SET Theory

Basic Concepts of Set Theory, Relations & ordering, the principle of inclusion – exclusion, Pigeon code principle & its applications, Functions, composition of functions, inverse functions, Recursive functions, Lattices & its properties, Algebric Structures : Algebric System examples and general properties, Semigroups & Monoids, Groups, Subgroups, Homomorphism & Isomorphism,

→ Basic concepts of set theory :-

A Set is a collection of well defined Objects. Any object belonging to a set is called a Member (or) An Element of that set. Elements of a set are usually denoted by lowercase letters. While sets are denoted by capital letters.

Example :-

$P = \{1, 2, 3, 4\}$

$A = \{a, b, c, \ldots z\}$

$N = \{1, 2, 3, \ldots \infty\}$

the symbol epsilon "$\in$" indicates that the membership in a set.

If an element $x \in A$, Then we write "x belongs to A". Which is read as "x is an element of set A" (or) "x belongs to the set A" (or) "x is in A"

If there exist an object 'y' which doesnot belong to the set A then we write "$y \notin A$" :

Which is equivalent to the Negation of the Statement is $y \notin x$ if and only if $\sim(y \in A)$

$$y \notin x \Leftrightarrow \sim(y \in A)$$

Subset :-

Let A & B be any two sets. If every element of A is an element of B then A is called a Subset of B. (or)

A is said to be included in B.

(or)

B includes A.

We use the notation $A \subseteq B$ (A subset are equal to B).

$$A \subseteq B \Leftrightarrow (\forall x)(x \in A \rightarrow x \in B)$$

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Equal Sets :-

Two sets A & B are equal if and only if $A \subseteq B$ and $B \subseteq A$ $\quad (P \rightarrow Q) \wedge (Q \rightarrow P) \Leftrightarrow P \rightleftarrows Q$

$$\therefore A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$$

$$\Leftrightarrow (x)(x \in A \rightarrow x \in B) \wedge (x)(x \in B \rightarrow x \in A)$$

$$\Leftrightarrow (x)(x \in A \rightleftarrows x \in B)$$

Proper Subset :-

Let A & B be any two sets then set A called as Proper subset of B if

$A = \{1, 2, 3\}$ in subset
then
$A = \{1, 2, 3, 4\}$ one extra
properset

$A \subseteq B$ and $A \neq B$ which is equivalent to $A \subset B$ (A contain B)

Universal Set :-

A set is called a universal set if it includes every set under discussion.

A Universal set is denoted by "U" (or) "E" mostly us

For Example :-

$A \vee \sim A$

$A \vee \phi \quad A$

$E = \{ x \mid P(x) \vee \sim P(x) \} \; P$

x such that

Null Set :-

A set which does not contain any element is called an Empty set (or) Null set.

Here $\wedge$ means intersection

An Empty set is denoted by ϕ. $A \cap \phi = \phi$

$\phi = \{ x \mid P(x) \wedge \sim P(x) \}$

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Power set :-

For a set A, a collection (or) family of all subsets of A is called the Power set of A.

The power set of A is denoted by $P(A)$ (or) 2^A.

For Example :- $A = \{a\}$, $2^1 = 2$

$\phi, \{a\}$

$A = \{a, b\}$ $2^2 = 4$

$\phi, \{a\}, \{b\}, \{a, b\}$

$A = \{a, b, c\}$ $2^3 = 8$

$\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}$

$A = \{a, b, c, d\}$ $2^4 = 16$

$\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{a, b, c, d\}, \{b, d\}, \{c, a\}, \{a, b, c, d\}$

$\{a, c, d\}, \{b, c, d\}, \{a, b, d\}$

Some Operations On Sets :-

The Intersection :-

The intersection of any two sets A & B is denoted by $A \cap B$, which is the set containing all the elements which belonging to both A & B

i.e $A \cap B = \{x \mid x \in A \wedge x \in B\}$

$A \cap B = \phi$

Properties :-

$A \cap B = B \cap A$

$A \cap A = A$

$A \cap \phi = \phi$

$A \cap E = A$

Disjoint sets :-

Two sets A & B are called Disjoint sets if there is no common element in both A & B.

i.e $A \cap B = \phi$

Mutually Disjoint sets :-

A collection of sets is called a Disjoint collection if for every pair of sets in the collection. The two sets are Disjoint.

The elements of a Disjoint collection are said to be mutually disjoint.

$A = \{1,2\}, B = \{3,4\}, C = \{5,6\}$

$A \cap B = \phi$

$B \cap C = \phi$

$A \cap C = \phi$

For Example :-

$A = \{\{1,2\}, \{3\}\}$

$B = \{\{1\}, \{2,3\}\}$

$C = \{\{1,2,3\}\}$

$A \cap B = \phi$

$B \cap C = \phi$ $A \cap C = \phi$

Union :-

For any two sets A & B, the union of A & B are written as $A \cup B$, which is the set containing all elements which are members of set A (or) the set B (or) Both A & B.

$$A \cup B = \{ x \mid x \in A \vee x \in B \}$$

Properties :-

$$A \cup B = B \cup A$$
$$A \cup A = A$$
$$A \cup \emptyset = A$$
$$A \cup E = E$$

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Relative Compliment :-

Let A & B be any two sets. The relative compliment of B in A written as A−B which is the set contain all the elements of A which are not elements of B.

i.e $A - B = \{ x \mid x \in A \wedge x \notin B \}$

For Example :-

$A = \{1, 2, 3, 4\}$

$B = \{3, 4, 5, 6\}$

$A - B = \{1, 2\}$

$A = \{1, 2, 3, 4\}$

$B = \{5, 6, 7, 8\}$

$A - B = \{1, 2, 3, 4\}$

Absolute Compliment :-

Let E be the Universal set, for any set A the relative compliment of A with respect to E

i.e E−A is called the Absolute compliment of A.

i.e A' (or) $A^c = E - A = \{ x \mid x \notin A \}$

Symmetric Difference / Boolean Sum :-

Let A & B be any two sets. The symmetric difference of A & B is the set A+B is denoted by

$$A+B = (A-B) \cup (B-A)$$

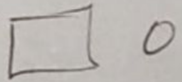
For Example :- $A = \{1,2,3,4\}$ $B = \{3,4,5,6\}$

$$A+B = (A-B) \cup (B-A)$$

$$= \{1,2\} \cup \{5,6\}$$

$$= \{1,2,5,6\}$$

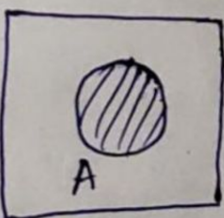
Venn Diagram :-



Only these two we use in V.D

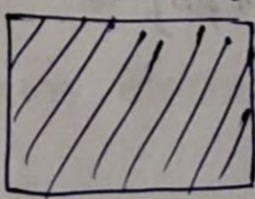
A Venn Diagram is a pictorial representation of set. The universal set E is represented by a set of points in a rectangle & a subset A of E is represented by the interior of a circle.

The below are some of the venn diagrams

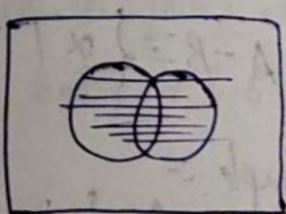


A

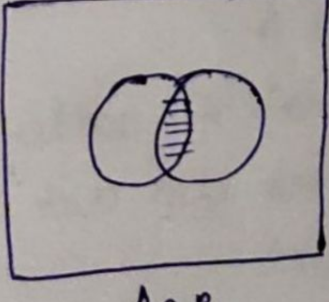
universal set



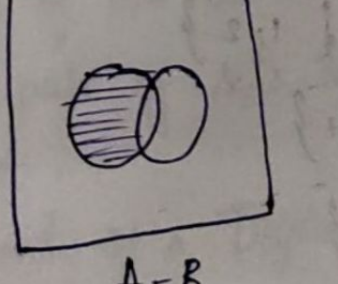
E



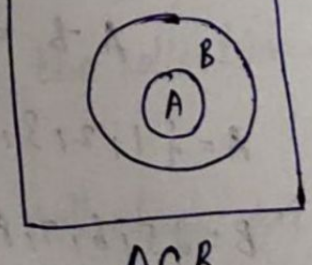
$A \cup B$



$A \cap B$



$A - B$



$A \subset B$

A Contained B

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Cartesian Product :-

Let A & B be any two sets. The set of all ordered pairs such that the first member of the ordered pair is an element of A and the second member is an element of B is called Cartesian Product of A and B and it is written by $A \underset{cross}{\times} B$

$$A \times B = \{ \langle x, y \rangle / x \in A \wedge y \in B \}$$

For Example :-

Let $A = \{1, 2, 3\}$

$B = \{\alpha, \beta\}$

$A \times B = \{(1,\alpha), (1,\beta), (2,\alpha), (2,\beta), (3,\alpha), (3,\beta)\}$

$B \times A = \{(\alpha,1), (\alpha,2), (\alpha,3), (\beta,1), (\beta,2), (\beta,3)\}$

$A \times A = \{(1,1) (1,2), (1,3), (2,1), (2,2) (2,3)$
$(3,1), (3,2), (3,3)\}$

$A \times B \cap B \times A = \phi$ $\quad \{(1,2) \neq (2,1)\}$

For Example :-

If $A = \phi$, $B = \{1, 2, 3\}$. Find $A \times B$ and $B \times A$

$A \times B = B \times A = \phi$ (There is no mapping)

① Let $X = \{1,2,3,4\}$ if $R = \{\langle x,y \rangle / x \in X \wedge y \in X \wedge (x-y)$ is an integral non-zero multiple of 2$\}$

$S = \{\langle x,y \rangle / x \in X \wedge y \in X \wedge (x-y)$ is an integral non-zero multiple of 3$\}$.

Sol:- $R = \{\langle 1,3 \rangle, \langle 3,1 \rangle, \langle 2,4 \rangle, \langle 4,2 \rangle\}$

$S = \{\langle 1,4 \rangle, \langle 4,1 \rangle\}$

$R \cap S = \{\langle 1,3 \rangle, \langle 3,1 \rangle, \langle 2,4 \rangle, \langle 4,2 \rangle, \langle 1,4 \rangle, \langle 4,1 \rangle\}$

$R \cap S = \phi$

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→ Relations & Ordering :- 2nd topic

Binary relation :- Any set of ordered pair defines a binary relation. We shall call a binary relation simply a relation. An ordered pair $< x y > \in R$ (or) x Related y xRy which may be read as "x is in relation R to y.

$F = \{ < x\ y > / x \text{ is Father of } y \}$

$= \{ <2,4>, <1,3>, <\lambda,6>, <John, \mu> \}$

Domain :-
Let S be a binary relation. The set D(S) of all objects x such that for some y ~~ie~~
i.e x, y belongs to S is called the Domain of S.

$\therefore D(S) = \{ x \mid (\exists y) < x, y > \in S \}$

~~Range~~ :- Let S be a binary relation.

Range :-
The set R(S) of all object y such that for some x
i.e x, y belongs to S is called Range of S

$\therefore R(S) = \{ y \mid (\forall x) < x, y > \in S \}$

$\therefore S = \{ <2,4>, <1,3>, <\lambda,6>, <John, \mu> \}$

$D(S) = \{ 2, 1, \lambda, John \}$

$R(S) = \{ 4, 3, 6, \mu \}$

Properties of Binary Relations :-

1. Reflexive :-

A binary relation R in a set X is reflexive if for every $x \in X$, xRx i.e $<x,x> \in R$

For example :- Let $X = \{1,2,3,4\}$

$\therefore R = \{<1,1>, <2,2>, <3,3>, <4,4>\}$

2. Symmetric :-

A relation R in a set X (cap) is symmetric if for every x and y in X whenever xRy and yRx

i.e $R = \{<1,2>, <2,1>, <1,1>\}$

3. Transitive :-

A relation R in a set X is transitive if for every x, y and z in X (capital) whenever $xRy \wedge yRz$ the xRz

$\therefore R = \{<1,1>, <2,2>, <1,2>, <2,1>\}$

4. Irreflexive :-

A relation R in a set X is irreflexive if for every $x \in X$, $x \not R x$

i.e $<x,x> \notin R$

$\therefore R = \{<1,2>, <1,3>\}$

5. Antisymmetric :-

A relation R in a set X is antisymmetric if for every x and y in X whenever xRy and $yRx \Rightarrow x = y$

$\therefore R = \{<1,2>, <2,1>, <1,1>\}$

if one exists then the set is Antisymmetric

Relation Matrix & Graph of relation :-

Matrix Representation :-

The Relation R from a finite set X to a finite set Y can also be represensed by a matrix which is called as Relation Matrix of R.

The Matrix can be defined in the following manner.

$$r_{ij} = \begin{cases} 1 & \text{if } x_i R y_j \\ 0 & \text{if } x_i \not R y_j \end{cases}$$

Example :-

consider the relation $R = \{ \langle x_1, y_1 \rangle, \langle x_2, y_1 \rangle, \langle x_3, y_2 \rangle, \langle x_2, y_2 \rangle \}$

	y_1	y_2
x_1	1	0
x_2	1	1
x_3	0	1

Graph Representation :-

A relation cannot be represented pictorially by drawing it's graph. Let R be a relation in a set $X = \{x_1, x_2 - - - x_n\}$

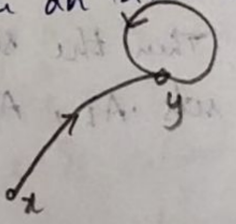
The elements of X are reprented by circles called Nodes.

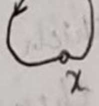
The Nodes are also called as vertices.

~~If x_i & x_j be~~

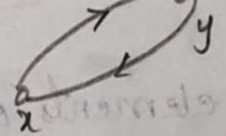
If $\langle x_i, x_j \rangle \in R$ then we connect nodes x_i & x_j by means of an arc & put an arrow on the arc in direction from x_i to x_j

If $\langle x_i, x_i \rangle \in R$ we get an arc which starts from node x_i & returns to node x_i such an arc is called as loop.





xRx



$(xRy) \wedge (yRx)$

$(xRy) \wedge (yRy)$

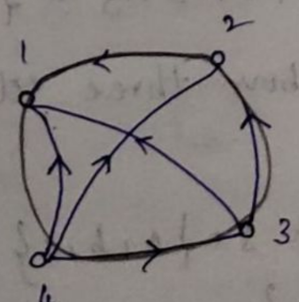
Let $X = \{1, 2, 3, 4\}$ and $R = \{\langle x, y \rangle \mid x > y\}$ draw the graph of R and also give its matric.

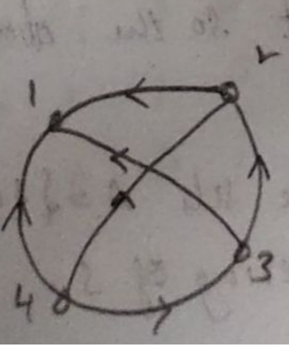
$X = \{1, 2, 3, 4\}$

$R = \{\langle x, y \rangle \mid x > y\}$

$R = \{\langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 4, 1 \rangle, \langle 3, 2 \rangle, \langle 4, 2 \rangle, \langle 4, 3 \rangle\}$

	1	2	3	4
1				
2	1	0	0	0
3	1	1	0	0
4	1	1	1	0





Partition & Covering of a Set :-

Let "S" be a given set and $A = \{A_1, A_2 \cdots A_m\}$ where each A_i, $i = \{1, 2 \cdots m\}$ is a subset of S and

$$\boxed{\bigcup_{i=1}^{m} A_i = S}$$

Then the set A is called a covering of S and the sets $A_1, A_2 \cdots A_m$ are said to be covers of "S".

In addition the elements of which are subsets of S which are mutually disjoint then A is called a Partition of S & the sets $A_1, A_2 \cdots A_m$ are called the Blocks of S.

Example :-

Let $S = \{a, b, c\}$ and the subsets are

$A_1 = \{a, b\}$, $A_2 = \{b\}$, $A_3 = \{c\}$

$A_1 \cup A_2 \cup A_3 = \{a, b, c\}$ but $A_1 \cap A_2 = b$

so The above three sets are covering of S.

Example :-

Let $S = \{a, b, c\}$ & let $A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{c\}$

All are mutually disjoint so the above 3 sets are partions of S.

Example :- Let $S = \{a, b, c\}$ consider the sets $A = \{\{a, b\}, \{b, c\}\}$ check A is the partition (or) covering of S.

Let $A_1 = \{a, b\}$

$A_2 = \{b, c\}$

$A_1 \cup A_2 = \{a, b, c\}$ and $A_1 \cap A_2 = \{b\}$

A_1 and A_2 are covering of S.

Let $S = \{a, b, c\}$ & $B = \{\{a\}, \{b, c\}\}$

$B_1 = \{a\}$

$B_2 = \{b, c\}$

$B_1 \cup B_2 = \{a, b, c\}$, $B_1 \cap B_2 = \phi$

$\therefore$ B_1 and B_2 partition of S

Equivalence Relation :-

Properties 1. 2. 3. Satisfy

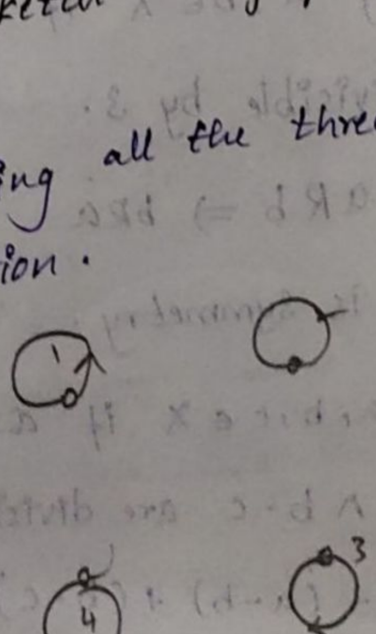
A relation ~~are~~ R in a set X is called an equivalence relation if it is Reflexsive, symmetric and transitive.

Example :-

Let $X = \{1, 2, 3, 4\}$ and $R = \{<1,1>, <1,4>, <2,2>, <2,3>, <3,2>, <3,3>, <4,1>, <4,4>\}$ and sketch it's graph and also write the matric of R and check equivalence relation.

$\therefore$ The above relation satisfying all the three properties

so R is an equivalent relation.



	1	2	3	4
1	1	0	0	1
2	0	1	1	0
3	0	1	1	0
4	1	0	0	1

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Example:-

Let $X = \{1, 2, 3 - - - 7\}$ and

$R = \{<x, y> / x - y$ is divisible by $3\}$ Show that

R is an Equivalence relation and draw the graph of R.

$R = \{<1,1>, - - - <7,7>, <1,4>, <4,1>,$
$<1,7>, <7,1>, <2,5>, <5,2>, <3,6>, <6,3>,$
$<4,7>, <7,4>\}$

To show an R is an Equivalent relation by using the properties of binary relations.

1) For every $a \in X$ $a - a$ is divisible by 3.

Hence aRa a is related to a

$\therefore$ R is Reflexive.

2) For every $a, b \in X$ if $a - b$ is divisible by 3 then $b - a$ is divisible by 3.

Hence $aRb \xRightarrow{\text{implies}} bRa$ b is related to a

$\therefore$ R is Symmetry.

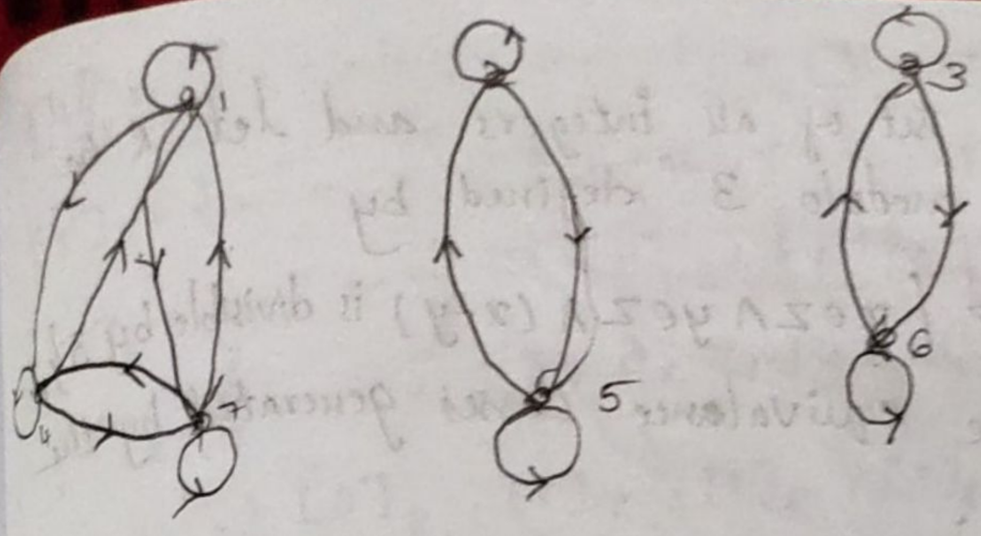
3) For any $a, b, c \in X$ if ~~aRbRc~~: $aRb \wedge bRC$ then both $a - b \wedge b - c$ (and) are divisible by 3. So that

$a - C = (a - b) + (b - c)$ is also divisible by 3.

Hence aRC

$\therefore$ R is Transitive

So R satisfying all the three properties then R is an Equivalence Relation.



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Note :-

Let us define a relation

$R = \{ \langle x, y \rangle \mid x - y$ is divisible by $m \}$

In the above relation both x & y have same remainder. When each is divided by m. It is customary to denote R by '$\equiv$' and to write xRy as $x \equiv y$ (x congriant y)

(or) $x \equiv y \pmod{m}$.

The relation '$\equiv$' is called as congruence.

Equivalence Classes :-

Let R be an equivalance relation on a set X. For any $x \in X$ the set $[x]_R \subseteq X$ is ~~given by~~ subset or equal to

$$[x]_R = \{ y \mid y \in X \wedge xRy \}$$

Is called as an R equivalance classes generated $x \in X$ which is also denoted by X/R X divides R

** Problem :-

1) Let Z be the set of all integers and let R be the "congriance modulo 3" defined by

$R = \{ \langle x, y \rangle \mid x \in Z \wedge y \in Z \wedge (x-y)$ is divisible by $3 \}$.

Determine the equivalance classes generated by the elements of Z.

$Z = \{ -\alpha, \ldots -3, -2, -1, 0, 1, 2, 3 \ldots \}$

Sol:- The equivalance classes are

$[0]_R = \{ \ldots -6, -3, 0, 3, 6 \ldots \}$

$[1]_R = \{ \ldots -5, -2, 1, 4, 7 \ldots \}$

$[2]_R = \{ \ldots -4, -1, 2, 5, 8 \ldots \}$

$Z|_R = \{ [0]_R, [1]_R, [2]_R \}$

Problem :-

2) Let Z be the set of all the integers and let R be the "congriance modulo 7" defined by

$R = \{ \langle x, y \rangle \mid x \in Z \wedge y \in Z \wedge (x-y)$ is divisible by $7 \}$.

Determine the equivalance classes generated by the elements of Z.

Sol:- $Z = \{ -\alpha \ldots -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7 \ldots \}$

The equalance classes are

$[0]_R = \{ \ldots -14, -7, 0, 7, 14 \ldots \}$

$[1]_R = \{ \ldots -13, -6, 1, 8, 15 \ldots \}$

$[2]_R = \{ \ldots -12, -5, 2, 9, 16 \ldots \}$

$[3]_R = \{ \ldots -11, -4, 3, 10, 17 \ldots \}$

$[4]_R = \{ - - - - , -10 , -3 , 4 , 11 , 18 - - - \}$

$[5]_R = \{ - - - - -9 , -2 , 5 , 12 , 19 - - - \}$

$[6]_R = \{ - - - -8 , -1 , 6 , 13 , 20 - - - \}$

$[7]_R = \{ - - - -7 , 0 , 7 , 14 , 21 - - - \}$

$Z/R = \{ [0]_R , [1]_R , [2]_R , [3]_R , [4]_R , [5]_R ,$
$[6]_R , [7]_R \}$

Compatability ~~Duration~~ Relation :- $\sim$

A Relation R in a set X is said to be compatibility relation if it satisfying

1) Reflexive
2) Symmetric

Note :- All equivalenas relations are compatability relations but all compatibility relations are not equivalances Relation.

Example :-

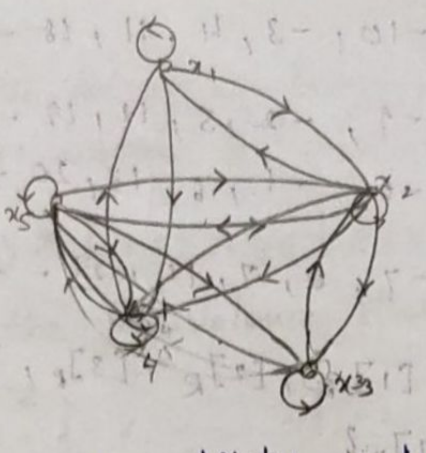
$X = \{ ball, bed, dog, let ; egg \}$ and let the relation R be given by

$R = \{ \langle x, y \rangle / x, y \in X \wedge xRy$ if $x \& y$ contains some common letter $\}$.

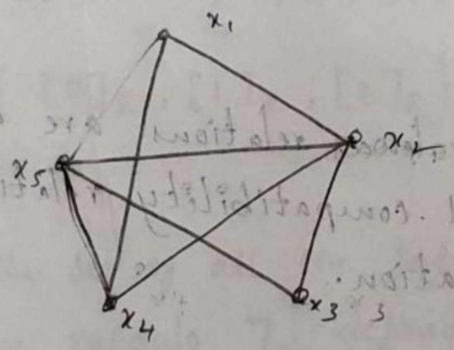
Let us assume that $x_1 = ball$, $x_2 = bed$, $x_3 = dog$, $x_4 = let$, $x_5 = egg$.

Now, the relation of R is

$R = \{ \langle x_1 , x_1 \rangle - - - \langle x_5 , x_5 \rangle , \langle x_1 , x_2 \rangle , \langle x_2 , x_1 \rangle ,$
$\langle x_1 , x_4 \rangle , \langle x_4 , x_1 \rangle , \langle x_2 , x_3 \rangle , \langle x_3 , x_2 \rangle ,$
$\langle x_3 , x_5 \rangle , \langle x_5 , x_3 \rangle , \langle x_4 , x_5 \rangle , \langle x_5 , x_4 \rangle ,$
$\langle x_2 , x_5 \rangle , \langle x_5 , x_2 \rangle , \langle x_2 , x_4 \rangle , \langle x_4 , x_2 \rangle \}$



Since '$\approx$' is a compatibility relation it is not necessary to draw the loops at each element nor is it necessary to draw both xRy and yRx we simplify the graph as show in the below figure.



The relation matrix of a compatibility relation is Symmetric & has its diagonal elements unity. It is therefore sufficient to give only the elements of the relation matrix.
lower triangular part of the relation matrix.
For the compatibility relation we have been discussing the relation matrix can be optained as shown in below.

Delete x_5 in x-axis

Delete x_1 in y-axis

	x_1	x_2	x_3	x_4
x_2	1			
x_3	0	1		
x_4	1	1	0	
x_5	0		1	1

Maximal compatability Block :-

Let X be a set and $\approx$ be a compatibility relation on X. $A \subseteq X$ is called a maximal compatability block if any element of A is compatable to every other element of A and no element of X - A is compatable to all the elements of A.

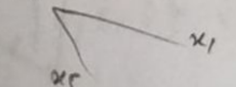
Processor to find Maximal compatability Block :-

Steps :-

1) To find Maximal compatability block first we draw a simplified graph of the compatability relation.

2) Pick from this graph the largest complete polygons. By a "largest complete polygon" we mean a polygon in which every vertex is connected to every other vertex.

3 triangles



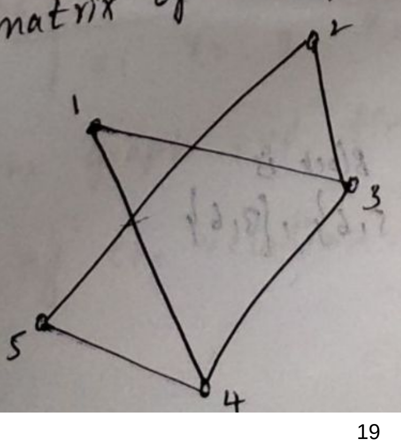
Example :-

A triangle is always a complete polygon. $\{x_1, x_6\}, \{x_5, x_6\}$

3) Any two elements which are compatable one another but to know other elements also forms a maximal compatability block.

Problem :-

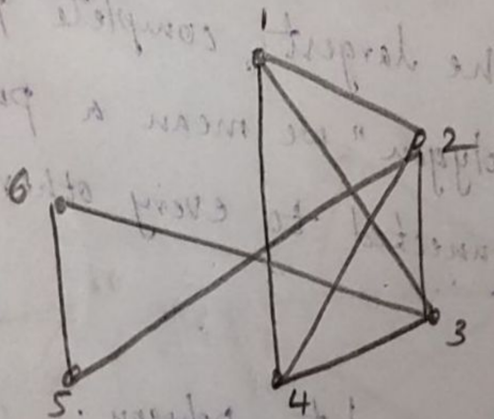
1) Find the Maximal compatability Block and draw the relation matrix of the following Graph.



2	0			
3		1		
4	1	0	1	
5	0	1	0	1
	1	2	3	4

The Maximal compatability Blocks are :-
{1, 4, 3}, {2, 5}, {5, 4}, {2, 3}.

2) Find matrix & Maximal compatability Block.

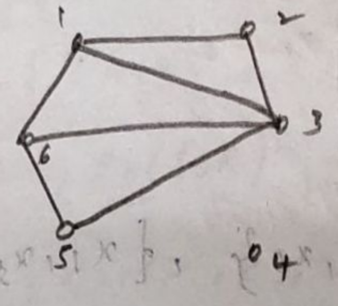


Sol:-

2	1				
3	1	1			
4	1	1	1		
5	0	1	0	0	
6	0	0	1	0	1
	1	2	3	4	5

~~{1, 4, 3}, {1, 2, 3}~~,
∴ The Maximal compatability Block is
{1, 2, 3, 4}, {2, 5}, {5, 6}, {3, 6}

3) Find the Maximal compatability Block and matrix



6, 4 is deleted in x-axis

Sol:

$$\begin{array}{c|cccc} 2 & 1 & & & \\ 3 & 1 & 1 & & \\ 5 & 0 & 0 & 1 & \\ 6 & 1 & 0 & 1 & 1 \\ \hline & 1 & 2 & 3 & 5 \end{array}$$

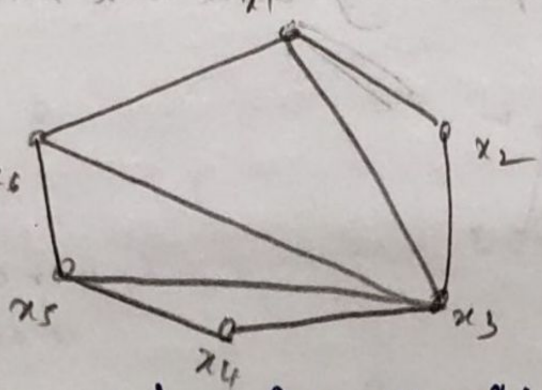
∴ The compatability (Maximal) Block is $\{1,2,3\}$, $\{1,3,6\}$, $\{3,5,6\}$ $\{4\}$

~~$\{1,2\}$, $\{3,6\}$~~

4) Let the compatability relation on a set $\{x_1, x_2 - - x_6\}$ be given by the matrix

$$\begin{array}{c|ccccc} x_2 & 1 & & & & \\ x_3 & 1 & 1 & & & \\ x_4 & 0 & 0 & 1 & & \\ x_5 & 0 & 0 & 1 & 1 & \\ x_6 & 1 & 0 & 1 & 0 & 1 \\ \hline & x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

Draw the graph and find the maximal compatability blocks of the relation.



$\{x_3, x_4, x_5\}$, $\{x_3, x_5, x_6\}$, $\{x_1, x_3, x_6\}$,
$\{x_1, x_2, x_3\}$,

- **Composition of Binary Relations** :-

Let R be a Relation from x to y and S be a relation from y to z. Then a relation written as Rcircles RoS is called a composite relation of R and S where

$$RoS = \{\langle x, z\rangle / x \in X \wedge z \in Z \wedge (\exists y) (y \in Y \wedge \langle x R y\rangle \in R \wedge \langle y R z\rangle \in S)\}$$

The operation of optaining RoS from R and S is called composition of relations.

$$R \qquad S \qquad \underset{X \to Z}{RoS}$$

$$\therefore \; x \longrightarrow y \longrightarrow z$$

Example:-

Let $R = \{<1,2>, <3,4>, <2,2>\}$ and

$S = \{<4,2>, <2,5>, <3,1>, <1,3>\}$. Find

$R \circ S$, $S \circ R$, $R \circ (S \circ R)$, $(R \circ S) \circ R$, $R \circ R$, $S \circ S$, $R \circ R \circ R$.

Sol:- $R \circ S = \{<1,5>, <3,2>, <2,5>\}$

$S \circ R = \{<4,2>, <3,2>, <1,4>\}$

$R \circ (S \circ R) = \{<3,2>\}$

$(R \circ S) \circ R = \{<3,2>\}$

$R \circ R = \{<1,2>, <2,2>\}$

$S \circ S = \{<4,5>, <3,3>, <1,1>\}$

$R \circ R \circ R = \{<1,2>, <2,2>\}$

3/1/22

1) If R and S be two relations on a set of positive integers.

$R = \{<x, 2x> / x \in I\}$, $S = \{<x, 7x> / x \in I\}$

Find $R \circ S$, $R \circ R$, $R \circ R \circ R$ and $R \circ S \circ R$.

Sol:- Let $I = \{1, 2, \ldots \alpha\}$

$R = \{<1,2>, <2,4>, <3,6>, <4,8>, <5,10>, <6,12>, <7,14> \ldots\}$

$S = \{<1,7>, <2,14>, <3,21>, <4,28> \ldots\}$

$R \circ S = \{<1,14>, <2,28>, <3,42> \ldots\}$

$= \{<x, 14x>\} = \{<x, 14x> / x \in I\}$

$R \circ R = \{<x, 4x> / x \in I\}$

$R \circ R \circ R = \{<x, 8x> / x \in I\}$

$R \circ S \circ R = \{<x, 28x> / x \in I\}$

→ Relation Matrix :-

We know that the relation matrix of a relation R from a set $X = \{x_1, x_2 - - - x_m\}$ and $Y = \{Y_1, Y_2 - - - Y_n\}$ is given by a matrix having 'm' rows & 'n' columns. We shall denote the relation matrix of R is "M_R"

$$\therefore \quad M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4 \times 4}$$

→ Converse :-

Given a relation R from $X \to Y$, the relation $\tilde{R}$ from $Y \to X$ is called the converse of R where the ordered pair of $\tilde{R}$ are obtained by interchanging the members in each of the ordered pair of R.

This means $x \in X$ and $y \in Y$ then xRy and $y\tilde{R}x$

$\therefore \langle x, y \rangle \in R$ & $\langle y, x \rangle \in \tilde{R}$

From the definition of $\tilde{R}$ it follows that

$$\boxed{\tilde{\tilde{R}} = R}$$

The relation matrix of M_R of $\tilde{R}$ can be obtained by simply interchanging the rows and columns of M_R such a matrix is called as transpose of M_R.

$\therefore M_{\tilde{R}}$ = transpose of M_R

** Transitive closure:-

Let x be any finite set and R be a relation in x. The relation

$R^+ = R \cup R^2 \cup R^3 \cup R^4 \cup \ldots$ is called the transitive closure R in x

Ex:- $R_1 = \{\langle a,b\rangle, \langle a,c\rangle, \langle c,b\rangle\}$

$R_1^2 = R_1 o R_1 = \{\langle a,b\rangle\}$

$R_1^3 = R_1 o (R_1 o R_1) = \phi$

$R_1^+ = R_1 \cup R_1^2 \cup R_1^3$

$= \{\langle a,b\rangle, \langle a,c\rangle, \langle c,b\rangle\}$

→ $R_2 = \{\langle a,b\rangle, \langle b,c\rangle, \langle c,a\rangle\}$;

$R_2 = \{\langle a,b\rangle, \langle b,c\rangle, \langle c,a\rangle\}$

$R_2^2 = R_2 o R_2 = \{\langle a,c\rangle, \langle b,a\rangle, \langle b,c\rangle\}$

$R_2^3 = R_2 o (R_2 o R_2) = \{\langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle\}$

$R_2^4 = \{\langle a,b\rangle, \langle b,c\rangle, \langle c,a\rangle\} = R_2$

$R_2^5 = \{\langle a,c\rangle, \langle b,a\rangle, \langle b,c\rangle\} = R_2^2$

$\therefore R_2^+ = \{\langle a,b\rangle, \langle b,c\rangle, \langle c,a\rangle; \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle\}$

$R_2^+ = R_2 \cup R_2^2 \cup R_2^3 \cup R_2^4 \cup R_2^5$

→ Partial ordering :-

A Binary relation 'R' in a set P is called a partial ordered relation (or) a partial ordering in P if and only if R is reflexive, antisymmetric and transitive.

It is conventional to denote a partial ordering by the symbol "$\leq$". If $\leq$ is a partial ordering then the ordering pair $\langle P, \text{"}\leq\text{"}\rangle$ is called as Poset (or) Partial ordered set.

→ **Totally ordered Set (or) Simply ordered Set :-**

Let $\langle P, \leq \rangle$ be a partially ordered set. For every $x, y \in P$ we have either $x \leq y \lor y \leq x$. Then "$\leq$" is called a partial ordering (or) simple ordering (or) linear ordering in P and $\langle P, \leq \rangle$ is called a simply ordered set (or) totally ordered set (or) chain.

→ **Partially Ordered Set : Representation and associated terminology**

* **Covers :-**

In a partially Ordered set $\langle P, \leq \rangle$, and element $y \in P$ is said to cover an element $x \in P$. if $x < y$ and if there does not exist any element $z \in P$ Such that $x \leq z$, & $z \leq y$.

y covers x if and only if

$$(x < y \land / x \leq z \leq y \Rightarrow x = z \lor z = y)$$

*** **Hasse Diagrams :-**

A partially ordering $\leq$ on a set P can be represented by means of a diagram known as Hasse diagram. (or) partially ordered set diagram of $\langle P, \leq \rangle$.

In such a diagram each element is represented by a small circle (or) a dot. The circle for $x \in P$ is drawn below the circle for $y \in P$, if $x < y$ and a line is drawn between x and y, if y covers x.

If $x < y$ but y doesnot covers x then x and y are not connected directly by a single line. However they are connected through one (or) more element of P.

Problem 1 :-

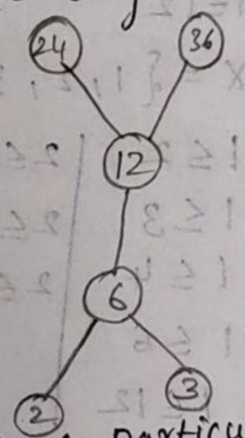
Let $P = \{1, 2, 3, 4\}$ and $\leq$ be the relation "less than (or) equal to". Draw the Hasse diagram.

Sol :-

$1 \leq 2$	$2 \leq 3$	$3 \leq 4$
$1 \leq 3$	$2 \leq 4$	
$1 \leq 4$		

Problem 2 :-

Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation '$\leq$' be such that $x \leq y$ if x divides y. Draw the Hasse diagram of $\langle X, \leq \rangle$



Sol :-

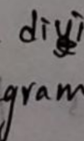
$2 \leq 6$	$3 \leq 6$	$6 \leq 12$	$12 \leq 24$
$2 \leq 12$	$3 \leq 12$	$6 \leq 24$	$12 \leq 36$
$2 \leq 24$	$3 \leq 24$	$6 \leq 36$	
$2 \leq 36$	$3 \leq 36$		

Problem 3 :- Let A be the set of factors of a particular positive integers m and let $\leq$ be the relation divides i.e

$\leq \; = \{\langle x, y \rangle \,/\, x \in A \wedge x \text{ divides } y\}$

Draw the Hasse diagrams for $m = 2$, $m = 6$, $m = 30$, $m = 12$, $m = 45$, $m = 120$.

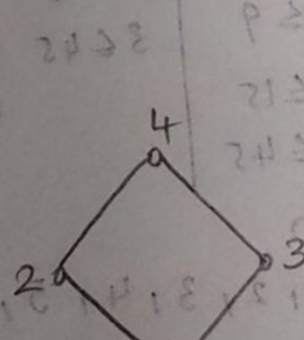
Sol :- $m = 2$



$X = \{1, 2\}$

$m = 6$

$X = \{1, 2, 3, 6\}$

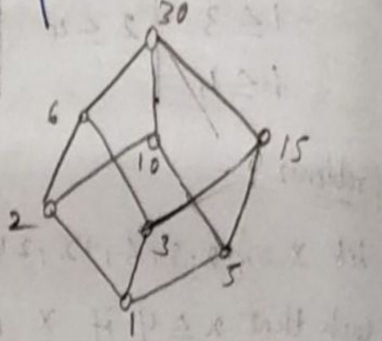


$1 \leq 2$	$2 \leq 6$	$3 \leq 6$
$1 \leq 3$		
$1 \leq 6$		

$m = 30$

$X = \{1, 2, 3, 5, 6, 10, 15, 30\}$

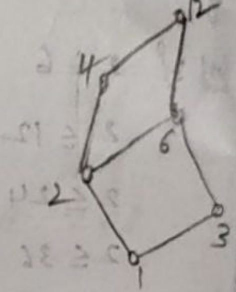
$1 \le 2$	$2 \le 6$	$3 \le 6$	$5 \le 10$	$6 \le 30$	$10 \le 30$	$15 \le 30$
$1 \le 3$	$2 \le 10$	$3 \le 15$	$5 \le 15$			
$1 \le 5$	$2 \le 30$	$3 \le 30$	$5 \le 30$			
$1 \le 6$						
$1 \le 10$						
$1 \le 15$						
$1 \le 30$						



$m = 12$

$X = \{1, 2, 3, 4, 6, 12\}$

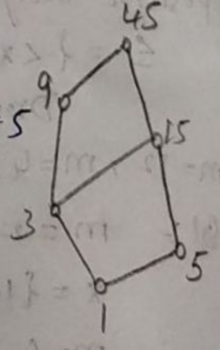
$1 \le 2$	$2 \le 4$	$3 \le 6$	$4 \le 12$	$6 \le 12$
$1 \le 3$	$2 \le 6$	$3 \le 12$		
$1 \le 4$	$2 \le 12$			
$1 \le 6$				
$1 \le 12$				



$m = 45$

$X = \{1, 3, 5, 9, 15, 45\}$

$1 \le 3$	$3 \le 9$	$5 \le 15$	$9 \le 45$	$15 \le 45$
$1 \le 5$	$3 \le 15$	$5 \le 45$		
$1 \le 9$	$3 \le 45$			
$1 \le 15$				
$1 \le 45$				



$m = 120$

$X = \{1, 2, 3, 4, 5, 6, 8, \cancel{9}, 10, 12, 15, 20, 24, 30, 40, 60, 120\}$

$1 \le 2$	$1 \le 15$	$2 \le 4$	$3 \le 6$	$4 \le 8$
$1 \le 3$	$1 \le 20$	$2 \le 6$	$3 \le 12$	$4 \le 12$
$1 \le 4$	$1 \le 24$	$2 \le 8$	$3 \le 15$	$4 \le 20$
$1 \le 5$	$1 \le 30$	$2 \le 10$	$3 \le 24$	$4 \le 24$
$1 \le 6$	$1 \le 40$	$2 \le 12$	$3 \le 30$	$4 \le 40$
$1 \le 8$	$1 \le 60$	$2 \le 20$	$3 \le 60$	$4 \le 60$
$1 \le 10$	$1 \le 120$	$2 \le 24$	$3 \le 120$	$4 \le 120$
$1 \le 12$		$2 \le 30$		
		$2 \le 40$		
		$2 \le 60$		
		$2 \le 120$		

$5 \le 10$	$6 \le 12$	$8 \le 24$	$10 \le 20$	$12 \le 24$
$5 \le 15$	$6 \le 24$	$8 \le 40$	$10 \le 30$	$12 \le 60$
$5 \le 20$	$6 \le 30$	$8 \le 120$	$10 \le 40$	$12 \le 120$
$5 \le 30$	$6 \le 60$		$10 \le 60$	
$5 \le 40$	$6 \le 120$		$10 \le 120$	
$5 \le 60$				
$5 \le 120$				

$15 \le 30$	$20 \le 40$	$24 \le 120$	$30 \le 60$	$40 \le 60$
$15 \le 60$	$20 \le 60$		$30 \le 120$	$40 \le 120$
$15 \le 120$	$20 \le 120$			

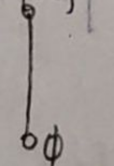
$60 \le 120$

Problem :-

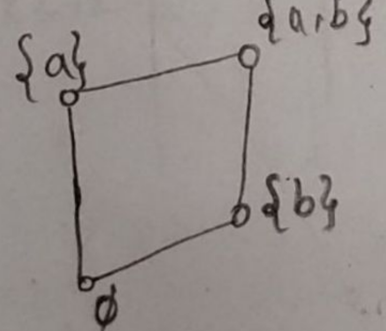
Let 'A' be a give finite Set and $P(A)$ its power Set. Let $\subseteq$ be the inclusion relation on the elements of $P(A)$. Draw Hasse diagrams of $\langle P(A), \subseteq \rangle$

(a) $A = \{a\}$ (b) $A = \{a, b\}$ (c) $A = \{a, b, c\}$

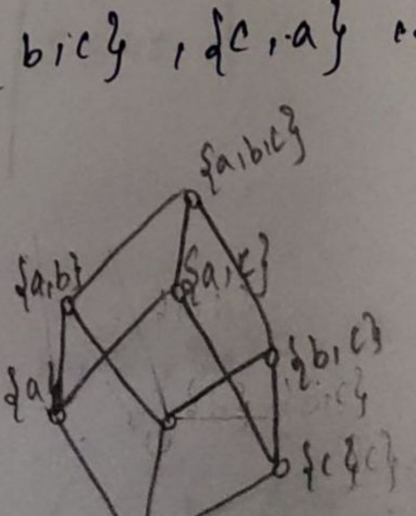
Sol:- (a) $A = \{a\}$ $P(A) = \{\phi, \{a\}\}$



(b) $A = \{a, b\}$ $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$



(c) $A = \{a, b, c\}$ $P(A) = \{\phi, \{a\}, \{b\}, \{c\},$
$\{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$



Upper bound and Lower bound :- 2nd topic

Let $\langle P, \leq \rangle$ be a partially ordered set and let $A \subseteq P$ any element $x \in P$ is an upper bound for A if for all $a \in A$, $a \leq x$

Similarly any element $x \in P$ is a lower bound for A if for all $a \in A$, $x \leq a$.

Least upper bound and Greater lower bound :-

Let $\langle P, \leq \rangle$ be a partially ordered set and $A \subseteq P$ and element $x \in P$ is a least upper bound for A if x is an upper bound for A and $x \leq y$ where y is any upper bound for A.

Similarly the greatest lower bound for an element $x \in P$ such that x is a lower bound and $y \leq x$ for all lower bounds y.

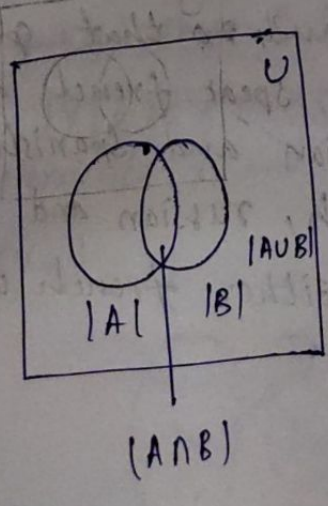
3rd topic $|A|$ condinology of A

→ Principle of inclusion - exclusion :-

If A and B are subsets of some universal set U, then

$$|A \cup B| = |A| + |B| - |A \wedge B|$$

This is fairly cleark from a venn diagram which is shown in below.



Problem :-

Suppose that 200 faculty members can speak french and 50 can speak russian while only 20 can speak both ~~fet~~ french and russion. How man faculty members can speak either french (or) Russian.

Faculty can speak French $|F| = 200$

Faculty can Speak Russian $|R| = 50$

Faculty can speak both russian & french $|F \cap R| = 20$

$$|F \cup R| = |F| + |R| - |F \cap R|$$

$$= 200 + 50 - 20$$

$$\therefore |F \cup R| = 230$$

- If A, B and C are subsets of some universal set U then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Problem :-

If there are 200 faulty members ~~they~~ that speak french, 50 that speak russian, 100 that speaks Spanish and 20 that speak french and russian, 60 that speak french and spanish, 35 that speaks russian and spanish. While only 10 can speak french, russian and spanish. How many speak either french or russian or Spanish.

Sol :- Faculty can speak French $|F| = 200$

Faculty can speak Russian $|R| = 50$

Faculty can speak Spanish $|S| = 20$

Faculty can speak french and russian $|F \cap R| = 20$

Faculty can speak French and spanish $= |F \cap S| = 60$

Faculty can speak rusian and Spanish $|R \cap S| = 35$,

Faulty can speak french, russian, Spanish

$|F \cap R \cap S| = 10$

$$|F \cup R \cup S| = |F| + |R| + |S| - |F \cap R| - |R \cap S| - |F \cap S| + |F \cap R \cap S|$$

$$= 200 + 50 + 100 - 20 - 60 - 35 + 10$$

$$= 245$$

4th topic

→ Pigeon hole principle :-

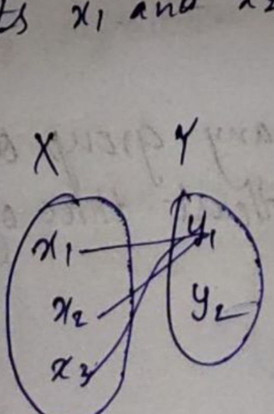
Suppose 'm' pigeons fly into 'n' pigeon poles to roost. Where $m > n$ the obviously atleast two pigeons must roost in the same pigeon hole. This property is called pigeon hole principle.

Theorem :-

Statement :- f from x to y

Let $f : X \to Y$, where X and Y are finite sets, $|X| = m$, $|Y| = n$ & $m > n$ then there exist atleast two distinct elements x_1 and x_2 in X such that

$f(x_1) = f(x_2)$



Problem :-

Suppose we select 367 students from compus. Show that atleast two of them must have the same birthday.

Sol :- The maximum no: of days in a year 366, and this occurs in a leap year.

Assume students as pigeon and days of the year as pigeon holes.

Let A be the set of students and B be the set of ~~stude~~ days.

Where $|A| = m = 367$

$|B| = n = 366$

$\therefore f : A \to B$ is defined by

$f(a) =$ Birthday of student A

Since $m > n$, by the pigeon hole principle there should be atleast two students a_1 & $a_2 \to f(a_1) = f(a_2)$ i.e atleast two student have the same birthday.

Generalised Pigeon Hole Principle :-

If 'm' pigeons are arranged to n pigeon holes there must be a pigeon hole containing atleast

$$\left[\frac{m-1}{n}\right] + 1 \text{ Pigeons}$$

Example :-

If we select any group of 1000 students on campus, show that atleast three of them must have the same birthday.

Su Assume students are pigeons and days of the year as pigeon holes.

$\therefore$ $m = 1000$ $n = 366$ $\therefore$ By using the generalised pigeon hole principle.

$$\left[\frac{m-1}{n}\right] + 1$$

$$\left[\frac{1000-1}{366}\right] + 1$$

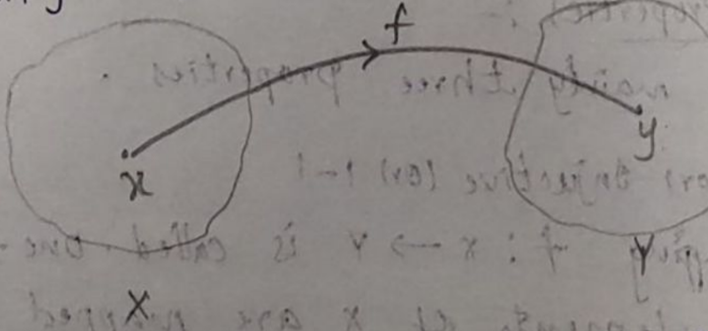
$= 2.7$

$= 3$

every element of x is maped with unique element in Y

→ **Functions** :- } 5th topic

Let X and Y be any two sets. A relation f from X to Y ($f : X \rightarrow Y$) is called a function if for every $x \in X$ there is a unique element in $y \in Y$ such that $\langle x, y \rangle \in f$



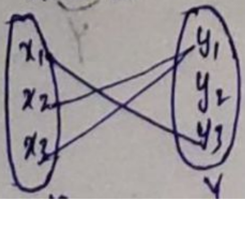
For a function $f : X \rightarrow Y$, if $\langle x, y \rangle \in f$ then x is called as Argument and the corresponding y is called as Image of x and f.

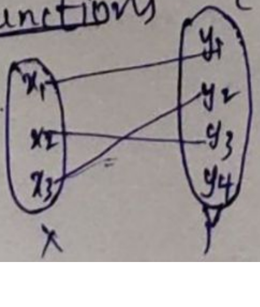
Instead of writing $\langle x, y \rangle \in f$, it is customary to write $y = f(x)$.

By the above definition $D_f = X$,

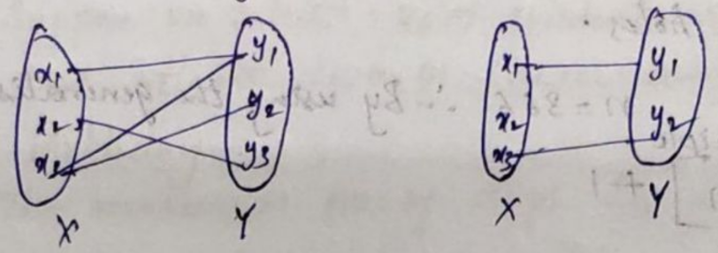
$R_f \subseteq Y$.

The below are some of the functions (Example)





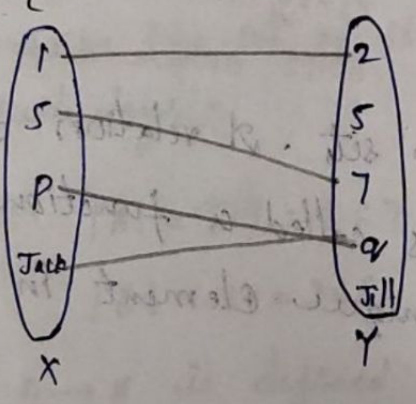
The below diagrams are not functions



Problem :-

Let $X = \{1, 5, P, \text{jack}\}$, $Y = \{2, 5, 7, q, \text{jill}\}$ and

$f = \{<1,2>, <5,7>, <P,q>, <\text{Jack}, q>\}$



$Df = X$

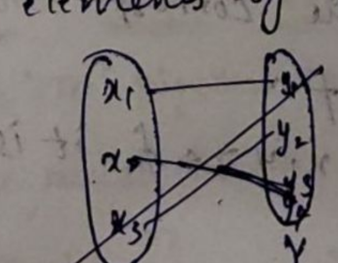
$Rf = \{2, 7, q\} \subseteq Y$

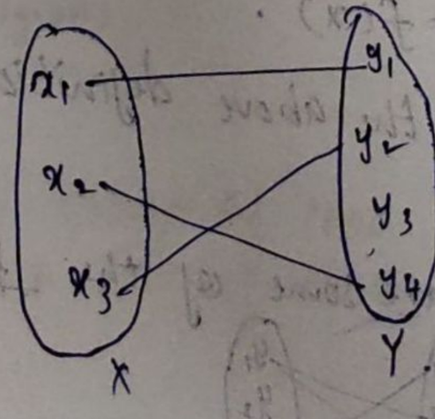
— Function Properties :-

There are mainly three properties.

1) One-One (or) Injective (or) 1-1

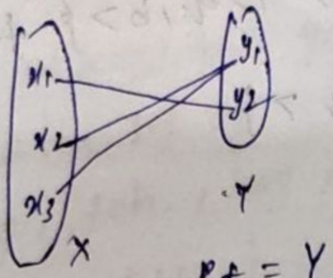
A mapping $f : X \to Y$ is called one-one if distinct elements of X are mapped into distinct elements of Y.





2) Onto (Surjective)

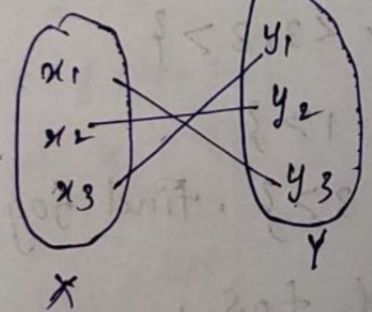
A mapping $f : X \to Y$ is called onto if the range $R_f = Y$



$R_f = Y$

3) Bijective :-

A mapping $f : X \to Y$ is called Bijuctive if it is both the one to one and onto.



6th topic

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→ Composition of functions :-

Let $f : X \to Y$ and $g : Y \to Z$ be two function the composite relation $g \circ f$ such that

$$g \circ f : \{ \langle x, z \rangle \mid x \in X \wedge z \in Z \wedge (\exists y) (y \in Y \wedge y = f(x) \wedge z = g(y) \}$$

is called the Composition of functions, $g \circ f$ is called the left composition of g with f.

Problem :-

Let $X = \{1, 2, 3\}$, $Y = \{p, q\}$ and $Z = \{a, b\}$. Also let $f : X \to Y$ be $f = \{\langle 1, p\rangle, \langle 2, p\rangle, \langle 3, q\rangle\}$ and $g : Y \to Z$ be given by $g = \{\langle p, \cancel{a}b\rangle, \langle q, b\rangle\}$ find gof.

$f = \{\langle 1, p\rangle, \langle 2, p\rangle, \langle 3, q\rangle\}$

$g = \{\langle p, b\rangle, \langle q, b\rangle\}$

$gof = \{\langle 1, b\rangle, \langle 2, b\rangle, \langle 3, b\rangle\}$

Example :-

Let $X = \{1, 2, 3\}$ and f, g, h & s be functions from $X \to X$ given by $f = \{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle\}$,

$g = \{\langle 1, 2\rangle, \langle 2, 1\rangle, \langle 3, 3\rangle\}$

$h = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 1\rangle\}$

$s = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle\}$. Find fog, gof, $fohog$, hog, gos, sos and fos.

$fog = \{\langle 1, 3\rangle, \langle 2, 2\rangle, \langle 3, 1\rangle\}$

$gof = \{\langle 1, 1\rangle, \langle 2, 3\rangle, \langle 3, 2\rangle \cancel{\}}, \langle 1, 3\rangle\}$ $fog \neq gof$

$fohog = \cancel{\{}$

$hog = \{\langle 1, 2\rangle, \langle 2, 1\rangle, \langle 3, 1\rangle\}$

$fohog = \{\langle 1, 3\rangle, \langle 2, 2\rangle, \langle 3, 2\rangle\}$

$hog = \{\langle 1, 2\rangle, \langle 2, 1\rangle, \langle 3, 3\rangle\}$

$gos = \{\langle 1, 2\rangle, \langle 3, 3\rangle, \langle 2, 1\rangle\}$

$sos = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle\}$

$fos = \{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 2\rangle\}$

In functions $fo(g(x))$ is written as $f(g(x))$

$\therefore fo(g(x)) = f(g(x))$

Problem

** Let $f(x) = x+2$, $g(x) = x-2$, $h(x) = 3x$ for $x \in R$. where R is the set of real numbers. Find gof, fog, fof, gog, foh, hog, hof and fohog.

Sol:- $gof = g(f(x))$
$= g(x+2)$
$= x+2-2$
$= x$

$fog = f(g(x))$
$= f(x-2)$
$= x-2+2$
$= x$

$fof = f(f(x))$
$= f(x+2)$
$= x+2+2$
$= x+4$

$gog = g(g(x))$
$= g(x-2)$
$= x-2-2$
$= x-4$

$foh = f(h(x))$
$= f(3x)$
$= 3x+2$

$hog = h(g(x))$
$= h(x-2)$
$= 3(x-2)$
$= 3x-6$

$hof = h(f(x))$
$= h(x+2)$
$= 3(x+2)$
$= 3x+6$

$fohog = f(h(g(x)))$
$= f(h(x-2))$
$= f(3(x-2))$
$= f(3x-6)$
$= 3x-6+2$
$= 3x-4$

Let $f : R \to R$ and $g : R \to R$ where 'R' is the set of the real number. fog, gof, where $f(x) = x^2 - 2$ and $g(x) = x + 4$.

$$\begin{aligned} fog &= f(g(x)) \\ &= f(x+4) \\ &= (x+4)^2 - 2 \\ &= x^2 + 16 - 8x - 2 \\ &= x^2 - 8x + 14 \end{aligned}$$

$$\begin{aligned} gof &= g(f(x)) \\ &= g(x^2 - 2) \\ &= x^2 - 2 + 4 \\ &= x^2 + 2 \end{aligned}$$

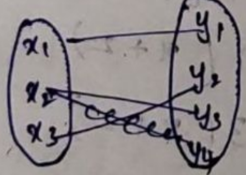
******

→ **Inverse functions :-**

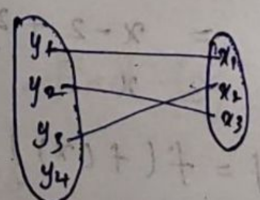
Checking condition :-

One-one

$f : X \to Y$

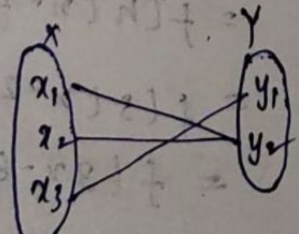


$f^{-1} : Y \to X$

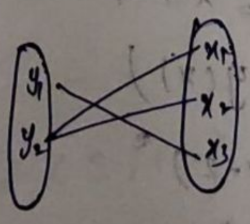


$\therefore f^{-1}$ is not a function.

Onto :- $f : X \to Y$

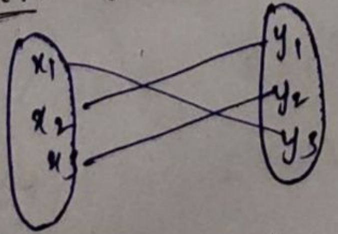


$f^{-1} : Y \to X$

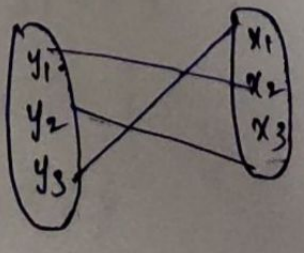


$\therefore f^{-1}$ is not a function

Bijunctive :- $f : X \to Y$



$f^{-1} : Y \to X$



$\therefore f^{-1}$ is a function.

In the above three properties for a given function $f: X \to Y$ is a function, f^{-1} is a function only when ever f is Bijuctive (one-one, onto it should satisfy).

$\therefore f^{-1}$ is a function from Y to X and f^{-1} is called the inverse function. $f^{-1}: Y \to X$

If f^{-1} exist then f is called invertible.

Indentity Mapping :-

The Mapping $I_X : X \to X$ is called an identity map

if $I_X = \{ \langle x, x \rangle \mid x \in X \}$

For any function f

$$f \circ I_X = I_X \circ f = f$$

Theorem :-

Statement :-

A mapping $f: X \to Y$ is invertible then

$f \circ f^{-1} = f^{-1} \circ f = I_X$

Let f and g be any two functions and $f \circ g$ is a function then the inverse function of $f \circ g$ is $(f \circ g)^{-1}$

$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Properties :- (4)

We consider mappings which are Bijuctive and from a set X onto X itself. Let F_X denotes the collection of all Bijuctive functions from X onto X so that the elements F_X for all invertible functions; the following properties hold.

1) For any $f, g \in F_x$, $f o g$ and $g o f$ are also in F_x. This is called the closure property of the operation of compostion.

2) For any $f, g, h \in F_x$

$$(f o g) o h = f o (g o h)$$

It's called Associative property.

3) Identity property.

There exist a function $I_x \in F_x$ called the Identity map such that for any $f \in F_x$

$$f o I_x = I_x o f = f$$

It's called the Indentity property.

4) For every $f \in F_x$ there exist an inverse function $f^{-1} \in F_x$ such that

$$f o f^{-1} = f^{-1} o f = I_x$$

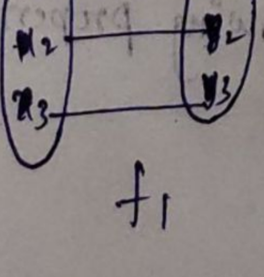
It's called the Inverse property.

Problem 1 many times *** $F_x : x \rightarrow x$

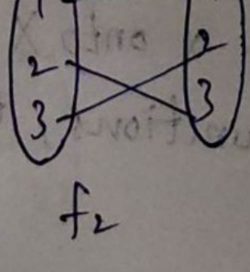
Let F_x be the set of one-one, onto mappings from x onto x) where $x = \{1, 2, 3\}$. Find all the elements of F_x and find the inverse of each element.

Here $x = \{1, 2, 3\}$ the possible functions are $3! = \underline{6}$ (3×2)

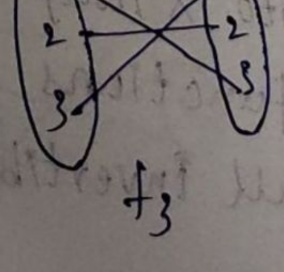
$\therefore F_x = \{f_1, f_2, f_3, f_4, f_5, f_6\}$



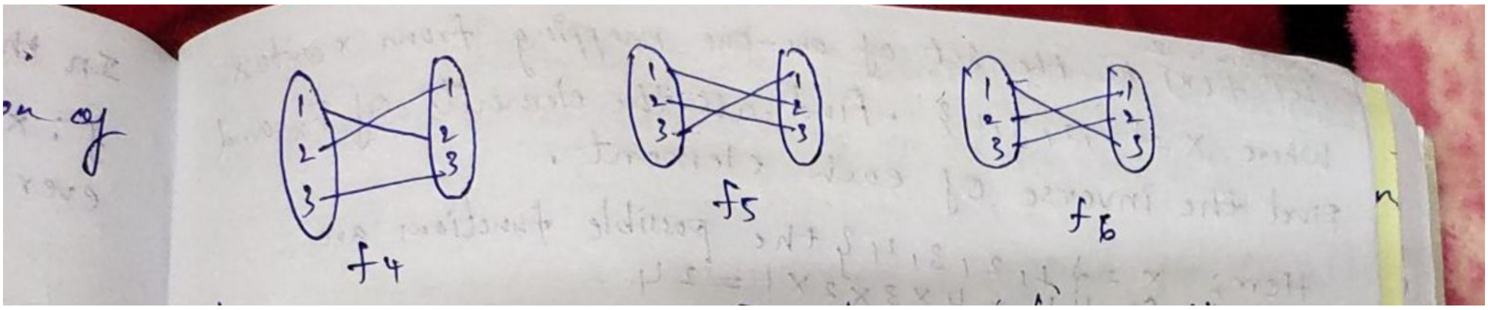
f_1



f_2



f_3


f4
f5
f6

Recursive functions :- Any function $f : N^n \to N$ is called (f from N^n to N)

Totalled. It is defined from every n-tuple in N^n.

Example :-

$f < x, y > = x + y$, which is defined for all $\forall x, y \in N$

and Hence it is a Totalled Function .

Partial Function :-

It $f : D \to N$ is called Partial function if

$f\ D \subseteq N^n$ (Natural number)

Example :-

$f < x, y > = x - y$, which is defined $\forall x, y \in N$,

which satisfy $x > y$. ~~Hence~~ ~~of <x,y>~~

Hence $f(x, y)$ is partial .

We Now give a 3-set of functions called the initial functions , which are used in defining other functions by induction .

1) Zero function (Z)

$Z : Z(x) = 0$

2) Successor function (S)

$S : S(x) = x + 1$

3) Projection Function (U_i^n)

$U_i^n : U_i^n (x_1, x_2 \ldots x_i \ldots x_n) = x_i$

The projection function is called as Identity function .

Ex :- $U_1^2 < x, y > = x$

Here 2 variable x & y

1 means x

2 means y

$U_2^3 < 2, 4, 6 > = 4$

Premitive Recursion :-

A function f is called premitive recursive if and only if it can be obtained from the initial functions by a finite number of operations of composition and recursion.

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1) Show that the function $f(x, y) = x + y$ is premitive recursive.

Sol :- $f\langle x, y\rangle = x, y$

$f\langle x, 0\rangle = x + 0 = x = U_1^3 \langle x, y, f\langle x, y\rangle\rangle$

Assume $(x+y)+1 = x + (y+1)$

$f\langle x, y+1\rangle = x + (y+1) = (x+y)+1 = f\langle x, y\rangle + 1$

$= S(f\langle x, y\rangle)$

$= S(U_3^3 \langle x, y, f\langle x, y\rangle\rangle)$

Inorder to see how we can use the above definition to actually compute the value of $f\langle 2, 4\rangle$

initially $f\langle 2, 0\rangle = 2$

$f\langle 2, 4\rangle = f\langle 2, 3+1\rangle$

$= S(f\langle 2, 3\rangle)$

$= S(S(f\langle 2, 2\rangle))$

$= S(S(S(f\langle 2, 1\rangle)))$

$= S(S(S(S(f\langle 2, 0\rangle))))$

$= S(S(S(S(2))))$

$= S(S(S(3)))$

$= S(S(4))$

$= S(5)$

$= 6$

2) Using recursion define the multiplication function given $g<x,y> = x+y$ is primitive recursive.

$g<x,y> = x+y$

Sol:- $g<x,y> = x+y$

$g<x,0> = x * 0 = 0$

$g<x,y+1> = x*(y+1) = x*y + x = g<x,y> + x$

$= S(U_3^3 <x,y,g<x,y>, U_1^3 <x,y,g<x,y>)$

Inverse Function :-

1) Let $f : R \to R$ is given by $f(x) = x^3 - 2$, find f^{-1}

Sol:- First we show that f is one to one

Let $x_1, x_2 \in R$ such that

$f(x_1) = f(x_2)$

$x_1^3 - \cancel{2} = x_2^3 - \cancel{2}$

$x_1^3 = x_2^3$

$x_1 = x_2$

$\therefore f$ is one to one

Now to show that f is onto ($\because y = f(x)$)

$f(x) = x^3 - 2$

$y = x^3 - 2$

$x^3 = y + 2$

$x = \sqrt[3]{y+2}$

$f^{-1}(y) = \sqrt[3]{y+2}$

$\because f$ is onto

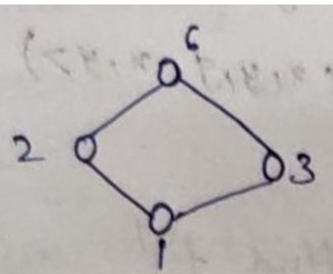
$\therefore$ Since f is invertible and $f^{-1}(x) = \sqrt[3]{x+2}$

Lattices and its properties :-

Lattice :- A lattice is a poset $<L, \leq>$ in which every pair of elements $a, b \in L$ as a greatest lower bound and least upper bound (LUB). The greatest lower bound is denoted by $a*b$ and least upper bound is denoted by $a \oplus b$

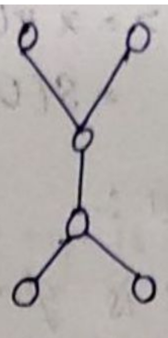
$\therefore GLB(a,b) = a * b$ (Meet or product)

$LUB(a,b) = a \oplus b$ (Join or Sum)



Lattice

$\{2,3\}$
GLB $=1$
LUB $=6$



It is not a lattice

Properties :-

Let $\langle L, \leq \rangle$ be a lattice and $*$ and $\oplus$ denotes the two binary operations meet and join on $\langle L, \leq \rangle$. Then for any $a, b, c \in L$ we have

(1) Idempotent law :-

(i) $a * a = a$ (ii) $a \oplus a = a$

(2) Commutative law :-

(i) $a * b = b * a$ (ii) $a \oplus b = b \oplus a$

(3) Associative law :-

(i) $a * (b * c) = (a * b) * c$

(ii) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

(4) Absorption law :-

(i) $a * (a + b) = a$

(ii) $a \oplus (a * b) = a$

Theorem :- Let $(L, \leq)$ be a lattice in which $*$ and $\oplus$ denote the operations of meet and join respectively. For any $a, b \in L$,

$$a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$$

Proof :- We shall first prove that $a \leq b \Leftrightarrow a * b = a$

In order to do this, let us assume that $a \leq b$.

Also, we know that $a \leq a$.

Therefore $a \leq a * b$. From the definition of $a * b$,

we have $a * b \leq a$.

Hence $a \leq b \Rightarrow a * b = a$

Next, assume that $a * b = a$; but it is only possible

if $a \leq b$, that is,

$$a * b = a \Rightarrow a \leq b.$$

Combining these two results we get required equivalent.

It is possible to show that $a \leq b \Leftrightarrow a \oplus b = b$ in a similar manner.

Alternating, from $a * b = a$, we have

$$b \oplus (a * b) = b \oplus a = a \oplus b$$

but $b \oplus (a * b) = b$

Hence $a \oplus b = b$ follows from $a * b = a$

By repeating similar steps, we can show that

$a * b = a$ follows from

$$a \oplus b = b$$

$$\therefore a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$$

Theorem:-

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$ the following properties called isotonicity hold.

$$b \leq c \Rightarrow \begin{cases} a * b \leq a * c \\ a \oplus b \leq a \oplus c \end{cases}$$

Proof:-

To prove $(a*b)*(a*c)$ we shall show that $(a*b)*(a*c) = (a*b)$

By the above theorem we know that

~~Now consider~~ $\cancel{(a*b)*(a*c) \Leftrightarrow a*(b*a)}$

$a \leq b \Leftrightarrow a*b = a$, lly $b \leq c \Leftrightarrow b*c = b$

Now, consider $(a*b)*(a*c) \Leftrightarrow a*(b*a)*c$

$\Leftrightarrow a*(a*b)*c$

$\Leftrightarrow (a*a)*(b*c)$

$\Leftrightarrow a*(b*c)$

$\Leftrightarrow a*b$

Similarly we can prove the isotonicity.

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→ Algebric Structures :

Algebric Systems : Examples & it's Properties :-

Let S be any set and the binary operations ~~(+, ×, ...)~~ $(+, \times, \cdot, \circ, *, \Delta)$ then $\langle S, +, * \rangle$ is said to be Algebric System.

7 properties

1) Closure

$a, b \in I$

2) Associative $\quad a, b, c \in I$

$$a+(b+c) = (a+b)+c$$

3) Identity

$a+0=0+a=a$

4) Inverse for any $a \in I$ there exist an element $-a \in I$

$a+(-a)=(-a)+a=0$

5) commitative for any $a, b \in I$

$a+b=b+a$

6) Distributive for any $a, b, c \in I$

$a \times (b+c) = (a \times b) + (a \times c)$

7) concillation for any $a, b, c \in I$ & $a \neq 0$ $a+b = a+c \Rightarrow b=c$

Ex:- Let I be the set of integers consider the algebraic system $< I, +, \times >$ are the operations of addition and multiplication on I. A list of import properties of this operations will now be given as

1) closure:- for any two element $a, b \in I$

$a, b \in I$

2) Associatity:- for any three element $a, b, c \in I$

$a+(b+c) = (a+b)+c$

3) Identity:- Identity element $a \in I$ such that for any $a \in I$

$a+0=0+a=a$

4) Inverse:- For any elements $a \in I$ there exist an element $-a \in I$

$a+(-a) = (-a)+a = 0$

5) Commitative:-

For any two element $a, b \in I$

$a+b=b+a$

6) Distributive :- For any three element $a, b, c \in I$

$$a \times (b+c) = (a \times b) + (a \times c)$$

7) Cancillation :- For any three elements $a, b, c \in I$ &
$a \neq 0$

$$a+b = a+c \Rightarrow b$$

1) closure

** Homomorphism :-

Let $\langle X, o \rangle$ and $\langle Y, * \rangle$ be two algebric system of the same time and $o, *$ are binary operations.

A mapping $g: X \to Y$ is called a homomorphism from $\langle X, o \rangle$ to $\langle Y, * \rangle$ if for any $(x_1, x_2 \in X)$

$$g(x_1 o x_2) = g(x_1) * g(x_2)$$

where $g(x_1), g(x_2) \in Y$

** Epimorphism :-

Let g be a Homomorphism from $\langle X, o \rangle$ & $\langle Y, * \rangle$ if g ~~from~~ $g: X \to Y$ is onto then g is called an Epimorphism.

* Monomorphism :-

If $g: X \to Y$ is one to one then g is called an Monomorphism.

** Isomorphism :-

If g from X to Y $(g: X \to Y)$ is one to one & onto then g is called an Isomorphism.

→ Semigroup & Moniod

Semigroup:-

Let S be a non empty set and (circle) o be a binary operation on S. The algebraic system $\langle S, o \rangle$ is called a Semigroup, if the operation 'o' is associative.

i.e for any $x\ y\ z \in S$

$$x\, o\, (y\, o\, z) = (x\, o\, y)\, o\, z$$

Monoid:-

A semigroup $\langle M, o \rangle$ with an identity element with respect to the operation o (circle) is called Monoid, if for any $x, y, z \in M$.

$$x o (y o z) = (x o y) o z$$

And there exist an element $e \in M$ such that for any $x \in M$

$$x o e = e o x = x$$

Subpart

Homomorphism of Semigroups and Monoids :-

Let $\langle S, * \rangle$ & $\langle T, \Delta \rangle$ be any two semigroups. A Mapping $g: S \rightarrow T$ is called a Semigroup homomorphism if for any $x_1, x_2 \in S$

$$g(x_1 * x_2) = g(x_1) \Delta g(x_2)$$

Where;

$$g(x_1), g(x_2) \in T$$

Monoid Homomorphism :-

Let $\langle M, *, e_m \rangle$ & $\langle T, \Delta, e_r \rangle$ be two monoids. A mapping $g: M \rightarrow T$ is called a Monoid Homomorphism, if for any $x_1, x_2 \in M$

$$g(x_1 * x_2) = g(x_1) \Delta g(x_2) \text{ \&}$$

$$g(e_m) = e_r$$

Theorem:- Let $\langle S, * \rangle$, $\langle T, \Delta \rangle$, $\langle V, \oplus \rangle$ be the semigroups and $g: S \to T$ & $h: T \to V$ be semigroup homomorphism then $h \circ g: S \to V$ is a semigroup homomorphism from $\langle S, * \rangle$ to $\langle V, \oplus \rangle$

Proof:- for any $x_1, x_2 \in S$

$$h \circ g(x_1 * x_2) = h(g(x_1 * x_2))$$
$$= h(g(x_1) \Delta g(x_2))$$
$$= h(g(x_1)) \oplus h(g(x_2))$$
$$h \circ g(x_1 * x_2) = h \circ g(x_1) \oplus h \circ g(x_2)$$

$\therefore$ $h \circ g$ is a semigroup homomorphism

→ **Groups**

1) **Group :-**

Let $\langle G, * \rangle$ be an Algebraic System which is said to be a group. which satisfying the following properties:-

(1) **Closure Property :-**

For any $x, y \in G$ such that

$x * y \in G$

(2) **Associative Property :-**

For any $x, y, z \in G$ such that

$x * (y * z) = (x * y) * z$

(3) **Identity Property :-**

There exist an identity element $e \in G$ such that for any $x \in G$

$x * e = e * x = x$

(4) **Inverse Property :-**

For any $x \in G$ there exists an inverse element $x^{-1} \in G$ such that

$x * x^{-1} = x^{-1} * x = e$

2) Abelian Group :- 4 + 1 = 5 properties

A group $<G, *>$ is called Abelian group if it is Commitative.

For any $a, b \in G$ such that

$a * b = b * a$

** 3) Order of the group :-

The number elements of a group is called as Order of the group.

The Order of the group is denoted by $O(G)$ (or) $|G|$

Ex:- Let $G = \{\pm 1, \pm i\}$, $i = \sqrt{-1}$ consider usual multiplication as binary operation. Find G is group (or) not.

Sol First we construct composition table for the above set.

$\times$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

$i^2 = -1$

From the above table clearly it satisfies closure and Associative properties.

Here '1' is the Identity element.

Every element has it's own inverse.

$\therefore <G, *>$ is a group.

clearly the above table ~~st~~ satisfyies commitative property

$\therefore G, *$ is a Abelian group.

Ex 2:- Let $G = \{1, w, w^2\}$. Prove the set G is an Abelian group under multiplication. The set G is known as the set of cube roots of unity.

i.e $w^3 = 1$.

Sol:- First construct composition table for the above set.

$\times$	1	w	w^2
1	1	w	w^2
w	w	w^2	1
w^2	w^2	1	w

$x^0 = 1$

$w^3 = 1$

$w^4 = w$

$\boxed{\begin{array}{l} w \text{ inverse is } w^2 \\ w^2 \text{ inverse is } w \\ 1 \text{ inverse is } 1 \end{array}}$

clearly the above table satisfy

$\therefore \langle G, * \rangle$ is an Abelian group.

Theorem:- If $\langle G, * \rangle$ is an Abelian group, then for all $a, b \in G$ show that $(a * b)^n = a^n * b^n$

$x^0 = 1$

e Identity

Proof:-

Case-1:- If $n = 0$ then

$$(a+b)^0 = a^0 + b^0$$
$$e = e * e$$
$$e = e$$

Hence theorem is true for $n = 0$

Case-2:- If $n > 0$, we prove this theorem by using mathematical induction.

Let $n = 1$ then

$$(a * b)^1 = a^1 * b^1$$
$$a * b = a * b$$

$\therefore$ Hence theorem is true for $n = 1$

Assume that theorem is true for $n=k$

$$(a*b)^k = a^k * b^k$$

Now to prove theorem is true for $n=k+1$

$$(a*b)^{k+1} = (a*b)^k * (a*b)$$

$$= (a^k * b^k) * (a*b)$$

$$= a^k * (b^k * a) * b$$

$$= (a^k * a) * (b^k * b)$$

$$(a*b)^{k+1} = a^{k+1} * b^{k+1}$$

$\therefore$ Hence theorem is true for $n=k+1$

Hence therem is true for all $n>0$

~~Case-3 : Let If $n>0$ let~~

Case-3 : If $n<0$ let $n=-m$ where m is a positive integer

then ;

$$(a*b)^n = (a*b)^{-m}$$

$$= ((a*b)^m)^{-1} \qquad \cancel{(a*b)^{-1} = b^{-1}*a^{-1}}$$

$$= (a^m * b^m)^{-1}$$

$$= (b^m * a^m)^{-1} \qquad (G \text{ is Abelian})$$

$$= (a^m)^{-1} * (b^m)^{-1} \qquad (a*b)^{-1} = b^{-1} * a^{-1}$$

$$= a^{-m} * b^{-m}$$

$$\therefore (a*b)^n = a^{-m} * b^{-m}$$

Therefore theorem is true for all $n<0$

$$\therefore (a*b)^n = a^n * b^n \text{ for all } n$$

Theorem :- Show that in a group $<G, *>$ if for any $a, b \in G$ $(a*b)^2 = a^2 * b^2$ then $<G, *>$ must be abelian.

Proof :- Given $(a*b)^2 = a^2 * b^2$ $a^{-1}*a = 1$ identity

$\Leftrightarrow (a*b)(a*b) = (a*a)*(b*b)$

Now multiplying both sides by a^{-1} on left side and b^{-1} on right side.

$\Leftrightarrow a^{-1}*[(a*b)(a*b)]*b^{-1} = a^{-1}*(a*a)*(b*b)*b^{-1}$

$\Leftrightarrow (a^{-1}*a)*(b*a)*(b*b^{-1}) = (a^{-1}*a)*(a*b)*(b*b^{-1})$

$\Leftrightarrow e*(b*a)*e = e*(a*b)*e$

$\Leftrightarrow b*a = a*b$

→ Subgroup :-

Let $<G, *>$ be a group & $S \subseteq G$ be such that it satisfies the following conditions.

1) Closure :- For any $a, b \in S$, $a*b \in S$

2) Identity :- Let $e \in S$, where e is the identity of $<G, *>$

3) Inverse :- For any $a \in S$, $a^{-1} \in S$ then

$<S, *>$ is called a Subgroup of $<G, *>$

V.imp ***

Theorem :-

A Subset (S is not equal to ϕ) $S \neq \phi$ Of G is a Subgroup of $<G, *>$ if and only if for any pair of elements $a, b \in S$, $a * b^{-1} \in S$

3 properties

prove both sides

Proof :- Assuming that S is a subgroup of $<G, *>$

It is clear that if $a, b \in S$ then $b^{-1} \in S$ and $a * b^{-1} \in S$

To prove the converse, let us assume that

$a, b \in S$ and $a * b^{-1} \in S$

For any pair a, b

Taking $b = a$ then $a * a^{-1} \in S \Rightarrow e \in S$ Identity

$e * a^{-1} \in S \Rightarrow a^{-1} \in S$,

From $e, a, b \in S$ we have

Similarly $b^{-1} \in S$ Inverse

Finally a, b are in S, $b^{-1} \in S$ then we have

$a * (b^{-1})^{-1} \in S \Rightarrow a * b \in S$ closure

$\therefore$ $<S, *>$ is a subgroup of $<G, *>$

→ Homomorphism & Isomorphism :-

1) Group Homomorphism :-

Let $<G, *>$ and $<H, \Delta>$ be two groups : A mapping $g : G \rightarrow H$ is called a group homomorphism from $<G, *>$ to $<H, \Delta>$ if for any $a, b \in G$

$g(a * b) = g(a) \Delta g(b)$ ✓

$g(a), g(b) \in H$

Def:- A Homomorphism from a group $\langle G, * \rangle$ to $\langle G, * \rangle$ is called as Endomorphism.

An isomorphism of $\langle G, * \rangle$ to $\langle G, * \rangle$ is called as automorphism.

Kernel of the Homomorphism :-

Let g be a group homomorphism from g $\langle G, * \rangle$ to $\langle H, \Delta \rangle$. The set of element of G which are mapped into e_H, the identity of H is called the kernel of the homomorphism g and is denoted by

$$Ker(g)$$

We shall now show that the above equation guartees that a group homomorphism preserves the identity, inverse and subgroups inother words if e_G and e_H are the identies of $\langle G, * \rangle$ and $\langle H, \Delta \rangle$ respectively then

$$g(e_G) = e_H$$

Also for any $a \in G$

$$g(a^{-1}) = [g(a)]^{-1}$$

And finally it is a subgroup.

Theorem :- The kernel of a homomorphism g from a group $\langle G, * \rangle$ to $\langle H, \Delta \rangle$ is a subgroup of $\langle G, * \rangle$ — Prove

Proof :- Since $g(e_G) = e_H$, $e_G \in \ker(g)$

$\therefore \ker(g)$ satisfies Identity Property.

Also if $a, b \in \ker(g)$ i.e $g(a) = g(b) = e_H$

$$g(a * b) = g(a) \Delta g(b)$$
$$= e_H \Delta e_H$$
$$= e_H$$

$\therefore a * b \in \ker(g)$

$\therefore \ker(g)$ satisfies Closure Property.

Finally if $a \in \ker(g)$ then

$$g(a^{-1}) = [g(a)]^{-1}$$
$$= [e_H]^{-1}$$
$$= e_H$$

$\therefore a^{-1} \in \ker(g)$

$\therefore \ker(g)$ satisfies Inverse Property.

$\therefore \ker(g)$ is a Subgroup of $\langle G, * \rangle$

Cyclic group

A group $\langle G, * \rangle$ is said to be cyclic if there exist an element $a \in G$ such that every element of G can be written in the form of a^n for some integer n.

*** Theorem:- S.T Every cyclic group of order 'n' is isomorphic to the group $< Z_n , +_n >$

Proof :- Let the cyclic group $< G , * >$ of order n generated by an element $a \in G$.

So, that the elements of G are $a^1, a^2 - - - a^n, e$

Define $g : Z_n \to G$ such that

$$g[1] = a$$

Note that $[1]$ is the generator of $< Z_n , +_n >$

$\therefore g[j] = a^j$, for $j = 0, 1, 2 - - - n-1$ and

hence the isomorphism is established.