

## Laplace Transforms

The knowledge of Laplace Transforms is an essential

part of mathematics required by engineers.

The Laplace transform is an essential tool for solving the linear differential equations with given initial values of an unknown function and its derivatives without the necessity of first finding the general solution (CF+PI) and then evaluating from it the particular solution satisfying the given conditions. This technique is used to solve some partial differential equations as well.

By using this Laplace transform we can directly find the solution of differential equation without finding the general solution i.e.,  $G.S = C.F + P.I.$

### Definition:-

Let  $f(t)$  be a given function defined for all  $t \geq 0$  then the Laplace transform of  $f(t)$

is denoted by  $L\{f(t)\}$  or  $\bar{f}(s)$  and is defined by

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t) dt \quad \rightarrow (1)$$

→ The symbol  $L$  transform  $f(t)$  into  $\bar{f}(s)$  is called the Laplace transform operator

→ The Relation (1) can also be written as

$$f(t) = L^{-1}\{\bar{f}(s)\}$$

In such case  $f(t)$  is said to be inverse Laplace

transform of  $\bar{f}(s)$ .

→ The symbol  $L^{-1}$  transforms  $\bar{f}(s)$  into  $f(t)$  is called Inverse Laplace transform operator.

Sufficient conditions for the existence of the

Laplace transform of a function :-

→ The function  $f(t)$  must satisfy the following

conditions for the existence of the Laplace transform

1. The function  $f(t)$  must be piece-wise continuous

(i.e.) sectionally continuous in any limited interval

or  $a \leq t \leq b$

2. The function  $f(t)$  is of the exponential order.

Piece-wise continuous function :-

A function  $f(t)$  is said to be piece-wise continuous over the closed interval  $[a, b]$  if it is defined on that interval and is such that the interval can be divided into finite no. of sub intervals in each of which  $f(t)$  is continuous.

Ex:- 1)  $f(t) = \begin{cases} 1 & \text{if } 0 < t < \pi \\ 0 & \text{if } t \geq \pi \end{cases}$

2)  $f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 2 \\ 1 & \text{if } 2 \leq t < 4 \\ 2 & \text{if } 4 \leq t < 6 \\ 3 & \text{if } t \geq 6 \end{cases}$

Function of exponential order :-

A function  $f(t)$  is said to be of exponential order  $\alpha$  if  $\int_0^\infty e^{\alpha t} |f(t)| dt = \text{a finite quantity}$ .

Linearity property :-

i)  $L\{c f(t)\} = c L\{f(t)\}$   
where  $c$  is a constant

(ii) If  $L\{f(t)\} = \bar{f}(s)$  and  $L\{g(t)\} = \bar{g}(s)$  then

$$\begin{aligned} L\{af(t) + bg(t)\} &= L[af(t)] + L[bg(t)] \\ &= aL\{f(t)\} + bL\{g(t)\} \\ &= a\bar{f}(s) + b\bar{g}(s) \\ \therefore L\{af(t) + bg(t)\} &= a\bar{f}(s) + b\bar{g}(s) \end{aligned}$$

Laplace transforms of some standard functions :-

$$1. L\{1\} = \frac{1}{s}$$

$\text{S.H. :- we know that } L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$

$$\begin{aligned} L\{1\} &= \int_0^\infty e^{-st} \cdot 1 dt \\ &= \left( \frac{e^{-st}}{-s} \right)_0^\infty \\ &= \frac{e^{-s\infty}}{-s} + \frac{e^{-s\cdot 0}}{-s} \\ &= 0 + \frac{1}{s} \\ &= \frac{1}{s} \end{aligned}$$

$$2. L\{t\} = \frac{1}{s^2}$$

$\text{S.H. :- } L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$

$$\begin{aligned} L\{t\} &= \int_0^\infty e^{-st} \cdot t dt \\ &= \left[ -\frac{t e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^\infty \\ &= 0 - \left[ 0 - \frac{1}{s^2} \right] = \frac{1}{s^2} \end{aligned}$$

$$3. L\{e^{at}\} = \frac{1}{s-a} \text{ where } s-a>0 \text{ i.e., } s>a$$

$\text{S.H. :- By the definition}$

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt \\ L\{e^{at}\} &= \int_0^\infty e^{-st} e^{at} dt \\ &= \int_0^\infty e^{(a-s)t} dt \\ &= \left[ \frac{e^{(a-s)t}}{-(s-a)} \right]_0^\infty \\ &= 0 - \left[ \frac{e^{0}}{-(s-a)} \right] = 0 - \left[ \frac{1}{s-a} \right] \\ &= \frac{1}{s-a} \end{aligned}$$

$$4. L\{e^{-at}\} = \frac{1}{s+a}, \text{ where } s > 0$$

Sol:-  $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$

$$\begin{aligned} L\{e^{-at}\} &= \int_0^\infty e^{-st} e^{-at} dt \\ &= \int_0^\infty e^{-(s+a)t} dt \end{aligned}$$

$$= \left[ \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

$$= 0 - \left[ \frac{e^0}{-(s+a)} \right]$$

$$= \frac{1}{s+a}$$

Note:-

$$\begin{aligned} i) \sinhx &= \frac{e^x - e^{-x}}{2} & ii) \coshx &= \frac{e^x + e^{-x}}{2} \\ iii) \sinh(at) &= \frac{e^{at} - e^{-at}}{2} & iv) \cosh(at) &= \frac{e^{at} + e^{-at}}{2} \end{aligned}$$

5.  $L\{\sinh(at)\} = \frac{a}{s^2 - a^2}$

Sol:- we know that  $\sinh(at) = \frac{e^{at} - e^{-at}}{2}$

$$\begin{aligned} L\{\sinh(at)\} &= L\left\{\frac{e^{at} - e^{-at}}{2}\right\} \\ &= \frac{1}{2} L\{e^{at} - e^{-at}\} \end{aligned}$$

$$= \frac{1}{2} [L\{e^{at}\} - L\{e^{-at}\}]$$

$$= \frac{1}{2} \left[ \frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[ \frac{s+a - s-a}{(s-a)(s+a)} \right]$$

$$= \frac{1}{2} \frac{2a}{s^2 - a^2} = \frac{a}{s^2 - a^2}$$

6.  $L\{\cosh(at)\} = \frac{s}{s^2 - a^2}$

Sol:- w.k.t  $\cosh(at) = \frac{e^{at} + e^{-at}}{2}$

$$\begin{aligned} L\{\cosh(at)\} &= L\left\{\frac{e^{at} + e^{-at}}{2}\right\} \\ &= \frac{1}{2} L\{e^{at} + e^{-at}\} \\ &= \frac{1}{2} [L\{e^{at}\} + L\{e^{-at}\}] \end{aligned}$$

$$= \frac{1}{2} \left[ \frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[ \frac{s+a + s-a}{(s-a)(s+a)} \right]$$

$$= \frac{1}{2} \left[ \frac{2s}{s^2 - a^2} \right] = \frac{s}{s^2 - a^2}$$

7.  $L\{\sin(at)\} = \frac{a}{s^2 + a^2}$  and  $L\{\cos(at)\} = \frac{s}{s^2 + a^2}$

Sol:- w.k.t  $L\{e^{at}\} = \frac{1}{s-a}$   
 $L\{e^{iat}\} = \frac{1}{s-ia}$   
 $= L\{e^{iat}\} = \frac{1}{s-ia} = \frac{1}{s-ia} \times \frac{s+ia}{s+ia} = \frac{s+ia}{s^2 + a^2}$

$$\therefore L\{e^{iat}\} = \frac{s+ia}{s^2+a^2}$$

$$L\{\cos at + i \sin at\} = \frac{s+ia}{s^2+a^2}$$

$$L\{\cos at\} + i L\{\sin at\} = \frac{s}{s^2+a^2} + \frac{ia}{s^2+a^2}$$

Comparing real and imaginary parts

$$L\{\cos at\} = \frac{s}{s^2+a^2} \quad \text{and} \quad L\{\sin at\} = \frac{a}{s^2+a^2}$$

=

$$8. \quad L\{t^n\} = \frac{n!}{s^{n+1}} \quad \text{where } n \text{ is the integer}$$

Sol:- By the definition of Laplace transform

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$$L\{t^n\} = \int_0^\infty t^n e^{-st} dt$$

$$= \left[ t^n \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s} dt$$

$$= 0 - 0 + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt$$

$$\therefore L\{t^n\} = \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt$$

$$= \frac{n}{s} L\{t^{n-1}\}$$

$$\text{By } L\{t^{m-1}\} = \frac{n-1}{s} L\{t^{m-2}\}$$

$$L\{t^{m-2}\} = \frac{n-2}{s} L\{t^{m-3}\}$$

$$L\{t^{m-3}\} = \frac{n-3}{s} L\{t^{m-4}\}$$

$$L\{t^{m-4}\} = \frac{n-4}{s} L\{t^{m-5}\}$$

$$L\{t^{m-5}\} = \frac{n-5}{s} L\{t^{m-6}\}$$

$$= \frac{n-2}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{1}{s} L\{t^1\}$$

$$= \frac{1}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{1}{s} \cdot \frac{n!}{s} L\{t^1\}$$

$$= \frac{n!}{s^{n+1}} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

### Problems:-

1. Find the Laplace transform of  $e^{3t} - 2e^{-2t} + \sin at + \cos at$

+  $\sinh 3t - 2\cosh 4t + 9$ .

Sol:- Given  $f(t) = e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9$

taking Laplace transform on both sides

$$L\{f(t)\} = L\{e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9\}$$

$$= L\{e^{3t}\} - 2 L\{e^{-2t}\} + L\{\sin 2t\} + L\{\cos 3t\} + L\{\sinh 3t\}$$

$$= \frac{1}{s-3} - 2 \left( \frac{1}{s+2} \right) + \frac{2}{s^2+4} + \frac{s}{s^2+9} - 2 \left( \frac{s}{s^2-9} \right) + 9 \left( \frac{1}{s} \right)$$

=

Q. Find the Laplace transform of  $e^{2t} + 4t^2 - 2\sin 3t + 3\cos 3t$ .

$$= \cos \frac{\omega}{s^2+\omega^2} - \sin \frac{\omega}{s^2+\omega^2}$$

Sol:- Given  
 $f(t) = e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

$$\begin{aligned} L\{f(t)\} &= L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\} \\ &= L\{e^{2t}\} + 4 L\{t^3\} - 2 L\{\sin 3t\} + 3 L\{\cos 3t\} \\ &= \frac{1}{s-2} + 4 \left[ \frac{3!}{s^4} \right] - 2 \left( \frac{3}{s^2+3^2} \right) + 3 \left( \frac{s}{s^2+3^2} \right) \\ &= \end{aligned}$$

3. Find the Laplace transform of  $\sin(\omega t + \alpha)$

Sol:- Given  
 $f(t) = \sin(\omega t + \alpha)$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\begin{aligned} L\{f(t)\} &= L\{\sin(\omega t + \alpha)\} \\ &= L\{\sin \omega t \cos \alpha + \cos \omega t \sin \alpha\} \\ &= \cos \omega t L\{\sin \omega t\} + \sin \omega t L\{\cos \omega t\} \\ &= \cos \omega t \left[ \frac{\omega}{s^2+\omega^2} \right] + \sin \omega t \left[ \frac{s}{s^2+\omega^2} \right] \\ &= \end{aligned}$$

4. Find the Laplace transform of  $\cos(\omega t + \alpha)$

Sol:- Given  
 $f(t) = \cos(\omega t + \alpha) = \cos \omega t \cos \alpha - \sin \omega t \sin \alpha$

$$\begin{aligned} L\{\cos(\omega t + \alpha)\} &= L\{\cos \omega t \cos \alpha - \sin \omega t \sin \alpha\} \\ &= L\{\cos \omega t \cos \alpha\} - L\{\sin \omega t \sin \alpha\} \\ &= \cos \omega t L\{\cos \alpha\} - \sin \omega t L\{\sin \alpha\} \end{aligned}$$

5. Find the L.T of  $\cos 5t \cdot \cos 3t$

$$\begin{aligned} \text{Sol:- } f(t) &= \cos 5t \cos 3t \\ &= \frac{1}{2} [2 \cos 5t \cos 3t] \\ &= \frac{1}{2} [\cos(5t+3t) + \cos(5t-3t)] \\ &= \frac{1}{2} [\cos 8t + \cos 2t] \\ &= \end{aligned}$$

$$\begin{aligned} \therefore L\{f(t)\} &= \frac{1}{2} L[\cos 8t] + L[\cos 2t] \\ &= \frac{1}{2} \left[ L\{\cos 8t\} + L\{\cos 2t\} \right] \\ &= \frac{1}{2} \left[ \frac{s}{s^2+8^2} + \frac{s}{s^2+2^2} \right] \\ &= \end{aligned}$$

6. Find the L.T of  $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

Note:- 1)  $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$

$$2) \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$$

By the definition of Laplace Transform

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st} f(t) \, dt \\ &= \int_0^\pi e^{-st} f(t) \, dt + \int_\pi^\infty e^{-st} f(t) \, dt \end{aligned}$$

$\cos \alpha + \cos \beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

$$\begin{aligned}
&= \int_0^{\pi} e^{-st} \cos t dt + \int_0^{\infty} e^{-st} \sin t dt \\
&= \left[ \frac{e^{-st}}{(s-1)^2} \left[ -s \cos t + \sin t \right] \right]_0^{\pi} + \left[ \frac{e^{-st}}{(s-1)^2} \left[ s \sin t - \cos t \right] \right]_0^{\infty} \\
&= \left[ \frac{e^{-s\pi}}{s^2+1} \left[ -s \cos \pi + \sin \pi \right] \right] - \left[ \frac{1}{s^2+1} \left[ -s \cos 0 + \sin 0 \right] \right] \\
&+ 0 - \left[ \frac{e^{-s\pi}}{s^2+1} \left[ -s \sin \pi - \cos \pi \right] \right] \\
&= - \frac{e^{-s\pi}}{s^2+1} (-s) \\
&= \left( \frac{e^{-s\pi}}{s} \right)^2 + 2 \left( \frac{e^{-s\pi}}{s} \right)^4 + 3 \left( \frac{e^{-s\pi}}{s} \right)^6 + 0 \\
&\quad (\because \cos n\pi = (-1)^n) \\
&= \left[ \frac{e^{-2s}}{-s} \otimes -\frac{e^0}{s} \right] + 2 \left[ \frac{e^{-4s}}{-s} - \frac{e^{-2s}}{-s} \right] + 3 \left[ \frac{e^{-6s}}{-s} - \frac{e^{-4s}}{-s} \right] \\
&= \frac{e^{-2s}}{-s} + \frac{1}{s} + 2 \frac{e^{-4s}}{-s} + \frac{2e^{-2s}}{-s} + \frac{3e^{-6s}}{-s} + \frac{3e^{-4s}}{-s} \\
&= -\frac{1}{s} \left[ e^{-2s} - 1 + 2e^{-4s} - 2e^{-2s} + 3e^{-6s} - 3e^{-4s} \right] \\
&= -\frac{1}{s} \left[ -e^{-2s} - 1 - e^{-4s} + 3e^{-6s} \right] \\
&= \frac{e^{-2s} + 1 + e^{-4s} - 3e^{-6s}}{s} \\
&= \frac{s e^{-s\pi} + s - e^{-s\pi}}{s^2+1} \\
&= \frac{e^{-s\pi} (s-1) + s}{s^2+1} \\
&\quad (\text{Q.E.D.})
\end{aligned}$$

$$\begin{aligned}
&= \int_0^2 e^{-st} f(t) dt + \int_2^4 e^{-st} f(t) dt + \int_4^6 e^{-st} f(t) dt + \int_6^\infty e^{-st} f(t) dt \\
&= \int_0^2 e^{-st} 1 dt + \int_2^4 e^{-st} 2 dt + \int_4^6 e^{-st} 3 dt + \int_6^\infty e^{-st} 0 dt \\
&= \left( \frac{e^{-st}}{s} \right)_0^2 + 2 \left( \frac{e^{-st}}{s} \right)_2^4 + 3 \left( \frac{e^{-st}}{s} \right)_4^6 + 0 \\
&= \left[ \frac{e^{-2s}}{-s} - \frac{e^0}{s} \right] + 2 \left[ \frac{e^{-4s}}{-s} - \frac{e^{-2s}}{-s} \right] + 3 \left[ \frac{e^{-6s}}{-s} - \frac{e^{-4s}}{-s} \right] \\
&= \frac{e^{-2s}}{-s} + \frac{1}{s} + 2 \frac{e^{-4s}}{-s} + \frac{2e^{-2s}}{-s} + \frac{3e^{-6s}}{-s} + \frac{3e^{-4s}}{-s} \\
&= -\frac{1}{s} \left[ e^{-2s} - 1 + 2e^{-4s} - 2e^{-2s} + 3e^{-6s} - 3e^{-4s} \right]
\end{aligned}$$

8. Find the Laplace transform of  $f(t) = \begin{cases} 1 & 0 < t < 2 \\ 2 & 2 < t < 4 \\ 3 & 4 < t < 6 \\ 0 & t > 6 \end{cases}$

$\therefore$  Given  $f(t) = t^3$  is exponential

The function  $f(t)$  is said to be an exponential

$$\text{Sol:- } \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

7. Find the Laplace transform of  $f(t) = \begin{cases} 1 & 0 < t < 2 \\ 2 & 2 < t < 4 \\ 3 & 4 < t < 6 \\ 0 & t > 6 \end{cases}$

8. Verify whether the function  $f(t) = t^3$  is exponential

order and find the transform.

$\therefore$  Given  $f(t) = t^3$

The function  $f(t)$  is said to be an exponential

order if it satisfies the condition

If  $e^{-at} f(t) = \text{a finite value}$   
 $t \rightarrow \infty$

$$\text{Now } e^{-at} f(t) = \lim_{t \rightarrow \infty} e^{-at} t^3$$

$$= \lim_{t \rightarrow \infty} \frac{t^3}{e^{at}} \quad [\text{L'Hospital rule}]$$

By using L'Hospital rule

$$= \lim_{t \rightarrow \infty} \frac{3t^2}{ae^{at}} \quad [\text{as form}]$$

$$= \lim_{t \rightarrow \infty} \frac{6t}{a^2 e^{at}} \quad [\text{as form}]$$

$$= \lim_{t \rightarrow \infty} \frac{6}{a^3 e^{at}}$$

$$= \lim_{t \rightarrow \infty} \frac{6}{a^3} = 0$$

$$= \lim_{t \rightarrow \infty} \frac{2}{a^3 e^{3t}}$$

$$= \lim_{t \rightarrow \infty} \frac{2}{a^3} = 0 \quad (\text{a finite quantity})$$

Hence  $t^2$  is an exponential order 3.

First Translation or First shifting Theorem :-

Statement:- If  $L\{f(t)\} = F(s)$  then  $L\{e^{at} f(t)\} = \bar{F}(s-a)$

$$L\{e^{at} f(t)\} = \bar{F}(s+a)$$

$\therefore f(t) = t^3$  is of exponential order 3.

The Laplace transform of  $t^3$  is

$$L\{t^3\} = \frac{3!}{s^4} = \frac{3!}{s^4}$$

$\therefore$

$$\begin{aligned} L\{e^{at} f(t)\} &= \int_0^\infty e^{-st} f(t) dt = \bar{F}(s) \\ L\{e^{at} f(t)\} &= \int_0^\infty e^{-st} e^{at} f(t) dt \\ &= \int_0^\infty e^{-(s-a)t} f(t) dt \\ &= \int_0^\infty e^{-(s-a)t} f(t) dt \end{aligned}$$

10. P.T. the function  $f(t) = t^2$  is of exponential order 3.

Proof:- By the definition of exponential order

$$L\{e^{at} f(t)\} = \lim_{t \rightarrow \infty} e^{-at} t^2$$

$$= \lim_{t \rightarrow \infty} \frac{t^2}{e^{at}} \quad (\text{L'Hospital rule})$$

$$= \lim_{t \rightarrow \infty} \frac{2t}{a^2 e^{at}}$$

$$= \lim_{t \rightarrow \infty} \frac{2}{a^2} = 0 \quad (\text{a finite quantity})$$

Put  $s-a=u$

$$\therefore L\{e^{at} f(t)\} = \int_0^\infty e^{-ut} f(t) dt = \bar{f}(u)$$
$$= \bar{f}(s-a)$$

"By  
 $L\{f(t)\} = \bar{f}(s)$ "  
 $L\{e^{at} f(t)\} = \bar{f}(s+a)$

Based on above theorem

$$1. L\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}$$

$$2. L\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}$$

$$3. L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

$$4. L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

$$5. L\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2 - b^2}$$

$$6. L\{e^{at} \cosh bt\} = \frac{b}{(s-a)^2 - b^2}$$

$$7. L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2}$$

$$8. L\{e^{at} \sin bt\} = \frac{s+a}{(s-a)^2 + b^2}$$

(By def  
 $L\{f(t)\} = \bar{f}(s)$ )

$$9. L\{e^{at} \cosh bt\} = \frac{s+a}{(s-a)^2 - b^2}$$

$$10. L\{e^{at} \sinh bt\} = \frac{s+a}{(s-a)^2 - b^2}$$

Note:- By Heat shifting theorem  $s \rightarrow s-a$  when  $e^{at}$  occurs

when  $e^{at}$  occurs

Problems :-

$$1. \text{Find } L\{e^{-3t} (2\cos 5t - 3\sin 5t)\}$$

$$\begin{aligned} & L\{e^{-3t} (2\cos 5t - 3\sin 5t)\} = L\{2e^{-3t} \cos 5t - 3e^{-3t} \sin 5t\} \\ & = 2 L\{e^{-3t} \cos 5t\} - 3 L\{e^{-3t} \sin 5t\} \end{aligned}$$

$$= 2 \left( \frac{s+3}{(s+3)^2 + 25} \right) - 3 \left( \frac{5}{(s+3)^2 + 25} \right)$$

$$= 2 \left[ \frac{s+3}{(s+3)^2 + 25} \right] - 3 \left[ \frac{5}{(s+3)^2 + 25} \right]$$

$$= \frac{2s+6-15}{(s+3)^2 + 25} = \frac{2s-9}{s^2 + 6s + 9 + 25} = \frac{2s-9}{s^2 + 6s + 34}$$

$$2. \text{Find } L\{e^{-t} (3\sin 2t - 5\cos 2t)\}$$

$$\begin{aligned} & L\{e^{-t} (3\sin 2t - 5\cos 2t)\} \\ & = L\{3e^{-t} \sin 2t - 5e^{-t} \cos 2t\} \end{aligned}$$

$$= 3L\{e^t \sin 2t\} - 5L\{e^t \cos 2t\}$$

$$= 3 \left[ \frac{2}{(s+4)^2 + 4} \right] - 5 \left[ \frac{s+1}{(s+1)^2 - 4} \right]$$

$$= 3 \left[ \frac{2}{s^2 + 2s + 1 + 4} \right] - 5 \left[ \frac{s+1}{s^2 + 2s + 1 - 4} \right]$$

$$= \frac{6}{s^2 + 2s + 5} - \frac{5s+5}{s^2 + 2s - 3}$$

$$3. L\{e^{3t} \sin t\}$$

$$= L\{e^{3t} \sin t\} = L\{e^{3t} \frac{1-\cos 2t}{2}\}$$

$$= \frac{1}{2} L\{e^{3t} (1 - \cos 2t)\}$$

$$= \frac{1}{2} L\{e^{3t} - e^{3t} \cos 2t\}$$

$$(\because L\{e^{at} \cos bt\} = \frac{sa}{(s-a)^2 + b^2})$$

$$= \frac{1}{2} \left[ L\{e^{3t}\} - L\{\frac{(s+3)}{(s+3)^2 + 2^2} (s+3)\} \right]$$

$$= \frac{1}{2} \left[ L\{e^{3t}\} - \frac{s+3}{s^2 + 2s + 5} \right]$$

$$= \frac{1}{2} \left[ L\{e^{3t}\} - \frac{s-3}{s^2 + 2s + 9 - 12} \right]$$

$$= \frac{1}{2} \left[ L\{e^{3t}\} - \frac{s-3}{s^2 - 6s + 13} \right]$$

$$\left( \because L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2} \right)$$

$$4. L\{e^t \cos^2 t\}$$

$$(\because L\{e^{at} \cos^2 t\} = \frac{s+a}{(s-a)^2 + b^2})$$

$$= \frac{1}{2} L\{e^t + e^{-t} \cos 2t\}$$

$$= \frac{1}{2} \left[ L\{e^t\} + L\{e^{-t} \cos 2t\} \right]$$

$$= \frac{1}{2} \left[ \frac{1}{s+1} + \frac{s+1}{s^2 + 2s + 1 + 4} \right]$$

$$= \frac{1}{2} \left[ \frac{1}{s+1} + \frac{(s+1)}{s^2 + 2s + 5} \right]$$

$$5. \text{ show that i) } L\{t + \sin at\} = \frac{2as}{(s^2 + a^2)^2}$$

$$\text{ii) } L\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

$$\text{Sol: we have } L\{t\} = \frac{1}{s^2}$$

$$L\{e^{at} t\} = \frac{1}{(s-a)^2}$$

$$L\{t e^{iat}\} = \frac{1}{(s-ia)^2} \times \frac{(s+ia)^2}{(s+ia)^2} = \frac{(s+ia)^2}{(s^2 + a^2)^2}$$

$$L\{t e^{iat}\} = \frac{s^2 - a^2 + 2isa}{(s^2 + a^2)^2}$$

$$L\{t(\cos \alpha t + i \sin \alpha t)\} = \frac{s^2 - \alpha^2 + i\alpha s}{(s^2 + \alpha^2)^2}$$

Equating real and imaginary parts

$$L\{t \cos \alpha t\} = \frac{s^2 - \alpha^2}{(s^2 + \alpha^2)^2} \quad L\{t \sin \alpha t\} = \frac{2\alpha s}{(s^2 + \alpha^2)^2}$$

Gamma function :- The integral  $\int_0^\infty e^{-tx} x^n dx$  is called the Gamma function. It is denoted by  $\Gamma(n)$ .

$$\Gamma(n) = \int_0^\infty e^{-tx} x^{n-1} dt \quad \text{where } n > 0$$

Properties of Gamma-function :-

1.  $\Gamma(n+1) = n!$  where  $n$  is true integer

2.  $\Gamma(n+1) = n \Gamma(n)$  where  $n > 0$  (integers or fractional parts also)

$$\Gamma(1) = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$$

6.  $\Gamma$  function is not defined for negative values and for zero, i.e.,  $\Gamma(0), \Gamma(-1), \Gamma(-2), \dots$  are not defined

$$7. \Gamma(n) = (n-1) \Gamma(n-1)$$

P.T.  $L\{t^n\} = \frac{n!}{s^{n+1}}$  by using Gamma-function. also P.T.  $L\{t^n\} = \frac{n!}{s^{n+1}}$  where  $n > 0$

By def of Laplace transform

$$L\{t^n\} = \int_0^\infty e^{-st} t^n dt$$

$$\text{put } st = x \Rightarrow t = \frac{x}{s}$$

$$s dt = dx \Rightarrow dt = \frac{dx}{s}$$

$$\text{when } t=0 \Rightarrow x=0$$

$$t=\infty \Rightarrow x=\infty$$

$$\therefore L\{t^n\} = \int_0^\infty e^{-sx} \left(\frac{x}{s}\right)^n \left(\frac{dx}{s}\right)$$

$$= \int_0^\infty e^{-sx} \cdot \frac{x^n}{s^{n+1}} dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-sx} x^n dx$$

$$\left( \because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \right)$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$$

$$\boxed{L\{t^n\} = \frac{1}{s^{n+1}} \Gamma(n+1)}$$

$$= \frac{1}{s^{n+1}} n! = \frac{n!}{s^{n+1}}$$

also from ①

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n \Gamma(n)}{s^{n+1}}$$

where  $n > 0$

2. Find  $L^{-1} \left[ \frac{1}{s^2 + \frac{1}{s}} \right]$

$$\begin{aligned}
 L\left\{\left(\sqrt{s} + \frac{1}{\sqrt{s}}\right)^3\right\} &= L\left\{(\sqrt{s})^3 + \left(\frac{1}{\sqrt{s}}\right)^3 + 3(\sqrt{s})^2 \cdot \frac{1}{\sqrt{s}} + 3\sqrt{s} \left(\frac{1}{\sqrt{s}}\right)^2\right\} \\
 &= L\left\{t^{\frac{3}{2}}\right\} + L\left\{t^{-\frac{3}{2}}\right\} + 3 L\left\{t^{\frac{1}{2}}\right\} + 3 L\left\{t^{-\frac{1}{2}}\right\} \\
 \text{or. R.T. } L\{t^n\} &= \frac{s^{n+1}}{s^{n+1}} = \frac{s^{n+1}}{s^{n+1}} \\
 L\left\{\left(\sqrt{s} + \frac{1}{\sqrt{s}}\right)^3\right\} &= \frac{\frac{3}{2} H\left(\frac{3}{2}\right)}{s^{\frac{3}{2}+1}} + \frac{H\left(-\frac{3}{2}\right)}{s^{-\frac{3}{2}+1}} - \frac{\frac{3}{2} H\left(\frac{1}{2}\right)}{s^{\frac{1}{2}+1}} \\
 &= \frac{\frac{3}{2} \cdot \frac{1}{2} H\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} + \frac{H\left(-\frac{1}{2}\right)}{s^{-\frac{1}{2}}} + \frac{\frac{3}{2} H\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} + \frac{3 H\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} \\
 &= \frac{\frac{3}{4} \sqrt{\pi}}{s^{\frac{5}{2}}} - 2\sqrt{\pi} + \frac{\frac{3}{2} \sqrt{\pi}}{s^{\frac{3}{2}}} + 3 \sqrt{\frac{\pi}{s}}
 \end{aligned}$$

3. Find  $L^{-1} e^{st} t^{\frac{n}{2}}$

$$\begin{aligned}
 L\{e^{st} t^{\frac{n}{2}}\} &= L\left\{t^{\frac{n}{2}}\right\} \frac{1}{s-t} \quad \left( \begin{array}{l} \text{By Gamma-function} \\ L\{t^n\} = \frac{H(n+1)}{s^{n+1}} \end{array} \right) \\
 &= \frac{H\left(\frac{n}{2}+1\right)}{(s-3)^{\frac{n}{2}+1}} = \frac{\frac{1}{2} \cdot \frac{n}{2} H\left(\frac{n}{2}\right)}{(s-3)^{\frac{n}{2}}} \\
 L\left\{t^{\frac{n}{2}}\right\} &= \frac{\frac{1}{2} \cdot \frac{n}{2} H\left(\frac{n}{2}\right)}{(s-3)^{\frac{n}{2}}} \\
 &= \frac{\frac{1}{2} \cdot \frac{n}{2} \cdot \frac{1}{2} H\left(\frac{1}{2}\right)}{(s-3)^{\frac{n}{2}}} \\
 &= \frac{\frac{1}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} H\left(\frac{3}{2}\right)}{(s-3)^{\frac{9}{2}}} = \frac{\frac{1}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} H\left(\frac{1}{2}\right)}{(s-3)^{\frac{9}{2}}} \\
 \text{or. R.T. } L\{t^n\} &= \frac{H(n+1)}{s^{n+1}} \\
 L\left\{\left(s+\frac{1}{s}\right)^3\right\} &= \frac{H\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}} + \frac{H\left(-\frac{3}{2}+1\right)}{s^{-\frac{3}{2}+1}} + \frac{3 H\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} + \frac{3 H\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} \\
 &= \frac{105\sqrt{\pi}}{16(s-3)^{\frac{9}{2}}}
 \end{aligned}$$

Q. Using the equation  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\text{S.T } L\{\sin t\} = \frac{\sqrt{\pi}}{2s^{3/2}} [e^{-1/4s}]$$

Sol:- Given  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\sin t = \pi t - \frac{1}{3!} (\pi t)^3 + \frac{1}{5!} (\pi t)^5 - \frac{1}{7!} (\pi t)^7 + \dots$$

$$L\{\sin t\} = L\left[\pi t - \frac{1}{3!} L\{t^3\} + \frac{1}{5!} L\{t^5\} - \frac{1}{7!} L\{t^7\}\right]$$

$$= \frac{\pi t}{s^{1/2+1}} - \frac{1}{3!} \frac{J\left(\frac{3}{2}+1\right)}{s^{3/2+1}} + \frac{1}{5!} \frac{J\left(\frac{5}{2}+1\right)}{s^{5/2+1}} - \frac{1}{7!} \frac{J\left(\frac{7}{2}+1\right)}{s^{7/2+1}}$$

$$= \frac{\frac{1}{2} J\left(\frac{1}{2}\right)}{s^{3/2}} - \frac{1}{3!} \frac{\frac{3}{2} \cdot \frac{1}{2} J\left(\frac{1}{2}\right)}{s^{5/2}} + \frac{1}{5!} \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} J\left(\frac{1}{2}\right)}{s^{7/2}} - \frac{1}{7!} \frac{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} J\left(\frac{1}{2}\right)}{s^{9/2}}$$

$$= \frac{1}{2} \frac{\pi}{s^{3/2}} - \frac{3}{8} \frac{\pi}{s^{5/2}} + \frac{15}{64} \frac{\pi}{s^{7/2}} - \frac{35}{448} \frac{\pi}{s^{9/2}} + \dots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \left[ 1 - \frac{1}{4s} + \frac{1}{32s^2} - \dots \right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$$

$$( \because e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots )$$

$$e^{-1/4s} = 1 - \frac{1}{4s} + \frac{1}{32s^2} - \dots )$$

### UNIT STEP FUNCTION :-

The unit step function at  $t=a$  is defined as

$$u(t-a) \text{ (or) } H(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

This function is also known as Heaviside unit function :-

### Laplace Transform of unit step function :-

We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a) dt$$

$$= \int_a^\infty e^{-st} u(t-a) dt + \int_a^\infty e^{-st} u(t-a) dt$$

$$= \int_a^\infty e^{-st} (0) dt + \int_a^\infty e^{-st} (1) dt$$

$$= \int_a^\infty e^{-st} dt = \left( \frac{e^{-st}}{-s} \right)_a^\infty = 0 - \left( \frac{e^{-sa}}{-s} \right)$$

$$\therefore L\{u(t-a)\} = \frac{e^{-sa}}{s}$$

Second translation or second shifting theorem :-

Statement:-

$$\text{If } L\{f(t)\} = \bar{f}(s) \text{ and } f(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t \leq a \end{cases}$$

$$= e^{-sa} \int_0^\infty e^{-su} f(u) du$$

$$= e^{-sa} \int_0^\infty e^{-st} f(t) dt \quad (\text{replacing } u \text{ by } t)$$

$$= \int_0^\infty e^{-su} e^{-sa} f(u) du$$

$$= e^{-sa} \int_0^\infty e^{-su} f(u) du$$

$$= \int_0^\infty e^{-st} f(t) dt \quad (\text{replacing } u \text{ by } t)$$

$$= e^{-sa} L\{f(t)\}$$

$$= e^{-sa} \bar{f}(s)$$

$$\text{then } L\{g(t)\} = e^{-sa} \bar{f}(s)$$

Proof:- By the definition of Laplace transform

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t) dt$$

$$= \int_0^\infty e^{-st} g(t) dt + \int_a^\infty e^{-st} g(t) dt$$

$$= a + \int_a^\infty e^{-st} f(t-a) dt$$

$$\text{put } t-a=u$$

$$dt = du$$

$$\text{If } t=a \Rightarrow u=0$$

$$t=\infty \Rightarrow u=\infty$$

$$\therefore L\{g(t)\} = \int_0^\infty e^{-su} f(u) du$$

$$= \int_0^\infty e^{-su - sa} f(u) du$$

Another form of second shifting theorem :-  
Statement:- If  $L\{f(t-a) \cdot H(t-a)\} = e^{-as} \bar{f}(s)$   
then  $L\{f(t-a) \cdot H(t-a)\} = \begin{cases} 1 & \text{if } t > a \\ 0 & \text{if } t \leq a \end{cases}$  and  $H(t)$  is called heavy side cosine  $\frac{d}{dt} H(t) = \begin{cases} 1 & \text{if } t > a \\ 0 & \text{if } t \leq a \end{cases}$  and  $H(t)$  is called heavy side unit step function.

Proof:- By the definition

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$$L\{f(t-a) H(t-a)\} = \int_0^\infty e^{-st} f(t-a) H(t-a) dt$$

$$\text{put } t-a=u \Rightarrow t=u+a$$

$$dt = du$$

$$\text{If } t=0 \quad u=-a$$

$$t=\infty \quad u=\infty$$

$$\therefore L\{f(t-a)H(t-a)\} = \int_a^\infty e^{-su(a)} f(u) H(u) du$$

$$= \int_a^\infty e^{-su(a)} f(u) H(u) du + \int_a^\infty e^{-su(a)} f(u) H(u) du$$

$$= \int_a^\infty e^{-su(a)} f(u) du + \int_a^\infty e^{-su(a)} f(u) H(u) du$$

$$= \int_0^\infty e^{-su(a)} f(u) du$$

$$\therefore L\{f(t-a) H(t-a)\} = e^{-sa} \bar{f}(s)$$

Problem :-

$$1. \text{ Find } L\{t \cdot g(t)\} \quad g(t) = \begin{cases} \cos(t - \frac{2\pi}{3}) & \text{if } t > \frac{2\pi}{3} \\ 0 & \text{if } t < \frac{2\pi}{3} \end{cases}$$

Sol:- By second shifting theorem

$$\text{we know that } g(t) = \begin{cases} t \cdot f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$$

$$\therefore \text{here } f(t-a) = \cos(t - \frac{2\pi}{3})$$

$$\Rightarrow a = \frac{2\pi}{3} \quad \text{and } t-f(t) = \cos t$$

$$\text{Also } -L\{f(t)\} = L\{\cos t\} = \frac{s}{s^2+1}$$

$$\text{Now } L\{f(t)\} = L\{\cos(t-a)\} = e^{-as} \frac{s}{s^2+1}$$

By second shifting theorem we have

$$L\{g(t)\} = e^{-as} \bar{f}(s)$$

$$= e^{-as} L\{f(t)\}$$

$$= e^{-as} \frac{s}{s^2+1}$$

$$= e^{-\frac{2\pi}{3}s} \frac{s}{s^2+1}$$

$$= \frac{s e^{-\frac{2\pi}{3}s}}{s^2+1}$$

2. Find the Laplace transform of the following functions

$$i) \quad g(t) = \begin{cases} \cos(t - \frac{\pi}{3}) & \text{if } t > \frac{\pi}{3} \\ 0 & \text{if } t < \frac{\pi}{3} \end{cases}$$

$$ii) \quad g(t) = \begin{cases} \sin(t - \frac{\pi}{3}) & \text{if } t > \frac{\pi}{3} \\ 0 & \text{if } t < \frac{\pi}{3} \end{cases}$$

Ex:- By second shifting theorem

$$\text{or } \mathcal{L}f(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$$

$$\therefore f(t-a) = \sin(t-\pi/3)$$

$$\therefore a = \pi/3$$

and let  $f(t) = \sin t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

By second shifting theorem

$$\mathcal{L}\{g(t)\} = e^{-as} \mathcal{F}(s)$$

$$= e^{-as} \mathcal{L}\{f(t)\}$$

$$= e^{-as} \frac{1}{s^2+1}$$

$$= e^{-\pi s/3} \frac{1}{s^2+1}$$

$$\begin{aligned} \text{Now by another form of second shifting theorem} \\ \mathcal{L}\{3\cos 4(t-2) \cdot u(t-2)\} &= e^{-as} \mathcal{F}(s) \\ &= e^{-as} \mathcal{L}\{f(t)\} \\ &= e^{-2s} \cdot \frac{3s}{s^2+16} \\ &= \frac{3se^{-2s}}{s^2+16} \end{aligned}$$

4. Find the L.T of i)  $\mathcal{L}\{(t-2)^3 u(t-2)\}$  ii)  $e^{3t} u(t-2)$

Ex:- on composing with  $\mathcal{L}\{f(t-a) H(t-a)\}$

$$f(t-a) = (t-2)^3, \quad H(t-a) = H(t-2)$$

$$\text{here } a=2 \text{ and let } f(t) = t^3$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^3\} = \frac{6}{s^4}$$

3. Find the Laplace transform of  $3\cos 4(t-2) \cdot u(t-2)$

Ex:- By the another form of second shifting theorem

$$\mathcal{L}\{f(t-a) H(t-a)\} = e^{-as} \mathcal{L}\{f(t)\} = e^{-as} \mathcal{F}(s)$$

$$\begin{aligned} \text{By second shifting theorem} \\ \mathcal{L}\{(t-2)^3 u(t-2)\} &= e^{-as} \mathcal{F}(s) \\ &= e^{-as} \mathcal{L}\{t^3\} \\ &= e^{-2s} \frac{6}{s^4} = \frac{6e^{-2s}}{s^4} \end{aligned}$$

$$\begin{aligned} \text{on composing we have } f(t-a) &= 3\cos 4(t-2) \\ \mathcal{L}\{f(t-a)\} &= u(t-2) \end{aligned}$$

$$\text{here } a=2 \quad \text{let } f(t) = 3\cos 4t$$

$$\mathcal{L}\{f(t)\} = 3 \mathcal{L}\{\cos 4t\} = 3 \frac{2}{s^2+16}$$

$$\text{iii) } e^{-3t} u(t-2)$$

$$L[e^{-3t} u(t-2)] = e^{-3t+6} u(t-2)$$

$$= e^{-3(t-2)} e^{-6} u(t-2)$$

$$L[e^{-3t} u(t-2)] = e^{-6} L\{e^{-3(t-2)} u(t-2)\}$$

By second shifting theorem (another form)

$$f(t-\alpha) = e^{-3(t-2)}$$

$$u(t-2) = H(t-\alpha)$$

$$\text{Note } \alpha = 2$$

$$\text{Let } f(t) = e^{-3t}$$

$$L\{f(t)\} = L\{e^{-3t}\} = \frac{1}{s+3}$$

$$\therefore L\{e^{-3t} u(t-2)\} = e^{-6} L\{e^{-3(t-2)} u(t-2)\}$$

$$= e^{-6} \cdot e^{-6s} \bar{f}(s)$$

$$= e^{-6} \cdot e^{-2s} L\{f(t)\}$$

$$= e^{-6} \cdot e^{-2s} \frac{1}{s+3}$$

$$= \frac{e^{-2(s+3)}}{s+3}$$

$$\begin{aligned} \text{Note :-} \\ \rightarrow \text{If } L\{f(t)\} = \bar{f}(s) \text{ then } L\{f(\frac{x}{a})\} = a \bar{f}(as) \\ \rightarrow L\{f(ax)\} = \frac{1}{a} L\{f(t)\} \xrightarrow{s \rightarrow \frac{s}{a}} \\ \rightarrow L\{f(x)\} = a L\{f(t)\} s \rightarrow as \end{aligned}$$

change of scale property :-

statement :-

$$\text{If } L\{f(t)\} = \bar{f}(s) \text{ then } L\{f(at)\} = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$$

Proof :- By the def of L.T

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \bar{f}(s)$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at) dt$$

$$\text{Put } at = x \Rightarrow t = \frac{x}{a}$$

$$at - dx = dt - \frac{dx}{a}$$

$$dt = 0 \quad \text{if } t=0 \text{ then } x=0$$

$$dt = 0 \quad \text{if } t=\infty \text{ then } x=\infty$$

$$\therefore L\{f(at)\} = \int_0^\infty e^{-sx} f(x) \frac{1}{a} dx$$

$$= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a}x} f(x) dx$$

$$= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a}t} f(t) dt = \frac{1}{a} L\{f(t)\}_{s \rightarrow \frac{s}{a}}$$

$$= \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$$

$$\rightarrow \text{If } L\{f(t)\} = \bar{f}(s) \text{ then } L\{f(\frac{x}{a})\} = a \bar{f}(as)$$

$$\rightarrow L\{f(ax)\} = \frac{1}{a} L\{f(t)\} \xrightarrow{s \rightarrow \frac{s}{a}}$$

$$\rightarrow L\{f(x)\} = a L\{f(t)\} s \rightarrow as$$

$$1. \text{ If } L\{f(t)\} = \frac{9s^2 - 12s + 15}{(s-1)^3} \text{ then find } L\{f(3t)\} \text{ using}$$

charge of scale property.

$$\text{Sol: - Given } L\{f(t)\} = \frac{9s^2 - 12s + 15}{(s-1)^3} = \bar{f}(s)$$

$$\text{L}\{K.T\} f(at) = \frac{1}{a} \text{L}\{f(t)\} \quad s \rightarrow s/a$$

$$\text{L}\{f(3t)\} = \frac{1}{3} \left[ \frac{9s^2 - 12s + 15}{(s-1)^3} \right] \quad s \rightarrow s/3$$

$$= \frac{1}{3} \left[ \frac{9 \frac{s^2}{9} - 12 \frac{s}{3} + 15}{(\frac{s}{3}-1)^3} \right]$$

$$= \frac{1}{3} \left[ \frac{9s^2 - 4s + 15}{(s-3)^3} \right]$$

$$= \frac{1}{3} \left[ \frac{9s^2 - 4s + 15}{27} \right] \times 27^9$$

$$= 9 \left[ \frac{s^2 - 4s + 15}{(s-3)^3} \right]$$

=

$$\text{L}\{\sinh 3t\} = \frac{1}{3} \left[ \frac{1}{s^2 - 1} \right] \quad s \rightarrow s/3$$

$$= \frac{1}{3} \left[ \frac{1}{(\frac{s}{3})^2 - 1} \right]$$

$$= \frac{1}{3} \left[ \frac{1}{\frac{s^2}{9} - 1} \right]$$

$$= \frac{1}{3} \left[ \frac{9}{s^2 - 9} \right]$$

$$= \frac{3}{s^2 - 9}$$

$$\text{L}\{e^{-3t} \sinh 3t\} = \frac{3}{(s+3)^2 - 9} = \frac{3}{s^2 + 9 + 6s - 9} = \frac{3}{s^2 + 6s}$$

~~L{te^{-at}}~~ ~~then find~~

$$3. \text{ If } L\{f(t)\} = \frac{1}{3} e^{-ts} \text{ prove that } L\{e^{t} f(3t)\}$$

$$\text{Sol: - Given } L\{f(t)\} = \frac{1}{3} e^{-ts} = \bar{f}(s)$$

By charge of scale property

$$L\{f(at)\} = \frac{1}{a} \bar{f}(\frac{s}{a})$$

$$2. \text{ Find } L\{e^{-st} \sinh 3t\} \text{ using charge of scale property}$$

Sol: - By the charge of scale property

$$L\{f(at)\} = \frac{1}{a} L\{f(t)\} \quad s \rightarrow s/a$$

$$L\{\sinh 3t\} = \frac{1}{3} [L\{\sinh t\}] \quad s \rightarrow s/3$$

$$= \frac{1}{3} \left[ \frac{1}{s^2 - 1} \right] \quad s \rightarrow s/3$$

$$= \frac{1}{3} \left[ \frac{1}{(\frac{s}{3})^2 - 1} \right]$$

$$= \frac{1}{3} \left[ \frac{1}{\frac{s^2}{9} - 1} \right]$$

$$= \frac{1}{3} \left[ \frac{9}{s^2 - 9} \right]$$

$$= \frac{3}{s^2 - 9}$$

$$\text{L}\{e^{-3t} \sinh 3t\} = \frac{3}{(s+3)^2 - 9} = \frac{3}{s^2 + 9 + 6s - 9} = \frac{3}{s^2 + 6s}$$

$$3. \text{ If } L\{f(t)\} = \frac{1}{3} e^{-ts} \text{ prove that } L\{e^{t} f(3t)\}$$

$$\text{Sol: - Given } L\{f(t)\} = \frac{1}{3} e^{-ts} = \bar{f}(s)$$

By charge of scale property

$$L\{f(at)\} = \frac{1}{a} \bar{f}(\frac{s}{a})$$

$$\begin{aligned} L\{f(3t)\} &= \frac{1}{3} \bar{f}\left(\frac{s}{3}\right) e^{-3/s} \\ &= \frac{1}{3} \cdot \frac{2}{3} e^{-3/s} \\ &= \frac{2}{9} e^{-3/s} \end{aligned}$$

Now  $L\{e^{-t} f(3t)\} = \frac{e^{-st} \bar{f}(s)}{(s+1)}$  (By 1st shifting theorem)  
then from L{f(3t)} and

$$4. L\{e^{-t} f(t)\} = \frac{20 - 4s}{s^2 - 4s + 20}$$

$$L\{e^{st} f(s)\}$$

$$\begin{aligned} \text{Given } L\{f(t)\} &= \frac{20 - 4s}{s^2 - 4s + 20} = \bar{f}(s) \\ L\{f(3t)\} &= \frac{1}{3} \bar{f}\left(\frac{s}{3}\right) \end{aligned}$$

By change of scale property  
 $L\{f(3t)\} = \frac{1}{3} \bar{f}\left(\frac{s}{3}\right)$

$$L\{f(3t)\} = \frac{1}{3} \bar{f}\left(\frac{s}{3}\right)$$

$$= \frac{20 - 4\left(\frac{s}{3}\right)}{\left(\frac{s}{3}\right)^2 - 4\left(\frac{s}{3}\right) + 20} \cdot \frac{1}{3}$$

$$\begin{aligned} &= \frac{1}{3} \cdot \frac{60 - 4s}{s^2 - 12s + 180} \\ &= \frac{s^2 - 4s}{s^2 - 12s + 180} \end{aligned}$$

$$= \frac{1}{3} \cdot \frac{60 - 4s}{s^2 - 12s + 180}$$

$$\frac{1}{3} \cdot \frac{60 - 4s}{s^2 - 12s + 180}$$

$$= \frac{1}{3} \frac{60 - 4s}{s^2 - 12s + 180} = \frac{60 - 4s}{s^2 - 12s + 180}$$

$$(i) L\{e^{-t} f(2t)\}$$

$$\begin{aligned} L\{f(2t)\} &= \frac{1}{2} \bar{f}\left(\frac{s}{2}\right) \\ &= \frac{1}{2} \cdot \frac{20 - 4\frac{s^2}{4}}{\left(\frac{s}{2}\right)^2 - 4\frac{s^2}{4} + 20} \\ &= \frac{20 - 2s^2}{s^2 - 2s + 20} = \frac{1}{2} \frac{20 - 2s^2}{s^2 - 8s + 80} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \frac{20 - 2s^2}{s^2 - 8s + 80} \times s^2 \\ &= \frac{40 - 4s^2}{s^2 - 8s + 80} \end{aligned}$$

$$\text{Now } L\{e^{-t} f(2t)\} = \frac{40 - 4(s+1)}{s^2 - 8(s+1) + 80}$$

$\Rightarrow$

## Laplace transform of derivatives

Statement :- If  $f(t)$  is continuous and of exponential

order and  $f'(t)$  is piecewise continuous then the Laplace transform of  $f'(t)$  is given by

$$L\{f'(t)\} = s \bar{f}(s) - f(0) \text{ where } \bar{f}(s) = L\{f(t)\}$$

Proof :- By the definition  $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt$

$$\begin{aligned} \text{Now } L\{f'(t)\} &= \int_0^\infty e^{-st} f'(t) dt \\ &= \int_0^\infty e^{-st} d[f(t)] \\ &= [e^{-st} f(t)]_0^\infty - \int_0^\infty s e^{-st} f(t) dt \\ &= [e^{-st} f(t)]_0^\infty + \int_0^\infty s e^{-st} f'(t) dt \\ &= [e^{-st} f(t)]_0^\infty + s \int_0^\infty e^{-st} f'(t) dt \\ &= 0 - 1 \cdot f(0) + s \int_0^\infty e^{-st} f'(t) dt \\ &= -f(0) + s L\{f'(t)\} \\ &= s L\{f(t)\} - f(0) \\ &= s \bar{f}(s) - f(0) \\ &= \end{aligned}$$

Theorem :- Prove that  $L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$

Proof :-

$$\begin{aligned} L\{f'(t)\} &= s \bar{f}(s) - f(0) \\ L\{f''(t)\} &= s L\{f'(t)\} - f'(0) \\ L\{f'''(t)\} &= s L\{f''(t)\} - f''(0) \\ &= s^2 \bar{f}(s) - s f(0) - f'(0) - f''(0) \\ &= s^3 \bar{f}(s) - s^2 f(0) - s f'(0) - f''(0) \end{aligned}$$

$$L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$$

- Using the theorem of transformation of derivatives find the L.T of the following functions.

- $e^{at}$
- $\cos at$
- $t \sin at$

Sol:- Given  
 $i) L\{f(t)\} = e^{at}$

$$\Rightarrow f(0) = 1$$

$$f'(t) = e^{at}, \alpha$$

Applying L.T on both sides

$$L\{f'(t)\} = L\{ae^{at}\}$$

$$sL\{f(t)\} - f(0) = aL\{e^{at}\}$$

$$sL\{e^{at}\} + aL\{e^{at}\} = aL\{e^{at}\}$$

$$sL\{e^{at}\} - aL\{e^{at}\} = 1$$

$$L\{e^{at}\}(s-a) = 1$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

$$\text{ii)} \quad f(t) = e^{at}$$

$$\Rightarrow f(0) = 1$$

$$f'(t) = (-\sin at) \cdot a = -a\sin at$$

$$L\{f'(t)\} = L\{a(-\sin at)\} \Rightarrow f'(0) = 0$$

$$f''(t) = -a \cos at, \alpha$$

$$\equiv -a^2 \cos at$$

Applying Laplace transform on both sides

$$L\{f''(t)\} = -L\{a^2 \cos at\}$$

$$\Rightarrow s^2 L\{f(t)\} - sf(0) - f'(0) = -L\{a^2 \cos at\}$$

$$\Rightarrow s^2 L\{f(t)\} - s \cdot 1 - f'(0) = -a^2 L\{e^{at}\} + 2a L\{e^{at}\}$$

$$s^2 L\{e^{at}\} - sf(0) - f'(0) = -a^2 L\{e^{at}\} + 2a L\{e^{at}\}$$

$$s^2 L\{e^{at}\} + a^2 L\{e^{at}\} = 2a \frac{s}{s^2 + a^2}$$

$$L\{e^{at}\}(s^2 + a^2) = \frac{2as}{s^2 + a^2}$$

$$L\{e^{at}\} = \frac{2as}{(s^2 + a^2)^2} //$$

$$\text{Q1:- Let } f(t) = \sin \frac{\pi t}{2} e^{-\lambda t} \text{ then find } L\left\{\frac{\cos \frac{\pi t}{2}}{t}\right\}$$

$$\Rightarrow g'(t) = f(t)$$

$$\begin{aligned} L\{g'(t)\} &= L\{f(t)\} \\ f'(t) &= \frac{\cos \frac{\pi t}{2}}{2\sqrt{s}} \\ f'(0) &= \frac{\cos 0}{2\sqrt{s}} \\ S L\{g(0)\} - g(0) &= \bar{f}(s) \\ S L\left\{\int_0^t f(\tau) d\tau\right\} - 0 &= \bar{f}(s) \\ L\left\{\int_0^t f(\tau) d\tau\right\} &= \frac{1}{s} \bar{f}(s) \end{aligned}$$

$$L\left\{\frac{\cos \frac{\pi t}{2}}{t}\right\} = S L\{\sin \frac{\pi t}{2}\} - 0$$

$$\frac{1}{s} L\left\{\frac{\cos \frac{\pi t}{2}}{t}\right\} = \frac{s \sqrt{\pi}}{2s^3 \cdot 2} e^{-\lambda s}$$

$$L\left\{\frac{\cos \frac{\pi t}{2}}{t}\right\} = \frac{\sqrt{\pi}}{s^2} e^{-\lambda s}$$

$$L\left\{\frac{\cos \frac{\pi t}{2}}{t}\right\} = \frac{\sqrt{\pi}}{s^2} e^{-\lambda s} = \sqrt{\frac{\pi}{s}} e^{-\lambda s}$$

Inverse transform of integrals

$$\text{Theorem:- If } L\{f(t)\} = \bar{f}(s) \text{ then } L\left\{\int_0^t f(t) dt\right\} = \frac{1}{s} \bar{f}(s)$$

$$\text{Proof:- Let } g(t) = \int_0^t f(t) dt \Rightarrow g(0) = \int_0^0 f(t) dt = 0$$

diff w.r.t to 't'

$$g'(t) = \frac{d}{dt} \int_0^t f(t) dt$$

Note:-  
If  $L\{f(t)\} = \bar{f}(s)$  then

$$\text{i)} L\left\{\int_0^t \int_0^t f(t) dt\right\} = \frac{1}{s^2} \bar{f}(s)$$

$$\text{ii)} L\left\{\int_0^t \int_0^t \int_0^t f(t) dt\right\} = \frac{1}{s^3} \bar{f}(s)$$

$$\text{iii)} L\left\{\int_0^t \int_0^t \dots n \text{ times} f(t) dt\right\} = \frac{1}{s^n} \bar{f}(s)$$

Find  $L\left\{\int_0^t e^{-\lambda t} f(t) dt\right\}$

$$\text{These } f(t) = e^{-\lambda t}$$

$$L\{f(t)\} = L\{e^{-\lambda t}\} = \frac{s+1}{(s+\lambda)^2} = \frac{s+1}{s^2 + 2s + 2}$$

$$\therefore L\left\{\int_0^t e^{-\lambda t} f(t) dt\right\} = \frac{1}{s} \bar{f}(s) = \frac{1}{s} \left[ \frac{s+1}{s^2 + 2s + 2} \right]$$

### Multiplication by $t^n$ :

If  $L\{f(t)\} = \bar{F}(s)$  then  $L\{t \cdot f(t)\} = (-1) \frac{d}{ds} \bar{F}(s)$

then for  $t^0$

$$L\{t^0 \cdot f(t)\} = (-1)^0 \frac{d^0}{ds^0} \bar{F}(s)$$

Proof:-

$$L\{f(t)\} = \bar{F}(s) = \int_0^\infty e^{-st} f(t) dt$$

Diff w.r.t to s

$$\frac{d}{ds} \bar{F}(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^\infty \frac{d}{ds} \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^\infty e^{-st} \frac{d}{dt} f(t) dt$$

$$= \int_0^\infty e^{-st} (-t) f(t) dt$$

$$= - \int_0^\infty e^{st} t \cdot f(t) dt$$

$$= (-1) \int_0^\infty e^{-st} t f(t) dt$$

$$= (-1) L\{t f(t)\}$$

$$\Rightarrow (-1) \frac{d}{ds} \bar{F}(s) = L\{t f(t)\}$$

By the method of mathematical induction  
it is true for  $n=1$

Let us assume that the theorem is true for  $n=k$

$$L\{t^k f(t)\} = (-1)^k \frac{d^k}{ds^k} \bar{F}(s)$$

$$\int_0^\infty e^{-st} t^k f(t) dt = (-1)^k \frac{d^k}{ds^k} \bar{F}(s)$$

diff w.r.t to s

$$\frac{d}{ds} \int_0^\infty e^{-st} t^k f(t) dt = (-1)^k \frac{d^{k+1}}{ds^{k+1}} \bar{F}(s)$$

$$\int_0^\infty e^{-st} (-t) t^k f(t) dt = (-1)^k \frac{d^{k+1}}{ds^{k+1}} \bar{F}(s)$$

$$\int_0^\infty e^{-st} t^{k+1} f(t) dt = (-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}} \bar{F}(s)$$

$$L\{t^{k+1} f(t)\} = (-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}} \bar{F}(s)$$

$\therefore$  The statement is true for  $n=k+1$

$$\therefore L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \bar{F}(s)$$

1. Evaluate  $L\{t \sin t\}$

Sol:- Let  $\sin at$ 's is in term of  $L\{f(t)\}$

Here  $f(t) = \sin at$   
 $L\{f(t)\} = L\{\sin at\} = \frac{a}{s^2+a^2} = \bar{F}(s)$

$$L\{t + f(t)\} = (-1) \frac{d}{ds} \bar{F}(s)$$
$$= (-1) \frac{d}{ds} \left[ \frac{1}{2} \left[ \frac{5}{s^2+25} + \frac{1}{s^2+1} \right] \right]$$

$$= (-1)^{\frac{1}{2}} \left[ 5 \frac{d}{ds} \left( \frac{1}{s^2+25} \right) + \frac{d}{ds} \left( \frac{1}{s^2+1} \right) \right]$$

$$= (-1) \left[ \frac{(s^2+a^2)(0) - a(2s)}{(s^2+a^2)^2} \right] = \frac{2as}{(s^2+a^2)^2}$$

$$= -\frac{5}{2} \left[ \frac{-2s}{(s^2+25)^2} \right] + \frac{1}{2} \left[ \frac{-2s}{(s^2+1)^2} \right]$$
$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

2. Find  $L\{t \sin st \cos st\}$

Sol:- Here  $f(t) = \sin st \cos st$

$$= \frac{1}{2} \sin(2st) \\ = \sin(A+B) + \sin(A-B)$$

$$= \frac{1}{2} [\sin 2st + \sin 0]$$

$$L\{f(t)\} = \frac{1}{2} [\sin st + L\{sint\}]$$

$$= \frac{1}{2} \left[ \frac{5}{s^2+25} + \frac{1}{s^2+1} \right]$$

3. Find  $L\{t^2 \sin st\}$

Sol:- Here  $f(t) = \sin 2t$   
 $L\{f(t)\} = L\{\sin 2t\} = \frac{2}{s^2+4} = \bar{F}(s)$

L.K.T  $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \bar{F}(s)$

$$L\{t^2 \sin 2t\} = (-1)^2 \frac{d^2}{ds^2} \left[ \frac{2}{s^2+4} \right]$$

$$= \frac{d}{ds} \left[ \frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2} \right] = \frac{d}{ds} \left[ \frac{-4s}{(s^2+4)^2} \right]$$
$$= \frac{(s^2+4)^2(-4) + 4s \cdot 2(s^2+4)(2s)}{(s^2+4)^4}$$
$$= \frac{4s(s^2+4)^2 + 16s^2(s^2+4)}{(s^2+4)^4}$$

$$= s^2 + 4 [-4(s^2 + 4) + 16s^2]$$

$$= \frac{s^2 + 4 [-4(s^2 + 4) + 16s^2]}{(s^2 + 4)^4}$$

$$= \frac{-4s^2 - 16s^2}{(s^2 + 4)^3} = \frac{-16 + 12s^2}{(s^2 + 4)^3}$$

$$= 4 \left[ \frac{3s^2 - 4}{(s^2 + 4)^3} \right]$$

=

4. Find  $L\{t^{\frac{1}{2}} t e^t \sin 2t\}$

Sol: By using Laplace transform of integral

$$L\{ \int_0^t t e^t \sin 2t dt \} = \frac{1}{s} L\{ t e^t \sin 2t \}$$

$$L\{ e^t t e^t \sin 2t \} = \frac{2}{(s+1)^2 + 4} = \frac{2}{s^2 + 2s + 5} \quad f(s)$$

$$\text{Now } L\{ t e^t \sin 2t \} = (-1) \frac{d}{ds} L\{ e^t \sin 2t \}$$

$$= (-1) \frac{d}{ds} \left[ \frac{2}{s^2 + 2s + 5} \right]$$

$$= (-1) \left[ \frac{s^2 + 2s + 5 - 2(s^2 + 2s)}{(s^2 + 2s + 5)^2} \right]$$

$$= (-1) \left[ \frac{-s^2 + 4s + 4}{(s^2 + 2s + 5)^2} \right]$$

$$= \frac{4s + 4}{(s^2 + 2s + 5)^2} = \frac{4(s+1)}{(s^2 + 2s + 5)^2}$$

$$\therefore L\left\{ \int_0^t t e^t \sin 2t dt \right\} = \frac{1}{s} \left[ \frac{4(s+1)}{(s^2 + 2s + 5)^2} \right]$$

=

5. Find  $L^{-1}\{ t^2 e^{-st} + t \cos at \}$

6. Find  $L\{t e^t \sin ht\}$

7. Find  $L^{-1}\{ t^2 e^{-st} \cos at \}$

$$\text{Sol: Let } f(t) = e^{-2t} \cos 2t$$

$$L\{f(t)\} = L\{e^{-2t} \cos 2t\} = \frac{s+2}{(s+2)^2 + 4} = \frac{s+2}{s^2 + 4s + 8}$$

$$L\{ t^2 e^{-2t} \cos 2t \} = (-1)^2 \frac{d^2}{ds^2} \left[ \frac{s+2}{s^2 + 4s + 8} \right]$$

$$= (1) \frac{d}{ds} \left[ \frac{(s^2 + 4s + 8)(1) - (2s + 2)(2s + 4)}{(s^2 + 4s + 8)^2} \right]$$

$$= \frac{d}{ds} \left[ \frac{s^2 + 4s + 8 - 2s^2 - 8s}{(s^2 + 4s + 8)^2} \right]$$

$$= \frac{d}{ds} \left[ \frac{-s^2 + 4s}{(s^2 + 4s + 8)^2} \right]$$

$$= \frac{d}{ds} \left[ -\frac{s^2 + 4s}{(s + 4s + 8)^2} \right]$$

$$= \frac{(s^2 + 4s + 8)^2(-2s + 4) - (-s^2 + 4s) \cancel{2(s+4)} \cancel{2(s+4s+8)(2s+4)}}{(s^2 + 4s + 8)^4}$$

$$(s^2 + 4s + 8)^4$$

$$= \frac{(s^2 + 4s + 8) \left[ (s^2 + 4s + 8)(-2s + 4) - (-s^2 + 4s) \cancel{2(2s+4)} \right]}{(s^2 + 4s + 8)^4}$$

$$= \frac{-2s^3 - 4s^2 - 16s + 4s^2 + 16s + 32 - (-4s^3 + 16s^2 - 8s^2 + 32s)}{(s^2 + 4s + 8)^3}$$

$$= \frac{-2s^3 - 4s^2 + 4s^2 + 16s + 32 + 4s^3 - 16s^2 + 8s^2 - 32s}{(s^2 + 4s + 8)^3}$$

$$= \frac{2s^3 - 8s^2 - 32s + 32}{(s^2 + 4s + 8)^3} = 2 \frac{(s^3 - 4s^2 - 16s + 16)}{(s^2 + 4s + 8)^3}$$

$$=$$

Division by  $t^0$ :

$$\text{Theorem: If } \mathcal{L}\{f(t)\} = F(s) \text{ then } \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s) ds$$

Proof:-

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s)$$

$$\begin{aligned} \text{Now } \int_0^\infty \frac{f(t)}{t} dt &= \int_s^\infty \int_0^\infty e^{-st} f(t) dt ds \\ &= \int_s^\infty \int_0^\infty e^{-st} + f(t) dt ds \end{aligned}$$

$$= \int_s^\infty \int_0^\infty e^{-st} + f(t) dt ds$$

$$= \int_{t=0}^{t=\infty} \int_s^\infty e^{-st} f(t) dt ds$$

$$= \int_{t=0}^{t=\infty} \left[ \int_s^\infty e^{-st} \frac{f(t)}{t} dt \right] ds$$

$$= \int_{t=0}^{t=\infty} \left[ \frac{e^{-st}}{-t} \right]_s^\infty f(t) dt$$

$$= \int_{t=0}^{t=\infty} \left[ \frac{e^{-s0} - e^{-st}}{-t} \right] f(t) dt$$

$$= \int_{t=0}^{t=\infty} \frac{0 - e^{-st}}{-t} f(t) dt$$

$$= \int_{t=0}^{t=\infty} \frac{e^{-st}}{t} f(t) dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t} dt$$

$$= L\left\{ \frac{f(t)}{t} \right\}$$

$$\text{Hence } L\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty f(s) ds$$

$\therefore$

$$= \int_s^\infty \left[ \log \frac{s+a}{s+b} \right] ds$$

$$= \left[ \log \left( \frac{s+a}{s+b} \right) \right]_s^\infty$$

$$1. \text{ Find } L\left\{ \frac{\sin t}{t} \right\}$$

$$\text{Sol: } L\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty L\left\{ \sin t \right\} ds$$

$$= \int_s^\infty \frac{1}{s+1} ds$$

$$= \left[ \tan^{-1} s \right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$\left( \because \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2} \right)$$

$$\left( Q.S \right) \tan \frac{\pi}{2} - \tan^{-1}s = x$$

$$\frac{\pi}{2} - x = \tan^{-1}s$$

$$\tan \left( \frac{\pi}{2} - x \right) = s$$

$$\cot x = s$$

$$\cot x = \cot^{-1}s$$

$$3. \text{ Find } L\left\{ \frac{\sin t \cos t}{t} \right\}$$

$$\text{Sol: } L\left\{ \frac{\sin t \cos t}{t} \right\} = \int_s^\infty L\left\{ \sin t \cos t \right\} dt$$

$$L\left\{ \sin 2st \right\} = \frac{1}{2} L\left\{ 2 \sin t \cos t \right\}$$

$$L\left\{ \sin st \right\} = \frac{1}{2} \left[ L\left\{ \sin t \right\} + L\left\{ \cos t \right\} \right]$$

$$= \frac{1}{2} \left[ \frac{4}{s^2+16} + \frac{2}{s^2+4} \right]$$

$$\therefore L \left\{ \sin \frac{x}{2} \right\} = \int_s^\infty \frac{1}{2} \left[ \frac{4}{s^2+16} + \frac{2}{s^2+4} \right] ds$$

$$= \int_s^\infty \left( \frac{2}{s^2+16} + \frac{1}{s^2+4} \right) ds$$

$$= \frac{1}{a} \tan^{-1} \left( \frac{s}{a} \right)$$

$$= \int_s^\infty \left( \frac{2}{s^2+a^2} + \frac{1}{s^2+2^2} \right) ds$$

$$= \left[ \tan^{-1}(as) \right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$= \left[ \frac{1}{2} \tan^{-1} \left( \frac{s}{4} \right) + \frac{1}{2} \tan^{-1} \left( \frac{s}{2} \right) \right]_s^\infty$$

$$= \left( \frac{1}{2} \cdot \frac{\pi}{2} + \frac{1}{2} \cdot \frac{\pi}{2} \right) - \left[ \frac{1}{2} \tan^{-1} \left( \frac{s}{4} \right) + \frac{1}{2} \tan^{-1} \left( \frac{s}{2} \right) \right]$$

$$= \left( \frac{\pi}{4} + \frac{\pi}{4} \right) - \frac{1}{2} \left( \tan^{-1} \left( \frac{s}{4} \right) + \frac{1}{2} \tan^{-1} \left( \frac{s}{2} \right) \right)$$

$$= \frac{\pi}{2} - \frac{1}{2} \left( \tan^{-1} \left( \frac{s}{4} \right) + \frac{1}{2} \tan^{-1} \left( \frac{s}{2} \right) \right)$$

4. Find  $L \left\{ \frac{1}{t} \right\}$

5. Find  $L \left\{ \int_0^t e^{st} dt \right\}$

$$\begin{aligned} \therefore L \left\{ \int_0^t e^{st} dt \right\} &= \frac{1}{s} L \left\{ e^{st} \right\} \\ &= \frac{1}{s} L \left\{ e^{\frac{st}{s}} \right\} \end{aligned}$$

$$= \frac{1}{s} \left[ L \left\{ \frac{e^{st}}{s} \right\} \right]_{s \rightarrow s-1}$$

$$L \left\{ \frac{e^{st}}{s} \right\} = \int_s^\infty e^{st} ds$$

$$= \int_s^\infty \frac{1}{s} \left( s e^{st} \right) ds$$

$$= \frac{1}{s} \left[ e^{st} \right]_s^\infty = \frac{1}{s} [e^{s(s-1)}]$$

$$\therefore L \left\{ \int_0^t e^{st} dt \right\} = \frac{1}{s} [e^{s(s-1)}]$$

(Ans)

$$\begin{aligned} \text{Q.E.D.} - L \left\{ \int_0^t e^{st} dt \right\} &= \frac{1}{s} L \left\{ e^{st} \right\} \\ &= \frac{1}{s} L \left\{ e^{st} \right\} \end{aligned}$$

$$L \left\{ e^{st} \right\} = \int_s^\infty L \left\{ e^{st} \right\} ds$$

$$= \int_s^\infty \left( \frac{1}{(s-1)^2+1} \right) ds$$

$$= \left[ \tan^{-1}(s-1) \right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s-1)$$

$$= \cot^{-1}(s-1)$$

$$\therefore L \left\{ \int_0^t e^{st} dt \right\} = \frac{1}{s} \left[ \cot^{-1}(s-1) \right]$$

## Evaluation of Integrals by Laplace Transforms :-

Sometimes evaluation of improper integrals i.e., integrals having lower limit 0 and upper limit  $\infty$  can be done easily by using Laplace transform technique

- using Laplace transform evaluate  $\int_0^\infty t e^{-st} \sin t dt$

sol:- we know that  $\int_0^\infty e^{-st} f(t) dt = L\{f(t)\}$

$$\therefore \int_0^\infty t e^{-st} \sin t dt = L\{t \sin t\} \quad (\text{where } s=)$$

$$= (-1) \frac{d}{ds} L\{\sin t\}$$

$$= (-1) \frac{d}{ds} \frac{1}{s^2+1}$$

$$= - \left[ \frac{(s^2+1)^{-1}(2s)}{(s^2+1)^2} \right] = \frac{2s}{(s^2+1)^2}$$

$$= \frac{2s}{s^4+2s^2+1}$$

$$( \because s=1 )$$

$$= \frac{2}{1+2+1}$$

$$= \frac{2}{4} = \frac{1}{2}$$

- using Laplace transform evaluate  $\int_0^\infty e^{st} \frac{e^t - e^{-2t}}{t} dt$

sol:- we know  $\int_0^\infty e^{-st} f(t) dt = L\{f(t)\}$

$$\begin{aligned} & \int_0^\infty e^{-st} \left[ \frac{e^t - e^{-2t}}{t} \right] dt = L\left\{ \frac{e^t - e^{-2t}}{t} \right\} \\ & = \int_0^\infty L\left\{ \frac{e^t - e^{-2t}}{t} \right\} ds \\ & = \int_0^\infty \left[ \log(s+1) - \log(s+2) \right] ds \\ & = \left[ \log\left(\frac{s+1}{s+2}\right) \right]_0^\infty \\ & = \log 1 - \log\left(\frac{s+1}{s+2}\right) \\ & = - [\log(s+1) - \log(s+2)] \\ & = \log(s+2) - \log(s+1) = \log\left(\frac{s+2}{s+1}\right) \\ & = \log 2 \quad (\text{where } s=0) \end{aligned}$$

- Evaluate  $\int_0^\infty e^{-4t} \left[ \frac{2 \sin t - 3 \sinh t}{t} \right] dt$

sol:- we know  $\int_0^\infty e^{-st} f(t) dt = L\{f(t)\}$

$$\begin{aligned} & \int_0^\infty e^{-st} \left[ \frac{2 \sin t - 3 \sinh t}{t} \right] dt = L\left\{ \frac{2 \sin t - 3 \sinh t}{t} \right\} \\ & = \int_0^\infty L\left\{ 2 \sin t - 3 \sinh t \right\} ds \end{aligned}$$

$$= \int_s^0 \left[ \frac{1}{s^2+1} - 3 \frac{1}{s^2+1} \right] ds$$

[ $\int \frac{1}{x^2+a^2} dx = \frac{1}{2a} \log\left(\frac{x^2+a^2}{a^2}\right)$ ]

$$= \int_0^\infty \left( \frac{2}{s^2+1} - \frac{3}{s^2+1} \right) ds$$

$$= 2 \left[ \tan^{-1}(s) \right]_0^\infty - 3 \left[ \frac{1}{2} \log\left(\frac{s^2+1}{1^2+1}\right) \right]_0^\infty$$

$$= 2 \left[ \tan^{-1}(0) - \tan^{-1}(\infty) \right] - \frac{3}{2} \left[ \log\left(\frac{s^2+1}{1^2+1}\right) \right]_0^\infty$$

$$= 2 \left[ \frac{\pi}{2} - \tan^{-1}(\infty) \right] - \frac{3}{2} \left[ \log\left[\frac{1}{2}\left(1-\frac{1}{2}\right)\right] - \log\left(\frac{1}{2}\right) \right]$$

$$= 2 \left[ \frac{\pi}{2} - \tan^{-1}(\infty) \right] - \frac{3}{2} \left[ \log(1) - \log\left(\frac{1}{2}\right) \right]$$

where  $s = 4$

$$= 2 \left[ \frac{\pi}{2} - \tan^{-1}(4) \right] - \frac{3}{2} \left[ -\log\left(\frac{1}{2}\right) \right]$$

$$4. \int_0^\infty t^2 e^{-4t} \sin 2t dt$$

Replace transforms of periodic functions

Periodic function:— A function  $f(x)$  is said to be  $\omega$  periodic.

$P$  iff  $f(x+P) = f(x)$  where  $P$  is least true integer.

Ex:-  $\sin x$  and  $\cos x$  are periodic functions with period  $2\pi$ .

2.  $\tan x$  and  $\cot x$  are periodic functions with period  $\pi$

③  $\sin x$  and  $\cos x$  are periodic functions with period  $2\pi$

Note:- If  $P$  is a period of  $f(x)$  then the period of  $f(ax+b)$  is  $\frac{P}{a}$

Theorem:— If  $f(t)$  is a periodic function with period  $T$  then  $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$

Proof:— By the def of L.T

$$L[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^T e^{-st} f(t) dt + \int_T^{sT} e^{-st} f(t) dt + \int_{sT}^{3T} e^{-st} f(t) dt + \dots \dots$$

Put  $t = u+T$

$dt = du$

If  $t=T$  then  $u=0$

$t=2T$  then  $u=T$

Put  $t=u+2T$

$dt = du$

If  $t=2T$  then  $u=0$

$t=3T$  then  $u=T$

$$= \int_0^T e^{-st} f(t) dt + \int_0^T e^{-su} e^{-sT} f(u+T) du + \int_0^T e^{-su} e^{-2sT} f(u+2T) du + \dots$$

$$= \int_0^T e^{-st} f(t) dt + \int_0^T e^{-su} e^{-sT} f(u+T) du + \int_0^T e^{-su} e^{-2sT} f(u+2T) du + \dots$$

$$= \int_0^T e^{-st} f(t) dt + e^{-sT} \int_0^T e^{-su} f(u+T) du + e^{-2sT} \int_0^T e^{-su} f(u+2T) du + \dots$$

$$= \int_0^T e^{-st} f(t) dt + e^{-sT} \int_0^T e^{-su} f(u) du + e^{-2\pi s} \int_0^T e^{-su} f(u) du \dots$$

$\therefore f(t)$  is periodic with period  $T$   
 $f(t) = f(t+T) = f(t+2\pi) \dots$

Replacing  $u$  by  $t$

$$\{f(t)\} = \int_0^T e^{-st} f(t) dt + e^{-sT} \int_0^T e^{-st} f(t) dt + e^{-2\pi T} \int_0^T e^{-st} f(t) dt + \dots$$

$$= [1 + e^{-sT} + e^{-2\pi T} + \dots] \{f(t)\}$$

$$= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

$$\boxed{\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt}$$

Hence the theorem is proved

1. Find  $\{f(t)\}$  where  $f(t)$  is a periodic function of period  $2\pi$  and it is given by

$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

$$\text{Sol: } \text{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{-s(2\pi)}} \int_0^{2\pi} e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{-sT}} \left[ \int_0^T e^{-st} f(t) dt + \int_T^{2\pi} e^{-st} f(t) dt \right]$$

$$= \frac{1}{1 - e^{-sT}} \left[ \int_0^T e^{-st} \sin t dt + \int_T^{2\pi} e^{-st} \cos t dt \right]$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[ \frac{e^{-st}}{s^2 + 1} [(-s \sin t - \cos t)] \right]_0^{\pi}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[ \frac{e^{-s\pi}}{s^2 + 1} (0 - (-\cos \pi)) - \frac{e^0}{s^2 + 1} (0 - \cos 0) \right]$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[ \frac{e^{-s\pi}}{s^2 + 1} (1) - \frac{e^0}{s^2 + 1} (1) \right]$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[ \frac{e^{-s\pi}}{s^2 + 1} + \frac{1}{s^2 + 1} \right] = \frac{e^{-s\pi} + 1}{(s^2 + 1)(1 - e^{-2\pi s})} //$$

2. Find  $\{f(t)\}$  where  $f(t) = \begin{cases} t & 0 < t < b \\ 2b-t & b < t < 2b \end{cases}$

$ab$  being the period of  $f(t)$ .

$\text{Sol: } \text{Given } f(t) \text{ is a periodic function with period } 2b$

i.e.,  $T = 2b$

$$\text{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-sb}} \int_0^{sb} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-sb}} \left[ \int_0^b e^{-st} t dt + \int_b^{2b} e^{-st} (2b-t) dt \right]$$

$$= \frac{1}{1-e^{-sb}} \left\{ t \left[ -\frac{e^{-st}}{s} \right] \Big|_0^b + \left[ (2b-t) \frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right] \Big|_b^{2b} \right\}$$

$$= \frac{1}{1-e^{-sb}} \left\{ \left[ \left( b \frac{e^{-sb}}{s} - \frac{e^{-sb}}{s^2} \right) - \left( 0 - \frac{1}{s} \right) \right] + \left[ \left( 0 + \frac{e^{-2bs}}{s^2} \right) - \frac{b e^{-sb}}{s} + \frac{e^{-sb}}{s^2} \right] \right\}$$

$$= \frac{1}{1-e^{-2bs}} \left\{ -b e^{-sb} - \frac{e^{-sb}}{s^2} + \frac{1}{s} + \frac{e^{-2bs}}{s^2} + b e^{-sb} - \frac{e^{-sb}}{s^2} \right\}$$

$$= \frac{1}{1-e^{-2bs}} \left[ \frac{1}{s^2} - 2 \frac{e^{-sb}}{s^2} + \frac{e^{-2bs}}{s^2} \right]$$

Q. If  $f(t) = \int_{-1}^t e^{-st} f(t) dt$  is a periodic function with period  $T$ , find its L.T.

- 1)  $L\{1\} = \frac{1}{s} \Rightarrow L^{-1}\{\frac{1}{s}\} = 1$
- 2)  $L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{n!}$
- 3)  $L\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{1}{a} \sin at$
- 4)  $L\{\cos at\} = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$
- 5)  $L\{\sinh at\} = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{\sinh at}{a}$

$$6) L\{\cosh at\} = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\{\frac{s}{s^2+a^2}\} = \cosh at$$

## Inverse Laplace Transforms

The inverse Laplace transforms are useful for solving differential equations without finding general solutions and arbitrary constants. We know that  $L\{f(t)\} = \bar{F}(s)$   $\Rightarrow f(t) = L^{-1}\{\bar{F}(s)\}$ . Here  $f(t)$  is called inverse Laplace transform of  $\bar{F}(s)$  and the symbol  $L^{-1}$  stands for the inverse Laplace transform operators.

Standard formulae :-

$$L^{-1}\{\frac{1}{s}\} = 1$$

$$L^{-1}\{\frac{1}{s^n}\} = \frac{t^{n-1}}{(n-1)!}$$

$$L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{1}{a} \sin at$$

$$L^{-1}\{\frac{s}{s^2+a^2}\} = \cosh at$$

$$7) \quad L\{e^{at}\} = \frac{1}{s-a} \Rightarrow L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$8) \quad L\{e^{-at}\} = \frac{1}{s+a} \Rightarrow L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

Note:- Linearity Property

$$L^{-1}\left\{c_1 f(s) + c_2 g(s)\right\} = c_1 L^{-1}\{f(s)\} + c_2 L^{-1}\{g(s)\}$$

where  $c_1$  and  $c_2$  are constants and  $f(s)$  and  $g(s)$  are

Laplace transforms of  $f(t)$  and  $g(t)$  respectively.

Q) Find  $L^{-1}\left\{\frac{s^2 - 3s + 4}{s^3}\right\}$

$$s^2 - 3s + 4 = s^2 - \frac{3s}{s^2} + \frac{4}{s^3}$$

$$= L^{-1}\left\{s^2 - \frac{3}{s^2} + \frac{4}{s^3}\right\}$$

$$= L^{-1}\left\{s^2\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 1 - 3 \cdot \frac{t}{1!} + 4 \cdot \frac{t^2}{2!}$$

$$= 1 - 3t + \frac{4t^2}{2}$$

$$= 1 - 3t + 2t^2$$

Q) Find  $L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\}$

$$s^2 - 2 = L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\} = L^{-1}\left\{\frac{3(s^4+4-4s^2)}{2s^5}\right\}$$

$$= L^{-1}\left\{\frac{3s^4}{2s^5} + \frac{12}{2s^5} - \frac{12s^2}{2s^5}\right\}$$

$$= L^{-1}\left\{\frac{3}{2s^2} + \frac{6}{s^5} - \frac{6}{s^3}\right\}$$

$$= L^{-1}\left\{\frac{3}{2} \cdot \frac{1}{s^2} + 6L^{-1}\left\{\frac{1}{s^5}\right\} - 6L^{-1}\left\{\frac{1}{s^3}\right\}\right\}$$

$$= \frac{3}{2}(1) + 6\left(\frac{t^4}{4!}\right) - 6\left(\frac{t^2}{2!}\right)$$

$$= \frac{3}{2} + \frac{6t^4}{24} - \frac{6t^2}{2}$$

$$= \frac{3}{2} + \frac{t^4}{4} - 3t^2$$

Sol:- Inverse Laplace Transform by using Partial fractions:-

1. Find the inverse L.T of  $\frac{4}{(s+1)(s+2)}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{4}{(s+1)(s+2)}\right\}$$

$$= 4 L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} \rightarrow \textcircled{1}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \rightarrow \textcircled{2}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$$

$$1 = A(s+2) + B(s+1)$$

$$As + 2A + Bs + B = 1$$

composing coefficients

$$\begin{array}{l} A+B=0 \\ \cancel{A+B=1} \\ -A=-1 \end{array}$$

$$A=1$$

$$A+B=0 \\ 1+B=0 \Rightarrow B=-1$$

$$\text{from } \textcircled{1} \\ \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

from \textcircled{1}

$$\begin{aligned} L^{-1}\{\bar{f}(s)\} &= 4 L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} \\ &= 4 \left[ L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\} \right] \end{aligned}$$

$$= 4 [e^{-t} - e^{-2t}]$$

Q. Find  $L^{-1}\left\{\frac{s^2+s-2}{s(s+3)(s-2)}\right\}$

$$\text{given } L^{-1}\int \frac{s^2+s-2}{s(s+3)(s-2)} \int e^{-st}$$

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$$

$$s^2+s-2 = A(s+3)(s-2) + B(s(s+3)) + C(s(s-2))$$

$$s^2+s-2 = A[s^2-2s+3s-6] + B(s^2) + C(s^2)$$

$$s^2+s-2 = A[s^2-2s+3s-6] + B(s^2) + C(s^2)$$

$$s^2+s-2 = As^2 + Bs^2 + Cs^2 - 6A + Bs^2 - 2Bs + Cs^2 + 3Cs$$

composing coefficients

$$A+B+C = 4 \rightarrow \textcircled{1}$$

$$A-2B+3C = 1 \rightarrow \textcircled{2}$$

$$-6A = -2 \rightarrow \textcircled{3}$$

$$\Rightarrow A = \frac{1}{3}$$

$$\text{Solving } \textcircled{1} \text{ and } \textcircled{2}$$

$$\begin{array}{l} \textcircled{1} \Rightarrow 3A + 3B + 3C = 1 \\ \textcircled{2} \Rightarrow A - 2B + 3C = 1 \\ \hline 2A + 5B = 2 \end{array}$$

=

$$\frac{2}{3} + \frac{5B}{5B} = 2 - \frac{2}{3} = \frac{4}{3}$$

$$B = \frac{4}{15}$$

from \textcircled{1}

$$\frac{1}{3} + \frac{4}{15} + C = 1$$

$$\frac{15}{15} + \frac{4}{15} + C = 1$$

$$\frac{9}{15} + C = 1$$

$$C = 1 - \frac{9}{15} = \frac{2}{5}$$

$$\therefore L^{-1}\left\{\frac{s^2+s-2}{s(s+3)(s-2)}\right\} = L^{-1}\left\{\frac{1}{3s}\right\} + \frac{4}{15(s+3)} + \frac{2}{5(s-2)}$$

$$= \frac{1}{3} L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{15} L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{2}{5} L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \frac{1}{3}(1) + \frac{4}{15} e^{-3t} + \frac{2}{5} e^{2t}$$

$$= \frac{1}{3} + \frac{4}{15} e^{-3t} + \frac{2}{5} e^{2t}$$

$$3. L^{-1} \left\{ \frac{s^2+2s-4}{(s^2+9)(s-5)} \right\}$$

$$s^2 - \frac{s^2+2s-4}{(s^2+9)(s-5)} = \frac{As+B}{(s^2+9)} + \frac{C}{(s-5)}$$

$$s^2+2s-4 = (As+B)(s-5) + C(s^2+9)$$

$$(s^2+2s-4) = As^2 + Bs - 5As + 5B + Cs^2 + 9C$$

comparing coefficients

$$A+C=1 \rightarrow ①$$

$$B-5A=2 \rightarrow ②$$

$$\cancel{As^2} - 4 \rightarrow ③$$

$$9C-5B= -4 \rightarrow ④$$

solve ① and ②

$$\begin{aligned} ① \times ④ 5 &\Rightarrow 5A+5C=5 \\ ② &\Rightarrow -5A+B=2 \end{aligned}$$

$$\frac{-5A+B=2}{B+5C=7} \rightarrow ⑤$$

from ③ & ④

$$9C-5B=-4$$

$$(⑤) \times 5 \Rightarrow 25C+5B=35$$

$$\frac{34C}{34C}=31$$

$$C=31/34$$

from ①

$$A+C=1$$

$$A+\frac{31}{34}=1$$

$$A=1-\frac{31}{34}=\frac{3}{34}$$

from ②

$$B-5A=2$$

$$B-\cancel{5(\frac{3}{34})}=2$$

$$B-\frac{15}{34}=2$$

$$B=2+\frac{15}{34}=\frac{83}{34}$$

$$\therefore L^{-1} \left\{ \frac{s^2+2s-4}{(s^2+9)(s-5)} \right\} = L^{-1} \left\{ \frac{As+B}{s^2+9} + \frac{C}{s-5} \right\}$$

$$= L^{-1} \left\{ \frac{\frac{3}{34}s + \frac{83}{34}}{s^2+9} \right\} + L^{-1} \left\{ \frac{\frac{31}{34}}{s-5} \right\}$$

$$= \frac{3}{34} L^{-1} \left\{ \frac{s}{s^2+9} \right\} + \frac{83}{34} L^{-1} \left\{ \frac{1}{s^2+9} \right\} + \frac{31}{34} L^{-1} \left\{ \frac{1}{s-5} \right\}$$

$$= \frac{3}{34} L^{-1} \left\{ \frac{s^2}{s^2+3^2} \right\} + \frac{83}{34} L^{-1} \left\{ \frac{1}{s^2+3^2} \right\} + \frac{31}{34} L^{-1} \left\{ \frac{1}{s-5} \right\}$$

$$= \frac{3}{34} (\cos 3t) + \frac{83}{34} \left( \frac{\sin 3t}{3} \right) + \frac{31}{34} (e^{5t})$$

=

$$4. \text{ Find } L^{-1} \left\{ \frac{s^2}{(s^2+4)(s^2+9)} \right\}$$

$$\begin{aligned} s^2 &= \frac{s^2}{(s^2+4)(s^2+9)} = \frac{As+B}{(s^2+4)} + \frac{Cs+D}{(s^2+9)} \end{aligned}$$

$$s^2 = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$s^2 = As^3 + Bs^2 + 9As + 9B + Cs^3 + Ds^2 + 4Cs + 4D$$

$$A+C=0 \rightarrow ①$$

$$B+D=1 \rightarrow ②$$

$$\begin{aligned} q_A + 4C &= 0 \rightarrow ③ \\ q_B + 4D &= 0 \rightarrow ④ \end{aligned}$$

$$\begin{aligned} \text{Solving } ① \& ③ \\ ① \times q &\Rightarrow q_B + qC = 0 \\ q_B + 4C &= 0 \\ \cancel{q_B - qC} \\ 5C &= 0 \Rightarrow C = 0 \end{aligned}$$

$$\begin{aligned} \text{Solving } ② \& ④ \\ ② \times q &\Rightarrow q_B + qD = 0 \\ q_B + 4D &= 0 \\ \cancel{q_B - qD} \\ 5D &= 0 \end{aligned}$$

$$\begin{aligned} D &= 0 \\ q_B &= 0 \end{aligned}$$

$$\begin{aligned} \text{from } ① \\ A+C &= 0 \Rightarrow A = 0 \end{aligned}$$

$$\begin{aligned} \text{from } ② \\ B+D=1 &\Rightarrow B+\frac{1}{5}=1 \Rightarrow B=1-\frac{1}{5}=-\frac{4}{5} \\ B+D &= 1 \Rightarrow B+\frac{1}{5}=1 \Rightarrow B=1-\frac{1}{5}=-\frac{4}{5} \end{aligned}$$

$$\begin{aligned} L^{-1} \left\{ \frac{s^2}{(s^2+4)(s^2+9)} \right\} &= L^{-1} \left\{ (s^2 + \frac{(-4)}{5}) \right\} + L^{-1} \left\{ (s^2 + \frac{9}{5}) \right\} \\ &= -\frac{4}{5} L^{-1} \left\{ \frac{1}{s^2+2^2} \right\} + \frac{9}{5} L^{-1} \left\{ \frac{1}{s^2+3^2} \right\} \end{aligned}$$

$$\begin{aligned} &= -\frac{4}{5} L^{-1} \left\{ \frac{\sin 2t}{2} \right\} + \frac{9}{5} L^{-1} \left\{ \frac{\sin 3t}{3} \right\} \\ &= -\frac{2}{5} \sin 2t + \frac{3}{5} \sin 3t \\ &= \end{aligned}$$

$$\begin{aligned} 5. \text{ Find } L^{-1} \left\{ \frac{s}{s^2+4s+5} \right\} & \quad [\text{Ans}: -s^2+4s+5 = (s+2)^2+1] \end{aligned}$$

First shifting theorem :-

$$\text{If } L^{-1}\{f(s)\} = f(t), \text{ then } L^{-1}\{f(s-a)\} = e^{at}f(t)$$

$$\text{If } L^{-1}\{f(s+a)\} = e^{-at}f(t).$$

Solving ② & ④

$$\begin{aligned} ② \times q &\Rightarrow q_B + qD = 0 \\ q_B + 4D &= 0 \\ \cancel{q_B - qD} \\ 5D &= 0 \end{aligned}$$

Problems :-

- Find  $L^{-1}\left\{ \frac{s+2}{(s-2)^3} \right\}$  using first shifting theorem.

$$\begin{aligned} L^{-1} \left\{ \frac{s+2}{(s-2)^3} \right\} &= \frac{s+2-2+2}{(s-2)^3} = \frac{(s-2)+4}{(s-2)^3} \\ L^{-1}\left\{ \frac{s+2}{(s-2)^3} \right\} &= L^{-1}\left\{ \frac{4}{(s-2)^3} \right\} + 4 L^{-1}\left\{ \frac{1}{(s-2)^3} \right\} \end{aligned}$$

$$= e^{2t} L\left\{\frac{s}{s^2 + 3}\right\} + 4 e^{2t} L\left\{\frac{1}{s^2 + 3}\right\}$$

$$= e^{2t} L^{-1}\left\{ \frac{1}{s^2} \right\} + 4 e^{2t} L^{-1}\left\{ \frac{1}{s^2 + 3} \right\}$$

$$= e^{2t} \frac{t}{1!} + 4 e^{2t} \frac{t^2}{2!}$$

$$= e^{2t} \frac{t}{1!} + 4 e^{2t} \frac{t^2}{2!} \\ = t e^{2t} + 2 e^{2t} t^2 = e^{2t} [t + 2t^2]$$

2. Find  $L^{-1}\left\{ \frac{s}{(s+3)^2} \right\}$

$$\text{Sol: } \frac{s}{(s+3)^2} = \frac{s+3-3}{(s+3)^2} = \frac{s+3}{(s+3)^2} - \frac{3}{(s+3)^2}$$

$$L^{-1}\left\{ \frac{s}{(s+3)^2} \right\} = L^{-1}\left\{ \frac{s+3}{(s+3)^2} \right\} - 3 L^{-1}\left\{ \frac{1}{(s+3)^2} \right\}$$

$$= e^{-3t} L^{-1}\left\{ \frac{s}{s^2} \right\} - 3 L^{-1}\left\{ \frac{1}{s^2} \right\} e^{-3t} \\ = e^{-3t} L^{-1}\left\{ \frac{1}{s^2} \right\} - 3 L^{-1}\left\{ \frac{1}{s^2} \right\} e^{-3t}$$

$$= e^{-3t} [1] - 3e^{-3t} \frac{t}{1!} \\ = e^{-3t} [1 - 3t]$$

3. Find  $L\left\{ \frac{s}{s^4 + 4s^4} \right\}$  using first shifting theorem.

$$\text{Sol: } s^4 + 4s^4 = (s^2)^2 + (2s^2)^2 \\ = (s^2 + 2s^2)^2 - 2s^2 \cdot 2s^2 \\ = (s^2 + 2s^2)^2 - (2s^2)^2 \\ = (s^2 + 2s^2)(s^2 + 2s^2 - 2s^2)$$

$$= (s^2 + 2s^2)(s^2 + 2s^2 - 2s^2) \\ = (s^2 + 4s^4)$$

$$\text{Now } \frac{s}{s^4 + 4s^4} = \frac{s}{(s^2 + 2s^2 + 2s^2)(s^2 + 2s^2 - 2s^2)} \\ s = \frac{As + B}{s^2 + 2s^2 + 2s^2} + \frac{Cs + D}{s^2 + 2s^2 - 2s^2}$$

$$\begin{aligned} s &= (As + B)(s^2 + 2s^2 + 2s^2) + (Cs + D)(s^2 + 2s^2 - 2s^2) \\ &= As^3 + Bs^2 + 2s^2 As + 2s^2 B - 2s^2 Bs + Cs - 2s^2 Bs \\ &\quad + Cs^3 + Ds^2 + 2s^2 Cs + 2s^2 D + 2s^2 Cs + 2s^2 D \end{aligned}$$

Comparing coefficients

$$A + C = 0 \rightarrow \textcircled{1} \Rightarrow A = -C$$

$$B + D = 0 \rightarrow \textcircled{2}$$

$$\begin{aligned} -2s^2 A + B + 2s^2 C + D &= 0 \rightarrow \textcircled{3} \\ \text{where } A &= -C \\ 2s^2 C + B + 2s^2 C + D &= 0 \rightarrow \textcircled{3} \\ \Rightarrow 4s^2 C + B + D &= 0 \rightarrow \textcircled{3} \\ 2s^2 C - 2s^2 B + 2s^2 C + 2s^2 D &= 1 \rightarrow -2Ba + 2Da = 1 \rightarrow \textcircled{3} \\ -2s^2 B - 2s^2 B + 2s^2 C + 2s^2 D &= 1 \Rightarrow -2Ba + 2Da = 1 \rightarrow \textcircled{3} \end{aligned}$$

Comparing coefficients of  $s^2$

$$A+C=0 \rightarrow ①$$

$$\text{we have } A = -C$$

$$\begin{aligned} \text{Comparing coefficients of } s^2 \\ -2Aa + B + 2Ac + D = 0 & \quad (A = -C) \\ -2a^2 + B + 2a^2 + D = 0 \end{aligned}$$

$$B + 2D = 0 \rightarrow ②$$

Comparing coefficients of  $s$

$$2Aa^2 - 2aB + 2a^2C + 2aD = 1$$

$$\text{sub } A = -C$$

$$-2a^2 - 2aB + 2(-Ca^2) + 2aD = 1$$

$$-2Ba + 2Da = 1 \rightarrow ③$$

Comparing constants

$$2Ba^2 + 2Da^2 = 0$$

$$a^2(2B + 2D) = 0$$

$$2a^2(B + D) = 0$$

$$B + D = 0 \rightarrow ④$$

$$\Rightarrow B = -D$$

$$2Da + 2Da = 1$$

$$4Da = 1$$

$$D = \frac{1}{4a}$$

$$\therefore B = -\frac{1}{4a}$$

from ②

$$4ac + B + D = 0 \quad (\because B = -D)$$

$$\Rightarrow C = 0$$

from ①

$$A + C = 0 \Rightarrow A = 0$$

$$\frac{s}{s^4 + 4a^4} = -\frac{\frac{1}{4a}}{s^2 + 2a^2 + 2a^2} + \frac{\frac{1}{4a}}{s^2 + 2a^2 - 2a^2}$$

$$\begin{aligned} L^{-1}\left\{\frac{s}{s^4 + 4a^4}\right\} &= \frac{1}{4a} L^{-1}\left\{\frac{1}{s^2 + 2a^2 + 2a^2}\right\} + \frac{1}{4a} L^{-1}\left\{\frac{1}{s^2 - 2a^2 + 2a^2}\right\} \\ &= -\frac{1}{4a} L^{-1}\left\{\frac{1}{s^2 + 2a^2 + a^2 + a^2}\right\} + \frac{1}{4a} L^{-1}\left\{\frac{1}{s^2 - 2a^2 + a^2 + a^2}\right\} \end{aligned}$$

$$= -\frac{1}{4a} L^{-1}\left\{\frac{1}{(s+a)^2 + a^2}\right\} + \frac{1}{4a} L^{-1}\left\{\frac{1}{(s-a)^2 + a^2}\right\}$$

$$= -\frac{1}{4a} e^{-at} L^{-1}\left\{\frac{1}{s^2 + a^2}\right\} + \frac{1}{4a} e^{at} L^{-1}\left\{\frac{1}{s^2 + a^2}\right\}$$

$$= -\frac{1}{4a} e^{-at} \frac{\sin at}{a} + \frac{1}{4a} e^{at} \frac{\sin at}{a}$$

$$= \frac{1}{4a} \frac{\sin at}{a} [e^{at} - e^{-at}]$$

$$= \frac{\sin at}{2a^2} \left[ \frac{e^{at} - e^{-at}}{2} \right] \quad (\because \frac{e^{at} - e^{-at}}{2} \sin at)$$

$$= \frac{\sin at}{2a^2} [\sin 2at]$$

$$4. \text{ Find } L^{-1} \left\{ \frac{s}{s^2 + 4s + 5} \right\}$$

$$\begin{aligned}
 & \stackrel{\text{Q.E.D.}}{=} L^{-1} \left\{ \frac{s+2-2}{s^2 + 4s + 5} \right\} = L^{-1} \left\{ \frac{s+2}{(s+2)^2 + 1} \right\} \\
 & = L^{-1} \left\{ \frac{s+2}{(s+2)^2 + 1} \right\} - 2 L^{-1} \left\{ \frac{1}{(s+2)^2 + 1} \right\} \\
 & = e^{-2t} L^{-1} \left\{ \frac{s}{s^2 + 1} \right\} - 2 e^{-2t} L^{-1} \left\{ \frac{1}{s^2 + 1} \right\} \\
 & = e^{-2t} L^{-1} \left[ \frac{s}{s^2 + 1} \right] - 2 e^{-2t} L^{-1} \left[ \frac{1}{s^2 + 1} \right] \\
 & = e^{-2t} \cos t - 2 e^{-2t} \frac{\sin t}{t} \\
 & = e^{-2t} [\cos t - 2 \sin t]
 \end{aligned}$$

$$5. \text{ Find } L^{-1} \left\{ \frac{s+3}{s^2 - 10s + 29} \right\}$$

$$\begin{aligned}
 & \stackrel{\text{Q.E.D.}}{=} \frac{s+3}{s^2 - 10s + 29} = \frac{s+3}{s^2 - 10s + 25 + 4} \\
 & = \frac{s+3}{(s-5)^2 + 4} \\
 & L^{-1} \left\{ \frac{s+3}{(s-5)^2 + 4} \right\} = L^{-1} \left\{ \frac{s+3}{(s-5)^2 + 4} \right\}
 \end{aligned}$$

Problems :-

second shifting theorem :-

$$\text{If } L^{-1}\{f(s)\} = f(t) \text{ then } L^{-1}\{e^{at}f(s)\} = g(t) \\
 \text{where } g(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t \leq a \end{cases}$$

$$4. \text{ Find } L^{-1} \left\{ \frac{1}{(st+1)^3} \right\}$$

$$\begin{aligned}
 & = L^{-1} \left\{ \frac{(s-5)}{(s-5)^2 + 4} \right\} + 8 L^{-1} \left\{ \frac{1}{(s-5)^2 + 4} \right\} \\
 & = e^{5t} L^{-1} \left\{ \frac{s}{s^2 + 2^2} \right\} + 8 e^{5t} L^{-1} \left\{ \frac{1}{s^2 + 2^2} \right\} \\
 & = e^{5t} \cos 2t + 8 e^{5t} \frac{\sin 2t}{2} \\
 & = e^{5t} [\cos 2t + 4 \sin 2t]
 \end{aligned}$$

1) Find  $L^{-1} \left\{ \frac{1+e^{-\pi s}}{s+1} \right\}$  using second shifting theorem.

$$\stackrel{\text{Q.E.D.}}{=} L^{-1} \left\{ \frac{1+e^{-\pi s}}{s+1} \right\} = L^{-1} \left\{ \frac{1}{s^2+1} \right\} + L^{-1} \left\{ \frac{e^{-\pi s}}{s^2+1} \right\} \rightarrow ①$$

$$\begin{aligned}
 & \text{Now } L^{-1} \left\{ \frac{e^{-\pi s}}{s^2+1} \right\} \\
 & \text{here } a = -\pi \quad f(s) = \frac{1}{s^2+1} \\
 & \therefore L^{-1}\{f(s)\} = \frac{1}{s^2+1} = \sin t = f(t)
 \end{aligned}$$

$$\therefore f(t) = \sin t$$

$f(t-a) = \sin(t-a)$  and we know that

$$g(t) = \begin{cases} -f(t-a) & \text{if } t > a \\ 0 & \text{if } t \leq a \end{cases}$$

hence

$$g(t) = \begin{cases} -f(t-\pi) \sin(t-\pi) & \text{if } t > \pi \\ 0 & \text{if } t \leq \pi \end{cases}$$

$\therefore$  from ①

$$L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{e^{-\pi s}\right\}$$

$$= \sin t + g(t)$$

$$\text{hence } g(t) = \begin{cases} \sin(t-\pi) & \text{if } t > \pi \\ 0 & \text{if } t \leq \pi \end{cases}$$

$$2. \text{ Find } L^{-1}\left\{\frac{e^{4t-3s}}{(s+4)^{5/2}}\right\}$$

$$\text{Ans:- } L^{-1}\left\{\frac{e^{4ts}}{(s+4)^{5/2}}\right\} = e^{-4t} L^{-1}\left\{\frac{e^{4s}}{(s+4)^{5/2}}\right\}$$

$$= e^{-4t} L^{-1}\left\{\frac{e^4 e^{-3s}}{s^{5/2}}\right\}$$

$$L^{-1}\left\{\frac{e^{4-3s}}{s^{5/2}}\right\} = e^4 L^{-1}\left\{\frac{e^{-3s}}{s^{5/2}}\right\}$$

$$\text{let } F(s) = \frac{1}{(s+4)^{5/2}}$$

$$(\because L^{-1}\left\{\frac{1}{s^{m+1}}\right\} = \frac{t^m}{m!})$$

$$L^{-1}\left\{F(s)\right\} = e^{-4t} L^{-1}\left\{\frac{1}{s^{5/2}}\right\}$$

$$= e^{-4t} \frac{t^{3/2}}{\Gamma(\frac{3}{2}+1)}$$

$$= e^{-4t} \frac{t^{3/2}}{\frac{3}{2} \cdot \frac{1}{2} \Gamma(\frac{1}{2})} = e^{-4t} \frac{4}{3} \frac{t^{3/2}}{\sqrt{\pi}} = f(t)$$

$$\text{Now } L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^4 L^{-1}\left\{\frac{e^{-3s}}{(s+4)^{5/2}}\right\} \rightarrow ①$$

keep  $a=3$

$$\text{Consider } L^{-1}\left\{e^{-3s} \frac{1}{(s+4)^{5/2}}\right\} = -e^4 f(e^{-4(t-3)})$$

$$\text{Consider } L^{-1}\left\{e^{-3s} \frac{1}{(s+4)^{5/2}}\right\}$$

$$\text{hence } a=3 \quad F(s) = \frac{1}{(s+4)^{5/2}}$$

$$\text{so } K.T \quad L^{-1}\left\{F(s)\right\} = L^{-1}\left\{\frac{1}{(s+4)^{5/2}}\right\} = e^{-4t} \frac{4}{3} \frac{t^{3/2}}{\sqrt{\pi}} = f(t)$$

$$\therefore \text{from ①} \quad L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^{-4t} L^{-1}\left\{\frac{e^{-3s}}{(s+4)^{5/2}}\right\}$$

①

$\text{if } t > 3$

Note:- If  $f(t) = L^{-1}\{F(s)\}$  then  $L^{-1}\{e^{-as}F(s)\} = f(t-a)H(t-a)$   
(another form)  
 $H(t-a)$  is unit step function

3. Find  $L^{-1}\left\{\frac{3+5s}{s^2+s}\right\}$

Let  $\bar{f}(s) = \frac{3+5s}{s^2+s} = \frac{3}{s^2} + \frac{5}{s}$   
 $L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{3}{s^2} + \frac{5}{s}\right\} = 3\left[L^{-1}\left(\frac{1}{s^2}\right)\right] + 5L^{-1}\left(\frac{1}{s}\right)$   
 $= 3t + 5 = f(t)$

Now  $L^{-1}\left\{\frac{3+5s}{s^2+s}\right\} = L^{-1}\left\{e^{-2s} \frac{3+5s}{s^2}\right\}$   
here  $a=2$

$$= f(t-2)H(t-2) \quad (\text{as})$$

$$= [3(t-2)+5] H(t-2)$$

$$= \begin{cases} 3(t-2)+5 & \text{if } t \geq 3 \\ 0 & \text{if } t < 3 \end{cases}$$

$$= (3t-6+5) H(t-2)$$

$$= (3t-1) H(t-2)$$

The problems ① & ② can also be expressed in the  
another form of 2nd shifting theorem

Problem 1 :-  
 $f(t) = \sin t$   
 $f(t-\alpha) = \sin(t-\alpha)$  here  $\alpha = -\pi$   
 $f(t-\pi) = \sin(t-\pi)$   
 $f(t-\pi) = f(t-\alpha) + \sin t$   
 $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\} = f(t-\pi) + f(t-\pi) = \sin(t-\pi) + \sin t + \sin t$

Problem 2 :-  $f(t) = e^{-4t} \frac{4}{3} \frac{t^{3/2}}{\sqrt{\pi}}$

Now  $L^{-1}\left\{\frac{e^{4-3s}}{(s^2+4)^{5/2}}\right\} = e^4 \cdot f(t-\alpha) H(t-\alpha)$  here  $\alpha = 3$   
 $= e^4 e^{-4(t-3)} \frac{4}{3} \frac{(t-3)^{3/2}}{\sqrt{\pi}} H(t-3)$

Find  $L^{-1}\left\{\frac{e^{-3s}}{(s+4)^2}\right\}$

Change of scale property :-

If  $f(t) = L^{-1}\{F(s)\}$  then  $L^{-1}\{f(\alpha s)\} = \frac{1}{\alpha} f\left(\frac{t}{\alpha}\right)$

Problem :-  
If  $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t$  then find  $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$

Sol:- we have  $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t = f(t)$  say

writing as  $t \sin t$

$$L^{-1}\left\{\frac{as}{(as^2+1)^2}\right\} = \frac{1}{a} \frac{1}{2} \frac{t}{a} \sin \frac{t}{a} \quad (\text{by C.S.P})$$

$$= \frac{1}{2a^2} \sin \frac{t}{a}$$

put  $a = 2$

$$L^{-1}\left\{\frac{2s}{(4s^2+1)^2}\right\} = \frac{1}{8} \sin \frac{t}{2}$$

$$= L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\} = \frac{1}{2} \sin \frac{t}{2}$$

$$20. \text{ If } L^{-1} \left\{ \frac{e^{-ts}}{\sqrt{s^2 + t^2}} \right\} = \frac{\cos st}{\sqrt{t^2}} \text{ find } L^{-1} \left\{ \frac{e^{-ts}}{s\sqrt{s^2 + t^2}} \right\}$$

Sol:- we have  $L^{-1} \left\{ \frac{e^{-ts}}{s\sqrt{s^2 + t^2}} \right\} = \frac{\cos st}{s\sqrt{t^2}} = f(t)$

Dividing by  $t$  we get

$$L^{-1} \left\{ \frac{e^{-ts}}{(ts)\sqrt{s^2 + t^2}} \right\} = \frac{1}{t} \frac{\cos st}{\sqrt{t^2 + s^2}}$$

$$L^{-1} \left\{ \frac{e^{-ts}}{\sqrt{ts}} \right\} = \frac{1}{t} \frac{\cos st\sqrt{t^2 + s^2}}{\sqrt{t^2 + s^2}} \quad (\text{By L.S.P})$$

$$\left( = \frac{1}{t} \frac{\cos st\sqrt{t^2 + s^2}}{\sqrt{t^2 + s^2}} \cdot \frac{t}{t} \right)$$

$$\Rightarrow L^{-1} \left\{ \frac{e^{-ts}}{\sqrt{s}} \right\} = \frac{1}{t\sqrt{s}} \frac{\cos st\sqrt{t^2 + s^2}}{\sqrt{t^2 + s^2}}$$

$$\text{Put } s = \frac{1}{a}$$

$$\text{we get } L^{-1} \left\{ \frac{e^{-at}}{\sqrt{s}} \right\} = \frac{\cos 2\sqrt{at}}{\sqrt{t}}$$

=

Inverse Laplace transform of Derivatives :-

If  $L^{-1} \{ f(s) \} = f(t) \rightarrow \text{then } L^{-1} \{ f^{(n)}(s) \} = (-1)^n t^n f(t)$

$$\text{where } \bar{f}^{(n)}(s) = \frac{d^n}{ds^n} [\bar{f}(s)]$$

$$\text{Note:-} \quad \begin{aligned} & L^{-1} \{ t^n f(t) \} = (-1)^n \frac{d^n}{ds^n} \bar{f}(s) \\ & t^n f(t) = (-1)^n L^{-1} \left\{ \frac{d^n}{ds^n} \bar{f}(s) \right\} \\ & L^{-1} \{ t^n f(t) \} = L^{-1} \{ \bar{f}^{(n)}(s) \} \end{aligned}$$

$$1. \text{ Find } L^{-1} \left\{ \log \left( \frac{1+s}{s^2} \right) \right\}$$

Sol:-  $L^{-1} \left\{ \log \left( \frac{1+s}{s^2} \right) \right\}$

This is of the form  $L^{-1} \{ \bar{f}(s) \}$

$$\text{where } \bar{f}(s) = \log \left( \frac{1+s}{s^2} \right) = \log(1+s) - \log(s^2)$$

$$\bar{f}'(s) = \frac{1}{1+s} - \frac{1}{s^2} = \frac{1}{1+s} - \frac{2}{s^2}$$

$$L^{-1} \{ \bar{f}'(s) \} = L^{-1} \left\{ \frac{1}{1+s} - \frac{2}{s^2} \right\}$$

$$(1) \quad t \bar{f}'(t) = L^{-1} \left\{ \frac{1}{1+t} \right\} - 2 L^{-1} \left\{ \frac{1}{t^2} \right\}$$

$$-t \bar{f}(t) = e^{-t} - 2t$$

$$\bar{f}(t) = e^{-t+2}$$

$$\therefore L^{-1} \{ \bar{f}(s) \} = \frac{e^{-t}}{t}$$

$$L^{-1} \left\{ \log \left( \frac{1+s}{s^2} \right) \right\} = \frac{e^{-t}}{t}$$

$$2. \text{ Find } L^{-1} \left\{ \log \left( \frac{s^2 + 4}{s^2 + 9} \right) \right\}$$

$$\text{Sol:- } \bar{f}(s) = \log \left( \frac{s^2 + 4}{s^2 + 9} \right) = \log(s^2 + 4) - \log(s^2 + 9)$$

$$\begin{aligned} \bar{f}'(s) &= \frac{1}{s^2 + 4} (2s) - \frac{1}{s^2 + 9} (2s) \\ &= \frac{2s}{s^2 + 4} - \frac{2s}{s^2 + 9} \end{aligned}$$



### Inverse Laplace transform of Integrals :-

$$f^{(1)}(s) = \frac{2s}{s^2 + 16} - \frac{2}{s}$$

$$L^{-1}\{f^{(1)}(s)\} = 2 L^{-1}\left\{\frac{s}{s^2 + 4^2}\right\} - 2 L^{-1}\left\{\frac{1}{s}\right\}$$

$$(1) f(t) = 2 \cos 4t - 2$$

$$\Rightarrow f(t) = \frac{2(\cos 4t - 1)}{-t} = \frac{2(1 - \cos 4t)}{t}$$

$$L^{-1}\{f(s)\} = \frac{2}{t}(1 - \cos 4t)$$

$$L^{-1}\{\log(1 + \frac{16}{s^2})\} = \frac{2}{t}(1 - \cos 4t)$$

2. Find  $L^{-1}\{ct^{-1}(s)\}$

$$f(s) = ct^{-1}(s)$$

$$f^{(1)}(s) = -\frac{1}{t^2}s^2$$

$$L^{-1}\{f^{(1)}(s)\} = L^{-1}\left\{-\frac{1}{t^2}s^2\right\}$$

$$(1) f(t) = L^{-1}\left\{\frac{1}{t^2 + s^2}\right\}$$

$$f(t) = \frac{\sin t}{t}$$

$$L^{-1}\{f(s)\} = \frac{\sin t}{t}$$

$$L^{-1}\{f^{(1)}(s)\} = \frac{\sin t}{t} //$$

$$\text{Note:- 1. } L^{-1}\{f(s)\} = \int_s^\infty f(\tau) d\tau$$

$$\Rightarrow L^{-1}\left\{\frac{t}{s}\right\} = L^{-1}\left\{\int_s^\infty f(\tau) d\tau\right\}$$

$$2. L^{-1}\left\{\frac{s+1}{(s^2 + 2s + 2)^2}\right\} = L^{-1}\left\{\frac{s+1}{((s+1)^2 + 1)^2}\right\}$$

1. Find  $L^{-1}\left\{\frac{s+1}{(s^2 + 2s + 2)^2}\right\}$  using Inverse L-T & integrals

$$= L^{-1}\left\{\frac{s+1}{(s^2 + 2s + 2)^2}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{s}{(s+1)^2}\right\}$$

$$= e^{-t} t \frac{1}{2} L^{-1}\left\{\int_s^\infty \frac{ds}{(s+1)^2}\right\} = t e^{-t} L^{-1}\left\{\int_s^\infty \frac{1}{s^2 + 1} ds\right\}$$

$$= t e^{-t} L^{-1}\left[\frac{-1}{s^2 + 1}\right]_s^\infty$$

$$= t e^{-t} L^{-1}\left\{\frac{1}{s^2 + 1}\right\}$$

$$= e^{t - \frac{1}{2}} \sin t$$

$$= e^{t - \frac{1}{2}} \sin t$$

2. Find  $L^{-1} \left\{ \frac{s}{(s-a)^2} \right\}$  by inverse L.T of integrals.

$$\begin{aligned}
 & \text{Soln:- } L^{-1} \left\{ \frac{1}{(s-a)^2} \right\} = t L^{-1} \left\{ \frac{1}{s} \int_0^\infty \frac{2s}{(s-a)^2} ds \right\} \\
 & = t L^{-1} \left\{ \int_0^\infty \frac{ds}{s} \left( \frac{1}{s-a} \right)^2 \right\} \\
 & = t L^{-1} \left\{ \left( \frac{1}{s-a} \right)_s^\infty \right\} \\
 & = t L^{-1} \left\{ \left( \frac{1}{s-a} \right)_0^\infty \right\} \\
 & = t L^{-1} \left\{ 0 - \left( \frac{1}{s-a} \right)_0 \right\} \\
 & = t L^{-1} \left\{ \frac{1}{s-a} \right\} \\
 & = t \frac{1}{a} L^{-1} \left\{ \frac{1}{s-a} \right\} \\
 & = t \frac{1}{a} \sinh at \\
 & = \frac{t}{a} \sinh at
 \end{aligned}$$

4.  $L^{-1} \left\{ \frac{1}{(s+1)^2} \right\}$

$$\begin{aligned}
 & \text{Soln:- } L^{-1} \left\{ \frac{1}{(s+1)^2} \right\} = t L^{-1} \left\{ \int_s^\infty \frac{1}{(s+1)^2} ds \right\} \\
 & = t L^{-1} \left\{ \int_s^\infty \frac{1}{s^2} \left( \frac{-1}{s+1} \right) ds \right\} \\
 & = t L^{-1} \left\{ \left( \frac{-1}{s+1} \right)_s^\infty \right\} \\
 & = t L^{-1} \left\{ \frac{1}{s+1} \right\} = t e^{-t} \\
 & = \cancel{t e^{-t}} \cancel{\frac{1}{s+1}}
 \end{aligned}$$

Multiplication by the powers of s :-

If  $f(t) = L^{-1}\{f(s)\}$  and  $f'(0) = 0$  then  $L^{-1}\{s f(s)\} = f'(t)$

Note:- In general

$$\begin{aligned}
 L^{-1}\{s^n f(s)\} &= f^{(n)}(t) \\
 L^{-1}\{s f(s)\} &= f'(t) = [L^{-1}\{f(s)\}]' \\
 L^{-1}\{s^2 f(s)\} &= f''(t) = [L^{-1}\{f(s)\}]'' \\
 &\vdots
 \end{aligned}$$

Problems :-

1. Find  $L^{-1} \left\{ \frac{3}{(s+3)^2} \right\}$

$$\begin{aligned}
 & \text{Soln:- } L^{-1} \left\{ \frac{3}{(s+3)^2} \right\} = L^{-1} \left\{ 3 \cdot \frac{1}{(s+3)^2} \right\} \\
 & \quad \text{This is of the form } L^{-1} \left\{ s \cdot \bar{f}(s) \right\} \\
 & \therefore L^{-1} \left\{ s \cdot \frac{1}{(s+3)^2} \right\} = \left[ L^{-1} \left\{ \frac{1}{(s+3)^2} \right\} \right]' \\
 & = t e^{-3t} L^{-1} \left\{ \frac{1}{(s+3)^2} \right\} = t e^{-3t} \sinh 3t
 \end{aligned}$$

$$\begin{aligned}
&= \left[ e^{-3t} L^{-1} \left\{ \frac{1}{s+2} \right\} \right]' \\
&= (e^{-3t} t)' \\
&= e^{-3t} (-3) + t e^{-3t} (-3) \\
&= e^{-3t} - 3te^{-3t} = e^{-3t} [1 - 3t]
\end{aligned}$$

2. Find  $L^{-1} \left\{ s \cdot \log \left( \frac{s-1}{s+1} \right) \right\}$

$\text{Sol: } L^{-1} \left\{ s \log \left( \frac{s-1}{s+1} \right) \right\}$

This is of form  $L^{-1} \left\{ s \cdot \bar{f}(s) \right\}$

$$\begin{aligned}
\therefore L^{-1} \left\{ s \log \left( \frac{s-1}{s+1} \right) \right\} &= \left[ L^{-1} \left\{ \log \left( \frac{s-1}{s+1} \right) \right\} \right]' \rightarrow ①
\end{aligned}$$

To find  $L^{-1} \left\{ \log \left( \frac{s-1}{s+1} \right) \right\}$

$$\begin{aligned}
\bar{f}(s) &= \log \left( \frac{s-1}{s+1} \right) = \log \left( \frac{s-1}{s+1} \right) \\
\bar{f}'(s) &= \frac{1}{s-1} - \frac{1}{s+1} \\
L^{-1} \left\{ \bar{f}'(s) \right\} &= L^{-1} \left\{ \frac{1}{s-1} \right\} - L^{-1} \left\{ \frac{1}{s+1} \right\}
\end{aligned}$$

$$\begin{aligned}
(s-1) + f(t) &= e^{st} \\
f(t) &= e^{st} - (s-1)
\end{aligned}$$

$$\begin{aligned}
L^{-1} \left\{ \bar{f}(s) \right\} &= \frac{e^t - e^{-t}}{t} = \frac{2}{t} e^{st} - \frac{2}{t} e^{-st} \\
L^{-1} \left\{ \log \left( \frac{s-1}{s+1} \right) \right\} &= \frac{2}{t} e^{st} - \frac{2}{t} e^{-st} = -\frac{2}{t} \sin ht
\end{aligned}$$

∴ from ①

$$\begin{aligned}
L^{-1} \left\{ s \log \left( \frac{s-1}{s+1} \right) \right\} &= \left[ L^{-1} \left\{ \frac{1}{s+2} \right\} \right]' = \left[ -\frac{2}{t} \sin ht \right]' \\
&= -2 \left[ \frac{1}{t} \cos ht + \sin ht \left( -\frac{1}{t^2} \right) \right] \\
&= -2 \left[ \frac{\cos ht}{t} - \frac{\sin ht}{t^2} \right]
\end{aligned}$$

3. Find  $L^{-1} \left\{ \frac{s-3}{s^2+4s+13} \right\}$

$\text{Sol: } L^{-1} \left\{ \frac{s-3}{s^2+4s+13} \right\} = L^{-1} \left\{ \frac{s}{s^2+4s+13} \right\} - 3$

$$\text{Let } \bar{f}(s) = \frac{1}{s^2+4s+13} = \frac{1}{(s+2)^2+3^2}$$

$$L^{-1} \left\{ \bar{f}(s) \right\} = L^{-1} \left\{ \frac{1}{(s+2)^2+3^2} \right\} = e^{-2t} \frac{\sin 3t}{3} = f(t)$$

Clearly  $f(0) = 0$

$$\begin{aligned}
L^{-1} \left\{ \frac{s-3}{s^2+4s+13} \right\} &= L^{-1} \left\{ \frac{s}{s^2+4s+13} \right\} - 3 L^{-1} \left\{ \frac{1}{s^2+4s+13} \right\} \\
&= L^{-1} \left\{ s \bar{f}(s) \right\} - 3 L^{-1} \left\{ \bar{f}(s) \right\}
\end{aligned}$$

$$\begin{aligned}
&= f(t) - 3 + (t) \\
&= f'(t) - 3 + (t)
\end{aligned}$$

$$\begin{aligned}
&= \frac{d}{dt} \left\{ e^{2t} \frac{\sin 3t}{3} \right\} - 3 e^{2t} \frac{\sin 3t}{3} - e^{2t} \sin 3t \\
&= \frac{1}{3} \left[ 1 e^{-2t} 3 \cos 3t + (-2e^{2t}) 3 \sin 3t \right] - e^{2t} \sin 3t \\
&= \frac{1}{3} [3e^{-2t} \cos 3t - 2e^{2t} \sin 3t] - e^{2t} \sin 3t \\
&=
\end{aligned}$$

### Division by $\frac{e^{-st}}{s}$ -

If  $L^{-1}\{\bar{f}(s)\} = f(t)$  then  $L^{-1}\{\bar{f}\frac{1}{s}\} = \int_0^t f(\tau) d\tau$

$$\text{Note:- } L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \int_0^t f(t) dt dt$$

$$\text{If } L^{-1}\left\{\frac{\bar{f}(s)}{s^n}\right\} = \int_0^t \int_0^t \dots \int_0^t f(t) dt dt \dots dt$$

1. Find  $L^{-1}\left\{\frac{1}{s(s+1)(s^2-1)}\right\}$

$$\text{Sol:- } L^{-1}\left\{\frac{1}{s(s^2+1)(s^2-1)}\right\} = L^{-1}\left\{\frac{1}{s(s+1)(s^2-1)}\right\}$$

$$\text{where } \bar{f}(s) = \frac{1}{(s+1)(s^2-1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-1}$$

$$1 = (As+B)(s^2-1) + (Cs+D)(s^2+1)$$

$$1 = As^3 + Bs^2 - As - B + Cs^3 + Ds^2 + Cs + D$$

$$A+C=0 \rightarrow \textcircled{1}$$

$$B+D=0 \rightarrow \textcircled{2}$$

$$-B+C=0 \rightarrow \textcircled{3}$$

$$-B+D=1 \rightarrow \textcircled{4}$$

Solving  $\textcircled{1} \& \textcircled{3}$

$$\begin{aligned} A+C &= 0 \\ A+C &= 0 \end{aligned}$$

$$\begin{aligned} B+D &= 0 \\ B+D &= 0 \end{aligned}$$

$$\begin{aligned} -B+C &= 0 \\ -B+C &= 0 \end{aligned}$$

$$\begin{aligned} C &= 0 \\ C &= 0 \end{aligned}$$

$$\therefore A=0$$

$$\therefore \frac{1}{(s^2+1)(s^2-1)} = \frac{-\frac{1}{2}}{s^2+1} + \frac{\frac{1}{2}}{s^2-1}$$

$$\text{Now } L^{-1}\left\{\frac{1}{(s^2+1)(s^2-1)}\right\} = -\frac{1}{2} L^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{s^2-1}\right\}$$

$$= -\frac{1}{2} \sin t + \frac{1}{2} \csc t$$

$$= \frac{1}{2} [\csc t - \sin t] = f(t)$$

2.  $L^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

$$\text{Sol:- } L^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \int_0^t f(t) dt$$

$$\text{where } \bar{f}(s) = \frac{1}{s^2+9} = \frac{1}{s^2+3^2}$$

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = L^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \int_0^t \frac{\sin 3t}{3} dt$$

$$= \frac{1}{3} \int_0^t \sin 3t dt$$

$$= \frac{1}{3} \left[ -\frac{\cos 3t}{2} \right]_0^t$$

$$= \frac{1}{3} [-\cos 3t + \cos 0]$$

$$= \frac{1}{3} [1 - \cos 3t]$$

$$3. \text{ Find } L^{-1} \left\{ \frac{4s+5}{(s-1)^2(s+2)} \right\}$$

$$L^{-1} \left\{ \frac{4s+5}{(s-1)^2(s+2)} \right\} = L^{-1} \left\{ \frac{4(s-1)+9}{(s-1)^2((s-1)+3)} \right\}$$

$$= e^t L^{-1} \left\{ \frac{4s+9}{s^2(s+3)} \right\}$$

$$= e^t L^{-1} \left\{ \frac{4s+9}{s(s^2+3s)} \right\} + 9 L^{-1} \left\{ \frac{1}{s^2(s+3)} \right\} \rightarrow (1)$$

$$= e^t \left[ \frac{1}{3} - \frac{9}{2} e^{-3t} + 3t + e^{3t} - 1 \right]$$

$\therefore$

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$1 = As + 3A + Bs^2$$

$$A+B=0 \Rightarrow A=-B$$

$$3A=1 \Rightarrow A=\frac{1}{3} \quad \therefore B=-\frac{1}{3}$$

$$\frac{1}{s(s+3)} = \frac{1}{3} \left[ \frac{1}{s} - \frac{1}{s+3} \right]$$

$$L^{-1} \left\{ \frac{1}{s(s+3)} \right\} = \frac{1}{3} \left[ L^{-1} \left\{ \frac{1}{s} \right\} - L^{-1} \left\{ \frac{1}{s+3} \right\} \right] \\ = \frac{1}{3} [1 - e^{-3t}] \rightarrow (2)$$

$$\text{Now } L^{-1} \left\{ \frac{1}{s^2(s+3)} \right\} = L^{-1} \left\{ \frac{s}{s^2+3s} \right\}$$

$$= \frac{1}{s} \int_0^t (t-e^{-3t}) dt$$

$$= \frac{1}{3} \left[ t - \frac{e^{-3t}}{3} \right]_0^t$$

$$= \frac{1}{3} \left[ t + \frac{e^{-3t}}{3} - \frac{1}{3} \right]$$

$$= \frac{1}{3} \left[ t + \frac{1}{3}(e^{-3t}-1) \right] \rightarrow (3)$$

from (1) & (2) and (3) we have

$$L^{-1} \left\{ \frac{4s+5}{(s-1)^2(s+2)} \right\} = e^t \left[ \frac{4}{3} (1-e^{-3t}) + \frac{9}{2} (t + \frac{1}{3}(e^{-3t}-1)) \right]$$

$$= e^t \left[ \frac{4}{3} - \frac{9}{2} e^{-3t} + 3t + e^{3t} - 1 \right]$$

$$4. \text{ Find } L^{-1} \left\{ s \frac{1}{(s^2+2s+2)} \right\}$$

$$L^{-1} \left\{ s \frac{1}{(s^2+2s+2)} \right\} = L^{-1} \left\{ \frac{1}{(s+1)^2+1} \right\}$$

$$\text{Let } f(s) = \frac{1}{(s+1)^2+1} \Rightarrow L^{-1} \{ f(s) \} = e^t L^{-1} \left\{ \frac{1}{t^2+1} \right\} \\ = e^t \sin t = f(t)$$

$$\therefore L^{-1} \left\{ \frac{1}{(s^2+2s+2)} \right\} = \int_0^t f(t) dt = \int_0^t e^{-t} \sin t dt$$

$$= \left[ \frac{e^{-t}}{t^2+1} (-\sin t - \cos t) \right]_0^t$$

$$= \left[ \frac{e^{-t}}{2} (-\sin t - \cos t) \right] - \left[ \frac{1}{2} (-1) \right]$$

$$= \left[ \frac{e^{-t}}{2} (-\sin t - \cos t) + \frac{1}{2} \right]$$

$$= \left[ -\frac{e^{-t}}{2} (\sin t + \cos t) + \frac{1}{2} \right]$$

$$= \frac{1}{2} [1 - e^{-t} (\sin t + \cos t)]$$

$\therefore$

5. Find  $L^{-1}\left\{ \frac{s+1}{s^2(s^2+1)} \right\}$

Sol :- Let  $\bar{f}(s) = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1}$

$$L\{\bar{f}(s)\} = L\left\{ \frac{s}{s^2+1} \right\} + L\left\{ \frac{1}{s^2+1} \right\}$$

$$= \cos t + \sin t = f(t)$$

$$\text{Now } L^{-1}\left\{ \frac{s+1}{s^2+1} \right\} = \int_0^t (\cos t + \sin t) dt$$

$$= (\sin t - \cos t)_0^t \\ = (\sin t - \cos t) - (\sin 0 - \cos 0) \\ = 1 + (\sin t - \cos t)$$

$$L^{-1}\left\{ \frac{s+1}{s^2+1} \right\} = \int_0^t (1 + \sin t - \cos t) dt \\ = (t - \cos t - \sin t)_0^t \\ = (t - \cos t - \sin t) - (0 - \cos 0 - \sin 0) \\ = 1 + t - \cos t - \sin t \\ \equiv$$

### Convolution theorem :-

Convolution is used for obtaining inverse L.T of a product of two transformations.

### Convolution theorem :-

Let  $f(t)$  and  $g(t)$  be the two functions defined for  $t > 0$ .

$$\text{If } f(t) = L\{\bar{f}(s)\} \text{ and } g(t) = L\{\bar{g}(s)\}, \text{ then } L\{f(t) \cdot g(t)\}$$

$$= f(t) * g(t) = \int_0^t f(\omega) g(t-\omega) d\omega$$

Where  $*$  represents the convolution operator.

1. By using convolution theorem find  $L^{-1}\left\{ \frac{1}{(s^2+\alpha^2)^2} \right\}$

Sol :-  $L^{-1}\left\{ \frac{1}{(s^2+\alpha^2)^2} \right\} = L\left\{ \frac{1}{s^2+\alpha^2} \cdot \frac{1}{s^2+\alpha^2} \right\}$

This is in the form  $L\{\bar{f}(s) \cdot \bar{g}(s)\}$

By where  $\bar{f}(s) = \frac{1}{s^2+\alpha^2}$   $\bar{g}(s) = \frac{1}{s^2+\alpha^2}$

$$L\{\bar{f}(s)\} = L\left\{ \frac{1}{s^2+\alpha^2} \right\}$$

$$f(t) = \frac{\sin at}{a}$$

$$\Rightarrow f(\omega) = \frac{1}{a} \sin a\omega$$

$$L\{\bar{g}(s)\} = L\left\{ \frac{1}{s^2+\alpha^2} \right\}$$

$$g(t) = \frac{\sin at}{a}$$

$$\Rightarrow g(t-a) = \frac{1}{a} \sin a(t-a)$$

By convolution theorem

$$\begin{aligned}
 L^{-1}\{f(s) \cdot \bar{g}(s)\} &= f(t) * g(t) \\
 &= \int_0^t f(u) g(t-u) du \\
 &= \int_0^t \frac{1}{\alpha^2} \sin \omega u \frac{1}{\alpha} \sin \omega(t-u) du \\
 &= \frac{1}{\alpha^2} \int_0^t \sin \omega u \sin(\omega t - \omega u) du \\
 &= \frac{1}{\alpha^2} \int_0^t 2 \sin \omega u \sin(\omega t - \omega u) du \\
 &= \frac{1}{\alpha^2} \int_0^t 2 \sin \omega u [\sin(\omega t - \omega u) - \cos(\omega t - \omega u) + \cos(\omega t - \omega u)] du \\
 &= \frac{1}{\alpha^2} \int_0^t (\cos(\omega u - \omega t) - \cos \omega t + \cos(\omega u - \omega t) - \cos \omega t) du \\
 &= \frac{1}{\alpha^2} \left[ \frac{\sin(2\omega u - \omega t)}{2\alpha} - \cos \omega t \right]_0^t \\
 &= \frac{1}{\alpha^2} \left[ \frac{\sin(2\omega u - \omega t)}{2\alpha} - \cos \omega t \right] \\
 &\quad - \frac{1}{4\alpha^3} \left[ \sin \omega t - \cos \omega t - \left[ \frac{\sin(2\omega u - \omega t)}{2\alpha} - \cos \omega t \right] \right]
 \end{aligned}$$

Q. Find  $L^{-1}\left\{\frac{1}{(s^2+5^2)^2}\right\}$  using convolution theorem.

$$\begin{aligned}
 \text{Sol: } L^{-1}\{f(s)\} &= L^{-1}\left\{\frac{1}{s^2+5^2}\right\} = \frac{1}{5} \sin \omega s \\
 \text{here } \bar{f}(s) &= \frac{1}{s^2+5^2} \quad \bar{g}(s) = \frac{1}{s^2+5^2} \\
 L^{-1}\{f(s)\} &= L^{-1}\left\{\frac{1}{s^2+5^2}\right\} = \frac{1}{5} \sin \omega s \\
 f(t) &= \frac{\sin 5t}{5} \quad g(t) = \frac{\sin 5t}{5} \\
 f(u) &= \frac{1}{5} \sin \omega u \quad g(t-u) = \frac{1}{5} \sin \omega(t-u)
 \end{aligned}$$

By convolution theorem

$$\begin{aligned}
 L^{-1}\{f(s) \cdot \bar{g}(s)\} &= f(t) * g(t) = \int_0^t f(u) g(t-u) du \\
 &= \int_0^t \frac{1}{5} \sin \omega u \cdot \frac{1}{5} \sin \omega(t-u) du \\
 &= \frac{1}{25} \int_0^t 2 \sin \omega u \sin \omega(t-u) du \\
 &= \frac{1}{25} \int_0^t [\cos(5\omega u - 5\omega t + 5\omega u) - \cos(5\omega u + 5\omega t - 5\omega u)] du \\
 &= \frac{1}{25} \left[ \frac{\sin 10\omega u}{2\omega} - t \cos 5\omega t + \frac{\sin 10\omega u}{2\omega} \right]_0^t \\
 &= 2x \frac{1}{25} \int_0^t \cos(10\omega u - 5\omega t) du
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{50} \left[ \sin\left(\frac{10\omega-5t}{10}\right) - (\cos 5t) \cdot u \right] \\
&= \frac{1}{50} \left[ \frac{\sin 5t - t \cos 5t}{10} - \left( -\frac{\sin 5t}{10} - 0 \right) \right] \\
&= \frac{1}{50} \left[ \frac{\sin 5t}{10} + t \cos 5t \right] \\
&= \frac{1}{50} \left[ \frac{\sin 5t}{5} - t \cos 5t \right] \\
&= \frac{\sin 5t}{250} - \frac{t \cos 5t}{50}
\end{aligned}$$

3. find  $L^{-1} \left\{ \frac{1}{s^2(s+1)^2} \right\}$  by using convolution theorem.

$$\text{Sol:- } L^{-1} \left\{ \frac{1}{s^2(s+1)^2} \right\} = L^{-1} \left\{ \frac{1}{s^2} \cdot \frac{1}{(s+1)^2} \right\} = L^{-1} \left\{ \frac{1}{s^2} \right\} \cdot L^{-1} \left\{ \frac{1}{(s+1)^2} \right\}$$

$$\begin{aligned}
&L^{-1} \left\{ \frac{1}{s^2} \right\} = \frac{1}{s+1} \cdot \frac{1}{s+b} \\
&L^{-1} \left\{ \frac{1}{(s+1)^2} \right\} = L^{-1} \left\{ \frac{1}{s^2} \right\} \cdot L^{-1} \left\{ \frac{1}{(s+1)^2} \right\} \\
&\text{Sol:- } L^{-1} \left\{ \frac{1}{s^2(s+1)^2} \right\} = L^{-1} \left\{ \frac{1}{s+1} \cdot \frac{1}{s+b} \right\} = L^{-1} \left\{ f(s) \cdot g(s) \right\} \\
&f(s) = e^{-as} \quad \text{and} \quad g(s) = e^{-bs} \\
&f(t) = e^{-at} \quad \text{and} \quad g(t) = e^{-bt} \\
&= f(t) * g(t) \\
&= \int_0^t f(u) g(t-u) du \\
&= \int_0^t e^{a u} e^{-b(t-u)} du \\
&= \int_0^t e^u a (t-u) du \\
&= t \int_0^t u e^{-u} du - \int_0^t u^2 e^{-u} du \\
&= t \left[ u e^u - e^{-u} \right]_0^t - \left[ -u^2 e^{-u} - 2u e^{-u} - 2e^{-u} \right]_0^t \\
&= t \left[ u e^u - e^{-u} \right]_0^t - \left[ -u^2 e^{-u} - 2u e^{-u} - 2e^{-u} \right]_0^t
\end{aligned}$$

4. Find  $L^{-1} \left\{ \frac{1}{(s+a)(s+b)} \right\}$  using convolution theorem.

$$\begin{aligned}
&\text{Sol:- } L^{-1} \left\{ \frac{1}{(s+a)(s+b)} \right\} = L^{-1} \left\{ \frac{1}{s+a} \cdot \frac{1}{s+b} \right\} = L^{-1} \left\{ \bar{f}(s) \cdot \bar{g}(s) \right\} \\
&\text{where } \bar{f}(s) = \frac{1}{s+a} \quad \bar{g}(s) = \frac{1}{s+b} \\
&L^{-1} \left\{ \bar{f}(s) \right\} = e^{-at} = f(t) \quad L^{-1} \left\{ \bar{g}(s) \right\} = e^{-bt} = g(t) \\
&f(t) = e^{-at} \quad \text{and} \quad g(t) = e^{-bt}
\end{aligned}$$

$$\begin{aligned}
&\text{By convolution theorem} \\
&L^{-1} \left\{ \frac{1}{(s+a)(s+b)} \right\} = \int_0^t f(u) g(t-u) du \\
&= \int_0^t e^{-au} e^{-b(t-u)} du \\
&= e^{-bt} \int_0^t e^{-au} e^{bu} du \\
&= e^{-bt} \int_0^t e^{(b-a)u} du \\
&= e^{-bt} \int_0^t e^{(b-a)u} du = e^{-bt} \left[ \frac{e^{(b-a)t}}{b-a} - \frac{1}{b-a} \right] \\
&= \frac{e^{-bt}}{b-a} \left[ e^{(b-a)t} - 1 \right]
\end{aligned}$$

5. Find  $L^{-1} \left\{ \frac{s^2}{(s^2+a^2)(s^2+b^2)} \right\}$  using convolution theorem.

$$\text{Sol: } L^{-1} \left\{ \frac{1}{(s^2+a^2)(s^2+b^2)} \right\} = L^{-1} \left\{ \frac{\frac{a}{s^2+a^2}}{s^2+b^2} \right\}$$

$$\bar{f}(s) = \frac{s}{s^2+a^2}$$

$$\bar{g}(s) = \frac{s}{s^2+b^2}$$

$$L^{-1}\{\bar{f}(s)\} = \cos at = f(t)$$

$$g(t-\omega) = \cos b(t-\omega)$$

$$f(\omega) = \cos \omega t$$

$$L^{-1} \left\{ \frac{s^2}{(s^2+a^2)(s^2+b^2)} \right\} = \int_0^t f(\omega) g(t-\omega) d\omega$$

$$= \int_0^t \cos \omega u \cos b(t-u) du$$

$$= \frac{1}{2} \int_0^t \cos \omega u \cos b(t-u) du$$

$$= \frac{1}{2} \int_0^t \cos(a\omega u + bt - bu) + \cos(a\omega u - bt + bu) du$$

$$= \frac{1}{2} \left[ \frac{\cos[(a-b)u + bt]}{ab} + \frac{\sin[(a+b)u - bt]}{ab} \right]_0^t$$

$$= \frac{1}{2} \left[ \frac{\sin(at - bt + kt)}{a-b} + \frac{\sin[(at+bt) - abt]}{a+b} \right]$$

$$- \left( \frac{\sin(bt)}{a-b} + \frac{\sin(-bt)}{a+b} \right)$$

$$= \frac{1}{2} \left[ \frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b} \right]$$

6. Find  $L^{-1} \left\{ \frac{1}{s(s^2+4)} \right\}$  using convolution theorem.

$$\text{Sol: } L^{-1} \left\{ \frac{1}{s(s^2+4)} \right\} = L^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{s^2+4} \right\}$$

$$\text{Rese (let } \bar{f}(s) = \frac{1}{s^2+2^2}$$

$$L^{-1}\{\bar{f}(s)\} = L^{-1} \left\{ \frac{1}{s^2+2^2} \right\}$$

$$\bar{f}(s) = \frac{1}{s}$$

$$L^{-1}\{\bar{g}(s)\} = L^{-1} \left\{ \frac{1}{s} \right\} = 1 = g(t)$$

$$g(t-\omega) = 1$$

$$= \frac{1}{2} \sin 2t$$

$$= f(t)$$

$$f(\omega) = \frac{1}{2} \sin \omega u$$

By convolution theorem

$$L^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{s^2+2^2} \right\} = \int_0^t f(\omega) g(t-\omega) d\omega$$

$$= \int_0^t \frac{1}{2} \sin \omega u \cdot 1 d\omega$$

$$= \frac{1}{2} \int_0^t \sin 2u du = \frac{1}{2} \left[ -\frac{\cos 2u}{2} \right]_0^t$$

$$= \frac{1}{4} [-\cos 2t - (-\cos 0)]$$

$$= \frac{1}{4} [1 - \cos 2t] //$$

## Applications to ordinary differential equations :-

solution of ordinary differential equations with constant coefficients  
 the solutions of ODE can be done easily by using Laplace transform method without finding the general solution by the following procedure.

step1:- Take the Laplace transform of both sides of the given D.E.

step2:- Use the formulae

$$\begin{aligned} i) L\{y'(t)\} &= sY(s) - y(0) \\ ii) L\{y''(t)\} &= s^2Y(s) - sy(0) - y'(0) \\ iii) L\{y'''(t)\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \end{aligned}$$

Replace  $y(0)$ ,  $y'(0)$ ,  $y''(0)$  with the given initial conditions

step3:- Transpose terms to the right hand side and divide by the coefficient of  $L\{y\}$  obtained in the above step

step4:- Resolve the function of 'z' obtained into partial fractions

step5:- Take the inverse L.T on both sides this gives  $y$  as a function of  $t$  which is the required solution of the given equation.

### Problems :-

1. solve  $(D^2 + 4D + 5)y = 5$  given that  $y(0) = 0$  &  $y'(0) = 0$

$$\begin{aligned} \text{Sol:- Given } (D^2 + 4D + 5)y = 5 \\ \Rightarrow D^2y + 4Dy + 5y = 5 \end{aligned}$$

$$\frac{dy}{dx^2} + 4 \frac{dy}{dx} + 5y = 5$$

$$y'' + 4y' + 5y = 5$$

Applying L.T on both sides

$$L\{y'' + 4y' + 5y\} = L\{5\}$$

$$\begin{aligned} L\{y''\} + 4L\{y'\} + 5L\{y\} &= 5 \\ \{s^2L\{y\} - s^2y(0) - y'(0)\} + 4\{sL\{y\} - y(0)\} + 5L\{y\} &= \frac{5}{s} \\ s^2L\{y\} - s(0) - 0 + 4sL\{y\} - 4(0) + 5L\{y\} &= \frac{5}{s} \\ s^2L\{y\} + 4sL\{y\} + 5L\{y\} &= \frac{5}{s} \end{aligned}$$

$$L\{y\} (s^2 + 4s + 5) = \frac{5}{s} \quad \rightarrow ①$$

$$L\{y\} = \frac{5}{s(s^2 + 4s + 5)} \quad \rightarrow ②$$

$$\text{Now } \frac{1}{s(s^2 + 4s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 5}$$

$$1 = A s^2 + 4As + 5A + Bs^2 + Cs$$

$$A + B = 0 \quad \rightarrow ③$$

$$4A + C = 0 \quad \rightarrow ④$$

$$5A = 1 \quad \rightarrow ⑤ \Rightarrow A = \frac{1}{5}$$

$$4A + C = 0 \Rightarrow \frac{4}{5} + C = 0 \Rightarrow C = -\frac{4}{5}$$

$$A + B = 0 \Rightarrow B = -\frac{1}{5}$$

$$L\{x\} = 5 \left[ \frac{1}{s} - \frac{s+4}{s^2+4s+5} \right]$$

$$L\{x\} = 5 \left[ \frac{1}{5s} - \frac{s+4}{5(s^2+4s+5)} \right]$$

Apply inverse L.T on both sides

$$y = 5 L^{-1} \left\{ \frac{1}{5s} - \frac{s+4}{5(s^2+4s+5)} \right\}$$

$$y = \frac{1}{5} L^{-1} \left\{ \frac{1}{s} \right\} - \frac{1}{5} L^{-1} \left\{ \frac{s+4}{s^2+4s+5} \right\}$$

$$y = 1 - L^{-1} \left\{ \frac{(s+2)+2}{(s+2)^2+1} \right\}$$

$$y = 1 - e^{-2t} L^{-1} \left\{ \frac{s+2}{s^2+1} \right\}$$

$$y = 1 - e^{-2t} \left[ L^{-1} \left\{ \frac{s}{s^2+1} \right\} + 2 L^{-1} \left\{ \frac{1}{s^2+1} \right\} \right]$$

$$y = 1 - e^{-2t} [ \cos t + 2 \sin t ]$$

$$y =$$

$$2. \text{ solve } L\{x\} - s x(0) - x'(0) - 4 [ L\{x\} - x(0) ] + 8 L\{x\} = \frac{1}{s-2}$$

$$L\{x\} - s x(0) - x'(0) - 2 + 4 L\{x\} = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 4s + 8) L\{x\} - s(2) - 2 + 4 L\{x\} = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 4s + 8) L\{x\} = \frac{1}{s-2} + 2s - 6$$

$$\Rightarrow s^2 L\{x\} - s x(0) - x'(0) - 4 [ L\{x\} - x(0) ] + 8 L\{x\} = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 4s + 8) L\{x\} - s(2) - 2 + 4 L\{x\} = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 4s + 8) L\{x\} = \frac{1+8s-6}{s-2}$$

$$= \frac{1+6s-6}{s-2}$$

$$\Rightarrow L\{x\} = \frac{1+6s-6}{(s-2)(s^2-4s+8)} = \frac{1}{(s-2)(s^2-4s+8)} + \frac{6s-6}{s^2-4s+8}$$

$$\frac{1}{(s-2)(s^2-4s+8)}$$

Apply inverse L.T on both sides

$$x = L^{-1} \left\{ \frac{1}{(s-2)(s^2-4s+8)} \right\} + L^{-1} \left\{ \frac{6s-6}{s^2-4s+8} \right\} \rightarrow \textcircled{1}$$

$$= \frac{1}{s-2} \left\{ \frac{1}{(s-2)(s^2-4s+8)} \right\} + \frac{6s-6}{s^2-4s+8} \left\{ \frac{1}{s-2} \right\}$$

$$\text{Now } L^{-1} \left\{ \frac{1}{(s-2)(s^2-4s+8)} \right\} = L^{-1} \left\{ \frac{1}{(s-2)(s^2-4s+4)} \right\}$$

$$= e^{2t} L^{-1} \left\{ \frac{1}{s(s^2+4)} \right\}$$

$$= e^{2t} L^{-1} \left\{ \frac{1}{s} \right\} \quad f(s) = \frac{1}{s^2+4} \quad f(t) = \int_0^t f(s) ds = \frac{1}{2} \sin 2t$$

$$= e^{2t} \int_0^t \frac{1}{s} \sin 2t dt$$

$$= e^{2t} \left[ -\frac{\cos 2t}{2} \right]_0^t$$

$$= \frac{e^{2t}}{2} [ 1 - \cos 2t ] \rightarrow \textcircled{2}$$

$$L^{-1} \left\{ \frac{8s-6}{s^2-4s+8} \right\} = L^{-1} \left\{ \frac{2s-6}{(s-2)^2+4} \right\}$$

$$= L^{-1} \left\{ \frac{2(s-2)-2}{(s-2)^2+4} \right\}$$

$$= e^{2t} L^{-1} \left\{ \frac{2s-2}{s^2+4} \right\}$$

$$= e^{2t} L^{-1} \left\{ s - 2 L^{-1} \left\{ \frac{1}{s^2+4} \right\} \right\}$$

$$= e^{2t} \left[ s \cos 2t - 2 \sin 2t \right]$$

$$= e^{2t} \left[ 2 \cos 2t - 2 \sin 2t \right] \rightarrow ③$$

Sub ② & ③ in Eqn ①

$$x = \frac{e^{2t}}{4} [1 - e^{2t}] + e^{2t} [2 \cos 2t - \sin 2t]$$

3. solve  $(D^2+1)y = 6 \cos 2t$  given that  $y=0$  at  $t=0$

$$\text{Ans} : \text{Given } y(0) = 3 \quad y'(0) = 1$$

$$y'' + y = 6 \cos 2t$$

$$L\{y''\} + L\{y\} = 6 L\{ \cos 2t \}$$

$$s^2 Y + sY = 6 \frac{s^2}{s^2+4}$$

$$(s^2+1) L\{y\} - sY - Y(0) = 6 \frac{s^2}{s^2+4}$$

$$(s^2+1) L\{y\} - sY - 6 = 6 \frac{s^2}{s^2+4}$$

$$(s^2+1) L\{y\} = \frac{6s}{s^2+4} + 6s + 6$$

$$L\{y\} = \frac{6s}{(s^2+4)(s^2+1)} + \frac{3s}{s^2+1} + \frac{1}{s^2+1} \rightarrow ①$$

$$\text{But } \frac{s}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$s = As^3 + Bs^2 + Cs + D + Cs^3 + Ds^2 + 4Cs + 4D$$

$$A + C = 0 \rightarrow ①$$

$$B + D = 0 \rightarrow ②$$

$$A + 4C = 1 \rightarrow ③$$

$$B + 4D = 0 \rightarrow ④$$

solving ① & ②

$$\frac{A+4C=1}{-3C=-1} \Rightarrow C = \frac{1}{3} \quad \therefore A = -\frac{1}{3}$$

solving ② & ④

$$\begin{aligned} B+D=0 \\ -B+4D=0 \\ \hline -3D=0 \Rightarrow D=0 \end{aligned} \quad \therefore B=0$$

$$\frac{\frac{1}{3}(s^2+4)(s^2+1)}{-\frac{1}{3}(s^2+4) + \frac{1}{3}(s^2+1)} = \frac{1}{3} \left[ \frac{\frac{1}{3}}{s^2+1} - \frac{1}{s^2+4} \right]$$

$$\therefore L\{y\} = \frac{1}{3} \left[ \frac{1}{s^2+1} - \frac{1}{s^2+4} \right] + 3 \frac{s}{s^2+4} + \frac{1}{s^2+1}$$

$$y = L^{-1} \left\{ \frac{1}{s^2+1} \right\} - L^{-1} \left\{ \frac{1}{s^2+4} \right\} + 3 L^{-1} \left\{ \frac{s}{s^2+4} \right\} + L^{-1} \left\{ \frac{1}{s^2+1} \right\}$$

$$y = 2 \sin t - \frac{3 \cos 2t}{2} + 3 \cos t + \sin t$$

$$\frac{s}{(s+4)(s^2+1)} = \frac{-s}{3(s+4)} + \frac{s}{3(s^2+1)}$$

$$= \frac{4}{3} \left[ \frac{s}{s^2+1} - \frac{s}{s+4} \right]$$

from ①

$$L\{y\} = \frac{2}{s} \left[ \frac{s}{s^2+1} - \frac{s}{s^2+4} \right] + 3 \left[ \frac{s}{s^2+1} \right] + \frac{1}{s+1}$$

$$y = 2 L^{-1} \left\{ \frac{s}{s^2+1} \right\} - 2 L^{-1} \left\{ \frac{s}{s^2+4} \right\} + 3 L^{-1} \left\{ \frac{s}{s^2+1} \right\} + L^{-1} \left\{ \frac{1}{s+1} \right\}$$

$$= 2 \text{cost} - 2 \cos 2t + 3 \text{cost} + \sin t$$

$$= 5 \text{cost} - 2 \cos 2t + \sin t$$

$$= 3e^{-t} L^{-1} \left\{ \frac{1}{s^2+4} \right\} + 4e^{-t} L^{-1} \left\{ \frac{1}{s^2+1} \right\} + 8e^{-t} L^{-1} \left\{ \frac{1}{s+1} \right\}$$

4. solve  $\frac{d^2x}{dt^2} + 2 \frac{dx}{dt} + x = 3t e^{-t}$  given  $x(0)=4$

$$\frac{dx}{dt} = 0 \text{ at } t=0$$

$$x^{(1)} + 2x^{(1)} + x = 3t e^{-t}$$

$$L\{x^{(1)}\} + 2L\{x^{(1)}\} + L\{x\} = 3L\{t e^{-t}\}$$

$$s^2 L\{x\} - s x(0) - x'(0) + 2 [s L\{x\} - x(0)] + 4x = 3 \frac{1}{(s+1)^2}$$

$$s^2 L\{x\} - 4s - 0 + 2s L\{x\} - 8 + 4x =$$

$$L\{x\} + s^2 + 2 + 1 = \frac{3}{(s+1)^2} + 4s + 8$$

$$L\{x\} (s+1)^2 = \frac{3}{(s+1)^2} + 4s + 8$$

$$L\{x\} = \frac{\frac{3}{(s+1)^2}}{(s^2+2s+8)} + \frac{\frac{4s}{(s+1)}}{(s^2+2s+8)} + \frac{\frac{8}{(s+1)}}{(s^2+2s+8)}$$

$$x = L^{-1} \left\{ \frac{3}{(s+1)^4} \right\} + 4 L^{-1} \left\{ \frac{4s}{(s+1)^2} \right\} + 8 L^{-1} \left\{ \frac{8}{(s+1)^2} \right\}$$

$$= 3 L^{-1} \left\{ \frac{1}{(s+1)^4} \right\} + 4 L^{-1} \left\{ \frac{(s+1)^{-1}}{(s+1)^2} \right\} + 8 L^{-1} \left\{ \frac{1}{(s+1)^2} \right\}$$

$$= 3e^{-t} L^{-1} \left\{ \frac{1}{s^4} \right\} + 4e^{-t} L^{-1} \left\{ \frac{s-1}{s^2} \right\} + 8e^{-t} L^{-1} \left\{ \frac{1}{s^2} \right\}$$

$$= 3e^{-t} L^{-1} \left\{ \frac{1}{s^4} \right\} + 4e^{-t} [L^{-1} \left\{ \frac{1}{s^2} \right\} - L^{-1} \left\{ \frac{1}{s^2} \right\}] + 8e^{-t} L^{-1} \left\{ \frac{1}{s^2} \right\}$$

$$= 3e^{-t} L^{-1} \left\{ \frac{1}{s^4} \right\} + 4e^{-t} [L^{-1} \left\{ \frac{1}{s^2} \right\} - L^{-1} \left\{ \frac{1}{s^2} \right\}] + 8e^{-t} L^{-1} \left\{ \frac{1}{s^2} \right\}$$

5. solve  $y''' + 2y'' - y' - 2y = 0$  given  $y(0) = y'(0) = 0$  and  $y''(0) = 6$ .

$$L\{y'''\} + 2L\{y''\} - L\{y'\} - 2L\{y\} = 0$$
~~$$s^3 L\{y\} - s^2 y(0) - s y'(0) - y''(0) - 2 [s^2 L\{y\} - s y(0) - y'(0)] + 2 [s^3 L\{y\} - s^2 y(0) - s y'(0) - y''(0)] = 0$$~~

$$- s^3 L\{y\} - s^2 y(0) - s y'(0) - y''(0) = 0$$

$$L\{y\} (s^3 + 2s^2 - s - 2) - 6 = 0$$

$$L\{y\} = \frac{6}{(s^3 + 2s^2 - s - 2)} = \frac{6}{(s+1)(s-1)(s+2)}$$

$$L\{y\} = \frac{6}{(s+1)(s-1)(s+2)} \rightarrow \textcircled{1}$$

$$\text{Now } \frac{1}{(s+1)(s-1)(s+2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$1 = A(s-1)(s+2) + B(s+1)(s+2) + C(s+1)(s-1)$$

$$\begin{aligned} 1 &= A(s^2+s-2) + B(s^2+3s+2) + C(s^2-1) \\ &= As^2+As-2A+Bs^2+3Bs+2B+Cs^2-C \\ &= As^2+As-2A+Bs^2+3Bs+2B+Cs^2-C \end{aligned}$$

$$\begin{aligned} A+B+C &= 0 \rightarrow \textcircled{1} \\ A+3B &= 0 \rightarrow \textcircled{2} \\ -2A+2B-C &= 0 \rightarrow \textcircled{3} \end{aligned}$$

Solving \textcircled{1} & \textcircled{3}

$$\begin{aligned} A+B+C &= 0 \\ -2A+2B-C &= 0 \\ -A+3B &= 1 \end{aligned} \rightarrow \textcircled{4}$$

Solving \textcircled{2} & \textcircled{4}

$$\begin{aligned} A+3B &= 0 \\ -A+3B &= 1 \\ B &= \frac{1}{2} \\ A &= -\frac{3}{2}B \\ A &= -\frac{3}{2} \cdot \frac{1}{2} \\ A &= -\frac{3}{4} \end{aligned}$$

$$\begin{aligned} -\frac{3}{4} + \frac{3}{2} + C &= 0 \\ -\frac{3}{4} + C &= 0 \\ C &= \frac{3}{4} \\ C &= \frac{1}{3} \end{aligned}$$

$$\therefore \frac{1}{(s+1)(s-1)(s+2)} = \frac{-\frac{1}{2}(s+1)}{2(s+1)} + \frac{\frac{1}{6}(s-1)}{6(s-1)} + \frac{\frac{1}{3}(s+2)}{3(s+2)}$$

from \textcircled{1}

$$\begin{aligned} L\{y\} &= 6 \left\{ -\frac{1}{2}(s+1) + \frac{1}{6}(s-1) + \frac{1}{3}(s+2) \right\} \\ &= -\frac{3}{s+1} + \frac{1}{s-1} + \frac{2}{s+2} \end{aligned}$$

$$y = L^{-1} \left\{ -\frac{3}{s+1} + \frac{1}{s-1} + \frac{2}{s+2} \right\}$$

$$\begin{aligned} y &= -3 L^{-1} \left\{ \frac{1}{s+1} \right\} + L^{-1} \left\{ \frac{1}{s-1} \right\} + 2 L^{-1} \left\{ \frac{1}{s+2} \right\} \\ y &= -3 e^{-t} + e^t + 2 e^{-2t} \end{aligned}$$

6. solve  $4y'' + \pi^2 y = 0$ ,  $y(0) = 2$ ,  $y'(0) = 0$

$$\text{Sol: } 4L\{y''\} + \pi^2 L\{y\} = 0$$

$$4 \left[ s^2 L\{y\} - s y(0) - y'(0) \right] + \pi^2 L\{y\} = 0$$

$$(4s^2 + \pi^2) L\{y\} - 8s = 0$$

$$L\{y\} = \frac{8s}{4s^2 + \pi^2} = \frac{8s}{4(s^2 + (\frac{\pi}{2})^2)} = \frac{2s}{s^2 + (\frac{\pi}{2})^2}$$

$$\begin{aligned} y &= 2 L^{-1} \left\{ \frac{s}{s^2 + (\frac{\pi}{2})^2} \right\} = 2 L^{-1} \left\{ \frac{\frac{s}{\pi}}{(\frac{s}{\pi})^2 + 1} \right\} \\ &\Rightarrow y = 2 \operatorname{cosec} \frac{\pi t}{2} \end{aligned}$$