UNIT V

FOURIER TRANSFORMS

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- **FOURIER TRANSFORMS**

Fourier Integral Theorem:-

Statement : If f(x) is a given function defined in (-l,l) and satisfies Dirichlet's condition then f(x) = $\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$.

The representation of f(x) is known as Fourier Integral of f(x)

Problems on integral theorem:

(1) Express the function f(x) = 1 , $|x| \leq 1$

$= 0$, $-\infty < x < -1 =$

0 , $1 < x < \infty$

as fourier integral and hence evaluate (i) $\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda}\, d\lambda$

(ii) $\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

- **Solution:** The Fourier Integral theorem is given by f(x)

$= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x)\, dt\, d\lambda.$

$= \frac{1}{\pi} \int_0^{\infty} \left[\int_{-1}^{1} 1 . \cos \lambda(t-x)\, dt \right] d\lambda$

$= \frac{1}{\pi} \int_0^{\infty} \left[\frac{\sin \lambda(t-x)}{\lambda}\right] d\lambda$ limits (-1 to 1) for t

$= \frac{1}{\pi} \int_0^{\infty} \left[\frac{\sin \lambda(1-x) - \sin \lambda(-1-x)}{\lambda}\right] d\lambda$

$= \frac{1}{\pi} \int_0^{\infty} \left[\frac{\sin(\lambda-\lambda x) + \sin(\lambda+\lambda x)}{\lambda}\right] d\lambda$

$= \frac{1}{\pi} \int_0^{\infty} 2. \left[\frac{\sin \lambda . \cos \lambda x}{\lambda}\right] d\lambda$

therefore $f(x) = \frac{2}{\pi} \int_0^{\infty} \left[\frac{\sin \lambda . \cos \lambda x}{\lambda}\right] d\lambda$ -----(1)

Deduction :

(I) $\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda}\, d\lambda = \frac{\pi}{2}$ f(x)

$= \frac{\pi}{2}$, $|x| \leq 1$

$= 0$, $|x| > 1$ -------- (2)

Put x = 0

(2) $\Rightarrow \int_0^\infty \frac{\sin\lambda \cos 0}{\lambda}\, d\lambda = \frac{\pi}{2}$

$\Rightarrow \int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda = \frac{\pi}{2}$

$\Rightarrow \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

Fourier cosine & sine Integrals:

1) Fourier cosine Integral of f(x) is

$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda_t\, dt\, d\lambda$

2) Fourier sine Integral of f(x) is

$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda_t\, dt\, d\lambda$

Problems:-

2) Express f(x) = 1 , $0 \leq x \leq \pi$

0 , x > π as a fourier sine integral and

Hence evaluate $\int_0^\infty (\frac{1-\cos\lambda\pi}{\lambda})$ sin λx dλ

Solution : Fourier sine integral of f(x) is given by

f(x) = $\frac{2}{\pi}\int_0^\infty sin$ λx [$\int_0^\infty f(t)\ sin$ λt dt] dλ

= $\frac{2}{\pi}\int_0^\infty sin$ λx [$\int_0^\pi$ sin λt dt] dλ

= $\frac{2}{\pi}\int_0^\infty sin$ λx ($\frac{-\cos\lambda t}{\lambda}$) (0 to π) dλ

$\frac{2}{\pi}\int_0^\infty (\frac{1-\cos\lambda\pi}{\lambda})$ sin λx dλ f(x) =

⇒ $\int_0^\infty (\frac{1-\cos\lambda\pi}{\lambda})$ sin λx dλ π = f(x) .

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= $\frac{\pi}{2}$. 1 , 0 $\leq$ x $\leq$ π

0 , x > π

Problem : 3) Using Fourier Integral show that

$\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}$ sinx λ dλ = $\frac{\pi}{2}$, 0 < x < π

0 , x > π

Solution : Let f(x) =1 , $0 \leq x \leq \pi$

0 , x > π

then write above solution (problem.(2) solution).

Problem :4) Using Fourier Integral , show that $e^{-ax} = \frac{2a}{\pi}\int_0^{\infty}\frac{\cos\lambda x}{\lambda^2+a^2}\,d\lambda$

Solution : Let f(x) = e^{-ax}

The Fourier Cosine Integral is given by f(x)

$= \frac{2}{\pi}\int_0^{\infty} \cos\lambda x\ [\int_0^{\infty} f(t)\cos\lambda t\,dt\,]\,d\lambda$

Now f(t) = e^{-at}

$$e^{-ax} = \frac{2}{\pi}\int_0^{\infty} \cos\lambda x\ [\int_0^{\infty} e^{-at}\cos\lambda t\,dt\,]\,d\lambda \text{ ----(1)}$$

at

$\int_0^{\infty} e^{-at}\cos\lambda t\,dt = [\frac{e^{-}}{a^2+\lambda^2}($

$= 0 - \frac{e^0}{a^2+\lambda^2}($

Therefore

Now -a cos λt + λ sin λt)(0 to ∞)]

a

-a.1 + 0) = $\overline{a^2+\lambda^2}$

sub in (1)

$$(1) \Rightarrow e^{-ax} = \frac{2}{\pi}\int_0^\infty \cos \lambda x \,.\, \frac{a}{a^2+\lambda^2}\, d\lambda$$

$$= \frac{2a}{\pi}\int_0^\infty \frac{\cos \lambda x}{a^2+\lambda^2}\, d\lambda$$

Problem 5): Prove that $\frac{\pi}{2} e^{-x} = \int_0^\infty \frac{\cos \lambda x}{a^2+\lambda^2}\, d\lambda$, put a = 1 in above problem(4)

Solution : Let f(x) = e^{-x}

Problem 6): Using Fourier Integral , show that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\, d\lambda \quad (a,b > 0)$$

Solution : Let f(x) = e^{-ax}

The Fourier Sine integral is given by f(x) = $\frac{2}{\pi}\int_0^\infty \sin \lambda x \,[\int_0^\infty f(t) \sin \lambda t \, dt]\, d\lambda$

f(x) = $\frac{2}{\pi}\int_0^\infty \sin \lambda x \;[\int_0^\infty e^{-at} \sin \lambda t \, dt]\, d\lambda$ ----(1)

$\int_0^\infty e^{-at} \sin \lambda t \, dt = [\frac{e^{-at}}{a^2+\lambda^2}($-a sin λt - λ cos λt)(0 to ∞)]

$$= 0 - \frac{1}{a^2+\lambda^2}(-\lambda) = \frac{\lambda}{a^2+\lambda^2}$$

sub in (1)

$$(1) \Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \sin \lambda x \,.\, \frac{\lambda}{a^2+\lambda^2}\, d\lambda$$

$$\Rightarrow e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{\lambda^2+a^2}\, d\lambda \text{ -----------(2)}$$

similarly ,

$$e^{-bx} = \frac{2}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{\lambda^2+b^2}\, d\lambda \text{ --------(3)}$$

$$(2) - (3) = e^{-ax} - e^{-bx} = \frac{2}{\pi}\int_0^\infty \lambda \sin \lambda x \left(\frac{1}{\lambda^2+a^2} - \frac{1}{\lambda^2+b^2}\right) d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \lambda \sin \lambda x\left[\frac{b^2-a^2}{(\lambda^2+a^2)(\lambda^2+b^2)}\right] d\lambda$$

$$= \frac{2}{\pi}(b^2 - a^2)\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\, d\lambda$$

There fore ,

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\, d\lambda$$

FOURIER TRANSFORMATION:

Definition : 1)The fourier transform of f(x) , $-\infty < x < \infty$ is denoted by f(s) or F{f(x)} and is defined as ,

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx = f(s) \text{ -------(1)}$$

The inverse fourier transform is given by

$$f(x) = F^{-1}\{f(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds \text{ -----(2)}$$

F{f(x)} = f(s)

Note 2): Some authors also defined as

F{f(x)} = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}$ f(x) dx

and inverse fourier transform as f(x) = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}$ f(s) ds

Def : 3) : F{f(x)} = $\int_{-\infty}^{\infty} e^{-isx}$ f(x) dx and

Inverse Fourier Transform as f(x) = $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}$ f(s) ds

Def: ***Fourier Sine Transform***:-

The Fourier Sine Transform of f(x), 0 < x < ∞ is denoted by fs(s) or Fs{f(x)} and defined by

Fs{f(x)} = $\int_{0}^{\infty} f(x)$ sin sx dx = fs(s) -----(3)

Fs{f(x)} = $\int_{0}^{\infty} f(x)$ sin sx dx = fs(s) -----(3) The inverse Fourier Sine Transform is given by

f(x) = $\frac{2}{\pi}\int_{0}^{\infty} fs(s)$ sin sx ds -------(4)

Note : Some authors also defined as

Fs{f(x)} = $\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)$ sin sx dx = fs(s) and inverse fourier sine transform as f(x) = $\sqrt{\frac{2}{\pi}} \int_0^{\infty} fs(s) \sin sx \, ds$

Def : ***Fourier Cosine Transform :-***

The Fourier Cosine Transform of f(x) , 0 < x < ∞ is denoted by fc(s) or Fc{f(x)} and defined by

Fc{f(x)} = $\int_0^{\infty} f(x) \cos sx \, dx$ = fc(s) -----(5) and

The inverse Fourier Cosine Transform is given by,

f(x) = $\frac{2}{\pi} \int_0^{\infty} fc(s) \cos sx \, ds$ -------(6)

Note : Some authors also defined as

Fc{f(x)} = $\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$

and inverse fourier cosine transform as f(x) = $\sqrt{\frac{2}{\pi}} \int_0^{\infty} fc(s) \cos sx \, ds$

Linear Property: If f(s) , g(s) are Fourier Transform of f(x) & g(x) then

$F\{c_1 f(x) + c_2 g(x)\} = c_1 F\{f(x)\} + c_2 F\{g(x)\}$

$= c_1 f(S) + c_2 g(s)$

Proof:- The definition of Fourier Transform is

F{f(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx = f(s) -----(1)

By definition F{c_1 f(x) + c_2 g(x)} = $\int_{-\infty}^{\infty} e^{isx}$ [c_1 f(x) + c_2 g(x)] dx

= $c_1 \int_{-\infty}^{\infty} e^{isx}$ f(x) dx + $c_2 \int_{-\infty}^{\infty} e^{isx}$ g(x) dx

= c_1 f(s) + c_2 g(s) by (1) ***Note:-***

Linear Property:

(I) Fs{c_1 f(x) + c_2 g(x)} = c_1 fs(s) + c_2 gs(s)

(II) Fc{c_1 f(x) + c_2 g(x)} = c_1 fc(s) + c_2 gc(s)

Proof:- (I) The definition of Fourier Sine Transform is

Fs{f(x)} = $\int_0^{\infty} f(x)$ sin sx dx = fs(s) -----(1)

By the definition , Fs{c_1 f(x) + c_2 g(x)} = $\int_0^{\infty}$ [c_1 f(x) + c_2 g(x)] sin sx dx

= $c_1 \int_0^{\infty} f(x)$ sin sx dx + $c_2 \int_0^{\infty} g(x)$ sin sx dx

= c_1 fs(s) + c_2 gs(s) by (1) ***Change***

of scale property:

Statement : If F{f(X)} = f(s) then F{f(ax)} = $\frac{1}{a}$ f($\frac{s}{a}$)

Proof :- The definition of Fourier Transform of f(x) is

F{f(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx = f(s) --------(1)

By definition F{f(ax)} $= \int_{-\infty}^{\infty} e^{isx}$ f(ax) dx let ax =t x = t/a

$= \int_{-\infty}^{\infty} e^{is\frac{t}{a}}$ 1 1

f(t) dt dx = dt

$= \frac{1}{a} \int_{-\infty}^{\infty} e^{i\left(\frac{s}{a}\right)t}$ f(t) dt

a a

$= \frac{1}{a} \int_{-\infty}^{\infty} e^{i\left(\frac{s}{a}\right)x}$

$= \frac{1}{a}$ f($\frac{s}{a}$)

f(x) dx (by property of def. integral)

Note : 1) If Fs{f(x)} = fs(s) then Fs {f(ax)} = $\frac{1}{a}$ fs($\frac{s}{a}$)

2) If Fc{f(x)} = fc(s) then Fc{f(ax)} = $\frac{1}{a}$ fc($\frac{s}{a}$)

Proof: (I) The definition of Fourier Sine Transform is

Fs{f(x)} $\int_0^{\infty} f(x)$ sin sx dx = fs(s) -----(1)

= {f(ax)} = $\int_0^{\infty} f(ax)$ sin

By definition Fssx dx let ax =t

$= \int_0^{\infty} f(t)$ sin s($\frac{t}{a}$). $\frac{1}{a}$ dt dx = dt

1

)t. dt $= \frac{1}{a}\int_0^{\infty} f(t)\sin(\frac{s}{a}$

$= \frac{1}{a}\int_0^{\infty} f(x)\sin(\frac{s}{a}$)x. dx =

$\frac{1}{a}$ fs($\frac{s}{a}$) by (1)

Shifting Property:-

If F{f(x)} = f(s) then F{f(x-a)} = $e^{isa} f(s)$

Proof : F{f(x)} $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx =f(s)----(1)

= $\int_{-\infty}^{\infty} e^{isx}$ f(x

By definition $= \int_{-\infty}^{\infty} e^{is(t+a)}$ F{f(x-a)} = -a) dx let

x-a=t f(t) dt $= \int_{-\infty}^{\infty} e^{ist}\, e^{isa}$ x=t+a

$= e^{isa} \int_{-\infty}^{\infty} e^{isx}$ f(t) dt dx= dt

f(x) dx

= e^{isa} f(s) by (1)

Modulation Theorem :-

If F{f(x)} = f(s) then F{f(x)cos ax} = $\frac{1}{2}$ {f(s-a) + f(s+a)}

$= \frac{1}{2}[\int^{\infty} e^{i(s+a)x}$ $\int^{\infty} e^{i(s-a)x}$

Proof: The defination of Fourier Transform is $\cos ax\} = \int_{-\infty}^{\infty} e^{isx}$ f(x) cos ax dx F{f(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx =f(s)----(1) By $= \int_{-\infty}^{\infty} e^{isx} \frac{e^{iax}+e^{-iax}}{2}$ definition F{f(x)

f(x) dx

f(x) dx + f(x) dx

$= \frac{1}{2}$ {f(s-a) + f(s+a)}

Note: If Fs(s) & Fc(s) are Fourier Sine & Cosine Transform of f(x) respectively

Then (i) Fs{f(x) cos ax} = $\frac{1}{2}$ {Fs(s+a) + Fs(s -a)}

(ii) Fs{f(x) sin ax} $\frac{1}{2}$ {Fs(s+a) - Fs(s = -a)}

(iii) Fs{f(x) sin ax} $\frac{1}{2}$ {Fc(s+a) - Fc(s = -a)}

Proof: The definition of Fourier Sine Transform of f(x) is

Fs{f(x)} = $\int_0^{\infty} f(x)$ sin sx dx = fs(s) -----(1)

By definition Fs{f(x) cos ax} = $\int_0^{\infty} f(x)$ cos ax sin sx dx

sx. Cos ax) dx

$= \int_0^\infty f(x) \cdot \frac{1}{2} .(2.\sin$ sin (s-a)x dx]

$= \frac{1}{2}$ f(x) $\int_0^\infty [\sin(sx + ax)$ + sin(sx-ax)]dx

$= \frac{1}{2} [\int_0^\infty f(x)$ sin (s+a)x dx + $\int_0^\infty f(x)$

$= \frac{1}{2}$ [Fs(s+a) + Fs(s-a)]

Similarly we get (ii) & (iii) Problems:

1) Find Fourier Transform of f(x) = e^{ikx}, a < x < b

0 , x <a , x > b

Solution : By definition , F{f(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx -∞ ∞

$= \int_a^b e^{isx} e^{ikx}$ dx

$= \int_a^b e^{i(s+k)x}$ dx

$= [\frac{e^{i(s+k)x}}{i(s+k)}]$ (apply limits a to b)

$= \frac{e^{i(s+k)b} - e^{i(s+k)a}}{i(s+k)}$

2) Find , F{f(x)} if f(x) = x, |x| < a

0 , |x| > a |x| < a means –a < x < a

Solution : By definition , F{f(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx

= $\int_{-a}^{a} e^{isx}$ x dx

use = $\int_{-a}^{a} x.e^{isx}$ dx , integration by parts ,

dx $\int udv =$ = $(\frac{xe^{isx}}{is})$ - $\frac{1}{is}\int_{-a}^{a} e^{isx}$ $uv - \int vdu$

(apply –a to a) u=x, dv= e^{isx}dx

= $\frac{1}{is}$(a. e^{ias} + a. e^{-ias}) - $\frac{1}{is}$ $(\frac{e^{isx}}{is}$

= $\frac{2a\cos as}{is} + \frac{1}{s^2}$ $(e^{ias} - e^{-ias})$ = $\frac{e^{isx}}{is}$

= $\frac{-2ia\cos as}{s} + \frac{2i\sin as}{s^2}$) (apply –a to a) du=dx, v= $\int$. e^{isx} dx

3) If f(x) = 1 , |x| < a

0 , |x| > a , Find Fourier Transform of f(x)

Deduce that $\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds$ (ii) $\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds$

(i) $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx

Solution : F{f(X)} = |x| < a means –a < x < a

= $\int_{-a} e^{isx}$ .1. dx

= $\frac{e^{isx}}{is}$ (-a to a)

$= \frac{1}{is}(e^{ias} - e^{-ias})$

$= \frac{1}{is}$

$2 \sin as$(2i sin as)

f(s) = F{f(x)} = f(s)

s

Deduction :

Inverse Fourier Transform is defined by f(x) = $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}$ f(s) ds

$\frac{1}{2\pi}\int_{-\infty}^{\infty}(\cos sx - i \sin sx)\ \frac{2\sin as}{s}$ ds f(x) =

$= \frac{2}{2\pi}[\int_{-\infty}^{\infty}(\cos sx)\ \frac{\sin as}{s}$ ds – i $\int_{-\infty}^{\infty}(\sin sx)\ \frac{\sin as}{s}$

ds]

(even) (odd)

$\Rightarrow f(x) = \frac{1}{\pi}[2\int^{\infty}(\cos sx)\ \frac{\sin as}{s}$ ds - 0]

(i) $\int_0^{\infty} \frac{\sin as \cos sx}{s}\, ds = \frac{\pi}{2} . f_{(x)}$

$=. \frac{\pi}{2}$ 1, |x| < a

0, |x| > a

(ii) Put a = 1 , x = 0 in (i) we get

$$∫_0^\infty \frac{\sin s}{s}\, ds = \frac{\pi}{2}.1$$

$$\Rightarrow \quad ∫_0^\infty \frac{\sin s}{s}\, ds = \frac{\pi}{2}$$

4) Find Fourier Transform of f(x) = 1 - x^2 , |x| ≤ 1

0, |x| > 1

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx$$ Evaluate

Solution:- $F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$

$= \int_{-1}^{1} e^{isx}(1 - x^2)\, dx$

$= \int_{-1}^{1} (1 - x^2) e^{isx}\, dx$ $\qquad$ $∫ udv = uv - ∫ vdu$

$=[\{(1 - x^2).\frac{e^{isx}}{is}\} - \int_{-1}^{1} \frac{e^{isx}}{is}(-2x)dx]$

(limits -1 to 1) $\qquad$ u= $(1 - x^2)$ dv= e^{isx}dx

$0 + \frac{2}{is}\int_{-1}^{1} x.e^{isx}$ $\qquad$ = [0 – dx $\qquad$ du =-2x

dx, v= ∫. e^{isx} dx

$= \frac{e^{isx}}{is}$

$\int_{-1}^{1} \frac{e^{isx}}{is}$ dx]

$= \frac{2}{is}[(\frac{xe^{isx}}{is})$ (-1 to 1) -

= $is2$ [1.($e\ is + ise\ \ -is$) - $is1\ e\ isisx$] (- 1 to 1) is - 2i sin s)

= $\frac{4}{s^3}$

[sin s – s = $is2$ [$2\cos is\ s$ - $is1$ ($e\ is - ise\ -is$)] cos s] = f(s)

Deduction : = $is2$. $is1$ (2 cos s 1 Inverse

$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}$

Fourier = - $s2^2$. 2[cos s - $\sin s.\ s$]

Transform is defined by f(x) = f(s) ds

. $\frac{4}{s^3}$ [sin s – s cos s] ds

= $\frac{1}{2\pi}.4\int_{-\infty}^{\infty}(\cos sx\ -\ i\sin sx)\frac{(\sin s - s\cos s)}{s^3}$ ds = $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}$

= $\frac{2}{\pi}$ [$\int_{-\infty}^{\infty}\cos sx\ \frac{(\sin s - s\cos s)}{s^3}$ ds - i $\int_{-\infty}^{\infty}\sin sx\ \frac{(\sin s - s\cos s)}{s^3}$ ds]

(even function) (odd function)

f(x) = $\frac{2}{\pi}$ [$\int_{-\infty}^{\infty}\cos sx\ \frac{(\sin s - s\cos s)}{s^3}$ ds - 0

$\Rightarrow \int_{-\infty}^{\infty}\cos sx\ \frac{(\sin s - s\cos s)}{s^3}$ π ds = f(x)

2

$$= \frac{\pi}{2}(1 - x^2), \ |x| \leq 1$$

$$0, \ |x| > 1$$

At $x = \frac{1}{2}$, $\Rightarrow \int_{-\infty}^{\infty} \cos\frac{s}{2} \frac{(\sin s - s\cos s)}{s^3} ds = \frac{\pi}{2}(1 - \frac{1}{4})$ put

s = x

$$\Rightarrow \int_{-\infty}^{\infty} \cos\frac{x}{2} \frac{(\sin x - x\cos x)}{x^3} dx = \frac{\pi}{2}(1 - \frac{1}{4}) = \frac{3\pi}{8}$$

$$\Rightarrow 2\int_{0}^{\infty} \cos\frac{x}{2} \frac{(\sin x - x\cos x)}{x^3} dx = \frac{3\pi}{8}$$

$$\int_{0}^{\infty} \cos\frac{x}{2} \left[\frac{(x\cos x - \sin x)}{x^3}\right] dx = -\frac{3\pi}{16}$$

5) Find Fourier Transform of $f(x) = \frac{1}{2a}$ if $|x| \leq a$

0, if $|x| > a$

Solution : By definition,

$$F\{f(x)\} = f(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$= \int_{-\infty}^{-a} e^{isx} f(x)dx + \int_{-a}^{a} e^{isx} f(x)dx + \int_{a}^{\infty} e^{isx} f(x)dx$$

$$= \int_{-a}^{a} \frac{1}{2a} e^{isx}\,dx = \frac{1}{2a}\frac{e^{isx}}{is} \quad (apply\ limits) = \frac{1}{2a}\frac{(e^{isa}-e^{-isa})}{is}$$

$$= \frac{\sin as}{ias}$$

6) Find Fourier Transform of f(x) = sin x , if $0 < x < \pi$

0 , otherwise

Solution : By definition,

$$F\{f(x)\} = f(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$= \int_{-\infty}^{0} e^{isx} f(x)dx + \int_{0}^{\pi} e^{isx} f(x)dx + \int_{\pi}^{\infty} e^{isx} f(x)dx$$

$$= \int_{0}^{\pi} e^{isx} \sin x\,dx$$

$$= \frac{e^{isx}}{(is)^2+1^2}[is \sin x - 1.\cos x] \quad \text{apply 0 to } \pi$$

$$= \frac{1}{1-s^2}[e^{is\pi}(0-\cos\pi) - e^{0}(0-1)]$$

$$= \frac{1}{1-s^2}[e^{is\pi}(1) - 1(0-1)]$$

$$= \frac{e^{is\pi}+1}{1-s^2}$$

7) Find Fourier Transform of f(x) = xe^{-x} , $0 < x < \infty$

Solution : By definition,

F{f(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx f(s) =

$= \int_0^{\infty} e^{isx}\, xe^{-x}\, dx$

$= \int_0^{\infty} x\, e^{(is-1)x}\, dx$

$= \left[\frac{x\, e^{(is-1)x}}{is-1} - 1.\frac{e^{(is-1)x}}{(is-1)^2}\right]$ (0 to ∞)

$= \left[\frac{x\{e^{isx} - e^{-x}\}}{is-1}\right]$(0 to ∞) $- \frac{1}{(is-1)^2}(e^{isx} - e^{-x})$

$= [(0-0) - \frac{1}{(is-1)^2}(0-1)$

$= \frac{1}{(is-1)^2}$

$= \frac{1}{(is-1)^2} \cdot \frac{(is+1)^2}{(is+1)^2}$

$= \frac{(1+is)^2}{(1+s)^2}$

$-x^2$ $-x^2$

8) Find Fourier Transform of $e^{\frac{-x^2}{2}}$. Show that $e^{\frac{-x^2}{2}}$ is reciprocal Solution : By definition,

$$F\{f(x)\} = f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} e^{\frac{-x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x^2-2isx)} dx \qquad (x-is)^2/\sqrt{2} = y^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}[(x-is)^2+s^2} dx \qquad x-is = \sqrt{2}y$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-is)^2} dx \qquad dx = \sqrt{2}dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{2}$$

$$= \frac{1}{\sqrt{\pi}} e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{\frac{-s^2}{2}} 2\int_{0}^{\infty} e^{-y^2}$$

$$= e^{\frac{-s^2}{2}} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}$$

$$= e^{\frac{-s^2}{2}} = f(s)$$

dy dy

Therefore Function is self reciprocal

9) Find the inverse Fourier Transform of f(x) of f(s) = $e^{-|s|y}$

Solution : We have |s| = -s , if s < 0

s , if s > 0

From inverse Fourier Transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-isx} f(s)\, ds + \int_{0}^{\infty} e^{-isx} f(s)\, ds\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-isx}\, e^{sy}\, ds + \int_{0}^{\infty} e^{-isx}\, e^{-sy}\, ds\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{(y-ix)s}\, ds + \frac{1}{2\pi}\int_{0}^{\infty} e^{-(y+ix)s}\, ds$$

$$= \frac{1}{2\pi}\left[\frac{e^{(y-ix)s}}{y-ix}\right](-\infty\ to\ 0) + \frac{1}{2\pi}\left[\frac{e^{-(y-ix)s}}{-(y+ix)}\right](0\ to\ \infty)$$

$$= \frac{1}{2\pi}\left[\frac{1}{y-ix}\right] + \frac{1}{2\pi}\left[\frac{1}{y+ix}\right]$$

$$= \frac{1}{2\pi}\left[\frac{y+ix+y-ix}{(y-ix)(y+ix)}\right] = \frac{1}{2\pi}\frac{2y}{y^2-i^2x^2}$$

$$= \frac{1}{\pi}\frac{y}{y^2+x^2}.$$

Problems on sine and cosine Transform:-

1) Find Fourier cosine Transform of f(x) defined by f(x) = cos x , 0 < x < a

= 0 , x > a

Solution : Fc{f(x)} = $\int_0^\infty f(x)\cos$ sx dx

= $\int_0^a \cos x \cos$ sx dx = $\frac{1}{2}\int_0^a 2\cos x \cos$ sx dx

= $\frac{1}{2}\int_0^a [\cos(x + sx)$

= $\frac{1}{2}[\int_0^a \cos(1+s)x$ dx + $\int_0^a \cos(1-s)x$

= $\frac{1}{2}[\frac{\sin(1+s)x}{1+s} + \frac{\sin(1-s)x}{1-s}]$ (apply 0 to a)

= $\frac{1}{2}[\frac{\sin(1+s)a}{1+s} + \frac{\sin(1-s)a}{1-s}]$

2cosAcosB=cos(A+B)+cos(A-B)

+ cos (x-sx)] dx

A=x, B=sx

dx]

2) Find Fourier cosine Transform of f(x) defined by f(x) = x , $0 < x < 1$

2-x , $1 < x < 2$

0 , $x > 2$

Solution : Fc{f(x)} = $\int_0^\infty f(x) \cos sx\, dx$

$= \int_0^1 f(x) \cos sx\, dx + \int_1^2 f(x)\cos sx\, dx + \int_2^\infty f(x)\cos sx\, dx$

$= \int_0^1 x \cos sx\, dx + \int_1^2 (2-x)\cos sx\, dx + 0$

$= [\, x \frac{\sin sx}{s} - 1 \, (-\frac{\cos sx}{s^2})]$ (apply 0 to 1) + $[(2-x)\frac{\sin sx}{s} - (-1)\,(-\frac{\cos sx}{s^2})]$

$= (\frac{\sin s}{s} + \frac{\cos s}{s^2} - 0 - \frac{1}{s^2}) + (0 - \frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2})$

$= \frac{2\cos s - \cos 2s - 1}{s^2}$

$= \frac{2\cos s - (2\cos^2 s - 1) - 1}{s^2}$

$= \frac{1}{s^2}(2\cos s - 2\cos^2 s)$

$= \frac{2}{s^2}\cos s(1 - \cos s)$

)] (1to 2)

3)Find Fourier sine & cosine Transform of $2e^{-5x} + 5e^{-2x}$

Solution : Given f(x) = $2e^{-5x} + 5e^{-2x}$

$$Fs\{f(x)\} = \int_0^\infty f(x)\ \sin_{sx}\ dx$$

$$= \int_0^\infty (2e^{-5x} + 5e^{-2x})\ \sin$$

$$= [2\int_0^\infty e^{-5x}\ \sin_{sx\ dx} + 5\int_0^\infty e^{-2x}\ \sin_{sx\ dx}$$

$$=[\ 2\ \{\frac{e^{-5x}}{25+s^2}\ (-_{5\ \sin sx - s\cos sx})\}\ (\text{apply } 0 \text{ to } \infty)\}\ sx$$

dx

$+ \quad 5 \{\frac{e^{-2x}}{4+s^2}$ (- 2 sin sx – s cos sx)} (apply 0 to ∞)}

0)} + 5 {0 $\frac{e^0}{-4+s^2}$(-s)}]

= [2 { 0 - $\frac{e^0}{25+s^2}$ (0 – s cos

$= [\frac{2s}{25+s^2} + \frac{5s}{4+s^2}]$

Similarly

(ii) Fc{f(x)} = [$\frac{10}{s^2+25} + \frac{10}{s^2+4}$]

4) Find Fourier cosine Transform of (i) e^{-ax} cos ax , (ii) e^{-ax} sin ax Solution

: Given f(x) = e^{-ax} cos ax (i)

$\int_0^{\infty} f(x)$ cos sx dx

Fc{f(x)}

= $\int_0^{\infty} e^{-ax}$ cos ax cos sx dx

=

= $\frac{1}{2}\int_0^{\infty} e^{-ax}$ 2 cos ax cos sx dx

= $\frac{1}{2}$ [$\int_0^{\infty} e^{-ax}$ cos (a+s)x dx + $\int_0^{\infty} e^{-ax}$ cos (a−s)

= $\frac{1}{2}$.$\frac{e^{-ax}}{a^2+(a+s)^2}$ {-a cos (a+s)x + (a+s) sin (a+s

x dx]

+ $\frac{e^{-ax}}{a^2+(a-s)^2}$ {-a cos (a

)x} (apply

-s)x + (a-s) sin (a-s)x} (apply 0 to ∞)

0 to ∞) = $\frac{1}{2}$[{0 - $\frac{e^0}{a^2+(a+s)^2}$ (− a cos

= $\frac{1}{2}$ [$\frac{a}{a^2+(a+s)^2}$ + $\frac{a}{a^2+(a-s)^2}$] 0)} + {0 - $\frac{e^0}{a^2+(a-s)^2}$ (− a cos

0)}]

(ii) Similarly Fs{f(x)} = Fs{(e^{-ax} sin ax) = $\frac{1}{2}$ [$\frac{a}{a^2+(s-a)^2}$ - $\frac{a}{a^2+(a+s)^2}$]

5) Find Fourier cosine & sine Transform of e^{-ax}, a > 0 hence

deduce (i) $\int_0^\infty \frac{\cos sx}{a^2+s^2}$ ds (ii) $\int_0^\infty \frac{s \sin sx}{a^2+s^2}$ ds

Solution : Let f(x) = e^{-ax}

Fc{f(x)} = $\int_0^\infty f(x) \cos$ sx dx

= $\int_0^\infty e^{-ax} \cos$ sx dx

= [$\frac{e^{-ax}}{a^2+s^2}$ (-a cos sx + s sin sx)] (apply 0 to ∞)

= [0 - $\frac{e^0}{a^2+s^2}$ (– a + 0)] = $\frac{a}{a^2+s^2}$= Fc(s)-----------(1)

Fs{f(x)} = $\int_0^\infty f(x)$ sin sx dx

= $\int_0^\infty e^{-ax}$ sin sx dx

= [$\frac{e^{-ax}}{a^2+s^2}$ (-a sin sx - s cos sx)] (apply 0 to ∞) $\frac{s}{a^2+s^2}$ -------(2)

Fs{f(x)} =

By Inverse cosine Transform

f(x) = $\frac{2}{\pi} \int_0^\infty fc(s) \cos$ sx ds

= $\frac{2}{\pi} \int_0^\infty \frac{a}{a^2+s^2} \cos$ sx ds

$\Rightarrow \int_0^\infty \frac{1}{a^2+s^2} \cos sx \, ds = \frac{e^{-ax}}{a} \cdot \frac{\pi}{2}$

By inverse sine Transform ,

$f(x) = \frac{2}{\pi} \int_0^\infty fs(s) \sin sx \, ds$

$= \frac{2}{\pi} \int_0^\infty \frac{s}{a^2+s^2} \sin sx \, ds$

$\Rightarrow \int_0^\infty \frac{s}{a^2+s^2} \sin sx \, ds = \frac{\pi}{2} \cdot e^{-ax}$

6) Find Fourier sine Transform of $f(x) = \frac{1}{x}$

Solution : $Fs\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx$

$= \int_0^\infty \frac{\sin sx}{x} dx$ ----(1)

$= \frac{\pi}{2}$

7) Find Fourier sine Transform of $\frac{e^{-ax}}{x}$, hence deduce that

Solution : Fs{f(x)} = $\int_0^\infty f(x)$ $\sin_{sx}$ dx

= $\int_0^\infty \frac{e^{-ax}}{x}$ sin sx dx = I ---(1)

$\frac{dt}{ds} = \int_0^\infty \frac{e^{-ax}}{x}$ .x.cos sx dx

= $\int_0^\infty e^{-ax}$ cos sx dx

= [$\frac{e^{-ax}}{a^2+s^2}$ (-a cos sx

\+ s sin sx)] (apply 0

to = [0 - $\frac{e^0}{a^2+s^2}$ ∞)

$\Rightarrow \frac{dt}{ds} = \frac{a}{a^2+s^2}$ (− a + 0)]

Integrate on both sides w.r.t. s we get

I= a $\int \frac{1}{a^2+s^2}$ ds = a .$\frac{1}{a}.Tan^{-1}\frac{s}{a}$+ c

= $Tan^{-1}(\frac{s}{a})$ + c ------(2)

put s = 0 on both sides we get {in (1) & (2)}

0 = Tan^{-1}(0) + c $\Rightarrow 0 = 0 + c \Rightarrow$ c=0

$I = Tan^{-1}(\frac{s}{a})$ = Fs{f(x)}

8)Find Fourier cosine Transform of $\frac{1}{1^2+x^2}$, and

(ii) Fourier sine Transform of $\frac{x}{1^2+x^2}$

Solution : Let f(x) = $\frac{1}{1^2+x^2}$, We will find Fc{f(x)} = Fc{$\frac{1}{1^2+x^2}$}

Fc{f(x)}

$= \int_0^\infty f(x)\cos$ sx dx

$= \int_0^\infty \frac{1}{1^2+x^2}\cos$ sx dx = I -------(1)

Differentiate on both sides w.r.t s

$\frac{dI}{ds} = \int_0^\infty - \frac{x \sin sx}{1+x^2}$ dx ----(2)

$= - \int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)}$ dx

$= - \int_0^\infty \frac{(1+x^2-1) \sin sx}{x(1+x^2)}$ dx

$= -[\int_0^\infty \frac{\sin sx}{s}$ dx - $\int_0^\infty \frac{\sin sx}{x(1+x^2)}$

$\frac{dI}{ds} = - \frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}$ dx ----(3) dx Diff

on both sides w.r.t 's'

We get $\frac{d^2 I}{ds^2} = \int_0^\infty \frac{x \ \cos sx}{x(1+x^2)}$ dx

$\Rightarrow \frac{d^2I}{ds^2} = I$ by (1) $\Rightarrow \frac{d^2I}{ds^2} - I = 0$

$\Rightarrow (D^2-1)I = 0$ This is D.E

A.E. is $m^2-1 = 0$

$m = \pm 1$

solution is $I = c_1e^s + c_2\, e^{-s}$ ----- (4)

$\frac{dI}{ds} = c_1e^s - c_2\, e^{-s}$ ------(5)

From (1) & (4) , $c_1e^s + c_2\, e^{-s} = \int_0^\infty \frac{1}{1+x^2} . \cos sx\, dx$

Put s = 0 on both sides

$\Rightarrow c_1 + c_2 = \int_0^\infty \frac{1}{1+x^2}$ dx

$= (tan^{-1})(0\ to\ \infty)$ $= tan^{-1} \infty - tan^{-1} 0$

$= \frac{\pi}{2} - 0$

there fore , $c_1 + c_2 = \frac{\pi}{2}$ ------(6)

From (3) & (5) , $c_1 e^s - c_2\, e^{-s} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}$ dx

$\Rightarrow c_1 - c_2 = -\frac{\pi}{2}$ -----(7)

solve (6) & (7) we get $c_1 = 0\, ,\, c_2 = \frac{\pi}{2}$ sub in (4)

(4) $\Rightarrow\ \mathrm{I} = \frac{\pi}{2} . e^{-s}$

i.e., Fc {f(x)} = Fc$\{\frac{1}{1+x^2}\} = \frac{\pi}{2} . e^{-s}$

Now I = $\frac{\pi}{2} . e^{-s}$

$\frac{dI}{ds} = -\frac{\pi}{2} . e^{-s}$ ----------(8)

From (2) & (8) , we have

$-\int_0^\infty \frac{x\ \sin sx}{1+x^2}$ dx = $-\frac{\pi}{2} . e^{-s}$

$\Rightarrow \int_0^\infty \left(\frac{x}{1+x^2}\right) \sin sx \, dx = \frac{\pi}{2} . e^{-s}$

There fore $F_s\{\frac{x}{1+x^2}\} = \frac{\pi}{2} . e^{-s}$

9) Find the Inverse Fourier Cosine Transform of f(x) of fc(s) = $\frac{1}{2a}$ (a - $\frac{s}{2}$) , s < 2a

0 , s $\geq$ 2a

Solution : From the inverse Fourier Cosine Transform , we have

f(X) = $\frac{2}{\pi} \int_0^\infty fc(x) \cos sx \, ds$

= $\frac{2}{\pi}$ [$\int_0^{2a} fc(x) \cos sx \, ds$ + $\int_{2a}^\infty fc(x) \cos sx \, ds$]

= $\frac{2}{\pi} \frac{1}{2a} \int_0^{2a}$ (a – $\frac{s}{2}$) cos sx ds

= $\frac{1}{\pi a}$ [{(a - $\frac{s}{2}$) . $\frac{\sin sx}{x}$ }(0 to 2a) -$\int_0^{2a} \frac{\sin sx}{x}$ (- $\frac{1}{2}$) ds]

= $\frac{1}{\pi a}$ [(0-0) + $\frac{1}{2}$. $\frac{1}{x^2}$ (- cos sx)(0 to 2a)]

= $\frac{1}{2\pi a x^2}$ (- cos 2ax + cos 0)

= $\frac{1-\cos 2ax}{2\pi a x^2} = \frac{\sin^2 ax}{\pi a x^2}$

10) Find f(x) if its Fourier Sine Transform is e^{-as}

Solution : Given f(s) = e^{-as}

By definition of inverse sine transform

f(x) = $\frac{2}{\pi}\int_0^\infty fs(x)$ sin sx ds

= $\frac{2}{\pi}\int_0^\infty e^{-as}$ sin sx ds

= $\frac{2}{\pi}$ [$\frac{e^{-as}}{a^2+x^2}$ (-a sin sx – x cos sx)(0 to ∞)

= $\frac{2}{\pi}$ [0 - $\frac{1}{a^2+x^2}$ (-x)]

= $\frac{2x}{\pi(a^2+x^2)}$

11) Find the Inverse Fourier Sine Transform f(x) of Fs(s) = $\frac{s}{1+s^2}$

(or)

Find f(x) if its Fourier sine Transform is $\frac{s}{1+s^2}$

Solution : By Fourier Inverse sine Transform f(x) = f(x) = $\frac{2}{\pi}\int_0^\infty fs(x)$ sin sx ds = I f(x) = $\frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}$ sin sx ds = I -------(1)

= $\frac{2}{\pi}\int_0^\infty (\frac{1}{s} - \frac{1}{s(s^2+1)})$ sin sx as

= $\frac{2}{\pi}$ [$\int_0^\infty \frac{\sin sx}{s}\, ds$ - $\int_0^\infty \frac{\sin sx}{s(s^2+1)}\, ds$

= $\frac{2}{\pi}$ [$\frac{\pi}{2}$ - $\int_0^\infty \frac{\sin sx}{s(s^2+1)}\, ds$]

f(x) = 1 - $\frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(s^2+1)}\, ds$ = I --------(2)

diff on both sides w.r.t. X

We get $\frac{dI}{dx}$ = - $\frac{2}{\pi}\int_0^\infty \frac{s\cos sx}{s(s^2+1)}\, ds$ ---------(3)

Diff w.r.t. x

$\frac{d^2I}{dx^2}$ = - $\frac{2}{\pi}\int_0^\infty -s\,\frac{\cos sx}{(s^2+1)}\, ds$

= $\frac{2}{\pi}\int_0^\infty s\,\frac{\cos sx}{(s^2+1)}\, ds$

$\frac{d^2I}{dx^2}$= I from (1) $\Rightarrow$ $(D^2 - 1)I = 0$ ------(4) is D.E.

Solution of (4) is I = $c_1e^x + c_2\, e^{-x}$-----------(5)

$\frac{dI}{dx}$ = c_1e^x - $c_2\, e^{-x}$------------(6)

From (2) & (5)

If x = 0 ,I = 1 ,

$\Rightarrow c_1 + c_2 = 1$ (5) From

Substitute in (5)

(3) & (6)

(5) $\Rightarrow f(x) =$ $\Rightarrow \frac{dI}{dx} = -\frac{2}{\pi}\int_0^\infty \frac{1}{1+s^2}\,ds$ $I = 0 +$

$c_2 e^{-x}$

If x = 0 , (3)

$\Rightarrow f(x) =$ e^{-x}

if x = 0 , (6) $\Rightarrow c_1 - c_2 = -\frac{2}{\pi}\,(tan^{-1} s)(0 \text{ to } \infty)$

$= -\frac{2}{\pi}\frac{\pi}{2} = -1$

Now solve $c_1 + c_2 = 1$ &

$c_1 - c_2 = -1$ we get $c_1 = 0$ & $c_2 = 1$

Convolution : The convolution of two functions f(x) & g(x) over the interval

$(-\infty, \infty)$ *is defined as* $f*g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, g(x-u)\, du$

CONVOLUTION THEOREM: If F{f(x)} and F{g(x)} are Fourier Transform of functions f(x) and g(x) , then

$$F\{f(x) * g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \{f(x) * g(x)\}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u)du\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} g(x-u)dx\right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is(u+y)} g(y)dy\right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isu} f(u)du \, . \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isy} g(y)dy$$

$$= F\{f(x)\} * F\{g(x)\}$$

Relation between Fourier and Laplace Transform:

Statement: If $f(t) = \begin{cases} e^{-xt} g(t), & t > 0 \\ 0, & t < 0 \end{cases}$ then F{f(t)} = L{g(t)}

Proof : $F\{f(t)\} = \int_{-\infty}^{\infty} e^{ist}\, f(t)\, dt$

$$= \int_{-\infty}^{0} e^{ist}\ f(t)\,dt + \int_{0}^{\infty} e^{ist}\ f(t)\,dt$$

$$= 0 + \int_{0}^{\infty} e^{ist}\ e^{-xt}\ g(t)\,dt$$

$$= \int_{0}^{\infty} e^{-(x-is)t}\ g(t)\,dt$$

$$= \int_{0}^{\infty} e^{\rho t}\ g(t)\,dt$$

$$= L\{g(t)\}$$

Fourier Transform of derivatives of a function:

Statement : If $F\{(f(x)\} = f(s)$ then $F\{f^{n}(x)\} = (-is)^{n} f(s)$, *if* the 1^{st} (n-1) derivatives of f(x) vanish identically as $x \to \pm\infty$

Proof : By definition $F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx}\ f(x)\,dx$ -------(1)

$$F\{f'(x)\} = F\{\frac{d}{dx} f(x)\}$$

$$= \int_{-\infty}^{\infty} e^{isx}\ f'(x)\,dx$$

$$= [e^{isx}\ f(x)]_{(-\infty\ to\ \infty)} - \int_{-\infty}^{\infty} f(x).is.\ e^{isx}\,dx$$

$$= 0 - is\int_{-\infty}^{\infty} e^{isx}\ f(x)\,dx$$

There fore F{f'(x)} = - is F{f(x)}

F{f'(x)} = - is f(x) --------(2)

Now $F\{f''(x)\} = \int_{-\infty}^{\infty} e^{isx} f''(x)\, dx$

$= [e^{isx} f'(x)](-\infty \; to \; \infty) - \int_{-\infty}^{\infty} f'(x).is.\; e^{isx}\, dx$

$= 0 - is \int_{-\infty}^{\infty} e^{isx} f'(x)\, dx$

= - is . F{f'(x)}

= -is (-is) f(s) by (2)

There fore $F\{f''(x)\} = (-is)^2 f(s)$

Similarly we can show that $F\{f^n(x)\} = (-is)^n f(s)$

Finite Fourier Transforms :-

Definition : The Finite Fourier sine Transform of f(x) , 0 < x < l is defined by

Fs{f(x)} = fs (s) = $\int_0^l f(x) \sin \frac{s\pi x}{l}\, dx$

If $0 < x < \pi$, Fs{f(x)} = fs (s) = $\int_0^{\pi} f(x) \sin sx\, dx$

The function f(x) is called the inverse finite Fourier sine transform of fs(s) and is

given by f(x) = ds

If $0 < x < \pi$, f(x) $\frac{2}{l} \sum_{s=1}^{\infty} fs(s) \sin \frac{s\pi x}{l}$ = sx

Definition : The finite Fourier sine Transform of f(x) , 0 < x < l is $\frac{2}{\pi} \sum_{s=1}^{\infty} fs(s) \sin$ defined by

Fc{f(x)} = fc(s) = $\int_0^l f(x) \cos \frac{s\pi x}{l}$ dx

If $0 < x < \pi$, Fc{f(x)} = $\int_0^\pi f(x) \cos$sx dx

The function f(x) is called inverse finite Fourier cosine transform of f(x) and is given

by f(x) = Fc^{-1}{fc(s)} = $\frac{1}{l}$ fc(0) + $\frac{2}{l} \sum_{s=1}^{\infty} fc(s) \cos \frac{s\pi x}{l}$ ds f(x)

= Fc^{-1}{fc(s)} = ^{1} fc(0) + $\frac{2}{\pi}$ $\sigma_{s=1}^{\infty} fc(s) \cos$ sx , $(0, \pi)$

π

Problem :

1) Find the Fourier Finite cosine transform of f(x) = x , $0 < x < \pi$ **Solution** : Fc{f(x)} = fc(s) = $\int_0^\pi f(x) \cos$sx dx

= $\int_0^\pi x \cos$sx dx = $\left(\frac{x \sin sx}{s}\right)$ (0 to π) - $\frac{1}{s} \int_0^\pi \sin sx$ dx

$= (0 - 0) - \frac{1}{s}\left(\frac{-\cos sx}{s}\right)(0\ to\ \pi)$

s = 1,2,3,........ $= \frac{1}{s^2}[\cos s\pi - 1]$

If s = 0 , fc(s) = $= \frac{1}{s^2}[(-1)^s - 1]$

Therefore $\int_0^{\pi} x$ dx = $\frac{x^2}{2}$ (0 to π) $= \frac{\pi^2}{2}$ fc(s) =

$\frac{1}{s^2}[[(-1)^s - 1]$, s > 0

2) Find the Fourier Finite sine transform of f(x) = $\frac{x}{\pi}$, $0 < x < \pi$ $\frac{\pi^2}{2}$, s = 0

Solution : Fs(n) = $\int_0^{\pi} \frac{x}{\pi}$ sin nx dx $= \frac{1}{\pi}\int_0^{\pi} x$ sin nx dx

$\frac{1}{\pi}[x(\frac{-\cos nx}{n})]$ (0 to π) - $1(\frac{-\sin nx}{n^2})$ (0 to π)

$= \frac{1}{\pi}[-\frac{\pi}{n}\cos n\pi + 0 - 0 - 0] = -\frac{1}{n}\cos n\pi = -\frac{1}{n}(-1)^n = \frac{(-1)^{n+1}}{n}$

3) Find the Fourier Finite sine transform of f(x) = x^3 in $(0, \pi)$ Solution : By definition the finite Fourier sine Transform is

Fs{f(x)} = $\int_0^{\pi} f(x)$ sin sx dx

$= \int_0^{\pi} x^3$ sin sx dx

u = x^3 $3x^2$ 6x 6 0 dv= sin nx dx $\frac{-\cos nx}{n}$ $\frac{-\sin nx}{n^2}$ $\frac{\cos nx}{n^3}$ $\frac{\sin nx}{n^4}$

$= [-x^3 \frac{\cos nx}{n} - 3x^2(\frac{-\sin nx}{n^2}) + 6x(\frac{\cos nx}{n^3}) - 6(\frac{\sin nx}{n^4})]$ (0 to π

$= [-\pi^3 \frac{\cos n\pi}{n} - 0 + 6\pi \frac{\cos n\pi}{n^3} - 0] - 0$

$= \frac{-\pi^3}{n}(-1)^n + \frac{6\pi}{n^3}(-1)^n$

$= (-1)^n \frac{\pi}{n} [\frac{6}{n^2} - \pi_2]$, n = 1,2,3......)

4) Find Finite sine Transform of f(x) = x in 0 < x < 4

Solution : Let f(x) is Fs{f(x)} = $\int_0^4 f(x) \sin \frac{n\pi x}{4} dx$

$= [x (\frac{-\cos \frac{n\pi x}{4}}{\frac{n\pi}{4}}) (0 \text{ to } 4) - (\frac{-\sin \frac{n\pi x}{4}}{\frac{n^2\pi^2}{16}}) (0 \text{ to } 4)]$

$= -\frac{4}{n\pi} . 4 . \cos n\pi - 0 + \frac{16}{n^2\pi^2}(0 - 0)$

$= -\frac{16}{n\pi} \cos n\pi = -\frac{16}{n\pi}(-1)^n$

Similarly Fc{f(x)} = $\frac{16}{n^2\pi^2}[(-1)^n - 1]$ = fc(n) if n = 0 , fc(0) = $\int_0^4 x$ dx = $(\frac{x^2}{2})$ (0 to 4) = 8

<u>Parseval's Identity for Fourier Transforms</u> :-

Statement : If f(s) & g(s) are Fourier Transform of f(x) & g(x) respectively then

(i) $\frac{1}{2\pi}$ = $\int_{-\infty}^{\infty} f(x)$ g(x) dx

(ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(s)|_2$ ds = $\int_{-\infty}^{\infty} |f(x)|_2$ dx

Now (iii) $\frac{2}{\pi}\int_{-\infty}^{\infty} fc(s)$ gc(s) ds = $\int_0^{\infty} f(x)$ g(x) dx

Proof : By the inverse Fourier Transform we have

g(x) = $\frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)\, e^{-isx}$ ds ------(1)

Taking cojugate Complex on both sides in (1)

(1) $\Rightarrow$ g(x) = $\frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)\, e^{isx}$ ds

$\int_{-\infty}^{\infty} f(x)$ g(x) dx = $\int_{-\infty}^{\infty} f(x)\, [\frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)\, e^{isx}$ ds

= $\frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)\, [\int_{-\infty}^{\infty} f(x)\, e^{isx}$]

= $\frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)$ f(s) ds

$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)$ g(s) ds = $\int_{-\infty}^{\infty} f(x)$ g(x) dx ---------(2)

dx dx]

ds

(ii) Putting g(x) = f(x) in (2) we get

$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)$ f(s) ds = $\int_{-\infty}^{\infty} f(x)$ f(x) dx

$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(s)|^2$ ds = $\int_{-\infty}^{\infty} |f(x)|^2$ dx --------- Therefore (3)

For Sine Transform:

(2) ⇒ $\frac{2}{\pi}\int_{0}^{\infty} fs(s)$ gs(s) ds = $\int_{0}^{\infty} f(x)$ g(x) dx

$\frac{2}{\pi}$ $\int_{0}^{\infty} |fs(s)|^2$ ds = $\int_{0}^{\infty} |f(x)|^2$ dx

Similarly for Cosine

Problem 1):) If f(x) = 1 , |x| < a

0 , |x| > a , Find Fourier Transform of f(x)

$\int_{0}^{\infty} \frac{\sin\ ax}{x^2}\, dx = \frac{\pi a}{2}$ 2

Deduce that

Solution : F{f(X)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx |x| < a means –a < x < a

= ?$_{-a}$ e_{isx} .1. dx

$= \frac{e^{isx}}{is}$ (-a to a)

$= \frac{1}{is}(e^{ias} - e^{-ias}) = \frac{1}{is}$ (2i sin as)

$= \frac{2\sin as}{s} =$ f(s) F{f(x)} = f(s)

By parseval's identity for Fourier Transform

$\int_{-\infty}^{\infty} |f(x)|_2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(s)|_2 \, ds$

$\Rightarrow \int_{-a}^{a} 1 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\frac{2\sin as}{s})_2 \, ds$

$\Rightarrow x(\text{-a to a}) = \frac{1}{2\pi} 2^2 \int_{-\infty}^{\infty} \frac{\sin^2 as}{s^2} \, ds$

$\Rightarrow 2a = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 as}{s^2} \, ds$

$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 as}{s^2} \, ds = a\pi$

$\Rightarrow \int_{-\infty}^{\infty} (\frac{\sin as}{s})_2 \, ds = a\,\pi$

$\Rightarrow 2 \, . \int_{0}^{\infty} \frac{\sin^2 as}{s^2} \, ds = a\pi$

$\int_{0}^{\infty} \frac{\sin \ as}{s^2}$ $\qquad \frac{2}{2}$ aπ

Therefore ds =

2)Find Fourier Transform of f(x) = 1 - x^2 , $|x| \leq 1$

0, $|x| > 1$ is $\frac{4}{s^3}$[sin s – s cos s]

Using Parseval's Identity Prove That

$\int_0^\infty [\frac{(\sin x - x\cos x)}{x^3}]^2 dx = \frac{\pi}{15}$

Solution : :- F{f(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx

$= \int_{-1}^{1} e^{isx}(1 - x^2)\, dx$

$= \int_{-1}^{1} (1 - x^2)e^{isx}\, dx$

$\text{?}\ udv = uv - \text{?}\ vdu$ $=[\{(1 - x^2).\frac{e^{isx}}{is}\} - \int_{-1}^{1} \frac{e^{isx}}{is}(-2x)dx]$

(limits -1 to 1)

u= (1 – x^2) dv= e^{isx}dx

du =-2x dx, v= ?. e^{isx} dx

$= [\ 0 - 0 + \frac{2}{is} \int_{-1}^{1} x.\, e^{isx}\ dx\ ]$

By parseval's identity for Fourier Transform

$= \frac{e^{isx}}{is}$

$= \frac{2}{is} [(\frac{xe^{isx}}{is})$ (-1 to 1) $- \int_{-1}^{1} \frac{e^{isx}}{is}$ dx]

$= \frac{2}{is}[1.(\frac{e^{is}+e^{-is}}{is}) - \frac{1}{is} \frac{e^{isx}}{is}]$ (-1 to 1)

$= \frac{2}{is}\ [\frac{2\cos s}{is} - \frac{1}{is}\ (\frac{e^{is}-e^{-is}}{is})\ ]$

$= \frac{2}{is}.\frac{1}{is}$ (2 cos s $- \frac{1}{is}$ 2i sin s)

$= -\frac{2}{s^2}$ .2[cos s $- \frac{\sin s}{s}$

$= \frac{4}{s^3}$ [sin s – s cos s] = f(s)

$\int_{-\infty}^{\infty} |f(x)|_2\ dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(s)|_2\ ds\ \int_{-1}^{1} (1 - x^2)^2\ dx =$

$\frac{1}{2\pi}\int_{-\infty}^{\infty}$ [$\frac{4}{s^3}$ (sin s – s cos s)]² ds 2. $\int_0^1(1-x^2)^2$ dx = $\frac{1}{2\pi}$ $\Rightarrow \int_0^\infty [\frac{(\sin s - s\cos s)}{s^3}]^2$

. 2. 16 $\int_0^\infty [\frac{(\sin s - s\cos s)}{s^3}]^2$ ds

ds =

15

$\Rightarrow \frac{16}{\pi}\int_0^\infty [\frac{(\sin s - s\cos s)}{s^3}$

8

]² ds = 2 . —

$\Rightarrow \int_0^\infty [\frac{(\sin x - x\cos x)}{x^3}$

15

π

π

]² dx =

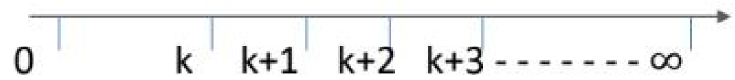
15

Shifting Properties:-

1.Shifting f(n) to the right :-

If Z[f(n)]=F(Z) then Z[f(n-k)]=Z^{-k}F(Z)

Proof: we know that



Z[f(n)]=

k)]=$\sum_{n=0}^{\infty} f(n-k)Z^{-n}$ (k,n are different forms)

= $\sum_{n=k}^{\infty} f(n-k)Z^{-n}$ (since we are shifting f(n) to right)

=f(0)z^{-k}+ f(1)$z^{-(k+1)}$+ f(2)$z^{-(k+2)}$+ - - - - - - - -

= $Z^{-k}[f(0)$ + f(1)z^{-1}+f(2)z^{-2}+ − − − − − −]

=$Z^{-k}\sum_{n=0}^{\infty} f(n)Z^{-n}$

=$Z^{-k}F(Z)$

$\sum_{n=0}^{\infty} f(n)Z^{-n}$ consider Z[f(n-

Z[f(n-k)]=$Z^{-k}F(Z)$

NOTE :- Z[f(n-k)]=Z^{-k}F(Z) putting k=1 ,we have

Z[f(n-1)]=$Z^{-1}F(Z)$ putting k=2 ,we have Z[f(n-2)]=$Z^{-2}F(Z)$

putting k=3 ,we have

Z[f(n-3)]=$Z^{-3}F(Z)$

<u>2.Shifting f(n) to left :-</u>

If Z[f(n)]=F(Z) then Z[f(n+k)]=Z^k[F(Z)-f(0)-f(1)$Z^{-1}-$ f (2)Z^{-2} $-$ $-------$ $-$ f(k-1)$Z^{-(k-1)}$]

Proof: we know that Z[f(n)]=$\sum_{n=0}^{\infty} f(n)Z^{-n}$ $[Z^{-n} = Z^k . Z^{-(n+k)}]$

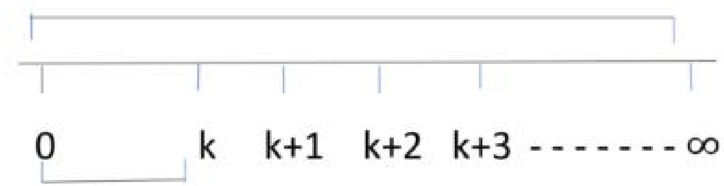
)]=$\sum_{n=0}^{\infty} f(n+k)Z^{-n}$

consider Z[f(n+k =$Z^k \sum_{n=k}^{\infty} f(n+k)Z^{-(n+k)}$

=$Z^k \sum_{n=k}^{\infty} f(n)Z^{-n}$ (replace (n+k) by n)

= $Z^k [\sigma_{n=0}^{\infty} f(n)Z^{-n}$ - $\sigma_{n=0}^{(k-1)} f(n)Z^{-n}]$

=$Z^k [Z[f(n)] - \sigma_{n=0}^{(k-1)} f(n)Z^{-n}]$



Z[f(n+k)]=Z^k[F(Z)-f(0)-f(1)$Z^{-1}-$f (2) Z^{-2} $-$ $------$ $-$f(k-1)$Z^{-(k-1)}$] which is Recurrence formula ∴

In particular

(a)If k=1 then Z[f(n+1)]=Z[F(Z)-f(0)]

(b) If k=2 then Z[f(n+2)]=Z^2[F(Z)-f(0)-f(1)Z^{-1}]

(c) If k=3 then Z[f(n+3)]=Z^3[F(Z)-f(0)-f(1)Z^{-1}-f(2)Z^2] - - - - - - - - - - - - - - and so on.

Problems:1.Prove Z(

$\frac{1}{(n+1}$)=Zlog($\frac{Z}{Z-1}$)

Solution- let f(n)=Z($\frac{1}{n+1}$)

we know that Z[f(n)]=$\sum_{n=0}^{\infty} f(n)Z^{-n}$

Z[$\frac{1}{n+1}$]=$\sum_{n=0}^{\infty}\frac{1}{n+1}Z^{-n}$

=$\sum_{n=0}^{\infty}\frac{1}{n+1}\cdot\frac{1}{Z^n}$

=$\frac{1}{1}\cdot\frac{1}{1}+\frac{1}{2}\cdot\frac{1}{Z}+\frac{1}{3}\cdot\frac{1}{Z^2}+$ - - - - - - -

- expansion needs 'Z' in denominator's, for this,multiply &divide with 'Z'

] [x + $\frac{x^2}{2}+\frac{x^3}{3}+$ - - - - - - - =-log(1 -x)]

$=Z[\frac{1}{Z} + \frac{1}{2}\cdot\frac{1}{Z^2} + \frac{1}{3}\cdot\frac{1}{Z^3} + \frac{1}{4}\cdot\frac{1}{Z^4} + ------$ evaluate (a)Z(

$=Z[\frac{\frac{1}{Z}}{1} + \frac{(\frac{1}{Z})^2}{2} + \frac{(\frac{1}{Z})^3}{3} + ------ --]$

$=Z[-\log(1-\frac{1}{Z})]$

$=Z[\log(1-\frac{1}{Z})^{-1}]$

$=Z\log(\frac{Z-1}{Z})^{-1}$

$=Z\log(\frac{Z}{Z-1})$

$\therefore$ hence proved

2.Find $Z[\frac{1}{n!}]$ and using shifting theorem

Solution- we know that $Z[f(n)]=\sum_{n=0}^{\infty} f(n)Z^{-n}$

let $f(n)=\frac{1}{n!}$ for n=0,1,2,3 - - -

$Z[\frac{1}{n!}] = \sum_{n=0}^{\infty}\frac{1}{n!}Z^{-n}$

$\frac{1}{(n+1)!})$ and (b)$Z(\frac{1}{(n+2)!})$

$=1 + \frac{1}{1!}Z^{-1} + \frac{1}{2!}Z^{-2} + \frac{1}{3!}Z^{-3} + -------$

$=1+\frac{1}{Z} + \frac{(\frac{1}{Z})^2}{2!} + \frac{(\frac{1}{Z})^3}{3!} + --------------$

$=e^{\frac{1}{Z}}$ $\quad [e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + -----]$

=F(Z) (say) By

shifting theorem

-> Z[f(n+1)]=Z[F(Z)-F(0)]

2 [( ) ( ) (1])

(1) $Z[\frac{1}{(n+1)!}] = Z[e^{\frac{1}{Z}} - 1]$ $[f(0)=\frac{1}{0!} = 1]$

(2) $Z[\frac{1}{(n+2)!}] = Z^2[e^{\frac{1}{Z}} - 1 - \frac{1}{1!}Z^{-1}]$

$= Z^2[e^{\frac{1}{Z}} - 1 - Z^{-1}]$

-

f(n)= 1 / $n!$ >Z[f(n+2)]=$Z\ F\ Z - F\ 0 - F\ 1\ Z$

f(n+ 1)=$\frac{1}{(n+1)}$

! f(n+ 2)=$\frac{1}{(n+2)}$

!

$(n)]=-Z\frac{d}{dZ}[F(Z)]$

Proof:- we know that Z[f(n)]=$\sum_{n=0}^{\infty} f(n)Z^{-n}$

∴ Z[nf(n)]=−Z Z[nf(n)] =$\sigma_{n=0}^{\infty} nf(n)Z^{-n}$

$\frac{d}{dZ}[F(Z)]$ $=-Z\,\sigma_{n=0}^{\infty} f(n)(-n)Z^{-n-1}$

$=-Z\,\sigma_{n=0}^{\infty}\frac{d}{dz}[f(n)Z^{-n}]$

pb)If F(Z)=__________ $= -Z\frac{d}{dz}[\,\sigma_{n=0}^{\infty} f(n)Z^{-n}\,]$ $(Z-1)4$ then find the values of f(2) and f(3)

$= -Z\frac{d}{dz}[Zf(n)]$

F(Z) $= -Z\frac{d}{dz}$ $\overline{(Z-1)^4}$

$=\frac{Z^2(5+3Z^{-1}+12Z^{-2})}{Z^4(1-Z^{-1})^4}$

$=\frac{1}{Z^2}\frac{(5+3Z^{-1}+12Z^{-2})}{(1-z^{-1})^4}$

Solution: Given F(Z)=

By Intial value theorem we have

<u>Multiplication by 'n'</u>:If Z[f(n)]=F(Z) then

$[\,Z^{-n} = Z^1.Z^{-n-1}\,]$

Z[nf

$[\frac{d}{dz}(Z^{-n}) = (-n)Z^{-n-1}]$

$5Z^2+3Z+12$

$5Z^2+3Z+12$

f(0)= $\lim_{Z\to\infty} F(Z) = 0 \quad (\frac{1}{\infty} = 0)$ $\longrightarrow$ 1

f(1)= $\lim_{Z\to\infty} Z[f(Z)$ -f(0)] = 0

f(2)= $\lim_{Z\to\infty} Z^2[F(Z) - f(0) - f(1)Z^{-1}]$

=5 - 0 – 0

=5

f(3)= $\lim_{Z\to\infty} Z^3$[F(Z)-f(0)-f(1)Z^{-1} –f(2)Z^{-2}]

= $\lim_{Z\to\infty} Z^3$[F(Z)- (0) –(0.Z^{-1}) $- 5Z^{-2}$]

= $\lim_{Z\to\infty} Z^3[\frac{5Z^2+3Z+12}{(Z-1)^4}$ - $\frac{5}{Z^2}]$

= $\lim Z^3 5Z^4+3Z^3+^2(12Z-Z1^2)-^4 5\ Z-1\ ^4$

$z\to\infty$ Z ( )

= lim $Z^3$5Z^4+3Z^3+12Z^2−^{2}5((ZZ−^{4}1−)4^4Z^3+6Z^2−4Z+1)

$z\to\infty$ Z

= lim $Z^3$5Z^4+3Z^3+12Z^2−5^2Z(Z^4+−201)$^4Z^3$−30Z^2+20Z−5

$z\to\infty$ Z ( )

= lim $Z^3$23Z2^3. Z18+1Z ^{2}Z+ 201Z1−5 ( )

$z\to\infty$ Z

= lim $Z^3$23−18Z3$^-$[^{1}1+−20Z−Z1^{-2}4−5Z^{-3} $z\to\infty$ z

= 23

$\rightarrow (Z-1)^4 = (z-1)^2.(z-1)^2$

$=(Z^2+1-2Z)(Z^2+1-2Z)$

$=Z^4+Z^2-2Z^3+Z^2+1-2Z-2Z^3-2Z+4Z^2 = Z^4 +$

$6Z^2-4Z^3-4Z+1$

INVERSE Z-TRANSFORM

$[g(0) + g(1)Z^{-1} + g(2)Z^{-2} + g(3)Z^{-3} + \text{-----} + g(n)Z^{-n} + \text{-----}]$

$= \sum_{n=0}^{\infty}[f(0)g(n) + f(1)g(n-1) + f(2)g(n-2) + \text{----} + f(n)g(0)]Z^{-n}$

$*$We have Z[f(n)]=F(Z) which can be also written as f(n)=$Z^{-1}[F(Z)]$.

Then f(n) is called inverse Z-transform of F(Z)

$*$Thus finding the sequence {f(n)} from F(Z) is defined as <u>Inverse Z-Transform.</u>

$*$The symbol Z^{-1} is the Inverse Z – Transform.

<u>CONVOLUTION THEOREM(v.v.imp):-</u>

If $Z^{-1}[F(Z)] = f(n)$ and $Z^{-1}[G(Z)] = g(n)$ then

$Z^{-1}[F(Z).G(Z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m)g(n-m)$

[where * is convolution operator]

Proof:- We have F(Z)=$\sum_{n=0}^{\infty} f(n)Z^{-n}$ $and\ G(Z) = \sum_{n=0}^{\infty} g(n)Z^{-n}$ then

F(Z).G(Z) = $[f(0) + f(1)Z^{-1} + f(2)Z^{-2} + f(3)Z^{-3} + \text{-----} + f(n)Z^{-n} + \text{-----}]$

=Z[f(0)g(n)+f(n)g(n-1)+ ------- +f(n)g(0)] Z^{-1}[F(Z).G(Z)]

=f(0)g(n)+f(n)g(n-1)+ ------- +f(n)g(0)

$$= \sum_{m=0}^{n} f(m) g(n-m)$$

$$\therefore Z^{-1}[F(Z).G(Z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m) g(n-m)$$

Problems:-

1.Evaluate (a)$Z^{-1}\left[\left(\frac{Z}{Z-a}\right)^2\right]$ ( ) $b\ Z^{-1}\left[\frac{Z^2}{(Z-a)(Z-b)}\right]$

Solution:-

(a) $Z_{-1}\left[\left(\frac{Z}{Z-a}\right)^2\right]$

$= Z^{-1}\left[\frac{Z}{Z-a} . \frac{Z}{Z-a}\right]$

F(Z)=$\frac{Z}{Z-a} => f\ n = Z_{-1} \frac{Z}{Z-a} = a_n$

G(Z)=$\frac{Z}{Z-a} => g\ n = Z_{-1} \frac{Z}{Z-a} = a_n$

by convolution theorem , Z $g((n)) = \sum_{m=0}^{n} Z_{-1} F$

$Z . G\ Z = Z_{-1}\left[\frac{Z}{Z-a} . \frac{Z}{Z-a}\right]$

$= \sigma_{nm=0}\ a_m . a_{n-m}$

$Z^{-1}[F(Z).G(Z)] = f(n) * g(n) = \sum_{m=0}^{n} a^n$

$= a^n \sum_{m=0}^{n} 1$

$= a^n[1 + 1 + 1 + - - - - - +1]$ (n+1)times

$= (n+1)a^n$

$f(m)g(n-m)$

(b) $Z^{-1}\left[\frac{Z^2}{(Z-a)(Z-b)}\right]$

$= Z^{-1}\left[\frac{Z}{Z-a} \cdot \frac{Z}{Z-b}\right]$

$= Z^{-1}$ F(Z)= $\frac{Z}{Z-a}$ => f(n) $= Z^{-1}\left[\frac{Z}{Z-a}\right] = a^n$

G(Z)= $\frac{Z}{Z-b}$ => g(n) $= Z^{-1}\left[\frac{Z}{Z-b}\right] = b^n$

by convolution theorem ,

$$Z^{-1}[F(Z).G(Z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m)g(n-m)$$

$$Z^{-1}[F(Z).G(Z)] = Z^{-1}\left[\frac{Z}{Z-a} \cdot \frac{Z}{Z-b}\right]$$

$= \sum_{m=0}^{n} a^m . b^{n-m}$

$= \sigma nm{=}0\ bn.\ (ab)m$

$= bn\ \sigma nm{=}0\ (ab)m$

$$= b^n\left[\left(\frac{a}{b}\right)^0 + \left(-\frac{a}{b}\right)^1 + \left(-\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^3 + \text{---------} + \left(\frac{a}{b}\right)^n\right]$$

this is in geometric progression,

$$^2+ar^3+\text{----}+ar^{n-1}+\text{----}\ \ \frac{}{1-r} = a(1-r^n),\ r<1\ a+ar$$

$$= \frac{a(r^n-1)}{1-r},\ r>1$$

$$= \frac{bn\left[1-a(\pi+)1\right]}{\begin{matrix} ba \\ 1- \end{matrix}}$$

b

$$\frac{bn\left[1-a\overline{bnn+}+11\right]}{\rule{2em}{0.4pt}}$$

$= \quad b-a$

b

Put a=3 and b=4 we get

$$Z^{-1}\left[\frac{Z^2}{(Z-3)(Z-4)}\right] = \frac{4^{n+1}-3^{n+1}}{4-3} = 4^{n+1}-3^{n+1}$$

$$= \frac{b^n\left[\frac{b^{n+1}-a^{n+1}}{b^{n+1}}\right]}{\frac{b-a}{b}}$$

$$= b^n.\frac{b^{n+1}-a^{n+1}}{b^{n+1}}.\frac{b}{b-a}$$

$$= \frac{b^{n+1}-a^{n+1}}{b-a}$$

2.Using Convolution theorem show that

$$Z^{-1}\left[\frac{Z^2}{(Z-a)(Z-b)}\right] = \frac{b^{n+1}-a^{n+1}}{b-a}$$

$$Z^{-1}\left[\frac{1}{n!} * \frac{1}{n!}\right] = \frac{2^n}{n!}$$ where $*$ is convolution operator

Solution: $f(n)=\frac{1}{n!}$ $g(n)=\frac{1}{n!}$

$$f(n) * g(n) = \sum_{m=0}^{n} f(m)g(n-m)$$

$$= \sum_{m=0}^{n} \frac{1}{m!} \cdot \frac{1}{(n-m)!}$$

$$= 1.\frac{1}{n!} + \frac{1}{1!} \cdot \frac{1}{(n-1)!} + \frac{1}{2!} \cdot \frac{1}{(n-2)!} + \text{- - - -} + \frac{1}{n!} \cdot \frac{1}{(0)}$$

$$= \frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{2!} \cdot \frac{1}{(n-2)!} + \text{- - - - - -} + \frac{1}{n!}$$

$$\left[\frac{1}{(n-1)!} = \frac{n}{n(n-1)!} = \frac{1}{(n-1)!}\right]$$

$$= \frac{1}{n!} + \frac{1}{1!}\frac{n}{n!} + \frac{1}{2!}\frac{n(n-1)}{n!} + \text{- - - - - -} + \frac{1}{n!}$$

!

$$= \frac{1}{n!}\left[1 + \frac{n}{1!} + \frac{n(n-1)}{2!} + \text{- - -}\right.$$

$$= \frac{1}{n!}(1+1)^n$$

$$= \frac{2^n}{n!}$$

- to (n+1) terms]

3. Evaluate $Z^{-1}\left[\frac{Z^2}{(Z-4)(Z-5)}\right]$

Z

$Z-4$

Z

$Z-5$

Solution- Given $Z^{-1}\left[\frac{Z}{Z-4}\cdot\frac{Z}{Z-5}\right]$

F(Z)=— => $f(n) = Z^{-1}\left[\frac{Z}{Z-4}\right] = 4^n$ [$G(Z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m)g(n-m)$

[(G(Z)=( )] => $g(n) = Z^{-1}\left[\frac{Z}{Z-5}\right] = 5^n$ $Z-1\,F$

$Z-5$

by convolution theorem , $Z^{-1}[F(Z)$. $=\sigma nm=0\; 4m\,.\,5n-m$

$Z\;.\,G\;Z \quad = Z^{-1}\left[\frac{Z}{Z-4}\cdot\frac{Z}{}\right]$

$=\sigma nm=0\; 5n.\,(45)m$

$= 5n\;\sigma nm=0\;(45)m$

$=5^n[(\frac{4}{5})^0 + (\frac{4}{5})^1 + (\frac{4}{5})^1 + (\frac{4}{5})^3 + \text{---------} + (\frac{4}{5})^n]$

$=5^n[1 + \frac{4}{5} + (\frac{4}{5})^2 + (\frac{4}{5})^3 + \text{---------} + (\frac{4}{5})^n]$

this is in geometric progression,

$^1 + ar^3 + \text{----} + ar^{n-1} + \text{----}\frac{}{1-r} = a(1-r^n)$, r<1 a+ar

$a(r^n-1)$

$= \qquad \frac{}{1-r}$, r>1

$= \frac{5^n\left[1-\left(\frac{4}{5}\right)^{n+1}\right]}{1-\frac{4}{5}}$

$= \frac{5^n\left[1-\frac{4^{n+1}}{5^{n+1}}\right]}{\frac{5-4}{5}}$

$= \frac{5^n\left[\frac{5^{n+1}-4^{n+1}}{5^{n+1}}\right]}{\frac{5-4}{5}}$

$= 5^n . \frac{5^{n+1}-4^{n+1}}{5^{n+1}} . \frac{5}{1}$

$\frac{5^{n+1}-4^{n+1}}{1}$

$-1\left[\frac{Z^2}{(Z-4)(Z-5)}\right]$

=

= $\qquad \therefore Z5_{n+1} - 4_{n+1}$

Partial Fractions Method:-

$Z\left[\frac{}{2+11^{Z}Z+24}\right]$ (non repeated linear factors) $(v. imp)$

1.Find Z^{-1} $\left[\frac{}{}\right]$ $\frac{}{}$

Solution:- let F(Z) = $Z-1\ Z2+11ZZ+24$ = $(Z+3)Z(Z+8)$

then $\frac{F(Z)}{Z} = \frac{1}{(Z+3)(Z+8)} = \frac{A}{(Z+3)} + \frac{B}{(Z+8)} \rightarrow 1$

$= \frac{1}{(Z+3)(Z+8)} = \frac{A(Z+8)+B(Z+3)}{(Z+3)(Z+8)}$

$= 1 = A(Z+8)+B(Z+3) \rightarrow 2$

put Z=-8 $\Rightarrow 1 = A(-8+8) + B(-8+3)$

1 = B(-5)

B= $\frac{-1}{5}$

put Z=-3 $\Rightarrow 1 = A(-3+8) + B(-3+3)$

1 =A(5)

A= $\frac{1}{5}$

$\{ Z+8=0 \Rightarrow Z=-8 \ \& \ Z+3=0 \Rightarrow z=-3 \}$

now substitute A and B values in equation -1 we get

$\left[\frac{F(Z)}{Z}\right] = \frac{1}{5(Z+3)} - \frac{1}{5(Z+8)}$

$F(Z) = \frac{Z}{5(Z+3)} - \frac{Z}{5(Z+8)}$

$= Z^{-1}\left[\frac{Z}{5(Z+3)} - \frac{Z}{5(Z+8)}\right]$

$= \frac{1}{5}\left[Z^{-1}\left[\frac{Z}{Z+3}\right] - Z^{-1}\left[\frac{Z}{Z+8}\right]\right]$

$= \frac{1}{5}[(-3)^n - (-8)^n]$

$\therefore Z^{-1}\left[\frac{Z}{Z^2+11Z+24}\right] = \frac{1}{5}[(-3)^n - (-8)^n]$

$Z^{-1}\left[\frac{Z}{Z-a}\right] = a^n \Rightarrow Z^{-1}\left[\frac{Z}{Z-a}\right] = a^n$

2.Find the Inverse Z-Transform of $\frac{Z}{(Z-1)(Z-2)}$

Solution:- let $F(Z) = \frac{Z}{(Z-1)(Z-2)}$ here we can resolve F(Z) into partial fractions directly as follows

$F(Z) = Z\left[\frac{1}{(Z-1)(Z-2)}\right] = Z\left[\frac{1}{Z-2} - \frac{1}{Z-1}\right]$

$F(Z) = \frac{Z}{Z-2} - \frac{Z}{Z-1}$

hence $Z^{-1}[F(Z)] = Z^{-1}\left[\frac{Z}{Z-2}\right] - Z^{-1}\left[\frac{Z}{Z-1}\right]$

$= 2^n - 1^n$

$\left[\frac{}{(5Z-1)(5Z+2)}\right]$

Solution:- let F(Z) = $\frac{Z(3Z+1)}{(5Z-1)(5Z+2)}$ then

$$\frac{F(Z)}{Z} = \frac{3Z+1}{(5Z-1)(5Z+2)} = \frac{A}{5Z-1} + \frac{B}{5Z+2}$$ → 1 (by partial fractions)

$$\frac{3Z+1}{(5Z-1)(5Z+2)} = \frac{A(5Z+2)+B(5Z-1)}{(5Z-1)(5Z-2)}$$

3Z+1 = A(5Z+2)+B(5Z-1)

put Z= $\frac{-}{5} \Rightarrow A = \frac{}{152}$ 8

put Z= $\frac{1}{5} \Rightarrow B = \frac{1}{15}$substituting A and B values in equation-1 we get

$$\frac{F(Z)}{Z} = \frac{8}{15}\frac{1}{5Z-1} + \frac{1}{15}\frac{1}{5Z+2}$$

$$\frac{F(Z)}{Z} = \frac{8}{15}\frac{1}{\left(Z-\frac{1}{5}\right)} + \frac{1}{15}\frac{1}{\left(Z+\frac{2}{5}\right)}$$

3.Find Z_{-1} $3Z2+Z$

hence F(Z) $= \frac{8}{75} \cdot \frac{Z}{\left(Z-\frac{1}{5}\right)} + \frac{1}{75} \cdot \frac{Z}{\left(Z+\frac{2}{5}\right)}$

Z^{-1}[F(Z)] $= Z^{-1}\left[\frac{8}{75}\left(\frac{Z}{Z-0.2}\right) + \frac{1}{75}\left(\frac{Z}{Z+0.4}\right)\right]$

$\frac{8}{75} Z^{-1}\left(\frac{Z}{Z-0.2}\right) + \frac{1}{75} Z^{-1}\left(\frac{Z}{Z-(-0.4)}\right)$

$\frac{8}{75}(0.2)^n + \frac{1}{75}(-0.4)^n$

$\therefore Z^{-1}\left[\frac{3Z^2+Z}{(5Z-1)(5Z+2)}\right] = \frac{8}{75}(0.2)^n + \frac{1}{75}(-0.4)^n$

=

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<u>Geometric Progression</u>:a)

Finite –

a $+ \text{ar} + \text{a}r^2 + ar^3 + \text{- - - - - - - -} + \text{a}r^{n-1} + ar^n = \frac{a(1-r^{n+1})}{1-r}$

b) Infinite –

a $+ \text{ar} + \text{a}r^2 + ar^3 + \text{- - - - - - - -} + \text{a}r^{n-1} + ar^n + \text{- - -} = \frac{a}{a-r}$

$+ \text{r} + r^2 + r^3 + \text{- - - - - - - - - -} + r^n + \text{- - - - - - - -} = \frac{1}{1-r}$

eg; 1

4.Find $Z^{-1}\left[\frac{Z}{(Z+3)^2(Z-2)}\right]$ (repeated Linear factor of form (ax + b)2 times)

Solution:-let F(Z) = $\frac{Z}{(Z+3)^2(Z-2)}$

$$\frac{F(Z)}{Z}=\frac{1}{(Z+3)^2(Z-2)}$$

$\frac{F(Z)}{Z} = \frac{1}{(Z+3)^2(Z-2)} = \frac{A}{Z-2}+\frac{B}{Z+3}+\frac{c}{(Z+3)^2} \qquad \rightarrow 1$

$$\frac{1}{(Z+3)^2(Z-2)} = \frac{A(Z+3)^2+B(Z-2)(Z+3)+c(Z-2)}{(z-2)(z+3)^2}$$

1 $=A(Z+3)^2+B(Z-2)(Z+3)+c(Z-2)$ { $Z-2=0 \Rightarrow Z=2$ & Z+3 =0 => Z=-3 } put Z=2 => 1=A(2 + 3)^2

1 =A(25)

A = $\frac{1}{25}$

put Z=-3 =>1=c(-3-2)

1 = -5c c= $\frac{-1}{5}$

now comparing the co-efficients of Z^2 on both sides

0=A+B

$B=\frac{-1}{25}$ substituting A,B and C

values in equation-1,we get

$$\frac{F(Z)}{Z} = \frac{1}{(Z+3)^2(Z-2)} = \frac{1}{25}\cdot\frac{1}{Z-2} - \frac{1}{25}\cdot\frac{1}{Z+3} - \frac{1}{5}\cdot\frac{1}{(Z+3)^2}$$

$$F(Z) = \frac{1}{25}\cdot\frac{Z}{Z-2} - \frac{1}{25}\frac{Z}{Z+3} - \frac{1}{5}\cdot\frac{Z}{(Z+3)^2}$$

$$Z^{-1}\left[\frac{Z^2}{(Z+3)^2(Z-2)}\right] = Z^{-1}\left[\frac{1}{25}\cdot\frac{Z}{Z-2} - \frac{1}{25}\frac{Z}{Z+3} - \frac{1}{5}\cdot\frac{Z}{(Z+3)^2}\right]$$

$$= \frac{1}{25}2^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^n$$

$$\therefore Z^{-1}\left[\frac{Z^2}{(Z+3)^2(Z-2)}\right] = \frac{1}{25}2^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^n$$

Solutions Of Difference Equations

Difference Equations:-

Just as the Differential equations are used for dealing with continuous process in nature , the difference equations are used for dealing of discrete process.

Definition:-

A difference equation is a relation between the difference of an unknown function at one (or) more

general value of the argument.

thus $\Delta y_n + 2y_n = 0$ *and*

$\Delta^2 y_n + 5\Delta y_n + 6y_n = 0$ are difference equations

Solution:-

The solution of a difference equation is an expression for y_n which satisfies the given difference equation

General Solution:-

The general solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

Linear Difference Equation:-

The Linear difference equation is that in which $y_{n+1}, y_{n+2}, y_{n+3}$ ------- etc occur to the 1^{st} degree only and are not multiplied together.

The difference equation is called Homogeneous if f(n)=0,Otherwise it is called as NonHomogeneous equation (i.e:-f(n)$\neq 0$)

Working rule (or) Working Procedure:-

To solve a given linear difference equation with constant co-efficient by Z-transforms.

Step-1 :- Let Z(y_n)=Z[y(n)] =Y(Z)

Step-2 :-Take Z-Transform on bothsides of the given difference equation.

Step-3 :-Use the formulae $Z(y_n) = Y(Z)$

$Z[y_n + 1] = Z[Y(Z) - y_0]$

$Z[y_n + 2] = Z^2[Y(Z) - y_0 - y_1 Z^{-1}]$

Step-4:-Simplify and find Y(Z) by transposing the terms to the right and dividing by the co-efficient of y(Z).

Step-5:-Take the Inverse Z-Transform of Y(Z) and find the solution y_n

This gives y_n as a function of n which is the desired solution. Problems:-

1.Solve $y_{n+1} - 2y_n = 0$ using Z –Transforms.

Solution:-let $Z[y_n] = Y(Z)$

$Z[y_{n+1}] = Z[Y(Z) - y_0]$ taking Z-Transform of the given equation we get $Z[y_{n+1}] - 2Z[y_n] = 0$

$Z[Y(Z) - y_0] - 2Y(Z) = 0$

$[\ Z(a^n) = \frac{Z}{Z-a}\ ]$

Y(Z)[Z-

$Y(Z) = \frac{Z}{Z-2} y_0$

$Z^{-1}[Y(Z)] = Z^{-1}\left[\frac{Z}{Z-2}\right] y_0$

=> Z[Y(n)]=Y(Z)

$y_n = 2^n y_0$

$Z^{-1}[Y(Z)] = y_n$

2.Solve the difference equation using Z-Transforms

$\mu_n+2 - 3\mu_n+1 + 2\mu_n = 0$ Given that

μ_0=0 , $\mu_1 = 1$

Solution:-let Z($\mu_{n)} = \mu\, Z(\ )$

Z($\mu_{n+1}) = Z[\mu (Z\ ) - \mu_0]$

Z($\mu_{n+2}) = Z^2\,\mu\left[Z(\ ) - \mu_0 - {}^{\mu}Z^1\right]$ now taking Z-Transform on both sides of

the given equation we get

$Z(\mu_{n+2}) - 3Z(\mu_{n+1}) + 2Z(\mu_n) = 0\ Z2 - \mu_0 - \mu Z^1$

$\left[\mu(Z)\right.$ - $3Z[\mu Z - \mu_0]$ +2$\mu(\ )\ Z(\ ) = 0$ using the given

$\left[\mu(Z)\right]$ conditions it reduces to

$(\ ) \qquad (\ )$

$Z2 - 0 - 1$ - $3Z\mu[\mu Z\ Z(\ ) [\ Z) - 2\ \ 0] - 3$+2 $Z\mu + Z\ 2\ \ ] = 0$

$z \qquad (\ ) \qquad \overline{\quad z \quad}$

$\mu \qquad \overline{\qquad\qquad}$

$Z = z\,2 - 3\,Z + 2 \quad \frac{1}{-2} \quad \frac{1}{-1}$ = Z (or)

= $(Z-1)^Z (Z-2)$

= Z [$z \quad - z \quad$]

$Z \quad Z$

$= Z - 2 - Z - 1$

on taking Inverse Z-Transform on both sides we get

$$Z^{-1}[\mu(Z)] = Z^{-1}\left[\frac{Z}{Z-2} - \frac{Z}{Z-1}\right]$$

$$\mu^n = Z^{-1}\left|\frac{Z}{Z-2}\right| - Z^{-1}\left|\frac{Z}{Z-1}\right|$$

$$\mu_n = 2^n - 1$$

3.Solve the difference equation using Z-Transform

$$y_{n+2} - 4y_{n+1} + 3y_n = 0$$

Given that $y_0 = 2$ and $y_1 = 4$

Solution:- let $Z[y_n] = Y(Z)$

$Z[y_{n+1}] = Z[Y(Z) - y_0]$ $Z[y_{n+2}] = Z^2[Y(Z) - y_0 - y_1 Z^{-1}]$

taking Z-Transform of the given equation we get

$Z(y_{n+2}) - 4Z(y_{n+1}) + 3Z(y_n) = 0$

$Z^2[Y(Z) - y_0 - y_1 Z^{-1}] - 4Z[Y(Z) - y_0] + 3Y(Z) = 0$ using the given conditions it reduces to

$Z^2[Y(Z) - 2 - 4Z^{-1}] - 4Z[Y(Z) - 2] + 3Y(Z) = 0$

i.e:- $Y(Z)[Z^2 - 4Z + 3] - 2Z^2 - 4Z + 8Z = 0$

$$Y(Z)[Z^2 - 4Z + 3] = Z(2Z-4)$$

$$\frac{Y\ Z}{Z} = \frac{2Z-4}{[Z^2-4Z+3]}$$

$$= \frac{2Z-4}{(Z-1)(Z-3)}(\quad)$$

$\frac{Y(Z)}{Z} = \frac{1}{Z-1} + \frac{1}{Z-3}$ (reducing by partial fractions)

$Y(Z) = \frac{Z}{Z-1} + \frac{Z}{Z-3}$ on taking Inverse Z-Transform on both sides we obtain

$$Z^{-1}[Y(Z)] = Z^{-1}\left[\frac{Z}{Z-1}\right] + Z^{-1}\left[\frac{Z}{Z-3}\right]$$

$$y_n = 1 + 3^n$$