Unit 1 : Vector Analysis & vector calculus.

Introduction :-

Electromagnetics (EM) may be regarded as the study of interactions between electric charges at rest and in motion.

→ It is a branch of physics (or) electrical engineering in which electric and magnetic phenomena are studied.

Scalar's and vectors :-

Vector Analysis is a mathematical tool with electromagnetic concepts are mostly expressed and best comprehened.

Quantity :- quantity is a property that can exist as a multitude (or) magnitude, which illustrate as discontinuity and continuity.

Classification of Quantity :-

Tensors :- The quantaties do not have any specified direction but it have different directions with a different values.

If a physical have only one component then it is scalar, if it have two components then it is vector.

Scalars :- The quantities have only magnitude but not direction. It is called as scalars.

* It is Represented by captial letters.

* A scalar quantity can hold only one value at a time.

* Scalar particle in physics is a bosan sub atomic particle whose spin is equal to zero.

Eg :- A, B are used to represent.

Vectors :- The quantities have not only magnitude, but also direction. It is called as vectors.

* It is Represented by letters by arrow mark.

* A vector quantity can also hold only one value at a time.

Eg :- $\vec{a}, \vec{b}$, -- etc.

* The magnitude value of a vector is scalar and it will be represented as

$$|\vec{A}| = 1.$$

where | | is represent as magnitude.

* Types of vectors are unit vectors, position vectors, negative vectors, null vector, axial vectors, --- etc.

Unit vector :-

* A vector $\bar{A}$ has both magnitude and direction. the direction. The magnitude of A is a scalar written as A (or) $|\bar{A}|$.

* A unit vector $\bar{a}_A$ along A is defined as a vector and its magnitude is unity (i.e 1) and its direction is along A.

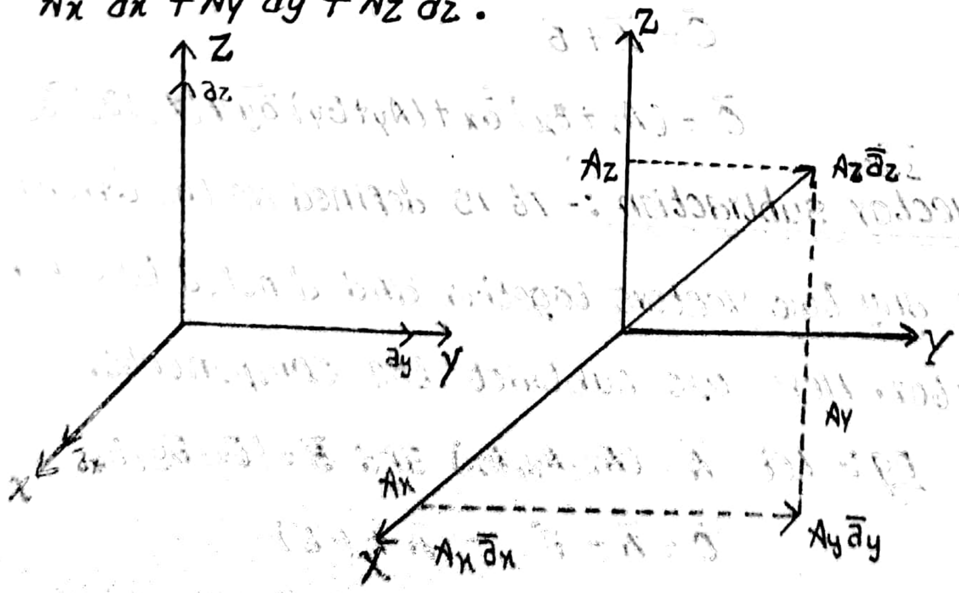
$$\boxed{\therefore \bar{a}_A = \frac{\bar{A}}{|\bar{A}|} = \frac{\bar{A}}{A}}$$

* Note that $|\bar{a}_A| = 1$, then we write $\bar{A}$ as $A\bar{a}_A$.

$$\boxed{\therefore \bar{A} = A\,\bar{a}_A}$$

which completely specifies $\bar{A}$ in terms of its magnitude A and its direction $\bar{a}_A$.

* A vector A in cartesian (or rectangular) coordinate may be represented as (A_x, A_y, A_z) (or) $A_x\,\bar{a}_x + A_y\,\bar{a}_y + A_z\,\bar{a}_z$.



→ where A_x, A_y, A_z are called components of A in the x, y and z directions respectively $\bar{a}_x, \bar{a}_y$ and $\bar{a}_z$ are unit vectors in x, y and z axes directions respectively.

→ The magnitude of vector $\bar{A}$ is given by A and $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ and the unit vector along $\bar{A}$ is given by

$$\bar{a}_A = \frac{A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}.$$

Vector operation :-

→ **Vector addition :-** It is defined as the sum of any two vectors together and produce another vector. Here the procedure is simple, i.e adding the components.

Eg :- Let $\bar{A} = (A_x, A_y, A_z)$ and $\bar{B} = (B_x, B_y, B_z)$

$$\bar{C} = \bar{A} + \bar{B}$$

$$\bar{C} = (A_x + B_x)\bar{a}_x + (A_y + B_y)\bar{a}_y + (A_z + B_z)\bar{a}_z.$$

→ **Vector subtraction :-** It is defined as the difference of any two vectors together and denoted by other vector. Here we subtract the components.

Eg :- Let $\bar{A} = (A_x, A_y, A_z)$ and $\bar{B} = (B_x, B_y, B_z)$

$$\bar{C} = \bar{A} - \bar{B} = \bar{A} + (-\bar{B})$$

$$= (A_x - B_x)\bar{a}_x + (A_y - B_y)\bar{a}_y + (A_z - B_z)\bar{a}_z.$$

Rules and Types to Represent vectors :-

* vector addition and vector subtraction can be represented By mainly two ways. They are

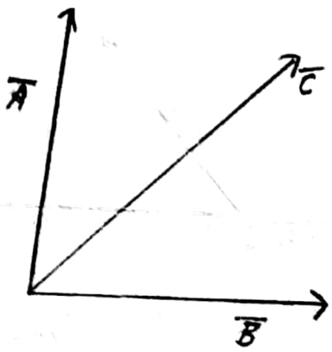
  a) parallelogram rule

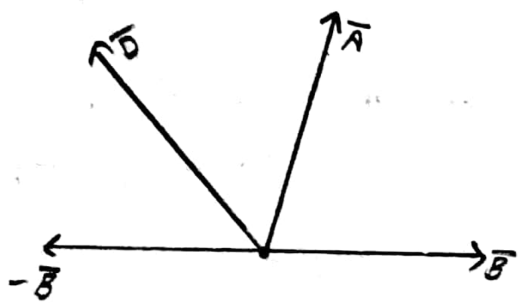
  b) head to tail rule

parallelogram Rule :-

If the vectors are represented by the sides of the parallelogram, then the operation of the vector was represented by the diagonal of parallelogram.

Eg :- $\bar{C} = \bar{A} + \bar{B}$ (ii) $\bar{D} = \bar{A} - \bar{B}$

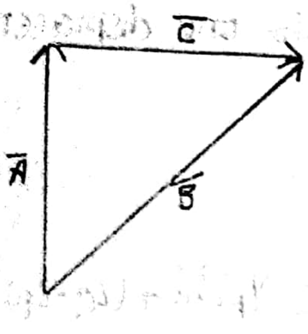


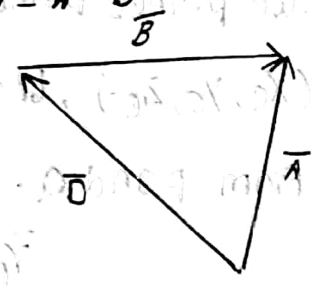


Head to tail Rule :-

If the head of the vector is connected to the tail of the another vector, then the operation of the vector also connected, then it is Head-tail Rule.

Eg :- (i) $\bar{C} = \bar{A} + \bar{B}$ (ii) $\bar{D} = \bar{A} - \bar{B}$





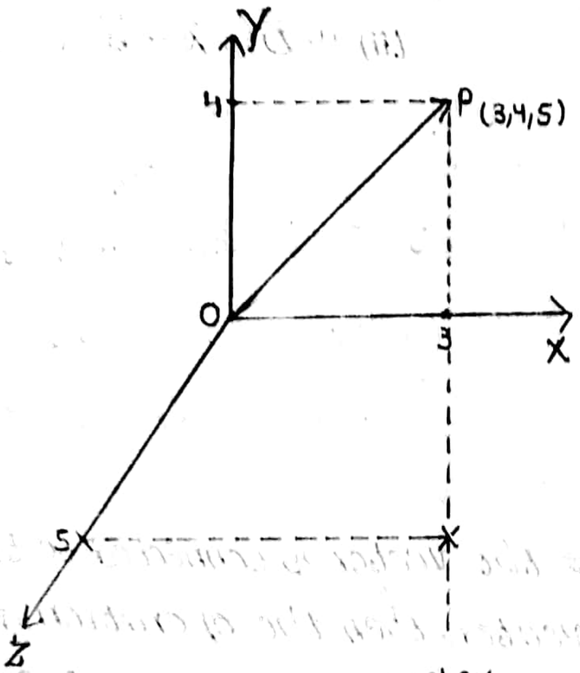
Position and Distance vectors :-

* A point p in cartesian coordinates may be represented as $p(x, y, z)$.

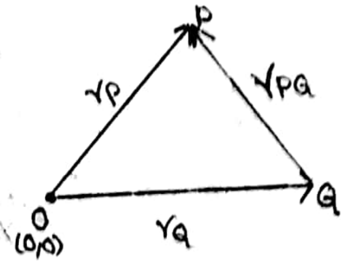
* The position vector (or) radius vector (r_p) of point p it is the direct distance from origin to p.

$$r_p = O_p = x\,a_x + y\,a_y + z\,a_z .$$

* The distance vector is the displacement from one point to another point.



Position vector



Distance vector

* Two points p and Q are given by (x_p, y_p, z_p) and (x_Q, y_Q, z_Q), the distance vector is the displacement from p and Q.

$$\overline{r_{PQ}} = \overline{r_Q} - \overline{r_P}$$

$$= (x_Q - x_P)\,\overline{a_x} + (y_Q - y_P)\,\overline{a_y} + (z_Q - z_P)\,\overline{a_z} .$$

→ Vector multiplication :-

* When two vectors are multiplied and its result may be scalar (or) vector.

* Depending on how they multiplied. They are divided as two types. They are :-

  (i) scalar (dot) product : $\bar{A} \cdot \bar{B}$

  (ii) vector (cross) product : $\bar{A} \times \bar{B}$.

* Multiplication of three vectors results as

  (i) Scalar triple product : $\bar{A} \cdot (\bar{B} \times \bar{C})$

  (ii) Vector triple product : $\bar{A} \times (\bar{B} \times \bar{C})$

1. DOT product :-

→ The dot product of a vector A and B is written as $A \cdot B$ is defined geometrically as product of their magnitudes of A and B and cosine of the angle between them.

Thus : $A \cdot B = AB \cos\theta_{(AB)}$

→ If $\bar{A} = (A_x, A_y, A_z)$ and $\bar{B} = (B_x, B_y, B_z)$

Then $\bar{A} \cdot \bar{B} = (A_x B_x)\bar{a} + (A_y B_y)\bar{b} + (A_z B_z)\bar{c}$

→ If $\bar{A} \cdot \bar{B} = 0$, only when $\bar{A} \perp \bar{B}$.

[This is orthagonal condition].

properties :-

(i) commutative law

$$\bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A}$$

(ii) Distributive law

$$\bar{A}\cdot(\bar{B}+\bar{C}) = \bar{A}\cdot\bar{B} + \bar{A}\cdot\bar{C}$$

→ if $\bar{A}\cdot\bar{A} = |A|^2 = A^2$.

(iii) $a_x \cdot a_y = a_y \cdot a_z = a_z \cdot a_x = 0$.

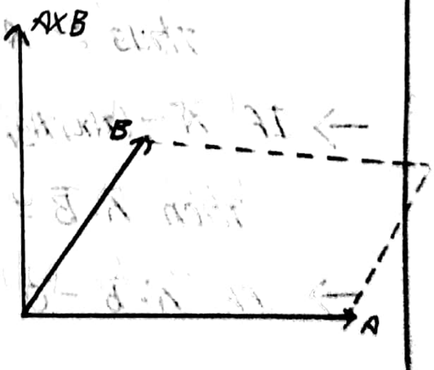
(iv) $a_x \cdot a_x = a_y \cdot a_y = a_z \cdot a_z = \underline{1}$.

Cross product :-

→ The cross product of two vectors $\bar{A}$ and $\bar{B}$ can be written as $\bar{A}\times\bar{B}$ and it is a vector quantity.

→ The magnitude is the area of parallelogram / parallelopiped formed by A and B.

* The cross product of A and B is the vector with the magnitude equal to Area of parallelogram.

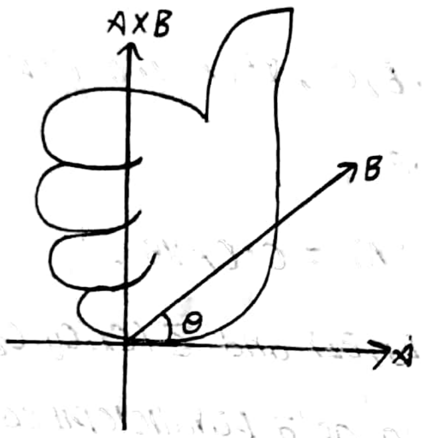


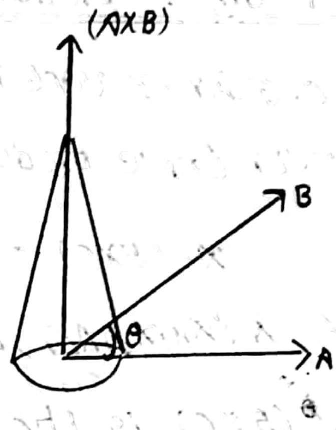
→ Thus $A \times B = AB \sin\theta_{AB}\, a_n$

→ Where a_n is a unit vector normal to the plane containing A and B.

→ The direction of $\bar{a}_n$ is taken as the direction of right hand thumb. when the fingers of the right hand rotate from A to B.

→ The direction of $\bar{a}_n$ is taken as that of the advance of right-hand screw as A turned into B.





→ If $\bar{A} = (A_x, A_y, A_z)$ and $\bar{B} = (B_x, B_y, B_z)$ then

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \bar{a}_x (A_y B_z - A_z B_y) + \bar{a}_y (A_x B_z - B_x A_z) + \bar{a}_z (A_x B_y - A_z B_x)$$

→ Here it is obtained by "crossing" terms in cyclic permutation. Hence it is known as cross product.

→ **Properties :-**

* $\bar{A} \times \bar{B} \neq \bar{B} \times \bar{A}$ → $\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$

* $\bar{A} \times (\bar{B} \times \bar{C}) \neq (\bar{A} \times \bar{B}) \times \bar{C}$ →

$\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C}$

* $\bar{A} \times \bar{A} = 0$.

* $\bar{a}_x \times \bar{a}_y = \bar{a}_z$; $\bar{a}_y \times \bar{a}_z = \bar{a}_x$; $\bar{a}_z \times \bar{a}_x = \bar{a}_y$

* $\bar{a}_x \times \bar{a}_x = \bar{a}_y \times \bar{a}_y = \bar{a}_z \times \bar{a}_z = 0$.

Scalar triple product :-

→ consider 3 vectors $\bar{A}, \bar{B}, \bar{C}$, then we define the scalar triple product as

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B).$$

→ If $\bar{A}(A_x, A_y, A_z)$, $\bar{B}(B_x, B_y, B_z)$ and $\bar{C}(C_x, C_y, C_z)$ then $A \cdot (B \times C)$ is the volume of a parallelopiped having A, B and C as edges and is easily obtained by finding the determinant as

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = \begin{vmatrix} \bar{A}_x & \bar{A}_y & \bar{A}_z \\ \bar{B}_x & \bar{B}_y & \bar{B}_z \\ \bar{C}_x & \bar{C}_y & \bar{C}_z \end{vmatrix}$$

$$= \bar{A}_x(\bar{B}_y \bar{C}_z - \bar{C}_y \bar{B}_z) + \bar{A}_y(\bar{B}_x \bar{C}_z - \bar{C}_x \bar{B}_z) + \bar{A}_z(\bar{B}_x \bar{C}_y - \bar{B}_y \bar{C}_z)$$

vector triple product :-

→ For the vectors $\bar{A}, \bar{B}$ and $\bar{C}$, the triple product is defined as

$$\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})$$

* It is obtained using "bac-cab" rule.

properties

→ It doesn't obey associative law

$$(\bar{A} \cdot \bar{B}) \cdot \bar{C} \neq (\bar{A})(\bar{B} \cdot \bar{C}).$$

→ It obeys commutative law

$$(\bar{A} \cdot \bar{B}) \cdot \bar{C} = \bar{C}(\bar{A} \cdot \bar{B}).$$

Components of a vector :-

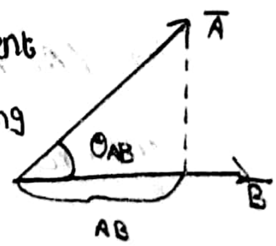
* A direct application of a vector product is its use in determining the projection (or) component of a vector in a given direction. The projection may be Scalar / vector.

* Given A vector A, the scalar component A_B of A along vector B is defined as

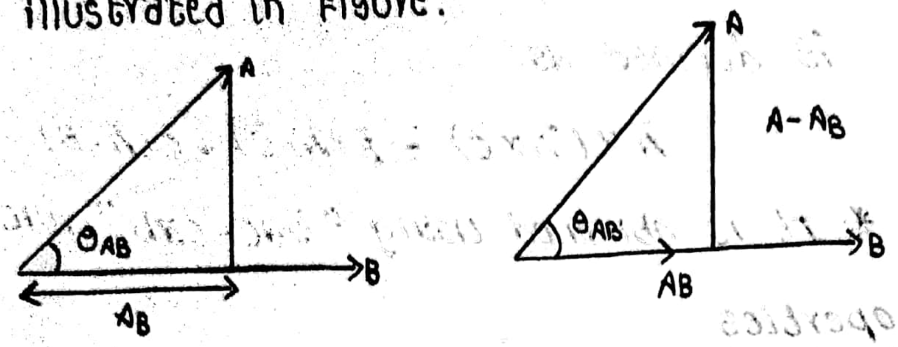
$$A_B = A \cos \theta_{AB} = |A||a_B| \cos \theta_{AB}$$

$$(\text{or}) \quad A_B = A * a_B$$

* The vector component of A_B of A along B is simply the scalar component of A_B multiplied by a unit vector along B, i.e $A_B = A_B \cdot a_B = (A \times a_B)$



→ Both the scalar and vector components of A are illustrated in figure.

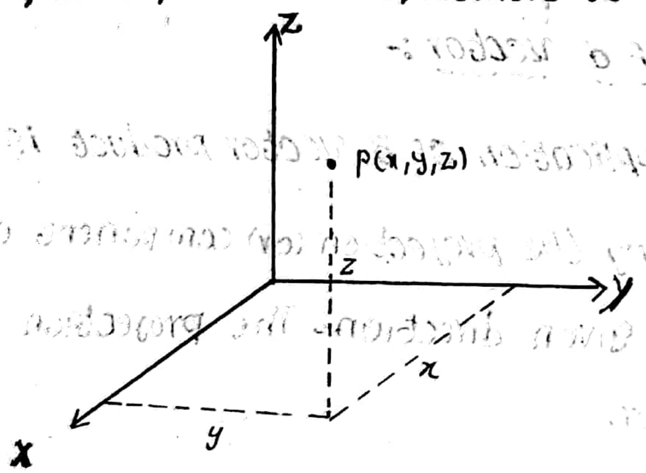


a) scalar component A_B b) vector component $\bar{A}_B$.

Coordinate systems and transformation :-

Cartesian / Rectangular coordinates (x,y,z) :-

→ A point p can be represented as (x,y,z) in below.



→ The ranges of coordinate variables x, y, z are

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty .$$

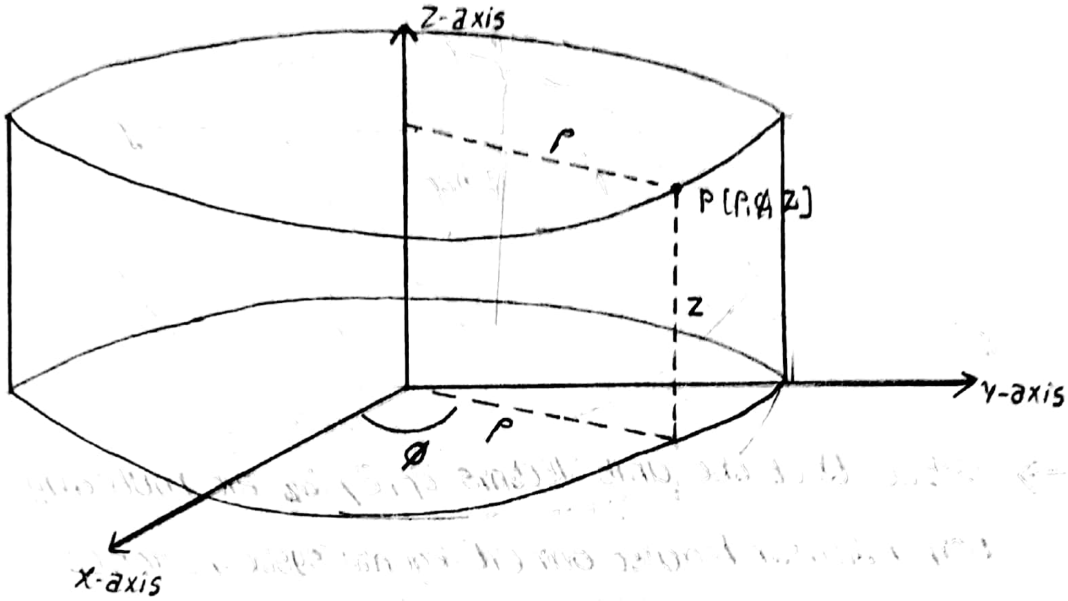
→ A vector A in cartesian coordinates can be written as (A_x, A_y, A_z) (or) $A_x a_x + A_y a_y + A_z a_z$.

where a_x, a_y and a_z are unit vectors along their axis.

Circular cylindrical coordinates (ρ, ϕ, z) :-

→ The circular cylindrical coordinate system is very convinient whenever we are dealing with problems having cylindrical symmetry.

→ A point p in cylindrical coordinates can be represented as (ρ, ϕ, z) in below diagram.



→ From above figure, we observe that point p in the cylindrical coordinate system and ρ is radius of cylinder and ϕ is azimutal angle measured from x axis in xy plane. z is same as in cartesian system.

→ The ranges of coordinate varibles ρ, ϕ, z are

$$0 \leq \rho < \infty$$

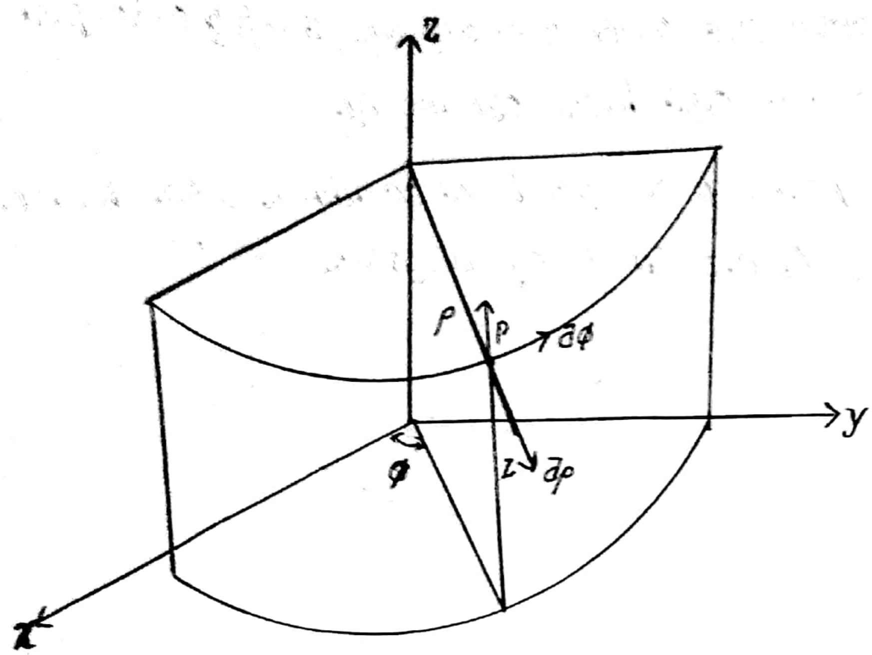
$$0 \leq \phi < 2\pi$$

$$-\infty < z < \infty$$

→ A vector A in cylindrical coordinates can be write as (A_ρ, A_ϕ, A_z) (or) $A_\rho a_\rho + A_\phi a_\phi + A_z a_z$. where a_ρ, a_ϕ, a_z are unit vectors along their axis.

→ The magnitude of A is

$$|A| = (A_\rho^2 + A_\phi^2 + A_z^2)^{1/2}$$



→ Notice that the unit vectors a_ρ, a_ϕ, a_z are mutually perpendicular beacuse our orthagonal system. Hence, a_ρ points in the direction of increasing ρ, a_ϕ in the direction of increasing ϕ, and a_z in the positive z-direction.

→ Thus

$$a_\rho \cdot a_\rho = a_\phi \cdot a_\phi = a_z \cdot a_z = 1$$

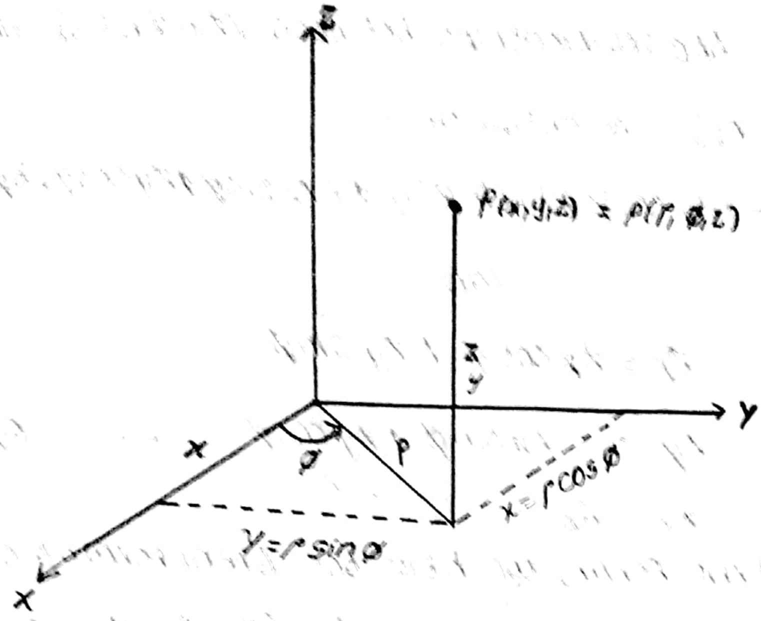
$$a_\rho \cdot a_\phi = a_\phi \cdot a_z = a_z \cdot a_\rho = 0$$

$$a_\rho \times a_\phi = a_z$$

$$a_\phi \times a_z = a_\rho$$

$$a_z \times a_\rho = a_\phi$$

→ The relationships between the variables (x, y, z) of the Cartesian coordinate system and those of cylindrical system (ρ, ϕ, z) are obtained from below figure.



From the above figure,

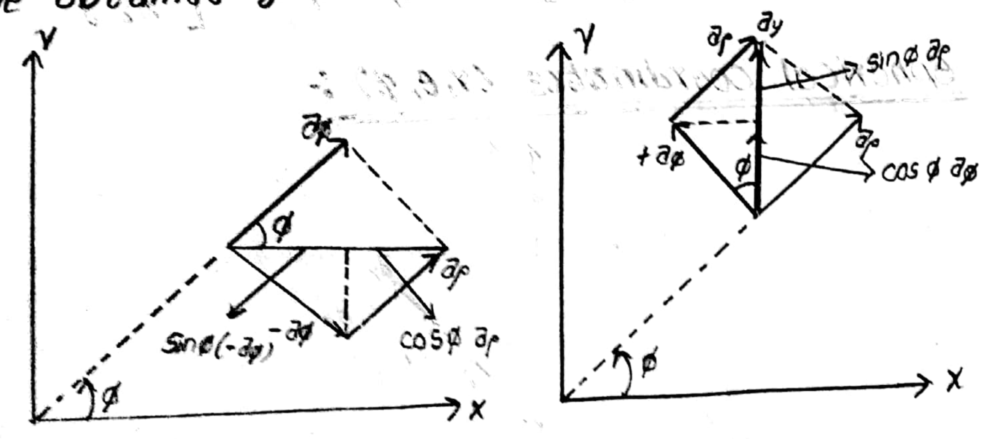
$$\rho = \sqrt{x^2+y^2}\ , \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)\ , \quad z = z$$

The above conversion is cartesian to cylindrical

$$x = \rho\cos\phi\ , \quad y = \rho\sin\phi\ , \quad z = z$$

The above conversion is cylindrical to cartesian

→ The relationships between (a_x, a_y, a_z) and (a_ρ, a_ϕ, a_z) are obtained geometrically from below figure.



(a) (b)

→

$$a_x = \cos\phi\, a_\rho - \sin\phi\, a_\phi$$
$$a_y = \sin\phi\, a_\rho + \cos\phi\, a_\phi \quad \text{——— (1)}$$
$$a_z = a_z$$

→ Finally, the relationships between (A_x, A_y, A_z) and (A_ρ, A_ϕ, A_z) are obtained as

$$\vec{A} = (A_x \cos\phi + A_y \sin\phi)\,\hat{a}_\rho + (-A_x \sin\phi + A_y \cos\phi)\,\hat{a}_\phi + A_z \hat{a}_z \quad \text{--- (3)}$$

(or)

$$A_\rho = A_x \cos\phi + A_y \sin\phi$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi \quad \text{--- (4)}$$

$$A_z = A_z$$

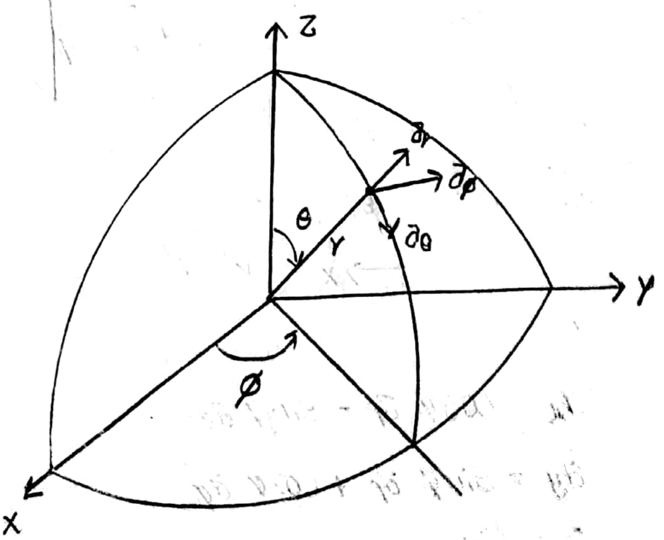
→ In matrix form, we have the transformation of vector $\vec{A}$ from (A_x, A_y, A_z) to (A_ρ, A_ϕ, A_z) as

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad \text{--- (5)}$$

→ For that inverse of transform we can make a vector from (A_ρ, A_ϕ, A_z) to (A_x, A_y, A_z) and that is

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \quad \text{--- (6)}$$

Spherical coordinates (r, θ, ϕ) :-



→ From above figure, r is defined as distance from origin to point P (or) the radius of sphere centered at the origin and passing through P. θ is the angle between z-axis and vector P and ϕ is angle measured from x-axis

→ The ranges of variables are

$$0 \leq r < \infty$$
$$0 \leq \theta \leq \pi$$
$$0 \leq \phi < 2\pi.$$

→ A vector A in spherical coordinates may be written as (A_r, A_θ, A_ϕ) (or) $A_r a_r + A_\theta a_\theta + A_\phi \cdot a_\phi$. where the terms a_θ, a_ϕ, a_r are unit vectors along r, θ, ϕ directions.

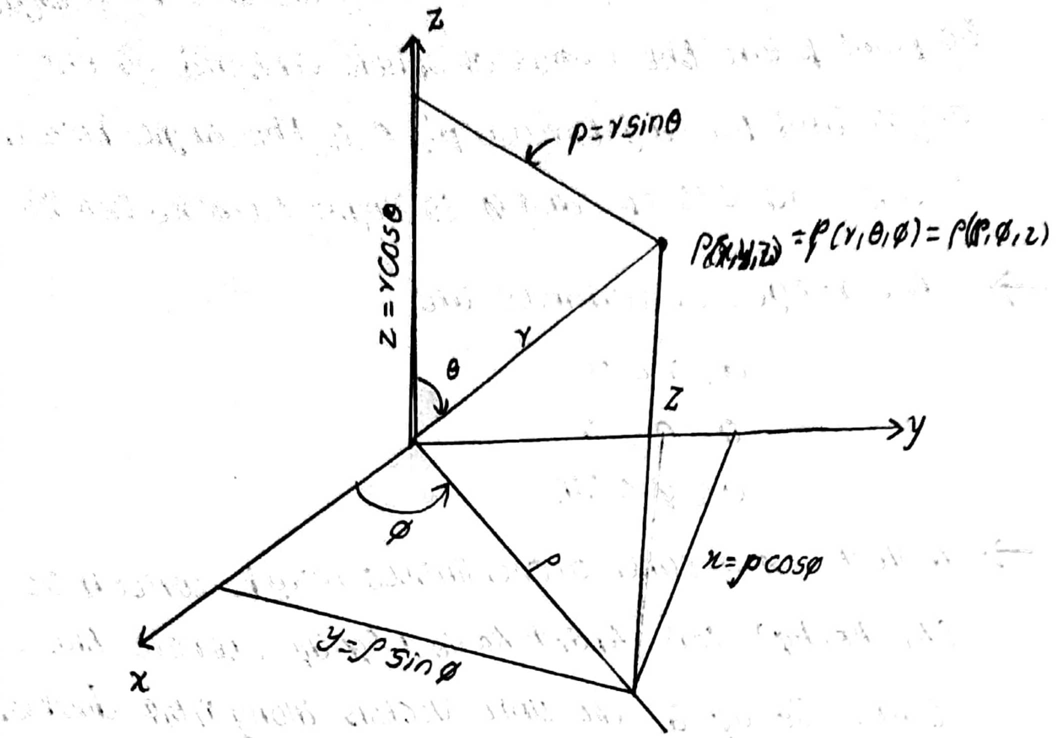
→ The magnitude of A is

$$|A| = (A_r^2 + A_\theta^2 + A_\phi^2)^{1/2}.$$

→ The unit vectors are a_r, a_θ and a_ϕ are mutually orthogonal, a_r is along radius and a_θ is along θ, a_ϕ is along ϕ.

→ Thus,

$$a_r \cdot a_r = a_\theta \cdot a_\theta = a_\phi \cdot a_\phi = 1$$
$$a_r \cdot a_\theta = a_\theta \cdot a_\phi = a_\phi \cdot a_r = 0$$
$$a_r \times a_\theta = a_\phi$$
$$a_\theta \times a_\phi = a_r$$
$$a_\phi \times a_r = a_\theta.$$



→ The space variables (x, y, z) in cartesian coordinates can be related to variables (r, θ, φ) of spherical coordinate.

$$r = \sqrt{x^2+y^2+z^2}\ ,\ \theta = \tan^{-1}\left[\frac{\sqrt{x^2+y^2}}{z}\right],\ \phi = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{—①}$$

$$x = r\sin\theta\cos\phi\ ,\ y = r\sin\theta\cdot\sin\phi\ ,\ z = r\cos\theta \quad \text{—②}$$

→ The unit vectors a_x, a_y, a_z and a_r, a_θ, a_ϕ related as

$$a_x = \sin\theta\cdot\cos\phi\, a_r + \cos\theta\cdot\cos\phi\, a_\theta - \sin\phi\, a_\phi$$

$$a_y = \sin\theta\cdot\sin\phi\, a_r + \cos\theta\cdot\sin\phi\, a_\theta + \cos\phi\, a_\phi \quad \text{—③}$$

$$a_z = \cos\theta\cdot a_r - \sin\theta\, a_\theta .$$

(or)

$$a_r = \sin\theta\cdot\cos\phi\, a_x + \sin\theta\cdot\sin\phi\, a_y + \cos\theta\, a_z$$

$$a_\theta = \cos\theta\cdot\cos\phi\, a_x + \cos\theta\cdot\sin\phi\cdot a_y + \sin\theta\, a_z \quad \text{—④}$$

$$a_\phi = -\sin\phi\, a_x + \cos\phi\, a_y$$

→ The components of vector $A = (A_x, A_y, A_z)$ and $A = (A_r, A_\theta, A_\phi)$ are related as

$$A = (A_x \sin\theta \cdot \cos\phi + A_y \sin\theta \cdot \sin\phi + A_z \cos\theta)\, a_r +$$
$$(A_x \cos\theta \cdot \cos\phi + A_y \sin\phi \cos\theta - A_z \sin\theta)\, a_\theta + (-A_x \sin\phi +$$
$$A_y \cos\phi)\, a_\phi \quad \text{—— (5)}$$

and from this,

$$A_r = A_x \sin\theta \cdot \cos\phi + A_y \sin\theta \cdot \sin\phi + A_z \cos\theta$$
$$A_\theta = A_x \cos\theta \cdot \cos\phi + A_y \cos\theta \cdot \sin\phi - A_z \sin\theta \quad \text{—— (6)}$$
$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

→ In matrix form :-

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta \cdot \cos\phi & \sin\theta \cdot \sin\phi & \cos\theta \\ -\cos\theta \cdot \cos\phi & \cos\theta \cdot \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad \text{(7)}$$

Then

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta \cdot \cos\phi & \cos\theta \cdot \cos\phi & -\sin\phi \\ \sin\theta \cdot \sin\phi & \cos\theta \cdot \sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \quad \text{(8)}$$

Constant-coordinate surfaces :-

Cartesian coordinate system :-

In the cartesian, if x is kept constant and allow y and z to vary, an infinite plane is generated. Thus 3 infinite planes are obtained as

x = constant , y = constant , z = constant

→ which are perpendicular to x, y and z axes as shown in figure below.

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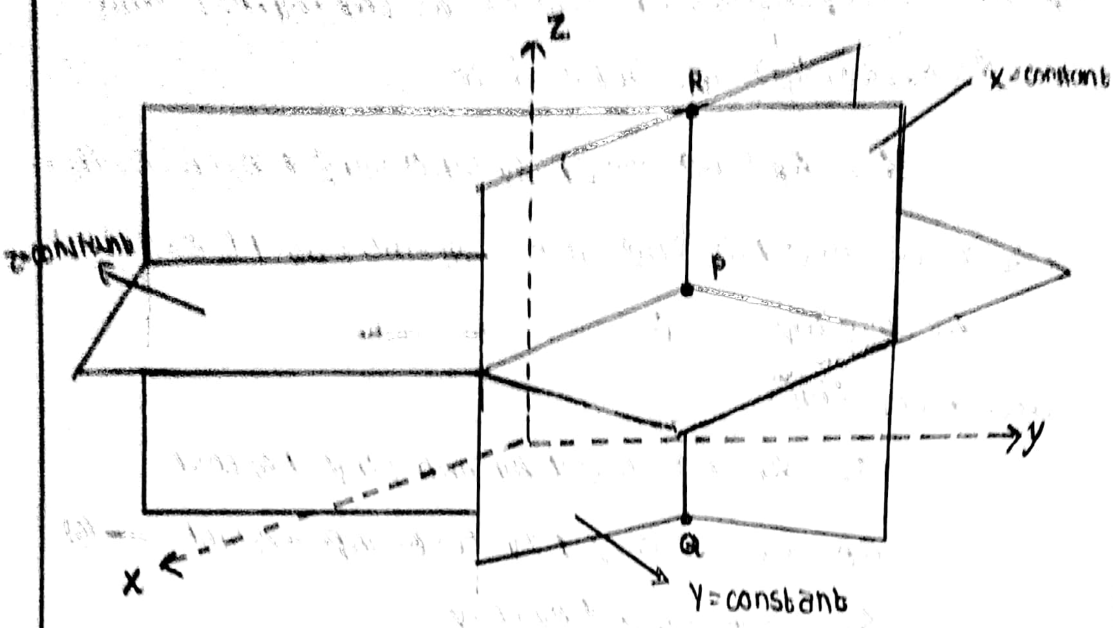
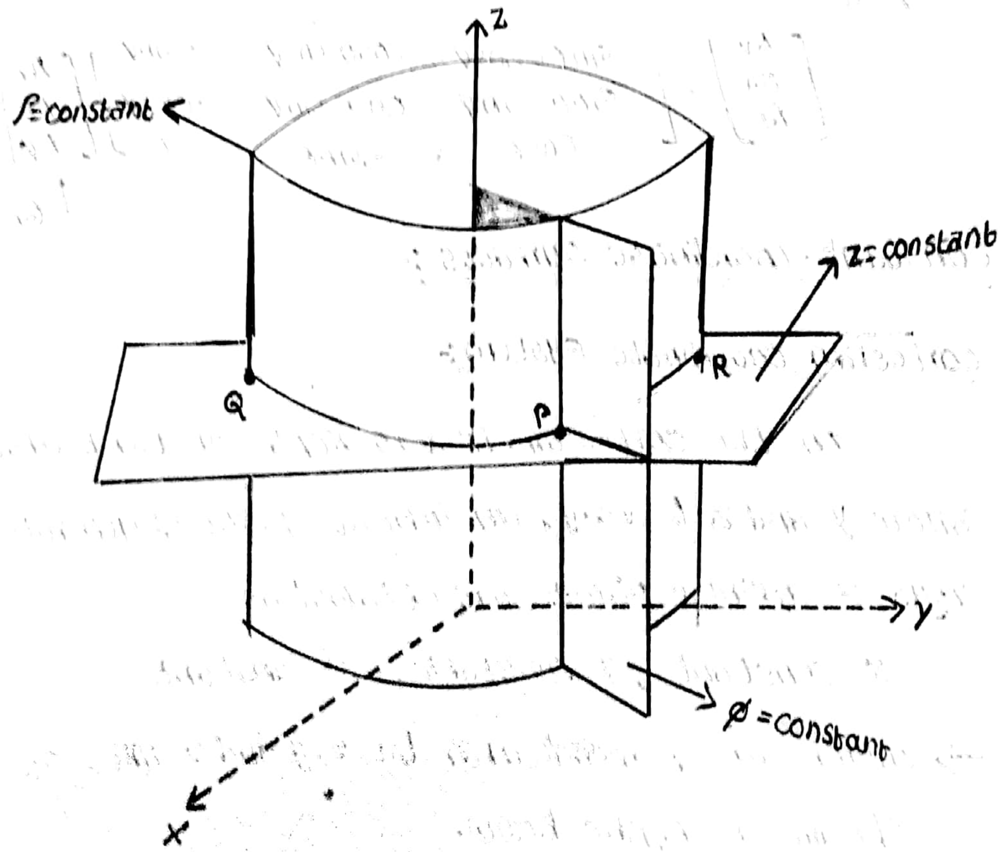


Cylindrical coordinate system :-

→ Orthagonal surfaces in cylindrical coordinates can likewise be generated. The surfaces

ρ = constants ; ϕ = constant ; z = constant.



→ From above Figure, we observe that ρ= constant and ϕ is constant and z= constant. Here ρ is a circular cylindrical plane, ϕ = plane with edge along z-axis, z is a plane as in cartesian system.

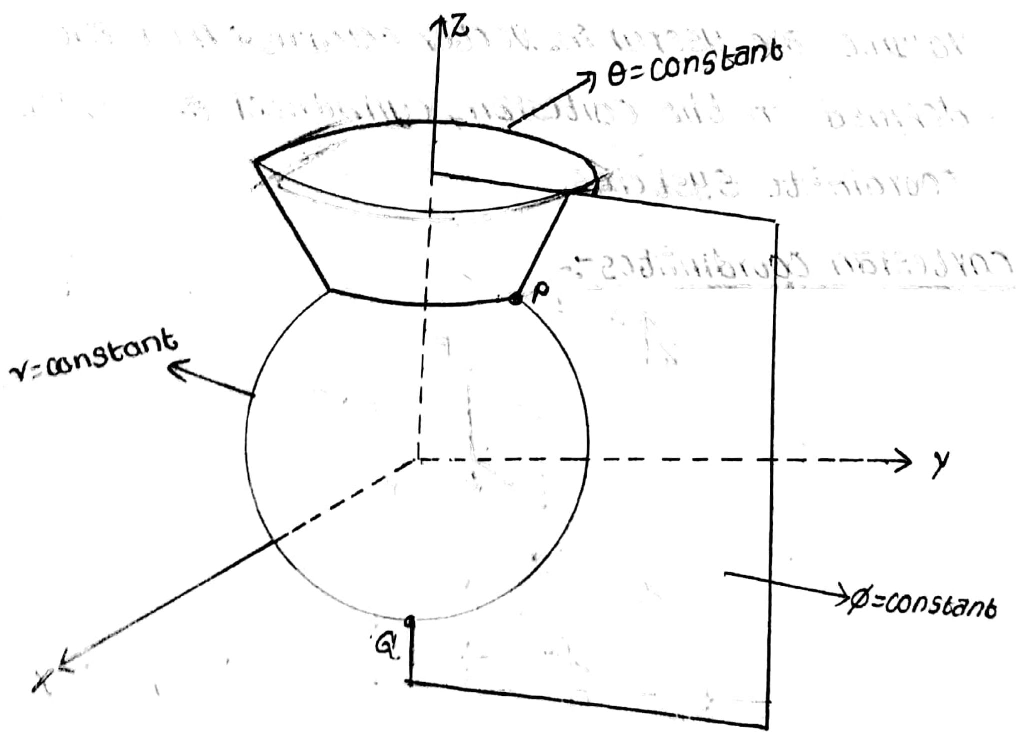
→ When two surfaces meet in either line (or) a circle. Thus z = constant ; ρ= constant.

→ If a circle QPR of radius ρ, whereas z= constant, ϕ= constant is a semi-infinite line. The intersection of three surfaces gives a point.

Spherical coordinate system :-

→ The orthogonal nature of the spherical coordinate system is evident by considering the three surfaces.

r= constant; θ = constant ; ϕ = constant.



→ From the above figure, we notice that the r = constant is a sphere with centre at origin.

θ = constant is a circular cone with z-axis.

ϕ = constant is a semi-infinite plane as cylindrical system.

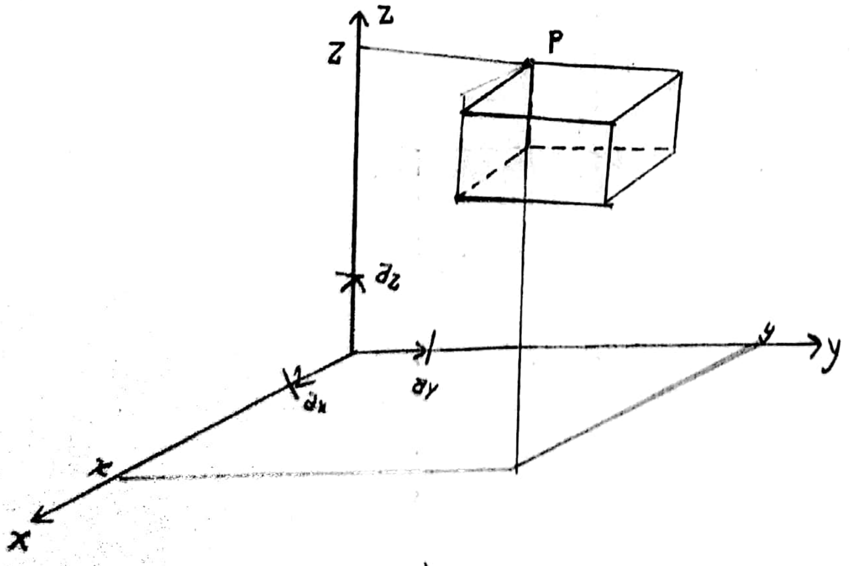
→ A line is formed by intersection of two surfaces for example : r = constant ; ϕ = constant. is a semicircle passing through Q and P. The point of intersection of three surfaces gives a point.

Vector calculus :-

Differential length, Area and Volume :-

→ The differential elements in length, area and volume are useful in vector calculus. They are defined in the cartesian, cylindrical and spherical coordinate systems.

Cartesian coordinates :-



Differential elements in right-handed cartesian system.

→ From above figure, we get

(i) Differential displacement is given by

$$dl = dx\, a_x + dy\, a_y + dz\, a_z \quad \text{--- (1)}$$

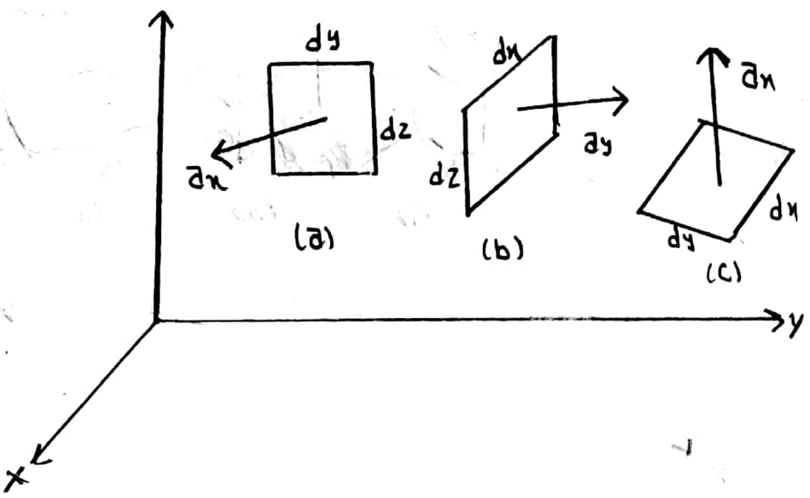
(ii) Differential normal area is given by

$$ds = dy\,dz\, a_x$$
$$dx\,dz\, a_y \quad \text{--- (2)}$$
$$dx\,dy\, a_z$$

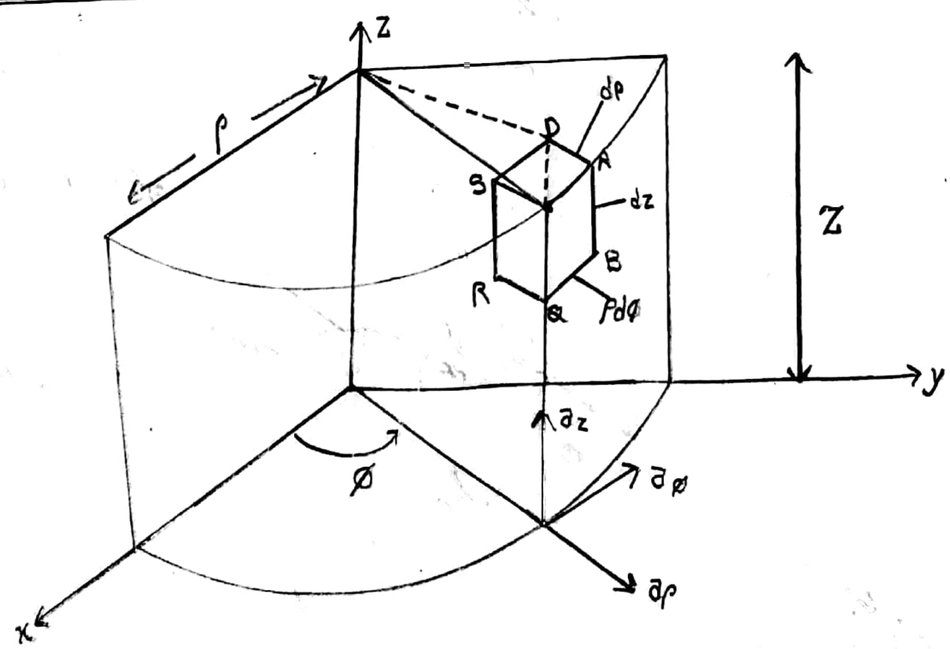
it is shown in below figure.

(iii) Differential volume is given by

$$dv = dx\,dy\,dz \quad \text{--- (3)}$$



Cylindrical coordinates :-



→ From the figure that in cylindrical coordinates.

(i) Differential displacement is given by

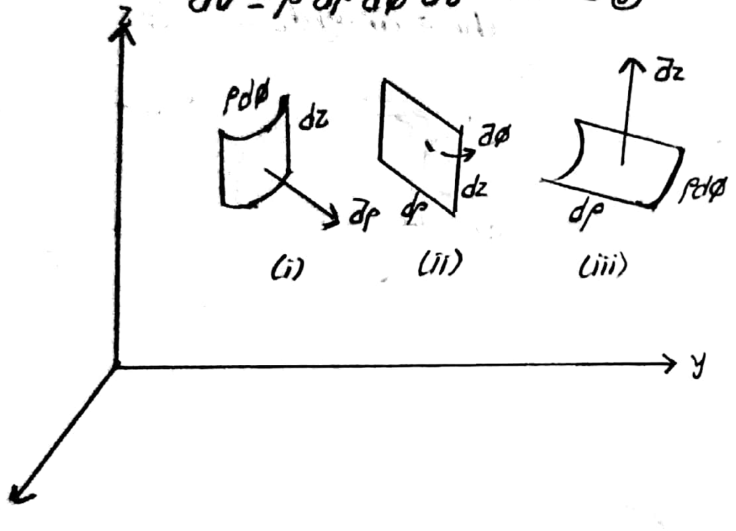
$$d\bar{l} = d\rho\, \bar{a}_\rho + \rho d\phi\, \bar{a}_\phi + dz\, \bar{a}_z \quad \text{——} (4)$$

(ii) Differential normal area is given

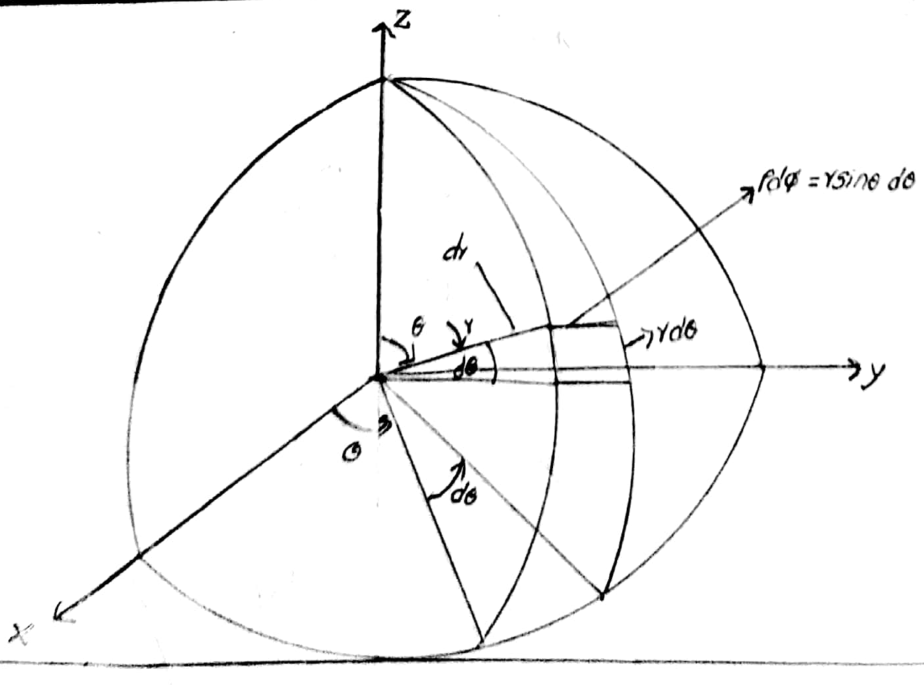
$$ds = \rho d\phi\, dz\, \bar{a}_\rho$$
$$= d\rho\, dz\, \bar{a}_\phi \quad \text{——} (5)$$
$$= \rho\, d\phi\, d\rho\, \bar{a}_z$$

(iii) Differential volume is given by

$$dv = \rho\, d\rho\, d\phi\, dz \quad \text{——} (6)$$



Spherical Coordinates :-



→ From the figure that in spherical coordinates.

(i) Differential displacement is

$$dl = dr\,a_r + r\,d\theta\,a_\theta + r\sin\theta\,d\phi\,a_\phi \quad \text{—(7)}$$

(ii) Differential normal area is

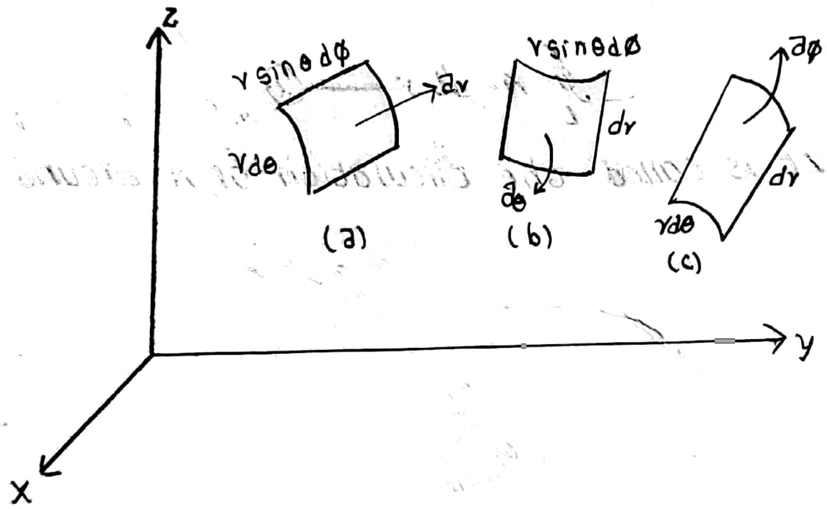
$$ds = r^2 \sin\theta \cdot d\theta \cdot d\phi \cdot a_r$$

$$= r\sin\theta\,dr\,d\phi\,a_\theta. \quad \text{—(8)}$$

$$= r\,dr\,d\theta\,a_\phi$$

(iii) Differential volume is given by

$$dv = r^2 \sin\theta\,dr\,d\theta\,d\phi \quad \text{—(9)}$$

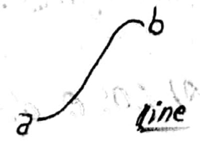


Line, surface and volume Integrals :-

→ The line integral $\oint A\,dL$ is the integral of tangential components of A along curve L.

line integral =

$$\int_L A\,dL = \int_a^b |A| \cos\theta\,dl \quad \text{—(1)}$$



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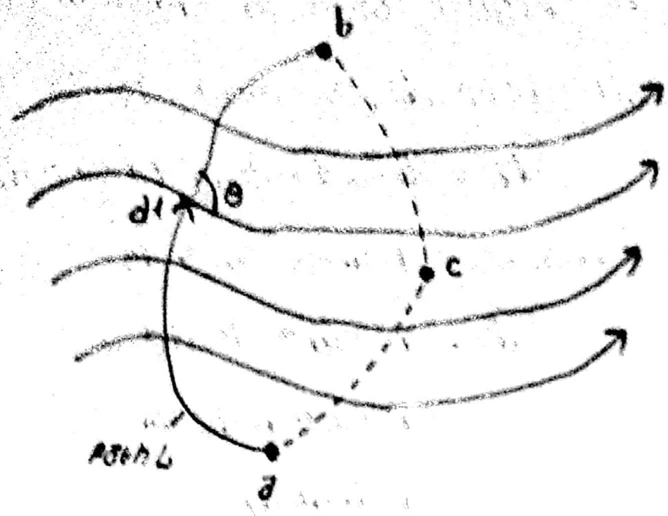
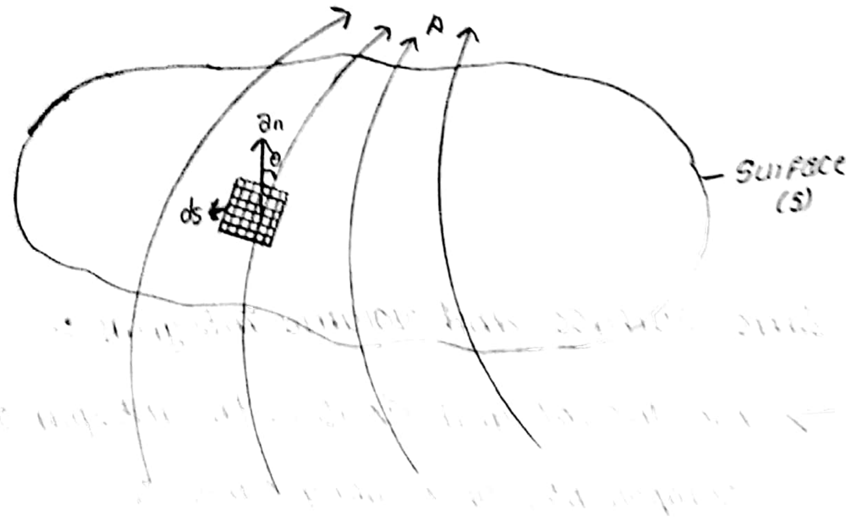


→ If the path of integration is a closed curve such as abca in above figure. Hence ① becomes

$$\oint_L A \cdot dl \quad \text{——— ②}$$

It is called the circulation of A around L.

*



→ Given a vector feild A, continous in a region which containing a smooth surface S, the surface integral or the flux of A through S.

$$\psi = \int_S |A| \cos\theta \, ds = \int_S A \cdot a_n \, ds$$

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$$\Rightarrow \psi = \int_s A \cdot ds \quad (3)$$

→ where, at any point on S, a_n is the unit normal to S. For a closed surface.

$$\Rightarrow \psi = \oint_S A \cdot ds \quad (4)$$

→ which is refered to as the net outward flux of A from S. Notice that a closed path defines an open surface where as a closed surface define volume.

$$\int_v \rho_v \, dv \quad (5)$$

→ The above integral of the scalar ρ_v over the volume. The physical meaning of a line, surface or volume integral depends on the nature of physical quantity represented by A (or) ρ_v.

Del operator :-

→ The del operator, written as "∇", and is the vector differential operator.

→ In cartesian coordinates,

$$\nabla = \partial/\partial x \, \bar{a}_x + \partial/\partial y \, \bar{a}_y + \partial/\partial z \, \bar{a}_z .$$

→ The operator is useful in defining

1. gradient of a scalar V, written as ∇V.
2. divergence of a vector A, written as $\nabla \cdot \bar{A}$.
3. The curl of a vector A, written as $\nabla \times \bar{A}$.
4. The laplacian of a scalar V, written as $\nabla^2 V$.

The gradient :-

→ The gradient of a scalar feild V is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V.

- For cartesian coordinates :-

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$$

- For cylindrical coordinates :-

$$\nabla V = \frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}\bar{a}_z$$

- For sperical coordinates :-

$$\nabla . V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi$$

Divergence :-

→ The divergence of A at a given point P is the outward flux per unit volume as volume shrinks about P.

$$\text{div } A = \nabla\cdot A = \underset{\Delta v\to 0}{Lt}\ \frac{\oint_s A\cdot ds}{\Delta v}$$

Divergence in cartesian coordinates

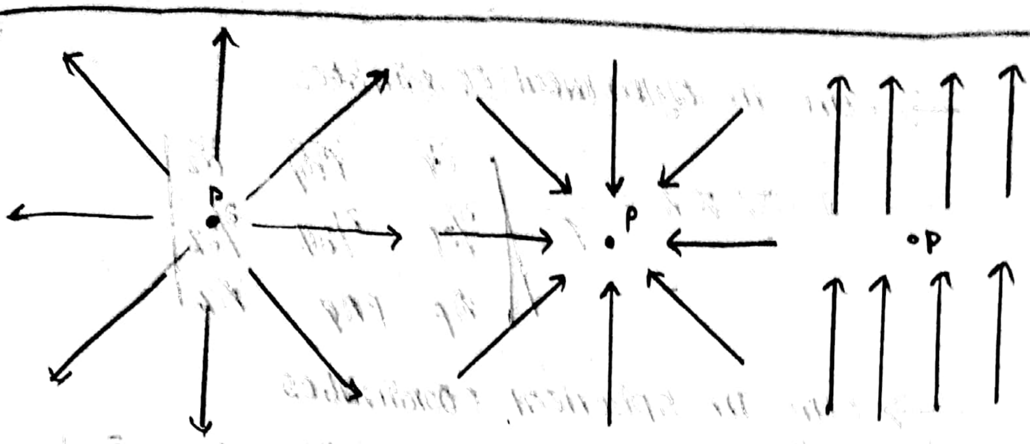
$$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence in cylindrical coordinates

$$\nabla\cdot\bar{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Divergence in spherical coordinates

$$\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}.$$



a) positive divergence b) negative divergence c) zero divergence.

Divergence Theorem :-

→ The divergence theorem states that the total outward flux of a vector feild A through the closed surface S is same as volume integral of div of A.

$$\oint_S A \cdot ds = \int_V \nabla \cdot A \, dv.$$

Curl of a vector :-

→ The curl of A is an axial (or rotational) vector whose magnitude is the maximum circulation of A per unit area tends to zero and whose direction is the normal direction of the area.

$$\text{curl } A = \nabla \times A = \left(\underset{\Delta S \to 0}{Lt} \frac{\oint_L A \cdot dl}{\Delta S} \right)_{max} a_n$$

→ curl in cartesian coordinates

$$\nabla \times \bar{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

→ curl in cylindrical coordinates

$$\nabla \times \bar{A} = \frac{1}{\rho}\begin{vmatrix} \bar{a}_\rho & \rho\bar{a}_\phi & \bar{a}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

→ curl in spherical coordinates

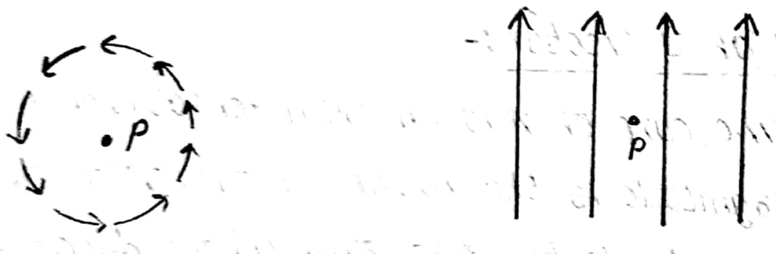
$$\nabla \times \bar{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

→ The divergence of curl of vector feild vanishes

$$\nabla \cdot \nabla \times \bar{A} = 0$$

→ The curl of gradient of a scalar feild vanishes

$$\nabla \times \nabla V = 0$$



(a) (b)

→ The above images shown illustration of a curl.

Stroke's Theorem :-

→ It states that the circulation of a vector feild around a closed path L is equal to surface integral of the curl of A over the open surface S bounded by L provided that A and $\nabla \times A$

$$\oint_L A \cdot dL = \int_S (\nabla \times A) \cdot dS$$

laplacian of a scalar :-

→ The laplacian of a scalar feild V, written as $\nabla^2 V$ is the divergence of the gradient of V.

- In cartesian coordinates,

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

- In spherical coordinates,

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 V}{\partial \phi^2}\right)}$$

- In cylindrical coordinates

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}}$$

- laplacian of a vector is given by

$$\boxed{\nabla^2 A = \nabla \cdot (\nabla \cdot A) - \nabla \times \nabla \times A}$$

- A vector feild A is said to be solenoidal if

$$\boxed{\nabla \cdot \bar{A} = 0}$$

- A vector feild A is said to be irrotational if

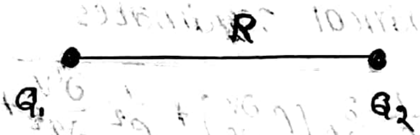
$$\boxed{\nabla \times \bar{A} = 0}$$

Electrostatics

Columb's law :-

→ Columb's law states that the force F between two point charges Q_1 and Q_2 is :

1. along the line joining them.
2. Directly proportional to the product $Q_1 \cdot Q_2$ of charges.
3. inversely proportional to square of distance R between them.



→ Expressed mathematically,

$$F \propto \frac{Q_1 \cdot Q_2}{R^2} \Rightarrow F = K \cdot \frac{Q_1 \cdot Q_2}{R^2}.$$

* where K is proportionality constant, The units of Q_1 and Q_2 are columbs (i-e charges). The distance between them is in meters (M) and force in newtons (N).

$$\therefore K = \frac{1}{4\pi\epsilon_0}$$

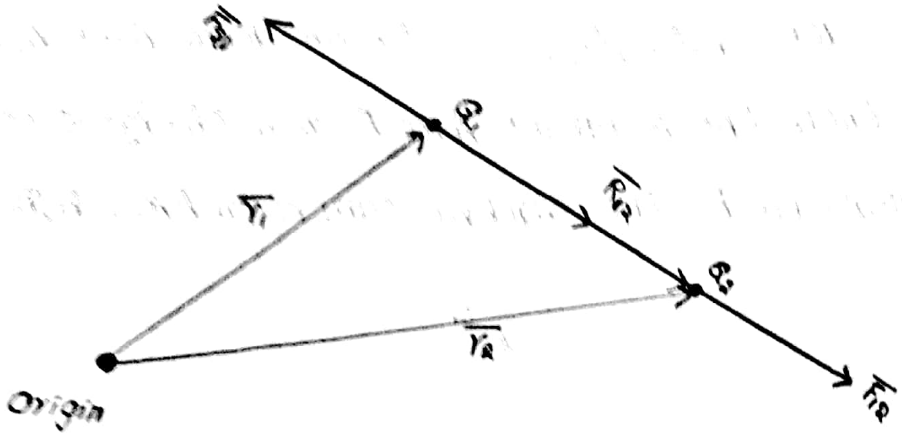
→ The constant ϵ_0 is called permitivity of free space and has value

$$\epsilon_0 = 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi} F/m$$

$$K = \frac{1}{4\pi\epsilon_0} \simeq 9 \times 10^9 F/m$$

$$F = \frac{Q_1 \cdot Q_2}{4\pi\epsilon_0 R^2}$$

Vector form of columb's law :-



→ Columb Vector force on point charges Q_1 and Q_2.

→ If point charges Q_1 and Q_2 are located at points having position vectors r_1 and r_2 then the force F_{12} on Q_2 due to Q_1. It is given by

$$\overline{F_{12}} = \frac{Q_1 \cdot Q_2}{4\pi\epsilon_0 R^2} \, \overline{a}_{R_{12}} \quad \text{—— ①}$$

where,

$$\overline{R_{12}} = \overline{r_2} - \overline{r_1} \quad \text{—— ②}$$

$$R = |\overline{R_{12}}| = |\overline{r_2} - \overline{r_1}| \quad \text{—— ③}$$

$$\overline{a}_{R_{12}} = \frac{\overline{R_{12}}}{R} \quad \text{—— ④}$$

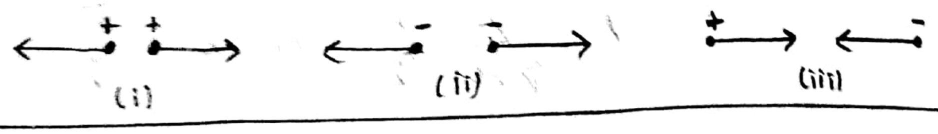
→ By substituting eqn ②, ③, ④ in ①, then

$$\overline{F_{12}} = \frac{Q_1 \cdot Q_2}{4\pi\epsilon_0 R^3} \overline{R_{12}}$$

$$\overline{F_{12}} = \frac{Q_1 \cdot Q_2 \, (\overline{r_2} - \overline{r_1})}{4\pi\epsilon_0 |\overline{r_2} - \overline{r_1}|^3}$$

→ Similarly the force F_{21} on Q_1 due to Q_2 is given by

$$\overline{F_{21}} = -\overline{F_{12}}.$$



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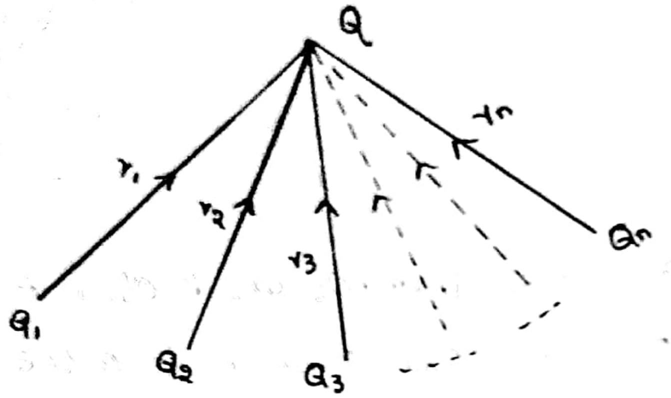
Multiple Charges :-

let $Q_1, Q_2, Q_3, \ldots, Q_n$ are located at $r_1, r_2, r_3, \ldots r_n$ then the resultant force F on a charge Q can be calculated by the algebric sum of all forces $F_1, F_2, \ldots F_n$.



$$F = F_1 + F_2 + F_3 + \ldots + F_n$$

But $$F_{12} = \frac{Q_1 \cdot Q_2}{4\pi\epsilon_0 R^2} \cdot \bar{a}R_{12}$$

$$F = \frac{QQ_1(r-r_1)}{4\pi\epsilon_0|r-r_1|^3} + \frac{QQ_2(r-r_2)}{4\pi\epsilon_0|r-r_2|^3} + \frac{QQ_3(r-r_3)}{4\pi\epsilon_0|r-r_3|^3} + \ldots + \frac{QQ_n(r-r_n)}{4\pi\epsilon_0|r-r_n|^3}$$

$$\boxed{F = \frac{Q}{4\pi\epsilon_0} \sum_{k=1}^{N} \frac{Q_k(r-r_k)}{|r-r_k|^3}}$$

Electric Feild Intensity :-

→ The electric feild intensity E is the force per unit charge when placed in electric feild.

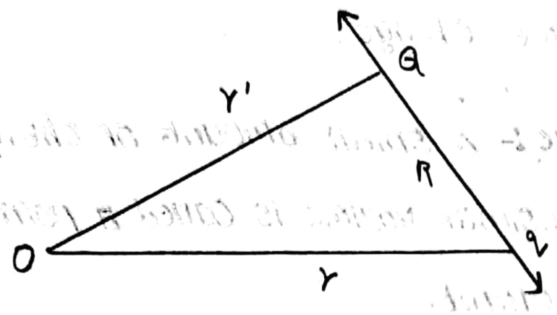
$$E = \lim_{q \to 0} \frac{\bar{F}}{q} \quad (or) \quad \bar{E} = \frac{\bar{F}}{q}$$

→ E is measured in newtons / columbs. (or) volt / meter.

→ The electric feild intensity at point r due to a point charge 'Q' located at 'r'' is given by

$$\bar{F} = \frac{qQ}{4\pi\epsilon_0 R^2}\,\bar{a}_R$$

→



$$\bar{E} = \bar{F}/q$$

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 R^2}\,\bar{a}_R$$

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 R^3}\cdot\bar{R} = \frac{Q(\bar{r}-\bar{r}')}{4\pi\epsilon_0 |\bar{r}-\bar{r}'|^3}$$

For n point charges $Q_1, Q_2, Q_3, --- Q_n$ are located at $r_1, r_2, --- r_n$, the electric feild intensity at point r is obtained as

$$E = \frac{Q_1(r-r_1)}{4\pi\epsilon_0|r-r_1|^3} + \frac{Q_2(r-r_2)}{4\pi\epsilon_0|r-r_2|^3} + \text{------} + \frac{Q_N(r-r_n)}{4\pi\epsilon_0|r-r_N|^3}$$

$$E = \frac{1}{4\pi\epsilon_0}\sum_{K=1}^{N}\frac{Q_K(r-r_K)}{|r-r_K|^3}$$

Electric feilds due to continous charge distribution :-

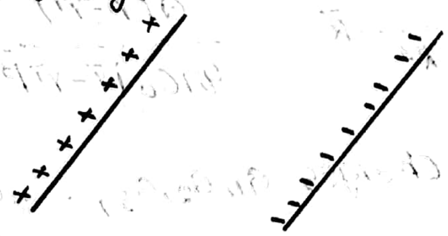
→ The electric charge may be any of following form

* point charge
* line charge
* Surface charge
* volume charge.

point charge :- A small amount of charge Q occupying an infinatesimal volume is called a point charge. It unit is columb.

$$\underset{+Q}{\oplus} \qquad \underset{-Q}{\ominus}$$

line charge :- A uniform charge distributed along a line is called line charge.



$+\rho_L$ C/m $-\rho_L$ C/m

→ line charge density is denoted by ρ_L. Its units are C/m.

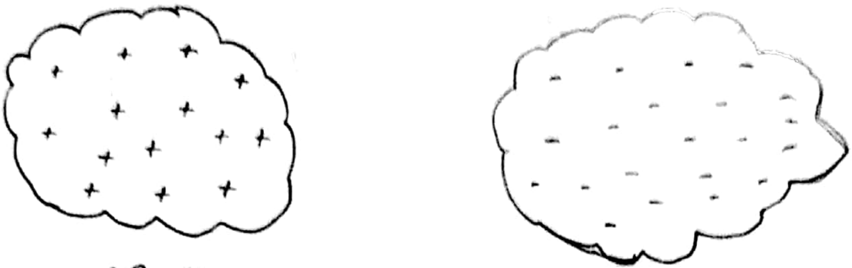
$$\rho_L = \frac{\text{Total charge in line}}{\text{length of the line}} \; C/m$$

Total charge in line charge

$$Q = \int_L \rho_L \, dL \quad \text{(columbs)}$$

$$\bar{E} = \int_L \frac{\rho_L \, dL}{4\pi \epsilon_0 R^2} \bar{a}_R .$$

<u>Surface Charge</u> :- A charge uniformly distributed over a surface or a sheet is called surface charge. It is denoted by ρ_S. It's units are C/m^2.



$+\rho_S$ C/m^2 $-\rho_S$ C/m^2

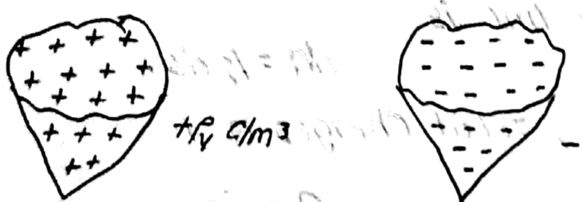
$$\rho_S = \frac{\text{Total charge in sheet}}{\text{Area of sheet}} \; C/m^2$$

Total surface charge

$$Q = \int \rho_S \, ds \quad (\text{columbs}/m^2)$$

$$E = \int_S \frac{\rho_S \cdot ds}{4\pi\epsilon_0 R^2} \, a_R$$

<u>Volume Charge</u> :- A charge uniformly distributed over a volume is called volume charge. It is denoted by ρ_V. It's units are C/m^3.



$+\rho_V$ C/m^3 $-\rho_V$ C/m^3

$$\rho_V = \frac{\text{Total charge in volume}}{\text{Total volume}} \; C/m^3$$

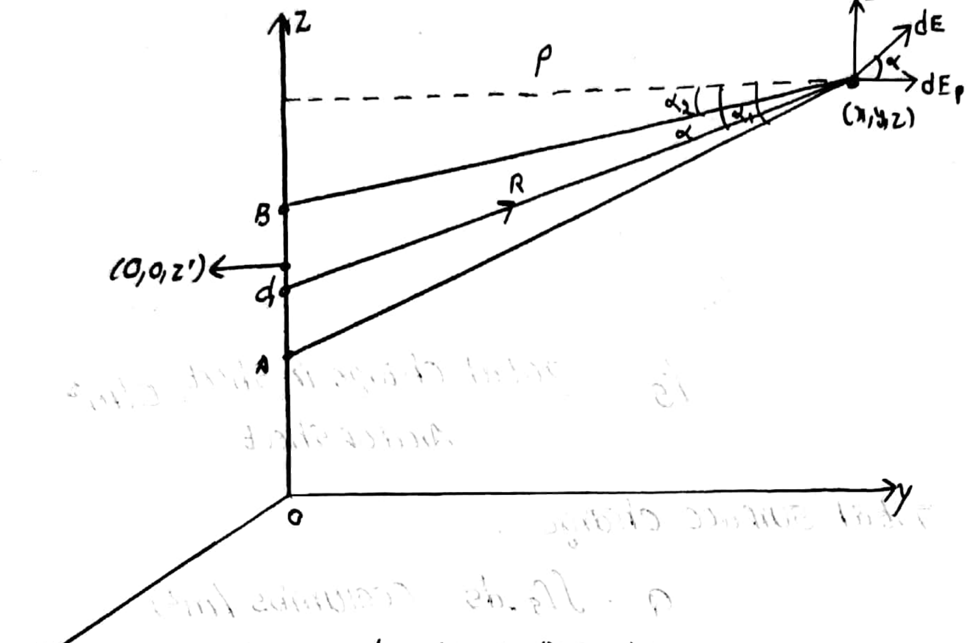
Total volume charge

$$Q = \int \rho_V \, dv \quad (\text{columbs}/m^3)$$

$$E = \int_V \frac{\rho_V \cdot dv}{4\pi\epsilon_0 R^2} \, a_R$$

Electric feild intensity due to finite line charges :-

→ Consider a line charge with uniform charge density ρ_L extending from A to B along z-axis as shown.



E feild due to a line charge.

→ Let us consider differential element element dl on the line at 'z''.

→ The charge element dQ associated with element dL of the line is

$$dQ = \rho_L \, dL.$$

→ Hence, Total charges Q is

$$Q = \int \rho_L \, dL.$$

→ The electric feild intensity E at an arbitary point $p(x,y,z)$ can be found using equation.

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 R^2} \bar{a}_R.$$

$$\bar{E} = \int \frac{\rho_L \, dl}{4\pi\epsilon_0 R^2} \bar{a}_R \quad \text{—— (1)}$$

* If the dl is located at $(0,0,z')$ then $dl = dz'$

$$\bar{R} = \rho \bar{a}_\rho + (z-z')\bar{a}_z \quad \text{——} ②$$

$$R = \sqrt{\rho^2 + (z-z')^2}$$

$$\bar{a}_R = \frac{\bar{R}}{R} = \frac{\rho \bar{a}_\rho + (z-z')\bar{a}_z}{\sqrt{\rho^2 + (z-z')^2}} \quad \text{——} ③$$

Substitute the above equations in eqn ①, then

$$\bar{E} = \frac{\rho_L}{4\pi\epsilon_0} \int_A^B \frac{\rho \bar{a}_\rho + (z-z')\bar{a}_z}{[\rho^2 + (z-z')^2]^{3/2}} dz'$$

To evaluate this, it is convenient that we define α, α_1 and α_2 as in figure

$$z - z' = \rho \tan\alpha$$

$$dz' = -\rho \sec^2\alpha \, d\alpha$$

$$\bar{E} = \frac{-\rho_L}{4\pi\epsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\rho \bar{a}_\rho + \rho \tan\alpha \, \bar{a}_z}{[\rho^2 + \rho^2 \tan^2\alpha]^{3/2}} \rho \sec^2\alpha \, d\alpha$$

$$\rho^2 + \rho^2 \tan^2\alpha = \rho^2 \sec^2\alpha$$

$$\bar{E} = \frac{-\rho_L}{4\pi\epsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\bar{a}_\rho + \tan\alpha \, \bar{a}_z}{\rho^3 \sec^3\alpha} \rho^2 \sec^2\alpha \, d\alpha$$

$$\bar{E} = \frac{-\rho_L}{4\pi\epsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\bar{a}_\rho + \tan\alpha \, \bar{a}_z}{\rho \sec\alpha} d\alpha$$

$$\bar{E} = \frac{-\rho_L}{4\pi\epsilon_0 \rho} \int_{\alpha_1}^{\alpha_2} [\cos\alpha \, \bar{a}_\rho + \sin\alpha \, \bar{a}_z] d\alpha$$

$$\bar{E} = \frac{-\rho_L}{4\pi\epsilon_0 \rho} [(\sin\alpha_2 - \sin\alpha_1)\bar{a}_\rho - (\cos\alpha_2 - \cos\alpha_1)] \cdot \bar{a}_z .$$

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→ Thus for a finite line charge

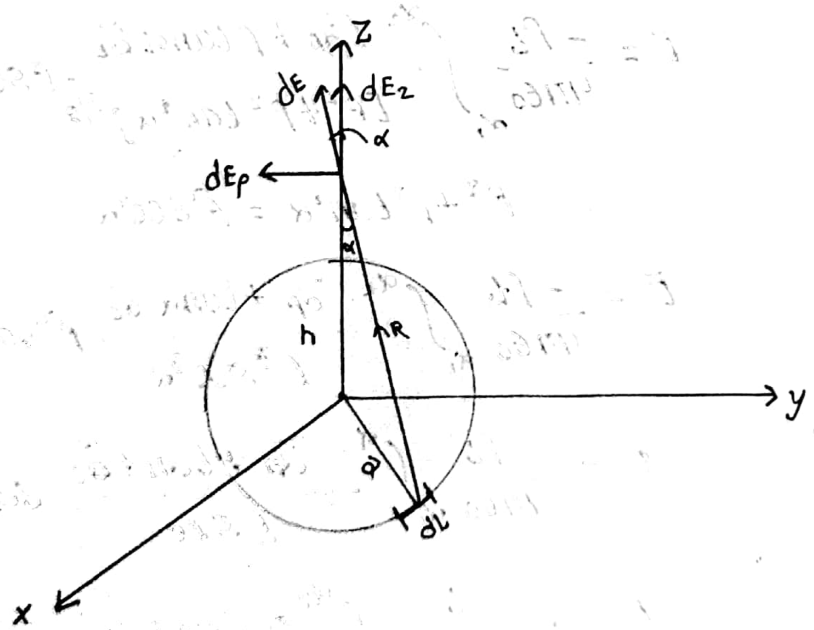
$$E = \frac{\rho_L}{4\pi\epsilon_0 \rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\bar{a}_\rho + (\cos\alpha_2 - \cos\alpha_1)\bar{a}_z\right] \quad \text{— (4)}$$

As a special case, for an infinite line charge, point B is at $(0,0,\infty)$ and A at $(0,0,-\infty)$. So that $\alpha_1 = \pi/2$, $\alpha_2 = -\pi/2$; then z-component varnishes and equ (4) will

$$\boxed{\bar{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\bar{a}_\rho}$$

E due to circular ring of charge :-

Let us consider a circular ring of radius a carrying a uniform charge ρ_L c/m and is placed on the x-y plane with the axis same as the z-axis as shown.



→ The electric feild intensity at point P(0,0,h) is given by

$$\bar{E} = \int \frac{\rho_L\, dL}{4\pi\epsilon_0 R^2}\bar{a}_R \quad \text{— (1)}$$

→ In this case, $dl = a\,d\phi$

$$\bar{R} = -a\,\bar{a}_\rho + h\,\bar{a}_z$$

$$R = \sqrt{a^2+h^2}$$

$$\bar{a}_R = \frac{\bar{R}}{R} = \frac{-a\,\bar{a}_\rho + h\,\bar{a}_z}{\sqrt{a^2+h^2}}$$

→ Substitute the above equations in equ ①

$$\bar{E} = \frac{\rho_L}{4\pi\epsilon_0}\int_0^{2\pi}\frac{(-a\,\bar{a}_\rho + h\,\bar{a}_z)}{[a^2+h^2]^{3/2}}\,a\,d\phi.$$

→ Due to symmetry all radial components are concelled to each other.

$$\bar{E} = \frac{\rho_L}{4\pi\epsilon_0}\int_0^{2\pi}\frac{h\,\bar{a}_z}{[a^2+h^2]^{3/2}}\,a\,d\phi.$$

$$\bar{E} = \frac{\rho_L\,a\,h\,\bar{a}_z}{4\pi\epsilon_0[a^2+h^2]^{3/2}}\int_0^{2\pi}d\phi$$

$$= \frac{\rho_L\,a\,h\,\bar{a}_z}{4\pi\epsilon_0\,[a^2+h^2]^{3/2}}\;2\pi$$

→ Electric feild intensity due to circular ring of charge is given by.

$$\bar{E} = \frac{\rho_L\,a\,h}{2\epsilon_0\,[a^2+h^2]^{3/2}}\;\bar{a}_z\,.$$

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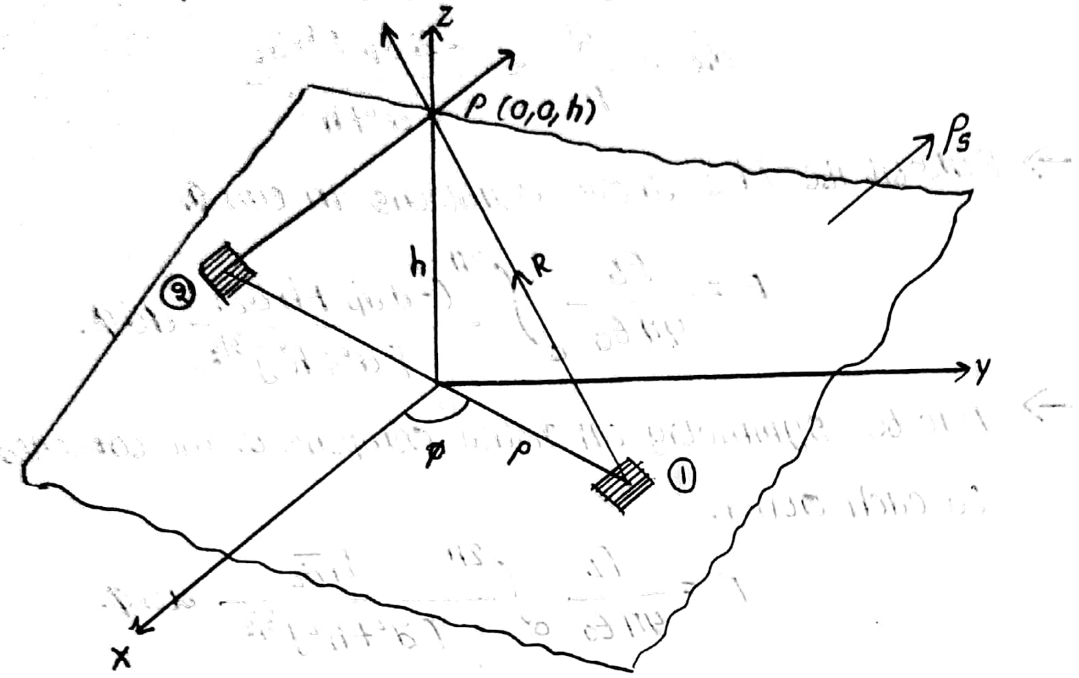
Electric Feild Intensity due to an infinate charge sheet :-

→ Consider an infinate charge sheet in xy plane with uniform charge density ρ_s as shown in figure.



→ The charges associated with an elemental area ds is $dQ = \rho_s\, ds$.

→ Total charge on sheet,

$$Q = \int_s \rho_s\, ds$$

→ The electric feild intensity at point P(0,0,h) is given by

$$\bar{E} = \int_s \frac{\rho_s\, ds}{4\pi\epsilon_0 R^2}\, \bar{a}_R \quad \text{—— ①}$$

→ In this case,

$$\bar{R} = -\rho\bar{a}_\rho + h\bar{a}_z$$

$$R = \sqrt{\rho^2+h^2}$$

$$\bar{a}_R = \frac{\bar{R}}{R} = \frac{-\rho\bar{a}_\rho + h\bar{a}_z}{\sqrt{\rho^2+h^2}}$$

→ Substitute the above equations in equ ①.

$$\bar{E} = \frac{\rho_s}{4\pi\epsilon_0}\int_s \frac{(-\rho\bar{a}_\rho + h\bar{a}_z)}{[\rho^2+h^2]^{3/2}}\, ds$$

→ Due to symmetry all radial components are cancelled.

$$\bar{E} = \frac{\rho_s}{4\pi\epsilon_0}\int_s \frac{h\bar{a}_z}{[\rho^2+h^2]^{3/2}}\, ds$$

where

$$ds = \rho d\rho d\phi$$

$$\bar{E} = \frac{\rho_s h}{4\pi\epsilon_0}\int_0^{2\pi}\int_{\rho=0}^{\infty} \frac{\rho d\rho d\phi}{[\rho^2+h^2]^{3/2}}\, \bar{a}_z$$

$$= \frac{\rho_s h}{4\pi\epsilon_0}\int_0^{2\pi} d\phi \int_{\rho=0}^{\infty} \frac{\rho d\rho}{[\rho^2+h^2]^{3/2}}\, \bar{a}_z$$

Let $t^2 = \rho^2+h^2$ then

$$\cancel{2}t\, dt = \cancel{2}\rho\, d\rho$$

$$t\, dt = \rho\, d\rho$$

$$\bar{E} = \frac{\rho_s h}{4\pi\epsilon_0} 2\pi \int_{t=h}^{\infty} \frac{t\, dt}{t^3}\, \bar{a}_z$$

$$\bar{E} = \frac{\rho_s h}{2\epsilon_0}\int_{t=h}^{\infty} \frac{dt}{t^2}\, \bar{a}_z$$

$$\bar{E} = \frac{\rho_s h}{2\epsilon_0}\left[\frac{-1}{t}\right]_h^{\infty},\ \bar{a}_z = \frac{\rho_s h}{2\epsilon_0}\frac{1}{h}\bar{a}_z$$

→ The electric feild intensity due to infinate charge sheet if point P is on positive z-axis is

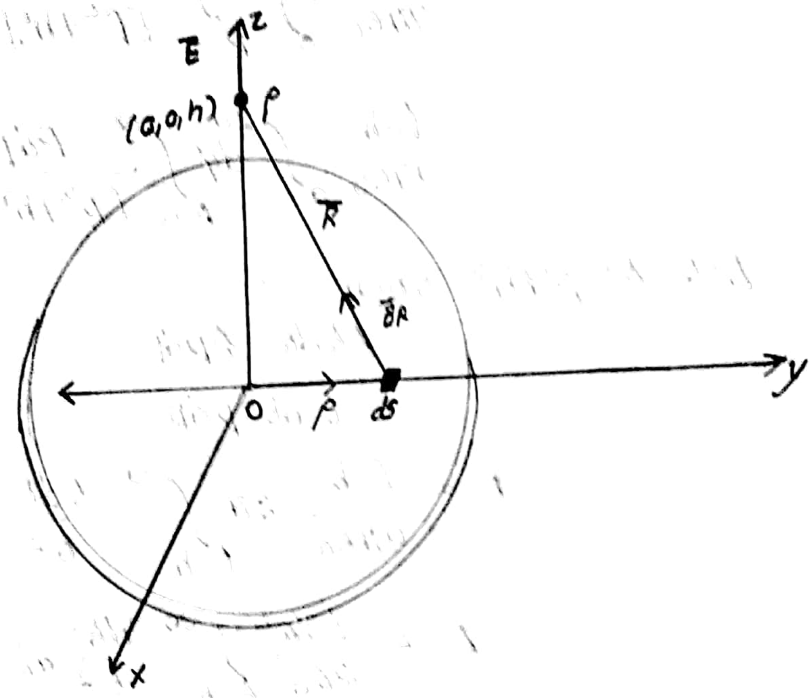
$$\bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_z$$

→ In a parallel plate capacitor, the electric feild exist between two plates having equal and opposite charges is

$$\bar{E} = \frac{\rho_s}{\epsilon_0} \cdot \bar{a}_z .$$

Electric Feild Intensity due to circular charged disc :-

→ Let us consider a circular charged disc of radius ρ carries a uniform charge ρ_s C/m^2 and is placed on xy-plane with the axis same as z-axis.



→ The charge associated with an elemental area ds is

$$dQ = \rho_s \, ds.$$

→ Total charge on disc

$$Q = \int_S \rho_s \, ds$$

→ The electric feild intensity at point $P(0,0,h)$ is given by

$$\bar{E} = \int_S \frac{\rho_s \, ds}{4\pi\epsilon_0 R^2} \bar{a}_R \quad \text{——— (1)}$$

In this case,

$$\bar{R} = -\rho\bar{a}_\rho + h\bar{a}_z$$

$$R = \sqrt{\rho^2+h^2}$$

$$\bar{a}_R = \frac{\bar{R}}{R} = \frac{-\rho\bar{a}_\rho + h\bar{a}_z}{\sqrt{\rho^2+h^2}}$$

→ substitute that in above equations (1).

$$\bar{E} = \frac{\rho_s}{4\pi\epsilon_0}\int_s \frac{(-\rho\bar{a}_\rho + h\bar{a}_z)}{[\rho^2+h^2]^{3/2}}\,ds$$

→ Due to symmetry all radial components are cancelled

$$\bar{E} = \frac{\rho_s}{4\pi\epsilon_0}\int_s \frac{h\bar{a}_z}{[\rho^2+h^2]^{3/2}}\,ds$$

where, $\qquad ds = \rho\, d\rho\, d\phi$

$$\bar{E} = \frac{\rho_s h}{4\pi\epsilon_0}\cdot\int_0^{2\pi}\int_0 \frac{\rho\cdot d\rho\, d\phi}{[\rho^2+h^2]^{3/2}}\,\bar{a}_z$$

$$\bar{E} = \frac{\rho_s h}{4\pi\epsilon_0}\int_0^{2\pi} d\phi\int_0^{\rho}\frac{\rho\, d\rho}{[\rho^2+h^2]^{3/2}}\,\bar{a}_z$$

Let $t^2 = \rho^2+h^2$ then $t\,dt = \rho\, d\rho$

$$\bar{E} = \frac{\rho_s h}{4\pi\epsilon_0}\,2\pi\int_{t=h}^{\sqrt{\rho^2+h^2}}\frac{t\,dt}{t^3}\,\bar{a}_z$$

$$= \frac{\rho_s h}{2\epsilon_0}\int_{t=h}^{\sqrt{\rho^2+h^2}}\frac{dt}{t^2}\,\bar{a}_z$$

$$= \frac{\rho_s h}{2\epsilon_0}\left[\frac{-1}{t}\right]_h^{\sqrt{\rho^2+h^2}}\bar{a}_z$$

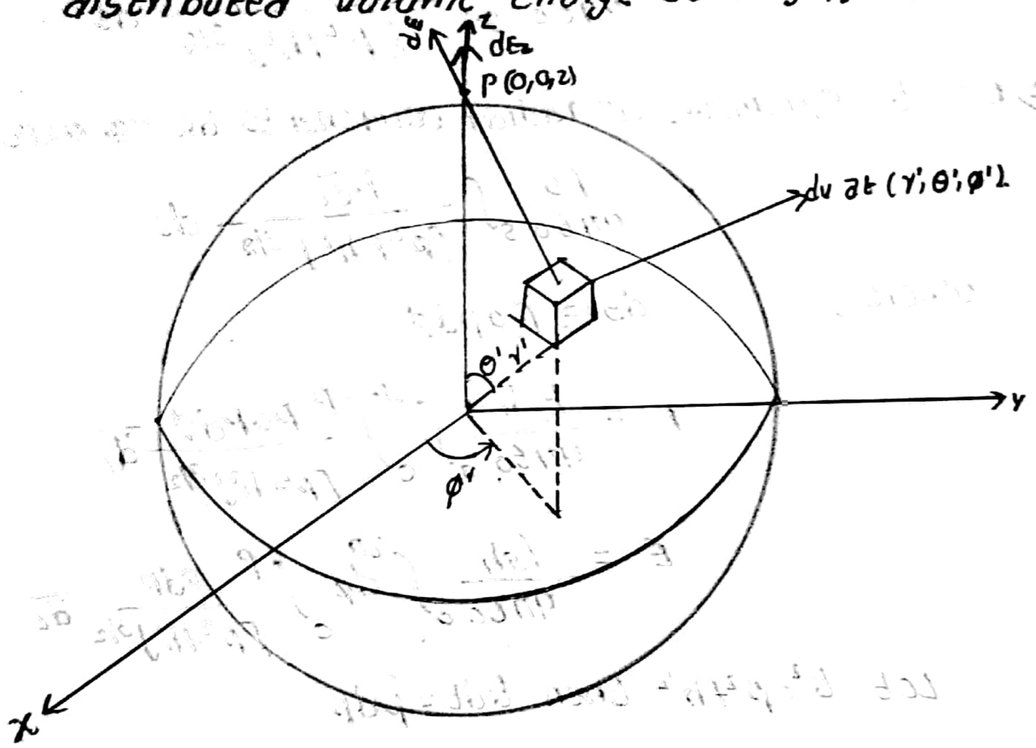
$$= \frac{\rho_s h}{2\epsilon_0}\left[\frac{1}{h} - \frac{1}{\sqrt{\rho^2+h^2}}\right]\bar{a}_z$$

→ The electric feild intensity at point p (0,0,h) due to circular disc is given by

$$\overline{E} = \frac{\rho_S}{2\epsilon_0}\left[1 - \frac{h}{\sqrt{\rho^2+h^2}}\right]\overline{a_z}$$

Electric feild intensity of a volume charge :-

→ Let us consider a sphere of radius 'a' with uniform distributed volume charge density ρ_v as shown.



→ we know that,

$$dE = \frac{\rho_v \, dv}{4\pi\epsilon_0 R^2}\,\hat{a}_R$$

now,

$$\overline{E} = \frac{Q}{4\pi\epsilon_0 R^2}\,\overline{a_R}$$

Electric flux density :-

→ Electric feild intensity depends on the medium in which the charges are placed.

→ To avoid this electric flux density is introduced.

Electric flux (ψ) :-

→ The total number of lines of forces in any particular electric feild is known as "electric flux".

→ It is denoted by ψ. The units is columbs (C).

→ The electric flux displaces charge from one place to other place.

Electric flux Density (D) :-

→ The total electric flux per unit area is defined as Electric flux Density.

$$\rightarrow D = \frac{d\psi}{ds} \text{ (columbs/m}^2\text{)}.$$

$$\rightarrow d\psi = D\,ds$$

$$\rightarrow \psi = \int_s \bar{D}\,\bar{ds}$$

$$\rightarrow \bar{D} = \epsilon_0 \bar{E}$$

$$\rightarrow \bar{D} = \frac{Q}{4\pi R^2}\,\bar{a}_R$$

→ It is also called the electric displacement.

Gauss's law :-

Gauss's law states that the total electric flux ψ passing through any closed surface is equal to the total charge enclosed by surface.

$$\psi = Q \quad \text{——} (1)$$

* Total flux $\psi = \oint_S \bar{D} \cdot \bar{ds}$ —— (2)

* Total charge enclosed $Q = \int_V \rho_v \, dv$ —— (3)

$$\oint_S \bar{D} \cdot \bar{ds} = \int_V \rho_v \, dv \quad \text{——} (4)$$

* This equation is called integral form of Gauss law.

* By applying divergence theorem of first term in (4),

$$\oint_S \bar{D} \cdot \bar{ds} = \int_V \nabla \cdot \bar{D} \, dv$$

→ Therefore,

$$\int_V \nabla \cdot \bar{D} \, dv = \int_V \rho_v \, dv$$

$$\boxed{\nabla \cdot \bar{D} = \rho_v}$$

→ The above equation is called point form of gauss law.

→ This equation is also known as maxwell's first eqn.

Procedure for find E (or) D using Gauss law:-

→ Verify whether charge symmetry exist (or) not.

→ Construct closed surface (Gaussian surface) around the charge distribution.

→ The surface is chosen such that D is either normal (or) tangential to Gaussian surface.

(a) When D is normal, $D \cdot ds = D \, ds$

(b) When D is tangential, $D \cdot ds = 0$.

→ Use the $\oint D \cdot ds = Q$ to find E (or) D.

Applications & limitations :-

Ap :- Gauss law provides an easy way to find E and D for symmetric charge distributions such as point charge, an infinate line charge, and a spherical charge.

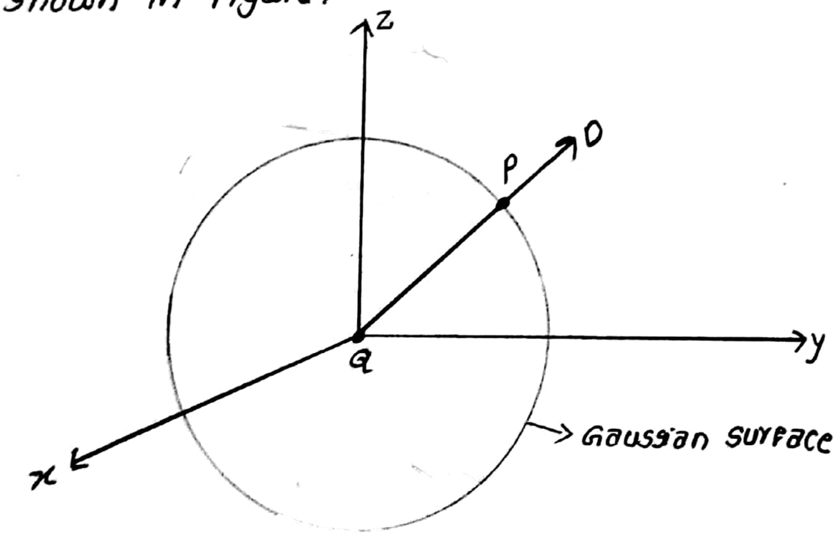
Lim :- * It is used only for symmetric charge distribution.

* It is applicable for closed surfaces only.

* It is applicable when charge is enclosed.

Electric feild intensity due to point charge :-

→ Suppose a point charge Q is located at origin as shown in figure.



→ To determine D at a point p, the spherical surface with radius r is chosen around point charge as Gaussian plane. From Gauss law, we know that

$$\oint_S \vec{D} \cdot \vec{ds} = Q$$

$$\oint_S D \cdot ds = Q$$

$$\oint_S D\,ds = Q$$

$$D \oint_S ds = Q$$

$$D(4\pi r^2) = Q \Rightarrow D = \frac{Q}{4\pi r^2}$$

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→ Electric feild intensity due to point charge

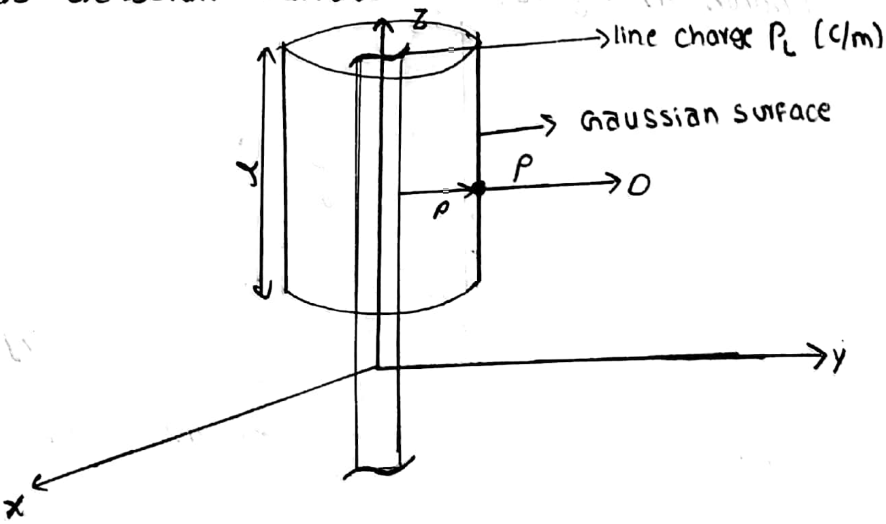
$$\bar{D} = \frac{Q}{4\pi r^2}\,\bar{a}_r$$

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\bar{a}_r.$$

Electric Feild Intensity due to infinate line charge:-

→ Let us consider the infinate line of uniform charge ρ_l c/m lies along z-axis.

→ To Determine D at point p, the cylindrical surface with radius ρ is chosen around infinate line charge as Gaussian surface.



→ From Gauss law, we know that

$$\oint_S \bar{D}\cdot\bar{ds} = Q$$

$$\oint_S D\,ds = Q$$

$$D\oint_S ds = Q$$

where $\oint ds = 2\pi\rho l$ is the surface area of Gaussian surface.

→ $\oint D\cdot ds$ evaluated on the top and bottom surface of cylinder is zero. since D is tangential to the top and bottom surfaces.

$$D(2\pi \rho L) = \rho_L L$$

$$D = \frac{\rho_L}{2\pi\rho}$$

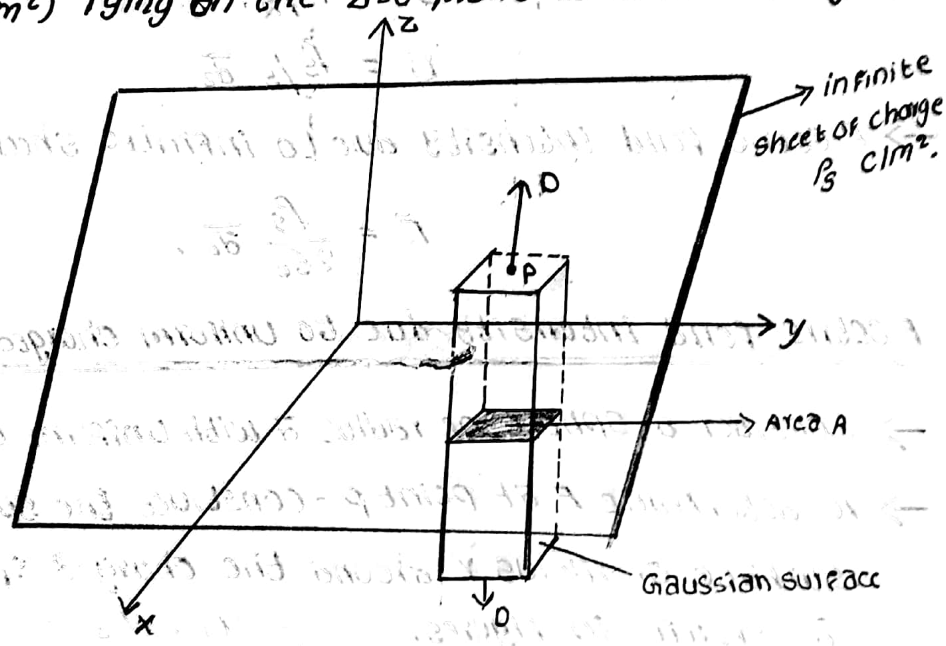
→ Electric feild intensity due to infinite line charge is

$$\bar{D} = \frac{\rho_L}{2\pi\rho}\,\vec{a}_\rho$$

$$\bar{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\,\vec{a}_\rho$$

Electric feild intensity due to infinite sheet of charge:-

→ Consider the infinite sheet of uniform charge ρ_s (C/m^2) lying on the z=0 plane as shown in figure.



→ To determine E at a point p, construct rectangular box as shown in figure.

→ From Gauss's law we know that

$$\oint \bar{D}\cdot \bar{ds} = Q$$

$$\int_{top} \bar{D}\cdot \bar{ds} + \int_{bottom} \bar{D}\cdot \bar{ds} = Q.$$

→ Note that D·ds evaluated on the sides of the box is zero, Because D is tangential to sides.

$$\int_{top} D\,ds + \int_{bottom} D\,ds = Q$$

$$D\left(\int_{top} ds + \int_{bottom} ds\right) = Q$$

→ If the top and bottom area of box each has area A, then above equation becomes

$$D(A+A) = \rho_s A$$

$$D(2A) = \rho_s A$$

$$2D = \rho_s$$

$$D = \rho_s/2$$

$$\bar{D} = \rho_s/2\ \bar{a}_z$$

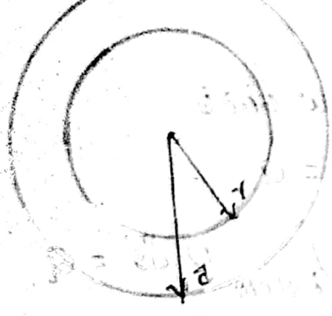
→ Electric feild intensity due to infinite sheet chrg is

$$\bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_z .$$

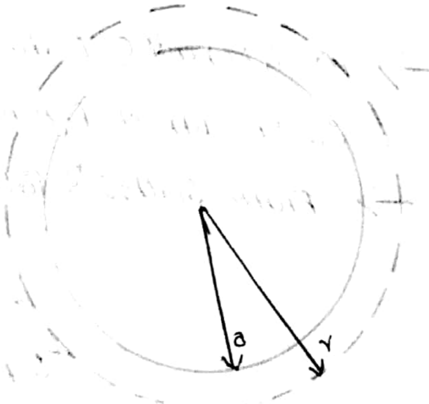
<u>**Electric feild intensity due to uniform charged sphere :-**</u>

→ Consider a sphere of radius a with uniform charge ρ.

→ To determine E at point P - construct the spherical surface of radius r around the charged sphere as shown in figures.



(i) $r \le a$



(ii) $r \ge a$

Case (1) :- $r \leq a$

From Gauss law we know that $\oint_S \bar{D} \cdot \bar{ds} = Q$

$$\oint_S D\,ds = Q$$

$$D \oint_S ds = Q$$

$$D\,4\pi r^2 = Q$$

→ For $r \leq a$, the total charge enclosed by spherical surface of radius r, as shown in figure (i) is

$$Q = \rho_v \frac{4}{3}\pi r^3$$

$$D\,4\pi r^2 = \rho_v \frac{4}{3}\pi r^3$$

$$D = \frac{\rho_v}{3} r$$

$$\bar{D} = \frac{r}{3}\rho_v \bar{a}_r .$$

→ Electric feild intensity inside the charged sphere is

$$\bar{E} = \frac{r}{3\epsilon_0} \rho_v \cdot \bar{a}_r .$$

Case (2) :- $r \geq a$

From Gauss law we know that

$$\oint_S \bar{D} \cdot \bar{ds} = Q \Rightarrow \oint_S D\,ds = Q$$

$$D \oint_S ds = Q \Rightarrow D\,4\pi r^2 = Q$$

→ In this case total charge enclosed by surface, is

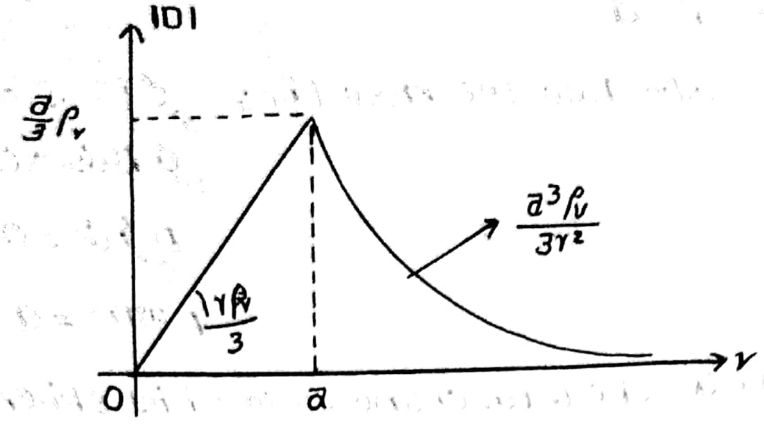
$$Q = \rho_v \frac{4}{3}\pi a^3 \text{ , then } D\,4\pi r^2 = \rho_v \frac{4}{3}\pi a^3$$

$$D = \frac{a^3}{3r^2}\rho_v$$

$$\bar{D} = \frac{a^3}{3r^2}\rho_v \bar{a}_r$$

→ Electric feild intensity outside the sphere is

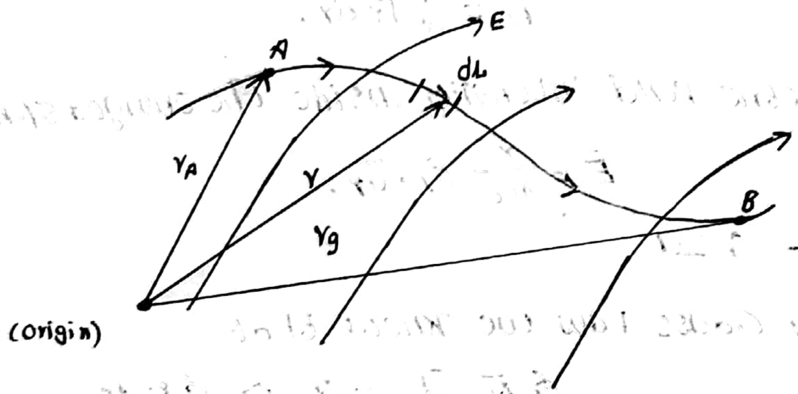
$$\bar{E} = \frac{a^3}{3\epsilon_0 r^2}\rho_v \bar{a}_r .$$



Sketch of D against r for a uniformly charged sphere.

Electric potential :-

→ Let us move a point charge from A to B in an electric feild having electric feild intensity E as shown.



Displacement of point charge in an electrostatic feild E, let the charge be Q.

→ To workdone to move a point charge by an elemental distance dL is

$$dw = -\bar{F}\cdot\bar{dL}.$$

→ The total workdone in moving a point charge from A to B is

$$W = -\int_A^B \bar{F}\cdot\bar{dL}.$$

→ The negative sign indicates that the work is done against the feild.

→ From Columb's law, the force Q is

$$\bar{F} = Q\bar{E}.$$

→ Thus, the total work done, in moving Q from A to B is

$$W = -Q \int_A^B \bar{E} \cdot \bar{dl}.$$

→ The work done per unit charge is given by

$$\frac{W}{Q} = -\int_A^B \bar{E} \cdot \bar{dl}$$

→ It is also determines potential difference between points A and B and is denoted by

$$V_{AB} = -\int_A^B \bar{E} \cdot \bar{dl}.$$

* In determining V_{AB}, A is intial point and B is final point.

* If V_{AB} is negative, there is a loss in potential energy in moving Q from A to B, this implies work is being done by feild.

* If V_{AB} is positive, there is a gain in potential energy in movement; an external agent performs the work.

* V_{AB} is independent of the path taken.

* V_{AB} is measured in Joules per columb; commonly referred as volts (V).

→ The electric feild E due to point charge Q, located at origin is given by

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2} \cdot \bar{a}_r$$

The elemental length

$$\bar{dl} = dr\,\bar{a}_r$$

→ The potential difference between points A & B is

$$V_{AB} = -\int_A^B \bar{E} \cdot \bar{dl}$$

$$* \quad V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} \bar{a}_r \cdot dr\,\bar{a}_r$$

$$* \quad V_{AB} = -\frac{Q}{4\pi\epsilon_0} \int_{r_A}^{r_B} \frac{dr}{r^2}$$

$$V_{AB} = -\frac{Q}{4\pi\epsilon_0} \left[\frac{-1}{r}\right]_{r_A}^{r_B}$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r}\right]_{r_A}^{r_B}$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$V_{AB} = \left[\frac{Q}{4\pi\epsilon_0 r_B} - \frac{Q}{4\pi\epsilon_0 r_A}\right]$$

$$V_{AB} = V_B - V_A.$$

→ where V_B and V_A are absolute potentials of A & B.

→ The potential difference V_{AB} may be regarded as the potential at B with reference to A.

→ If $V_A = 0$ as $r_A \to \infty$, then the potential at any point ($r_B \to r$) due to point charge Q at origin is

$$V = \frac{Q}{4\pi\epsilon_0 r}.$$

The Electric potential due to point charge :-

→ The potential at a distance r from the point charge is defined as the workdone in moving unit positive charge from infinity to point against to feild.

$$V = -\int_{\infty}^{r} \vec{E} \cdot \vec{dl}$$

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

→ The potential difference between two points A & B is

$$V_{AB} = -\int_{A}^{B} \vec{E} \cdot \vec{dl}$$

→ If the point charge Q is not located at origin but at a point whose position vector is r', the potential V at r becomes

$$V(r) = \frac{Q}{4\pi\epsilon_0 |r - r'|}$$

→ For n point charges $Q_1, Q_2, \ldots, Q_n$ located at points with position vectors $r_1, r_2, \ldots r_n$ the potential at r is

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 |r - r_1|} + \frac{Q_2}{4\pi\epsilon_0 |r - r_2|} + \cdots + \frac{Q_n}{4\pi\epsilon_0 |r - r_n|}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum_{K=1}^{n} \frac{Q_K}{|r - r_K|}$$

→ Potential for different charges distribution are given as follow

$$V = \int_L \frac{\rho_L \, dl}{4\pi\epsilon_0 r} \quad \text{(line charge)}$$

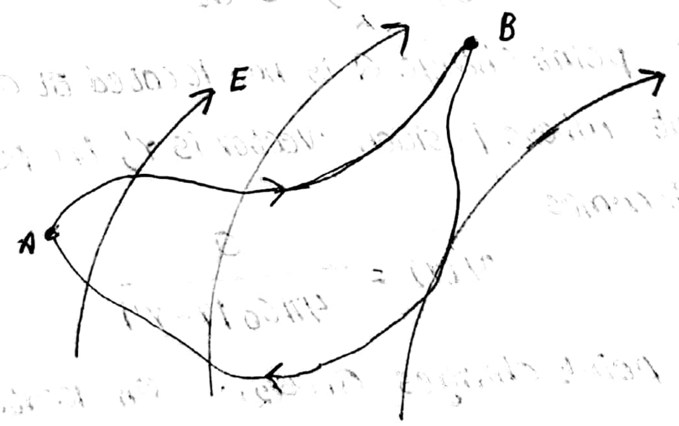
$$V = \int_S \frac{\rho_S \, ds}{4\pi\epsilon_0 r} \quad \text{(surface charge)}$$

$$V = \int_V \frac{\rho_V \, dv}{4\pi\epsilon_0 r} \quad \text{(volume charge).}$$

→ If any other non-zero potential is taken as reference point then,

$$V = \frac{Q}{4\pi\epsilon_0 r} + C$$

Conservative nature of electrostatic feild:-



Conservative nature of an electrostatic feild.

→ The potential difference between points A and B is independent of path taken. Hence

$$V_{AB} = -V_{BA}.$$

→ That is $V_{AB} + V_{BA} = \oint E \cdot dL = 0$ ——— ①

→ The above equation shows that the line integral of E along a closed path as shown in figure must be zero.

→ Applying strokes theorem to equ ①, then

$$\oint \bar{E}\cdot d\bar{l} = \int_S (\nabla \times E)\cdot d\bar{s} = 0$$

(or)

$$\nabla \times E = 0 \quad \text{——— ②}$$

→ A feild whose line integral does not depend on the path of integration is called conservative feild. Hence Electrostatic feild is conserved.

<u>Relation between E and V :-</u>

→ From the definition of potential

$$V = -\int \bar{E}\cdot d\bar{l}$$

→ It follows that

$$dv = -\bar{E}\cdot d\bar{L}$$

where

$$\bar{E} = E_x \bar{a}_x + E_y \bar{a}_y + E_z \bar{a}_z$$

$$d\bar{L} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$

$$dv = -E_x\,dx - E_y\,dy - E_z\,dz \quad \text{——— ①}$$

But from calculus,

$$du = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \quad \text{——— ②}$$

Comparing the two expressions for du, we get

$$E_x = -\frac{\partial v}{\partial x}, \quad E_y = -\frac{\partial v}{\partial y}, \quad E_z = -\partial v/\partial z$$

Thus

$$\bar{E} = -\frac{\partial v}{\partial x}\bar{a}_x - \frac{\partial v}{\partial y}\bar{a}_y - \partial v/\partial z\,\bar{a}_z$$

$$\boxed{\bar{E} = -\nabla V}$$

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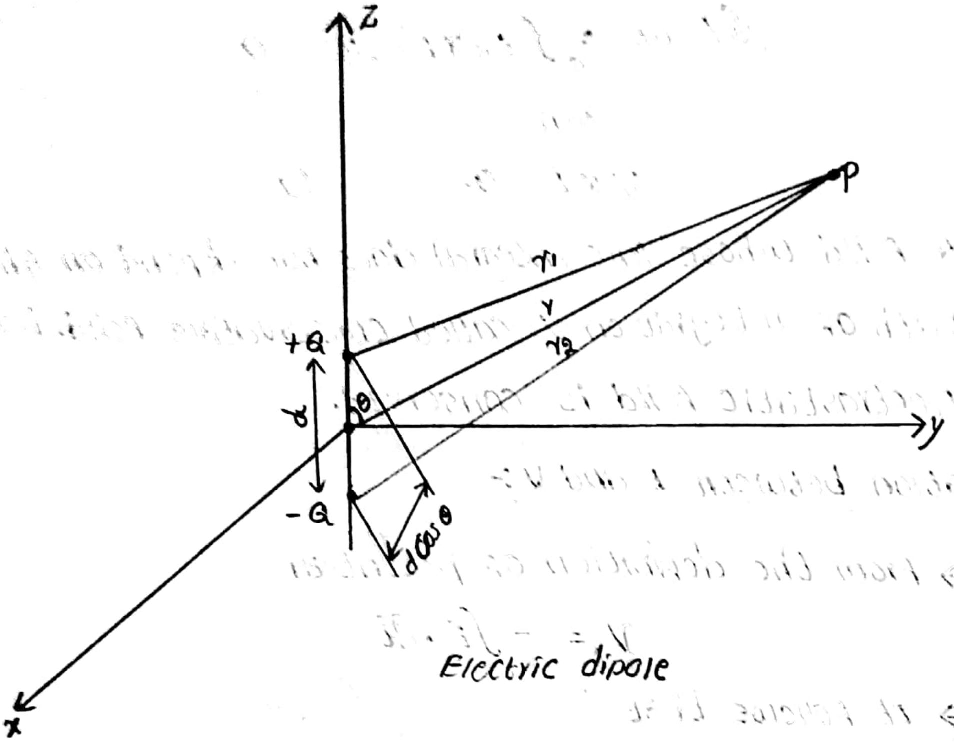
Electric dipole :-

→ An electric dipole is formed when two point charges of equal magnitude but opposite sign are separated by small distance as shown.



Electric dipole

→ The potential at point $p(r,\theta,\phi)$ is given by

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{Q}{4\pi\epsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \quad (1)$$

→ where r_1 and r_2 are distance between p and $+Q$, p and $-Q$ respectively.

→ If $r \gg d$, $r_2 - r_1 = d\cos\theta$, $r_2 r_1 = r^2$ and equ (1) will be

$$V = \frac{Q}{4\pi\epsilon_0} \cdot \frac{d\cos\theta}{r^2} \quad (2)$$

→ If we define $p = Qd$ as dipole moment, equ (2) will written as

$$V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \quad (3)$$

→ The electric feild due to dipole with center at the origin, shown in figure. can be obtained as follows.

$$\bar{E} = -\nabla \cdot V = -\left[\frac{\partial V}{\partial r} \bar{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \bar{a}_\theta \right]$$

$$\bar{E} = -\left[\frac{\partial}{\partial r} \left(\frac{P\cos\theta}{4\pi\epsilon_0 r^2} \right) \bar{a}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{P\cos\theta}{4\pi\epsilon_0 r^2} \right) \bar{a}_\theta \right]$$

$$\bar{E} = -\left[\left(\frac{P\cos\theta}{4\pi\epsilon_0} \right) \frac{\partial}{\partial r} \left(\frac{1}{r^2} \right) \bar{a}_r + \left(\frac{P}{4\pi\epsilon_0 r^3} \right) \frac{\partial}{\partial \theta} (\cos\theta) \bar{a}_\theta \right]$$

$$\bar{E} = \frac{P}{4\pi\epsilon_0 r^3} \left[2\cos\theta \, \bar{a}_r + \sin\theta \cdot \bar{a}_\theta \right]$$

→ A point charge is a monopole and its electric feild varies inversely as r^2 while its potential feild varies inversely as t.

→ The electric feild due to a dipole varies inversely as r^3 while its potential varies inversely as r^2.

→ the electric feild due to successive heigher-order multiples vary inversely as $r^4, r^5, r^6, \ldots$ while their potentials vary inversely as $r^3, r^4, r^5, \ldots$.

An equipotent surface :-

→ Any surface on which the potential is the same throughout is known as an equipotential surface.

→ No work is done in moving a charge from one point to another along an equipotential surface and hence

$$\oint \bar{E} \cdot d\bar{l} = 0.$$

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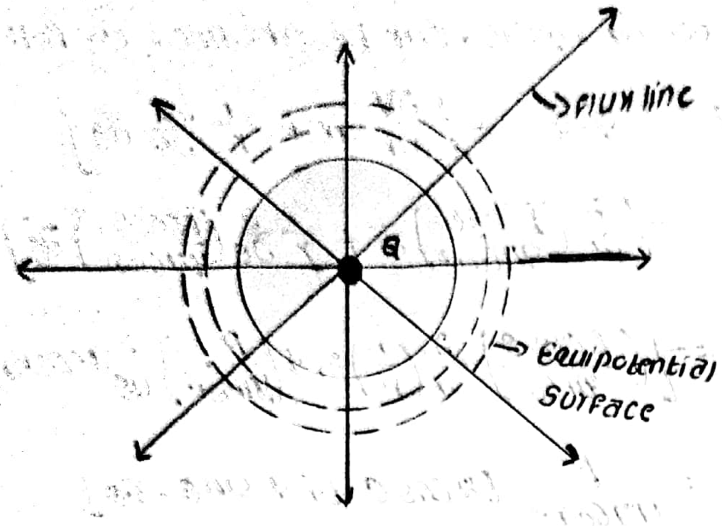
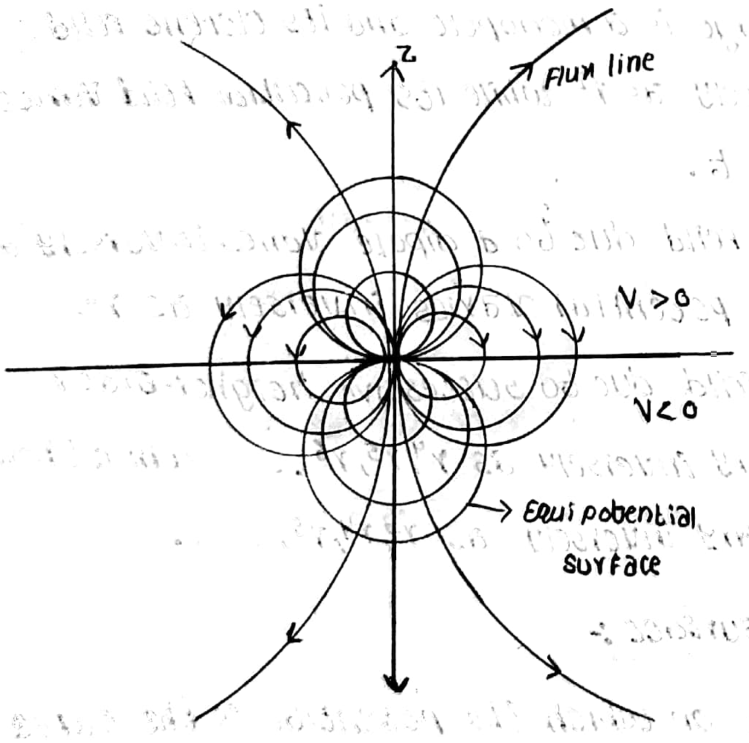


(a) point charge

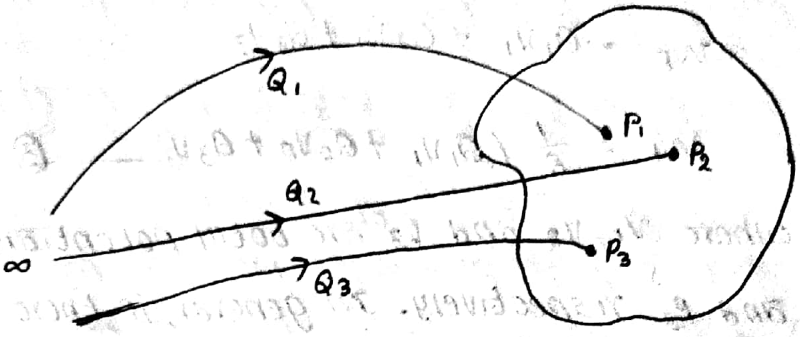


(b) an electric dipole.

Energy Density in Electrostatic Feilds :-

→ To determine the energy present in an assembly of charge, we must first determine the amount of work necessary to assemble them.

→ Three point charges Q_1, Q_2 and Q_3 are moved to an empty space as shown in figure.



Assembling of charges.

→ No work is required to move Q_1 from infinity to P_1 because there is no electric feild intially $[W=0]$.

→ The work done in moving Q_2 from infinity to P_2 is equal to product of Q_2 and potential V_{21} at P_2 due to Q_1.

→ Similarly, the work done in moving Q_3 to P_3 is equal to $Q_3(V_{32}+V_{31})$. where V_{32} and V_{31} are potentials at P_3 due to Q_2 and Q_1 respectively.

→ Hence the total work done in positioning the three charges is $W_E = W_1 + W_2 + W_3$

$$W_E = 0 + Q_2 V_{21} + Q_3(V_{31}+V_{32}) \quad \text{—— ①}$$

→ If the charges were moved in reverse order

$$W_E = W_3 + W_2 + W_1$$

$$W_E = 0 + Q_2 V_{23} + Q_1(V_{12}+V_{13}) \quad \text{—— ②}$$

→ where V_{23} is potential at P_2 due to Q_2. V_{12} and V_{13} are potentials at P_1 due to Q_2 and Q_3.

→ Adding equ ① and equ ②, gives

$$2W_E = Q_1(V_{12}+V_{13}) + Q_2(V_{21}+V_{23}) + Q_3(V_{31}+V_{32})$$

$$2W_E = Q_1V_1 + Q_2V_2 + Q_3V_3$$

$$W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3) \quad \text{———} ③$$

→ where V_1, V_2 and V_3 are total potentials at P_1, P_2 and P_3 respectively. In general, if there are n points charges equ ③ becomes

$$W_E = \frac{1}{2}\sum_{K=1}^{n} Q_K V_K \text{ (in Joules)} \quad \text{———} ④$$

→ If instead of point charges, the region has a continous charge distribution, the summation in equ ④, becomes integration

$$W_E = \frac{1}{2}\int \rho_L V dL \text{ (line integral)} \quad \text{———} ⑤$$

$$W_E = \frac{1}{2}\int \rho_S V ds \text{ (surface integral)} \quad \text{———} ⑥$$

$$W_E = \frac{1}{2}\int \rho_V V dv \text{ (volume charge)} \quad \text{———} ⑦$$

→ we know that

$$\rho_V = \nabla\cdot\bar{D}$$

Then equ ⑦ becomes

$$W_E = \frac{1}{2}\int(\nabla\cdot\bar{D}) V dv \quad \text{———} ⑧$$

→ From vector identity

$$\nabla\cdot V\bar{D} = \bar{D}\cdot\nabla V + V(\nabla\cdot\bar{D})$$

$$(\nabla\cdot\bar{D})V = \nabla\cdot V\bar{D} - \bar{D}\nabla V \quad \text{———} ⑨$$

→ subs equ (9) in equ (8), then

$$W_E = \frac{1}{2}\int_V (\nabla \cdot V\bar{D})\, dv - \frac{1}{2}\int_V (\bar{D}\cdot \nabla V)\, dv$$

→ By applying divergence theorem to first term on the right-hand side of above equation we have

$$W_E = \frac{1}{2}\oint_S (V\cdot\bar{D})\, ds - \frac{1}{2}\int_V (\bar{D}\cdot\nabla V)\, dv \quad \text{—— (10)}$$

→ The first term VD in equ (10) must vary at least as $1/r^3$ while ds varies as r^2.

→ consequently, the first integral in equ (10) must tend to zero as surface s becomes large.

→ Hence equ (10) reduces to

$$W_E = -\frac{1}{2}\int_V (\bar{D}\cdot\nabla V)\, dv$$

we know that

$$E = -\nabla V \text{ and } \bar{D} = \epsilon_0 \bar{E}.$$

then above equation becomes

$$W_E = \frac{1}{2}\int_V (\bar{D}\cdot\bar{E})\, dv = \frac{1}{2}\int_V \epsilon_0 E^2\, dv = \frac{1}{2}\int_V \frac{D^2}{\epsilon_0}\, dv$$

→ From this we can define electrostatic energy density w_E as

$$w_E = \frac{dW_E}{dv} = \frac{1}{2}(\bar{D}\cdot\bar{E}) = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\frac{D^2}{\epsilon_0}$$

That can also write as

$$W_E = \int_V w_E\, dv.$$

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Current :-

→ The current through a given area is a electric charge passing through the area per unit time.

→ The current is the rate of flow of charge.

$$I = dq/dt \text{ Amperes.}$$

→ one ampere means charge is being transferred at a rate of one columb per second.

Current density (J) :-

→ current density at a given point is the current passing through a unit normal area at that point.

→ If current ΔI flows through a surface ds, the current density is.

$$J = \frac{\Delta I}{\Delta S} \quad A/m^2$$

$$\Delta I = J \Delta S$$

$$I = \int_s I \cdot ds$$

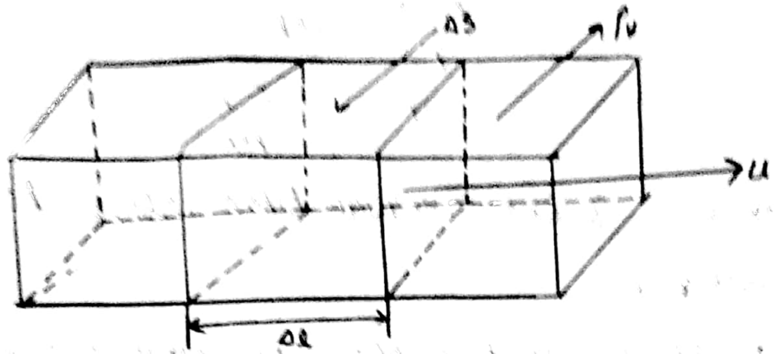
Convection current :-

→ convection current occurs when current flows through an insulating medium such as liquid, rarefeild gas, (or) a vacum.

→ A beam of electrons in a vaccum tube, for example, is a convention current.

→ Convection current does not involve conductors and hence does not satisfy ohm's law.

→ Consider a filament shown in figure.



current in a filament

→ If there is a flow of charge of density ρ_v at the velocity $u = u_y a_y$, then the current through the filament is

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta l}{\Delta t} = \rho_v \Delta S u_y$$

→ The y-directed current density J_y is given by,

$$J_y = \frac{\Delta I}{\Delta S} = \rho_v u_y$$

→ Hence, convection current density

$$J = \rho_v u.$$

Conduction current :-

→ Conduction current is defined as the current which is produced due to flow of electrons in the conductors on application of electric feild.

→ It requires a conductor.

→ When an electric feild E is applied, the force on an electron with charge $-e$ is

$$F = -eE.$$

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→ If the electron with mass m is moving in an electric field E with an average drift velocity u, according to Newtons law

$$\frac{mu}{\tau} = -eE$$

$$u = -\frac{e\tau}{m}E$$

→ where τ is the average time interval between the collisions.

→ If there are n electrons per unit volume, the electronic change density is given by

$$\rho_v = -ne$$

→ Thus the conductance current density is

$$J = \rho_v u = \frac{ne^2\tau}{m}E = \sigma E$$

where $\sigma = \frac{ne^2\tau}{m}$

→ The conduction current density is given by

$$\boxed{J = \sigma E}$$ (point from ohms law).

Dielectric constant :-

→ The dielectric constant (ϵ_r) is the ratio of the permitivity of the dielectric to that free space.

$$\boxed{\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e}$$

where χ_e is electric susceptability.

Dielectric Strength :-

→ The dielectric strength is the maximum electric feild that a dielectric can tolerate (or) withstand without breakdown.

Linear, Isotropic and homogeneous Dielectrics :-

→ A dielectric material is linear if ϵ does not change with the applied electric feild, isotropic if ϵ doesn't change with direction and homogeneous, if ϵ doesn't change from point to point.

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Continuity equation :-

→ According to the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume.

→ Thus current I comming out of closed surface is

$$I = \oint \bar{J} \cdot d\bar{s} = \frac{-dQ}{dt} \quad \text{——} (1)$$

where Q is the total charge enclosed by closed surface.

→ According to divergence theorem,

$$\oint_S \bar{J} \cdot d\bar{s} = \int_V \nabla \cdot \bar{J} dv \quad \text{——} (2)$$

→ But

$$-\frac{dQ}{dt} = -\frac{d \int_V \rho_v \, dv}{dt} = -\int_V \frac{\partial \rho_v}{\partial t} dv \quad \text{——} (3)$$

→ Subs ② and ③ in ①, we get

$$\int_V \nabla \cdot \bar{J}\, dv = -\int_V \frac{\partial \rho_v}{\partial t} dv$$

$$\boxed{\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}}$$

which is called the continuty of current equation.

→ For steady currents,

$$\frac{\partial \rho_v}{\partial t} = 0$$

and hence

$$\boxed{\nabla \cdot \bar{J} = 0}$$

Relaxation Time :-

→ Consider a linear and homogeneous medium with constants σ and ϵ for which current is flowing with current density.

$$\bar{J} = \sigma \bar{E} = \frac{\sigma}{\epsilon} \bar{D}$$

→ From continuity equation, we know that

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

$$\nabla \cdot \left(\frac{\sigma}{\epsilon} \bar{D}\right) = -\frac{\partial \rho_v}{\partial t}$$

$$\frac{\sigma}{\epsilon} \nabla \cdot \bar{D} = -\frac{\partial \rho_v}{\partial t}$$

→ From Gauss law, we know that

$$\nabla \cdot \bar{D} = \rho_v$$

$$\frac{\sigma}{\epsilon} \rho_v = -\frac{\partial \rho_v}{\partial t}$$

$$\frac{\partial}{\partial t}\rho_v + \frac{\sigma}{\epsilon}\rho_v = 0$$

→ This is a first order differential equation in ρ_v, whose solution is given by

$$\rho_v = \rho_{v0}\, e^{-t/\tau_r}$$

where ρ_{v0} is the intial charge density.

→ The charge density inside the material decays exponentially with time.

→ The time constants τ_r, is known as the relaxation time and is given by

$$\tau_r = \frac{\epsilon}{\sigma}$$

→ Relaxation time is time it takes a charge placed in the interior of a material to decay to 36.8 % of its intial value.

POISSON'S AND LAPLACE'S EQUATIONS :-

→ poisson's and laplace equations are easily derived from Gauss law (for a linear medium).

$$\nabla \cdot \bar{D} = \rho_v$$

$$\nabla \cdot \epsilon \bar{E} = \rho_v \quad \text{——} ①$$

$$\text{But } \bar{E} = -\nabla V \quad \text{——} ②$$

→ Substitute eqn ② in eqn ①, we get

$$\nabla \cdot \epsilon(-\nabla V) = \rho_v$$

$$-\epsilon \nabla^2 V = \rho_v$$

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}}$$

→ when $\rho_v = 0$, the above equation becomes

$$\boxed{\nabla^2 V = 0}$$

→ which is also known as laplacian equation

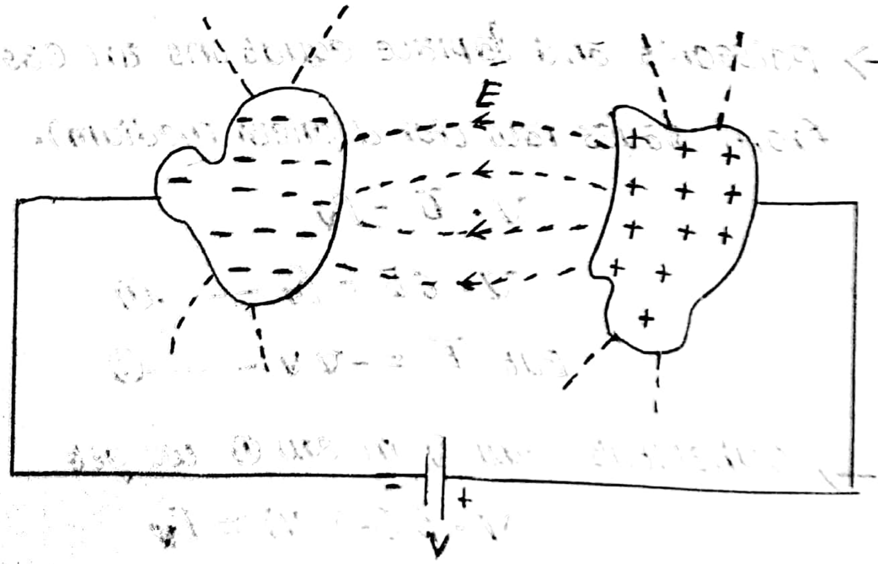
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0.$$

Capacitor :-

→ capacitor is a system which consists of two conducting surfaces carrying equal and opposite charges separated by a dielectric medium and giving rise to capacitance.



A two-conductor capacitor.

Capacitance :-

→ The capacitance C of the capacitor is defined as the ratio of magnitude of charge on one plate to potential difference between them.

$$\boxed{C = \frac{Q}{V} = \frac{\oint_s D \cdot ds}{-\int E \cdot dl}} \quad \text{(Farads)}.$$

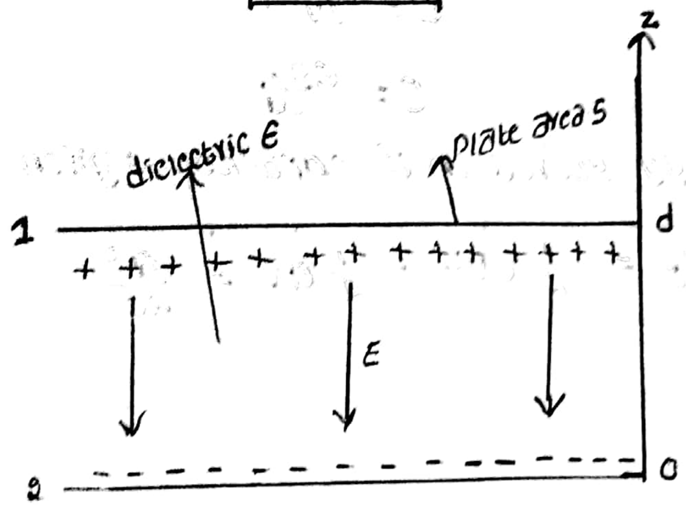
Parallel plate capacitor :-

→ Consider the parallel-plate capacitor as shown.

→ Let S be the area of each plate and d be the distance between two plates.

→ We assume that plates 1 and 2 carry charges +Q and -Q. so that

$$\boxed{\rho_s = Q/S}$$



parallel plate capacitor.

→ If the space between the plates is filled with a homogeneous dielectric with permitivity ε, then the electric feild intensity inside the place is given by

$$\overline{E} = -\frac{\rho_s}{\epsilon}\,\overline{a_z}$$

$$= -\frac{Q}{\epsilon S}\,\overline{a_z}$$

→ The potential difference between the plates is given by

$$V = -\int_2^1 \overline{E}\cdot \overline{dl}$$

$$V = -\int_0^d \left[-\frac{Q}{\epsilon S}\,\overline{a_z}\right]\cdot dz\,\overline{a_z}$$

$$V = \int_0^d \frac{Q}{\epsilon S}\,dz$$

$$V = \frac{Q}{\epsilon S}\int_0^d dz$$

$$V = \frac{Qd}{\epsilon S}$$

→ the capacitance for a parallel-plate capacitor is given by

$$C = Q/V$$

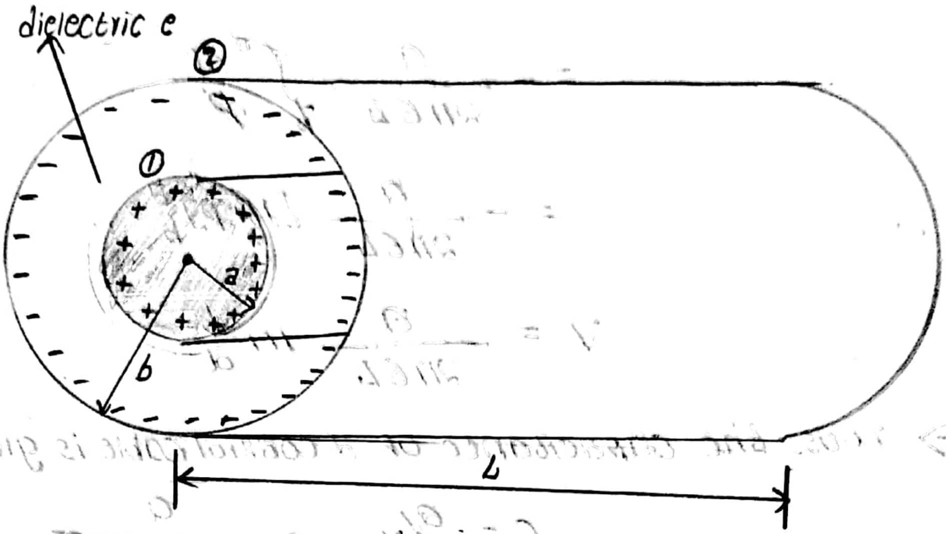
$$C = \epsilon S/d$$

→ The energy stored in a capacitor is given by

$$\boxed{W_E = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}}$$

Coaxial capacitor :-

→ consider the coaxial cable as shown in figure.



Coaxial capacitor.

→ Let a is the inner radius and b is the outer radius.

→ Length of co-axial cable is L.

→ Let the space between the conductors be filled with a homogeneous dielectric with permittivity ϵ.

→ We assume that conductors 1 and 2 carry charges $+Q$ and $-Q$ respectively.

→ The electric feild intensity due to infinite line charge is given by $\bar{E} = \frac{\rho_L}{2\pi\epsilon\rho}\,\bar{a}_\rho$

But, $\rho_L = Q/L$

$$\bar{E} = \frac{Q}{2\pi\epsilon\rho L}\,\bar{a}_\rho$$

→ The potential difference V is given by

$$V = -\int_2^1 \bar{E}\cdot d\bar{l}$$

$$V = -\int_b^a \left[\frac{Q}{2\pi\epsilon\rho L}\bar{a}_\rho\right] \cdot d\rho\, \bar{a}_\rho$$

$$= -\int_b^a \frac{Q}{2\pi\epsilon\rho L}\, d\rho$$

$$= -\frac{Q}{2\pi\epsilon L}\int_b^a \frac{d\rho}{\rho}$$

$$= -\frac{Q}{2\pi\epsilon L}\left[\ln\rho\right]_b^a$$

$$V = \frac{Q}{2\pi\epsilon L}\ln\frac{b}{a}$$

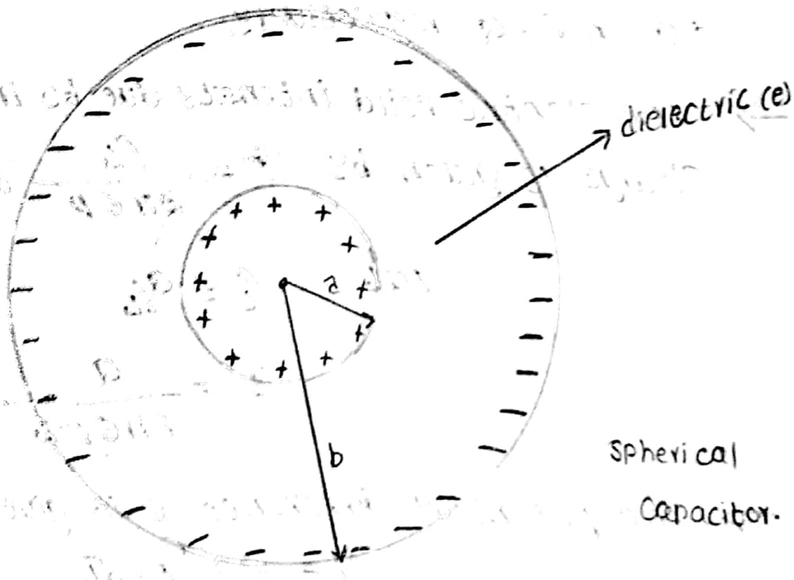
→ Thus the capacitance of a coaxial cable is given by

$$C = Q/V = \frac{Q}{\frac{Q}{2\pi\epsilon L}\cdot\ln\frac{b}{a}}$$

$$C = \frac{2\pi\epsilon L}{\ln b/a}$$

Spherical capacitor :-

→ Consider a spherical capacitor as shown in figure.



Spherical Capacitor.

→ The inner radius of sphere is a and outer radius of sphere is b. $(b>a)$.

→ Two spheres are seperated by a dielectric medium with permitivity ϵ.

→ Let us assume charges $+Q$ and $-Q$ on the inner and outer spheres respectively.

→ The electric feild intensity between two concentric circles (spheres) is given by

$$\bar{E} = \frac{Q}{4\pi\epsilon r^2} \cdot \bar{a_r}$$

→ The potential difference V between spheres is

$$V = -\int_2^1 \bar{E} \cdot d\bar{L}$$

$$= -\int_b^a \left[\frac{Q}{4\pi\epsilon r^2} \bar{a_r}\right] \cdot dr\, \bar{a_r}$$

$$= -\int_b^a \frac{Q}{4\pi\epsilon r^2} dr$$

$$= -\frac{Q}{4\pi\epsilon} \int_b^a \frac{dr}{r^2}$$

$$= -\frac{Q}{4\pi\epsilon} \left[\frac{-1}{r}\right]_b^a$$

$$= \frac{Q}{4\pi\epsilon} \left[\frac{1}{a} - \frac{1}{b}\right]$$

→ Thus the capacitance of spherical capacitor is

$$C = Q/V$$

$$C = \frac{Q}{\frac{Q}{4\pi\epsilon}\left[\frac{1}{a} - \frac{1}{b}\right]}$$

$$\boxed{C = \frac{4\pi\epsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]}}$$

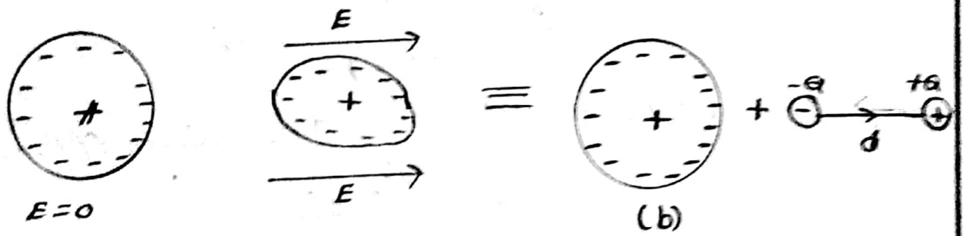
when $b \to \infty$, $C = 4\pi\epsilon a$

which is the capacitance of a spherical capacitor, whose outer plate is infinitely large i.e isolated sphere.

polarization in dielectrics :-

→ The dielectric is said to be polarized when a dipole results from displacements of the charges on the application of electric feild.

→ Dipole moment $p = qd$.



(a)

(b)

non-polar dielectrics :-

→ Nonpolar dielectric molecules do not posses dipoles until the application of electric. feild.

Eg :- hydrogen, oxygen, - - etc.

polar dielectrics :-

→ The polar dielectric molecules have built-in permanent dipoles that are randomly oriented.

Eg :- water, hcl acid, - - - etc.

Di-electric materials properties :-

→ The dielectrics do not have any free charges, but it have bounded charges.

→ The dielectrics are polarized under the application.

→ The dielectrics stores energy due to polarization.

UNIT 1 :- STATIC ELECTRIC FIELDS

Problem:1.1

Point charges 1mC and –2mC are located at (3, 2, -1) and (-1, -1, 4) respectively. Calculate the electric force on a 10nC charge located at (0, 3, 1) and the electric field intensity at that point.

Solution:

We know

$$\overline{F} = \frac{Q}{4\pi \in_0} \sum_{K=1}^{2} Q_K \frac{\overline{r} - \overline{r}_K}{\left|\overline{r} - \overline{r}_K\right|^3}$$

$$= \frac{10\times10^{-9}}{4\pi \in_0}\left[1\times10^{-3}\frac{\left(-3\overline{a}_x + \overline{a}_y + 2\overline{a}_z\right)}{\left(\sqrt{9+1+4}\right)^3} - 2\times10^{-3}\frac{\left(\overline{a}_x + 4\overline{a}_y - 3\overline{a}_z\right)}{\left(\sqrt{1+16+9}\right)^3}\right]$$

$$= 90\left[\frac{\left(-3\overline{a}_x + \overline{a}_y + 2\overline{a}_z\right)\times10^{-3}}{52.38} - 10^{-3}\frac{\left(2\overline{a}_x + 8\overline{a}_y - 6\overline{a}_z\right)}{132.57}\right]$$

$$= 90\times10^{-3}\left[\overline{a}_x\left(\frac{-3}{52.38} - \frac{2}{132.57}\right) + \overline{a}_y\left(\frac{1}{52.38} - \frac{8}{132.57}\right) + \overline{a}_z\left(\frac{2}{52.38} + \frac{6}{132.57}\right)\right]$$

$$= 90\times10^{-3}\left[-0.0723\overline{a}_x - 0.0413\overline{a}_y + 0.0834\overline{a}_z\right]$$

$$= -0.0065\,\overline{a}_x - 0.0037\,\overline{a}_y + 0.0075\,\overline{a}_z \quad \text{N.}$$

Also we know $\overline{E} = \dfrac{\overline{F}}{Q}$

$$= -\frac{0.0065}{10\times10^{-9}}\overline{a}_x - \frac{0.037}{10\times10^{-9}}\overline{a}_y + \frac{0.0075}{10\times10^{-9}}\overline{a}_z$$

$$= -650\,\overline{a}_x - 370\,\overline{a}_y + 750\,\overline{a}_z \quad \text{KV/m.}$$

Problem :1.2

Point charges 5nC and –2nC are located at $2\overline{a}_x + 4\overline{a}_z$ and $-3\overline{a}_x + 5\overline{a}_z$ respectively. (a) Determine the force on a 1nC point charge located at $\overline{a}_x - 3\overline{a}_y + 7\overline{a}_z$.(b) Find the electric field $\overline{E}$ *at* $\overline{a}_x - 3\overline{a}_y + 7\overline{a}_z$.

Solution:

(a) We know

$$\overline{F} = \frac{Q}{4\pi \in_0} \sum_{K=1}^{2} Q_K \frac{\overline{r} - \overline{r}_K}{\left|\overline{r} - \overline{r}_K\right|^3}$$

$$= 10^{-9} \times 9 \times 10^{9} \times 10^{-9} \left[5 \frac{\left(-\overline{a}_x - 3\overline{a}_y + 3\overline{a}_z\right)}{\left(\sqrt{1+9+9}\right)^3} - \frac{2\left(4\overline{a}_x - 3\overline{a}_y + 2\overline{a}_z\right)}{\left(\sqrt{16+9+4}\right)^3} \right]$$

$$= 9 \times 10^{-9} \left[\overline{a}_x \left(\frac{-5}{82.81} - \frac{8}{156.169} \right) + \overline{a}_y \left(\frac{-15}{82.81} + \frac{6}{156.169} \right) + \overline{a}_z \left(\frac{15}{82.81} - \frac{4}{156.169} \right) \right]$$

$$= 9 \times 10^{-9} \left[\overline{a}_x(-0.112) + \overline{a}_y(-0.143) + \overline{a}_z(0.155) \right]$$

$$= -1.008\,\overline{a}_x - 1.287\,\overline{a}_y + 1.395\,\overline{a}_z \quad nN$$

(b) $\overline{E} = \frac{\overline{F}}{Q}$, here Q=1nC

$$\therefore \overline{E} = -1.008\overline{a}_x - 1.287\overline{a}_y + 1.395\overline{a}_z\, V/m$$

***Problem:1.3**

Point charges Q_1 and Q_2 are respectively located at(4,0,-3) and (2,0,1). If Q_2=4nC, Find Q_1 such that (a) The $\overline{E}$ at (5,0,6) has no Z-component. (b) The force on a test charge at (5,0,6) has no X-component.

Solution:

We have $\overline{F} = \frac{Q}{4\pi \in_0} \sum_{K=1}^{2} Q_K \frac{\overline{r} - \overline{r}_K}{\left|\overline{r} - \overline{r}_K\right|^3}$

(a) $$\overline{E} = \frac{\overline{F}}{Q} = \frac{1}{4\pi \in_0} \left[\frac{Q_1 \left|(5,0,6) - (4,0,-3)\right|}{\left(\sqrt{1+81}\right)^3} + \frac{4 \times 10^{-9} \left|(5,0,6) - (2,0,1)\right|}{\left(\sqrt{9+25}\right)^3} \right]$$

Given $\overline{E}$ has no Z-component $$0 = \frac{1}{4\pi \in_0} \left[\frac{Q_1 \times 9}{\left(\sqrt{82}\right)^3} + \frac{4 \times 10^{-9} \times 5}{\left(\sqrt{34}\right)^3} \right]$$

$$\frac{Q_1 \times 9}{\left(\sqrt{82}\right)^3} = -\frac{4 \times 10^{-9} \times 5}{\left(\sqrt{34}\right)^3}$$

$$Q_1 = -\frac{20}{9} \left(\sqrt{\frac{41}{17}} \right)^3 nC = -8.3nC$$

(b) Given the force on test charge has no X-component

$$0=\frac{Q}{4\pi\in_0}\left[\frac{Q_1}{\left(\sqrt{82}\right)^3}+\frac{4\times10^{-9}\times3}{\left(\sqrt{34}\right)^3}\right]$$

$$\frac{Q_1}{\left(\sqrt{82}\right)^3}=-\frac{4\times10^{-9}\times3}{\left(\sqrt{34}\right)^3}$$

$$Q_1=-12\left(\sqrt{\frac{41}{17}}\right)^3 nC=-44.95nC$$

Problem:1.4

Two point charges of equal mass 'm', charge 'Q' are suspended at a common point by two threads of negligible mass and length 'l'. Show that at equilibrium the inclination angle 'α' of each thread to the vertical is given by $Q^2 = 16\ \pi\in_0\ mgl^2\ \sin^2\alpha$ $\tan\alpha$, (or) $\frac{\tan^3\alpha}{1+\tan^2\alpha}=\frac{Q^2}{16\pi\in_0 mgl^2}$,

if 'α' is very small

Show that $\alpha=\sqrt[3]{\frac{Q^2}{16\pi\in_0 mgl^2}}$

Solution:

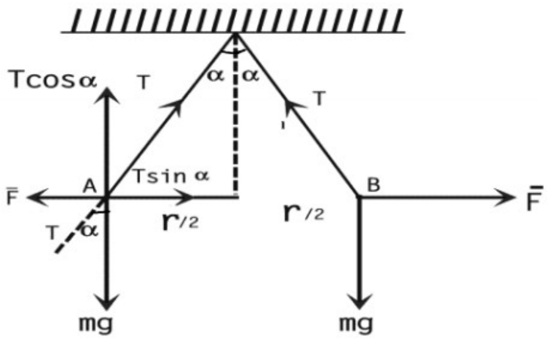


Fig: 1.3 suspended charge particles

When two charges are suspended from a common point with threads of length 'l', we can represent graphically as sown in Fig:1.3, where T is the tension in thread 'mg' is the weight of charge towards ground due to gravitational force and $\overline{F}$ is force on charge at 'A'(B) due to charge at 'B'(A). T cos α is the vertical component of 'T' which is upwards and T sin α is the horizontal component of 'T' which is opposite to $\overline{F}$. To form equilibrium either at 'A' or 'B'

$$T\cos\alpha = mg \tag{1.3.1}$$

$$T\sin\alpha = \overline{F} \tag{1.3.2}$$

$$\frac{(1.3.1)}{(1.3.2)} = \frac{T\sin\alpha}{T\cos\alpha} = \frac{\overline{F}}{mg}$$

$$\Rightarrow Tan\,\alpha = \frac{\overline{F}}{mg}$$

where $\overline{F} = \dfrac{Q^2}{4\pi \in_0 r^2}$ from Fig:1.3 $\sin\alpha = \dfrac{r/2}{l}$

$$\Rightarrow r = 2l\sin\alpha$$

$$Tan\,\alpha = \frac{Q^2}{4mg\pi \in_0 r^2}$$

$$= \frac{Q^2}{4mg\pi \in_0 4l^2 \sin^2\alpha}$$

$$Tan\alpha = \frac{Q^2}{16mgl^2\pi \in_0 \sin^2\alpha}$$

$$Sin^2\alpha\, Tan\,\alpha = \frac{Q^2}{16mgl^2\pi \in_0} \qquad (1.3.3)$$

$$\Rightarrow Q^2 = 16\pi \in_0 mgl^2 Sin^2\alpha\, Tan\,\alpha \qquad (1.3.4)$$

From (1.3.3)

$$Cos^2\alpha \frac{Sin^2\alpha}{Cos^2\alpha} Tan\,\alpha = \frac{Q^2}{16\pi \in_0 mgl^2}$$

$$\frac{Tan^3\alpha}{Sec^2\alpha} = \frac{Q^2}{16\pi \in_0 mgl^2}$$

$$\frac{Tan^3\alpha}{1+Tan^2\alpha} = \frac{Q^2}{16\pi \in_0 mgl^2}$$

If α is very small, $\sin\alpha = \text{Tan}\,\alpha = \alpha$

From (1.3.4) $Q^2 = 16\pi\in_0 \text{mg}\, l^2 \alpha^3$

$$\alpha^3 = \frac{Q^2}{16\pi \in_0 mg\, l^2}$$

$$\alpha = \sqrt[3]{\frac{Q^2}{16\pi \in_0 mg\, l^2}}$$

Problem:1.5

Two small identical conducting spheres have charges of $2x10^{-9}$ and $-0.5x10^{-9}C$ respectively. (a) When they are placed 4cm apart what is the force between them? (b) If they are brought into contact and then separated by 4cm. What is the force between them?

Solution:

(a) We know

$$\overline{F} = \frac{Q_1 Q_2}{4\pi \in_0 R^2}$$

$$= \frac{-2\times10^{-9}\times0.5\times10^{-9}\times9\times10^{9}}{4\times10^{-4}\times4}$$

$= -5.625\ \mu N$

(b) when they are brought into contact, charges will be added and again when they are separated charge will be distributed equally

$Q_1 = 0.758x10^{-9}C$ $Q_2 = 0.75x10^{-9}C$

$$\overline{F} = 3.164\ \mu N$$

Problem:1.6

If the charges in the above problem are separated with the same distance in a kerosene ($\in_r = 2$), then find (a) and (b) as in the previous problem.

Solution:

(a)

$$\overline{F_k} = \frac{-5.625}{2}\ \mu N$$

$= -2.8125\ \mu N$

(b) $\overline{F_k} = \frac{3.164}{2}$ $= 1.582\ \mu N$

Problem:1.7

Three equal +Ve charges of $4x10^{-9}C$ each are located at 3 corners of a square, side 20cm. Determine the magnitude and direction of the electric field at the vacant corner point of the square.

Solution:

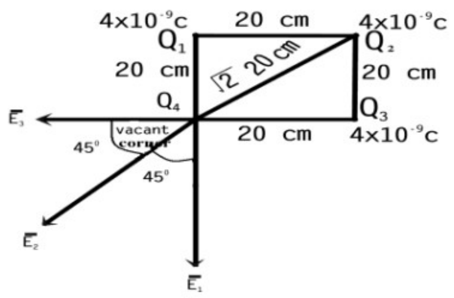


Fig:1.4

$\overline{E}_1$ = Electric field intensity at Q_4 due to Q_1

$$= \frac{Q_1}{4\pi \in_0 R^2}$$

= 900 V/m

$\overline{E}_2$ = 450 V/m

$\overline{E}_3$ = 900 V/m

The electric field intensity at vacant point is

$$\overline{E} = \overline{E}_2 + \overline{E}_1 \cos 45^0 + \overline{E}_3 \cos 45^0$$

$$= 450 + \frac{900}{\sqrt{2}} + \frac{900}{\sqrt{2}}$$

$$= 450 + 900\sqrt{2}$$

= 1722.792206 V/m

Problem: 1.8

A circular ring of radius 'a' carries a uniform charge ρ_L C/m and is placed on the XY plane with axis the same as the Z-axis.

(a) Show that $\overline{E}(0,0,h) = \dfrac{\rho_L ah}{2\in_0 \left(h^2 + a^2\right)^{3/2}} \overline{a}_z$.

(b) What values of h gives the maximum value of $\overline{E}$

(c) If the total charge on the ring is Q. Find $\overline{E}$ as 'a' tends to zero.

Solution:

(a)

Here dl = a dϕ

dQ = ρ_L dl

= ρ_L a$d\phi$

$$\therefore d\bar{E} = \frac{dQ}{4\pi \in_0 R^2}\bar{a}_r$$

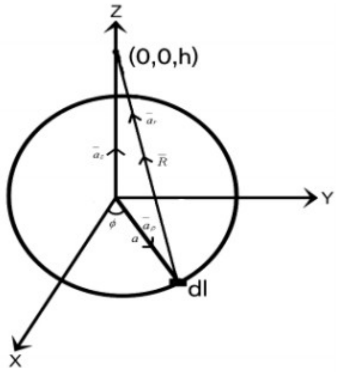


Fig:1.15

$$\bar{a}_r = \frac{\bar{R}}{|R|};\ \frac{\bar{a}_r}{R^2} = \frac{\bar{R}}{R^3}$$

$$\therefore d\bar{E} = \frac{dQ}{4\pi \in_0} \frac{\left[-a\,\bar{a}_\rho + h\bar{a}_z\right]}{\left(a^2+h^2\right)^{3/2}}$$

$dQ = \rho_L\ a\ d\phi$

$$Q = \int \rho_L\, a\, d\phi$$

when we add up electric fields, the electric field in ρ direction gets cancelled.

$$\therefore\ \bar{E} = \frac{dQ}{4\pi \in_0} \frac{h\bar{a}_z}{\left(a^2+h^2\right)^{3/2}}$$

$$= \int \frac{\rho_L a\, d\phi}{4\pi \in_0} \frac{h\bar{a}_z}{\left(a^2+h^2\right)^{3/2}}$$

$$= \frac{\rho_L a}{4\pi \in_0} \frac{h\bar{a}_z}{\left(a^2+h^2\right)^{3/2}} \int_0^{2\pi} d\phi = \frac{\rho_L a h}{2 \in_0 \left(a^2+h^2\right)^{3/2}}\bar{a}_z$$

(b) $\frac{d\bar{E}}{dh} = 0$

$$\frac{\rho_L a}{2\in_0}\bar{a}_z \frac{\left(a^2+h^2\right)^{3/2}.1 - h\frac{3}{2}(a^2+h^2)^{1/2}\,2h}{\left(a^2+h^2\right)^3} = 0$$

$(a^2 + h^2) - 3h^2 = 0$

$a^2 - 2h^2 = 0$

$2h^2 = a^2$

$$h = \pm \frac{a}{\sqrt{2}}$$

(c) When 'a' tends to zero, it becomes a point charge 'Q' located at origin and we have to find electric field at (0,0,h) due to point charge 'Q' located at origin.

$$\therefore \quad \overline{E} = \frac{Q}{4\pi \in_0 h^2} \overline{a}_z$$

***Problem: 1.9**

Derive an expression for the electric field strength due to a circular ring of radius 'a' and uniform charge density ρ_L C/m. Obtain the value of height 'h' along Z-axis at which the net electric field becomes zero. Assume the ring to be placed in X-Y plane.

Solution:

Derivation is as in Problem: 1.8.

$$\overline{E} = \frac{\rho_L a h}{2 \in_0 \left(a^2 + h^2\right)^{3/2}} \overline{a}_z$$

Which can be written as

$$\overline{E} = \frac{\rho_L a}{2 \in_0 h^2 \left(\frac{a^2}{h^2} + 1\right)^{3/2}} \overline{a}_z$$

From the above equation we can say that for h=∞, the net electric field becomes zero.

***Problem: 1.10**

A circular ring of radius 'a' carries uniform charge ρ_L C/m and is in XY-plane. Find the Electric field at point (0,0,2) along its axis.

Solution:

Replacing 'h' in problem:1.8 with '2' and solving, we get

$$\overline{E} = \frac{\rho_L a 2}{2 \in_0 \left(a^2 + 4\right)^{3/2}} \overline{a}_z$$

Problem: 1.11

A circular disk of radius 'a' is uniformly charged with ρ_s C/m². If the disk lies on the Z=0 plane with it's axis along the Z-axis

(a) Show that at point (0, 0, h), $\bar{E} = \frac{\rho_s}{2\in_0}\left[1-\frac{h}{\left(h^2+a^2\right)^{1/2}}\right]\bar{a}_z$

(b) From this derive the $\bar{E}$ due to an infinite sheet of charge on the Z=0 plane.

(c) If a <<h, Show that $\bar{E}$ is similar to the field due to a point charge.

Solution:

(a)

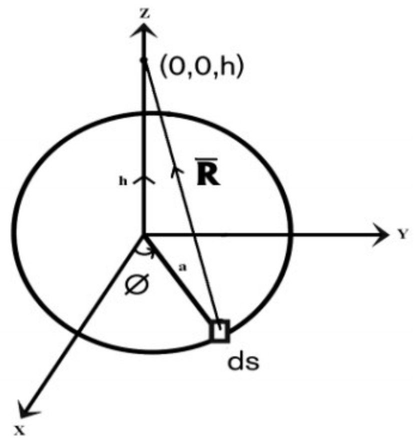


Fig:1.17

$$d\bar{E} = \frac{dQ}{4\pi \in_0 R^2}\bar{a}_r$$

dQ = ρ_s ds ; ds=dρ.ρdϕ,

= ρ_s ρdρ dϕ

$$\rho\bar{a}_\rho + \bar{R} = h\bar{a}_z$$

$$\bar{R} = h\bar{a}_z - \rho\bar{a}_\rho$$

$$\bar{E} = \int_S \frac{\rho_s \rho\, d\rho\, d\phi}{4\pi \in_0}\frac{\left(h\bar{a}_z - \rho\bar{a}_\rho\right)}{\left(h^2+\rho^2\right)^{3/2}}$$

$$\bar{E} = \frac{\rho_s}{4\pi \in_0}\bar{a}_z\int_0^{2\pi} d\phi\int_0^a \frac{\rho h}{\left(h^2+\rho^2\right)^{3/2}}d\rho$$

$$= \frac{\rho_s}{4\pi \in_0}\bar{a}_z 2\pi h\int_0^a \frac{1}{2}\left(h^2+\rho^2\right)^{-3/2} d\left(\rho^2\right)$$

$$= \frac{\rho_s h}{2\in_0}\bar{a}_z \frac{1}{2}\left[\frac{\left(h^2+\rho^2\right)^{\frac{-3}{2}+1}}{\frac{-3}{2}+1}\right]_0^a$$

$$= \frac{\rho_s h}{4\in_0}\bar{a}_z \left\{-2\left[\left(h^2+a^2\right)^{-1/2}-\left(h^2\right)^{-1/2}\right]\right\}$$

$$= \frac{-\rho_s h\bar{a}_z}{2\in_0}\left[\frac{1}{\sqrt{\left(h^2+a^2\right)}}-\frac{1}{h}\right]$$

$$\bar{E} = \frac{\rho_s}{2\in_0}\left[1-\frac{h}{\left(h^2+a^2\right)^{1/2}}\right]\bar{a}_z$$

(b) a $\rightarrow \infty$; $\therefore \bar{E} = \dfrac{\rho_s}{2\in_0}\bar{a}_z$

(c) when a<<h, the volume charge density becomes a point charge located at origin, $\therefore \bar{E} = \dfrac{Q}{4\pi\in_0 h^2}\bar{a}_z$

Problem: 1.12

The finite sheet 0 < x < 1, 0 < y < 1 on the Z=0 plane has a charge density $\rho_s = xy(x^2 + y^2 + 25)^{3/2}$ nC/m^2.
Find (a) the total charge on the sheet

(b) the electric field at (0, 0, 5)

(c) the force experienced by a –1nC charge located at (0, 0, 5)

Solution:

(a) dQ = ρ_s ds

Q = $\int_s \rho_s ds$

$$= \int_{x=0}^{1}\int_{y=0}^{1} xy\left(x^2+y^2+25\right)^{3/2} ndxdy$$

$$= n\int_{x=0}^{1} x\int_{y=0}^{1}\left(x^2+y^2+25\right)^{3/2}\frac{1}{2}d(y^2)dx$$

$$= n \int_{x=0}^{1} x \left[\left(x^2 + y^2 + 25\right)^{5/2} \right]_0^1 \frac{2}{5} \frac{1}{2} dx$$

$$= \frac{n}{5} \int_{x=0}^{1} \left[\left(x^2 + 26\right)^{5/2} - \left(x^2 + 25\right)^{5/2} \right] \frac{1}{2} d(x^2)$$

$$= \frac{n}{5} \left[\left(x^2 + 26\right)^{7/2} - \left(x^2 + 25\right)^{7/2} \right]_0^1 \frac{1}{7}$$

$$= \frac{n}{35} \left[(27)^{7/2} - 2(26)^{7/2} + (25)^{7/2} \right]$$

$$= \frac{n}{35} [102275.868136 - 179240.733942 + 78125]$$

Q = 33.15nC

(b) Electric field at (0, 0, 5)

$$d\bar{E} = \frac{\rho_s ds}{4\pi \in_0 R^2} \bar{a}_R$$; on Z-plane point is *(x,y,0)*

$$\therefore \bar{R} = (0,0,5) - (x, y, 0) = -x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z$$

$$\frac{\bar{a}_R}{R^2} = \frac{\bar{R}}{|\bar{R}|^3} = \frac{-x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z}{\left(\sqrt{x^2 + y^2 + 25}\right)^3}$$

$$\bar{E} = \int_S \frac{\rho_S ds}{4\pi \in_0} \frac{\bar{R}}{|\bar{R}|^3}$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \frac{xy\left(x^2 + y^2 + 25\right)^{3/2} \times 10^{-9}}{4\pi \in_0} \left(\frac{-x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z}{\left(\sqrt{x^2 + y^2 + 25}\right)^3} \right) dx\, dy$$

$$= \frac{1}{4\pi \in_0} \int_{x=0}^{1} \int_{y=0}^{1} -x^2 y\bar{a}_x - xy^2\bar{a}_y + 5xy\bar{a}_z dx\, dy \times 10^{-9}$$

$$= \frac{1}{4\pi \in_0} \int_{x=0}^{1} -x^2 \left[\frac{y^2}{2} \right]_0^1 \bar{a}_x - x\left[\frac{y^3}{3} \right]_0^1 \bar{a}_y + 5x\left[\frac{y^2}{2} \right]_0^1 \bar{a}_z\, dx \times 10^{-9}$$

$$= \frac{1}{4\pi \epsilon_0} \int_{x=0}^{1} -\frac{x^2}{2}\bar{a}_x - \frac{x}{3}\bar{a}_y + \frac{5}{2}x\bar{a}_z \, dx \times 10^{-9}$$

$$= \frac{1}{4\pi \epsilon_0} \left[\left[-\frac{x^3}{6} \right]_0^1 \bar{a}_x - \left[\frac{x^2}{6} \right]_0^1 \bar{a}_y + \frac{5}{2}\left[\frac{x^2}{2} \right]_0^1 \bar{a}_z \right] \times 10^{-9}$$

$$= \frac{1}{4\pi \epsilon_0} \left[-\frac{1}{6}\bar{a}_x - \frac{1}{6}\bar{a}_y + \frac{5}{4}\bar{a}_z \right] \times 10^{-9}$$

$$= 9 \times 10^9 \left[-\frac{1}{6}\bar{a}_x - \frac{1}{6}\bar{a}_y + \frac{5}{4}\bar{a}_z \right] \times 10^{-9}$$

$$= -1.5\bar{a}_x - 1.5\bar{a}_y + 11.25\bar{a}_z \text{ V/m}$$

(c) $\bar{\text{F}} = \text{q}\bar{E}$

$$= (-1nC)\left[-1.5\bar{a}_x - 1.5\bar{a}_y + 11.25\bar{a}_z \right]$$

$$= 1.5\bar{a}_x + 1.5\bar{a}_y - 11.25\bar{a}_z \text{ nN}$$

Problem: 1.13

A square plane described by $-2 < x < 2$, $-2 < y < 2$, $z = 0$ carries a charge density $12|y|$ mC/m^2. Find the total charge on the plate and the electric field intensity at (0, 0, 10)

Solution:

$dQ = \rho_s \, ds$

$$Q = \int_s \rho_s ds$$

$$= \int_{x=-2}^{2} \int_{y=-2}^{2} 12|y| \times 10^{-3} dx dy$$

$$= 10^{-3} \int_{x=-2}^{2} \left[\int_{y=-2}^{0} -12y \, dy + \int_{y=0}^{2} 12y \, dy \right] dx$$

$$= 10^{-3} \int_{x=-2}^{2} -12\left[\frac{y^2}{2} \right]_{-2}^{0} + 12\left[\frac{y^2}{2} \right]_0^2 dx$$

$$= 10^{-3} \int_{x=-2}^{2} 12(2)+12(2)dx$$

$$= 48\times10^{-3} \int_{x=-2}^{2} dx = 48\times10^{-3}\times4 = 192mC$$

$$d\overline{E} = \frac{\rho_s ds}{4\pi \in_0 R^2}\overline{a}_R ;\ \overline{R} = (0,0,10)-(x,y,0) = -x\overline{a}_x - y\overline{a}_y + 10\overline{a}_z$$

$$d\overline{E} = \frac{\rho_s ds}{4\pi \in_0}\frac{\overline{R}}{R^3}$$

$$\overline{E} = \int_s \frac{\rho_s ds}{4\pi \in_0}\frac{\overline{R}}{R^3}$$

$$= \int_{x=-2}^{2}\int_{y=-2}^{2} \frac{12\,|\,y\,|\times10^{-3}}{4\pi \in_0}\left(\frac{-x\overline{a}_x - y\overline{a}_y + 10\overline{a}_z}{\left(\sqrt{x^2+y^2+100}\right)^3}\right)dx\,dy$$

$$= 9\times10^6\times12\int_{x=-2}^{2}\left[\int_{y=-2}^{0}\frac{xy\overline{a}_x + y^2\overline{a}_y - 10y\overline{a}_z}{\left(x^2+y^2+100\right)^{3/2}}dy + \int_{y=0}^{2}\frac{-xy\overline{a}_x - y^2\overline{a}_y + 10y\overline{a}_z}{\left(x^2+y^2+100\right)^{3/2}}dy\right]dx$$

Replacing y with –y in the first integral and simplifying

$$\overline{E} = 108\times10^6\int_{x=-2}^{2}\left[\int_{y=0}^{2}\frac{-2xy\overline{a}_x + 20y\overline{a}_z}{\left(x^2+y^2+100\right)^{3/2}}dy\right]dx$$

$$= 108\times10^6\int_{x=-2}^{2}\left[-x\int_{y=0}^{2}2y\overline{a}_x\left(x^2+y^2+100\right)^{-3/2}dy + 10\int_{y=0}^{2}2y\overline{a}_z\left(x^2+y^2+100\right)^{-3/2}dy\right]dx$$

$$= 108\times10^6\int_{x=-2}^{2}\left[-x\int_{y=0}^{2}\overline{a}_x\left(x^2+y^2+100\right)^{-3/2}d(y^2) + 10\int_{y=0}^{2}\overline{a}_z\left(x^2+y^2+100\right)^{-3/2}d(y^2)\right]dx$$

$$= 108\times10^6\int_{x=-2}^{2}\left[-x\left[\frac{\left(x^2+y^2+100\right)^{-1/2}}{-1/2}\right]_0^2\overline{a}_x + 10\left[\frac{\left(x^2+y^2+100\right)^{-1/2}}{-1/2}\right]_0^2\overline{a}_z\right]dx$$

$$= 108\times10^6 \int_{x=-2}^{2} \left\{ \left[2x\left(x^2+104\right)^{-1/2} - 2x\left(x^2+100\right)^{-1/2} \right]\bar{a}_x - 20\left[\left(x^2+104\right)^{-1/2} - \left(x^2+100\right)^{-1/2} \right]\bar{a}_z \right\} dx$$

$$\because x\left(x^2+104\right)^{-1/2} \;\&\; x\left(x^2+100\right)^{-1/2} \; are\, odd\;\; functions$$

$$and \quad \left(x^2+104\right)^{-1/2} \;\&\; \left(x^2+100\right)^{-1/2} \; are\, even\; functions$$

$$\int_{-a}^{a} f(x)dx \quad = \quad 0 \quad if \quad f \quad is \quad odd$$

$$= \quad 2\int_{0}^{a} f(x)dx \quad if \quad f \quad is \quad even$$

$$\therefore \bar{E} = -20\times108\times10^6\times2\int_{x=0}^{2} \left[\frac{1}{\sqrt{x^2+\left(\sqrt{104}\right)^2}} - \frac{1}{\sqrt{x^2+10^2}} \right]\bar{a}_z dx$$

$$= -40\times108\times10^6 \left[\sinh^{-1}\left(\frac{x}{\sqrt{104}}\right) - \sinh^{-1}\left(\frac{x}{10}\right) \right]_0^2 \bar{a}_z$$

$$= -40\times108\times10^6 \left[\sinh^{-1}\left(\frac{2}{\sqrt{104}}\right) - \sinh^{-1}\left(\frac{1}{5}\right) \right]\bar{a}_z$$

$$= -40\times108\times10^6 \left[0.19488 - 0.19869\right]\bar{a}_z$$

$\bar{E} = 16.46\,\bar{a}_z$ MV/m.

Problem: 1.14

Determine $\bar{D}$ at (4, 0, 3) if there is a point charge -5π mC at (4, 0, 0) and a line charge 3π mC/m along the Y-axis

Solution:

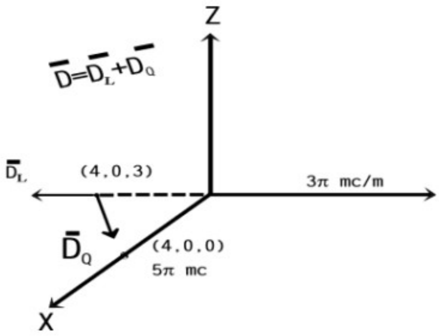


Fig: 1.20

$$\overline{D}_Q = \frac{Q}{4\pi}\frac{\overline{a}_R}{R^2} \qquad \text{Where } \overline{R} = (4,0,3)-(4,0,0) = (0,0,3)$$

$$= \frac{-5\pi}{4\pi}\frac{3\overline{a}_z \times 10^{-3}}{(9)^{3/2}}$$

$$= \frac{-5}{4}\frac{3\overline{a}_z \times 10^{-3}}{27} = \frac{-5\overline{a}_z \times 10^{-3}}{36} = -0.139\overline{a}_z \times 10^{-3}\ \text{C/m}^2.$$

$$\overline{a}_\rho = \frac{\overline{\rho}}{|\overline{\rho}|}$$

$$\overline{\rho} = (4,0,3)-(0,0,0) = 4\overline{a}_x + 3\overline{a}_z$$

$$\overline{D}_L = \frac{\rho_L}{2\pi\rho}\overline{a}_\rho$$

$$= \frac{3\pi}{2\pi}\times 10^{-3}\,\frac{4\overline{a}_x + 3\overline{a}_z}{25}$$

$$= 0.24\overline{a}_x + 0.18\overline{a}_z\ \text{mC/m}^2.$$

$$\overline{D} = \overline{D}_Q + \overline{D}_L = 240\overline{a}_x + 42\overline{a}_z \quad \mu C/m^2$$

Problem:1.15

Find the gradient of the following scalar fields

(a) $V = e^{-z}\sin 2x \cos hy$
(b) $U = \rho^2 z \cos 2\phi$
(c) $W = 10r \sin^2\theta \cos\phi$

Solution:

(a) Since given V is in x and y, consider gradient in Cartesian co-ordinate system

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$$

$$= e^{-z}\cosh y \cos 2x\, 2\bar{a}_x + e^{-z}\sin 2x \sinh y\,\bar{a}_y + \sin 2x \cosh y\, e^{-z}(-1)\bar{a}_z$$

$$= 2\cos 2x \cosh y\, e^{-z}\bar{a}_x + \sin 2x \sinh y\, e^{-z}\,\bar{a}_y - \sin 2x \cosh y\, e^{-z}\bar{a}_z$$

(b) Since given U is in ρ, z and ϕ, consider gradient in cylindrical co-ordinate system

$$\nabla U = \frac{\partial U}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\bar{a}_\phi + \frac{\partial U}{\partial z}\bar{a}_z$$

$$= Z\cos 2\phi\, 2\rho\,\bar{a}_\rho + \rho z(-\sin 2\phi)2\bar{a}_\phi + \rho^2\cos 2\phi\,\bar{a}_z$$

(c) Since given W is in r,θ and ϕ, consider gradient in spherical co-ordinate system

$$\nabla W = \frac{\partial W}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\bar{a}_\phi$$

$$= 10\sin^2\theta\cos\phi\,\bar{a}_r + \left(\frac{10r}{r}\right)2\sin\theta\cos\theta\cos\phi\,\bar{a}_\theta + 10r\sin^2\theta(-\sin\phi)\bar{a}_\phi.\frac{1}{r\sin\theta}$$

Problem:1.16

Determine the divergence of the following vector fields.

(a) $\overline{P} = x^2 yz\,\bar{a}_x + x^3 zy\,\bar{a}_y + xy^2 z^3\bar{a}_z$

(b) $\overline{Q} = \rho\sin\phi\,\bar{a}_\rho + \rho^2 z\,\bar{a}_\phi + z\cos\phi\,\bar{a}_z$

(c) $\overline{T} = \frac{1}{r^2}\cos\theta\;\bar{a}_r + r\sin\theta\cos\phi\,\bar{a}_\theta + \cos\theta\,\bar{a}_\phi$

(d) $\overline{N} = r^3\sin\theta\;\bar{a}_r + \sin 2\theta\cos^2\phi\,\bar{a}_\theta + \cos\theta\, r^2\,\bar{a}_\phi$

Solution:

(a) Given $\overline{P} = x^2 yz\,\bar{a}_x + x^3 zy\,\bar{a}_y + xy^2 z^3\bar{a}_z$

$$\nabla\cdot\overline{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$$

$= 2xyz + x^3z + 3xy^2z^2$

(b)Given $\overline{Q} = \rho \sin\phi\,\overline{a}_\rho + \rho^2 z\,\overline{a}_\phi + z\cos\phi\,\overline{a}_z$

$$\nabla \cdot \overline{Q} = \frac{1}{\rho}\frac{\partial(\rho Q_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(Q_\phi)}{\partial \phi} + \frac{\partial Q_z}{\partial z}$$

$$= \frac{1}{\rho} 2\rho \sin\phi + \frac{1}{\rho}(0) + \cos\phi$$

$= 2\sin\phi + \cos\phi$

(c) Given $\overline{T} = \frac{1}{r^2}\cos\theta\ \overline{a}_r + r\sin\theta\cos\phi\,\overline{a}_\theta + \cos\theta\,\overline{a}_\phi$

$$\nabla \cdot \overline{T} = \frac{1}{r^2}\frac{\partial(r^2 T_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, T_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_\phi}{\partial \phi}$$

$$= \frac{1}{r^2}(0) + \frac{1}{r\sin\theta} r\, 2\sin\theta\cos\theta\cos\phi + \frac{1}{r\sin\theta}(0)$$

$= 2\cos\theta\cos\phi$

(d) Given $\overline{N} = r^3\sin\theta\ \overline{a}_r + \sin 2\theta\cos^2\phi\,\overline{a}_\theta + \cos\theta\, r^2\,\overline{a}_\phi$

$$\nabla \cdot \overline{N} = \frac{1}{r^2}\frac{\partial(r^2 N_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, N_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial N_\phi}{\partial \phi}$$

$$= \frac{1}{r^2} 5r^4 \sin\theta + \frac{1}{r\sin\theta}\frac{1}{2}\left(-\sin\theta + \frac{\sin 3\theta}{3}\right)\cos^2\phi + \frac{1}{r\sin\theta}(0)$$

$$= 5r^2\sin\theta - \frac{1}{2r}\cos^2\phi + \frac{\sin 3\theta}{6r\sin\theta}\cos^2\phi$$

Problem:1.17

Given $\bar{D} = z\rho\cos^2\phi\,\bar{a}_z$ C/m². Calculate the charge density at (1, π/4, 3) and the total charge enclosed by the cylinder of radius 1m with $-2 \le z \le 2$ m.

Solution:

We know

$$\rho_v = \nabla \cdot \bar{D}$$

in cylindrical co-ordinate system the divergence can be written as

$$\rho_v = \frac{1}{\rho}\frac{\partial(\rho D_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(D_\phi)}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$\rho_v = \dfrac{\partial D_z}{\partial z}$ since $\bar{D}$ has only Z- component

$$\rho_v = \rho \cos^2 \phi$$

$$(\rho_v)_{\left(1,\frac{\pi}{4},3\right)} = (1)\cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2} \text{ C/m}^3$$

change enclosed = Q_{enc} = $\int_v \rho_v dv$ where dv = $\rho d\rho d\phi dz$

$$Q_{enc} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} \int_{z=-2}^{2} \rho \cos^2 \phi \rho \, d\rho \, d\phi \, dz$$

$$= \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} \rho^2 \cos^2 \phi (4) d\rho \, d\phi$$

$$= 4 \int_{\rho=0}^{1} \rho^2 \left[\frac{1}{2}(2\pi) + \frac{1}{2}\sin 4\phi\right] d\rho$$

$$= 4\pi \int_{\rho=0}^{1} \rho^2 d\rho = 4\pi \left[\frac{\rho^3}{3}\right]_0^1 = \frac{4\pi}{3} \text{ C.}$$

Problem:1.18

If $\bar{D} = \left(2y^2 + z\right)\bar{a}_x + 4xy\bar{a}_y + x\bar{a}_z$ C/m². Find

(a) the volume charge density at (-1, 0, 3)

(b) the flux through the cube defined by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

(c) the total charge enclosed by the cube

Solution:

According to Maxwell's I equation

$$\rho_v = \nabla \cdot \bar{D}$$

$$\rho_v = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$= 0 + 4x + 0$$

= 4x C/m3

(a) $\left(\rho_v\right)_{(-1,0,3)} = 4(-1) = -4$ C/m2

(b) & (c) $\psi = \int_v \rho_v dv = Q_{\text{enc}}$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} 4x\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 4x(1)\,dx\,dy$$

$$= \int_{x=0}^{1} 4x(1)\,dx$$

$$= 4\left[\frac{x^2}{2}\right]_0^1 = \frac{4}{2} = 2C$$

Problem: 1.19

Given the electric flux density $\bar{D} = 0.3r^2\bar{a}_r$ nC/m2, in free space. Find (a) $\bar{E}$ at point (2, 25^0, 90^0)

(b) the total charge within the sphere r = 3

(c) the total electric flux leaving the sphere r = 4

Solution:

(a)

Given $\bar{D} = 0.3r^2\bar{a}_r$ nC/m2

$$\therefore \quad \bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{0.3r^2\bar{a}_r}{8.854\times10^{-12}}$$

$$\left(\bar{E}\right)_{(2,25^0,90^0)} = \frac{0.3(4)}{8.854\times10^{-12}}\bar{a}_r = 1.355\times10^{11}\bar{a}_r\times10^{-9} = 135.5\ \bar{a}_r \text{ V/m}$$

(b) we know $\rho_v = \nabla\cdot\bar{D}$

$$= \frac{1}{r^2}\frac{\partial\left(r^2 D_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta\, D_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} =$$

$$\frac{1}{r^2}0.3(4)r^3 = 1.2r$$

Also know $Q = \int_v \rho_v dv$ where dv = r sinθ dϕ r dθ dr

= r^2 sinθ dθ dϕ dr

$$\therefore\ Q = \int_{r=0}^{3}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} 1.2r\ r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= \int_{r=0}^{3}\int_{\theta=0}^{\pi} 1.2r^3 \sin\theta\,(2\pi)\, d\theta\, dr$$

$$= 2.4\pi\int_{r=0}^{3} r^3\left[-\cos\theta\right]_0^\pi dr$$

$$= 2.4\pi\,(2)\left[\frac{r^4}{4}\right]_0^3 = 305.4\,nC$$

(c) $$Q = \int_{r=0}^{4}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} 1.2\ r^3 \sin\theta\, d\theta\, d\phi\, dr$$

Upon simplifying, we get

Q= 965.09nC

Problem: 1.20

A charge distribution with spherical symmetry has density

$$\rho_v = \begin{cases} \rho_0 \dfrac{r}{R}, & 0 \le r \le R \\ 0, & r > R \end{cases}$$ Determine $\bar{E}$ everywhere

Solution:

Replace ‘a’ with ‘R’ in Fig:1.27, Then

Case I: Inside the sphere of radius ‘R’

The charge enclosed by the sphere of radius 'r' is $Q_{enc} = \int_v \rho_v dv$

$$Q_{enc} = \int_v \rho_0 \frac{r}{R} dv$$

$$= \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} \frac{r^3}{R} \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{\rho_0}{R} \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{r=0}^{r} r^3 dr$$

$$= \frac{4\pi r^4 \rho_0}{4R}$$

$$Q_{enc} = \frac{\rho_0}{R} \pi r^4$$

The flux flowing through the spherical surface

$$\psi = \oint_s \overline{D}.d\overline{s}$$

As the flux density is normal to the surface it will have components only in 'r' direction.

$$= D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$\psi = D_r \, 4\pi \, r^2$$

According to Gauss's law charge enclosed = flux flowing through the surface i.e., $Q_{enc} = \psi$

$$\frac{\rho_0}{R} \pi r^4 = D_r 4\pi r^2$$

$$D_r = \frac{\rho_0}{4R} r^2$$

$$\overline{D} = \frac{\rho_0}{4R} r^2 \, \overline{a}_r \text{ and}$$

$$\overline{E} = \frac{\overline{D}}{\epsilon_0} = \frac{\rho_0}{4R \epsilon_0} r^2 \, \overline{a}_r$$

Case II: outside the sphere of radius 'R'

Charge enclosed by the sphere of radius 'r' is

$$Q_{enc} = \int_v \rho_v dv$$

$$Q_{enc} = \int_v \rho_0 \frac{r}{R} dv$$

$$= \frac{\rho_0}{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{R} r^3 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \rho_0 \pi R^3$$

Flux flowing through the surface

$$\psi = D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$= D_r \, 4\pi \, r^2$

$Q_{enc} = \psi$ according to Gauss's law

$\rho_0 \, \pi \, R^3 = D_r \, 4\pi \, r^2$

$$D_r = \frac{\rho_0 R^3}{4r^2}$$

$$\bar{D} = \frac{\rho_0 R^3}{4r^2} \bar{a}_r \text{ and}$$

$$\bar{E} = \frac{\rho_0 R^3}{4r^2 \in_0} \bar{a}_r$$

***Problem: 1.21**

A sphere of radius 'a' is filled with a uniform charge density of ρ_v C/m^3 . Determine the electric field inside and outside the sphere.

Solution:

The answer is as derived in section "Uniformly charged sphere" case-I(inside the sphere) and case-II(outside the sphere).

Problem: 1.22

A charge distribution in free space has $\rho_v = 2r$ nC/m^3 for $0 < r < 10m$ and '0' otherwise. Determine $\bar{E}$ at r=2m and r=12m

Solution:

Replace 'a' with '10m' in Fig:1.27, Then

At r=2m:

The charge enclosed by the sphere of radius '2m' is $Q_{enc} = \int_v \rho_v dv$

$$Q_{enc} = \int_v 2rndv$$

$$= 2n \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{2} r^3 \sin\theta \, d\theta \, d\phi \, dr$$

$$= 32\pi nC$$

The flux flowing through the spherical surface

$$\psi = \oint_s \overline{D}.d\overline{s}$$

As the flux density is normal to the surface it will have components only in 'r' direction.

$$= D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \, 16\pi$$

According to Gauss's law charge enclosed = flux flowing through the surface i.e., $Q_{enc} = \psi$

$$32\pi n = D_r \, 16\pi$$

$$D_r = 2n$$

$$\overline{D} = 2n\,\overline{a}_r \text{ and}$$

$$\overline{E} = \frac{\overline{D}}{\epsilon_0} = 226\overline{a}_r V / m$$

At r=12m:

Charge enclosed by the sphere of radius '12m' is

$$Q_{enc} = \int_v \rho_v dv$$

$$Q_{enc} = \int_v 2rndv$$

$$= 2n \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{r=0}^{10} r^3 \sin\theta \, d\theta \, d\phi \, dr$$

$= 20\pi \, \mu C$

Flux flowing through the surface

$$\psi = D_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$= D_r \, 4\pi \, 12^2$

$Q_{enc} = \psi$ according to Gauss's law

$20\pi \, \mu = D_r \, 4\pi \, 12^2$

$D_r = 0.0347\mu$

$\bar{D} = 0.0347\mu \, \bar{a}_r$ and

$\bar{E} = 3.92 \, \bar{a}_r kV / m$

Problem: 1.23

Two point charges -4μC and 5μC are located at (2,-1,3) and (0,4,-2) respectively. Find the potential at (1, 0, 1). Assuming '0' potential at infinity.

Solution:

$$V = \frac{Q_1}{4\pi \in_0 |r - r_1|} + \frac{Q_2}{4\pi \in_0 |r - r_2|}$$

$$V = \frac{-4\times10^{-6}}{4\pi \in_0 |(1,0,1)-(2,-1,3)|} + \frac{5\times10^{-6}}{4\pi \in_0 |(1,0,1)-(0,4,-2)|}$$

Simplifying, we get

V= -5.872KV

Problem:1.24

A point charge 3μC is located at the origin in addition to the two charges of previous problem. Find the potential at (-1, 5, 2). Assuming V(∞) = 0.

Solution:

$r - r_1 = \sqrt{1+25+4} = 5.478$

$r - r_2 = \sqrt{9+36+1} = 6.782$

$r - r_3 = \sqrt{16+1+1} = 4.243$

$$V = \left[\frac{3\times10^3}{5.478} + \frac{-4\times10^3}{6.782} + \frac{5\times10^3}{4.243}\right]\times 9$$

= 10.23KV

Problem:1.25

A point charge of 5nC is located at the origin if V=2V at (0, 6, -8) find (a) the potential at A (-3, 2, 6)

(b) the potential at B (1, 5, 7)

(c) the potential difference V_{AB}

Solution:

(a) $$V_A - V = \frac{Q}{4\pi \in_0}\left(\frac{1}{r_A} - \frac{1}{r}\right)$$

r_A = (-3, 2, 6) – (0, 0, 0)= $\sqrt{3^2+2^2+6^2} = 7$

r = (0, 6, -8) – (0, 0, 0)= $\sqrt{0+6^2+8^2} = 10$

$$V_A - 2 = \frac{5\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}}\left(\frac{1}{7} - \frac{1}{10}\right)$$

V_A=3.929 V.

(b) $$V_B - V = \frac{Q}{4\pi \in_0}\left(\frac{1}{r_B} - \frac{1}{r}\right)$$

r_B=(1,5,7)-(0,0,0,)= $\sqrt{1+5^2+7^2}$ = $\sqrt{74}$

$$V_B - 2 = \frac{5\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}}\left(\frac{1}{\sqrt{74}} - \frac{1}{10}\right)$$

V_B=2.696 V.

(c)V_{AB} = V_B - V_A=-1.233 V.

***Problem:1.26**

A point of 5nC is located at (-3,4,0), while line y=1,z=1 carries uniform charge 2nC/m.

(a) If V=0V at O(0,0,0), find V at A(5,0,1).

(b) If V=100V at B(1,2,1), find V at C(-2,5,3).

(c) If V=-5V at O, find V_{BC}.

Solution:

Let the potential at any point be

$V=V_Q+V_L$

Where V_Q is potential due to point charge

$$i.e. \quad V_Q = \frac{Q}{4\pi \in_0 r}$$

by neglecting constant of integration

and V_L is potential due to line charge distribution,

for infinite line, we have

$$\bar{E} = \frac{\rho_L}{2\pi \in_0 \rho}\bar{a}_\rho$$

$$\therefore V_L = -\int \bar{E}.d\bar{l} = -\int \frac{\rho_L}{2\pi \in_0 \rho}\bar{a}_\rho.d\rho\bar{a}_\rho$$

$$\therefore V_L = -\frac{\rho_L}{2\pi \in_0}\ln\rho$$

by neglecting constant of integration.

Here ρ is the perpendicular distance from the line y=1, z=1(which is parallel to the x-axis) to the field point.

Let the field point be (x,y,z), then

$$\rho = |(x,y,z)-(x,1,1)| = \sqrt{(y-1)^2+(z-1)^2}$$

$$\therefore V = -\frac{\rho_L}{2\pi \in_0}\ln\rho + \frac{Q}{4\pi \in_0 r}$$

by neglecting constant of integration.

(a)

$$\rho_O = |(0,0,0)-(0,1,1)| = \sqrt{2}$$

$$\rho_A = |(5,0,1)-(5,1,1)| = 1$$

$$r_O = |(0,0,0)-(-3,4,0)| = 5$$

$$r_A = |(5,0,1)-(-3,4,0)| = 9$$

$$\therefore V_O - V_A = -\frac{\rho_L}{2\pi \in_0}\ln \rho_O + \frac{\rho_L}{2\pi \in_0}\ln \rho_A + \frac{Q}{4\pi \in_0 r_O} - \frac{Q}{4\pi \in_0 r_A}$$

$$V_O - V_A = -\frac{\rho_L}{2\pi \in_0}\ln \frac{\rho_O}{\rho_A} + \frac{Q}{4\pi \in_0}\left[\frac{1}{r_O} - \frac{1}{r_A}\right]$$

$$0 - V_A = -\frac{2\times 10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}}\ln \frac{\sqrt{2}}{1} + \frac{5\times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{1}{5} - \frac{1}{9}\right]$$

$$-V_A = -36\ln\sqrt{2} + 45\left[\frac{1}{5} - \frac{1}{9}\right]$$

$$V_A = 36\ln\sqrt{2} + 4 = 8.477V$$

(b)

$$\rho_B = |(1,2,1)-(1,1,1)| = 1$$

$$\rho_C = |(-2,5,3)-(-2,1,1)| = \sqrt{20}$$

$$r_B = |(1,2,1)-(-3,4,0)| = \sqrt{21}$$

$$r_C = |(-2,5,3)-(-3,4,0)| = \sqrt{11}$$

$$\therefore V_C - V_B = -\frac{\rho_L}{2\pi \in_0}\ln \frac{\rho_C}{\rho_B} + \frac{Q}{4\pi \in_0}\left[\frac{1}{r_C} - \frac{1}{r_B}\right]$$

$$V_C - 100 = -36\ln\frac{\sqrt{21}}{1} + 45\left[\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{21}}\right]$$

$$V_C - 100 = -50.175$$

$$V_C = 49.825V$$

(c)

$$V_{BC} = V_C - V_B = 49.825 - 100 = -50.175V$$

Problem: 1.27

Given the potential $V = \frac{10}{r^2} \sin\theta \cos\phi$

(a) Find the electric flux density $\overline{D}\, at\left(2, \pi/2, 0\right)$

(b) Calculate the work done in moving a 10mC charge from point A(1, 30⁰, 120⁰) to B(4, 90⁰, 60⁰)

Solution:

(a)

We have

$$\bar{E} = -\nabla V$$

Since V is given in spherical co-ordinate system, consider ∇V in spherical co-ordinate system

$$\therefore \overline{E} = -\frac{\partial v}{\partial r}\overline{a}_r + \frac{1}{r}\frac{\partial v}{\partial \theta}\overline{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi}\overline{a}_\phi$$

$$= -\left(10\left(-2r^{-3}\right)\sin\theta\cos\phi\,\overline{a}_r + \frac{1}{r}\frac{10\cos\theta\cos\phi}{r^2}\overline{a}_\theta + \frac{1}{r\sin\theta}\frac{10\sin\theta\left(-\sin\phi\right)}{r^2}\overline{a}_\phi\right)$$

$$= -\left(-20r^{-3}\sin\theta\cos\phi\,\overline{a}_r + \frac{10\cos\theta\cos\phi}{r^3}\overline{a}_\theta + \frac{-10\sin\phi}{r^3}\overline{a}_\phi\right)$$

$$= \left(\frac{20\sin\theta\cos\phi}{r^3}\overline{a}_r + \frac{-10\cos\theta\cos\phi}{r^3}\overline{a}_\theta + \frac{10\sin\phi}{r^3}\overline{a}_\phi\right)$$

$$= \frac{10}{r^3}\left(2\sin\theta\cos\phi\,\overline{a}_r - \cos\theta\cos\phi\,\overline{a}_\theta + \sin\phi\,\overline{a}_\phi\right)$$

$$\overline{D} = \overline{E} \in_0$$

$$= \frac{8.825\times10^{-11}}{r^3}\left[2\sin\theta\cos\phi\,\overline{a}_r - \cos\theta\cos\phi\,\overline{a}_\theta + \sin\phi\,\overline{a}_\phi\right]$$

$$= \frac{8.825\times10^{-11}}{r^3}\left[2.1.1\overline{a}_r - 0 + 0\right]$$

$$\overline{D}_{(2,\frac{\pi}{2},0)} = 22.1\,\overline{a}_r \text{ pC/m}^2$$

(b) Work done = $-Q\int_A^B \overline{E}\cdot d\overline{L} = -Q\left(-V_{AB}\right)$

= Q (V_B – V_A)

$$V_B = \frac{10}{16} 1 . \frac{1}{2} = 0.3125v$$

$$V_A = \frac{10}{1}\frac{1}{2}(-0.5) = -5 \times 0.5 = -2.5v$$

V_B - V_A = 2.8125V

W = 10^{-3} x 10 x (V_B-V_A) = 28.125mJ.

Problem:1.28

Given that $\overline{E} = (3x^2 + y)\overline{a}_x + x\overline{a}_y kV/m$. Find the work done in moving a –2μC charge from (0, 5, 0) to (2, -1, 0) by taking the path

(a) (0, 5, 0) →(2, 5, 0) → (2, -1, 0)

(b) y = 5-3x

Solution:

(a)

Line equation for (0, 5, 0) to (2, 5, 0) is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$$

$$\frac{x - 0}{0 - 2} = \frac{y - 5}{5 - 5} = \frac{z - 0}{0 - 0}$$

y = 5 z =0

dy = 0 dz = 0

$$W_1 = -QK \int_{(0,5,0)}^{(2,5,0)} ((3x^2 + y)\overline{a}_x + x\overline{a}_y)(dx\,\overline{a}_x + dy\overline{a}_y + dz\overline{a}_z)$$

$$= -QK \int_{(0,5,0)}^{(2,5,0)} (3x^2 + y)dx + xdy$$

$$= 2 \times 10^{-3} \int_{(0)}^{(2)} (3x^2 + 5)dx + 0$$

$$= 2\times10^{-3}\left(3\left(\frac{x^3}{3}\right)_0^2 + 5(2)\right)$$

= 36mJ

Line equation for (2, 5, 0) to (2, -1, 0)

Z = 0 dz = 0 $\frac{x-2}{2-2} = \frac{y-5}{5+1} = \frac{z-0}{0-0}$

x=2 dx=0

$$W_2 = -QK\int_{(2,5,0)}^{(2,-1,0)} (3x^2 + y)dx + xdy$$

$$W_2 = -QK\int_5^{-1} 2dy = -2Qk(-1-5) = -24mJ$$

$W=W_1+W_2$=12mJ.

(b)

Line equation for (0, 5, 0) to (2, 5, 0) is y=5-3x

dy=-3dx

$$W = -QK\int_{(0,5,0)}^{(2,-1,0)} (3x^2 + y)dx + xdy$$

$$W = 2\times10^{-3}\int_0^2 \left(3x^2 + 5 - 3x\right)dx - 3xdx = 12\text{mJ}.$$

Problem:1.29

An electric dipole located at the origin in free space has a moment $\overline{p} = 3\overline{a}_x - 2\overline{a}_y + \overline{a}_z nCm$

(a) Find V at P_A (2,3,4)

(b) Find V at r = 2.5, θ = 30⁰, ϕ = 40⁰

Solution:

(a) We have

$$V = \frac{1}{4\pi \in_0 \left|r - r^1\right|^2} \bar{p} . \frac{r - r^1}{\left|r - r^1\right|}$$

$r^1 = (0, 0, 0)$

$$\left|r - r^1\right| = \sqrt{4+9+16} = \sqrt{29}$$

$$V = 9 \times 10^9 \frac{(3\bar{a}_x - 2\bar{a}_y + \bar{a}_z) \cdot (2\bar{a}_x + 3\bar{a}_y + 4\bar{a}_z)}{29\sqrt{29}} \times 10^{-9}$$

$$= \frac{9 \times 10^9 \times (4)}{(29)^{3/2}} = 0.235V$$

(b) $r = 2.5$ $\theta = 30^0$ $\phi = 40^0$

$x = r \sin\phi \cos\theta = 0.958$

$y = r \sin\phi \sin\theta = 0.8035$

$z = r \cos\theta = 2.165$

upon simplifying we get

V= 1.97V

Problem:1.30

Three point charges –1nC, 4nC and 3nC are located at (0, 0, 0), (0, 0, 1) and (1, 0, 0) respectively. Find the energy in the system.

Solution:

$$W_E = W_1 + W_2 + W_3$$

$$= 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32})$$

$$= Q_2 . \frac{Q_1}{4\pi \in_0 \left|r_2 - r_1\right|} + \frac{Q_3}{4\pi \in_0} \left[\frac{Q_1}{\left|r_3 - r_1\right|} + \frac{Q_2}{\left|r_3 - r_2\right|} \right]$$

$$= \frac{1}{4\pi \in_0} \left(Q_1 Q_2 + Q_1 Q_3 + \frac{Q_2 Q_3}{\sqrt{2}} \right)$$

$$= \frac{1}{4\pi . \frac{10^{-9}}{36\pi}} \left(-4 - 3 + \frac{12}{\sqrt{2}} \right) . 10^{-18}$$

$$= 9\left(\frac{12}{\sqrt{2}} - 7 \right) nJ = 13.37 nJ$$

Problem:1.31

Point charges Q_1 = 1nC, Q_2 = -2nC, Q_3 = 3nC and Q_4 = -4nC are positioned one at a time and in that order at (0, 0, 0), (1, 0, 0), (0, 0, -1) and (0, 0, 1) respectively. Calculate the energy in the system after each charge is positioned.

Solution:

Energy after Q_1 is positioned is $W_1=0$

$$W_2=Q_2V_{21}=Q_2.\frac{Q_1}{4\pi\in_0|r_2-r_1|}=\frac{-2\times1\times10^{-18}}{4\pi.\frac{10^{-9}}{36\pi}|(1,0,0)-(0,0,0)|}=-18\text{nJ}$$

Energy after Q_2 is positioned $W_2' = W_1+W_2=-18\text{nJ}$

Energy after Q_3 is positioned

$$W_3' = W_2' + Q_3(V_{32}+V_{31}) = -18nJ + \frac{3\times10^{-9}}{4\pi.\frac{10^{-9}}{36\pi}}\left[\frac{-2\times10^{-9}}{|(0,0,-1)-(1,0,0)|}+\frac{1\times10^{-9}}{|(0,0,-1)-(0,0,0)|}\right]$$

$$=-29.18\text{nJ}$$

Energy after Q_4 is positioned

$$W_4' = W_3' + Q_4(V_{43}+V_{42}+V_{41}) = -68.27\text{nJ}.$$

Problem: 1.32

If $\bar{J}=\frac{1}{r^3}\left(2\cos\theta\,\bar{a}_r+\sin\theta\,\bar{a}_\theta\right)$ A/m^2. Calculate the current passing through

(a) Hemispherical shell of radius 20cm.

(b) A spherical shell of radius 20cm.

Solution:

$$I=\int\bar{J}.\bar{ds}$$

Since it is sphere $\bar{ds}=r^2\sin\theta\,d\theta\,d\phi\,\bar{a}_r$

(a) ϕ = 0 to 2π, θ = 0 to $\pi/2$ and r=0.2m for hemispherical shell

$$I=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\frac{1}{r^3}\left(2\cos\theta\,\bar{a}_r+\sin\theta\,\bar{a}_\theta\right)\cdot r^2\sin\theta\,d\theta\,d\phi\,\bar{a}_r$$

$$= \frac{1}{r}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} 2\cos\theta\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{r}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \sin 2\theta\, d\theta\, d\phi$$

$$= \frac{1}{r}\int_{\phi=0}^{2\pi}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} d\phi$$

$$= -\frac{1}{2r}(-1-1)(2\pi) = \frac{2\pi}{0.2} = 10\pi = 31.4A$$

(b) ϕ = 0 to 2π ,θ = 0 to π and r=0.2m for spherical shell

$$I = \frac{1}{r}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin 2\theta\, d\theta\, d\phi$$

$$= \frac{1}{r}\int_{\phi=0}^{2\pi}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi} d\phi$$

$$= -\frac{1}{2r}\int_{\varphi=0}^{2\pi}[1-1]\, d\phi = 0A$$

Problem: 1.33

For the current density $\overline{J} = 10z\sin^2\phi\,\overline{a}_\rho$ A/m². Find the current through the cylindrical surface ρ = 2, $1 \le z \le 5$ m .

Solution:

Since it is cylinder $d\overline{s} = \rho\, d\phi\, dz\, \overline{a}_\rho$

We have

$$I = \int \overline{J}\cdot d\overline{s}$$

$$= \int_{z=1}^{5}\int_{\phi=0}^{2\pi} 10z\sin^2\phi\rho\, d\phi\, dz$$

$$=10\rho\int_{z=1}^{5} z(1-\cos\phi)$$

=754A

***Problem: 1.34**

In a cylindrical conductor of radius 2mm, the current density varies with distance from the axis according to $J=10^3 e^{-400r} A/m^2$. Find the total current I.

Solution:

Since it is cylinder $d\overline{s}=\rho\, d\phi\, dz\, \overline{a}_\rho$

Here r=ρ=0.02m, $\therefore \overline{J}=10^3 e^{-400\rho}\overline{a}_\rho A/m^2$

We know the total current $I=\int_s \overline{J}.d\overline{s}$

$$\therefore I=\int_{\phi=0}^{2\pi}\int_{z=0}^{z} 10^3 e^{-400\rho}\rho d\phi dz$$

$$I=2\pi z 10^3 e^{-400\rho}\rho$$

$$I=4\pi z e^{-0.8}=z5.65A$$

Problem:1.35

If the current density $\overline{J}=\frac{1}{r^2}(\cos\theta\overline{a}_r+\sin\theta\overline{a}_\theta)A/m^2$, find the current passing through a sphere of radius 1.0m.

Solution:

We know the total current $I=\int_s \overline{J}.d\overline{s}$

Since it is spherical symmetry $d\overline{s}=r^2\sin\theta d\theta d\phi\overline{a}_r$

$$\overline{J}.d\overline{S}=\frac{r^2}{r^2}\cos\theta\sin\theta d\phi d\theta$$

$$I=\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\cos\theta\sin\theta d\phi d\theta$$

$$I=\pi\int_0^{\pi}\sin 2\theta d\theta$$

$$= \pi\left(\frac{-\cos 2\theta}{2}\right)_0^\pi = 0A$$

Problem:1.36

Write Laplace's equation in rectangular co-ordinates for two parallel planes of infinite extent in the X and Y directions and separated by a distance 'd' in the Z-direction. Determine the potential distribution and electric field strength in the region between the planes.

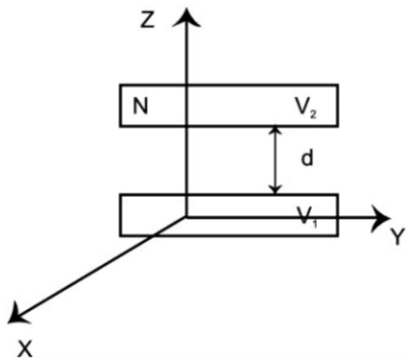


Fig: 2.15

Solution:

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$ since the potential is constant in X and Y directions

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial V}{\partial z} = A$$

$$V = Az + B$$

At Z =0 $\quad V = V_1$

$V_1 = 0 + B$

At Z =d $\quad V = V_2$

$V_2 = Ad + B$

$V_2 = Ad + V_1$

$$A = \frac{V_2 - V_1}{d}$$

The Potential distribution is $V = \frac{V_2 - V_1}{d} z + V_1$

The Electric field strength is

$$\overline{E} = -\nabla V = -\frac{\partial V}{\partial z}\overline{a}_z = -\frac{V_2 - V_1}{d}\overline{a}_z = \frac{V_1 - V_2}{d}\overline{a}_z$$

***Problem:1.37**

Calculate the capacitance of a parallel plate capacitor with a dielectric, mica filled between plates. $\in_r$ of mica is 6. The plates of the capacitor are square in shape with 0.254cm side. Separation between the two plates is, 0.254cm.

Solution:

We have $C = \frac{\in A}{d}$

Here $\in = \in_0 \in_r = 8.854 \times 10^{-12} \times 6$

$$C = \frac{8.854 \times 10^{-12} \times 6 \times 0.254 \times 0.254 \times 10^{-4}}{0.254 \times 10^{-2}} = 0.1349\, pF$$

***Problem:1.38**

A parallel plate capacitance has 500mm side plates of square shape separated by 10mm distance. A sulphur slab of 6mm thickness with $\in_r = 4$ is kept on the lower plate find the capacitance of the set-up. If a voltage of 100volts is applied across the capacitor, calculate the voltages at both the regions of the capacitor between the plates.

Solution:

Given

Area of parallel plates, A=500mmX500mm=500X500X10^{-6} m^2.

Distance of separation d=10mm=10X10^{-3}m.

Thickness of sulphur slab d_2=6mm=6X10^{-3}m.

Relative permittivity of sulphur slab $\in_r = 4$.

Voltage applied across the capacitor V=100v.

Here the capacitor has two dielectric media,

One medium is the sulphur slab of thickness (d_2)6mm,

since the distance between the plates(d) is 10mm

The remaining distance is air $d_1 = d - d_2$=4mm.

∴ the other dielectric medium is air with thickness(d_1) 4mm.

The capacitance of the parallel plate capacitor with two dielectric media is

$$C = \frac{\epsilon_0 A}{\left(\frac{d_1}{\epsilon_{r_1}} + \frac{d_2}{\epsilon_{r_2}}\right)} \quad F$$

Here ϵ_{r_1} (air)=1, $\epsilon_{r_2} = \epsilon_r = 4$

$$C = \frac{8.854 \times 10^{-12} \times 500 \times 500 \times 10^{-6}}{\left(\frac{4 \times 10^{-3}}{1} + \frac{6 \times 10^{-3}}{4}\right)} = 0.402nF$$

The charge Q=CV=$0.402X10^{-9}X100=4.02x10^{-8}$ C

The value of capacitance(C_1) in delectric-1 i.e., air is

$$C_1 = \frac{\epsilon_0 A}{d_1} = \frac{8.854 \times 10^{-12} \times 500 \times 500 \times 10^{-6}}{4 \times 10^{-3}} = 0.55nF$$

Similarly, The value of capacitance(C_2) in delectric-2 i.e., sulphur is

$$C_2 = \frac{\epsilon A}{d_2} = \frac{4 \times 8.854 \times 10^{-12} \times 500 \times 500 \times 10^{-6}}{6 \times 10^{-3}} = 1.48nF$$

We have $V=V_1+V_2$

Where V_1 is the voltage at the region of the capacitor plate near dielectric-1 i.e., air.

And V_2 is the voltage at the region of the capacitor plate near dielectric-2 i.e., sulphur.

$$V_1 = \frac{Q_1}{C_1} = \frac{Q}{C_1} = \frac{4.02 \times 10^{-8}}{0.55 \times 10^{-9}} = 73.1V$$

∴ V_2=100-73.1=26.9V.

EXERCISE QUESTIONS

1.1 State the Coulomb's law in SI units and indicate the parameters used in the equations with the aid of a diagram.

1.2 State Gauss's law. Using divergence theorem and Gauss's law, relate the density D to the volume charge density ρ_v.

1.3 Explain the following terms:

(ii) Homogeneous and isotropic medium and
(iii) Line, surface and volume charge distributions.

1.4 State and Prove Gauss's law. List the limitations of Gauss's law

1.5 Express Gauss's law in both integral and differential forms. Discuss the salient features of Gauss's law.

1.6 Derive Poisson's and Laplace's equations starting from Gauss's law.

1.7 Using Gauss's law derive expressions for electric field intensity and electric flux density due to an infinite sheet of conductor of charge density ρ C/cm.

1.8 Find the force on a charge of -100mC located at P(2,0,5) in free space due to another charge 300 μC located at Q(1,2,3).

1.9 Find the force on a 100μC charge at(0,0,3)m, if four like charges of 20μC are located on X and Y axes at $\pm$4m.

1.10 Derive an expression for the electric field intensity due to a finite length line charge along the Z-axis at an arbitrary point Q(x,y,z).

1.11 A point charge of 15nC is situated at the origin and another point charge of -12nC is located at the point(3,3,3)m. Find $\overline{E}$ and V at the point(0,-3,-3).

1.12 Obtain the expressions for the field and the potential due to a small Electric dipole oriented along Z-axis.

1.13 Define conductivity of a material. Explain the equation of continuity for time varying fields.

1.14 As an example of the solution of Laplace's equation, derive an expression for capacitance of a parallel plate capacitors.

1.15 In a certain region $\overline{J} = 3r^2\cos\theta\overline{a}_r - r^2\sin\theta\overline{a}_\theta A/m$, find the current crossing the surface defined by $\theta = 30^0, 0 < \phi < 2\pi, 0 < r < 2m$.