COURSE MATERIAL	
SUBJECT	NUMERICAL METHODS AND PROBABILITY THEORY (20A54402)
UNIT	3
COURSE	B.TECH
SEMESTER	2 - 2
DEPARTMENT	HUMANITIES & SCIENCE
PREPARED BY (Faculty Name/s)	Department of Mathematics

7. ACTIVITY BASED LEARNING

1. To Solve Ordinary Differential Equations by using Numerical Techniques

8. LECTURE NOTES

3.1 Numerical Integration

This is the most popular and widely used numerical integration formula. It forms the basis for a number of numerical integration methods.

3.2 Trapezodial Rule

Here the function f(x) is approximated by a first-order polynomial, h is the length of interval and n is number of sub-intervals

Then $\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots\cdots + y_{n-1})]$

=> $\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}[Sum\ of\ first\ and\ last\ ordinates + 2 \times The\ remaining\ ordinates]$

First divide the interval $[x_0, x_n]$ into n sub-intervals with the length of interval h, then substitute all values of y and simply which gives the value of $\int_{x_0}^{x_n} f(x)\,dx$

Problems

1. Evaluate $\int_0^1 x^3 dx$ with five sub-intervals by Trapezodial rule

Sol: Here $a = 0, b = 1, n = 5$ and $h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$

The values of x and y are tabulated below

x	0	0.2	0.4	0.6	0.8	1
y	0	0.008	0.064	0.216	0.512	1

Trapezodial rule

$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$

$\int_0^1 x^3 dx = \frac{0.2}{2}[(0 + 1) + 2(0.008 + 0.064 + 0.216 + 0.512)]$

$= 0.26$

2. Solve $\int_0^{\pi} t\ \sin t\ dt$ using the Trapezodial rule

Sol: Divide the interval $[0, \pi]$ into 6 parts each of width $h = \frac{\pi}{6}$

The values of $f(t) = t\ \sin t$ are given below

t	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
Y=t sin t	0	0.2618	0.9069	1.5708	1.8138	1.309	0

Trapezodial rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$\int_0^1 x^3 dx = \frac{\pi}{12}[(0 + 0) + 2(0.2618 + 0.9069 + 1.5708 + 1.8138 + 1.309)]$$

=3.07

3.3 Simpson's 1/3 Rule:

This is another popular method. Here the function f(x) is approximated by a second order polynomial

Then

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots\cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots\cdots + y_{n-2})]$$

First divide the interval $[x_0, x_n]$ into n(even) sub-intervals with the length of interval h, then substitute all values of y and simply which gives the value of $\int_{x_0}^{x_n} f(x)\,dx$.

1. Find the value of $\int_1^2 \frac{1}{x} dx$ by Simpson's 1/3 rule.

Sol: Divide the interval $[1, 2]$ into 8 parts each of width $h = 0.125$

The values of $f(x) = \frac{1}{x}$ are given below

x	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
y	1	0.8888	0.8	0.7272	0.6666	0.6153	0.5714	0.5333	0.5

Then by Simpsons 1/3 rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$\int_{1}^{2} \frac{1}{x}\,dx = \frac{0.125}{3}[(1 + 0.5) + 4(0.8888 + 0.7272 + 0.6153 + 0.5333) + 2(0.8 + 0.6666 + 0.5714)]$$

= 0.6931

2. Evaluate $\int_0^6 \frac{1}{1+x}dx$ by Simpson's 1/3 rule.

Sol: Divide the interval $[0, 6]$ into 6 parts each of width $h=1$

The values of $f(x) = \frac{1}{1+x}$ are given below

x	0	1	2	3	4	5	6
$Y=f(x) = \frac{1}{1+x}$	1	0.5	0.3333	0.25	0.2	0.1666	0.1428

Then by Simpsons 1/3 rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$\int_{1}^{2} \frac{1}{x}\,dx = \frac{1}{3}[(1 + 0.1428) + 4(0.5 + 0.25 + 0.1666) + 2(0.3333 + 0.2)]$$

=1.9659

3.4 Simpson's 3/8 Rule:

This is another popular method. Here the function f(x) is approximated by a second order polynomial

Then

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots\cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots\cdots + y_{n-2})]$$

First divide the interval $[x_0, x_n]$ into n(even) sub-intervals with the length of interval h, then substitute all values of y and simply which gives the value of $\int_{x_0}^{x_n} f(x)\, dx$.

1. Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ by Simpson's 1/3 rule.

Sol: Divide the interval $[0, 6]$ into 6 parts each of width $h = 1$

The values of $f(x) = \frac{1}{1+x^2}$ are given below

x	0	1	2	3	4	5	6
$Y = f(x) = \frac{1}{1+x^2}$	1	0.5	0.2	0.1	0.0588	0.0385	0.0027

Then by Simpsons 3/8 rule

$$\int_{x_0}^{x_n} f(x)\, dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$\int_1^2 \frac{1}{x}\, dx = \frac{1}{3}[(1 + 0.0027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)]$$

=1.3571

2. Evaluate $\int_0^1 \sqrt{1 + x^4}\, dx$ using Simpson's 3/8 rule

Sol: Sol: Divide the interval $[0, 1]$ into 6 parts each of width $h = \frac{1}{6}$

The values of $f(x) = \sqrt{1 + x^4}$ are given below

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$Y = f(x) = \sqrt{1 + x^4}$	1	1.0004	1.0062	1.0301	1.0943	1.2175	1.4142

Then by Simpsons 3/8 rule

$$\int_{x_0}^{x_n} f(x)\, dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$\int_{1}^{2} \frac{1}{x}\, dx = \frac{1}{3}[(1 + 1.4142) + 3(1.0004 + 1.0062 + 1.0943 + 1.2175) + 2(1.0301)]$$

=1.0894

Numerical solutions of ordinary differential equations

1. The important methods of solving ordinary differential equations of first order numerically are as follows
   1) Taylor's series method
   2) Picard's method
   3) Euler's method
   4) Modified Euler's method of successive approximations
   5) Runge- kutta method

To describe various numerical methods for the solution of ordinary differential eqn's, we consider the general 1st order differential eqn

Given O.D.Eqn. dy/dx=f(x,y)-------(1)

with the initial condition $y(x_0)=y_0$, $X_1=X_0+h$, $X_2=X_1+h$,we have to evaluate Y_1,Y_2---- etc

The methods will yield the solution in one of the two forms:

i) A series for y in terms of powers of x, from which the value of y can be obtained by direct substitution.

ii) A set of tabulated values of y corresponding to different values of x

The methods of Taylor and Picard belong to class(i)

The methods of Euler, Runge - kutta method, Adams, Milne etc, belong to class (ii)

3.4 TAYLOR'S SERIES METHOD

To find the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x,y) \rightarrow (1)$$

With the initial condition $y(x_0) = y_0 \rightarrow (2)$

$y(x)$ can be expanded about the point x_0 in a Taylor's series in powers of $(x - x_0)$ as

$$y(x) = y(x_0) + \frac{(x-x_0)}{1} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + ... + \frac{(x-x_0)^n}{n!} y^n(x_0) \rightarrow (3)$$

In equ3, $y(x_0)$ is known from I.C equ2. The remaining coefficients $y'(x_0), y''(x_0), ...y^n(x_0)$ etc are obtained by successively differentiating equ1 and evaluating at x_0. Substituting these values in equ3, $y(x)$ at any point can be calculated from equ3. Provided $h = x - x_0$ is small.

When $x_0 = 0$, then Taylor's series equ3 can be written as

$$y(x) = y(0) + x.y'(0) + \frac{x^2}{2!} y''(0) + ... + \frac{x^n}{n!} y^n(0) + ... \rightarrow (4)$$

1. Using Taylor's expansion evaluate the value of $y' - 2y = 3e^x, y(0) = 0$**, at a)** $x = 0.2$

b) Compare the numerical solution obtained with exact solution .

Sol: Given equation can be written as $2y + 3e^x = y', y(0) = 0$ $x_0=0$, $y_0=0$

Differentiating repeatedly w.r.t to 'x' and evaluating at $x = 0$

$$y'(x) = 2y + 3e^x, y'(0) = 2y(0) + 3e^0 = 2(0) + 3(1) = 3$$
$$y''(x) = 2y' + 3e^x, y''(0) = 2y'(0) + 3e^0 = 2(3) + 3 = 9$$
$$y'''(x) = 2.y''(x) + 3e^x, y'''(0) = 2y''(0) + 3e^0 = 2(9) + 3 = 21$$
$$y^{iv}(x) = 2.y'''(x) + 3e^x, y^{iv}(0) = 2(21) + 3e^0 = 45$$
$$y^v(x) = 2.y^{iv} + 3e^x, y^v(0) = 2(45) + 3e^0 = 90 + 3 = 93$$

In general, $y^{(n+1)}(x) = 2.y^{(n)}(x) + 3e^x$ or $y^{(n+1)}(0) = 2.y^{(n)}(0) + 3e^0$

The Taylor's series expansion of $y(x)$ about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + \frac{x^5}{5!}y'''''(0) + ...$$

Substituting the values of $y(0), y'(0), y''(0), y'''(0), ...$

$$y(x) = 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \frac{93}{120}x^5 + ...$$

$$y(x) = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \frac{31}{40}x^5 + ... \rightarrow \text{equ1}$$

Now put $x = 0.1$ in equ1

$$y(0.1) = 3(0.1) + \frac{9}{2}(0.1)^2 + \frac{7}{2}(0.1)^3 + \frac{15}{8}(0.1)^4 + \frac{31}{40}(0.1)^5 = 0.34869$$

Now put $x = 0.2$ in equ1

$$y(0.2) = 3(0.2) + \frac{9}{2}(0.2)^2 + \frac{7}{2}(0.2)^3 + \frac{15}{8}(0.2)^4 + \frac{31}{40}(0.2)^5 = 0.811244$$

$$y(0.3) = 3(0.3) + \frac{9}{2}(0.3)^2 + \frac{7}{2}(0.3)^3 + \frac{15}{8}(0.3)^4 + \frac{31}{40}(0.3)^5 = 1.41657075$$

Analytical Solution:

The exact solution of the equation $\frac{dy}{dx} = 2y + 3e^x$ with $y(0) = 0$ can be found as follows

$\frac{dy}{dx} - 2y = 3e^x$ Which is a linear in y.

Here $P = -2, Q = 3e^x$

I.F = $\int_e^{pdx} = \int_e^{-2dx} = e^{-24}$

General solution is $y.e^{-2x} = \int 3e^x.e^{-2x}dx + c = -3e^{-x} + c$,dividing by e^{-2x} on b.s.

$\therefore y = -3e^x + ce^{2x}$ where $x = 0, y = 0$, $0 = -3 + c \Rightarrow c = 3$

The particular solution is $y = 3e^{2x} - 3e^x$ or $y(x) = 3e^{2x} - 3e^x$

Put $x = 0.1$ in the above particular solution,

$$y = 3.e^{0.2} - 3e^{0.1} = 0.34869$$

Similarly put $x = 0.2$

$y = 3e^{0.4} - 3e^{0.2} = 0.811265$

put $x = 0.3$

$y = 3e^{0.6} - 3e^{0.3} = 1.416577$

There is negligible error between numerical solution and analytical solution.

2. Using Taylor's series method, solve the equation $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$ given that $y = 0$ when $x = 0$

Sol: Given that $\frac{dy}{dx} = x^2 + y^2$ and $y = 0$ when $x = 0$ i.e. $y(0) = 0$

Here $y_0 = 0$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at $x = 0$

$$\begin{aligned}
&y'(x) = x^2 + y^2, y'(0) = 0 + y^2(0) = 0 + 0 = 0 \\
&y''(x) = 2x + y'.2y, y''(0) = 2(0) + y'(0)2.y = 0 \\
&y'''(x) = 2 + 2yy'' + 2y'.y', y'''(0) = 2 + 2.y(0).y''(0) + 2.y'(0)^2 = 2 \\
&y''''(x) = 2.y.y''' + 2.y''.y' + 4.y''.y', y''''(0) = 0
\end{aligned}$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + ...$$

Substituting the values of $y(0), y'(0), y''(0), ...$

$y(x) = 0 + x(0) + 0 + \frac{2x^3}{3!} + 0 + ... = \frac{x^3}{3} +$ (Higher order terms are neglected)

$$\therefore y(0.4) = \frac{(0.4)^3}{3} = \frac{0.064}{3} = 0.02133$$

3. Solve $y' = x - y^2, y(0) = 1$ using Taylor's series method and compute y(0.1),y(0.2)

Sol: Given that $y' = x - y^2, y(0) = 1$

Here $y_0 = 1$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at x=0

$$y'(x) = x - y^2, y'(0) = 0 - y(0)^2 = 0 - 1 = -1$$
$$y''(x) = 1 - 2y.y', y''(0) = 1 - 2.y(0)y'(0) = 1 - 2(-1) = 3$$
$$y'''(x) = 1 - 2yy' - 2(y')^2, y'''(0) = -2.y(0).y''(0) - 2.(y'(0))^2 = -6 - 2 = -8$$
$$y''''(x) = -2.y.y''' - 2.y''.y' - 4.y''.y', y''''(0) = -2.y(0).y'''(0) - 6.y''(0).y'(0) = 16 + 18 = 34$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!}y^1(0) + \frac{x^2}{2!}y^{11}(0) + \frac{x^3}{3!}y^{111}(0) + \ldots..$$

Substituting the value of $y(0)$, $y^1(0)$, $y^{11}(0)$,…..

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{8}{6}x^3 + \frac{34}{24}x^4 + \ldots..$$

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 + \ldots.. \quad \rightarrow (1)$$

now put x = 0.1 in (1)

$$y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 + \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4 + \ldots..$$

$$= 0.91380333 \simeq 0.91381$$

Similarly put x = 0.2 in (1)

$$y(0.2) = 1 - 0.2 + \frac{3}{2}(0.2)^2 - \frac{4}{3}(0.2)^3 + \frac{17}{12}(0.2)^4 + \ldots..$$

$$= 0.8516.$$

4. Solve $y^1 = x^2 - y$, $y(0) = 1$, using Taylor's series method and compute y(0.1), y(0.2), y(0.3) and y(0.4) (correct to 4 decimal places).

Sol. Given that $y^I = x^2 - y$ and $y(0) = 1$

Here $x_0 = 0$, $y_0 = 1$ or y =1 when x=0

Differentiating repeatedly w.r.t 'x' and evaluating at x = 0.

$Y^I(x) = x^2 - y$, $y^I(0) = 0 - 1 = -1$

$y^{II}(x) = 2x - y^I$, $y^{II}(0) = 2(0) - y^I(0) = 0 - (-1) = 1$

$y^{III}(x) = 2 - y^{II}$, $y^{III}(0) = 2 - y^{II}(0) = 2 - 1 = 1$,

$y^{IV}(x) = - y^{III}$, $y^{IV}(0) = -y^{III}(0) = -1$.

The Taylor's servies for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!}y^{I}(0) + \frac{x^2}{2!}y^{II}(0) + \frac{x^3}{3!}y^{III}(0) + \frac{x^4}{4!}y^{IV}(0) + \ldots\ldots$$

substituting the values of $y(0)$, $y^1(0)$, $y^{11}(0)$, $y^{111}(0)$,……

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(-1) + \ldots\ldots$$

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \ldots\ldots \qquad \rightarrow(1)$$

Now put x = 0.1 in (1),

$$y(0.1) = 1 - 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} - \frac{(0.1)^4}{24} + \ldots.$$

$= 1 - 0.1 + 0.005 + 0.01666 - 0.0000416 -0.905125 \sim 0.9051$

(4 decimal places)

Now put x = 0.2 in eq (1),

$$y(0.2) = 1 - 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} - \frac{(0.2)^4}{64}$$

$= 1 - 0.2 + 0.02 + 0.001333 - 0.000025$

$= 1.021333 - 0.200025$

$= 0.821308 \sim 0.8213$ (4 decimals)

Similarly y(0.3) = 0.7492 and y (0.4) = 0.6897 (4 decimal places).

5. Solve $\frac{dy}{dx}$ -1 = xy and y(0) = 1 using Taylor's series method and compute y(0.1).

Sol. Given that $\frac{dy}{dx} - 1 = xy$ and $y(0) = 1$

Here $\frac{dy}{dx} = 1 + xy$ and $y_0 = 1$, $x_0 = 0$.

Differentiating repeatedly w.r.t 'x' and evaluating at $x_0 = 0$

$y^{I}(x) = 1 + xy$, $\qquad y^{I}(0) = 1+0(1) = 1$.

$y^{II}(x) = x.y' + y$, $\qquad y^{II}(0) = 0+1=1$

$y^{III}(x) = x.y'' + y^{I} + y^{I}$, $\qquad y^{III}(0) = 0.(1) + 2\sqcup(1) = 2$

$y^{IV}(x) = xy^{III} + y^{II} + 2y^{II}$, $\qquad y^{IV}(0) = 0+3(1) = 3$.

$y^{V}(x) = xy^{IV} + y^{III} + 2y^{III}$, $\qquad y^{V}(0) = 0 + 2 + 2(3) = 8$

The Taylor series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + x.y^{I}(0) + \frac{x^2}{2!} y^{II}(0) + \frac{x^3}{3!} y^{III}(0) + \frac{x^4}{4!} y^{IV}(0) + \frac{x^5}{5!} y^{V}(0) + \ldots..$$

Substituting the values of $y(0)$, $y^{I}(0)$, $y^{II}(0)$, ….

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}(2) + \frac{x^4}{24}(3) + \frac{x^5}{120}(8) + \ldots.$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \ldots. \qquad \rightarrow (1)$$

Now put x = 0.1 in equ (1),

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \ldots..$$

$= 1 + 0.1 + 0.005 + 0.000333 + 0.0000125 + 0.0000006$

$= 1.1053461$

H.W

6. Given the differential equ $y^1 = x^2 + y^2$, $y(0) = 1$.Obtain y(0.25), and y(0.5) by Taylor's Series method.

Ans: 1.3333, 1.81667

7. Solve $y^1 = xy^2 + y$, $y(0) =1$ using Taylor's series method and compute y(0.1) and y(0.2).

Ans: 1.111, 1.248.

Note: We know that the Taylor's expansion of y(x) about the point x_0 in a power of $(x – x_0)$is.

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y^{I}(x_0) + \frac{(x-x_0)^2}{2!} y^{II}(x_0) + \frac{(x-x_0)^3}{3!} y^{III}(x_0) + \ldots \rightarrow (1)$$

Or

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0^{I} + \frac{(x-x_0)^2}{2!} y_0^{II} + \frac{(x-x_0)^3}{3!} y_0^{III} + \ldots..$$

If we let $x - x_0 = h$. (i.e. $x = x_0 + h = x_1$) we can write the Taylor's series as

$$y(x) = y(x_1) = y_0 + \frac{h}{1!} y_0^{I} + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots.$$

$$\text{i.e. } y_1 = y_0 + \frac{h}{1!} y_0^{I} + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^{IV}}{4!} y_0^{IV} + \ldots.. \quad \rightarrow (2)$$

Similarly expanding y(x) in a Taylor's series about $x = x_1$. We will get.

$$y_2 = y1 + \frac{h}{1!} y_1^{I} + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots\ldots \quad \rightarrow (3)$$

Similarly expanding y(x) in a Taylor's series about $x = x_2$ We will get.

$$y_3 = y_2 + \frac{h}{1!} y_2^{I} + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots\ldots \quad \rightarrow (4)$$

In general, Taylor's expansion of y(x) at a point $x = x_n$ is

$$y_{n+1} = y_n + \frac{h}{1!} y_n^{I} + \frac{h^2}{2!} y_n^{II} + \frac{h^3}{3!} y_n^{III} + \frac{h^4}{4!} y_n^{IV} + \ldots.. \quad \rightarrow (5)$$

8. Solve $y^1 = x - y^2$, $y(0) = 1$ using Taylor's series method and evaluate y(0.1), y(0.2) by step size h=0.1.

Sol: Given $y^I = x - y^2$ → (1)

and $y(0) = 1$ → (2)

Here $x_0 = 0$, $y_0 = 1$.

Differentiating (1) w.r.t 'x', we get.

$y^{II} = 1 - 2yy^{I}$ → (3)

$y^{III} = -2(y.\, y^{II} + (y^{I})^2)$ → (4)

$y^{IV} = -2[y.\, y^{III} + y^{I}.\, y^{II} + 2y^{I}.\, y^{II}]$ → (5)

$= -2(3y^{I}.\, y^{II} + y.\, y^{III})$ ……

Put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5),

We get

$y_0^{I} = 0\text{-}1 = -1,$

$y_0^{II} = 1\text{-}2(1)\,(-1) = 3,$

$y_0^{III} = -2[(-1)^2) + (1)\,(3)] = -8$

y_0^{IV} = -2[3(-1) (3) + (1) (-8)] = -2(-9 -8) = 34.

Take h=0.1

Step1: By Taylor's series, we have

$$y_1 = y_0 + \frac{h}{1!}y_0^I + \frac{h^2}{2!}y_0^{II} + \frac{h^3}{3!}y_0^{III} + \frac{h^4}{4!}y_0^{IV} + \ldots \quad \rightarrow (6)$$

on substituting the values of y_0, y_0^I, y_0^{II}, etc in equ (6) we get

$$y(0.1) = y_1 = 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34) + \ldots$$

$$= 1 - 0.1 + 0.015 - 0.00133 + 0.00014 + \ldots$$

$$= 0.91381$$

Step2: Let us find y(0.2), we start with (x_1,y_1) as the starting value.

Here $x_1 = x_0 + h = 0+0.1 = 0.1$ and $y_1 = 0.91381$

Put these values of x_1 and y_1 in (1),(3),(4) and (5),we get

$y_1^I = x_1 - y_1^2 = 0.1 - (0.91381)^2 = 0.1 - 0.8350487 = -0.735$

$y_1^{II} = 1 - 2y_1 \cdot y_1^I = 1- 2(0.91381)(-0.735) = 1 + 1.3433 = 2.3433$

$y_1^{III} = -2[(y_1^I)^2 + y_1 \cdot y_1^{II}] = -2[(-0.735)^2 + (0.91381)(2.3433)] = -5.363112$

$y_1^{IV} = -2[3.\,y_1^I\, y_1^{II} + y_1\, y_1^{III}]$

$= -2[3.(-0.735)(2.3433) + (0.91381)(-5.363112)]$

$= -2[(-5.16697) - 4.9] = 20.133953$

By Taylor's series expansion,

$$y_2 = y_1 + \frac{h}{1!}y_1^I + \frac{h^2}{2!}y_1^{II} + \frac{h^3}{3!}y_1^{III} + \frac{h^4}{4!}y_1^{IV} + \ldots$$

$$\therefore y(0.2) = y_2 = 0.91381 + (0.1)(-0.735) + \frac{(0.1)^2}{2}(2.3433) +$$

$$\frac{(0.1)^3}{6}(-5.363112) + \frac{(0.1)^4}{24}(20.133953) + \ldots$$

$y(0.2) = 0.91381 - 0.0735 + 0.0117 - 0.00089 + 0.00008 = 0.8512$

9. Tabulate y(0.1), y(0.2) and y(0.3) using Taylor's series method given that $y^1 = y^2 + x$ and y(0) = 1

Sol: Given $y^{I} = y^2 + x$ → (1)

and $y(0) = 1$ → (2)

Here $x_0 = 0$, $y_0 = 1$.

Differentiating (1) w.r.t 'x', we get

$y^{II} = 2y \cdot y^{I} + 1$ → (3)

$y^{III} = 2[y \cdot y^{II} + (y^{I})^2]$ → (4)

$y^{IV} = 2[y \cdot y^{III} + y^{I} y^{II} + 2 y^{I} y^{II}]$

$= 2[y \cdot y^{III} + 3 y^{I} y^{II}]$ → (5)

Put $x_0 = 0$, $y_0 = 1$ in (1), (3), (4) and (5), we get

$y_0^{I} = (1)^2 + 0 = 1$

$y_0^{II} = 2(1)(1) + 1 = 3,$

$y_0^{III} = 2((1)(3) + (1)^2) = 8$

$y_0^{IV} = 2[(1)(8) + 3(1)(3)]$

$= 34$

Take h = 0.1.

<u>Step1</u>: By Taylor's series expansion, we have

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0^{I} + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots \quad \rightarrow (6)$$

on substituting the values of y_0, y_0^{I}, y_0^{II} etc in (6),we get

$$y(0.1) = y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(34) + \ldots$$

$= 1 + 0.1 + 0.015 + 0.001333 + 0.000416$

$y_1 = 1.116749$

<u>Step2</u>: Let us find y(0.2),we start with (x_1,y_1) as the starting values

Here $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = 1.116749$

Putting these values in (1),(3),(4) and (5), we get

$y_1^{I} = y_1^2 + x_1 = (1.116749)^2 + 0.1 = 1.3471283$

$y_1^{II} = 2y_1 y_1^{I} + 1 = 2(10116749)(1.3471283) + 1 = 4.0088$

$y_1^{III} = 2(y_1 y_1^{II} + (y_1^{I})^2) = 2((1.116749)(4.0088) + (1.3471283)^2]$

= 12.5831

y_1^{IV} = 2y₁ y_1^{III} + 6 y_1^{I} y_1^{II} = 2(1.116749) (12.5831) + 6(1.3471283) (4.0088)

= 60.50653

By Taylor's expansion

$$y(x_2) = y_2 = y_1 + \frac{h}{1!}y_1^{I} + \frac{h^2}{2!}y_1^{II} + \frac{h^3}{3!}y_1^{III} + \frac{h^4}{4!}y_1^{IV} + \ldots.$$

$\therefore$ y(0.2) = y_2 = 1.116749 + (0.1) (1.3471283)

$$+ \frac{(0.1)^2}{2}(4.0088) + \frac{(0.1)^3}{6}(12.5831) + \frac{(0.1)^4}{24}(60.50653)$$

y_2 = 1.116749 + 0.13471283 + 0.020044 + 0.002097 + 0.000252

= 1.27385

y(0.2) = 1.27385

Step3: Let us find y(0.3),we start with (x_2,y_2) as the starting value.

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 1.27385$

Putting these values of x_2 and y_2 in eq (1), (3), (4) and (5),

we get

$y_2^{I} = y_2^2 + x_2 = (1.27385)^2 + 0.2 = 1.82269$

$y_2^{II} = 2y_2\, y_2^{1} + 1 = 2(1.27385)\,(1.82269) + 1 = 5.64366$

$y_2^{III} = 2[y_2\, y_2^{II} + (\, y_2^{I}\,)^2\,] = 2[(1.27385)\,(5.64366) + (1.82269)^2]$

= 14.37835 + 6.64439 = 21.02274

$y_2^{IV} = 2y_2 + \; y_2^{III} + 6\; y_2^{I} \cdot y_1^{II}$

= 2(1.27385) (21.00274) + 6(1.82269)+(5.64366)

= 53.559635 + 61.719856 = 115.27949

By Taylor's expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!}y_2^{I} + \frac{h^2}{2!}y_2^{II} + \frac{h^3}{3!}y_2^{III} + \frac{h^4}{4!}y_2^{IV} + \ldots..$$

y(0.3) = y_3 = 1.27385 + (0.1) (1.82269)

$$+ \frac{(0.1)^2}{2}(5.64366) + \frac{(0.1)^3}{6}(21.02274)$$

$$+ \frac{(0.1)^4}{24}(115.27949)$$

= 1.27385 + 0.182269 + 0.02821

$+ 0.0035037 + 0.00048033 \quad = 1.48831$

$y(0.3) = 1.48831$

10. Solve $y^1 = x^2 - y$, $y(0) = 1$ using Taylor's series method and evaluate

$y(0.1), y(0.2), y(0.3)$ and $y(0.4)$ (correct to 4 decimal places)

Sol: Given $y^1 = x^2 - y \quad \rightarrow (1)$

and $y(0) = 1 \quad \rightarrow (2)$

Here $x_0 = 0$, $y_0 = 1$

Differentiating (1) w.r.t 'x', we get

$y^{II} = 2x - y^1 \rightarrow (3)$

$y^{III} = 2 - y^{II} \quad \rightarrow (4)$

$y^{IV} = -y^{III} \rightarrow (5)$

put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5), we get

$y_0^I = x_0^2 - y_0 = 0-1 = -1,$

$y_0^{II} = 2x_0 - y_0^I = 2(0) - (-1) = 1$

$y_0^{III} = 2 - y_0^{II} = 2-1 = 1,$

$y_0^{IV} = -y_0^{III} = -1 \qquad$ Take $h = 0.1$

<u>Step1</u>: by Taylor's series expansion

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0^I + \frac{h^2}{2!} y_0^{II} + \frac{h^3}{3!} y_0^{III} + \frac{h^4}{4!} y_0^{IV} + \ldots \quad \rightarrow (6)$$

On substituting the values of y_0, y_0^I, y_0^{II} etc in (6), we get

$$y(0.1) = y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1) + \ldots$$

$= 1 - 0.1 + 0.005 + 0.01666 - 0.0000416$

$= 0.905125 \simeq 0.9051$ (4 decimal place).

<u>Step2</u>: Let us find $y(0.2)$ we start with (x_1, y_1) as the starting values

Here $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = 0.905125$,

Putting these values of x_1 and y_1 in (1), (3), (4) and (5), we get

From 1,3,4,5 we get

$y_1^1 = x_1^2 - y_1 = (0.1)^2 - 0.905125 = -0.895125$

$y_1^{II} = 2x_1 - y_1^1 = 2(0.1) - (-0.895125) = 1.095125,$

$y_1^{III} = 2 - y_1^{II} = 2 - 1.095125 = 0.90475,$

$y_1^{IV} = - y_1^{III} = -0.904875,$

By Taylor's series expansion,

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1^I + \frac{h^2}{2!} y_1^{II} + \frac{h^3}{3!} y_1^{III} + \frac{h^4}{4!} y_1^{IV} + \ldots.$$

$$y(0.2) = y_2 = 0.905125 + (0.1)(-0.895125) + \frac{(0.1)^2}{2}(1.09125)$$

$$+ \frac{(0.1)^3}{6}(1.095125) + \frac{(0.1)^4}{24}(-0.904875) + \ldots.$$

$y(0.2) = y_2 = 0.905125 - 0.0895125 + 0.00547562 + 0.000150812$

$= 0.8212351 \simeq 0.8212$ (4 decimal places)

Step3: Let us find y(0.3), we start with (x_2,y_2) as the starting value

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 0.8212351$

Putting these values of x_2 and y_2 in (1),(3),(4), and (5) we get

$y_2^1 = x_2^2 - y_2 = (0.2)^2 - 0.8212351 = 0.04 - 0.8212351 = - 0.7812351$

$y_2^{II} = 2x_2 - y_2^1 = 2(0.2) + (0.7812351) = 1.1812351,$

$y_2^{III} = 2 - y_2^{II} = 2 - 1.\ 1812351 = 0.818765,$

$y_2^{IV} = - y_2^{III} = -0.818765,$

By Taylor's series expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!} y_2^I + \frac{h^2}{2!} y_2^{II} + \frac{h^3}{3!} y_2^{III} + \frac{h^4}{4!} y_2^{IV} + \ldots.$$

$$y(0.3) = y_3 = 0.\ 8212351 + (0.1)(-0.7812351) + \frac{(0.1)^2}{2}(1.1812351)$$

$$+ \frac{(0.1)^3}{6}(0.818765) + \frac{(0.1)^4}{24}(-0.818765) + \ldots.$$

$y(0.3) = y_3 = 0.\ 8212351 - 0.07812351 + 0.005906 + 0.000136 -$

$= 0.749150 \simeq 0.7492$ (4 decimal places)

Step4: Let us find y(0.4), we start with (x_3,y_3) as the starting value

Here $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$ and $y_3 = 0.749150$

Putting these values of x_3 and y_3 in (1),(3),(4), and (5) we get

$y_3^1 = x_3^2 - y_3 = (0.3)^2 - 0.749150 = -0.65915,$

$y_3^{II} = 2x_3 - y_3^1 = 2(0.3) + (0.65915) = 1.25915,$

$y_3^{III} = 2 - y_3^{II} = 2 - 1.25915 = 0.74085,$

$y_3^{IV} = - y_3^{III} = -0.74085,$

By Taylor's series expansion,

$$y(x_4) = y_4 = y_3 + \frac{h}{1!} y_3^I + \frac{h^2}{2!} y_3^{II} + \frac{h^3}{3!} y_3^{III} + \frac{h^4}{4!} y_3^{IV} + \ldots.$$

$$y(0.4) = y_4 = 0.749150 + (0.1)(-0.65915) + \frac{(0.1)^2}{2}(1.25915) + \frac{(0.1)^3}{6}(0.74085) + \frac{(0.1)^4}{24}(-0.74085) + \ldots.$$

$y(0.4) = y_4 = 0.749150 - 0.065915 + 0.0062926 + 0.000123475 - 0.0000030$

$= 0.6896514 \simeq 0.6896$ (4 decimal places)

11. Solve $y^1 = x^2 - y$, $y(0) = 1$using T.S.M and evaluate y(0.1),y(0.2),y(0.3) and y(0.4) (correct to 4 decimal place) Ans : 0.9051, 0.8212, 07492, 0.6896

12. Given the differentiating equation $y^1 = x^1 + y^2$, $y(0) = 1$. Obtain y(0.25) and y(0.5) by T.S.M.

    Ans: 1.3333, 1.81667

13. Solve $y^1 = xy^2 + y$, $y(0) = 1$ using Taylor's series method and evaluate y(0.1) and y(0.2)

    Ans: 1.111, 1.248.

3.6 Picard's Method

Consider the differential equation $\frac{dy}{dx} = f(x, y)$

Given that $y = y_0$ for $x = x_0$

Then $y^{(n)} = y_0 + \int_{x_0}^{x_n} f(x, y^{(n-1)})\, dx, \quad n = 1, 2, 3, \cdots$

Problems

1. Find the value of y for x=0.4 by Picard's method, given that $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 0.$

Sol: Given $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 0$

By Picard's method $y^{(n)} = y_0 + \int_{x_0}^{x_n} f\left(x, y^{(n-1)}\right) dx, \quad n = 1, 2, 3, \cdots$

For the first approximation, replace y_0 by 0

$$y^{(1)} = 0 + \int_0^x (x^2 + 0)\, dx \ = \frac{x^3}{3}$$

Second approximation is $y^{(2)} = \int_0^x (x^2 + \left(\frac{x^3}{3}\right)^2)\, dx \ = \int_0^x (x^2 + \frac{x^6}{9})\, dx \ = \frac{x^3}{3} + \frac{x^7}{63}$

Calculation of $y^{(3)}$ is tedious and hence approximate value is $y^{(2)}$

For x=0.4, $y = \frac{(0.4)^3}{3} + \frac{(0.4)^7}{63} = 0.02133 + 0.00026 = 0.0214$

2. Solve Find the value of y at x=0.1 by Picard's method, given that

$\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$

Sol: Given $\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$

By Picard's method $y^{(n)} = y_0 + \int_{x_0}^{x_n} f\left(x, y^{(n-1)}\right) dx = y_0 + \int_0^x \frac{y-x}{y+x}\, dx$

For the first approximation, replace y_0 by 1

$$y^{(1)} = 1 + \int_0^x \frac{1-x}{1+x}\, dx = 1 + \int_0^x -1 + \frac{2}{1+x}\, dx$$

$$y^{(1)} = 1 + [-x + 2\log(1+x)]_0^x$$

$$y^{(1)} = 1 - x + 2\log(1+x)$$

Second approximation is $y^{(2)} = 1 + \int_0^x \frac{1-x+2\log(1+x)-x}{1-x+2\log(1+x)+x}\, dx$

Which is very difficult to integrate

Hence we use the first approximation itself a s the value of y

$$\therefore y(x) = y^{(1)} = 1 - x + 2\log(1+x)$$

Put x=0.1, we get

$$y(0.1) = 1 - 0.1 + 2\log(1 + 0.1) = 1.0906$$

3.6 EULER'S METHOD

It is the simplest one-step method and it is less accurate. Hence it has a limited application.

Consider the differential equation $\frac{dy}{dx} = f(x,y) \quad \rightarrow(1)$

With $y(x_0) = y_0 \rightarrow (2)$

Consider the first two terms of the Taylor's expansion of y(x) at $x = x_0$

$$y(x) = y(x_0) + (x - x_0)\, y^1(x_0) \qquad \rightarrow(3)$$

from equation (1) $y^1(x_0) = f(x_0, y(x_0)) = f(x_0, y_0)$

Substituting in equation (3)

$\therefore y(x) = y(x_0) + (x - x_0)\, f(x_0, y_0)$

At $x = x_1$, $y(x_1) = y(x_0) + (x_1 - x_0)\, f(x_0, y_0)$

$\therefore y_1 = y_0 + h\, f(x_0, y_0)$ where $h = x_1 - x_0$

Similarly at $x = x_2$, $y_2 = y_1 + h\, f(x_1, y_1)$,

Proceeding as above, $\mathbf{y_{n+1} = y_n + h\, f(x_n, y_n)}$

This is known as Euler's Method

1. **Using Euler's method solve for x = 2 from $\frac{dy}{dx} = 3x^2 + 1$, y(1) = 2, by taking step size**

(I) h = 0.5 and (II) h=0.25

Sol: Here $\frac{dy}{dx} = f(x,y) = 3x^2 + 1$, $x_0 = 1$, $y_0 = 2$

Euler's algorithm is $y_{n+1} = y_n + h\; f(x_n, y_n)$, n = 0,1,2,3,….. $\rightarrow(1)$

(I) h = 0.5 $\qquad \therefore x_1 = x_0 + h = 1+0.5 = 1.5$

Taking n = 0 in (1) , we have $\qquad x_2 = x_1 + h = 1.5 + 0.5 = 2$

$y_1 = y_0 + h\, f(x_0, y_0)$

i.e. $y_1 = y(1.5) = 2 + (0.5)\, f(1,2) = 2 + (0.5)\,(3 + 1) = 2 + (0.5)(4) = 4$

Here $x_1 = x_0 + h = 1 + 0.5 = 1.5$

$\therefore y(1.5) = 4 = y_1$

Taking n = 1 in (1),we have

$y_2 = y_1 + h\, f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 4 + (0.5)\, f(1.5,4) = 4 + (0.5)[3(1.5)^2 + 1] = 7.875$

Here $x_2 = x_4 + h = 1.5 + 0.5 = 2$

$\therefore y(2) = 7.875$

(II) $h = 0.25$ $x_0 = 1, y_0 = 2$ $\therefore x_1 = 1.25,\ x_2 = 1.50,\ x_3 = 1.75,\ x_4 = 2$

Taking n = 0 in (1), we have

$y_1 = y_0 + h\, f(x_0,y_0)$

i.e. $y(x_1) = y_1 = 2 + (0.25)\, f(1,2) = 2 + (0.25)\ (3 + 1) = 3 = y(1.25)$

$y(x_2) = y_2 = y_1 + h\, f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 3 + (0.25)\, f(1.25,3)$

$= 3 + (0.25)[3(1.25)^2 + 1]$

$= 4.42188$

Here $x_2 = x_1 + h = 1.25 + 0.25 = 1.5$

$\therefore y_2 = y(1.5) = 4.42188$

Taking n = 2 in (1), we have

i.e. $y(x_3) = y_3 = y_2 + h\, f(x_2,y_2)$

$= 4.42188 + (0.25)\, f(1.5,2)$

$= 4.42188 + (0.25)\ [3(1.5)^2 + 1]$

$= 6.35938$

Here $x_3 = x_2 + h = 1.5 + 0.25 = 1.75$

$\therefore y(1.75) = 6.35938 = y_3$

Taking n = 4 in (1), we have

$y(x_4) = y_4 = y_3 + h\, f(x_3,y_3)$

i.e. $y(x_4) = y_4 = 6.35938 + (0.25)\, f(1.75,2)$

$= 6.35938 + (0.25)[3(1.75)^2 + 1]$

$y(x_4) = 8.90626 = y(2)$

Note that the difference in values of y(2) in both cases (i.e. when h = 0.5 and when h = 0.25).The accuracy is improved significantly when h is reduced to 0.25 (Example significantly of the eqn is $y = x^3 + x$ and with this $y(2) = y_2 = 10$

2. **Solve by Euler's method, $y^1 = x + y$, $y(0) = 1$ and find y(0.3) taking step size h = 0.1. compare the result obtained by this method with the result obtained by analytical solution**

Sol: Given D.E. is $y^1 = f(x,y) = x + y$, $y(0) = 1, h=0.1, x_0=0, x_1=0.1, x_2=0.2, x_3=0.3, y_0=1$

From Euler's method

$y_1 = y_0 + h\, f(x_0,y_0) = 1+0.1(0+1) = 1.1$

$y(x_2) = y_2 = y_1 + h\, f(x_1,y_1) = 1.1+0.1(0.1+1.1) = 1.22$

$y(x_3) = y_3 = y_2 + h\, f(x_2,y_2) = 1.22+0.1(0.2+1.22) = 1.362$

$y_1 = 1.1 = y(0.1)$,

$y_2 = y(0.2) = 1.22$

$y_3 = y(0.3) = 1.362$

Analytical method (linear d.e. method)

$y^1 = x + y$

dy/dx-y=x

P=-1,Q=x

I.F.=e^int(Pdx)=e^int(-dx)=e^-x

Sol. Y(I.F.)=int(Q(i.f.)dx+c

Y(e^-x)=int(xe^-x)dx+c=e^-x(-x-1)+c

Divide by e^-x on b.s.

Solution y=-x-1+ce^x

Put x=0,y=1 then

1=-0-1+c

C=2

General solution y=-x-1+2e^x

Particular solution is $y(x) = 2e^x – (x + 1)$

Hence analytical values y(0.1) = 1.11034, y(0.2) = 1.3428, y(0.3) = 1.5997

We shall tabulate the result as follows

X	0	X_1=0.1	X_2=0.2	X_3=0.3
Euler y(numerical)	1	1.1	1.22	1.362
Linear y(analytical)	1	1.11034	1.3428	1.3997

The value of y deviate from the execute value as x increases. This indicate that the method is not accurate

3. **Solve by Euler's method $y^1 + y = 0$ given y(0) = 1 and find y(0.04) taking step size**

   h = 0.01 Ans: 0.9606

4. **Using Euler's method, solve y at x = 0.1 from $y^1 = x+ y +xy$, y()) = 1 taking step size h = 0.025.**

5. **Given that $\frac{dy}{dx}$ = xy ,y(0) = 1 determine y(0.1),using Euler's method.** h =0.1

Sol: The given differentiating equation is $\frac{dy}{dx}$ = xy, y(0) = 1 ,a = 0,b=0.1

Here f(x,y) = xy , x_0 = 0 and y_0 = 1

Since h is not given much better accuracy is obtained by breaking up the interval (0,0.1) in to five steps.

i.e. h = $\frac{b-a}{5} = \frac{0.1}{5}$ = 0.02

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$ → (1)

∴ From (1) form = 0, we have

$y_1 = y_0 + h\, f(x0,y_0)$

= 1 + (0.02) f(0,1)

= 1 + (0.02) (0)

= 1

Next we have $x_1 = x_0 + h = 0 + 0.02 = 0.02$

∴ From (1), form = 1,we have

$y_2 = y_1 + h\, f(x_1,y_1)$

= 1 + (0.02) f(0.02,1)

= 1 + (0.02) (0.02)

= 1.0004

Next we have $x_2 = x_1 + h = 0.02 + 0.02 = 0.04$

∴ From (1), form = 2,we have

$y_3 = y_2 + h\, f(x_2,y_2)$

= 1.004 + (0.02) (0.04) (1.0004)

= 1.0012

Next we have $x_3 = x_2 + h = 0.04 + 0.02 = 0.06$

∴ From (1), form = 3,we have

$y_4 = y_3 + h\, f(x_3,y_3)$

= 1.0012 + (0.02) (0.06) (1.00012)

= 1.0024.

Next we have $x_4 = x_3 + h = 0.06 + 0.02 = 0.08$

∴ From (1), form = 4,we have

$y_5 = y_4 + h\, f(x_4, y_4)$

$= 1.0024 + (0.02)\ (0.08)\ (1.00024)$

$= 1.0040.$

Next we have $x_5 = x_4 + h = 0.08 + 0.02 = 0.1$

When $x = x_5$, $y \simeq y_5$

$\therefore y = 1.0040$ when $x = 0.1$

6. **Solve by Euler's method $y^1 = \frac{2y}{x}$ given y(1) = 2 and find y(2).**

7. **Given that $\frac{dy}{dx} = 3x^2 + y$, y(0) = 4. Find y(0.25) and y(0.5) using Euler's method**

Sol: given $\frac{dy}{dx} = 3x^2 + y$ and $y(0) = 4$.

Here $f(x,y) = 3x^2 + y$, $x_0 = 0$, $y_0 = 4$

Consider $h = 0.25$

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n, y_n)$ $\rightarrow (1)$

$\therefore$ From (1), for n = 0, we have

$y_1 = y_0 + h\, f(x_0, y_0)$

$= 2 + (0.25)[0 + 4]$

$= 2 + 1$

$= 3$

Next we have $x_1 = x_0 + h = 0 + 0.25 = 0.25$

When $x = x_1$, $y_1 \simeq y$

$\therefore y_1 = 3$ when $x_1 = 0.25$

$\therefore$ From (1), for n = 1, we have

$y_2 = y_1 + h\, f(x_1, y_1)$

$= 3 + (0.25)[3.(0.25)^2 + 3]$

$= 3.7968$

Next we have $x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

When $x = x_2$, $y \simeq y_2$

$\therefore y = 3.7968$ when $x = 0.5$.

8. **Solve first order differential equation $\frac{dy}{dx}$ = $\frac{y-x}{y+x}$, y(0) = 1 and estimate y(0.1) using Euler's method (5 steps).h=0.02**

   Ans: 1.0928

9. **Use Euler's method to find approximate value of solution of $\frac{dy}{dx}$ = y-x + 5 at x = 2-1 and 2-2with initial contention y(0.2) = 1**

3.7 Modified Euler's method

It is given by $y^{(i)}{}_{k+1} = y_k + h/2f\left[(x_k, y_k) + f(x_{k+1}, 1)_{k+1}{}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$

Working rule :

i)Modified Euler's method

$$y^{(i)}{}_{k+1} = y_k + h/2f\left[(x_k, y_k) + f(x_{k+1}, 1)_{k+1}{}^{(i-1)}\right], i = 1, 2....., ki = 0, 1.....$$

ii) When $i = 1$ $y^{0}{}_{k+1}$ can be calculated from Euler's method

iii) K=0, 1……… gives number of iteration. $i = 1, 2...$

gives number of times, a particular iteration k is repeated

Suppose consider dy/dx=f(x, y) -------- (1) with $y(x_0) = y_0$----------- (2)

To find $y(x_1) = y_1$ at $x = x_1 = x_0 + h$

Now take k=0 in modified Euler's method

We get $y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(i-1)}\right)\right]$........................... (3)

Taking i=1, 2, 3...k+1 in eqn (3), we get

$y_1^{(0)} = y_0 + h/2\left[f(x_0, y_0)\right]$ (By Euler's method)

$$y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$y_1^{(k+1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(k)}\right)\right]$$

If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are sufficiently close to one another, we will take the common value as $y_2 = y(x_2) = y(x_1 + h)$

We use the above procedure again

1) using modified Euler's method find the approximate value of x when $x = 0.3$ given that $dy/dx = x + y$ and $y(0) = 1$

sol: Given $dy/dx = x + y$ and $y(0) = 1$

Here $f(x, y) = x + y, x_0 = 0,$ and $y_0 = 1$

Take h = 0.1 which is sufficiently small

Here $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

The formula for modified Euler's method is given by

$$y_{k+1}{}^{(i)} = y_k + h/2\left[f(x_k + y_k) + f\left(x_{k+1}, y_{k+1}{}^{(i-1)}\right)\right] \to (1)$$

Step1: To find $y_1 = y(x_1) = y\ (0.1)$

Taking k = 0 in eqn(1)

$$y_{k+1}{}^{(i)} = y_0 + h/2\left[f(x_0 + y_0) + f\left(x_1, y_1{}^{(i-1)}\right)\right] \to (2)$$

when i = 1 in eqn (2)

$$y_1^{(i)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

First apply Euler's method to calculate $y_1^{(0)}$ = y_1

$$\therefore\ y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

= 1+(0.1)f(0.1)

= 1+(0.1)

= 1.10

$$now\left[x_0 = 0, y_0 = 1, x_1 = 0.1, y_1(0) = 1.10\right]$$

$$\therefore\ y_1^{(1)} = y_0 + 0.1/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

= 1+0.1/2[f(0,1) + f(0.1,1.10)

= 1+0.1/2[(0+1)+(0.1+1.10)]

= 1.11

When i=2 in eqn (2)

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

= 1+0.1/2[f(0.1)+f(0.1,1.11)]

= 1 + 0.1/2[(0+1)+(0.1+1.11)]

= 1.1105

$$y_1^{(3)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

= 1+0.1/2[f(0,1)+f(0.1 , 1.1105)]

= 1+0.1/2[(0+1)+(0.1+1.1105)]

= 1.1105

Since $y_1^{(2)} = y_1^{(3)}$

$\therefore$ y_1 = 1.1105

Step:2 To find $y_2 = y(x_2) = y(0.2)$

Taking k = 1 in eqn (1) , we get

$$y_2^{(i)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(i-1)}\right)\right] \to (3) \qquad i = 1,2,3,4,\ldots..$$

For i = 1

$$y_2^{(1)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$$

$y_2^{(0)}$ is to be calculate from Euler's method

$$y_2^{(0)} = y_1 + h\ f(x_1, y_1)$$

= 1.1105 + (0.1) f(0.1 , 1.1105)

= 1.1105+(0.1)[0.1+1.1105]

= 1.2316

$$\therefore\ y_2^{(1)} = 1.1105 + 0.1/2\left[f(0.1, 1.1105) + f(0.2, 1.2316)\right]$$

= 1.1105 +0.1/2[0.1+1.1105+0.2+1.2316]

= 1.2426

$$y_2^{(2)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(1)}\right)\right]$$

= 1.1105 + 0.1/2[f(0.1 , 1.1105) , f(0.2 . 1.2426)]

= 1.1105 + 0.1/2[1.2105 + 1.4426]

= 1.1105 + 0.1(1.3266)

= 1.2432

$$y_2^{(3)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(2)}\right)\right]$$

= 1.1105+0.1/2[f(0.1,1.1105)+f(0.2 , 1.2432)]

= 1.1105+0.1/2[1.2105+1.4432)]

= 1.1105 + 0.1(1.3268)

= 1.2432

Since $y_2^{(3)} = y_2^{(3)}$

Hence y_2 = 1.2432

Step:3

To find $y_3 = y(x_3)$ = y y(0.3)

Taking k =2 in eqn (1) we get

$$y_3^{(1)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(i-1)}\right)\right] \rightarrow (4)$$

For i = 1 ,

$$y_3^{(1)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(0)}\right)\right]$$

$y_3^{(0)}$ is to be evaluated from Euler's method .

$$y_3^{(0)} = y_2 + h f\left(x_2, y_2\right)$$

= 1.2432 +(0.1) f(0.2 , 1.2432)

= 1.2432+(0.1)(1.4432)

= 1.3875

$\therefore\ y_3^{(1)}$ = 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3, 1.3875)]

= 1.2432 + 0.1/2[1.4432+1.6875]

= 1.2432+0.1(1.5654)

= 1.3997

$$y_3^{(2)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(1)}\right)\right]$$

= 1.2432+0.1/2[1.4432+(0.3+1.3997)]

= 1.2432+ (0.1) (1.575)

= 1.4003

$$y_3^{(3)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(2)}\right)\right]$$

= 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3 , 1.4003)]

= 1.2432 + 0.1(1.5718)

= 1.4004

$$y_3^{(4)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(3)}\right)\right]$$

= 1.2432 + 0.1/2[1.4432+1.7004]

= 1.2432+(0.1)(1.5718)

= 1.4004

Since $y_3^{(3)} = y_3^{(4)}$

Hence $y_3 = 1.4004$ $\therefore$ The value of y at x = 0.3 is 1.4004

2 . Find the solution of $\frac{dy}{dx}$ = x-y , y(0)=1 at x =0.1 , 0.2 ,0.3 , 0.4 and 0.5 . Using modified Euler's method

Sol . Given $\frac{dy}{dx}$ = x-y and y(0) = 1

Here f(x,y) = x-y , $x_0 = 0$ and $y_0 = 1$

Consider h = 0.1 so that

x = 0.1 , x_2 = 0.2 , x_3 =0.3 , x_4 = 0.4 and x_5 = 0.5

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2\ f(x_k, y_k) + fx_{k+1}, y_{k+1}^{(i-1)} \rightarrow (1)$$

Where k = 0,1, 2, 3,….. i = 1, 2, 3,…..

x	$f(x_k, y_k) = x_k - y_k$	$\frac{1}{2}\left[f(x_k, y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right]$	$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k, y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right]$
K = 0			
0.0	0-1=-1	-	1+(0.1)(-1)=0.9 = $y_1^{(0)}$

0.1(i=1)	0-1=-1	½(-1-0.8) = -0.9	1+(0.1)(-0.9)=0.91
0.1(i=2)	0-1=-1	½(-1-0.81)= -0.905	1+(0.1)(-0.905)=0.9095
0.1(i=3)	0-1=-1	½(-1-0.80.95)= -0.90475	1+(0.1)(-0.90475)=0.9095
K=1			
0.1	0.1-0.9095= -0.8095	-	0.9095+(0.1)(-0.8095)=0.82855
0.2(i=1)	-0.8095	½(-0.8095-0.62855)	0.9095+(0.1)(-0.719025)=0.8376
0.2(i=2)	-0.8095	½(-0.8095-0.6376)	0.9095+(0.1)(-0.72355)=0.8371
0.2(i=3)	-0.8095	½(-0.8095-0.6371)	0.9095+(0.1)(-0.7233)=0.8372
0.2(i=4)	-0.8095	½(-0.8095-0.6372)	0.9095+(0.1)(-0.72355)=0.8371
K=2			
0.2	0.2-0.8371=-0.6371	-	0.8371+(0.1)(-0.6371)=0.7734

0.3(i=1)	= -0.6371	½(-0.6371-0.4734)	0.8371+(0.1)(-0.555)=0.7816
0.3(i=2)	= -0.6371	½(-0.6371-0.4816)	0.8371-0.056=0.7811
0.3(i=3)	= -0.6371	½(-0.6371-0.4811)	0.8371-0.05591=0.7812
0.3(i=4)	= -0.6371	½(-0.6371-0.4812)	0.8371-0.055915 = 0.7812
K =3			
0.3(i=1)	0.3-0.7812	-	0.7812+(0.1)(-0.4812) = 0.7331
0.4(i=1)	-0.4812	½(-0.4812-0.4311)	0.7812-0.0457 = 0.7355
0.4(i=2)	-0.4812	½(-0.4812-0.4355)	0.7812-0.0458 = 0.7354
0.4(i=3)	-0.4812	½(-0.4812-0.4354)	0.7812-0.0458 = 0.7354
K=4			
0.4	-0.3354	-	0.7354-0.03354 = 0.70186
0.5	-0.3354	½(-0.3354-0.301816)	0.7354-0.03186 = 0.7035
0.5	-0.3354	½(-0.3354-0.30354)	0.7354-0.0319 = 0.7035

3. Find y(0.1) and y(0.2) using modified Euler's formula given that $dy/dx=x^2-y, y(0)=1$

[consider h=0.1, $y_1=0.90523, y_2=0.8214$]

4. Given $dy/dx = -xy^2, y(0) = 2$ compute y(0.2) in steps of 0.1

Using modified Euler's method

[h=0.1, y_1=1.9804, y_2=1.9238]

5. Given y^1 = x+siny, y(0)=1 compute y(0.2) and y(0.4) with h=0.2 using modified Euler's

method

[y_1=1.2046, y_2=1.4644]

3.8 Runge – Kutta Methods

I. Second order R-K Formula

$y_{i+1} = y_i + 1/2\ (K_1+K_2)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i+h, y_i+k_1)$

For i= 0,1,2-------

II. Third order R-K Formula

$y_{i+1} = y_i + 1/6\ (K_1+4K_2+ K_3)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i+h/2, y_0+k_1/2)$

$K_3 = h\ (x_i+h, y_i+2k_2-k_1)$

For i= 0,1,2-------

III. Fourth order R-K Formula

$y_{i+1} = y_i + 1/6\ (K_1+2K_2+ 2K_3+K_4)$,

Where $K_1 = h\ (x_i, y_i)$

$K_2 = h\ (x_i+h/2, y_i+k_1/2)$

$K_3 = h\ (x_i+h/2, y_i+k_2/2)$

$K_4 = h\ (x_i+h, y_i+k_3)$

For i= 0,1,2-------

1. Using Runge-Kutta method of second order, find $y(2.5)$ from $\frac{dy}{dx}$ = $\frac{x+y}{x}$, y(2)=2 , h = 0.25 .

Sol: Given $\frac{dy}{dx}$ = $\frac{x+y}{x}$, y(2) = 2 .

Here f(x, y) = $\frac{x+y}{x}$, x_0 = 0 , y_0=2 and h = 0.25

$\therefore$ $x_1 = x_0+h = 2+0.25 = 2.25$, $x_2 = x_1+h = 2.25+0.25 = 2.5$

By R-K method of second order,

$$y_{i+1} = y_i + 1/2(k_1 + k_2), k_1 - hf(x_i + h, y_i + k_1), i = 0,1.... \rightarrow (1)$$

Step -1:-

To find $y(x_1)$i.e y(2.25) by second order R - K method taking i=0 in eqn(i)

We have $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

Where k_1= hf (x_0,y_0), $k_{2=}$ hf (x_0+h,y_0+k_1)

f (x_0,y_0)=f(2,2)=2+2/2=2

k_1=hf (x_0,y_0)=0.25(2)=0.5

k_2= hf (x_0+h,y_0+k_1)=(0.25)f(2.25,2.5)

=(0.25)(2.25+2.5/2.25)=0.528

$\therefore$ y_1=y(2.25)=2+1/2(0.5+0.528)

=2.514

Step2:

To find $y(x_2)$ i.e., y(2.5)

i=1 in (1)

x_1=2.25,y_1=2.514,and h=0.25

$y_2=y_1+1/2(k_1+k_2)$

where $k_1=h\,f((x_1,y_1)=(0.25)f(2.25,2.514)$

$=(0.25)[2.25+2.514/2.25]=0.5293$

$$k_2 = h\,f\left(x_0+h, y_0+k_1\right) = \left(0.1\right) f\left(0.1, 1-0.1\right) = \left(0.1\right)\left(-0.9\right) = -0.09$$

$=(0.25)[2.5+2.514+0.5293/2.5]$

$=0.55433$

$y_2 = y\ (2.5)=2.514+1/2(0.5293+0.55433)$

$=3.0558$

$\therefore$ y =3.0558 when x = 2.5

Obtain the values of y at x=0.1,0.2 using R-K method of

(i)second order (ii)third order (iii)fourth order for the diff eqn $y^1+y=0, y(0)=1$

Sol: Given dy/dx = -y, y(0)=1

$f(x,y) = -y$, $x_0 = 0$, $y_0 = 1$

Here $f(x,y) = -y$, $x_0 = 0$, $y_0 = 1$ take h = 0.1

$\therefore\ x_1 = x_0+h = 0.1$,

$x_2 = x_1+h = 0.2$

Second order:

step1: To find $y(x_1)$ i.e y(0.1) or y_1

by second-order R-K method,we have

$y_1 = y_0+1/2(k_1+k_2)$

where $k_1=hf(x_0,y_0)=(0.1)\ f(0,1) = (0.1)(-1)= -\ 0.1$

$k_2= hf\ (x_0+h,\ y_0+k_1)= (0.1)\ f\ (0.1,\ 1-0.1) = (0.1)(-0.9) = -0.09$

$y_1=y(0.1)=1+1/2(-0.1-0.09)=1-0.095=0.905$

$\therefore$ y =0.905 when x=0.1

Step2:

To find y_2 i.e $y(x_2)$ i.e y(0.2)

Here x_1 = 0.1, y_1 = 0.905 and h=0.1

By second-order R-K method, we have

$y_2 = y(x_2) = y_1 + 1/2(k_1+k_2)$

Where $k_1 = h\ f(x_1, y_1)$ =(0.1)f(0.1,0.905)=(0.1)(-0.905)=-0.0905

$$\begin{aligned} k_2 &= h\ f(x_1+h, y_1+k_1) = (0.1) f(0.2, 0.905-0.0905) \\ &= (0.1) f(0.2, 0.8145) = (0.1)(-0.8145) \\ &= -0.08145 \end{aligned}$$

y_2 = y(0.2)=0.905+1/2(-0.0905-0.08145)

= 0.905- 0.085975 = 0819025

<u>**Third order**</u>

Step1:

To find y_1 i.e $y(x_1)$= y(0.1)

By Third order Runge kutta method

$$y_1 = y_0 + 1/6(k_1 + 4k_2 + k_3)$$

where $k_1 = h\ f(x_0, y_0)$ = (0.1) f (0.1) = (0.1) (-1) = -0.1

$$\begin{aligned} k_2 = h f(x_0 + h/2, y_0 + k_1/2) = (0.1) f(0.1/2, 1-0.1/2) = (0.1) f(0.05, 0.95) \\ = (0.1)(-0.95) = -0.095 \end{aligned}$$

and $k_3 = h\ f((x_0+h, y_0+2k_2-k_1)$

(0.1) f (0.1,1+2(-0.095)+0.1)= -0.905

Hence y_1 = 1+1/6(-0.1+4(-0.095)-0.09) = 1+1/6 (-0.57) = 0.905

y_1=0.905 i.e y(0.1)= 0.905

Step2:

To find y_2,i.e $y(x_2)= y(0.2)$

Here $x_1=0.1, y_1=0.905$ and $h = 0.1$

Again by 2nd order R-K method

$y_2 = y_1+1/6(k_1+4k_2+k_3)$

Where $k_1=h\ f(x_1, y_1) = (0.1)f\ (0.1,0.905)= -0.0905$

$k_2 = h\ f\ (x_1+h/2,y_1+k_1/2)=(0.1)f(0.1+0.2,0.905 - 0.0905)= -(0.1)\ f\ (0.15, 0.85975)= (0.1)\ (-0.85975)$

and $k_3 = h\ f((x_1+h,y_1+2k_2-k_1)=(0.1)f(0.2,0.905+2(0.08975)+0.0905= -0.082355$

hence $y_2 = 0.905+1/6(-0.0905+4(-0.085975)-0.082355)=0.818874$

$\therefore y = 0.905$ when $x = 0.1$

And $y =0.818874$ when $x =0.2$

fourth order:

step1:

$x_0=0,y_0=1,h=0.1$ To find y_1 i.e $y(x_1)=y(0.1)$

By 4th order R-K method, we have

$y_1 = y_0+1/6(k_1+2k_2+2k_3+k_4)$

Where $k_1=h\ f(x_0,y_0)=(0.1)f(0.1)= -0.1$

$k_2= h\ f\ (x_0+h/2,\ y_0+k1/2) = -0.095$

and $k_3= h\ f((x_0+h/2,y_0+k_2/2)=(0.1)f\ (0.1/2,1-0.095/2)$

$= (0.1)f(0.05,0.9525)$

$= -0.09525$

and $k_4= h\ f(x_0+h,y_0+k_3)$

$= (0.1)\ f(0.1,1-0.09525)=(0.1)f(0.1,0.90475)$

$=-0.090475$

Hence $y_1=1+1/6(-0.1)+2(-0.095)+2(0.09525)-0.090475)$

=1+1/6(-0.570975)+1-0.951625 = 0.9048375

Step2:

To find $y_2, i.e., y(x_2) = y(0.2), y_1 = 0.9048375, i.e., y(0.1) = 0.9048375$

Here x_1 = 0.1, y_1=0.9048375 and h = 0.1

Again by 4th order R-K method, we have

$y_2 = y_1+1/6(k_1+2k_2+2k_3+k_4)$

Where k_1=h f(x_1,y_1)=(0.1)f(0.1,0.9048375)=-0.09048375

k_2= hf (x_1+h/2,y_1+k_1/2)=(0.1)f(0.1+0.1/2,0.9048375 -0.09048375 /2)=-0.08595956

and k_3=hf(x_1+h/2, y_1+k_2/2)=(0.1)f(0.15,0. 8618577)= -0.08618577

k_4 =h f(x_1+h,y_1+k_3)=(0.1)f(0.2,0.86517)

= -0.08186517

Hence y_2 = 0.09048375+1/6(-0.09048375-2(0.08595956)-2(0.08618577)-0.08186517

=0.9048375-0.0861065

=0.818731

y = 0.9048375 when x =0.1 and y =0.818731

3. Apply the 4th order R-K method to find an approximate value of y when x=1.2 in steps of 0.1,given that $y^1 = x^2+y^2$,y (1)=1.5

sol. Given $y^1= x^2+y^2$,and y(1)=1.5

Here f(x,y)= x^2+y^2, y_0 =1.5 and x_0=1,h=0.1

So that x_1=1.1 and x_2=1.2

Step1:

To find y_1 i.e., $y(x_1)$

by 4th order R-K method we have

$y_1=y_0+1/6\ (k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_0,y_0)=(0.1)f(1,1.5)=(0.1)\ [1^2+(1.5)^2]=0.325$

$k_2=$ hf $(x_0+h/2,y_0+k_1/2)=(0.1)f(1+0.05,1.5+0.325)=0.3866$

and $k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(1.05,1.5+0.$
$3866/2)=(0.1)[(1.05)^2+(1.6933)^2]$

$=0.39698$

$k_4=hf(x_0+h,y_0+k_3)=(0.1)f(1.0,1.89698)$

$=0.48085$

Hence

$$y_1 = 1.5 + \frac{1}{6}\left[0.325 + 2\left(0.3866\right) + 2\left(0.39698\right) + 0.48085\right]$$
$$= 1.8955$$

Step2:

To find y_2, i.e., $y(x_2) = y(1.2)$

Here $x_1=0.1, y_1=1.8955$ and $h=0.1$

by 4th order R-K method we have

$y_2= y_1+1/6(k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_1,y_1)=(0.1)f(0.1,1.8955)=(0.1)\ [1^2+(1.8955)^2]=0.48029$

$k_2=$ hf $(x_1+h/2,y_1+k_1/2)=(0.1)f(1.1+0.1,1.8937+0.4796)\ =0.58834$

and $k_3=hf((x_1+h/2,y_1+k_2/2)=(0.1)f(1.5,1.8937+0.58743)$
$=(0.1)[(1.05)^2+(1.6933)^2]$

$=0.611715$

$k_4=hf(x_1+h,y_1+k_3)=(0.1)f(1.2,1.8937+0.610728)$

$=0.77261$

Hence $y_2=1.8937+1/6(0.4796+2(0.58834)+2(0.611715)+0.7726)\ =2.5043$

$\therefore$ y =2.5043 where $x = 0.2$

4. Using R-K method, find y(0.2) for the eqn dy/dx=y-x,y(0)=1,take h=0.2

Ans:1.15607

5.Given that y^1=y-x,y(0)=2 find y(0.2) using R- K method take h=0.1

Ans: 2.4214

6. Apply the 4th order R-K method to find $y(0.2)$ *and* $y(0.4)$ **for one equation** $10\frac{dy}{dx} = x^2 + y^2, y(0) = 1$ **take h = 0.1** Ans. 1.0207, 1.038

7. using R-K method, estimate y(0.2) and y(0.4) for the eqn dy/dx=y^2-x^2/ y^2+x^2,y(0)=1,h=0.2

Ans:1.19598,1.3751

8. use R-K method, to approximate y when x=0.2 given that y^1=x+y,y(0)=1

Sol: Here f(x,y)=x+y,y_0=1,x_0=0

Since h is not given for better approximation of y

Take h=0.1

$\therefore$ x_1=0.1, x_2=0.2

Step1

To find y_1 i.e $y(x_1)$=y(0.1)

By R-K method,we have

$y_1=y_0+1/6\ (k_1+2k_2+2k_3+k_4)$

Where $k_1=hf(x_0,y_0)=(0.1)f(0,1)=(0.1)\ (1)=0.1$

$k_2=$ hf $(x_0+h/2,y_0+k_1/2)=(0.1)f(0.05,1.05)=0.11$

and k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(0.05,1+0. 11/2)=(0.1)[(0.05) +(4.0.11/2)]

=0.1105

k_4=h f (x_0+h,y_0+k_3)=(0.1)f(0.1,1.1105)=(0.1)[0.1+1.1105]

=0.12105

Hence $\therefore y_1 = y(0.1) = 1 + \frac{1}{6}(0.1 + 0.22 + 0.240 + 0.12105)$

y = 1.11034

Step2:

To find y_2 i.e y(x_2) = y(0.2)

Here x_1=0-1, y_1=1.11034 and h=0.1

Again By R-K method,we have

y_2=y_1+1/6(k_1+2k_2+2k_3+k_4)

k_1=h f(x_1,y_1)=(0.1)f(0.1,1.11034)=(0.1) [1.21034]=0.121034

k_2= h f (x_1+h/2, y_1+k_1/2)=(0.1)f(0.1+0.1/2,1.11034+0.121034/2)

=0.1320857

and k_3=h f((x_1+h/2,y_1+k_2/2)=(0.1)f(0.15,1.11034+0.1320857/2)

=0.1326382

k_4=h f(x_1+h,y_1+k_3)=(0.1)f(0.2,1.11034+0.1326382)

(0.1)(0.2+1.2429783)=0.1442978

Hence y_2=1.11034+1/6(0.121034+0.2641714+0.2652764+0.1442978

=1.11034+0.1324631 =1.242803

$\therefore$ y =1.242803 when x=0.2

9.Using Runge-kutta method of order 4,compute y(1.1) for the eqn y^1=3x+y^2,y(1)=1.2 h = 0.05

Ans:1.7278

10. Using Runge-kutta method of order 4,compute y(2.5) for the eqn dy/dx = x+y/x, y(2)=2 [hint h = 0.25(2 steps)]

Ans:3.058

9. Practice Quiz

1. The trapezoidal rule is $\int_{x_0}^{x_n} y dx =$ [a]

a) $\frac{h}{2}[y_0 + 2(y_1 + y_2 + - - - - + y_{n-1}) + y_n]$

b) $h[y_0 + 2(y_1 + y_2 + - - - = y_{n-1}) + y_n]$

c) $\frac{h}{2}[y_0 + (y_1 + y_2 + - - - - - - + y_{n-1}) + y_n]$

d) $h[y_0 + 3(y_1 + y_2 + - - - - - - + y_{n-1}) + y_n]$

2. To apply Simpson's 1/3rd and 3/8th rules both the interval must be divided into minimum of _________ intervals. [c]

a) 10

b) 12

c) 6

d) 4

3. Simpson's 1/3 rule for n = 2 is [a]

a) $y = \frac{h}{3}[y_0 + 4y_1 + y_2]$

b) $y = \frac{3h}{8}[y_0 + 4y_1 + y_2]$

c) $y = \frac{4}{3}[y_0 + 2y_1 + y_2]$

d) $y = \frac{3h}{8}[y_0 + 2y_1 + y_2]$

4. To evaluate $\int_0^4 \frac{1}{1+x^2}dx$ for h = 1, we take n = [c]

a) 2

b) 3

c) 4

d) 5

5. In the case of Simpson's $\frac{1}{3}$ rule, the number of sub-intervals must be [b]

a) Small

b) Even

c) Odd

d) Larger

6. In Trapezoidal rule, if the interval of $\int_2^9 f(x)dx$ is divided into 7 equal sub-intervals, then h is ………… [c]

a) 2

b) ½

c) 1

d) 3/2

7. The Taylor's series for f(x) = log(1 + x) is [b]

a) $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\ldots\ldots\ldots$

b) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots\ldots$

c) $x + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots\ldots\ldots\ldots$

d) $x - x^2 + x^3 - x^4 + \ldots\ldots\ldots\ \ldots\ldots$

8. If $\frac{dy}{dx} = x + y$, $y = 1$ at $x = 1$, using picard method the first approximation is [b]

a) $1 + x$

b) $1 + x + \frac{x^2}{2}$

c) $1 - x - x^2$

d) none.

9. R. K method 2nd order formula for y_1 = [a]

a) $y = y_0 + \frac{1}{2}(K_1 + K_2)$

b) $y_1 = \frac{K_1 + K_2}{2}$

c) $y_1 = y_0 - \frac{1}{2}(K_1 + K_2)$

d) none

10. The first order Range-Kutta method is equal to______ [a]

a) Taylors series

b) Eulers method

c) Picards

d) none