UNIT-I

Linear differential equations of Higher order (Constant Coefficients)

An equation is of the form $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = Q(x)$ $\longrightarrow (1)$

Where $P_1, P_2, \ldots P_n$ are 'n' Constants, $Q(x)$ is a function of 'x' is called a linear differential equation with Constant coefficients of order 'n'.

Let $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2 \ldots \frac{d^n}{dx^n} = D^n$

from (1)

$$D^n y + P_1 D^{n-1} y + \cdots + P_n y = Q(x)$$

$$(D^n + P_1 D^{n-1} + \cdots + P_n) \cdot y = Q(x) \longrightarrow (2)$$

Eqn (2) is called differential equation in operator-D form.

$\because$ D = differential operator

where $f(D) = D^n + P_1 D^{n-1} + \cdots + P_n$

from (2) $\boxed{f(D) \cdot y = Q(x)} \longrightarrow (3)$

$\therefore$ The general solution of given diff eqn $f(D) \cdot y = Q(x)$ is. $\boxed{y = C \cdot F + P \cdot I}$ (or) $\boxed{y = y_c + y_p}$

$C \cdot F = y_c$ = Complimentary function

$P \cdot I = y_p$ = Particular Integral.

To find C.F:

The given diff equation $f(D) \cdot y = Q(x)$

write $\boxed{f(D) \cdot y = 0}$ $\because Q(x) = 0$ (make)

Write the Auxillary equation $f(m) = 0$, Solve for 'm', we get roots. Based on the roots we write the Complimentary function (C.F).

suppose $m_1, m_2, \ldots m_n$ are roots of $f(m)=0$

case(i): If $m_1, m_2, \ldots m_n$ are all real and different roots of $f(m)=0$ then $C\cdot F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$

case(ii) Two roots are equal and the rest are real and different.

$$C\cdot F = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

case(iii) three roots are real and equal and the remaining are real and different $(m_1 = m_2 = m_3,\ m_4, m_5, \ldots m_n)$

$$C\cdot F = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

case(iv): If Two roots are complex (imaginary) say $(\alpha + i\beta)$ and $(\alpha - i\beta)$ and the remaining roots are real and different.

$$C\cdot F = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

Ex ①: Solve $\frac{d^2y}{dx^2} - a^2 y = 0 \quad a \neq 0$

Soln: write the given d.e in D-form

$\because \frac{d^2}{dx^2} = D^2$

$$D^2 y - a^2 y = 0$$
$$(D^2 - a^2)\cdot y = 0$$

The Auxillary eqn $(A\cdot E) = f(m) = 0$

$$m^2 - a^2 = 0$$

Roots are real and different. $\quad m^2 = a^2 \Rightarrow m = \pm a$

$$\therefore C\cdot F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\boxed{\text{General solution } (y) = c_1 e^{ax} + c_2 e^{-ax}}$$

② Solve: $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$

Soln: write D-form, $(D^3 - 9D^2 + 23D - 15)y = 0$

The A.E $f(m) = 0 \Rightarrow m^3 - 9m^2 + 23m - 15 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -9 & 23 & -15 \\ & 0 & 1 & -8 & 15 \\ \hline & 1 & -8 & 15 & \boxed{0} \end{array}$$

$m^2 - 8m + 15 = 0$

$m = 1$ is one root of $f(m) = 0$

$$(m-1)[m^2 - 8m + 15] = 0$$
$$(m-1)[m^2 - 3m - 5m + 15] = 0$$
$$(m-1)[m(m-3) - 5(m-3)] = 0$$

roots are real and different

$$(m-1)(m-5)(m-3) = 0 \Rightarrow m = 1, 5, 3.$$

General solution

$$y = C\cdot F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} \Rightarrow c_1 e^{x} + c_2 e^{5x} + c_3 e^{3x}.$$

(3) Solve : $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Soln: The given d.eqn. $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

write differential operator D-form, $(D^3 - 3D + 2)\cdot y = 0$

$f(D)\cdot y = 0$

$\because \frac{d^3}{dx^3} = D^3, \ \frac{d}{dx} = D$ The A.E is $f(m) = 0$

$m^3 - 3m + 2 = 0$

$(m-1)(m^2+m-2) = 0$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$m^2 + m - 2 = 0$

$(m-1)\left[m^2 + 2m - m - 2\right] = 0$

$(m-1)\left[m(m+2) - 1(m+2)\right] = 0$

$(m-1)\ (m-1)(m+2) = 0$

$\therefore m = 1, 1, -2$ are roots, two real and equal and other real and different.

$m_1 = m_2 = 1$

$m_3 = -2$

$\therefore$ C.F $= e^{m_1 x}(c_1 + c_2 x) + c_3 e^{m_3 x}$

General solution

$$\boxed{y_c = C.F = e^{x}(c_1 + c_2 x) + c_3 e^{-2x}}$$

(4) Solve : $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Soln: the given diffal equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

write diff operator D-form, $(D^2 + D + 1)\cdot y = 0$

$\because D^2 = \frac{d^2}{dx^2}, \ D = \frac{d}{dx}$ i.e $f(D)\cdot y = 0$

The Auxillary eqn (A.E) $= f(m) = 0$

$m^2 + m + 1 = 0$

$am^2 + bm + c = 0$

$a = b = c = 1$

$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2}$

$\sqrt{-3} = \sqrt{-1}\cdot\sqrt{3}$

$= i\sqrt{3}$

$= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\because i^2 = -1$

$i = \sqrt{-1}$

$= \alpha \pm i\beta$

$\therefore$ roots are Complex

C.F $= e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$

General solution (y) $= e^{-\frac{1}{2}x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$.

(5) Solve : $y'' + 6y' + 9y = 0, \quad y(0) = -4, \quad y'(0) = 14$

Soln: Given diffal equation $y'' + 6y' + 9y = 0$

write diffal operator D-form $(D^2 + 6D + 9)y = 0 \rightarrow (1)$

$\because y'' = \frac{d^2}{dx^2} = D^2; \quad y' = \frac{d}{dx} = D$ i.e $f(D) \cdot y = 0$

The A-E is $f(m) = 0$

$$m^2 + 6m + 9 = 0$$
$$(m+3)^2 = 0$$
$$(m+3)(m+3) = 0$$

$\therefore m = -3, -3$ are two real & equal roots.

The general solution $y = e^{m_1 x}(c_1 + c_2 x)$ $\quad m_1 = m_2 = -3$

$$\boxed{y = e^{-3x}(c_1 + c_2 x)} \rightarrow (2)$$

$y = e^{-3x}(c_1 + c_2 x)$

$y(0) = -4 \quad \because y(x) = -4$

when $x = 0, \ y = -4$

$-4 = e^{-3(0)}[c_1 + c_2(0)]$

$\boxed{c_1 = -4}$

$y' = e^{-3x}(0 + c_2(1)) + (c_1 + c_2 x)(-3e^{-3x})$

$y' = c_2 e^{-3x} + (-3)e^{-3x}(c_1 + c_2 x)$

$y'(0) = 14$

when $x = 0, \ y' = 14$

$14 = c_2 e^0 + (-3)e^0[c_1 + c_2(0)]$

$14 = c_2 - 3c_1$ $\quad c_1 = -4$

$14 = c_2 - 3(-4)$

$14 - 12 = c_2$

$\therefore \boxed{c_2 = 2}$

from (2) substitute c_1, c_2 values,

General solution $\boxed{y = e^{-3x}(-4 + 2x)}$

Exercise : solve the following differential equations.

(1) $4y''' + 4y'' + y' = 0$

(2) $(D^2 - 3D + 4)y = 0$

(3) $2\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 15y = 0$

Method to find particular integral : (P.I)

The operator D^{-1} is called the inverse differential operator D

i.e D = Differential operator, $\frac{1}{D} = D^{-1}$ = Integral operator.

EX: $\frac{1}{D}(\cos 3x) = \int \cos 3x \, dx = \frac{\sin 3x}{3}$

Formula:

P.I of $\frac{1}{D-\alpha} \cdot Q(x) = e^{\alpha x} \int Q(x) \cdot e^{-\alpha x} dx$

P.I of $\frac{1}{D+\alpha} \cdot Q(x) = e^{-\alpha x} \int Q(x) \cdot e^{\alpha x} dx$

Complimentary function	(C.F) =	$f(D) \cdot y = 0$
Particular Integral	(P.I) =	$\frac{1}{f(D)} \cdot Q(x)$

EX: ① $\frac{1}{D+1} \cdot x = e^{-x} \int x \cdot e^{x} dx = e^{-x}\left[xe^{x} - e^{x}\right] = x - 1$

$\because \frac{1}{D+\alpha} Q(x) = e^{-\alpha x} \int Q(x) \cdot e^{\alpha x} dx$

$\because \alpha = 1$

Integration By parts.

$\int \underset{f(x)\cdot g(x)}{x e^{x}} dx = f(x)\int g(x)dx - \int\left[f'(x) \cdot \int g(x) dx\right] \cdot dx$

$= x \cdot \int e^{x} dx - \int \frac{d}{dx}(x) \cdot \int e^{x} dx \cdot dx$

$= xe^{x} - \int (1) e^{x} dx$

$= xe^{x} - e^{x}$

② Find the particular integral of $\frac{1}{(D-2)(D-3)} \cdot e^{2x}$

Soln: Given $\frac{1}{(D-2)(D-3)} \cdot e^{2x} = \frac{1}{(D-2)} \cdot \left[\frac{1}{(D-3)} \cdot e^{2x}\right]$ $\alpha = 3$

$= \frac{1}{(D-2)}\left[e^{3x} \int e^{2x} \cdot e^{-3x} dx\right]$

$\because \frac{1}{D-\alpha} Q(x) = e^{\alpha x} \int Q(x) \cdot e^{-\alpha x} dx.$

$= \frac{1}{D-2}\left[e^{3x} \int e^{-x} dx\right]$

$= \frac{1}{D-2}\left[e^{+3x}\left(\frac{e^{-x}}{-1}\right)\right]$

$= \frac{1}{D-2}\left(-e^{2x}\right)$ $\alpha = 2$

$= -\left[e^{2x} \int e^{2x} \cdot e^{-2x} dx\right]$

$= -e^{2x} \int (1) dx = -e^{2x} \cdot x$

Rules to find Particular integral (P.I) : Special cases.

Case (i): P.I of $f(D).y = Q(x)$ when $Q(x) = e^{ax}$ (a = constant)

The given differential equation $f(D).y = Q(x)$.

Particular integral $P.I = \frac{1}{f(D)}.Q(x)$.

Now $P.I = \frac{1}{f(D)}.Q(x) = \underset{\text{put } \boxed{D=a}}{\frac{1}{f(D)}.e^{ax}} = \frac{1}{f(a)}.e^{ax}$ $\quad f(a) \neq 0$

if $f(a) = 0$ then $D = a$ (or) $(D-a)$ is a factor of $f(D)$. (root)

i.e $\frac{1}{D-a}.e^{ax} = x.e^{ax}$

$\frac{1}{(D-a)^2}.e^{ax} = \frac{x^2}{2!}e^{ax}$

$\vdots$

If 'a' is a root repeated 'K' times then

$$\boxed{\frac{1}{(D-a)^K}.e^{ax} = \frac{x^K}{K!}.e^{ax}} \quad \text{if } f(a) = 0$$

problems:

① Solve : $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$

Soln: The given diffal equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$

To find C.F : operator D-form, $(D^2+4D+3).y = e^{2x}$

i.e $f(D).y = Q(x)$

when $Q(x) = e^{ax} = e^{2x}$

$f(D).y = 0$

$(D^2+4D+3).y = 0$

The A.E is $f(m) = 0 \Rightarrow$ $m^2+4m+3 = 0$

$m^2+3m+m+3 = 0$

$m(m+3)+1(m+3) = 0$

$(m+3)(m+1) = 0$

$\therefore m = -3, -1$ are real & different roots of $f(D)$.

$\therefore C.F = c_1e^{m_1x} + c_2e^{m_2x} = c_1e^{-3x} + c_2e^{-x}$.

$P.I = \frac{1}{f(D)}.Q(x) = \frac{1}{D^2+4D+3}.e^{2x}$ $(D=a=2)$ $= \frac{1}{2^2+4(2)+3}.e^{2x} = \frac{e^{2x}}{15}$

$\therefore$ General solution $y = C.F + P.I = [c_1e^{-3x} + c_2e^{-x}] + \frac{e^{2x}}{15}$

② Solve: $(4D^2-4D+1)y = 100$

Soln: The given diffal equation $(4D^2-4D+1)\cdot y = 100$

i.e $f(D)\cdot y = Q(x)$.

$(4D^2-4D+1)\cdot y = 100\cdot e^{(0)x}$

TO find C.F:

$f(D)\cdot y = 0 \quad \because Q(x) = 0$

$(4D^2-4D+1)\cdot y = 0$

the Auxillary equation $f(m)=0 \Rightarrow 4m^2-4m+1 = 0$

$(2m)^2 - 2\cdot(2m) + 1^2 = 0$

$(2m-1)^2 = 0$

$(2m-1)\ (2m-1) = 0$

$\therefore \quad m = 1/2, 1/2$ are repeated roots (equal)

$\therefore \quad C\cdot F = (C_1+C_2x)\,e^{m_1 x} = (C_1+C_2x)\,e^{\frac{1}{2}x}$ $\Big| \because m_1 = m_2 = \frac{1}{2}$

To find P.I:

$P\cdot I = \frac{1}{f(D)}\cdot Q(x) = \frac{1}{4D^2-4D+1}\cdot 100\cdot e^{(0)x} = 100\cdot\frac{1}{4(0)^2-4(0)+1}\cdot e^{(0)x}$

$\boxed{\because \text{ when } Q(x) = e^{ax}}$ put $\boxed{D=a=0}$

$= 100.$

$\therefore$ the general solution $y = C\cdot F + P\cdot I$

$y = (C_1+C_2x)\,e^{\frac{1}{2}x} + 100$

③ Solve: $(D^2+16)\cdot y = e^{-4x}$

Soln: The given d.e $(D^2+16)\,y = e^{-4x} = f(D)\cdot y = Q(x)$.

To find C.F: $f(D)\cdot y = 0 \Rightarrow (D^2+16)\cdot y = 0$

The A.E $f(m) = 0 \Rightarrow m^2+16 = 0 \Rightarrow m^2 = -16 \Rightarrow m = \pm 4i$ (complex) $\alpha \pm i\beta$

$\therefore C\cdot F = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x) = C_1\cos 4x + C_2\sin 4x.$

$\boxed{\alpha=0,\ \beta=4}$

To find P.I: $\frac{1}{f(D)}\cdot Q(x) = \frac{1}{D^2+16}\cdot e^{-4x} = \frac{1}{(-4)^2+16}\,e^{-4x} = \frac{e^{-4x}}{32}$

$\boxed{\text{when } Q(x) = e^{ax}}$ put $\boxed{D=a=-4}$ $\quad f(-4)\neq 0$

$\therefore$ General Solution $(y) = C\cdot F + P\cdot I = (C_1\cos 4x + C_2\sin 4x) + \frac{e^{-4x}}{32}$

4. Solve: $(D^3-5D^2+8D-4)\cdot y = e^{2x}$

Soln: The given diffal eqn $(D^3-5D^2+8D-4)\cdot y = e^{2x}$

$f(D)\cdot y = Q(x)$

To find C.F:

$C.F = f(D)\cdot y = 0$

$(D^3-5D^2+8D-4)\cdot y = 0$

the Auxillary equation $f(m)=0 \Rightarrow m^3-5m^2+8m-4=0$

by obselavation $m=1$ is one root.

$$\begin{array}{c|cccc} 1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

m^2-4m+4

$\therefore m^3-5m^2+8m-4=0$

$(m-1)(m^2-4m+4)=0$

$(m-1)(m^2-2m-2m+4)=0$

$(m-1)[m(m-2)-2(m-2)]=0$

$(m-1)(m-2)(m-2)=0$

$\therefore m = 1, 2, 2$. Here two roots are real & equal and the remaining real and different.

$\therefore C.F = (C_1+C_2x)e^{m_1x} + C_3e^{m_3x}$ | $m_1 = m_2 = 2$, $m_3 = 1$

$= (C_1+C_2x)e^{2x} + C_3e^{x}$

To find P.I: $P.I = \frac{1}{f(D)}\cdot Q(x)$ | $Q(x) = e^{ax} = e^{2x}$

$= \frac{1}{(D^3-5D^2+8D-4)}\cdot e^{2x}$ | $D=a=2$, $f(D)=f(a)=0$ fails. | $(D-2)$ is a factor of $f(D)$

$= \frac{1}{(D-1)(D-2)^2}\cdot e^{2x}$

$f(a)=f(2)=0$

$= \frac{1}{(D-1)}\left[\frac{1}{(D-2)^2}\cdot e^{2x}\right]_{(D=a=2)}$

$= \frac{1}{(2-1)}\left[\frac{x^2}{2!}e^{2x}\right]$

$= \frac{x^2}{2}\cdot e^{2x}$

$\frac{1}{(D-a)^k}e^{ax} = \frac{x^k}{k!}e^{ax}$ | $(D-a)$ is a factor

$\therefore$ General solution $y = C.F + P.I$

$y = \left[(C_1+C_2x)e^{2x} + C_3e^{x}\right] + \frac{x^2}{2}\cdot e^{2x}$

4. Solve: $(D^2-3D+2)y = \cosh x$

Soln: The given diff equation $(D^2-3D+2)\cdot y = \cosh x$

$\cosh x = \dfrac{e^x+e^{-x}}{2}$ i.e $(D^2-3D+2)\cdot y = \dfrac{e^x+e^{-x}}{2}$

$\sinh x = \dfrac{e^x-e^{-x}}{2}$ $f(D)\cdot y = Q(x)$ form

To find (C.F) (y_c): $C.F = f(D)\cdot y = 0$ | $Q(x)=0$

$(D^2-3D+2)\cdot y = 0$

The Auxillary equation $f(m)=0 \Rightarrow m^2-3m+2=0$

$m^2-2m-m+2=0$

$m(m-2)-1(m-2)=0$

$(m-1)(m-2)=0$

$m=1,2$

$\therefore$ $m=1,2$ are real & different roots of $f(m)=0$

$\therefore$ Complimentary function $(C.F) = c_1e^{m_1x}+c_2e^{m_2x}$

$$\boxed{y_c = c_1e^{x}+c_2e^{2x}}$$

To find Particular Integral (P.I): $y_p = \dfrac{1}{f(D)}\cdot Q(x)$

$$= \frac{1}{(D^2-3D+2)}\cdot\frac{e^x+e^{-x}}{2}$$

$$= \frac{1}{2}\left[\frac{1}{D^2-3D+2}\cdot e^{x} + \frac{1}{(D^2-3D+2)}\cdot e^{-x}\right]$$

$$= \frac{1}{2}\left\{\frac{1}{(D-2)}\left[\frac{1}{(D-1)}\cdot e^{x}\right] + \frac{1}{(D-2)(D-1)}\cdot e^{-x}\right\}$$

fail

put $\boxed{D=a=1}$ put $\boxed{D=a=-1}$

$$= \frac{1}{2}\left[\frac{1}{(1-2)}\cdot\left(\frac{x}{1!}e^{x}\right) + \frac{1}{(-1-2)(-1-1)}\cdot e^{-x}\right]$$

$$y_p = \frac{1}{2}\left[-xe^{x}+\frac{e^{-x}}{6}\right]$$

$\therefore$ The General solution $y = y_c+y_p$

$$y = \left[c_1e^{x}+c_2e^{2x}\right]+\left[-\frac{1}{2}xe^{x}+\frac{1}{12}e^{-x}\right]$$

⑤ Solve: $(D+2)(D-1)^2 \cdot y = e^{-2x} + 2\sinh x$

Soln: The given diff^{al} equation $(D+2)(D-1)^2 \cdot y = e^{-2x} + 2\sinh x$

$$(D+2)(D-1)^2 \cdot y = e^{-2x} + \cancel{2}\cdot\left(\frac{e^{x} - e^{-x}}{\cancel{2}}\right)$$

$$(D+2)(D-1)^2 \cdot y = e^{-2x} + e^{x} - e^{-x}$$

i.e $f(D) \cdot y = Q(x)$ form.

To find C.F : y_c : $f(D) \cdot y = 0$

$$(D+2)(D-1)^2 \cdot y = 0$$

The Auxillary eqn $f(m) = 0 \Rightarrow (m+2)(m-1)^2 = 0$

$$m = -2, 1, 1$$

Here the roots are real and one root repeated twice.

$\therefore$ $y_c = (C_1 + C_2 x)e^{m_1 x} + C_3 e^{m_3 x} = \boxed{(C_1 + C_2 x)e^{x} + C_3 e^{-2x}}$

$\because m_1 = m_2 = 1$

To find P.I : y_p : $\dfrac{1}{f(D)} \cdot Q(x)$ when $Q(x) = e^{ax}$ Model

$$y_p = \frac{1}{(D+2)(D-1)^2} \cdot e^{-2x} + e^{x} + e^{-x}$$

$$= \frac{1}{(D+2)(D-1)^2} \cdot e^{-2x} + \frac{1}{(D+2)(D-1)^2} \cdot e^{x} + \frac{1}{(D+2)(D-1)^2} \cdot e^{-x}$$

put $\boxed{D = a = -2}$ fail   put $\boxed{D = a = 1}$ fail   put $\boxed{D = a = -1}$

$$= \frac{1}{(D-1)^2}\left[\frac{1}{D+2} e^{-2x}\right] + \frac{1}{(D+2)}\left[\frac{1}{(D-1)^2} \cdot e^{x}\right] + \frac{1}{(-1+2)(-1-1)^2} \cdot e^{-x}$$

$\boxed{D = a = -2}$ $\boxed{D = a = 1}$

$$= \frac{1}{(-2-1)^2}\left(\frac{x}{1!} e^{-2x}\right) + \frac{1}{(1+2)}\left(\frac{x^2}{2!} e^{x}\right) + \frac{e^{-x}}{4}$$

formula $\because$ $\dfrac{1}{(D-a)^K} e^{ax} = \dfrac{x^K}{K!} e^{ax}$

$$y_p = \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^{x} + \frac{e^{-x}}{4}$$

$\therefore$ The general solution $y = y_c + y_p$

$$y = \left[(C_1 + C_2 x)e^{x} + C_3 e^{-2x}\right] + \left[\frac{x}{9} e^{-2x} + \frac{x^2}{6} e^{x} + \frac{e^{-x}}{4}\right]$$

7 Solve: $(D^3-1)\cdot y = (e^x+1)^2$

Soln: The given differential eqn $(D^3-1)\cdot y = (e^x+1)^2$

$\boxed{a^3-b^3=(a-b)(a^2+ab+b^2)}$ i.e $f(D)\cdot y = Q(x)$

To find C.F: $y_c = f(D)\cdot y = 0$

$(D^3-1)\cdot y = 0$ | $Q(x)=0$

The Auxillary equation $f(m)=0 \Rightarrow (m^3-1)=0$

$(m-1)(m^2+m+1)=0$

$m-1=0 \quad m^2+m+1=0$

$m=1, \quad m = \dfrac{-1 \pm \sqrt{1^2-4(1)(1)}}{2(1)}$

$= \dfrac{-1 \pm i\sqrt{3}}{2} = \alpha \pm i\beta$ (complex) roots

$\therefore \quad y_c = c_1 e^x + e^{-\frac{1}{2}x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$

To find P.I: $y_p = \dfrac{1}{f(D)}\cdot Q(x)$

$= \dfrac{1}{D^3-1}\cdot(e^x+1)^2$ | $(a+b)^2 = a^2+b^2+2ab$

$= \dfrac{1}{D^3-1}\left(e^{2x}+1+2e^x\right)$

$= \dfrac{1}{D^3-1}e^{2x} + \dfrac{1}{D^3-1}1\cdot e^{(0)x} + 2\dfrac{1}{D^3-1}e^x$

$\boxed{D=a=2}$ $\boxed{D=a=0}$ $\boxed{D=a=1}$ fail

$= \dfrac{1}{(2)^3-1}e^{2x} + \dfrac{1}{(0)^3-1}e^{(0)x} + 2\cdot\dfrac{1}{(D-1)(D^2+D+1)}\cdot e^x$

$= \dfrac{1}{7}e^{2x} - 1 + 2\cdot\dfrac{1}{(D^2+D+1)}\left[\dfrac{1}{D-1}e^x\right]_{\text{fail}}$ $\boxed{D=a=1}$

$= \dfrac{1}{7}e^{2x} - 1 + 2\dfrac{1}{(1+1+1)}\left(\dfrac{x}{1!}e^x\right)$

$= \dfrac{1}{7}e^{2x} - 1 + \dfrac{2x}{3}e^x$

$\therefore$ the general solution $y = y_c + y_p$

$$y = c_1 e^x + e^{-\frac{1}{2}x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] + \left(\frac{e^{2x}}{7} - 1 + \frac{2x}{3}e^x\right)$$

8 Solve: $(D^3-3D^2+4)\cdot y = (1+e^{-x})^3$

Soln: To find, the given d.e $(D^3-3D^2+4)\cdot y = (1+e^{-x})^3$

i.e $f(D)\cdot y = Q(x)$ form.

To find C.F: y_c: $f(D)\cdot y = 0$

$$(D^3-3D^2+4)\cdot y = 0$$

The A.E is $f(m)=0 \Rightarrow m^3-3m^2+4=0$

$$(m+1)(m^2-4m+4)=0$$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & 0 & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$m^2-4m+4=0$

$$(m+1)(m-2)^2=0$$

$\therefore m = -1, 2, 2$

$\therefore$ one root is real and other two roots real and equal.

$$y_c = c_1 e^{-x} + (c_2+c_3x)e^{2x}$$

To find (P.I): y_p: $\frac{1}{f(D)}\cdot Q(x)$.

$$= \frac{1}{(D^3-3D^2+4)}\cdot(1+e^{-x})^3$$

$(a+b)^3 = a^3+b^3+3a^2b+3ab^2$

$$= \frac{1}{(D^3-3D^2+4)} 1+e^{-3x}+3e^{-x}+3e^{-2x}$$

$$= \frac{1}{(D+1)(D-2)^2}1.e^{(0)x} + \frac{1}{(D+1)(D-2)^2}e^{-3x} + 3.\frac{1}{(D+1)(D-2)^2}e^{-x} + 3.\frac{1}{(D+1)(D-2)^2}e^{-2x}$$

$$= y_{p_1} + y_{p_2} + y_{p_3} + y_{p_4}$$

$$y_{p_1} = \frac{1}{(D+1)(D-2)^2}e^{(0)x} \quad \boxed{D=a=0} = \frac{1}{(0+1)(0-2)^2}e^{(0)x} = \frac{1}{4}$$

$$y_{p_2} = \frac{1}{(D+1)(D-2)^2}e^{-3x} \quad \boxed{D=a=-3} = \frac{1}{(-3+1)(-3-2)^2}\cdot e^{-3x} = \frac{e^{-3x}}{-50}$$

$$y_{p_3} = 3.\frac{1}{(D-2)^2}\left[\frac{1}{D+1}e^{-x}\right] \text{ fail} \quad \boxed{D=a=-1} = \frac{3}{(-1-2)^2}\left(\frac{x}{1!}e^{-x}\right) = \frac{x}{3}e^{-x}$$

$$y_{p_4} = 3.\frac{1}{(D+1)(D-2)^2}\cdot e^{-2x} \quad \boxed{D=a=-2} = \frac{3}{(-2+1)(-2-2)^2}\cdot e^{-2x} = \frac{3e^{-2x}}{-16}$$

$$\therefore y_p = y_{p_1}+y_{p_2}+y_{p_3}+y_{p_4} = \frac{1}{4} - \frac{e^{-3x}}{50} + \frac{x}{3}e^{-x} - \frac{3}{16}e^{-2x}$$

$\therefore$ General soln $y = y_c + y_p$

$$y = \left[c_1e^{-x} + (c_2+c_3x)e^{2x}\right] + \left[\frac{1}{4} - \frac{e^{-3x}}{50} + \frac{x}{3}e^{-x} - \frac{3}{16}e^{-2x}\right]$$

Type 2: Particular integral when $Q(x) = \sin bx$ (or) $\cos bx$
$b = \text{constant}$.

The given differential equation $f(D) \cdot y = Q(x)$.

Particular Integral:

$$P.I = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{f(D)} \sin bx \text{ (or) } \cos bx$$

put $\boxed{D^2 = -b^2}$

when $D^2 = -b^2$, $f(D) = 0$ then

① $\frac{1}{f(D)} \cdot Q(x) = \frac{1}{D^2+b^2} \sin bx = \frac{x}{f'(D)} \cdot Q(x)$

put $\boxed{D^2 = -b^2}$ fail

$f(D) = D^2+b^2$
$f'(D) = 2D$
$b = \text{const.}$

$$= \frac{x}{2D} \sin bx$$

$$= \frac{x}{2} \cdot \frac{1}{D}(\sin bx)$$

$$= \frac{x}{2} \int \sin bx \, dx$$

$$= \frac{x}{2}\left(-\frac{\cos bx}{b}\right)$$

$$\boxed{\frac{1}{D^2+b^2} \sin bx = \frac{-x}{2b} \cos bx}$$

② $\frac{1}{f(D)} \cdot Q(x) = \frac{1}{D^2+b^2} \cos bx = \frac{x}{f'(D)} \cdot \cos bx$

put $\boxed{D^2 = -b^2}$ fail

$$= \frac{x}{2D} \cdot \cos bx$$

$$= \frac{x}{2} \frac{1}{D}(\cos bx)$$

$$= \frac{x}{2} \int \cos bx \, dx$$

$$= \frac{x}{2}\left(\frac{\sin bx}{b}\right)$$

$$\boxed{\frac{1}{D^2+b^2} \cos bx = \frac{x}{2b} \sin bx}$$

① Solve : $(D^2+3D+2)y = \sin 3x$

Soln: The given diffal eqn $(D^2+3D+2)\cdot y = \sin 3x$

$$f(D)\cdot y = Q(x).$$

To find C.F : y_c : $f(D)\cdot y = 0$

$$(D^2+3D+2)\cdot y = 0$$

The Auxillary eqn $f(m)=0 \Rightarrow m^2+3m+2=0$

$$m^2+2m+m+2=0$$

$$m(m+2)+1(m+2)=0$$

$$(m+1)(m+2)=0$$

$\therefore$ $m=-1,-2$ are two real & distinct roots.

$\therefore$ $C.F = y_c = \boxed{c_1 e^{-x} + c_2 e^{-2x}}$

To find P.I : $y_p = \frac{1}{f(D)}\cdot Q(x)$ | $Q(x) = \sin bx$

$$= \frac{1}{D^2+3D+2}\cdot \sin 3x \quad \text{put } \boxed{D^2 = -b^2 = -3^2 = -9}$$

$$= \frac{1}{-9+3D+2}\sin 3x$$

$$= \frac{1}{3D-7}\sin 3x$$

$$= \frac{(3D+7)}{(3D-7)(3D+7)}\cdot \sin 3x \quad \text{(Rationalize)}$$

$$= \frac{(3D+7)\sin 3x}{9D^2-49} \quad \text{put } \boxed{D^2 = -b^2 = -3^2 = -9}$$

$$= \frac{3\frac{d}{dx}(\sin 3x) + 7\sin 3x}{9(-9)-49}$$

$$= -\frac{1}{130}\left[3(3\cos 3x) + 7\sin 3x\right] = -\frac{1}{130}\left[9\cos 3x + 7\sin 3x\right]$$

$\therefore$ General Solution $y = y_c + y_p$

$$y = \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left[-\frac{1}{130}\left(9\cos 3x + 7\sin 3x\right)\right]$$

~~② $(D^2-4)\cdot y = 2\cdot\cos^2 x$ solve the diffal eqn.~~

②. Solve the differential equation $(D^2-4)\cdot y = 2\cdot\cos^2 x$

Soln: The given diff eqn $(D^2-4)\cdot y = 2\cos^2 x$

$$f(D)\cdot y = Q(x).$$

To find C.F: y_c : $f(D)\cdot y = 0$

$$(D^2-4)\cdot y = 0$$

The Auxillary equation $f(m)=0 \Rightarrow m^2-4=0$

$$m^2=4 \Rightarrow m=\pm 2$$

$\therefore$ $m=2,-2$ are real and diffent roots.

$\therefore$ $y_c = c_1 e^{2x} + c_2 e^{-2x}$

To find P.I: y_p : $\dfrac{1}{f(D)}\cdot Q(x)$

$$= \frac{1}{D^2-4}\cdot 2\cos^2 x \qquad \Big|\; 2\cos^2 x = 1+\cos 2x$$

$$= \frac{1}{D^2-4}(1+\cos 2x)$$

$$= \frac{1}{D^2-4}\,1\, e^{(0)x} + \frac{1}{D^2-4}\cos 2x.$$

$$\boxed{D=a=0} \qquad \boxed{D^2=-b^2=-2^2=-4}$$

$$= -\frac{1}{4} + \frac{1}{-8}\cos 2x.$$

$\therefore$ $y_p = -\frac{1}{4} - \frac{1}{8}\cos 2x.$

$\therefore$ The general solution $y = y_c + y_p$

$$y = (c_1 e^{2x} + c_2 e^{-2x}) + \left(-\frac{1}{4} - \frac{1}{8}\cos 2x\right)$$

Formulae:

$2\cdot\sin A\cdot\cos B = \sin(A+B) + \sin(A-B)$

$2\cdot\cos A\cdot\sin B = \sin(A+B) - \sin(A-B)$

$2\cdot\cos A\cdot\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\cdot\sin B = \cos(\cancel{A+B}) - \cos(A+B)$

4. Solve: $(D^2+1)\cdot y = \sin x \cdot \sin 2x$

3 soln: The given diff eqn $(D^2+1)y = \sin x \cdot \sin 2x$

$$f(D)\cdot y = Q(x)$$

To find C.F: y_c : $f(D)\cdot y = 0$

$$(D^2+1)\cdot y = 0$$

The Auxillary equation $f(m)=0 \Rightarrow m^2+1=0$

$$m^2=-1$$

$$m=\pm i \quad (\alpha \pm i\beta)$$

$\therefore m = i, -i$ are Complex roots, $\alpha=0, \beta=1$

$\therefore C.F = e^{\alpha x}[C_1\cos\beta x + C_2 \sin\beta x] = \boxed{C_1\cos x + C_2\sin x}$

To find P.I: y_p : $\frac{1}{f(D)}\cdot Q(x)$

$$= \frac{1}{(D^2+1)}\cdot \sin x\cdot \sin 2x$$

$$= \frac{1}{2}\frac{1}{D^2+1}(\cos x - \cos 3x)$$

$\sin A\cdot \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\sin x \cdot \sin 2x = \frac{1}{2}[\cos x - \cos 3x]$

$$= \frac{1}{2}\frac{1}{D^2+1}\cos x - \frac{1}{2}\frac{\cos 3x}{D^2+1}$$

$$= y_{p_1} + y_{p_2}$$

$y_{p_1} = \frac{1}{2}\frac{1}{D^2+1}\cos x$ $\boxed{D^2=-1^2=-1}$ fail $= \frac{1}{2}\left[\frac{x}{f'(D)}\cdot Q(x)\right] = \frac{1}{2}\left[\frac{x}{2D}\cdot\cos x\right]$

$$= \frac{x}{4}\int \cos x\,dx$$

$$= \frac{x}{4}\sin x$$

$y_{p_2} = -\frac{1}{2}\frac{1}{D^2+1}\cos 3x$ $\boxed{D^2=-3^2=-9}$ $= -\frac{1}{2}\cdot\frac{1}{(-9+1)}\cos 3x$

$$= \frac{1}{16}\cos 3x.$$

$\therefore y_p = y_{p_1} + y_{p_2} = \frac{x}{4}\sin x + \frac{\cos 3x}{16}$

$\therefore$ General Solution $y = y_c + y_p$

$$y = (C_1\cos x + C_2\sin x) + \left(\frac{x}{4}\sin x + \frac{\cos 3x}{16}\right)$$

(4) Solve: $(D^2-4D+3)\cdot y = \sin 3x \cdot \cos 2x$

Soln: The given diff equation $(D^2-4D+3)\cdot y = \sin 3x \cdot \cos 2x$

$$f(D)\cdot y = Q(x).$$

To find C.F: y_c: $f(D)\cdot y = 0$

$$(D^2-4D+3)\cdot y = 0$$

the Auxillary equation $f(m)=0 \Rightarrow m^2-4m+3=0$

$$m^2-3m-m+3=0$$
$$m(m-3)-1(m-3)=0$$
$$(m-1)(m-3)=0$$

$\therefore m=1,3$, are two real & distinct roots.

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = \boxed{c_1 e^{x} + c_2 e^{3x}}$$

To find P.I: $y_p = \frac{1}{f(D)} Q(x) = \frac{1}{(D^2-4D+3)} \cdot \sin 3x \cdot \cos 2x.$

$\sin 3x \cdot \cos 2x = \sin A \cdot \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\} = \frac{1}{2}(\sin 5x + \sin x)$

$$= \frac{1}{2} \cdot \frac{1}{D^2-4D+3} \sin 5x + \frac{1}{2} \frac{1}{D^2-4D+3} \cdot \sin x.$$

$$= P.I_1 + P.I_2.$$

P.I$_1$: $\frac{1}{2} \frac{1}{D^2-4D+3} \cdot \sin 5x = \frac{1}{f(D)} \cdot Q(x)$ $\quad Q(x) = \sin bx$

$$\boxed{D^2 = -5^2 = -25}$$

$$= \frac{1}{2} \frac{1}{-25-4D+3} \cdot \sin 5x$$

$$= \frac{1}{2} \frac{1}{-4D-22} \sin 5x$$

$$= \frac{1}{2} \frac{1}{-2(2D+11)} \sin 5x = -\frac{1}{4} \frac{(2D-11)}{(2D-11)(2D+11)} \cdot \sin 5x$$

$$= -\frac{1}{4} \frac{(2D-11) \sin 5x}{4D^2-121} \quad \boxed{D^2=-25}$$

$$= -\frac{1}{4} \frac{\left[2 \frac{d}{dx}(\sin 5x) - 11 \sin 5x\right]}{-100-121}$$

$$\boxed{P.I_1 = \frac{1}{884}\left[10 \cos 5x - 11 \sin 5x\right]}$$

$P.I_2$: $\frac{1}{2} \frac{1}{D^2-4D+3} \cdot \sin x$ $\quad$ $Q(x) = \sin bx = \sin x$
$b = 1$

$D^2 = -1^2$

$$= \frac{1}{2} \frac{1}{-1-4D+3} \sin x$$

$$= \frac{1}{2} \frac{1}{2-4D} \sin x$$

$$= \frac{1}{2} \frac{1}{-2(2D-1)} \sin x$$

$$= -\frac{1}{4} \frac{(2D+1)}{(2D-1)(2D+1)} \cdot \sin x \quad \text{(Rationalize)}$$

$$= -\frac{1}{4} \frac{(2D+1)}{4D^2-1} \cdot \sin x \qquad \boxed{D^2 = -1^2 = -1}$$

$$= -\frac{1}{4} \frac{(2D+1)\sin x}{4(-1)-1}$$

$$= \frac{1}{20} \left[2 \frac{d}{dx}(\sin x) + \sin x\right]$$

$$P.I_2 = \frac{1}{20}(2\cos x + \sin x)$$

$$\therefore \; P.I = P.I_1 + P.I_2 = \frac{1}{884}[10\cos 5x - 11\sin 5x] + \frac{1}{20}[2\cos x + \sin x]$$

$\therefore$ General solution $y = y_c + y_p$.

$$y = (c_1 e^{x} + c_2 e^{3x}) + \left(\frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}[2\cos x + \sin x]\right)$$

(5) Solve the differential equation $(D^2+9)\cdot y = \cos^3 x$

Soln: The given diff equation is of the form $f(D)\cdot y = Q(x)$

i.e $(D^2+9)\cdot y = \cos^3 x \longrightarrow (1)$

The general solution of eqn (1) is $y = y_c + y_p$

<u>To find</u> <u>C.F</u> : $y_c = f(D)\cdot y = 0$ $\quad Q(x) = 0$

$(D^2+9)\cdot y = 0$

The Auxillary eqn $f(m) = 0 \Rightarrow m^2+9=0$

$m^2 = -9$

$m = \sqrt{-9} = \pm 3i$

The roots are complex, $\alpha \pm i\beta = (0) \pm 3i$ $\quad \boxed{\alpha=0, \beta=3}$

$\therefore$ The complimentary function $y_c = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

$y_c = c_1 \cos 3x + c_2 \sin 3x \longrightarrow (2)$

<u>To find</u> <u>P.I</u> : $y_p = \frac{1}{f(D)}\cdot Q(x)$

$= \frac{1}{D^2+9}\cdot \cos^3 x$ $\quad \cos 3x = 4\cos^3 x - 3\cos x$

$= \frac{1}{D^2+9}\left(\frac{1}{4}\cos 3x + 3\cos x\right)$ $\quad \cos^3 x = \frac{1}{4}[\cos 3x + 3\cos x]$

$= \frac{1}{4}\cdot\frac{1}{D^2+9}\cos 3x + \frac{3}{4}\frac{1}{D^2+9}\cos x$

$$\because \boxed{\frac{1}{f(D)}\cdot Q(x) = \frac{1}{D^2+b^2}\cos bx \quad \boxed{D^2=-b^2} \text{ fail} \quad = \frac{x}{f'(D)}\cdot \cos bx}$$

$= \frac{1}{4}\frac{1}{D^2+9}\cos 3x + \frac{3}{4}\frac{1}{D^2+9}\cos x$

$\boxed{D^2 = -b^2 = -9}$ fail $\quad \boxed{D^2 = -b^2 = -1}$

$\frac{x}{2D}(\cos 3x)$

$\frac{x}{2}\int \cos 3x\, dx$

$\frac{x}{2}\left(\frac{\sin 3x}{3}\right)$

$= \frac{1}{4}\left[\frac{x}{2D}\cos 3x\right] + \frac{3}{4}\cdot\frac{1}{(-1)+9}\cos x$

$y_p = \frac{1}{4}\left(\frac{x}{6}\sin 3x\right) + \frac{3}{32}\cos x \longrightarrow (3)$

$\therefore$ The general solution $y = y_c + y_p$

$$\boxed{y = c_1 \cos 3x + c_2 \sin 3x + \frac{x}{24}\sin 3x + \frac{3}{32}\cos x}$$

(6) Solve: $(D^3-1)\cdot y = e^x + \sin^3 x + 2$

Soln: The given diff equation $f(D)\cdot y = Q(x)$ form

i.e $(D^3-1)\cdot y = e^x + \sin^3 x + 2 \rightarrow ①$

The general solution of eqn ① is $y = C\cdot F + P\cdot I$.

To find C·F : $y_c = f(D)\cdot y = 0$

$$(D^3-1)\cdot y = 0$$

The Auxillary eqn $f(m) = 0 \Rightarrow m^3 - 1 = 0$

by observation $m=1$ is one root.

$$\begin{array}{r|rrrr} m=1 & 1 & 0 & 0 & -1 \\ & 0 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

$m^2+m+1=0$

$\therefore m^3 - 1 = 0$

$(m-1)(m^2+m+1) = 0$

$(m-1) = 0$, $m = 1$,

$m = \dfrac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$

$= \dfrac{-1}{2} \pm i\dfrac{\sqrt{3}}{2} = \alpha \pm i\beta$

$\therefore$ one root is real and other two roots are complex.

The complimentary fun $y_c = c_1 e^{m_1 x} + e^{\alpha x}[c_2 \cos\beta x + c_3 \sin\beta x]$

$$y_c = c_1 e^x + e^{-\frac{1}{2}\cdot x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right] \rightarrow ②$$

To find P·I : $y_p = \dfrac{1}{f(D)}\cdot Q(x)$

$$= \frac{1}{D^3-1}\cdot e^x + \sin^3 x + 2$$

$$= \frac{1}{D^3-1} e^x + \frac{1}{D^3-1}\sin^3 x + \frac{1}{D^3-1} 2\cdot e^{(0)x}$$

$$= P\cdot I_1 + P\cdot I_2 + P\cdot I_3$$

P·I₁ : $\dfrac{1}{(D-1)(D^2+D+1)}\cdot e^x = \dfrac{1}{(D^2+D+1)}\left[\dfrac{x}{1!} e^x\right] = \dfrac{x e^x}{3}$

Put $\boxed{D=a=1}$ fail

$f(D) = f(a) = f(1) = 0$

type I: $\dfrac{1}{f(D)}\cdot e^{ax} = \dfrac{x}{1!} e^{ax}$

$\boxed{D=a}$ fail

$f(a)=0$

P.I$_2$: $\frac{1}{D^3-1}\cdot\sin^3 x = \frac{1}{D^3-1}\frac{1}{4}(3\sin A-\sin 3A)$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\sin^3 A = -\frac{1}{4}[\sin 3A - 3\sin A]$

$= \frac{1}{4}[3\sin A-\sin 3A]$

$= \frac{3}{4}\cdot\frac{1}{D^3-1}\sin x - \frac{1}{4}\frac{1}{D^3-1}\sin 3x$

$= \frac{3}{4}\cdot\frac{1}{D^2\cdot D-1}\sin x - \frac{1}{4}\frac{1}{D^2\cdot D-1}\sin 3x$

$\boxed{D^2=-1}$ $\boxed{D^2=-9}$

$= \frac{3}{4}\cdot\frac{1}{-D-1}\sin x - \frac{1}{4}\frac{1}{-9D-1}\sin 3x$

$= \frac{3}{4}\frac{1}{-(D+1)}\sin x - \frac{1}{4}\frac{1}{-(9D+1)}\sin 3x$

$= -\frac{3}{4}\frac{(D-1)}{(D+1)(D-1)}\sin x + \frac{1}{4}\frac{(9D-1)}{(9D+1)(9D-1)}\sin 3x$

(Rationalize)

$= -\frac{3}{4}\frac{(D-1)}{D^2-1}\sin x + \frac{1}{4}\frac{(9D-1)\sin 3x}{81D^2-1}$

$\boxed{D^2=-1}$ $\boxed{D^2=-9}$

$= -\frac{3}{4}\frac{\frac{d}{dx}(\sin x)-\sin x}{-2} + \frac{1}{4}\frac{9\frac{d}{dx}(\sin 3x)-\sin 3x}{81(-9)-1}$

$= \frac{3}{8}(\cos x-\sin x) - \frac{1}{2920}[9(3\cos 3x)-\sin 3x]$

$= \frac{3}{8}(\cos x-\sin x) - \frac{1}{2920}(27\cos 3x-\sin 3x)$

P.I$_3$: $\frac{1}{D^3-1}2\cdot e^{(0)x} = \frac{1}{(0)-1}\cdot 2e^{(0)x} = -2$

$\boxed{D=a=0}$

$\therefore$ P.I = P.I$_1$ + P.I$_2$ + P.I$_3$

$\therefore$ The general solution $y = C.F + P.I$

$y = c_1e^x + e^{-\frac{x}{2}}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}\right) + \frac{x}{3}e^x$

$+ \frac{3}{8}(\cos x-\sin x) - \frac{1}{2920}(27\cos 3x-\sin 3x) - 2$

Type : 3 : Particular Integral of $f(D)\cdot y = Q(x)$, when $Q(x) = x^K$

K = positive integer.

$$P.I = \frac{1}{f(D)}\cdot Q(x) = \frac{1}{f(D)}\cdot x^K$$

Reduce $\frac{1}{f(D)}$ into of the form $\frac{1}{1\pm\phi(D)}$ by taking out the Lowest degree term from $f(D)$. Now write $\frac{1}{f(D)} = [1\pm\phi(D)]^{-1}$ and expand it in ascending (Increasing) powers of 'D' using Binomial theorem upto the term containing D^K. Then operate x^K with the terms of the expansion $[1\pm\phi(D)]^{-1}$

Use the following formulae for expansion :

1. $\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots\cdots$
2. $\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots\cdots$
3. $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots\cdots$
4. $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots\cdots$
5. $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \cdots\cdots$
6. $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \cdots\cdots$

Problems: Solve : $(D^2+D+1)\cdot y = x^3$

Soln: The given diff equation $f(D)\cdot y = Q(x)$ form.

$$(D^2+D+1)\cdot y = x^3$$

To find C.F : y_c : $f(D)\cdot y = 0$

$$(D^2+D+1)\cdot y = 0$$

The Auxillary equation $f(m) = 0 \Rightarrow m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

The roots are complex.

Complimentary fun $y_c = e^{-\frac{1}{2}x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$ $\boxed{\alpha = -\frac{1}{2},\ \beta = \frac{\sqrt{3}}{2}}$

To find P.I. $y_p = \frac{1}{f(D)} \cdot Q(x)$

$= \frac{1}{D^2+D+1} \cdot x^3$ when $Q(x) = x^K = x^3$

$= \frac{1}{[1+(D^2+D)]} \cdot x^3$

$= [1+(D^2+D)]^{-1} \cdot x^3$

$= \{1-(D^2+D)+(D^2+D)^2-(D^2+D)^3+\ldots\ldots\} \cdot x^3$

$= \{1-(D^2+D)+(D^4+D^2+2D^3)-(D^6+D^3+3(D^2)^2 \cdot D+3D^2 \cdot (D)^2)+\ldots\} x^3$

$= (1-\cancel{D^2}-D+D^4+\cancel{D^2}+2D^3-D^6-D^3-3D^5-3D^4) x^3$

$= (1-D+D^3) \cdot x^3$

$= x^3-3x^2+6$ $\quad \because D = \frac{d}{dx}$

$D(x^3) = 3x^2$
$D^2(x^3) = 6x$
$D^3(x^3) = 6$
$D^4(x^3) = D^5(x^3) = 0$
$D^6(x^3) = 0$

$\therefore$ The general solution $y = y_c + y_p$

$$y = e^{-\frac{1}{2}x}\left[c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right] + x^3 - 3x^2 + 6$$

(2) Solve : $(2D^2+2D+3) \cdot y = x^2+2x-1$

Soln: The given diff eqn $f(D) \cdot y = Q(x)$ form

$(2D^2+2D+3) \cdot y = x^2+2x-1$

To find C.F : $y_c = f(D) \cdot y = 0$

$(2D^2+2D+3) \cdot y = 0$

the Auxillary eqn $f(m) = 0 \Rightarrow 2m^2+2m+3 = 0$

$a = 2$
$b = 2$
$c = 3$

$m = \frac{-2 \pm \sqrt{(2)^2-4(2)(3)}}{2(2)}$

$= \frac{-2 \pm \sqrt{-20}}{4} = \frac{-2 \pm i 2\sqrt{5}}{4} = -\frac{1}{2} \pm i\frac{\sqrt{5}}{2}$

$\alpha = -\frac{1}{2}$, $\beta = \frac{\sqrt{5}}{2}$

$\therefore$ C.F $= e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x] = e^{-\frac{1}{2}x}\left[c_1 \cos \frac{\sqrt{5}}{2}x + c_2 \sin \frac{\sqrt{5}}{2}x\right]$

<u>To find P.I:</u> $y_p = \frac{1}{f(D)} \cdot Q(x)$

$$= \frac{1}{(2D^2+2D+3)} \cdot (x^2+2x-1)$$

when $Q(x) = x^K$

Take out lowest degree term from $f(D)$.

$$= \frac{1}{3\left[1+\left(\frac{2D^2+2D}{3}\right)\right]} \cdot (x^2+2x-1)$$

$$= \frac{1}{3}\left[1+\left(\frac{2D^2+2D}{3}\right)\right]^{-1} \cdot (x^2+2x-1)$$

expand by using $(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots\cdots$

(upto $x^K = x^2 = D^2$)

$$= \frac{1}{3}\left[1 - \left(\frac{2D^2+2D}{3}\right) + \left(\frac{2D^2+2D}{3}\right)^2 - \cdots\cdots\right](x^2+2x-1)$$

$$= \frac{1}{3}\left[1 - \frac{1}{3}(2D^2+2D) + \frac{1}{9}(4D^4+4D^2+8D^3)\right](x^2+2x-1)$$

$\because D^4 = D^3 = 0$

$$= \frac{1}{3}\left[(x^2+2x-1) - \frac{2}{3}D^2(x^2+2x-1) - \frac{2}{3}D(x^2+2x-1) + \frac{4}{9}D^2(x^2+2x-1)\right]$$

$$= \frac{1}{3}\left[(x^2+2x-1) - \frac{2}{3}(2) - \frac{2}{3}(2x+2) + \frac{4}{9}(2)\right]$$

$$= \frac{1}{3}\left[(x^2+2x-1) - \frac{2}{3}(2x+2+2) + \frac{8}{9}\right]$$

$$= \frac{1}{3}\left[\frac{(9x^2+18x-9) - 6(2x+4) + 8}{9}\right]$$

$$= \frac{1}{27}\left[9x^2+6x-25\right]$$

$\therefore$ General solution $y = y_c + y_p$

$$y = e^{-\frac{x}{2}}\left[c_1 \cos\frac{\sqrt{5}}{2}x + c_2 \sin\frac{\sqrt{5}}{2}x\right] + \frac{1}{27}(9x^2+6x-25)$$

③ Solve: $(D^3+2D^2+D)y = e^{2x} + x^2+x + \sin 2x$

Soln: The given diff al eqn $f(D)\cdot y = Q(x)$ form

$$(D^3+2D^2+D)\cdot y = e^{2x} + x^2+x+\sin 2x$$

To find C.F: $y_c = f(D)\cdot y = 0$

$$(D^3+2D^2+D)\cdot y = 0$$

The Auxillary equation, $f(m)=0 \Rightarrow m^3+2m^2+m=0$

$$m(m^2+2m+1)=0$$

$$m=0,\ m^2+2m+1=0$$

$$m^2+m+m+1=0$$

$$m(m+1)+1(m+1)=0$$

$$(m+1)(m+1)=0$$

$$m=-1,-1$$

$\therefore$ $m=0,-1,-1$ one root is real and other two roots are real and equal.

$\therefore$ The Complementary fun $y_c = c_1 e^{m_1 x} + (c_2+c_3x)e^{m_2 x}$

$$\therefore\ y_c = c_1 e^{(0)x} + (c_2+c_3x)e^{(-1)x}$$

$m_1=0$

$m_2=m_3=-1$

$$\boxed{y_c = c_1 + (c_2+c_3x)e^{-x}}$$

To find P.I: $y_p = \frac{1}{f(D)}\cdot Q(x)$

$$= \frac{1}{(D^3+2D^2+D)}\cdot e^{2x} + (x^2+x) + \sin 2x$$

$$= \frac{1}{D^3+2D^2+D}\cdot e^{2x} + \frac{1}{D^3+2D^2+D}\cdot(x^2+x) + \frac{1}{D^3+2D^2+D}\cdot\sin 2x$$

$$= y_{p_1} + y_{p_2} + y_{p_3}$$

y_{p_1}: $\frac{1}{D^3+2D^2+D}\cdot e^{2x} = \frac{1}{(2)^3+2(2)^2+2}\cdot e^{2x} = \frac{e^{2x}}{18}$

put $\boxed{D=a=2}$

Type-I, P.I $\frac{1}{f(D)}\cdot e^{ax}$ put $\boxed{D=a}$

$\underline{y_{P_2}}$: $\dfrac{1}{D^3+2D^2+D}(x^2+x)$ \ P.I, Type-3 $Q(x)=x^K$

Taking the Lowest degree term outside from $f(D)$.

$$= \frac{1}{D\left[1+\left(\frac{D^3+2D^2}{D}\right)\right]} \cdot x^2+x$$

$$= \frac{1}{D}\left[1+(D^2+2D)\right]^{-1} \cdot (x^2+x)$$

$$= \frac{1}{D}\left[1-(D^2+2D)+(D^2+2D)^2-\ldots\ldots\right](x^2+x)$$

$$= \frac{1}{D}\left[1-D^2-2D+D^4+4D^2+4D^3-\ldots\right](x^2+x)$$

$$D^3=D^4=0$$

$$= \frac{1}{D}\left[(x^2+x)-D^2(x^2+x)-2D(x^2+x)+0+4D^2(x^2+x)+0\right]$$

$$= \frac{1}{D}\left[x^2+x-2-2\cdot(2x+1)+4(2)\right]$$

$$= \frac{1}{D}\left[x^2-3x+4\right]$$ \ $\frac{1}{D}$ stands for integral

$$= \int(x^2-3x+4)\,dx$$

$$= \frac{x^3}{3}-3\frac{x^2}{2}+4x$$

$\underline{y_{P_3}}$: $\dfrac{1}{D^3+2D^2+D}\cdot\sin 2x = \dfrac{1}{D^2\cdot D+2D^2+D}\cdot\sin 2x = \dfrac{1}{f(D)}\cdot\sin bx$

$\boxed{D^2=-b^2=-4}$ \ $Q(x)=\sin bx$ (Type-2)

$$= \frac{1}{(-4)D+2(-4)+D}\sin 2x$$

$$= \frac{1}{-3D-8}\sin 2x$$

$$= -\frac{1}{(3D+8)}\sin 2x$$

$$= -\frac{(3D-8)}{9D^2-64}\sin 2x$$ (Rationalize)

$$= \frac{-(3D-8)\sin 2x}{9(-4)-64}$$ $\boxed{D^2=-b^2=-4}$

$$= \frac{1}{100}\left[3\frac{d}{dx}(\sin 2x) - 8\sin 2x\right]$$

$$= \frac{1}{100}\left[3(2\cos 2x) - 8\sin 2x\right]$$

$$= \frac{1}{100}\left[6\cos 2x - 8\sin 2x\right]$$

$$= \frac{1}{50}\left[3\cos 2x - 4\sin 2x\right]$$

$\therefore$ Particular Integral $y_p = y_{p_1} + y_{p_2} + y_{p_3}$

$$y_p = \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x + \frac{1}{50}(3\cos 2x - 4\sin 2x)$$

$\therefore$ General Solution $y = y_c + y_p$

$$y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x + \frac{1}{50}(3\cos 2x - 4\sin 2x)$$

Solve the following differential equations.

(1.) $y''' + 2y'' - y' - 2y = 1 - 4x^3$

(2) $(D^2 + 3D + 2)\cdot y = 2\cos(2x+3) + 2e^x + x^2$

(3) $(D^3 - 3D^2 + 3D - 1) = \sin x + x^3$

$\because$ apply $(D-1)^3 = D^3 - 3D^2 + 3D - 1$

$$\frac{1}{D^3 - 3D^2 + 3D - 1}\cdot x^3 = \frac{1}{(D-1)^3}\cdot x^3$$

$$= \frac{1}{-(1-D)^3}\cdot x^3$$

$$= -(1-D)^{-3}\cdot x^3$$

$$= -\left[1 + 3D + 6D^2 + 10D^3 + \cdots\right]x^3$$

simplify.

Type : 4 : P.I of $f(D)\cdot y = Q(x)$, when $Q(x) = e^{ax}\cdot V(x)$:

(Here $V(x) = \sin bx$ (or) $\cos bx$ (or) x^k polynomial of degree 'k')

$$P.I = \frac{1}{f(D)}\cdot Q(x) = \frac{1}{f(D)}\cdot e^{ax}\cdot V(x)$$

put $\boxed{D = D+a}$

$$= e^{ax}\cdot \frac{1}{f(D+a)}\cdot V(x)$$

Solve $\frac{1}{f(D+a)}\cdot V(x)$ by using previous method.

Note: $P.I = \frac{1}{f(D)}\cdot Q(x) = \frac{1}{f(D)}\cdot e^{ax}\cdot V(x) = e^{ax}\frac{1}{f(D+a)}\cdot \sin bx$ (or) $\cos bx$ (or) x^k

---

Problem: ① Solve : $(D^2-5D+6)\cdot y = x e^{4x}$

Soln: The given diff'al eqn $(D^2-5D+6)\cdot y = x e^{4x}$

To find C.F : $f(D)\cdot y = 0$ i.e $f(D)\cdot y = Q(x)$.

The Auxillary equation $f(m)=0 \Rightarrow m^2-5m+6=0$

$$m^2-3m-2m+6=0$$
$$m(m-3)-2(m-3)=0$$
$$(m-3)(m-2)=0$$
$$m = 3, 2$$

$\therefore$ roots are real & different.

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} = \boxed{c_1 e^{3x} + c_2 e^{2x}}$$

To find P.I : $P.I = \frac{1}{f(D)}\cdot Q(x) = \frac{1}{D^2-5D+6}\cdot x\cdot e^{4x} = \frac{1}{f(D)}\cdot e^{ax}\cdot V(x)$.

put $\boxed{D = D+a = D+4}$

$$\boxed{P.I = \frac{1}{f(D)}\cdot e^{ax}\cdot V(x) = e^{ax}\frac{1}{f(D+a)}\cdot V(x)}$$

$$= e^{4x}\cdot \frac{1}{(D+4)^2-5(D+4)+6}\cdot x$$

$$= e^{4x}\cdot \frac{1}{D^2+3D+2}\cdot x \quad \text{(Type-3)}$$

$$= e^{4x}\frac{1}{2\left[1+\left(\frac{D^2+3D}{2}\right)\right]}\cdot x$$

$$\boxed{(1+D)^{-1} = 1-D+D^2-D^3+\cdots}$$

$$= \frac{e^{4x}}{2}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1}\cdot x$$

$$= \frac{e^{4x}}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\cdots\right]\cdot x$$

$$= \frac{e^{4x}}{2}\left[x - \frac{1}{2}\left[D^2(x) + 3D(x)\right]\right]$$

$$= \frac{e^{4x}}{2}\left[x - \frac{1}{2}(0 + 3(1))\right]$$

$$= \frac{e^{4x}}{2}\left[x - \frac{3}{2}\right]$$

$$P.I = \frac{e^{4x}}{4}(2x-3)$$

$\therefore$ The general solution $y = y_c + y_p$

$$y = c_1 e^{3x} + c_2 e^{2x} + \frac{e^{4x}}{4}(2x-3)$$

② solve: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 8e^{3x}\sin 2x$

soln: The given diff equation $(D^2 - 6D + 13)\cdot y = 8e^{3x}\sin 2x$

i.e $f(D)\cdot y = Q(x)$.

To find C.F: $y_c = f(D)\cdot y = 0$

$(D^2 - 6D + 13)\cdot y = 0$

The Auxillary eqn $f(m) = 0 \Rightarrow m^2 - 6m + 13 = 0$

$$m = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(13)}}{2(1)} = 3 \pm i2$$

$\therefore$ The roots are complex $\alpha \pm i\beta = 3 \pm 2i$ $\alpha = 3$, $\beta = 2$

Complimentary fun $y_c = e^{3x}[c_1\cos 2x + c_2 \sin 2x]$

To find P.I :- $y_p = \frac{1}{f(D)}\cdot Q(x) = \frac{1}{(D^2 - 6D + 13)}\cdot 8e^{3x}\sin 2x$

$$= 8\cdot e^{3x}\frac{1}{(D+3)^2 - 6(D+3) + 13}\cdot \sin 2x$$

$$\frac{1}{f(D)}\cdot e^{ax}\cdot v(x) = e^{ax}\cdot\frac{1}{f(D+a)}\cdot v(x)$$

put $\boxed{D = D+a}$ $v(x) = \sin 2x$

put $\boxed{D = D+a = D+3}$ write e^{ax} outside

$$= 8e^{3x}\cdot\frac{1}{D^2+4}\sin 2x \quad \boxed{D^2 = -b^2 = -4} \text{ fail}$$

$\therefore \frac{1}{D^2+b^2}\sin bx$ $\boxed{D^2 = -b^2}$ fail

$f(D) = 0$

$\frac{x}{f'(D)}\cdot\sin bx$

$$= 8e^{3x}\left[\frac{x}{2D}\cdot\sin 2x\right]$$

$$= \frac{8}{2}\cdot xe^{3x}\cdot\frac{1}{D}(\sin 2x)$$

$$= 4xe^{3x}\int \sin 2x\, dx$$

$$= 4xe^{3x}\left[-\frac{\cos 2x}{2}\right] = -2xe^{3x}\cos 2x$$

Hence the general solution $y = y_c + y_p$

$$y = e^{3x}[C_1 \cos 2x + C_2 \sin 2x] - 2x e^{3x} \cdot \cos 2x$$

(3) Solve: $(D^3 - 7D^2 + 14D - 8) \cdot y = e^{x} \cdot \cos 2x$

Soln: The given diff equation $(D^3 - 7D^2 + 14D - 8) \cdot y = e^{x} \cos 2x$

i.e $f(D) \cdot y = Q(x)$.

To find C.F: y_c : $f(D) \cdot y = 0$

$$(D^3 - 7D^2 + 14D - 8) \cdot y = 0$$

The Auxillary equation $f(m) = 0 \Rightarrow m^3 - 7m^2 + 14m - 8 = 0$

by observation $m = 1$ is one root.

$$\begin{array}{c|cccc} 1 & 1 & -7 & 14 & -8 \\ & 0 & 1 & -6 & 8 \\ \hline & 1 & -6 & 8 & \boxed{0} \end{array}$$

$m^2 - 6m + 8 = 0$

$$m^3 - 7m^2 + 14m - 8 = 0$$

$$(m-1)(m^2 - 6m + 8) = 0$$

$m - 1 = 0$ $\quad m^2 - 4m - 2m + 8 = 0$

$m = 1$ $\quad m(m-4) - 2(m-4) = 0$

$$(m-2)(m-4) = 0$$

$$m - 2 = 0 \quad m - 4 = 0$$

$$m = 2, m = 4$$

$\therefore$ $m = 1, 2, 4$ are real and distict roots of $f(m) = 0$

$\therefore$ Complementary fuction $\boxed{y_c = C_1 e^{x} + C_2 e^{2x} + C_3 e^{4x}}$

To find P.I: $y_p = \frac{1}{f(D)} \cdot Q(x)$

$$= \frac{1}{D^3 - 7D^2 + 14D - 8} \cdot e^{x} \cdot \cos 2x$$

$\because V(x) = \cos bx = \cos 2x$

$$\boxed{\frac{1}{f(D)} \cdot e^{ax} \cdot V(x) = e^{ax} \cdot \frac{1}{f(D+a)} \cdot V(x)}$$

put $\boxed{D = D + a}$ write e^{ax} outside from $f(D)$.

$e^{ax} = e^{x}$ $\boxed{a = 1}$

$$= e^{x} \cdot \frac{1}{(D+1)^3 - 7(D+1)^2 + 14(D+1) - 8} \cdot \cos 2x$$

$$= e^{x} \cdot \frac{1}{D^3 - 4D^2 + 3D} \cdot \cos 2x.$$

put $\boxed{D^2 = -b^2 = -4}$

$$= e^x \cdot \frac{1}{D^2 \cdot D - 4D^2 + 3D} \cdot \cos 2x \qquad \boxed{D^2 = -b^2 = -4}$$

$$= e^x \cdot \frac{1}{(-4)D - 4(-4) + 3D} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{16 - D} \cos 2x \qquad \text{(Rationalize)}$$

$$= e^x \cdot \frac{(16+D)}{(16-D)(16+D)} \cdot \cos 2x$$

$\frac{1}{f(D)} \cdot \cos bx \quad D^2 = -b^2$

$$= e^x \cdot \frac{(16+D)}{256 - D^2} \cdot \cos 2x$$

put $\boxed{D^2 = -b^2 = -4}$

$$= e^x \cdot \frac{(16+D)\cos 2x}{256 - (-4)}$$

$$= \frac{e^x}{260}\left[16\cos 2x + \frac{d}{dx}(\cos 2x)\right]$$

$$= \frac{e^x}{260}\left[16\cos 2x + (-2\sin 2x)\right]$$

$$y_p = \frac{e^x}{130}\left[8\cos 2x - \sin 2x\right]$$

$\therefore$ General solution $y = y_c + y_p$

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{4x} + \frac{e^x}{130}(8\cos 2x - \sin 2x)$$

Solve the following differential equations

1. $(D^2 - 2D + 1) \cdot y = e^{3x} \cdot x^2$
2. $(D^4 - 1) \cdot y = e^x \cdot \cos x$
3. $(D^2 + 2D + 4) \cdot y = e^x \cdot \sin 2x$.

④. Solve : $\frac{d^4y}{dx^4} - y = \cos x \cdot \cosh x + 2x^4 + x - 1$

Soln: The given diff eqn $(D^4 - 1) \cdot y = \cos x \cdot \cosh x + 2x^4 + x - 1$

i.e $f(D) \cdot y = Q(x)$ form.

To find C.F : $y_c = f(D) \cdot y = 0$

$(D^4 - 1) \cdot y = 0$

The Auxillary eqn $f(m) = 0 \Rightarrow m^4 - 1 = 0$

$(m^2)^2 - 1^2 = 0$

$\boxed{a^2 - b^2 = (a+b)(a-b)}$ $= (m^2+1)(m^2-1) = 0$

$\therefore m^2 + 1 = 0$ | $m^2 - 1 = 0$

$m^2 = -1$ | $m^2 = 1$

$m = \pm i$ | $m = \pm 1$

Two roots are real and different other two roots are complex, conjugate numbers.

Complementary fun $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + e^{\alpha x}[c_3 \cos\beta x + c_4 \sin\beta x]$

$\boxed{y_c = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x}$ $\alpha = 0$, $\beta = 1$

$\because e^0 = 1$

To find P.I : $y_p = \frac{1}{f(D)} \cdot Q(x)$

$= \frac{1}{D^4 - 1} \cos x \cdot \cosh x + 2x^4 + x - 1$

$= \frac{1}{D^4 - 1} \cos x \left(\frac{e^{x} + e^{-x}}{2}\right) + \frac{1}{D^4 - 1}(2x^4 + x - 1)$

$= \frac{1}{2} \cdot \frac{1}{D^4 - 1} e^{x} \cdot \cos x + \frac{1}{2} \cdot \frac{1}{D^4 - 1} e^{-x} \cdot \cos x + \frac{1}{D^4 - 1}(2x^4 + x - 1)$

P.I₁ + P.I₂ + P.I₃

P.I₁ : $\frac{1}{2} \cdot \frac{1}{D^4 - 1} e^{x} \cdot \cos x = \frac{e^x}{2} \frac{1}{(D+1)^4 - 1} \cdot \cos x$

$\boxed{D = D + a = D + 1}$

$= \frac{e^x}{2} \frac{1}{[D^4 + 4D^3 + 6D^2 + 4D + \cancel{1}] - \cancel{1}} \cdot \cos x$

$$= \frac{e^x}{2} \cdot \frac{1}{D^2 \cdot D^2 + 4 \cdot D^2 \cdot D + 6D^2 + 4D} \cdot \cos x$$

$$\boxed{D^2 = -b^2 = -1}$$

$$= \frac{e^x}{2} \cdot \frac{1}{(-1)(-1) + 4(-1)D + 6(-1) + 4D} \cdot \cos x$$

$$= \frac{e^x}{2} \cdot \frac{1}{(-5)} \cdot \cos x$$

$$= -\frac{e^x}{10} \cos x.$$

P.I$_2$: $\frac{1}{2} \cdot \frac{1}{D^4 - 1} e^{-x} \cdot \cos x = \frac{e^{-x}}{2} \cdot \frac{1}{(D-1)^4 - 1} \cos x$

$$\boxed{D = D + a = D - 1}$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{(D^4 - 4D^3 + 6D^2 - 4D + \cancel{1}) - \cancel{1}} \cdot \cos x$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{D^2 \cdot D^2 - 4D^2 \cdot D + 6D^2 - 4D} \cdot \cos x$$

$$\boxed{D^2 = -b^2 = -1}$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{(-1)(-1) - 4(-1) \cdot D + 6(-1) - 4D} \cdot \cos x$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{(-5)} \cdot \cos x.$$

$$= -\frac{e^{-x}}{10} \cos x.$$

P.I$_3$: $\frac{1}{D^4 - 1} \cdot 2x^4 + x - 1 = \frac{1}{-1[1 - D^4]} \cdot (2x^4 + x - 1)$

$$= -[1 - D^4]^{-1} \cdot (2x^4 + x - 1)$$

$$= -[1 + D^4 + (D^4)^2 + \cdots](2x^4 + x - 1)$$

$$= -[(2x^4 + x - 1) + D^4(2x^4 + x - 1)]$$

$$= -[(2x^4 + x - 1) + 48]$$

$$= -(2x^4 + x + 47)$$

$D(2x^4 + x - 1) = 2(4x^3) + 1$
$= 8x^3 + 1$

$D^2(2x^4 + x - 1) = 8(3x^2) + 0$
$= 24x^2$

$D^3(2x^4 + x - 1) = 24(2x)$
$= 48x$

$D^4(2x^4 + x - 1) = 48$

$\therefore$ The General solution $y = y_c + y_p$

$$y = c_1e^{x} + c_2e^{-x} + c_3\cos x + c_4\sin x - \frac{e^{x}}{10}\cos x - \frac{e^{-x}}{10}\cos x - (2x^4+x+47)$$

$$= c_1e^{x} + c_2e^{-x} + c_3\cos x + c_4\sin x - \left(\frac{e^{x}+e^{-x}}{2}\right)\cdot\frac{\cos x}{5} - (2x^4+x+47)$$

$$y = c_1e^{x} + c_2e^{-x} + c_3\cos x + c_4\sin x - \frac{1}{5}\cos x\cdot\cosh x - (2x^4+x+47)$$

Type-5: P.I of $f(D)\cdot y = Q(x)$, when $Q(x) = x\cdot V(x)$

$$P.I = \frac{1}{f(D)}\cdot Q(x) = \frac{1}{f(D)}\cdot x\cdot V(x)$$

$$= \left[x - \frac{f'(D)}{f(D)}\right]\cdot\frac{1}{f(D)}\cdot V(x).$$

Problem: ① Solve: $(D^2+2D+1)\cdot y = x\cdot\cos x$

Soln: the given diff equation $f(D)\cdot y = Q(x)$ form.

i.e $(D^2+2D+1)\cdot y = x\cdot\cos x$.

The general solution $y = y_c + y_p$

To find C.F: y_c: $f(D)\cdot y = 0$

$$(D^2+2D+1)\cdot y = 0$$

the Auxillary equation $f(m) = 0 \Rightarrow m^2+2m+1 = 0$

$$(m+1)^2 = 0$$

$$(m+1)(m+1) = 0$$

$$m = -1, -1$$

The roots are $-1, -1$, which are real and equal.

$\therefore$ Complementary function $\boxed{y_c = (c_1 + c_2x)e^{-x}}$

To find P.I: $y_p = \frac{1}{f(D)}\cdot Q(x)$

$$= \frac{1}{(D^2+2D+1)}\cdot x\cdot\cos x$$

$\therefore \frac{1}{D^2+2D+1} \cdot x \cdot V(x)$ $\quad V(x) = \cos x$.

$= \left[x - \frac{(2D+2)}{(D^2+2D+1)}\right] \cdot \frac{1}{D^2+2D+1} \cdot \cos x$ $\quad \boxed{D^2 = -b^2 = -1}$

$\because \frac{1}{f(D)} \cdot x V(x) = \left[x - \frac{f'(D)}{f(D)}\right] \cdot \frac{1}{f(D)} \cdot V(x)$

$= \left[x - \frac{2\cancel{(D+1)}}{(D+1)^{\cancel{2}}}\right] \cdot \frac{1}{(-1)+2D+1} \cos x$

$= \left[x - \frac{2}{D+1}\right] \cdot \frac{1}{2D} \cos x$ $\quad \frac{1}{D} \cos x = \int \cos x \, dx = \sin x$

$= \left(x - \frac{2}{D+1}\right) \cdot \frac{1}{2} \sin x$

$= \frac{x}{2} \sin x - \frac{1}{D+1} \sin x$

$= \frac{x}{2} \sin x - \frac{(D-1)}{D^2-1^2} \sin x$ (Rationalize) $\quad \boxed{D^2 = -b^2 = -1}$

$= \frac{x}{2} \sin x - \frac{(D-1) \sin x}{-2}$

$= \frac{x}{2} \sin x + \frac{1}{2} \left[\frac{d}{dx}(\sin x) - \sin x\right]$

$y_p = \frac{x}{2} \sin x + \frac{1}{2} [\cos x - \sin x]$

$\therefore$ The general solution $y = y_c + y_p$

$y = (c_1 + c_2 x) e^{-x} + \frac{x}{2} \sin x + \frac{1}{2} (\cos x - \sin x)$

<u>Note:</u> $e^{iax} = \cos ax + i \sin ax$

$\cos ax =$ Real Part of e^{iax}

$\sin ax =$ Imaginary Part of e^{iax}

① $\frac{1}{f(D)} \cdot x \cdot \cos ax =$ Real part of $\frac{1}{f(D)} x \cdot e^{iax}$

② $\frac{1}{f(D)} x \cdot \sin ax =$ Imaginary Part of $\frac{1}{f(D)} \cdot x \cdot e^{iax}$

② Solve: $(D^2-4) \cdot y = x \cdot \sin\lambda x$

Soln: the given diffal equation $(D^2-4) \cdot y = x \cdot \sin\lambda x \rightarrow ①$

i.e $f(D) \cdot y = Q(x)$ form.

The general solution of eqn ① is $y = y_c + y_p$.

To find C.F: $y_c = f(D) \cdot y = 0$

$$(D^2-4) \cdot y = 0$$

The AUXillary eqn $f(m) = 0 \Rightarrow m^2-4=0$

$$m^2 = 4 \Rightarrow m = \pm 2$$

The roots are real and different.

$\therefore$ Complementary function $y_c = c_1 e^{2x} + c_2 e^{-2x}$

To find P.I: $y_p = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{D^2-4} \cdot x \sin\lambda x$

$$= \text{Imaginary Part of } \frac{1}{D^2-4} \cdot x \cdot e^{i\lambda x}$$

$e^{i\lambda x} = \underset{R.P}{\cos\lambda x} + i \underset{I.P}{\sin\lambda x}$

put $\boxed{D = D+a = D+i\lambda}$

$$= I.P \text{ of } e^{i\lambda x} \frac{1}{(D+i\lambda)^2-4} \cdot x$$

$$= I.P \text{ of } e^{i\lambda x} \cdot \frac{1}{D^2+i^2\lambda^2+2i\lambda D-4} \cdot x$$

$\because i^2 = 1$

$$= I.P \text{ of } e^{i\lambda x} \frac{1}{D^2+2i\lambda D-(\lambda^2+4)} \cdot x$$

Take out the Lowest degree term outside λ^2+4 from $f(D)$.

$$= I.P \text{ of } e^{i\lambda x} \cdot \frac{1}{-(\lambda^2+4)\left[1-\frac{D^2+2i\lambda D}{\lambda^2+4}\right]} \cdot x$$

$$= I.P \text{ of } \frac{-e^{i\lambda x}}{\lambda^2+4}\left[1-\left(\frac{D^2+2i\lambda D}{\lambda^2+4}\right)\right]^{-1} \cdot x$$

$\because (1-D)^{-1} = 1+D+D^2+ \cdots\cdots$

$$= \text{I.P of } -\frac{e^{i\lambda x}}{\lambda^2+4}\left[1+\left(\frac{D^2+2i\lambda D}{\lambda^2+4}\right)\right]\cdot x$$

$$= \text{I.p of } -\frac{e^{i\lambda x}}{\lambda^2+4}\left[x+\frac{D^2(x)+2i\lambda D(x)}{\lambda^2+4}\right]$$

$D(x)=1$
$D^2(x)=0$

$$= \text{I.P of } -\frac{e^{i\lambda x}}{\lambda^2+4}\left[x+\frac{2i\lambda}{\lambda^2+4}\right]$$

$$= \text{I.P of } -\frac{(\cos\lambda x+i\sin\lambda x)}{\lambda^2+4}\left(x+\frac{2i\lambda}{\lambda^2+4}\right)$$

$$= \text{I.P of }\left[-\frac{x\cos\lambda x}{\lambda^2+4} - \underbrace{\frac{ix\sin\lambda x}{\lambda^2+4}}_{\text{I.P}} - \underbrace{\frac{2i\lambda\cos\lambda x}{(\lambda^2+4)^2}}_{\text{I.P}} - \frac{i^2\,2\lambda\sin\lambda x}{(\lambda^2+4)^2}\right]$$

$i^2=-1$

$$y_p = -\frac{x\sin\lambda x}{\lambda^2+4} - \frac{2\lambda\cos\lambda x}{(\lambda^2+4)^2}$$

$\therefore$ General solution $y = y_c + y_p$

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{x\sin\lambda x}{\lambda^2+4} - \frac{2\lambda\cos\lambda x}{(\lambda^2+4)^2}$$

③ Solve: $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y = xe^x\sin x$

Soln: the given diff eqn $(D^2+3D+2)\cdot y = xe^x\sin x$
i.e $f(D)\cdot y = Q(x)$ form

The general solution $y = y_c + y_p$

To find C.F: y_c: $f(D)\cdot y = 0$
$(D^2+3D+2)\cdot y = 0$

The Auxillary eqn $f(m)=0 \Rightarrow m^2+3m+2=0$
$m^2+2m+m+2=0$
$m(m+2)+1(m+2)=0$
$(m+1)(m+2)=0$
$m=-1,-2$

$\therefore$ Two roots are real & different.

Complementary function $\boxed{y_c = c_1e^{-x}+c_2e^{-2x}}$

To find P.I: $y_p = \frac{1}{f(D)} \cdot Q(x)$

$$= \frac{1}{D^2+3D+2} x e^x \sin x$$

First eliminate e^{ax}, put $\boxed{D = D+a = D+1}$ write e^x outside

$$= e^x \frac{1}{(D+1)^2+3(D+1)+2} \cdot x \sin x$$

$$= e^x \cdot \frac{1}{D^2+5D+6} \cdot x \cdot \sin x$$

$$\frac{1}{f(D)} \cdot x \cdot V(x) = \left[x - \frac{f'(D)}{f(D)}\right] \cdot \frac{1}{f(D)} \cdot V(x)$$

$$= e^x \cdot \left[x - \frac{(2D+5)}{D^2+5D+6}\right] \cdot \frac{1}{D^2+5D+6} \sin x$$

put $\boxed{D^2 = -b^2 = -1}$

$$= e^x \left[x - \frac{2D+5}{D^2+5D+6}\right] \cdot \frac{1}{(-1)+5D+6} \sin x$$

$$= e^x \left[x - \frac{(2D+5)}{D^2+5D+6}\right] \frac{1}{5D+5} \cdot \sin x$$

$$= e^x \left[x - \frac{(2D+5)}{D^2+5D+6}\right] \cdot \frac{1}{5(D+1)} \sin x$$

$$= \frac{e^x}{5} \left[x - \frac{(2D+5)}{D^2+5D+6}\right] \cdot \frac{(D-1)}{D^2-1} \sin x \text{ (Rationalize)}$$

$\boxed{D^2 = -b^2 = -1}$

$$= \frac{e^x}{5} \left[x - \frac{(2D+5)}{D^2+5D+6}\right] \cdot \frac{\frac{d}{dx}(\sin x) - \sin x}{(-1)-1}$$

$$= \frac{e^x}{5} \left[x - \frac{2D+5}{D^2+5D+6}\right] \cdot \frac{\cos x - \sin x}{-2}$$

$$= \frac{e^x}{5} \left[\frac{x(\cos x - \sin x)}{-2} - \frac{(2D+5)\cos x - \sin x}{-2(D^2+5D+6)}\right]$$

$$= \frac{e^x}{10} \left[x(\sin x - \cos x) + \frac{2\frac{d}{dx}(\cos x - \sin x) + 5(\cos x - \sin x)}{D^2+5D+6}\right]$$

$$= \frac{xe^x}{10}(\sin x - \cos x) + \frac{e^x}{10}\left[\frac{2(-\sin x - \cos x) + 5(\cos x - \sin x)}{D^2+5D+6}\right]$$

$$= \frac{xe^x}{10}(\sin x - \cos x) + \frac{e^x}{10}\left[\frac{3\cos x - 7\sin x}{D^2+5D+6}\right]$$

$\boxed{D^2 = -b^2 = -1}$

$$= \frac{xe^{x}}{10}(\sin x-\cos x) + \frac{e^{x}}{10}\left[\frac{3\cos x - 7\sin x}{-1+5D+6}\right]$$

$$= \frac{xe^{x}}{10}(\sin x-\cos x) + \frac{e^{x}}{10}\left[\frac{1}{5(D+1)}(3\cos x-7\sin x)\right]$$

$$= \frac{xe^{x}}{10}(\sin x-\cos x) + \frac{e^{x}}{50}\left[\frac{(D-1)(3\cos x-7\sin x)}{D^2-1^2}\right]$$

$$\boxed{D^2=-b^2=-1}$$

$$= \frac{xe^{x}}{10}(\sin x-\cos x) + \frac{e^{x}}{50}\left[\frac{\frac{d}{dx}(3\cos x-7\sin x)-(3\cos x-7\sin x)}{(-1)-1}\right]$$

$$= \frac{xe^{x}}{10}(\sin x-\cos x) + \frac{e^{x}}{50}\left[\frac{3(-\sin x)-7\cos x-3\cos x+7\sin x}{-2}\right]$$

$$y_p = \frac{xe^{x}}{10}(\sin x-\cos x) - \frac{e^{x}}{100}(4\sin x-10\cos x)$$

$\therefore$ General solution $y = y_c + y_p$

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{x}\left[\frac{x}{10}(\sin x-\cos x) - \frac{1}{25}\sin x + \frac{1}{10}\cos x\right]$$

(4) * Solve the differential equation $(D^3-3D^2+3D-1)\cdot y=\sin x+x^3$

* solve : $(D^4+2D^2+1)\cdot y = x^2\cos^2 x$

Soln: the given diff equation $(D^4+2D^2+1)\cdot y = x^2\cdot\cos^2 x \rightarrow (1)$

i.e $f(D)\cdot y = Q(x)$ form.

The general solution of (1) is $y = y_c + y_p$

To find C.F: $y_c = f(D)\cdot y = 0$

$(D^4+2D^2+1)\cdot y = 0$

The Auxillary equation $f(m)=0 \Rightarrow m^4+2m^2+1=0$

$(m^2+1)^2=0$

$m^2+1=0 \quad | \quad m^2+1=0$

$m^2=-1 \quad | \quad m^2=-1$

$m=\pm i \quad | \quad m=\pm i$

The roots are Conjugate Complex numbers and Repeated (equal)

Complementary fun $y_c = e^{\alpha x}[(c_1+c_2x)\cos\beta x+(c_3+c_4x)\sin\beta x]$

$$\boxed{y_c = (c_1+c_2x)\cos x + (c_3+c_4x)\sin x} \qquad \alpha=0,\ \beta=1$$

To find P.I: $y_p = \frac{1}{f(D)}\cdot Q(x)$

$= \frac{1}{D^4+2D^2+1}\cdot x^2\cos^2 x$

$= \frac{1}{D^4+2D^2+1} x^2\left(\frac{1+\cos 2x}{2}\right)$

$= \frac{1}{2}\frac{1}{D^4+2D^2+1}x^2 + \frac{1}{2}\frac{1}{D^4+2D^2+1}x^2\cdot\cos 2x$

$= y_{p_1} + y_{p_2}$

$y_{p_1}: \frac{1}{2}\frac{1}{D^4+2D^2+1}x^2 = \frac{1}{2}\cdot\frac{1}{(D^2+1)^2}x^2 = \frac{1}{2}\cdot(1+D^2)^{-2}\cdot x^2$

$= \frac{1}{2}[1-2D^2+3(D^2)^2-\ldots\ldots]\cdot x^2$

$[1+D]^{-2} = 1-2D+3D^2-\ldots\ldots$

$= \frac{1}{2}[x^2-2D^2(x^2)+3D^4(x^2)-\ldots]$

$= \frac{1}{2}[x^2-4] = \frac{x^2}{2}-2$

y_{P_2} : $\frac{1}{2} \frac{1}{D^4+2D^2+1} x^2 \cos 2x = \frac{1}{2} \cdot \frac{1}{D^4+2D^2+1} x^2 \left(\text{Real part of } e^{i2x}\right)$

put $\boxed{D = D+a = D+i2}$

$$= \text{Real part of } \frac{e^{i2x}}{2} \cdot \frac{1}{(D+i2)^4 + 2(D+i2)^2 + 1} \cdot x^2$$

$$= \text{R.P of } \frac{e^{i2x}}{2} \cdot \frac{1}{\left[(D+i2)^2+1\right]^2} \cdot x^2$$

$$= \text{R.P. of } \frac{e^{i2x}}{2} \cdot \frac{1}{(D^2+4i^2+4iD+1)^2} \cdot x^2$$

$$= \text{R.P of } \frac{e^{i2x}}{2} \cdot \frac{1}{(D^2+4iD-3)^2} \cdot x^2 \qquad \Big| \, i^2 = -1$$

$$= \text{R.P of } \frac{e^{i2x}}{2} \cdot \frac{1}{(-3)^2\left[1 - \frac{D^2+4iD}{3}\right]^2} \cdot x^2$$

$$= \text{R.P of } \frac{e^{i2x}}{18} \left[1 - \left(\frac{D^2+4iD}{3}\right)\right]^{-2} \cdot x^2$$

$$= \text{R.P of } \frac{e^{i2x}}{18} \left[1 + 2\left(\frac{D^2+4iD}{3}\right) + 3\left(\frac{D^2+4iD}{3}\right)^2 + \cdots\right] x^2$$

$$= \text{R.P of } \frac{e^{i2x}}{18} \left(1 + \frac{2}{3}D^2 + \frac{8i}{3}D + \frac{1}{3}\left(D^4 + 16i^2D^2 + 8iD^3\right)\right) x^2$$

$$= \text{R.P of } \frac{e^{i2x}}{18} \left[x^2 + \frac{2}{3}D^2(x^2) + \frac{8i}{3}D(x^2) + \frac{1}{3}\left(0 - 16D^2(x^2) + 0\right)\right]$$

$$\because D^4(x^2) = D^3(x^2) = 0$$

$$= \text{R.P of } \frac{(\cos 2x + i \sin 2x)}{18} \left[x^2 + \frac{2}{3}(2) + \frac{8i}{3}(2x) - \frac{16}{3}(2)\right]$$

$$= \frac{1}{18}\left[x^2 \cos 2x + \frac{4}{3}\cos 2x - \frac{32}{3}\cos 2x - \frac{16x}{3}\sin 2x\right]$$

$$y_{P_2} = \frac{1}{18}\left[x^2 \cos 2x - \frac{28}{3}\cos 2x - \frac{16}{3} x \cdot \sin 2x\right]$$

particular Integral $y_p = y_{p_1} + y_{p_2}$

$\therefore$ General solution $y = y_c + y_p$

$$y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + \frac{x^2}{2} - 2 + \frac{1}{18}\left[x^2 \cos 2x - \frac{28}{3}\cos 2x - \frac{16}{3} x \cdot \sin 2x\right]$$

General solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q \cdot y = R$ by Method of Variation of parameters:

Procedure:

The given equation $\frac{d^2y}{dx^2} + P(x) \cdot \frac{dy}{dx} + Q(x) \cdot y = R(x)$.
where $P(x), Q(x), R(x)$ are real valued functions of 'x' is called the linear diff equation of second order with variable coefficients

① Reduce the given equation to the standard form. if necessary

② Find the solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q \cdot y = 0$

Let the solution of above eqn = C.F + P.I.

Let $y_c = c_1\, u(x) + c_2 \cdot v(x)$

Take P.I: $y_p = A \cdot u(x) + B \cdot v(x)$

A, B are functions of 'x'

③ Find $W(u,v) = \begin{vmatrix} u & v \\ \frac{du}{dx} & \frac{dv}{dx} \end{vmatrix} = u \cdot \frac{dv}{dx} - v\frac{du}{dx}$

④ Find A, B Using the formulae

$$A = -\int \frac{v(x) \cdot R(x)}{W(u,v)}\, dx \quad ; \quad B = \int \frac{u(x) \cdot R(x)}{W(u,v)}\, dx$$

⑤ The general solution $y = y_c + y_p$

$$\boxed{y = [c_1 \cdot u(x) + c_2 \cdot v(x)] + [A(x) \cdot u(x) + B(x) \cdot v(x)]}$$

c_1, c_2 are constants.

problems :

(1) solve by method of variation of parameters $\frac{d^2y}{dx^2}+y=\operatorname{cosec}x$

Soln: the given diff equation $\frac{d^2y}{dx^2}+y=\operatorname{cosec}x \rightarrow (1)$

i.e $\cancel{(D^2+1)\cdot y = \operatorname{cosec}x}$ $(D^2+1)\cdot y=\operatorname{cosec}x$

i.e $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)\cdot y=R(x)$. form.

$P=0,\ Q=1,\ R=\operatorname{cosec}x$

write the Auxillary eqn $f(m)=0 \Rightarrow m^2+1=0$

$m^2=-1$

$m=\pm i=\alpha\pm i\beta$

Roots are complex, conjugate, $\alpha=0,\ \beta=1$

Complementary fn $\boxed{y_c=c_1\cos x+c_2\sin x}$

Let $y_c=c_1\cdot u(x)+c_2\cdot v(x)$. $\boxed{\begin{array}{l}u(x)=\cos x\\ v(x)=\sin x\end{array}}$

Let $y_p=A\cdot u(x)+B\cdot v(x) \longrightarrow (2)$

$$\therefore A=-\int\frac{v\cdot R}{W(u,v)}dx=-\int\frac{\sin x\cdot\operatorname{cosec}x}{1}dx=-\int dx=-x$$

$$W(u,v)=\begin{vmatrix}u & v\\ \frac{du}{dx} & \frac{dv}{dx}\end{vmatrix}=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=\cos^2x+\sin^2x=1$$

$$B=\int\frac{u\cdot R}{W(u,v)}dx$$

$$=\int\frac{\cos x\cdot\operatorname{cosec}x}{1}dx$$

$$=\int\frac{\cos x}{\sin x}dx=\int\cot x\,dx$$

$$=\log|\sin x|$$

$\therefore$ from (2) $y_p=A\cdot u(x)+B\cdot v(x)$

$y_p=-x\cdot\cos x+\sin x\cdot\log|\sin x|$

$\therefore$ the general solution $y=y_c+y_p$

$$y=c_1\cos x+c_2\sin x-x\cos x+\sin x\cdot\log|\sin x|$$

② solve by method of Variation of Parameters

$(D^2+a^2)\cdot y = \sec ax$

Soln: The given diffal eqn $(D^2+a^2)\cdot y = \sec ax$

i.e $\frac{d^2y}{dx^2} + P(x)\cdot\frac{dy}{dx} + Q(x)\cdot y = R(x)$. form.

$P(x)=0;\ Q(x)=a^2;\ R(x)=\sec ax$

The AUXillary eqn $f(m)=0 \Rightarrow m^2+a^2=0$

$m^2=-a^2$

$m=\pm ia = \alpha \pm i\beta$

The roots are complex, conjugate numbers.

Complementary function $y_c = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x]$

$\boxed{y_c = c_1\cos ax + c_2\sin ax}$

$\alpha=0,\ \beta=a$

Let $y_c = c_1\cdot u(x) + c_2\cdot v(x)$

$P.I = y_p = A\cdot u(x) + B\cdot v(x)$

$\boxed{\begin{array}{l} u(x)=\cos ax \\ v(x)=\sin ax \end{array}}$

$A = -\int \frac{v(x)\cdot R(x)}{W(u,v)}dx = -\int \frac{\sin ax\cdot\sec ax}{a}dx$

$W(u,v) = \begin{vmatrix} u & v \\ \frac{du}{dx} & \frac{dv}{dx}\end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax\end{vmatrix} = a\cos^2 ax + a\sin^2 ax$

$= a(\cos^2 ax+\sin^2 ax)$

$= a(1)$

$= a$

$= -\frac{1}{a}\int \tan ax\,dx$

$= -\frac{1}{a}\left[\frac{-\log|\cos ax|}{a}\right] = \frac{\log|\cos ax|}{a^2}$

$B = \int \frac{u\cdot R}{W(u,v)}dx = \int \frac{\cos ax\cdot\sec ax}{a}dx = \frac{1}{a}\int (1)\,dx = \frac{x}{a}$

$\therefore y_p = A\cdot u(x) + B\cdot v(x) = \frac{\log|\cos ax|}{a^2}\cdot\cos ax + \frac{x}{a}\cdot\sin ax$

$\therefore$ General solution $y = y_c + y_p$

$y = c_1\cdot\cos ax + c_2\sin ax + \frac{\cos ax}{a^2}\log|\cos ax| + \frac{x}{a}\sin ax$

③. solve the differential equation $y''+4y=\tan 2x$ by method of variation of parameters.

Soln: The given diff eqn $y''+4y=\tan 2x$

i.e $(D^2+4)\cdot y=\tan 2x$ | $y''=\frac{d^2y}{dx^2}=D^2y$

It is of the form $\frac{d^2y}{dx^2}+P\cdot\frac{dy}{dx}+Q\cdot y=R(x)$

by method of Variation of Parameters

Here $P=0,\ Q=4,\ R=\tan 2x$.

The Auxillary eqn $f(m)=0 \Rightarrow m^2+4=0$

$m^2=-4 \Rightarrow m=\pm 2i$

The roots are complex, conjugate numbers

Complementary fun $y_c=c_1\cos 2x+c_2\sin 2x$ $\boxed{\begin{matrix}\alpha=0\\ \beta=2\end{matrix}}$

i.e $y_c=c_1\cdot u(x)+c_2\cdot v(x)$.

Let $y_p=A\cdot u(x)+B\,v(x)$. $\boxed{\begin{matrix}u(x)=\cos 2x\\ v(x)=\sin 2x\end{matrix}}$

$$A=-\int\frac{v(x)\cdot R(x)}{W(u,v)}dx=-\int\frac{\sin 2x\cdot\tan 2x}{2}dx=-\frac{1}{2}\int\sin 2x\cdot\frac{\sin 2x}{\cos 2x}dx$$

$$W(u,v)=\begin{vmatrix}u & v\\ \frac{du}{dx} & \frac{dv}{dx}\end{vmatrix}=\begin{vmatrix}\cos 2x & \sin 2x\\ -2\sin 2x & 2\cos 2x\end{vmatrix}=2(\cos^2 2x+\sin^2 2x)=2(1)=2$$

$$=-\frac{1}{2}\int\frac{\sin^2 2x}{\cos 2x}dx$$

$$=-\frac{1}{2}\int\frac{1-\cos^2 2x}{\cos 2x}dx$$

$$=-\frac{1}{2}\left\{\int\frac{1}{\cos 2x}dx-\int\frac{\cos^2 2x}{\cos 2x}dx\right\}$$

$$=-\frac{1}{2}\left\{\int\sec 2x\,dx-\int\cos 2x\,dx\right\}$$

$$=-\frac{1}{2}\left[\frac{1}{2}\log|\sec 2x+\tan 2x|-\frac{\sin 2x}{2}\right]$$

$$A=-\frac{1}{4}\log|\sec 2x+\tan 2x|+\frac{1}{4}\sin 2x$$

$$B = \int \frac{u(x)\cdot R(x)}{W(u,v)}\,dx = \int \frac{\cos 2x\cdot \tan 2x}{2}\,dx$$

$$= \frac{1}{2}\int \cancel{\cos 2x}\cdot \frac{\sin 2x}{\cancel{\cos 2x}}\,dx$$

$$= \frac{1}{2}\int \sin 2x\,dx$$

$$= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)$$

$$B = -\frac{1}{4}\cos 2x$$

$$\therefore\ y_p = A\cdot u(x) + B\cdot v(x)$$

$$y_p = \left[-\frac{1}{4}\log|\sec 2x + \tan 2x| + \frac{\sin 2x}{4}\right]\cdot \cos 2x + \left(-\frac{1}{4}\cos 2x\right)\cdot \sin 2x$$

$\therefore$ the general solution $y = y_c + y_p$

$$y = \left[c_1\cos 2x + c_2\sin 2x\right] + \left[-\frac{1}{4}\log|\sec 2x + \tan 2x| + \frac{\sin 2x}{4}\right]\cos 2x + \left[-\frac{1}{4}\cos 2x\right]\sin 2x.$$

NOTE:

① $$\int e^{ax}\cdot \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

② $$\int e^{ax}\cdot \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

④ Solve by method of Variation of Parameters
$(D^2-2D)\cdot y = e^x\cdot \sin x$

Soln: The given diff eqn $(D^2-2D)\cdot y = e^x\cdot \sin x$

It of the form $\frac{d^2y}{dx^2}+P\cdot\frac{dy}{dx}+Q\cdot y = R(x)$. by Method of variation of parameters

Given $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x\cdot\sin x$

i.e $P=-2,\ Q=0,\ R(x) = e^x\cdot \sin x$.

The Auxillary eqn $f(m)=0 \Rightarrow m^2-2m=0$

$m(m-2)=0$

i.e $m=0,\ m-2=0$
$m=2$

∴ The roots are real and different.

Complementary function $y_c = c_1 e^{(0)x} + c_2 e^{2x}$

$\boxed{y_c = c_1(1) + c_2 e^{2x}}$ $\quad e^0 = 1$

Let $y_c = c_1\cdot u(x) + c_2\cdot v(x)$ $\quad$ ∴ $\boxed{\begin{array}{l} u(x)=1 \\ v(x)=e^{2x}\end{array}}$

Let particular integral

$P.I = y_p = A\cdot u(x) + B\cdot v(x)$

$A = -\int \frac{v(x)\cdot R(x)}{W(u,v)}dx = -\int \frac{\cancel{e^{2x}}\cdot(e^x \sin x)}{2\cancel{e^{2x}}}dx$

$W(u,v) = \begin{vmatrix} u & v \\ \frac{du}{dx} & \frac{dv}{dx}\end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x}\end{vmatrix} = 2e^{2x} - 0 = 2e^{2x}$

$= -\frac{1}{2}\int e^x \sin x\, dx$

$\int e^{ax}\cdot \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

∵ $a=1$, $b=1$

$= -\frac{1}{2}\left[\frac{e^x}{(1)^2+(1)^2}\left((1)\sin x - (1)\cos x\right)\right]$

$= -\frac{1}{2}\frac{e^x}{2}(\sin x - \cos x)$

$\boxed{A = -\frac{e^x}{4}(\sin x - \cos x)}$

$$B = \int \frac{u(x)\cdot R(x)}{W(u,v)}\,dx = \int \frac{(1)\cdot e^{x}\sin x}{2e^{2x}}\,dx$$

$$= \frac{1}{2}\int e^{x-2x}\cdot \sin x\,dx$$

$$= \frac{1}{2}\int e^{-x}\cdot \sin x\,dx \qquad \boxed{\begin{matrix} a=-1 \\ b=1 \end{matrix}}$$

by $\int e^{ax}\cdot\sin bx\,dx$ formula

$$= \frac{1}{2}\left[\frac{e^{-x}}{(-1)^2+1^2}\left[(-1)\sin(1)x-(1)\cos(1)x\right]\right]$$

$$= \frac{1}{2}\cdot\frac{e^{-x}}{2}(-\sin x-\cos x)$$

$$\boxed{B = -\frac{e^{-x}}{4}(\sin x+\cos x)}$$

$$\therefore \text{P.I} = y_p = A\cdot u(x) + B\cdot v(x)$$

$$= \left[-\frac{e^{x}}{4}(\sin x-\cos x)\right](1) + \left[-\frac{e^{-x}}{4}(\sin x+\cos x)\right]\cdot e^{2x}$$

$$= -\frac{e^{x}}{4}(\sin x-\cos x) - \frac{e^{x}}{4}(\sin x+\cos x)$$

$$= -\frac{e^{x}}{4}\left[\sin x-\cancel{\cos x}+\sin x+\cancel{\cos x}\right]$$

$$= -\frac{e^{x}}{4}(2\sin x)$$

$$y_p = -\frac{e^{x}}{2}\sin x$$

$\therefore$ General soln $y = y_c + y_p \Rightarrow y = c_1 + c_2 e^{2x} - \frac{e^{x}}{2}\sin x$

$\Rightarrow$ 5 solve $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^{x}}$ by method of variation of parameters

soln: Given $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^{x}}$ is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q\cdot y = R$ form

by method of variation of parameters $P=0,\ Q=-1$

$R = \frac{2}{1+e^{x}}$

A.E $f(m)=0 \Rightarrow (D^2-1)\cdot y = 0 \Rightarrow m^2-1=0$

$m^2=1 \Rightarrow m=\pm 1$

Let $\boxed{y_c = c_1 e^{x} + c_2 e^{-x}} = \boxed{c_1\cdot u(x) + c_2\cdot v(x)}$

Let $P.I = y_p = A\cdot u(x) + B\cdot v(x)$

where $A = -\int \frac{v(x)\cdot R(x)}{W(u,v)} = -\int \frac{e^{-x}\cdot\frac{2}{1+e^x}\,dx}{-2}$

$$W(u,v) = \begin{vmatrix} u & v \\ \frac{du}{dx} & \frac{dv}{dx} \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = e^{x}(-e^{-x}) - e^{-x}\cdot e^{x}$$

$$= -e^{0} - e^{0}$$

$$= -1-1 = -2$$

$$= +\int \frac{e^{-x}}{1+e^{x}}\,dx$$

$$= +\int \frac{1}{e^{x}(1+e^{x})}\,dx \quad \text{(partial fractions)}$$

$$= +\int \left[\frac{1}{e^{x}} - \frac{1}{1+e^{x}}\right] dx$$

$$= +\int e^{-x}\,dx + \int \frac{1}{1+e^{x}}\,dx$$

$$= +\left(\frac{e^{-x}}{-1}\right) + \log(1+e^{-x})$$

$$\boxed{A = -e^{-x} + \log(1+e^{x})}$$

$$\int \frac{1}{1+e^{x}}\,dx = \int \frac{1}{1+\frac{1}{e^{x}}}\,dx$$

$$= \int \frac{e^{-x}}{e^{-x}+1}\,dx$$

$$= -\log(1+e^{-x})$$

$$\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$$

$$B = \int \frac{u(x)\cdot R(x)}{W(u,v)}\,dx = \int \frac{e^{x}\cdot\left(\frac{2}{1+e^{x}}\right)}{-2}\,dx = -\int \frac{e^{x}}{1+e^{x}}\,dx$$

put $1+e^{x} = t$, $0+e^{x}\,dx = dt$

$$= -\int \frac{dt}{t}$$

$$= -\log t$$

$$\boxed{B = -\log(1+e^{x})}$$

$$y_p = A\cdot u(x) + B\cdot v(x) = \left[-e^{-x} + \log(1+e^{-x})\right]\cdot e^{x} + \left[-\log(1+e^{x})\right]\cdot e^{-x}$$

$$= -1 + e^{x}\log(1+e^{-x}) - e^{-x}\cdot\log(1+e^{x})$$

$\therefore$ General Solution $y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{-x} - 1 - e^{x}\cdot\log(1+e^{-x}) - e^{-x}\log(1+e^{x})$$

(6) Solve $(D^2-2D+2)\cdot y = e^x \tan x$ by Method of Variation of Parameters.

<u>Soln</u>: The given diff eqn $(D^2-2D+2)\cdot y = e^x \tan x \rightarrow (1)$

By Method of Variation of Parameters it is of the form

$$\frac{d^2y}{dx^2} + P(x)\cdot\frac{dy}{dx} + Q(x)\cdot y = R(x)$$

$$P=-2,\quad Q(x)=2,\quad R(x)=e^x \tan x$$

$Dy = \frac{dy}{dx}$

$D^2y = \frac{d^2y}{dx^2}$

The general solution of (1) is $y = y_c + y_p$

<u>To find y_c</u>: $(D^2-2D+2)\cdot y = 0$

$$f(D)\cdot y = 0$$

The Auxillary equation $f(m)=0 \Rightarrow m^2-2m+2=0$

$$m = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)}$$

$$= \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm i2}{2} = 1\pm i$$

$$= \alpha \pm i\beta$$

$\therefore$ Roots are Complex, Conjugate numbers.

Complementary fun $y_c = e^{\alpha x}[c_1\cos\beta x + c_2 \sin\beta x]$

$\alpha=\beta=1$

$$y_c = e^x[c_1\cos x + c_2\sin x]$$

$$\boxed{y_c = c_1(e^x\cdot\cos x) + c_2(e^x\cdot\sin x)} \rightarrow (2)$$

Let $y_c = c_1\cdot u(x) + c_2\cdot v(x)$

$$\boxed{\begin{aligned} u(x) &= e^x\cos x \\ v(x) &= e^x\sin x \end{aligned}}$$

Let particular integral (P.I): $y_p = A\cdot u(x) + B\cdot v(x) \rightarrow (3)$

$$W(u,v) = \begin{vmatrix} u & v \\ \frac{du}{dx} & \frac{dv}{dx} \end{vmatrix} = \begin{vmatrix} e^x\cos x & e^x\sin x \\ \frac{d}{dx}(e^x\cos x) & \frac{d}{dx}(e^x\sin x) \end{vmatrix}$$

$$\frac{d}{dx}(e^x\cos x) = e^x(-\sin x) + \cos x\cdot e^x = e^x(\cos x - \sin x)$$

$$\frac{d}{dx}(e^x\sin x) = e^x(\cos x) + \sin x\cdot e^x = e^x(\cos x + \sin x)$$

$$W(u,v)=\begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x(\cos x-\sin x) & e^x(\cos x+\sin x)\end{vmatrix} = e^{2x}\left[(\cos^2 x+\cancel{\cos x\cdot\sin x}) - (\cancel{\cos x\sin x}-\sin^2 x)\right]$$

$$= e^{2x}(\cos^2 x+\sin^2 x)$$

$$= e^{2x}(1) = e^{2x}$$

$$\therefore \quad A = -\int \frac{v(x)\cdot R(x)}{W(u,v)}dx = -\int \frac{e^x\cdot\sin x\cdot e^x.\tan x}{e^{2x}}dx$$

$$= -\int \frac{\cancel{e^{2x}}\cdot\sin x\cdot\frac{\sin x}{\cos x}}{\cancel{e^{2x}}}dx$$

$$= -\int \frac{\sin^2 x}{\cos x}dx = -\int\frac{1-\cos^2 x}{\cos x}dx$$

$$= -\int\frac{1\,dx}{\cos x} - \int\frac{\cos^2 x\,dx}{\cos x}$$

$$= -\int \sec x\,dx - \int\cos x\,dx$$

$$= -\left[\log(\sec x+\tan x)\right]-(\sin x)$$

$$\boxed{A = -\left[\log(\sec x+\tan x)+\sin x\right]}$$

$$B = \int\frac{u(x)\cdot R(x)}{W(u,v)}dx = \int\frac{e^x\cos x\cdot e^x\tan x}{e^{2x}}dx = \int\frac{\cancel{e^{2x}}\cdot\cancel{\cos x}\cdot\frac{\sin x}{\cancel{\cos x}}}{\cancel{e^{2x}}}dx$$

$$= \int\sin x\,dx$$

$$= -\cos x$$

$\therefore$ Particular integral $y_p = A\cdot u(x)+B\cdot v(x)$

$$y_p = -\left[\log(\sec x+\tan x)+\sin x\right]\cdot e^x\cos x + (-\cos x)e^x\sin x$$

$$= -e^x\cos x\,\log(\sec x+\tan x) - e^x\cos x\,\sin x - e^x\cos x\cdot\sin x$$

$$= -e^x\cos x\left[\log(\sec x+\tan x)+2\sin x\right]$$

$\therefore$ General soln $y = y_c + y_p$

$$y = \left[c_1\cdot u(x)+c_2\cdot v(x)\right] + \left[A\cdot u(x)+B\cdot v(x)\right]$$

$$y = c_1(e^x\cos x)+c_2(e^x\sin x) - e^x\cos x\left[\log(\sec x+\tan x)+2\sin x\right]$$

Simultaneous linear differential equations:

Solve: $\frac{dx}{dt} = x-2y$; $\frac{dy}{dt} = 5x+3y$

Soln: The given simultaneous diff eqns

$$\frac{dx}{dt} = x-2y \quad \text{and} \quad \frac{dy}{dt} = 5x+3y$$

$\because \frac{d}{dt} = D$

$$Dx - x + 2y = 0 \qquad Dy - 5x - 3y = 0$$

$$(D-1)x + 2y = 0 \qquad -5x + (D-3)\cdot y = 0$$

$$5x - (D-3)\cdot y = 0$$

$$\therefore \quad (D-1)\cdot x + 2y = 0 \longrightarrow ①$$

$$5x - (D-3)\cdot y = 0 \longrightarrow ②$$

To eliminate 'y' Multiply eqn ① by $(D-3)$ and equation ② by '2'.

$$(D-3)\cdot(D-1)x + \cancel{2(D-3)\cdot y} = 0$$

$$10x \qquad - \cancel{2(D-3)\cdot y} = 0$$

Adding

$$(D-3)(D-1)x + 10x = 0$$

$$[(D-3)(D-1) + 10]x = 0$$

$$(D^2 - 4D + 13)\cdot x = 0$$

i.e $f(D)\cdot x = 0$

The Auxillary eqn $f(m) = 0 \Rightarrow m^2 - 4m + 13 = 0$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm i6}{2} = 2 \pm i3$$

$$= \alpha \pm i\beta$$

Roots are Complex, Congugate numbers.

$$\therefore \quad x = e^{\alpha t}[c_1 \cos\beta t + c_2 \sin\beta t]$$

$$x = e^{2t}[c_1 \cos 3t + c_2 \sin 3t] \longrightarrow ③$$

Now $\frac{dx}{dt} = e^{2t}\left[c_1(-3\sin 3t) + c_2(3\cos 3t)\right]$
$+ (c_1\cos 3t + c_2\sin 3t).\, 2e^{2t}$

$= e^{2t}\cos 3t(2c_1+3c_2) + e^{2t}\sin 3t(2c_2-3c_1)$

Substitute 'x' and $\frac{dx}{dt}$ values in eqn ① we get

from ①
$$(D-1)\cdot x + 2y = 0$$
$$\frac{dx}{dt} - x + 2y = 0$$
$$2y = x - \frac{dx}{dt}$$

$\therefore 2y = e^{2t}\left[c_1\cos 3t + c_2\sin 3t\right] - \left[(2c_1+3c_2)e^{2t}\cos 3t + (2c_2+3c_1)e^{2t}\sin 3t\right.$

$= e^{2t}\cos 3t(c_1 - 2c_1 - 3c_2) + e^{2t}\sin 3t(c_2 - 2c_2 - 3c_1)$

$= e^{2t}\left[(-c_1-3c_2)\cos 3t + (-3c_1-c_2)\sin 3t\right]$

$\therefore y = -\frac{e^{2t}}{2}\left[(c_1+3c_2)\cos 3t + (3c_1+c_2)\sin 3t\right]$

Alternative Method

Eliminate 'x',
$$(D-1)x + 2y = 0 \rightarrow ①$$
$$5x + (D-3)\cdot y = 0 \rightarrow ②$$
Given

multiply eqn ① by '5' and eqn ② by $(D-1)$

$$\cancel{5(D-1)x} + 10y = 0$$
$$\cancel{5(D-1)\cdot x} \underset{(+)}{-} (D-1)(D-3)\cdot y = 0$$

Subtracting

$$10y + (D-1)(D-3)\cdot y = 0$$
$$\left[10 + (D^2-4D+3)\right]\cdot y = 0$$
$$(D^2-4D+13)\cdot y = 0$$

Auxillary eqn $f(m) = 0 \Rightarrow m^2-4m+13 = 0$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2-4(1)(13)}}{2(1)} = \frac{4\pm\sqrt{-36}}{2} = \frac{4\pm i6}{2} = 2\pm i(3)$$

$\therefore$ General solution $\boxed{y = y_c = e^{2t}\left[c_1\cos 3t + c_2\sin 3t\right]}$ $\because \boxed{y_p = 0}$

(2). Solve: $\frac{dx}{dt}+5x-2y=t$ and $\frac{dy}{dt}+2x+y=0$

Soln: The given equations can be written as

$$\frac{dx}{dt}+5x-2y=t \quad ; \quad \frac{dy}{dt}+2x+y=0 \qquad \frac{d}{dt}=D$$

$$(D+5)x-2y=t \rightarrow (1) \quad ; \quad 2x+(D+1)y=0 \rightarrow (2)$$

To eliminate 'x', Multiply eqn (1) by '2' and eqn (2) by (D+5) we get

$$\cancel{2(D+5)x}-4y=2t \rightarrow (3)$$

$$\cancel{2(D+5)x}+(D+1)(D+5)\cdot y=0 \rightarrow (4)$$

subtracting $(-) \quad (-)$

$$-4y-\left[(D+1)\cdot(D+5)\right]\cdot y=2t$$

$$-4y-(D^2+6D+5)\cdot y=2t$$

$$-[4+(D^2+6D+5)]\cdot y=2t$$

$$(D^2+6D+9)\cdot y=-2t \rightarrow (5)$$

This is a diff eqn with const coefficients $f(D)\cdot y=Q(t)$.

The general solution $y=y_c+y_p$.

To find y_c: $f(D)\cdot y=0$

$$(D^2+6D+9)\cdot y=0$$

The Auxillary eqn $f(m)=0 \Rightarrow m^2+6m+9=0$

$$(m+3)^2=0$$

$$(m+3)(m+3)=0$$

$$m=-3,-3$$

$\therefore$ The roots are real and equal.

Complementary fun $\boxed{y_c=(C_1+C_2t)e^{-3t}}$

To find y_p:

$$y_p=\frac{1}{f(D)}\cdot Q(t)$$

$$=\frac{1}{D^2+6D+9}\cdot(-2t) \quad (\text{polynomial in } t' = t^k)$$

$$y_p = \frac{1}{(D+3)^2}(-2t)$$

$$= -2 \cdot \frac{1}{(D+3)^2} \cdot t$$

Take out the lowest degree term ~~from~~ from. $f(D)$.

$$= -2 \cdot \frac{1}{(3)^2\left[1+\frac{D}{3}\right]^2} \cdot t$$

$$= -\frac{2}{9}\left[1+\left(\frac{D}{3}\right)\right]^{-2} \cdot t$$

$$= -\frac{2}{9}\left[1-2\left(\frac{D}{3}\right)+\cdots\right] \cdot t$$

$$\because (1+D)^{-2} = 1-2D+3D^2-4D^3+\cdots\cdots$$

$$= -\frac{2}{9}\left[t-\frac{2}{3}D(t)+\cdots\right]$$

$$= -\frac{2}{9}\left[t-\frac{2}{3}(1)\right]$$

$$P.I = y_p = -\frac{2t}{9}+\frac{4}{27}$$

$\therefore$ The general solution $y = y_c + y_p$

$$\boxed{y = (C_1+C_2 t)e^{-3t} - \frac{2t}{9}+\frac{4}{27}} \longrightarrow (6)$$

$\therefore$ substituting the value of 'y' in eqn (2)

$$2x+(D+1)y = 0$$

$$2x = -(D+1)y$$

$$x = -\frac{1}{2}(Dy+y)$$

$$x = -\frac{1}{2}\left[\frac{d}{dt}\left[(C_1+C_2 t)e^{-3t}-\frac{2t}{9}+\frac{4}{27}\right]+\left[(C_1+C_2 t)e^{-3t}-\frac{2t}{9}+\frac{4}{27}\right]\right]$$

$$= -\frac{1}{2}\left[\left((C_1+C_2 t)(-3e^{-3t})+e^{-3t}(0+C_2(1))-\frac{2}{9}(1)+0\right)\right.$$
$$\left.+ e^{-3t}(C_1+C_2 t) - \frac{2t}{9}+\frac{4}{27}\right]$$

General solution.

$$\boxed{x = e^{-3t}\left[C_1-\frac{C_2}{2}+C_2 t\right]+\frac{t}{9}+\frac{1}{27}} \longrightarrow (7)$$

$\therefore$ The general solution is given by (6) and (7) equation

③ Solve: $Dx = 3x+4y$; $Dy = 4x-3y$ with $x(0)=1$, $y(0)=3$; $D = \frac{d}{dt}$

Soln: The given System of Linear diff equations

$$Dx = 3x+4y \; ; \quad Dy = 4x-3y$$

i.e
$$(D-3)x - 4y = 0 \rightarrow ①$$
$$4x - (D+3)\cdot y = 0 \rightarrow ②$$

To find x, eliminate 'y' from ① & ②, by operating on both sides of ① by (D+3) and eqn ② by '4' we get

$$(D+3)(D-3)\cdot x - 4\cancel{(D+3)}\cdot y = 0$$
$$16x - 4\cancel{(D+3)}\cdot y = 0$$
$$(-) \qquad (+)$$
subtracting

$$(D+3)(D-3)x - 16x = 0$$
$$(D^2-9)x - 16x = 0$$
$$(D^2-9-16)x = 0$$
$$(D^2-25)\cdot x = 0$$

The Auxillary eqn $f(m)=0 \Rightarrow m^2-25=0$
$$m^2 = 25$$
$$m = \pm 5$$

Roots are real & different.

$$\boxed{\therefore\ x = c_1 e^{5t} + c_2 e^{-5t}} \rightarrow ③$$

Now $\frac{dx}{dt} = c_1(5e^{5t}) + c_2(-5e^{-5t})$

$$\frac{dx}{dt} = 5c_1e^{5t} - 5c_2e^{-5t}$$

Substitute 'x' and '$\frac{dx}{dt}$' in eqn ①

$$(D-3)x - 4y = 0$$
$$\therefore 4y = (D-3)x$$
$$4y = Dx - 3x$$

$$\therefore\ 4y = \frac{d}{dt}(x) - 3x$$

$$4y = (5c_1 e^{5t} - 5c_2 e^{-5t}) - 3(c_1 e^{5t} + c_2 e^{-5t})$$

$$4y = 2c_1 e^{5t} - 8c_2 e^{-5t}$$

$$\therefore\ \boxed{y = \frac{c_1}{2} e^{5t} - 2c_2 e^{-5t}} \rightarrow (4)$$

Given initial condition $x(0)=1$; $y(0)=3$

To eliminate consts (c_1, c_2)

from (3) $\quad x = c_1 e^{5t} + c_2 e^{-5t}$ $\qquad x(0) = 1$ when $t=0$

$x(t) = 1 \qquad x = 1$

$$1 = c_1 e^{(0)} + c_2 e^{-(0)}$$

$$\boxed{1 = c_1 + c_2} \longrightarrow (5)$$

from (4) $\quad y = \frac{c_1}{2} e^{5t} - 2c_2 e^{-5t}$ $\qquad y(0) = 3$

$y(t) = 3$

when $t = 0 \Rightarrow y = 3$

$$3 = \frac{c_1}{2} e^{(0)} - 2c_2 e^{-(0)}$$

$$3 = \frac{c_1}{2} - 2c_2$$

$$3 = \frac{c_1 - 4c_2}{2} \Rightarrow \boxed{c_1 - 4c_2 = 6} \longrightarrow (6)$$

from (5) $\quad c_1 + c_2 = 1 \Rightarrow \boxed{c_1 = 1 - c_2}$

from (6) $\quad c_1 - 4c_2 = 6$

$$(1 - c_2) - 4c_2 = 6$$

$$1 - 5c_2 = 6$$

$$-5c_2 = 5 \Rightarrow \boxed{c_2 = -1}$$

$c_1 = 1 - c_2$

$= 1 - (-1)$

$\boxed{c_1 = 2}$

Substitute $\boxed{c_1 = 2}$ and $\boxed{c_2 = -1}$ in eqns (3) & (4)

$x = c_1 e^{5t} + c_2 e^{-5t} \qquad y = \frac{c_1}{2} e^{5t} - 2c_2 e^{-5t}$

$x = 2e^{5t} - e^{-5t}$; $\quad y = e^{5t} + 2e^{-5t}$ is the solution.

④ Solve: $\frac{dx}{dt}+2y=\sin 2t$; $\frac{dy}{dt}-2x=\cos 2t$

Soln: The given simultaneous linear diff eqns

$\frac{dx}{dt}+2y=\sin 2t$ $\quad$ $\frac{dy}{dt}-2x=\cos 2t$ $\quad$ $\frac{d}{dt}=D$

$Dx+2y=\sin 2t \rightarrow ①$ $\quad$ $Dy-2x=\cos 2t \rightarrow ②$

To eliminate 'x', Multiply eqn ① by '2' and eqn ② by 'D'

$$2\cancel{Dx}+4y = 2\sin 2t$$
$$-\cancel{2Dx}+D^2y = D(\cos 2t)$$

Add

$$4y+D^2y = 2\sin 2t + D(\cos 2t)$$
$$(D^2+4)\cdot y = 2\sin 2t + (-2\sin 2t)$$
$$(D^2+4)\cdot y = 0 \rightarrow ③$$

The general solution of ③ is only y_c, $y_p=0$

The Auxillary eqn $f(m)=0 \Rightarrow m^2+4=0$

$$m^2=-4$$
$$m=\pm 2i$$

Roots are complex, conjugate numbers.

G.S of eqn ② $\boxed{y=y_c=c_1\cos 2t+c_2\sin 2t} \rightarrow ④$

Substitute the value of 'y' in eqn ② we get

$$Dy-2x=\cos 2t$$
$$2x = Dy-\cos 2t$$
$$= \frac{d}{dt}(c_1\cos 2t+c_2\sin 2t)-\cos 2t$$
$$2x = c_1(-2\sin 2t)+c_2(2\cos 2t)-\cos 2t$$
$$x = \frac{1}{2}\left[-2c_1\sin 2t+(2c_2-1)\cos 2t\right]$$

General soln of ①

$$\boxed{x = -c_1\sin 2t+\left(c_2-\tfrac{1}{2}\right)\cos 2t} \rightarrow ⑤$$

Equations ④ & ⑤ are the general solutions of given equations ② & ① respectively.

Simultaneous linear equations:

Solve: $\frac{dx}{dt} = x-2y$; $\frac{dy}{dt} = 5x+3y$

Soln: The given simultaneous linear diff eqns

$\frac{dx}{dt} = x-2y$; $\frac{dy}{dt} = 5x+3y$

$Dx = x-2y$ $Dy = 5x+3y$ write $D = \frac{d}{dt}$

$(D-1)x+2y=0$ $5x+(3-D)y=0$

$\therefore$ $(D-1)x+2y=0 \rightarrow (1)$

$5x+(3-D)\cdot y=0 \rightarrow (2)$

To eliminate 'y' Multiply eqn (1) by $(3-D)$ and eqn (2) by '2'

Solve the following linear diff equations

(1) $\frac{dx}{dt}+5x-2y=t$; $\frac{dy}{dt}+2x+y=0$

(2) $Dx+y = \cos t-\sin t$

$Dy+x = \cos t+\sin t$

(3) $\frac{dx}{dt} = 2x+3y$

$3\frac{dy}{dt} = x+6y$

(4) $Dx = 3x+4y$; $Dy = 4x-3y$ with $x(0)=1$, $y(0)=3$

(5) $(D+5)x+(D+3)y = e^{-t}$

$(2D+1)x+(D+1)y = 3$.

Applications of second order Linear differential equations:

L-C-R Circuit:

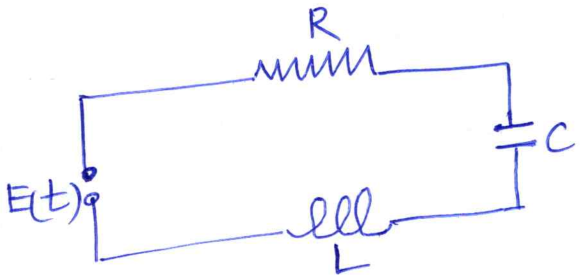
L-C-R circuits have many applications as oscillator circuits such as radio receiver and T.V sets. It is also used as a band pass filter, Low pass filter (or) high pass filter.

The LCR circuit flow is described as a second order circuit which can be described as a second order ordinary differential equation.

Consider the discharge of a condenser 'C' through an induction 'L' ~~E(t)~~ and resistance R. Since the Voltage drop across L, C and R are $L\frac{di}{dt} = L\frac{d^2q}{dt^2}$, $\frac{q}{c}$ and

$Ri = R\frac{dq}{dt}$ $\quad \boxed{\because i = \frac{dq}{dt}}$

(The algebric sum of all voltage drops around a closed circuit is zero)



Case(i) Voltage source E = const

i.e $E = E_0$ (or) $E_0 \cos wt$ (or) $E_0 \sin wt$

By Kirchhoff's Law

$$\boxed{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E} \longrightarrow ①$$

$\because E = E_0$ (or) $E_0 \sin wt$ (or) $E_0 \cos wt$

The general solution $\boxed{q = q_c + q_p}$

i.e $q = C.F + P.I$

Eqn ① is called the linear differential eqn of second order with constant coefficients. We can write i.e

$$\boxed{\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E}{L}}$$

Case ②: The circuit without source i.e $E=0$

Now eqn ① becomes,

$$L\frac{d^2q}{dt^2}+R\cdot\frac{dq}{dt}+\frac{q}{c}=0$$

on dividing by 'L'

$$\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=0$$

Take $\frac{R}{L}=2\lambda$; $\frac{1}{LC}=\mu^2$ we get

$$\frac{d^2q}{dt^2}+2\lambda\frac{dq}{dt}+\mu^2\cdot q=0$$

(or) $(D^2+2\lambda D+\mu^2)q=0 \rightarrow ②$ $\quad \frac{d}{dt}=D;\ \frac{d^2}{dt^2}=D^2$

The general solution of the eqn ② is only ~~Comple~~ Complementary function. $q_p=0\ (P.I=0)$

To find C.F: the Auxillary equation $f(m)=0$

$$m^2+2\lambda m+\mu^2=0 \qquad a=1,\ b=2\lambda,\ c=\mu^2$$

$$\therefore\ m=\frac{-(2\lambda)\pm\sqrt{(2\lambda)^2-4(1)(\mu^2)}}{2(1)}$$

$$=\frac{-2\lambda\pm\sqrt{4\lambda^2-4\mu^2}}{2}=\frac{-2\lambda\pm2\sqrt{\lambda^2-\mu^2}}{2}=-\lambda\pm\sqrt{\lambda^2-\mu^2}$$

i.e $\boxed{m=-\lambda\pm\sqrt{\lambda^2-\mu^2}}$, Now Three cases arises

Case(i) when $\lambda>\mu$ roots are real and distinct (m_1, m_2)

The solution is $q=c_1e^{m_1t}+c_2e^{m_2t}$

Case(ii) when $\lambda=\mu$ the roots are real and equal

the solution is $q=(c_1+c_2t)e^{-\lambda t}$

Case(iii) when $\lambda<\mu$ the roots are complex conjugate

say $-\lambda\pm i(\alpha)$

The solution is $q=e^{-\lambda t}(c_1\cos\alpha t+c_2\sin\alpha t)$

Problems:

① Consider an electrical circuit containing an inductance 'L' Resistance 'R' and capacitance 'C'. Let 'q' be the electrical charge on the condenser plate and 'i' be the current in the circuit at any time. Given that $L = 0.25$ henries, $R = 250$ ohms, $C = 2\times10^{-6}$ farads and there is no applied E.M.F in the circuit. At time zero the current is zero and the charge is 0.002 coulomb. Then find the charge (q) and current (i) at any time.

Soln: The given differential equation containing L, R, C is

$$L\frac{d^2q}{dt^2} + R\cdot\frac{dq}{dt} + \frac{q}{C} = 0 \quad \text{(or)} \quad \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0 \longrightarrow ①$$

Given $L = 0.25$; $R = 250$ ohms $C = 2\times10^{-6}$ farads

$\therefore$ from ①

$$\frac{d^2q}{dt^2} + \frac{250}{0.25}\frac{dq}{dt} + \frac{q}{0.25\times2\times10^{-6}} = 0$$

$$\frac{d^2q}{dt^2} + 1000\frac{dq}{dt} + (2\times10^{6})q = 0$$

(or) $\left[D^2 + 1000D + (2\times10^{6})\right]q = 0$

The Auxillary eqn $f(m) = 0 \Rightarrow m^2 + 1000m + 2\times10^6 = 0$

$$m = \frac{-1000 \pm \sqrt{10^6 - 4(1)(2\times10^6)}}{2(1)} \quad \Big| \ (1000)^2 = 10^6$$

$$= \frac{-1000 \pm \sqrt{10^6(1-8)}}{2} = \frac{-1000 \pm 1000\sqrt{7}\,i}{2}$$

$$= -500 \pm i(1323)$$

$\therefore$ the general solution $q = e^{\alpha t}[c_1\cos\beta t + c_2\sin\beta t]$

$$\therefore q = e^{-500t}[c_1\cos(1323)t + c_2\sin(1323)t] \longrightarrow ②$$

Given time $t=0$, charge $q = 0.002$

i.e $q = e^{-500t}[c_1 \cos(1323)t + c_2 \sin(1323)t]$

$0.002 = e^{(0)}[c_1 \cos(0) + c_2 \sin(0)]$

$\boxed{c_1 = 0.002}$

$\sin(0) = 0$
$\cos(0) = 1$

Now to find $\frac{dq}{dt} = i$ (current)

$q = e^{-500t}[c_1 \cos(1323)t + c_2 \sin(1323)t]$

$$\frac{dq}{dt} = e^{-500t}\left[c_1\left(-1323 \times \sin(1323)t\right) + c_2\left(1323 \times \cos(1323)t\right)\right] + \left(c_1 \cos(1323)t + c_2 \sin(1323)t\right)\left(-500 e^{-500t}\right) \rightarrow (3)$$

At time $t=0$, $\frac{dq}{dt} = 0$

$0 = e^{(0)}\left[\left(0 + 1323 c_2 \cos(0)\right)\right] + \left(c_1 \cos(0) + 0\right)\left(-500 e^{(0)}\right)$

$0 = 1323 c_2 + c_1(-500)$

$1323 c_2 = 500 c_1$
$= 500(0.002)$

$1323 c_2 = 1$

$c_2 = \frac{1}{1323} = 0.0008$

$\therefore \boxed{c_2 = 0.0008}$

from (2)
$\therefore$ The charge $(q) = e^{-500t}[(0.002)\cos(1323)t + (0.0008)\sin(1323)t$

from (3)
Current $(i) = \frac{dq}{dt} =$ substitute $c_1 = 0.002$, $c_2 = 0.0008$ write eqn (3)

② An Uncharged Condenser of capacity 'c' is charged by applying an e.m.f $E.\sin\left(\frac{t}{\sqrt{LC}}\right)$ through leads of self inductance 'L' and negligible resistance. prove that at any time 't' the charge on one of the plates is $\frac{EC}{2}\left\{\sin\frac{t}{(LC)} - \frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right\}$.

Soln: If 'q' is the charge on the Condenser, the diffal eqn of circuit is $L\frac{d^2q}{dt^2} + R\cdot\frac{dq}{dt} + \frac{q}{c} = E.$

$$L\frac{d^2q}{dt^2} + 0 + \frac{q}{c} = E.\sin\frac{t}{\sqrt{LC}}$$

$$\frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{E}{L}\sin\frac{t}{\sqrt{LC}} \longrightarrow (1)$$

$$\boxed{\left(D^2+\frac{1}{LC}\right)q = \frac{E}{L}\sin\frac{t}{\sqrt{LC}}}$$

The general solution of eqn (1) $q = C.F + P.I.$

To find C.F : q_c : $f(D).q = 0$

$$\left(D^2+\frac{1}{LC}\right)q = 0$$

The Auxillary equation $f(m)=0 \Rightarrow m^2+\frac{1}{LC}=0$

$$m^2 = -\frac{1}{LC}$$

$$m = \pm i\frac{1}{\sqrt{LC}} \qquad \boxed{\alpha=0,\ \beta=\frac{1}{\sqrt{LC}}}$$

Roots are complex, Conjugate numbers

Complementary fun $q_c = c_1\cos\frac{1}{\sqrt{LC}}t + c_2\sin\frac{1}{\sqrt{LC}}t$

To find P.I : q_p : $\frac{1}{f(D)}\cdot Q(t)$ $\qquad Q(t) = \frac{E}{L}.\sin\frac{t}{\sqrt{LC}}$

$$\frac{1}{D^2+\frac{1}{LC}}\cdot\frac{E}{L}.\sin\frac{t}{\sqrt{LC}}$$

put $\boxed{D^2 = -b^2 = -\left(\frac{1}{\sqrt{LC}}\right)^2 = -\frac{1}{LC}}$

$\boxed{f(D^2) = f(-b^2) = 0}$ fail.

$$= \frac{t}{f'(D)}\cdot Q(t) \Rightarrow \frac{t}{2D}\cdot\frac{E}{L}\cdot\sin\frac{t}{\sqrt{LC}}$$

$$= \frac{t}{2} \cdot \frac{E}{L} \frac{1}{D} \sin\left(\frac{\cancel{t}}{\sqrt{LC}}\right) . t$$

$$= \frac{1}{2} \frac{Et}{L} \int \sin \frac{1}{\sqrt{LC}} t \, dt$$

$$= \frac{1}{2} \frac{Et}{L} \frac{\left(-\cos \frac{1}{\sqrt{LC}} t\right).}{\frac{1}{\sqrt{LC}}}$$

$$= -\frac{Et}{2L} \times \sqrt{LC} \cos \frac{1}{\sqrt{LC}} t .$$

$$q_p = -\frac{Et}{2} \cdot \sqrt{\frac{C}{L}} \cdot \cos\left(\frac{t}{\sqrt{LC}}\right)$$

$\therefore$ The general soln $q = q_c + q_p$

$$q = c_1 \cos \frac{1}{\sqrt{LC}} t + c_2 \sin \frac{1}{\sqrt{LC}} t - \frac{Et}{2} \sqrt{\frac{C}{L}} \cos \frac{t}{\sqrt{LC}} .$$

when time $t=0$, Charge$(q)=0$, $0 = c_1(1) + 0 - 0$

$$\boxed{c_1 = 0}$$

$$\therefore q = c_2 \sin \frac{1}{\sqrt{LC}} t - \frac{Et}{2} \sqrt{\frac{C}{L}} \cos \frac{t}{\sqrt{LC}} \longrightarrow (2)$$

Now $\frac{dq}{dt} = c_2 \cdot \frac{1}{\sqrt{LC}} \cos \frac{1}{\sqrt{LC}} t - \frac{E}{2} \sqrt{\frac{C}{L}} \left\{ t\left(-\sin \frac{t}{\sqrt{LC}} \cdot \frac{1}{\sqrt{LC}}\right) + \cos \frac{t}{\sqrt{LC}} \cdot (1) \right\}$

when time $t=0$, $\frac{dq}{dt} = i = \text{current} = 0$

$$0 = \frac{c_2}{\sqrt{LC}} (1) - \frac{E}{2} \sqrt{\frac{C}{L}} \left\{ 0 + (1) \right\}$$

$$\frac{c_2}{\sqrt{LC}} - \frac{E}{2} \sqrt{\frac{C}{L}} = 0$$

Multiply by $\sqrt{LC}$, $\frac{\cancel{\sqrt{LC}}\, c_2}{\cancel{\sqrt{LC}}} - \frac{E}{2} \cdot \frac{\sqrt{C}}{\cancel{\sqrt{L}}} \cdot \cancel{\sqrt{L}} \cdot \sqrt{C} = 0$

$$c_2 - \frac{EC}{2} = 0$$

$$\boxed{c_2 = \frac{EC}{2}}$$

Substitute $\boxed{c_1 = 0, \; c_2 = \frac{EC}{2}}$ in eqn (2)

$\therefore$ Charge $q = \frac{EC}{2} \cdot \sin \frac{t}{\sqrt{LC}} - \frac{Et}{2} \sqrt{\frac{C}{L}} \cdot \cos \frac{t}{\sqrt{LC}}$

$$q = \frac{EC}{2} \left\{ \sin \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}} \right\}$$

(3) The charge $q(t)$ on the capacitor is given by D.E, $10\frac{d^2q}{dt^2}+120\frac{dq}{dt}+1000q = 17.\sin 2t$. At time $t=0$ the current(i) is zero and charge on the capacitor is $\frac{1}{2000}$ c coulomb. Find the charge on the capacitor for $t>0$.

Soln: The given diff of LCR circuit, $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E$.

i.e $$10\frac{d^2q}{dt^2}+120\frac{dq}{dt}+1000q = 17.\sin 2t.$$

(or) $$\frac{d^2q}{dt^2}+12\frac{dq}{dt}+100q = \frac{17}{10}\sin 2t$$

$$(D^2+12D+100)q = \frac{17}{10}\sin 2t \rightarrow (1)$$

It is of the form $f(D).q = Q(t)$.

The general solution of (1) is $q = q_c + q_p$

To find C.F: q_c: $f(D).q=0$

$$(D^2+12D+100)q=0$$

The A.E, $f(m)=0 \Rightarrow m^2+12m+100=0$

$$m = \frac{-12\pm\sqrt{(12)^2-4(1)(100)}}{2(1)} = \frac{-12\pm 16i}{2}$$

$$= -6\pm 8i$$

The roots are complex, conjugate numbers. $= \alpha \pm i\beta$

$$\therefore \quad q_c = e^{-6t}[c_1\cos 8t + c_2\sin 8t]$$

To find P.I: q_p: $\frac{1}{f(D)}.Q(t)$

$$= \frac{1}{D^2+12D+100}.\frac{17}{10}\sin 2t$$

$$= \frac{17}{10}.\frac{1}{D^2+12D+100}.\sin 2t$$

put $D^2=-b^2=-4$

$$= \frac{17}{10}.\frac{1}{(-4)+12D+100}.\sin 2t$$

Type: 2

$\frac{1}{f(D)}.\sin bt$

$D^2=-b^2$

$\boxed{b=2}$

$$= \frac{17}{10} \cdot \frac{1}{12D+96} \cdot \sin 2t$$

$$= \frac{17}{10} \cdot \frac{1}{12(D+8)} \sin 2t$$

$$= \frac{17}{120} \cdot \frac{(D-8)}{(D+8)(D-8)} \sin 2t \quad \text{(Rationalize)}$$

$$= \frac{17}{120} \frac{(D-8)\sin 2t}{D^2-64} \quad \boxed{D^2=-b^2=-4}$$

$$= \frac{17}{120} \frac{\frac{d}{dt}(\sin 2t)-8\sin 2t}{-4-64}$$

$$= \frac{17}{120\times(-68)} \cdot (2\cos 2t - 8\sin 2t)$$

$$= \frac{17}{\cancel{120}\times(-68)} \cancel{2}(\cos 2t - 4\sin 2t)$$

$$q_c = \frac{-17}{4080}(\cos 2t - 4\sin 2t) = \frac{17}{4080}(4\sin 2t - \cos 2t)$$

$\therefore$ The general solution $q = q_c + q_p$

$$q = e^{-6t}[c_1\cos 8t + c_2\sin 8t] + \frac{17}{4080}(4\sin 2t - \cos 2t) \longrightarrow (2)$$

when time $t=0$, $q = \frac{1}{2000}$

$$\frac{1}{2000} = e^{0}[c_1\cos(0) + c_2\sin(0)] + \frac{17}{4080}(4\sin(0) - \cos(0))$$

$$= c_1 + \frac{17}{4080}[0-1]$$

$$= \boxed{c_1 = \frac{17}{4080}} + \frac{1}{2000} \Rightarrow \boxed{c_1 = \frac{7}{1500}}$$

$$\frac{dq}{dt} = e^{-6t}[c_1(-8\sin 8t) + c_2\, 8\cos 8t] + (c_1\cos 8t + c_2\sin 8t)(-6e^{-6t})$$
$$+ \frac{17}{4080}[4(2\cos 2t) - (-2\sin 2t)] \longrightarrow (3)$$

$t=0$, $\frac{dq}{dt} = i = 0$

$$\underset{0}{i} = \frac{dq}{dt} = e^0\left[0 + 8c_2(1)\right] + \left(c_1(1)+0\right) - 6\,e^{(0)}\Big]$$

$$+ \frac{17}{4080}\left[8(1) + 2(0)\right]$$

$$0 = 8c_2 - 6c_1 + \frac{17}{510}$$

$$8c_2 = 6c_1 - \frac{17}{510}$$

$$8c_2 = 6\left(\frac{7}{1500}\right) - \frac{17}{510}$$

$$\boxed{c_2 = -\frac{1}{1500}}$$

from (2)

$$\therefore q(t) = e^{-6t}\left[\frac{7}{1500}\cos 8t - \frac{1}{1500}\sin 8t\right] + \frac{\cancel{17}}{\cancel{4080}_{240}}\left[4\sin 2t - \cos 2t\right]$$

$$q = \frac{e^{-6t}}{1500}\left[7\cos 8t - \sin 8t\right] + \frac{\cancel{17}}{\cancel{4080}_{240}}\left(4\sin 2t - \cos 2t\right)$$

similary substitute c_1, c_2, in eqn (3) To find

$$i = \frac{dq}{dt}.$$

Mass spring systems:

Linear ordinary differential equations with constant coefficients have important applications in Mechanics.

Construction of Mass spring systems:

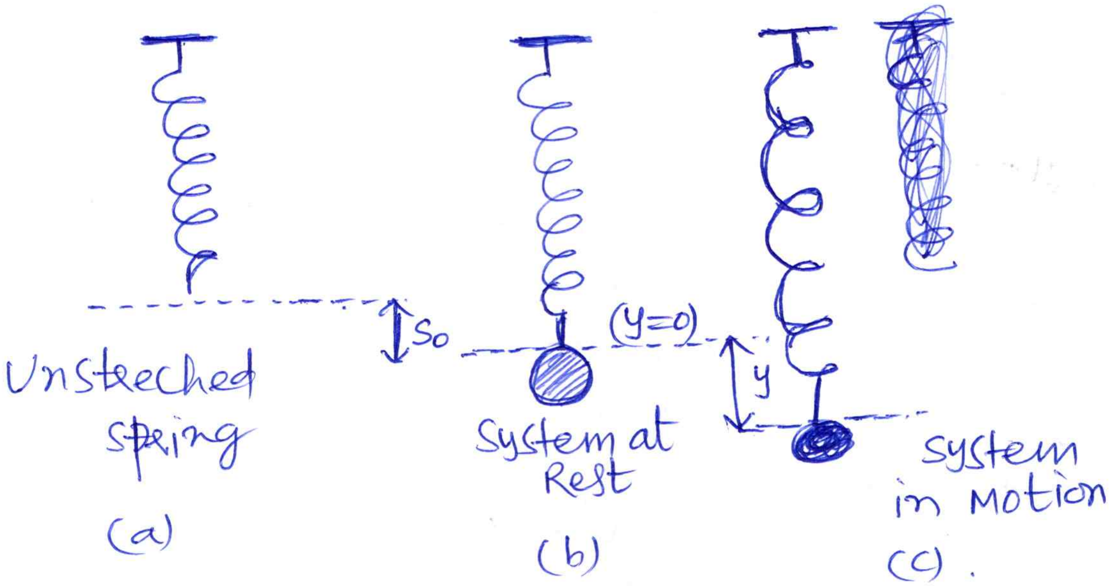
This is a basic Mechanical system; a mass on an elastic spring which moves up and down. Initially we take an ordinary spring that resists compression as well as extension. It is suspended vertically from a fixed support as shown in figure below. At the Lower end of the spring a body of mass 'm' is attached 'm' assumed so large that the mass of the spring can be neglected. If we pull the body down a certain distance and then release it, it starts moving, the moment is supposed as strictly vertical.

The motion of the body i-e, the displacement $y(t)$ is a function of time 't'. this motion is determined by Newton's second law.

$$\text{mass} \times \text{acceleration} = \text{Force}$$

$$\boxed{m \cdot y'' = \text{Force}} \longrightarrow (1)$$

$$m \cdot \frac{d^2y}{dt^2} = \text{Resultant of all forces acting on the body}$$



Initially the spring is unstreched. we now attach the body this streches the spring by S_0 as shown in the figure. It causes an upward force F_0 in the spring.

By Hooke's Law, F_0 is proportional to S_0

i.e $\boxed{F_0 = -KS_0}$ → ② ($K>0$ spring is constant (or) spring modulus)

the negative sign indicates that F_0 is upwards

The Extension S_0 is such that F_0 in the spring balances the weight $W=mg$ of the body.

Hence $F_0+W = -KS_0+mg = 0$

These forces do not effect the motion spring and body are again in rest. this is called "static equilibrium" of the system. we measure the displacement $y(t)$ of the body from this equilibrium point as origin $y=0$.

From the position $y=0$ the body is pulled downward this further stretches the spring by some amount $y>0$

By Hooke's Law this causes an additional upward force F_1 in the spring.

$F_1 = -Ky$ F_1 = restoring force That is to pull the body back to y_0

Undamped System:

~~oscillato~~ Damping is an influence with (or) upon an oscillatory system that has the effect of reducing restricting (or) preventing its oscillations. Every system has damping, otherwise it will keep on moving forever. Practically the effect of damping may often be negligible. Then F_1 is the only force in eqn ① causing motion.

From eqn ① $my'' = \text{Force} = -K\cdot y$

$$my''+Ky=0$$

$$y'' + \frac{K}{m}\cdot y = 0$$

writing operator 'D' form, $\left(D^2 + \frac{K}{m}\right)\cdot y = 0$

The Auxillary equation $f(m) = 0 \Rightarrow m^2 + \frac{K}{m} = 0$

$$m^2 = -\frac{K}{m}$$

$$m = \pm i\sqrt{\frac{K}{m}}$$

The general solution $y(t) = e^{\alpha t}\left[A\cos\beta t + B\sin\beta t\right]$

$$y(t) = e^{(0)t}\left[A\cos\sqrt{\frac{K}{m}}\,t + B\sin\sqrt{\frac{K}{m}}\,t\right] \quad \backslash\ \alpha = 0,\ \beta = \sqrt{\frac{K}{m}}$$

$$y(t) = A\cos w_0 t + B\sin w_0 t \quad \backslash\ \sqrt{\frac{K}{m}} = w_0$$

The corresponding motion is called Harmonic motion. (oscillation)

Problems:

① An Iron ball of weight $W = 98$ Newtons (about 22 lb) stretches a spring 1.09 m (43 inches). How many cycles per minute will this mass spring system execute?

② If the weight is pulled down an additional 16 cm (about 6 inches) and let it start with zero initial velocity, what will be the motion?

Soln: ① Hooke's Law with 'W' as force and 1.09 m as the stretch gives

$$W = 1.09K \Rightarrow K = \frac{W}{1.09} = \frac{98}{1.09} = 90\ \mathrm{kg/sec^2}$$

$$= 90\ \text{newton/mtr}$$

The mass $m = \frac{W}{g} = \frac{98}{9.8} = 10\ \mathrm{kg} \quad \backslash \because W = mg$

this gives frequency $= \frac{w_0}{2\pi} = \frac{\sqrt{K/m}}{2\pi} = \frac{\sqrt{\frac{90}{10}}}{2\pi} = \frac{\sqrt{9}}{2\pi}$

$$= \frac{3}{2\pi}$$

$$= 0.48\ (\mathrm{Hz})$$

$$= 29\ \text{cycles/min}$$

(2) We have $y(t) = A\cos W_0 t + B\sin W_0 t$

We have $y(0) = 16\text{ cm} = 0.16\text{ mtrs} = A$

and $y'(0) = W_0 B = 0 \Rightarrow B = 0$

$y(t) = A\cos W_0 t + B\sin W_0 t$

$y(t) = (0.16)\cos 3t + 0$

$y(t) = 0.16\cdot\cos 3t$ (meter) $= 0.52\cdot\cos 3t$ feet.

(3) Solve the initial value problem $y'' + 4y = 0$, $y(0) = 10$ and $y'(0) = 0$ and give physical interpretation.

Soln: From the given eqn, pulled the mass on a rest spring down to 10 units below the equilibrium position and then release it from rest at $t = 0$.

Given $y'' + 4y = 0$, compare with $y'' + \frac{K}{m}\cdot y = 0$

$\therefore \frac{K}{m} = 4$; $W_0 = \sqrt{\frac{K}{m}} = \sqrt{4} = 2$;

$\therefore y(t) = A\cos W_0 t + B\sin W_0 t$

$y(t) = A\cos 2t + B\sin 2t \rightarrow (1)$ $y(0) = \boxed{A = 10}$

$\frac{dy}{dt} = A(-2\sin 2t) + B(2\cos 2t)$

$= 10(-2\sin 2t) + B(2\cos 2t)$

$\frac{dy}{dt} = -20\sin 2t + 2B\cos 2t$

since at $t = 0$, $y'(0) = 0$

$0 = -20\sin(0) + 2B\cos(0)$

$0 = 0 + 2B$

$\boxed{0 = B}$

from (1) $y(t) = A\cos 2t + B\sin 2t$

$= 10\cos 2t + (0)\sin 2t$

$\therefore y(t) = 10\cos 2t$.

Period of oscillation is $\frac{2\pi}{W_0} = \frac{2\pi}{2} = \pi$ Units of time.