Frequency response Analysis

The ratio of sinusoidal response and sinusoidal i/p is called sinusoidal transfer function. In general, it is denoted by $T(j\omega)$. The sinusoidal transfer function is the frequency domain representation of the system and so it is also called "frequency domain transfer function"

→ If the s-domain transfer function $T(s)$ is known the frequency domain transfer function $T(j\omega)$ can be obtained directly from $T(s)$ by replacing s by '$j\omega$'

i.e $T(s) = T(j\omega)$

$$r(t) \longrightarrow \boxed{T(j\omega) = |T(j\omega)| \angle T(j\omega)} \longrightarrow c(t) = B \angle \phi$$

$r(t) = A \sin(\omega t + \theta)$

$= A \angle \theta$

$B = A \times |T(j\omega)|$

$\phi = \theta + \angle T(j\omega)$

Frequency response :-

The frequency domain transfer function $T(j\omega)$ is a complex function of ω. Hence it can be separated into magnitude function and phase function. Now, the magnitude and phase functions will be real functions of ω, and they are called frequency response.

Open loop transfer function $G(s) \xrightarrow{s=j\omega} G(j\omega) \angle G(j\omega)$

Loop transfer function $G(s)\,H(s) \xrightarrow{s=j\omega} G(j\omega)\,H(j\omega)$

$$= |G(j\omega)\,H(j\omega)| \angle G(j\omega)H(j\omega)$$

Closed loop transfer function $M(s) \xrightarrow{s=j\omega} M(j\omega)$

$$= |M(j\omega)| \angle M(j\omega)$$

For unity feedback system $H(s) = 1$ and open loop and open loop transfer functions are same.

Advantages of frequency response analysis :-

→ The absolute and relative stability of closed loop system can be estimated from the knowledge of open loop frequency response.

→ The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipments.

→ The transfer function of complicated systems can be determined experimentally by frequency response tests.

→ The design and parameter adjustment of the open loop transfer function of a system for specified closed loop performance is carried out more easily in frequency domain.

→ When the system is designed by the use of frequency response analysis the effect of noise disturbance and parameters variations are relatively easy to visualize and incorporate corrective measures.

→ The frequency response analysis and designs can be extended to certain nonlinear control systems.

Frequency domain specifications:-

Resonant peak (M_r):- The maximum value of the magnitude of closed loop transfer function is called Resonant peak.

Resonant frequency (ω_r):- The frequency at which resonant peak occurs is called Resonant frequency.

Bandwidth (ω_b):-

The Bandwidth is the range of frequencies for which the system normalized gain is more than -3db. The frequency at which the gain is -3db is called cut off frequency. Bandwidth is usually defined for closed loop system and it transmits the signals whose frequencies are less than the cut off frequency. The Bandwidth is a measure of the ability of a feedback system to reproduce the i/p signal, noise rejection characteristics and rise time. A large bandwidth corresponds to a small rise time or fast response.

Cut-off rate :-

The slope of the log-magnitude curve near the cut off frequency is called cut-off rate. The cut-off rate indicates the ability of the system to distinguish the signal from noise.

Gain Margin K_g :-

The gain margin K_g is defined as the value of gain, to be added to system, in order to bring the system to the verge of instability.

The gain margin K_g is given by the reciprocal of the magnitude of open loop transfer function at phase cross over frequency ω_{pc}.

$$\text{Gain Margin } K_g = \frac{1}{|G(j\omega_{pc})|}$$

The gain margin in db can be expressed as

$$K_g \text{ in db} = 20 \log K_g = 20 \log \frac{1}{|G(j\omega_{pc})|}$$

The Gain margin in db is given by the -ve of the db magnitude of $G(j\omega)$ at phase cross over frequency. The gain margin indicates the additional gain that can be provided to system without affecting the stability of the system.

Phase Margin (γ) :-

The phase margin γ is defined as the additional phase lag to be added at the gain cross over frequency in order to bring the system to the verge of instability. The gain cross over frequency ω_{gc} is the frequency at which the magnitude of the open loop transfer function is unity. (or it is the frequency at which the db magnitude is zero).

The phase margin γ is obtained by adding $180°$ to the phase angle ϕ of the open loop transfer function at the gain cross over frequency.

$$\text{phase margin } \gamma = 180° + \phi_{gc}$$

$$\text{where } \phi_{gc} = \angle G(j\omega_{gc})$$

Note :- $\angle G(j\omega_{gc})$ is the phase angle of $G(j\omega)$ at $\omega = \omega_{gc}$.

Bode Plot :-

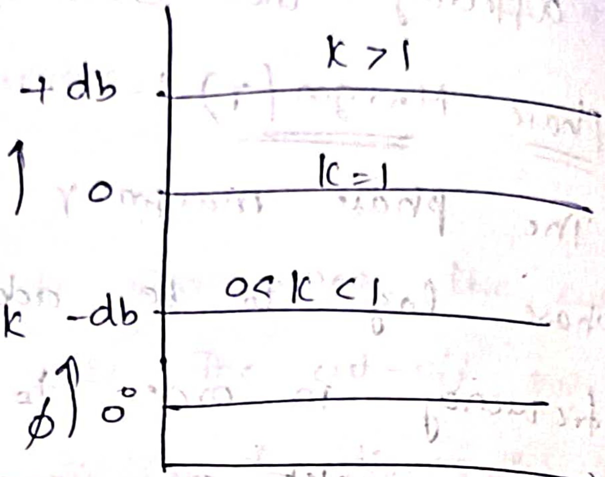
Constant Gain k :-

Let $G(s) = k$

$G(j\omega) = k = k\angle 0^\circ$

$A = |G(j\omega)|$ in db $= 20\log k$

$\phi = \angle G(j\omega) = 0^\circ$



The magnitude plot for a constant gain k is a horizontal straight line at the magnitude of 20 log k db. The phase plot is straight line at 0°.

When $k > 1$, $20\log k$ is +ve

When $0 < k < 1$, $20\log k$ is −ve

When $k = 1$, $20\log k$ is zero.

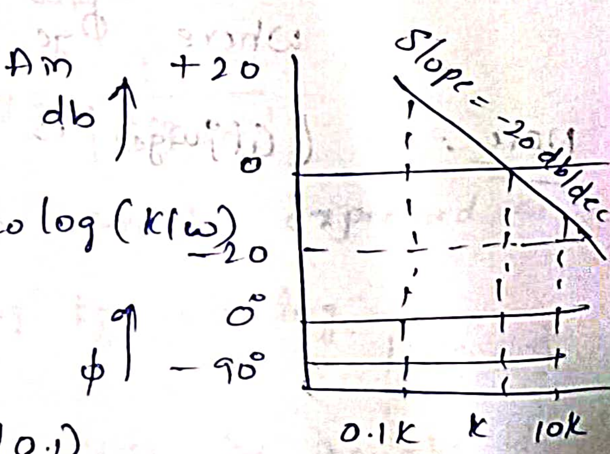
Integral factor :-

Let $G(s) = \frac{k}{s}$

$G(j\omega) = \frac{k}{j\omega} = \frac{k}{\omega}\angle -90^\circ$

$A = |G(j\omega)|$ in db $= 20\log(k/\omega)$

$\phi = \angle G(j\omega) = -90^\circ$



when $\omega = 0.1k$ $A = 20\log(1/0.1)$

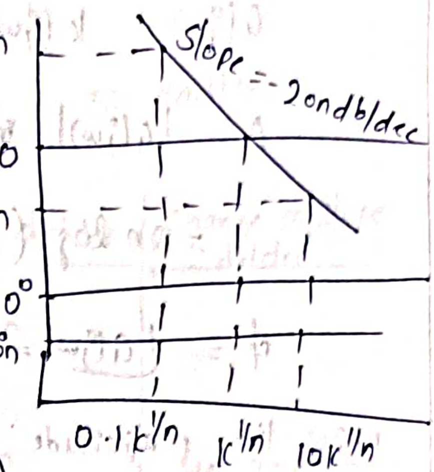
$= 20$ db

when $\omega = k$ $A = 20\log 1 = 0$ db

when $\omega = 10k$, $A = 20\log(1/10) = -20$db when integral factor has multiplicity of n, then

$G(s) = k/s^n$

$G(j\omega) = k/(j\omega)^n = k/\omega^n \angle -90^\circ n$



$A = |G(j\omega)|$ in db

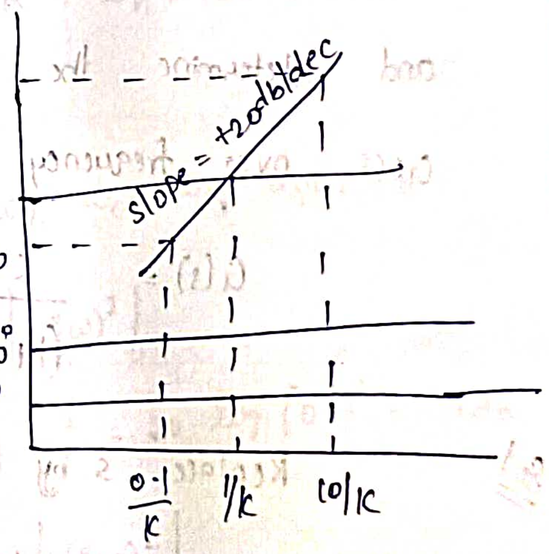
$= 20 \log \dfrac{k}{\omega^n}$

$= 20 \log \left(\dfrac{k^{1/n}}{\omega}\right)^n = 20n \log\left(\dfrac{k^{1/n}}{\omega}\right)$

$\phi = \angle G(j\omega) = -90^\circ n^\circ$

Now the magnitude plot of the integral factor is straight line with a slope of $-20n$ db/dec and passing through zero db when $\omega = k^{1/n}$. The phase plot is a straight line at $-90n^\circ$.

<u>Derivative factor</u>:-



Let $G(s) = ks$

$G(j\omega) = kj\omega = k\omega \angle 90^\circ$

$A = |G(j\omega)|$ in db

$= 20 \log (k\omega)$

$\phi = \angle G(j\omega) = +90^\circ$

when $\omega = 0.1/k$ $A = 20 \log (0.1) = -20db$

when $\omega = 1/k$ $A = 20 \log 1 = 0 db$

when $\omega = 10/k$ $A = 20 \log 10 = +20db$

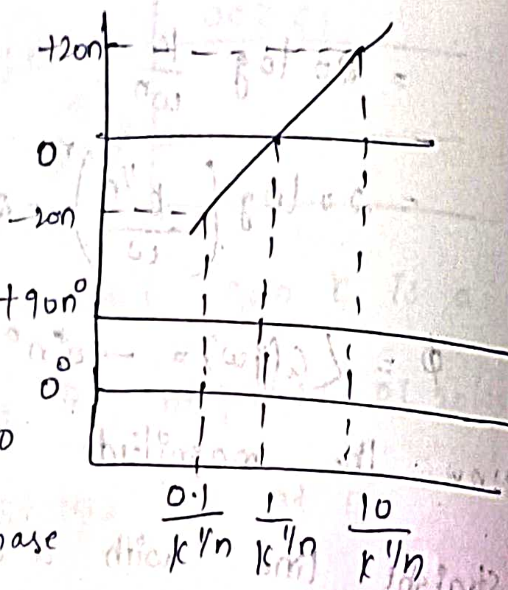
$G(s) = k s^n$

$G(j\omega) = k (j\omega)^n = k\omega^n \angle 90n^\circ$

$A = |G(j\omega)|$ in db $= 20 \log (k\omega^n)$

$= 20 \log (k^{1/n}\omega)^n = 20\, n \log (k^{1/n}\omega)$

$\phi = \angle G(j\omega) = 90n^\circ$



Now the magnitude plot of the derivative factor is a st. line with a slope of +20n db/dec and passing through zero db when $\omega = (1/k)^{1/n}$. The phase plot is a st. line at $+90n^\circ$.

* Sketch Bode plot for the following transfer function and determine the system gain k for the gain cross over frequency to be 5 rad/sec.

$$G(s) = \frac{k s^2}{(1+0.2s)(1+0.02s)}$$

Sol Replace s by $j\omega$

$$G(j\omega) = \frac{k (j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

$$G(j\omega) = \frac{(j\omega)^2 \cdot 1}{(1+j0.2\omega)(1+j0.02\omega)}$$

Magnitude plot :-

The corner frequencies $\omega_{c1} = \frac{1}{0.2} = 5$ rad/sec

$\omega_{c2} = \frac{1}{0.02} = 50$ rad/sec

Term	Corner frequency	Slope db/dec	Change in slope db/dec
$(j\omega)^2$		+40	
$\frac{1}{1+j0.2\omega}$	$\omega_{c1} = \frac{1}{0.2} = 5$	-20	40-20=20
$\frac{1}{1+j0.02\omega}$	$\omega_{c2} = \frac{1}{0.02} = 50$	-20	20-20=0

choose a low frequency ω_l such that $\omega_l < \omega_{c1}$

and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$

Let $\omega_l = 0.5$ rad/sec and $\omega_h = 100$ rad/sec

Let $A = |G(j\omega)|$ in db.

Let us calculate A at ω_l, ω_{c1} ω_{c2} and ω_h

At $\omega = \omega_l$, $A = 20 \log |(j\omega)^2|$

$= 20 \log \omega^2 = 20 \log (0.5)^2 = -12$ db

At $\omega = \omega_{c1}$, $A = 20 \log |(j\omega)^2|$

$= 20 \log \omega^2 = 20 \log 5^2 = 28$ db

At $\omega = \omega_{c2}$, $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}}\right]$

$+ A(\text{at } \omega = \omega_{c1})$

$= 20 \log \frac{50}{5} + 28 = 48$ db

At $\omega = \omega_h$, $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c2}}\right]$

$+ A \left(\text{at } \omega = \omega_{c2}\right)$

$= 0 \times \log \frac{100}{80} + 48 = 48\,db$

<u>phase plot</u> :-

$$\phi = \angle G(j\omega) = 180^\circ - \tan^{-1} 0.2\omega - \tan^{-1} 0.02\omega$$

ω	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} 0.02\omega$ deg	$\phi = \angle G(j\omega)$ deg	Point in Phase plot
0.5	5.7	0.6	$173.7 \approx 174$	e
1	11.3	1.1	$167.6 \approx 168$	f
5	45	5.7	$129.3 \approx 130$	g
10	63.4	11.3	$105.3 \approx 106$	h
50	84.3	45	$50.7 \approx 50$	i
100	87.1	63.4	$29.5 \approx 30$	j

<u>calculation of k</u> :-

$\therefore\ 20 \log k = -28\,db$

$\log k = \frac{-28}{20} = 10^{-28/20} = 0.0398$

*

* Sketch the bode plot for the following transfer function and determine phase margin & gain margin.

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}$$

Sol

$\therefore s^2+16s+100 = s^2+2\delta\omega_n s+\omega_n^2$

on comparing we get

$\omega_n^2 = 100 \Rightarrow \omega_n = 10$

$2\delta\omega_n = 16 \Rightarrow \delta = \frac{16}{2\omega_n} = \frac{16}{2\times10} = 0.8$

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)} = \frac{75(1+0.2s)}{s\times100\left[\frac{s^2}{100}+\frac{16s}{100}+1\right]}$$

$$= \frac{0.75(1+0.2s)}{s[1+0.01s^2+0.16s]}$$

The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$ in $G(s)$

$$\therefore G(j\omega) = \frac{0.75(1+0.2j\omega)}{j\omega[1+0.01(j\omega)^2+0.16j\omega]}$$

$$= \frac{0.75(1+j0.2\omega)}{j\omega[1-0.01\omega^2+j0.16\omega]}$$

Magnitude Plot :-

The corner frequencies are $\omega_{c1} = \frac{1}{0.2} = 5$ rad/sec

and $\omega_{c2} = \omega_n = 10$ rad/sec

Table 1

Term	corner frequency	slope db/dec	change in slope db/dec
$\frac{0.75}{j\omega}$	-	-20	
$1+j0.2\omega$	$\omega_{c1} = \frac{1}{0.2} = 5$	20	$-20+20=0$
$\frac{1}{1-0.01\omega^2+j0.16\omega}$	$\omega_{c2} = \omega_n = 10$	-40	$0-40=-40$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and

Choose a high frequency ω_h such that $\omega_h > \omega_{c2}$

Let $\omega_l = 0.5$ rad/dec and $\omega_h = 20$ rad/sec

Let $A = |G(j\omega)|$ in db

At $\omega = \omega_l$, $A = 20 \log \left|\frac{0.75}{j\omega}\right|$

$= 20 \log \frac{0.75}{0.5} = 3.5\,db$

At $\omega = \omega_{c1}$ $A = 20 \log \left|\frac{0.75}{j\omega}\right| = 20 \log \left[\frac{0.75}{5}\right]$

$= -16.5\,db$

At $\omega = \omega_{c2}$, $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}}\right]$

$+ A \text{ (at } \omega = \omega_{c1})$

$= 0 \times \log \frac{10}{5} + (-16.5)$

$= -16.5\,db$

At $\omega = \omega_h$, $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c2}}\right]$

$+ A \text{ (at } \omega = \omega_{c1})$

$= -40 \times \log \frac{20}{10} + (-16.5) = -28.5 \text{ db}$

phase plot :-

The phase angle of $G(j\omega)$ as a function of ω is given by

$$\phi = \angle G(j\omega) = \tan^{-1} 0.2\omega - 90^0 - \tan^{-1} \frac{0.16\omega}{1 - 0.01\omega^2}$$

for $\omega \leq \omega_n$

$$\phi = \angle G(j\omega) = \tan^{-1} 0.2\omega - 90^0 - \left[\tan^{-1} \frac{0.16\omega}{1 - 0.01\omega^2} + 180^0\right]$$

for $\omega > \omega_n$

ω rad/sec	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} \frac{0.16\omega}{1-0.01\omega^2}$ deg	$\phi = \angle G(j\omega)$ deg	Points in phase plot
0.5	5.7	4.6	$-88.9 \approx -88$	e
1	11.3	9.2	$-87.9 \approx -88$	f
5	45	46.8	$-91.8 \approx -92$	g
10	63.4	90	$-116.6 \approx -116$	h
20	75.9	$-46.8 + 180 = 133.2$	$-147.3 \approx -148$	i
50	84.3	$-18.4 + 180 = 161.6$	$-167.3 \approx -168$	j
100	87.1	$-9.2 + 180 = 170.8$	$-173.7 \approx -174$	k

Let ϕ_{gc} be the phase of $G(j\omega)$ at gain cross over frequency

$\phi_{gc} = 88^\circ$ (from the figure)

$\therefore$ phase margin $\gamma = 180^\circ + \phi_{gc}$

$= 180^\circ - 88 = 92^\circ$

The phase plot crosses -180° only at ∞.

The $|G(j\omega)|$ at ∞ is $-\infty$ db

Hence Gain margin is $+\infty$

* Given $G(s) = \frac{k e^{-0.2s}}{s(s+2)(s+8)}$. Find k so that the system is stable with

a) Gain margin equal to 2db

b) phase margin equal to 45°

Sol let us take $k=1$ & convert the given transfer function to time constant form or bode form.

$$\therefore G(s) = \frac{e^{-0.2s}}{s(s+2)(s+8)} = \frac{e^{-0.2s}}{s \times 2\left(1+\frac{s}{2}\right) \times 8\left(1+\frac{s}{8}\right)}$$

$$= \frac{0.0625\, e^{-0.2s}}{s(1+0.5s)(1+0.125s)}$$

$$G(j\omega) = \frac{0.0625\, e^{-j0.2\omega}}{j\omega(1+j0.5\omega)(1+j0.125\omega)}$$

Magnitude plot :-

The corner frequencies are $\omega_{c1} = \frac{1}{0.5} = 2$ rad/sec

and $\omega_{c2} = \frac{1}{0.125} = 8$ rad/sec

Table-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{0.0625}{j\omega}$		-20	
$\frac{1}{1+j0.5\omega}$	$\omega_{c1} = \frac{1}{0.5} = 2$	-20	$-20-20=-40$
$\frac{1}{1+j0.125\omega}$	$\omega_{c2} = \frac{1}{0.125} = 8$	-20	$-40-20=-60$

choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and

choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

let $\omega_l = 0.5$ rad/sec and $\omega_h = 50$ rad/sec

let $A = |G(j\omega)|$ in db

At $\omega = \omega_l$, $A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\frac{0.0625}{0.5}$

$= -18\,db$

At $\omega = \omega_{c1}$, $A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\frac{0.0625}{2}$

$= -30\,db$

At $\omega = \omega_{c2}$, $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega=\omega_{c1})}$

$= -40 \times \log\frac{8}{2} + (-30) = -54\,db$

At $\omega = \omega_h$, $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega=\omega_{c2})}$

$$= -60 \times \log \frac{50}{8} + (-54) = -102 \text{ db}$$

Phase plot :-

The phase angle of $G(j\omega)$ as a function of ω is given by

$$\phi = -0.2\omega \times \frac{180}{\pi} - 90^\circ - \tan^{-1} 0.5\omega - \tan^{-1} 0.125\omega$$

ω rad/sec	$-0.2\omega(180^\circ/\pi)$ deg	$\tan^{-1} 0.5\omega$ deg	$\tan^{-1} 0.125\omega$ deg	$\phi = \angle G(j\omega)$ deg	Points in Phase Plot
0.01	-0.1145	0.2864	0.0716	$-90.4 \approx -90$	e
0.1	-1.145	2.862	0.716	$-94.7 \approx -94$	f
0.5	-5.7	14	3.6	$-113.3 \approx -114$	g
1	-11.4	26	7.12	$-134.4 \approx -134$	h
2	-22.9	45	14	$-171.9 \approx -172$	i
3	-34.37	56.30	20.56	$-201.2 \approx -202$	j
4	-45.84	63.43	26.57	$-225.8 \approx -226$	k

Calculation of k :-

Phase margin $\gamma = 180^\circ + \phi_{gc}$

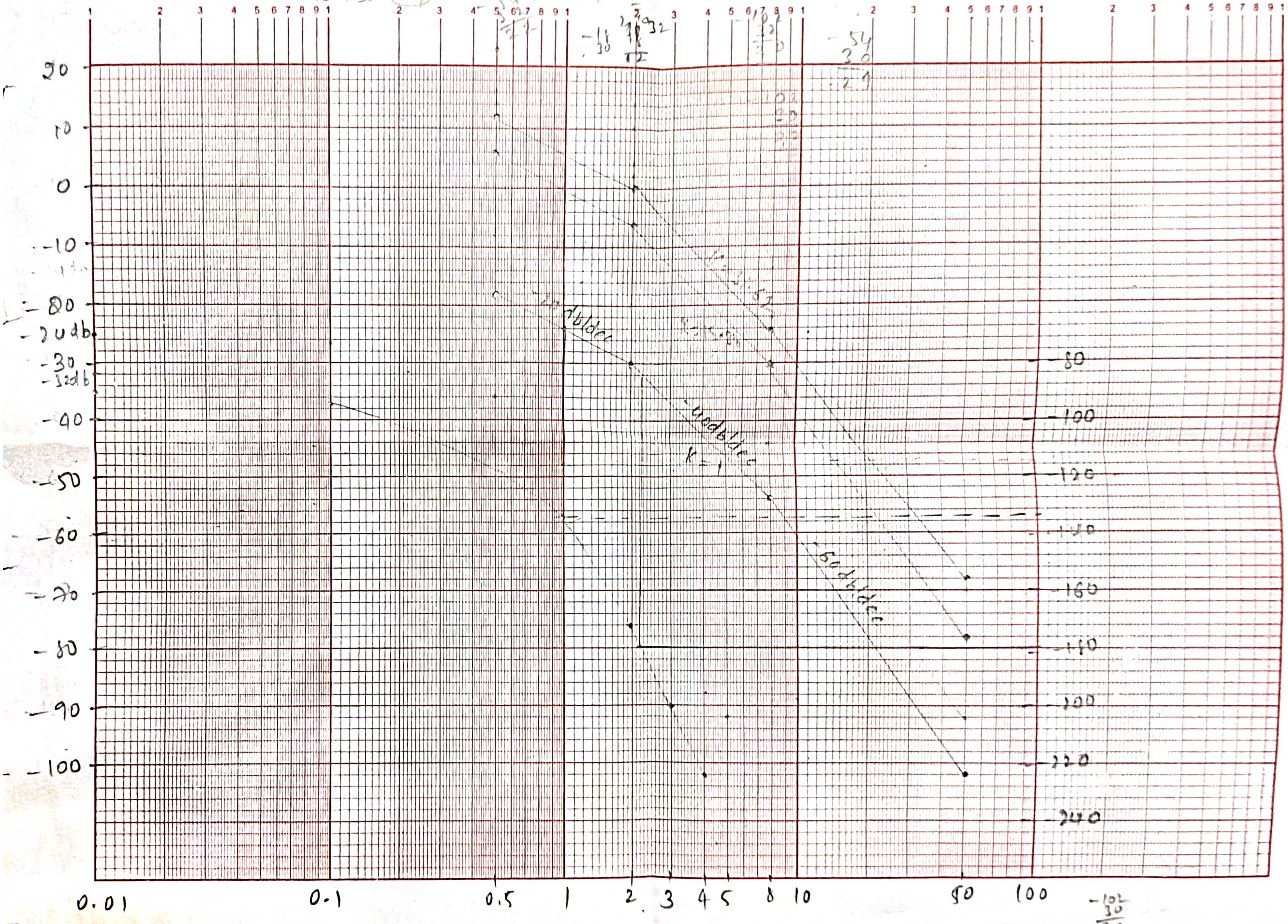
ϕ_{gc} is the phase of $G(j\omega)$ at $\omega = \omega_{gc}$

When $\gamma = 45^\circ$, $\phi_{gc} = \gamma - 180^\circ = 45^\circ - 180^\circ = -135^\circ$

With $k = 1$, the db gain at $\phi = -135^\circ$ is -24 db.

This gain should be made zero to have to PM

$-30+32$ $\quad -32+32$ $\quad -18+32$ (14) $\quad -18+24 = 6$ $\quad 54+32$ $\quad -18+10$ $\quad -30$



of 45°. Hence to every pt. of magnitude plot a db gain of 24db should be added. The corrected magnitude plot is obtained by shifting the plot with $K=1$ by 24 db upwards. The magnitude correction is independent of frequency.

$$\therefore \; 20 \log K = 24$$

$$K = 10^{24/20} = 15.84$$

with $K=1$, the gain margin $= -(-32) = 32$ db. But the required gain margin is 2db. Hence to every point of magnitude plot a db gain of 30db should be added. The addition of 30 db shifts the plot upwards. The magnitude correction is independent of frequency. Hence the magnitude of 30db is contributed by the term k.

$$\therefore \; 20 \log K = 30$$

$$K = 10^{30/20} = 31.62$$

4.4 Plot the Bode diagram for the following transfer function and obtain the gain and phase cross over frequencies

$$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$$

$$G(j\omega) = \frac{10}{j\omega(1+0.4j\omega)(1+j0.1\omega)}$$

Magnitude plot :-

Corner frequencies are

$$\omega_{c1} = \frac{1}{0.4} = 2.5 \text{ rad/sec} \quad \omega_{c2} = \frac{1}{0.1} = 10 \text{ rad/sec}$$

Table - 1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{10}{j\omega}$		-20	
$\frac{1}{1+j0.4\omega}$	$\omega_{c1} = \frac{1}{0.4} = 2.5$	-20	$-20-20=-40$
$\frac{1}{1+j0.1\omega}$	$\omega_{c2} = \frac{1}{0.1} = 10$	-20	$-40-20=-60$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and

Choose a high frequency ω_h such that $\omega_h > \omega_{c2}$

Let $\omega_l = 0.1$ rad/sec and $\omega_h = 50$ rad/sec.

Let $A = |G(j\omega)|$ in db

At $\omega = \omega_l$, $A = 20 \log \left|\frac{10}{j\omega}\right| = 20 \log \frac{10}{0.1} = 40$ db

At $\omega = \omega_{c1}$ $A = 20 \log \left|\frac{10}{j\omega}\right| = 20 \log \frac{10}{2.5} = 12$ db

At $\omega = \omega_{c2}$, $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(at\ \omega = \omega_{c1})}$

$$= -40 \times \log \frac{10}{2.5} + 12 = -12 \text{ db}$$

At $\omega = \omega_h$, $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c2}}\right] + A\,(\text{at } \omega = \omega_{c2})$

$$= -60 \times \log \frac{50}{10} + (-12) = -54\,db$$

phase plot :-

$$\phi = -90^\circ - \tan^{-1} 0.4\omega - \tan^{-1} 0.1\omega$$

ω rad/sec	$\tan^{-1} 0.4\omega$ deg	$\tan^{-1} 0.1\omega$ deg	$\phi = \angle G(j\omega)$ deg	Points in Phase plot
0.1	2.29	0.57	$-92.86 \approx -92$	e
1	21.80	5.71	$-117.5 \approx -118$	f
2.5	45.0	14.0	$-149 \approx -150$	g
4	57.99	21.8	$-169.79 \approx -170$	h
10	75.96	45.0	$-210.96 \approx -210$	i
20	82.87	63.43	-236.3	j

Result :-

Gain cross over frequency = 5 rad/sec

phase cross over frequency = 5 rad/sec

Example 4.5 :- For the following transfer function draw bode plot and obtain gain cross over frequency

$$G(s) = \frac{20}{s(1+3s)(1+4s)}$$

Solution

Put s by $j\omega$

$$G(j\omega) = \frac{20}{j\omega(1+3j\omega)(1+j4\omega)}$$

Magnitude Plot :-

The corner frequencies $\omega_{c1} = \frac{1}{4} = 0.25$ rad/sec

$\omega_{c2} = \frac{1}{3} = 0.333$ rad/sec

Table-1 :-

Term	Corner frequency	Slope db/dec	Change in slope db/dec
$\frac{20}{j\omega}$	-	-20	
$\frac{1}{1+j4\omega}$	$\omega_{c1} = \frac{1}{4} = 0.25$	-20	$-20-20=-40$
$\frac{1}{1+j3\omega}$	$\omega_{c2} = \frac{1}{3} = 0.33$	-20	$-40-20=-60$

Choose a frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$

Let $\omega_l = 0.15$ rad/sec and $\omega_h = 1$ rad/sec

At $\omega = \omega_l$, $A = |G(j\omega)| = 20 \log \left| \frac{20}{0.15} \right| = 42.5$ db

At $\omega = \omega_{c1}$ $A = |G(j\omega)| = 20 \log \left| \frac{20}{0.25} \right| = 38$ db

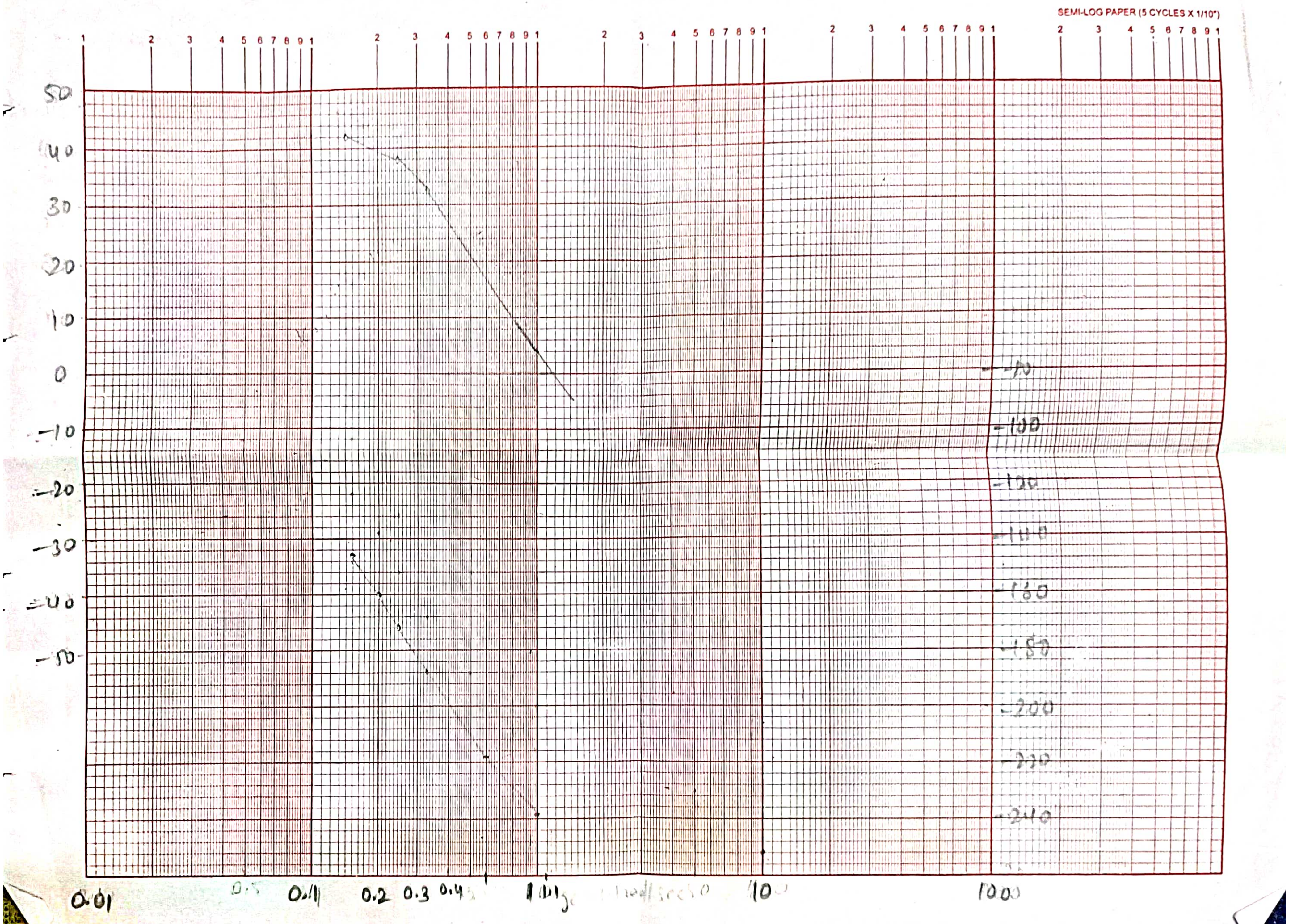
At $\omega = \omega_{c2}$ $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}} \right]$

$+ A \text{ (at } \omega = \omega_{c1})$

$= -40 \log \frac{0.33}{0.25} + 38 = 33$ db

At $\omega = \omega_h$, $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c2}} \right] +$

$A \text{ (at } \omega = \omega_{c2})$


SEMI-LOG PAPER (5 CYCLES X 1/10")
50
40
30
20
10
0
-10
-20
-30
-40
-50
0.01
0.1
0.2
0.3
0.4
1
10
100
1000
-100
-120
-140
-160
-180
-200
-220
-240

$= -60 \log \frac{1}{0.33} + 33 = 4 db$

<u>phase plot</u> :-

The phase angle of $G(j\omega)$; $\phi = -90° - \tan^{-1} 3\omega - \tan^{-1} 4\omega$

ω, rad/sec	$-\tan^{-1} 3\omega$, deg	$-\tan^{-1} 4\omega$, deg	$\phi = \angle G(j\omega)$	points on phase plot
0.15	24.22	30.96	$-145.18 \approx -146$	e
0.2	30.96	38.66	$-159.61 \approx -160$	f
0.25	36.86	45.0	$-171.86 \approx -172$	g
0.33	44.7	52.8	$-187.5 \approx -188$	h
0.6	60.14	67.38	$-218.32 \approx -218$	i
1	71.56	75.96	$-237.56 \approx -238$	j

<u>4.6</u> For the function $G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25s)}$ draw the bode plot.

<u>Sol</u> Replace s by $j\omega$

$$G(j\omega) = \frac{5(1+j2\omega)}{(1+j4\omega)(1+j0.25\omega)}$$

<u>Magnitude plot</u> :-

The corner frequencies are $\omega_{c1} = \frac{1}{4} = 0.25$ rad/sec

$\omega_{c2} = \frac{1}{2} = 0.5$ rad/sec

$\omega_{c3} = \frac{1}{0.25} = 4$ rad/sec

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$

and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

let $\omega_l = 0.1$ rad/sec and $\omega_h = 10$ rad/sec

Term	Corner frequency rad/sec	Slope db/dec	Change in Slope db/dec
5	-	0	
$\frac{1}{1+j4\omega}$	$\omega_{c1} = \frac{1}{4} = 0.25$	-20	$0-20=-20$
$1+2j\omega$	$\omega_{c2} = \frac{1}{2} = 0.5$	20	$-20+20=0$
$\frac{1}{1+j0.25\omega}$	$\omega_{c3} = \frac{1}{0.25} = 4$	-20	$0-20=-20$

At $\omega = \omega_l$, $A = |G(j\omega)| = 20 \log 5 = 14db$

At $\omega = \omega_{c1}$, $A = |G(j\omega)| = 20 \log 5 = 14db$

At $\omega = \omega_{c2}$, $A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}}\right] + A \text{ (at } \omega = \omega_{c1})$

$$= -20 \log \frac{0.5}{0.25} + 14 = +8db$$

At $\omega = \omega_{c3}$, $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log \frac{\omega_{c3}}{\omega_{c2}}\right] + A \text{ (at } \omega = \omega_{c2})$

$$= 0 \times \log \frac{4}{0.5} + 8 = 8 db$$

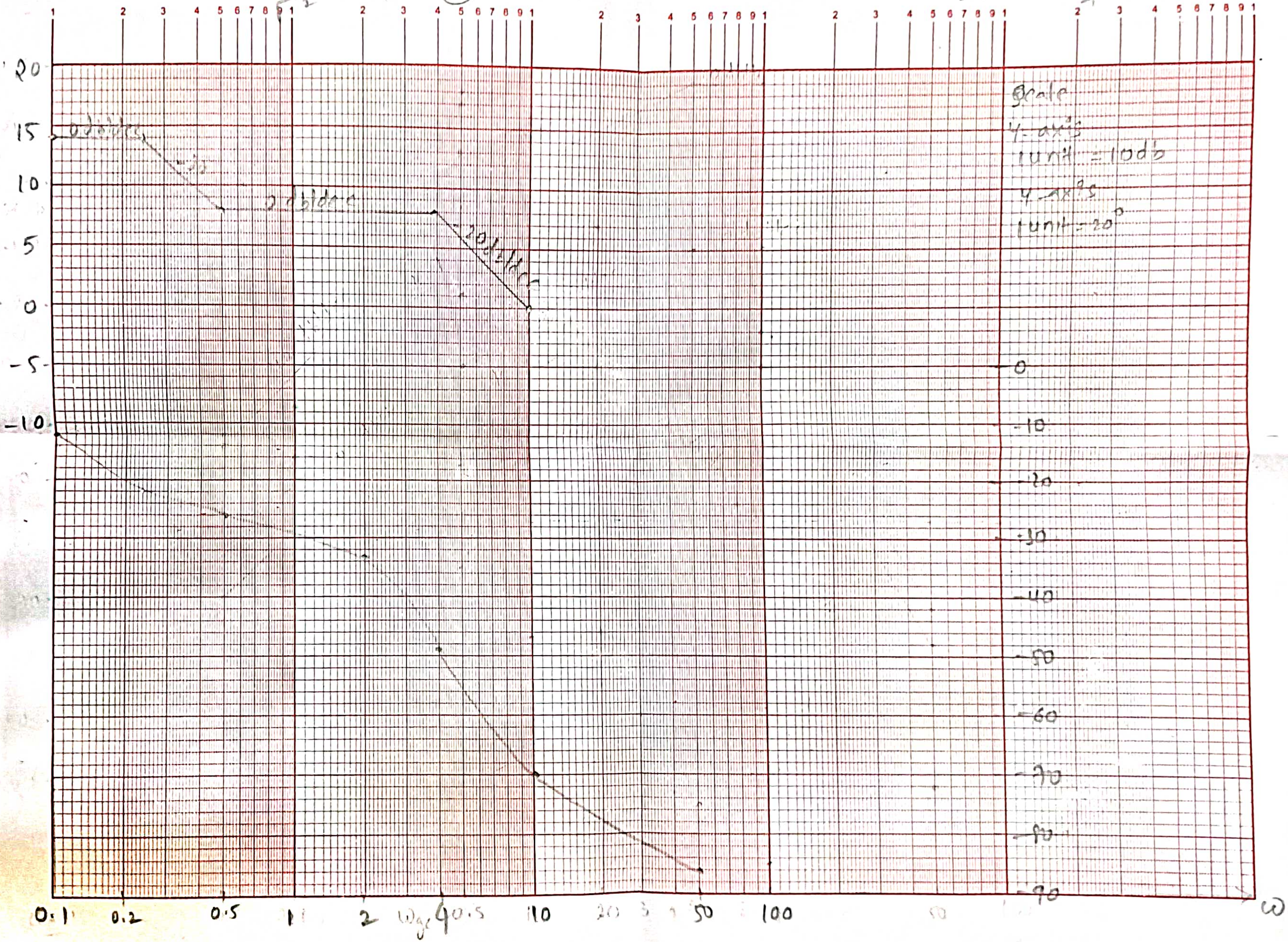
At $\omega = \omega_h$, $A = \left[\text{Slope from } \omega_{c3} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c3}}\right] + A \text{ (at } \omega = \omega_{c3})$

$$= -20 \log \frac{10}{4} + 8 = 0db$$

-18 + 24

$20 \log K = -28$

$K = 10^{-28/20} =$



Phase plot :-

The phase angle of $G(j\omega)$, $\phi = \tan^{-1}(2\omega) - \tan^{-1}(4\omega) - \tan^{-1}(0.25\omega)$

ω	$\tan^{-1} 2\omega$ deg	$\tan^{-1} 4\omega$ deg	$\tan^{-1} 0.25\omega$ deg	$\phi = \angle G(j\omega)$	points in Phase plot
0.1	11.3	21.8	1.43	$-11.93 \approx -12$	f
0.25	26.56	45.0	3.5	$-21.94 \approx -22$	g
0.5	45.0	63.43	7.1	$-25.53 \approx -26$	h
2	75.96	82.87	26.56	$-33.47 \approx -33$	i
4	82.87	86.42	45.0	$-48.55 \approx -49$	j
10	87.13	88.56	68.19	$-69.62 \approx -70$	k
50	89.42	89.71	85.42	$-85.71 \approx -86$	l

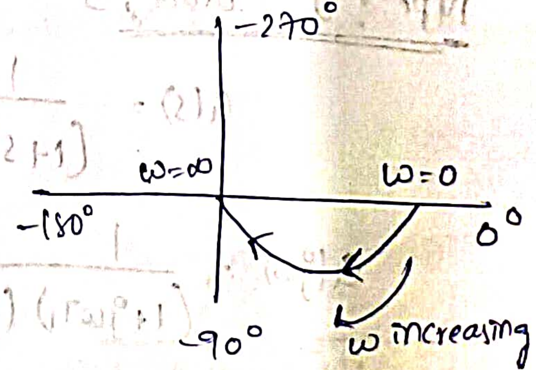
Polar plot :-

Type : 0, order : 1 $\qquad G(s) = \frac{1}{1+sT}$

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}\ \angle \tan^{-1}\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T$$

As $\omega \to 0$ $\quad G(j\omega) = 1\angle 0^\circ$

As $\omega \to \infty$ $\quad G(j\omega) = 0\angle -90^\circ$

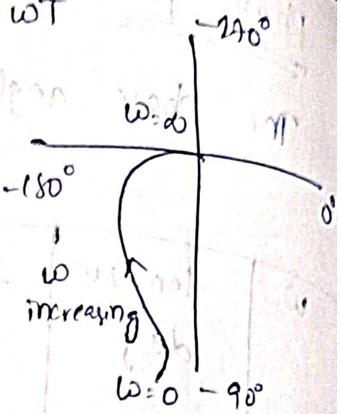


Type : 1, order : 2

$$G(s) = \frac{1}{s(1+sT)}$$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)} = \frac{1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2T^2}} \cdot \frac{1}{\angle \tan^{-1}\omega T}$$

$$= \frac{1}{\omega\sqrt{1+\omega^2T^2}} \angle -90^\circ - \tan^{-1}\omega T$$



As $\omega \to 0$, $G(j\omega) \to \infty \angle -90^\circ$

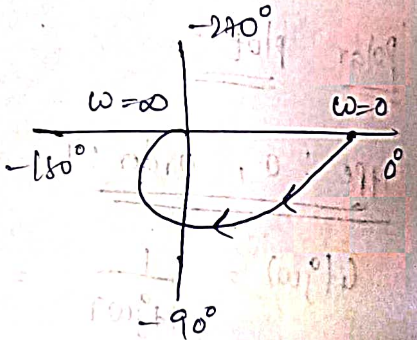
As $\omega \to \infty$, $G(j\omega) \to 0 \angle -180^\circ$

Type : 0, order : 2

$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$$

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)} = \frac{1}{\sqrt{1+\omega^2T_1^2}\angle \tan^{-1}\omega T_1 \cdot \sqrt{1+\omega^2T_2^2}\angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}} \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$



As $\omega \to 0$, $G(j\omega) \to 1 \angle 0^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0 \angle -180^\circ$

Type : 0, order : 3

$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$$

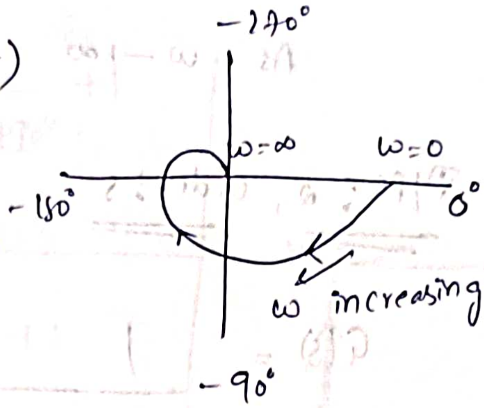
$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

$$= \frac{1}{\sqrt{1+\omega^2T_1^2}\angle \tan^{-1}\omega T_1 \sqrt{1+\omega^2T_2^2}\angle \tan^{-1}\omega T_2 \sqrt{1+\omega^2T_3^2}\angle \tan^{-1}\omega T_3}$$

$$= \frac{1}{\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)(1+\omega^2T_3^2)}} \angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$$

As $\omega \to 0$, $G(j\omega) \to 1 \angle 0^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0 \angle -270^\circ$



Type : 1, order : 3

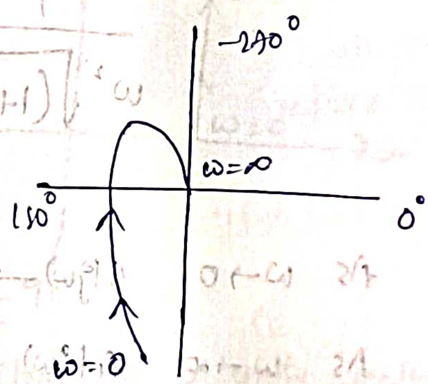
$$G(s) = \frac{1}{s(1+sT_1)(1+sT_2)}$$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T_1)(1+j\omega T_2)} = \frac{1}{\omega\angle 90^\circ \sqrt{1+\omega^2T_1^2}\angle \tan^{-1}\omega T_1 \sqrt{1+\omega^2T_2^2}\angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\omega\sqrt{(1+\omega T_1^2)(1+\omega T_2^2)}} \angle -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

As $\omega \to 0$, $G(j\omega) \to \infty \angle -90^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0 \angle -270^\circ$



Type : 2, order : 4 :-

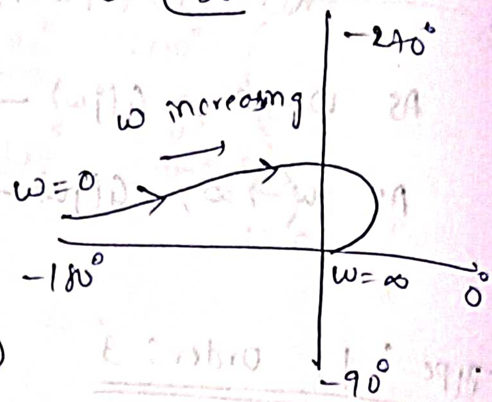
$$G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)}$$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)} = \frac{1}{\omega^2\angle -180^\circ \sqrt{1+\omega^2T_1^2}\angle \tan^{-1}\omega T_1 \sqrt{1+\omega^2T_2^2}\angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\omega^2\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}} \angle -180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

As $\omega \to 0$, $G(j\omega) \to \infty \angle -180^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0 \angle -360^\circ$



Type: 2, Order: 5

$$G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$$

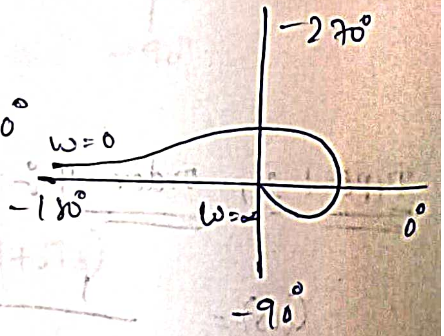
$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

$$= \frac{1}{\omega^2 \angle -180^\circ \sqrt{1+\omega^2T_1^2} \angle \tan^{-1}\omega T_1 \sqrt{1+\omega^2T_2^2} \angle \tan^{-1}\omega T_2 \sqrt{1+\omega^2T_3^2} \angle \tan^{-1}\omega T_3}$$

$$= \frac{1}{\omega^2\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)(1+\omega^2T_3^2)}} \angle -180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$$

As $\omega \to 0$ $G(j\omega) \to \infty \angle -180^\circ$

As $\omega \to \infty$ $G(j\omega) \to 0 \angle -450^\circ = 0 \angle -90^\circ$

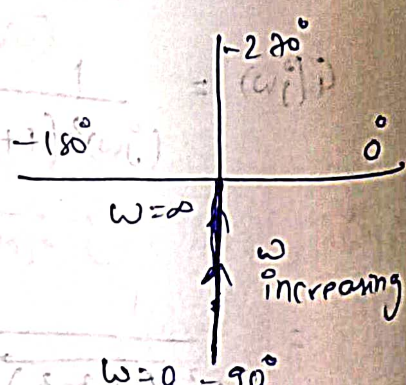


Type: 1 order: 1 :- $G(s) = 1/s$

$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega\angle 90^\circ} = \frac{1}{\omega}\angle -90^\circ$$

As $\omega \to 0$, $G(j\omega) \to \infty \angle -90^\circ$

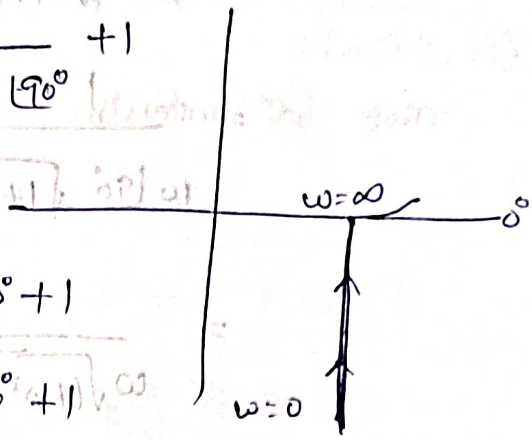
As $\omega \to \infty$, $G(j\omega) \to 0 \angle -90^\circ$



$$G(s) = \frac{1+sT}{sT}$$

$$G(j\omega) = \frac{1+j\omega T}{j\omega T} = \frac{1}{j\omega T} + 1 = \frac{1}{\omega T \angle 90^\circ} + 1$$

$$= \frac{1}{\omega T}\angle -90^\circ + 1$$

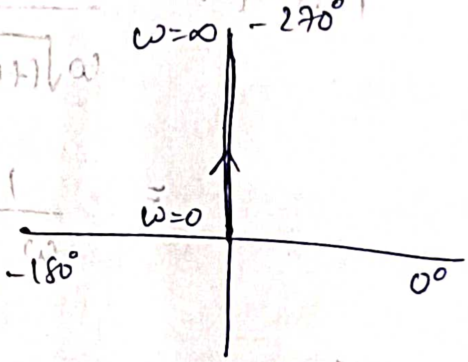


As $\omega \to 0$, $G(j\omega) \to \infty \angle -90^\circ + 1$

As $\omega \to \infty$, $G(j\omega) \to 0 \angle -90^\circ + 1$

$G(s) = s$

$G(j\omega) = j\omega = \omega \angle 90^\circ$

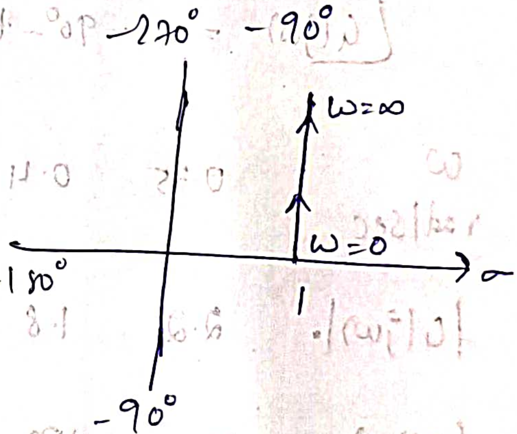


As $\omega \to 0$, $G(j\omega) \to 0 \angle 90^\circ$

As $\omega \to \infty$, $G(j\omega) \to \infty \angle 90^\circ$

$G(s) = 1 + sT$

$G(j\omega) = 1 + j\omega T = 1 + \omega T \angle 90^\circ$



As $\omega \to 0$, $G(j\omega) = 1 + 0 \angle 90^\circ$

As $\omega \to \infty$, $G(j\omega) = 1 + \infty \angle 90^\circ$

Ex: 4.7 The open loop transfer function of a unity feedback system is given by $G(s) = \frac{1}{s(1+s)(1+2s)}$. Sketch the polar plot and determine the Gain Margin & Phase Margin.

Sol

$$G(s) = \frac{1}{s(1+s)(1+2s)}$$

Puts $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ \sqrt{1+\omega^2}\angle\tan^{-1}\omega \sqrt{1+4\omega^2}\angle\tan^{-1}2\omega}$$

$$= \frac{1}{\omega\sqrt{(1+\omega^2(1+4\omega^2)}} \angle -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2(1+4\omega^2)}} = \frac{1}{\omega\sqrt{1+\omega^2+4\omega^2+4\omega^4}}$$

$$= \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

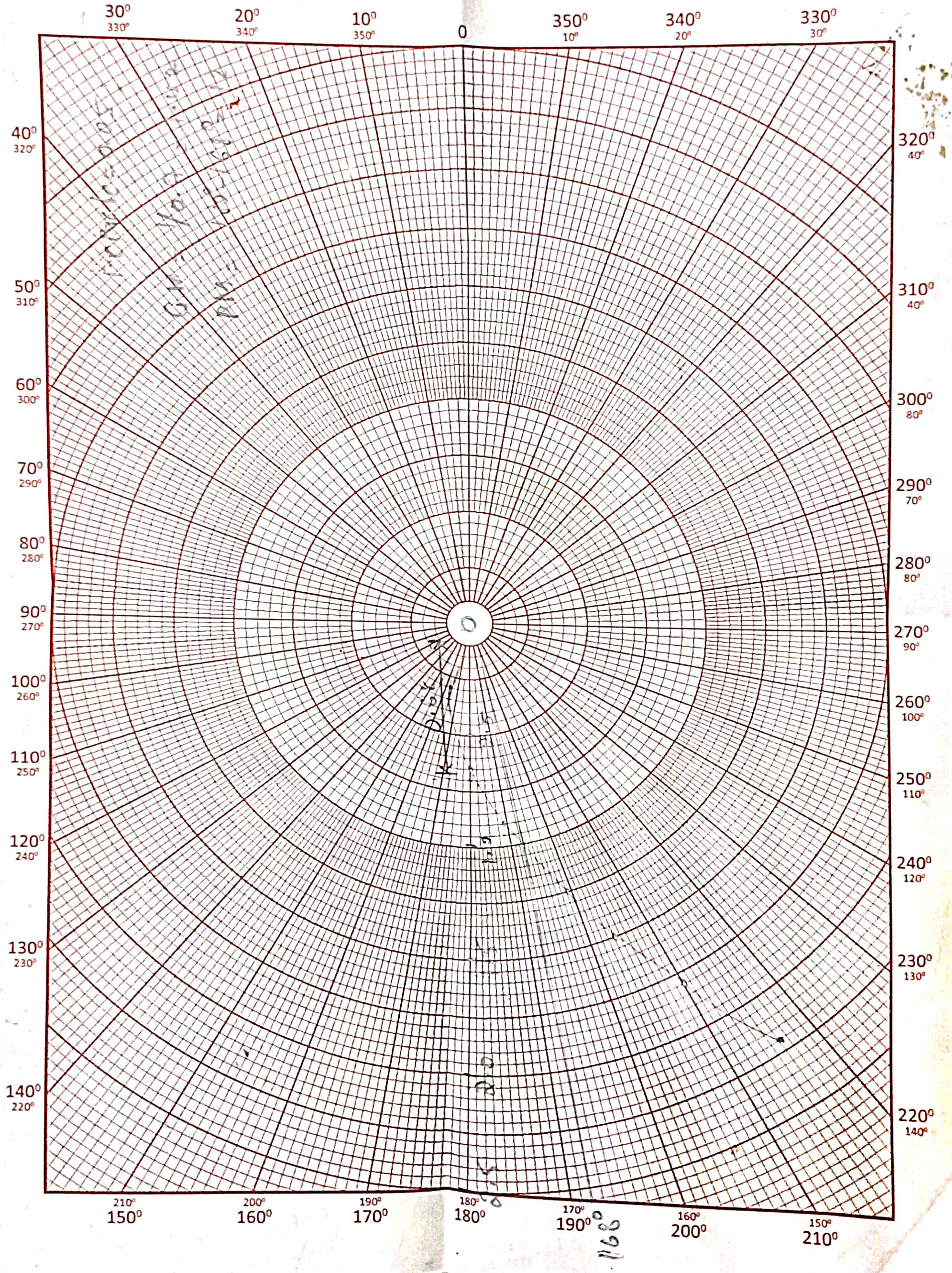
ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$\lvert G(j\omega)\rvert$	2.2	1.8	1.5	1.2	0.9	0.7	0.3
$\angle G(j\omega)$	-144	-150	-156	-162	-171	-179.5 ≈ -180	-198

Result :-

Gain Margin $K_g = \frac{1}{|G(j\omega_{pc})|} = \frac{1}{0.7} = 1.4286$

Phase Margin $\gamma = 180^\circ + \phi_{gc}$

$= 180^\circ - 168^\circ = 12^\circ$


30° 330°
20° 340°
10° 350°
0
350° 10°
340° 20°
330° 30°
40° 320°
50° 310°
60° 300°
70° 290°
80° 280°
90° 270°
100° 260°
110° 250°
120° 240°
130° 230°
140° 220°
320° 40°
310° 40°
300° 80°
290° 70°
280° 80°
270° 90°
260° 100°
250° 110°
240° 120°
230° 130°
220° 140°
210° 150°
200° 160°
190° 170°
180° 180°
170° 190°
160° 200°
150° 210°

* The open loop transfer function of a unity feed back system is given by $G(s) = \frac{1}{s^2(1+s)(1+2s)}$ sketch the polar plot and determine the gain Margin and phase Margin

Sol

$$G(s) = \frac{1}{s^2(1+s)(1+2s)}$$

put $s = j\omega$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega)(1+2j\omega)}$$

$$= \frac{1}{\omega^2 \angle 180^\circ \sqrt{1+\omega^2} \tan^{-1}\omega \sqrt{1+4\omega^2} \tan^{-1}(2\omega)}$$

$$G(j\omega) = \frac{1}{\omega^2\sqrt{1+\omega^2}\sqrt{1+4\omega^2}} \angle -180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

$$|G(j\omega)| = \frac{1}{\omega^2\sqrt{1+\omega^2}\sqrt{1+4\omega^2}} = \frac{1}{\omega^2\sqrt{(1+\omega^2)(1+4\omega^2)}}$$

$$= \frac{1}{\omega^2\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

Magnitude and phase plot of $G(j\omega)$ at various frequencies

ω rad/sec	0.45	0.5	0.55	0.6	0.65	0.7	0.75	1.0
$\lvert G(j\omega)\rvert$	3.3	2.5	1.9	1.5	1.2	0.97 ≈ 1	0.8	0.3
$\angle G(j\omega)$	-246	-251	-256	-261	-265	-269	-273	-288

Result :-

Gain margin $K_g = 1/|G(j\omega_{pc})| = 1/\infty = 0$

Phase margin $\gamma = 180^\circ + \phi_{gc} = 180^\circ - 270^\circ = -90^\circ$

4.9 The open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{(1+0.2s)(1+0.025s)}{s^3(1+0.005s)(1+0.001s)}$$ Sketch the polar plot & determine the phase margin.

Sol)

$$G(s) = \frac{(1+0.2s)(1+0.025s)}{s^3(1+0.005s)(1+0.001s)}$$

$$G(j\omega) = \frac{(1+0.2j\omega)(1+0.025j\omega)}{(j\omega)^3(1+j0.005\omega)(1+j0.001\omega)}$$

$$= \frac{\sqrt{1+(0.2\omega)^2}\,\angle\tan^{-1}0.2\omega\;\sqrt{1+(0.025\omega)^2}\,\angle\tan^{-1}0.025\omega}{\omega^3\angle 270^\circ\,\sqrt{1+(0.005\omega)^2}\,\angle\tan^{-1}0.005\omega\,\sqrt{1+(0.001\omega)^2}\,\angle\tan^{-1}0.001\omega}$$

$$|G(j\omega)| = \frac{\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.025\omega)^2}}{\omega^3\sqrt{1+(0.005\omega)^2}\sqrt{1+(0.001\omega)^2}}$$

$$\angle G(j\omega) = \tan^{-1}0.2\omega + \tan^{-1}0.025\omega - 270^\circ - \tan^{-1}0.005\omega - \tan^{-1}0.001\omega$$

Magnitude & phase of $G(j\omega)$

ω rad/sec	0.9	0.95	1.0	1.1	1.2	1.4	1.7
$\lvert G(j\omega)\rvert$	1.4	1.2	1.0	0.8	0.6	0.4	0.2
$\angle G(j\omega)$	-259	-258	-257	-256	-255	-253	-249

Result :-

phase margin $\gamma = -77°$ $(180° - 257° = -77°)$

4.10 The open loop transfer function of a unity feedback system is given by $G(s) = \frac{1}{s(1+s)^2}$. Sketch the polar plot and determine the gain & phase margin

Sol

$$G(s) = \frac{1}{s(1+s)^2}$$

Put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)^2} = \frac{1}{j\omega(1+j\omega)(1+j\omega)}$$

$$G(j\omega) = \frac{1}{\omega\angle 90° \sqrt{1+\omega^2}\angle \tan^{-1}\omega \sqrt{1+\omega^2}\angle \tan^{-1}\omega}$$

$$= \frac{1}{\omega\left(\sqrt{1+\omega^2}\right)^2} \angle\left(-90° - 2\tan^{-1}\omega\right)$$

$$|G(j\omega)| = \frac{1}{\omega(1+\omega^2)} = \frac{1}{\omega + \omega^3}$$

$$\angle G(j\omega) = -90° - 2\tan^{-1}\omega$$

Magnitude and phase of $G(j\omega)$ at various frequencies

ω rad/sec	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
$\lvert G(j\omega)\rvert$	2.2	1.6	1.2	1	0.8	0.6	0.5	0.4
$\angle G(j\omega)$	-134	-143	-151	-159	-167	-174	-180	-185

Result :-

Gain Margin $K_g = 2 = \frac{1}{|G(j\omega_{pc})|} = \frac{1}{0.5} = 2$

Phase Margin $\gamma = 21° = 180° + \phi_{gc}$

$= 180° - 159° = 21°$

4.11 Consider a unity feedback system having an open loop transfer function $G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$. Sketch the polar plot and determine the value of K so that (i) Gain margin is 18 db
(ii) phase margin is 60°

Sol) Given that $G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$

Polar plot is sketched by taking $K = 1$

put $K = 1$ and $s = j\omega$ in $G(s)$

$$G(j\omega) = \frac{1}{j\omega(1+0.2j\omega)(1+0.05j\omega)}$$

$$= \frac{1}{\omega \angle 90^\circ \sqrt{1+(0.2\omega)^2} \angle \tan^{-1} 0.2\omega \sqrt{1+(0.05\omega)^2} \angle \tan^{-1} 0.05\omega}$$

$$= \frac{1}{\omega\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}} \angle -90^\circ - \tan^{-1} 0.2\omega - \tan^{-1} 0.05\omega$$

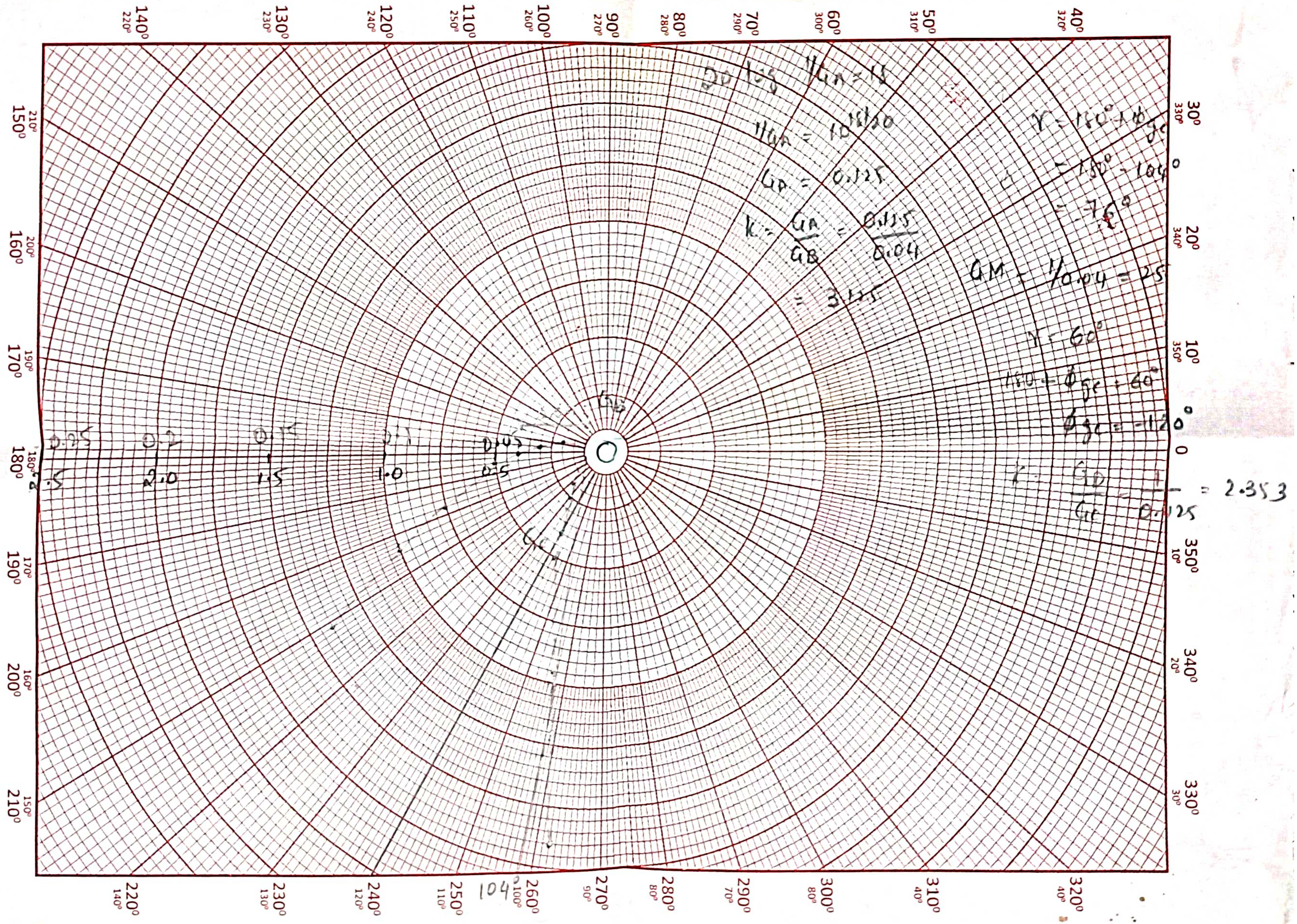
$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1} 0.2\omega - \tan^{-1} 0.05\omega$$

Magnitude and phase of $G(j\omega)$ at various frequencies

ω rad/sec	0.6	0.8	1	2	3	4	5	6	7
$\lvert G(j\omega)\rvert$	1.65	1.23	1.0	0.5	0.3	0.2	0.14	0.1	0.07
$\angle G(j\omega)$	-98	-101	-104	-117.5	-129.4	-140	-149	-157	-164

0.01



20 log 1/Ga = 18
1/Ga = 10^(18/20)
Ga = 0.125
K = Ga/Gb = 0.125/0.04
= 3.125
GM = 1/0.04 = 25
γ = 180° + φgc
= 180° - 104°
= 76°
γ = 60°
180 + φgc = 60°
φgc = -120°
K = Gd/Gc = 1/0.425 = 2.353
Gb
Gc
104°
0.25
0.2
0.15
0.1
0.05
2.5
2.0
1.5
1.0
0.5

ω	9	10	11	14
$\lvert G(j\omega)\rvert$	0.05	0.04	0.03	0.02
$\angle G(j\omega)$	-176	-180	-184	-195

From the polar plot with $K=1$

Gain Margin $K_g = 1/0.04 = 25$

Gain Margin in db $= 20\log 25 = 28$db

phase Margin $\gamma = 76^\circ$

Case (i) :-

with $K=1$, let $G(j\omega)$ cut the -180° axis at point B and gain corresponding to that point be G_B. From the polar plot $G_B = 0.04$. The gain Margin of 28db with $K=1$ has to be reduced to 18db and so K has to be increased to a value greater than one.

Let G_A be the gain at -180° for a gain Margin of 18db.

Now, $20\log\frac{1}{G_A} = 18 \Rightarrow \log\frac{1}{G_A} = \frac{18}{20} \Rightarrow \frac{1}{G_A} = 10^{18/20}$

$\therefore G_A = \frac{1}{10^{18/20}} = 0.125$

The value of K is given by $K = \frac{G_A}{G_B} = \frac{0.125}{0.04} = 3.125$

Case (ii) :-

with $K=1$, the gain margin is 76°. This has to be reduced to 60°. Hence gain has to be increased. let

ϕ_{gc_2} be the phase of $G(j\omega)$ for a phase margin of 60°.

$\therefore 60^\circ = 180^\circ + \phi_{gc_2}$

$\phi_{gc_2} = -180^\circ + 60^\circ = -120^\circ$

In the polar plot the -120° line cut the locus of $G(j\omega)$ at point C and cut the unity circle at point D.

let G_C = Magnitude of $G(j\omega)$ at point C.

G_D = Magnitude of $G(j\omega)$ at point D.

From the polar plot, $G_C = 0.425$ and $G_D = 1$

Now $k = \frac{G_D}{G_C} = \frac{1}{0.425} = 2.353$

Result:-

(i) when $k = 1$, Gain Margin $k_g = 25$

Gain Margin in db = 28 db

(ii) when $k = 1$, phase margin $\gamma = 76^\circ$

(iii) for a gain margin of 18 db, $k = 3.125$

(iv) For a phase margin of 60°, $k = 2.353$.

Ex:-4.12:- consider a unity feedback system having an open loop transfer function $G(s) = \frac{k}{s(1+0.5s)(1+4s)}$

Sketch the polar plot and determine the value of k so that (i) Gain Margin is 20db &

(ii) phase margin is 30°.

Sol: Given that $G(s) = \frac{k}{s(1+0.5s)(1+4s)}$

polar plot is sketched by taking $k=1$

$$G(j\omega) = \frac{1}{j\omega(1+j0.5\omega)(1+j4\omega)}$$

$$G(j\omega) = \frac{1}{j\omega(1+j0.5\omega)(1+j4\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ \sqrt{1+(0.5\omega)^2}\angle \tan^{-1} 0.5\omega \sqrt{1+(4\omega)^2}\angle \tan^{-1} 4\omega}$$

$$= \frac{1}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+16\omega^2}} \angle -90^\circ - \tan^{-1} 0.5\omega - \tan^{-1} 4\omega$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+16\omega^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1} 0.5\omega - \tan^{-1} 4\omega$$

Magnitude and phase of $G(j\omega)$ at various frequencies

ω rad/sec	0.3	0.4	0.5	0.6	0.8	1.0	1.2
$\lvert G(j\omega)\rvert$	2.11	1.3	0.87	0.61	0.35	0.22	0.15
$\angle G(j\omega)$ deg	-149	-159	-167	-174	-184	-193	-199

From the polar plot with $k=1$

Gain Margin $k_g = 1/0.44 = 2.27$

Gain margin in db $= 20 \log 2.27 = 7.12$ db

Phase margin $\gamma = 180^\circ + \phi_{gc}$

$= 180^\circ - 165^\circ = 15^\circ$

Case (i) :-

with $k=1$, let $G(j\omega)$ cut the -180° axis at point B and gain corresponding to that point be G_B. From the polar plot $G_B = 0.44$. The gain margin of 7.12db with $k=1$ has to be increased to 20db and so k has to be decreased to a value less than one.

Let G_A be the gain at -180° for a gain margin of 20db

$$20 \log \frac{1}{G_A} = 20$$

$$\log \frac{1}{G_A} = 1 \Rightarrow \frac{1}{G_A} = 10^1$$

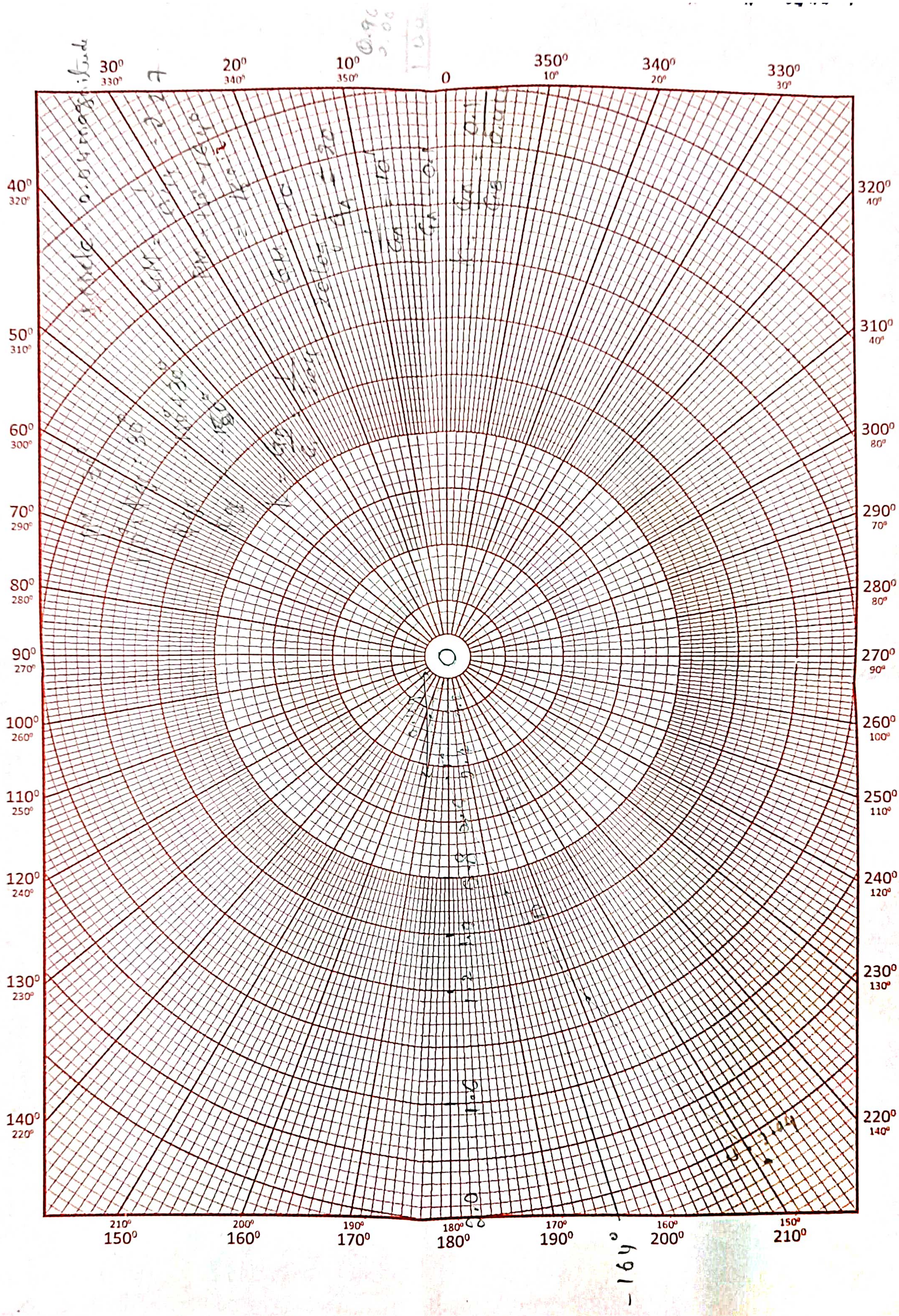
$$G_A = \frac{1}{10} = 0.1$$

Value of k is given by $k = \frac{G_A}{G_B} = \frac{0.1}{0.44} = 0.227$

Case (ii) :- with $k=1$, the phase margin is 15°. This has to be increased to 30°. Hence the gain has to be decreased.

Let ϕ_{gc_2} be the phase of $G(j\omega)$ for a phase margin of 30°.

$$\therefore 30^\circ = 180^\circ + \phi_{gc_2}$$


0
10⁰
20⁰
30⁰
350⁰
340⁰
330⁰
40⁰
50⁰
60⁰
70⁰
80⁰
90⁰
100⁰
110⁰
120⁰
130⁰
140⁰
320⁰
310⁰
300⁰
290⁰
280⁰
270⁰
260⁰
250⁰
240⁰
230⁰
220⁰
150⁰
160⁰
170⁰
180⁰
190⁰
200⁰
210⁰

$$\phi_{gcn} = 30^\circ - 180^\circ = -150^\circ$$

In the polar plot, the -150° line cuts the locus of $G(j\omega)$ at point C and cut the unity circle at point D.

Let G_C = Magnitude of $G(j\omega)$ at point C.

G_D = Magnitude of $G(j\omega)$ at point D.

From the polar plot, $G_C = 2.04$ and $G_D = 1$

Now $k = \frac{G_D}{G_C} = \frac{1}{2.04} = 0.49$

<u>Result :-</u>

(i) When $K = 1$, Gain Margin $k_g = 2.27$

Gain Margin in db = 7.12db

(ii) When $K = 1$, phase margin $\gamma = 150$

(iii) For a gain margin of 20db $k = 0.227$

(iv) For a phase margin of 30° $k = 0.49$

<u>Nyquist plot :-</u>

Draw the Nyquist plot for the system whose open loop transfer function is $G(s)H(s) = \frac{K}{s(s+2)(s+10)}$. Determine the range of K for which closed loop system is stable

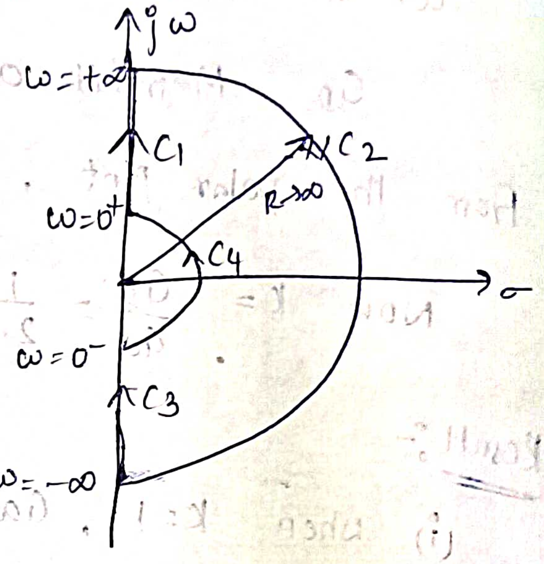
<u>Sol</u>

$$G(s)H(s) = \frac{K}{s(s+2)(s+10)} = \frac{K}{s \times 2(1+\frac{s}{2})10(\frac{s}{10}+1)}$$

$$= \frac{0.05K}{S(1+0.5S)(1+0.1S)}$$

The open loop transfer function has a pole at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane except the origin

Nyquist contour has four sections C_1 C_2 C_3 and C_4.



Mapping of section C_1 :-

In section C_1 ω varies from 0 to +∞. The mapping of section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$.

$$G(S)H(S) = \frac{0.05K}{S(1+0.5S)(1+0.1S)}$$

Put $S = j\omega$

$$G(j\omega)H(j\omega) = \frac{0.05K}{j\omega(1+j0.5\omega)(1+j0.1\omega)}$$

$$= \frac{0.05K}{j\omega(1+j0.6\omega - 0.05\omega^2)}$$

$$= \frac{0.05K}{-0.6\omega^2 + j\omega(1-0.05\omega^2)}$$

When the locus of $G(j\omega) H(j\omega)$ crosses real axis the imaginary term will be zero and the corresponding frequency is the phase cross over frequency ω_{pc}.

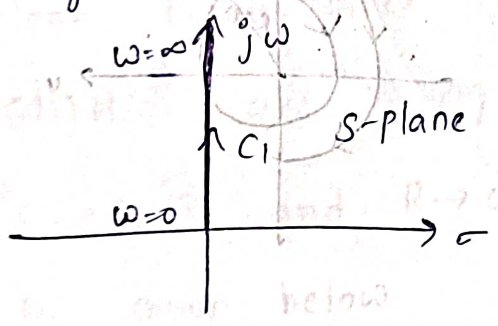
At $\omega = \omega_{pc}$, $\omega_{pc}(1 - 0.05\omega^2_{pc}) = 0$

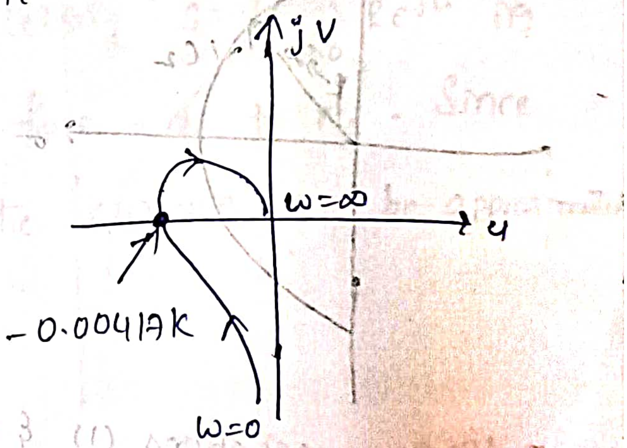
$$1 - 0.05\omega^2_{pc} = 0 \Rightarrow \omega_{pc} = \sqrt{\frac{1}{0.05}} = 4.472 \text{ rad/sec}$$

At $\omega = \omega_{pc} = 4.472$ rad/sec

$$G(j\omega) H(j\omega) = \frac{0.05k}{-0.6\omega^2} = \frac{0.05k}{0.6(4.472)^2} = -0.00417k$$

The open loop system is type -1 and third order system. Also it is a minimum phase system with all poles. Hence the polar plot of $G(j\omega) H(j\omega)$ starts at $-90°$ axis at ∞, crosses real axis at $-0.00417k$ and ends at origin in second quadrant.





Mapping of Section C_2 :-

The mapping of section C_2 from s-plane to $G(s)H(s)$-plane is obtained by letting $s = \lim_{R \to \infty} Re^{j\theta}$ in $G(s)H(s)$ and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to Re^{j\theta}$ and $R \to \infty$, the $G(s)H(s)$ can be approximated as shown below.

$$G(s)H(s) = \frac{0.05K}{s(1+0.5s)(1+0.1s)} \approx \frac{0.05K}{s\times 0.5s\times 0.1s} = \frac{K}{s^3}$$

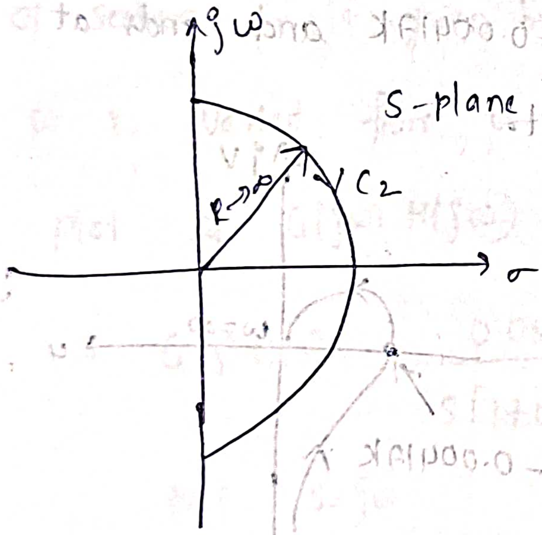
let $s = \lim_{R\to\infty} Re^{j\theta}$

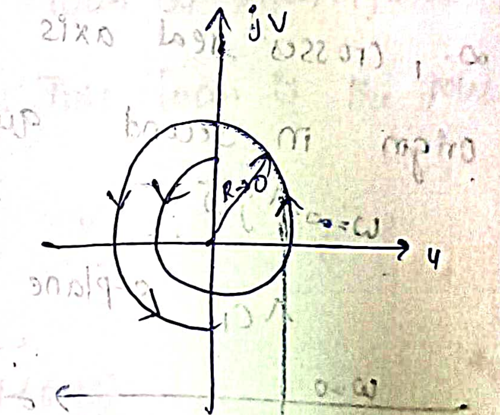
$$\therefore G(s)H(s)\Big|_{s=\lim_{R\to\infty} Re^{j\theta}} = \frac{K}{s^3}\Big|_{s=\lim_{R\to\infty} Re^{j\theta}}$$

$$= \frac{K}{\lim_{R\to\infty}(Re^{j\theta})^3} = 0e^{-j3\theta}$$

when $\theta = \frac{\pi}{2}$, $G(s)H(s) = 0e^{-j\frac{3\pi}{2}} \rightarrow (1)$

when $\theta = -\frac{\pi}{2}$, $G(s)H(s) = 0e^{+j3\pi/2} \rightarrow (2)$



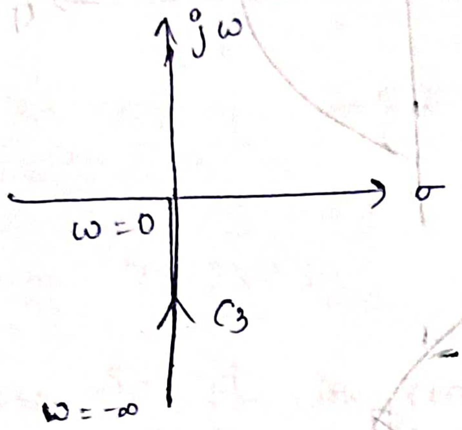


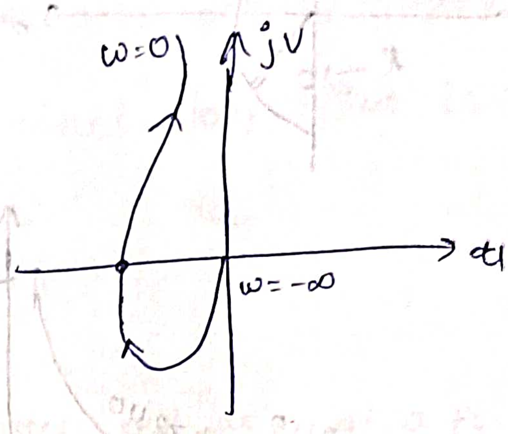
From the equations (1) & (2) we can say that section C_2 in S-plane is mapped as circular arc of zero radius around origin in $G(s)H(s)$ plane with argument (phase) varying from $-\frac{3\pi}{2}$ to $+\frac{3\pi}{2}$

Mapping of section C_3:-

Mapping of Section C_3 :-

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from $-\infty$ to 0. This locus is inverse polar plot of $G(j\omega)H(j\omega)$





Mapping of Section C_4 :-

The mapping of section C_4 from s-plane to $G(s)H(s)$ plane is obtained by letting $s = \lim_{R\to 0} Re^{j\theta}$ in $G(s)H(s)$ and varying θ from $-\pi/2$ to $\pi/2$. Since $s \to Re^{j\theta}$ and $R \to 0$, the $G(s) \cdot H(s)$ can be approximat as shown below

$$G(s)H(s) = \frac{0.05k}{s(1+0.5s)(1+0.1s)} \approx \frac{0.05k}{s \times 1 \times 1} = \frac{0.05k}{s}$$

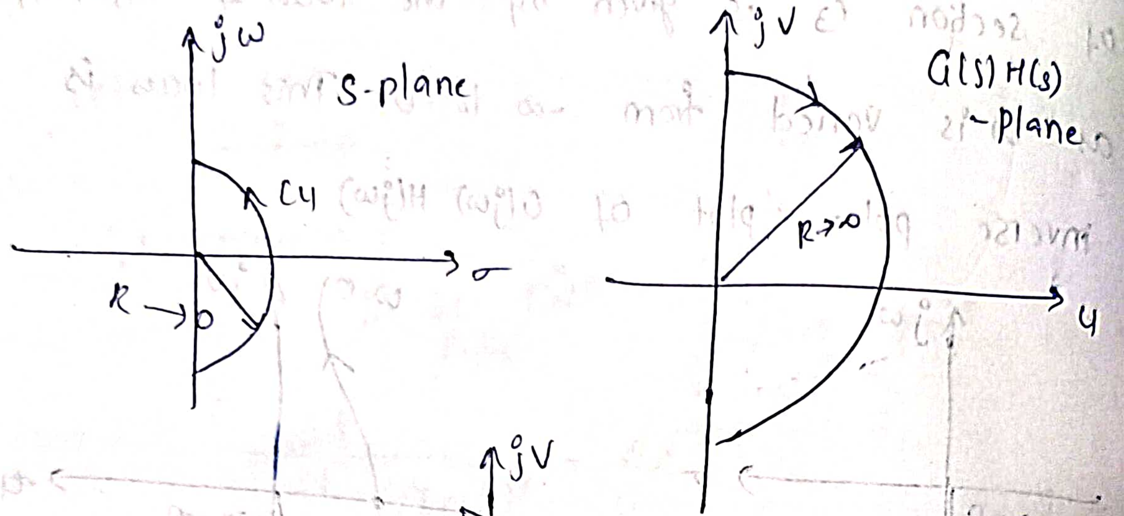
let $s = \lim_{R\to 0} Re^{j\theta}$

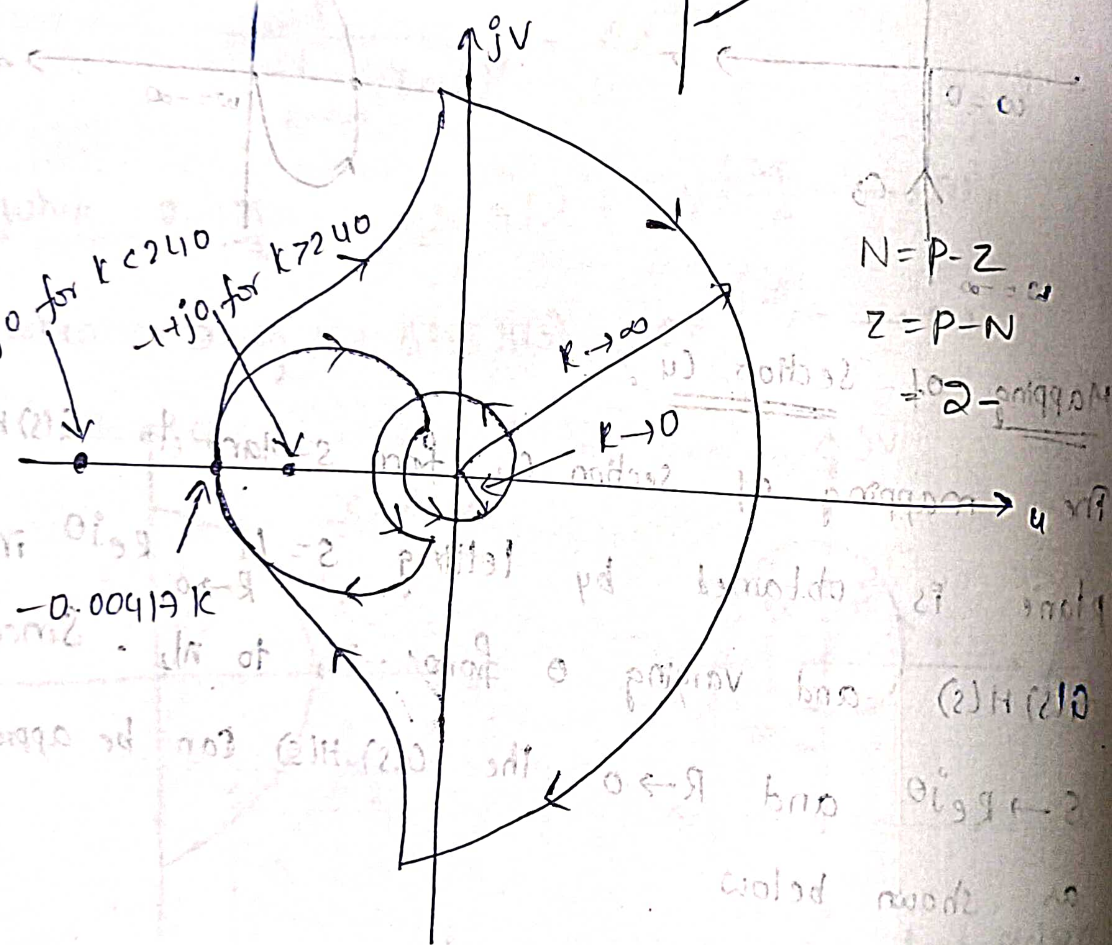
$$\therefore G(s)H(s)\Big|_{s=\lim_{R\to 0} Re^{j\theta}} = \frac{0.05k}{s}\Big|_{s=\lim_{R\to 0} Re^{j\theta}}$$

$$= \frac{0.05k}{\lim_{R\to 0} Re^{j\theta}} = \infty e^{-j\theta}$$

when $\theta = -\pi/2$, $G(s)H(s) = \infty e^{j\pi/2}$

when $\theta = \pi/2$, $G(s)H(s) = \infty e^{-j\pi/2}$





$N = P - Z$

$Z = P - N$

Stability analysis:-

when $-0.00417K = -1$ the contour passes through $-1+j0$ point and corresponding value of K is the limiting value of K for stability.

$\therefore$ Limiting value of $K = \frac{1}{0.00417} = 240$

When $k<240$:-

When $k<240$, the contour crosses real axis at a point b/w -0 & $-1+j0$. On travelling through Nyquist plot along the indicated direction it is found that $-1+j0$ is not encircled. Also the open loop transfer function has no poles on the right half of s-plane. Therefore the closed loop system is stable.

When $k>240$:-

When $k>240$, the contour crosses real axis at a pt. b/w $-1+j0$ & $-\infty$. On travelling through Nyquist plot along the indicated dir^n it is found that the pt. $-1+j0$ is encircled in clockwise dir^n two time.

$\therefore$ closed loop system is unstable.

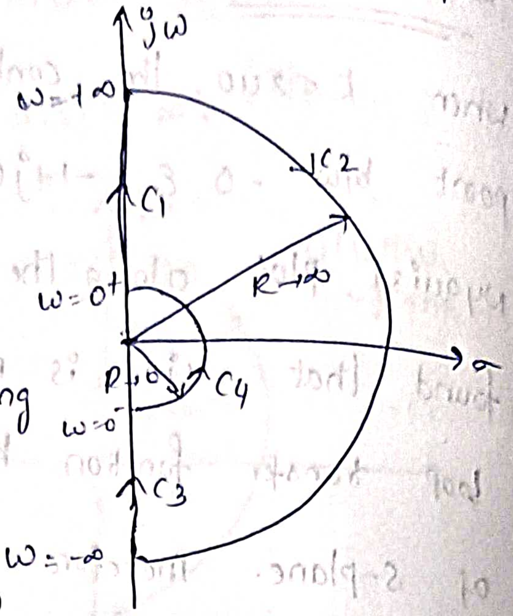
Result:-

The value of k for stability is $0<k<240$.

5.14 Construct the Nyquist plot for a system whose open loop transfer function is given by $G(s)H(s) = \dfrac{K(1+s)^2}{s^3}$. find the range of K for stability.

Sol Given that $G(s)H(s) = \dfrac{k(1+s)^2}{s^3}$

The open loop transfer function has 3 poles at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half of plane except the origin.



Mapping of section C_1

In section C_1, ω varies from 0 to $+\infty$. The mapping of section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is polar plot of $G(j\omega)H(j\omega)$

$$G(s)H(s) = \frac{k(1+s)^2}{s^3}$$

Let $s = j\omega$

$$\therefore G(j\omega)H(j\omega) = \frac{k(1+j\omega)^2}{(j\omega)^3} = \frac{k(1-\omega^2+2j\omega)}{-j\omega^3}$$

$$= \frac{k(1-\omega^2)}{-j\omega^3} + \frac{k\,2j\omega}{-j\omega^3} = \frac{-2k}{\omega^2} + j\,\frac{k(1-\omega^2)}{\omega^3}$$

When $G(j\omega)H(j\omega)$ locus crosses real axis the imaginary term will be zero and the corresponding frequency is the phase cross over frequency ω_{pc}

At $\omega = \omega_{pc}$, $k(1-\omega^2_{pc}) = 0 \Rightarrow 1-\omega^2_{pc} = 0$

$\omega_{pc} = 1$ rad/sec.

At $\omega = \omega_{pc} = 1$ rad/sec

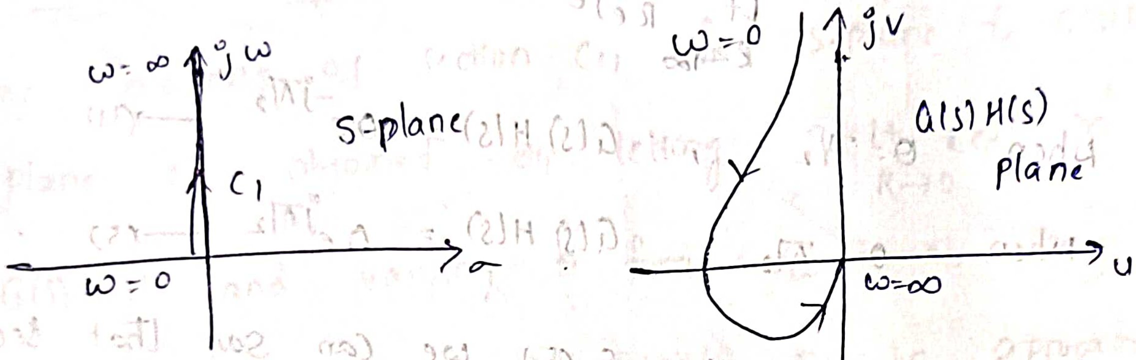
$$G(j\omega)H(j\omega) = \frac{-2k}{\omega^2} = \frac{-2k}{1^2} = -2k \longrightarrow (1)$$

$$G(j\omega)H(j\omega) = \frac{k(1+j\omega)^2}{(j\omega)^3} = \frac{k\sqrt{1+\omega^2}\,\angle \tan^{-1}\omega\,\sqrt{1+\omega^2}\,\angle \tan^{-1}\omega}{\omega^3 \angle 270^\circ}$$

$$= \frac{k(1+\omega^2)}{\omega^3}\angle 2\tan^{-1}\omega - 270^\circ$$

As $\omega \to 0$, $G(j\omega)H(j\omega) \to \infty \angle -270^\circ \longrightarrow (2)$

As $\omega \to \infty$, $G(j\omega)H(j\omega) \to 0 \angle -90^\circ \longrightarrow (3)$



From the equations (1), (2) and (3) we can say that the polar plot starts at -270° axis at ∞, crosses real axis at $-2k$ and ends at origin in 3rd quadrant.

Mapping of section C_2 :-

The mapping of section C_2 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt} Re^{j\theta}$ in G(s) H(s) and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to Re^{j\theta}$ and $R \to \infty$, the G(s)H(s) can be approximated as shown below.

$$G(s)\,H(s) = \frac{K(1+s)^2}{s^3} \approx \frac{K s^2}{s^3} = \frac{K}{s}$$

Let $s = \lim_{R\to\infty} Re^{j\theta}$

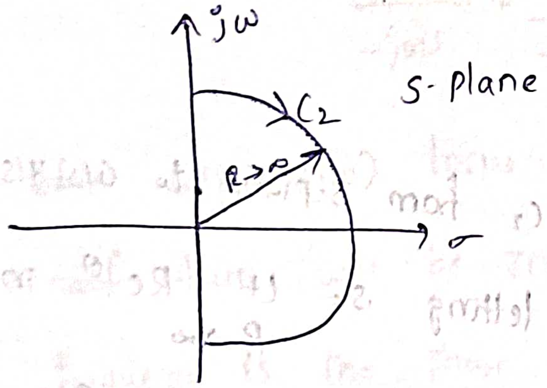
$$\therefore G(s)H(s)\Big|_{s=\lim_{R\to\infty} Re^{j\theta}} = \frac{K}{s}\Big|_{s=\lim_{R\to\infty} Re^{j\theta}}$$

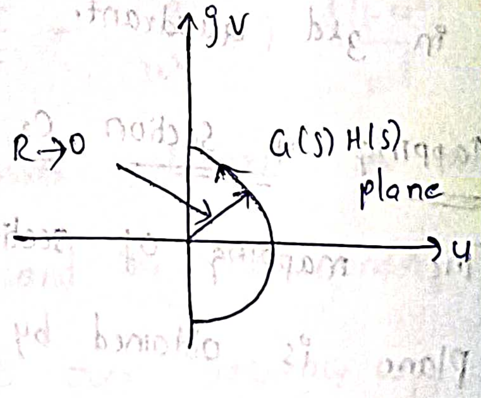
$$= \frac{K}{\lim_{R\to\infty} Re^{j\theta}} = 0e^{-j\theta}$$

When $\theta = \pi/2$, $G(s)H(s) = 0e^{-j\pi/2}$ → (4)

When $\theta = -\pi/2$, $G(s)H(s) = 0e^{j\pi/2}$ → (5)

From equations (4) & (5) we can say that section C_2 in s-plane is mapped as circular arc of zero radius around origin in $G(s)H(s)$-plane with argument varying from $-\pi/2$ to $+\pi/2$



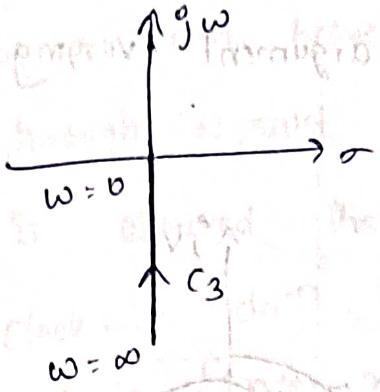


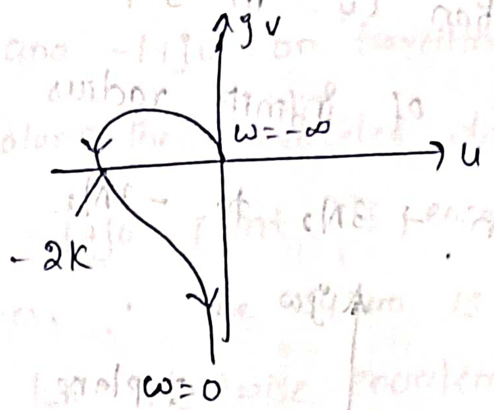
Mapping of section C_3 :-

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by locus of

$G(j\omega)H(j\omega)$ as ω is varied from $-\infty$ to 0.

This locus is the inverse polar plot of $G(j\omega)H(j\omega)$





Mapping of Section C_4 :-

The mapping of section C_4 from s-plane to $G(s)H(s)$ plane is obtained by letting $s = \lim_{R\to 0} Re^{j\theta}$ in $G(s)H(s)$ and varying θ from $-\pi/2$ to $\pi/2$. Since $s \to Re^{j\theta}$ and $R \to 0$ the $G(s)H(s)$ can be approximated as shown below.

$$G(s)H(s) = \frac{K(1+s)^2}{s^3} \approx \frac{K \times 1}{s^3} \approx \frac{K}{s^3}$$

Let $s = \lim_{R\to 0} Re^{j\theta}$

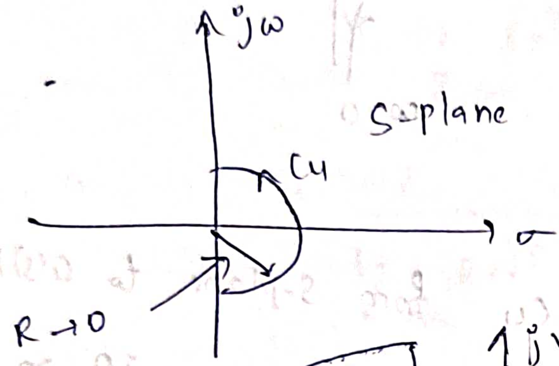
$$G(s)H(s)\Big|_{s=\lim_{R\to 0} Re^{j\theta}} = \frac{K}{s^3}\Big|_{s=\lim_{R\to 0} Re^{j\theta}}$$

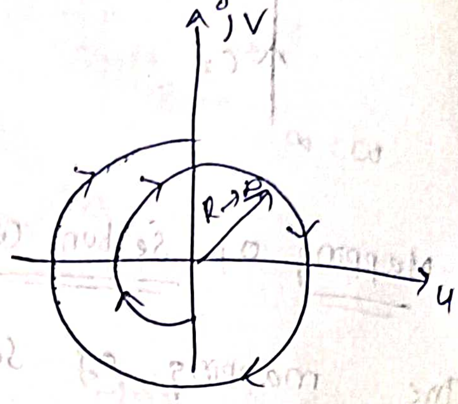
$$= \lim_{R\to 0} \frac{K}{(Re^{j\theta})^3} = \infty e^{-j3\theta}$$

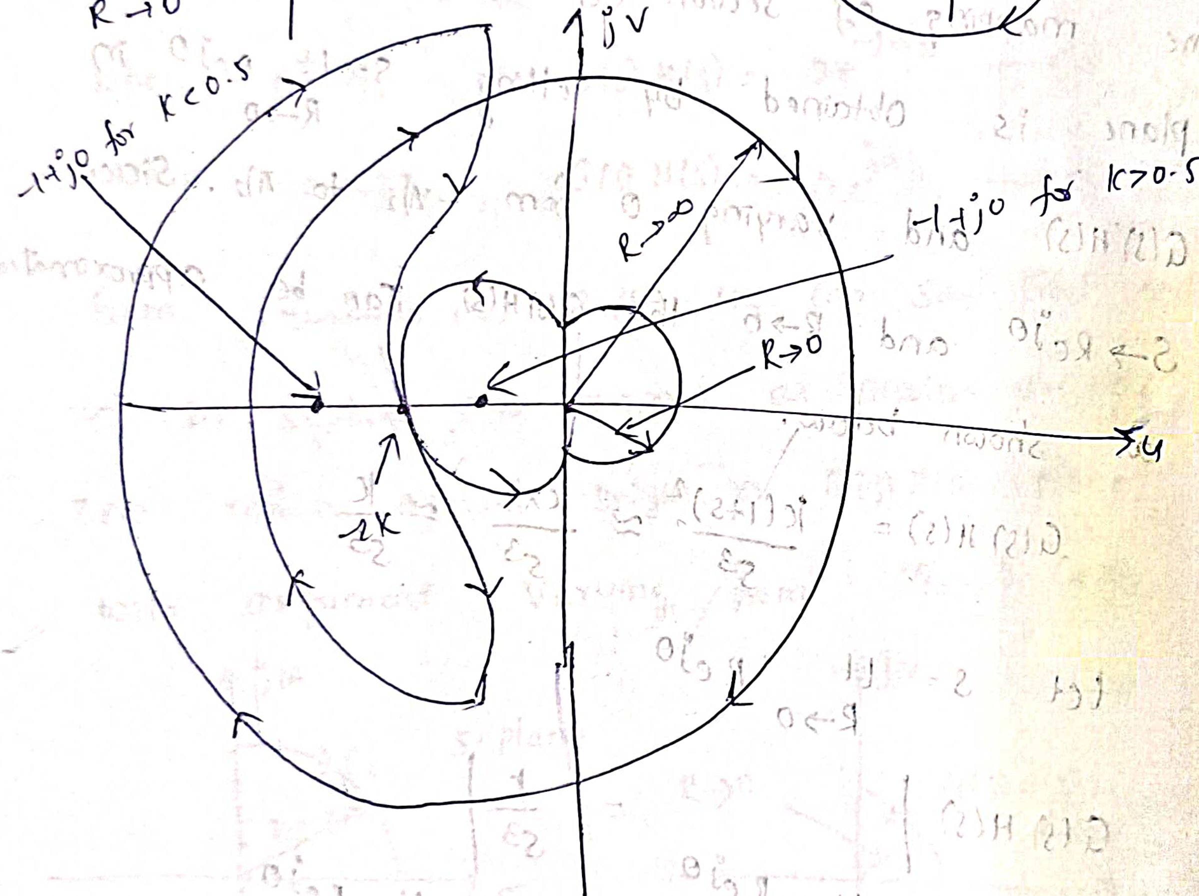
When $\theta = -\pi/2$, $G(s)H(s) = \infty e^{+j3\pi/2}$ $\rightarrow (6)$

When $\theta = +\pi/2$, $G(s)H(s) = \infty e^{-j3\pi/2}$ $\rightarrow (7)$

From the equations (6) and (7) we can say that section C_4 in s-plane is mapped as a circular arc of infinite radius with argument varying from $+3\pi/2$ to $-3\pi/2$







<u>Stability analysis:-</u>

When $-2K = -1$ the contour passes through $-1+j0$ point and corresponding value of k is the limiting value of k for stability.

$\therefore$ Limiting value of $k = \frac{1}{2} = 0.5$

when $k < 0.5$:-

when k is lessthan 0.5, the contour crosses real axis at a point between 0 and $-1+j0$. on travelling through Nyquist plot along the indicated dr^n it is observed that the $-1+j0$ point is encircled in clock wise dr^n two times. ∴ The system is unstable. [Since there are two clockwise encirclement and no right half open loop poles, the closed loop system will have two poles on right half of s-plane].

when $k > 0.5$:-

when $k > 0.5$, the contour crosses real axis at a pt b/w $-1+j0$ & $-\infty$. on travelling through Nyquist plot along the indicated dr^n it is observed that $-1+j0$ pt is encircled in both clockwise & anti-clockwise dr^n one time. Hence net encirclement is zero. Also the open loop system has no poles at the right half of s-plane.

∴ closed loop system is stable.

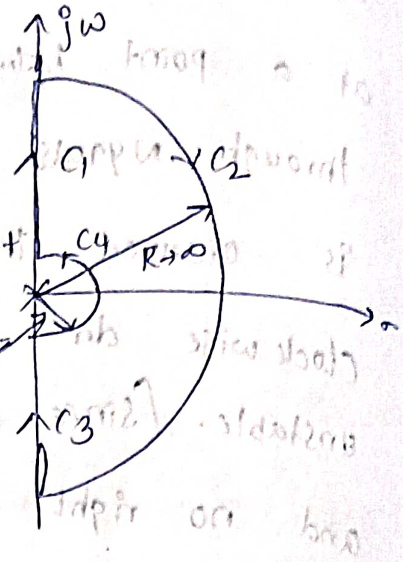
Result:- system is stable when $k > 0.5$.

* The open loop transfer function of a system is $G(s)H(s) = \frac{1+4s}{s^2(1+s)(1+2s)}$. Determine the stability of closed loop system. If the closed loop system is not stable then find the no. of closed loop poles lying

on the right half of s-plane.

<u>Sol)</u> $G(s)H(s) = \frac{1+4s}{s^2(1+s)(1+2s)}$

The open loop transfer function has two poles at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane except the origin.



<u>Mapping of Section C1 :-</u>

In section C1, ω varies from 0 to +∞. The mapping of section C1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$.

$$G(s)H(s) = \frac{1+4s}{s^2(1+s)(1+2s)}$$

Let $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{1+j4\omega}{(j\omega)^2(1+j\omega)(1+2j\omega)}$$

$$= \frac{\sqrt{1+16\omega^2}\,\angle\tan^{-1}(4\omega)}{\omega^2\angle 180^\circ\,\sqrt{1+\omega^2}\,\angle\tan^{-1}\omega\,\sqrt{1+4\omega^2}\,\angle\tan^{-1}2\omega}$$

$$= \frac{\sqrt{1+16\omega^2}}{\omega^2\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}\angle\tan^{-1}4\omega - 180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

$$\therefore |G(j\omega)H(j\omega)| = \frac{\sqrt{1+16\omega^2}}{\omega^2\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$$

When the $G(j\omega)H(j\omega)$ locus crosses real axes, the phase will be $-180°$ and the corresponding frequency is the phase cross over frequency ω_{pc}.

At $\omega = \omega_{pc}$, $\angle G(j\omega)H(j\omega) = -180°$

$$\tan^{-1} 4\omega_{pc} - \cancel{180°} - \tan^{-1}\omega_{pc} - \tan^{-1} 2\omega_{pc} = -\cancel{180}°$$

$$\tan^{-1} 4\omega_{pc} = \tan^{-1}\omega_{pc} + \tan^{-1} 2\omega_{pc}$$

On taking tan on both sides

$$\tan[\tan^{-1} 4\omega_{pc}] = \tan[\tan^{-1}\omega_{pc} + \tan^{-1} 2\omega_{pc}]$$

$$4\omega_{pc} = \frac{\tan\tan^{-1}\omega_{pc} + \tan\tan^{-1} 2\omega_{pc}}{1 - \tan\tan^{-1}\omega_{pc} * \tan\tan^{-1} 2\omega_{pc}}$$

$$4\omega_{pc} = \frac{\omega_{pc} + 2\omega_{pc}}{1 - 2\omega^2_{pc}}$$

$$1 - 2\omega^2_{pc} = \frac{3\omega_{pc}}{4\omega_{pc}}$$

$$\therefore \omega_{pc} = \sqrt{\frac{-0.25}{-2}} = 0.354 \text{ rad/sec}$$

$$|G(j\omega)H(j\omega)| = \frac{\sqrt{1+16\omega^2_{pc}}}{\omega^2\sqrt{1+\omega^2_{pc}}\sqrt{1+4\omega^2_{pc}}}$$

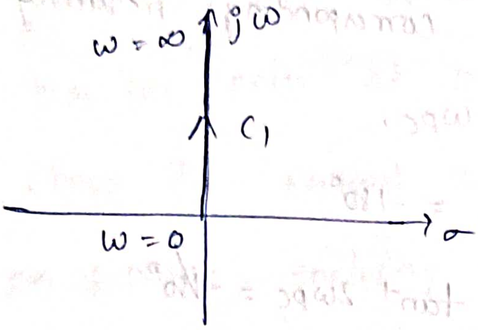
$$= \frac{\sqrt{1+16(0.354)^2}}{(0.354)^2\sqrt{1+(0.354)^2}\sqrt{1+4\times(0.354)^2}}$$

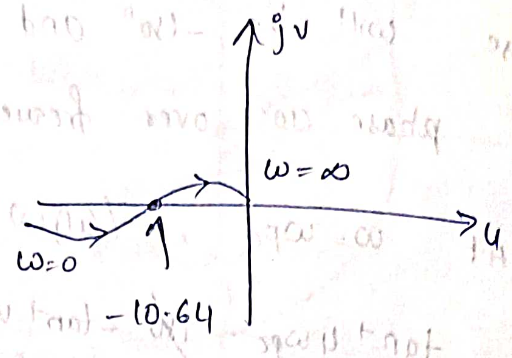
$$= 10.64 \longrightarrow (1)$$

Hence $G(j\omega)H(j\omega)$ locus crosses the real axis at -10.64

At $\omega \to 0$, $G(j\omega)H(j\omega) \to \infty \angle -180^\circ \longrightarrow (2)$

At $\omega \to \infty$, $G(j\omega)H(j\omega) \to 0 \angle -270^\circ \longrightarrow (3)$





Mapping of Section C_2 :-

The mapping of section C_2 from s-plane to $G(s)H(s)$-plane is obtained by letting $s = \lim_{R\to\infty} Re^{j\theta}$ in $G(s)H(s)$ and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to Re^{j\theta}$ and $R \to \infty$ the $G(s)H(s)$ can be approximated as shown below

$$G(s)H(s) = \frac{1+4s}{s^2(1+s)(1+2s)} = \frac{4s}{s^2 \times s \times 2s} = \frac{2}{s^3}$$

Let $s = \lim_{R\to\infty} Re^{j\theta}$

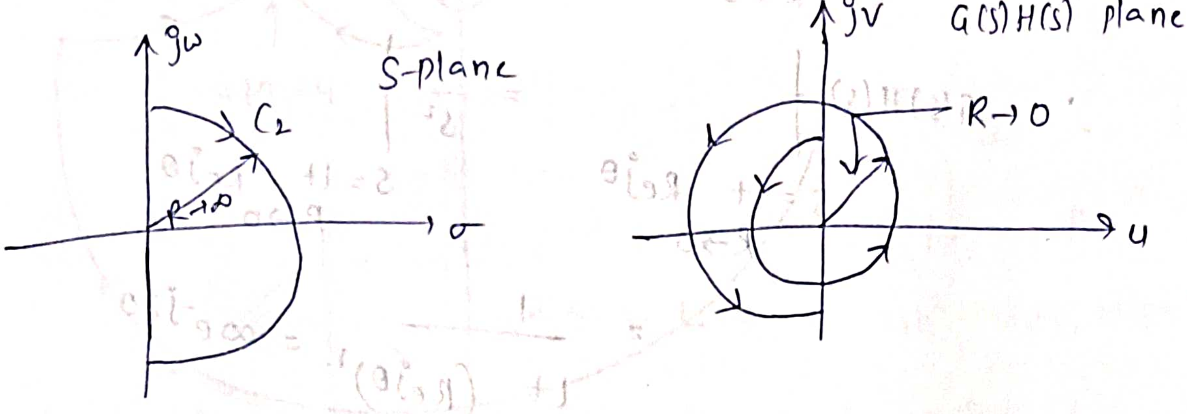
$$\therefore G(s)H(s)\Big|_{s=\lim_{R\to\infty} Re^{j\theta}} = \frac{2}{s^3}\Big|_{s=\lim_{R\to\infty} Re^{j\theta}} = \frac{2}{\lim_{R\to\infty} (Re^{j\theta})^3}$$

$$= 0\, e^{-j3\theta}$$

when $\theta = \pi/2$, $G(s)H(s) = 0\, e^{-j3\pi/2} \longrightarrow (4)$

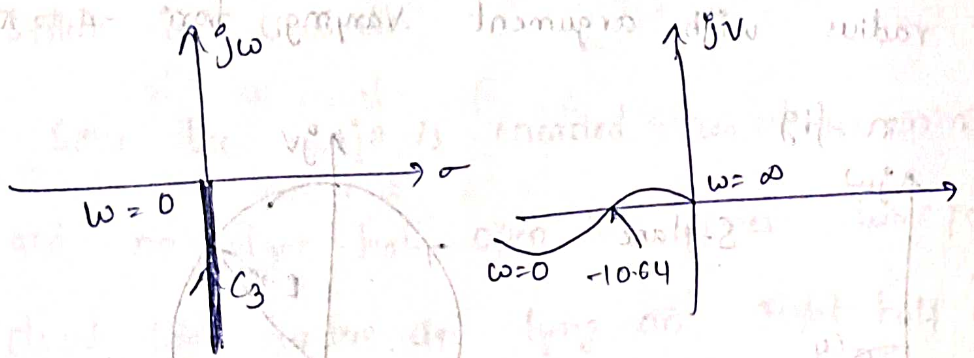
when $\theta = -\pi/2$, $G(s)H(s) = 0\, e^{j3\pi/2} \longrightarrow (5)$

From the equations (4) and (5) we can say that section C_2 in s-plane is mapped as circular arc of zero radius around origin in $G(s)H(s)$ - plane with argument varying from $-3\pi/2$ to $3\pi/2$



Mapping of section C_3 :-

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of $G(j\omega)H(j\omega)$



Mapping of Section C_4 :-

The mapping of section C_4 from s-plane to $G(s)H(s)$ plane is obtained by letting $s = \lim_{R\to 0} Re^{j\theta}$ in $G(s)H(s)$ and varying θ from $-\pi/2$ to $\pi/2$. Since $s \to Re^{j\theta}$ and $R \to 0$, the $G(s)H(s)$ can be approximated as shown below

$$G(s)H(s) = \frac{1+4s}{s^2(1+s)(1+2s)} \approx \frac{1}{s^2\times1\times1} = \frac{1}{s^2}$$

let $s = \lim_{R\to 0} Re^{j\theta}$

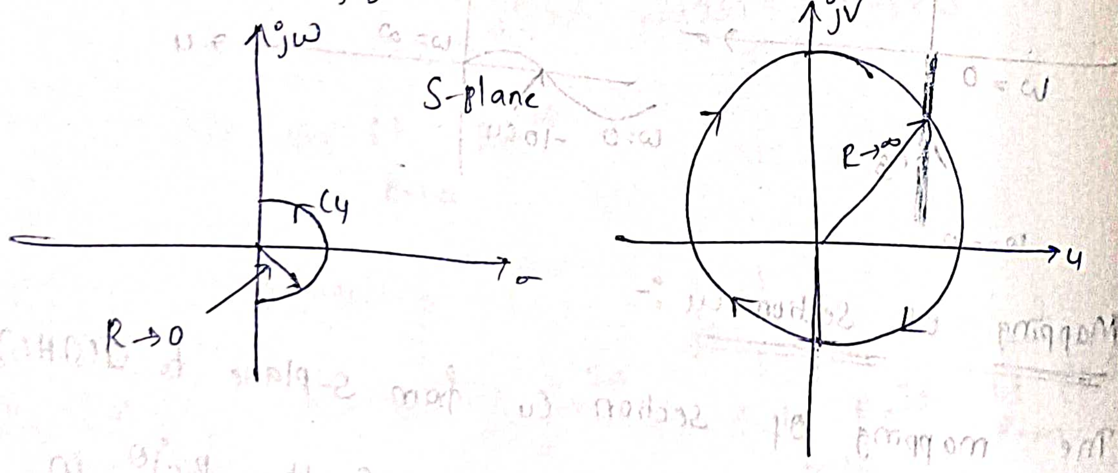
$$\therefore G(s)H(s)\Big|_{s=\lim_{R\to0} Re^{j\theta}} = \frac{1}{s^2}\Big|_{s=\lim_{R\to0} Re^{j\theta}}$$

$$= \frac{1}{\lim_{R\to0}(Re^{j\theta})^2} = \infty e^{-j2\theta}$$

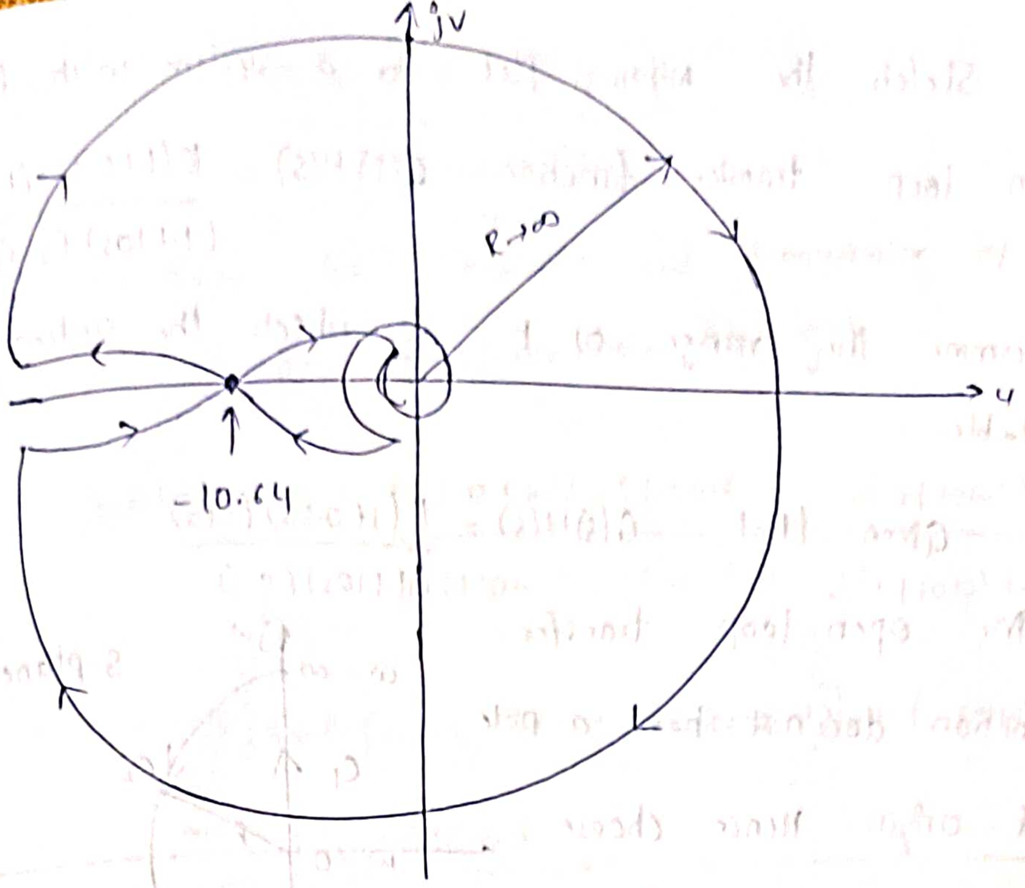
when $\theta = -\pi/2$, $G(s)H(s) = \infty e^{+j\pi} \longrightarrow (6)$

when $\theta = \pi/2$, $G(s)H(s) = \infty e^{-j\pi} \longrightarrow (7)$

From the equations (6) and (7) we can say that section C_4 in s-plane is mapped as a circle of infinite radius with argument varying from $-\pi$ to π as shown in fig



Complete Nyquist plot:-



Stability analysis :-

On travelling through Nyquist contour in G(s) H(s) plane it is observed that (-1+j0) point is encircled in clock direction two times. Therefore the closed loop system is unstable.

Since the -1+j0 is encircled two times in clockwise and no right half open loop poles, two poles of closed loop system are lying on right half of S-plane.

Result :-

→ Closed loop System is Stable.

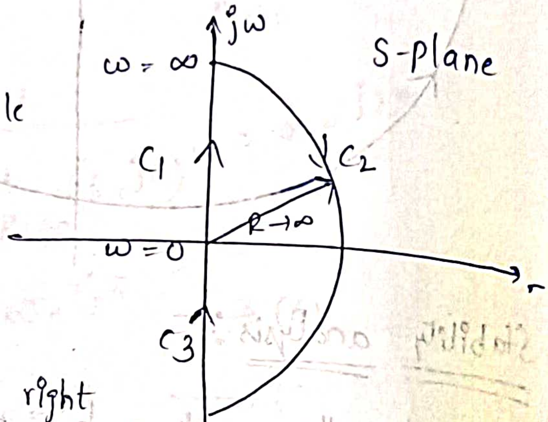
→ Two poles of closed loop System are lying on the right half of S-plane.

5.16 Sketch the Nyquist plot for a system with the open loop transfer function $G(s)H(s) = \frac{K(1+0.5s)(1+s)}{(1+10s)(s-1)}$

Determine the range of K for which the system is stable.

Sol Given that $G(s)H(s) = \frac{K(1+0.5s)(1+s)}{(1+10s)(s-1)}$

The open loop transfer function does not have a pole at origin. Hence choose the Nyquist contour on S-plane enclosing the entire right half plane.



Mapping of Section C_1 :-

In section C_1, ω varies from 0 to +∞. The mapping of Section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$.

$$G(s)H(s) = \frac{K(1+0.5s)(1+s)}{(1+10s)(s-1)}$$

Let $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{K(1+j0.5\omega)(1+j\omega)}{(1+j10\omega)(-1+j\omega)}$$

$$= \frac{K(1+j1.5\omega - 0.5\omega^2)}{-1-j9\omega-10\omega^2}$$

$$= \frac{K(1-0.5\omega^2) + j1.5\omega K}{-(1+10\omega^2) - j9\omega}$$

On multiplying the numerator and denominator by the complex conjugate of Denominator we get

$$G(j\omega)H(j\omega) = \frac{K(1-0.5\omega^2) + j1.5\omega K}{-(1+10\omega^2) - j9\omega} \times \frac{-(1+10\omega^2) + j9\omega}{-(1+10\omega^2) + j9\omega}$$

$$= \frac{j[9\omega K(1-0.5\omega^2) - 1.5\omega K(1+10\omega^2)] - K(1-0.5\omega^2)(1+10\omega^2) - 13.5\omega^2 K}{(1+10\omega^2)^2 + (9\omega)^2}$$

When the $G(j\omega)H(j\omega)$ locus crosses real axis the imaginary term is zero and the corresponding frequency is the phase cross over frequency.

At $\omega = \omega_{pc}$, $9\omega_{pc}K(1-0.5\omega^2_{pc}) - 1.5\omega_{pc}K(1+10\omega^2_{pc}) = 0$

$$9\omega_{pc}K(1-0.5\omega^2_{pc}) = 1.5\omega_{pc}K(1+10\omega^2_{pc})$$

$$1-0.5\omega^2_{pc} = \frac{1.5}{9}(1+10\omega^2_{pc})$$

$$1-0.5\omega^2_{pc} = 0.167 + 1.67\omega^2_{pc}$$

$$2.17\omega^2_{pc} = 0.833$$

$$\omega_{pc} = \sqrt{\frac{0.833}{2.17}} = 0.62 \text{ rad/sec}$$

At $\omega = \omega_{pc} = 0.62$ rad/sec

$$G(j\omega)H(j\omega) = \frac{-K(1-0.5\omega^2_{pc})(1+10\omega^2_{pc}) - 13.5\,\omega^2_{pc}K}{(1+10\omega^2_{pc})^2 + (9\omega_{pc})^2}$$

$$= -K\left[\frac{(1-0.5\times0.62^2)(1+10\times0.62^2)+13.5\times0.62^2}{(1+10\times0.62^2)^2+(9\times0.62)^2}\right]$$

$$= -K\left[\frac{3.913+5.189}{23.464+31.136}\right] = -0.1667K$$

$\therefore$ $G(j\omega)H(j\omega)$ locus crosses real axis at a point $-0.1667K$

$$G(j\omega)H(j\omega) = \frac{K(1+j0.5\omega)(1+j\omega)}{(1+j10\omega)(-1+j\omega)}$$

$$= \frac{K\sqrt{1+(0.5\omega)^2}\,\angle\tan^{-1}0.5\omega\,\sqrt{1+\omega^2}\,\angle\tan^{-1}\omega}{\sqrt{1+(10\omega)^2}\,\angle\tan^{-1}10\omega\,\sqrt{1+\omega^2}\,\angle180^\circ-\tan^{-1}\omega}$$

$$= \frac{K\sqrt{1+0.25\omega^2}}{\sqrt{1+100\omega^2}}\,\angle\tan^{-1}0.5\omega+2\tan^{-1}\omega-\tan^{-1}10\omega-180^\circ$$

$$\therefore |G(j\omega)H(j\omega)| = \frac{K\sqrt{1+0.25\omega^2}}{\sqrt{1+100\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = \tan^{-1}0.5\omega + 2\tan^{-1}\omega - \tan^{-1}10\omega - 180^\circ$$

As $\omega \to 0$, $|G(j\omega)H(j\omega)| = K$

As $\omega \to 0$, $\angle G(j\omega)H(j\omega) = -180^\circ$

As $\omega \to \infty$, $|G(j\omega)H(j\omega)| = \lim_{\omega \to \infty} K \frac{\sqrt{1+0.25\omega^2}}{\sqrt{1+100\omega^2}}$

$$= K \lim_{\omega \to \infty} \sqrt{\frac{\omega^2\left(\frac{1}{\omega^2}+0.25\right)}{\omega^2\left(\frac{1}{\omega^2}+100\right)}} = K \lim_{\omega \to \infty} \sqrt{\frac{\left(\frac{1}{\omega^2}+0.25\right)}{\left(\frac{1}{\omega^2}+100\right)}}$$

$$= K\sqrt{\frac{0+0.25}{0+100}} = 0.05K$$

As $\omega \to \infty$, $\angle G(j\omega) = \tan^{-1}\infty + 2\tan^{-1}\infty - \tan^{-1}\infty - 180^\circ$

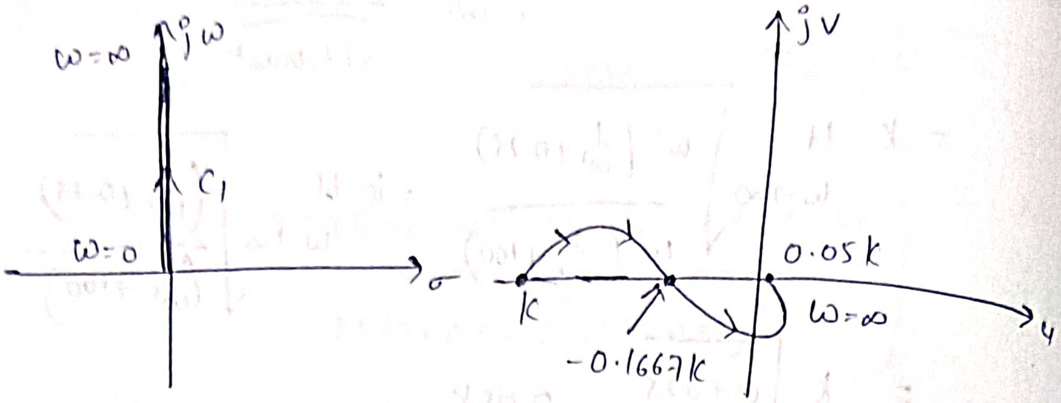
$$= 90^\circ + \cancel{180^\circ} - 90^\circ - \cancel{180^\circ} = 0$$

ω rad/sec	0	0.1	0.5	1.5	2.0	5.0	∞
$\lvert G(j\omega)H(j\omega)\rvert$	K	0.707K	0.202K	0.053K	0.07K	0.054K	0.05K
$\angle G(j\omega)H(j\omega)$	-180°	210°	-191	-116	-95	-43	0

From the above analysis, the following conclusions are made –

1. The locus of $G(j\omega)H(j\omega)$ starts at $K\angle -180^\circ$ when $\omega = 0$ and travels in 2nd Quadrant.
2. The locus crosses real axis at $-0.1667K$ and enters 3rd quadrant.
3. Then the locus crosses $-$ve imaginary axis and enters 4th quadrant
4. Finally the locus ends at $0.05K \angle 0^\circ$ when $\omega = \infty$

The section C_1 in s-plane and its corresponding mapping, the in $G(s)H(s)$ plane are shown in figure

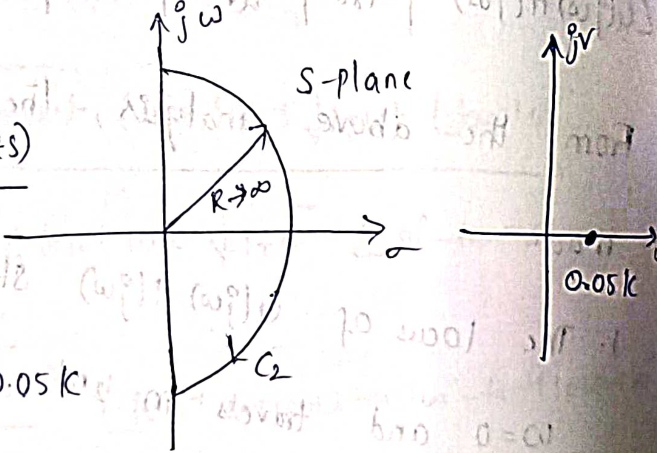


Mapping of Section C_2 :-

The mapping of Section C_2 from s-plane to $G(s)H(s)$ plane is obtained by letting $s = \lim_{R\to\infty} Re^{j\theta}$ in $G(s)H(s)$ and varying θ from $-\pi/2$ to $\pi/2$. Since $s \to Re^{j\theta}$ and $R \to \infty$, $G(s)H(s)$ can be approximated as shown below.

$$G(s)H(s) = \frac{k(1+0.5s)(1+s)}{(1+10s)(s-1)}$$

$$\simeq \frac{k\,0.5\,s \times s}{10s \times s} = 0.05k$$

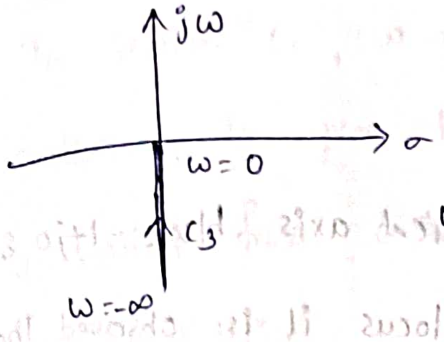


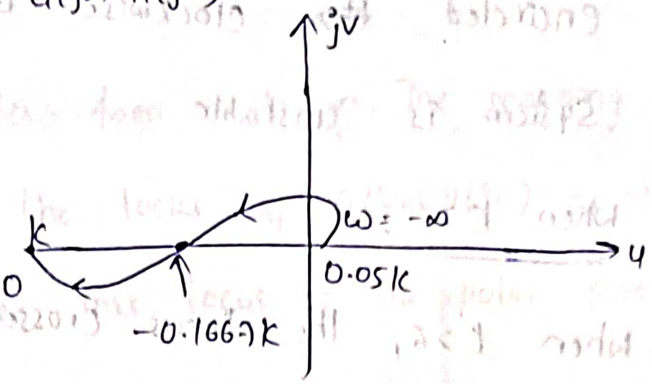
The approximate $G(s)H(s)$ is independent of s and so the contour of section C_2 in s-plane is mapped as a point at $0.05k$ in $G(s)H(s)$ plane.

Mapping of Section C_3 :-

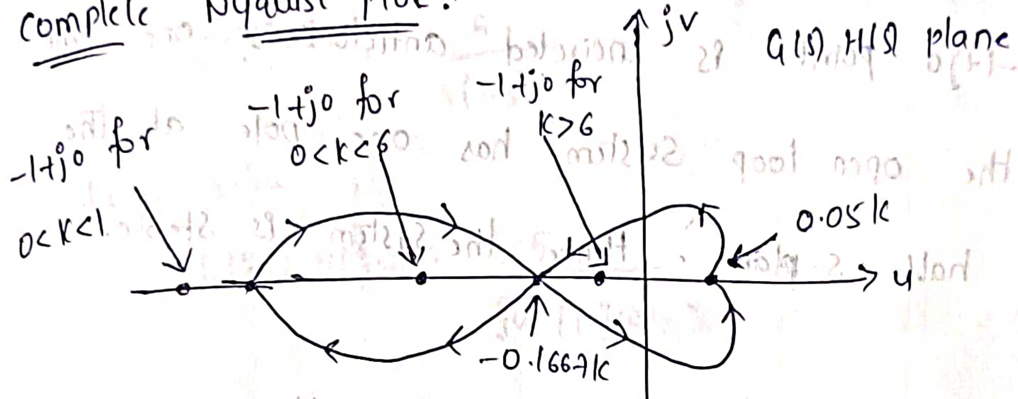
Section C_3, ω varies from $-\infty$ to 0. The mapping Section C_3 is given by the locus of $G(j\omega)H(j\omega)$

as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of $G(j\omega)H(j\omega)$





Complete Nyquist Plot :-



Stability analysis :-

When $-0.1667k = -1$ The contour passes through $-1+j0$ point & this value of k is the limiting value for stability

The limiting value of $k = \frac{1}{0.1667} = 6$

When $0<k<1$:-

When $0<k<1$, the $-1+j0$ point is not encircled, but there is one open loop right half pole and so system is unstable

When $1<k<6$:-

When $1<k<6$, the locus crosses real axis b/w 0 & $-1+j0$. on travelling through the locus it is observed that the $-1+j0$ point is encircled clockwise and so the closed loop system is unstable.

When $K>6$:-

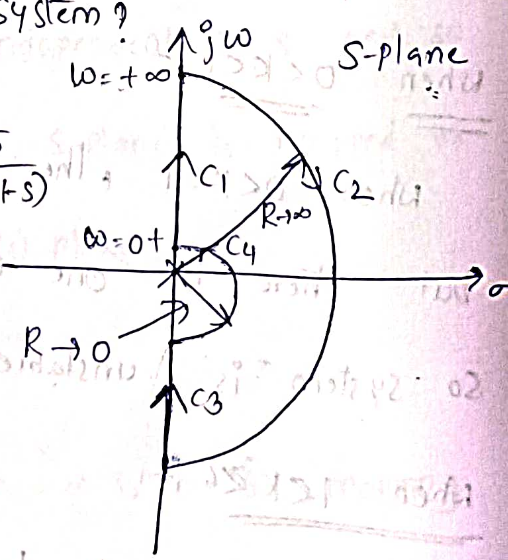
When $K>6$, the locus crosses real axis b/w $-1+j0$ & $-\infty$. on travelling through the locus it is observed that the $-1+j0$ point is encircled anticlockwise one time. Also the open loop system has one pole at the right half s-plane. Hence the system is stable.

Result:-

The open loop system is unstable.

For stability of the closed loop system $K>6$.

5.17 Construct Nyquist plot for a feed back control system whose open loop transfer function is given by $G(s)H(s) = \frac{5}{s(1-s)}$; Comment on the stability of open loop & closed loop system?



Sol Given that $G(s)H(s) = \frac{5}{s(1-s)}$

The open loop transfer function has a pole at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half of s-plane. except the origin.

<u>Mapping of section C_1</u> :-

In Section C_1, ω varies from 0 to ∞. The mapping of section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$.

$$G(s)H(s) = \frac{5}{s(1-s)}$$

Let $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{5}{j\omega(1-j\omega)}$$

$$= \frac{5}{\omega\angle 90^\circ \sqrt{1+\omega^2}\angle -\tan^{-1}\omega}$$

$$= \frac{5}{\omega\sqrt{1+\omega^2}}\left(-90^\circ + \tan^{-1}\omega\right)$$

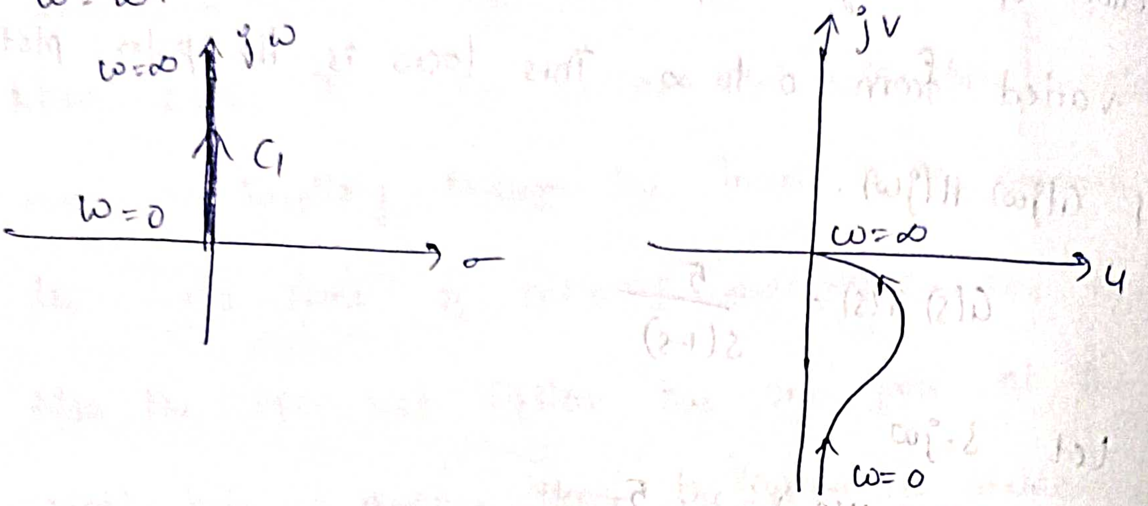
$$|G(j\omega)H(j\omega)| = \frac{5}{\omega\sqrt{1+\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = -90^\circ + \tan^{-1}\omega$$

The exact shape of $G(j\omega)H(j\omega)$ is determined by calculating the magnitude & phase of $G(j\omega)H(j\omega)$ for various values of ω.

ω rad/sec	0	0.6	1.0	2.0	10.0	∞
$\lvert G(j\omega)H(j\omega)\rvert$	∞	7.15	3.53	1.12	0.05	0
$\angle G(j\omega)H(j\omega)$	-90	-59	-45	-26	-5	0

From the above analysis, we can conclude that $G(j\omega)H(j\omega)$ locus starts at -90° axis at ∞ for $\omega = 0$ and meets the origin along 0° axes when $\omega = \infty$.



Mapping of section C_2 :-

The mapping of section C_2 from s-plane to $G(s)H(s)$ plane is obtained by letting $s = \lim_{R\to\infty} Re^{j\theta}$ in $G(s)H(s)$ and varying θ from $\pi/2$ to $-\pi/2$. Since $s \to Re^{j\theta}$ and $R \to \infty$, the $G(s)H(s)$ can be approximated as shown below.

$$G(s)H(s) = \frac{5}{s(1-s)} \approx \frac{5}{s(-s)} \simeq \frac{5}{s^2 e^{j\pi}}$$

let $s = \lim_{R\to\infty} Re^{j\theta}$

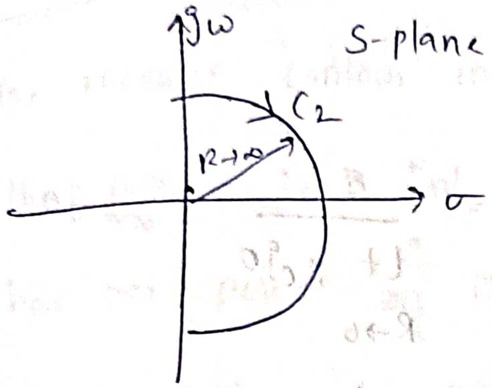
$$\therefore G(s)H(s)\Big|_{s=\lim_{R\to\infty} Re^{j\theta}} = \frac{5}{\lim_{R\to\infty} (Re^{j\theta})^2 e^{j\pi}}$$

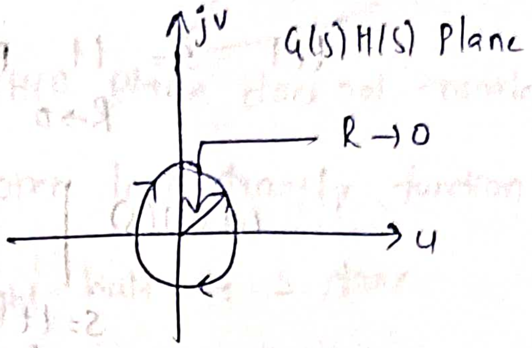
$$= 0\, e^{-j(2\theta+\pi)}$$

When $\theta = \pi/2$ $\quad G(s)H(s) = 0e^{-j2\pi} \longrightarrow (1)$

When $\theta = \pi/2$, $G(s)H(s) = 0 e^{j0}$ ——(2)

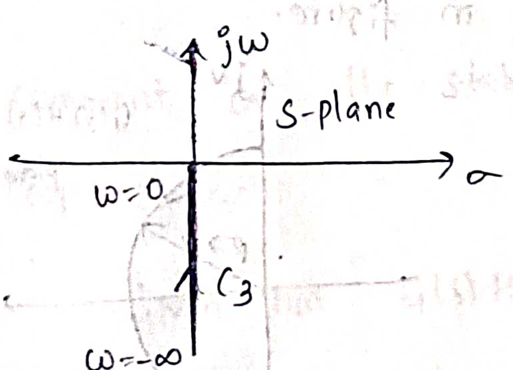
From the eq (1) & (2) we can say that section C_2 in S-plane is mapped as circular arc of zero radius around origin in $G(s)H(s)$ plane with argument varying from $-\pi$ to $+0$ as shown in fig.

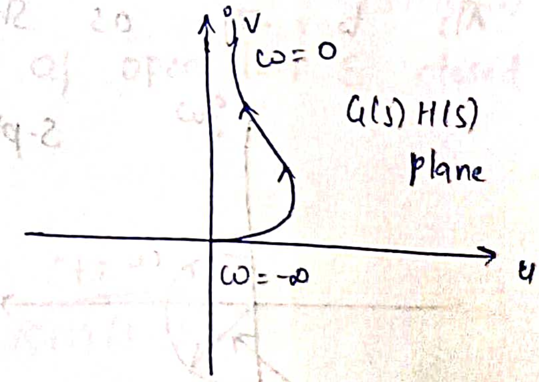




Mapping of Section C_3 :-

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of $G(j\omega)H(j\omega)$





Mapping of Section C_4 :-

The mapping of section C_4 from S-plane to $G(s)H(s)$ plane is obtained by letting $s = \lim_{R \to 0} Re^{j\theta}$ in

$G(s)H(s)$ and varying θ from $-\pi/2$ to $\pi/2$. Since $s \to Re^{j\theta}$ and $R \to 0$, $G(s)H(s)$ can be approximated as shown below.

$$G(s)H(s) = \frac{5}{s(1-s)} \approx \frac{5}{s \times 1} \approx 5/s$$

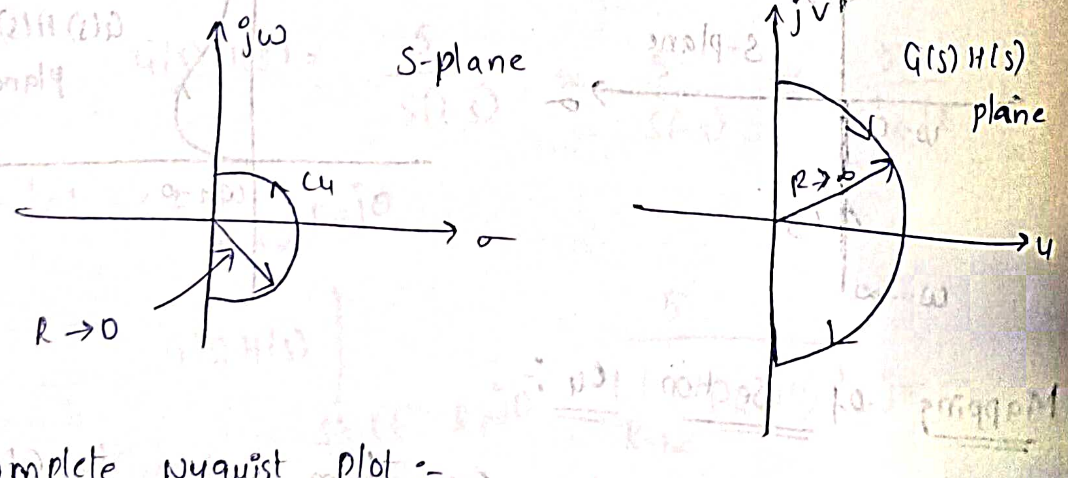
let $s = 1 + Re^{j\theta}$
$R \to 0$

$$\therefore G(s)H(s) \Big|_{\substack{s = 1+Re^{j\theta} \\ R \to 0}} = \frac{5}{1 + Re^{j\theta}} \Big|_{R \to 0} = \infty e^{-j\theta}$$

When $\theta = -\pi/2$, $G(s)H(s) = \infty e^{j\pi/2}$ $\longrightarrow (3)$

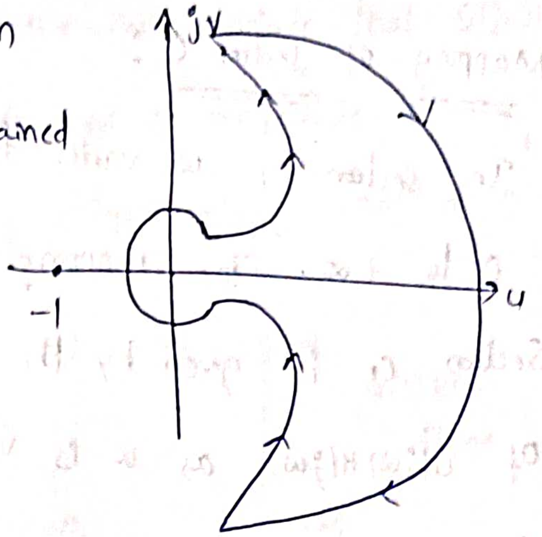
When $\theta = \pi/2$, $G(s)H(s) = \infty e^{-j\pi/2}$ $\longrightarrow (4)$

From the equations (3) & (4) we can say that section C_4 is in s-plane is mapped as circular arc of ∞ radius with argument varying from $\pi/2$ to $-\pi/2$ as shown in figure.



Complete Nyquist plot :-

The entire Nyquist plot in $G(s)H(s)$ plane can be obtained by combining the mappings of individual sections.



Stability analysis :-

The Nyquist contour in $G(s)H(s)$ plane does not encircle the point $-1+j0$ but the open loop transfer function has one pole on the right half of s-plane.

∴ The system is unstable.

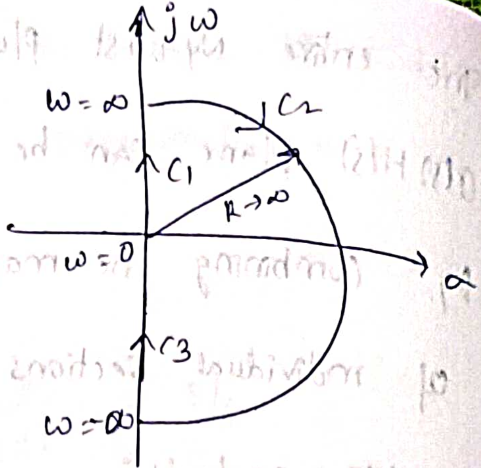
Result :- Both open loop & closed loop systems are unstable.

Ex: 5.18 By Nyquist stability criterion, determine the stability of closed loop system, whose open loop transfer function is given by $G(s)H(s) = \frac{s+2}{(s+1)(s-1)}$ Comment on the stability of open loop & closed loop system?

Sol Given that $G(s)H(s) = \frac{s+2}{(s+1)(s-1)}$

The open loop transfer function does not have a pole at origin. Hence choose the Nyquist Contour on s-plane enclosing the entire right half plane as shown in figure.

Mapping of section C_1 :-



In section C_1 ω varies from 0 to $+\infty$. The mapping of Section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$

$$G(s)H(s) = \frac{s+2}{(s+1)(s-1)} = \frac{2(1+0.5s)}{(1+s)(-1+s)}$$

let $s = j\omega$

$$\therefore G(j\omega)H(j\omega) = \frac{2(1+j0.5\omega)}{(1+j\omega)(-1+j\omega)} = \frac{2\sqrt{1+0.25\omega^2}\,\angle\tan^{-1}0.5\omega}{\sqrt{1+\omega^2}\,\angle\tan^{-1}\omega\,\sqrt{1+\omega^2}\,\angle 180^\circ - \tan^{-1}\omega}$$

$$= \frac{2\sqrt{1+0.25\omega^2}}{1+\omega^2}\,\angle -180^\circ + \tan^{-1}0.5\omega$$

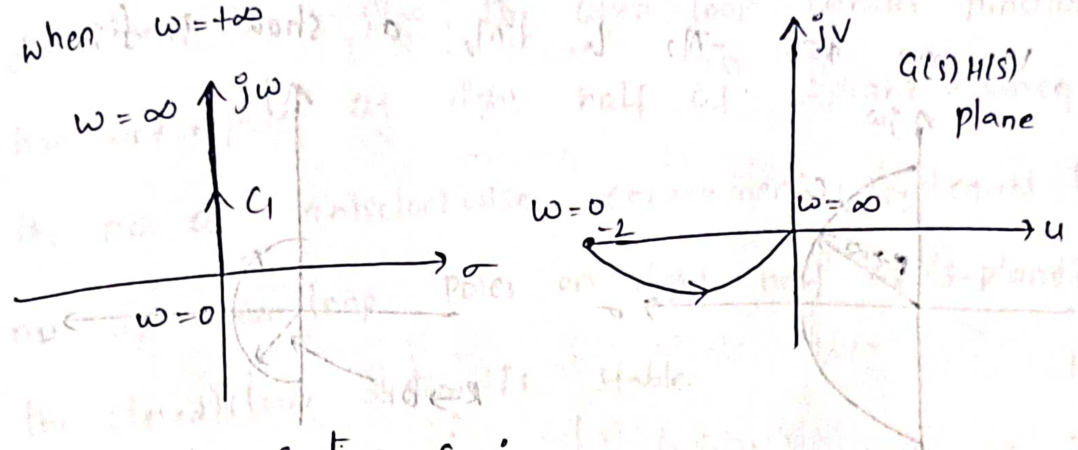
$$\therefore |G(j\omega)H(j\omega)| = \frac{2\sqrt{1+0.25\omega^2}}{1+\omega^2}$$

$$\angle G(j\omega)H(j\omega) = -180^\circ + \tan^{-1}0.5\omega$$

The exact shape of $G(j\omega)H(j\omega)$ locus is determined by calculating the magnitude & phase of $G(j\omega)H(j\omega)$ for various values of ω.

ω rad/sec	0	0.4	1.0	2.0	10.0	∞
$\lvert G(j\omega)H(j\omega)\rvert$	2	1.76	1.12	0.57	0.1	0
$\angle G(j\omega)H(j\omega)$	-180	-168	-153	-135	-101	-90

From the above analysis, we can conclude that $G(j\omega)H(j\omega)$ locus starts at $-180°$ axis at a magnitude of -2 for $\omega = 0$ and meets the origin along $-90°$ axis when $\omega = +\infty$



Mapping of section C_2 :-

The mapping of section C_2 from s-plane to $G(s)H(s)$ plane is obtained by letting $s = \lim_{R\to\infty} Re^{j\theta}$ in $G(s)H(s)$ and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to Re^{j\theta}$ and $R \to \infty$, $G(s)H(s)$ can be approximated as shown below.

$$G(s)H(s) = \frac{2(1+0.5s)}{(1+s)(1+s)} \approx \frac{2\times 0.5s}{s\times s} = \frac{1}{s}$$

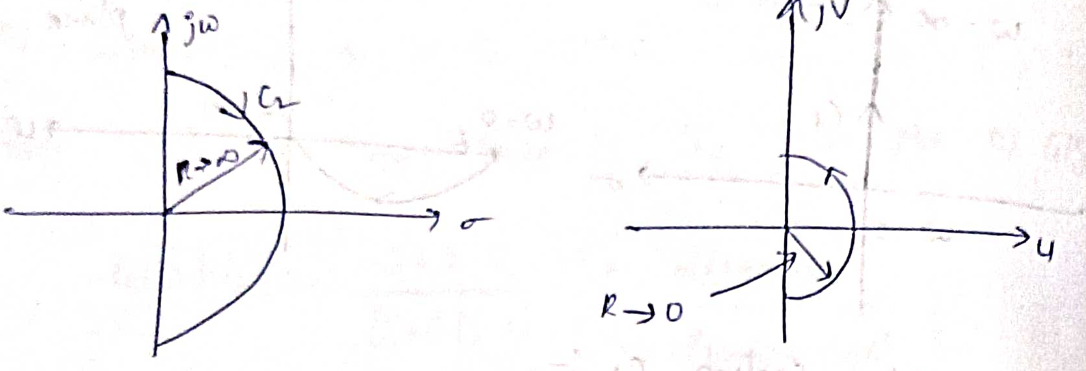
Let $s = \lim_{R\to\infty} Re^{j\theta}$

$$\therefore G(s)H(s)\Big|_{s=\lim_{R\to\infty} Re^{j\theta}} = \lim_{R\to\infty}\frac{1}{Re^{j\theta}} = 0e^{-j\theta}$$

When $\theta = \pi/2$, $G(s)H(s) = 0e^{-j\pi/2}$ → (1)

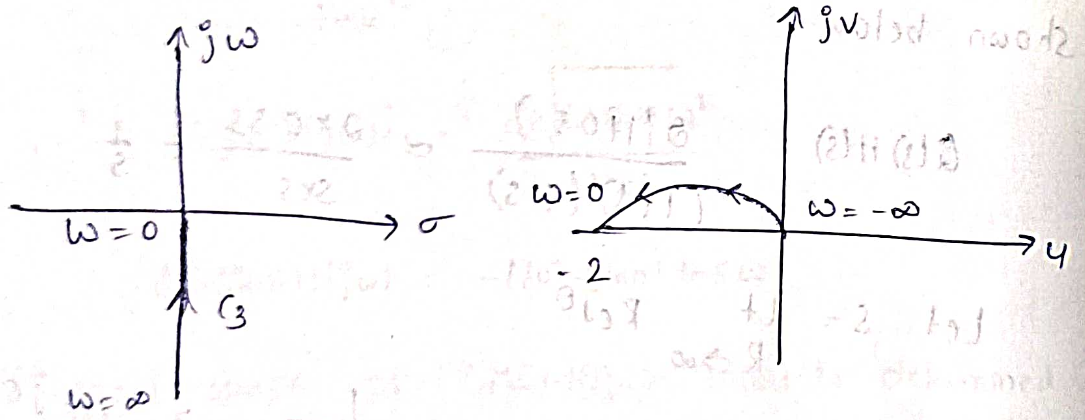
When $\theta = -\pi/2$, $G(s)H(s) = 0e^{j\pi/2}$ → (2)

From the equations (1) & (2) we can say that Section C_2 in s-plane is mapped as circular arc of zero radius around origin in $G(s)H(s)$ - plane with argument varying from $-\pi/2$ to $+\pi/2$ as shown in figure.

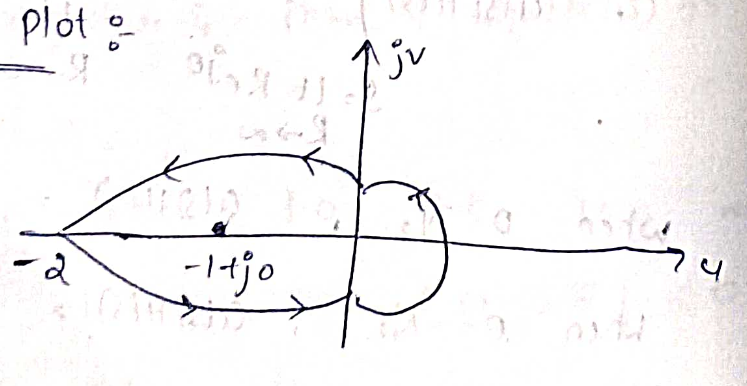


Mapping of section C_3 :-

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of $G(j\omega)H(j\omega)$.



Complete Nyquist plot :-



Stability analysis :-

on travelling through Nyquist contour it is observed that $-1+j0$ point is encircled in anticlock wise direction one time. Also the open loop transfer function has one pole at right half of s-plane. Since the no. of anticlockwise encirclement is equal to no. of open loop poles on right half of s-plane, the closed loop system is stable.

Result :-

1. open loop system is unstable.
2. Closed loop system is stable.