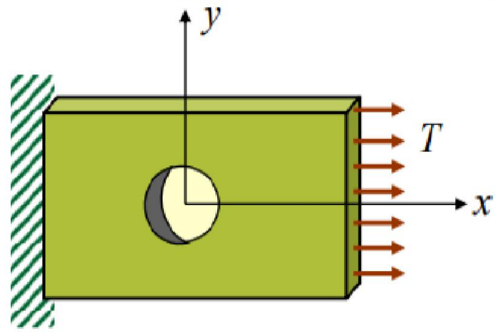
# UNIT –III
# GENERATION OF ELEMENT

A thin plate of thickness $t$, with a hole in the middle, is subjected to a uniform traction load, $T$ as shown. This 3-D plate can be analyzed as a **two-dimensional** problem.

2-D problems generally fall into two categories: *plane stress* and *plane strain*.




A *plane stress* problem

## a) Plane Stress

The thin plate can be analyzed as a *plane stress* problem, where the normal and shear stresses perpendicular to the $x$-$y$ plane are *assumed* to be zero, i.e.

$$\sigma_z = 0;\ \tau_{xz} = 0;\ \tau_{yz} = 0$$

The *nonzero* stress components are

$$\sigma_x \neq 0;\ \sigma_y \neq 0;\ \tau_{xy} \neq 0$$

## b) Plane Strain

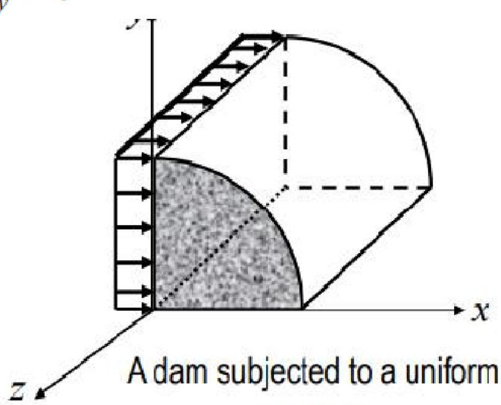
A dam subjected to uniform pressure and a pipe under a uniform internal pressure can be analyzed in two-dimension as *plain strain* problems.

The strain components perpendicular to the $x$-$y$ plane are assumed to be zero, i.e.

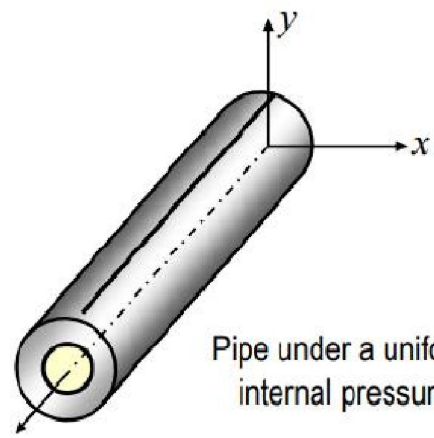
$$\varepsilon_z = 0;\ \gamma_{xz} = 0;\ \gamma_{yz} = 0$$

Thus, the *nonzero* strain components are $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$.

$$\varepsilon_x \neq 0;\ \varepsilon_y \neq 0;\ \gamma_{xy} \neq 0$$




A dam subjected to a uniform pressure




Pipe under a uniform internal pressure

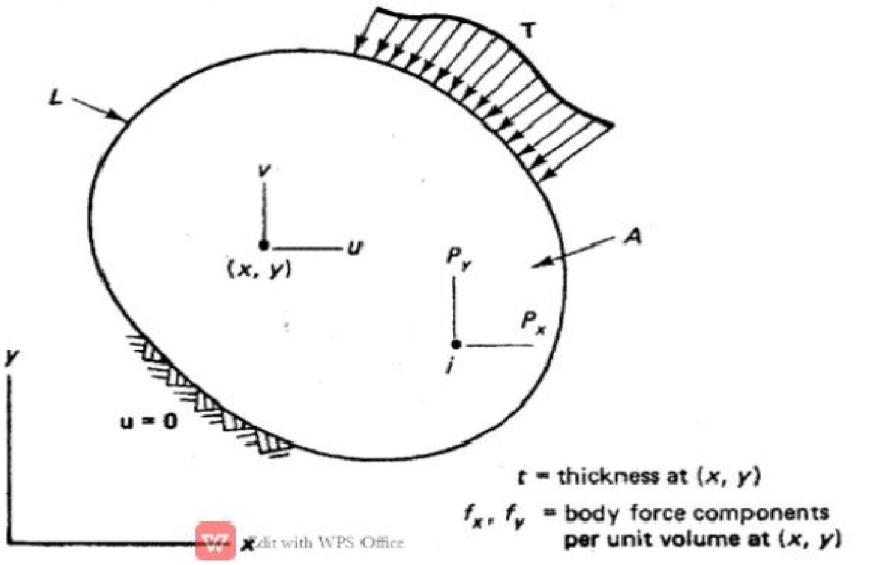
## 8-2 General Loading Condition

A two-dimensional body can be subjected to **three** types of forces:

a) Concentrated forces, $P_x$ & $P_y$ at a point, $i$;

b) Body forces, $f_{b,x}$ & $f_{b,y}$ acting at its *centroid*;

c) Traction force, $T$ (i.e. force per unit length), acting along a *perimeter*



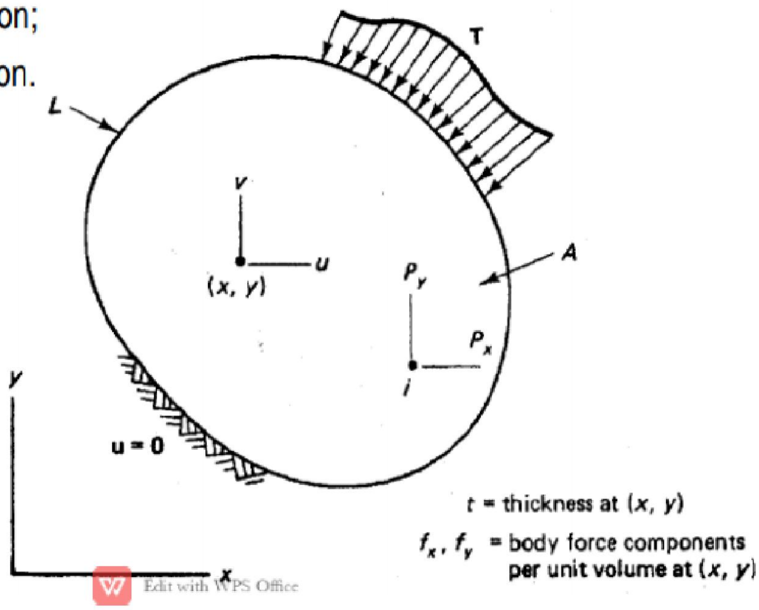



The 2-dimensional body experiences a deformation due to the applied loads.

At any point in the body, there are two components of displacement, i.e.

$u$ = displacement in $x$-direction;

$v$ = displacement in $y$-direction.



## Stress-Strain Relation

Recall, at any point in the body, there are three components of strains, i.e.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

The corresponding stress components at that point are

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

The stresses and strains are related through,

$$\{\sigma\} = [D]\{\varepsilon\}$$

where $[D]$ is called the *material matrix*, given by

$$[D] = \frac{E}{1-\nu^2} \cdot \begin{Bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{Bmatrix}$$

for *plane stress* problems and

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \cdot \begin{Bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{Bmatrix}$$

for *plane strain* problems.

## 8-3 Finite Element Modeling

The two-dimensional body is transformed into finite element model by subdividing it using triangular elements.

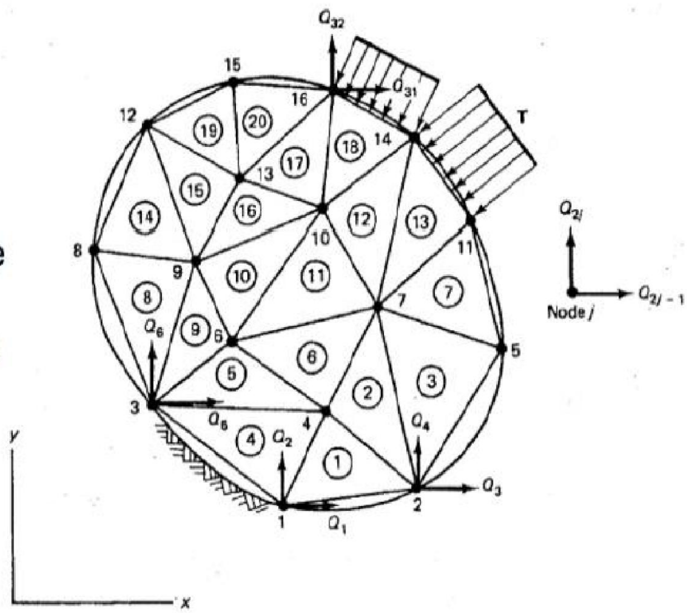
**Note**:

1. *Unfilled* region exists for curved boundaries, affecting accuracy of the solution. The accuracy can be improved by using smaller elements.

2. There are **two** displacement components at a node. Thus, at a node *j*, the displacements are:

$Q_{2j-1}$ in $x$-direction

$Q_{2j}$ in $y$-direction

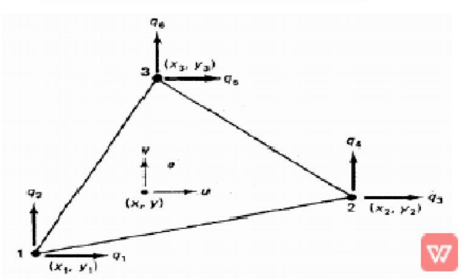


# Difference B/W CST & LST elements

## CST

- CST - Constant Strain Triangle
- 3 nodes per Triangle
- First order Triangle Element
- Strain in the element won't vary. Through out the element surface **constant strain** is observed.
- Displacement function is Linear
- Hence the displacement model is

$$\begin{cases} u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y \\ v(x,y) = \beta_1 + \beta_2 x + \beta_3 y \end{cases}$$

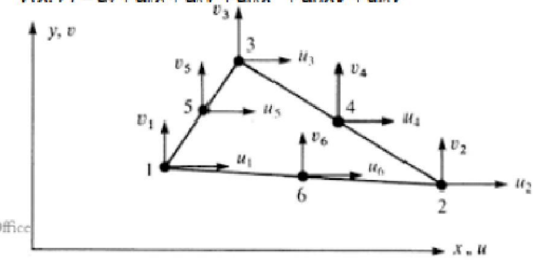


## LST

- LST - Linear Strain Triangle
- 6 nodes in Triangle
- Second order Triangle Element
- Strain will vary in the element as **Linear**
- Displacement function is quadratic
- The variation of the displacements over the element may be expressed as:

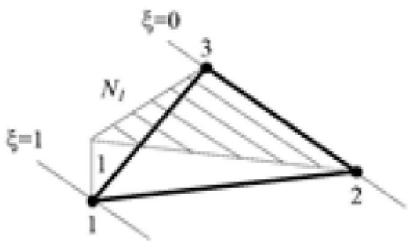
$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

$$v(x,y) = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$$

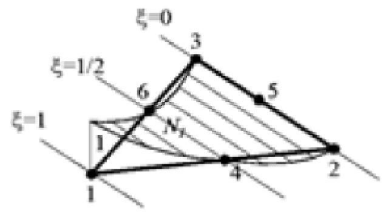


- CST elements are poor in Capturing the bending behaviour
- For same number of elements, true displacement and stresses not obtained in CST elements
- Fig below shows the variation of shape function N1 for the CST element

- LST elements are good in Capturing the bending behaviour
- For same number of elements, true displacement and stresses obtained better in LST elements
- Fig below shows the variation of shape function N1 for the LST element




Shape Function $N_1$ for CST




Shape Function $N_1$ for LST

**Example 5.1**

Evaluate the shape functions $N_1$, $N_2$, and $N_3$ at the interior point $P$ for the triangular element shown in Fig. E5.1.

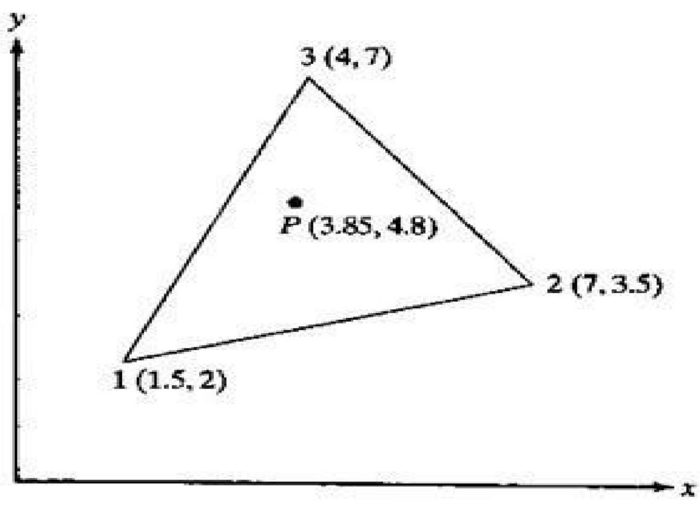



FIGURE E5.1 Examples 5.1 and 5.2.

**Solution** Using the isoparametric representation (Eqs. 5.15), we have

$$3.85 = 1.5N_1 + 7N_2 + 4N_3 = -2.5\xi + 3\eta + 4$$
$$4.8 = 2N_1 + 3.5N_2 + 7N_3 = -5\xi - 3.5\eta + 7$$

These two equations are rearranged in the form

$$2.5\xi - 3\eta = 0.15$$
$$5\xi + 3.5\eta = 2.2$$

Solving the equations, we obtain $\xi = 0.3$ and $\eta = 0.2$, which implies that

$$N_1 = 0.3 \qquad N_2 = 0.2 \qquad N_3 = 0.5$$ ∎

In evaluating the strains, partial derivatives of $u$ and $v$ are to be taken with respect to $x$ and $y$. From Eqs. 5.12 and 5.15, we see that $u, v$ and $x, y$ are functions of $\xi$ and $\eta$. That is, $u = u(x(\xi, \eta), y(\xi, \eta))$ and similarly $v = v(x(\xi, \eta), y(\xi, \eta))$. Using the chain rule for partial derivatives of $u$, we have

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$
$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

which can be written in matrix notation as

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} \tag{5.16}$$

where the $(2 \times 2)$ square matrix is denoted as the *Jacobian* of the transformation, $\mathbf{J}$:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{5.17}$$

Some additional properties of the Jacobian are given in the appendix. On taking the derivative of $x$ and $y$,

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \tag{5.18}$$

Also, from Eq. 5.16,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{5.19}$$

where $\mathbf{J}^{-1}$ is the inverse of the Jacobian $\mathbf{J}$, given by

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \tag{5.20}$$

$$\det \mathbf{J} = x_{13}y_{23} - x_{23}y_{13} \tag{5.21}$$

**Example 5.2**

Determine the Jacobian of the transformation **J** for the triangular element shown in Fig. E5.1.

**Solution** We have

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} -2.5 & -5.0 \\ 3.0 & -3.5 \end{bmatrix}$$

Thus, $\det \mathbf{J} = 23.75$ units. This is twice the area of the triangle. If 1, 2, 3 are in a clockwise order, then $\det \mathbf{J}$ will be negative. ■

From Eqs. 5.19 and 5.20, it follows that

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}\dfrac{\partial u}{\partial \xi} - y_{13}\dfrac{\partial u}{\partial \eta} \\ -x_{23}\dfrac{\partial u}{\partial \xi} + x_{13}\dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{5.23a}$$

Replacing $u$ by the displacement $v$, we get a similar expression

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}\dfrac{\partial v}{\partial \xi} - y_{13}\dfrac{\partial v}{\partial \eta} \\ -x_{23}\dfrac{\partial v}{\partial \xi} + x_{13}\dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{5.23b}$$

Using the strain–displacement relations (5.5) and Eqs. 5.12b and 5.23, we get

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$= \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}(q_1 - q_5) - y_{13}(q_3 - q_5) \\ -x_{23}(q_2 - q_6) + x_{13}(q_4 - q_6) \\ -x_{23}(q_1 - q_5) + x_{13}(q_3 - q_5) + y_{23}(q_2 - q_6) - y_{13}(q_4 - q_6) \end{Bmatrix} \tag{5.24a}$$

From the definition of $x_{ij}$ and $y_{ij}$, we can write $y_{31} = -y_{13}$ and $y_{12} = y_{13} - y_{23}$, and so on. The foregoing equation can be written in the form

$$\boldsymbol{\epsilon} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}q_1 + y_{31}q_3 + y_{12}q_5 \\ x_{32}q_2 + x_{13}q_4 + x_{21}q_6 \\ x_{32}q_1 + y_{23}q_2 + x_{13}q_3 + y_{31}q_4 + x_{21}q_5 + y_{12}q_6 \end{Bmatrix} \tag{5.24b}$$

This equation can be written in matrix form as

$$\boldsymbol{\epsilon} = \mathbf{Bq} \tag{5.25}$$

where **B** is a $(3 \times 6)$ element strain–displacement matrix relating the three strains to the six nodal displacements and is given by

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{5.26}$$

It may be noted that all the elements of the **B** matrix are constants expressed in terms of the nodal coordinates.

**Example 5.3**

Find the strain–nodal displacement matrices $\mathbf{B}^e$ for the elements shown in Fig. E5.3. Use local numbers given at the corners.

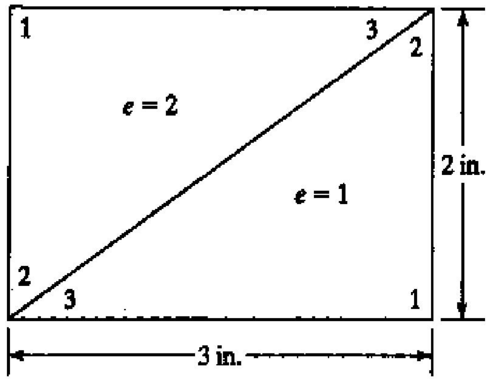



FIGURE E5.3

**Solution** We have

$$\mathbf{B}^1 = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

where det **J** is obtained from $x_{13}y_{23} - x_{23}y_{13} = (3)(2) - (3)(0) = 6$. Using the local numbers at the corners, $\mathbf{B}^2$ can be written using the relationship as

$$\mathbf{B}^2 = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix} \qquad \blacksquare$$

### *Potential-Energy Approach*

The potential energy of the system, $\Pi$, is given by

$$\Pi = \frac{1}{2}\int_A \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}t\,dA - \int_A \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}t\,d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \qquad (5.27)$$

In the last term in Eq. 5.27, $i$ indicates the point of application of a point load $\mathbf{P}_i$ and $\mathbf{P}_i = [P_x, P_y]_i^{\mathrm{T}}$. The summation in $i$ gives the potential energy due to all point loads.

Using the triangulation shown in Fig. 5.2, the total potential energy can be written in the form

$$\Pi = \sum_e \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}t\,dA - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}t\,d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \qquad (5.28a)$$

or

$$\Pi = \sum_e U_e - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA - \sum \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}t\,d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \qquad (5.28b)$$

where $U_e = \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}t\,dA$ is the element strain energy.

## THE FOUR-NODE QUADRILATERAL

Consider the general quadrilateral element shown in Fig. 7.1. The local nodes are numbered as 1, 2, 3, and 4 in a *counterclockwise* fashion as shown, and $(x_i, y_i)$ are the coordinates of node $i$. The vector $\mathbf{q} = [q_1, q_2, \ldots, q_8]^T$ denotes the element displacement vector. The displacement of an interior point $P$ located at $(x, y)$ is represented as $\mathbf{u} = [u(x, y), v(x, y)]^T$.

### Shape Functions

Following the steps in earlier chapters, we first develop the shape functions on a master element, shown in Fig. 7.2. The master element is defined in $\xi$-, $\eta$-coordinates (or *natural* coordinates) and is square shaped. The Lagrange shape functions where $i = 1, 2, 3$, and 4, are defined such that $N_i$ is equal to unity at node $i$ and is zero at other nodes. In particular, consider the definition of $N_i$:

$$\begin{aligned} N_1 &= 1 \quad \text{at node 1} \\ &= 0 \quad \text{at nodes 2, 3, and 4} \end{aligned} \tag{7.1}$$

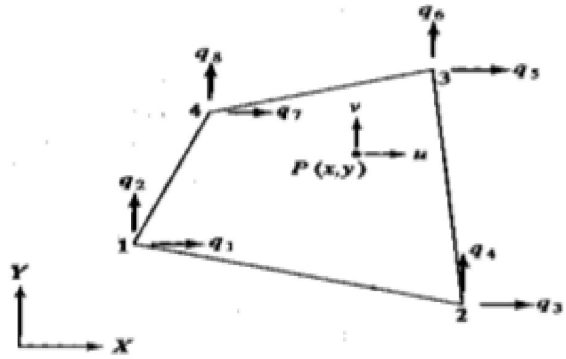



FIGURE 7.1 Four-node quadrilateral element.

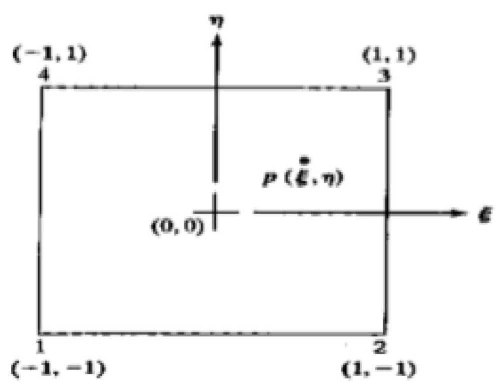



FIGURE 7.2 The quadrilateral element in $\xi$, $\eta$ space (the *master* element).

Now, the requirement that $N_1 = 0$ at nodes 2, 3, and 4 is equivalent to requiring that $N_1 = 0$ along edges $\xi = +1$ and $\eta = +1$ (Fig. 7.2). Thus, $N_1$ has to be of the form

$$N_1 = c(1 - \xi)(1 - \eta) \tag{7.2}$$

where $c$ is some constant. The constant is determined from the condition $N_1 = 1$ at node 1. Since $\xi = -1, \eta = -1$ at node 1, we have

$$1 = c(2)(2) \tag{7.3}$$

which yields $c = \frac{1}{4}$. Thus,

$$N_1 = \tfrac{1}{4}(1 - \xi)(1 - \eta) \tag{7.4}$$

All the four shape functions can be written as

$$\begin{aligned} N_1 &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned} \tag{7.5}$$

While implementing in a computer program, the compact representation of Eqs. 7.5 is useful

$$N_i = \tfrac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \tag{7.6}$$

where $(\xi_i, \eta_i)$ are the coordinates of node $i$.

We now express the displacement field within the element in terms of the nodal values. Thus, if $\mathbf{u} = [u, v]^{\mathrm{T}}$ represents the displacement components of a point located at $(\xi, \eta)$, and $\mathbf{q}$, dimension $(8 \times 1)$, is the element displacement vector, then

$$\begin{aligned} u &= N_1 q_1 + N_2 q_3 + N_3 q_5 + N_4 q_7 \\ v &= N_1 q_2 + N_2 q_4 + N_3 q_6 + N_4 q_8 \end{aligned} \tag{7.7a}$$

which can be written in matrix form as

$$\mathbf{u} = \mathbf{N}\mathbf{q} \tag{7.7b}$$

where

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \tag{7.8}$$

In the isoparametric formulation, we use the *same* shape functions $N_i$ to also express the coordinates of a point within the element in terms of nodal coordinates. Thus,

$$\begin{aligned} x &= N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \\ y &= N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 \end{aligned} \tag{7.9}$$

Subsequently, we will need to express the derivatives of a function in $x$-, $y$-coordinates in terms of its derivatives in $\xi$-, $\eta$-coordinates. This is done as follows: A function $f = f(x, y)$, in view of Eqs. 7.9, can be considered to be an implicit function of $\xi$ and $\eta$ as $f = f[x(\xi, \eta), y(\xi, \eta)]$. Using the chain rule of differentiation, we have

$$\begin{aligned} \frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} \\ \frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta} \end{aligned} \tag{7.10}$$

or

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} \tag{7.11}$$

where $\mathbf{J}$ is the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{7.12}$$

In view of Eqs. 7.5 and 7.9, we have

$$\mathbf{J} = \frac{1}{4}\left[\begin{array}{c|c} -(1-\eta)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4 & -(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4 \\ -(1-\xi)x_1-(1+\xi)x_2+(1+\xi)x_3+(1-\xi)x_4 & -(1-\xi)y_1-(1+\xi)y_2+(1+\xi)y_3+(1-\xi)y_4 \end{array}\right] \tag{7.13a}$$

$$\equiv \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{7.13b}$$

Equation 7.11 can be inverted as

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} \tag{7.14a}$$

or

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} \tag{7.14b}$$

These expressions will be used in the derivation of the element stiffness matrix.

An additional result that will be needed is the relation

$$dx\,dy = \det \mathbf{J}\, d\xi\, d\eta \tag{7.15}$$

The proof of this result, found in many textbooks on calculus, is given in the appendix.

### *Element Stiffness Matrix*

The stiffness matrix for the quadrilateral element can be derived from the strain energy in the body, given by

$$U = \int_V \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dV \tag{7.16}$$

or

$$U = \sum_e t_e \int_e \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dA \tag{7.17}$$

where $t_e$ is the thickness of element $e$.

The strain–displacement relations are

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{7.18}$$

By considering $f \equiv u$ in Eq. 7.14b, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{7.19a}$$

Similarly,

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{7.19b}$$

Equations 7.18 and 7.19a,b yield

$$\boldsymbol{\epsilon} = \mathbf{A} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{7.20}$$

where $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \tag{7.21}$$

Now, from the interpolation equations Eqs. 7.7a, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \mathbf{G}\mathbf{q} \tag{7.22}$$

where

$$\mathbf{G} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \tag{7.23}$$

Equations 7.20 and 7.22 now yield

$$\boxed{\boldsymbol{\epsilon} = \mathbf{Bq}} \tag{7.24}$$

where

$$\mathbf{B} = \mathbf{AG} \tag{7.25}$$

The relation $\boldsymbol{\epsilon} = \mathbf{Bq}$ is the desired result. The strain in the element is expressed in terms of its nodal displacement. The stress is now given by

$$\boxed{\boldsymbol{\sigma} = \mathbf{DBq}} \tag{7.26}$$

where $\mathbf{D}$ is a $(3 \times 3)$ material matrix. The strain energy in Eq. 7.17 becomes

$$U = \sum_e \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\left[t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{DB} \det \mathbf{J}\, d\xi\, d\eta\right]\mathbf{q} \tag{7.27a}$$

$$= \sum_e \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q} \tag{7.27b}$$

where

$$\boxed{\mathbf{k}^e = t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{DB} \det \mathbf{J}\, d\xi\, d\eta} \tag{7.28}$$

is the element stiffness matrix of dimension $(8 \times 8)$.

We note here that quantities $\mathbf{B}$ and $\det \mathbf{J}$ in the integral in Eq. (7.28) are involved functions of $\xi$ and $\eta$, and so the integration has to be performed numerically. Methods of numerical integration are discussed subsequently.

### *Element Force Vectors*

**Body Force** A body force that is distributed force per unit volume, contributes to the global load vector $\mathbf{F}$. This contribution can be determined by considering the body force term in the potential-energy expression

$$\int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\, dV \tag{7.29}$$

Using $\mathbf{u} = \mathbf{Nq}$, and treating the body force $\mathbf{f} = [f_x, f_y]^{\mathrm{T}}$ as constant within each element, we get