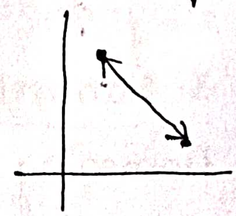
I. Data representation

Distance measures : Some of the distance measures are

1. Euclidean distance :

Euclidean distance measures the length of a segment connecting two points



formula :

$$\text{Euclidean distance} = D(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Euclidean distance is not scale in-varient, the data has to be normalized before using this measure. It is less useful as the dimension increased i.e; for high dimensional points. euclidean distance is very useful for low dimensional data.

2. Cosine similarity :

cosine similarity is used to find the distance between two vectors with high dimensionality. the cosine similarity is the cosine of the angle b/w two vectors. If the cosine similarity is 1, then the two vectors have exactly the same orientation. where as two vectors diametrically opposite to each other have similarity of -1.

formula :

$$D(x,y) = \cos\theta = \frac{\bar{x} \cdot \bar{y}}{\|\bar{x}\| \, \|\bar{y}\|}$$

The main disadvantage is this measure does not consider the magnitude of the vectors (only direction is taken into account) It can used in case of high dimensional data & when the magnitude of vectors for example, when a word occurs more frequently in one document over another, this does not mean that one document is more related to that word. It could be the case that documents have uneven lengths and the magnitude of the count is less important.

3. Hamming distance:

Hamming distance is the number of values that are different between two vectors.

It is used to compare two binary strings of equal length. It is difficult to use when the two vectors are not of equal length.

A	1	0	1	1	0	0
B	1	1	1	0	0	0

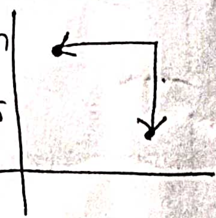
It is not advised to use this measure when the magnitude is an important measure.

formula:

$$D(x,y) = \sum_{1}^{x} abs(x_i - y_i)$$

4. Manhattan distance: (Taxicab or city block distance)

Manhattan distance calculates the distance between real valued vectors. It measures the distance between two vectors if they could only move right angles and there is no diagonal movement.



Ex: objects on a uniform grid such as a chessboard.

formula: $D(x,y) = \sum_{i=1}^{k} |x_i - y_i|$

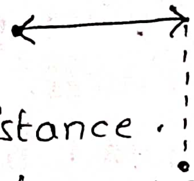
Manhattan distance works quite well if the dataset has discrete or binary attributes, since it takes into account the paths that realistically could be taken within values of those attributes.

path possible.

5. chebyshev distance :

It is defined as the greatest of diffuence between two vectors along any coordinate dimension or simply it is the maximum distance along one axis

formula. $D(x,y) = \max_i (|x_i - y_i|)$



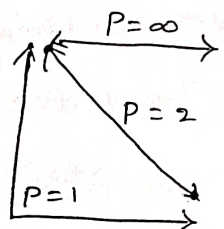
It is often referred to as chessboard distance. Since the minimum number of moves needed by a king to go from one square to another is equal to chebyshev distance.

This distance measure is typically used in very specific use-cases.

It can be a useful measure in games that allow unristricted 8-way movement.

6. Minkowski distance ;

Minkowski distance is a bit more intricate measure than most.



It is a metric used in normed vector space (n-dimensional real space) which means that it can be used in a space where distances can be represented as a vector that has a length.

This measure have, three requirements.

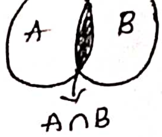
1. zero vector – a vector of length zero
2. Scalar factor – multiplying a vector with a positive number its length is changed and the direction remains same.
3. Triangular Inequality – the shortest distance between two points is a straight line

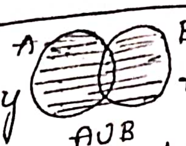
formula. $D(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$

common values of p are

p – chebyshev distance

7. jaccard index distance : The jaccard index is a metric used to calculate the similarity and diversity of sample sets. It is the size of the intersection divided by the size of the union of the sample sets. (or) it is the total number of similar entities between sets divided by the total number of entities.



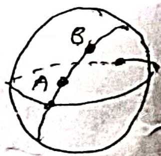


formula : $D(x,y) = 1 - \dfrac{|x \cap y|}{|x \cup y|}$ (jaccard distance)

where $\dfrac{|x \cap y|}{|x \cup y|}$ gives jaccard index.

The jaccard index is often used in applications where binary or binarized data are used. Also it can be used in text similarity analysis to measure how much word choice overlap there is between documents. If the size of the data is large then this index does not give accurate measure.

8. Haversine distance : It is the distance between two points on a sphere given their longitudes & lattitudes



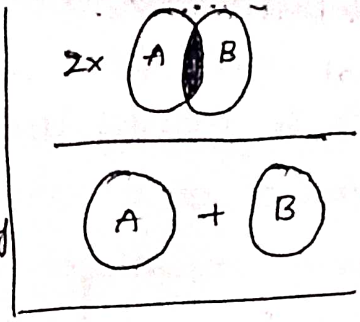
formula

$$d = 2r \arcsin\left(\sqrt{\sin^2\left(\frac{\phi_2-\phi_1}{2}\right) + \cos\phi_1 \cos\phi_2 \sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}\right)$$

Haversine distance is often used in navigation. for example, it can use to calculate the distance between two contries when flying b/w them. It is much less suited if the distances by themselves are already not that large.

9. Sorensen-dice Index : It is very similar to jaccard ... similarity & diversity of sample

formula $D(x,y) = \dfrac{2|x \cap y|}{|x|+|y|}$



It is typically used in either image segmentation tasks or text similarity analysis.

Distance measure : Distance measure plays an important role in machine learning. A distance measure is an objective score that summarizes the relative difference between two objects in a problem domain.

Most commonly the two objects are rows of data that describe a subject. An effective distance measure improves the performance of our machine learning model, whether that is for classification tasks or clustering. mostly the distance measures are used in a specific machine learning algorithm that uses distance measures at its core. The most famous algorithm of this type is the k-nearest neighbors algorithm or KNN algorithm. Another related one is self-organizing map (SOM) algorithm in which data measures are used for supervised or unsupervised learning. Some other popular machine algorithms that use machine measures at their core one.

learning vector quantization (LVQ)

k-means clustering KMC : Numerical values may have different scales. This can greatly impact the calculation of distance measure and it is often a good practice to normalize or standardize numerical values prior to calculating the distance measure.

Most commonly used distance measures in machine learning are

1. Euclidean distance
2. Hamming distance

4. Minkowski distance.

1. Euclidean distance :- It calculates the distance b/w two real-value vectors.

- It is mostly used in calculating the distance between two rows of data that have numerical values, such a floating point or integer values.

If columns have values with different scales, it is common to normalize or standardize the numerical values accross all the columns prior to calculating Euclidean distance, otherwise columns that have large values will dominate the distance measure.

$$\text{Euclidean distance} = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$$

where n = number of dimensions p_i, q_i = data points

Example

```
# calculating Euclidean distance b/w vectors
from math import sqrt

# calculating euclidean distance
def euclidean_distance(a,b):
        return sqrt(sum((e1-e2)**2 for e1,e2 in zip(a,b)))

# define data
row1 = [10, 20, 15, 10, 5]
row2 = [12, 24, 18, 8, 7]
# calculate distance
dist = euclidean_distance(row1, row2)
print(dist)
```

ii. Hamming distance :- It calculates the distance between two binary vectors (or binary strings or bitstrings).

Ex :- * red = [1, 0, 0]

The distance between red and green could be calculated as the sum or the average number of bit differences between the two bitstrings, this is the hamming distance.

$$\text{Hamming distance} = \sum_{i=1}^{n} abs(v_{1i} - v_{2i})$$

$$\text{Hamming distance} = \frac{\sum_{i=1}^{n} abs(v_{1i} - v_{2i})}{N}$$

Ex :

```
#calculating hamming distance b/w bit strings

#calculate hamming distance
def hamming-distance(a,b):
        return sum(abs(e1-e2) for e1,e2 in zip(a,b)) / len(a)

# define data
row1 = [0,0,0,0,0,1]
row2 = [0,0,0,0,1,0]
#calculate distance
dist = hamming-distance(row1, row2)
print (dist)
```

Running this example reports the hamming distance between the two bitstrings clearly there are two differences between the two strings.

$D = \text{Average of } (2/6) = 1/3 = 0.333$

iii. Manhatton distance :- (Taxi cab or city block distance)

Manhatton distance calculates the distance between two real-valued vectors.

It is more useful to vectors that describe objects on a uniform gril, like a chess board

It is also called taxi cab distance, because it measures the shortest path that a taxicab would take between differences b/w the two vectors

$\therefore$ Manhatton distance $= D = \sum_{i=1}^{N} |v_{1i} - v_{2i}|$

Example

```
calculating manhatton distance between vectors
from math import sqrt

#calculate manhatton distance
def manhatton-distance(a,b):
    return sum(abs(e1-e2) for e1, e2 in zip(a,b))

# define data
row1 = [10,20,15,10,5]
row2 = [12,24,18,8,7]
# calculate distance
dist = manhatton-distance(row1, row2)
print(dist)
```

iv. Minkowski distance :- It calculates the distance b/w two real-valued vectors. It is a generalization of the Euclidean & manhatton distance measures and adds a parameter 'p', called the 'order'.

The minkowski distance measure is calculated as follows

$$D = \left(\sum_{i=1}^{n} abs\,(v_{1i} - v_{2i})^{p}\right)^{1/p}$$

where p is the order parameter.

If $p=1$, it is same as manhatton distance and if $p=2$, it is same as Euclidean distance

It is common to use minkowski distance when implementing a machine learning algorithm that uses distance measures.

Example of calculating the minkowski distance b/w two real vectors.

```
#calculating minkowski distance b/w vectors
```

```
#calculate minkowski distance
def minkowski-distance (a,b,p):
    return sum (abs(e1-e2)**p for e1,e2 in zip (a,b))**(1/p)

# define data
row1 = [10,20,15,10,5]
row2 = [12,24,18,8,7]
#calculate distance (p=1)
dist = minkowski-distance(row1,row2,1).
print (dist)
#calculat distance (p=2)
dist = minkowski-distance (row1, row2, 2)
print (dist)
```

The sample example using Scipy :-

```
#calculating minkowski distance b/w vectors
from scipy.spatial import minkowski_distance
define data
row1 = [10, 20, 15, 10, 5]
row2 = [12, 24, 18, 8, 7]
#calculate distance (p=1)
dist = minkowski-distance (row1, row2,1)
print (dist)
#calculate distance (p=2)
dist = minkowski-distance (row1, row2, 2)
print (dist)
```

Problems

1. find the Hamming distance b/w the code words of
$C = [(0000), (0101), (1011), (0111)]$

a. Let $x = 0\ 0\ 0\ 0$ $\quad w = 0\ 1\ 1\ 1$

$d(x,y) = \begin{matrix} 0 \\ 0 \end{matrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{matrix} 0 \\ 0 \end{matrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2$; $d(x,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{matrix} 0 \\ 0 \end{matrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} =$

$d(x,w) = \begin{matrix} 0 \\ 0 \end{matrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3$; $d(y,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{matrix} 1 \\ 1 \end{matrix} = 3$

$d(y,w) = \begin{matrix} 0 \\ 0 \end{matrix} \begin{matrix} 1 \\ 1 \end{matrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{matrix} 1 \\ 1 \end{matrix} = 1$; $d(z,w) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{matrix} 1 \\ 1 \end{matrix} \begin{matrix} 1 \\ 1 \end{matrix} = 2$

2. find the chebyshev distance b/w the points (0, 3, 4, 5) and (7, 6, 3, −1)

a. Let $A = (0, 3, 4, 5)$

$B = (7, 6, 3, -1)$

The chebyshev distance b/w A & B is $d_{AB} = \max\{|0-7|, |3-6|, |4-3|, |5+1|\}$

$\max\{7, 3, 1, 6\} = 7$

3. If the two objects represented by 4-tuples (22, 1, 42, 10) and (20, 0, 36, 8), then

a. Compute the Euclidean distance b/w two objects

b. compute the manhattan distance b/w two objects

c. compute the minkowski distance b/w two objects using p=

a. a. $D(x,y) = \sqrt{|22-20|^2 + |1-0|^2 + |42-36|^2 + |10-8|^2} = 6.71$

b. $D(x,y) = |22-20| + |1-0| + |42-36| + |10-8| = 1$

c. $D(x,y) = \left[|22-20|^3 + |1-0|^3 + |42-36|^3 + |10-8|^3\right]^{1/3}$

$= 6.15$

Hyper plane :- Geometrically, a hyperplane is a Geometric entity whose dimension is 1 < that of its ambient space

for example if you take the three 3D space then hyperp

2D entity in a 3D space would be a plane. Now, if you take 2 Dimensions, then 1D less would be a single dimensional Geometric entity, which would be a line.
The hyperplane is usually described by an equation $x^T n + b = 0$.

A hyperplane in 'n' Dimensional vector plane space R^n is defined to be the set of vectors $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$ satisfy the lineal eqn of the form

$$x_1 n_1 + x_2 n_2 + x_3 n_3 - - - - + x_n n_n + b = 0$$

The eqn of a hyperplane in 2 dimensional space is $x_1 n_1 + x_2 n_2 + b = 0$, which is a eqn of line.

Hyperplanes are decision boundaries that helps to Classify the data points. data points falling on a either side of hyperplane can be attributed to different classes. In other words, it is ability of our machine learning model to correctly differentiate or classify b/w different groups of data.

Example :-
let us consider a 2D Geometry with $n = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $b = 4$

a. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then

the eqn of hyperplane $x^T n + b = 0$

$$x_1 n_1 + x_2 n_2 + b = 0$$

$$x_1 + 3x_2 + 4 = 0$$

$$x_1 = -3x_2 - 4$$

This is an eqn of a line

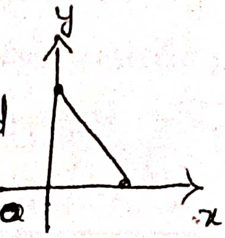
Sub spaces :

* hyperplanes in General or not Subspaces however if we ... x^T ... if the plane

subspace.

Half space :-

Consider the 2D figure given below



here we have 2D space in x_1 and x_2, and as we have discussed before an eqn in 2D would be a line, which would be a hyperplane, so eqn of line is written as $x^T n + b = 0$.

Since, it is a 2D space the equation of the hyperplane is $x_1 n_1 + x_2 n_2 + b = 0$ we can notice from the above graph that this hold 2D space is divided into 2 spaces, 1 on side i.e; +ve half of the plane and the other is on -ve half of the plane. These 2 spaces are called Half spaces.

Ex :- Let us consider the eqn of the hyperplane as $x_1 + 3x_2 + 4 = 0$

There may arise three cases

Let us discuss each case with an example

Case i :- $x_1 + 3x_2 + 4 = 0$ on the line

let us consider two points $(-1, -1)$

This point satisfies the line equation. therefore, we can say that this point lies on the hyper plane of the line.

Case ii :- $x_1 + 3x_2 + 4 > 0$: above the line (: +ve half space)

Consider the two points $(1, -1)$ which satisfies the eqn $x_1 + 3x_2 + 4 > 0$, therefore, the point $(1, -1)$ lies on the +ve half space

Case iii :- $x_1 + 3x_2 + 4 < 0$ below the line (-ve half space)

Consider the points $(1, -2)$ and this point satisfies the condition $x_1 + 3x_2 + 4 < 0$. Therefore the point $(1, -2)$ lies on the -ve half space

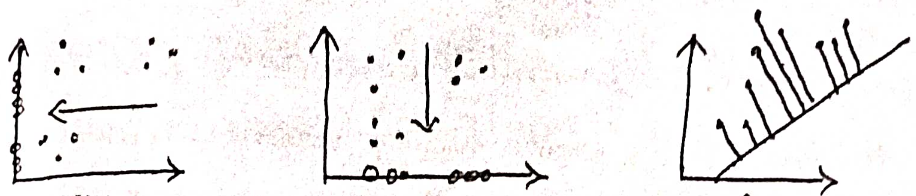
projections

presenting a high dimensional object onto a lower dimension

2. projecting the shadow puppets

one of the simplest form of projection is that when we look at object from the side we essentially ignoring one of the dimensions.

for example if we look at the 3D cube from the side it looks like a 2D cube.



(1) (2) (3)

from the above figures ① & ② we notice that the 2D data set appears to be made up of 3 groups of three data points but when we project onto the vertical or horizontal axis, we see one group of six and one group of three.

This is because in each of these directions, one of the group of three is hiding behind one of the others so both projections are deceptive and in both projections we lose information.

So the things can be improved a little by projecting along an angle rather than directly along the features. This is called a linear-projection. for example, if we project onto the diagonal line shown in the fig (3), the three groups of three data points actually look like three seperate groups in the projection.

from any angle, the projection is the result of crushing / flattening the data, so we still lose information and the shape of the data will be distorted in some way. The way to minimize the distortion is to cursh along the directions where the data is least spread out and to preserve the dimensions along which it is the widest.

projection.

Representing n-dimensional object into (n-1) dimension is known as projection.

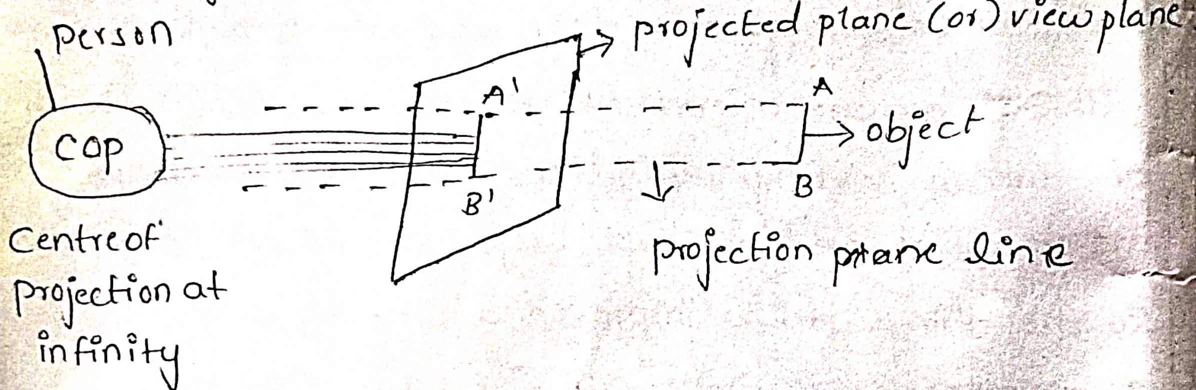
It is also defined as mapping (or) transformation of the object in projection plane (or) view plane.

projections are of two types :-

① parallel projection :- In this, coordinate positions are transformed to the view plane along parallel lines.

→ A projection is said to be parallel if centre of projection is at infinite distance from the projected plane.

→ A projection lines are parallel to each other and extended from the object and intersect the view plane.

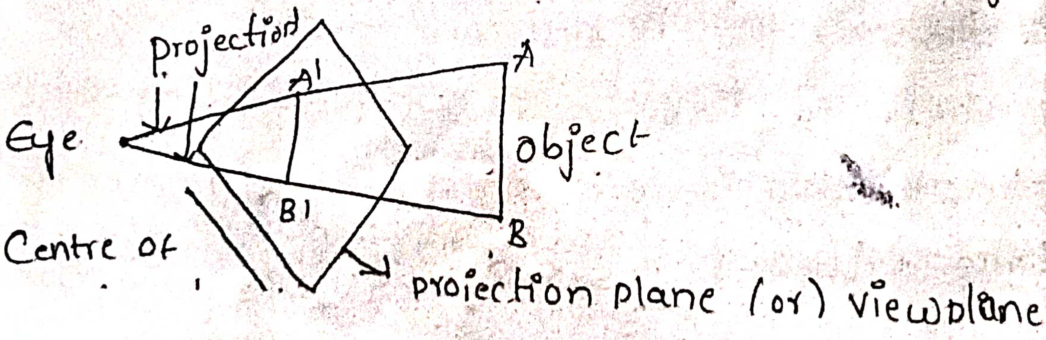


② perspective projections :- The projection is said to be perspective projection, if the centre of projection is at finite distance from the projected plane.

→ visual effect is similar to human visual system

→ The objects appear small as distance from centre of projection increases

→ difficult to determine exact size & shape of the object



Covariance :- Covariance is a statistical formula which is used to assess the relationship b/w two variables.
In other words, covariance is one of the statistical measurement to find the relationship of the variance b/w the two variables.
The covariance integrate how the two variables are related and also helps to know whether the two variables vary together or change together. It is denoted by cov(X,Y) and it's formula is given by;

population covariance formula :-

$$\text{cov}(x,y) = \frac{\sum_{i=1}^{n} (x_i-\bar{x})(y_i-\bar{y})}{n}$$

Sample covariance formula :-

$$\text{cov}(x,y) = \frac{\sum_{i=1}^{n} (x_i-\bar{x})(y_i-\bar{y})}{n-1}$$

Where x_i = data value of X

y_i = data value of Y

$\bar{x}$ = mean of x

$\bar{y}$ = mean of y

n = number of data values

Properties of covariance :-

1. $\text{cov}(x,y) = \text{cov}(y,x)$
2. $\text{cov}(x,x) = \text{var}(x)$
3. $\text{cov}((ax, by)) = ab\,\text{cov}(x,y)$
4. $\text{cov}(x+a, y+b) = \text{cov}(x,y)$

6. $cov(x,y) = \cancel{E(xy)}$, if x & y are independent, $E(xy) = E(x)E(y)$
0

7. $cov\left(\frac{x-\bar{x}}{\sigma_x}, \frac{y-\bar{y}}{\sigma_y}\right) = \frac{1}{\sigma_x \sigma_y} cov(x,y)$

8. $cov(x+y, z) = cov(x,z) + cov(y,z)$

9. $cov(x,y) = \frac{1}{n}\sum xy - \bar{x}\bar{y}$

Correlation coefficient formula :-

Correlation :- $r = \frac{cov(x,y)}{\sigma_x \sigma_y} = \frac{cov(x,y)}{\sqrt{var(x)}\sqrt{var(y)}}$

problems

1. Daily closing prices of two stocks arranged as per returns. Calculate covariance.

Day	ABC returns	xyz returns
1	1.8	2.5
2	1.5	4.3
3	2.1	4.5
4	2.4	4.1
5	0.2	2.2

Sol :- formula

$$Cov(x,y) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{n-1}$$

x	$\bar{x}$	y	$\bar{y}$	$(x_i-\bar{x})$	$(y_i-\bar{y})$	$(x_i-\bar{x})(y_i-\bar{y})$
1.8		2.5		0.2	-1.02	-0.204
1.5		4.3		-0.1	0.78	-0.078
2.1		4.5		0.5	0.98	0.49
2.4		4.1		0.8	0.58	0.464

Here $\bar{x} = \frac{\Sigma x}{n} = \frac{1.8 + 1.5 + 2.1 + 2.4 + \cdots}{5} = \frac{8}{5} = 1.6$

$\bar{x} = 1.6$

$\bar{y} = \frac{\Sigma y}{n} = \frac{2.5 + 4.3 + 4.5 + 4.1 + 2.2}{5} = \frac{17.6}{5} = 3.52$

$$cov(x, y) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{n-1}$$

$$= \frac{2.52}{5-1} = \frac{2.52}{4} = 0.63$$

returns

$cov(x, y) = 0.63$

$\therefore$ The two stocks will move together in a +ve direction

2. The Given table discribes the rate of economic growth, (x_i) and the rate of return (y_i) on the S&P 500 with the help of the covariance formula, determine whether economic growth and S&P 500 returns have a positive on inverse relationship

Economic Growth (%)	S&P 500 returns
2	8
2.8	11
4	12
3.2	8

a. formula for population covariance

$$cov(x, y) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{n}$$

$\bar{x} = \frac{\Sigma x}{-} = \frac{2 + 2.8 + 4 + 3.2}{} = \frac{12}{-} = 3$

$$\bar{y} = \frac{\sum y}{n} = \frac{8+11+12+8}{4} = \frac{39}{4} = 9.75$$

x	y	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$
2	8	-1	-1.75	1.75
2.8	11	-0.2	1.25	-0.25
4	12	1	2.25	2.25
3.2	8	0.2	-1.75	-0.35 / 3.4

$$\operatorname{cov}(x,y) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{n} = \frac{3.4}{4} = 0.85$$

$\operatorname{cov}(x,y) = 0.85$

∴ Economic growth will moving on +ve direction.

3. Suppose that 5 students were asked their grades on two statistics classes, stat-1 and stat-2.

Student	Stat-1	Stat-2
A	95	70
B	77.5	55
C	100	87.5
D	62.5	47.5
E	82.5	62.5

find i. whether stat-1 & stat-2 grades related A/c to the data.

ii. How are they related

a. formulas; $\operatorname{cov}(x,y) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{n-1}$

$$\text{Variance}(x) = \frac{\sum_{i}^{n} (x_i - \bar{x})^2}{n-1}$$

Correlation cov(x,y) coefficient $r = \frac{cov(x,y)}{\sigma_x \sigma_y}$

$\bar{x} = \frac{\Sigma x}{n} = \frac{95+77.5+100+62.5+82.5}{5} = \frac{417.5}{5} = 83.5$

$\bar{y} = \frac{\Sigma y}{n} = \frac{70+55+87.5+47.5+62.1}{5} = \frac{322.1}{5} = 64.5$

Calculation of Covariance.

x	y	$(x_i-\bar{x})$	$(y_i-\bar{y})$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$	$(x_i-\bar{x})(y_i-\bar{y})$
95	70	11.5	5.5	132.25	30.25	63.25
77.5	55	−6	−9.5	36	90.25	57
100	87.5	16.5	23	272.25	529	379.5
62.5	47.5	−21	−17	441	289	357
82.5	62.5	−1	−2	1	~~5.78~~ 4	2
				882.5	942.5	858.75

$variance(x) = \frac{(x_i-\bar{x})^2}{n-1} = \frac{882.5}{4} = 220.625$

$var(y) = \frac{(y_i-\bar{y})^2}{n-1} = \frac{942.5}{4} = 235.625$

$cov(x,y) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{n-1} = \frac{858.75}{4} = 214.6875$

∴ stat-1 & stat-2 will have the +ve relationship.

$r = \frac{cov(x,y)}{\sigma_x \sigma_y} = \frac{214.6875}{\sqrt{220.625}\sqrt{235.625}} = 0.9416$

$r = 0.9416$

∴ The relationship between stat-1 & stat-2 is strong

4. Consider two sample data sets

x: 65·21, 64·75, 65·56, 66·45, 65·34

y: 67·15, 66·29, 66·20, 64·70, 66·54

Calculate the covariance b/w x & y

a. formula for sample covariance.

$$\text{cov}(x,y) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{n-1}$$

$$\bar{x} = \frac{\sum x}{n} = \frac{65{\cdot}21 + 64{\cdot}75 + 65{\cdot}56 + 66{\cdot}45 + 65{\cdot}34}{5} = 65{\cdot}462$$

$$\bar{y} = \frac{\sum y}{n} = \frac{67{\cdot}15 + 66{\cdot}29 + 66{\cdot}20 + 64{\cdot}70, 66{\cdot}54}{5} = 66{\cdot}176$$

x	y	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$
65·21	67·15	−0·252	0·974	−0·245
64·75	66·29	−0·712	0·114	−0·081
65·56	66·20	0·098	0·024	0·00235
66·45	64·70	0·988	−1·476	−1·458
65·34	66·54	−0·122	0·364	−0·04
				−1·708

$$\text{cov}(x,y) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{n-1} = \frac{-1{\cdot}708}{4} = -0{\cdot}427$$

∴ The two datas moving in inverse direction.

5. The following ... the ... of ... students, of no
of hours they spent studying for Gate extreme and the marks that they received.

Hours (x) :	9	15	25	14	10	18	0	16	5	19	16	20
marks (y) :	39	56	93	61	50	75	32	85	42	70	66	80

find the covariance between the hours of study done and marks received.

Sol :- formula : $cov(x,y) = \dfrac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{n-1}$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{9+15+25+14+10+18+0+16+5+19+16+20}{12}$$

$$\boxed{\bar{x} = 13.97}$$

$$\bar{y} = \frac{39+56+93+61+50+75+32+85+42+70+66+80}{12}$$

$$\boxed{\bar{y} = 62.42}$$

covariance :-

x	y	$x_i-\bar{x}$	$y_i-\bar{y}$	$(x_i-\bar{x})(y_i-\bar{y})$
9	39	-4.92	-23.42	115.23
15	56	1.08	-6.42	-6.93
25	93	11.08	30.58	338.83
14	61	0.08	-1.42	-0.11
10	50	-3.92	-12.42	48.69
18	75	4.08	12.58	51.33
0	32	-13.92	-30.42	423.45
16	85	2.08	22.58	46.97
5	42	-8.92	-20.42	182.15
19	70	5.08	7.58	38.51
16	66	2.08	3.58	7.45
20	80	6.08	17.58	106.89
				1149.89

Covariance = 1149.89 ...

6. What is the covariance in relation to variance ... of two data sets given

x	2	4	6	8	10
y	1	3	8	11	12

Sol :- Here $\bar{x} = \dfrac{2+4+6+8+10}{5} = 6$

$\bar{y} = \dfrac{1+3+8+11+12}{5} = 7$

To calculate variance of covariance.

x	y	$(x_i-\bar{x})$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})$	$(y_i-\bar{y})^2$	$(x_i-\bar{x})(y_i-\bar{y})$
2	1	-4	16	-6	36	24
4	3	-2	4	-4	16	8
6	8	0	0	1	1	0
8	11	2	4	4	16	8
10	12	4	16	5	25	20
30	35		40		94	60

$$\text{Variance}(x) = \frac{\sum(x_i-\bar{x})^2}{n} = \frac{40}{5} = 8.$$

$$\text{var}(y) = \frac{94}{5} = 18.8$$

$$\text{cov}(x,y) = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{n} = \frac{60}{5} = 12$$

$$\text{cor}(r) = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y} = \frac{12}{\sqrt{8}\sqrt{18.8}} = 0.98$$

$\therefore$ There is a strong relation b/w the two data sets.

7. Find the variance, covariance and correlation coefficient of the given data. Hence write the covariance matrix

i. $x = 2, 4, 6, 8, 10$ ii. $x = 2, 4, 6, 8, 10$

$y = 1, 3, 8, 11, 12$ $y = 10, 8, 6, 4, 2$

iii. $x = 2, 4, 6, 8, 10$

Sol:- Here $\bar{x} = \frac{2+4+6+8+10}{5} = 6$

$\bar{y} = \frac{1+3+8+11+12}{5} = 7$

x	y	$x_i-\bar{x}$	$y_i-\bar{y}$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$	$(x_i-\bar{x})(y_i-\bar{y})$
2	1	-4	-6	16	36	24
4	3	-2	-4	4	16	8
6	8	0	1	0	1	0
8	11	2	4	4	16	8
10	12	4	5	16	25	20
30	35			40	94	60

$var(x) = \frac{\sum(x_i-\bar{x})^2}{n} = \frac{40}{5} = 8$

$var(y) = \frac{\sum(y_i-\bar{y})^2}{n} = \frac{94}{5} = 18.8$

$cov(x,y) = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{n} = \frac{60}{5} = 12$

$M = \begin{bmatrix} 8 & 12 \\ 12 & 18.8 \end{bmatrix}$

$cor(r) = \frac{cov(x,y)}{\sigma_x \sigma_y} = \frac{12}{\sqrt{8}\sqrt{18.8}} = 0.98$

Covariance matrix is a nxn matrix such that the diagonal elements represent the variances of each data set and all other elements represent the cov b/w the data sets

Cov matrix for 2D data ; $M = \begin{bmatrix} var(x) & cov(x,y) \\ cov(y,x) & var(y) \end{bmatrix}$

cov matrix for 3D data ; $M = \begin{bmatrix} var(x) & cov(x,y) & cov(x,z) \\ cov(y,x) & var(y) & cov(y,z) \\ cov(z,x) & cov(z,y) & var(z) \end{bmatrix}$

Cov matrix is a symmetric matrix

ii. $\bar{x} = \dfrac{2+4+6+8+10}{5} = 6$

$\bar{x} = 6$

$\bar{y} = \dfrac{10+8+6+4+2}{5} = 6$

$\bar{y} = 6$

x	y	$(x_i-\bar{x})$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})$	$(y_i-\bar{y})^2$	$(x_i-\bar{x})(y_i-\bar{y})$
2	10	-4	16	4	16	-16
4	8	-2	4	2	4	-4
6	6	0	0	0	0	0
8	4	2	4	-2	4	-4
10	2	4	16	-4	16	-16
			40		40	-40

$var(x) = \dfrac{\Sigma (x_i-\bar{x})^2}{n} = \dfrac{40}{5} = 8$

$var(y) = \dfrac{\Sigma (y_i-\bar{y})^2}{n} = \dfrac{40}{5} = 8$

$cov(x,y) = \dfrac{\Sigma (x_i-\bar{x})(y_i-\bar{y})}{n} = \dfrac{-40}{5} = -8$

$r = \dfrac{cov(x,y)}{\sigma_x \sigma_y} = \dfrac{-8}{\sqrt{8}\sqrt{8}} = -1$

Covariance matrix: $\begin{bmatrix} var(x) & cov(x,y) \\ cov(y,x) & var(y) \end{bmatrix}$

$= \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$

iii. $x = \frac{2+4+6+8+10}{5}$ $\bar{x} = 6$

$\bar{y} = \frac{7+3+5+1+9}{5} = \bar{y} = 5$

x	y	$(x_i-\bar{x})$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})$	$(y_i-\bar{y})^2$	$(x_i-\bar{x})(y_i-\bar{y})$
2	7	-4	16	2	4	-8
4	3	-2	4	-2	4	4
6	5	0	0	0	0	0
8	1	2	4	-4	16	-8
10	9	4	16	4	16	16
			40		40	4

$$\mathrm{var}(x) = \frac{\sum (x_i-\bar{x})^2}{n} = \frac{40}{5} = 8$$

$$\mathrm{var}(y) = \frac{\sum (y_i-\bar{y})^2}{n} = \frac{40}{5} = 8$$

$$\mathrm{cov}(x,y) = \frac{\sum (x_i-\bar{x})(y_i-\bar{y})}{n} = \frac{4}{5} = 0.8$$

$$(r) = \frac{\mathrm{cov}(x,y)}{\sigma_x \sigma_y} = \frac{0.8}{\sqrt{8}\sqrt{8}} = 0.1$$

Covariance matrix: $\begin{bmatrix} \mathrm{var}(x) & \mathrm{cov}(x,y) \\ \mathrm{cov}(y,x) & \mathrm{var}(y) \end{bmatrix}$

$$= \begin{bmatrix} 8 & 0.8 \\ 0.8 & 8 \end{bmatrix}$$

... for the datasets

x	y	z
2	3	9
4	6	7
6	9	5
8	12	3
10	15	1

Sol :- covariance matrix for 3D data.

$$M = \begin{pmatrix} var(x) & cov(x,y) & cov(x,z) \\ cov(y,x) & var(y) & cov(y,z) \\ cov(z,x) & cov(z,y) & var(z) \end{pmatrix}$$

$$\bar{x} = \frac{\sum x}{n} = \frac{2+4+6+8+10}{5} = 6$$

$$\boxed{\bar{x} = 6}$$

$$\bar{y} = \frac{\sum y}{n} = \frac{3+6+9+12+15}{5} = 9$$

$$\boxed{\bar{y} = 9}$$

$$\bar{z} = \frac{\sum z}{n} = \frac{9+7+5+3+1}{5} = 5$$

$$\boxed{\bar{z} = 5}$$

x	y	z	$(x_i-\bar{x})$	$(y_i-\bar{y})$	$(z_i-\bar{z})$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$	$(z_i-\bar{z})^2$
2	3	9	-4	-6	4	16	36	16
4	6	7	-2	-3	2	4	9	4
6	9	5	0	0	0	0	0	0
8	12	3	2	3	-2	4	9	4
10	15	1	4	6	-4	16	36	16
						40	90	40

$(x_i-\bar{x})(y_i-\bar{y})$ 24 24 | $(y_i-\bar{y})(z_i-\bar{z})$ -24 | $(x_i-\bar{x})(z_i-\bar{z})$

$$var(x) = \frac{\sum (x_i - \bar{x})^2}{n} = \frac{40}{5} = 8$$

$$var(y) = \frac{\sum (y_i - \bar{y})^2}{n} = \frac{90}{5} = 18$$

$$var(z) = \frac{\sum (z_i - \bar{z})^2}{n} = \frac{40}{5} = 8$$

$$cov(x,y) = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{n} = \frac{60}{5} = 12$$

$cov(x, y) = 12$

$$cov(y, z) = \frac{\sum (y_i - \bar{y})(z_i - \bar{z})}{n} = \frac{-60}{5} = -12$$

$$cov(x, z) = \frac{\sum (x_i - \bar{x})(z_i - \bar{z})}{n} = \frac{-40}{5} = -8$$

covariance matrix $M = \begin{bmatrix} 8 & 12 & -8 \\ 12 & 18 & -12 \\ -8 & -12 & 8 \end{bmatrix}$

2. find the covariance matrix for given sample data

x : 1 2 3 4

y : 2 4 1 1

z : 3 1 1 2

$$\bar{x} = \frac{\sum x}{n} = \frac{1+2+3+4}{4} = \frac{10}{4} = 2.5 \quad \boxed{\bar{x} = 2.5}$$

$$\bar{y} = \frac{\sum y}{n} = \frac{2+4+1+1}{4} = \frac{8}{4} = 2 \quad \boxed{\bar{y} = 2}$$

$$\bar{z} = \frac{\sum z}{n} = \frac{3+1+1+2}{4} = \frac{7}{4}\ 1.75 \quad \boxed{\bar{z} = 1.75}$$

$cov(x, y) = \sum (x_i - \bar{x}$

x	y	z	$(x_i-\bar{x})$	$(y_i-\bar{y})$	$(z_i-\bar{z})$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$	$(z_i-\bar{z})^2$	$(x_i-\bar{x})(y_i-\bar{y})$
1	2	3	-1.5	0	1.25	2.25	0	1.56	0
2	4	1	-0.5	2	-0.75	0.25	4	0.56	-1
3	1	1	0.5	-1	-0.75	0.25	1	0.56	-0.5
4	1	2	1.5	-1	0.25	2.25	1	0.06	-1.5
						5	6	2.74	-3

$(y_i-\bar{y})(z_i-\bar{z})$	$(x_i-\bar{x})(z_i-\bar{z})$
0	-1.87
-1.5	0.370
0.75	-0.37
-0.25	0.37
-1	-1.5

$$var(x) = \frac{\sum(x_i-\bar{x})^2}{n-1} = \frac{5}{4-1} = \frac{5}{3} = 1.66$$

$$var(y) = \frac{\sum(y_i-\bar{y})^2}{n-1} = \frac{6}{3} = 2$$

$$var(z) = \frac{\sum(z_i-\bar{z})^2}{n-1} = \frac{2.74}{3} = 0.91$$

$$cov(x,y) = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{n-1} = \frac{-3}{3} = -1$$

$$cov(x,z) = \frac{\sum(x_i-\bar{x})(z_i-\bar{z})}{n-1} = \frac{-1.5}{3} = -0.5$$

$$cov(y,z) = \frac{\sum(y_i-\bar{y})(z_i-\bar{z})}{n-1} = \frac{-1}{3} = -0.333$$

$$M = \begin{bmatrix} 1.66 & -1 & -0.5 \\ -1 & 2 & -0.33 \end{bmatrix}$$

② Calculate covariance matrix for 3D data sample

Item no	1	2	3
x	1	-1	4
y	2	1	3
z	1	3	-1

Sol :- covariance matrix

$$\bar{x} = \frac{\sum x}{n} = \frac{1-1+4}{3} = \frac{4}{3} = 1.33$$

$$\bar{y} = \frac{\sum y}{n} = \frac{2+1+3}{3} = \frac{6}{3} = 2$$

$$\bar{z} = \frac{\sum z}{n} = \frac{1+3-1}{3} = \frac{3}{3} = 1$$

x	y	z	$(x_i-\bar{x})$	$(y_i-\bar{y})$	$(z_i-\bar{z})$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$	$(z_i-\bar{z})^2$	$(x_i-\bar{x})(y_i-\bar{y})$
1	2	1	-0.33	0	0	0.1089	0	0	0
-1	1	3	-2.33	-1	2	5.4289	1	4	2.33
4	3	-1	2.67	1	-2	7.1289	1	4	2.67
						12.6667	2	8	5

$(y_i-\bar{y})(z_i-\bar{z})$	$(x_i-\bar{x})(z_i-\bar{z})$
0	0
-2	-4.66
-2	-5.34
-4	-10

$$Var(x) = \frac{\sum(x_i-\bar{x})^2}{n-1} = \frac{12.6667}{2} = 6.33$$

$$Var(y) = \frac{\sum(y_i-\bar{y})^2}{n-1} = \frac{2}{2} = 1$$

$$Var(z) = \frac{\sum(z_i-\bar{z})^2}{n-1} = \frac{8}{2} = 4$$

$$cov(x,y) = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{} = \frac{5}{2} = 2.5$$

$$cov(x,z) = \frac{\sum(x_i-\bar{x})(z_i-\bar{z})}{n-1} = \frac{-10}{2} = -5$$

$$cov(y,z) = \frac{\sum(y_i-\bar{y})(z_i-\bar{z})}{n-1} = \frac{-4}{2} = -2$$

$$M = \begin{bmatrix} 6.33 & 2.5 & -5 \\ 2.5 & 1 & -2 \\ -5 & -2 & 4 \end{bmatrix}$$

Principal Component Analysis

Principal component Analysis (PCA) is a statistical process that turns a set of correlated variables into a set of uncorrelated variables using an orthogonal transformation.

* In exploratory data analysis and machine learning for predictive models, PCA is the most extensively used tool.
* PCA is also an unsuperivised statistical tool for examining the the interrelationships between a set of variables.
* PCA is a dimensionality-reduction method for reducing the dimensionality of large data sets by transforming a large set of variables into a smaller one that retains the majority of the information in the large set.

Step by step evaluation of PCA :

Step - 1 : Standardization :

The importance of standardization prior to PCA is due to the latter's sensitivity to the original variables variances. i.e; If the ranges of initial variables differ significantly, the variables with wider ranges will dominate those with smaller ranges. resulting in biased results.

This difficulty can be avoided by converting the data to equivalent scales.

Mathematically, this can be done by subtracting the mean from each value of each variable step-2: covariance matrix computation :-

The goal of this stage is to find it there is any relationship between the input data set and the standardized data. because variables might be highly connected to the point where they include duplicated data. we compute the covariance matrix in order to find these associations. The covariance matrix summarises the correlations between all the possible pairs of variables

step-3: compute the eigen values and eigen vectors :-

of the covariance matrix to identify the principal components. The eigen vectors of the covariance matrix are actually the directions of the axes where there is the most variance and that we call principal components.

And the eigen values are simply the coefficients attached to eigen vectors, which give the amount of variance carried in each principal component. By ranking the eigen vectors in order to their eigen values, highest to lowest, we will get the principal components in order of significance.

Geometrically speaking, principal components represent the directions of the data that explain a maximal amount of variance, i.e; the lines that capture most information of the data. The relationship between variance and the given information is that, the larger the variance carried by a line, the larger the dispersion of data points along it, and the larger the dispersion along a line, the more the information it has.

Step 4: feature vector :-

In this step, what we do is, to choose whether to keep all these components or discard those of lesser significance and form with the remaining ones a matrix of vectors that we call 'feature vector'.

step 5: Recast the data along the principal components axes :-

the data from the original axes to the ones represented by the principal components.

This can be done by multiplying the standardized original data by the transpose of the feature vector.

final Data set = feature vector ^{T} x Standardized original data

PCA algorithm :-

Step 1 :- Get data

Step 2 :- Compute the mean vector

Step 3 :- Subtract mean from the given data

Step 4 :- Calculate the covariance matrix

Step 5 :- Calculate the eigen values & eigen vectors of the covariance matrix

Step 6 :- choosing components and forming a feature vector.

Step 7 :- deriving the new data

problem

Given data = {2, 3, 4, 5, 6, 7; 1, 5, 3, 6, 7, 8} compute the PCA. (or) consider the 2D patterns (2,1) (3,5) (4,3) (5,6) (6,7) (7,8)

a. Step 1 :- Get data

Let x : 2, 3, 4, 5, 6, 7

y : 1, 5, 3, 6, 7, 8

Step 2 :- calculation of mean

$$\bar{x} = \frac{\Sigma x}{n} = \frac{2+3+4+5+6+7}{6} = 4.5$$

$$\bar{y} = \frac{1+5+3+6+7+8}{} = 5.$$

Step 3 :- Subtract mean from the given data

x	y	$x-\bar{x}$	$y-\bar{y}$
2	1	-2.5	-4
3	5	-1.5	0
4	3	-0.5	-2
5	6	0.5	1
6	7	1.5	2
7	8	2.5	3

Step 4 :- Calculate the covariance matrix.

$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$	$(x_i-\bar{x})(y_i-\bar{y})$	$x_i-\bar{x}$	$y_i-\bar{y}$
6.25	16	10	-2.5	-4
2.25	0	0	-1.5	0
0.25	4	1	-0.5	-2
0.25	1	0.5	0.5	1
2.25	4	3	1.5	2
6.25	9	7.5	2.5	3
17.5	34	22		

$$Var(x) = \frac{\Sigma(x_i-\bar{x})^2}{n} = \frac{17.5}{6} = 2.92$$

$$Var(y) = \frac{34}{6} = 5.67$$

$$cov(x,y) = \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{n} = \frac{22}{6} = 3.67$$

Covariance matrix $M = \begin{bmatrix} var(x) & cov(x,y) \\ cov(y,x) & var(y) \end{bmatrix} = \begin{bmatrix} 2.92 & 3.67 \\ 3.67 & 5.67 \end{bmatrix}$

Step 5 :- Calculate the eigen values and eign vectors of the covariance matrix

$$M = \begin{bmatrix} 2.92 & 3.67 \\ 3.67 & 5.67 \end{bmatrix}$$

The characteristic equation of M is $|M - \lambda I| = 0$

i.e; $\begin{vmatrix} 2.92-\lambda & 3.67 \\ 3.67 & 5.67-\lambda \end{vmatrix} = 0$

$(2.92-\lambda)(5.67-\lambda) - 3.67 \times 3.67 = 0$

$16.5564 - 2.92\lambda - 5.67\lambda + \lambda^2 - 13.4689 = 0$

$16.5564 - 8.59\lambda + \lambda^2 - 13.4689 = 0$

$\lambda^2 - 8.59\lambda + 3.0875 = 0$

Solving $\lambda = 8.22, 0.37$ are the eigen values of M

Clearly the second eigen value (λ_2) is very small compared to the first eigen value (λ_1)

So, we can neglect the second eigen value λ_2.

Eigen vector

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to λ

Such that $[M - \lambda I]x = 0$

i.e; $\begin{bmatrix} 2.92-\lambda & 3.67 \\ 3.67 & 5.67-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

Where $\lambda = 8.22$

$$\begin{bmatrix} 2.92-8.22 & 3.67 \\ 3.67 & 5.67-8.22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5.3 & 3.67 \\ 3.67 & -2.55 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-5.3x_1 + 3.67x_2 = 0 \rightarrow (1)$

$3.67x_1 - 2.55x_2 = 0 \rightarrow (2)$

Solving (1) & (2)

$(1) \Rightarrow x_1 = \left(\frac{3.67}{5.3}\right)x_2 \Rightarrow x_1 = (0.69)x_2$

$(2) \Rightarrow x_1 = \left(\frac{2.55}{3.67}\right)x_2 \Rightarrow x_1 = (0.69)x_2$

from (2) $(3.67)x_1 = (2.55)x_2$

$\Rightarrow \frac{x_1}{2.55} = \frac{x_2}{3.67} = k$

$x_1 = 2.55k$, $x_2 = 3.67k$

Where $k = 1$

$x_1 = 2.55$, $x_2 = 3.67$

$x = \begin{bmatrix} 2.55 \\ 3.67 \end{bmatrix}$ is the eigen vector corresponding to λ_1

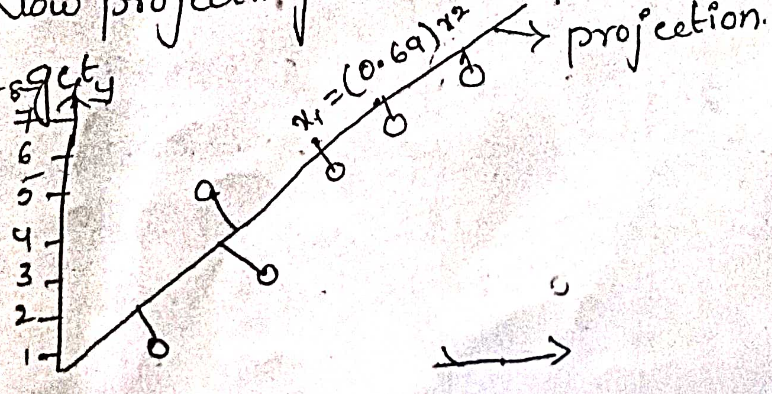
Step-6 :- feature vector

feature vector = eigen vector corresponding to λ_1

$= \begin{bmatrix} 2.55 \\ 3.67 \end{bmatrix}$

Hence the principal component is $\begin{bmatrix} 2.55 \\ 3.67 \end{bmatrix}$

Step-7 :- Now projecting the data points on to the new Sub space we get



2. Compute the PC using PCA algorithm for the following data.

x : 2.5, 0.5, 2.2, 1.9, 3.1, 2.3, 2, 1, 1.5, 1.1

y : 2.4, 0.7, 2.9, 2.2, 3.0, 2.7, 1.6, 1.1, 1.6, 0.9

Sol :- Step 1 :- Get data

x : 2.5, 0.5, 2.2, 1.9, 3.1, 2.3, 2, 1, 1.5, 1.1

y : 2.4, 0.7, 2.9, 2.2, 3.0, 2.7, 1.6, 1.1, 1.6, 0.9

Step 2 :- Calculate mean

$$\bar{x} = \frac{2.5+0.5+2.2+1.9+3.1+2.3+2+1+1.5+1.1}{10}$$

$\bar{x} = 1.81$

$$\bar{y} = \frac{2.4+0.7+2.9+2.2+3.0+2.7+1.6+1.1+1.6+0.9}{10}$$

$\bar{y} = 1.91.$

Step 3 :- Subtract mean from the given data

x	y	$x_i - \bar{x}$	$y_i - \bar{y}$
2.5	2.4	0.69	0.49
0.5	0.7	-1.31	-1.21
2.2	2.9	0.39	0.99
1.9	2.2	0.09	0.29
3.1	3.0	1.29	1.09
2.3	2.7	0.49	0.79
2	1.6	0.19	-0.31
1	1.1	-0.81	-0.81
1.5	1.6	-0.31	-0.31
1.1	0.9	-0.71	-1.01

Step 4 :- Calculate covariance matrix.

$(x_i-\bar{x})$	$(y_i-\bar{y})$	$(x_i-\bar{x})^2$	$(y_i-\bar{y})^2$	$(x_i-\bar{x})(y_i-\bar{y})$
0.69	0.49	0.476	0.240	0.3381
-1.31	-1.21	1.716	1.464	1.5851
0.39	0.99	0.152	0.980	0.3861
0.09	0.29	0.008	0.084	0.0261
1.29	1.09	1.664	1.188	1.4061
0.49	0.79	0.240	0.624	0.3871
0.19	-0.31	0.036	0.096	-0.0589
-0.81	-0.81	0.656	0.656	0.6561
-0.31	-0.31	0.096	0.096	0.0961
-0.71	-1.01	0.504	1.020	0.7171
		5.548	6.448	5.539

$$var(x) = \frac{\sum(x_i-\bar{x})^2}{n-1} = \frac{5.548}{9} = 0.616$$

$$var(y) = \frac{\sum(y_i-\bar{y})^2}{n-1} = \frac{6.448}{9} = 0.716$$

$$cov(x,y) = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{n-1} = \frac{5.539}{9} = 0.615$$

covariance matrix $M = \begin{bmatrix} 0.616 & 0.615 \\ 0.615 & 0.716 \end{bmatrix}$

Step 5 :- calculate the eigen values and eigen vectors of covariance matrix.

The characteristic equation of M is $|M - \lambda I| = 0$

i.e; $\begin{vmatrix} 0.616-\lambda & 0.615 \\ 0.615 & 0.716-\lambda \end{vmatrix} = 0$

$(0.616-\lambda)(0.716-\lambda) - 0.615\times0.615 = 0$

$0.616\times0.716 - 0.616\lambda - 0.716\lambda + \lambda^2 - 0.615\times0.615 = 0$

$0.441 - 1.332\lambda + \lambda^2 - 0.378 = 0$

$0.063 - 1.332\lambda + \lambda^2 = 0$

$\lambda^2 - 1.332\lambda + 0.063 = 0$

$\lambda_1 = 1.2828,\ 0.0491$ are the eigen values of M

Clearly the second eigen value(λ_2) is very small compared to the first eigen value (λ_1)

So, we ~~get~~ can neglect the second eigen value λ_2.

Eigen vector

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to λ such that $[M - \lambda I]x = 0$

$$\begin{bmatrix} 0.616-\lambda & 0.615 \\ 0.615 & 0.716-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $\lambda = 1.2828$

$$\begin{bmatrix} 0.616-1.2828 & 0.615 \\ 0.615 & 0.716-1.2828 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -0.666 & 0.615 \\ 0.615 & -0.566 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-0.666x_1 + 0.615x_2 = 0 \rightarrow ①$

Solving ① ~ ②

① $\Rightarrow x_1 = \left(\frac{0.615}{0.668}\right) x_2 \Rightarrow x_1 = (0.92) x_2$

② $\Rightarrow x_1 = \left(\frac{0.566}{0.615}\right) x_2 \Rightarrow x_1 = (0.92) x_2$

from ② $\quad 0.615 x_1 = 0.566 x_2$

$\Rightarrow \frac{x_1}{0.566} = \frac{x_2}{0.615} = k$

$x_1 = 0.566k, \quad x_2 = 0.615k$

where $k = 1$

$x_1 = 0.566, \quad x_2 = 0.615$

$x = \begin{bmatrix} 0.566 \\ 0.615 \end{bmatrix}$ is the eigen vector corresponding to λ_1

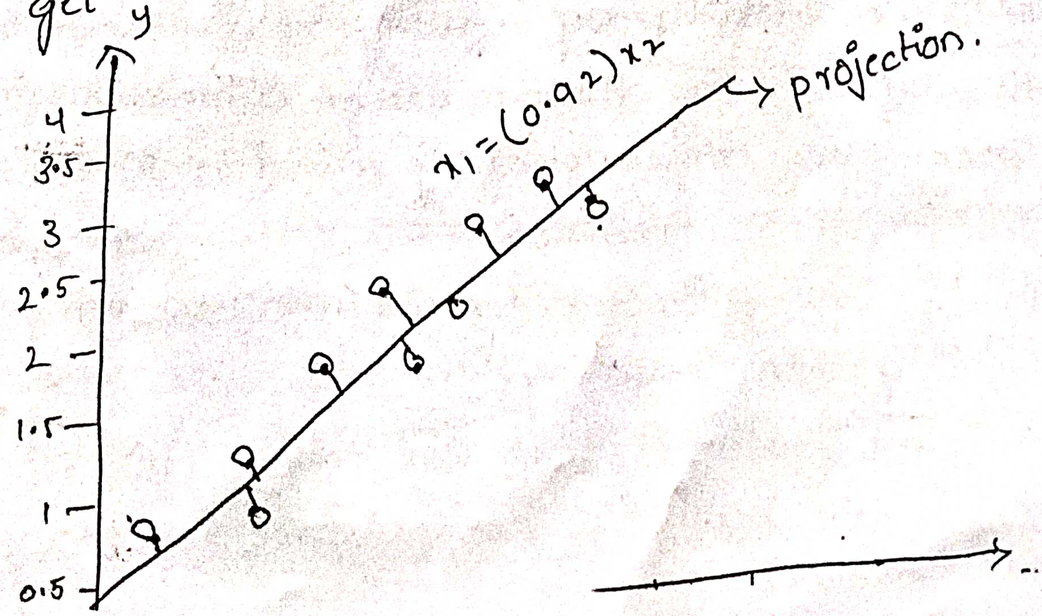
Step : 6 feature vector

feature vector = eigen vector corresponding to λ_1

$= \begin{bmatrix} 0.566 \\ 0.615 \end{bmatrix}$

Hence the principal component is $\begin{bmatrix} 0.566 \\ 0.615 \end{bmatrix}$

Step :- 7 Now projecting the data points on to the new sub space we get



Dimensionality reduction

Dimensionality reduction, or dimension reduction, is the transformation of data from a high-dimensional space so that the low-dimensional representation retains some meaningful properties of the original data, ideally close to its intrinsic dimension.

For example, hi-dimensional data is often over complete, that is many dimensions are redundant and can be explained by a combination of other dimension.

Also dimensions in hi-dimensional data are often correlated so that the data prosesses an intrinsic lower dimensional structure.

Dimensionality reduction exploits structure and correlation and allows us to work with a more compact representation of the data, ideally without loosing information.

If you have too many input variables, machine learning algorithm performance may degrade. If many dimensions reside in the feature space, that results in a large volume of space. Consequently the points in the space and rows of data may represent only a tiny, non-representative sample. This imbalance can negatively affect machine learning algorithm performance. This condition is known as "the curse of dimensionality. The problem caused by too many features are often referred to as the curse of dimensionality. By reducing the number of input features, thereby reducing the number of dimensions in the feature space. Hence dimensionality reduction means reducing your feature set's dimension.

Dimensionality reduction brings many advantages to your Machine learning data, including:

* fewer features mean less complexity.

* you will need less storage space because you have fewer data.

* Model accuracy improves due to less misleading data

ans. * Reducing the data set's feature dimensions helps visualize the data faster

* It removes noise and redundant features

* Dimension reduction process reduces the no. of random variables or features under consideration in a machine learning algorithm.

eg:- DR reduces the need for storage and time to perform a machine learning algorithm. It makes the algorithm more efficient.

DR removes the multi-colinearity between features it can improve the performance of the model.

DR makes it easy to visualize the model in 2D or 3D views.

Dimensionality Reduction Methods

feature selection.

feature selection is a means of selecting the input data set's optimal relevant features and removing irrelevant features.

* filter methods. This method filters down the data set into a relevant subset.

* Wrapper methods. This method uses the machine learning model to evaluate the performance of features fed into it. the performance determines whether it's better to keep or remove the features to improve the model's accuracy. This method is more accurate than filtering but is also more complex.

* Embedded methods. The embedded process checks the machine learning model's various training iterations and evaluates each feature's importance.

feature extraction :-

This method transforms the space containing too many dimensions into a space with fewer dimensions. This process is useful for keeping the whole information while using fewer resources during information processing. Here are three of the more common extraction techniques.

* learning discriminant analysis:- LDA is commonly used for ... data ... rotates and

features with maximum variance are designated the principal components.

*kernel PCA :- This process is a nonlinear extension of PCA, works for more complicated structures that cannot be represented in a linear subspace in an easy or appropriate manner. It uses the "kernel trick" to construct non linear mappings.

*Quadratic discriminant analysis :- This technique projects data in a way that maximizes class separability. the projection puts examples from the same class close together, and examples from different classes are placed farther apart.

Dimensionality reduction techniques :-

Here are some techniques machine learning professionals use.

Principal component Analysis :- PCA extracts a new set of variables from an existing, more extensive set. The new set is called "principal components."

·Backward feature Elimination :- This five-step technique defines the optimal number of features required for a machine learning algorithm by choosing the best model performance and the maximum tolerable error rate.

forward feature selection :- This technique follows the inverse of the backward feature elimination process. Thus, we don't eliminate the feature. Instead, we find the best features that produce the highest increase in the model's performance.

Missing value Ratio :- This technique sets a threshold level for missing values. If a variable exceeds the threshold, it's dropped.

low variance filter :- like the missing value ratio technique, low variance filter works with a threshold. However in this case, it's testing data columns. The method calculates the variance of each variable. All data columns with variances falling below the threshold are dropped since low variance features don't

High correlation filter: This method applies to two variables carrying the same information, thus potentially degrading the model. In this method, we identify the variables with high correlation and use the variance inflation factor to choose one. you can remove variables with a higher value.

Decision trees :- Decision trees are a popular supervised learning algorithm that splits data into homogenous sets based on input variables. This approach solves problems like data outliers, missing values, and identifying significant variables.

Random forest :- This method is like the decision tree strategy. However, in this case, we generate a large set of trees (hence "forest") against the target variable. Then we find feature subsets with the help of each attribute's usage statistics of each attribute.

factor analysis :- This method places highly correlated variable into their own group, symbolizing a single factor or construct.

1. Compute the PC from the following data

x_1 : 1.4000, 1.6000, −1.4000, −2.0000, −3.0000, −2.4000, 1.5000, 2.3000, −3.2000, −4.1000

x_2 : 1.6500, 1.9750, −1.7750, −2.5250, −3.9500, 3.0750, 2.0250, 2.7500, −4.0500, −4.8500

Sol :- Step 1 :- Get data

x_1 : 1.4000, 1.6000, −1.4000, −2.0000, −3.0000, −2.4000, 1.5000, 2.3000, −3.2000, −4.1000

x_2 : 1.6500, 1.9750, −1.7750, −2.5250, −3.9500, 3.0750, 2.0250, 2.7500, −4.0500, −4.8500

Step-2 calculate mean

$$\bar{x}_1 = \frac{\Sigma x}{n} = \frac{1.4000 + 1.6000 - 1.4000 - 2.0000 - 3.0000 - 2.4000 + 1.5000 + 2.3000 - 3.2000 - 4.1000}{10}$$

$$\bar{x}_1 = -0.93$$

$x_1 = -0.93$

$\bar{x}_2 = -0.56$

Step-3 :- subtract mean from given data

x_1	x_2	$x_1 - \bar{x}_1$	$x_2 - \bar{x}_2$
1.4000	1.6500	2.33	2.21
1.6000	1.9750	2.53	2.54
-1.4000	-1.7750	-0.47	-1.22
-2.0000	-2.5250	-1.07	-1.97
-3.0000	-3.9500	-2.07	-3.39
-2.4000	3.0750	-1.47	3.635
1.5000	2.0250	2.43	2.59
2.3000	2.7500	3.23	3.31
-3.2000	-4.0500	-2.27	-3.49
-4.1000	-4.8500	-3.17	-4.29

Step-4 calculate covariance matrix

$(x_1-\bar{x}_1)^2$	$(x_2-\bar{x}_2)^2$	$(x_1-\bar{x}_1)(x_2-\bar{x}_2)$
5.429	4.884	5.1493
6.400	6.451	6.4262
0.220	1.488	0.5734
1.144	3.880	2.1079
4.285	11.492	7.0173
2.161	13.177	-5.3361
5.905	6.7081	6.2937
10.432	10.958	10.6913
5.153	12.180	7.9223
10.049	18.404	13.5993
51.178	89.6201	54.4446

$$Var(x_1) = \frac{51.178}{10} = 5.12$$

$$Var(x_2) = \frac{89.6201}{10} = 89.62$$

$cov(x_1, x_2) = 54.4446$

$$M = \begin{bmatrix} 5.12 & 5.44 \\ 5.44 & 89.62 \end{bmatrix}$$

Step-5 :- calculate the eigen values and eigen vectors of covariance matrix.

The C.E of M is $|M - \lambda I| = 0$

i.e $\begin{vmatrix} 5.12-\lambda & 5.44 \\ 5.44 & 89.62-\lambda \end{vmatrix} = 0$

$(5.12-\lambda)(89.62-\lambda) - 5.44 \times 5.44 = 0$

$5.12 \times 89.62 - 5.12\lambda - 89.62\lambda + \lambda^2 - 29.63 = 0$

$458.8544 - 94.74\lambda + \lambda^2 - 29.63 = 0$

$\lambda^2 - 94.74\lambda + 429.2244 = 0$

$\lambda_1 = 89.96,\ 4.77$ are the eigen values of M

Clearly the second eigen value (λ_2) is very small compared to the first eigen valen (λ_1)

So, we can neglect the second eigen values λ_2

Eigen vector

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to λ such that

$[M - \lambda I]x = 0$

$$\begin{bmatrix} 5.12-\lambda & 5.44 \\ 5.44 & 89.62-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

~~5.12~~ Where $\lambda = 89.96$.

$$\begin{bmatrix} 5.12-89.96 & 5.44 \\ 5.44 & 89.62-89.96 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -84.84 & 5.44 \\ 5.44 & -0.34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving ① & ②

$$① = x_1 = \left(\frac{5.44}{84.84}\right) x_2 \Rightarrow x_1 = (0.064) x_2$$

$$② = x_1 = \left(\frac{0.34}{5.44}\right) x_2 \Rightarrow x_1 = (0.06) x_2$$

from ②

$$5.44 x_1 = 0.34 x_2$$

$$\frac{x_1}{0.34} = \frac{x_2}{5.44} = k$$

$$x_1 = 5.44k, \quad x_2 = 0.34k$$

where $k = 1$

$$x_1 = 0.34k, \quad x_2 = 5.44k$$

where $k = 1$

$x = \begin{bmatrix} 0.34 \\ 5.44 \end{bmatrix}$ is the eigen vector corresponding to λ_1

Step :- 6 :- feature vector

feature vector = eigen vector corresponding to λ_1

$$= \begin{bmatrix} 0.34 \\ 5.44 \end{bmatrix}$$

1. Use PCA algorithm to reduce the dimension from 2D - 1D for the following data.

x : 4 8 13 7

y : 11 4 5 14

Sol :- Step 1 :- Get data

x : 4 8 13 7

y : 11 4 5 14

Step 2 :- calculate mean

$$\bar{x} = \frac{\sum x}{n} = \frac{4+8+13+7}{4} = 8$$

$$\bar{y} = \frac{\sum y}{n} = \frac{11+4+5+14}{4} = 8.5$$

Step 3 :- subtract mean from the given values

x	y	$(x-\bar{x})$	$(y-\bar{y})$
4	11	-4	2.5
8	4	0	-4.5
13	5	5	-3.5
7	14	-1	5.5

Step 4 :- calculate covariance matrix.

$(x-\bar{x})$	$(y-\bar{y})$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
-4	2.5	16	6.25	-10
0	-4.5	0	20.25	0
5	-3.5	25	12.25	-17.5
-1	5.5	1	30.25	-5.5
		42	69	-33

$$Var(x) = \frac{\sum(x-\bar{x})^2}{n-1} = \frac{42}{3} = \cancel{10.5}\ 14$$

$\therefore$ $cov(x,y) = \frac{-33}{9} = -8.2511$

$$M = \begin{bmatrix} 19.5 & -8.25 \\ -8.25 & 17.25 \end{bmatrix}$$

$$M = \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

Step 5 :- calculate eigen values and eigen vectors of M

i.e; $|M - \lambda I| = 0$

$$\begin{vmatrix} 14-\lambda & -11 \\ -11 & 23-\lambda \end{vmatrix} = 0$$

$(14-\lambda)(23-\lambda) + 11 \times -11 = 0$

$14 \times 23 - 14\lambda - 23\lambda + \lambda^2 - 121 = 0$

$322 - 37\lambda + \lambda^2 - 121 = 0$

$\lambda^2 - 37\lambda + 201 = 0$

$\lambda_1 = 30.38$ $\quad \lambda_2 = 6.615$ are the eigen values of M

$\because \lambda_1 > \lambda_2$ $\therefore$ we can neglect λ_2

Eigen vector

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to λ such that $[M - \lambda I]x = 0$

i.e $\begin{bmatrix} 14-\lambda & -11 \\ -11 & 23-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

where $\lambda = 30.38$

$$\begin{bmatrix} 14-30.38 & -11 \\ -11 & 23-30.38 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -16.38 & -11 \\ -11 & -7.38 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-16.38x_1 - 11x_2 = 0 \rightarrow (1)$

$-11x_1 - 7.38x_2 = 0 \rightarrow (2)$

Solving (1) & (2)

$(1) \Rightarrow x_1 = \left(\frac{11}{16.38}\right)x_2 \Rightarrow x_1 = (0.670)x_2$

$(2) \Rightarrow x_1 = \left(\frac{7.38}{11}\right)x_2 \Rightarrow x_1 = (0.67)x_2$

from (2)

$11x_1 = 7.38x_2$

$\frac{x_1}{7.38} = \frac{x_2}{11} = k$

$x_1 = 7.38k, \quad x_2 = 11k$

where $k = 1$

$x_1 = 7.38, \quad x_2 = 11$

$x = \begin{bmatrix} 7.38 \\ 11 \end{bmatrix}$ is the eigen vector corresponding to λ_1

Singular value Decomposition (SVD):-

The Single value decomposition (SVD) of a matrix is a central matrix decomposition method in linear algebra

* It has been referred as the "fundamental theorem". of linear algebra, because it can be applied to all matrices and it always exists

SVD theorem :- Defn

Let $A \in \mathbb{R}^{m \times n}$ be a rectangular matrix of rank r.

$$M\boxed{A} = m\boxed{U}\; m\boxed{\Sigma}\; n\boxed{V^T}$$

(or)

$$A_{m\times n} = U_{m\times m}\,\Sigma_{m\times n}\,V^T_{n\times n} \rightarrow (1)$$

* with an orthogonal matrix $U \in R^{m\times m}$ with column vectors u_i, $i = 1 \dots m$ and an orthogonal matrix V belongs $R^{n\times n}$ with column vector v_j, $j = 1, \dots n$.

* More over, Σ is an $m\times n$ matrix with $\Sigma_{ii} = \sigma_i \geq 0$ and $\Sigma_{ij} = 0$, for $i \neq j$

* The diagonal entries σ_i, $i = 1, \dots r$ of Σ are called the singular values, u_i are called the left singular vectors, and v_j are called the right singular vectors.

* In general the singular values are ordered as $\sigma_1 \geq \sigma_2 > \dots \geq \sigma_r \geq 0$

Note :-

The singular value matrix is unique

* The matrix σ is a rectangular matrix which is similar to the order of A

* The SVD expresses a change of basis in both the domain and co-domain.

Construction of SVD :-

* computing the SVD of $A \in R^{m\times n}$ is equivalent to finding two sets of orthonormal bases $U = \{u_1 \dots u_m\}$ and $V = \{v_1 \dots v_n\}$ of the co-domain R^m, R^n respectively. ... ordered bases we will construct the matrices

* We will start with constructing the right singular vectors.

* We have that the eigen vectors of a symmetric matrix from an orthonormal basis (ONB) which means that it can be diagnolized.

* Also, we construct a symmetric, positive semi definite matrix $A^T A \in R^{m \times n}$ from any rectangular matrix $A \in R^{m \times m}$

* Thus we can dignolize $A^T A$ as $A^T A = PDP^T$

$$P = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & \lambda_n \end{bmatrix} P^T \rightarrow (2)$$

Where P is an orthogonal matrix, which is composed of the orthonormal eigen basis

Here $\lambda_i \geq 0$ or the eigen values of $A^T A$

* Let us assume that the SVD of A exists and by using eq (1) & (2) we get $A^T A = (U \Sigma V^T)^T (U \Sigma V^T)$

$= V \Sigma^T U^T U \Sigma V^T \rightarrow (3)$

Where U, V are orthogonal matrices

$\therefore A^T A = V \Sigma^T \Sigma V^T \quad (\because V^T U = I)$

$$\Rightarrow A^T A = V \begin{bmatrix} \sigma_1^2 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & & \cdots & \sigma_n^2 \end{bmatrix} V^T \rightarrow (4) \quad (\Sigma^T \Sigma = \Sigma^2)$$

(3) & (4) we get

$V^T = P^T \rightarrow (5)$

and $\sigma_i^2 = \lambda_i \rightarrow (6)$

• The eigen vectors of $A^T A$ that compose are

* The eigen values of $A'A$ are the squared singular values of Σ

* To obtain the left singular matrix U we follow the similar procedure by considering the matrix $AA^T \in R^{m \times n}$

Alternatively, we can find the left singular vectors that are orthonormal, by using the following formula

$$u_i = \frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sqrt{\lambda_i}}$$

$$\boxed{u_i = \frac{Av_i}{\sigma_i}} \text{ (or) } \boxed{v_i = \frac{A^T u_i}{\sigma_i}}$$

Working Rule to find SVD :-

1. find A^TA (or) AA^T
2. find the eigen values and eigen vector A^TA or AA^T
3. write Σ
4. Normalize the eigen vectors of A^TA (or) AA^T to find the vectors of the matrix V (or) U
5. find u or v using normalized vectors of V or U where $u_i = \frac{1}{\sigma_i} A v_i$ (or)

$$v_i = \frac{1}{\sigma_i} A^T u_i$$

1. find the single value decomposition of $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

a. Step:-1

Given $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The C.E of $|A^T A - \lambda I| = 0$

i.e; $\begin{vmatrix} 5-\lambda & -2 & 1 \\ -2 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$

$(5-\lambda)[(1-\lambda)^2 - 0] + 2[-2(1-\lambda)] + 1[-(1-\lambda)] = 0$

$(5-\lambda)(1+\lambda^2-2\lambda) + 2(-2+2\lambda) + 1[-1+\lambda] = 0$

$5+5\lambda^2-10\lambda-\lambda-\lambda^3+2\lambda^2-4+4\lambda-1+\lambda = 0$

$-\lambda^3+7\lambda^2-6\lambda = 0$

$\lambda^3-7\lambda^2+6\lambda = 0$

$\lambda(\lambda^2-7\lambda+6) = 0$

$\lambda(\lambda^2-6\lambda-\lambda+6) = 0$

$\lambda(\lambda(\lambda-6)-1(\lambda-6)) = 0$

$\lambda(\lambda-1)(\lambda-6) = 0$

$\lambda = 0, 1, 6$ are the eigen values of $A^T A$

eigen vectors :-

To find the eigen vectors, consider $[A^T A - \lambda I]x = 0$

$$\begin{bmatrix} 5-\lambda & -2 & 1 \\ -2 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 6 \Rightarrow \begin{bmatrix} -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

$x - 2y + z = 0 \rightarrow (1)$

$-2x - 5y = 0 \rightarrow (2)$

$x - 5z = 0 \rightarrow (3)$

Solve (1) & (2)

x	y	
-1	-2	1
-2	-5	0

$$\frac{x}{0+5} = \frac{-y}{0+2} = \frac{z}{5-4}$$

$$\frac{x}{5} = \frac{-y}{2} = \frac{z}{1}$$

$x = 5,\ y = -2,\ z = 1$

$v_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 6$

ii. $\lambda = 1$

$$\begin{bmatrix} 4 & -2 & 1 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x - 2y + z = 0 \rightarrow (1)$

$-2x = 0 \rightarrow (2)$

$x = 0 \rightarrow (3)$ put in (1)

$-2y + z = 0$

$+2y = +z$

$$\frac{y}{1} = \frac{z}{2}$$

$x = 0,\ y = 1,\ z = 2$

$v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

iii. $\lambda = 0$

$5x - 2y + z = 0 \rightarrow (1)$

Sove (1) & (2)

$-2x + y = 0 \rightarrow (2)$

$\frac{x}{0-1} = \frac{-y}{2} = \frac{z}{5-4}$

$x + z = 0 \rightarrow (3)$

$x = 1,\ y = 2,\ z = -1$

$V3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ are the eigen vector corresponding to $\lambda = 0$

Normalized eigen vector A^TA are

$$\left(\frac{V1}{\|v_1\|}, \frac{V2}{\|v_2\|}, \frac{V3}{\|v_3\|}\right) = \begin{bmatrix} 5/\sqrt{30} & 0 & 1/\sqrt{6} \\ -2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ -1/\sqrt{30} & 2/\sqrt{5} & -1/\sqrt{6} \end{bmatrix} = P$$

$A^TA = P \cdot D \cdot P^T$ $\quad D = P^{-1}AP.$

$$\begin{bmatrix} 5/\sqrt{30} & 0 & 1/\sqrt{6} \\ -2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ -1/\sqrt{30} & 2/\sqrt{5} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 5/\sqrt{30} & -2/\sqrt{30} & -1/\sqrt{30} \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

Step 2 :-

As the singular values are the square roots are the eigen vectors of A^TA we obtain straight from 'D' rank $A = 2$, there are only two non-zero singular values

$\sigma_1 = \sqrt{6}$ & $\sigma_2 = \sqrt{1} = 1$

$$\Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3}$$

Step :- 3

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{6}{\sqrt{180}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \frac{6}{6\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

2. find the singular values of $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and find SVD of A.

Sol Step :- 1

Given $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

then $A^T \cdot A = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

The C.E of A^TA is $|A^TA - \lambda I| = 0$

i.e; $\begin{vmatrix} 2-\lambda & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6-\lambda & 2 \\ 0 & 2 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)\left[(6-\lambda)(2-\lambda) - 4\right] - 2\sqrt{2}\left[2\sqrt{2}(2-\lambda) - 0\right] + 0 = 0$

$\Rightarrow (2-\lambda)\left[12 - 8\lambda + \lambda^2 - 4\right] - 8(2-\lambda) = 0$

$\Rightarrow (2-\lambda)\left[\lambda^2 - 8\lambda + 8 - 8\right] = 0$

$\Rightarrow (2-\lambda)\left[\lambda^2 - 8\lambda\right] = 0$

$\Rightarrow \lambda(2-\lambda)(\lambda - 8) = 0$

$\lambda = 0, 2, 8$

$\lambda_1 = 8, \lambda_2 = 2, \lambda_3 = 0$ are the eigen values of A^TA

eigen vectors :-

Eigen vector corresponding to λ is $[A^TA - \lambda I]x = 0$

$\Rightarrow \begin{bmatrix} 2-\lambda & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6-\lambda & 2 \\ \end{bmatrix} \begin{bmatrix} x \\ y \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 8$

$$\Rightarrow \begin{bmatrix} 2-8 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6-8 & 2 \\ 0 & 2 & 2-8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & -2 & 2 \\ 0 & 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-6x + 2\sqrt{2}y = 0 \rightarrow (1)$

$2\sqrt{2}x - 2y + 2z = 0 \rightarrow (2)$

$2y - 6z = 0 \rightarrow (3)$

Solve (1) & (3)

$$\begin{matrix} + & - & + \\ -6 & 2\sqrt{2} & 0 \\ 0 & 2 & -6 \end{matrix}$$

$2\sqrt{2} \quad 0 \quad -6 \quad 2\sqrt{2}$

-2

$4\sqrt{2}$

$$\frac{x}{-12\sqrt{2}} = \frac{-y}{36-0} = \frac{z}{-12-0}$$

$$\frac{x}{-12\sqrt{2}} = \frac{-y}{36} = \frac{z}{-12}$$

$$\frac{x}{\sqrt{2}} = \frac{y}{3} = \frac{z}{1}$$

$x = \sqrt{2}, \quad y = 3, \quad z = 1$

$\therefore v_1 = \begin{bmatrix} \sqrt{2} \\ 3 \\ 1 \end{bmatrix}$

Normalizing vector $= v_1 = \begin{bmatrix} \sqrt{2}/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}$ $\left[\frac{\sqrt{2}}{\sqrt{(\sqrt{2})^2 + 3^2 + 1^2}}\right]$

When $\underline{\lambda = 2}$

$\begin{bmatrix} 2-2 & 2\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

2. find the singular values of $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and SVD of A.

Sol Step :- 1

Given $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

then $A^T \cdot A = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

The c.E of A^TA is $|A^TA - \lambda I| = 0$

i.e; $\begin{vmatrix} 2-\lambda & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6-\lambda & 2 \\ 0 & 2 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)[(6-\lambda)(2-\lambda) - 4] - 2\sqrt{2}[2\sqrt{2}(2-\lambda) - 0] + 0 = 0$

$\Rightarrow (2-\lambda)[12 - 8\lambda + \lambda^2 - 4] - 8(2-\lambda) = 0$

$\Rightarrow (2-\lambda)[\lambda^2 - 8\lambda + 8 - 8] = 0$

$\Rightarrow (2-\lambda)[\lambda^2 - 8\lambda] = 0$

$\Rightarrow \lambda(2-\lambda)(\lambda - 8) = 0$

$\lambda = 0, 2, 8$

$\lambda_1 = 8$, $\lambda_2 = 2$, $\lambda_3 = 0$ are the eigen values of A^TA

eigen vectors :-

Eigen vector corresponding to λ is $[A^TA - \lambda I]X$

$\Rightarrow \begin{bmatrix} 2-\lambda & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6-\lambda & 2 \\ \cdot & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 8$

$$\Rightarrow \begin{bmatrix} 2-8 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6-8 & 2 \\ 0 & 2 & 2-8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -6 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & -2 & 2 \\ 0 & 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-6x + 2\sqrt{2}y = 0 \longrightarrow (1)$

$2\sqrt{2}x - 2y + 2z = 0 \rightarrow (2)$

$2y - 6z = 0 \longrightarrow (3)$

Solve (1) & (3)

$$\begin{matrix} + & - & + \\ -6 & 2\sqrt{2} & 0 \\ 0 & 2 & -6 \end{matrix}$$

$$\begin{matrix} 2\sqrt{2} & 0 & -6 & 2\sqrt{2} \\ -2 & -6 & 0 & -2 \end{matrix}$$

$4\sqrt{2}$

$$\frac{x}{-12\sqrt{2}} = \frac{-y}{36-0} = \frac{z}{-12-0}$$

$$\frac{x}{-12\sqrt{2}} = \frac{-y}{36} = \frac{z}{-12}$$

$$\frac{x}{\sqrt{2}} = \frac{y}{3} = \frac{z}{1}$$

$x = \sqrt{2}$, $y = 3$, $z = 1$

$$\therefore v_1 = \begin{bmatrix} \sqrt{2} \\ 3 \\ 1 \end{bmatrix}$$

$$\left[\frac{\sqrt{2}}{\sqrt{(\sqrt{2})^2 + 3^2 + 1^2}}\right]$$

Normalizing vector $= v_1 = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{12}} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}$

When $\underline{\underline{\lambda = 2}}$

$$\begin{bmatrix} 2-2 & 2\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\begin{pmatrix} 0 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 4 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2\sqrt{2}y = 0 \rightarrow (1)$

$2\sqrt{2}x + 4y + 2z = 0 \rightarrow (2)$

$2y = 0 \rightarrow (3)$

from (1) & (3)

$y = 0$ sub in (2)

$2\sqrt{2}x + 2z = 0$

$\cancel{2}\sqrt{2}x = -\cancel{2}z$

$x = \cancel{-z}$

$$\frac{x}{1} = \frac{z}{-\sqrt{2}}$$

$x = 1, \ z = -\sqrt{2}, \ y = 0$

$$V_2 = \begin{bmatrix} 1 \\ \cancel{-\sqrt{2}} \\ \cancel{0} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \end{bmatrix}$$

$$\text{Normalized vector} = \begin{bmatrix} \cancel{1/\sqrt{3}} \\ \cancel{-\sqrt{2}/\sqrt{3}} \\ \cancel{0/\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 0/\sqrt{3} \\ -\sqrt{2}/3 \end{bmatrix}$$

Here $\underline{\lambda = 0}$

$$\begin{bmatrix} 2-0 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6-0 & 2 \\ 0 & 2 & 2-0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x + 2\sqrt{2}y = 0 \rightarrow (1)$

$2\sqrt{2}x + 6y + 2z = 0 \rightarrow (2)$

$2y + 2z = 0 \rightarrow (3)$

Solve (1) & (3)

$$\begin{array}{ccc} + & - & + \\ 2 & 2\sqrt{2} & 0 \\ 0 & 2 & 2 \end{array}$$

$$\frac{x}{4\sqrt{2}} = \frac{-y}{4} = \frac{z}{4-0}$$

$$\frac{x}{\sqrt{2}} = \frac{-y}{1} = \frac{z}{1}$$

$x = \sqrt{2}, \quad y = -1, \quad z = 1$

$$v_3 = \begin{bmatrix} \sqrt{2} \\ -1 \\ 1 \end{bmatrix}$$

Normalized vector $v_3 = \begin{bmatrix} \sqrt{2}/2 \\ -1/2 \\ 1/2 \end{bmatrix}$

Normalized eigen vectors of A^TA are

$$\left(\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right) = \begin{bmatrix} \sqrt{2}/\sqrt{12} & 1/\sqrt{3} & \sqrt{2}/2 \\ 3/\sqrt{12} & 0/\sqrt{3} & -1/2 \\ 1/\sqrt{12} & -\sqrt{2}/\sqrt{3} & 1/2 \end{bmatrix} = P$$

$\therefore A^TA = P \cdot D \cdot P^T$

$$\begin{bmatrix} \sqrt{2}/\sqrt{12} & 1/\sqrt{3} & \sqrt{2}/2 \\ 3/\sqrt{12} & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/\sqrt{12} & 3/\sqrt{12} & 1/\sqrt{12} \\ 1/\sqrt{3} & 0 & -\sqrt{2}/\sqrt{3} \end{bmatrix}$$

The right singular vectors are the columns of P.

$$\therefore V = P = \begin{bmatrix} 2/\sqrt{12} & 1/\sqrt{3} & \sqrt{2}/2 \\ 3/\sqrt{12} & 0 & -1/2 \\ 1/\sqrt{12} & -\sqrt{2}/\sqrt{3} & 1/2 \end{bmatrix}$$

Step-2

The eigen values of A^TA are $\lambda_1 = 8,\ \lambda_2 = 2,\ \lambda_3 = 0$

We have $\sigma_i = \sqrt{\lambda_i}$

$\therefore$ The singular values of A are $\sigma_1 = \sqrt{8}$, $\sigma_2 = \sqrt{2}$, $\sigma_3 = \sqrt{0}$

$\therefore$ The singular matrix is given by

$$\Sigma = \begin{bmatrix} \sqrt{8} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step-3

To calculate left singular matrix then

$$\boxed{u_i = \frac{Av_i}{\sigma_i}}$$

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{8}} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}$$

$$= \frac{1}{\sqrt{8}} \cdot \frac{1}{\sqrt{12}} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{6}} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0+3+1 \\ 2+6+0 \\ 0+3+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \quad = \begin{matrix} 4/4\sqrt{6} \\ 8/4\sqrt{6} \end{matrix} \quad \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{bmatrix} 1/\sqrt{3} \\ 0 \\ -\sqrt{2}/\sqrt{3} \end{bmatrix}$

$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$

Since order of $v_{i_{3\times 3}}$ we need to find another vector which is orthogonal to v_1 and v_2

Let $u_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be orthogonal to u_1 & u_2

then $u_1 \cdot u_3 = 0 \Rightarrow \frac{x_1}{\sqrt{6}} + \frac{2x_2}{\sqrt{6}} + \frac{x_3}{\sqrt{6}} = 0$

$u_2 \cdot u_3 = 0 \Rightarrow -\frac{x_1}{\sqrt{3}} + \frac{x_2}{\sqrt{3}} + \frac{x_3}{\sqrt{3}} = 0$

Solving, we get

$\frac{x_1}{\frac{-2}{\sqrt{18}} - \frac{1}{\sqrt{18}}} = \frac{-x_2}{\frac{-1}{\sqrt{18}} + \frac{1}{\sqrt{18}}} = \frac{x_3}{\frac{1}{\sqrt{18}} + \frac{2}{\sqrt{18}}}$

$= \frac{x_1}{\frac{-3}{\sqrt{18}}} = \frac{-x_2}{0} = \frac{x_3}{\frac{3}{\sqrt{18}}}$

$x_1 = -1/\sqrt{2}, x_2 = 0$

$u_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

The matrix of left singular matrix is

$$U = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{3} & -1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

Hence $A = U \Sigma V^T$

3. find the singular values of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Given ; $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

The C.E of A^TA is $|A^TA - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = 0$

$= (5-\lambda)^2 - 16 = 0$

$25 + \lambda^2 - 10\lambda - 16 = 0$

$\lambda^2 - 10\lambda + 9 = 0$ is eqn of A^TA

$\lambda^2 - 9\lambda - \lambda + 9 = 0$

$\lambda(\lambda - 9) - 1(\lambda - 9) = 0$

$\lambda = 9, 1 \Rightarrow \lambda_1 = 9, \lambda_2 = 1$

$\sigma_1 = \sqrt{9}, \Rightarrow 3, \sigma_2 = \sqrt{1} = 1$ are the singular values

$\therefore \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

4. Find ...

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times 2}$$

Sol :-

Given $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

then

$A \cdot A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$A \cdot A^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

The C.E of $A \cdot A^T$ is $|AA^T - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)\left[(1-\lambda)(2-\lambda) - 1\right] - 1\left((2-\lambda) - 0\right) + 0(\;) = 0$

$\Rightarrow (2-\lambda)(2-\lambda-2\lambda+\lambda^2-1] - 1\cdot(2-\lambda) = 0$

$\Rightarrow (2-\lambda)(\lambda^2 - 3\lambda + 1) - (2+\lambda) = 0$

$\Rightarrow (2-\lambda)(\lambda^2 - 3\lambda + \cancel{1}) - \cancel{1} = 0$

$\Rightarrow (2-\lambda)(\lambda^2 - 3\lambda) = 0$

$2\lambda^2 - 6\lambda - \lambda^3 + 4\lambda$

$\Rightarrow \lambda(2-\lambda)(\lambda^2 - 3) = 0$

eigen vector

consider $[AA^T - \lambda I]x = 0$

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\lambda = 3$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$-x + y = 0 \rightarrow (1)$

$x - 2y + z = 0 \rightarrow (2)$

$y - z = 0 \rightarrow (3)$

$$\begin{matrix} + & - & + \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{matrix}$$

Solve (1) & (3)

$$\frac{x}{-1} = \frac{-y}{1} = \frac{z}{-1}$$

$x = 1,\ y = 1,\ z = 1$

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Normalized vector $u_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

Where $\lambda = 2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y = 0 \to (1)$

$x + y + z = 0 \to (2)$

$y = 0 \to (3)$ sub in (2)

$x + z = 0$

$x = -z$

$\frac{x}{1} = \frac{z}{-1}$

$x = 1, \ y = 0, \ z = -1$

$u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ Normalized vector $u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$

Where $\lambda = 0$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{ccc} + & - & + \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{array}$$

$2x + y = 0 \to (1)$

$x + y + z = 0 \to (2)$

$y + 2z = 0 \to (3)$

solve (1) & (3)

$\frac{x}{2} = \frac{-y}{4} = \frac{z}{2}$

$\frac{x}{1} = \frac{y}{-2} = \frac{z}{1}$

$x = 1, \ y = -2, \ z = 1$

$u_3 = \begin{bmatrix} 1 \\ -2 \\ \end{bmatrix}$ Normalised vector $u_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ \end{bmatrix}$

∴ The matrix of left singular vector

$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

Step 2 :-

The eigen values of AA^T are $\lambda_1 = 3$, $\lambda_2 = 2$, λ_3

We have $\sigma_i = \sqrt{\lambda_i}$

∴ The singular values of A are $\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{2}$

$\sigma_3 = 0$

The singular matrix is given by

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

Step 3 :-

To calculate right singular matrix then we have that

$$\boxed{v_i = \frac{A^T u_i}{\sigma_i}}$$

$$v_1 = \frac{A^T u_1}{\sigma_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$1 - 0 - 1$ $\quad 1 + 1 + 1$

$0 \quad 3$

$$= \frac{1}{3} \begin{bmatrix} 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$v_2 = \frac{A^T u_2}{\sigma_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \cdot & \cdot \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \cdot \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$$

Hence $\underset{3\times 2}{A} = \underset{3\times 3}{U} \cdot \underset{3\times 2}{\Sigma} \cdot V^T_{2\times 2}$

2022

5. Obtain the singular value decomposition of A where

$$A = \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}_{2\times 2}$$

Sol :- Given $A = \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}$

$$A^T = \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 \\ 5 & 29 \end{bmatrix}$$

The characteristic eq of $|A^T A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 5 \\ 5 & 29-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(29-\lambda) - 25 = 0$$

$$29 - \lambda - 29\lambda + \lambda^2 - 25 = 0$$

$\lambda_1 = 29.86$

$\lambda_2 = 0.13$ are the eigen values of $A^T A$

eigen vector

consider $[A^T A - \lambda I]\hat{x} = 0$

$$\begin{bmatrix} 1-\lambda & 5 \\ 5 & 29-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 29.86$

$$\begin{bmatrix} 1-29.86 & 5 \\ 5 & 29-29.86 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -28.86 & 5 \\ 5 & -0.86 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-28.86x + 5y = 0 \rightarrow (1)$

$5x - 0.86y = 0 \rightarrow (2)$

$$\begin{array}{cc} + & - \\ -28.86 & 5 \\ 5 & -0.86 \end{array}$$

$\frac{\cancel{x}}{\cancel{24}.82} = \cancel{-4y}$

$5x = 0.86y$

$\frac{x}{0.86} = \frac{\cancel{0.86}y}{5}$ & $\frac{y}{\cancel{0.86}\,5}$

$$v_1 = \begin{bmatrix} 0.86 \\ \frac{\cancel{0.86}}{5} \end{bmatrix}$$

$= 5.073$

$$Norm = \begin{bmatrix} \frac{0.86}{5.073} \end{bmatrix} = \begin{bmatrix} 0.17 \\ 0.98 \end{bmatrix}$$

$\lambda = 0.13$

$$\begin{bmatrix} 1-0.13 & 5 \\ 5 & 29-0.13 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.87 & 5 \\ 5 & 28.87 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0.87x + 5y = 0 \rightarrow (1)$

$5x + 28.87y = 0 \rightarrow (2)$

$0.87x = -5y$

$$\frac{x}{-5} = \frac{y}{0.87}$$

$$V_2 = \begin{bmatrix} -5 \\ 0.87 \end{bmatrix}$$

$= \cancel{2}5.075$

$$\text{Norm} = \begin{bmatrix} \frac{-5}{5.075} \\ \frac{0.87}{5.075} \end{bmatrix} = \begin{bmatrix} -0.98 \\ 0.17 \end{bmatrix}$$

right singular vectors are the columns of P

$$V = P = \begin{bmatrix} 0.17 & -0.98 \\ 0.98 & 0.17 \end{bmatrix}$$

Step-2

The eigen values of $A^T A$ are : $\lambda_1 = 29.86$, $\lambda_2 = 0.13$

We have $\sigma_i = \sqrt{\lambda_i}$

The singular values of A are $\sigma_1 = \sqrt{29.86}$, $\sigma_2 = \sqrt{0.13}$

$$\Sigma = \begin{bmatrix} \sqrt{29.86} & 0 \\ 0 & \sqrt{0.13} \end{bmatrix}$$

Step-3

To calculate left singular matrix then

$$u_1 = \frac{AV_1}{\sigma_1} = \frac{1}{\sqrt{29.86}} \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}_{2\times2} \begin{bmatrix} 0.17 \\ 0.98 \end{bmatrix}_{2\times1}$$

$$= \frac{1}{\sqrt{29.86}} \begin{bmatrix} 0 & -1.96 \\ 0.17 & 4.9 \end{bmatrix} \quad \frac{1}{\sqrt{29.86}} \begin{bmatrix} -1.96 \\ 5.07 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 0 & -0.35 \\ 0.03 & 0.89 \end{bmatrix} = \begin{bmatrix} -0.3586 \\ 0.9278 \end{bmatrix}$$

$$u_2 = \frac{AV_2}{\sigma_2} = \frac{1}{\sqrt{0.13}} \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -0.98 \\ 0.17 \end{bmatrix}$$

$$= \frac{1}{\sqrt{0.13}} \begin{bmatrix} 0 & -0.34 \\ -0.98 & 0.85 \end{bmatrix} \quad \frac{1}{\sqrt{0.13}} \begin{bmatrix} -0.34 \\ -0.13 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0 & -0.943 \\ -2.72 & 2.357 \end{bmatrix} = \begin{bmatrix} -0.9429 \\ -0.3605 \end{bmatrix}$$

Hence

$$\underset{2\times2}{A} = \underset{2\times2}{U}\,\underset{2\times2}{\Sigma}\,\underset{2\times2}{V^T} \qquad \begin{bmatrix} -0.3586 & -0.9429 \\ 0.9278 & -0.3605 \end{bmatrix}$$

6. find SVD of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}_{3\times2}$

a. Given; $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

$$A^{T} = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$AA^{T} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$\begin{matrix} 1+1 & -2-2 & 2+2 \\ -2-2 & 4+4 & -4-4 \\ 2+2 & -4-4 & 4+4 \end{matrix}$$

$$= \begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix}$$

The C.E of AA^{T} is $|AA^{T} - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -4 & 4 \\ -4 & 8-\lambda & -8 \\ 4 & -8 & 8-\lambda \end{vmatrix} = 0$$

$(2-\lambda)[(8-\lambda)^2 - 64] + 4(-4(8-\lambda)+32) + 4(32 - 4(8-\lambda)) = 0$

$(2-\lambda)(64+\lambda^2-16\lambda-64] + 4(-32+4\lambda+32) + 4(32-32+4\lambda) = 0$

$(2-\lambda)(64+\lambda^2-16\lambda-64) + 4(4\lambda) + 4(4\lambda) = 0$

$(2-\lambda)(\lambda^2-16\lambda) + 16\lambda + 16\lambda = 0$

$2\lambda^2 - \cancel{32\lambda} - \lambda^3 + 16\lambda^2 + \cancel{32\lambda} = 0$

$18\lambda^2 - \lambda^3 = 0$

$-\lambda^3 \times 1$

$\lambda^3 - 18\lambda^2 = 0$

$\lambda^2(\lambda - 18) = 0$

$\lambda^2 = 0, \ \lambda = 18$

$\lambda = 18, 0, 0$

<u>eigen vector</u> :-

Consider $[AA^{T} - \lambda I]x = 0$

$$\begin{bmatrix} 2-\lambda & -4 & 4 \\ -4 & 8-\lambda & -8 \\ 4 & -8 & 8-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $\lambda = 18$

$$\begin{bmatrix} 2-18 & -4 & 4 \\ -4 & 8-18 & -8 \\ 4 & -8 & 8-18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -16 & -4 & 4 \\ -4 & -10 & -8 \\ 4 & -8 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-16x - 4y + 4z = 0 \rightarrow$ (1)

$-4x - 10y - 8z = 0 \rightarrow$ (2)

$4x - 8y - 10z = 0 \rightarrow$ (3)

+	-	+
-16	-4	4
4	-8	-10

Solve (1) & (3)

$$\frac{x}{40+32} = \frac{-y}{160-16} = \frac{z}{128+16}$$

$$\frac{x}{72} = \frac{-y}{144} = \frac{z}{144}$$

$$\frac{x}{1} = \frac{-y}{2} = \frac{z}{2}$$

$x = 1, \; y = -2, \; z = 2$

$\sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$

$$u_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Normalized vector $u_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$

Here $\lambda = 0$

$$\begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x - 4y + 4z = 0 \rightarrow$ (1)

$-4x + 8y - 8z = 0 \rightarrow$ (2)

$4x - 8y + 8z = 0 \rightarrow$ (3)

solve (1) & (2)

$\frac{x}{32-32} = \frac{-y}{-16+16} = \frac{z}{16-16}$

+	-	T
2	-4	4
-4	8	-8

Gram-schmidt orthogonalization process - 2/10 M

Inner product :- The inner product of two vectors $\alpha = (a_1 - - - a_n)$ and $\beta = (b_1 - - - b_n)$ where $\alpha, \beta \in R^n$ is defined by $\langle \alpha, \beta \rangle = a_1b_1 + a_2b_2 + - - - + a_nb_n$

orthogonal vectors :- A vectors α and β are said to be orthogonal if $\langle \alpha, \beta \rangle = 0$

orthonormal set :- A set $S = \{\alpha_1, \alpha_2 - - - \alpha_n\} \in R^n$ is said to be an orthonormal set if

$\langle \alpha_i, \alpha_j \rangle = 0$, if $i \neq j$

$\langle \alpha_i, \alpha_j \rangle = 1$, if $i = j$

Gram - schmidt process :- 2M

Gram-schmidt process allows us to constructively transform any basis $(b_1, - - - b_n)$ of a n-dimensional vector space V into an orthogonal / orthonormal basis $(u_1 - - - u_n)$ of V

Steps :-

The Gram-schmidt orthogonalization method iteratively constrants an orthogonal basis $(v_1 - - - v_n)$ from any basis $(b_1, b_2 - - - b_n)$ of a vector space

This process contains the following steps :-

1. $u_1 = b_1$
2. $u_2 = b_2 - \frac{\langle b_2, u_1 \rangle}{\|u_1\|^2} u_1$
3. $u_3 = b_3 - \frac{\langle b_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle b_3, u_2 \rangle}{\|u_2\|^2} u_2$

and so on ... for all n-basis vectors $(b_1 - - b_n)$

if we normalize the v_k, we obtained an orthonormal ba
of V

Prblm

1. consider a basis (b_1, b_2) of R^2 where $b_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
Construct an orthonormal basis by using Grams-schmidt process.

a. Given

$b_1 = (2,0)$ and $b_2 = (1,1)$

Step-1 :- Let $u_1 = b_1 = (2,0)$

~~$\|u_1\|^2 = 2^2+0^2$~~

$\|u_1\|^2 = (4+0) = 4$

$\|u_1\| = 2$

Step-2 :- $u_2 = b_2 - \dfrac{\langle b_2, u_1 \rangle}{\|u_1\|^2} u_1$

Now $\langle b_2, u_1 \rangle = \langle (1,1)(2,0) \rangle$

$\langle b_2, u_1 \rangle = 2$

$= (1,1) - \dfrac{\cancel{2}}{\cancel{4}\,2}(2,0)$

$= (1,1) - \dfrac{1}{2}(2,0)$

$= (1,1) - (1,0)$

$= (1-1, 1-0)$

$u_2 = (0,1)$ $\|u_2\| = \sqrt{0+1} = 1$

∴ The orthogonal vectors are $u_1 (2,0)$ and $u_2 (0,1)$

The orthonormal vectors are

Hence the orthonormal basis of R^2 is $\{(1,0),(0,1)\}$

2. Given $\{u_1, u_2, u_3\}$ where $u_1 = (1,2,1)$, $u_2 = (1,1,3)$, $u_3 = (2,1,1)$ use the Gram-schmidt procedure to obtain an ortho-normal basis. Manu

Sol :- Given

$u_1 = (1,2,1)$, $u_2 = (1,1,3)$, $u_3 = (2,1,1)$

Step 1 :- Let $v_1 = u_1 = (1,2,1)$

$$\|v_1\|^2 = (1+4+1) = 6$$

$$\|v_1\| = \sqrt{6}$$

Step 2 :- $v_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

$\langle u_2, v_1 \rangle$

$\langle (1,1,3),(1,2,1) \rangle$

$\langle 1+2+3 \rangle \Rightarrow \langle 6 \rangle = 6$

$$v_2 = (1,1,3) - \frac{6}{6}(1,2,1)$$

$$= (1,1,3) - (1,2,1)$$

$$v_2 = (0,-1,2) \Rightarrow \|v_2\| = \sqrt{5}$$

Step 3 :- $v_3 = u_3 - \dfrac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \dfrac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

$\langle u_3, v_1 \rangle = (2,1,1)(1,2,1)$

$= (2+2+1) = 5$

$$v_3 = (2,1,1) - \frac{5}{6}(1,2,1) - \frac{1}{5}(0,-1,2)$$

$$= (2,1,1) - \left(\frac{5}{6}, \frac{10}{6}, \frac{1}{6}\right) - \left(\frac{0}{5}, -\frac{1}{5}, \frac{2}{5}\right)$$

$$= \left(2 - \frac{5}{6} - 0,\ 1 - \frac{10}{6} + \frac{1}{5},\ 1 - \frac{1}{6} - \frac{2}{5}\right)$$

$$v_3 = \left(7/6, -7/15, -\frac{7}{30}\right)$$

$$\|v_3\|^2 = \left(\left(\frac{7}{6}\right)^2 + \left(\frac{7}{15}\right)^2 + \left(\frac{7}{30}\right)^2\right)$$

$$= \frac{49}{36} + \frac{49}{225} + \frac{49}{900}$$

$$= \frac{49}{30}$$

$$\|v_3\| = \frac{7}{\sqrt{30}}$$

$\therefore$ The orthonormal vectors are $\{v_1, v_2, v_3\}$

$$= (1,2,1)(0,-1,2)(7/6, -7/15, -7/30)$$

$$= \left(1 + 0 - \frac{7}{6},\ 2 + 1 + 7/15,\ 1 - 2 + \frac{7}{30}\right)$$

The orthonormal vectors are given by

$$\frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \quad \frac{v_2}{\|v_2\|} = \left(0, \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\frac{v_3}{\|v_3\|} = \left(\frac{\cancel{7}}{6} \times \frac{\sqrt{30}}{\cancel{7}}, \frac{-7}{15} \times \frac{\sqrt{30}}{7}, \frac{-7}{30} \times \frac{\sqrt{30}}{7}\right)$$

Hence the orthonormal basis of B is

$$\left\{ \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{\sqrt{30}}{6}, -\frac{\sqrt{30}}{15}, -\frac{\sqrt{30}}{30}\right) \right\}$$

3. using Gram-schmidt process find an orthonormal basis for $S = \{(1,0,1), (1,1,1), (-1,1,0)\}$.

Sol :- Let

$b_1 = (1,0,1)$ $b_2 = (1,1,1)$, $b_3 = (-1,1,0)$

Step 1 :- Let $u_1 = b_1 \Rightarrow (1,0,1)$

$\|u_1\| = \sqrt{1+1} = \sqrt{2}$ $\|u_1\|^2 = 2$

Step 2 :- $u_2 = b_2 - \frac{\langle b_2, u_1 \rangle}{\|u_1\|^2} u_1$

$\langle b_2, u_1 \rangle \Rightarrow (1,1,1)(1,0,1)$
$= (1+0+1) = 2$

$u_2 = (1,1,1) - \frac{2}{2}(1,0,1)$

$= (1,1,1) - (1,0,1)$

$= (1-1, 1-0, 1-1)$

$= (0,1,0) \Rightarrow \|u_2\|^2 = 1 \Rightarrow \|u_2\| = \sqrt{1} = 1$

Step 3 :- $u_3 = b_3 - \frac{\langle b_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle b_3, u_2 \rangle}{\|u_2\|^2} u_2$

$\langle b_3, u_1 \rangle = (-1,1,0)(1,0,1)$
$= (-1+0+0) = -1.$

$\langle b_3, u_2 \rangle = (-1,1,0)(0,1,0)$

$u_3 = (-1,1,0) + \frac{1}{2}(1,0,1) - \frac{1}{1}(0,1,0)$

$= (-1,1,0) + \left(\frac{1}{2}, 0, \frac{1}{2}\right) - (0,1,0)$

$= \left(-1 + \frac{1}{2} - 0, 1 - 0 + 1, 0 + \frac{1}{2} - 0\right)$

$= \left(-1/2, 0, +1/2\right)$

$\|u_3\| = \frac{1}{\sqrt{16}}$

The orthogonal vectors are $\left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}\right\}$

$\left\{\left(-1/\sqrt{2}, 0, 1/\sqrt{2}\right), (0,1,0), \left(-\frac{1}{2} \times \sqrt{16}, 0, \frac{1}{2} \times \sqrt{16}\right)\right\}$

$\left\{\left(-1/\sqrt{2}, 0, 1/\sqrt{2}\right), (0,1,0), (-2, 0, 2)\right\}$

1. Given $S = \{(2,1,3)\ (1,2,3), (1,1,1)\}$ is a basis R^3. construct an orthonormal basis.

Sol Let; $b_1 = (2,1,3)$, $b_2 = (1,2,3)$, $b_3 = (1,1,1)$

Step-1 :- Let $u_1 = b_1 = (2,1,3)$

$\|u_1\|^2 = \sqrt{(4+1+9)} = 14$

$\|u_1\| = \sqrt{14} =$

Step-2 :- $u_2 = b_2 - \frac{\langle b_2, u_1 \rangle}{\quad} u_1$

$(b_2, u_1) = (1,2,3)(2,1,3)$

$= (2+2+9)$

$\langle b_2, u_1 \rangle = 13$

$u_2 = (1,2,3) - \frac{13}{14}(2,1,3)$

$= (1,2,3) - \left(\frac{26}{14}, \frac{13}{14}, \frac{39}{14}\right)$

$= \left(1 - \frac{26}{14}, 2 - \frac{13}{14}, 3 - \frac{39}{14}\right)$

$u_2 = \left(-\frac{6}{7}, \frac{15}{14}, \frac{3}{14}\right)$

$\|u_2\|^2 = \left(\left(-\frac{6}{7}\right)^2 + \left(\frac{15}{14}\right)^2 + \left(\frac{3}{14}\right)^2\right)$

$= \left(\frac{36}{49} + \frac{225}{196} + \frac{9}{196}\right)$

$= \frac{27}{14}$

$\|u_2\| = \sqrt{\frac{27}{14}} = \frac{3\sqrt{3}}{\sqrt{14}}$

or Step 3 :- $u_3 = b_3 - \frac{\langle b_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle b_3, u_2 \rangle}{\|u_2\|^2} u_2$

$\langle b_3, u_1 \rangle = \langle (1,1,1)(2,1,3) \rangle$

$= (2+1+3) = 6$

$\langle b_3, u_2 \rangle = (1,1,1)\left(-\frac{6}{7}, 15/14, 3/14\right)$

$= \left(-\frac{6}{7} + 15/14 + 3/14\right)$

$= \frac{3}{7}$

$$u_3 = (1,1,1) - \frac{6}{14}(2,1,3) - \frac{\frac{3}{\cancel{7}}}{\frac{27}{\cancel{14}2}}\left(-6/7, 15/14, 3/14\right)$$

$$= (1,1,1) - \frac{6}{14}(2,1,3) - \frac{\cancel{9}1}{\cancel{27}3}\left(-6/7, 15/14, 3/14\right)$$

$$= (1,1,1) - \left(\frac{12}{14}, \frac{6}{14}, \frac{18}{14}\right) - \left(\frac{-2}{7}, \frac{5}{14}, \frac{1}{14}\right)$$

$$= \left(1 - \frac{12}{14} + \frac{2}{7}, 1 - \frac{6}{14} - \frac{5}{14}, 1 - \frac{18}{14} - \frac{1}{14}\right)$$

$$u_3 = \left(3/7, 3/14, -5/14\right)$$

$$\|u_3\|^2 = \left((3/7)^2 + (3/14)^2 + (-5/14)^2\right)$$

$$= \left(\frac{9}{49} + \frac{9}{196} + \frac{25}{196}\right)$$

$$= 5/14$$

$$\|u_3\| = \sqrt{5/14}$$

The orthogonal vectors are $\left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}\right\}$

$$\left\{\left(2/\sqrt{14}, 1/\sqrt{14}, 3/\sqrt{14}\right), \left(\frac{-\cancel{6}^{2}}{7} \times \frac{\sqrt{14}}{\cancel{3}\sqrt{3}}, \frac{\cancel{15}^{5}}{14} \times \frac{\sqrt{14}}{\cancel{3}\sqrt{3}}, \frac{\cancel{3}}{14} \times \frac{\sqrt{14}}{\cancel{3}\sqrt{3}}\right)\right.$$

$$\left.\left(3/7 \times \frac{\sqrt{14}}{\sqrt{5}}, \frac{3}{14} \times \frac{\sqrt{14}}{\sqrt{5}}, \frac{-5}{14} \times \frac{\sqrt{14}}{\sqrt{5}}\right)\right\}$$

$$\left\{\left(2/\sqrt{14}, 1/\sqrt{14}, 3/\sqrt{14}\right), \left(\frac{-2\sqrt{14}}{7\sqrt{3}}, \frac{5\cancel{\sqrt{14}}}{\cancel{\sqrt{14}}\sqrt{14}\sqrt{3}}, \frac{\cancel{\sqrt{14}}}{\cancel{\sqrt{14}}\sqrt{14}\sqrt{3}}\right),\right.$$

$$\left.\left(\frac{3\sqrt{14}}{7\sqrt{5}}, \frac{3\cancel{\sqrt{14}}}{\cancel{\sqrt{14}}\sqrt{14}\sqrt{5}}, \frac{-5\cancel{\sqrt{14}}}{\cancel{\sqrt{14}}\sqrt{14}\sqrt{5}}\right)\right\}$$

$$\left\{\left(\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right), \left(\frac{-2\sqrt{14}}{7\sqrt{3}}, \frac{5}{\sqrt{42}}, \frac{1}{\sqrt{42}}\right)\left(\frac{3\sqrt{14}}{7\sqrt{5}}, \frac{3}{\sqrt{70}}, \frac{-5}{\sqrt{70}}\right.\right.$$

1. find an orthonormal basis for a [illegible]

where $u_1 = (2,2,1)$, $u_2 = (1,3,1)$, $u_3 = (1,2,2)$ using Gram-Schmidt process.

a. Given:- $u_1 = (2,2,1)$, $u_2 = (1,3,1)$, $u_3 = (1,2,2)$

Step-1 :- Let $v_1 = u_1 \Rightarrow (2,2,1)$

$$\|v_1\|^2 = (4+4+1) = 9$$

$$\|v_1\| = 3$$

Step 2 :- $v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$

$$\langle u_2, v_1\rangle = (1,3,1)(2,2,1)$$

$$= (2+6+1) = 9$$

$$v_2 = (1,3,1) - \frac{9}{9}(2,2,1)$$

$$= (1,3,1) - (2,2,1)$$

$$= (1-2, 3-2, 1-1)$$

$$v_2 = (-1,1,0) \qquad \|v_2\|^2 = 1+1 = 2 \qquad \|v_2\| = \sqrt{2}$$

Step 3 = $v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$

$$\langle u_3, v_1\rangle = (1,2,2)(2,2,1)$$

$$= (2+4+2) = 8$$

$$\langle u_3, v_2\rangle = (1,2,2)(-1,1,0)$$

$$= -1+2+0 = 1$$

$$v_3 = (1,2,2) - \frac{8}{9}(2,2,1) - \frac{1}{2}(-1,1,0)$$

$$v_3 = (1,2,2) - \left(\frac{16}{9}, \frac{16}{9}, \frac{8}{9}\right) + \frac{1}{2} - \frac{1}{2} - 0$$

$$= \cancel{(1,2,2)}$$

$$= \left(\left(1 - \frac{16}{9} + \frac{1}{2}\right), \left(2 - \frac{16}{9} - \frac{1}{2}\right), \left(2 - \frac{8}{9} - 0\right)\right)$$

$$= (-5/18, -5/18, 10/9)$$

~~The~~

$$\|v_3\|^2 = \left(\left(\frac{-5}{18}\right)^2 + \left(\frac{-5}{18}\right)^2 + \left(\frac{10}{9}\right)^2\right)$$

$$= \left(\frac{25}{324} + \frac{25}{324} + \frac{100}{81}\right)$$

$$= \frac{25}{18}$$

$$\|v_3\| = \frac{5}{3\sqrt{2}}$$

The orthonormal vectors $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$

$$\left\{ (2/3, 2/3, 1/3), (-1/2, 1/2, 0), \left(\frac{-5}{18} \times \frac{3\sqrt{2}}{5}, \frac{-5}{18} \times \frac{3\sqrt{2}}{5}, \frac{10}{9} \times \frac{3\sqrt{2}}{5}\right) \right\}$$

$$\left\{ (2/3, 2/3, 1/3), (-1/2, 1/2, 0), \left(\frac{-\sqrt{2}}{6}, \frac{-\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right) \right\}$$

$$\left\{ (2/3, 2/3, 1/3), (-1/2, 1/2, 0), \left(-\frac{\sqrt{2}}{6}, -\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right) \right\}$$

Schmidt process to construct an orthonormal basis.

Sol Let;

$u_1 = (1, 2, 0)$, $u_2 = (8, 1, -6)$, $u_3 = (0, 0, 1)$

Step :- 1) Let $v_1 = (1, 2, 0) = u_1$

$\|v_1\|^2 = (1+4+0) = 5$

$\|v_1\| = \sqrt{5}$

Step :- 2

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$$

$\langle u_2, v_1\rangle = (8, 1, -6)(1, 2, 0)$

$= (8 + 2 - 0)$

$= 10$

$v_2 = (8, 1, -6) - \frac{\cancel{10}^{2}}{\cancel{5}}(1, 2, 0)$

$= (8, 1, -6) - (2, 4, 0)$

$= (8-2, 1-4, -6-0)$

$= (6, -3, -6)$ $= \|v_2\|^2 = 36 + 9 + 36 = 81$ $\|v_2\| = 9$

Step 3 :- $v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$

$\langle u_3, v_1\rangle = (0, 0, 1)(1, 2, 0)$

$= 0 + 0 + 0 = 0$

$\langle u_3, v_2\rangle = (0, 0, 1)(6, -3, -6)$

$$v_3 = (0,0,1) - 0 + \frac{6}{81}(6,-3,-6)$$

$$= (0,0,1) + \left(\frac{36}{81}, \frac{-18}{81}, \frac{-36}{81}\right)$$

$$= \left(0 + \frac{36}{81}, 0 - \frac{18}{81}, 1 - \frac{36}{81}\right)$$

$$= \left(\frac{\cancel{36}^{4}}{\cancel{81}_{9}}, -\frac{\cancel{18}^{2}}{\cancel{81}_{9}}, 5/9\right)$$

$$= (4/9, -2/9, 5/9)$$

$$\|v_3\|^2 = \left(\left(\frac{4}{9}\right)^2 + \left(\frac{-2}{9}\right)^2 + \left(\frac{5}{9}\right)^2\right)$$

$$= \left(\frac{16}{81} + \frac{4}{81} + \frac{25}{81}\right)$$

$$= 5/9$$

$$\|v_3\| = \frac{\sqrt{5}}{3}$$

The orthonormal vectors $\left\{\left(\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\right)\right\}$

$$\left\{(1/\sqrt{5}, 2/\sqrt{5}, 0), \left(\frac{\cancel{6}^{2}}{\cancel{9}_{3}}, \frac{-\cancel{3}^{1}}{\cancel{9}_{3}}, \frac{-\cancel{6}^{2}}{\cancel{9}_{3}}\right), \left(\frac{4}{\cancel{9}_{3}} \times \frac{\cancel{3}}{\sqrt{5}}, \frac{-2}{\cancel{9}_{3}} \times \frac{\cancel{3}}{\sqrt{5}}, 5/\cancel{9}_{3} \times \frac{\cancel{3}}{\sqrt{5}}\right)\right\}$$

$$\left\{(1/\sqrt{5}, 2/\sqrt{5}, 0), (2/3, -1/3, -2/3), \left(\frac{4}{3\sqrt{5}}, \frac{-2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}\right)\right\}$$

The orthonormal basis are

2. $u_1 = (1,0,1)$, $u_2 = (0,1,1)$, $u_3 = (1,-1,3)$

Sol Given

$u_1 = (1,0,1)$, $u_2 = (0,1,1)$, $u_3 = (1,-1,3)$

Step :- 1

Let :- $v_1 = u_1 = (1,0,1)$

$||v_1||^2 = (1+1) = 2$

$||v_1|| = \sqrt{2}$

Step 2 :-

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{||v_1||^2} v_1$$

$\langle u_2, v_1 \rangle = (0,1,1)(1,0,1)$

$= (0+0+1) = 1$

$= (0,1,1) - \frac{1}{2}(1,0,1)$

$= (0,1,1) - (\frac{1}{2}, 0, \frac{1}{2})$

$= (0-\frac{1}{2}, 1-0, 1-\frac{1}{2})$

$= (-1/2, 1, 1/2)$

$||v_2||^2 = ((\frac{-1}{2})^2 + (1)^2 + (\frac{1}{2})^2)$

$= (\frac{1}{4} + 1 + \frac{1}{4})$

$= \frac{7}{4} = \frac{\sqrt{7}}{2} \Rightarrow ||v_2||$

Step :- 3 :-

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$\langle u_3, v_1 \rangle = (1, -1, 3)(1, 0, 1)$$

$$= (1 - 0 + 3) = 4$$

$$\langle u_3, v_2 \rangle = (1, -1, 3)(-1/2, 1, 1/2)$$

$$= (-1/2 - 1 + 3/2)$$

$$= 0$$

$$= (1, -1, 3) - \frac{\cancel{4}^{2}}{\cancel{2}}(1, 0, 1) - 0$$

$$= (1, -1, 3) - (2, 0, 2)$$

$$= (1 - 2, -1 - 0, 3 - 2)$$

$$v_3 = (-1, -1, 1)$$

$$\|v_3\|^2 = 1 + 1 + 1 = 3$$

$$\|v_3\| = \sqrt{3}$$

The orthonormal vectors are $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$

$$\left\{ (1/\sqrt{2}, 0, 1/\sqrt{2}), \left(-\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\sqrt{7}}, 1 \times \frac{2}{\sqrt{7}}, \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\sqrt{7}}\right), (-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}) \right\}$$

$$\left\{ (-1/\sqrt{2}, 0, 1/\sqrt{2}), (-1/\sqrt{7}, 2/\sqrt{7}, 1/\sqrt{7}), (-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}) \right\}$$

The orthonormal basis are

Q. $u_1 = (2,0,1)$, $u_2 = (3,-1,5)$, $u_3 = (0,4,2)$

Sol:- Given $u_1 = (2,0,1)$, $u_2 = (3,-1,5)$, $u_3 = (0,4,2)$

Step-1:- Let $v_1 = u_1 = (2,0,1)$

$\|v_1\|^2 = (4+1) = 5$

$\|v_1\| = \sqrt{5}$

Step:-2 :- $v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$

$\langle u_2, v_1\rangle = (3,-1,5)(2,0,1)$

$= (0+0+2)$, $= \cancel{2}$ $6+5=11$

$u_2 = (3,-1,5) - \frac{11}{5}(2,0,1)$

$= (3,-1,5) - \left(\frac{22}{5}, -0, \frac{11}{5}\right)$

$= \left(3 - 22/5, -1, 5 + \frac{11}{5}\right)$

$= (-4/5, 4, 8/5)$

$\|v_2\|^2 = \left((-4/5)^2 + (4)^2 + (8/5)^2\right)$

$= \left(\left(\frac{16}{25}\right) + 16 + \frac{64}{25}\right)$

$1/\sqrt{3}$

$=$