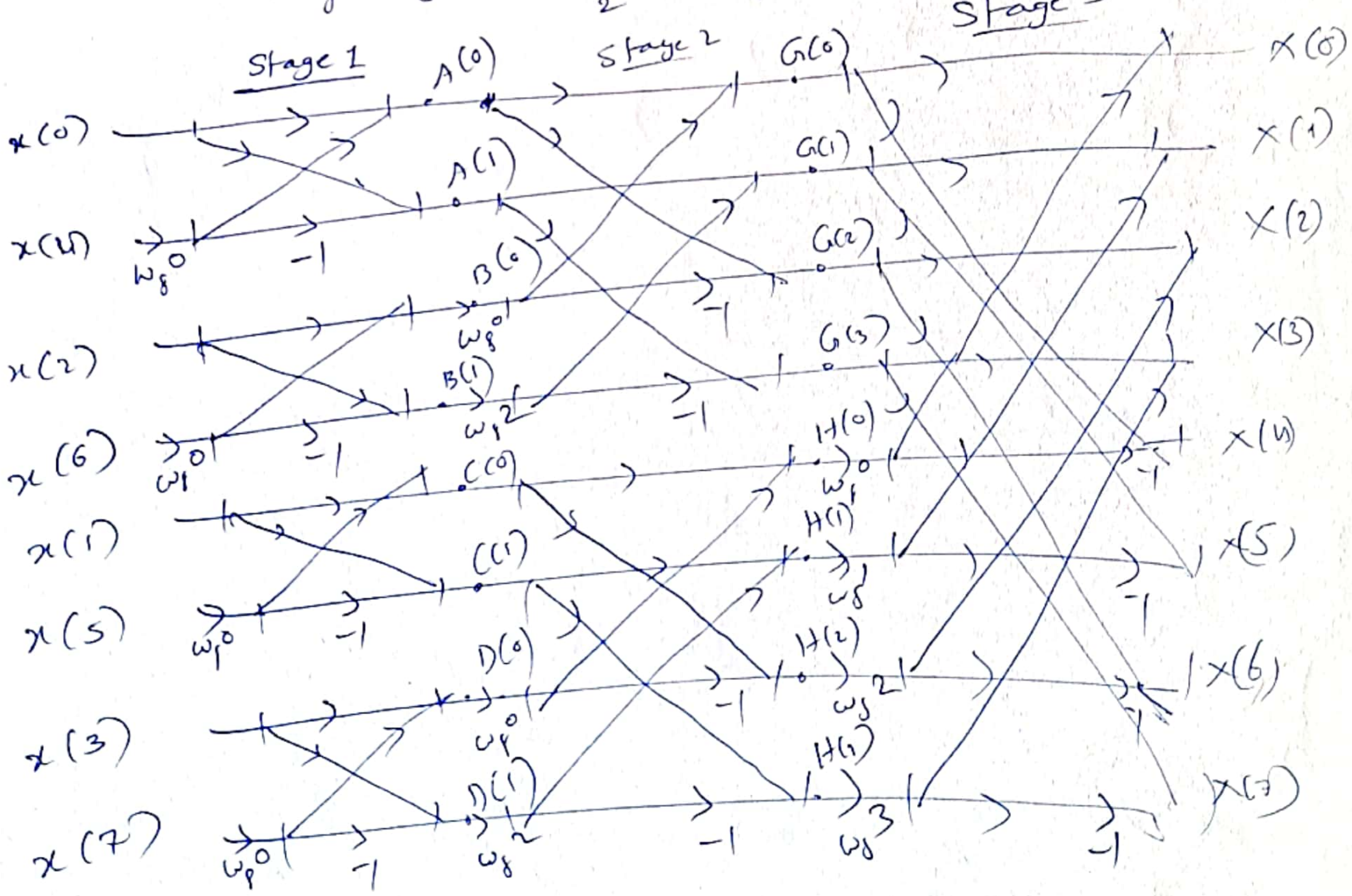



fig: 8-pt DIT-FFT algorithm

## * Decimation in Frequency Algorithm (DIF-FFT)

DIF FFT decomposes the DFT by recursively splitting the sequence elements $X(k)$ in the frequency domain into sets of smaller & smaller subsequences.

The input sequence $x(n)$ is divided into 1st & last half of the points

$$\therefore X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{nk} + \sum_{n=N/2}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2}) W_N^{(n+\frac{N}{2})k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{nk} + W_N^{\frac{N}{2}k} \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2}) W_N^{nk}$$

$$W_N^{(\frac{N}{2})k} = e^{\frac{-j2\pi}{N}\cdot\frac{N}{2}\cdot k} = e^{-j\pi k} = (-1)^k$$

$$\therefore X(k) = \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{nk} + (-1)^k \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2}) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[ x[n] + (-1)^k \cancel{x(n+N)} \right] W_N^{nk}$$

~~Since x[n] is periodic with~~

$$\boxed{X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[ x[n] + (-1)^k x[n+\tfrac{N}{2}] \right] W_N^{nk}}$$

When K is even

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} \left[ x(n) + (-1)^{2r} x\left(n+\frac{N}{2}\right) \right] W_N^{2rn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[ x(n) + x\left(n+\frac{N}{2}\right) \right] W_{N/2}^{rn} \quad ; 0 \le r \le \frac{N}{2}-1 \quad \text{— (1)}$$

when K is odd

$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[ x(n) + (-1)^{2r+1} x\left(n+\frac{N}{2}\right) \right] W_N^{(2r+1)n}$$

$$= \sum_{n=0}^{N/2-1} \left[ x(n) - x\left(n+\frac{N}{2}\right) \right] W_N^{n} \cdot W_{N/2}^{rn} \quad ; 0 \le r \le \frac{N}{2}-1 \quad \text{— (2)}$$

eq$^n$s (1) & (2) represent the $\frac{N}{2}$ point DFTs

eq$^n$(1) gives the sum of the 1st half & last half of the i/p sequence.

eq$^n$(2) gives the product of $W_N^n$ with the difference of the 1st half & the last half of the i/p sequence

Let $g(n) = x(n) + x\left(n+\frac{N}{2}\right)$ &

$h(n) = x(n) - x\left(n+\frac{N}{2}\right)$

For an 8-pt DFT i.e $N=8$

$g(0) = x(0) + x(4)$

$g(1) = x(1) + x(5)$

$g(2) = x(2) + x(6)$

$g(3) = x(3) + x(7)$

$h(0) = x(0) - x(4)$

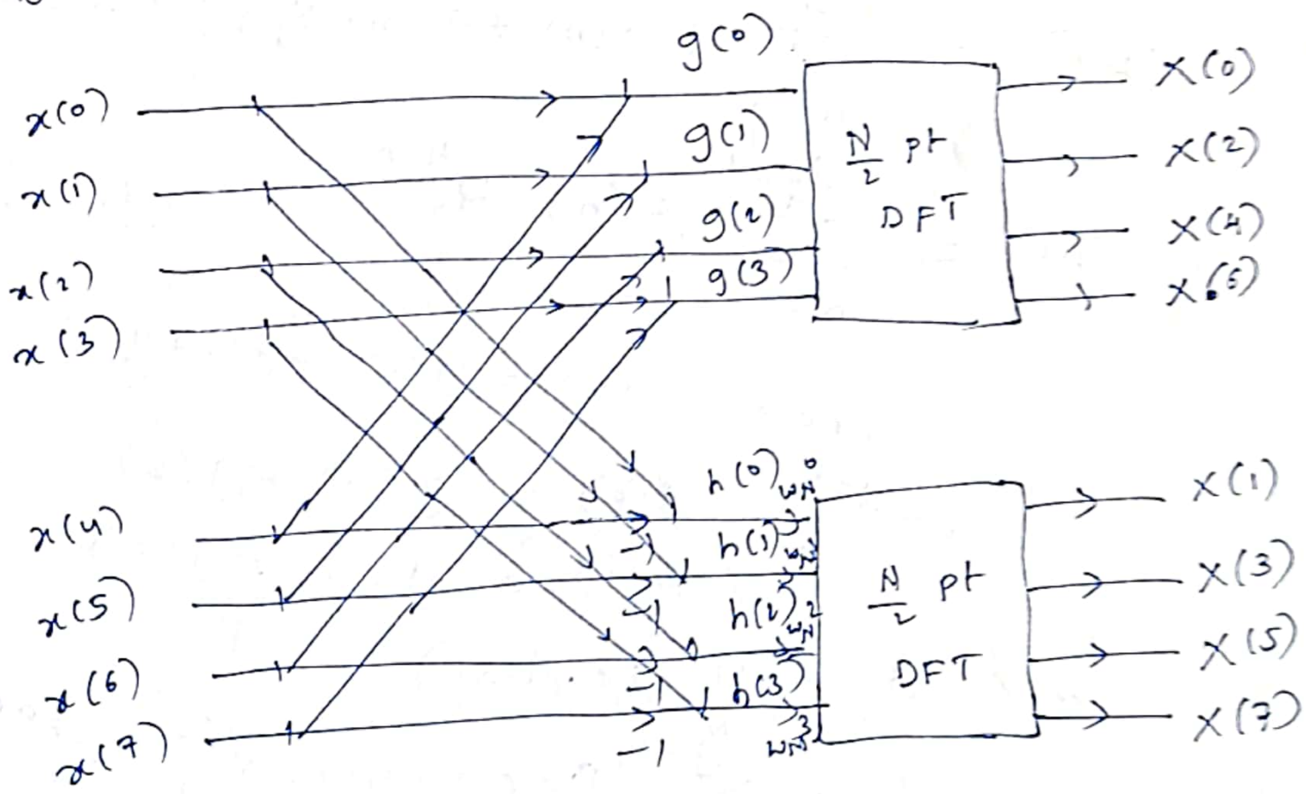
$h(1) = x(1) - x(5)$

$h(2) = x(2) - x(6)$

$h(3) = x(3) - x(7)$

The Computation of the DFT is done by first forming the sequences $g(n)$ & $h(n)$, then calculating $h(n) W_N^n$ & finally evaluating the $\frac{N}{2}$ point DFT of these two sequences to obtain the even numbered o/p units & the odd numbered o/p units.

The flow graph for the 1st stage of 8-pt DFT Computation is shown in figure below



$$① \Rightarrow \cancel{X}(2r) = \sum_{n=0}^{\frac{N}{2}-1} g(n)\, W_N^{2rn} \quad ; 0 \le r \le \frac{N}{2}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g(n)\, W_N^{2rn} + \sum_{n=\frac{N}{4}}^{N/2-1} g(n)\, W_N^{2rn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g(n)\, W_N^{2rn} + \sum_{n=0}^{\frac{N}{4}-1} g\left(n+\frac{N}{4}\right) W_N^{2r(n+\frac{N}{4})}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g(n)\, W_N^{2rn} + W_N^{2r.\frac{N}{4}} \sum_{n=0}^{\frac{N}{4}-1} g\left(n+\frac{N}{4}\right) W_N^{2rn}$$

WKT $W_N^{N/2} = -1$

$$\boxed{X(2r) = \sum_{n=0}^{\frac{N}{4}-1} \left[g(n) + (-1)^r g\left(n+\frac{N}{4}\right)\right] W_N^{2rn}}$$

when $r = 2l$ (i.e even)

$$X(4l) = \sum_{n=0}^{\frac{N}{4}-1} \left[g(n) + g\left(n+\frac{N}{4}\right)\right] W_N^{4ln} \quad ; l = 0,1,\ldots \frac{N}{4}-1$$

$$X(4l) = \sum_{n=0}^{\frac{N}{4}-1} A(n)\, W_N^{4ln} \quad \text{——— } ③$$

where $A(n) = g(n) + g\left(n+\frac{N}{4}\right)$

for $N=8$

$A(0) = g(0) + g(2)$

$A(1) = g(1) + g(3)$

when $r = 2l+1$ (i.e. odd)

$$④\; X(4l+2) = \sum_{n=0}^{\frac{N}{4}-1} \left[g(n) - g\left(n+\frac{N}{4}\right)\right] W_N^{2n(2l+1)}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} B[n]\, W_N^{2n}\, W_N^{4ln}$$

where $B(n) = g(n) - g(n+\frac{N}{4})$

For $N=8$ $\quad B(0) = g(0) - g(2)$

$B(1) = g(1) - g(3)$

(2) $\Rightarrow$

$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} h(n) W_N^n W_N^{2rn}$$

$$X(2r+1) = \sum_{n=0}^{\frac{N}{4}-1} h(n) W_N^n W_N^{2rn} + \sum_{n=\frac{N}{4}}^{\frac{N}{2}-1} h(n) W_N^n W_N^{2rn}$$

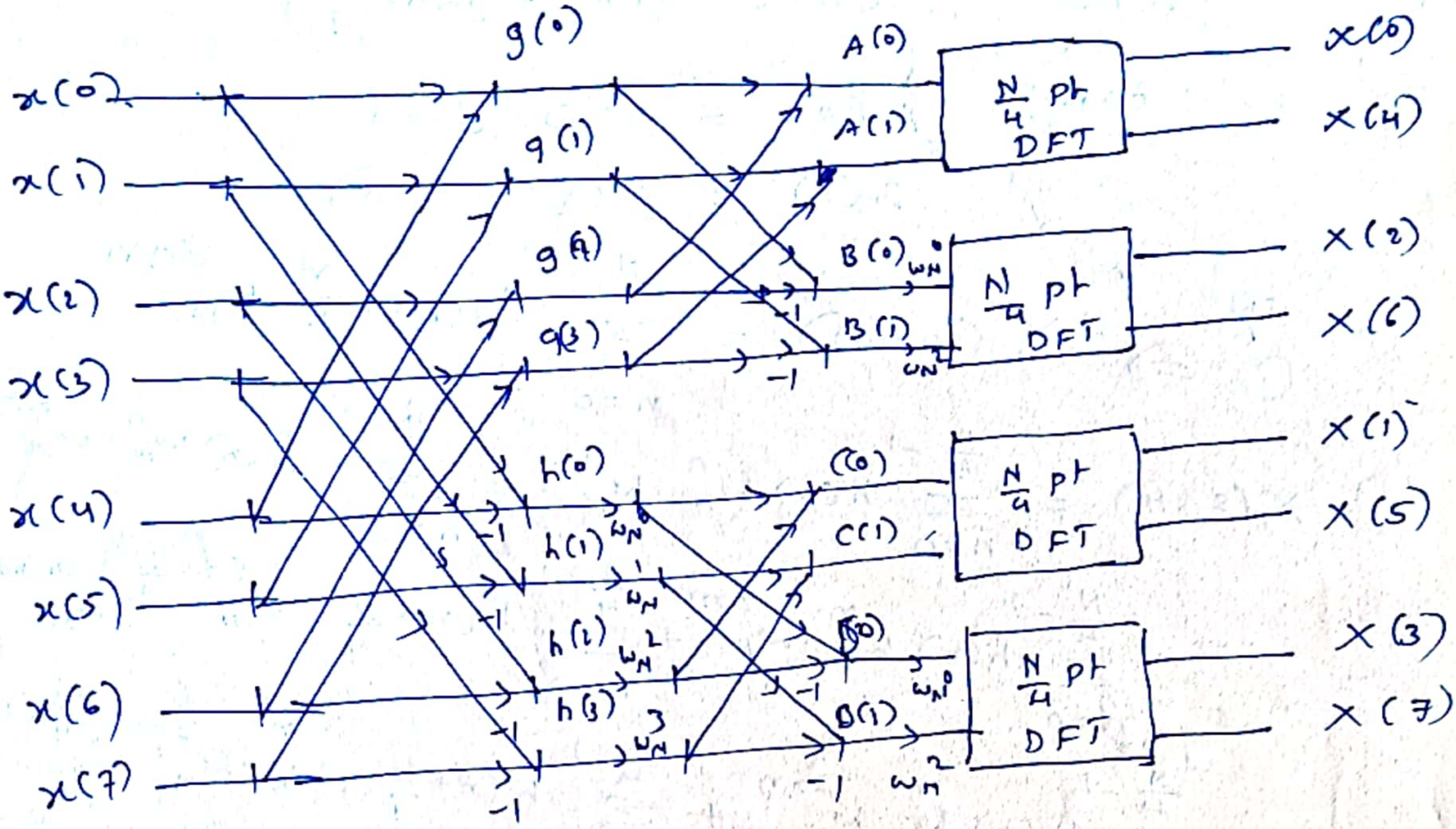
$$= \sum_{n=0}^{\frac{N}{4}-1} h(n) W_N^n W_N^{2rn} + \sum_{n=0}^{\frac{N}{4}-1} h(n+\tfrac{N}{4}) W_N^{(n+\frac{N}{4})} W_N^{2r(n+\frac{N}{4})}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} h(n) W_N^n W_N^{2rn} + \sum_{n=0}^{\frac{N}{4}-1} h(n+\tfrac{N}{4}) W_N^n W_N^{2rn} W_N^{\frac{N}{4}(2r+1)}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[h(n) + h(n+\tfrac{N}{4}) W_N^{\frac{N}{4}(2r+1)}\right] W_N^{(2r+1)n}$$

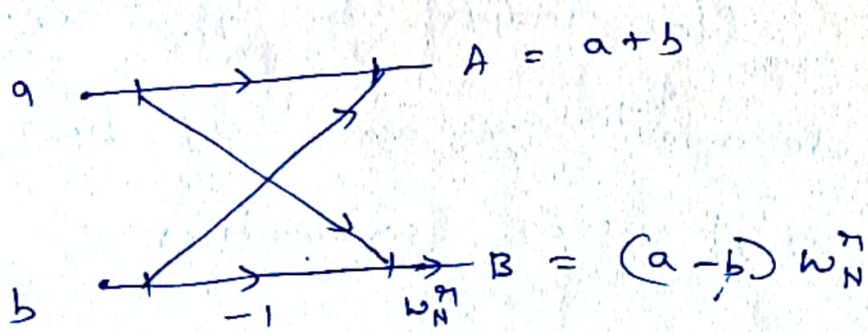
$$X(2r+1) = \sum_{n=0}^{\frac{N}{4}-1} \left[h(n) + (-1)^r W_N^{\frac{N}{4}} h(n+\tfrac{N}{4})\right] W_N^{(2r+1)n}$$

The following figure shows the flow graph of the second stage DIF decomposition of an 8 pt DFT into four 2 pt DFT computations




The above decomposition process can be continued through the decimation of $\frac{N}{2}$ pt DFTs $x(2n)$ & $x(2n+1)$.

The complete process consists of $L = \log_2 N$ stages of decimation where each stage involves $\frac{N}{2}$ butterflies of the type shown below



DIF FFT algorithm requires $\frac{N}{2}\log_2 N$ Complex multiplications

and $N \log_2 N$ Complex additions

The Complete flow graph of an 8-point DIF FFT algorithm is shown below

In DIF FFT the input sequence $x[n]$ appears in natural order while the output $X(K)$ appears in the bit reversed order

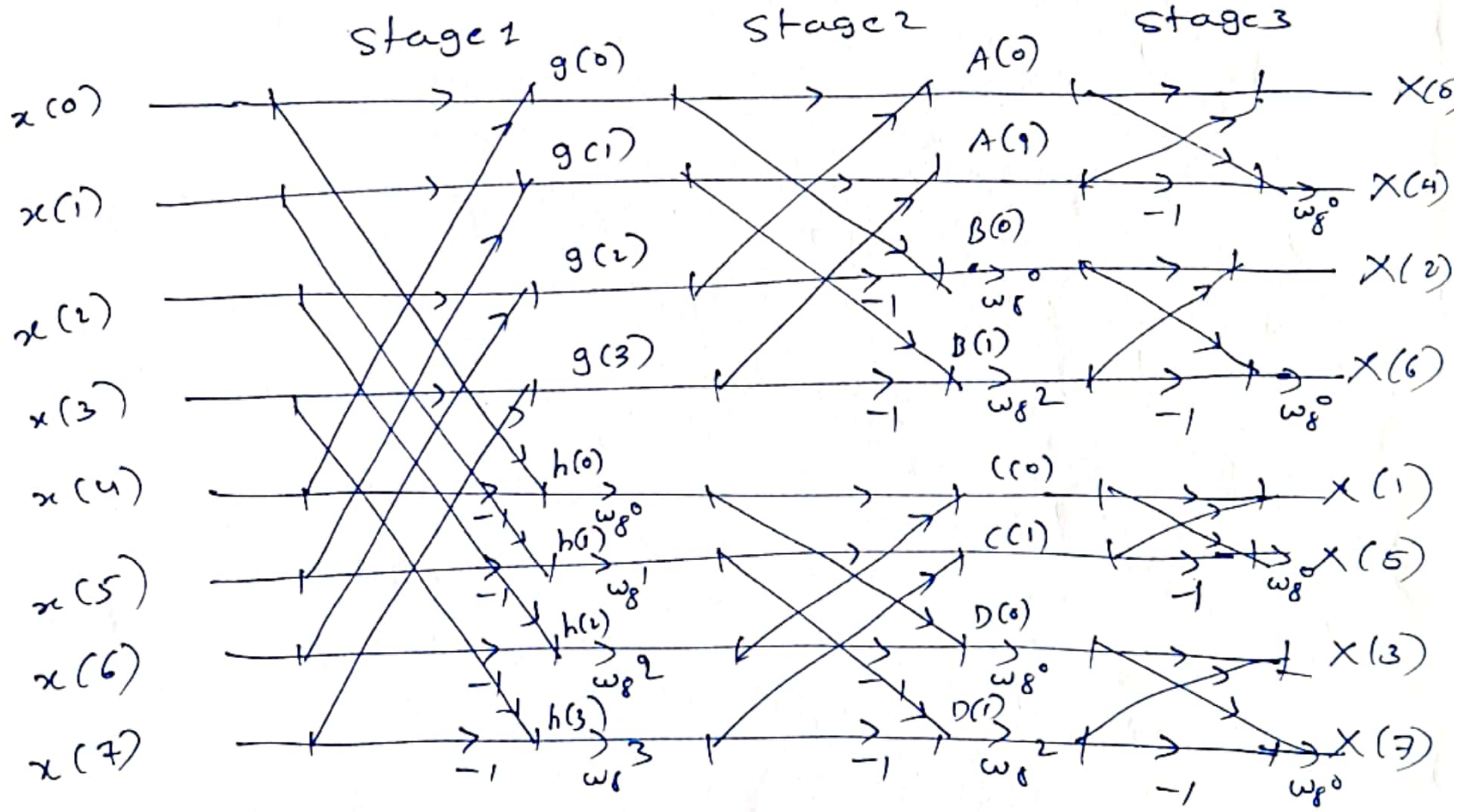



fig: 8 pt DIF - FFT algorithm

## * Inverse FFT

### IDFT Using FFT Algorithm

FFT algorithms can be used to compute an inverse DFT without any change in the algorithm. The inverse DFT of an N-point sequence is defined as

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-nK} \; ; \; 0 \le n \le N-1 \quad \text{--- (1)}$$

Multiply by N & take complex conjugate for eq (1)

$$N x^*[n] = \sum_{K=0}^{N-1} X^*(K)\, W_N^{nK} \quad \text{--- (2)}$$

The righ hand side of eq$^n$ (2) is DFT of the sequence $X^*(K)$ and may be computed using any FFT algorithm.

The desired o/p sequence $x(n)$ can be found by complex conjugating the DFT and dividing by N

i.e $$\boxed{x[n] = \frac{1}{N} \Big[ \underbrace{\sum_{k=0}^{N-1} X^*(K)\, W_N^{nk}}_{DFT[X^*(K)]} \Big]^*}$$

* Compute the 8-pt DFT of the sequence

$$x[n] = \begin{cases} 1 ; & 0 \le n \le 7 \\ 0 ; & \text{otherwise} \end{cases}$$ by using

DIT-FFT & DIF FFT algorithms

Sol: Given i.e $x[n] = \{1, 1, 1, 1, 1, 1, 1, 1\}$

$W_8^0 = 1$

$W_8^1 = 0.707 - j0.707$

$W_8^2 = -j$

$W_8^3 = -0.707 - j0.707$

$X(K) = \{8, 0, 0, 0, 0, 0, 0, 0\}$

* Given $x[n] = \{1, 2, 3, 4, 4, 3, 2, 1\}$ find $X(K)$ using DIT-FFT algorithm

$X(K) = \{20, -5.828 - j2.414, 0, 0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$

* $x[n] = \{0, 1, 2, 3, 4, 5, 6, 7\}$ find $X(K)$ using DIT-FFT Algorithm

$X(K) = \{28, -4 + j9.656, -4 + j4, -4 + j1.656, -4, -4 - j1.656, -4 - j4, -4 - j9.656\}$

* $x[n] = 2^n$ & $N = 8$

find $X(k)$ using DIT-FFT Algorithm

sol: $x[n] = \{1, 2, 4, 8, 16, 32, 64, 128\}$

$X(K) = \{255, 48.63 + j166.05, -51 + j102, -78.63 + j46.05, -85, -78.63 - j46.05, -51 - j102, 48.63 - j166.05\}$

* Given $x[n] = \{0, 1, 2, 3\}$ & $N = 4$ find $X(k)$ using DIT-FFT Algorithm

sol: $W_4^0 = 1$

$W_4^1 = -j$

$X(k) = \{6, -2 + j2, -2, -2 - j2\}$

* Give $x[n] = \{1, 0, 0, 0\}$ & $N = 4$

$X(k) = \{1, 1, 1, 1\}$

* Given $x[n] = \{1, 1, 1, 1\}$

$X(k) = \{4, 0, 0, 0\}$

* Use the 4-pt inverse FFT & Find $x[n]$ if $X(k) = \{6, -2 + j2, -2, -2 - j2\}$

$\cancel{X(k)} = x[n] = \{0, 1, 2, 3\}$

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