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Course Code	SIGNALS AND SYSTEMS		L	T	P	C
20A04301T			3	0	0	3
Pre-requisite	Mathematics - I	Semester	III			

Course Objectives:

- To introduce students to the basic idea of signal and system analysis and its characterization in time and frequency domains.
- To present Fourier tools through the analogy between vectors and signals.
- To teach concept of sampling and reconstruction of signals.
- To analyze characteristics of linear systems in time and frequency domains.
- To understand Laplace and z-transforms as mathematical tool to analyze continuous and discrete-time signals and systems.

Course Outcomes (CO):

CO1: Understand the mathematical description and representation of continuous-time and discrete-time signals and systems. Also understand the concepts of various transform techniques.

CO2: Apply sampling theorem to convert continuous-time signals to discrete-time signals and reconstruct back, different transform techniques to solve signals and system related problems.

CO3: Analyze the frequency spectra of various continuous-time and discrete-time signals using different transform methods.

CO4: Classify the systems based on their properties and determine the response of them.

UNIT - I **Signals and Systems**

Signals & Systems: Basic definitions and classification of Signals and Systems (Continuous time and discrete time), operations on signals, Concepts of Convolution and Correlation of signals, Analogy between vectors and signals-Orthogonality, mean square error.

UNIT - II **Fourier Series and Fourier Transform**

Fourier series: Trigonometric & Exponential, Properties of Fourier series, concept of discrete spectrum, Illustrative Problems.

Continuous Time Fourier Transform: Definition, Computation and properties of Fourier transform for different types of signals and systems, Inverse Fourier transform. Statement and proof of sampling theorem of low pass signals, Illustrative Problems.

UNIT - III **Laplace Transform**

Laplace Transform: Definition, ROC, Properties, Inverse Laplace transforms, the S-plane and BIBO stability, Transfer functions, System Response to standard signals, Solution of differential equations with initial conditions.

UNIT - IV **Signal Transmission through LTI systems**

Signal Transmission through Linear Systems: Linear system, impulse response, Response of a linear system for different input signals, linear time-invariant (LTI) system, linear time variant (LTV) system, Transfer function of a LTI system. Filter characteristics of linear systems. Distortion less transmission through a system, Signal bandwidth, System bandwidth, Ideal LPF, HPF and BPF characteristics, Causality and Paley-Wiener criterion for physical realization, Relationship between bandwidth and rise time, Energy and Power spectral densities, Illustrative Problems.

UNIT - V **DTFT & Z-Transform**

Discrete Time Fourier Transform: Definition, Computation and properties of Discrete Time Fourier transform for different types of signals and systems.

Z–Transform: Definition, ROC, Properties, Poles and Zeros in Z-plane, The inverse Z-Transform, System analysis, Transfer function, BIBO stability, System Response to standard signals, Solution of difference equations with initial conditions. Illustrative Problems.

Textbooks:

1. A.V. Oppenheim, A.S. Willsky and S.H. Nawab, “Signals and Systems”, 2nd Edition, PHI, 2009.
2. Simon Haykin and Van Veen, “Signals & Systems”, 2nd Edition, Wiley, 2005.

Reference Books:

1. BP Lathi, “Principles of Linear Systems and Signals”, 2nd Edition, Oxford University Press, 2015.
2. Matthew Sadiku and Warsame H. Ali, “Signals & Systems A primer with MATLAB”, 2016.

UNIT-I
SIGNALS & SYSTEMS

Introduction:

In a communication system, the word ‘signal’ is commonly used. Therefore we must know itsexact meaning.

- Mathematically, signal is described as a function of one or more independentvariables.
- Basically it is a physical quantity. It varies with some dependent or independentvariables.
- So the term signal is defined as “A physical quantity which contains someinformation and which is a function of one or more independent variables.”

Examples: speech signal, ECG signal, radio signal, TV

signal.These signals can be one dimensional or

multidimensional.

One dimensional signal: when the function depends on a single variable, the signal is said tobe one dimensional.

Example: speech signal

Multidimensional signal: When the function depends on two or more variables, the signal issaid to be multi dimensional.

Example: image

Definition of system:

A system is a physical device (or an algorithm) which performs required operation on a signal.

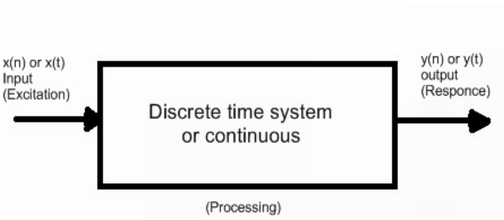


Fig. 1.1 Functional block diagram of system

A system is a set of elements or functional blocks that are connected together & produces an output in response to an input signal. The response of the system depends ontransfer function of the system.

The functional relationship between input & output is y(t)= T [

x(t)]Where x (t) is the input to the system or excitation.

Y (t) is the output or response of the

system.T is the transfer function of the

system.

Examples: communication filters, amplifiers, TV, audio amplifiers, transmitters, receivers etc.

Classification of signals

There are various types of signals. Every signal has its own characteristics. The processingof signals mainly depends on the characteristic of that particular signal. So classification of signal is necessary. Broadly the signals are classified as below:

- Continuous and discrete time signals
- Periodic and non-periodic signals
- Even and odd signals
- Energy and power signals
- Deterministic and random signals

Continuous Time and Discrete Time Signals

Continuous Time signal: A signal is said to be continuous when it is defined for all instants oftime. Or a signal of continuous amplitude & time is known as Continuous Time signal.

Examples: electrical signals derived in proportion with the physical quantities such as temperature, pressure, sound etc.

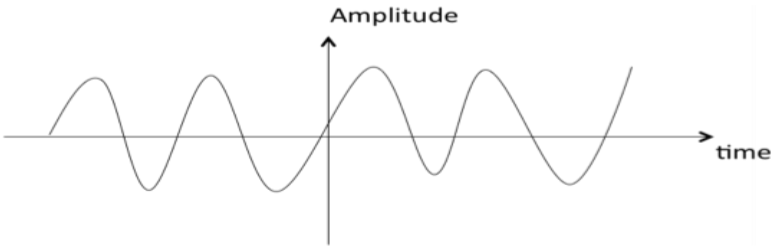


Fig. 1.2 Continuous time sin signal

Discrete Time Signals: A signal is said to be discrete when it is defined at only discreteinstants of time.

Example: if we take blood pressure readings of a patient after every one hour & plot the graphthen it is discrete signal.

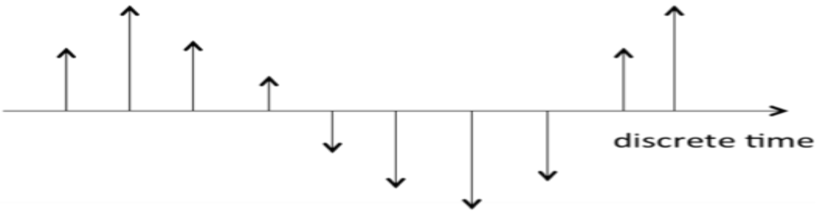


Fig. 1.3. Discrete time signal

Periodic and Aperiodic Signals

Periodic signal: A signal which repeats itself after a fixed time period is called as Periodicsignal. Alternatively a signal is said to be periodic if it satisfies the condition

$x(t) = x(t + T_0)$ for CT signal

$x(n) = x(n + N)$ for DT signal

Where T_0 = fundamental time period. The minimum possible interval over which a functionrepeats is called fundamental period T_0.

$1/T_0 = F =$ fundamental frequency.

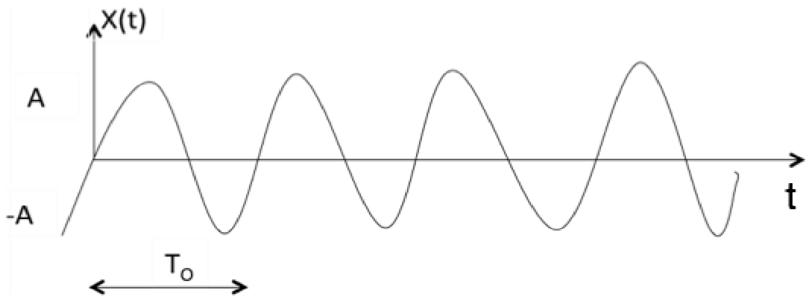


Fig 1.4 Periodic signal

The above signal will repeat for every time interval T_0 hence it is periodic with period T_0.

Examples: sine, cosine waves, square waves etc..

Aperiodic Signals: A signal which does not repeat at regular interval is called as AperiodicSignals. Alternatively a signal is said to be non periodic if it satisfies the condition

$x(t) \neq x(t + T_0)$ for CT signal

$x(n) \neq x(n + N)$ for DT signal

Examples : exponential signal, rectangular pulse, noise etc.

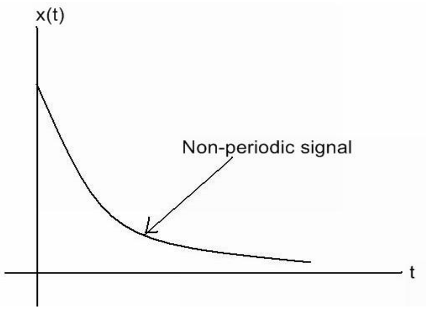


Fig 1.4 Aperiodic Signal

The above signal is a exponential signal which does not repeat for a particular time interval. Soit is a non periodic signal.

Condition for periodicity of DT signal

For DT signal, the condition of periodicity is x(n) = x(n + N)

$$\text{Let } x(n) = A\cos[2\pi f_0 n + \theta]$$

$$\text{So } x(n+N) = A\cos[2\pi f_0 (n+N) + \theta]$$

According to the condition of periodicity $A\cos[2\pi f_0 n + \theta] = A\cos[2\pi f_0 (n+N) + \theta]$

$$= A\cos[2\pi f_0 n + 2\pi f_0 N + \theta]$$

To satisfy this condition, $2\pi f_0 N = 2\pi k$

$$f_0 = k/N$$

Here k,N are integers. Thus DT signal is periodic if its frequency f_0 is rational.

Periodicity condition for $x(n) = x_1(n)+x_2(n)$

According to the Periodicity $f_1 = k_1/N_1$ $f_2 = k_2/N_2$

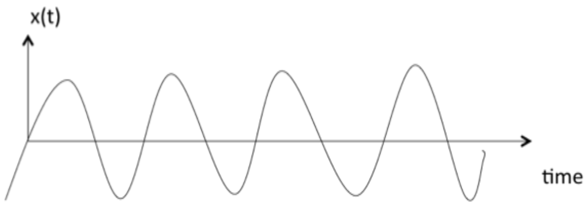
The resultant signal is periodic if N_1/N_2 is the ratio of two integers. The period of x(n) will beleast common multiple of N_1, N_2.

Deterministic and random Signals

Deterministic signal: A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time. Or, signals which can be defined exactly by a mathematical formula, look up table or some well defined rule are known as deterministic signals. Example: sine, cosine wave.

Random Signals: A signal is said to be non-deterministic if there is uncertainty with respect to its value at some instant of time. Non-deterministic signals are random in nature hence they are called random signals. Random signals cannot be described by a mathematical equation. They are modeled in probabilistic terms.

Example: noise generated in electronic components, transmission channels etc



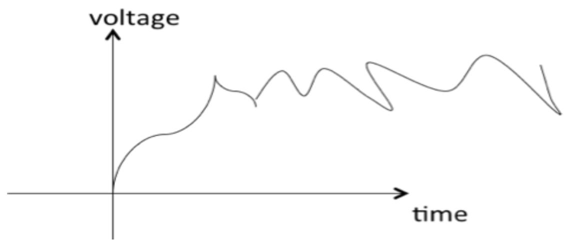


Fig 1.5 Deterministic signal **Fig 1.6 Random signal**

Deterministic signal	**Random Signal**
1. A deterministic signal is one which can be completely represented by Mathematical equation at any time.	A random signal is one which cannot be represented by any mathematical equation.
2.eg. Sine wave, cosine wave.	Noise generated in electronic components, transmission channels, cables.
3. These signals can be periodic or non periodic.	These signals are non periodic.

Even and odd signal

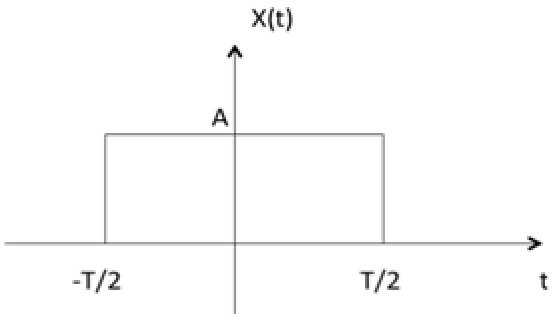
Even or symmetrical signal

A signal x(t) is said to be symmetrical (or) even if it satisfies the following condition:$X(t) = x(-t)$ for CT signal

$X(n) = x(-n)$ for DT signal

Example: cosine waveform, rectangular pulse.

As shown in the following diagram, rectangular function satisfies the condition $x(t) = x(-t)$ so it isalso even function.



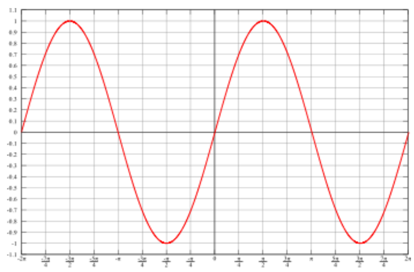


Fig 1.7(a) Example of Even signal **Fig 1.7(b) Example of odd signal**

Odd (or) Anti symmetrical signal

A signal x(t) is said to be anti symmetrical (or) odd if it satisfies the following condition:X(t) = -x(-t) for CT signal

X(n)= -x(-n) for DT signal

Example: sine waveform

Decomposing a CT signal into even & odd parts:

Any CT signal can be expressed as the summation of even part & odd part.

$X(t)= x_e(t)+ x_o(t)$

Even part of x(t) is expressed as, $x_{ev}(t) = \frac{x(t)+x(-t)}{2}$

Odd part of x(t) is expressed as,

$x_{od}(t) = \frac{x(t)-x(-t)}{2}$ The even part of a signal is an even signal, since

$$x_{ev}(-t) = \frac{x(-t)+x(t)}{2} = x_{ev}(t)$$

Similarly any DT signal can be expressed as the summation of even part & odd part.

$X(n)= x_e(n)+ x_o(n)$

Even part of x(t) is expressed as, $x_e(n)$ = [x(n)+ x(-n] / 2Odd part of x(t) is expressed as, $x_o(n)$ = [x(n)-x(-n)] / 2.

Steps to be followed to find even and odd component of the given signal is:

1) Draw the signal x (t).
2) Draw its folded version x (-t)
3) Add x (t) and x (-t) or subtract x (-t) from x (t)
4) Divide the addition by 2 to get $x_e(t)$& subtraction by 2 to get $x_o(t)$.

Energy and power signal

Power signal: A signal x (t) is said to be power signal, if and only if the normalized averagepower p is finite and non-zero. ($0 < P < \infty$, $E \to \infty$).

Almost all the periodic signals are power signals.The average normalized power is given by

$$\text{Power } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

,

For periodic signals, the power *P* can be computed using a simpler form based on theperiodicity of the signal as

$$\text{Power } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

,

where *T* is the period of the signal.

Energy signal

A signal x (t) is said to be energy signal if and only if the total normalized energy is finiteand non-zero. ($0 \le E < \infty$, $P = 0$)

The total energy contained in and average power provided by a signal x(t) (which is a function oftime) are defined as

$$\text{Energy } E = \int_{-\infty}^{\infty} x^2(t)dt$$

Comments:

1. The square root of the average power $\sqrt{P}$ of a power signal is what is usually defined as the RMS value of that signal.

2. If a signal approaches zero as t approaches ∞ then the signal is an energy signal.This is in most cases true but not always.

3. All periodic signals are power signals (but not all non–periodic signals are energysignals).

4. Any signal f that has limited amplitude ($|f| < \infty$) and is time

   limited($f = 0$ for $|t| > t_0$ for some $t_0 > 0$) is an energy signal .

Energy and Power: The *total energy* of a discrete-time signal is defined by

$$E_\infty = \sum_{n=-\infty}^{\infty} x^2[n] = \lim_{N\to\infty} \sum_{n=-N}^{N} x^2[n]$$

The time-average power is

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n]$$

Comparison of energy and power signals

Power signals	Energy signals
The normalized average power is finite and non zero.	Total normalized energy is finite and non- zero.
Almost all periodic signals are power signals.	Almost all non periodic signals are energy signals.
Energy of the power signal is infinite.	Power of the energy signal is zero.
Power signals can exist over an infinite time. They are not time limited.	Energy signals exist over a short period of time. They are time limited.

PROBLEMS BASED ON CLASSIFICATION OF SIGNALS

1. Prove that the sine wave is a periodic signal.

The sinewave is mathematically expressed as, $x(t) = A \sin w_0 t$

If it is a periodic signal it should satisfy the condition $x(t) = x(t+T_0)$Now let us calculate $x(t+T_0) = A \sin w_0 (t + T_0)$

$= A \sin (w_0 t + w_0 T_0)$

$= A \sin (w_0 t + 2 \pi T_0 / T_0)$

$= A \sin (w_0 t + 2 \pi)$

$= A [\sin w_0 t \cos 2 \pi + \cos w_0 t \sin 2 \pi)$

$= A [\sin w_0 t .1 + \cos w_0 t .0]$

$= A \sin w_0 t$

$= x(t)$

Therefore $x(t+T_0) = x(t)$.The sine wave is a periodic signal.

2. Prove that the exponential signal is non periodic.

The exponential signal is expressed as, $x(t) = e^{-\alpha t}$

Put $t = t+T_0$ in above equation, we get

$$x(t+T_0) = e^{-\alpha(t+T_0)}$$

$$= e^{-\alpha t}\, e^{-\alpha T_0}$$

But for exponential signal

$T_0=\infty$.

So $e^{-\alpha T_0} = 0$

$x(t+T_0) = e^{-\alpha.t}.0=0$

$x(t) \neq x(t+T_0)$.

Hence the exponential signal is non periodic signal.

3. State whether the following signals x(t) is periodic or not. If it is periodic find thecorresponding period.

(a) x(t)= 3 sin 4t

The given equation is a sine wave. So it is a periodic signal.

To find time period, compare the given equation with standard sine wave x(t)= A

sin wtw =4

$f = 4 / 2\pi = 2 / \pi$.

So time period $T = \pi / 2$.

(b) x(t)= 2 cos 100πt + 5 sin 50t

The given signal x(t) is the addition of two signals $x_1(t)$ &

$x_2(t)$.Let $x_1(t) = 2 \cos 100\pi t$ & $x_2(t)= 5 \sin 50t$

If x(t) is a periodic, the ratio T_1/T_2 is the ratio of two integers.(rational

no)So calculate T_1 & T_2 .

Compare $x_1(t)= 2 \cos 100\pi t$ with the standard cosine wave equation $x_1(t)=$ A cos

$w_1 t w_1 = 100\pi$

$$2\pi f_1 = 100\pi \quad 2\pi/T_1 = 100\pi T_1 = 1/50$$

Similarly Compare $x_2(t)= 5 \sin 50t$ with the standard sine wave equation $x_2(t)= A \sin$

$w_2t, W_2 = 50$

$2\pi f_2 = 50$

$2\pi/T_2 = 50$

$T_2 = 2\pi/50$

Therefore the ratio $T_1/T_2 = 1/2\pi$.

It is not the ratio of two integers. Thus x (t) is non periodic signal.

4. Find the fundamental period and frequency of the following signals

(a) $x(t) = 5\sin 24\pi t + 7\sin 36\pi t$

(b) $x(t) = 5\cos \pi t \sin 3\pi t$

(a)

$$\begin{aligned} x(t) &= 5\sin 24\pi t + 7\sin 36\pi t \\ &= x_1(t) + x_2(t) \end{aligned}$$

where

$$\begin{aligned} x_1(t) &= 5\sin 24\pi t \\ x_2(t) &= 7\sin 36\pi t \end{aligned}$$

Let T_1 & T_2 be the fundamental periods of $x_1(t)$ $x_2(t)$ respectively

$$\omega_1 = 24\pi$$
$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{24\pi} = \frac{1}{12} \text{ (rational)}$$
$$\omega_2 = 36\pi$$
$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{36\pi} = \frac{1}{18} \text{ (rational)}$$
$$\frac{T_1}{T_2} = \frac{1}{12} \times 18 = \frac{3}{2} \text{ (rational)}$$

The composite signal is periodic signal. Since T_1 &T_2 are rational,x(t) is periodic. The fundamental period is the LCM of T_1 &T_2.

In this case, T_1 &T_2 are fractions; they are made integers by multiplying by a least number. for T_1&T_2 thus obtained,LCM is found.T_0 is obtained by dividing by the same number which was chosen to make T_1 &T_2 as integers.

(1)

$$T_1 = \frac{1}{12} \quad \text{and} \quad T_2 = \frac{1}{18}$$

(2) By multiplying T_1 &T_2 by 36, $T_1 = 3$ $T_2 = 2$.

(3) The LCM for the new T_1 &T_2 is easily obtained

as 6.(4)T_0 is obtained by dividing LCM by 36.

$$T_0 = \frac{\text{LCM}}{36} = \frac{6}{36} = \frac{1}{6} \text{ sec.}$$

$$\boxed{\begin{aligned} T_0 &= \frac{1}{6} \text{ sec.} \\ f_0 &= 6\,\text{Hz.} \end{aligned}}$$

(b)

$$\begin{aligned} x(t) &= 5\cos \pi t \sin 3\pi t \\ &= x_1(t)x_2(t) \end{aligned}$$

Where

$$x_1(t) = 5\cos \pi t$$
$$x_2(t) = \sin \pi t$$

Let T_1 &T_2 be the fundamental periods of $x_1(t)$ $x_2(t)$ respectively.

$$\omega_1 = \pi$$
$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\pi} = 2 \text{ sec. (rational)}$$
$$\omega_2 = 3\pi$$
$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{3\pi} = \frac{2}{3} \text{ sec. (rational).}$$
$$\frac{T_1}{T_2} = 2 \times \frac{3}{2} = 3 \text{ (rational).}$$

The composite signal is periodic with fundamental period is

$$T_0 = T_1 = 3T_2 = 2 \text{ sec.}$$

$$\boxed{\begin{array}{l} T_0 = 2 \text{ sec.} \\ f_0 = 0.5 \text{ Hz} \end{array}}$$

5. Determine whether the following signal is periodic. If periodic find the fundamentaltime period.

$$x[n] = \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{4}\right)$$

Solution:

$$x[n] = \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{4}\right)$$
$$= x_1[n]x_2[n]$$
$$x_1[n] = \cos\frac{n\pi}{2}$$
$$\omega_1 = \frac{\pi}{2}$$
$$N_1 = \frac{2\pi}{\omega_1}m_1 = \frac{2\pi}{\pi}2m_1 = 4 \quad \text{for } m_1 = 1$$

$$
\begin{aligned}
x_2[n] &= \cos\frac{n\pi}{4} \\
\omega_2 &= \frac{\pi}{4} \\
N_2 &= \frac{2\pi}{\omega_2}m_2 = \frac{2\pi}{\pi}4m_2 = 8 \quad \text{for } m_2 = 1 \\
\frac{N_1}{N_2} &= \frac{4}{8} = \frac{1}{2} \quad \text{or} \\
2N_1 &= N_2 = N
\end{aligned}
$$

The signal is periodic with fundamental period N=8.

6. Find whether the following signals are even or odd. Find the even and oddcomponents.

(a) $\boldsymbol{x(t) = t^2 - 5t + 10}$

Put $t = -t$

$$
\begin{aligned}
x(-t) &= t^2 + 5t + 10 \\
&\neq x(t) \\
&\neq -x(t)
\end{aligned}
$$

The function is neither even nor odd.

$$
\begin{aligned}
x_e(t) &= \frac{1}{2}[x(t) + x(-t)] \\
&= \frac{1}{2}[t^2 - 5t + 10 + t^2 + 5t + 10]
\end{aligned}
$$

$$\boxed{x_e(t) = (t^2 + 10)}$$

$$
\begin{aligned}
x_0(t) &= \frac{1}{2}[x(t) - x(-t)] \\
&= \frac{1}{2}[t^2 - 5t + 10 - t^2 - 5t - 10]
\end{aligned}
$$

$$\boxed{x_0(t) = -5t}$$

(b) $x(t) = t^4 + 4t^2 + 6$
Put $t = -t$

$$x(-t) = t^4 + 4t^2 + 6 = x(t)$$
$$x(t) = x(-t)$$

The function is even. The odd part should be zero which can be verified as

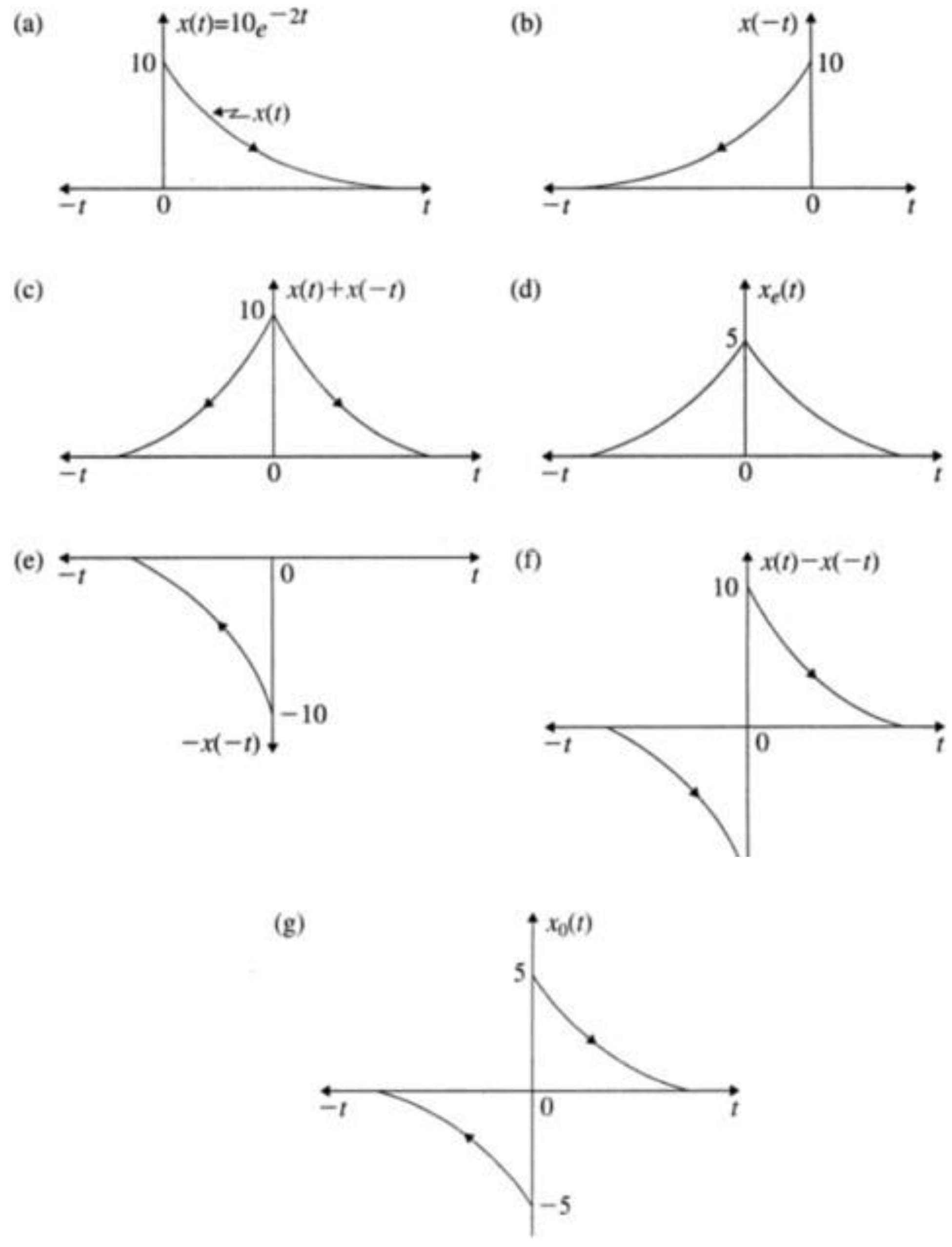
$$\begin{aligned} x_0(t) &= \frac{1}{2}[x(t) - x(-t)] \\ &= \frac{1}{2}[t^4 + 4t^2 + 6 - t^4 - 4t^2 - 6] \\ &= 0 \end{aligned}$$

$$\boxed{x_e(t) = x(t) = t^4 + 4t^2 + 6}$$

7. Sketch the even and odd components of exponential signal x(t)=10 e^{-2t}

1) Draw the signal x (t).
2) Draw its folded version x (-t).
3) Add x (t) and x (-t) or subtract x (-t) from x (t).
4) Even & odd component is calculated by,

$$x_{ev}(t) = \frac{x(t) + x(-t)}{2} \qquad x_{od}(t) = \frac{x(t) - x(-t)}{2}$$



8. Determine whether the following signals are even or odd.

(a) $x[n] = \sin 2\pi n$

$$x[-n] = \sin(-2\pi n) = -\sin 2\pi n$$
$$= -x[n]$$

$$\boxed{\text{This is an odd signal.}}$$

(b) $x[n] = \cos 2\pi n$

$$x[-n] = \cos(-2\pi n) = \cos 2\pi n$$
$$= x[n]$$

$$\boxed{\text{This is an even signal.}}$$

9. Determine the following signals are energy or power signal?

(a) x(t)= u(t)

The unit-step function, defined by

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

is a power signal, since

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} u^2(t)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} 1\, dt$$
$$= \lim_{T\to\infty} \frac{1}{T}\frac{T}{2} = \frac{1}{2}$$

P=1/2 watts.

(b) $x(t) = A\ \sin(\omega_0 t + \phi)$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} A^2 \sin^2(\omega_0 t + \phi)dt$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{2T}\int_{-T}^{T} \frac{[1 - \cos 2(\omega_0 t + \phi)]}{2}dt$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{4T}\left[\int_{-T}^{T} dt - \int_{-T}^{T} \cos 2(\omega_0 t + \phi)\right] dt$$

Since $\int_{-T}^{T} \cos 2(\omega_0 t + \phi)dt = 0$

$$P = \underset{T\to\infty}{Lt} \frac{A^2}{4T}[t]_{-T}^{T}$$

$$\boxed{P = \frac{A^2}{2}}$$

Since P is finite, $E = \infty$.

10.Determine whether the following signals are power or energy signal.

(a) $x[n] = A\delta[n]$

$$x[n] = A\delta[n]$$
$$= A \qquad n = 0$$
$$= 0 \qquad n \neq 0$$

$$\text{Energy } E = \sum_{n=0}^{0}(A)^2$$

$$\boxed{E = A^2}$$

For unit impulse, $A = 1$ and $E = 1$.

(b) $\boldsymbol{x[n] = u[n];\ n \geq 0}$

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{n=0}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{n=0}^{N} 1$$

But $\sum_{n=0}^{N} 1 = (N+1)$

$$P = \underset{N\to\infty}{Lt} \frac{(N+1)}{(2N+1)}$$

$$= \underset{N\to\infty}{Lt} \frac{N(1+\frac{1}{N})}{N(2+\frac{1}{N})} = \frac{1}{2}$$

$$\boxed{\begin{array}{c} P = \frac{1}{2} \\ E = \infty \end{array}}$$

(c) $\boldsymbol{x[n] =}$ **ramp** $\boldsymbol{n;\ n \geq 0}$

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{n=0}^{N} |x[n]|^2$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{n=0}^{N} n^2$$

But $\sum_{n=0}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$

$$P = \underset{N\to\infty}{Lt} \frac{N(N+1)(2N+1)}{(2N+1)6}$$

$$\boxed{P = \infty}$$

$$E = \underset{N\to\infty}{Lt} \sum_{n=0}^{N} n^2$$

$$= \underset{N\to\infty}{Lt} \frac{N(N+1)(2N+1)}{6} = \infty$$

$$\boxed{E = \infty}$$

The signal $x[n] = n$ is neither power signal nor energy signal.

(d) X (n) = $(1/3)^n$ n>0

Solution:

$$E = \underset{N\to\infty}{Lt} \sum_{n=0}^{N} \left(\frac{1}{3}\right)^{2n}$$

$$= \underset{N\to\infty}{Lt} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^{n}$$

$$= 1 + \frac{1}{9} + \left(\frac{1}{9}\right)^2 + \dots$$

$$= \frac{1}{1-\frac{1}{9}}$$

$$\boxed{\begin{matrix} E = \frac{9}{8} \\ P = 0 \end{matrix}}$$

Elementary signals

In the analysis of communication system, standard test signals play very important role.Such signals are used to check the performance of the system. Applying such

Property	Periodic Signal	Fourier Series Coefficients
	x(t), y(t) are periodic with period T	a_k for x(t) and b_k for y(t)
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$e^{-jk\omega_0 t_0} a_k = e^{-jk\frac{2\pi}{T}t_0} a_k$
Frequency Shifting	$e^{jM\omega_0 t} x(t) = e^{jM\frac{2\pi}{T}t} x(t)$	$a_k - M$
Conjugation	$x^*(t)$	$a^*_{(-k)}$
Time Reversal	$x(-t)$	$a_{(-k)}$
Time scaling	$x(ct), c < 0$, periodic with period T/c	a_k
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\,\frac{2\pi}{T}\,a_k$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd

signals at thesystem; the output is checked. Now depending on the input-output characteristic of that particular system study of different properties of a system can be done. Some standard test signals are as follows:

1) DC signal 2) Unit step signal 3) Delta or unit impulse function

4) Unit ramp signal 5) Sinusoidal signal 6) Rectangular pulse

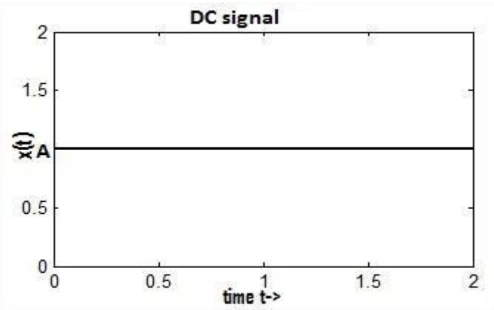
7) Exponential signal 8) Signum function 9) Sinc function

1. DC signal

The amplitude of DC signal remains constant & is independent of time.

Mathematically CT is expressed as $x(t) = A \quad -\infty < t < \infty$

DT is expressed as $x(n) = A \quad -\infty < n < \infty$



2. Unit delta function δ(t)

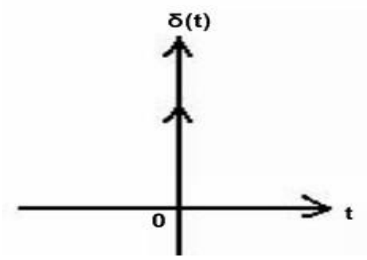
CT unit impulse δ(t) : A continuous time delta function is denoted by δ(t). Mathematically it isexpressed as follows:

δ(t)=1 for

t=0δ(t)=0

for t≠0

The graphical representation of delta function for C.T. signal is shown in figure



CT Unit impulse

The delta function is an extremely important function used for the analysis of communication systems.The unit impulse, $\delta(t)$, is a function that is zero for all $t \neq 0$ and for which

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

The area under the pulse is unity. Due to its unity area it is called as a unit impulse function.

Importance of impulse function:

1. By applying impulse signal to a system one can get the impulse response of the system.From the impulse response it is possible to get the transfer function of the system.

2. For a LTI system if the area under the impulse response curve is finite, then the system issaid to be stable.

3. From the impulse response of the system, one can easily get the step response by integrating it once &twice respectively.

4. It is easy to generate &apply to any system.

Properties of impulse response:

1. δ(at)= 1/a δ(t)
2. δ(-t)= δ(t)
3. $\int_{-\infty}^{\infty} \delta(t)dt = 1$
4. X(t)* δ(t) = X(t)
5. X(t))* δ(t-t_0) = X(t_0)

DT unit impulse δ(n) :

A discrete time unit impulse function is denoted by δ(n). Its amplitude is 1 at n=0 and for allother values of n; its amplitude is zero.

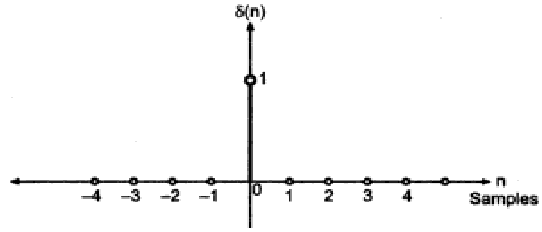
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

In the sequence form it can be represented as,

$$\delta(n) = \{\ldots 0,\ 0,\ 0,\ \underset{\uparrow}{1},\ 0,\ 0,\ 0 \ldots\}$$

In the above sequence the arrow represents 0^{th} sample. The above sequence can also bewritten as δ(n) = {1}

The graphical representation of delta function for D.T. signal is as shown in figure below:



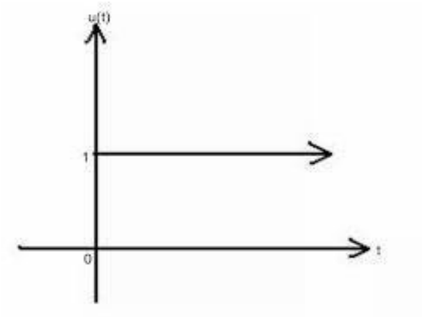
DT unit impulse

3. Unit step function

CT unit step : A continuous time unit step signal is denoted by u(t). Its value is unity (1) for all positive values of t. that means its value is one for t ≥ 0. While for other values of t, its value is zero. Unit step function is defined as U(t)= 1 for t ≥ 0

0 Otherwise

The graphical representation of CT unit step function is as shown in figure below:



Unit step function

DT unit step sequence

A discrete time unit step signal is denoted by u(n) and all its samples have value of 1 for n ≥ 0.While for other values of n, its value is zero.

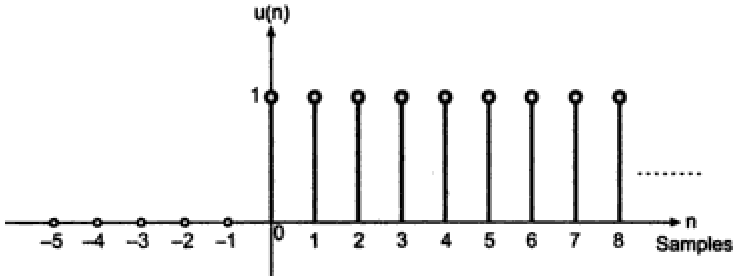
DT Unit step function is defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

In the form of sequence it can written as,

$$u(n) = \{0,\ 0,\ \underset{\uparrow}{\bar{1}},\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1\}$$

Graphically it can be represented as follows:



Graphical representation of DT unit step signal

The unit step and the impulse functions are related to one another by

$$u(t) = \int_{-\infty}^{t} \delta(t)dt$$

$$\delta(t) = \frac{du(t)}{dt}$$

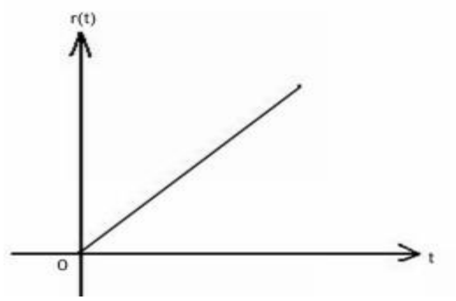
4. Unit Ramp signal

CT unit Ramp signal: A continuous time unit ramp signal is denoted by r(t). Its value is t for allpositive values of t. While for other values of t, its value is zero.

Unit ramp function is defined as r(t) = t for t ≥ 0

0 otherwise

The graphical representation of CT unit ramp signal is as shown in figure below:



CT unit Ramp signal

DT unit Ramp:

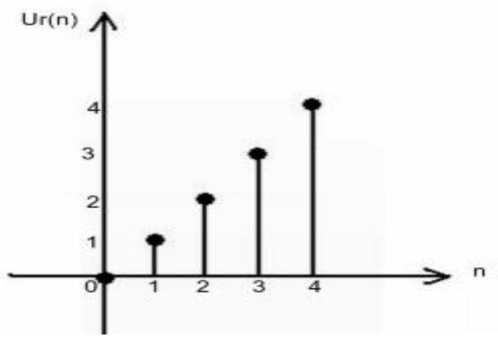
A discrete time unit ramp signal is denoted by r(n). Its value increases linearly with samplenumber n. mathematically it is defined as,

$$u_r(n) = n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

From above equation, it is clear that the value of signal at a particular interval is equal to thenumber of interval at that instant.

$$u_r(n) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Graphically it is represented in figure below:



DT unit Ramp sequence

Relationship between unit impulse, step and ramp signal

1. Integrating the unit step signal we get,

$$\boxed{\int u(t)dt = \int dt = t}$$

By integrating the unit step function, unit ramp function is obtained. In the reverse process, bydifferentiating unit ramp function, the unit step function is obtained.

2. The continuous time unit step function is the running integral of unit impulse function which isexpressed as,

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

$$\boxed{\frac{du(t)}{dt} = \delta(t)}$$

3. By differentiating the ramp function twice, the impulse function is obtained.

$$r(t) = t$$
$$\frac{dr(t)}{dt} = 1 = u(t)$$

$$\boxed{\frac{d^2r(t)}{dt^2} = \frac{du(t)}{dt} = \delta(t)}$$

Thus impulse function is obtained by differentiating the ramp function twice. In reverse process, By integrating unit impulse function twice, the ramp function is obtained which is mathematicallyexpressed as follows:

$$\boxed{r(t) = \iint \delta(t)\, dt}$$

The relationship between unit step, impulse and ramp signals are represented below

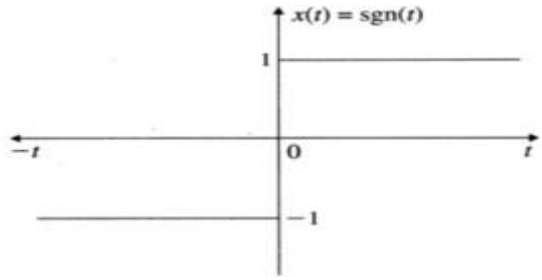
$$\delta(t) \xrightarrow{\text{integrate}} u(t) \xrightarrow{\text{integrate}} r(t)$$
$$r(t) \xrightarrow{\text{differentiate}} u(t) \xrightarrow{\text{differentiate}} \delta(t)$$

5. Signum function (sgn(t))

The CT signum function is defined as

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

The graphical representation of CT signum function is as shown in figure below:



Representation of unit Signum function

Signum function is an odd function.

The relation between sgn(t) and u(t) as follows: Sgn(t) = -1+ 2 u(t)

Sgn(t) = u(t)- u(-t)

DT Signum function

The DT signum function is defined as Sgn(n) = 1 for n > 0

-1 for n < 0

6. Sinusoidal signal

Continuous-Time Sinusoidal Signals: A simple harmonic oscillation is mathematicallydescribed by the following continuous-time sinusoidal signal as shown in figure.

$$x_a(t) = A\cos(\Omega t + \theta), -\infty < t < \infty$$

The subscript *a* used with x (t) denotes an analog signal. This signal is completelycharacterized by three parameters:

A is the *amplitude* of the sinusoid.
Ω is the *frequency* in radian s per second
(rad/s),ϴ is the *phase* in radians.

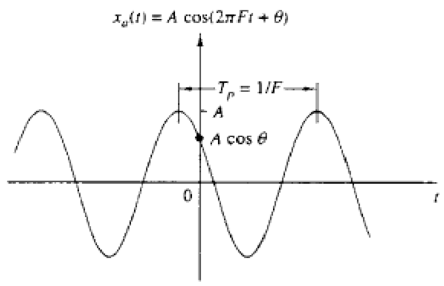
Instead of Ω, we often use the frequency *F* in cycles per second or hertz (Hz), where

$$\Omega = 2\pi F$$

X_a(t) can be written as

$$x_a(t) = A\cos(2\pi Ft + \theta), -\infty < t < \infty$$

We will use both forms for representing sinusoidal signals.



Analog sinusoidal signalDiscrete-Time Sinusoidal Signals

A discrete-time sinusoidal signal may be ex pressed as

$$x(n) = A\cos(\omega n + \theta), -\infty < n < \infty$$

Where *n* is an integer variable, called the sample number.

A is the amplitude of the sinusoid,

ώ is the frequency in radians per sample,

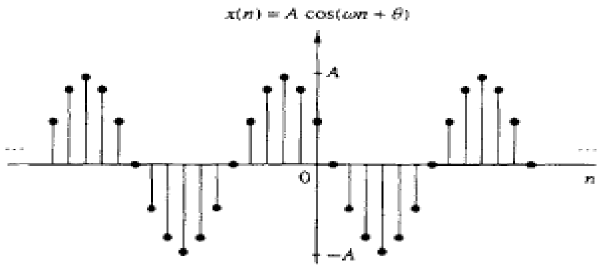
andϴ is the phase in radians.

If instead of ώ we use the frequency variable f defined by

$$\omega \equiv 2\pi f$$

The x(n) relation becomes

$$x(n) = A\cos(2\pi fn + \theta), -\infty < n < \infty$$



Discrete-time sinusoidal signal

In contrast to continuous time sinusoids, the discrete time sinusoids are characterized by thefollowing properties:

A discrete-time sinusoid is periodic only if its frequency f is a rational number.
By definition, a discrete time signal x(n) is periodic with period N (N > 0) if and only if

$$x(n+N) = x(n) \qquad \text{for all } n$$

The smallest value of N for which the above is true is called the fundamental period.
The proof of the periodicity property is simple. For a sinusoid with frequency to be periodic, weshould have

$$\cos[2\pi f_0(N+n)+\theta] = \cos(2\pi f_0 n+\theta)$$

This relation is true if an d only if there exists an integer k such that

$$2\pi f_0 N = 2k\pi$$

$$f_0 = \frac{k}{N}$$

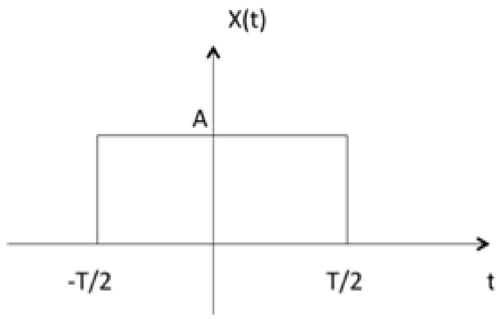
According to the above discrete-time sinusoidal signal is periodic only if its frequency f can beexpressed as the ratio of two integers (i.e.f_0 is rational).

7. Rectangular pulse

The Rectangular signal is having constant amplitude A for the time interval between –T/2 toT/2. Mathematically it is defined as , A rect(t/T) = A for –T/2 <t< T/2

= 0 Otherwise

The graphical representation of CT Rectangular function is as shown in figure below:



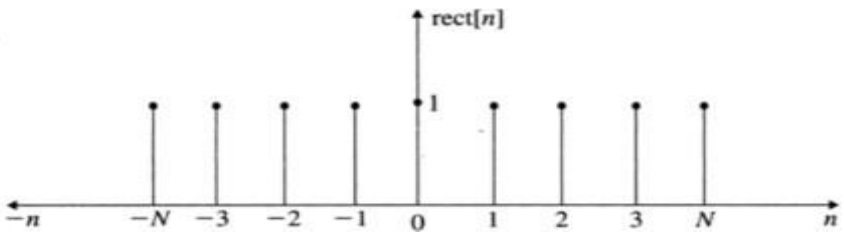
CT rectangular pulse

DT Rectangular signal is defined as

$$\text{rect}[n] = \begin{cases} 1 & |n| \le N \\ 0 & |n| > N \end{cases}$$

The above equation can also be written as,

$$\text{rect}[n] = 1 \qquad -N \le n \le N.$$



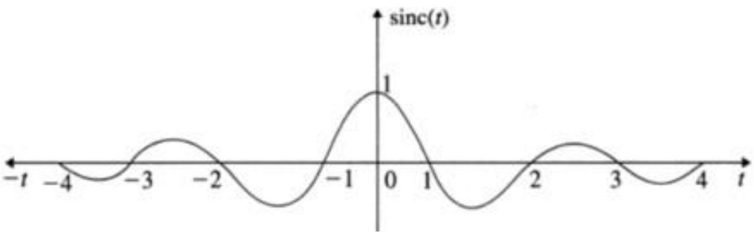
DT rectangular sequence

8. SINC Function

Sinc function is defined as follows:

$$\operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t} \qquad -\infty < t < \infty.$$

Sinc(t) = 1 at t=0 and Sinc(t) = 0 at t =1,-1,2,-2,3,-3 …..



Representation of SINC Function

9. CT complex exponential signal

The CT complex exponential signal is of the following form $x(t) = c\, e^{\alpha t}$

Where c,α are parameters.

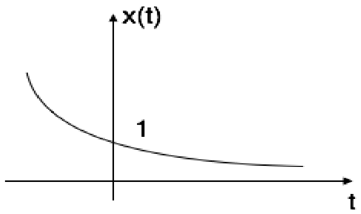
Types of CT complex exponential signal:

Depending on the type of c,α, the complex exponential signal can be classified as,

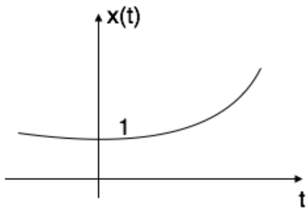
1. Real exponential signal: If c,α are real.

a) Decaying exponential $x(t)= e^{-\alpha t}$

b) Rising exponential $x(t)= e^{\alpha t}$



Decaying signal



exponential Rising

2. Periodic complex exponential signal.

The complex exponential signal is described as $X(t)=e^{jw_0 t}$

The most important property of this signal is that it is a periodic signal.

Discrete time exponential signal

A discrete time exponential signal is expressed as,

$$x(n) = a^n$$

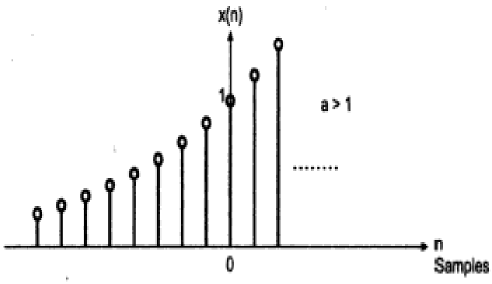
Here ‘a’ is a real constant. Depending upon the value of ‘a’ we have four different cases:

Case 1: if a > 1, then x (n) becomes the rising exponential sequence.

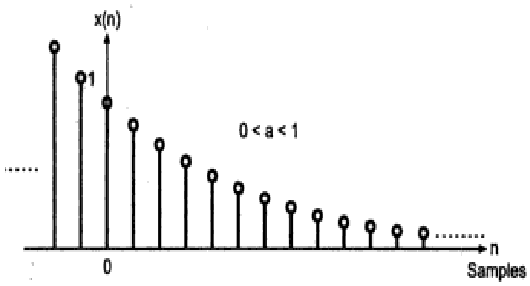
Case 2: if 0 < a < 1, then x (n) becomes decaying exponential sequence.

Case 3: if a < -1, then x (n) becomes double sided rising exponential sequence.

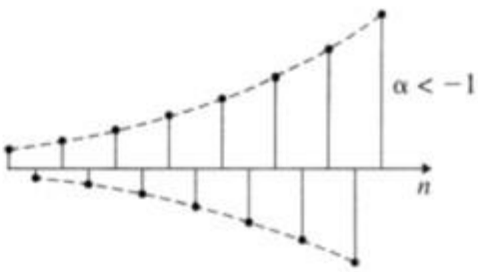
Case 4: if -1 < a < 0, then x (n) becomes double sided decaying exponential sequence



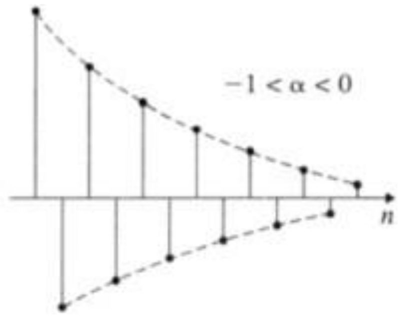
Rising exponential sequence



Decaying exponential sequence



Double sided rising exponential sequence



Double sided decaying exponential sequence

The exponential sequence can be real or complex valued. If ‘a’ is complex valued then it can berepresented as,

$$a = r\,e^{j\theta}$$

Here r is the magnitude of ‘a’ & θ is the phase of ‘a’. Hence the sequence x(n) becomes

$$\begin{aligned} x(n) &= a^n \\ &= \left[r\,e^{j\theta}\right]^n \qquad \text{Since } a = re^{j\theta} \\ &= r^n\,e^{j\theta n} \end{aligned}$$

Using Euler’s identity, the above equation becomes,

$$\begin{aligned} x(n) &= r^n\left[\cos(\theta n) + j\sin(\theta n)\right] \\ &= r^n\cos(\theta n) + j\,r^n\sin(\theta n) \end{aligned}$$

Thus each sample of sequence x(n) has real and imaginary part.ie

$$\begin{aligned} &\text{Real part of } x(n) \Rightarrow x_R(n) = r^n\cos(\theta n) \\ \text{and}\quad &\text{Imaginary part of } x(n) \Rightarrow x_I(n) = r^n\sin(\theta n) \end{aligned}$$

Similarly we can write magnitude and phase of x (n) as,

$$\begin{aligned} &\text{Magnitude of } x(n) \Rightarrow |x(n)| = A(n) = r^n \\ &\text{Phase of } x(n) \Rightarrow \angle x(n) = \phi(n) = \theta n \end{aligned}$$

1.Prove the following:

$$(i)\ \delta(n) = u(n) - u(n-1)$$

$$(ii)\ u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

$$(iii)\ u(n) = \sum_{k=0}^{\infty} \delta(n-k)$$

Sol. : **(i)** $\delta(n) = u(n) - u(n-1)$

We know that

$$u(n) = \begin{cases} 1 & \text{for} \quad n \geq 0 \\ 0 & \text{for} \quad n < 0 \end{cases}$$

And
$$u(n-1) = \begin{cases} 1 & \text{for} \quad n \geq 1 \\ 0 & \text{for} \quad n < 1 \end{cases}$$

Hence,
$$u(n) - u(n-1) = \begin{cases} 0 & \text{for} \quad n \geq 1 \text{ i.e. } n > 0 \\ 1 & \text{for} \quad n = 0 \\ 0 & \text{for} \quad n < 0 \end{cases}$$

The above equation is nothing but $\delta(n)$. i.e.,

$$u(n) - u(n-1) = \begin{array}{l} \delta(n) = 1 \quad \text{for} \quad n = 0 \\ \quad\;\;\;\; = 0 \quad \text{for} \quad n \neq 0 \end{array}$$

(ii) $u(n) = \sum_{k=-\infty}^{n} \delta(k)$:

$$\sum_{k=-\infty}^{n} \delta(k) = \begin{cases} 0 & \text{for} \quad n < 0 \\ 1 & \text{for} \quad n \geq 0 \end{cases}$$

The right side of above equation is unit impulse sequence u(n). Hence the given equation is proved.

(iii) $u(n) = \sum_{k=0}^{\infty} \delta(n-k)$:

$$\sum_{k=0}^{\infty} \delta(n-k) = \begin{cases} 0 & \text{for} \quad n < 0 \\ 1 & \text{for} \quad n \geq 0 \end{cases}$$

The right side of above equation is unit impulse sequence u (n). Hence the given equation is proved.

Basic operations on the signal

The basic operations performed on the signal are

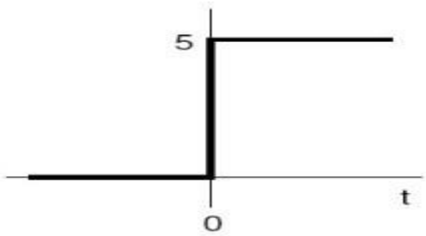
1) Amplitude scaling

Amplitude scaling means changing an amplitude of given continuous time signal. We will denotecontinuous time signal by x(t). If it is multiplied by some constant 'B' then resulting signal is,

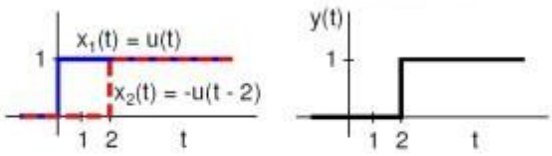
y(t)= B x(t)

Example: Sketch y(t) = 5u(t)

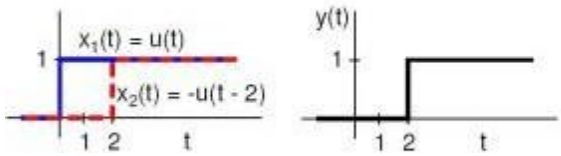
Solution: we know that u(t) is unit step function. So if we multiply it with 5, its amplitude willbecome 5 and it shown as follows:



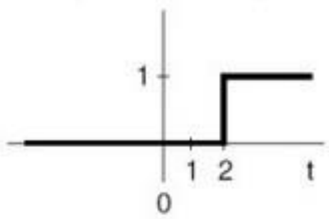
2) Sum and difference of two signals:
Consider two signals $x_1(t)$ and $x_2(t)$. Then addition of these signals is denoted by $y(t)=x_1(t)+x_2(t)$.Similarly subtraction is given by $y(t)=x_1(t)-x_2(t)$.
Example: Sketch $y(t) = u(t) – u(t – 2)$
Solution: First, plot each of the portions of this signal
separately$x_1(t) = u(t)$ …….Simply a step signal
$x_2(t) = –u(t-2)$ ……. Delayed step signal by 2 units and multiplied by -1.Then, move from one side to the other, and add their instantaneous values:



3) Product of two signals:
If $x_1(t)$ and $x_2(t)$ are two continuous signals then the product of $x_1(t)$ and $x_2(t)$ is,**Y(t) =** $x_1(t)\, x_2(t)$.
Example: Sketch $y(t) = u(t)·u(t – 2)$
Solution: First, plot each of the portions of this signal
separately$x_1(t) = u(t)$ Simply a step signal
$x_2(t) = u(t-2)$ Delayed step signal
Then, move from one side to the other, and multiply instantaneous values:



Operations performed on independent variables:
1) Time shifting:
A signal $x(t)$ is said to be 'shifted in time' if we replace t by (t-T). thus $x(t-T)$ represents the timeshifted version of $x(t)$ and the amount of time shift is 'T' sec. if T is positive then the shift is to right (delay) and if T is negative then the shift is to the left (advance).
Example: Sketch $y(t) = u(t – 2)$



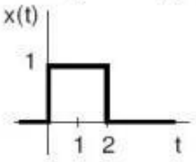
2) Time scaling:

The compression or expression of a signal in time is known as the time scaling. If x(t) is theoriginal signal then x(at) represents its time scaled version. Where a is constant.

- If a> 1 then x(at) will be a compressed version of x(t) and
- if a< 1 then it will be a expanded version of x(t).

Example: Let x(t) = u(t) – u(t – 2). Sketch y(t) = x(t/2)

First, plot x(t)

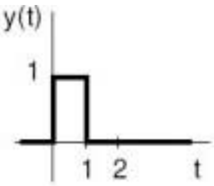


Replace all t's with 2t

y(t) = x(2t) = u(2t) – u(2t – 2)

Turns on at
2t ≥ 0
t ≥ 0
No change

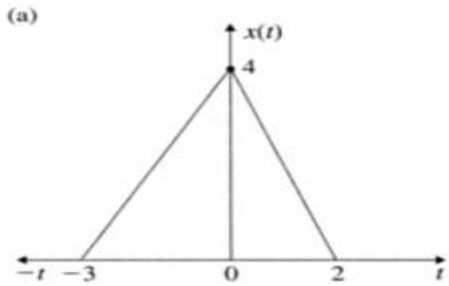
Turns on at
2t - 2 ≥ 0
t ≥ 1

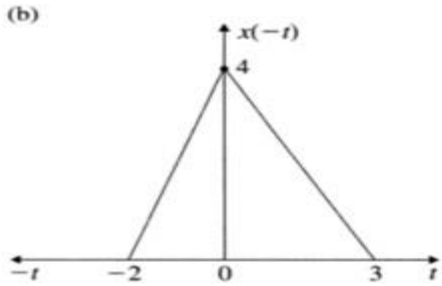


3) Time reversal (Time inversion):

Flips the signal about the y axis. y (t) = x(-t) .

Consider the signal x (t) shown in figure (a). The signal x (-t) is obtained by putting a mirror along the vertical axis. The signal to the right of the vertical axis gets reflected to the left andvice versa. Alternatively, if we make a folding across the vertical axis, the signal in the right ofthe axis is printed in left and vice versa. The signal so obtained is x (-t).



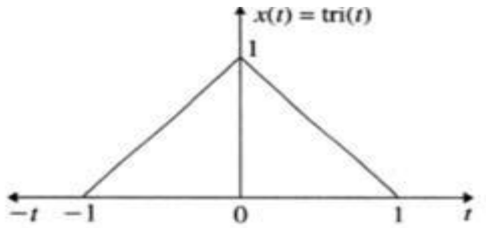


Original signal folded version x(-t)

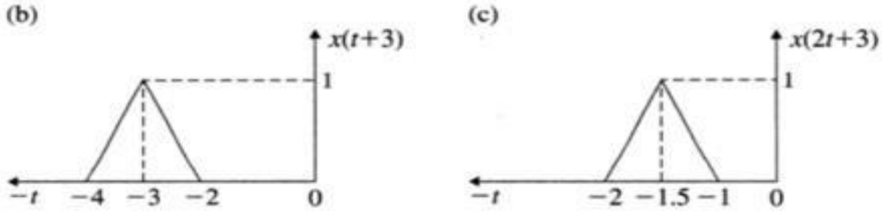
Q. Consider the triangular waveform x (t) shown in figure. Sketch the followingwaveforms. (a) x(2t+3) (b) x [(t+3)/2] (c) x (t/2-3) (d) x(-2t+3) (e) x (-2t-3)

(a) To sketch x(2t+3)

Figure (a) shows x (t) = tri(t).by shifting by t=3 towards left,x(t+3) is obtained and this is sketched in figure (b).x(t+3) is time compressed by a factor to get x(2t+3).this is sketched in figure(c).



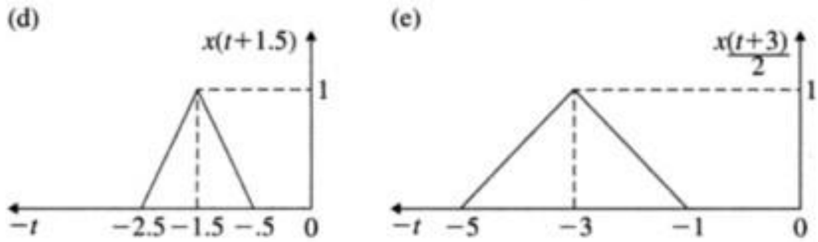
(a) $x(t) = \text{tri}(t)$.



(b) $x(t + 3)$; (c) $x(2t + 3)$.

(b)To sketch x [(t+3)/2]

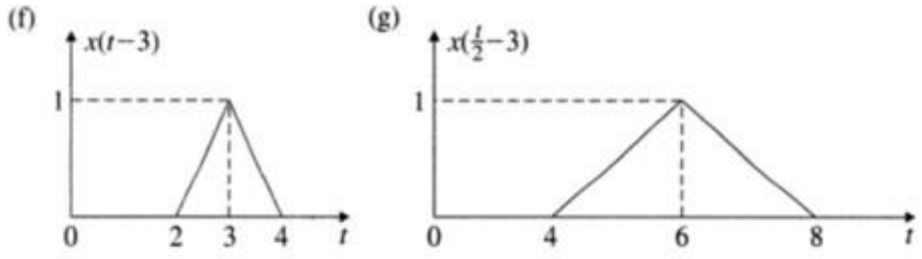
The signal x [(t+3)/2] is written as x(t/2+1.5). The signal x (t) is time shifted to the left by 1.5unit to get x(t+1.5) which is nothing but x[(t+3)/2].This is sketched in figure(e).



(d) $x(t + 1.5)$; (e) $x\left(\frac{t+3}{2}\right)$.

(c) To sketch x (t/2-3)

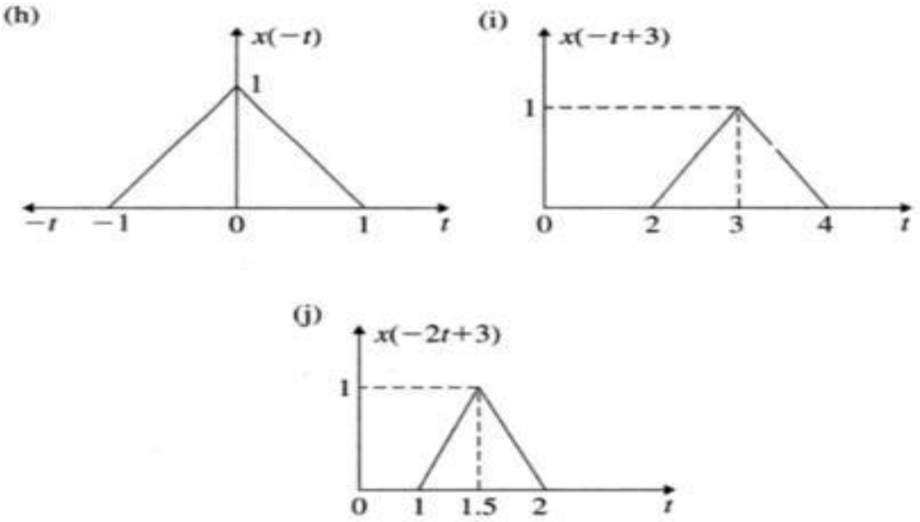
X(t-3) is obtained from x(t) by shifting the signal x(t) to the right by 3 unit and is shown in figure(f).by time expansion of x(t-3) by a factor 2, x(t/2-3) is obtained and sketched as shown infigure (g).



(f) $x(t-3)$; (g) $x\left(\frac{t}{2}-3\right)$.

(d) To sketch x(-2t+3):

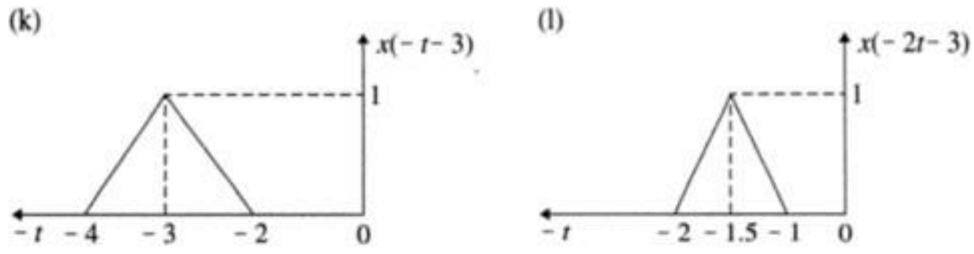
Signal x(-t) is obtained by folding and it is shown in figure (h).x(-t) is time shifted to the rightby 3 unit to get x(-t+3).this is shown in figure(i).the signal x(-t+3) is time compressed by a factor 2 to get x(-2t+3).



(h) $x(-t)$; (i) $x(-t+3)$; (j) $x(-2t+3)$.

(e) To sketch x (-2t-3)

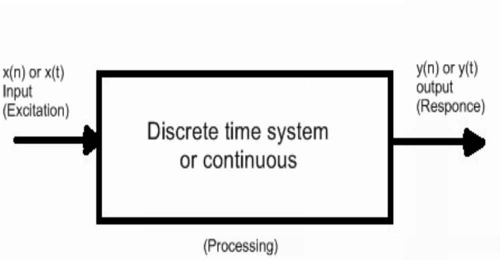
X(-t) is time shifted towards left by 3 units to get x(-t-3).this is shown in figure(k).x(t-3) istime compressed by a factor 2 to get x(-2t-3).this is sketched in figure (l)



(k) $x(-t-3)$; (l) $x(-2t-3)$.

Systems Definition:

A system is a physical device (or an algorithm) which performs required operation on a discrete time signal. Alternatively a system is defined as a set of elements or functional blocksthat are connected together and produces an output in response to an input signal. Eg: An audio amplifier, attenuator, TV set etc. The block diagram of CT/DT system is shown infigure below.



CT/DT system

Here x(t) (or) x(n) is the input signal applied to the system. It is also called as excitation.

Output of the system is denoted by y(t) (or) y(n). It is also called as response of the system.

Example: A filter is good example of a system. A signal containing noise is applied to the inputof the filter. This is an input signal to the system. The filter cancels or attenuates noise signal. This is the processing of the signal. A noise-free signal obtained at the output of the filter is called as response of the system.

Classification of systems

Generally systems are broadly classified into two categories, such as continuous timesystem(CT) and discrete time system(DT), depending upon the type of given input to the system.

CT system: if the input and output signals x(t) & y(t) are continuous time signals, then thesystem is continuous time system. The output y(t)= T[x(t)]

DT system: if the input and output signals x(n) & y(n) are discrete time signals, then thesystem is discrete time system. The output y(n)= T[x(n)]

Both CT System & DT system are classified into the following categories:

- Linear and Non-linear Systems
- Time Variant and Time Invariant Systems
- Linear Time variant and Linear Time invariant systems
- Static and Dynamic Systems
- Causal and Non-causal Systems
- Invertible and Non-Invertible Systems
- Stable and Unstable Systems

(a) Linear and Non-linear Systems

Linear Systems: A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as $x_1(t)$, $x_2(t)$, and outputs as $y_1(t)$, $y_2(t)$ respectively. Then, according to the superposition and homogenate principles,

$$T\,[a_1\, x_1(t) + a_2\, x_2(t)] = a_1\, T[x_1(t)] + a_2\, T[x_2(t)]$$

$$\therefore,\ T\,[a_1\, x_1(t) + a_2\, x_2(t)] = a_1\, y_1(t) + a_2\, y_2(t)$$

From the above expression, is clear that response of overall system is equal to response ofindividual system.

Communication channels and filters are example of linear systems.

Non-linear Systems: If the system does not satisfy the superposition theorem, then it is saidto be a nonlinear system.

How to determine whether the given system is Linear or not?

To determine whether the given system is Linear or not, we have to follow the following steps:**Step 1:** Apply zero input and check the output. If the output is zero then the system is linear. Ifthis step is satisfied then follow the remaining steps.

Step 2: Apply individual inputs to the system and determine corresponding outputs. Then addall outputs. Denote this addition by y'(n). This is the R.H.S. of the 1^{st} equation.

Step 3: Combine all inputs. Apply it to the system and find out y"(n). This is L.H.S. of equation(1).

Step 4: if y'(n) = y"(n) then the system is linear otherwise it is non-linear system.

1. Determine whether the following system is linear or not?

(a) Y(n)= n x(n)
Solution:

Step 1: When input x(n) is zero then output is also zero. Here first step is satisfied so we willcheck remaining steps for linearity.

Step 2: Let us consider two inputs x1(n) and x2(n) be the two inputs which produces outputsy1(t) and y2(t) respectively. It is given as follows:

$$x1(n) \xrightarrow{T} y1(n) = n\, x1(n)$$

$$\text{And } x2(n) \xrightarrow{T} y2(n) = n\, x2(n)$$

Now add these two output to get y'(n)

Therefore y'(n) = y1(n) + y2(n) = n x1(n) + n

x2(n)Therefore y'(n) = n [x1(n) + x2(n)]

Step 3: Now add x1(n) and x2(n) and apply this input to the system.

$$[x1(n) + x2(n)] \xrightarrow{T} y''(n) = T\,[x1(n) + x2(n)] = n\,[x1(n) + x2(n)]$$

Therefore We know that the function of system is to multiply input by ‘n’.

Here [x1(n) + x2(n)] acts as one input to the system. So the corresponding output is,

$$y''(n) = n\,[x1(n) + x2(n)]$$

Step 4: Compare y'(n) and y"(n).

Here y'(n) = y"(n). Hence the given system is linear.

(b) $y(t) = x^2(t)$

Solution: $y_1(t) = T[x_1(t)] = x_1^2(t)$

$y_2(t) = T[x_2(t)] = x_2^2(t)$

$T\,[a_1 x_1(t) + a_2 x_2(t)] = [a_1 x_1(t) + a_2 x_2(t)]^2$

Which is not equal to $a_1 y_1(t) + a_2 y_2(t)$. Hence the system is said to be non linear.

(b) Time Variant and Time Invariant Systems

A system is said to be Time Invariant if its input output characteristics do not change withtime. Otherwise it is said to be Time Variant system.

Explanation:

As already mentioned time invariant systems are those systems whose input output characteristics do not change with time shifting. Let us consider x(n) be the input to the systemwhich produces output y(n). Now delay input by k samples, it means our new input will becomex(n-k). Now apply this delayed input x(n-k) to the same system as shown in figure below.

Now if the output of this system also delayed by k samples (i.e. if output is equal to y(n-k)) thenthis system is said to be Time invariant (or shift invariant) system.

If we observe carefully, x(n) is the initial input to the system which gives output y(n), if we delayed input by k samples output is also delayed by same (k) samples. Thus we can say that input output characteristics of the system do not change with time. Hence it is Time invariantsystem.

Now let us discuss about How to determine that the given system is Time invariant or not?To determine whether the given system is Time Invariant or Time Variant, we have to follow the following steps:

Step 1: Delay the input x(n) by k samples i.e. x(n-k). Denote the corresponding output byy(n,k).That means x(n-k) → y(n,k)
Step 2: In the given equation of system y(n) replace 'n' by 'n-k' throughout. Thus the output isy(n-k).
Step 3: If y(n,k) = y(n-k) then the system is time invariant (TIV) and if y(n,k) ≠ y(n-k) thensystem is time variant (TV).
Same steps are applicable for the continuous time systems.

1) Determine whether the following system is time invariant or not.y(n) = x(n) – x(n-2)
Solution:
Step 1: Delay the input by 'k' samples and denote the output by
y(n,k)Therefore y(n,k) = x(n-k) – x(n-2-k)
Step 2: Replace 'n' by 'n-k' throughout the given equation.
Therefore y(n-k) = x(n-k) – x(n-k-2)
Step 3: Compare above two equations. Here y(n,k) = y(n-k).
Thus the system is Time Invariant.
2) Determine whether the following systems are time invariant or not?y(n) = x(n) + n x(n-2)
Solution:
Step 1: Delay the input by 'k' samples and denote the output by
y(n,k)Therefore y(n,k) = x(n-k) + n x(n-k-2)
Step 2: Replace 'n' by 'n-k' throughout the given equation.
Therefore y(n-k) = x(n-k) + (n-k) x(n-k-2)
Step 3: Compare above two equations. Here y(n,k) ≠ y(n-k).
Thus the system is Time Variant.
3) Determine whether the following systems are time invariant or not?y(n) = x(-n)

Step 1: Delay the input by 'k' samples and denote the output by
y(n,k)y(n, k) = T[x(n-k)] = x(-n-k)

Step 2: Replace 'n' by 'n-k' throughout the given

equationy(n-k) = x(-(n-k)) = x(-n + k)

Step 3: Compare above two equations. Here y(n,k) ≠ y(n-k).

Thus the system is Time Variant.

(c) Linear Time variant (LTV) and Linear Time Invariant (LTI) Systems

If the system satisfies both linearity and time variant property, then it is called linear timevariant (LTV) system.

If the system satisfies both linearity and time Invariant then that system is called liner timeinvariant (LTI) system.

(d) Static (memory less) and Dynamic system(system with memory)

Static system: A system is said to be static or memory less if its output depends upon the present input only.

Why static systems are memory less systems?

Observe the input output relations of static system. Output does not depend on delayed [x(n-k)] or advanced [x (n+k)] input signals. It only depends on present input (nth) input signal. Ifoutput depends upon delayed input signals then such signals should be stored in memory to calculate the output at n^{th} instant. This is not required in static systems. Thus for static systems,memory is not required. Therefore static systems are memory less systems.

Dynamic systems:

Definition: It is a system in which output at any instant of time depends on input sample at thesame time as well as at other times.

Here other time means, other than the present time instant. It may be past time or future time.Note that if x (n) represents input signal at present instant then,

1) x (n-k); that means delayed input signal is called as past signal.
2) x (n+k); that means advanced input signal is called as future signal.

Thus in dynamic systems, output depends on present input as well as past or future inputs.

Why dynamic system has a memory?

Observe input output relations of dynamic system. Since output depends on past or future inputsample; we need a memory to store such samples. Thus dynamic system has a memory.

1) Determine whether the following systems are static or dynamic?

a) y(t) = 2 x(t)

For present value t=0, the system output is y(0) = 2x(0). Here, the output is only dependentupon present input. Hence the system is memory less or static.

b) y(t) = 2 x(t) + 3 x(t-3)

For present value t=0, the system output is y(0) = 2x(0) + 3x(-3).

Here x(-3) is past value for the present input for which the system requires memory to get thisoutput. Hence, the system is a dynamic system.

b) y (n) = 9x(n)

In this example 9 is constant which multiplies input x(n). But output at nth instant that means y (n) depends on the input at the same (nth) time instant x(n). So this is static system.**d) y (n) = x^2(n) + 8x(n) + 17**

Here also output at n^{th} instant, y (n) depends on the input at nth instant. So this is staticsystem.

e) y (n) = x(n) + 6x(n-2)

Here output at n^{th} instant depends on input at n^{th} instant, x(n) as well as $(n-2)^{th}$ instant x(n-2)is previous sample. So the system is dynamic.

f) y (n) = 4x(n+7) + x(n)

Here x (n+7) indicates advanced version of input sample that means it is future sampletherefore this is dynamic system.

(e) Causal and non-Causal systems

A system is said to be causal if its output depends upon present and past inputs, and doesnot depend upon future input.

For non causal system, the output depends upon future inputs also.

Example 1: y (n) = 2 x (t) + 3 x(t-3)

For present value t=1, the system output is y (1) = 2x(1) + 3x(-2).Here, the system outputonly depends upon present and past inputs. Hence, the system is causal.

Example 2: y(n) = 2 x(t) + 3 x(t-3) + 6x(t + 3)

For present value t=1, the system output is y(1) = 2x(1) + 3x(-2) + 6x(4) Here, the systemoutput depends upon future input. Hence the system is non-causal system. .

(f) Stable and unstable systems

The system is said to be stable only when the output is bounded for bounded input. For abounded input, if the output is unbounded in the system then it is said to be unstable.

BIBO stable system: Any relaxed systems is said to be bounded input – output (BIBO) stableif and only if every bounded input yields a bounded outputs.

Here we will see how to determine whether the system is stable or unstable i.e. stability property. To define stability of a system we will use the term 'BIBO'. It stands for Bounded InputBounded Output. The meaning of word 'bounded' is some finite value. So bounded input meansinput signal is having some finite value. i.e. input signal is not infinite. Similarly bounded output means, the output signal attains some finite value i.e. the output is not reaching to infinite level.

Mathematical representation:

Let us consider some finite number M_x whose value is less than infinite. That means $M_x < \infty$, soit's a finite value. Then if input is bounded, we can write,

$$|x(n)| \leq M_x < \infty$$

Similarly for C.T. system

$$|x(t)| \leq M_x < \infty$$

Similarly consider some finite number M_y whose value is less than infinity. That means $M_y < \infty$,so it's a finite value. Then if output is bounded, we can write,

$$|y(n)| \leq M_y < \infty$$

Similarly for continuous time system

$$|y(t)| \leq M_y < \infty$$

Definition of Unstable system:

An initially system is said to be unstable if bounded input produces unbounded (infinite) output.

Significance: Unstable system shows erratic and extreme behavior.

When unstable system is practically implemented then it causes overflow.

Determine whether the following discrete time functions are stable or not.

1) **y(n) = x(-n)**

Solution: we have to check the stability of the system by applying bounded input. That means the value of x(-n) should be finite. So when input is bounded output will be bounded. Thus thegiven function is Stable system.

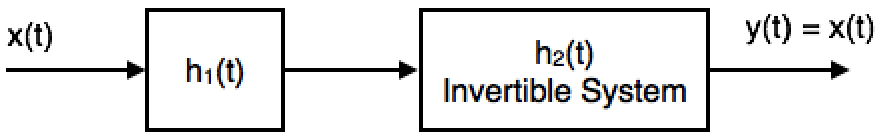
2) y (t) = x^2(t)

Let the input is u (t) (unit step bounded input) then the output y(t) = u^2(t) = u(t) = boundedoutput. Hence, the system is stable.

3) y (t) = ∫ x(t) dt
Let the input is u (t) (unit step bounded input) then the output y(t) = ∫u(t)dt = ramp signal (unbounded because amplitude of ramp is not finite it goes to infinite when t → infinite).
Hence, the system is unstable.

(g) **Invertible and noninvertible system**

A system is said to invertible if the input of the system appears at the output.



Y(S) = X(S) H1(S) H2(S)

= X(S) H1(S) · 1(H1(S))   *Since H2(S) = 1/(H1(S))*

∴ Y(S) = X(S)

→ y(t) = x(t)

Hence, the system is invertible.

If y(t) ≠ x(t), then the system is said to be non-invertible.

Solved Problems

1. Determine whether the system described by the following input output equations arelinear, time invariant, stable, memory less and casual or not.

(1) y(t)=cos[x(t)]

(i) Memory less / system with memory

A system is said to be static or memory less if its output depends upon the present input

only.at time t=0; y(0)= cos[x(0)] ; the present output y(0) depends only on present input

x(0).

at time t=1; y(1)= cos[x(1)] ; the present output y(1) depends only on present input x(1).

From the above equations, we can say that the present output is only dependent upon presentinput. Hence the system is memory less or static system.

(ii) Casual / non casual system

A system is said to be causal if its output depends upon present and past inputs, and doesnot depend upon future input.

For present value t=1, the system output is y(1)= cos[x(1)] ;.Here, the system output only depends upon present inputs. So the system is a causal system.

(iii) Time variant/ time invariant

A system is said to be time variant if its input and output characteristics vary with time. Otherwise, the system is considered as time invariant.

The condition for time invariant system is : y (t , T) = y(t-T)The condition for time variant

system is : y (t , T) ≠ y(t-T)

y (t , T) = cos[x(t-T)]

y(t-T) = cos[x(t-T)]

∴ y (t , T) = y(t-T) . Hence, the system is time invariant.

(iv) Stable

For the bounded input the system produces bounded output. So the system is said to bestable.

(v) linear / nonlinear

A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as $x_1(n)$, $x_2(n)$, and outputs as $y_1(n)$, $y_2(n)$ respectively. Then according to the superposition and homogenate principles,

$$T[a_1 x_1(t) + a_2 x_2(t)] = a_1 T[x_1(t)] + a_2 T[x_2(t)]$$

$$y_1(t) = T[x_1(t)] = \cos[x_1(t)]$$
$$y_2(t) = T[x_2(t)] = \cos[x_2(t)]$$
$$a_1 T[x_1(t)] + a_2 T[x_2(t)] = a_1 \cos[x_1(t)] + a_2 \cos[x_2(t)] \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$T[a_1 x_1(t) + a_2 x_2(t)] = \cos[a_1 x_1(t) + a_2 x_2(t)] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
$$(1) \neq (2)$$

Comparing the above two equations it does not satisfy the superposition property. So thesystem is said to be non linear.

(2) y(n)= x(-n)

(i) Memory less / system with memory

A system is said to be static or memory less if its output depends upon the present input

only.at time n=0; y(0)=x(0); the present output y(0) depends only on present input x(0).

at time n=1; y(1)=x(-1); the present output y(1) depends only on past input x(-1).

at time n=-1; y(-1)=x(1); the present output y(-1) depends only on future input x(1).

From the above equations, we can say that the present output is not only depends uponpresent input. Hence the system is said to be dynamic system.

(ii) Casual / non casual system

A system is said to be causal if its output depends upon present and past inputs, and does notdepend upon future input. Otherwise it is non casual.

at time n=0; y(0)=x(0); the present output y(0) depends only on present input

x(0).at time n=1; y(1)=x(-1); the present output y(1) depends only on past

input x(-1). at time n=-1; y(-1)=x(1); the present output y(-1) depends only on

future input x(1).

Here, the system output depends upon present input, past and future inputs. So the system issaid to be non causal system.

(iii) Time variant/ Time invariant

A system is said to be time variant if its input and output characteristics vary with time.Otherwise, the system is considered as time invariant.

The condition for time invariant system is : y (n , k) = y(n-k)The condition for time variant

system is : y (n , k) ≠ y(n-k)

$$y(n, k) = T[x(n-k)] = x(-n-k)$$

$$y(n-k) = x(-(n-k))=x(-n+k)$$

∴ y(n, k) ≠ y(n-k). Hence, the system is time variant.

(iv)Stable/unstable

For the bounded input this system produces bounded output. So the system is saidto be stable.

(v) Linear / nonlinear

A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as $x_1(n)$, $x_2(n)$, and outputs as $y_1(n)$, $y_2(n)$ respectively. Then according to the superposition and homogenate principles,

$$T\,[a_1\,x_1(n) + a_2\,x_2(n] = a_1\,T[x_1(n)] + a_2\,T[x_2(n)]$$

$$y_1\,(n) = T[x_1(n)] = x\,_1(-n)$$

$$y_2(n) = T[x_2(n)] = x\,_2(-n)$$

$$a_1\,T[x_1(n)] + a_2\,T[x_2(n)] = a_1\,x\,_1(-n) + a_2\,x\,_2(-n) \text{ .... } (1)$$

$$T\,[a_1\,x_1(n) + a_2\,x_2(n] = [a_1\,x\,_1(-n) + a_2\,x\,_2(-n)] \text{ .... } (2)$$

Comparing the above two equations it satisfy the superposition property. So the system is saidto be linear.

(3) y(t)= t x(t)

(i) Memory less / system with memory

A system is said to be static or memory less if its output depends upon the present input only.At

time t=0; y(0)= 0 x(0) ; the present output y(0) depends only on present input x(0).

At time t=1; y(1)= x(1) ; the present output y(1) depends only on present input x(1).

From the above equations, we can say that the present output is only dependent upon presentinput. Hence the system is memory less or static system.

(ii) Casual / non casual system

A system is said to be causal if its output depends upon present and past inputs, and does notdepend upon future input.

For present value t=1, the system output is y(1)= x(1) ;.Here, the system output only dependsupon present inputs. So the system is a causal system.

(iii) Time variant/ time invariant

A system is said to be time variant if its input and output characteristics vary with time.Otherwise, the system is considered as time invariant.

The condition for time invariant system is : y (t , T) = y(t-T)

The condition for time variant system is : y (t , T) ≠ y(t-T)

y (t , T) = t x(t-T) y(t-T) = (t-T) x(t-T)

∴ y (t , T) ≠ y(t-T) . Hence, the system is time variant.

(iv) Stable

For the bounded input the system produces bounded output. So the system is said to be stable.

(v) Linear / nonlinear

A system is said to be linear when it satisfies superposition and homogenate principles. Consider two systems with inputs as $x_1(t)$, $x_2(t)$, and outputs as $y_1(t)$, $y_2(t)$ respectively. Then according to the superposition and homogenate principles,

$$T[x_1(t) + x_2(t)] = T[x_1(t)] + T[x_2(t)]$$

$$y_1(t) = T[x_1(t)] = t\,x_1(t)$$
$$y_2(t) = T[x_2(t)] = t\,x_2(t)$$

$$T[x_1(t)] + T[x_2(t)] = t\,x_1(t) + t\,x_2(t) \ldots\ldots\ldots\ldots (1)$$

$$T[x_1(t) + x_2(t)] = t\,[x_1(t) + x_2(t)] \ldots\ldots\ldots (2) \qquad (1) = (2)$$

Comparing the above two equations it satisfy the superposition property. So the system is saidto be linear**.**

2. Determine whether the following systems are static, casual, time invariant, linear, andstable.

(a) $y[n] = x[4n + 1]$

1.

$$y[0] = x[1]$$

The output depends on future input. Hence

$$\boxed{\text{The system is Dynamic and Non-causal.}}$$

2. The output due to the delayed input is,

$$y[n, n_0] = x[4n - n_0 + 1]$$

The delayed output due to the input is,

$$\begin{aligned} y[n-n_0] &= x[4(n-n_0)+1] \\ &= x[4n-4n_0+1] \end{aligned}$$

$$y[n, n_0] \neq y[n-n_0]$$

$$\boxed{\text{The system is Time Variant.}}$$

3.

$$\begin{aligned} a_1y_1[n] &= a_1x_1[4n+1] \\ a_2y_2[n] &= a_2x_2[4n+1] \\ y_3[n] &= a_1y_1[n] + a_2y_2[n] \\ &= a_1x_1[4n+1] + a_2x_2[4n+1] \\ y_4[n] &= a_1x_1[4n+1] + a_2x_2[4n+1] \\ y_3[n] &= y_4[n] \end{aligned}$$

$$\boxed{\text{The system is Linear.}}$$

4. The input is shifted and time compressed signal. As long as the input is bounded the output isalso bounded.

$$\boxed{\text{The system is Stable.}}$$

The representation of signal with respect to time is called as its time domain representation. The time domain representation is not sufficient for its analysis. Hence we haveto use the frequency domain

The system is

$$\boxed{\text{(1) Dynamic, (2) Non-causal, (3) Time Variant, (4) Linear and (5) Stable.}}$$

representation of the signal. The signal represented in frequency domain is called as the line spectrum. The line spectrum consists of two graphs namely:

1) Frequency spectrum: A graph of amplitude VS frequency.
2) Phase spectrum: A graph of phase VS frequency.

The time domain representation gives us the following information:

- Shape of the signal
- Its frequency
- Type of signal
- One cycle period

But we cannot know anything about what frequency components are present. All this information can be obtained from the line spectrum of a signal. Line spectrum can be obtained by using either Fourier series or Fourier transform. Line spectrum enables us to analyze and synthesis a signal.

How to plot line spectrum?

Line spectrum is useful in understanding the existence and amplitude/ phases of variousfrequency components present in a waveform.

1) In all spectral drawing the independent variable plotted on the x axis is frequency f.

2) Phase angle is always measured with respect to cosine wave's .hence if it necessary toconvert sine waves to cosine waves using the following standard identity

$$\text{Sin } wt = \cos(wt-90)$$

3) The amplitude is always regarded as positive quantity. So if negative sign appears theyshould be absorbed in the phase change to keep amplitude positive.

$$-A\cos wt = A\cos(wt+-180)$$

Thus additional phase change of 180 converts the negative amplitude –A to positive amplitude +A.

PART-A QUESTIONS

1. What are the classifications of signals?
   - Continuous and discrete time signals
   - Periodic and non-periodic signals
   - Even and odd signals
   - Energy and power signals
   - Deterministic and random signals

2. Give examples of con continuous time signals.

Ans. sine wave, cosine wave, triangular wave etc.

3. Define Even and Odd Signals.

Ans. A signal x(t) is said to be symmetrical (or) even if it satisfies the following condition:X(t) = x(-t) for CT signal

X(n) = x(-n) for DT signal

A signal x(t) is said to be anti symmetrical (or) odd if it satisfies the following condition:X(t) = -x(-t) for CT signal

4. Differentiate between Signal & System.

Ans. A signal is a description of how one parameter varies with another parameter. For instance, voltage changing over time in an electronic circuit, or brightness varying with distance in an image. A system is any process that produces an output signal in response to an input signal.

PART-B QUESTIONS

(1) With regard to Fourier series representation justify the following statements:

Odd functions have only sine terms.

Functions with half wave symmetry have only odd harmonics.

Even functions have no sine terms

(2) Approximate the function described below by a wave form $\sin t$ over the interval (0, 2).The function is:

$$f(t) = -1 \qquad 0 < t < 1$$
$$= 1 \qquad 1 < t < 2.$$

Also sketch the original function and approximated function.

(3) Find Convolution integral for $x_1(t) = t^2\, u(t)$ & $x_2(t) = e^{-t}u(t)$?

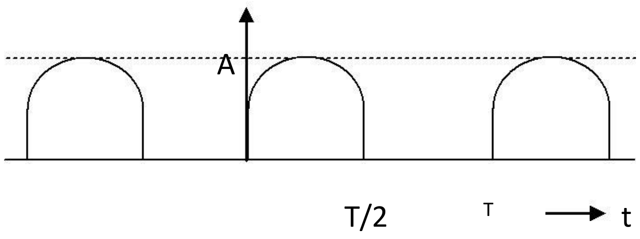
(4) Determine complex exponential Fourier series representation for the following signals

(i) $x(t) = \cos(2t+\pi/4)$ (ii) $x(t) = \cos4t+\sin6t$ (iii) $x(t) = \sin^2(t)$

(5) Derive the relations between exponential and trigonometric fourier series

coefficients.(6)Explain about Dirichlet's conditions.

(7) Determine the trigonometric Fourier series of the half rectified waveshown below: f (t)



(8)Write short notes on exponential Fourier spectrum.

(9) Verify the following signals $\cos n\,\omega_0 t$ and $\sin m\,\omega_0 t$ are the orthogonal or not over theinterval

$(t_0, t_0 + 2/\ \omega_0)$.

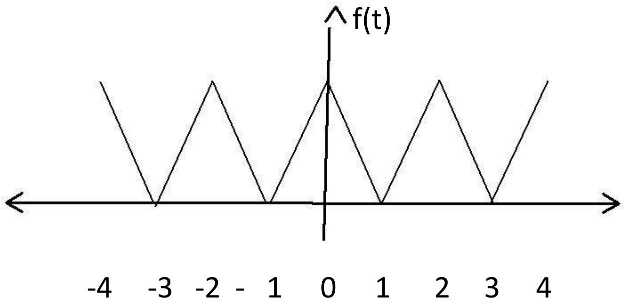
(10) Explain why mean square error is preferred in signal approximations. (11)Express the Impulse function in terms of sampling function and explain.

(12) Discuss the analogy between vectors and signals. Explain orthogonal vector space andorthogonal signal spaces.

(13)Explain the condition of Orthogonality between two signal $f_1(t)$ & $f_2(t)$.

(14) Show that the functions $Sin(nw_o t)$ and $Sin(mw_o t)$ are orthogonal to each other for allinteger values of m and n.

(15)Find the exponential Fourier series and plot the magnitude and phase spectrum for thetriangular waveform shown in figure.



(16)Expand following function f(t) by exponential Fourier series over the interval (0,1). In thisinterval f(t) is expressed as f(t) = At.

(17) Define a system. How the systems are classified? Define any four systems with examples?

(18) Sketch the following signals

(a)Y(t) = π(0.5t-1)+π(2t-3.5) (b)Y(t) = 3u(t)+2sin2t (c)Y(t)=u(t)+u(t-2)-3u(t-5)+u(t-7)

(19) (a) Define orthogonal subspace.

(b) What is orthonormal vector and orthonormal set of vectors.

(c) Prove that the complex exponential functions are orthogonal functions.

(20)Define and sketch the following signals:

(i) Impulse function. (ii) Unit step function. (iii) Ramp function. (iv) Signum function.

(21) Express discrete impulse function in terms of unit step function. Also show graphicalillusion.

(22) Describe BIBO stability of a system.(2M)

(23) Find which of the following signals are causal or non – causal:

(i) x(t) = e2t u(t – 1). (ii) x(t) = cos 2t. (iii) x(t) = 2 u(-t). (iv) x(n) = u(-n).(v) x(n) = u(n + 4) – u(n – 2).

(24) Check whether the following systems are:

(i) Static or dynamic. (ii) Linear on non-linear. (iii) Causal or non-causal.

(iv) Time invariant or time variant. (v) Stable on not stable. The given system is y(n) = an u(n).

UNIT II
FOURIER SERIES & FOURIER TRANSFORM

Continuous time Fourier series

Fourier series represents a periodic waveform in the form of sum of infinite number ofsine and cosine terms. It is a representation of a signal in a time domain series form. Fourier series is a tool used to analyze any periodic signal. After the analysis we obtain the followinginformation about the signal.

1) What are the freq. components are present in the signal.

2) Their amplitudes.

3) The relative phase difference between these frequency components.Types of Fourier series:

1. Trigonometric (or) Quadrature Fourier series
2. Polar Fourier series(or) cosine Fourier series
3. Exponential Fourier series

Trigonometric (or) Quadrature Fourier series

Consider any arbitrary continuous time signal x(t).This arbitrary signal can be split up as sine and cosines of fundamental frequency w_0 and all of its harmonics and expressed as givenbelow.

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t}$$

In above equation, a_0, corresponds to the zeroth harmonic or DC. The expression for the constant term a_0 and the amplitudes of the harmonic can be derived as,

$$\boxed{a_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt}$$

$$\boxed{a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos n\omega_0 t\, dt}$$

$$\boxed{b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin n\omega_0 t\, dt}$$

T_0 - fundamental period of x(t) in seconds
F_0 - fundamental frequency in Hz W_0 - radian frequency in rad/sec

Exponential Fourier series

By using Euler's identity, the complex sinusoids can always be expressed in terms ofexponentials. Thus the trigonometric Fourier series can be represented as

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where

$$D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

The above equation represents exponential Fourier series and D_n is the coefficient of the exponential Fourier series. The coefficient D_n is related to trigonometric Fourier series coefficients a_n,b_n as

$$\boxed{\begin{aligned} D_0 &= a_0 \\ D_n &= \frac{1}{2}(a_n - jb_n) \\ D_n^* &= \text{conjugate of } D_n \\ &= \frac{1}{2}(a_n + jb_n) \end{aligned}}$$

Polar Fourier series (or) Cosine Fourier series

In cosine polar series any periodic signal x(t) can be expressed as follows,

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos n(n\omega_0 t - \theta_n)$$

where

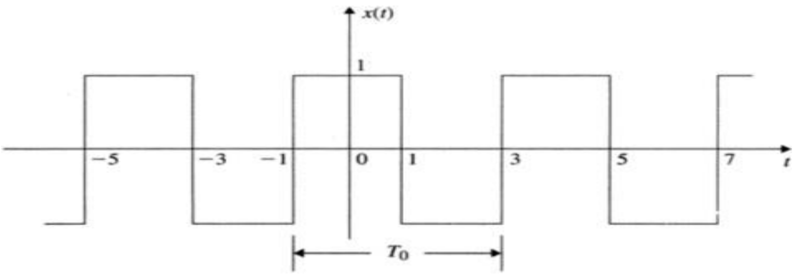
$$C_0 = a_0$$
$$C_n = \sqrt{a_n^2 + b_n^2}$$
$$\theta_n = \tan^{-1} \frac{b_n}{a_n}$$

The coefficients of compact form Fourier series and exponential form Fourier series are relatedas

$$D_0 = C_0$$
$$|D_n| = |D_n^*| = \frac{1}{2}C_n$$
$$\angle D_n = \theta_n; \quad \angle D_n^* = -\theta_n$$

FS Form	Coefficients	Equivalence
1. Trigonometric	$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$	$a_0 = C_0 = D_0$
$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$	$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\, dt$	$a_n - jb_n = C_n e^{j\theta_n} = 2D_n$
$+ b_n \sin n\omega_0 t$	$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\, dt$	$a_n + jb_n = C_n e^{-j\theta_n} = 2D_n^*$
2. Exponential		
$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$	$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt$	$C_n = 2\|D_n\| \quad n \geq 1$
3. Polar or compact cosine	$C_0 = a_0$	$\theta_n = \angle D_n$
$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n^{\omega_0 t} - \theta_n)$	$C_n = \sqrt{a_n^2 + b_n^2}$	
	$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$	

1. Find the trigonometric Fourier series of the periodic signal as shown in figure below.



1. From the figure, it is evident that the waveform is symmetrical with respect to the axis t.

So a_0=0.

2. By folding x(t) across the vertical axis, it is observed that x(t)=x(-t) which shows that thefunction of the signal is even. Hence b_n=0.

3. From figure it is obtained that the fundamental period T_0 = 4 sec and the fundamental radianfrequency w_0= π/2 rad/sec.

The trigonometric Fourier series is written as

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

The given signal is expressed as

$$\begin{aligned} x(t) &= 1 \qquad \text{for} \quad -1 \le t \le 1 \\ &= -1 \qquad \text{for} \quad 1 \le t \le 3 \end{aligned}$$

The signal is symmetrical with respect to time axis and hence a_0=0.Also from figure, it isevident that x(t) = x(-t) and therefore the signal is an even signal so b_n=0.

Substituting a_0=0 & b_n=0 in x(t) equation, we get

$$x(t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{2} t$$

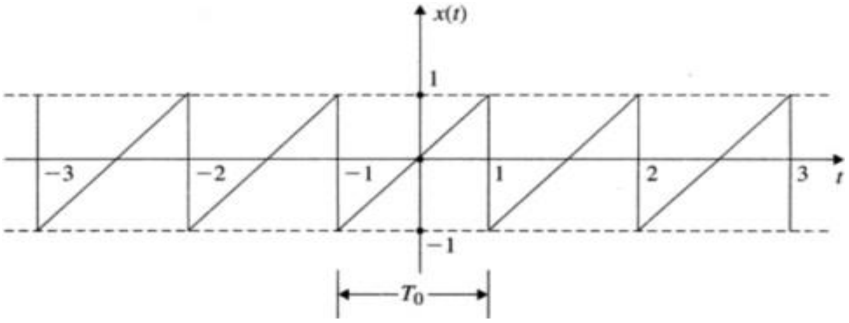
$$\begin{aligned} a_n &= \frac{2}{T_0} \int_{-1}^{3} x(t) \cos\left(\frac{n\pi}{2} t\right) dt \\ &= \frac{1}{2} \left[\int_{-1}^{1} \cos \frac{n\pi}{2} t + \int_{1}^{3} (-1) \cos \frac{n\pi}{2} t \, dt \right] \\ &= \frac{1}{2} \left[\left\{ \frac{2}{n\pi} \sin \frac{n\pi}{2} t \right\}_{-1}^{1} - \left\{ \frac{2}{n\pi} \sin \frac{n\pi}{2} t \right\}_{1}^{3} \right] \\ &= \frac{1}{n\pi} \left[\sin \frac{n\pi}{2} + \sin \frac{n\pi}{2} + \sin \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right] \\ &= \frac{4}{n\pi} \sin \frac{n\pi}{2} \\ &= 0 \qquad \text{for } n = \text{even} \\ &= \frac{4}{n\pi} \qquad \text{for } n = 1, 5, 9, 13, \ldots \\ &= -\frac{4}{n\pi} \qquad \text{for } n = 3, 7, 11, 15, \ldots \end{aligned}$$

$$x(t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{2} t$$

The signal x(t) can be expanded as follows

$$\boxed{x(t) = \frac{4}{\pi} \left[\cos \frac{\pi}{2} t - \frac{1}{3} \cos \frac{3\pi}{2} t + \frac{1}{5} \cos \frac{5\pi}{2} t - \frac{1}{7} \cos \frac{7\pi}{2} t \right]}$$

2. For the periodic signal shown in figure, determine the trigonometric Fourier series.



1. From figure $T_0 = 2$ sec and the fundamental radian frequency $w_0 = \pi$ rad/sec. The signal is symmetrical with respect to time axis and hence $a_0 = 0$.Also from figure, it is evident that x(t)=-x(-t) and therefore the signal is an odd signal and a_n=0.The Fourier series for such a signal is therefore

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t.$$

2. The coefficient b_n is determined as follows:

$$\begin{aligned} x(t) &= t \qquad -1 \le t \le 1 \\ b_n &= \frac{2}{T_0}\int_{-1}^{1} t \sin n\omega_0 t \, dt \\ &= \int_{-1}^{1} t \sin n\pi t \, dt \end{aligned}$$

The above integral is solved using the infinite integral

$$\int u dv = uv - \int v du$$

Let $u = t$, $du = dt$

$$dv = \int \sin n\pi t \, dt; \qquad v = -\frac{1}{n\pi} \cos n\pi t$$

$$b_n = \left[-\frac{t}{n\pi} \cos n\pi t \right]_{-1}^{1} + \frac{1}{n^2\pi^2} \left[\sin n\pi t \right]_{-1}^{1}$$

$$= -\frac{2}{n\pi} \cos n\pi + \frac{1}{n^2\pi^2} [\sin n\pi + \sin n\pi]$$

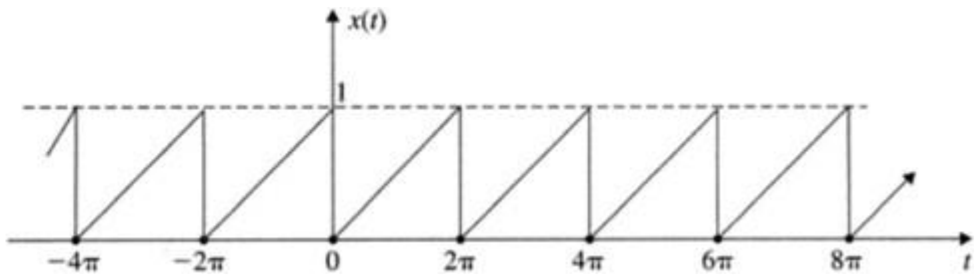
since $\sin n\pi = 0$,

$$\boxed{b_n = -\frac{2}{n\pi} \cos n\pi}$$

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$$

$$\boxed{x(t) = \frac{2}{\pi} \left[\sin \pi t - \frac{1}{2} \sin 2\pi t + \frac{1}{3} \sin 3\pi t + \ldots \right]}$$

3. Find the trigonometric Fourier series of the periodic signal.



Solution:

1. From figure $T_0 = 2\pi$ sec and the fundamental radian frequency $w_0 = 1$ rad/sec.The signal isneither odd nor even. Further it is not symmetrical with respect to the time axis. So the coefficients, a_0,a_n & b_n are to be evaluated.

$$x(t) = \frac{t}{2\pi} \qquad 0 \leq t \leq 2\pi$$

$$a_0 = \frac{1}{T_0}\int_0^{2\pi} \frac{t}{2\pi} dt = \frac{1}{4\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi}$$

$$\boxed{a_0 = \frac{1}{2}}$$

$$\begin{aligned} a_n &= \frac{2}{T_0}\int_0^{2\pi} \frac{t}{2\pi} \cos nt\, dt \\ &= \frac{1}{2\pi^2}\int_0^{2\pi} t \cos nt\, dt \end{aligned}$$

Let $u = t$; $du = dt$

$$\begin{aligned} dv &= \int \cos nt\, dt; \quad v = \frac{\sin nt}{n} \\ a_n &= uv - \int v du \\ &= \frac{1}{2\pi^2}\left[\frac{t \sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^{2\pi} \\ &= \frac{1}{2\pi^2}[0 + 0 + 1 - 1] \end{aligned}$$

$$\boxed{a_n = 0}$$

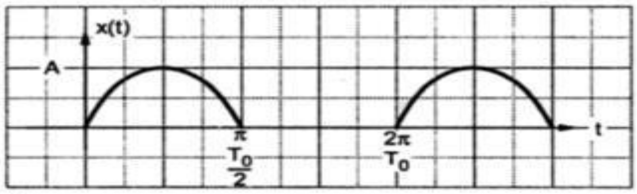
(This is due to half wave symmetry).

$$\begin{aligned} b_n &= \frac{2}{T_0}\int_0^{2\pi} \frac{t}{2\pi} \sin nt\, dt \\ &= \frac{1}{2\pi^2}\left[-\frac{t \cos nt}{n} + \frac{\sin nt}{n^2}\right]_0^{2\pi} \quad \text{[using } u\text{-}v \text{ method]} \\ &= \frac{1}{2\pi^2}\left[-\frac{2\pi}{n}\cos 2\pi n\right] \end{aligned}$$

$$\boxed{b_n = -\frac{1}{n\pi}}$$

$$\boxed{x(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin nt}$$

4. Find the exponential Fourier series of half wave rectified sine wave as shown in figure. Also plot magnitude and phase spectrum.



Half wave rectified sine wave

Solution: The given signal can be expressed as**,**

$$x(t) = A \sin \omega_0 t \qquad \text{for } 0 < t < \frac{T_0}{2}$$

$$= 0 \qquad \text{for } \frac{T_0}{2} < t < T_0$$

The Fourier series coefficient can be calculated as,

$$C_0 = \frac{1}{T_0} \int_t^{t+T_0} x(t)\ dt$$

$$= \frac{1}{2\pi} \int_0^{\pi} A \sin \omega_0 t\ \ dt$$

$$= \frac{A}{2\pi} \left[-\cos \omega_0 t\right]_0^{\pi}$$

$$= \frac{A}{\pi}$$

$$C_n = \frac{1}{T_0} \int_t^{t+T_0} x(t)\ e^{-j2\pi nt/T_0}\ dt$$

$$= \frac{1}{T_0} \int_0^{T_0/2} A \sin \omega_0 t\ e^{-jn\omega_0 t}\ dt \quad \text{since } \frac{2\pi}{T_0} = \omega_0$$

$$= \frac{A}{T_0} \int_0^{T_0/2} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-jn\omega_0 t}\ dt$$

$$= \frac{A}{2T_0 j} \int_0^{T_0/2} [e^{-j\omega_0 t(n-1)} - e^{-j\omega_0 t(n+1)}]\ dt$$

With these consideration, c_n becomes

$$= \frac{A}{2T_0 j}\left[\frac{e^{-j\omega_0 t(n-1)}}{-j\omega_0(n-1)} - \frac{e^{-j\omega_0 t(n+1)}}{-j\omega_0(n+1)}\right]_0^{T_0/2}$$

$$= \frac{A}{2T_0 j}\left\{\left[\frac{e^{-j2\pi f_0\cdot(n-1)T_0/2}}{-j2\pi n f_0(n-1)} - \frac{e^{-j2\pi f_0(n+1)T_0/2}}{-j2\pi f_0(n+1)}\right] - \left[\frac{1}{-j2\pi f_0(n-1)} - \frac{1}{-j2\pi f_0(n+1)}\right]\right\}$$

Since $\omega_0 = 2\pi f_0$

$$= \frac{A}{2T_0 j}\cdot\frac{1}{-j2\pi f_0}\left\{\left[\frac{e^{-j\pi(n-1)}}{n-1} - \frac{e^{-j\pi(n+1)}}{n+1}\right] - \left[\frac{1}{n-1} - \frac{1}{n+1}\right]\right\}$$

Since $f_0 \cdot T_0 = f_0 \cdot \frac{1}{f_0} = 1$

$$= \frac{A}{4\pi}\left\{\frac{e^{-j\pi(n-1)}-1}{n-1} - \frac{e^{-j\pi(n+1)}-1}{n+1}\right\}$$

$$\left.\begin{aligned} e^{j(n-1)\pi} = e^{j(n+1)\pi} &= 1 \text{ for odd `n'} \\ &= -1 \text{ for odd `n'} \end{aligned}\right\}$$

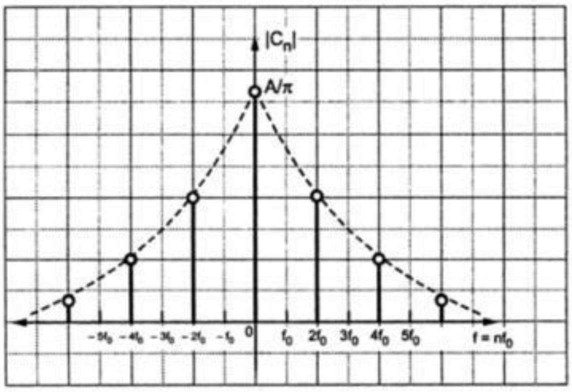
$$C_n = \frac{A}{4\pi}\left\{\frac{-2}{n-1} - \frac{-2}{n+1}\right\} \qquad \text{for even `n'}$$

$$\left.\begin{aligned} &= \frac{A}{\pi(n^2-1)} \text{ for even 'n'} \\ &= 0 \qquad \text{for odd 'n'} \end{aligned}\right\}$$

$\therefore$ $$|C_n| = \frac{A}{\pi(n^2-1)} \quad \text{for even `n'}$$

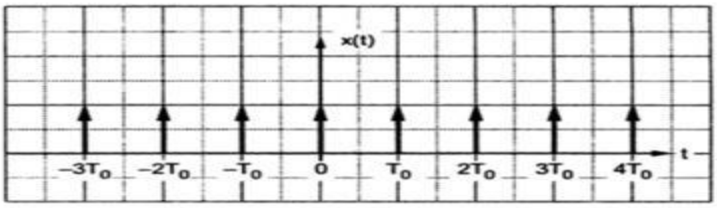
and $$C_0 = \frac{A}{\pi} \quad \text{as can be obtained from above equation also}$$

$$arg(C_n) \text{ or } \phi_n = 0$$



Magnitude spectrum of half wave rectified sine wave

5. Find the exponential Fourier series of unit impulse train as shown in figure. Also plotmagnitude and phase spectrum.



Unit impulse train

Solution: The unit impulse train can be written as,

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

C_n is calculated by,

$$C_n = \frac{1}{T_0} \int_{t}^{t+T_0} x(t)\, e^{-j2\pi nt/T_0}\, dt$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-j2\pi nt/T_0}\, dt$$

$$= \frac{1}{T_0} \int_{-\infty}^{\infty} \delta(t-0)\, e^{-j2\pi nt/T_0}\, dt$$

Assuming that only one impulse is present and T_0 tends to infinity.

The above equation is arranged for use of sifting property of delta

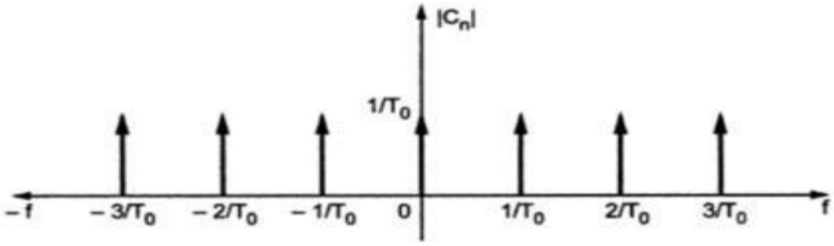
function.The shift property is given as,

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\, dt = x(t_0)$$

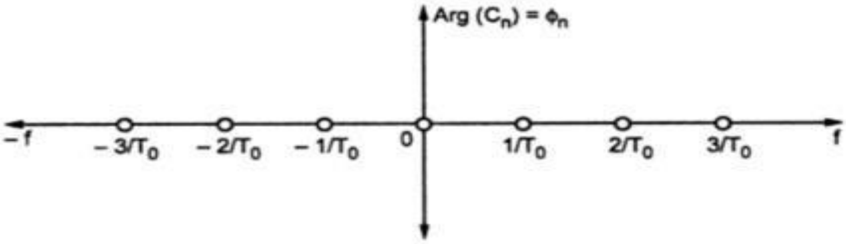
In the equation for C_n, $t_0 = 0$ and $x(t) = e^{-j2\pi nt/T_0}$. Therefore C_n will be,

$$C_n = \frac{1}{T_0} \cdot e^{-j2\pi nt/T_0}\Big|_{t=0} \quad \text{Using sifting property}$$

$$= \frac{1}{T_0}$$



(a) Amplitude spectrum of unit impulse train



(b) Phase spectrum of unit impulse train

Parseval's power theorem

It states that the total average power of the periodic signal x (t), is equal to the sum of theaverage powers of its phasor components.

Parseval's Power Theorem :

Total average power : $P = \sum_{n=-\infty}^{\infty} |C_n|^2$

Proof:

The total average normalized power of x(t) is given as,

$$P = \langle |x(t)|^2 \rangle$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt$$

$$P = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \; x^*(t) \, dt$$

Exponential Fourier series is given as,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \; e^{j2\pi n f_0 t}$$

Therefore Fourier series for $x^*(t)$ will be,

$$x^*(t) = \left[\sum_{n=-\infty}^{\infty} C_n \; e^{j2\pi n f_0 t} \right]^*$$

$$= \sum_{n=-\infty}^{\infty} C_n^* \; e^{-j2\pi n f_0 t}$$

$$P = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot \left[\sum_{n=-\infty}^{\infty} C_n^* \; e^{-j2\pi n f_0 t} \right] dt$$

Rearranging the above equation in terms of the order of summation and integration

$$P = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \; e^{-j2\pi n f_0 t} \; dt \right] C_n^*$$

In the above equation

$$\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \; e^{-j2\pi n f_0 t} \; dt = C_n$$

Therefore above equation will be,

$$P = \sum_{n=-\infty}^{\infty} C_n \cdot C_n^*$$

$$= \sum_{n=-\infty}^{\infty} |C_n|^2$$

Thus Parseval’s theorem is proved**.**

Properties of Fourier series:

Parseval's Relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Note: In Trigonometric Fourier series

- Odd functions[x(-t) =-x(t)] contains only <u>Sine</u> terms i.e. b_n Coefficients.
- Even functions[x(-t) =x(t)] contains only <u>Constant</u> and <u>Cos</u> terms i.e. a_0 ,a_n Coefficients.
- Half wave Symmetry[x(t)= -x(t ± T)] contains only odd harmonics i.e. a_n, b_n are defined for n is odd and a_0,a_n,b_n are zero for n is even

Fourier transform

The Fourier transform of any signal x(t) is given by

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt} \quad \text{for all } \omega$$

The inverse Fourier transform is calculated using the formula

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega} \quad \text{for all } t$$

$$\boxed{\begin{aligned} X(j\omega) &= F[x(t)] \\ x(t) &\overset{FT}{\longleftrightarrow} X(j\omega) \\ x(t) &= F^{-1}[X(j\omega)] \\ X(j\omega) &\overset{IFT}{\longleftrightarrow} x(t) \end{aligned}}$$

Fourier transform and inverse Fourier transform can be shown as below

(a) Fourier transformation and (b) Inverse Fourier transformation

Fourier transform X(f) is the complex function of frequency f. Therefore it can be expressedin the complex exponential form as follows,

$$X(f) = |X(f)| \cdot e^{j\theta(f)}$$

Here $|X(f)|$ is amplitude spectrum $\theta(f)$ is the phase spectrum. We know that for a real valued signal we can write,

$$X(-f) = X^*(f)$$

$$\boxed{\begin{aligned} |X(-f)| &= |X(f)| \\ \theta(-f) &= -\theta(f) \end{aligned}}$$

The following conditions should be satisfied by the signal to obtain its Fourier transform.

(i) The function x (t) should be single valued in any finite time interval T.

(ii) The function x (t) should have at the most finite number of discontinuities in any finitetime interval T.

(iii) The function x (t) should have finite number of maxima and minima in any finite timeinterval T.

(iv) The function x(t) should be absolutely integrable i.e.

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$$

The above conditions are applied to periodic as well as non periodic signals. The same conditions are also called Dirichlet's conditions. These conditions are sufficient but not necessary for the signal to be transformable. Physically realizable signal is always Fourier transformable. Thus physical realizability is the sufficient condition for the existence of Fouriertransform. We know that for all energy signals,

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$$

Properties of Fourier transform

The Fourier transform possesses the following properties and using the same results areeasily obtained .these properties are:

1. Linearity

2. Time scaling

3. Duality (or) symmetry

4. Time shifting

5. Frequency shifting

6. Area under x(t)

7. Area under X(f)

8. Differentiation in time domain

9. Integration in time domain

10. Conjugate

11. Multiplication in time domain

12. Convolution theorem

Linearity (Superposition)

Let $x_1(t) \leftrightarrow X_1(f)$ represent a Fourier transform pair and $x_2(t) \leftrightarrow X_2(f)$ represent another Fourier transform pair. Then for all constants like C_1 and C_2 we have,

$$C_1 x_1(t) + C_2 x_2(t) \quad \leftrightarrow \quad C_1 X_1(f) + C_2 X_2(f)$$

Proof : By definition of Fourier transform,

$$F[C_1 x_1(t) + C_2 x_2(t)] = \int_{-\infty}^{\infty} [C_1 x_1(t) + C_2 x_2(t)]\, e^{-j2\pi ft}\, dt$$

$$= C_1 \int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi ft}\, dt + C_2 \int_{-\infty}^{\infty} x_2(t)\, e^{-j2\pi ft}\, dt$$

$$= C_1 X_1(f) + C_2 X_2(f) \qquad \text{By definition of FT}$$

$$\boxed{\textbf{Linearity :}\ C_1 x_1(t) + C_2 x_2(t) \leftrightarrow C_1 X_1(f) + C_2 X_2(f)}$$

Time Scaling

Let $x(t)$ and $X(f)$ be a Fourier transform pair and 'a' is some constant.

Then by time scaling property,

$$\boxed{x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)}$$

Duality or Symmetry Property

Duality property of Fourier transform states that if

$$x(t) \leftrightarrow X(-f)$$

then $$X(t) \leftrightarrow x(-f)$$

Proof : By definition of inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

for $t = -t$

$$x(-t) = \int_{-\infty}^{\infty} X(f) e^{-j2\pi ft} df$$

Interchanging t and f we get,

$$x(-f) = \int_{-\infty}^{\infty} X(t) e^{-j2\pi ft} dt = F[X(t)]$$

$$\therefore \quad x(-f) \therefore \leftrightarrow X(t)$$

4 Time Shifting

If $x(t) \leftrightarrow X(f)$, then

$$\boxed{\textbf{Time shifting : } x(t-t_0) \leftrightarrow X(f)\ e^{-j2\pi f t_0}}$$

Proof : By definition of Fourier transform

$$f\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\ e^{-j2\pi ft}\ dt$$

Let $t - t_0 = \tau$

Then $t = t_0 + \tau \rightarrow dt = d\tau$

$$F\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(\tau)\ e^{-j2\pi f(t_0+\tau)}\ d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\ e^{-j2\pi ft_0}\ e^{-j2\pi f\tau}\ d\tau$$

$$= e^{-j2\pi ft_0} X(f) \qquad \text{By definition of FT.}$$

Thus a time shift 't_0' has no change on the amplitude spectrum but there is a phase shift of $-2\pi f t_0$.

.8 Differentiation in Time Domain

If $x(t) \leftrightarrow X(f)$ and first derivative of $x(t)$ is Fourier transformable, then

$$\boxed{\frac{d}{dt}\, x(t) \leftrightarrow (j2\pi f) X(f)}$$

$$\boxed{\textbf{Frequency shifting : } e^{j2\pi f_c t}\, x(t) \leftrightarrow X(f-f_c)}$$

Here f_c is real constant. This property is also called modulation theorem.

Proof : By definition of FT,

$$F\left[e^{j2\pi f_c}\, x(t)\right] = \int_{-\infty}^{\infty} e^{j2\pi f_c t}\, x(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f-f_c)t}\, dt$$

$$= X(f-f_c)$$

Multiplication of the function $x(t)$ by $e^{j2\pi f_c t}$ results in shifting of Fourier spectrum $X(f)$ in positive side by f_c.

.6 Area under *x* (*t*)

If $x(t) \leftrightarrow X(f)$, then

$$\boxed{\textbf{Area under } x(t) : \int_{-\infty}^{\infty} x(t)\, dt = X(0)}$$

That is area under $x(t)$ is equal to its Fourier transform at zero frequency.

Proof : By definition of FT,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

Let $\quad f = 0$

$$\therefore \quad X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

7 Area under *X* (*f*)

If $x(t) \leftrightarrow X(f)$, then

$$\boxed{\textbf{Area under } X(f) : \int_{-\infty}^{\infty} X(f)\, df = x(0)}$$

That is the area under Fourier spectrum of a signal is equal to its value at $t=0$

Proof : By definition of IFT,

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

Let $\quad t = 0,$

$$x(0) = \int_{-\infty}^{\infty} X(f)\, df$$

Differentiation of function $x(t)$ in time domain is equivalent to multiplying its Fourier transform by $(j2\pi f)$.

Proof : By definition of FT,

$$F\left[\frac{d}{dt}x(t)\right] = \int_{-\infty}^{\infty} \frac{d}{dt}x(t)\, e^{-j2\pi ft}\, dt$$

Integrating by parts,

$$F\left[\frac{d}{dt}x(t)\right] = e^{-j2\pi ft}\left[x(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)(-j2\pi f)\, e^{-j\pi ft}\, dt$$

$$= j2\pi f \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$= j2\pi f X(f)$$

9 Integration in Time Domain

If $x(t) \leftrightarrow X(f)$, and provided that $X(0) = 0$, then,

$$\boxed{\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{j2\pi f}X(f)}$$

Assuming that $X(0)=0$, the integration of $x(t)$ in time domain has the effect of dividing its Fourier transform by $(j2\pi f)$.

Proof : Let $x(t)$ be expressed as,

$$x(t) = \frac{d}{dt}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\}$$

We know that, $x(t) \leftrightarrow X(f)$

$$\therefore \quad F[x(t)] = F\left[\frac{d}{dt}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\}\right] = j2\pi f\left[F\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\}\right]$$

By differentiation property

i.e. $$X(f) = j2\pi f\left[F\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\}\right]$$

$$\therefore \quad F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j2\pi f}X(f)$$

10 Conjugate Functions

If $x(t) \leftrightarrow X(f)$, then for complex valued time function $x(t)$ we have

$$\boxed{x^*(t) \leftrightarrow X^*(-f)}$$

Proof : By definition of IFT,

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

By taking complex conjugates of both sides

$$x^*(t) = \int_{-\infty}^{\infty} X^*(f)\, e^{-j2\pi ft}\, df$$

Now by replacing f with – f gives,

$$x^*(t) = \int_{-\infty}^{\infty} X^*(-f)\, e^{j2\pi ft}\, df$$

$$= F^{-1}\,[X^*(-f)]$$

$$x^*(t) \leftrightarrow X^*(-f)$$

11 Multiplication in Time Domain (Multiplication Theorem)

Let the two Fourier transform pairs be $x_1(t) \leftrightarrow X_1(f)$ and $x_2(t) \leftrightarrow X_2(f)$, then

$$x_1(t)\, x_2(t) \leftrightarrow \int_{-\infty}^{\infty} X_1(\lambda)\, X_2(f-\lambda)\, d\lambda$$

That is multiplication of two signals in time domain is transformed into convolution of their Fourier transforms in frequency domain.

The short hand notation for this property is,

$$\boxed{x_1(t)\, x_2(t) \leftrightarrow X_1(f) * X_2(f)}$$

Proof : Let us write the RHS of equation

$$x_1(t)\, x_2(t) \leftrightarrow X_{12}(f)$$

i.e. $F[x_1(t)\, x_2(t)] = X_{12}(f)$

$$= \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-j2\pi ft}\, dt \quad \text{by definition of FT}$$

We know that $x_2(t)$ can be written by IFT as,

$$x_2(t) = \int_{-\infty}^{\infty} X_2(f')\, e^{-j2\pi f't}\, df' \qquad \text{since } f' \text{ and } f \text{ are not same.}$$

Let us substitute for $x_2(t)$ in equation

$$\therefore \qquad X_{12}(f) = \int_{-\infty}^{\infty} x_1(t) \int_{-\infty}^{\infty} X_2(f')\, e^{-j2\pi f't}\, df'\, e^{-j2\pi ft}\, dt$$

Let $\lambda = f - f'$, then by arranging above equation,

$$X_{12}(f) = \int_{-\infty}^{\infty} X_2(f-\lambda)\, d\lambda \int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi(f-f')t}\, dt$$

$$= \int_{-\infty}^{\infty} X_2(f-\lambda)\, d\lambda \int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi\lambda t}\, dt$$

The second integral above is $X_1(\lambda)$ from definition of FT

$$\therefore \qquad X_{12}(f) = \int_{-\infty}^{\infty} X_2(f-\lambda)\, d\lambda\, X_1(\lambda) \qquad \text{or}$$

$$= \int_{-\infty}^{\infty} X_1(\lambda)\, X_2(f-\lambda)\, d\lambda$$

$$\therefore \qquad x_1(t)\, x_2(t) \leftrightarrow \int_{-\infty}^{\infty} X_1(\lambda)\, X_2(f-\lambda)\, d\lambda$$

This property is some times called as multiplication theorem.

$$x_1(t)\, x_2(t) \leftrightarrow X_1(f) * X_2(f).$$

12 Convolution in Time Domain (Convolution Theorem)

If $x_1(t) \leftrightarrow X_1(f)$ and $x_2(t) \leftrightarrow X_2(f)$

then, $$\int_{-\infty}^{\infty} x_1(\tau)\ x_2\ (f-\tau)\ d\tau \leftrightarrow X_1(f)X_2(f)$$

This property states that convolution of two signals in time domain is transformed into multiplication of their individual Fourier transforms in frequency domain.

The short hand notation of convolution can be used to represent this property as follows,

i.e.;

$$x_1(t) * x_2(t) \leftrightarrow X_1(f)X_2(f)$$

Proof : Convolution of $x_1(t)$ *and* $x_2(t)$ as given equation 2.4.24 is,

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\ x_2(t-\tau)\, d\tau$$

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\right)e^{-j2\pi ft}\, dt \qquad \text{By definition of FT}$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j2\pi f\tau}\, d\tau \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{j2\pi f\tau}\, e^{-j2\pi ft}\, dt$$

$$= \int^{\infty} x_1(\tau)\, e^{-j2\pi f\tau}\, d\tau \int^{\infty} x_2(t-\tau)\, e^{-j2\pi f(t-\tau)} dt$$

Let $t-\tau=\alpha$ in the second integral.

$$\therefore F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j2\pi f t}\, d\tau \int_{-\infty}^{\infty} x_2(\alpha)\, e^{-j2\pi f\alpha}\, d\alpha$$

From definition of FT applied to RHS

$$F[x_1(t) * x_2(t)] = X_1(f)X_2(f)$$

i.e. $\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \leftrightarrow X_1(f)\ X_2\ (f)$

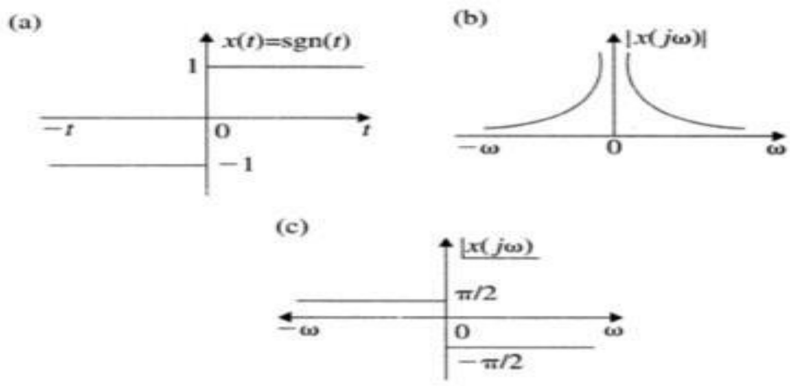
2.3.3 Fourier Transform Properties

PROPERTY	Time signal $x(t)$	Fourier transform $X(j\omega)$
Linearity	$x(t) = A\,x_1(t) + B\,x_2(t)$	$X(j\omega) = A\,X_1(j\omega) + B\,X_2(j\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Differentiation in time	$\dfrac{d^n x(t)}{dt^n}$	$(j\omega)^n X(j\omega)$
Differentiation in frequency	$tx(t)$	$j\dfrac{d}{d\omega} X(j\omega)$
Time integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\dfrac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
Time scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert} X\left(j\dfrac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-j\omega)$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X[j(\omega - \omega_0)]$
Duality	$X(t)$	$2\pi x(j\omega)$
Time convolution	$x(t) * h(t)$	$X(j\omega)H(j\omega)$
Parseval's theorem	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt$	$E = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\,d\omega$

SOLVED PROBLEMS

1. Find the Fourier transform of following and sketch their amplitude and phase spectra.

(a) X(t)= sgn(t)



Representation of sgn(t) and its spectra.

$$F[e^{-a|t|}\mathrm{sgn}(t)] = \int_{-\infty}^{0} -e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \underset{a\to 0}{Lt}\left[\frac{-1}{a-j\omega}\left\{e^{(a-j\omega)t}\right\}_{-\infty}^{0} - \frac{1}{(a+j\omega)}\left\{e^{-(a+j\omega)t}\right\}_{0}^{\infty}\right]$$

$$= \underset{a\to 0}{Lt}\left[\frac{-1}{(a-j\omega)} + \frac{1}{a+j\omega}\right] = \frac{1}{j\omega} + \frac{1}{j\omega}$$

$$= \frac{2}{j\omega}$$

$$\boxed{\mathrm{sgn}(t) \xleftrightarrow{\text{FT}} \frac{2}{j\omega}}$$

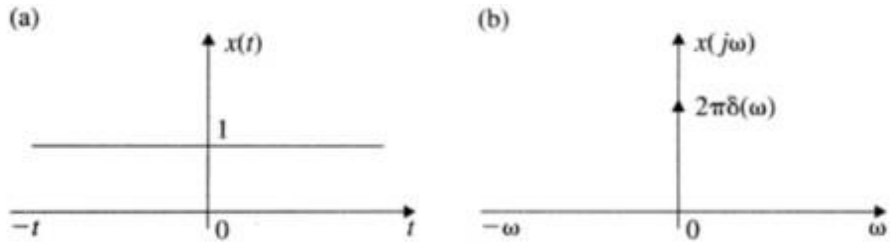
***Fourier Spectra of* sgn(t)**

$$X(j\omega) = \frac{2}{j\omega}\begin{cases}\dfrac{2}{\omega}\angle{-90^\circ} & \omega \geq 0\\[1ex] \dfrac{2}{\omega}\angle 90^\circ & \omega < 0\end{cases}$$

(b) x(t) = 1 for all values of t

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi} \qquad \delta(\omega) = \begin{cases}1 & \omega = 0\\ 0 & \text{otherwise}\end{cases}$$



Representation of $x(t) = 1$ and its FT.

$$\frac{1}{2\pi} \xleftrightarrow{\text{FT}} = \delta(\omega)$$

$$1 \xleftrightarrow{\text{FT}} = 2\pi\delta(\omega)$$

The above result shows that a constant signal x(t)=1 for all t,.x (t) and X(jw) arerepresented in figure(a) & (b) respectively.

(c) X (t) = u(t)

$$x(t) = \begin{cases} u(t) \\ 1 & t \geq 0 \end{cases}$$

To find the Fourier transform of unit step u(t) by integration yields an indeterminate value as isevident from the following equation because it has a jump discontinuity at t=0.

$$X(j\omega) = \int_0^{\infty} e^{-j\omega t}\, dt$$
$$= -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_0^{\infty}$$

So the problem is approached by considering

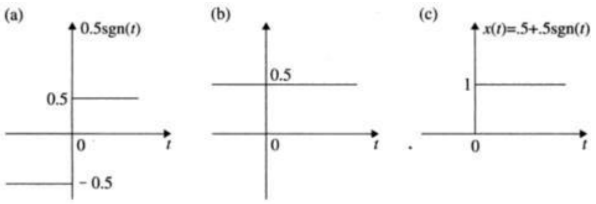
$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$$

$$F[u(t)] = F\left[\frac{1}{2}\right] + \frac{1}{2}F\operatorname{sgn}(t)$$

$$F\left[\frac{1}{2}\right] = \pi\,\delta(\omega)$$

$$F\left[\frac{1}{2}\operatorname{sgn}(t)\right] = \frac{1}{j\omega}$$

$$\boxed{F[u(t)] = \pi\,\delta(\omega) + \frac{1}{j\omega}}$$



Representation of $u(t)$ in terms of signum function.

$$F[u(-t)] = X(-j\omega)$$

$$\boxed{F[u(-t)] = \pi\,\delta(\omega) - \frac{1}{j\omega}}$$

2. Find the Fourier transform of decaying exponential signal.

The decaying exponential signal can be expressed as, x (t) = e^{-at}

u(t)By the definition of Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, u(t) \cdot e^{-j2\pi ft}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j2\pi f)t}\, dt$$

The lower limit is taken '0' since x(t) = 0, for t < 0.and u(t) =1 for t ≥ 0

$$X(f) = \frac{1}{-(a+j2\pi f)} \left[e^{-(a+j2\pi f)t}\right]_0^{\infty}$$

$$= \frac{1}{a+j2\pi f}$$

Thus the Fourier transform pair becomes,

$$\boxed{\textbf{Decaying exponential pulse}: e^{-at}\, u(t) \leftrightarrow \frac{1}{a+j2\pi f}}$$

To calculate magnitude and phase spectrum:

The function X (f) is expressed as,

$$X(f) = A(f) + jB(f)$$

Here A (f) is real part of X (f) and B (f) is the imaginary part of X(f).

Therefore magnitude spectrum of X (f) is given as,

$$|X(f)| = \sqrt{A^2(f) + B^2(f)}$$

And phase spectrum is given as,

$$\theta(f) = tan^{-1} \frac{B(f)}{A(f)}$$

$$X(f) = \frac{1}{a+j2\pi f}$$

Multiply and divide RHS by a-j2πf

$$X(f) = \frac{1}{a+j2\pi f} \times \frac{a-j2\pi f}{a-j2\pi f}$$

$$= \frac{a-j2\pi f}{a^2+(2\pi f)^2}$$

$$= \frac{a}{a^2+(2\pi f)^2} + j\,\frac{-2\pi f}{a^2+(2\pi f)^2}$$

……. (1)

$$\left.\begin{aligned} \text{Here real part } A(f) &= \frac{a}{a^2+(2\pi f)^2} \\ \text{and imaginary part } B(f) &= \frac{-2\pi f}{a^2+(2\pi f)^2} \end{aligned}\right\}$$

From the equation (1), The magnitude spectrum of X(f) will be

$$|X(f)| = \sqrt{\frac{a^2}{\left[a^2+(2\pi f)^2\right]^2} + \frac{(-2\pi f)^2}{\left[a^2+(2\pi f)^2\right]^2}}$$

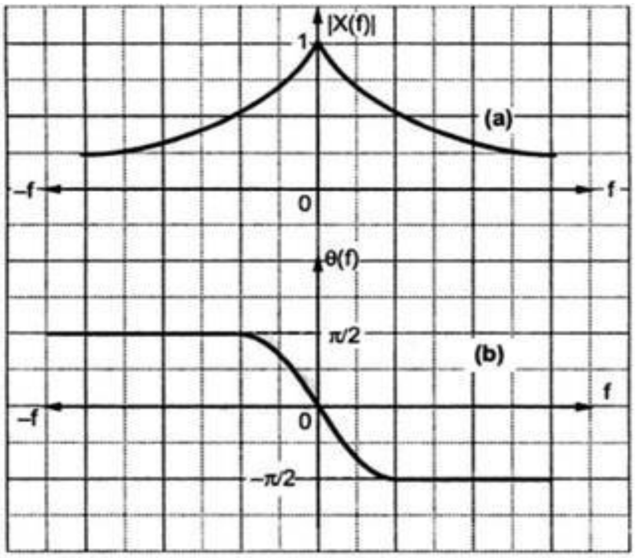
$$= \sqrt{\frac{1}{a^2+(2\pi f)^2}}$$

The phase spectrum will be

$$\theta(f) = tan^{-1}\left\{\frac{-2\pi f/[a^2+(2\pi f)^2]}{a/[a^2+(2\pi f)^2]}\right\}$$

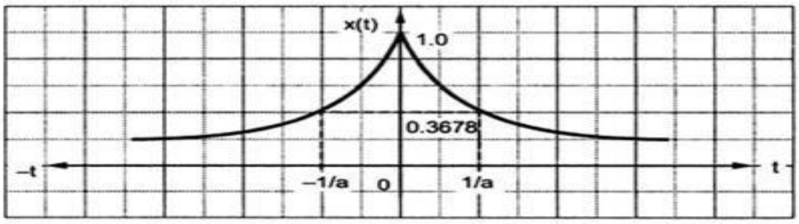
$$= tan^{-1}\left(\frac{-2\pi f}{a}\right)$$

$$= -tan^{-1}\left(\frac{2\pi f}{a}\right)$$



(a) Amplitude spectrum of decaying exponential pulse Here a = 1 (assumed). It is even function of frequency.
(b) Phase spectrum. It is odd function of frequency.

Find the Fourier transform of double sided exponential signal.



Symmetric double exponential pulse

Solution: The double exponential pulse of above figure can be represented as,

$$x(t) = e^{-at}\ ; \qquad t>0$$
$$= 1\ ; \qquad t=0$$
$$= e^{at}\ ; \qquad t<0$$

From the definition of Fourier transform

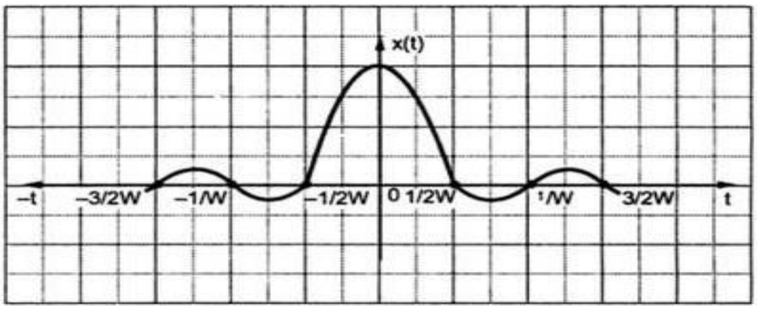
$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\ dt$$

$$= \int_{-\infty}^{0-} e^{at}\ e^{-j2\pi ft}\ dt + \int_{0-}^{0+} 1 \cdot e^{-j2\pi ft}\ dt + \int_{0+}^{\infty} e^{-at}\ e^{-j2\pi ft}\ dt$$

Or in other words integration at single point with upper and lower limits same is zero only.

$$X(f) = \frac{1}{a-j2\pi f} + 0 + \frac{1}{a+j2\pi f}$$
$$= \frac{2a}{a^2+(2\pi f)^2}$$

Find the Fourier transform of Sinc pulse as shown in figure below



Sinc pulse

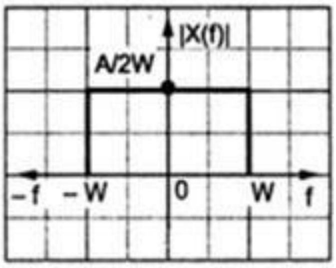
Solution:

$$A\,rect\left(\frac{t}{T}\right) \leftrightarrow \;A\,T\,sinc\,(f\,T)$$

By applying duality and time scaling properties of Fourier transform we have,

$$A\,sinc\,(2\,W\,t) \;\leftrightarrow\; \frac{A}{2W}\,rect\left(\frac{f}{2\,W}\right)$$

Figure shows the spectrum of Sinc pulse.



Spectrum of Sinc pulse

3. Find the Fourier transform of following signal

$\boldsymbol{x(t) = \cos(\omega_0 t + \phi)}$

$$\cos(\omega_0 t + \phi) = \frac{1}{2}\left[e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}\right]$$
$$= \frac{1}{2}\left[e^{j\phi}e^{j\omega_0 t} + e^{-j\phi}e^{-j\omega_0 t}\right]$$

By frequency shifting property

$$F\left[e^{j\omega_0 t}\right] = 2\pi\delta(\omega - \omega_0)$$
$$F\left[e^{-j\omega_0 t}\right] = 2\pi\delta(\omega + \omega_0)$$
$$F[x(t)] = X(j\omega) = \frac{2\pi}{2}\left[e^{j\phi}\delta(\omega - \omega_0) + e^{-j\phi}\delta(\omega + \omega_0)\right]$$

$$\boxed{X(j\omega) = \pi\left[e^{j\phi}\delta(\omega - \omega_0) + e^{-j\phi}\delta(\omega + \omega_0)\right]}$$

$\boldsymbol{x(t) = \sin(\omega_0 t + \phi)}$

$$\sin(\omega_0 t + \phi) = \frac{1}{2j}\left[e^{+j(\omega_0 t+\phi)} - e^{-j(\omega_0 t+\phi)}\right]$$
$$= \frac{1}{2j}\left[e^{j\phi}e^{j\omega_0 t} - e^{-j\phi}e^{-j\omega_0 t}\right]$$
$$F[x(t)] = X(j\omega) = \frac{2\pi}{2j}\left[e^{j\phi}\delta(\omega - \omega_0) - e^{-j\phi}\delta(\omega + \omega_0)\right]$$

$$\boxed{X(j\omega) = -j\pi\left[e^{j\phi}\delta(\omega - \omega_0) - e^{-j\phi}\delta(\omega + \omega_0)\right]}$$

$\boldsymbol{x(t) = e^{j\omega_0 t}u(t)}$

$$F\left[u(t)\right] = \frac{1}{j\omega} + \pi\delta(\omega)$$

By using frequency shifting property the FT of x (t) is obtained

$$\boxed{F\left[e^{j\omega_0 t}u(t)\right] = \frac{1}{j(\omega-\omega_0)} + \pi\delta(\omega-\omega_0)}$$

$\boldsymbol{x(t) = e^{-at}\cos\omega_0 t u(t)}$

$$\begin{aligned}
\cos\omega_0 t &= \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] \\
X(j\omega) &= \int_0^\infty e^{-at}\cos\omega_0 T e^{-j\omega t}\,dt \\
&= \frac{1}{2}\int_0^\infty e^{-at}e^{j\omega_0 t}e^{-j\omega t}\,dt + \frac{1}{2}\int_0^\infty e^{-at}e^{-j\omega_0 t}e^{-j\omega t}\,dt \\
&= \frac{1}{2}\int_0^\infty e^{-(a-j\omega_0+j\omega)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(a+j\omega_0+j\omega)t}\,dt \\
&= \frac{1}{2}\left[\frac{-1}{(a-j\omega_0+j\omega)}e^{-(a-j\omega_0+j\omega)t} - \frac{e^{-(a+j\omega_0+j\omega)t}}{(a+j\omega_0+j\omega)}\right]_0^\infty \\
&= \frac{1}{2}\left[\frac{1}{(a+j\omega)-j\omega_0} + \frac{1}{(a+j\omega)+j\omega_0}\right] \\
&= \frac{1}{2}\frac{[a+j\omega+j\omega_0+a+j\omega-j\omega_0]}{(a+j\omega)^2+\omega_0^2}
\end{aligned}$$

$$\boxed{X(j\omega) = \frac{(a+j\omega)}{(a+j\omega)^2+\omega_0^2}}$$

$\boldsymbol{x(t) = e^{-2|t|}\cos 5t}$

$$F[e^{-2|t|}] = \frac{4}{\omega^2+4}$$

$$F[x(t)\cos\omega_0 t] = \frac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)]$$

In the given problem $\omega_0 = 5$

$$\boxed{X(j\omega) = \frac{2}{[(\omega-5)^2+4]} + \frac{2}{[(\omega+5)^2+4]}}$$

$x(t) = e^{-3|t|} \sin 2t$

$$F\left[e^{-3|t|}\right] = \frac{6}{(9+\omega^2)}$$

$$x(t) \sin \omega_0 t \xleftrightarrow{\text{FT}} \frac{1}{2j}[X(\omega-\omega_0) - X(\omega+\omega_0)]$$

$$F\left[e^{-3|t|} \sin 2t\right] \xleftrightarrow{\text{FT}} \frac{1}{2j}\left[\frac{6}{[9+(\omega-2)^2]} - \frac{1}{[9+(\omega+2)^2]}\right]$$

$$\boxed{x(j\omega) = \frac{-j24}{[9+(\omega-2)^2][9+(\omega+2)^2]}}$$

$x(t) = u(t+2) - u(t-2)$

$$F[u(t+2)] = \frac{1}{j\omega}e^{j2\omega}$$

$$F[u(t-2)] = \frac{1}{j\omega}e^{-j2\omega}$$

$$F[u(t+2) - u(t-2)] = \frac{1}{j\omega}\left[e^{j2\omega} - e^{-j2\omega}\right]$$

$$= \frac{2}{\omega} \sin 2\omega$$

$$\boxed{X(j\omega) = 4\text{sinc } 2\omega}$$

$x(t) = e^{-3t}u(t-1)$

Method 1

$$F\left[e^{-3t}u(t)\right] = \frac{1}{(3+j\omega)}$$

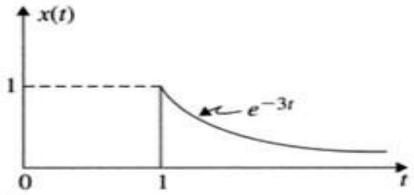
Using time shifting property, we get

$$F\left[e^{-3(t-1)}u(t-1)\right] = \frac{e^{-j\omega}}{(3+j\omega)}$$

$$e^3 F\left[e^{-3t}u(t-1)\right] = \frac{e^{-j\omega}}{(3+j\omega)}$$

$$\boxed{F\left[e^{-3t}u(t-1)\right] = \frac{e^{-(j\omega+3)}}{(3+j\omega)}}$$

Method 2



Representation of $x(t) = e^{-3t}u(t-1)$.

Using FT definition, from Figure , we get

$$F[x(t)] = \int_1^{\infty} e^{-3t}e^{-j\omega t}\,dt$$
$$= \int_1^{\infty} e^{-(3+j\omega)t}\,dt$$
$$= \frac{-1}{(3+j\omega)}\left[e^{-(3+j\omega)t}\right]_1^{\infty}$$

$$\boxed{F[x(t)] = \frac{e^{-(3+j\omega)}}{j\omega+3}}$$

$\boldsymbol{x(t) = te^{-at}u(t)}$

$$F[e^{-at}u(t)] = \frac{1}{(a+j\omega)}$$

Using the FT property of differentiation in frequency we get

$$F\left[te^{-at}u(t)\right] = j\frac{d}{d\omega}\left[\frac{1}{(a+j\omega)}\right]$$
$$= \frac{j(-j)}{(a+j\omega)^2}$$

$$\boxed{X(j\omega) = \frac{1}{(a+j\omega)^2}}$$

$\boldsymbol{x(t) = e^{-a(t-2)}u(t-2)}$

$$x(t) \xleftrightarrow{\text{FT}} X(j\omega)$$
$$x(t-t_0) \xleftrightarrow{\text{FT}} X(j\omega)e^{-j\omega t_0}$$

$$\boxed{F\left[e^{-a(t-2)}u(t-2)\right] = \frac{1}{(a+j\omega)}e^{-j2\omega}}$$

Method 2

Using the definition of FT we get

$$X(j\omega) = \int_2^{\infty} e^{-a(t-2)} e^{-j\omega t}\, dt$$
$$= e^{2a} \int_2^{\infty} e^{-(a+j\omega)t}\, dt$$
$$= \frac{-e^{2a}}{(a+j\omega)} \left[e^{-(a+j\omega)t} \right]_2^{\infty}$$
$$= \frac{+e^{2a}}{(a+j\omega)} e^{-(a+j\omega)2}$$

$$\boxed{X(j\omega) = \frac{e^{-j2\omega}}{(a+j\omega)}}$$

$\boldsymbol{x(t) = e^{-a|t-2|}}$

$$x(t) = \begin{cases} e^{-a(t-2)} & 0 \le t \le \infty \\ e^{a(t+2)} & -\infty \le t < 0 \end{cases}$$

Let $|t-2| = \tau$

$$x(\tau) = e^{-a|\tau|}$$

$$F\left[e^{-a|\tau|}\right] = \frac{2a}{a^2+\omega^2}$$

Using time shifting property,

$$\boxed{F\left[e^{-a|t-2|}\right] = \frac{2a}{a^2+\omega^2} e^{-j2\omega}}$$

$\boldsymbol{x(t) = \frac{d^2}{dt^2} x(t-2)}$

$$F\left[\frac{d^2x(t)}{dt^2}\right] = -\omega^2 X(j\omega)$$

For the time delay t_0,

$$F[x(t-t_0)] = e^{-j\omega t_0} X(j\omega)$$

Here $t_0 = 2$.

$$\boxed{F\left[\frac{d^2}{dt^2} x(t-2)\right] = -\omega^2 e^{-j2\omega} X(j\omega)}$$

$x(t) = x(2-t) + x(-2-t)$

$$x(t) = x_1(t) + x_2(t)$$

where

$$x_1(t) = x(2-t)$$
$$x_2(t) = x(-2-t)$$
$$F[x(-t)] = X(-j\omega)$$

Using time shifting property of FT we get

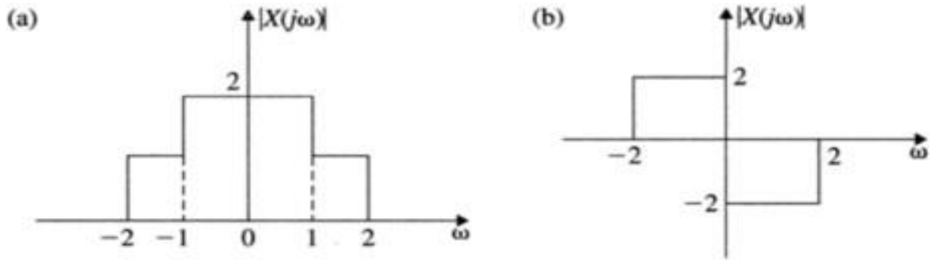
$$X_1(j\omega) = F[x(2-t)] = e^{-j2\omega} X(-j\omega)$$
$$X_2(j\omega) = F[x(-2-t)] = e^{j2\omega} X(-j\omega)$$
$$X(j\omega) = X_1(j\omega) + X_2(j\omega)$$
$$= X(-j\omega)\left[e^{-j2\omega} + e^{j2\omega}\right]$$

$$\boxed{X(j\omega) = 2X(-j\omega)\cos 2\omega}$$

Basic Fourier transform pairs

Signal	Fourier transform
1. $\delta(t)$	1
2. $u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
3. $\delta(t-t_0)$	$e^{-j\omega t_0}$
4. $te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$
5. $u(-t)$	$\pi\delta(\omega) - \frac{1}{j\omega}$
6. $e^{at}u(-t)$	$\frac{1}{(a-j\omega)}$
7. $e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$
8. $\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$
9. $\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$
10. $\frac{1}{(a^2+t^2)}$	$e^{-a\|\omega\|}$
11. $\mathrm{sgn}(t)$	$\frac{2}{j\omega}$
12. 1; for all t	$2\pi\,\delta(\omega)$

4. For the Fourier transforms shown in figure, find the energy of the signals usingParseval's theorem.



Solution:

Using Parseval's theorem, energy is calculated as,

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

(a)

$$E = \frac{1}{2\pi}\left\{\int_{-2}^{1} 1^2\, d\omega + \int_{-1}^{1} 2^2\, d\omega + \int_{1}^{2} 1^2\, d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\omega\right]_{-2}^{-1} + 4\left[\omega\right]_{-1}^{1} + \left[\omega\right]_{1}^{2}\right\}$$

$$= \frac{1}{2\pi}\{-1+2+4+4+2-1\}$$

$$\boxed{E = \frac{5}{\pi}}$$

(b)

$$E = \frac{1}{2\pi}\left\{\int_{-2}^{0} 2^2\, d\omega + \int_{0}^{2} (-2)^2\, d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{4\left[\omega\right]_{-2}^{0} + 4\left[\omega\right]_{0}^{2}\right\}$$

$$\boxed{E = \frac{8}{\pi}}$$

7. Find the Fourier transform of following using convolution theorem.

1. $x(t) = e^{-2t}u(t) * e^{-5t}u(t)$
2. $x(t) = \frac{d}{dt}\left[e^{-2t}u(t) * e^{-5t}u(t)\right]$
3. $x(t) = \left[e^{-2t}u(t) * e^{-5t}u(t-5)\right]$

1. $\boldsymbol{x(t) = e^{-2t}u(t) * e^{-5t}u(t)}$

$$X(j\omega) = F\left[e^{-2t}u(t)\right]F\left[e^{-5t}u(t)\right]$$

$$F\left[e^{-2t}u(t)\right] = \frac{1}{(j\omega+2)}$$

$$F\left[e^{-5t}u(t)\right] = \frac{1}{(j\omega+5)}$$

$$\boxed{X(j\omega) = \frac{1}{(j\omega+2)(j\omega+5)}}$$

$$X(j\omega) = \frac{1}{3}\left[\frac{1}{j\omega+2} - \frac{1}{(j\omega+5)}\right]$$

$$x(t) = F^{-1}[X(j\omega)] = \frac{1}{3}[e^{-2t}u(t) - e^{-5t}u(t)]$$

$$\boxed{x(t) = \frac{1}{3}\left[e^{-2t} - e^{-5t}\right]u(t)}$$

2. $\boldsymbol{x(t) = \frac{d}{dt}\left[e^{-2t}u(t) * e^{-5t}u(t)\right]}$

Let

$$x_1(t) = e^{-2t}u(t) * e^{-5t}u(t)$$

$$X_1(j\omega) = \frac{1}{(j\omega+2)(j\omega+5)}$$

Using time differentiation property of FT, we get

$$x(t) = \frac{d\,x_1(t)}{dt}$$

$$X(j\omega) = j\omega X_1(j\omega)$$

$$\boxed{X(j\omega) = \frac{j\omega}{(j\omega+2)(j\omega+5)}}$$

Putting into partial fraction we get

$$X(j\omega) = \frac{A_1}{j\omega + 2} + \frac{A_2}{j\omega + 5}$$

$$j\omega = A_1(j\omega + 5) + A_2(j\omega + 2)$$

Let $j\omega = -2$;

$$A_1 = -\frac{2}{3}$$

Let $j\omega = -5$;

$$A_2 = \frac{5}{3}$$

$$X(j\omega) = \frac{1}{3}\left[-\frac{2}{j\omega + 2} + \frac{5}{j\omega + 5}\right]$$

$$x(t) = F^{-1}[X(j\omega)] = \frac{1}{3}\left[-2e^{-2t} + 5e^{-5t}\right]u(t)$$

$$\boxed{x(t) = \frac{1}{3}\left[-2e^{-2t} + 5e^{-5t}\right]u(t)}$$

3. $\boldsymbol{x(t) = e^{-2t}u(t) * e^{-5t}u(t-5)}$

$$x(t) = x_1(t) * x_2(t)$$

$$X(j\omega) = X_1(j\omega)X_2(j\omega)$$

$$X_1(j\omega) = \frac{1}{(j\omega + 2)}$$

$$x_2(t) = e^{-5t}u(t-5)$$

$$= e^{-25}e^{-5(t-5)}u(t-5)$$

$$X_2(j\omega) = e^{-25}\frac{1}{(j\omega + 5)}$$

$$\boxed{X(j\omega) = e^{-25}\left[\frac{1}{(j\omega + 2)(j\omega + 5)}\right]}$$

$$X(j\omega) = \frac{1}{3}e^{-25}\left[\frac{1}{j\omega + 2} - \frac{1}{j\omega + 5}\right]$$

$$\boxed{x(t) = \frac{e^{-25}}{3}\left[e^{-2t} - e^{-5t}\right]u(t)}$$

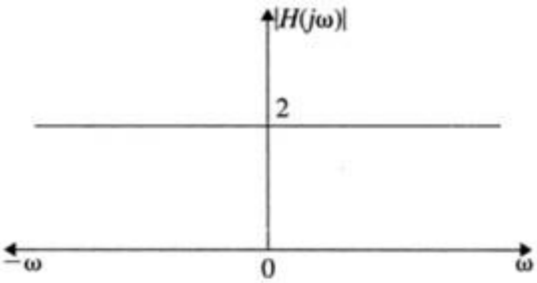
8. Find the magnitude spectrum for H(jw) and plot it. Where

$$H(j\omega) = \frac{(1 + 2e^{-j\omega})}{(1 + \frac{1}{2}e^{-j\omega})}.$$

Solution:

$$\begin{aligned}
H(j\omega) &= \frac{(1 + 2e^{-j\omega})}{(1 + \frac{1}{2}e^{-j\omega})} \\
&= \frac{(1 + 2\cos\omega) - j2\sin\omega}{(1 + \frac{1}{2}\cos\omega) - \frac{j}{2}\sin\omega} \\
|H(j\omega)| &= \frac{\sqrt{(1 + 2\cos\omega)^2 + 4\sin^2\omega}}{\sqrt{(1 + \frac{1}{2}\cos\omega)^2 + \frac{1}{4}\sin^2\omega}} \\
&= \frac{\sqrt{1 + 4\cos^2\omega + 4\cos\omega + 4\sin^2\omega}}{\sqrt{1 + \frac{1}{4}\cos^2\omega + \cos\omega + \frac{1}{4}\sin^2\omega}} \\
&= \frac{\sqrt{5 + 4\cos\omega}}{\sqrt{\frac{5}{4} + \cos\omega}} = 2
\end{aligned}$$

$$\boxed{|H(j\omega)| = 2}$$



Magnitude spectrum of $H(j\omega)$.

Using the properties of continuous time Fourier transform determine the time domain signalx (t), if the frequency domain signal

$$X(j\omega) = j\frac{d}{d\omega}\left[\frac{e^{j2\omega}}{(1 + \frac{j\omega}{3})}\right].$$

Solution: From inspection of X(jw), the given problem can be solved using differentiation in frequency, time shifting and scaling in the proper order.

First the time scaling property is applied. Let

$$X_1(j\omega) = \frac{1}{1+j\omega}$$
$$x_1(t) = e^{-t}u(t)$$
$$F\,[x_1[3t]] = 3e^{-3t}u(3t)$$
$$F\left[3e^{-3t}u(3t)\right] = \frac{1}{\left[1+\frac{j\omega}{3}\right]}$$
$$F^{-1}\left[\frac{1}{\left(1+\frac{j\omega}{3}\right)}\right] = 3e^{-3t}u(t) \quad [\because u(t) = u(3t)]$$

According to time shifting property

$$e^{j2\omega}Y(j\omega) = y(t+2)$$
$$F^{-1}\left[\frac{e^{j2\omega}}{\left(1+\frac{j\omega}{3}\right)}\right] = 3e^{-3(t+2)}u(t+2)$$

According to differentiating property,

$$j\frac{d}{d\omega}X(j\omega) = tx(t).$$

Applying the above property we have

$$F^{-1}\left[j\frac{d}{d\omega}\frac{e^{j2\omega}}{\left(1+\frac{j\omega}{3}\right)}\right] = 3te^{-3(t+2)}u(t+2)$$
$$\therefore \quad X(j\omega) = \frac{d}{j\omega}\left[\frac{e^{j2\omega}}{\left(1+\frac{j\omega}{3}\right)}\right]$$

$$\boxed{x(t) = 3te^{-3(t+2)}u(t+2)}$$

9. Find the inverse Fourier transform of following:

1. $X(j\omega) = \delta(\omega - \omega_0)$
   The IFT of $\delta(\omega) = \frac{1}{2\pi}\delta(\omega)$ is frequency shifted by ω_0.

$$F^{-1}[X(j\omega)] = e^{j\omega_0 t}\frac{1}{2\pi}$$

$$\boxed{F^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}e^{j\omega_0 t}}$$

The above result can also got from first principle of inverse Fourier transform

$$F^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega - \omega_0)e^{j\omega t}d\omega$$

Using the sampling property of the impulse function which exists only at $w = w_0$ we get

$$\boxed{F^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}e^{j\omega_0 t}}$$

2. $X(j\omega) = \frac{j\omega}{(2 + j\omega)^2}$

$$F\left[e^{-2t}\right] = \frac{1}{(2 + j\omega)}$$

By applying

$$F\left[te^{-2t}\right] = \frac{d}{d\omega}\frac{1}{(2 + j\omega)}.$$

$$F\left[te^{-2t}\right] = \frac{1}{(2 + j\omega)^2}$$

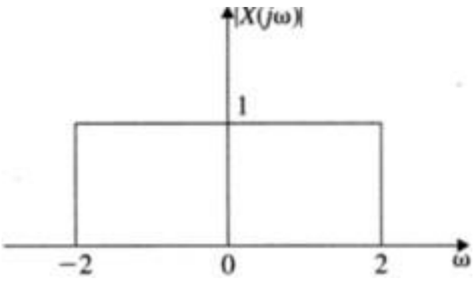
$$\therefore \quad F^{-1}\left[\frac{1}{(2 + j\omega)^2}\right] = te^{-2t}.$$

By applying time differentiation property,

$$\frac{dx(t)}{dt} = j\omega X(j\omega)$$

$$\boxed{F^{-1}\left[\frac{j\omega}{(2 + j\omega^2)}\right] = \frac{d}{dt}\left(te^{-2t}\right)}$$

3. $X(j\omega) = \begin{cases} 1 & |\omega| < 2 \\ 0 & \text{otherwise} \end{cases}$



Using the definition of inverse FT, we get

$$x(t) = \frac{1}{2\pi}\int_{-2}^{2} X(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi jt}\left[1e^{j\omega t}\right]_{2}^{2}$$

$$= \frac{1}{2\pi jt}\left[e^{j2t} - e^{-j2t}\right]$$

$$= \frac{1}{\pi t}\sin 2t$$

$$\boxed{x(t) = \frac{2}{\pi}\text{sinc}2t}$$

4. $X(j\omega) = \frac{6}{(\omega^2+9)}$

$$X(j\omega) = \frac{-6}{(j\omega+3)(j\omega-3)}$$

$$= \frac{A_1}{j\omega+3} + \frac{A_2}{j\omega-3}$$

$$-6 = A_1(j\omega-3) + A_2(j\omega+3).$$

Let $j\omega = -3$

$$A_1 = 1$$

Let $j\omega = 3$

$$A_2 = -1$$

$$X(j\omega) = \frac{1}{j\omega + 3} - \frac{1}{j\omega - 3}$$

$$x(t) = F^{-1}[X(j\omega)] = e^{-3t}u(t) + e^{3t}u(-t)$$

$$\boxed{x(t) = e^{-3t}u(t) + e^{3t}u(-t)}$$

5. $\boldsymbol{X(j\omega) = \frac{(j\omega+2)}{[(j\omega)^2+4j\omega+3]}}$

$$X(j\omega) = \frac{(j\omega + 2)}{(j\omega + 1)(j\omega + 3)}$$

$$= \frac{A_1}{(j\omega + 1)} + \frac{A_2}{(j\omega + 3)}$$

$$(j\omega + 2) = A_1(j\omega + 3) + A_2(j\omega + 1)$$

Let $j\omega = -1$,

$$1 = 2A_1$$

$$A_1 = \frac{1}{2}$$

Let $j\omega = -3$, $A_2 = \frac{1}{2}$

$$X(j\omega) = \frac{1}{2}\left[\frac{1}{j\omega + 1} + \frac{1}{j\omega + 3}\right]$$

$$\boxed{x(t) = \frac{1}{2}\left[e^{-t} + e^{-3t}\right]u(t)}$$

6. $\boldsymbol{X(j\omega) = \frac{(j\omega+1)}{(j\omega+2)^2(j\omega+3)}}$

$$X(j\omega) = \frac{A_1}{(j\omega + 2)^2} + \frac{A_2}{(j\omega + 2)} + \frac{A_3}{(j\omega + 3)}$$

$$(j\omega + 1) = A_1(j\omega + 3) + A_2(j\omega + 2)(j\omega + 3) + A_3(j\omega + 2)^2$$

Let $j\omega = -2$;

$$-1 = A_1$$

Let $j\omega = -3$;

$$-2 = A_3$$

$$(j\omega + 1) = A_1(j\omega + 3) + A_2\left[(j\omega)^2 + 5j\omega + 6\right] + A_3\left[(j\omega)^2 + 4j\omega + 4\right]$$

Compare the coefficients of jw on both sides,

$$
\begin{aligned}
1 &= A_1 + 5A_2 + 4A_3 \\
&= -1 + 5A_2 - 8 \\
A_2 &= 2 \\
X(j\omega) &= \frac{-1}{(j\omega+2)^2} + \frac{2}{(j\omega+2)} - \frac{2}{(j\omega+3)} \\
x(t) &= F^{-1}[x(j\omega)]
\end{aligned}
$$

$$\boxed{x(t) = \left[-te^{-2t} + 2e^{-2t} - 2e^{-3t}\right]u(t)}$$

SAMPLING THEOREM FOR LOW-PASS SIGNALS:

- A low pass signal contains frequencies from 1 Hz to some higher value.
- Statement of the sampling theorem

1) A band limited signal of finite energy , which has no frequency components higher than W hertz , is completely described by specifying the values of the signal at instants of time separated by1/2W seconds and
2) A band limited signal of finite energy, which has no frequency components higher than W hertz , may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second.

The first part of above statement tells about sampling of the signal and second part tells about reconstruction of the signal. Above statement can be combined and stated alternately as follows :

A continuous time signal can be completely represented ints samples and recovered back ifthe sampling frequency is twice of the highest frequency content ofthe signal i.e.,

$f_s \geq 2W$ Here fs is the sampling frequencyAnd W is the higher frequency content

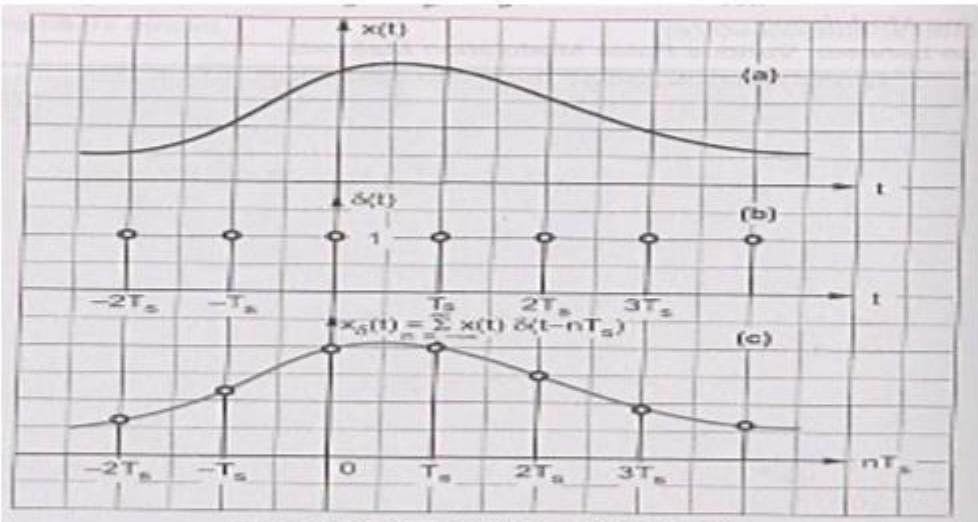


Fig. 4.1.1 CT and its DT signal

Proof of sampling theorem

There are two parts :

I)Representation of x(t) in terms of its samplesII)Reconstruction of x(t) from its samples

PART I: Representation of x(t) in its samples x(nTs)Step 1 : Define xδ(t)

Step 2 : Fourier transform of xδ(t) i.e. Xδ(f)Step 3: Relation between X(f) and Xδ(f) Step 4 : Relation between x(t) and x(nTs) Step 1 : Define xδ(t)

The sampled signal xδ(t) is given as ,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s))\delta(t - nT_s) \qquad \ldots\ldots(1)$$

$x(nT_s)$ is basically $x(t)$ sampled at t= nT_s , n=0,$\pm 1, \pm 2, \pm 3, \ldots$

Here, observe that xδ(t) is the product of x(t) and impulse train δ(t) as shown in figure.

In the above equation δ(t-nTs) indicates the samples placed at ±Ts,±2Ts,±3Ts…and so on

Step 2 : Fourier transform of xδ(t)i.e.

Xδ(f)Taking FT of equation (1)

$X_\delta(f) = FT\{ \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT_s) \}$ ....(2)

=FT {Product of $x(t)$ and impulse train}

We know that FT of product in time domain becomes convolution in frequency domain i.e.,

We know that FT of product in time domain becomes convolution in frequency domain i.e.,

$X_\delta(f) = FT\{x(t)\} * FT\{\delta(t-nT_s)\}$

By definitions $x(t) \leftrightarrow X(f)$ and

$$\delta(t-nT_s) \leftrightarrow f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Hence equation (2) becomes,

$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Since convolution is linear,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f-nfs)$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \text{ .... By shifting property of impulse function.}$$

$$= \ldots f_s X(f-2f_s) + f_s X(f-f_s) + f_s X(f) + f_s X(f-f_s) + f_s X(f-2f_s) + \ldots$$

Conclusions:

i)The RHS of above equation shows that X(f) is placed at±fs,±2fs,±3fs,..ii)This means X(f) is periodic in fs.

iii) If sampling frequency is fs=2W , then the spectrums X(f) just touch each other .

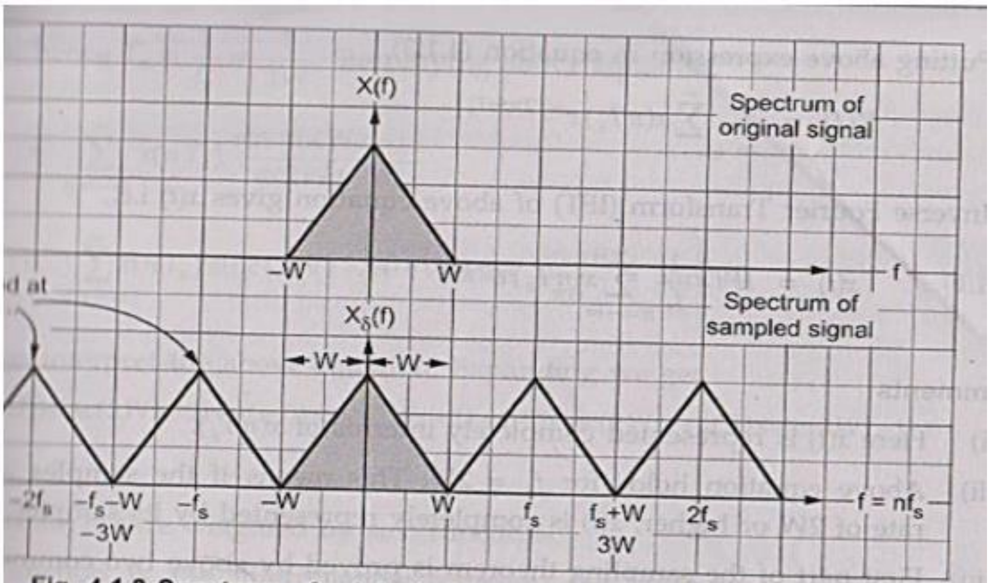


Fig. 4.1.2 Spectrum of original signal and sampled signal

Step 3:Relation between X(f) and Xδ **(f)**

Important assumption

Let us assume that fs=2W ,then as per above diagram .

$X_\delta(f) = f_s X(f) for - W <= f <= W and f_s = 2W$ ......(3)

Or

$$X(f) = \frac{1}{f_s} X_\delta(f)$$

Step 4 : Relation between $x(t) and x(nT_s)$

DTFT is $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$

$\therefore X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n}$ ....(4)

In above equation $'f'$ is the frequency of DT signal .If we replace $X(f)$ by $X_\delta(f)$, then 'f'' becomes frequency of CT signal .i.e.,

$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \frac{f}{f_s} n}$

In above equation 'f' is frequency of CT signal and fs = Frequency of DT signal in equation (4) .Since x(n)=x(nTs),i.e. samples of x(t), then we have,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s}$$

Since 1fs=Ts1fs=Ts

Putting above expression in equation (3),

$$\therefore X(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s}$$

Inverse Fourier Transform (IFT) of above equation gives $x(t) i.e.,$

$x(t) = IFT\left\{\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s}\right\}$..........(5)

Conclusions:

1) Here x(t) is represented completely in terms of x(nTs).

2) Above equation holds for fs=2W.This means if the samples are taken at the rateof 2W or higher, x(t) is completely represented by its samples.

3) First part of the sampling theorem is proved by above two conclusions.

II) Reconstruction of x(t)from its samples

Step 1 : Take inverse Fourier transform of X(f) which is in terms of $X_\delta(f)$

Step 2 : Show that x(t) is obtained back with the help of interpolation

function.Step 1 :Take inverse Fourier transform of equation (5) becomes ,

$$x(t) = \int_{-\infty}^{\infty} \{\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s}\} e^{j2\pi f t} df$$

Here the integration can be taken from -W≤ $f \leq W$. Since $\therefore X(f) = \frac{1}{f_s} X_\delta(f)$ for W≤ $f \leq W$

Refer fig.4.1.2.

$$\therefore x(t) = \int_{-W}^{W} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} e^{j2\pi f t} df$$

Interchanging the order of summation and integration,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{1}{f_s} \int_{-W}^{W} e^{j2\pi f(t-n\, T_s)} df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{1}{f_s} \left[\frac{e^{j2\pi f(t-nT_s)}}{j2\pi((t-nT_s)}\right] from - WTOW$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{1}{f_s} \{\frac{e^{j2\pi W(t-nT_s)} - e^{-j2\pi W(t-nT_s)}}{j2\pi(t-nT_s)}\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{1}{f_s} \frac{sin 2\pi W(t-nT_s)}{\pi(t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\pi(2Wt-2WnT_s)}{\pi(f_s t - f_s n T_s)}$$

Here f_s =2W, hence$T_s = \frac{1}{f_s} = \frac{1}{2W}$

Simplifying above equation,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\pi(2Wt-n)}{\pi(2Wt-n)}$$

$=\sum_{n=-\infty}^{\infty} x(nT_s)\, sinc(2Wt-n)$

Since $\frac{\sin\pi\theta}{\pi\theta} = sinc\theta$ ....(6)

Step 2:

Let us interpret the above equation .Expanding we get ,

$$x(t) = ..+x(-2T_s)sinc(2Wt+2) + x(-T_s)sinc(2Wt+1) + x(0)sinc(2Wt) + x(T_s)sinc(2Wt-1) + \cdots$$

Conclusions:

The samples x(nTs) are weighted by sinc functions.

The sinc function is the interpolating function .fig 4.1.3 shows, how x(t)is interpolated.

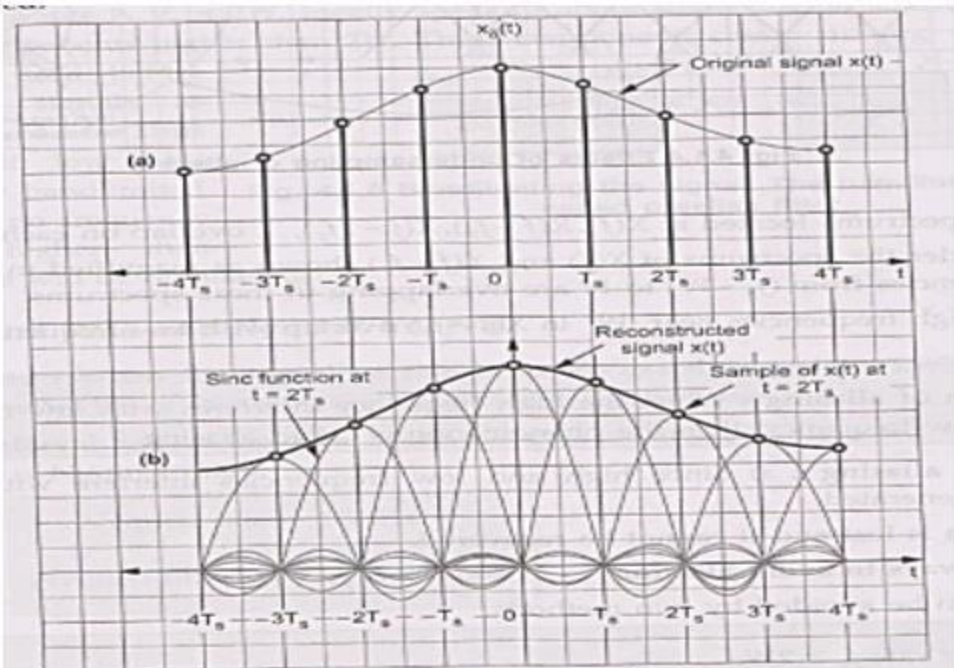


Fig. 4.1.3 (a) Sampled version of signal $x(t)$
(b) Reconstruction of $x(t)$ from its samples

Step 3:

Reconstruction of x(t) by low pass filter

When the interpolated signal of equation (6) is passed through the low pass filter of bandwidth $-W \leq f \leq W$, then the reconstructed waveform shown in fig.4.1.3(b) is obtained. The individual sinc functions are interpolated to get smooth x(t).

When high frequency interferes with low frequency and appears as low frequency , then the phenomenon is called aliasing.

Effects of aliasing:

i)since high and low frequencies interfere with each other , distortion is

generated.ii)The data is lost and it cannot be recovered.

Different ways to avoid aliasing :

Aliasing can be avoided by two

methodsi)sampling rate $f_s \geq 2W$

ii) Strictly band limit the signal to 'W'

PART-A QUESTIONS

1. Write Dirichlet's conditions.
   - The function x (t) should be single valued in any finite time interval T.
   - The function x (t) should have at the most finite number of discontinuities in any finitetime interval T.
   - The function x (t) should have finite number of maxima and minima in any finite timeinterval T.
   - The function x(t) should be absolutely integrable i.e.

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$$

2. Define Sampling Theorem.
   The sampling theorem specifies the minimum-sampling rate at which a continuous-time signal needs to be uniformly sampled so that the original signal can be completely recovered or reconstructed by these samples alone.

3. What is an aliasing Effect?

Aliasing occurs when you sample a signal (anything which repeats a cycle over time) too slowly (at a frequency comparable to or smaller than the signal being measured), and obtain an incorrect frequency and/or amplitude as a result.

4. What is the condition for existence of Fourier transform?

- Signal must be absolutely integrable over a period.
- Signal must be of bounded variation in any given bounded interval.
- Signal must have a finite number of discontinuities in any given bounded interval, and the discontinuities cannot be infinite

PART-B QUESTIONS

(1) Find IDFT X (e ^{jw}) =(1+coswt-2sin2wt)

(2) State and prove time convolution and time differentiation properties of Fourier

transform.(3)Obtain the Fourier transform of the following functions: (a)Unit step function

(b) DC signal

(4) Find the fourier transform of symmetrical triangular pulse and sketch the spectrum usingproperties.

(5)State and prove following properties of fourier transform:(1)Time shifting, (2)Time scaling.

(6) Show that the Fourier transform of a $Sin\ c$ functions; $y(t) = A\ Sin\ c\ (2\ wt)$ is a gatefunction. Also sketch both the functions.

(7)Obtain the Fourier transform of the following functions:

(8) Explain the following functions:
(i) Impulse function $\delta(t)$. (ii) DC signal with a value 'A'. (iii) Unit step function, u(t).

(9) State and prove time convolution property of Fourier transform.

(10) Find the correlation of symmetrical gate pulse with amplitude and time duration 'l' with itself. Evaluate $u(t) * e_{-2t}\, u(t)$

(11) Find Fourier transform of sin $\omega_0 t$.

(12)State and prove following properties of Fourier transform: (i) Time shifting. (ii) Scaling.

(13) Determine the Fourier transform of a two sided exponential pulse *x (t) = e*$^{-||}$.

UNIT III
Laplace Transform

Introduction

We know that Fourier transform exists if the signals have finite energy. But for the signals such as ramp, rising exponents etc. This condition of finite energy is not satisfied. ThusFT does not exist for such signals. By the use of Laplace transformation this limitation can be avoided.WKT in FT the variable s = jw.

But in Laplace transform variable s can be expressed as s = σ+jώ

σ - real part of which represents the attenuation factor.

Jώ- imaginary part, ώ- angular frequency.

Laplace transform exists for almost all signals of practical interest. Some of the advantages ofLaplace transform are as follows: Laplace transform can be used for the analysis of unstable systems.

There are two types of Laplace transform:

1) Bilateral (or) two sided Laplace transform
2) Unilateral (or) one sided Laplace transform

1) Bilateral Laplace Transform

The Laplace transform can be alternatively defined as the bilateral Laplace transform ortwo-sided Laplace transform by extending the limits of integration to be the entire real axis.

The bilateral Laplace transform is defined as follows:

$$F(s) = \mathcal{L}\left\{f(t)\right\}(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt.$$

The f(t) and F(s) are Laplace transform pairs. It is written as,

$$f(t) \leftrightarrow F(s)$$

Where‘s’ is complex frequency, it is given as

$$s = \sigma + j\omega$$

Here σ is the attenuation constant or damping factor and w is the angular frequency. With abovevalue of 's' we can write F(s) equation as,

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-\sigma t} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[f(t)\, e^{-\sigma t}\right] e^{-j\omega t}\, dt$$

The above equation shows that F(s) is basically a Fourier transform of f(t) $e^{-\sigma t}$.This is the relationship between Fourier transform and Laplace transform. The Fourier transform given by the above equation must exist, which is actually Laplace transform. Hence sufficient condition off(t) to be Laplace transformable is that

$$\int_0^{\infty} |f(t)|\, e^{-\sigma t}\, dt < \infty$$

For real and positive values of σ.

2) Unilateral Laplace Transform

The unilateral Laplace transform is the special case of Laplace transform and is defined as,

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt.$$

The unilateral Laplace transform has the following features:

1. The unilateral Laplace transform simplifies the system analysis considerably.
2. The signals are restricted to casual signals.
3. There is one to one correspondence between LT and ILT.
4. In view of above advantages Laplace transform means unilateral Laplace transform.

3) Inverse Laplace Transform

The inverse Laplace transform is given as

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)\, e^{st}\, ds$$

The above formula of inverse Laplace transform involves a complex integration. In this chapter we will use a partial fraction expansion method to evaluate inverse Laplace transforms.

Region of convergence

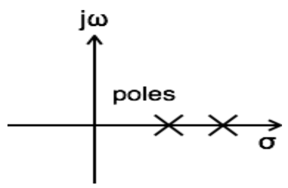
The range variation of σ for which the Laplace transform converges is called region of convergence.

Properties of ROC

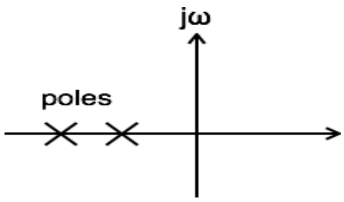
- ROC contains strip lines parallel to jω axis in s-plane.
- If x (t) is absolutely integral and it is of finite duration, then ROC is entire s-plane.
- If x (t) is a right sided sequence then ROC: $Re\{s\} > \sigma_o$.
- If x (t) is a left sided sequence then ROC: $Re\{s\} < \sigma_o$.
- If x (t) is a two sided sequence then ROC is the combination of two regions.

2.4.3 Causality and Stability

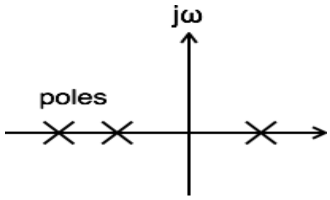
1. For a system to be causal, all poles of its transfer function must be right half of s-plane.



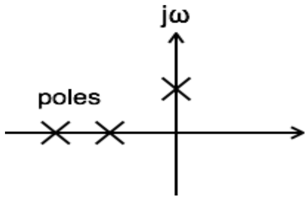
2. A system is said to be stable when all poles of its transfer function lay on the left half of s-plane.



3. A system is said to be unstable when at least one pole of its transfer function is shifted to theright half of s-plane.



4. A system is said to be marginally stable when at least one pole of its transfer function lies onthe jω axis of s-plane.



Properties of Laplace transform

1. Linearity

Let $x_1(t)$ be the two Laplace transform pairs. Then linearity property states that,

$$\mathcal{L}\ [a_1\, f_1\,(t) + a_2\, f_2\,(t)] = a_1\ F_1\,(s) + a_2\ F_2\,(s)$$

Here a_1 and a_2 are constants.

Proof: let us find the Laplace transform of $a_1 f_1(t) + a_2 f_2(t)$ by applying definition.ie

$$x_1(t) \xleftrightarrow{L} X_1(s)$$
$$x_2(t) \xleftrightarrow{L} X_2(s)$$
$$[a_1 x_1(t) + a_2 x_2(t)] \xleftrightarrow{L} [a_1 X_1(s) + a_2 X_2(s)]$$

2. Time scaling property

It states that

$$x(at) \xleftrightarrow{L} \frac{1}{a} X\left(\frac{s}{a}\right)$$

3. Scaling in s domain

It states that

$$\boxed{\frac{1}{a} x\left(\frac{t}{a}\right) \xleftrightarrow{L} X(as)}$$

Proof: According to time scaling property**,**

$$x(at) \xleftrightarrow{L} \frac{1}{a} X\left(\frac{s}{a}\right)$$

Let

$$b = \frac{1}{a}$$

$$x\left(\frac{t}{b}\right) \xleftrightarrow{L} bX(bs)$$

Replacing b by a we get

$$\boxed{\frac{1}{a} x\left(\frac{t}{a}\right) \xleftrightarrow{L} X(as)}$$

3. Time differentiation

It states that

$$x(t) \xleftrightarrow{L} X(s)$$

$$\frac{dx(t)}{dt} \xleftrightarrow{L} sX(s) - x(0^-)$$

$$\frac{d^2x(t)}{dt^2} \xleftrightarrow{L} s^2X(s) - sx(0^-) - \frac{d}{dt}x(0^-)$$

Proof: According to the definition of Laplace transform

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

The above integral is evaluated by parts using

$$\int udv = uv - \int vdu$$

Let $u = x(t)$ and $dv = e^{-st}dt$; $du = \frac{d}{dt}x(t)dt$ and $v = -\frac{1}{s}e^{-st}$

$$\int_0^\infty x(t)e^{-st}dt = \left[\frac{-1}{s}x(t)e^{-st}\right]_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\frac{d}{dt}x(t)dt$$

or

$$X(s) = \frac{1}{s}x(0) + \frac{1}{s}\int_0^\infty e^{-st}\frac{d}{dt}x(t)dt.$$

But

$$L\left[\frac{d}{dt}(x(t))\right] = \int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt$$

$$\boxed{\therefore \qquad L\frac{d}{dt}(x(t)) \overset{L}{\longleftrightarrow} sX(s) - x(0^-)}$$

The time differentiation twice is proved as follows:

$$\frac{d^2}{dt^2}(x(t)) = \frac{d}{dt}\left(\frac{d}{dt}(x(t))\right)$$

Using the property

$$\frac{d}{dt}(x(t)) \overset{L}{\longleftrightarrow} sX(s) - x(0^-)$$

We get

$$L\left[\frac{d^2(x(t))}{dt^2}\right] = sL\left[\frac{d}{dt}(x(t))\right] - \frac{d}{dt}(x(0^-))\Big|_{t=0}$$

$$\boxed{\frac{d^2(x(t))}{dt^2} \overset{L}{\longleftrightarrow} s^2X(s) - sx(0^-) - \frac{d}{dt}(x(0^-))}$$

In general

$$\boxed{\begin{gathered}\frac{d^n x(t)}{dt^n} \xleftrightarrow{L} s^n X(s) - s^{n-1}x(0^-) - s^{n-2}x(0^-)\cdots x^{n-1}(0^-) \\ \text{OR} \\ \frac{d^n x(t)}{dt^n} \xleftrightarrow{L} s^n X(s) - \sum_{k=1}^{n} s^n x^{k-1}(0^-)\end{gathered}}$$

5. Time integration

The time integration property states that if

$$x(t) \xleftrightarrow{L} X(s)$$
$$\int_0^t x(\tau)d\tau \xleftrightarrow{L} \frac{X(s)}{s}$$

Proof: We define

$$f(t) = \int_0^t x(\tau)d\tau.$$

Differentiating the above equation we get

$$\frac{df(t)}{dt} = x(t) \quad \text{and} \quad x(0^-) = 0$$

if

$$f(t) \xleftrightarrow{L} F(s)$$
$$X(s) = L\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0^-) = sF(s) \quad \text{if} \quad f(0^-) = 0$$
$$F(s) = \frac{X(s)}{s}$$

$$\boxed{\int_0^t x(\tau)d\tau \xleftrightarrow{L} \frac{X(s)}{s}}$$

6. Time convolution

The time convolution property states that if

$$x_1(t) \xleftrightarrow{L} X_1(s)$$
$$x_2(t) \xleftrightarrow{L} X_2(s)$$
$$x_1(t) * x_2(t) \xleftrightarrow{L} X_1(s)X_2(s)$$

Proof:

$$\begin{aligned} L[x_1(t) * x_2(t)] &= \int_{-\infty}^{\infty} e^{-st} \left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \right] dt \\ &= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} e^{-st} x_2(t-\tau)dt \right] d\tau \end{aligned}$$

The inner integral is the LT of $x_2(t-ς)$ with a time delay ς. substituting

$$\int_{-\infty}^{\infty} e^{-st} x_2(t-\tau)dt = X_2(s)e^{-\tau s}$$

In the above equation, we get

$$\begin{aligned} L[x_1(t) * x_2(t)] &= \int_{-\infty}^{\infty} x_1(\tau)X_2(s)e^{-\tau s} d\tau \\ &= X_2(s) \int_{-\infty}^{\infty} x_1(\tau)e^{-\tau s} d\tau \\ &= X_2(s)X_1(s) \end{aligned}$$

$$\boxed{[x_1(t) * x_2(t)] \xleftrightarrow{L} X_1(s)X_2(s)}$$

7. Complex frequency differentiation

According to this property,

$$-tx(t) \xleftrightarrow{L} \frac{d}{ds}(X(s))$$

Proof: By definition of LT,

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

Differentiating both sided with respect to s,

$$\begin{aligned}\frac{d}{ds}(X(s)) &= \frac{d}{ds}\int_0^\infty x(t)e^{-st}dt \\ &= -\int_0^\infty tx(t)e^{-st}dt \\ &= -L[tx(t)]\end{aligned}$$

$$\therefore \quad \boxed{-tx(t) \xleftrightarrow{L} \frac{d}{ds}(X(s))}$$

8. Frequency shifting

According to this property,

$$[e^{s_0t}x(t)] \xleftrightarrow{L} X(s-s_0)$$

$$L[e^{s_0t}x(t)] = \int_0^\infty e^{s_0t}x(t)e^{-st}dt \qquad \text{where } s_0 \text{ is a constant}$$

$$= \int_0^\infty x(t)e^{-(s-s_0t)}dt = X(s-s_0)$$

$$\boxed{[e^{s_0t}x(t)] \xleftrightarrow{L} X(s-s_0)}$$

9. Conjugation property

According to this property,

$$x(t) \overset{L}{\longleftrightarrow} X(s) \text{ then}$$

$$x^*(t) \overset{L}{\longleftrightarrow} X^*(-s)$$

Proof: By definition of LT

$$\begin{aligned} L[x^*(t)] &= \int_0^\infty x^*(t)e^{-st}dt \\ &= \int_0^\infty [x(t)e^{-(-s)t}dt]^* \\ &= X^*(-s) \end{aligned}$$

$$\boxed{x^*(t) \overset{L}{\longleftrightarrow} X^*(-s)}$$

10. Initial value theorem

According to this theorem,

$$\underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)$$

Proof:

$$L\left[\frac{d}{dt}x(t)\right] = \int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt = sX(s) - x(0)$$

Let $s \to \infty$; then

$$\begin{aligned} \underset{s\to\infty}{Lt}\int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt &= \underset{s\to\infty}{Lt}[sX(s) - x(0)] \\ 0 &= \underset{s\to\infty}{Lt}[sX(s) - x(0)] \end{aligned}$$

Since $x(0) = \underset{t\to 0}{Lt}\, x(t)$

$$\boxed{\underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)}$$

11. Final value theorem

According to this theorem,

$$\underset{t \to \infty}{Lt}\, x(t) = \underset{s \to 0}{Lt}\, sX(s)$$

Proof: The LT of d/dt(x (t)) could be written as

$$\int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt = [sX(s) - x(0)]$$

Taking s-> 0 on both sides of the above equation, we get

$$\int_0^\infty \frac{d}{dt}(x(t))dt = \underset{s \to 0}{Lt}[sX(s) - x(0)]$$

$$\underset{t \to \infty}{Lt}[x(t) - x(0)] = \underset{s \to 0}{Lt}[sX(s) - x(0)]$$

$$\boxed{\underset{t \to \infty}{Lt}\, x(t) = \underset{s \to 0}{Lt}\, sX(s)}$$

The above theorem is valid if X(s) has no poles in RHP of s-plane.

Solved Problems

1. Find the Laplace transform of ramp signal.

The ramp signal is given as,

$$\begin{aligned} r(t) &= t \qquad \text{for } t \geq 0 \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

$$or \qquad r(t) = t\,u(t)$$

By the definition of Laplace transform,

$$\mathcal{L}\,[r(t)] = \int_{0-}^{\infty} t\, e^{-st}\, dt$$

Integrating the above equation by parts we get,

$$\mathcal{L}\,[r(t)] = \left[t \cdot \frac{e^{-st}}{-s}\right]_{0-}^{\infty} - \int_{0-}^{\infty} 1 \cdot \frac{e^{-st}}{-s}\, dt$$

$$= \left[\frac{t\,e^{-st}}{-s}\right]_{0-}^{\infty} - \left[\frac{e^{-st}}{-s^2}\right]_{0-}^{\infty}$$

$$= \frac{1}{s^2}$$

Thus $\boxed{\mathcal{L}\; t\,u(t) = \frac{1}{s^2}}$

2. Find the Laplace transform of delayed ramp signal.
Solution:

If unit ramp function is delayed by time t_0, it is given as,

$$r(t-t_0) = t-t_0 \quad \text{for} \quad t \geq t_0$$
$$= 0 \quad \text{otherwise.}$$

By the shifting property of Laplace transform

$$\mathcal{L}\,[r(t-t_0)] = \frac{e^{-st_0}}{s^2}$$

Similarly

$$\mathcal{L}\,[Kr(t-t_0)] = \frac{Ke^{-st_0}}{s^2}$$

Find out the Laplace transform of impulse function.

We have evaluated the relationship between unit impulse function and step function.

Thedifferentiation of unit step function gives unit impulse function i.e.,

$$\delta(t) = \frac{d}{dt}u(t)$$

Taking Laplace transform on both sides,

$$\mathcal{L}\,[\delta(t)] = \mathcal{L}\left[\frac{d}{dt}u(t)\right]$$

By differentiation property, the RHS of above equation will be,

$$\mathcal{L}\,[\delta(t)] = \mathcal{L}\left[\frac{d}{dt}u(t)\right]$$
$$= s\,F(s) - f(0-)$$

In the above equation

$$f(0-) = u(t)\big|_{t=0}{}^{1} = 0$$

Therefore

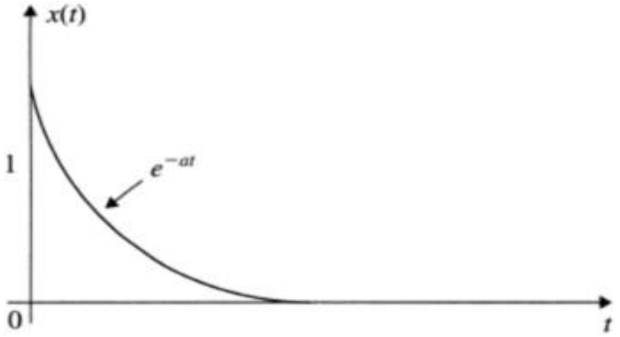
$$\mathcal{L}\,[\delta(t)] = s\cdot\frac{1}{s} - 0 = 1$$

Thus $\boxed{\mathcal{L}\,[\delta(t)] = 1}$

If the impulse function is delayed by t_0, then its Laplace transform can be obtained by shiftingproperty as,

$$\boxed{\mathcal{L}\,[\delta(t-t_0)] = e^{-st_0}}$$

3. Determine the LT of an exponential decay which is shown in figure.



Solution: The exponential decay is represented by

$$x(t) = e^{-at}u(t) \quad t \geq 0.$$

Taking LT for the above function we get

$$L[e^{-at}u(t)] = \int_0^\infty e^{-at}e^{-st}dt$$
$$= \int_0^\infty e^{-(s+a)t}dt$$

$$L[e^{-at}u(t)] = -\frac{1}{(s+a)}\left[e^{-(s+a)t}\right]_0^\infty$$
$$= \frac{1}{(s+a)} \quad \text{with ROC: Re } s > -a$$

$$\boxed{L[e^{-at}u(t)] = \frac{1}{(s+a)}}$$

4. Find out the Laplace transform of sine wave.
Solution:

A sine wave is given as,

$$f(t) = A \sin \omega_0 t$$

We know that sin w_0t can be represented using Euler's identity as,

$$\sin \omega_0 t = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

So f(t) becomes

$$f(t) = \frac{A}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

Taking Laplace transform on both sides,

$$\mathcal{L}[f(t)] = \frac{A}{2j}\left\{\mathcal{L}\left[e^{j\omega_0 t}\right] - \mathcal{L}\left[e^{-j\omega_0 t}\right]\right\}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

$$\therefore \quad \mathcal{L}\left[e^{j\omega_0 t}\right] = \frac{1}{s-j\omega_0}$$

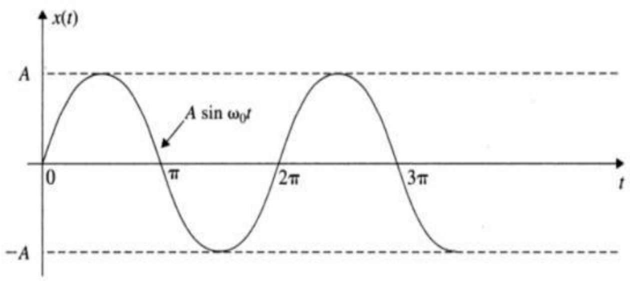
$$\text{and} \quad \mathcal{L}\left[e^{-j\omega_0 t}\right] = \frac{1}{s+j\omega_0}$$

Putting these values in L{f(t)} we get,

$$\begin{aligned}\mathcal{L}[f(t)] &= \frac{A}{2j}\left\{\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right\} \\ &= \frac{A}{2j}\cdot\frac{2j\omega_0}{s^2+\omega_0^2} \\ &= \frac{A\omega_0}{s^2+\omega_0^2}\end{aligned}$$

$$\text{Thus,} \quad \boxed{\mathcal{L}[A\sin\omega_0 t] = \frac{A\omega_0}{s^2+\omega_0^2}}$$

5.Determine the LT of a sine function which is shown in figure



Solution: A sinusoidal function shown in figure is mathematically expressed as follows:

$$x(t) = A \sin \omega_0 t \, u(t) \quad t \geq 0$$

The given sinusoidal function is written as follows using Euler's identity.

$$\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

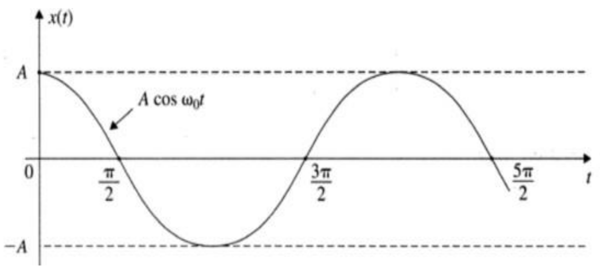
$$L[A \sin \omega_0 t] = \frac{A}{2j}[L(e^{j\omega_0 t}) - Le^{-j\omega_0 t}]$$

$$L[A \sin \omega_0 t] = \frac{A}{2j}\left[\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right]$$

$$= \frac{A}{2j}\frac{2j\omega_0}{(s^2 + \omega_0^2)}$$

$$\boxed{L[A \sin \omega_0 t] = \frac{A\omega_0}{(s^2 + \omega_0^2)}} \text{ ROC: Re } s > 0.$$

6. Determine the LT of a cosine function which is shown in figure



Solution: A cosine function shown in figure is mathematically expressed as follows:

$$x(t) = A \cos \omega_0 t u(t) \quad t \leq 0.$$

The given sinusoidal function is written as follows using Euler's identity

$$A \cos \omega_0 t = \frac{A}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

Taking LT for x (t), the following equation is written

$$L[A \cos \omega_0 t u(t)] = \frac{A}{2}[Le^{j\omega_0 t}u(t) + Le^{-j\omega_0 t}u(t)]$$

$$L[A\cos\omega_0 t u(t)] = \frac{A}{2}\left[\frac{1}{(s+j\omega_0)} + \frac{1}{(s-j\omega_0)}\right]$$
$$= \frac{As}{(s^2+\omega_0^2)}$$
$$\boxed{L[A\cos\omega_0 t u(t)] = \frac{As}{(s^2+\omega_0^2)}} \quad \text{ROC: Re}s > 0.$$

7. Find out the Laplace transform of sine wave.
Solution: A sine wave is given as,

$$f(t) = A\cos\omega_0 t$$

We know that sin w_0t can be represented using Euler's identity as,

$$\cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

So f(t) becomes

$$f(t) = \frac{A}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

Taking Laplace transform on both sides,

$$\mathcal{L}[f(t)] = \frac{A}{2}\mathcal{L}\left\{e^{j\omega_0 t} + e^{-j\omega_0 t}\right\}$$

$$\mathcal{L}\left[e^{j\omega_0 t}\right] = \frac{1}{s-j\omega_0}$$

$$\text{and} \quad \mathcal{L}\left[e^{-j\omega_0 t}\right] = \frac{1}{s+j\omega_0}$$

$$\mathcal{L}[f(t)] = \frac{A}{2}\left\{\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right\}$$
$$= \frac{A}{2}\cdot\frac{2s}{s^2+\omega_0^2}$$
$$= \frac{A\cdot s}{s^2+\omega_0^2}$$

$$\text{Thus,} \qquad \boxed{\mathcal{L}[A\cos\omega_0 t] = \frac{A\cdot s}{s^2+\omega_0^2}}$$

8. Find the Laplace transform of damped sine wave.

$$x(t) = e^{-at} \sin \omega_0 t.$$

We know that,

$$L[\sin \omega_0 t] = \frac{\omega_0}{(s^2 + \omega_0^2)}$$

Using the complex shifting property,

$$L[e^{-at} x(t)] = X(s + a)$$

Applying the above property we get,

$$\boxed{L[e^{-at} \sin \omega_0 t] = \frac{\omega_0}{(s + a)^2 + \omega_0^2}}$$

ROC: Re $s > -a$.

(Or)

$$f(t) = e^{-at} \left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right]$$

$$= \frac{1}{2j} \left\{e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right\}$$

Taking Laplace transform on both sides,

$$\mathcal{L}\, f(t) = \frac{1}{2j}\, \mathcal{L}\left\{e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right\}$$

$$= \frac{1}{2j} \left\{\frac{1}{s + (a - j\omega)} - \frac{1}{s + (a + j\omega)}\right\}$$

$$= \frac{1}{2j} \cdot \frac{2j\omega}{(s + a)^2 + \omega^2}$$

$$= \frac{\omega}{(s + a)^2 + \omega^2}$$

Thus, $$\boxed{\mathcal{L}\left[e^{-at} \sin \omega t\right] = \frac{\omega}{(s + a)^2 + \omega^2}}$$

9. Find the Laplace transform of damped cosine wave.

$$f(t) = e^{-at} \cos \omega t$$

Solution: With the help of Euler's identity,

$$f(t) = e^{-at}\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right]$$

$$= \frac{1}{2}\cdot\left[e^{-(a-j\omega)t} + e^{-(a+j\omega)t}\right]$$

Taking Laplace transform on both sides,

$$\mathcal{L}\, f(t) = \frac{1}{2} L\left\{e^{-(a-j\omega)t} + e^{-(a+j\omega)t}\right\}$$

$$\mathcal{L}\, f(t) = \frac{1}{2}\, \mathcal{L}\left\{e^{-(a-j\omega)t} + e^{-(a+j\omega)t}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s+(a-j\omega)} + \frac{1}{s+(a+j\omega)}\right\}$$

$$= \frac{1}{2}\cdot\frac{2\,(s+a)}{(s+a)^2+\omega^2}$$

$$= \frac{(s+a)}{(s+a)^2+\omega^2}$$

10. By applying the complex differentiation property, determine the LT of

$$x(t) = t \sin \omega_0 t.$$

Solution: We know that

$$L[\sin \omega_0 t] = \frac{\omega_0}{(s^2+\omega_0^2)}$$

According to the complex differentiation property,

$$L[-tx(t)] = \frac{d}{ds}X(s)$$

$$\therefore \quad L[\sin \omega_0 t] = \frac{d}{ds}\frac{\omega_0}{(s^2+\omega_0^2)}$$

$$\boxed{L[t \sin \omega_0 t] = \frac{2\omega_0 s}{(s^2+\omega_0^2)^2}}$$

11. Determine the LT of

$$x(t) = \cos at \sin bt.$$

Solution: The given signal x(t) is written in the following form.

$$x(t) = \frac{1}{2}[\sin(a+b)t - \sin(a-b)t]$$

$$L[\cos at \sin bt] = \frac{1}{2}[L \sin(a+b)t - L \sin(a-b)t]$$

$$\boxed{L[\cos at \sin bt] = \frac{1}{2}\left[\frac{(a+b)}{s^2+(a+b)^2} - \frac{(a-b)}{s^2+(a-b)^2}\right]}$$

12. Find the Laplace transform of x(t) = t^n u(t).

Solution: Using the definition of LT for the given function we get

$$L[x(t)] = \int_0^\infty t^n e^{-st} dt$$

Let

$$u = t^n \quad \text{and} \quad du = nt^{n-1} dt$$

$$dv = \int e^{-st} dt \quad \text{and} \quad v = \frac{e^{-st}}{(-s)}$$

Using the property

$$\int udv = uv - \int vdu$$

we get

$$L[t^n] = \left[t^n \frac{e^{-st}}{(-s)}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{(-s)} nt^{n-1} dt$$

$$= 0 + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt.$$

It can be shown that

$$\int_0^\infty t^{n-1} e^{-st} dt = \frac{(n-1)}{s} \int_0^\infty t^{n-2} e^{-st} dt.$$

Thus $L[t^n]$ is written as

$$L[t^n] = \frac{n}{s} \frac{(n-1)}{s} \frac{(n-2)}{s} \cdots \frac{2}{s} \frac{1}{s}$$

$$= \frac{n(n-1)(n-2)\ldots 2}{s^n} \frac{1}{s}$$

$$= \frac{\angle n}{s^{n+1}}$$

$$\boxed{L[t^n] = \frac{\angle n}{s^{n+1}}} \quad \text{ROC: Re } s > 0.$$

13. Determine the Laplace transform and sketch the ROC in the s- plane.

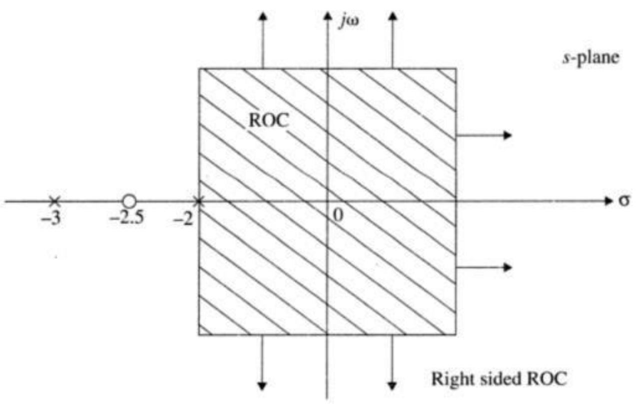
$$x(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

Solution:

1. X (t) is completely a right sided signal and hence the limit of the integration is from t=0 tot=∞.Thus the following equation is written for X(s).

$$\begin{aligned} X(s) &= \int_0^\infty e^{-2t}e^{-st}dt + \int_0^\infty e^{-3t}e^{-st}dt \\ &= \int_0^\infty e^{-(s+2)t}dt + \int_0^\infty e^{-(s+3)t}dt \\ &= \frac{1}{(s+2)} + \frac{1}{(s+3)} \\ &= \frac{(2s+5)}{(s+2)(s+3)} \end{aligned}$$

$$\boxed{X(s) = \frac{2(s+2.5)}{(s+2)(s+3)}}$$



Poles and zeros and ROC of $X(s) = \frac{2(s+2.5)}{(s+2)(s+3)}$.

2. The poles are at s= -2 and s= -3 and a zero is at s= -2.5 and are marked in figure.

3. For the pole 1/(s+2), the ROC is right sided to the vertical line passing through σ = -2.For the pole 1/(s+3), the ROC is also right sided passing through σ = -3.If ROC where σ > -2 is satisfiedthe ROC where σ > -3 is automatically satisfied. Further no pole of X(s) will be inside the ROC.

4. A strip to the right of σ =-2 is created and shaded. The strip is enlarged to ∞ in the direction ofreal and imaginary axis.

5. Thus, the ROC of a casual signal is to the right of the right most pole of X(s).

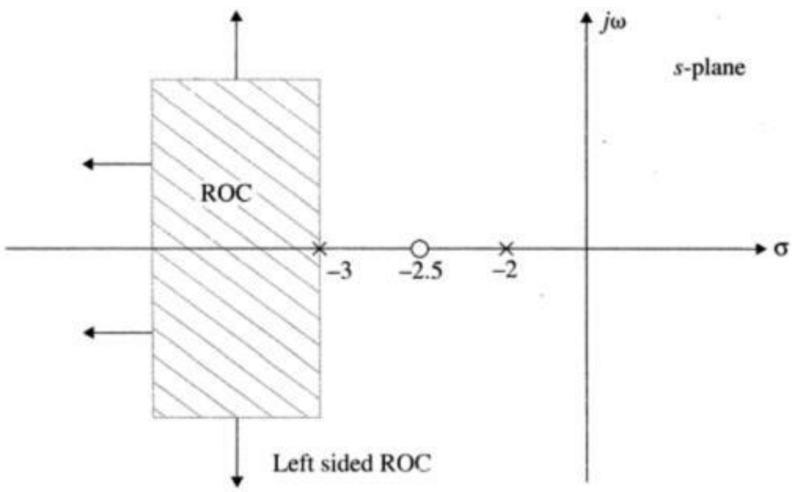
15. Determine the Laplace transform and locate the poles and zeros of X(s) and also theROC in the s- plane.

$$x(t) = e^{-2t}u(-t) + e^{-3t}u(-t)$$

Solution:

The given signal is fully a left sided signal and hence the limit of LT integration is from -∞ to 0.The LT of x (t) is obtained as follows:

$$\begin{aligned} X(s) &= \int_{-\infty}^{0} e^{-2t}e^{-st}dt + \int_{-\infty}^{0} e^{-3t}e^{-st}dt \\ &= \int_{-\infty}^{0} e^{-(s+2)t}dt + \int_{-\infty}^{0} e^{-(s+3)t}dt \\ &= \frac{-1}{(s+2)}\left[e^{-(s+2)t}\right]_{-\infty}^{0} - \frac{1}{(s+3)}\left[e^{-(s+3)t}\right]_{-\infty}^{0} \end{aligned}$$



Poles and zeros and ROC of $X(s) = \dfrac{-2(s+2.5)}{(s+2)(s+3)}$.

16. Determine the Laplace transform and locate the poles and zeros and ROC in thes- Plane for the following signal.

$$x(t) = Au(t)$$

Solution:

1. The given signal is right- sided signal. And hence the limit of LT integration is from 0 to ∞.The LT of x (t) is obtained as follows

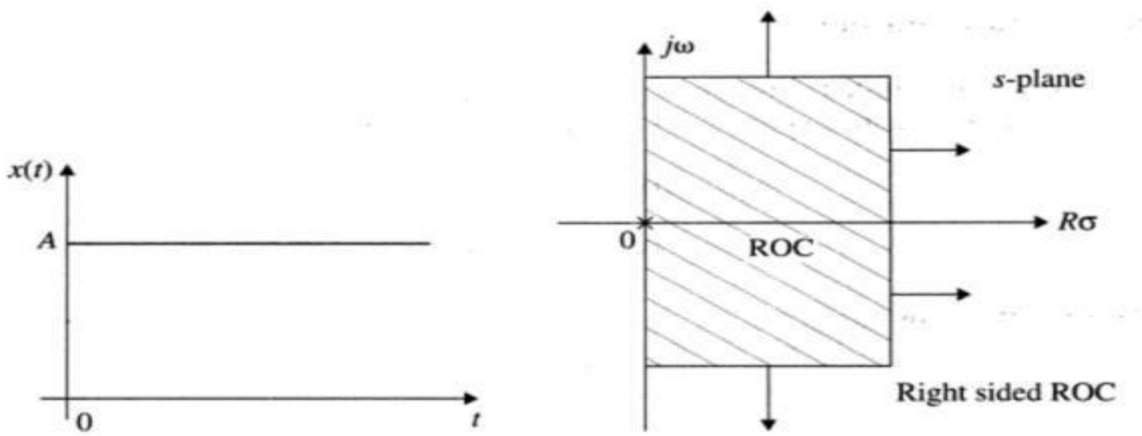
$$X(s) = \int_0^{\infty} Ae^{-st} dt$$

$$= \frac{-A}{s} [e^{-st}]_0^{\infty}$$

$$\boxed{X(s) = \frac{A}{s}} \quad \text{ROC Re } s > 0.$$

2. For the given signal, a pole at the origin exists and it is marked in figure (b).

3. The LT converges only σ >0.Thus the ROC is the entire right half of s-plane.



$x(t) = Au(t)$
(a)

$X(s) = \frac{A}{s}$ and ROC
(b)

Representation of $x(t)$ and ROC.

17. Find the Laplace transform and ROC of

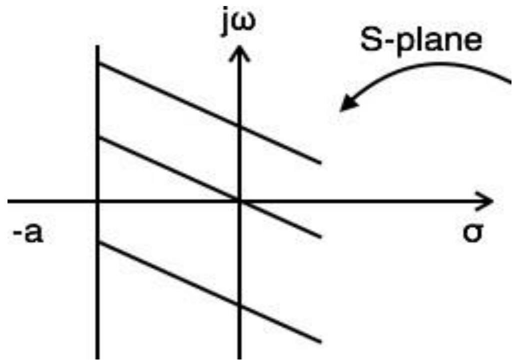
$$x(t) = e^{-at}\, u(t)$$

Solution:

$$L.T[x(t)] = L.T[e^{-at}\, u(t)] = \frac{1}{S+a}$$

$$Re > -a$$

$$ROC : Res >> -a$$



18. Find the Laplace transform and ROC of

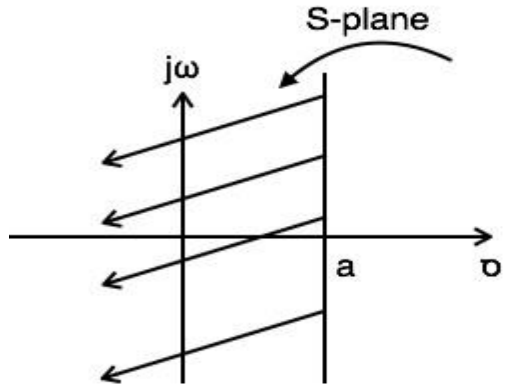
$$x(t) = e^{at} u(-t)$$

Solution:

$$L.T[x(t)] = L.T[e^{at} u(t)] = \frac{1}{S-a}$$

$$Res < a$$

$$ROC : Res < a$$



19. Find the Laplace transform and ROC of

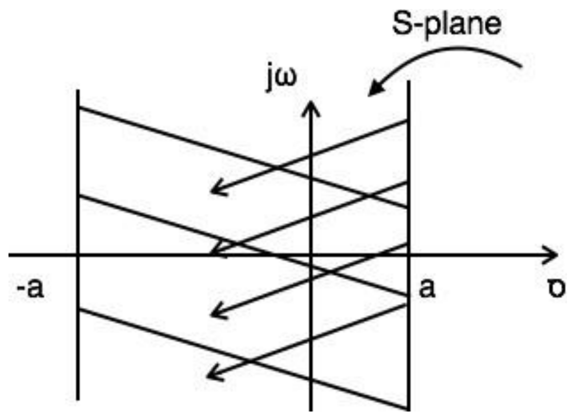
$$x(t) = e^{-at}u(t) + e^{at}u(-t)$$

Solution:

$$L.T[x(t)] = L.T[e^{-at}u(t) + e^{at}u(-t)] = \frac{1}{S+a} + \frac{1}{S-a}$$

For $\frac{1}{S+a} Re\{s\} > -a$

For $\frac{1}{S-a} Re\{s\} < a$



Referring to the above diagram, combination region lies from –a to a. Hence,

$$ROC: -a < Res < a$$

Laplace transform of periodic signal

If a signal x (t) is a periodic signal with period T, then the LT of X(s) is given as

$$X(s) = X_1(s)\left[1 + e^{-Ts} + e^{-2Ts} + \ldots\right] = \frac{X_1(s)}{(1 - e^{-Ts})}$$

Here $x_1(t)$ is the signal which is repeated for every T.

Inverse Laplace transform

The time signal x (t) is the Inverse LT of X(s).This is represented by the following mathematical equation.

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds \quad \text{...... (1)}$$

Use of the above equation to obtain x (t) from X(s) is really a tedious process. The alternative is to express X(s) in polynomial form both in the numerator and the denominator.Both these polynomials are factorized as

$$X(s) = \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \quad \text{......(2)}$$

The points in the s plane at which X(s) = 0 are called zeros. Thus $(s+z_1)$, $(s+z_2)$, $(s+z_3)$.....$(s+z_m)$ are the zeros of X(s). Similarly, the points in the s-plane at which X(s) = ∞ are called poles of X(s).

The zeros are identified by a small circle 0 and the poles by a small cross x in the s- plane. For m<n the degree of the numerator polynomial is less than the degree of the denominator polynomial. Under this condition X(s) in equation (2) is written in the following partial fractionform.

$$X(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3} + \cdots + \frac{A_n}{s+p_n} \quad \text{.... (3)}$$

In equation (3) $A_1, A_2, \dots. A_n$ are called the residues and are determined by any convenientmethod. Once the residues are determined, one can easily obtain x(t) which is the required inverse LT of X(s).

1. Find the inverse LT of

$$X(s) = \frac{10e^{-3s}}{(s-2)(s+2)}.$$

Solution: Consider the function

$$X_1(s) = \frac{10}{(s-2)(s+2)}$$

Putting this into partial fraction we get

$$X_1(s) = \frac{A_1}{(s-2)} + \frac{A_2}{(s+2)}$$
$$= \frac{A_1(s+2) + A_2(s-2)}{(s-2)(s+2)}$$
$$10 = A_1(s+2) + A_2(s-2)$$

Substitute $s = -2$

$$10 = 0 + A_2(-2-2)$$
$$A_2 = -2.5$$

Substitute $s = 2$

$$10 = A_1(2+2) + 0$$
$$A_1 = 2.5$$
$$X_1(s) = 2.5\left[\frac{1}{s-2} - \frac{1}{s+2}\right]$$

Taking inverse LT we get

$$x_1(t) = 2.5[e^{+2t} - e^{-2t}]u(t)$$

According to time shifting property of LT

$$X(s) = X_1(s)e^{-3s}$$

$$\boxed{x(t) = 2.5[e^{2(t-3)} - e^{-2(t-3)}]u(t-3)}$$

2. Find the inverse LT of

$$X(s) = \frac{(s+1) + 3e^{-4s}}{(s+2)(s+3)}.$$

Solution: The given function is written in the following form:

$$X(s) = \frac{(s+1)}{(s+2)(s+3)} + \frac{3e^{-4s}}{(s+2)(s+3)}$$
$$= X_1(s) + X_2(s)$$
$$X_1(s) = \frac{(s+1)}{(s+2)(s+3)}$$
$$= \frac{A_1}{(s+2)} + \frac{A_2}{(s+3)}$$
$$= \frac{A_1(s+3) + A_2(s+2)}{(s+2)(s+3)}$$
$$(s+1) = A_1(s+3) + A_2(s+2)$$

Put $s = -3$

$$(-3+1) = 0 + A_2(-3+2)$$
$$A_2 = 2$$

Put $s = -2$

$$(-2+1) = A_1(-2+3) + 0$$
$$A_1 = -1$$
$$X_1(s) = -\frac{1}{s+2} + \frac{2}{s+3}$$
$$x_1(t) = (-e^{-2t} + 2e^{-3t})u(t).$$

Now consider $X_2(s)$ without delay as $X_3(s)$

$$X_3(s) = \frac{3}{(s+2)(s+3)}$$
$$= \frac{A_1}{(s+2)} + \frac{A_2}{(s+3)}$$
$$3 = A_1(s+3) + A_2(s+2)$$

Put $s = -2$

$$3 = A_1$$

Put $s = -3$

$$3 = A_2(-3+2)$$

$$A_2 = -3$$

$$X_3(s) = 3\left[\frac{1}{s+2} - \frac{1}{s+3}\right]$$

$$X_2(s) = X_3(s)e^{-4s}$$

$$x_3(t) = 3[e^{-2t} - e^{-3t}]u(t)$$

$$x_2(t) = 3[e^{-2(t-4)} - e^{-3(t-4)}]u(t-4)$$

$$x(t) = x_1(t) + x_2(t)$$

$$\boxed{x(t) = \left[-e^{-2t} + 2e^{-3t}\right]u(t) + 3\left[e^{-2(t-4)} - e^{-3(t-4)}\right]u(t-4)}$$

PART-A QUESTIONS.

1. Define ROC.
   The set of signals that cause the system's output to converge lie in the region of convergence (ROC).
2. Write any 2 properties of Laplace Transform.
   (a) Linearity
   $$\mathcal{L}\ [a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$
   (b) Time Scaling
   $$x(at) \overset{L}{\longleftrightarrow} \frac{1}{a}X\left(\frac{s}{a}\right)$$
3. What are the properties of ROC?
   1. ROC is a connected region
   2. ROC does not contain any poles
   3. ROC of a finite duration signal is entire z plane except for origin and/or infinity
4. What is the condition for stability of a system in LT?
   A system is said to be stable when all poles of its transfer function lay on the left half of s-plane.

PART-B QUESTIONS

(1) Determine the LT and associated ROC and pole-zero plot for the following function of time $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$.

(2) Find the signal corresponding to $X(s) = \dfrac{3s^2 + 22S + 27}{(S+1)(S+2)(s^2+2S+5)}$.

(3) Find Laplace tran $e^{j\omega(t)}u(t)$ sketch their ROC of: x(t) = u(t - 5)

(4) A causal LTI system described by the differential equation $\frac{d^2y(t)}{} + \frac{3d(t)}{} + 2y(t) = x(t)$ with input x (t) =2u (t) and with initial conditions $\frac{d}{dt}y(0^-) = 3; y(0^-) = -5\ \&\ x(t) = 2u(t)$ find system y(t)

(5) State and prove Initial and Final value theorems in LT

UNIT IV

Signal Transmission Through Linear Systems

System: A system is defined as set of rules that associates an o/p time function to every i/p time function.

(Or)

A system is an interconnection of elements which produces expected o/p for available i/p.

i/p excitation $f(t)$ — $\boxed{\text{system}}$ — o/p Response $r(t)$

$$f(t) \longrightarrow r(t)$$

→ System is an mathematical operator which maps i/p into o/p

Classification of system

↓

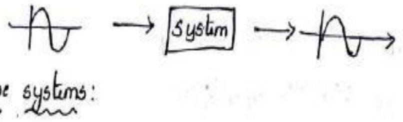
Continuous time system | Discrete time system.

↓

1. Static & Dynamic systems
2. Linear & Non-Linear
3. Time invariant & Time variant
4. Linear TIV & LTIV
5. Stable system
6. casual & non-causal systems.

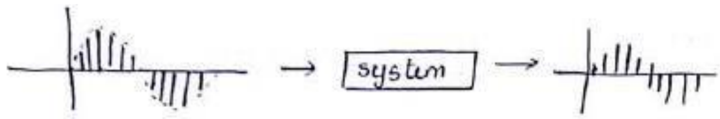
(i) Continuous time systems

→ A continuous time system operates on a continuous time i/p signal to produce a continuous time o/p signal



(ii) Discrete time systems:

A discrete time system operates on a discrete time i/p signal to produce a discrete time o/p signal.



classifications :

(i) static and Dynamic Systems:

→ A static system or system is said to be static if its o/p at any instant depends only on present values of i/p.

Ex: $y(t) = ax(t)$ (ii) $y(t) = a^r x(t)$

at $t=0$ $y(0) = ax(0)$ at $t=0$ $y(0) = a^r x(0)$

at $t=1$ $y(1) = ax(1)$ at $t=1$ $y(1) = a^r x(1)$

→ A system is said to be dynamic if its o/p depends on present & past values of i/p.

Ex: $y(t) = x(t-1) + x(t-2) + x(t)$

$$y(2) = x(2-1) + x(2-2) + x(2) = \underbrace{x(1) + x(0)}_{\text{past}} + \underbrace{x(2)}_{\text{present}}$$

(ii) Linear and Non Linear Systems :

→ A system is said to be linear if its satisfies the superposition principle.

→ It states that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of o/p's of the system to each of the individual i/p signal.

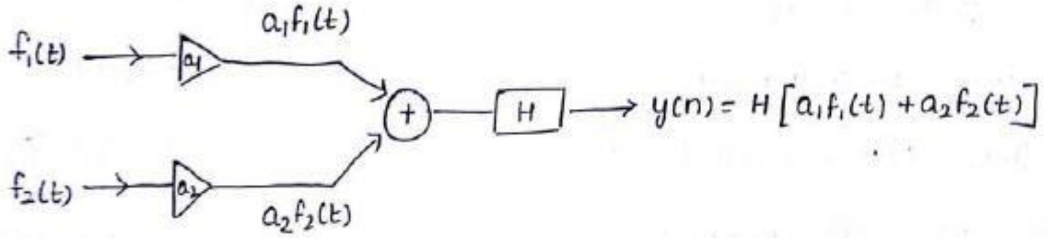
$$H[a_1 f_1(t) + a_2 f_2(t)] = a_1 H[f_1(t)] + a_2 H[f_2(t)]$$

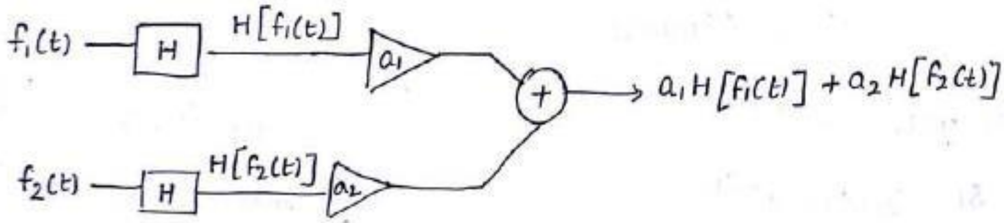
where a_1, a_2 are weighted constants.

$$a_1 f_1(t) \xrightarrow{\text{Response}} a_1 H[f_1(t)]$$

$$a_2 f_2(t) \xrightarrow{\text{Response}} a_2 H[f_2(t)]$$

$$H[a_1 f_1(t) + a_2 f_2(t)] \longrightarrow a_1 H[f_1(t)] + a_2 H[f_2(t)]$$





Block diagram.

→ Any system which does not obey the above principle is called as non-linear systems.

Check for Linearity:

Procedure:

1. Apply different i/p's separately and get the o/p.
2. Apply different i/p's simultaneously and get the output.
3. If both outputs are same it is linear otherwise non-linear.

Ex:

(i) $y(t) = 4\sin t\, x(t)$

step 1: $y_1(t) = 4\sin t\; x_1(t)$

$y_2(t) = 4\sin t\; x_2(t)$

$y_1(t) + y_2(t) = 4\sin t\,[x_1(t) + x_2(t)]$

step 2: $y(t) = 4\sin t\,[x_1(t) + x_2(t)]$

$S_1 = S_2$

(2) $y(t) = a x(t)$

Sol $S_1: y_1(t) = a x_1(t)$

$y_2(t) = a x_2(t)$

$y(t) = a x_1(t) + a x_2(t)$

$y(t) = a[x_1(t) + x_2(t)]$

$S_2: y(t) = a[x_1(t) + x_2(t)]$

$S_1 = S_2$ (Linear)

③ $y(t) = x^2(t)$

Sol $S_1: y_1(t) = x_1^2(t)$

$y_2(t) = x_2^2(t)$

$y(t) = x_1^2(t) + x_2^2(t)$

$S_2: y(t) = [x_1(t) + x_2(t)]^2 \rightarrow$

$S_1 \neq S_2$ (Non-Linear)

(4) $y(t) = e^{x(t)}$

Sol $S_1: y_1(t) = e^{x_1(t)}$

$y_2(t) = e^{x_2(t)}$

$y(t) = e^{x_1(t)} + e^{x_2(t)}$

$y(t) \neq e^{[x_1(t) + x_2(t)]}$

$S_2: e^{x_1(t)} \cdot e^{x_2(t)}$

$S_1 \neq S_2$ (Non-Linear)

⑤ $y(t) = t x(t)$

Sol S_1: $y_1(t) = t x_1(t)$

$y_2(t) = t x_2(t)$

$y(t) = t[x_1(t) + x_2(t)]$

$S_2: y(t) = t[x_1(t) + x_2(t)]$

$S_1 = S_2$ (Linear)

⑪ $y(t) = x(t^2)$

Sol $S_1: y_1(t) = x_1(t^2)$

$y_2(t) = x_2(t^2)$

$y(t) = x_1(t^2) + x_2(t^2)$

$S_2: y(t) = x_1(t^2) + x_2(t^2)$

$S_1 = S_2$ (Linear)

⑥ $y(t) = x(t - t_0)$

Sol $S_1: y_1(t) = x_1(t - t_0)$

$y_2(t) = x_2(t - t_0)$

$y(t) = x_1(t - t_0) + x_2(t - t_0)$

$S_2: y(t) = x_1(t - t_0) + x_2(t - t_0)$

$S_1 = S_2$ (Linear)

⑦ $y(t) = 3x(t+3)$

Sol $S_1: y_1(t) = 3x_1(t+3)$

$y_2(t) = 3x_2(t+3)$

$y(t) = 3[x_1(t+3) + x_2(t+3)]$

$S_2: 3[x_1(t+3) + x_2(t+3)]$

$S_1 = S_2$ (Linear)

⑫ $y(t) = Ax(t) + B$

Sol $y_1(t) = Ax_1(t) + B$

$y_2(t) = Ax_2(t) + B$

$y(t) = A[x_1(t) + x_2(t)] + 2B$

$y(t) = A[x_1(t) + x_2(t)] + B$

$S_1 \neq S_2$ = Non-linear

⑧ $y(t) = x(t+1)e^{-t}$

Sol $S_1: y_1(t) = x_1(t+1)e^{-t}$; $y_2(t) = x_2(t+1)e^{-t}$

$y(t) = e^{-t}[x_1(t+1) + x_2(t+1)]$

$S_2: y(t) = e^{-t}[x_1(t+1) + x_2(t+1)]$

$S_1 = S_2$ (Linear)

⑨ $y(t) = \cos[x(t)]$

Sol $S_1: y_1(t) = \cos[x_1(t)]$; $y_2(t) = \cos[x_2(t)]$

$y(t) = \cos[x_1(t)] + \cos[x_2(t)]$

$S_2: \cos[x_1(t) + x_2(t)]$

$S_1 \neq S_2$ (Non-Linear)

⑩ $y(t) = k\Delta x(t)$ where $\Delta x(t) = [x(t+1) - x(t)]$

Sol $S_1: y_1(t) = k[x_1(t+1) - x_1(t)]$; $y_2(t) = k\Delta x_2(t)$

$y(t) = [x_1(t+1) - x_1(t) + x_2(t+1) - x_2(t)] \cdot k$

$S_2: y(t) = k[x_1(t+1) - x_1(t) + x_2(t+1) - x_2(t)]$

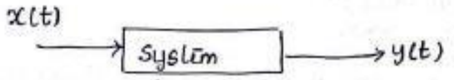
$S_1 = S_2$ (Linear)

⑬ $y(t) = 4x(t) + 2\dfrac{dx(t)}{dt} \rightarrow$ Linear.

Time Variant And Time Invariant Systems

→ A system is said to be time invariant if the system does not depend on time i.e system delay is not function of time.

Ex:



$$x(t) \longrightarrow y(t)$$
$$x(t-t_0) \longrightarrow y(t-t_0)$$

→ A time shift to in the input results in the same amount of time shift in the O/p but the waveshape does not change.

i.e the i/p and/ to o/p characteristics does not change with time.

→ Any system which doesnot obey the above principle is called as time varying system.

→ An electrical system is said to be time invariant if its component values (R, L, C) does not change with time.

2. Shift the entire system and get the O/p.

3. If both steps are identical for O/p then it is time invariant system.

Ex!

(1) $y(t) = 4x(t)$

Sol $S_1: y(t) = 4x(t-1)$
$S_2: y(t-1) = 4x(t-1)$ $\Big] \rightarrow S_1 = S_2$ (TIV)

(2) $y(t) = 4t\,x(t)$

Sol $S_1: y(t) = 4t\,x(t-1)$
$S_2: y(t-1) = 4(t-1)(x(t-1))$ $\Big] \rightarrow S_1 \neq S_2$ (TV)

(3) $y(t) = a x(t)$

Sol $S_1: y(t) = a x(t-1)$
S_2 $y(t-1) = a x(t-1)$ $\Big] \rightarrow S_1 = S_2$ (TIV)

(4) $y(t) = a x(t) + b$

Sol $S_1: y(t) = a x(t-1) + b$
$S_2: y(t-1) = a x(t-1) + b$ $\Big]$ $S_1 = S_2$ (TIV)

(5) $y(t) = 5t\,[x(t)]^2$

Sol $S_1: y(t) = 5t\,[x(t-1)]^2$
$S_2: y(t-1) = 5(t-1)[x(t-1)]^2$ $\Big] S_1 \neq S_2$ (TV)

(6) $y(t) = x(t+1)e^{-t}$

Sol $S_1: y(t) = x(t+1-1)e^{-t} = x(t)e^{-t}$
$S_2: y(t-1) = x(t+1-1)e^{-(t-1)}$
$= x(t) \cdot e^{-t} \, e^{1}$ → constant
$S_1 = S_2$ (TIV)

(7) $y(t) = x(t+3)$

Sol S1: $y(t) = x(t+3-1)$

S2: $y(t-1) = x(t+3-1)$ $\Big\} S_1 = S_2$ (TIV)

(8) $y(t) = x^2(t)$

Sol S1: $y(t) = x^2(t-1)$

S2: $y(t-1) = x^2(t-1)$

$\therefore S_1 = S_2$ (TIV)

(9) $y(t) = e^{x(t)}$

Sol S1: $y(t) = e^{x(t-1)}$

S2: $y(t-1) = e^{x(t-1)}$

$S_1 = S_2$ (TIV)

Stable System :

Linear Tin

→ System is absolutely integrable

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

→ Any sy

is called

Causal And Non Causal Systems:

Linear Tim

→ A system is said to be causal if o/p $y(t_0)$ depends only on the values of i/p $x(t)$ at $t < t_0$ {present, i/p, past i/ps, past o/ps} $\Big\{$ $x(t)=0$, for $t<0$ / noncausal $t \le 0$, or $t \le 0$ and t / $x(t) \neq 0$ for $t<0$

→ Any sys

property

Ex: $y(t) = 4x(t-1)$

Ex: y(t)=

$y(2) = 4x(2-1) \Rightarrow 4x(1)$

$y(t) = 4x(t-1) + x(t)$

Linearity :

$y(2) = 4x(1) + x(2)$

→ A system is said to be non-causal if the o/p depends on future values of i/p i.e future i/ps & o/ps.

S2

Ex: $y(t) = 4x(t+1)$

$y(2) = 4x(3)$

Examples whether it is causal & Non causal:

T.I :

$y(t) = ax(t)$

S1: $y(t) = ax(t-1)$

S2: $y(t-1) = ax(t-1)$ $\Big\} \rightarrow S_1 = S_2$ (TIV)

$\therefore$ It is a linear time invariant system (LTI)

$\parallel^{ly}$:

(2) $y(t) = t\,x(t) \rightarrow$ LTI

(3) $y(t) = a\,x(t) + b \rightarrow$ NLTI

(4) $y(t) = ax^2(t) \rightarrow$ NLTI

(5) $y(t) = e^{x(t)} \rightarrow$ NLTI

(6) $y(t) = x(t - t_0) \rightarrow$ LTI

(1) $y(t) = K[x(t+1) - x(t)]$

$y(0) = K[x(1) - x(0)] \rightarrow$ Non causal

(2) $y(t) = 3x(t+3)$

$y(0) = 3x(3) \longrightarrow$ Non causal

(3) $y(t) = (t+3)\,x(t-3)$

$y(0) = (0+3)\,x(0-3)$

$= 3x(-3) \rightarrow$ causal

(6) $y(t) = x(2t) \longrightarrow$ Non causal

(7) $y(t) = x(t) - x(t-1) \rightarrow$ causal

(8) $y(t) = x(t) + \int_0^t x(\lambda)\, d\lambda$

$= x(t) + z(\lambda)\big]_0^t \Rightarrow$ Causal.

At $t=0, t=1, t=2$

(4) $y(t) = x(t) + 3x(t+4)$

when $t=0$, $y(0) = x(0) + 3x(4)$

when $t=1$, $y(1) = x(1) + 3x(5)$

So here response at $t=0$, $y(0)$ depends on the present i/p & future i/p

Hence system is noncausal.

(5) $y(t) = x(t^2)$

$t=-1$, $y(-1) = x(1) \rightarrow$ future

$t=0$, $y(0) = x(0) \rightarrow$ present

$t=1$, $y(1) = x(1) \rightarrow$ present

$t=2$, $y(2) = x(4) \rightarrow$ future

Non causal.

Except at $t=0$, $t=1$, the response of any value of t depends on future i/p.

Impulse Response:

The response of a system for an impulse i/p is called a impulse response of the system and it is denoted by $h(t)$

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

$$\delta(t) \rightarrow h(t)$$

→ Every system is characterised by its impulse response.

$$\underset{f(t)}{\text{i/p}} \longrightarrow \boxed{h(t)} \longrightarrow \underset{r(t)}{\text{o/p}}$$

Response of a System for an arbitary i/p:

Response of $\delta(t) \rightarrow h(t)$

$\delta(t-t_0) \rightarrow h(t-t_0)$

$\delta(t) + \delta(t-t_0) = h(t) + h(t-t_0)$

The response of a system for a given i/p $f(t)$ is determined by using Superposition principle.

step 1: Resolve the i/p function interms of impulse functions.

step 2: Determine individually the response of LTI system for impulse function.

step 3: Find the sum of individual responses which will become the overall response $r(t)$.

Representation of a function $f(t)$ in terms of an impulse function.

Here the $$f(t) = \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t)\, \Delta t\, \delta(t - n\Delta t)$$

The rectangular of width Δt & height $f(n\Delta t)$ and area under the rectangles is $\Delta t \cdot f(n\Delta t)$ and this n^{th} element approached a delta function of strength $f(n\Delta t)\,\Delta t$ located at $t = n\Delta t$. and this delta function is represent as $f(n\Delta t)\,\Delta t\, \delta(t - n\Delta t)$

$$\boxed{f(t) = \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t \cdot \delta(t - n\Delta t)}$$

As $\Delta t \to 0$, the n^{th} element may be considered.

2) Determination of $r(t)$ for the input $f(t)$:

Let $h(t)$ be the impulse response of the system.

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

then $\delta(t) \longrightarrow h(t)$

$\delta(t - n\Delta t) \longrightarrow h(t - n\Delta t)$

$f(n\Delta t)\, \delta(t - n\Delta t) \longrightarrow f(n\Delta t) \cdot h(t - n\Delta t)$

$$\underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\, \delta(t - n\Delta t) \longrightarrow \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\, h(t - n\Delta t).$$

$$f(t) \longrightarrow \boxed{\text{system}} \longrightarrow r(t)$$

$$r(t) = \underset{\Delta t \to 0}{Lt} \sum_{n=-\infty}^{\infty} f(n\Delta t) \cdot \Delta t\, h(t - n\Delta t)$$

$\Delta t \to 0$ means summation becomes integration.

$$r(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t - \tau)\, d\tau$$

$$r(t) = f(t) \otimes h(t)$$

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow r(t) \Rightarrow f(t) \otimes h(t).$$

i.e if the response of a system is known then response to any other function f(t) can be obtained from the above eqn.

→ An unit impulse function is called as a Test function and it is used to characterise a system.

FILTER CHARACTERISTICS OF LINEAR SYSTEMS :-

In 1 IDEAL LOW PASS FILTERS :

→ It transmits all the signals below certain frequency 'B' Hz without any distortion.

→ The range of frequencies from 0Hz to 'B' Hz is called passband of lowpass filter.

→ The frequency 'B' Hz is called cut-off frequency of the ideal lowpass filter.

→ The transfer function of ideal low pass filter can be written as

$$H(f) = K e^{-j2\pi f t_0} \quad ; -B \leq f \leq B$$
$$= 0 \quad ; |f| > B.$$

$$H(\omega) = \underbrace{|H(\omega)|}_{\text{Amplitude response of the system}} e^{j\overbrace{\theta(\omega)}^{\text{phase response of the system}}}$$

$$\boxed{\begin{aligned} \ln[H(\omega)] &= \ln[|H(\omega)|] + j\theta(\omega) \\ &= L(\omega) + j\,\theta(\omega) \end{aligned}}$$

$L(\omega)$ → Gain of the system

$\theta(\omega)$ → phase shift introduced by system.

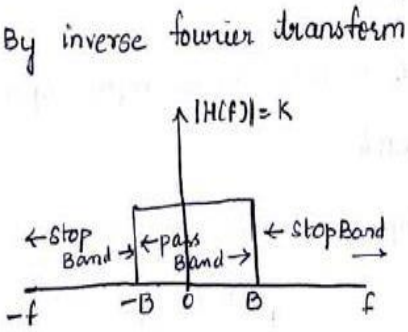
Note: An impulse function contains all frequencies in equal amount so we can use it as a test function.

$$H(\omega) = \frac{R(\omega)}{F(\omega)} \rightarrow \text{Transfer fn of LTI system.}$$

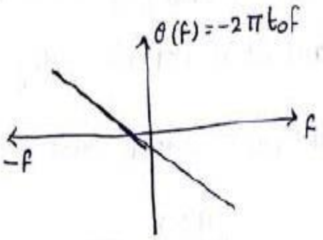
F.T $[h(t)]$ $\quad h(t) = \text{I.F.T}[H(\omega)]$

$= 0 \qquad ; \quad |f| > B$

→ By inverse fourier transform, $h(t)$ can be obtained for ideal LPF



a) Magnitude response.



(b) phase response

$$h(t) = \int_{-B}^{B} e^{-j2\pi f t_0} \cdot e^{j2\pi f t} \, df$$

$$= \int_{-B}^{B} \left[e^{j2\pi f (t-t_0)} \right] df = \frac{1}{j2\pi(t-t_0)} \left[e^{j2\pi f(t-t_0)} \right]_{-B}^{B}$$

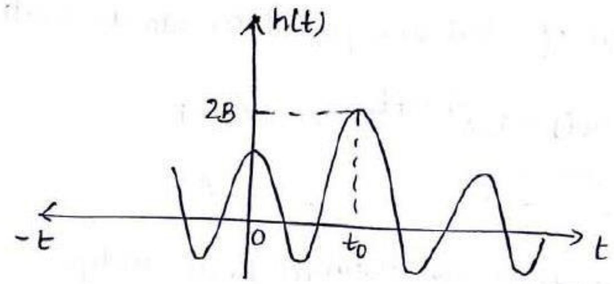
$$= \frac{1}{j2\pi(t-t_0)} \left[e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)} \right]$$

$$= \frac{1}{\pi(t-t_0)} \left[\frac{e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)}}{2j} \right]$$

$$= \frac{1}{\pi(t-t_0)} \sin\left[2\pi B(t-t_0)\right]$$

$$h(t) = 2B\left(\frac{\sin[2\pi B(t-t_0)]}{2\pi B(t-t_0)} \right) = 2B\,\text{Sinc}\left[2B(t-t_0)\right]$$

Response



→ Figure shows that impulse response exists for negative values of 't'. But actually unit impulse is applied at t=0 always.

→ Practically it is impossible to implement such a system.

By taking fourier transform

$$Y(f) = F[y(t)] = F\{K\,x(t-t_0)\}$$

From time shifting property of FT

$$Y(f) = K\,X(f)\,e^{-j2\pi f t_0}$$

→ All ideal filters are physically not

response is non-causal.

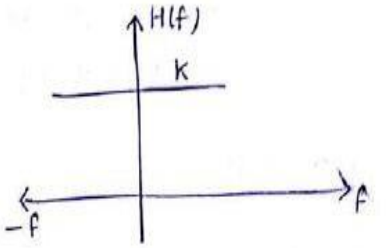
Transfer fn $H(f) = \frac{Y(f)}{X(f)}$

$$\therefore \; H(f) = \frac{Y(f)}{X(f)} = k \cdot e^{-j2\pi f t_0}$$

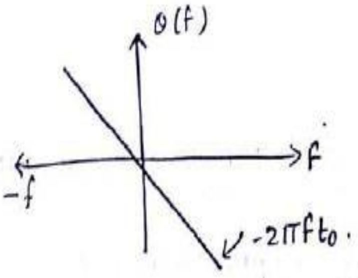
Magnitude of transfer fn independent of frequency.

→ Transfer function has constant amplitude at all frequencies. The phase shift is

$$\theta(f) = -2\pi f t_0$$



(a) Amplitude Spectrum



(b) phase spectrum passing through origin

→ By considering simple example

Let there be signal in time domain as

$$x(t) = \cos(2\pi f t)$$

Let the o/p sgl be same in amplitude but shifted in time by t_0 sec

$$y(t) = \cos[2\pi f(t - t_0)]$$

$$\therefore \; y(t) = \cos(2\pi f t - 2\pi f t_0) = \cos(2\pi f t - \theta(f))$$

$\therefore$ phase shift of $y(t)$ is

$\theta(f) = -2\pi f t_0$.

which is proportional to frequency 'f'.

Two types :

(i) Amplitude distortion

(ii) phase distortion.

AMPLITUDE DISTORTION

→ This distortion occurs when $|H(\omega)|$ is not constant over frequency band of interest and the frequency components present in I/P sgl are transmitted with different gain and attenuation.

PHASE DISTORTION :

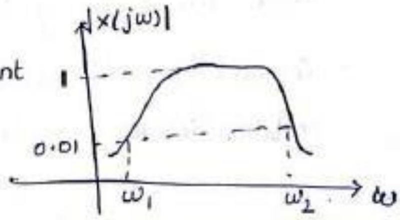
→ This distortion occurs when phase of $H(\omega)$ is not linearly changing with time and different frequency components in I/p are subjected to different time delays during transmission.

SIGNAL BANDWIDTH : The band of frequencies that contains most of signal energy is called B.W of signal denoted by fm.

→ It is the range of significant signal frequencies which are present in the signal.

→ observe in the waveform $x(t)$ has significant frequencies from ω_1 to ω_2.

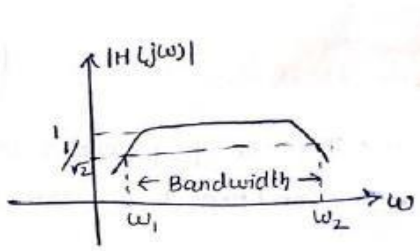
→ The B.W of this signal is $\omega_2 - \omega_1$



→ All the physically obtained signals have limited bandwidth.

SYSTEM BANDWIDTH

→ The B.W of a system is defined as range of frequencies over which $|H(\omega)|$ remains within $1/\sqrt{2}$ times of its mid-band value. For distortionless transmission the system must have infinite B.W but physical system are limited to finite BW.



→ so a system with finite BW can provide distortionless transmission for a band limited signal if $|H(\omega)|$ remains constant over BW of the signal.

→ The range of frequencies for which magnitude $|H(j\Omega)|$ of the systems remains within $\frac{1}{\sqrt{2}}$ of its maximum value.

→ The frequency domain statements can be interpreted as $|H(\omega)|$ of a physically realizable system may be zero for some discrete frequency but it can never be zero for a finite band of frequencies.

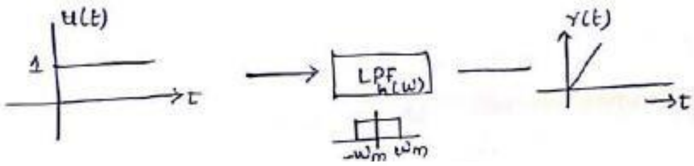
→ $H(\omega)$ for a realisable system cannot decay faster than a function of exponential order.

Ex: A system with T.F $e^{-\omega}$ is realisable whereas $e^{-\omega^2}$ is not as it decay faster

RELATIONSHIP BETWEEN RISE TIME AND BANDWIDTH:

→ If a unit step fn $u(t)$ is applied to an ideal LPF, the o/p will show a gradual rise instead of a sharp rise in the i/p.

→ The rise time (t_r) is the time required by the response to reach its final value from initial value.



Transfer function of ideal low pass filter is

$$H(\omega) = |H(\omega)|\, e^{j\theta(\omega)}$$

$$= G(\omega)\, e^{-j\omega t_0}$$

↓

Rectangular pulse with magnitude K.

for $-B \leq f \leq B$ i.e $-\omega_m \leq \omega \leq \omega_m$ where $\omega_m = 2\pi B$.

and $\theta(\omega) = -2\pi f t_0 = -\omega t_0$.

→ Fourier transform of unit step fn $u(t)$

$$FT\{u(t)\} \Rightarrow U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

→ Fourier transform of response $R(\omega)$, input and $H(\omega)$ related as

$$R(\omega) = \left[\pi\delta(\omega) + \frac{1}{j\omega}\right] H(\omega) = \pi\delta(\omega).H(\omega) + \frac{1}{j\omega} H(\omega).$$

$\delta(\omega)$ exists only for $\omega=0$ and $H(\omega)|_{\omega=0}=1$

$$R(\omega)=\pi\delta(\omega)+\frac{1}{j\omega}H(\omega).$$

By taking IFT for above eqn

$$r(t)=\text{IFt}\left[R(\omega)\right]=\text{IFT}\left\{\pi\delta(\omega)+\frac{1}{j\omega}H(\omega)\right\}$$

$$=\text{IFT}\left\{\pi\delta(\omega)+\frac{1}{j\omega}G(\omega)e^{-j\omega t_0}\right\}\qquad\left(\because H(\omega)=G(\omega)e^{-j\omega t_0}\right)$$

Inverse fourier transform of $\pi\delta(\omega)$ is $\frac{1}{2}$.

$$\left\{\begin{array}{l}\because 1\longrightarrow 2\pi\delta(\omega)\\ \frac{1}{2}\longleftarrow\pi\delta(\omega)\end{array}\right\}$$

$$r(t)=\frac{1}{2}+\text{IFT}\left\{\frac{1}{j\omega}G(\omega)e^{-j\omega t_0}\right\}$$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\omega}G(\omega)\,e^{-j\omega t_0}\cdot e^{j\omega t}\,d\omega$$

we know $G(\omega)=1$ for $-\omega_m\le\omega\le\omega_m$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{-\omega_m}^{\omega_m}\frac{e^{j\omega(t-t_0)}}{j\omega}\,d\omega$$

$$=\frac{1}{2}+\frac{1}{2\pi}\int_{-\omega_m}^{\omega_m}\frac{\cos\omega(t-t_0)+j\sin\omega(t-t_0)}{j\omega}\,d\omega$$

$$=\frac{1}{2}+\frac{1}{2\pi}\underbrace{\int_{-\omega_m}^{\omega_m}\frac{\cos\omega(t-t_0)}{j\omega}\,d\omega}_{=\text{ zero for its odd term}}+\frac{1}{2\pi}\underbrace{\int_{-\omega_m}^{\omega_m}\frac{\sin\omega(t-t_0)}{\omega}\,d\omega}_{\text{even}}.$$

$$r(t)=\frac{1}{2}+\frac{1}{2\pi}\times 2\int_{0}^{\omega_m}\frac{\sin\omega(t-t_0)}{\omega}\,d\omega=\frac{1}{2}+\frac{1}{\pi}\int_{0}^{\omega_m}\frac{\sin\omega(t-t_0)}{\omega}\,d\omega$$

$$=\frac{1}{2}+\frac{1}{\pi}\left[\text{Si }\omega(t-t_0)\right]_0^{\omega_m}$$

$$=\frac{1}{2}+\frac{1}{\pi}\text{Si }\omega_m(t-t_0)\longrightarrow\text{sine integral}$$

$$\rightarrow\frac{dr(t)}{dt}\Big|_{t=t_0}=\frac{1}{\pi}\cos\left[\omega_m(t-t_0)\right]\cdot\omega_m$$

The rise time is given as $t_r=\frac{2\pi}{\omega_m}=\frac{1}{B}$

$\frac{1}{t_r}=\frac{\omega_m}{\pi}\Rightarrow t_r=\frac{\omega_m}{\pi}$

B = cut off frequency of LPF

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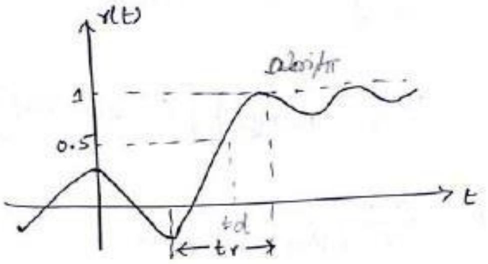


$\omega_c \to \infty$

$y(t) = 1$

$\omega_c \to -\infty$

$y(t) = 0$

Note: {Elements of block diagram)

① Adder:

$x_1(t)$, $x_2(t)$ → ⊕ → $y(t) = x_1(t) + x_2(t)$

which performs the addition of two signal sequences to form sum

② constant multiplier:

$x(t)$ → [a] → $y(t) = a\,x(t)$

It represents applying a scale factor on i/p $x(t)$.

③ Signal multiplier:

$x_1(t)$ → ⊗ → $y(t) = x_1(t) \cdot x_2(t)$.

↑ $x_2(t)$

The multiplication of two signal to form product sequence.

PROBLEMS:

① The impulse response of continuous time system is given as

$$h(t) = \frac{1}{RC}\, e^{-t/RC} \cdot u(t)$$

Determine the frequency response & plot the magnitude phase plots.

Sol

Take FT

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{RC} \cdot e^{-t/RC} \cdot u(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t/RC} \cdot e^{-j\omega t}\, dt \quad \left(\because u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}\right) \text{ step.}$$

$$= \frac{1}{RC} \int_0^\infty e^{-t\left(j\omega + \frac{1}{RC}\right)} dt$$

$$= \frac{1}{RC} \left(- \frac{1}{j\omega + \frac{1}{RC}} \right) \left[e^{-t(j\omega + 1/RC)} \right]_0^\infty$$

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC} = \frac{1}{1 + j\omega RC}$$

Magnitude & phase

$$H(\omega) = \frac{1}{1 + j\omega RC} \times \frac{1 - j\omega RC}{1 - j\omega RC} = \frac{1 - j\omega RC}{1 + (\omega RC)^2}$$

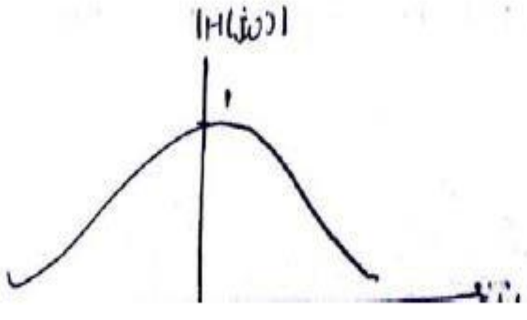
$$= \frac{1}{1 + (\omega RC)^2} + j\,\frac{-\omega RC}{1 + (\omega RC)^2}$$

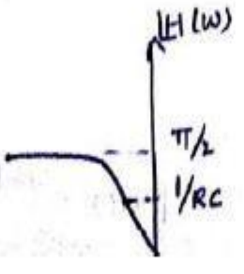
$$|H(\omega)| = \left\{ \frac{1}{[1 + (\omega RC)^2]^2} + \frac{(\omega RC)^2}{[1 + (\omega RC)^2]^2} \right\}^{1/2}$$

$$= \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$$\angle H(\omega) = \tan^{-1} \left\{ \frac{(-\omega RC)/1 + (\omega RC)^2}{1 + [1 + (\omega RC)^2]} \right\} = -\tan^{-1}(\omega RC)$$

If $RC = 1$, $|H(\omega)| = \frac{1}{\sqrt{1 + \omega^2}}$; $\angle H(\omega) = -\tan^{-1}(\omega)$.





② For the system shown find the T.T & impulse response of the system.

$$f(t) = e^{-at} \quad t>0$$
$$\quad\;\; 0 \quad \text{elsewhere} \qquad ; \quad Y(\omega) = \frac{1}{\alpha+j\omega}$$

Sol

$$H(\omega) = \frac{R(\omega)}{F(\omega)}$$

$R(\omega) = Y(\omega)$

$F(\omega) \longrightarrow \boxed{} \longrightarrow Y(\omega)$

$$F(t) = e^{-at}$$

$$F(\omega) = \frac{1}{a+j\omega} \quad ; \quad Y(\omega) = \frac{1}{\alpha+j\omega}$$

$$H(\omega) = \frac{1/\alpha+j\omega}{1/a+j\omega} = \frac{a+j\omega}{\alpha+j\omega}$$

$$F^{-1}\left[\frac{a+j\omega}{\alpha+j\omega}\right] \Rightarrow \frac{a+\alpha-\alpha+j\omega}{\alpha+j\omega} = \frac{a-\alpha}{\alpha+j\omega} + \frac{\alpha+j\omega}{\alpha+j\omega}$$

$$= \frac{a-\alpha}{\alpha+j\omega} + 1$$

$$\boxed{h(t) = (a-\alpha)\, e^{-\alpha t} u(t) + \delta(t)}$$

③ The linear system impulse response is $[e^{-2t}+e^{-3t}]u(t)$ find the excitation to produce an o/p of $t.e^{-2t}u(t)$?

Sol

$$h(t) = \left[e^{-2t}+e^{-3t}\right]u(t)$$

$F(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow R(\omega)$

$$r(t) = t\cdot e^{-2t}u(t)$$

$$H(\omega) = \frac{R(\omega)}{F(\omega)}$$

$$F(\omega) = \frac{R(\omega)}{H(\omega)}$$

$$r(t) = t\cdot e^{-2t}u(t) \xleftrightarrow{F.T} \frac{1}{(2+j\omega)^2} \qquad \left(\because\ t\cdot e^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2}\right)$$

$$R(\omega) = \frac{1}{(2+j\omega)^2}$$

$$h(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

$$H(\omega) = \frac{1}{2+j\omega} + \frac{1}{3+j\omega}$$

$$R(\omega) = \frac{1/(2+j\omega)}{\frac{3+j\omega+2+j\omega}{(2+j\omega)(3+j\omega)}} = \frac{1}{2+j\omega} \times \frac{3+j\omega}{5+2j\omega}$$

$$\frac{3+j\omega}{(2+j\omega)(5+2j\omega)} = \frac{A}{2+j\omega} + \frac{B}{5+2j\omega}$$

$$3+j\omega = A(5+2j\omega) + B(2+j\omega)$$

put $3+j\omega = 5A + 2B + j\omega(2A+B)$

put $j\omega = 0$; put $j\omega(-2)$

$(3 = 5A + 2B) \times 1$

$(1 = 2A + B) \times 2$

$A = 1, \; B = -1$

$$R(\omega) = \frac{1}{2+j\omega} - \frac{1}{5+2j\omega} = \frac{1}{2+j\omega} - \frac{1}{2[5/2+j\omega]}$$

$$\boxed{r(t) = e^{-2t}u(t) - \frac{1}{2}e^{-5/2t}u(t)}$$

DIFFERENTIAL EQUATION:

→ To obtain frequency response & impulse response.

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t).$$

differentiation property of FT is

$$\frac{d}{dt} x(t) \xleftrightarrow{FT} j\omega X(\omega).$$

$$\sum_{K=0}^{N} a_K (j\omega)^K Y(\omega) = \sum_{K=0}^{M} b_K (j\omega)^K X(j\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{K=0}^{M} b_K (j\omega)^K}{\sum_{K=0}^{N} a_K (j\omega)^K}.$$

System transfer fn.

PROBLEMS:

① The differential equation of system is given as $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = -\frac{dx(t)}{dt}$

Determine the frequency response & impulse response.

Sol

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = -\frac{dx(t)}{dt}$$

Taking F.T

$$(j\omega)^2 Y(\omega) + 5(j\omega)Y(\omega) + 6Y(\omega) = -j\omega X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{-j\omega}{(j\omega)^2 + 5j\omega + 6}$$

$$H(\omega) = \frac{-j\omega}{(j\omega+2)(j\omega+3)} = \frac{A}{j\omega+2} + \frac{B}{j\omega+3}$$

$$= \frac{2}{j\omega+2} - \frac{3}{j\omega+3}$$

$$h(t) = \left[2 \cdot e^{-2t} - 3e^{-3t}\right] u(t)$$

$$\left\{ \because e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega} \right\}$$

impulse response of the system.

② The input voltage to the RC circuit is given by $x(t) = te^{-t/RC}u(t)$ and impulse response of this circuit is given by $h(t) = \frac{1}{RC} e^{-t/RC}u(t)$. Find output $y(t)$

Sol) output $y(t) = x(t) * h(t)$

In frequency domain

$$Y(\omega) = X(\omega) H(\omega)$$

and $H(\omega) = F[h(t)]$

$$H(\omega) = F\left\{\frac{1}{RC} e^{-t/RC} \cdot u(t)\right\}$$

$$= \frac{1}{RC} \cdot \frac{1}{\frac{1}{RC} + j\omega} = \frac{1}{1+j\omega RC}$$

$$X(\omega) = F\left[t \cdot e^{-t/RC} u(t)\right]$$

$$= \int_{-\infty}^{\infty} t \cdot e^{-t/RC} e^{-j\omega t} dt = \frac{1}{\left(\frac{1}{RC} + j\omega\right)^2} = \frac{(RC)^2}{(1+j\omega RC)^2}.$$

$$\left[\because t e^{-at} u(t) \xleftrightarrow{FT} \left(\frac{1}{a+j\omega}\right)^2\right]$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$= \frac{(RC)^2}{(1+j\omega RC)^2} \cdot \frac{1}{(1+j\omega RC)} = \frac{(RC)^2}{(1+j\omega RC)^3}$$

$$y(t) = F^{-1}\{Y(\omega)\} = F^{-1}\left\{\frac{(RC)^2}{(1+j\omega RC)^3}\right\} = F^{-1}\left\{\frac{\cancel{(RC)^2}}{\cancel{(RC)^3}\left(\frac{1}{RC} + \right.}\right.$$

$$\boxed{y(t) = \frac{1}{RC} \cdot \frac{t^2 \cdot e^{-t/RC}}{2} u(t)}$$

3) $h(t) = e^{-4t}u(t)$ (stable)

4) $h(t) = t\cos t\,u(t)$ constable.

$\int_0^{\infty} t\cos t\,dt$

$|t|/\,dt = \int_{-\infty}^{\infty} e^{-5|t|}\,dt$

5) $h(t) = e^{-t}\sin t\,u(t)$ (stable.

$= \int_0^{\infty} e^{-t}\sin t\,dt$

$\int^{\infty} e^{-5t}\,dt = \left[\frac{e^{5t}}{5}\right]_{-\infty}^{0} + \left[\frac{e^{-5t}}{-5}\right]_{0}^{\infty}$

stant / so system is stable.

2) $h(t) = e^{4t}u(t)$

$= \int_{-\infty}^{\infty} |e^{4t}u(t)|\,dt = \int_{-\infty}^{\infty} e^{4t}u(t)\,dt$

$= \int_0^{\infty} e^{4t}\,dt = \frac{e^{\infty}}{4} - \frac{e^{0}}{4} = \infty - \frac{1}{4} = \infty$ (unstant.

3) $h(t) = e^{-4t}u(t)$ (stable)

4) $h(t) = t\cos t\,u(t)$ constable.

$\int_0^{\infty} t\cos t\,dt$

5) $h(t) = e^{-t}\sin t\,u(t)$ (stable.

$= \int_0^{\infty} e^{-t}\sin t\,dt$

(1) Th...

2) Determine the maximum bandwidth of signals that can be transmitted through low pass RC filter as shown in figure , if over this bandwidth,

the gain vc

7% of ideal

Determine it

But $B = 2\pi f$

$$f = \frac{B}{2\pi} = \frac{4.84\times10^3}{2\pi} = 770.8\,HZ$$

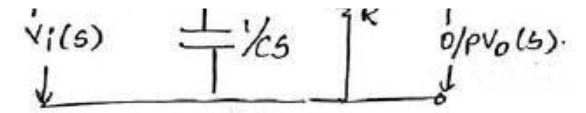
phase at frequency, $f = 770.8\,HZ$

$$\phi(\omega) = -\tan^{-1}\left(\frac{4.84}{10}\right) = -25.83\%$$

(2) There are several possible ways of estimating an essential bandwid of non-band limited signal. For a low pass signal, for example, the essential b.w may

Sol

R



$$H(s) = \frac{V_o(s)}{V_i(s)}$$

$$= \frac{[R \parallel (1/cs)]}{([R \parallel (1/cs)] + R]}$$

$$= \frac{(R/cs) / [R + 1/cs]}{[R/cs] / [[R + 1/cs)] + R}$$

$$= \frac{R/(1+sCR)}{[R/(1+sCR)] + R} = \frac{R}{R + R(1+sCR)}$$

$$H(s) = \frac{1}{2 + sCR}.$$

impulse response.

PART-A QUESTIONS

1. What are the characteristics of a filter?

An ideal low-pass filter **completely eliminates all frequencies above the cutoff frequency while passing those below unchanged**; its frequency response is a rectangular function and is a brick-wall filter. The transition region present in practical filters does not exist in an ideal filter.

2. Define LTI system?

A linear time-invariant system (LTI system) is a system that produces an output signal from any input signal subject to the constraints of linearity and time-invariance

3. State Poly-Wiener criterion for physical realization.

$$\int_{-\infty}^{\infty} \frac{\left|\log\left|H(jw)\right|\right|}{w^2+1} dw < \infty$$

4. Give the relation between Rise Time and Bandwidth.

$$BW = \frac{\omega_0}{2\pi} = \frac{2.2}{RT \times 2\pi} = \frac{0.35}{RT}$$

5. Differentiate between Group Delay and Phase Delay.

The phase delay of the filter is the amount of time delay each frequency component of the signal suffers in going through the filter. 2. The group delay is the average time delay the composite signal suffers at each frequency.

6. What is a distortion less Transmission?

Transmission is said to be distortion-less if the input and output have identical wave shapes. i.e., in distortion-less transmission, the input x(t) and output y(t) satisfy the condition: y (t) = Kx(t - td)

PART-B QUESTIONS

(1) State and prove sampling theorem for band limited signals using graphical approach.

(2)What is aliasing effect? How it can be eliminated? Explain with neat diagram.

(3) Explain the conditions required for distortion less transmission.

(4) Define system bandwidth and signal bandwidth, compare them with the help of examples.

(5) Derive an expression for the transfer function of an LTI system.

(6) Prove that the transmission of a pulse through a low pass filter causes the dispersion of pulse.

(7)Derive the expression for transfer function of flat top sampled signal.

(8) Define and derive how rise time and system bandwidth are related?

(9) Mention and compare different types of sampling techniques with neat waveforms.

UNIT V

Discrete Time Fourier Transform AND Z-Transform

Introduction

The discrete time Fourier transform (DTFT) is the member of the Fourier transform family that operates on Aperiodic, discrete signals. The best way to understand the DTFT is how it relates to the DFT. To start, imagine that you acquire an N sample signal, and want to find its frequency spectrum. By using the DFT, the signal can be decomposed into sine and cosine waves, with frequencies equally spaced between zero and one-half of the sampling rate. As padding in the time domain signal with zeros makes the period of the time domain longer, as well as making the spacing between samples in the frequency domain narrower. As N approaches infinity, the time domain becomes Aperiodic, and the frequency domain becomes a continuous signal. This is DTFT, the Fourier transform that relates an Aperiodic, discrete signal, with a periodic, continuous frequency spectrum.

Definition

The Discrete time Fourier transform of the discrete signal x[n] is given by,

$$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

The above equation is called analysis equation.

The time domain signal x[n] is obtained from $X(e^{jw})$ by taking inverse Discrete Time FourierTransform which is given by,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

The above equation is referred to as synthesis equation (or) Inverse DTFT.

Fourier transform of a signal in general is a complex valued function, we can write

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

Where $X_R(e^{jw})$ is the real part of $X(e^{jw})$ and $X_I(e^{jw})$ is imaginary part of the function $X(e^{jw})$.we canalso use a polar form

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{\angle X(e^{j\omega})}$$

Where I $X(e^{jw})$I is magnitude and < $X(e^{jw})$ is the phase of $X(e^{jw})$.we also use the term Fourier transform or simply, the spectrum to refer to $X(e^{jw})$.thus I $X(e^{jw})$I is called magnitude spectrumand < $X(e^{jw})$ is called the phase spectrum.

For simplicity e^{jw} is considered as Ω.so the Fourier transform pair can be represented as,

$$x(n) \xleftrightarrow{DTFT} X(\Omega)$$

Existence of DTFT

From the definition of DTFT observe that there is summation over infinite range of n.Hencefor DTFT to exist, the convergence of this summation is necessary. The above equation will converge if x(n) is absolutely summable.i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \qquad \textbf{.......(3)}$$

If x(n) is not absolutely summable,then it should have finite energy for DTFT to exist.i.e.,

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty \qquad ......(4)$$

Note that inverse DTFT does not have convergence problem since the integration is over limitedrange (-π to π).most of the physical signals satisfy above conditions.

Properties of DTFT

1) Periodicity

It states that

$$\boxed{X(\Omega+2\pi k) = X(\Omega)}$$

Proof: By definition of DTFT,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

Let $\Omega = \Omega + 2\pi k$ in above equation,

$$X(\Omega+2\pi k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\Omega+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}\, e^{-j2\pi kn}$$

Here $\quad e^{-j2\pi kn} = \cos(2\pi kn) - j\sin(2\pi kn)$

In the above equation k and n are integers. Hence cos (2πkn)=1 always and sin(2πkn)=0always. Hence $e^{-j2\pi kn}$ =1 and the above equation becomes,

$$X(\Omega+2\pi k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= X(\Omega)$$

2) Linearity

This property states that,

If $\quad x(n) \xleftrightarrow{DTFT} X(\Omega)$

and $\quad y(n) \xleftrightarrow{DTFT} Y(\Omega)$

then $\quad \boxed{z(n) = a\,x(n) + b\,y(n) \xleftrightarrow{DTFT} Z(\Omega) = a X(\Omega) + b Y(\Omega)}$

Proof: By definition of DTFT,

$$Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\Omega n}$$

Putting for z(n)=a x(n)+b y(n) i.e. linear combination of two inputs in above equation

$$\begin{aligned} Z(\Omega) &= \sum_{n=-\infty}^{\infty} [a\,x(n)+b\,y(n)]\, e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} a\,x(n)\, e^{-j\Omega n} + \sum_{n=-\infty}^{\infty} b\,y(n)\, e^{-j\Omega n} \\ &= a \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} + b \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n} \\ &= a\, X(\Omega) + b\, Y(\Omega) \end{aligned}$$

Thus the outputs are linearly related. This is superposition principle.

3.Time shifting

This property states that,

If $$x(n) \xleftrightarrow{DTFT} X(\Omega)$$

then $$\boxed{y(n) = x(n-n_0) \xleftrightarrow{DTFT} Y(\Omega) = e^{-j\Omega n_0}\, X(\Omega)}$$

Proof: By definition of DTFT,

$$\begin{aligned} Y(\Omega) &= \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x(n-n_0)\, e^{-j\Omega n} \end{aligned}$$

put $m=n-n_0$ .since n varies from $-\infty$ to ∞,m will also have the same range.
The above equationbecomes,

$$Y(\Omega) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\Omega(m+n_0)}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\Omega m}\; e^{-j\Omega n_0}$$

$$= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\Omega n}$$

$$= e^{-j\Omega n_0}\; X(\Omega)$$

Thus delaying sequence in time domain is equivalent to multiplying its spectrum by

$e^{-j\Omega n_0}$.

4. Frequency shifting

This property states that,

If $\quad x(n) \xleftrightarrow{DTFT} X(\Omega)$

then $\quad \boxed{y(n) = e^{j\Omega_0 n}\; x(n) \xleftrightarrow{DTFT} Y(\Omega) = X(\Omega - \Omega_0)}$

Proof: By definition of DTFT,

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n}\; x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\Omega-\Omega_0)n}$$

$$= X(\Omega - \Omega_0)$$

5. Differentiation in frequency domain

This property states that,

If $$x(n) \xleftrightarrow{DTFT} X(\Omega)$$

then $$\boxed{-j\,n x(n) \xleftrightarrow{DTFT} \frac{d}{d\Omega} X(\Omega)}$$

Proof: By definition of DTFT,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$\frac{d}{d\Omega} X(\Omega) = \frac{d}{d\Omega} \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

Changing the summation and differentiation,

$$\frac{d}{d\Omega} X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{d\Omega}\left[e^{-j\Omega n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-jn)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} [-jn\, x(n)]\, e^{-j\Omega n}$$

Comparing above equation with the definition of DTFT, we find that –jnx(n) has

DTFT ofd/dΩ [X(Ω)] .i.e,

$$-jnx(n) \xleftrightarrow{DTFT} \frac{d}{d\Omega} X(\Omega)$$

6. Time reversal

This property states that

$$\text{If} \qquad x(n) \xleftrightarrow{DTFT} X(\Omega)$$

$$\text{Then} \qquad \boxed{y(n) = x(-n) \xleftrightarrow{DTFT} Y(\Omega) = X(-\Omega)}$$

Proof: By definition of DTFT,

$$Y(\Omega) = \sum_{n=-\infty} y(n)\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\Omega n}$$

Put m=-n, the above equation becomes,

$$Y(\Omega) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\Omega(-m)}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j(-\Omega)m}$$

$$= X(-\Omega)$$

Thus the sequence is folded in time, then its spectrum is also folded.

7. Convolution in time domain

This property states that

$$\text{If} \qquad x(n) \xleftrightarrow{DTFT} X(\Omega)$$

$$\text{and} \qquad y(n) \xleftrightarrow{DTFT} Y(\Omega)$$

$$\text{then} \qquad \boxed{z(n) = x(n) * y(n) \xleftrightarrow{DTFT} Z(\Omega) = X(\Omega)\, Y(\Omega)}$$

Proof: By definition of DTFT,

$$Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\Omega n}$$

Putting for

$$z(n) = x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)$$

$$Z(\Omega) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, y(n-k)\right] e^{-j\Omega n}$$

Changing the order of summation,

$$Z(\Omega) = \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)\, e^{-j\Omega n}$$

Put n-k=m, the above equation becomes,

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\Omega(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\Omega m}\, e^{-j\Omega k}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, e^{-j\Omega k} \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\Omega m}$$

$$= X(\Omega)\, Y(\Omega)$$

Thus convolution of the two sequences is equivalent to multiplication of their spectrums.

8. Multiplication in time domain

This property states that

If $\quad x(n) \xleftrightarrow{DTFT} X(\Omega)$

and $\quad y(n) \xleftrightarrow{DTFT} Y(\Omega)$

then $\quad \boxed{z(n) = x(n)\, y(n) \xleftrightarrow{DTFT} Z(\Omega) = \frac{1}{2\pi}\left[X(\Omega) * Y(\Omega)\right]}$

Proof: By definition of DTFT,

$$Z(\Omega) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\Omega n}$$

Putting for z(n)= x(n) y(n) in above equation,

$$Z(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, y(n)\, e^{-j\Omega n}$$

From the inverse DTFT, we know that,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda)\, e^{j\lambda n}\, d\lambda$$

Here we have used separate frequency variable λ.putting the above expression of x(n) in z(Ω).

$$Z(\Omega) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) e^{j\lambda n}\, d\lambda \cdot y(n)\, e^{-j\Omega n}$$

Interchanging the order of summation and integration,

$$Z(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) \sum_{n=-\infty}^{\infty} y(n) e^{j\lambda n} e^{-j\Omega n} \, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) \left[\sum_{n=-\infty}^{\infty} y(n) \, e^{-j(\Omega-\lambda)n} \right] d\lambda$$

$$Z(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) \, Y(\Omega-\lambda) d\lambda$$

The above equation represents the convolution of X(Ω) and Y(Ω)

$$Z(\Omega) = \frac{1}{2\pi} [X(\Omega) * Y(\Omega)]$$

Thus multiplication of two sequences in time domain is equivalent to convolution of their spectrums.

9. Parseval's theorem

Parseval's theorem states that

If $x(n) \xleftrightarrow{DTFT} X(\Omega)$

Then energy of the signal is given as,

$$\boxed{E = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^2 \, d\Omega}$$

Proof: we know that energy of the signal is given as,,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E = \sum_{n=-\infty}^{\infty} x(n)\, x^{*}(n)$$

We can write the inverse DTFT of $x^{*}(n)$ as,

$$x^{*}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^{*}(\Omega)\, e^{-j\Omega n}\, d\Omega$$

Putting the above value in energy equation,

$$E = \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} X^{*}(\Omega)\, e^{-j\Omega n}\, d\Omega$$

Changing the order of summation and integration,

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^{*}(\Omega) \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^{*}(\Omega)\, X(\Omega)\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^{2}\, d\Omega$$

The energy of the discrete signal is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^{2}\, d\Omega$$

Z-transform

In mathematics and signal processing, the Z-transform converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency domain representation.

Definition

The Z-transform can be defined as either a one-sided or two-sided transform.

(i) Bilateral Z-transform

The bilateral or two-sided Z-transform of a discrete-time signal x[n] is the formal powerseries X(z) defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where *n* is an integer and *z* is, in general, a complex number:

$$z = Ae^{j\phi} = A(\cos\phi + j\sin\phi)$$

where *A* is the magnitude of *z*, *j* is the imaginary unit, and ϕ is the complex argument (also referred to as angle or phase) in radians.

(ii) Unilateral Z-transform

Alternatively, in cases where *x[n]* is defined only for *n* ≥ 0, the single-sided or unilateralZ-transform is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

Inverse Z-transform

The inverse Z-transform is calculated by using the formula,

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where C is a counter clockwise closed path encircling the origin and entirely in the region ofconvergence (ROC).

A special case of this contour integral occurs when C is the unit circle (and can be usedwhen the ROC includes the unit circle which is always guaranteed when $X(z)$ is stable, (i.e. all the poles are within the unit circle). The inverse Z-transform simplifies to the inverse discrete- time Fourier transform:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\omega}) e^{j\omega n} d\omega.$$

The Z-transform with a finite range of n and a finite number of uniformly spaced z values can becomputed efficiently via Bluestein's FFT algorithm.

Region of convergence

The region of convergence (ROC) is the set of points in the complex plane for which theZ-transform summation converges.

$$ROC = \left\{ z : \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty \right\}$$

The range of variation of z for which z-transform converges is called region of convergence ofz-transform.

The stability of a system can also be determined by knowing the ROC alone. If the ROCcontains the unit circle (i.e., $|z| = 1$) then the system is stable.

If you are provided a Z-transform of a system without an ROC .you can determine a unique x[n]provided you desire the following:

- Stability
- Causality

If you need stability then the ROC must contain the unit circle.

If you need a causal system then the ROC must contain infinity and the system function will bea right-sided sequence.

If you need an anti-causal system then the ROC must contain the origin and the system functionwill be a left-sided sequence.

If you need both, stability and causality, all the poles of the system function must be inside theunit circle.

Properties of ROC of Z-Transform

- ROC of z-transform is indicated with circle in z-plane.
- Roc is a ring, whose center is at origin.
- ROC does not contain any poles.
- If x(n) is a finite duration causal sequence or right sided sequence, then the ROC isentire z-plane except at z = 0.
- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC isentire z-plane except at z = ∞.
- If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. |z| > a.
- If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. |z| < a.
- If x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except at z = 0 & z = ∞.

CausalityandStability

Causality condition for discrete time LTI systems is as follows:

A discrete time LTI system is causal when

- ROC is outside the outermost pole.

- In The transfer function H[Z], the order of numerator cannot be greater than the order ofdenominator.

Stability Condition for Discrete Time LTI Systems

A discrete time LTI system is stable when

- Its system function H[Z] include unit circle |z|=1.
- All poles of the transfer function lay inside the unit circle |z|=1.

Properties of the *z*-Transform

The z-transform has a few very useful properties, and its definition extends to infinite signals/impulse responses.

(1) Linearity

This property states that

$$\mathcal{Z}[ax[n] + by[n]] = aX(z) + bY(z), \quad ROC \supseteq (R_x \cap R_y)$$

While it is obvious that the ROC of the linear combination of x(n) and y(n) should be the intersection of the their individual ROCs in which both X(Z) and Y(Z) exist.

(2) Time Shifting

This property states that

$$\mathcal{Z}[x[n - n_0]] = z^{-n_0} X(z), \quad ROC = R_x$$

Proof:

$$\mathcal{Z}[x[n - n_0]] = \sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n}$$

Let $m = n - n_0$, we have $n = m + n_0$ and

$$\sum_{m=-\infty}^{\infty} x[m]z^{-m}z^{-n_0} = z^{-n_0}X(z)$$

The new ROC is the same as the old one except the possible addition/deletion of the origin orinfinity as the shift may change the duration of the signal.

(3) Time Expansion (Scaling)

This property states that

$$\mathcal{Z}[x[n/k]] = X(z^k), \quad ROC = R_x^{1/k}$$

Proof: The z-transform of such an expanded signal is

$$\mathcal{Z}[x[n/k]] = \sum_{n=-\infty}^{\infty} x[n/k]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-km} = X(z^k)$$

Note that the change of the summation index from n to m has no effect as the terms skipped areall zeros.

(4) Convolution

This property states that

$$\mathcal{Z}[x[n] * y[n]] = X(z)Y(z), \quad ROC \supseteq (R_x \cap R_y)$$

The ROC of the convolution could be larger than the intersection of R_X and R_Y, due to the possible pole-zero cancellation caused by the convolution.

(5) Time Difference

This property states that

$$\mathcal{Z}[x[n] - x[n-1]] = (1 - z^{-1})X(z), \quad ROC = R_x$$

Proof:

$$\mathcal{Z}[x[n]-x[n-1]] = X(z) - z^{-1}X(z) = (1-z^{-1})X(z) = \frac{z-1}{z}X(z)$$

Note that due to the additional zero z=1 and pole z=0, the resulting ROC is the same as R_X except the possible deletion of z=0 caused by the added pole and/or addition of z = 1 caused bythe added zero which may cancel an existing pole.

(6) Time Accumulation

This property states that

$$\mathcal{Z}[\sum_{k=-\infty}^{n} x[k]] = \frac{1}{1-z^{-1}}X(z), \quad ROC \supseteq [R_x \cap (|z|>1)]$$

Proof: The accumulation of x[n] can be written as its convolution with u[n]:

$$u[n]*x[n] = \sum_{k=-\infty}^{\infty} u[n-k]x[k] = \sum_{k=-\infty}^{n} x[k]$$

Applying the convolution property, we get

$$\mathcal{Z}[\sum_{k=-\infty}^{n} x[k]] = \mathcal{Z}[u[n]*x[n]] = \frac{1}{1-z^{-1}}X(z)$$

$$\mathcal{Z}[u[n]] = 1/(1 - z^{-1})$$

.

(7) Time Reversal

This property states that

$$\mathcal{Z}[x[-n]] = X(1/z) \quad ROC = 1/R_x$$

Proof:

$$\mathcal{Z}[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{m=-\infty}^{\infty} x[m](\frac{1}{z})^{-m} = X(1/z)$$

where m=-n.

(8) Scaling in Z-domain

This property states that,

$$\mathcal{Z}[a^n x[n]] = X\left(\frac{z}{a}\right), \quad ROC = |a|R_x$$

Proof:

$$\mathcal{Z}[a^n x[n]] = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right)$$

In particular, if a=e^{jw0}, the above becomes

$$\mathcal{Z}[e^{jn\omega_0}x[n]] = X(e^{-j\omega_0}z) \quad ROC = R_x$$

This property is essentially the same as the frequency shifting property of discrete Fourier transform.

(9) Conjugation

$$\mathcal{Z}[x^*[n]] = X^*(z^*), \qquad ROC = R_x$$

Proof: Complex conjugate of the z-transform of $x[n]$ is

$$X^*(z) = [\sum_{n=-\infty}^{\infty} x[n]z^{-n}]^* = \sum_{n=-\infty}^{\infty} x^*[n](z^*)^{-n}$$

Replacing z by z^*, we get the desired result.

(10) Differentiation in z-Domain

$$\mathcal{Z}[nx[n]] = -\frac{d}{dz}X(z), \quad ROC = R_x$$

Proof:

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} x[n]\frac{d}{dz}(z^{-n}) = \sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1} = \frac{-1}{z}\sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

i.e.,

$$\mathcal{Z}[nx[n]] = -z\frac{d}{dz}X(z)$$

SOLVED PROBLEMS

1. Determine the Fourier transform of the unit sample sequence x(n) = δ(n).

Solution: The unit sample sequence is defined as,

$$\begin{aligned} x(n) &= 1 \quad \text{for} \quad n=0 \\ &= 0 \quad \text{for} \quad n \neq 0 \end{aligned}$$

By the definition of Fourier transform**,**

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} = 1 \cdot e^{0}$$

$$= 1 \quad \text{for all } \Omega$$

Thus the Fourier transform has the value 1 for all values of Ω.

2. Determine the Fourier transform of the unit step sequence x(n) =

u(n)Solution: The unit step sequence is defined as,

$$\begin{aligned} x(n) &= 1 \quad \text{for} \quad n \geq 0 \\ &= 0 \quad \text{elsewhere} \end{aligned}$$

By the definition of Fourier transform**,**

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} 1 \cdot e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(e^{-j\Omega}\right)^{n}$$

Now let us use the relation,

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

Hence X(Ω) becomes

$$X(\Omega) = \frac{\left(e^{-j\Omega}\right)^0 - \left(e^{-j\Omega}\right)^\infty}{1-e^{-j\Omega}}$$

$$= \frac{1}{1-e^{-j\Omega}}$$

This relation is not convergent for Ω=0.This is because x(n) is not absolutely summable sequence. However X(Ω) can be evaluated for other values of Ω. Let us rearrange equation

$$X(\Omega) = \frac{1}{e^{-j\Omega/2} \cdot e^{j\Omega/2} - e^{-j\Omega/2} \cdot e^{-j\Omega/2}}$$

$$= \frac{1}{e^{-j\Omega/2}\left[e^{j\Omega/2} - e^{-j\Omega/2}\right]}$$

By Euler's identity we can write,

$$X(\Omega) = \frac{1}{e^{-j\Omega/2} \cdot 2j \sin\frac{\Omega}{2}}$$

$$= \frac{e^{j\Omega/2}}{2j \sin\frac{\Omega}{2}}, \; \Omega \neq 0$$

3. Determine the Fourier transform of

$$x(n) = a^n u(n) \qquad for \; -1 < a < 1$$

Solution: Let us check whether the Fourier transform is convergent i.e.

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=0}^{\infty} |a|^n$$

$$= \frac{1}{1-|a|} \quad \text{By geometric series and } |a| < 1$$

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty.$$

Hence Fourier transform is convergent. By definition of Fourier transform we have

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} a^n\ e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} \left(a\,e^{-j\Omega}\right)^n$$

$$\text{Here } \left|a\,e^{-j\Omega}\right| = |\ a\ | < 1,$$

Hence we can apply geometric summation formula. i.e.,

$$X(\Omega) = \frac{1}{1-a\,e^{-j\Omega}}$$

This is the required Fourier transform.

4. Determine the Fourier transform of the discrete time rectangular pulse of amplitude Aand length L.ie

$$\begin{aligned} x(n) &= A && \text{for } 0 \le n \le L-1 \\ &= 0 && \text{otherwise} \end{aligned}$$

Solution: Let us check whether the Fourier transform is convergent. i.e.

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=0}^{L-1} |A|$$

$$= |A|\,L < \infty$$

Thus x(n) is absolutely summable and Fourier transform will exist. By definition of

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j\Omega n}$$

$$= \sum_{n=0}^{L-1} A\ e^{-j\Omega n}$$

Fouriertransform

Here let us use the standard relation,

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$$\therefore \qquad X(\Omega) = A \frac{1-e^{-j\Omega L}}{1-e^{-j\Omega}}$$

The above equation can be further simplified using Euler's identity

$$X(\Omega) = A\, e^{-j\Omega(L-1)/2} \frac{\sin(\Omega L/2)}{\sin(\Omega/2)}$$

5. Determine the discrete time sequence where DTFT is given as,

$$\begin{aligned} X(\Omega) &= 1 \quad \text{for} \quad -\Omega_c \le \Omega \le \Omega_c \\ &= 0 \quad \text{for} \quad \Omega_c < |\Omega| < \pi \end{aligned}$$

Solution: The inverse DTFT is given by,

$$\begin{aligned} x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} 1\ e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2\pi} \left[\frac{e^{j\Omega n}}{jn} \right]_{-\Omega_c}^{\Omega_c} = \frac{1}{2\pi} \left[\frac{e^{j\Omega_c n} - e^{-j\Omega_c n}}{jn} \right] \\ &= \frac{1}{n\pi} \left[\frac{e^{j\Omega_c n} - e^{-j\Omega_c n}}{2j} \right] \end{aligned}$$

By Euler's identity we can write above equation as,

$$x(n) = \frac{1}{n\pi} \sin(\Omega_c n) \quad \text{for} \quad n \ne 0$$

When n=0 in above equation, we get

$$x(n) = \frac{\Omega_c}{\pi} \quad \text{for} \quad n=0$$
$$= \frac{\sin \Omega_c n}{n\pi} \quad \text{for} \quad n \neq 0$$

$$= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} d\Omega = \frac{1}{2\pi} |\Omega|_{-\Omega_c}^{\Omega_c}$$

$$= \frac{\Omega_c}{\pi}$$

Thus the sequence x(n) is,

SOLVED PROBLEMS

1) Find the z transform of following:

$$x[n] = \delta[n] + \tfrac{1}{2}\delta(n+1) + \delta(n-3)$$

$$\delta[n] \xleftrightarrow{Z} 1$$
$$\frac{1}{2}\delta[n+1] \xleftrightarrow{Z} \frac{1}{2}z$$
$$\delta[n-3] \xleftrightarrow{Z} z^{-3}$$
$$\boxed{X[z] = 1 + \frac{1}{2}z + z^{-3}}$$

$$x[n] = nu[n-1]$$

Method 1

$$u[n-1] \xleftrightarrow{Z} \frac{1}{(z-1)}$$

Using differential property we get

$$nu[n-1] \xleftrightarrow{Z} -z\frac{d}{dz}\frac{1}{(z-1)}$$
$$\boxed{z[nu[n-1]] = \frac{z}{(z-1)^2}}$$

Method 2

$$nu[n-1] = (n-1)u[n-1] + u[n-1]$$
$$(n-1)u[n-1] \xleftrightarrow{Z} \frac{zz^{-1}}{(z-1)^2} = \frac{1}{(z-1)^2}$$

$$x[n] = 4^n \cos\left[\frac{2\pi n}{6} + \frac{\pi}{4}\right] u[-n-1]$$

$$\cos\left(\frac{2\pi n}{6}\right) = \cos\frac{\pi n}{3}$$

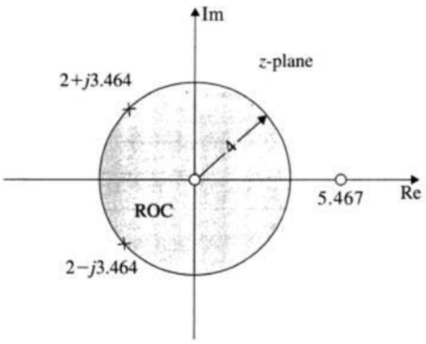
$$\cos\left(\frac{2\pi n}{6} + \frac{\pi}{4}\right) = \frac{e^{j(\frac{\pi}{4}+\frac{\pi n}{3})} + e^{-j(\frac{\pi}{4}+\frac{\pi n}{3})}}{2}$$

$$= \frac{1}{2}e^{j(\frac{\pi}{4})}e^{j(\frac{\pi n}{3})} + \frac{1}{2}e^{-j(\frac{\pi}{4})}e^{-j(\frac{\pi n}{3})}$$

$$4^n \cos\left(\frac{2\pi n}{6} + \frac{\pi}{n}\right) = \frac{1}{2}e^{j(\frac{\pi}{4})}\left(4e^{j\frac{\pi}{3}}\right)^n + \frac{1}{2}e^{-j\frac{\pi}{4}}\left(4e^{-j\frac{\pi}{3}}\right)^n$$

$$\left(4e^{j\frac{\pi}{3}}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{-z}{(z-4e^{j\frac{\pi}{3}})} \qquad \text{ROC: } |z| < 4$$

$$\left(4e^{-j\frac{\pi}{3}}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{-z}{(z-4e^{-j\frac{\pi}{3}})} \qquad \text{ROC: } |z| < 4$$



$$X[z] = -\frac{1}{2}z\left[\frac{e^{j\frac{\pi}{4}}}{(z-4e^{j\frac{\pi}{3}})} + \frac{e^{-j\frac{\pi}{4}}}{(z-4e^{-j\frac{\pi}{3}})}\right]$$

$$= -\frac{1}{2}z\left[\frac{ze^{j\frac{\pi}{4}} - 4e^{-j\frac{\pi}{12}} - 4e^{j\frac{\pi}{12}} + ze^{-j\frac{\pi}{4}}}{z^2 - z4(e^{j\frac{\pi}{3}} + e^{-j\frac{\pi}{3}}) + 16}\right]$$

$$= \frac{-\frac{1}{2}z[\sqrt{2}z - 7.73]}{(z^2 - 4z + 16)}$$

$$\boxed{X[z] = \frac{-0.707z[z-5.467]}{(z-2+j3.464)(z-2-j3.464)}} \quad \text{ROC: } |z| < 4$$

The pole zero diagram is shown in above figure. The ROC is the interior of the circle.

2. **Find the inverse z transform of following:**

Solution:

(a) $X[z] = \frac{4z}{(z^2-3z+2)}$; **ROC: $|z| > 2$**

$$X[z] = \frac{4z}{(z^2-3z+2)} = \frac{4z}{(z-1)(z-2)}$$

(b) $X[z] = \frac{4z}{(z^2-3z+2)}$; **ROC: $|z| < 1$**

$$\begin{array}{r l}
 & 4z^{-1} + 12z^{-2} + 28z^{-3} + \cdots \\
z^2 - 3z + 2 \,\big)\!\! & 4z \\
 & \underline{4z - 12 + 8z^{-1}} \\
 & \quad 12 - 8z^{-1} \\
 & \quad \underline{12 - 36z^{-1} + 24z^{-2}} \\
 & \qquad 28z^{-1} - 24z^{-2} \\
 & \qquad \underline{28z^{-1} - 84z^{-2} + 56z^{-3}}
\end{array}$$

$$X[z] = 4z^{-1} + 12z^{-2} + 28z^{-3} + \cdots$$

$$\boxed{x[n] = \{\underset{\uparrow}{0},\ 4,\ 12,\ 28, \ldots\}}$$

For ROC: IzI > 2, x[n] is a right-sided sequence where n ≥ 0.Hence , the long division isdone in such a way that X[z] is expressed in power of z^{-1}.

For ROC: IzI < 1, x[n] sequence is negative where n≤0.Hence , the long division is done insuch a way that X[z] is expressed in power of z.

$$\begin{array}{r l}
 & 2z + 3z^2 + \frac{7}{2}z^3 \\
2 - 3z + z^2 \,\big)\!\! & 4z \\
 & \underline{4z - 6z^2 + 2z^3} \\
 & \quad 6z^2 - 2z^3 \\
 & \quad \underline{6z^2 - 9z^3 + 3z^4} \\
 & \qquad 7z^3 - 3z^4 \\
 & \qquad \underline{7z^3 - \frac{21}{2}z^4 + \frac{7}{2}z^5}
\end{array}$$

$$X[z] = 2z + 3z^2 + \frac{7}{2}z^3 + \cdots$$

$$\boxed{x[n] = \left\{\cdots \frac{7}{2},\ 3,\ 2,\ \underset{\uparrow}{0}\right\}}$$

3. **Find the inverse z transform of following:**

(a)

$$X[z] = \frac{z(z+1)}{(z+3)(z+5)}$$

$$x[n] = \sum \text{Residue of } \frac{z(z+1)}{(z+3)(z+5)} z^{n-1}$$

$$= \text{Residue of } (z+3)\frac{z(z+1)}{(z+3)(z+5)} z^{n-1}\bigg|_{z=-3}$$

$$+ \text{ Residue of } (z+5)\frac{z(z+1)z^{n-1}}{(z+3)(z+5)}\bigg|_{z=-5}$$

$$\boxed{x[n] = -(-3)^n + 2(-5)^n}$$

(b) $X[z] = \frac{z^{-1}}{(1-10z^{-1}+24z^{-2})}$; $4 < |z| < 6$

$$X[z] = \frac{z}{(z^2 - 10z + 24)} = \frac{z}{(z-4)(z-6)}$$

For $n \geq 0$

$$x[n] = \text{Residue of } X[z]z^{n-1}\Big|_{z=4}$$

$$= (z-4)\frac{z(z^{n-1})}{(z-4)(z-6)}\bigg|_{z=4}$$

$$= -\frac{1}{2}(4)^n u[n]$$

For $n < 0$

$$x[n] = -\left[(z-6)\frac{zz^{n-1}}{(z-4)(z-6)}\right]_{z=6}$$

$$= -\frac{1}{2}(6)^n u(-n-1)$$

$$\boxed{x[n] = -\frac{1}{2}[(4)^n u[n] + (6)^n u(-n-1)]}$$

(c) $X[z] = \frac{z}{(z-\frac{1}{2})^2}$

$$x[n] = \frac{d}{dz}\left[\left(z-\frac{1}{2}\right)^2 \frac{zz^{n-1}}{\left(z-\frac{1}{2}\right)}\right]_{z=\frac{1}{2}}$$

$$= \frac{d}{dz}z^n\Big|_{z=1/2} = nz^{n-1}\Big|_{z=1/2}$$

$$\boxed{x[n] = 2n\left(\frac{1}{2}\right)^n u[n]}$$

PART-A Questions

1. Describe the ROC of the signal.
   The range of variation of z for which z-transform converges is called region of convergence ofz-transform.
2. Write any two properties of ZT.

   (a) Linearity

   $$\mathcal{Z}[ax[n] + by[n]] = aX(z) + bY(z), \quad ROC \supseteq (R_x \cap R_y)$$

   (b) Time Shifting

   $$\mathcal{Z}[x[n-n_0]] = z^{-n_0}X(z), \quad ROC = R_x$$
3. Find the Z-Transform of x[n] = {1, 3, 2, 1}.
   $X(Z)=1 + 3Z^{-1} + 2Z^{-2} + 1\, Z^{-3}$
4. What is the relation between LT &ZT.
   The Laplace transform evaluated at s=jω is equal to the Fourier transform if its region of convergence (ROC) contains the imaginary axis.
5. Write Initial and final value theorems in ZT.
   For a causal signal x(n), the initial value theorem states that. **x(0)=limz→∞X(z)**

Final Value Theorem: $\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$

PART-B Questions

1. Find z transform, ROC and pole zero locations of: X(n) = a^n u(n).

2. State and prove z – transform time reversal property.

3. Find the inverse z – transform of X(z) = (1/1 + z) (2z/z – 0.2).

4. Find the Laplace transform of: x(t) = $e^{-(t-2)}$ (t – 2) u(t – 2) (13)Find the inverse z – transform of:

   X(z) = (1/1 + 2z) + (2z/z – 0.25)

5. Prove Initial and Final value theorems of ZT.