Fluid Mechanics

2nd Year

Civil Engineering

Contents

1. Introduction

1.1 Course Outline

Goals

The goal is that you will:

1. Have fundamental knowledge of fluids:
    a. compressible and incompressible;
    b. their properties, basic dimensions and units;
2. Know the fundamental laws of mechanics as applied to fluids.
3. Understand the limitations of theoretical analysis and the determination of correction factors, friction factors, etc from experiments.
4. Be capable of applying the relevant theory to solve problems.

Syllabus

Basics:

- Definition of a fluid: concept of ideal and real fluids, both compressible and incompressible.
- Properties of fluids and their variation with temperature and pressure and the dimensions of these properties.

Hydrostatics:

- The variation of pressure with depth of liquid.
- The measurement of pressure and forces on immersed surfaces.

Hydrodynamics:

- Description of various types of fluid flow; laminar and turbulent flow; Reynolds's number, critical Reynolds's number for pipe flow.
- Conservation of energy and Bernoulli's theorem. Simple applications of the continuity and momentum equations.
- Flow measurement e.g. Venturi meter, orifice plate, Pitot tube, notches and weirs.
- Hagen-Poiseuille equation: its use and application.
- Concept of major and minor losses in pipe flow, shear stress, friction factor, and friction head loss in pipe flow.
- Darcy-Weisbach equation, hydraulic gradient and total energy lines. Series and parallel pipe flow.
- Flow under varying head.
- Chezy equation (theoretical and empirical) for flow in an open channel.
- Practical application of fluid mechanics in civil engineering.

1.2 Programme

Lectures

There are 4 hours of lectures per week. One of these will be considered as a tutorial class – to be confirmed.

The lectures are:

- Monday, 11:00-12:00, Rm. 209 and 17:00-18:00, Rm 134;
- Wednesday, to be confirmed.

Assessment

The marks awarded for this subject are assigned as follows:

- 80% for end-of-semester examination;
- 20% for laboratory work and reports.

1.3 Reading Material

Lecture Notes

The notes that you will take in class will cover the basic outline of the necessary ideas. It will be essential to do some extra reading for this subject.

Obviously only topics covered in the notes will be examined. However, it often aids understanding to hear/read different ways of explaining the same topic.

Books

Books on Fluid Mechanics are kept in Section 532 of the library. However, any of these books should help you understand fluid mechanics:

- Douglas, J.F., Swaffield, J.A., Gasiorek, J.M. and Jack, L.B. (2005), *Fluid Mechanics*, 5th Edn., Prentice Hall.
- Massey, B. and Ward-Smith, J. (2005), *Mechanics of Fluids*, 8th Edn., Routledge.
- Chadwick, A., Morfett, J. and Borthwick, M. (2004), *Hydraulics in Civil and Environmental Engineering*, 4th Edn., E & FN Spon.
- Douglas, J.F. and Mathews, R.D. (1996), *Solving Problems in Fluid Mechanics, Vols. I and II*, 3rd Edn., Longman.

The Web

There are many sites that can help you with this subject. In particular there are pictures and movies that will aid your understanding of the physical processes behind the theories.

If you find a good site, please let me know and we will develop a list for the class.

1.4 Fluid Mechanics in Civil/Structural Engineering

Every civil/structural engineering graduate needs to have a thorough understanding of fluids. This is more obvious for civil engineers but is equally valid for structural engineers:

- Drainage for developments;
- Attenuation of surface water for city centre sites;
- Sea and river (flood) defences;
- Water distribution/sewerage (sanitation) networks;
- Hydraulic design of water/sewage treatment works;
- Dams;
- Irrigation;
- Pumps and Turbines;
- Water retaining structures.
- Flow of air in / around buildings;
- Bridge piers in rivers;
- Ground-water flow.

As these mostly involve water, we will mostly examine fluid mechanics with this in mind.

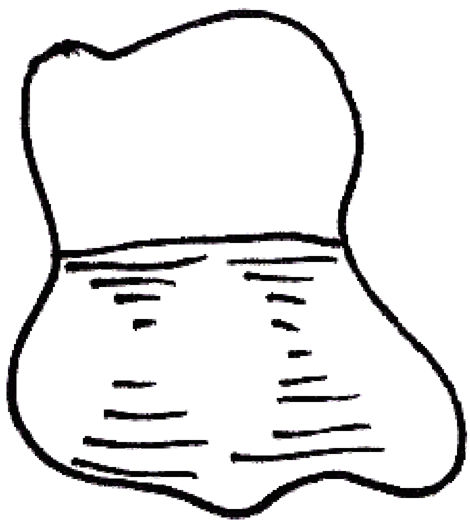
Remember: it is estimated that drainage and sewage systems – as designed by civil engineers – have saved more lives than all of medical science. Fluid mechanics is integral to our work.

2. Introduction to Fluids

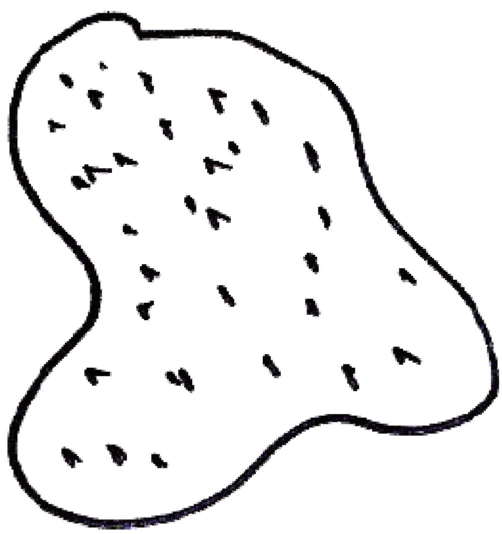
2.1 Background and Definition

Background

- There are three states of matter: solids, liquids and gases.
- Both liquids and gases are classified as fluids.
- Fluids do not resist a change in shape. Therefore fluids assume the shape of the container they occupy.
- Liquids may be considered to have a fixed volume and therefore can have a free surface. Liquids are almost incompressible.
- Conversely, gases are easily compressed and will expand to fill a container they occupy.
- We will usually be interested in liquids, either at rest or in motion.



Liquid showing free surface



Gas filling volume

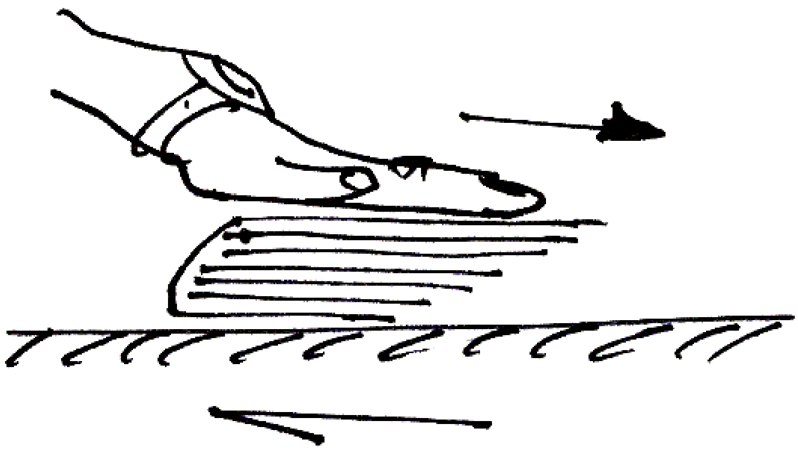
Behaviour of fluids in containers

Definition

The strict definition of a fluid is:

> *A fluid is a substance which conforms continuously under the action of shearing forces.*

To understand this, remind ourselves of what a shear force is:



Application and effect of shear force on a book

Definition Applied to Static Fluids

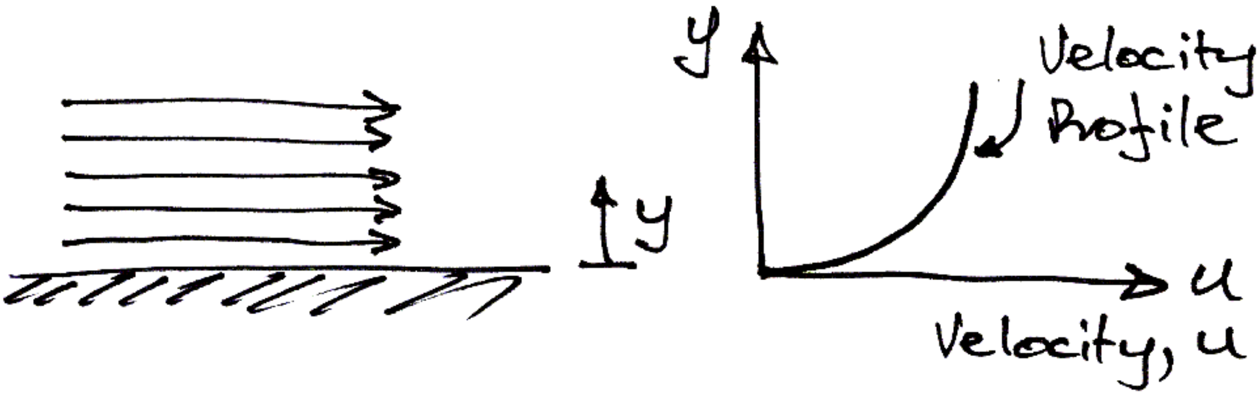
According to this definition, if we apply a shear force to a fluid it will deform and take up a state in which no shear force exists. Therefore, we can say:

> *If a fluid is at rest there can be no shearing forces acting and therefore all forces in the fluid must be perpendicular to the planes in which they act.*

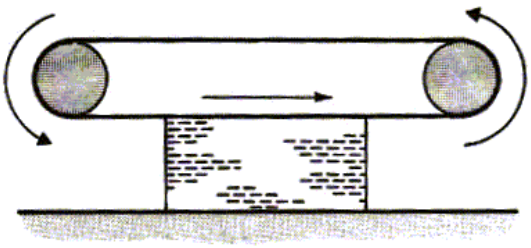
Note here that we specify that the fluid must be at rest. This is because, it is found experimentally that fluids in motion can have slight resistance to shear force. This is the source of *viscosity*.

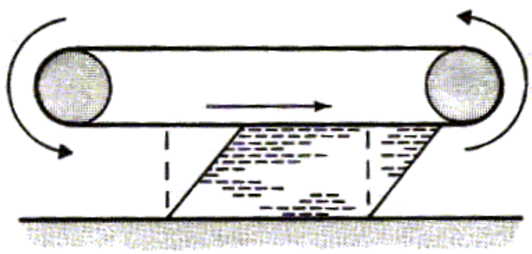
Definition Applied to Fluids in Motion

For example, consider the fluid shown flowing along a fixed surface. At the surface there will be little movement of the fluid (it will 'stick' to the surface), whilst further away from the surface the fluid flows faster (has greater velocity):



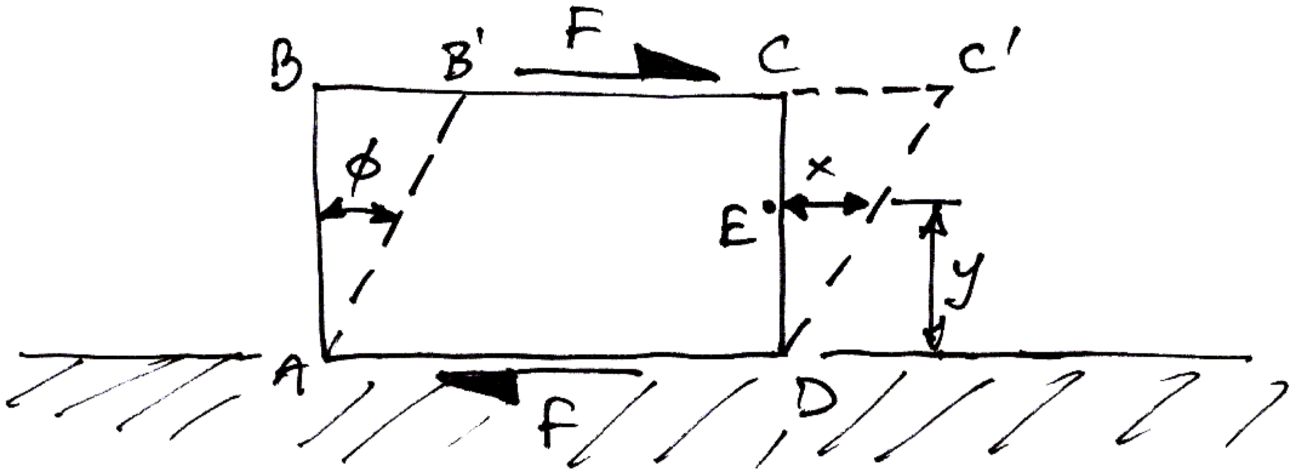
If one layer of is moving faster than another layer of fluid, there must be shear forces acting between them. For example, if we have fluid in contact with a conveyor belt that is moving we will get the behaviour shown:


Ideal fluid


Real (Viscous) Fluid

When fluid is in motion, any difference in velocity between adjacent layers has the same effect as the conveyor belt does.

Therefore, to represent real fluids in motion we must consider the action of shear forces.



Consider the small element of fluid shown, which is subject to shear force and has a dimension *s* into the page. The force *F* acts over an area A = BC×*s*. Hence we have a *shear stress* applied:

$$\text{Stress} = \frac{\text{Force}}{\text{Area}}$$
$$\tau = \frac{F}{A}$$

Any stress causes a deformation, or strain, and a shear stress causes a *shear strain*. This shear strain is measured by the angle ϕ.

Remember that a fluid *continuously* deforms when under the action of shear. This is different to a solid: a solid has a single value of ϕ for each value of τ. So the longer a shear stress is applied to a fluid, the more shear strain occurs. However, what is known from experiments is that the rate of shear strain (shear strain per unit time) is related to the shear stress:

$$\text{Shear stress} \propto \text{Rate of shear strain}$$
$$\text{Shear stress} = \text{Constant} \times \text{Rate of shear strain}$$

We need to know the rate of shear strain. From the diagram, the shear strain is:

$$\phi = \frac{x}{y}$$

If we suppose that the particle of fluid at E moves a distance x in time t, then, using $S = R\theta$ for small angles, the rate of shear strain is:

$$\frac{\Delta\phi}{\Delta t} = \left(\frac{x}{y}\right)\Big/ t = \frac{x}{t}\cdot\frac{1}{y}$$
$$= \frac{u}{y}$$

Where u is the velocity of the fluid. This term is also the change in velocity with height. When we consider infinitesimally small changes in height we can write this in differential form, du/dy. Therefore we have:

$$\tau = \text{constant} \times \frac{du}{dy}$$

This constant is a property of the fluid called its dynamic viscosity (dynamic because the fluid is in motion, and viscosity because it is resisting shear stress). It is denoted μ which then gives us:

Newton's Law of Viscosity:

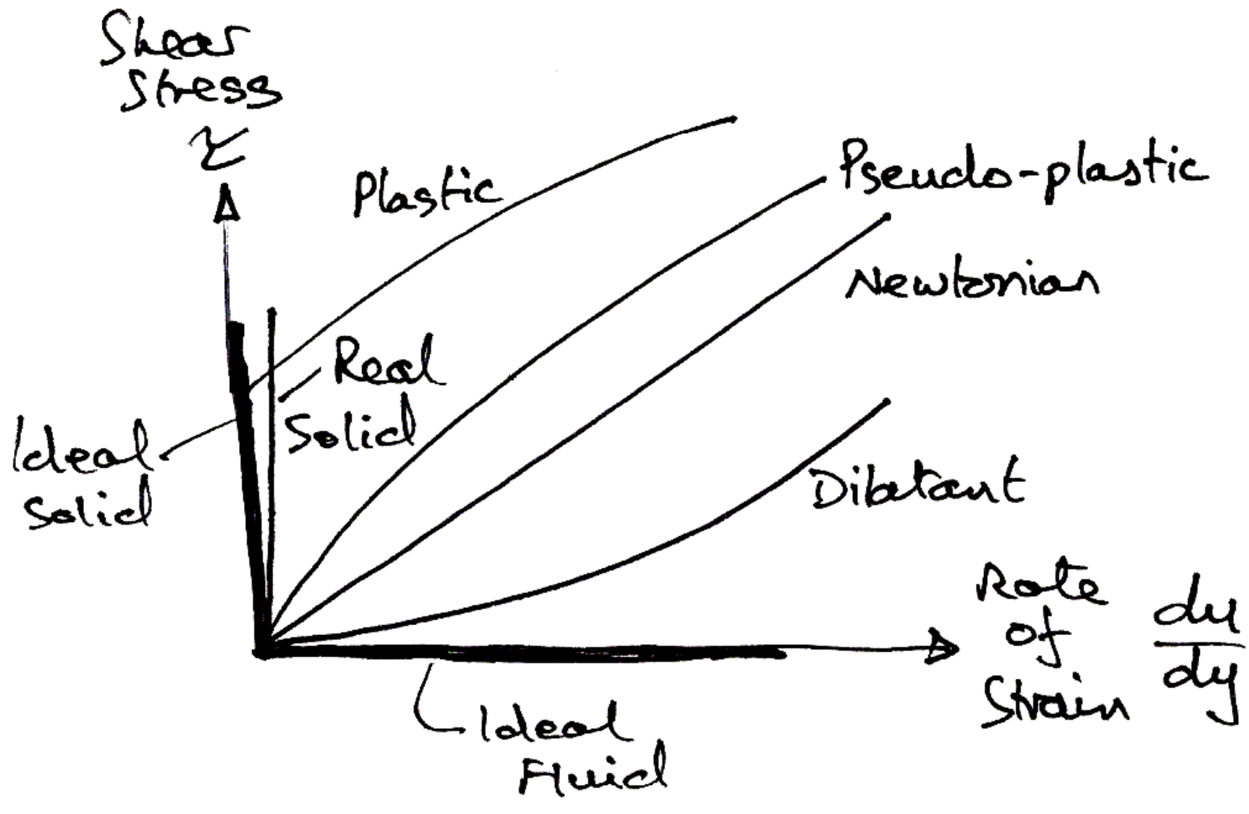
$$\boxed{\tau = \mu \frac{du}{dy}}$$

Generalized Laws of Viscosity

We have derived a law for the behaviour of fluids – that of *Newtonian fluids*. However, experiments show that there are *non-Newtonian* fluids that follow a *generalized law of viscosity*:

$$\tau = A + B\left(\frac{du}{dy}\right)^n$$

Where *A*, *B* and *n* are constants found experimentally. When plotted these fluids show much different behaviour to a Newtonian fluid:



Behaviour of Fluids and Solids

In this graph the Newtonian fluid is represent by a straight line, the slope of which is μ. Some of the other fluids are:

- *Plastic:* Shear stress must reach a certain minimum before flow commences.
- *Pseudo-plastic:* No minimum shear stress necessary and the viscosity decreases with rate of shear, e.g. substances like clay, milk and cement.
- *Dilatant substances;* Viscosity increases with rate of shear, e.g. quicksand.
- *Viscoelastic materials:* Similar to Newtonian but if there is a sudden large change in shear they behave like plastic.
- *Solids:* Real solids do have a slight change of shear strain with time, whereas ideal solids (those we idealise for our theories) do not.

Lastly, we also consider the *ideal fluid*. This is a fluid which is assumed to have no viscosity and is very useful for developing theoretical solutions. It helps achieve some practically useful solutions.

2.2 Units

Fluid mechanics deals with the measurement of many variables of many different types of units. Hence we need to be very careful to be consistent.

Dimensions and Base Units

The *dimension* of a measure is independent of any particular system of units. For example, velocity may be in metres per second or miles per hour, but dimensionally, it is always length per time, or $L/T = LT^{-1}$. The dimensions of the relevant base units of the Système International (SI) system are:

Unit-Free		**SI Units**	
Dimension	**Symbol**	**Unit**	**Symbol**
Mass	M	kilogram	kg
Length	L	metre	m
Time	T	second	s
Temperature	θ	kelvin	K

Derived Units

From these we have some relevant derived units (shown on the next page).

Checking the dimensions or units of an equation is very useful to minimize errors. For example, if when calculating a force and you find a pressure then you know you've made a mistake.

Quantity	**Dimension**	**SI Unit**	
		Derived	**Base**
Velocity	LT^{-1}	m/s	m s^{-1}
Acceleration	LT^{-2}	m/s^2	m s^{-2}
Force	MLT^{-2}	Newton, N	kg m s^{-2}
Pressure Stress	$ML^{-1}T^{2}$	Pascal, Pa N/m^2	kg m^{-1} s^{-2}
Density	ML^{-3}	kg/m^3	kg m^{-3}
Specific weight	$ML^{-2}T^{-2}$	N/m^3	kg m^{-2} s^{-2}
Relative density	Ratio	Ratio	Ratio
Viscosity	$ML^{-1}T^{-1}$	Ns/m^2	kg m^{-1} s^{-1}
Energy (work)	$ML^{2}T^{-2}$	Joule, J Nm	kg m^{2} s^{-2}
Power	$ML^{2}T^{-3}$	Watt, W Nm/s	kg m^{2} s^{-3}

Note: The acceleration due to gravity will always be taken as 9.81 m/s^2.

SI Prefixes

SI units use prefixes to reduce the number of digits required to display a quantity. The prefixes and multiples are:

Prefix Name	Prefix Unit	Multiple
Tera	T	10^{12}
Giga	G	10^{9}
Mega	M	10^{6}
Kilo	k	10^{3}
Hecto	h	10^{2}
Deka	da	10^{1}
Deci	d	10^{-1}
Centi	c	10^{-2}
Milli	m	10^{-3}
Micro	μ	10^{-6}
Nano	n	10^{-9}
Pico	p	10^{-12}

Be very particular about units and prefixes. For example:

- kN means kilo-Newton, 1000 Newtons;
- Kn is the symbol for knots – an imperial measure of speed;
- KN has no meaning;
- kn means kilo-nano – essentially meaningless.

Further Reading

- Sections 1.6 to 1.10 of *Fluid Mechanics* by Cengel & Cimbala.

2.3 Properties

Further Reading

Here we consider only the relevant properties of fluids for our purposes. Find out about surface tension and capillary action elsewhere. Note that capillary action only features in pipes of $\leq$ 10 mm diameter.

Mass Density

The mass per unit volume of a substance, usually denoted as ρ. Typical values are:

- Water: 1000 kg/m^3;
- Mercury: 13546 kg/m^3;
- Air: 1.23 kg/m^3;
- Paraffin: 800 kg/m^3.

Specific Weight

The weight of a unit volume a substance, usually denoted as γ. Essentially density times the acceleration due to gravity:

$$\gamma = \rho g$$

Relative Density (Specific Gravity)

A dimensionless measure of the density of a substance with reference to the density of some standard substance, usually water at 4°C:

$$\begin{aligned}\text{relative density} &= \frac{\text{density of substance}}{\text{density of water}} \\ &= \frac{\text{specific weight of substance}}{\text{specific weight of water}} \\ &= \frac{\rho_s}{\rho_w} = \frac{\gamma_s}{\gamma_w}\end{aligned}$$

Bulk Modulus

In analogy with solids, the bulk modulus is the modulus of elasticity for a fluid. It is the ratio of the change in unit pressure to the corresponding volume change per unit volume, expressed as:

$$\frac{\text{Change in Volume}}{\text{Original Volume}} = \frac{\text{Chnage in pressure}}{\text{Bulk Modulus}}$$

$$\frac{-dV}{V} = \frac{dp}{K}$$

Hence:

$$K = -V\frac{dp}{dV}$$

In which the negative sign indicates that the volume reduces as the pressure increases. The bulk modulus changes with the pressure and density of the fluid, but for liquids can be considered constant for normal usage. Typical values are:

- Water: 2.05 GN/m^3;
- Oil: 1.62 GN/m^3.

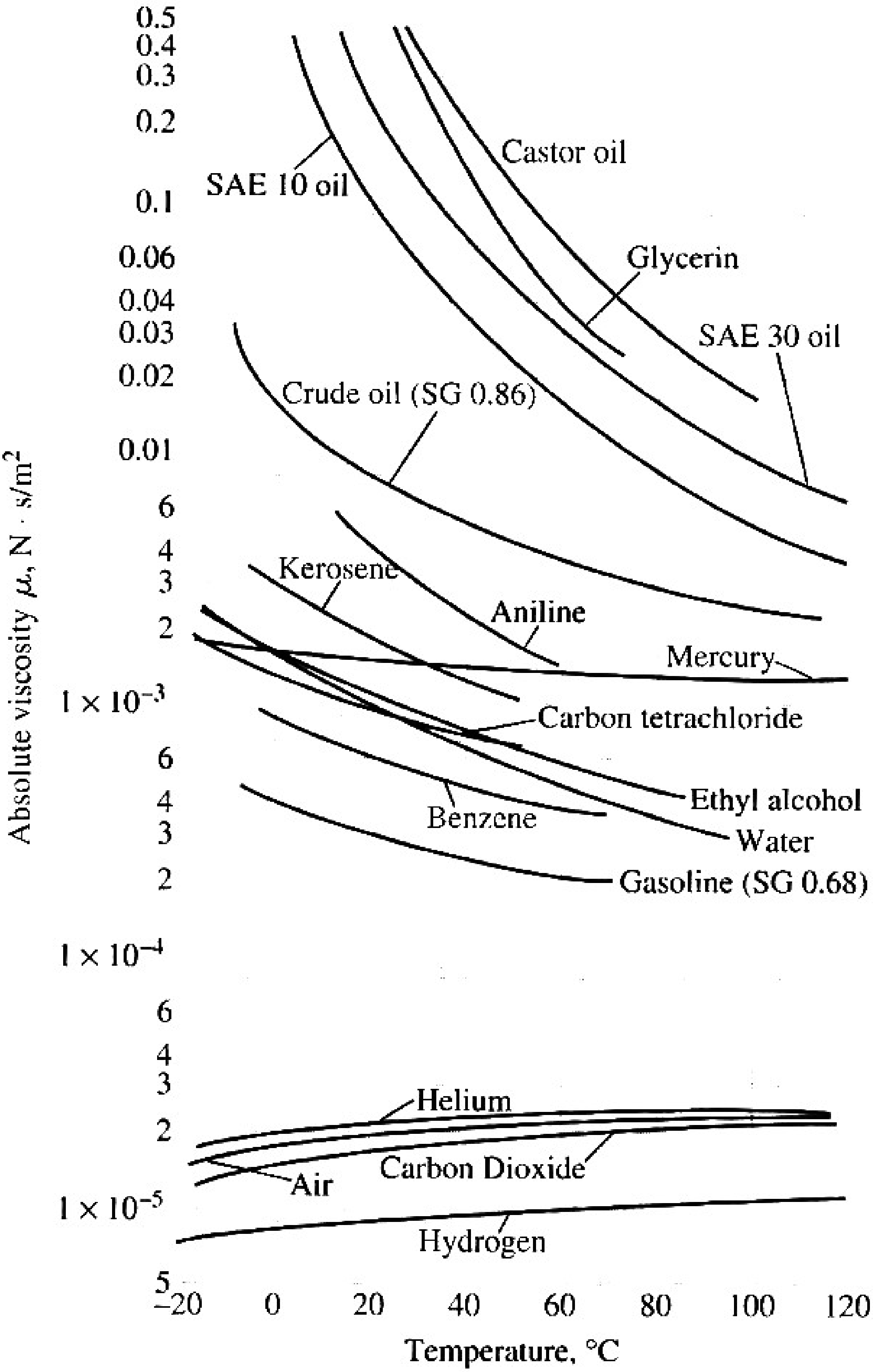
The units are the same as those of stress or pressure.

Viscosity

The viscosity of a fluid determines the amount of resistance to shear force. Viscosities of liquids decrease as temperature increases and are usually not affected by pressure changes. From Newton's Law of Viscosity:

$$\mu = \frac{\tau}{du/dy} = \frac{\text{shear stress}}{\text{rate of shear strain}}$$

Hence the units of viscosity are $\text{Pa}\cdot\text{s}$ or $\text{N}\cdot\text{s/m}^2$. This measure of viscosity is known as *dynamic viscosity* and some typical values are given:



Absolute viscosity μ, N · s/m²
0.5
0.4
0.3
0.2
0.1
0.06
0.04
0.03
0.02
0.01
6
4
3
2
1 × 10⁻³
6
4
3
2
1 × 10⁻⁴
6
4
3
2
1 × 10⁻⁵
5
SAE 10 oil
Castor oil
Glycerin
SAE 30 oil
Crude oil (SG 0.86)
Kerosene
Aniline
Mercury
Carbon tetrachloride
Ethyl alcohol
Water
Benzene
Gasoline (SG 0.68)
Helium
Air
Carbon Dioxide
Hydrogen
−20
0
20
40
60
80
100
120
Temperature, °C

Problems - Properties

a) If 6 m^3 of oil weighs 47 kN, find its specific weight, density, and relative density.

(Ans. 7.833 kN/m^3, 798 kg/m^3, 0.800)

b) At a certain depth in the ocean, the pressure is 80 MPa. Assume that the specific weight at the surface is 10 kN/m^3 and the average bulk modulus is 2.340 GPa. Find:

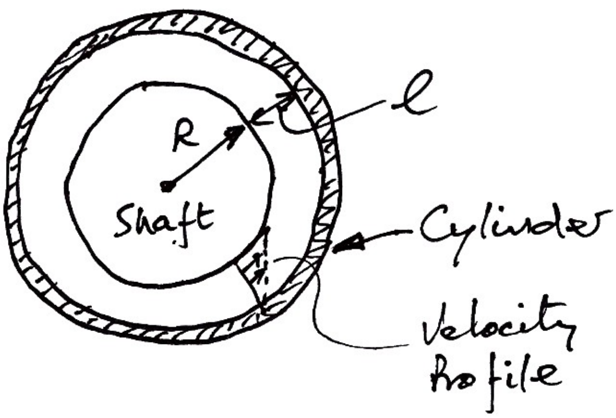
   a) the change in specific volume between the surface and the large depth;
   b) the specific volume at the depth, and;
   c) the specific weight at the depth.

(Ans. -0.335×10^{-4} m^3/kg, 9.475×10^{-4} m^3/kg, 10.35 kN/m^3)

c) A 100 mm deep stream of water is flowing over a boundary. It is considered to have zero velocity at the boundary and 1.5 m/s at the free surface. Assuming a linear velocity profile, what is the shear stress in the water?

(Ans. 0.0195 N/m^2)

d) The viscosity of a fluid is to be measured using a viscometer constructed of two 750 mm long concentric cylinders. The outer diameter of the inner cylinder is 150 mm and the gap between the two cylinders is 1.2 mm. The inner cylinder is rotated at 200 rpm and the torque is measured to be 10 Nm.



   a) Derive a generals expression for the viscosity of a fluid using this type of viscometer, and;
   b) Determine the viscosity of the fluid for the experiment above.

(Ans. 6×10^{-4} Ns/m^2)

3. Hydrostatics

3.1 Introduction

Pressure

In fluids we use the term pressure to mean:

The perpendicular force exerted by a fluid per unit area.

This is equivalent to stress in solids, but we shall keep the term pressure. Mathematically, because pressure may vary from place to place, we have:

$$p = \lim_{\Delta \to 0} \frac{\Delta F}{\Delta A}$$

As we saw, force per unit area is measured in N/m^2 which is the same as a pascal (Pa). The units used in practice vary:

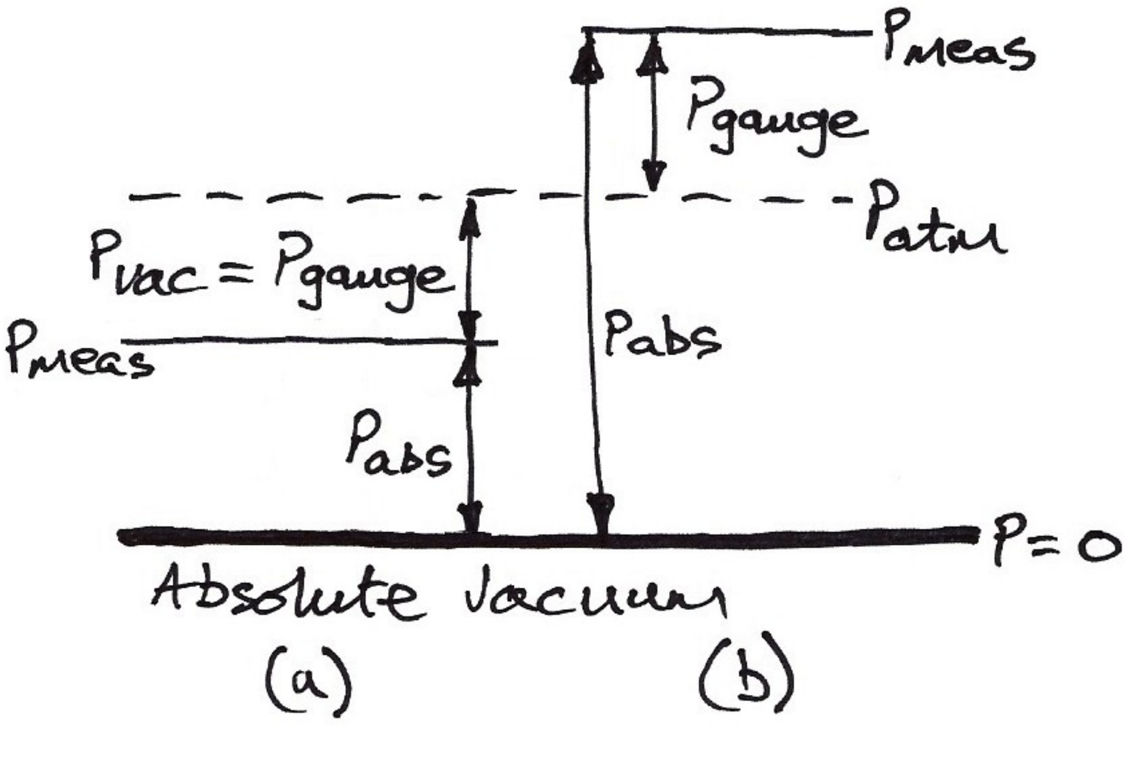
- 1 kPa = 1000 Pa = 1000 N/m^2
- 1 MPa = 1000 kPa = 1×10^6 N/m^2
- 1 bar = 10^5 Pa = 100 kPa = 0.1 MPa
- 1 atm = 101,325 Pa = 101.325 kPa = 1.01325 bars = 1013.25 millibars

For reference to pressures encountered on the street which are often imperial:

- 1 atm = 14.696 psi (i.e. pounds per square inch)
- 1 psi = 6894.7 Pa ≈ 6.89 kPa ≈ 0.007 MPa

Pressure Reference Levels

The pressure that exists anywhere in the universe is called the *absolute pressure*, P_{abs}. This then is the amount of pressure greater than a pure vacuum. The atmosphere on earth exerts *atmospheric pressure*, P_{atm}, on everything in it. Often when measuring pressures we will calibrate the instrument to read zero in the open air. Any measured pressure, P_{meas}, is then a positive or negative deviation from atmospheric pressure. We call such deviations a *gauge pressure*, P_{gauge}. Sometimes when a gauge pressure is negative it is termed a *vacuum pressure*, P_{vac}.



The above diagram shows:

(a) the case when the measured pressure is below atmospheric pressure and so is a negative gauge pressure or a vacuum pressure;

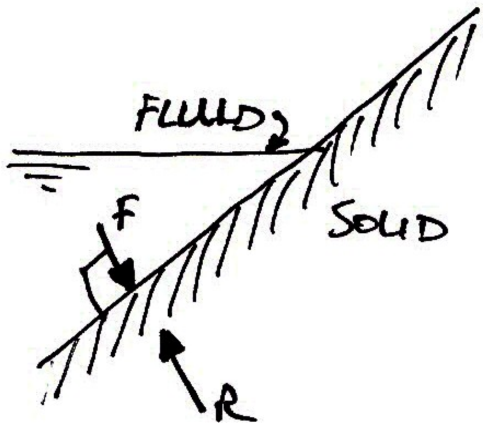
(b) the more usual case when the measured pressure is greater than atmospheric pressure by the gauge pressure.

3.2 Pressure in a Fluid

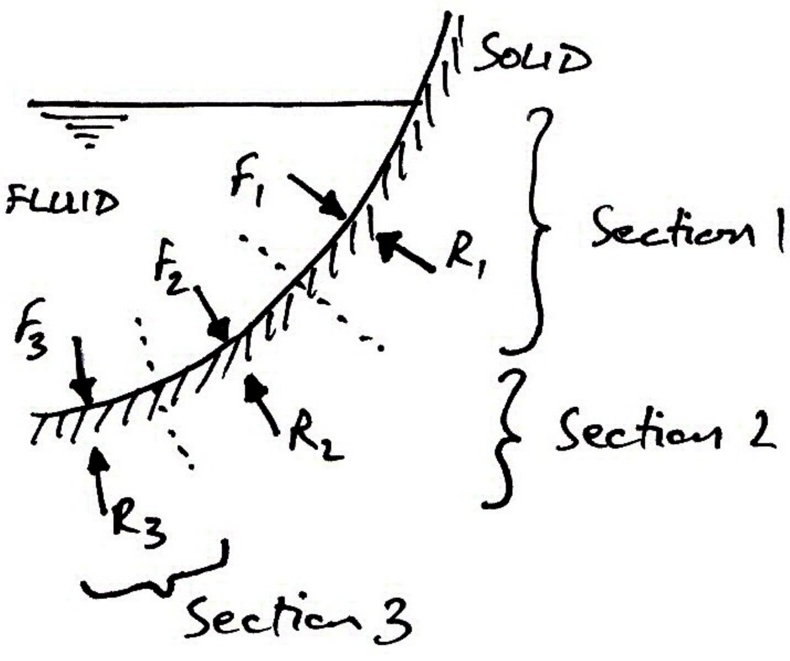
Statics of Definition

We applied the definition of a fluid to the static case previously and determined that there must be no shear forces acting and thus only forces normal to a surface act in a fluid.

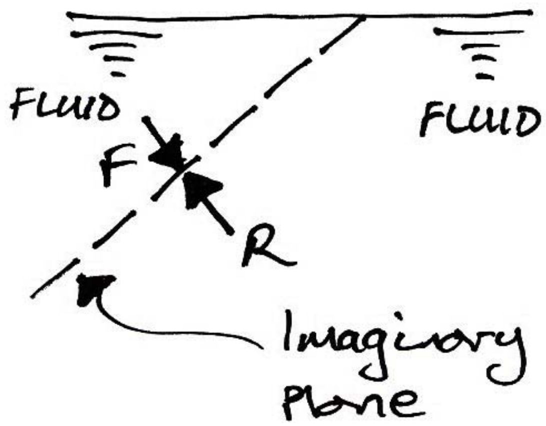
For a flat surface at arbitrary angle we have:



A curved surface can be examined in sections:



And we are not restricted to actual solid-fluid interfaces. We can consider imaginary planes through a fluid:

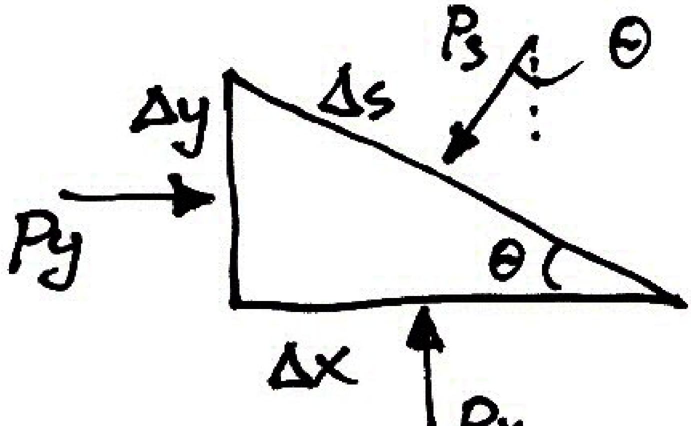


Pascal's Law

This law states:

The pressure at a point in a fluid at rest is the same in all directions.

To show this, we will consider a very small wedge of fluid surrounding the point. This wedge is unit thickness into the page:



As with all static objects the forces in the *x* and *y* directions should balance. Hence:

$$\sum F_x = 0: \qquad p_y \cdot \Delta y - p_s \cdot \Delta s \cdot \sin\theta = 0$$

But $\sin\theta = \dfrac{\Delta y}{\Delta s}$, therefore:

$$p_y \cdot \Delta y - p_s \cdot \Delta s \cdot \frac{\Delta y}{\Delta s} = 0$$
$$p_y \cdot \Delta y = p_s \cdot \Delta y$$
$$p_y = p_s$$

$$\sum F_y = 0: \qquad p_x \cdot \Delta x - p_s \cdot \Delta s \cdot \cos\theta = 0$$

But $\cos\theta = \dfrac{\Delta x}{\Delta s}$, therefore:

$$p_x \cdot \Delta x - p_s \cdot \Delta s \cdot \frac{\Delta x}{\Delta s} = 0$$
$$p_x \cdot \Delta x = p_s \cdot \Delta x$$
$$p_x = p_s$$

Hence for any angle:

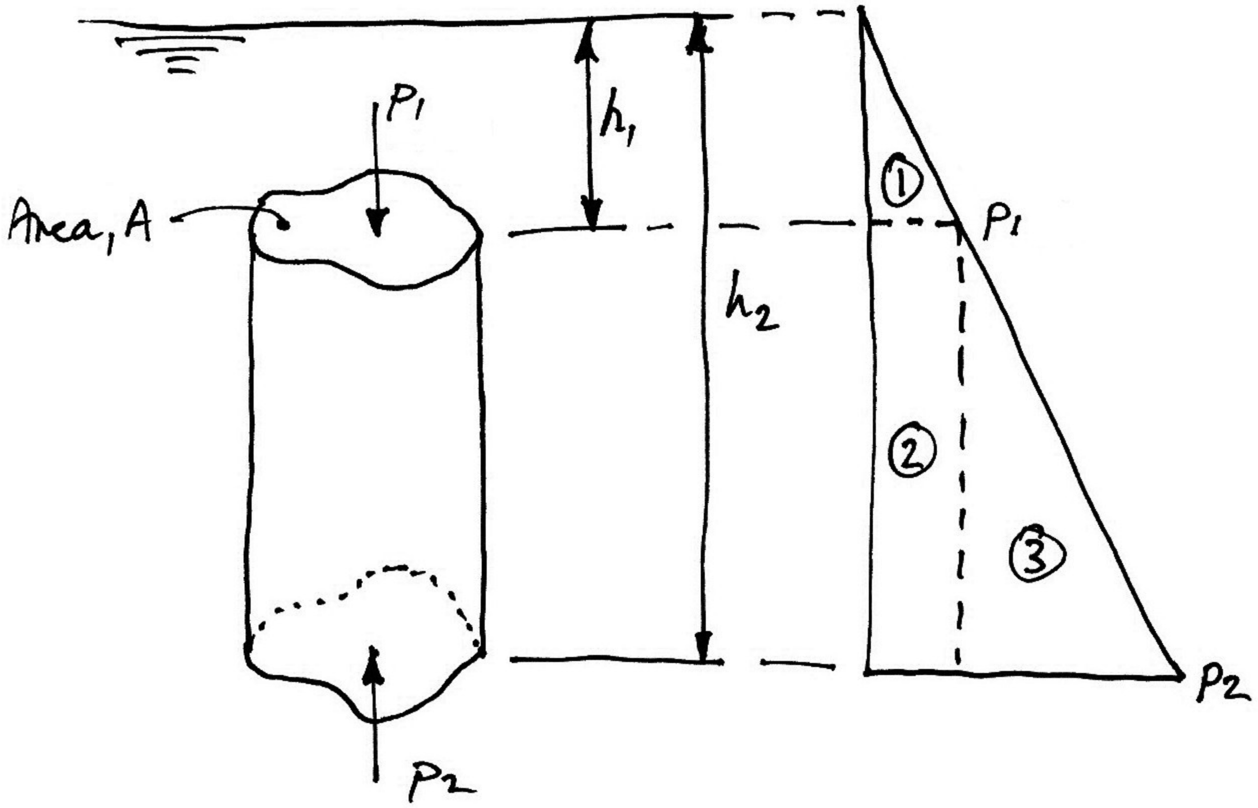
$$p_y = p_x = p_s$$

And so the pressure at a point is the same in any direction. Note that we neglected the weight of the small wedge of fluid because it is infinitesimally small. This is why Pascal's Law is restricted to the pressure at a point.

Pressure Variation with Depth

Pressure in a static fluid does not change in the horizontal direction as the horizontal forces balance each other out. However, pressure in a static fluid does change with depth, due to the extra weight of fluid on top of a layer as we move downwards.

Consider a column of fluid of arbitrary cross section of area, A:



Column of Fluid **Pressure Diagram**

Considering the weight of the column of water, we have:

$$\sum F_y = 0: \qquad p_1 A + \gamma A\left(h_2 - h_1\right) - p_2 A = 0$$

Obviously the area of the column cancels out: we can just consider pressures. If we say the height of the column is $h = h_2 - h_1$ and substitute in for the specific weight, we see the difference in pressure from the bottom to the top of the column is:

$$p_2 - p_1 = \rho g h$$

This difference in pressure varies linearly in *h*, as shown by the Area 3 of the pressure diagram. If we let $h_1 = 0$ and consider a gauge pressure, then $p_1 = 0$ and we have:

$$p_2 = \rho g h$$

Where *h* remains the height of the column. For the fluid on top of the column, this is the source of p_1 and is shown as Area 1 of the pressure diagram. Area 2 of the pressure diagram is this same pressure carried downwards, to which is added more pressure due to the extra fluid.

To summarize:

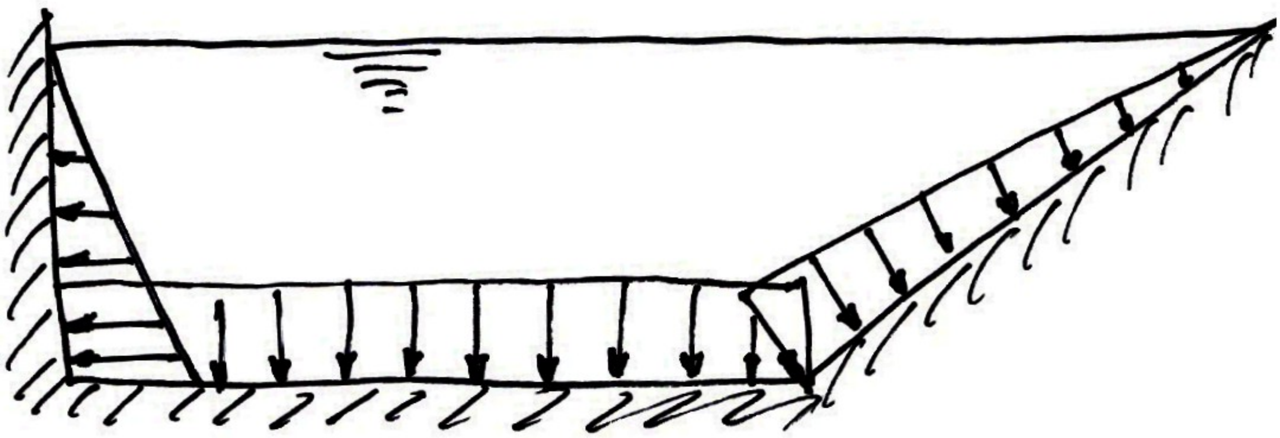
The gauge pressure at any depth from the surface of a fluid is:

$$\boxed{p = \rho g h}$$

Summary

1. Pressure acts normal to any surface in a static fluid;
2. Pressure is the same at a point in a fluid and acts in all directions;
3. Pressure varies linearly with depth in a fluid.

By applying these rules to a simple swimming pool, the pressure distribution around the edges is as shown:

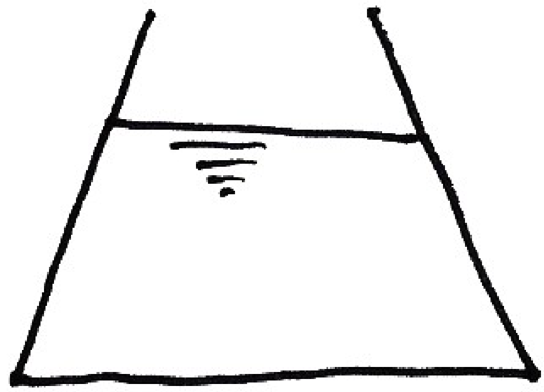


Note:

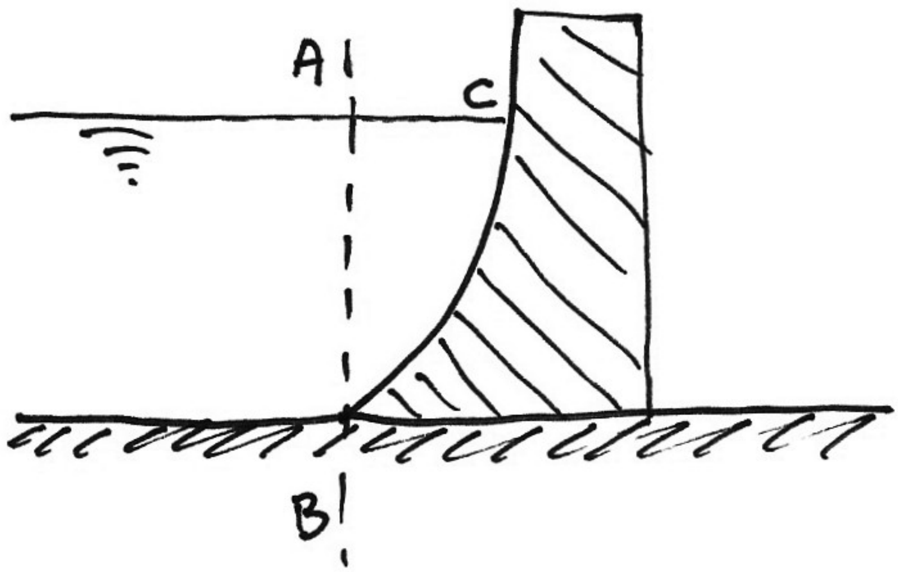
1. Along the bottom the pressure is constant due to a constant depth;
2. Along the vertical wall the pressure varies linearly with depth and acts in the horizontal direction;
3. Along the sloped wall the pressure again varies linearly with depth but also acts normal to the surface;
4. At the junctions of the walls and the bottom the pressure is the same.

Problems - Pressure

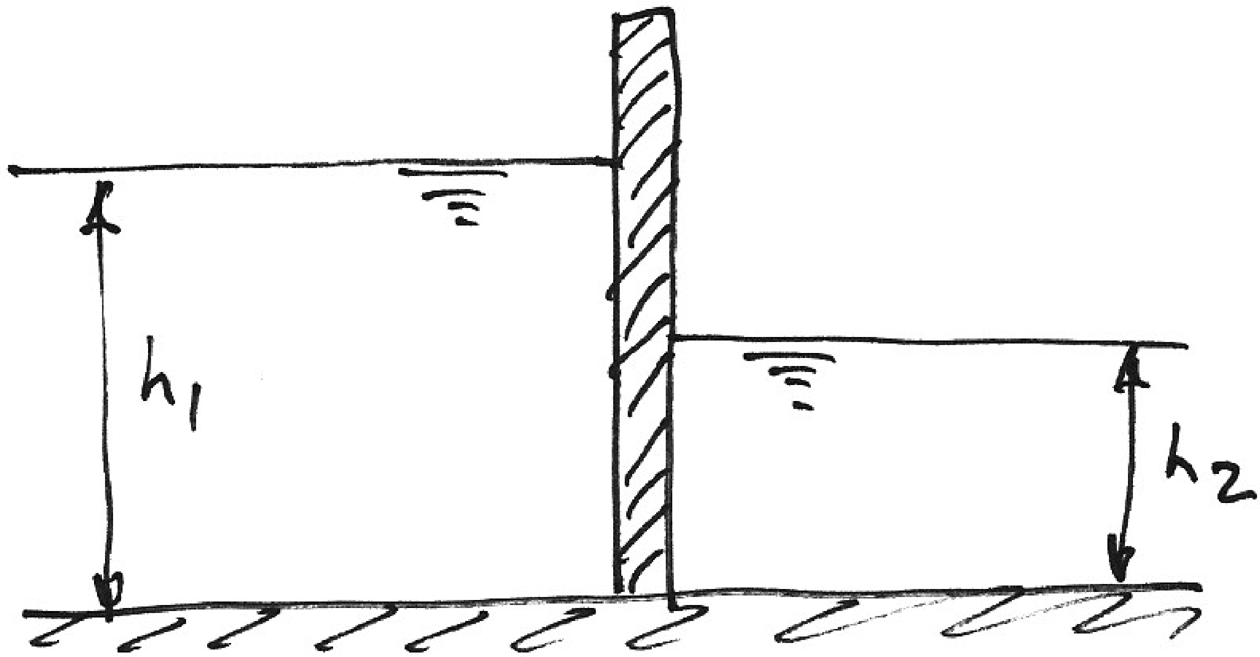
1. Sketch the pressure distribution applied to the container by the fluid:



2. For the dam shown, sketch the pressure distribution on line *AB* and on the surface of the dam, *BC*. Sketch the resultant force on the dam.



3. For the canal gate shown, sketch the pressure distributions applied to it. Sketch the resultant force on the gate? If $h_1 = 6.0$ m and $h_2 = 4.0$ m, sketch the pressure distribution to the gate. Also, what is the value of the resultant force on the gate and at what height above the bottom of the gate is it applied?



3.3 Pressure Measurement

Pressure Head

Pressure in fluids may arise from many sources, for example pumps, gravity, momentum etc. Since $p = \rho gh$, a height of liquid column can be associated with the pressure *p* arising from such sources. This height, *h*, is known as the pressure head.

Example:

The gauge pressure in a water mains is 50 kN/m^2, what is the pressure head?

The pressure head equivalent to the pressure in the pipe is just:

$$
\begin{aligned}
p &= \rho gh \\
h &= \frac{p}{\rho g} \\
&= \frac{50 \times 10^3}{1000 \times 9.81} \\
&\approx 5.1 \text{ m}
\end{aligned}
$$

So the pressure at the bottom of a 5.1 m deep swimming pool is the same as the pressure in this pipe.

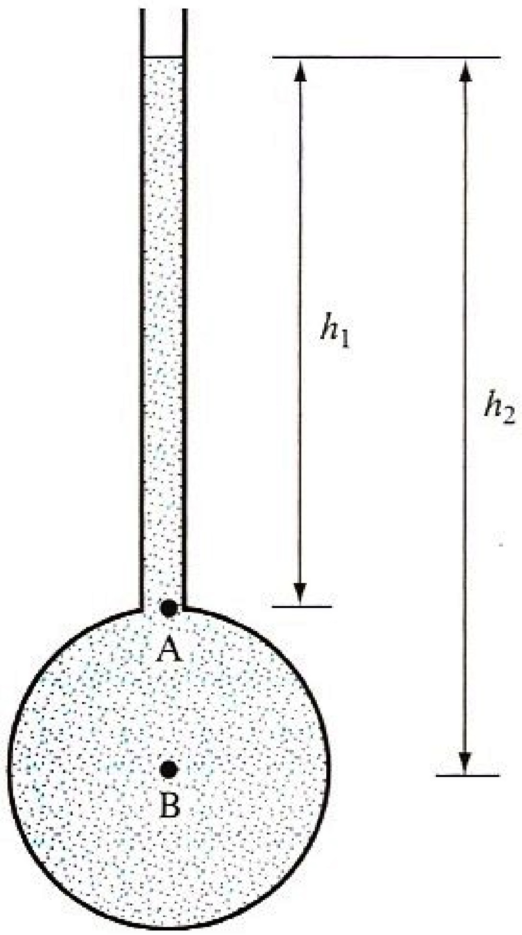
Manometers

A manometer (or liquid gauge) is a pressure measurement device which uses the relationship between pressure and head to give readings.

In the following, we wish to measure the pressure of a fluid in a pipe.

Piezometer

This is the simplest gauge. A small vertical tube is connected to the pipe and its top is left open to the atmosphere, as shown.



The pressure at *A* is equal to the pressure due to the column of liquid of height h_1:

$$p_A = \rho g h_1$$

Similarly,

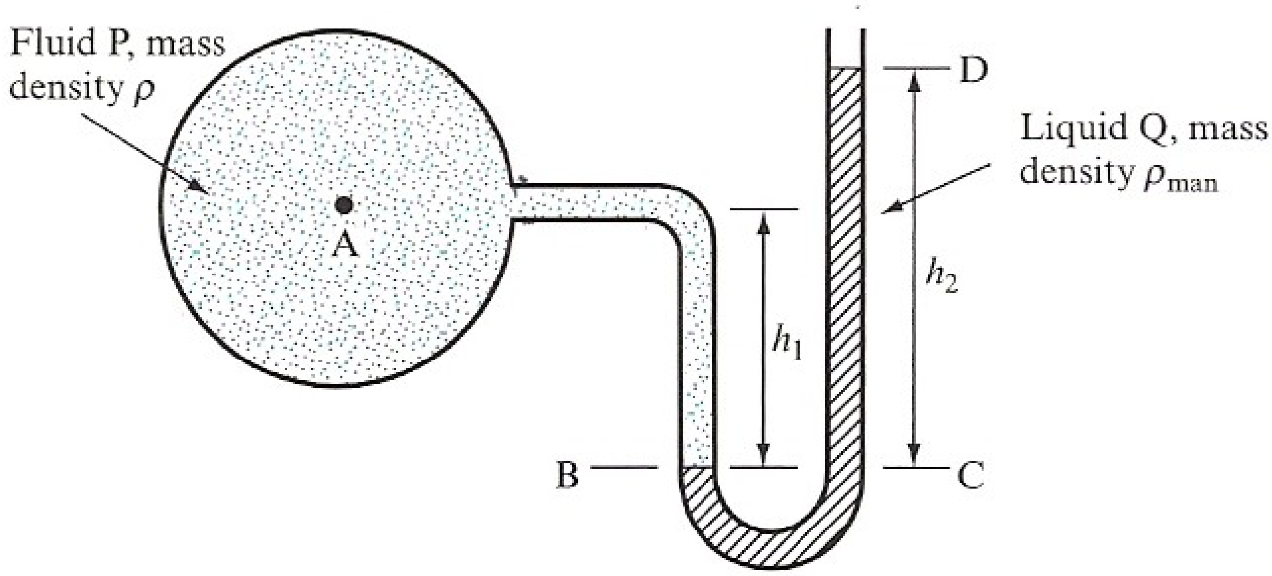
$$p_B = \rho g h_2$$

The problem with this type of gauge is that for usual civil engineering applications the pressure is large (e.g. 100 kN/m^2) and so the height of the column is impractical (e.g.10 m).

Also, obviously, such a gauge is useless for measuring gas pressures.

U-tube Manometer

To overcome the problems with the piezometer, the U-tube manometer seals the fluid by using a measuring (manometric) liquid:



Choosing the line *BC* as the interface between the measuring liquid and the fluid, we know:

$$\text{Pressure at } B,\ p_B = \text{Pressure at } C,\ p_C$$

For the left-hand side of the U-tube:

$$p_B = p_A + \rho g h_1$$

For the right hand side:

$$p_C = \rho_{man} g h_2$$

Where we have ignored atmospheric pressure and are thus dealing with gauge pressures. Thus:

$$p_B = p_C$$
$$p_A + \rho g h_1 = \rho_{man} g h_2$$

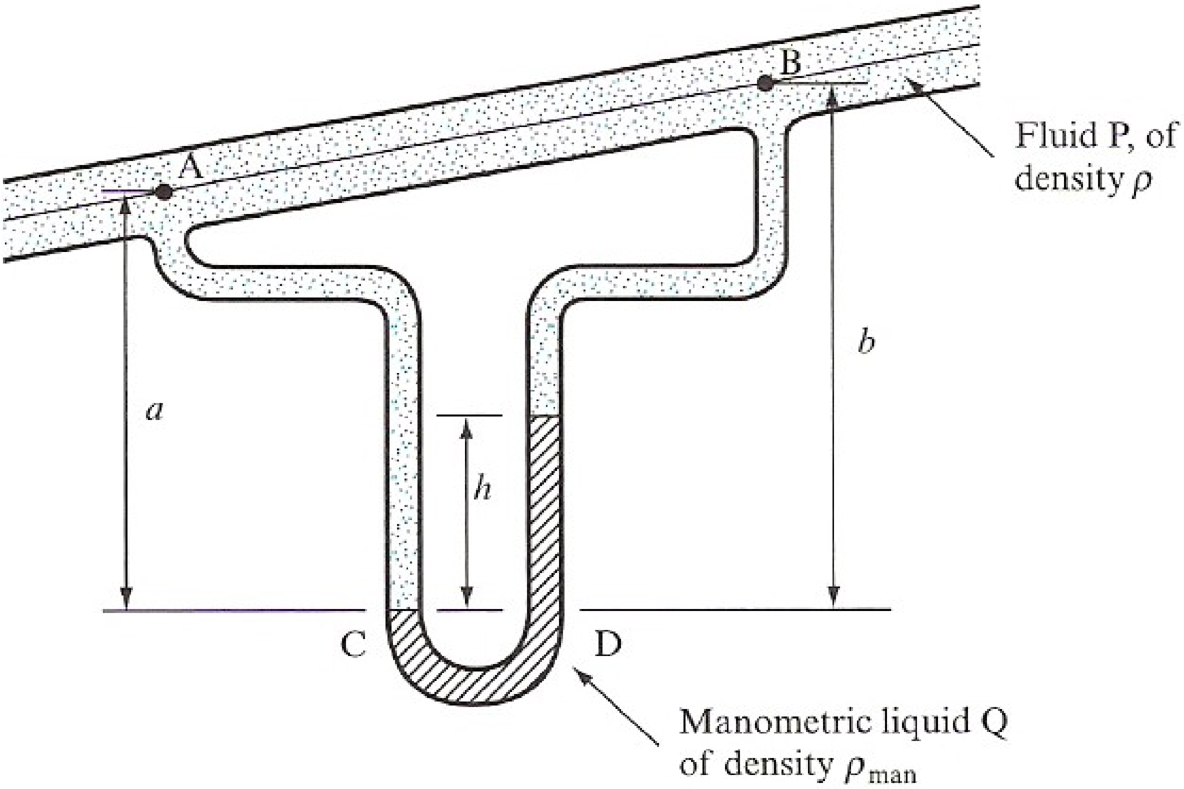
And so:

$$p_A = \rho_{man} g h_2 - \rho g h_1$$

Notice that we have used the fact that in any continuous fluid, the pressure is the same at any horizontal level.

Differential Manometer

To measure the pressure difference between two points we use a u-tube as shown:



Using the same approach as before:

$$\text{Pressure at } C,\ p_C = \text{Pressure at } D,\ p_D$$

$$p_A + \rho g a = p_B + \rho g\left(b - h\right) + \rho_{man} g h$$

Hence the pressure difference is:

$$p_A - p_B = \rho g\left(b - a\right) + hg\left(\rho_{man} - \rho\right)$$

Problems – Pressure Measurement

1. What is the pressure head, in metres of water, exerted by the atmosphere?

   (Ans. 10.3 m)

2. What is the maximum gauge pressure of water that can be measured using a piezometer 2.5 m high?

   (Ans. 24.5 kN/m²)

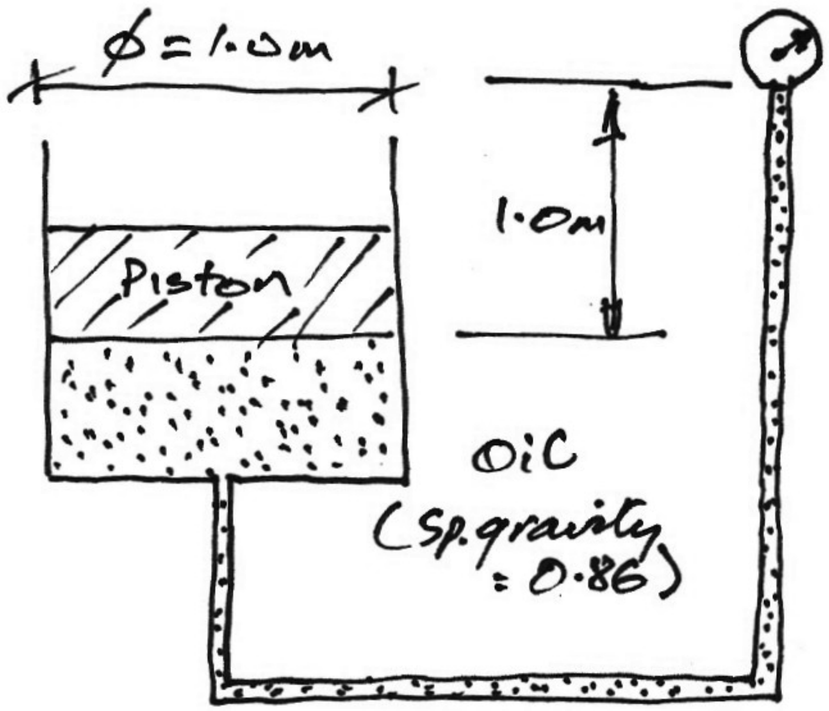
3. A U-tube manometer is used to measure the pressure of a fluid of density 800 kg/m^3. If the density of the manometric liquid is 13.6×10^3 kg/m^3, what is the gauge pressure in the pipe if
   (a) $h_1 = 0.5$ m and D is 0.9 m above BC;
   (b) $h_1 = 0.1$ m and D is 0.2 m below BC?

   (Ans. 116.15 kN/m^2, -27.45 kN/m^2)

4. A differential manometer is used to measure the pressure difference between two points in a pipe carrying water. The manometric liquid is mercury and the points have a 0.3 m height difference. Calculate the pressure difference when $h = 0.7$ m.

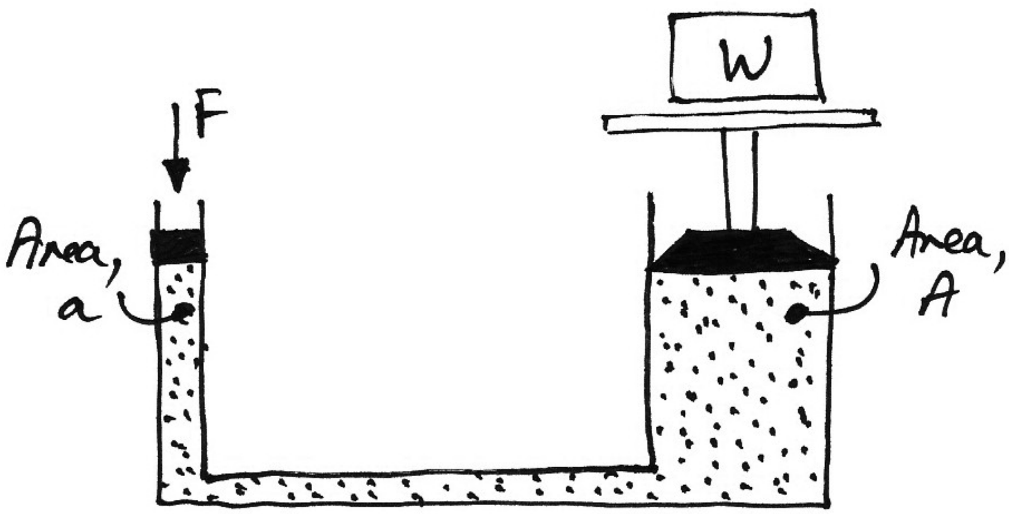
   (Ans. 89.47 kN/m^2)

5. For the configuration shown, calculate the weight of the piston if the gauge pressure reading is 70 kPa.



(Ans. 61.6 kN)

6. A hydraulic jack having a ram 150 mm in diameter lifts a weight W = 20 kN under the action of a 30 mm plunger. What force is required on the plunger to lift the weight?

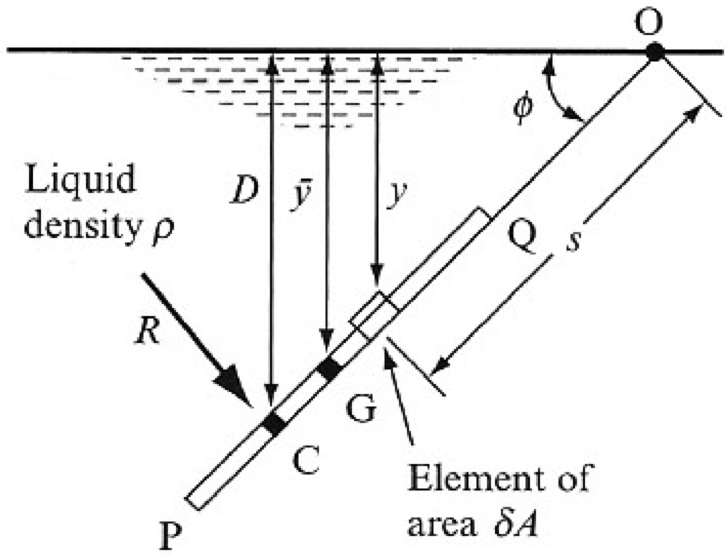


(Ans. 800 N)

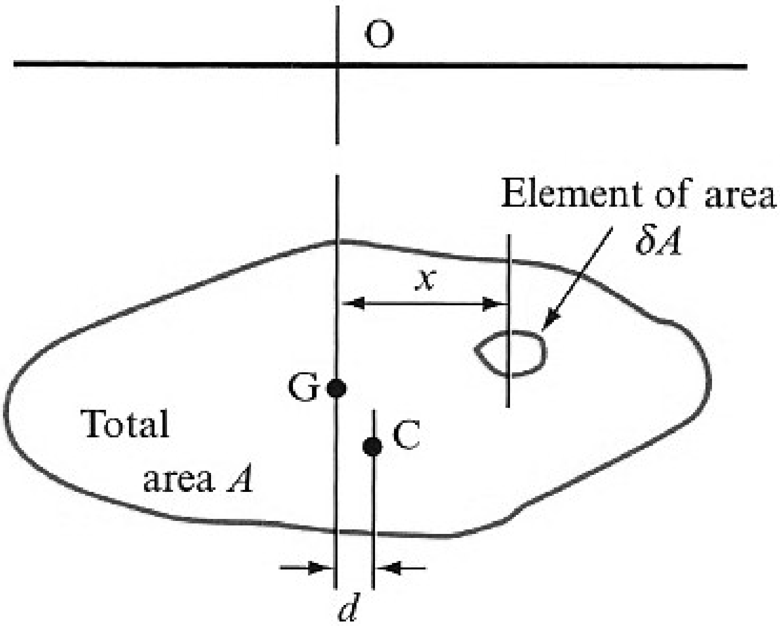
3.4 Fluid Action on Surfaces

Plane Surfaces

We consider a plane surface, *PQ*, of area *A*, totally immersed in a liquid of density ρ and inclined at an angle ϕ to the free surface:



Side Elevation



Front Elevation

If the plane area is symmetrical about the vertical axis *OG*, then $d = 0$. We will assume that this is normally the case.

Find Resultant Force:

The force acting on the small element of area, δA, is:

$$\delta R = p \cdot \delta A = \rho g y \cdot \delta A$$

The total force acting on the surface is the sum of all such small forces. We can integrate to get the force on the entire area, but remember that *y* is not constant:

$$\begin{aligned} R &= \int \rho g y \cdot \delta A \\ &= \rho g \int y \cdot \delta A \end{aligned}$$

But $\int y \cdot \delta A$ is just the first moment of area about the surface. Hence:

$$R = \rho g A \bar{y}$$

Where $\bar{y}$ is the distance to the centroid of the area (point *G*) from the surface.

Vertical Point Where Resultant Acts:

The resultant force acts perpendicular to the plane and so makes an angle $90° - \phi$ to the horizontal. It also acts through point *C*, the centre of pressure, a distance *D* below the free surface. To determine the location of this point we know:

$$\text{Moment of } R \text{ about } O \quad = \quad \begin{array}{l} \text{Sum of moments of forces} \\ \text{on all elements about } O \end{array}$$

Examining a small element first, and since $y = s\sin\phi$, the moment is:

$$\text{Moment of } \delta R \text{ about } O = \left[\rho g\left(s\sin\phi\right)\cdot\delta A\right]s$$
$$= \rho g\sin\phi\left(s^2\cdot\delta A\right)$$

In which the constants are taken outside the bracket. The total moment is thus:

$$\text{Moment of } R \text{ about } O = \rho g\sin\phi\cdot\int s^2\cdot\delta A$$

But $\int s^2\cdot\delta A$ is the second moment of area about point *O* or just I_O. Hence we have:

$$\begin{aligned}
\text{Moment of } R \text{ about } O &= \rho g\sin\phi\cdot I_O \\
\rho g A\bar{y}\times OC &= \rho g\sin\phi\cdot I_O \\
A\bar{y}\times\frac{D}{\sin\phi} &= \sin\phi\cdot I_O \\
D &= \frac{I_O}{A\bar{y}}\cdot\sin^2\phi
\end{aligned}$$

If we introduce the parallel axis theorem:

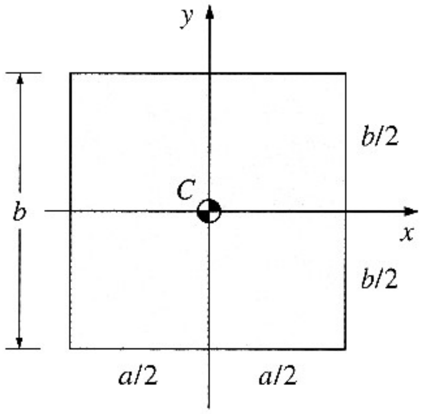
$$\begin{aligned}
I_O &= I_G + A\times\left(OG\right)^2 \\
&= I_G + A\cdot\left(\frac{\bar{y}}{\sin\phi}\right)^2
\end{aligned}$$

Hence we have:

$$\begin{aligned}
D &= \frac{I_G + A\bar{y}^2}{A\bar{y}}\cdot\frac{\sin^2\phi}{\sin^2\phi} \\
&= \bar{y} + \frac{I_G}{A\bar{y}}
\end{aligned}$$

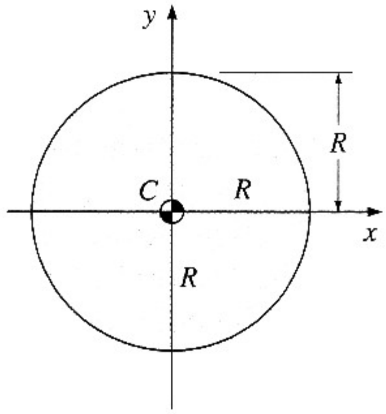
Hence, the centre of pressure, point *C*, always lies below the centroid of the area, *G*.

Plane Surface Properties



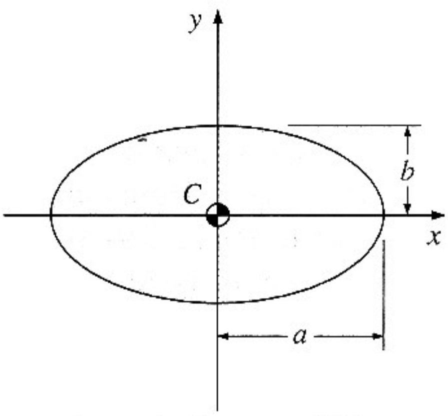
$A = ab,\ I_{xx,\,C} = ab^3/12$

(*a*) Rectangle



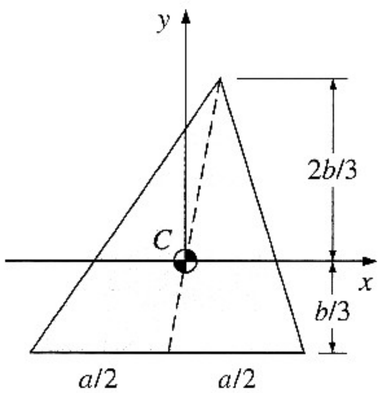
$A = \pi R^2,\ I_{xx,\,C} = \pi R^4/4$

(*b*) Circle



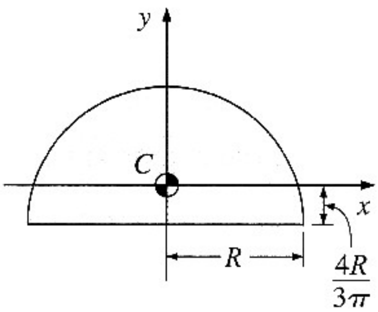
$A = \pi ab,\ I_{xx,\,C} = \pi ab^3/4$

(*c*) Ellipse



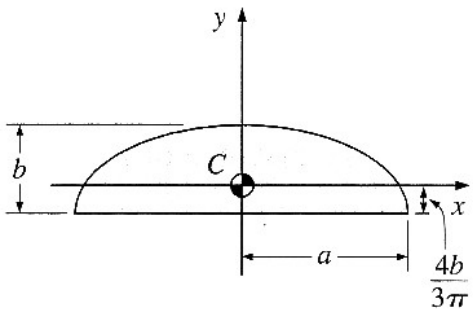
$A = ab/2,\ I_{xx,\,C} = ab^3/36$

(*d*) Triangle



$A = \pi R^2/2,\ I_{xx,\,C} = 0.109757R^4$

(*e*) Semicircle



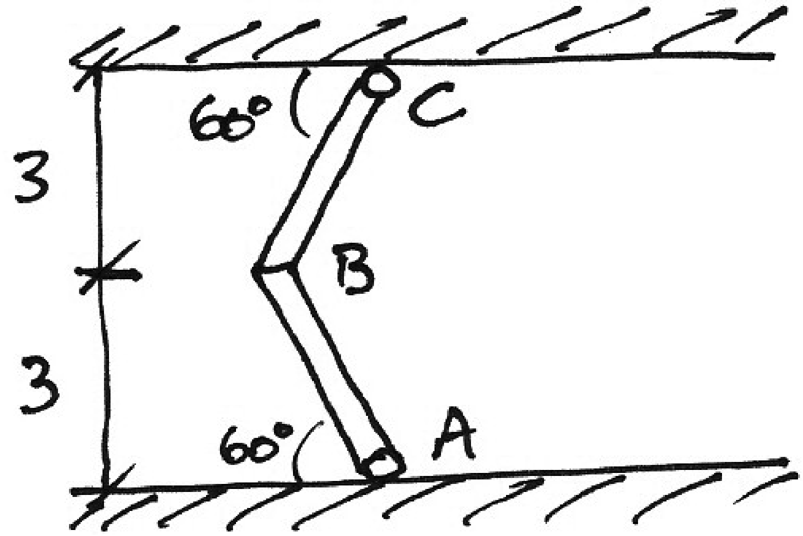
$A = \pi ab/2,\ I_{xx,\,C} = 0.109757ab^3$

(*f*) Semiellipse

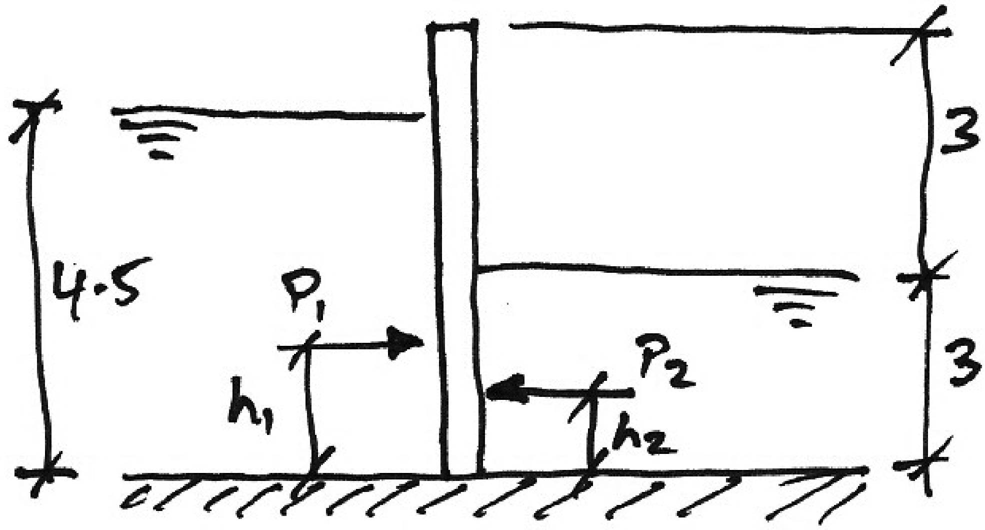
Plane Surfaces – Example

Problem

Calculate the forces on the hinges supporting the canal gates as shown. The hinges are located 0.6 m from the top and bottom of each gate.



Plan



Elevation

Solution

We will consider gate *AB*, but all arguments will equally apply to gate *BC*.

The length of the gate is $L = 3.0/\sin 30^\circ = 3.464\text{ m}$. The resultant pressure on the gate from the high water side is:

$$\begin{aligned} P_1 &= \rho g A_1 \bar{y}_1 \\ &= 10^3 \times 9.81 \times (3.464 \times 4.5) \times \frac{4.5}{2} \\ &= 344\text{ kN} \end{aligned}$$

Similarly for the low water side:

$$\begin{aligned} P_2 &= \rho g A_2 \bar{y}_2 \\ &= 10^3 \times 9.81 \times (3.464 \times 3.0) \times \frac{3.0}{2} \\ &= 153\text{ kN} \end{aligned}$$

The net resultant force on the gate is:

$$P = P_1 - P_2 = 344 - 153 = 191\text{ kN}$$

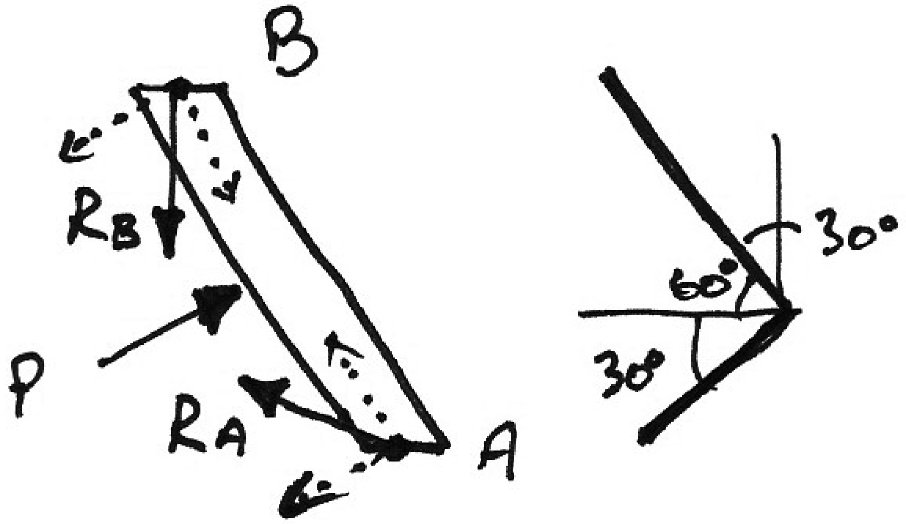
To find the height at which this acts, take moments about the bottom of the gate:

$$\begin{aligned} Ph &= P_1 h_1 + P_2 h_2 \\ &= 344 \times \frac{4.5}{3} - 153 \times \frac{3}{3} = 363\text{ kNm} \end{aligned}$$

Hence:

$$h = \frac{363}{191} = 1.900\text{ m}$$

Examining a free-body diagram of the gate, we see that the interaction force between the gates, R_B, is shown along with the total hinge reactions, R_A and the net applied hydrostatic force, *P*. Relevant angles are also shown. We make one assumption: the interaction force between the gates acts perpendicular on the contact surface between the gates. Hence R_B acts vertically downwards on plan.



From statics we have $\sum$Moments about $A = 0$:

$$P \cdot \frac{L}{2} + \left(R_B \sin 30^\circ\right) L = 0$$

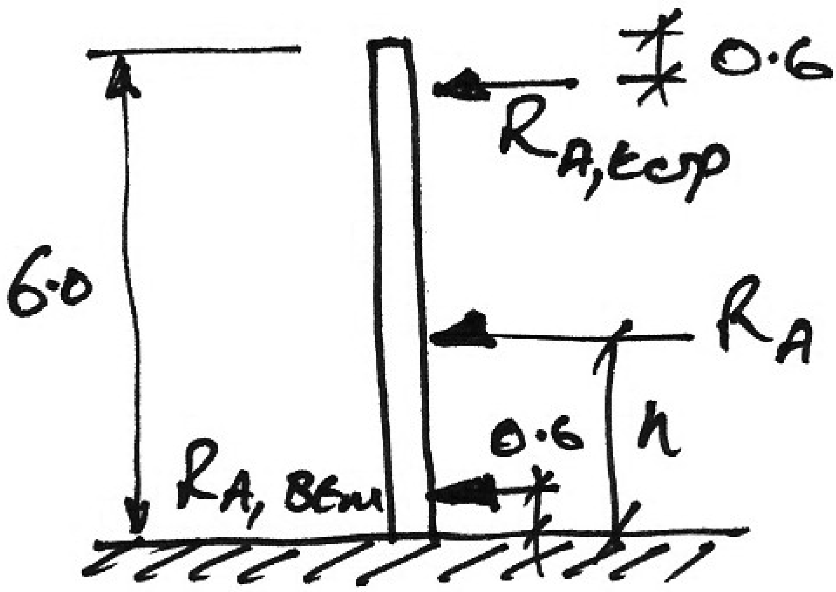
$$R_B \cdot \frac{1}{2} = \frac{P}{2}$$

$$R_B = P$$

Hence $R_B = 191$ kN and the component of R_B perpendicular to the gate is 95.5 kN.

By the sum of forces perpendicular to the gate, the component of R_A perpendicular to the gate must also equal 95.5 kN. Further, taking the sum of forces along the gate, the components of both R_A and R_B must balance and so $R_A = R_B = 191$ kN.

The resultant forces R_A and R_B must act at the same height as P in order to have static equilibrium. To find the force on each hinge at A, consider the following figure:



Taking moments about the bottom hinge:

$$R_A(h-0.6)-R_{A,top}(6-0.6-0.6)=0$$
$$R_{A,top}=\frac{191(1.900-0.6)}{4.8}=51.7 \text{ kN}$$

And summing the horizontal forces:

$$R_A=R_{A,top}+R_{A,btm}$$
$$R_{A,btm}=191-51.7=139.3 \text{ kN}$$

It makes intuitive sense that the lower hinge has a larger force. To design the bolts connecting the hinge to the lock wall the direct tension and shear forces are required. Calculate these for the lower hinge.

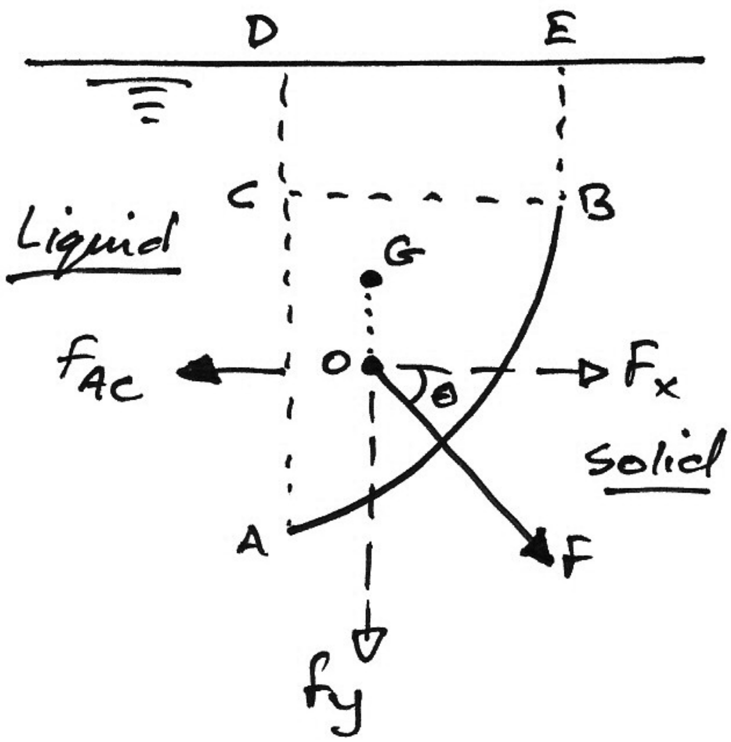
(Ans. T = 120.6 kN, V = 69.7 kN)

Curved Surfaces

For curved surfaces the fluid pressure on the infinitesimal areas are not parallel and so must be combined vectorially. It is usual to consider the total horizontal and vertical force components of the resultant.

Surface Containing Liquid

Consider the surface AB which contains liquid as shown below:



- Horizontal Component

Using the imaginary plane *ACD* we can immediately see that the horizontal component of force on the surface must balance with the horizontal force F_{AC}. Hence:

$$F_x = \text{Force on projection of surface onto a vertical plane}$$

F_x must also act at the same level as F_{AC} and so it acts through the centre of pressure of the projected surface.

- Vertical Component

The vertical component of force on the surface must balance the weight of liquid above the surface. Hence:

$$F_y = \text{Weight of liquid directly above the surface}$$

Also, this component must act through the centre of gravity of the area *ABED*, shown as *G* on the diagram.

- Resultant

The resultant force is thus:

$$F = \sqrt{F_x^2 + F_y^2}$$

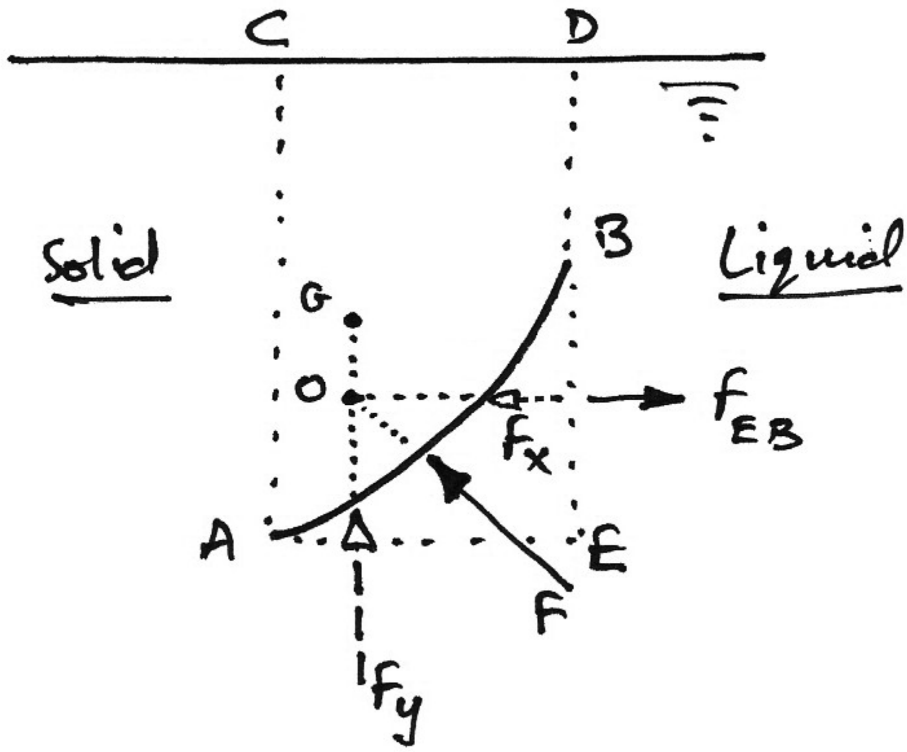
This force acts through the point *O* when the surface is uniform into the page, at an angle of:

$$\theta = \tan^{-1}\frac{F_y}{F_x}$$

to the horizontal. Depending on whether the surface contains or displaces water the angle is measured clockwise (contains) or anticlockwise (displaces) from the horizontal.

Surface Displacing Liquid

Consider the surface AB which displaces liquid as shown below:



- Horizontal Component

Similarly to the previous case, the horizontal component of force on the surface must balance with the horizontal force F_{EB}. Hence again:

$$F_x = \begin{array}{l}\text{Force on projection of surface} \\ \text{onto a vertical plane}\end{array}$$

This force also acts at the same level as F_{EB} as before.

- Vertical Component

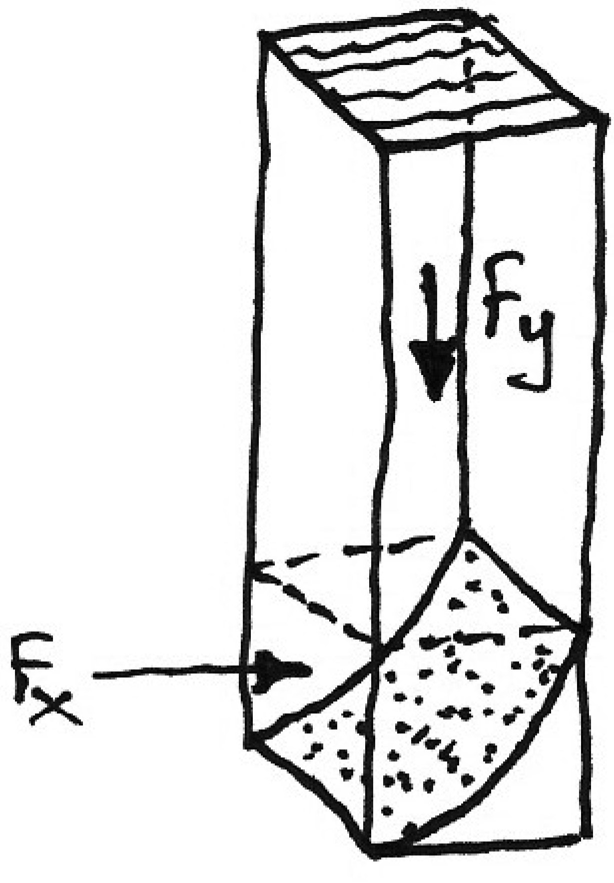
In this case we imagine that the area *ABDC* is filled with the same liquid. In this case F_y would balance the weight of the liquid in area *ABDC*. Hence:

$$F_y = \begin{array}{l}\text{Weight of liquid which}\\ \text{would lie above the surface}\end{array}$$

This component acts through the centre of gravity of the imaginary liquid in area *ABDC*, shown as *G* on the diagram.

The resultant force is calculated as before.

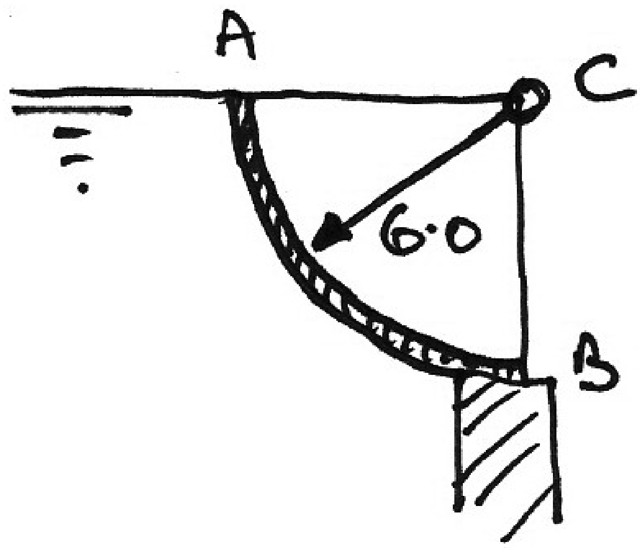
Both of these situations can be summed up with the following diagram:



Curved Surfaces – Example

Problem

Determine the resultant force and its direction on the gate shown:



Solution

The horizontal force, per metre run of the gate, is that of the surface projected onto a vertical plane of length *CB*:

$$F_x = \rho g A_{CB} \bar{y}_{CB}$$
$$= 10^3 \times 9.81 \times (6 \times 1) \times \left(\frac{6}{2}\right)$$
$$= 176.6 \text{ kN}$$

And this acts at a depth $h = \frac{2}{3} \cdot 6 = 4$ m from the surface. The vertical force is the weight of the imaginary water above *AB*:

$$F_y = 10^3 \times 9.81\left(\frac{\pi 6^2}{4} \times 1\right)$$
$$= 277.4 \text{ kN}$$

In which $\pi R^2/4$ is the area of the circle quadrant. The vertical force is located at:

$$x = \frac{4R}{3\pi} = \frac{4 \times 6}{3\pi} = 2.55 \text{ m}$$

to the left of line *BC*. The resultant force is thus:

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2} \\ &= \sqrt{176.6^2 + 277.4^2} \\ &= 328.8 \text{ kN} \end{aligned}$$

And acts at an angle:

$$\begin{aligned} \theta &= \tan^{-1}\frac{F_y}{F_x} \\ &= \tan^{-1}\frac{277.4}{176.6} \\ &= 57.5^\circ \end{aligned}$$

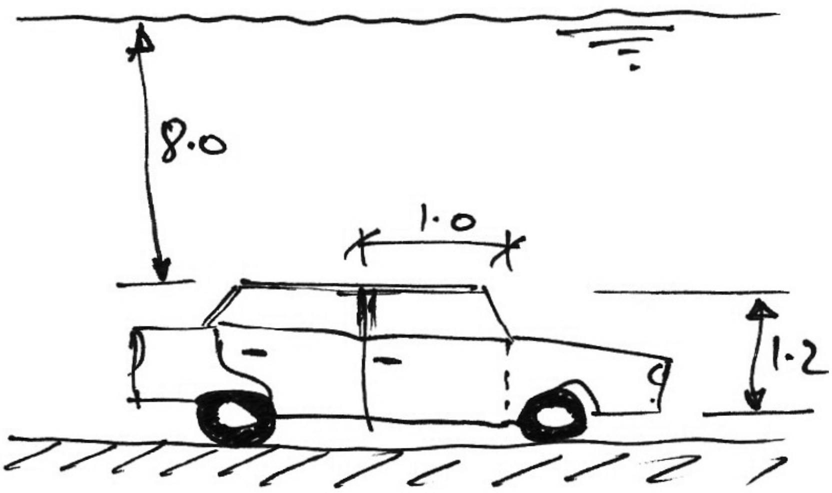
measured anticlockwise to the horizontal. The resultant passes through point *C*. Also, as the force on each infinitesimal length of the surface passes through *C*, there should be no net moment about *C*. Checking this:

$$\begin{aligned} \sum \text{Moments about } C &= 0 \\ 176.6 \times 4 - 277.4 \times 2.55 &= 0 \\ 706.4 - 707.4 &\approx 0 \end{aligned}$$

The error is due to rounding carried out through the calculation.

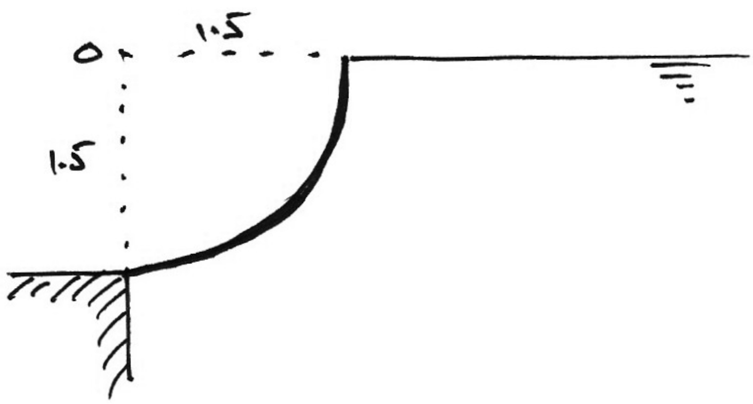
Problems – Fluid Action on Surfaces

1. You are in a car that falls into a lake to a depth as shown below. What is the moment about the hinges of the car door (1.0 × 1.2 m) due to the hydrostatic pressure? Can you open the door? What should you do?



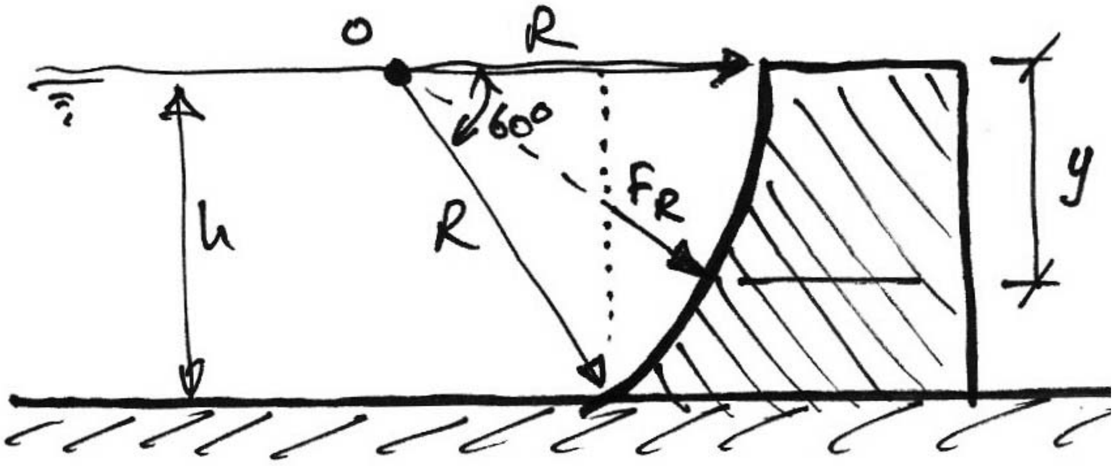
(Ans. 50.6 kNm, ?, ?)

2. A sluice gate consist of a quadrant of a circle of radius 1.5 m pivoted at its centre, O. When the water is level with the gate, calculate the magnitude and direction of the resultant hydrostatic force on the gate and the moment required to open the gate. The width of the gate is 3 m and it has a mass of 6 tonnes.



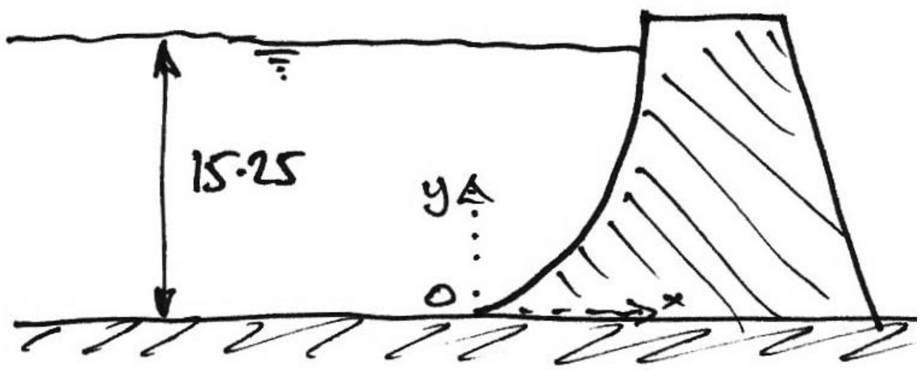
(Ans. 61.6 kN, 57˚, 35.3 kNm)

3. The profile of a masonry dam is an arc of a circle, the arc having a radius of 30 m and subtending an angle of 60° at the centre of curvature which lies in the water surface. Determine: (a) the load on the dam in kN/m length; (b) the position of the line of action to this pressure.



(Ans. 4280 kN/m, 19.0 m)

4. The face of a dam is curved according to the relation $y = x^2/2.4$ where y and x are in meters, as shown in the diagram. Calculate the resultant force on each metre run of the dam. Determine the position at which the line of action of the resultant force passes through the bottom of the dam.



(Ans. 1920 kN, 14.15 m)

4. Hydrodynamics: Basics

4.1 General Concepts

Introduction

Hydrostatics involves only a few variables: ρ, g, and h, and so the equations developed are relatively simple and experiment and theory closely agree. The study of fluids in motion is not as simple and accurate. The main difficulty is viscosity.

By neglecting viscosity (an ideal fluid), we do not account for the shear forces which oppose flow. Based on this, reasonably accurate and simple theories can be derived.. Using experimental results, these theories can then be calibrated by using experimental coefficients. They then inherently allow for viscosity.

As we will be dealing with liquids, we will neglect the compressibility of the liquid. This is incompressible flow. This is not a valid assumption for gases.

Classification of Flow Pattern

There are different patterns of fluid flow, usually characterized by time and distance:

- Time: A flow is *steady* if the parameters describing it (e.g. flow rate, velocity, pressure, etc.) do not change with time. Otherwise a flow is *unsteady*.

- Distance: A flow is *uniform* if the parameters describing the flow do not change with distance. In *non-uniform* flow, the parameters change from point to point along the flow.

From these definitions almost all flows will be one of:

Steady uniform flow

Discharge (i.e. flow rate, or volume per unit time) is constant with time and the cross section of the flow is also constant. Constant flow through a long straight prismatic pipe is an example.

Steady non-uniform flow

The discharge is constant with time, but the cross-section of flow changes. An example is a river with constant discharge, as the cross section of a river changes from point to point.

Unsteady uniform flow

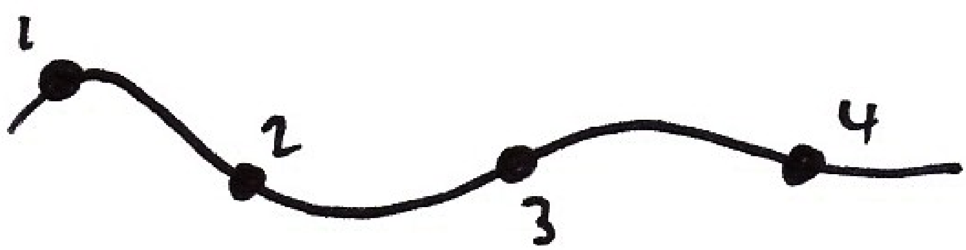
The cross-section is constant but the discharge changes with time resulting in complex flow patterns. A pressure surge in a long straight prismatic pipe is an example.

Unsteady non-uniform flow

Both discharge and cross section vary. A flood wave in a river valley is an example. This is the most complex type of flow.

Visualization

To picture the motion of a fluid, we start by examining the motion of a single fluid 'particle' over time, or a collection of particles at one instant. This is the flow path of the particle(s), or a *streamline*:

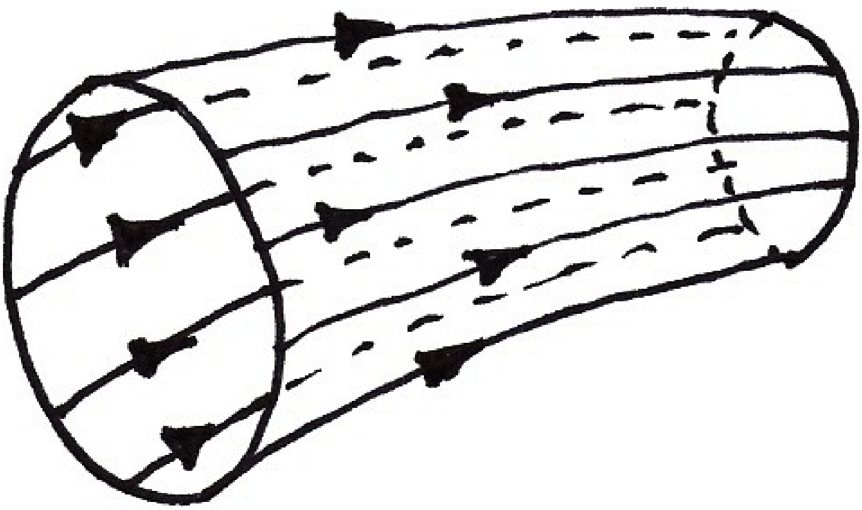


At each point, each particle has both velocity and acceleration vectors:

A streamline is thus tangential to the velocity vectors of the particles. Hence:

- there can be no flow across a streamline;
- therefore, streamlines cannot cross each other, and;
- once fluid is on a streamline it cannot leave it.

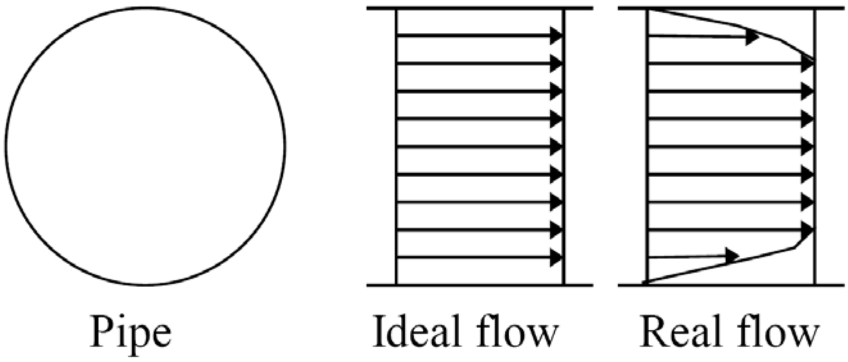
We extend this idea to a collection of paths of fluid particles to create a *streamtube*:



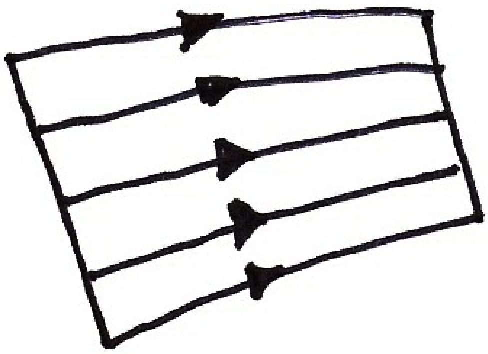
Streamlines and streamtubes are theoretical notions. In an experiment, a streakline is formed by injecting dye into a fluid in motion. A streakline therefore approximates a streamline (but is bigger because it is not an individual particle).

Dimension of Flow

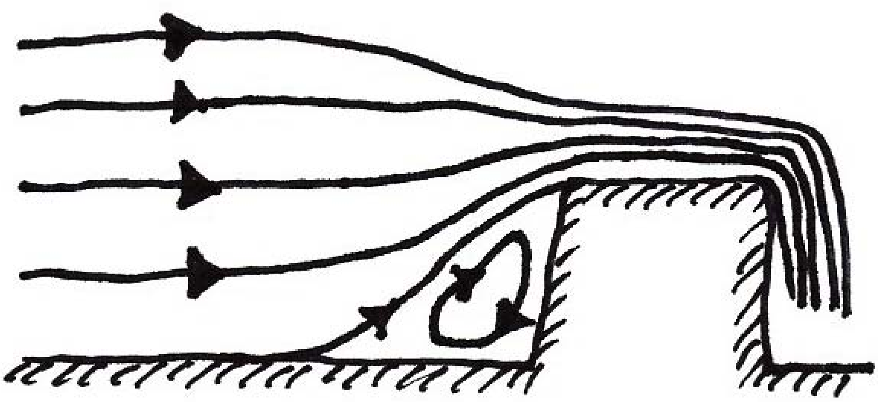
Fluid flow is in general three-dimensional in nature. Parameters of the flow can vary in the *x*, *y* and *z* directions. They can also vary with time. In practice we can reduce problems to one- or two-dimensional flow to simplify. For example:



One dimensional flow



A two-dimensional streamtube



Flow over an obstruction

Fundamental Equations

To develop equations describing fluid flow, we will work from relevant fundamental physical laws.

The Law of Conservation of Matter

Matter cannot be created nor destroyed (except in a nuclear reaction), but may be transformed by chemical reaction. In fluids we neglect chemical reactions and so we deal with the conservation of mass.

The Law of Conservation of Energy

Energy cannot be created nor destroyed, it can only be transformed from one form to another. For example, the potential energy of water in a dam is transformed to kinetic energy of water in a pipe. Though we will later talk of 'energy losses', this is a misnomer as none is actually lost but transformed to heat and other forms.

The Law of Conservation of Momentum

A body in motion remains in motion unless some external force acts upon it. This is Newton's Second Law:

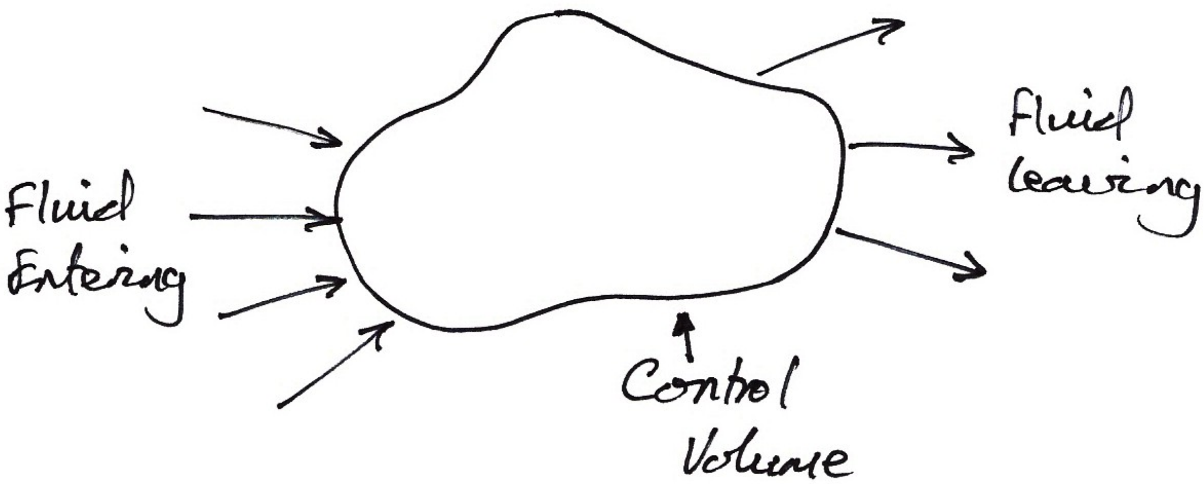
$$\text{Force} = \text{Rate of change of momentum}$$

$$\begin{aligned} F &= \frac{d(mv)}{dt} \\ &= m\frac{dv}{dt} \\ &= ma \end{aligned}$$

To apply these laws to fluids poses a problem, since fluid is a continuum, unlike rigid bodies. Hence we use the idea of a 'control volume'.

Control Volume

A control volume is an imaginary region within a body of flowing fluid, usually at fixed location and of a fixed size:



It can be of any size and shape so we choose shapes amenable to simple calculations. Inside the region all forces cancel out, and we can concentrate on external forces. It can be picture as a transparent pipe or tube, for example.

4.2 The Continuity Equation

Development

Applying the Law of Conservation of Mass to a control volume, we see:

$$\begin{array}{l} \text{Rate of mass} \\ \text{entering} \end{array} = \begin{array}{l} \text{Rate of mass} \\ \text{leaving} \end{array} + \begin{array}{l} \text{Rate of mass} \\ \text{increase} \end{array}$$

For steady incompressible flow, the rate of mass increase is zero and the density of the fluid does not change. Hence:

$$\begin{array}{l} \text{Rate of mass} \\ \text{entering} \end{array} = \begin{array}{l} \text{Rate of mass} \\ \text{leaving} \end{array}$$

The rate of mass change can be expressed as:

$$\begin{array}{l} \text{Rate of mass} \\ \text{change} \end{array} = \begin{array}{l} \text{Fluid} \\ \text{density} \end{array} \times \begin{array}{l} \text{Volume} \\ \text{per second} \end{array}$$

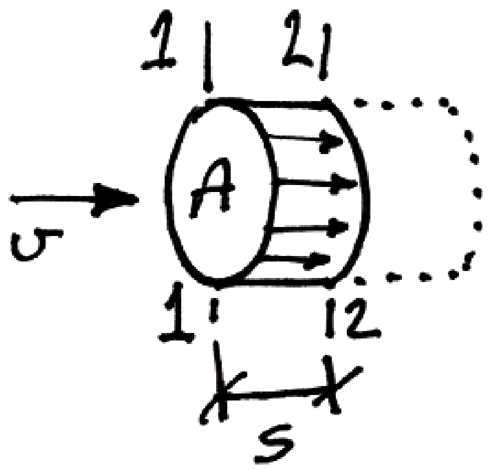
Using Q for flow rate, or volume per second (units: m^3/s, dimensions: L^3T^{-1}):

$$\rho Q_{in} = \rho Q_{out}$$

And as before, assuming that the flow is incompressible:

$$Q_{in} = Q_{out}$$

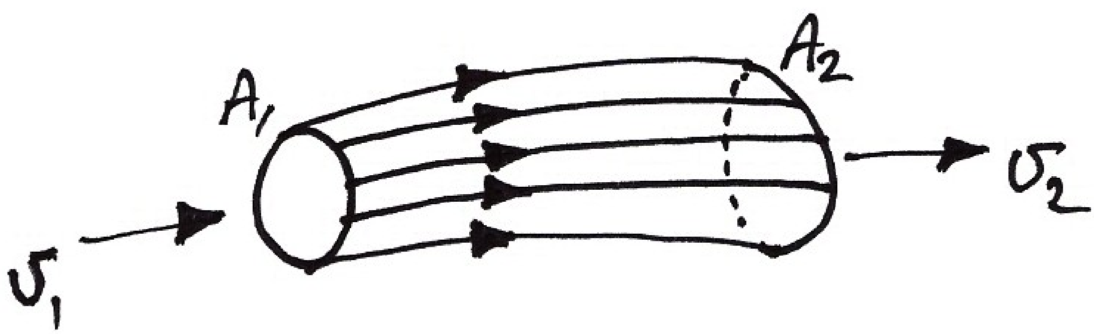
Consider a small length of streamtube:



The fluid at 1-1 moves a distance of $s = vt$ to 2-2. Therefore in 1 second it moves a distance of v. The volume moving per second is thus:

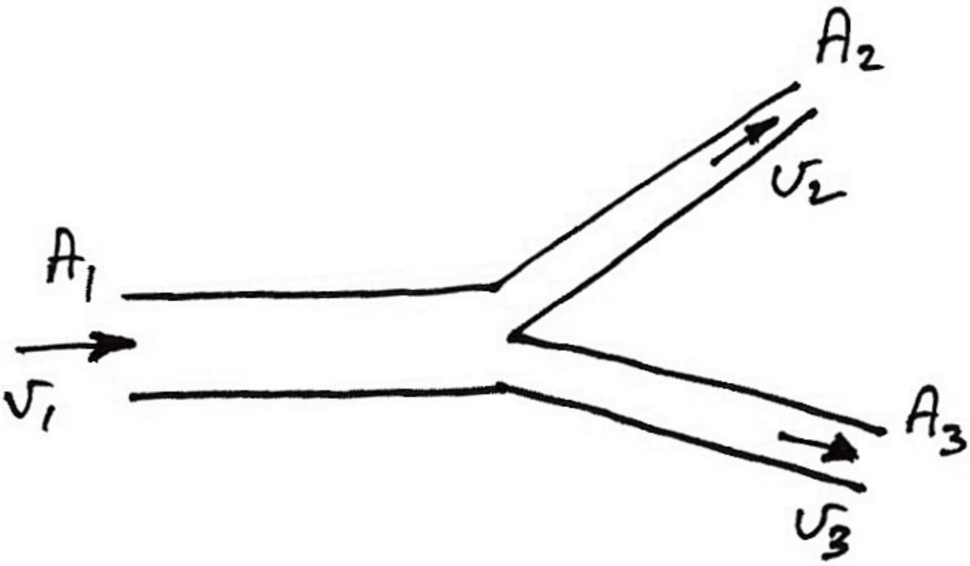
$$Q = Av$$

Thus, for an arbitrary streamtube, as shown, we have:



$$\boxed{A_1 v_1 = A_2 v_2}$$

A typical application of mass conservation is at pipe junctions:



From mass conservation we have:

$$Q_1 = Q_2 + Q_3$$
$$A_1 v_1 = A_2 v_2 + A_3 v_3$$

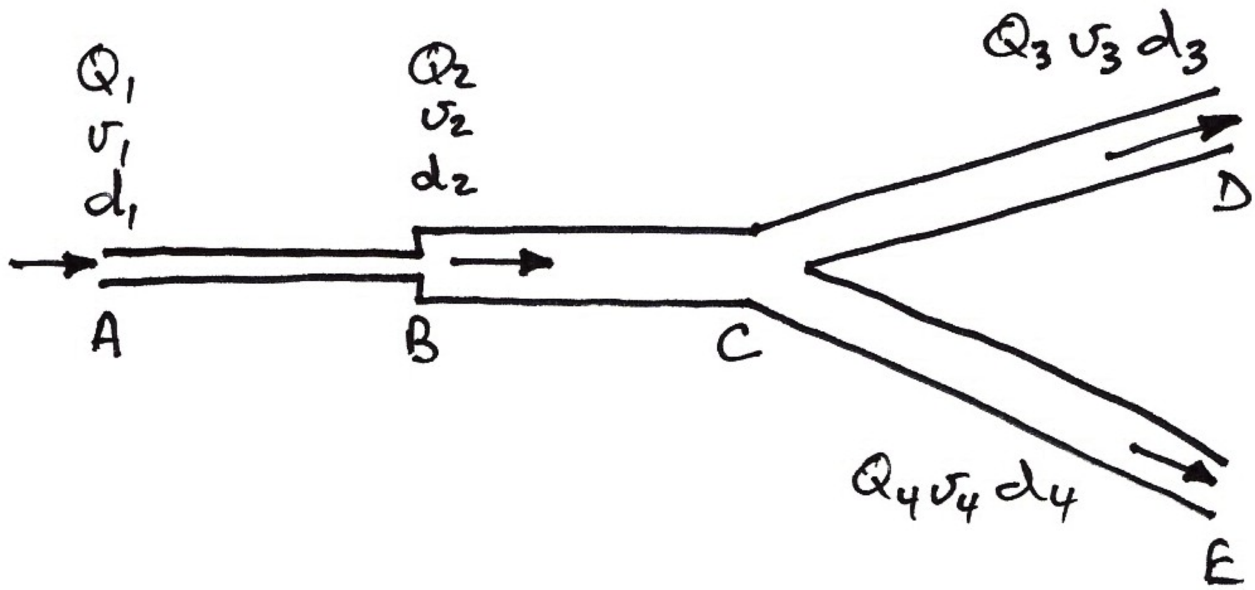
If we consider inflow to be positive and outflow negative, we have:

$$\sum_{i=1}^{\text{No. of Nodes}} A_i v_i = 0$$

Mass Conservation – Example

Problem

Water flows from point *A* to points *D* and *E* as shown. Some of the flow parameters are known, as shown in the table. Determine the unknown parameters.



Section	Diameter (mm)	Flow Rate (m^3/s)	Velocity (m/s)
AB	300	?	?
BC	600	?	1.2
CD	?	$Q_3 = 2Q_4$	1.4
CE	150	$Q_4 = 0.5Q_3$	?

Solution

From the law of mass conservation we can see:

$$Q_1 = Q_2$$

And as total inflow must equal total outflow:

$$\begin{aligned} Q_1 &= Q_{out} \\ &= Q_3 + Q_4 \\ &= Q_3 + 0.5Q_3 \\ &= 1.5Q_3 \end{aligned}$$

We must also work out the areas of the pipes, $A_i = \frac{\pi d_i^2}{4}$. Hence:

$A_1 = 0.0707 \text{ m}^3$ $A_2 = 0.2827 \text{ m}^3$ $A_4 = 0.0177 \text{ m}^3$

Starting with our basic equation, $Q = Av$, we can only solve for Q_2 from the table:

$$\begin{aligned} Q_2 &= (0.2827)(1.2) \\ &= 0.3393 \text{ m}^3/s \end{aligned}$$

We know that $Q_1 = Q_2$ and so we can now calculate Q_3 from previous:

$$\begin{aligned} Q_1 &= 1.5Q_3 \\ Q_3 &= \frac{Q_1}{1.5} = \frac{0.3393}{1.5} = 0.2262 \text{ m}^3\text{/s} \end{aligned}$$

And so,

$$Q_4 = \frac{Q_3}{2} = \frac{0.2262}{2} = 0.1131 \text{ m}^3/\text{s}$$

Thus we have all the flows. The unknown velocities are:

$$v_1 = \frac{Q_1}{A_1} = \frac{0.3393}{0.0707} = 4.8 \text{ m/s}$$

$$v_4 = \frac{Q_4}{A_4} = \frac{0.1131}{0.0177} = 6.4 \text{ m/s}$$

And lastly, the diameter of pipe *CD* is:

$$A_3 = \frac{Q_3}{v_3} = \frac{0.2262}{1.4} = 0.1616 \text{ m}^2$$

$$d_3 = \sqrt{\frac{4A_3}{\pi}} = 0.454 \text{ m}$$

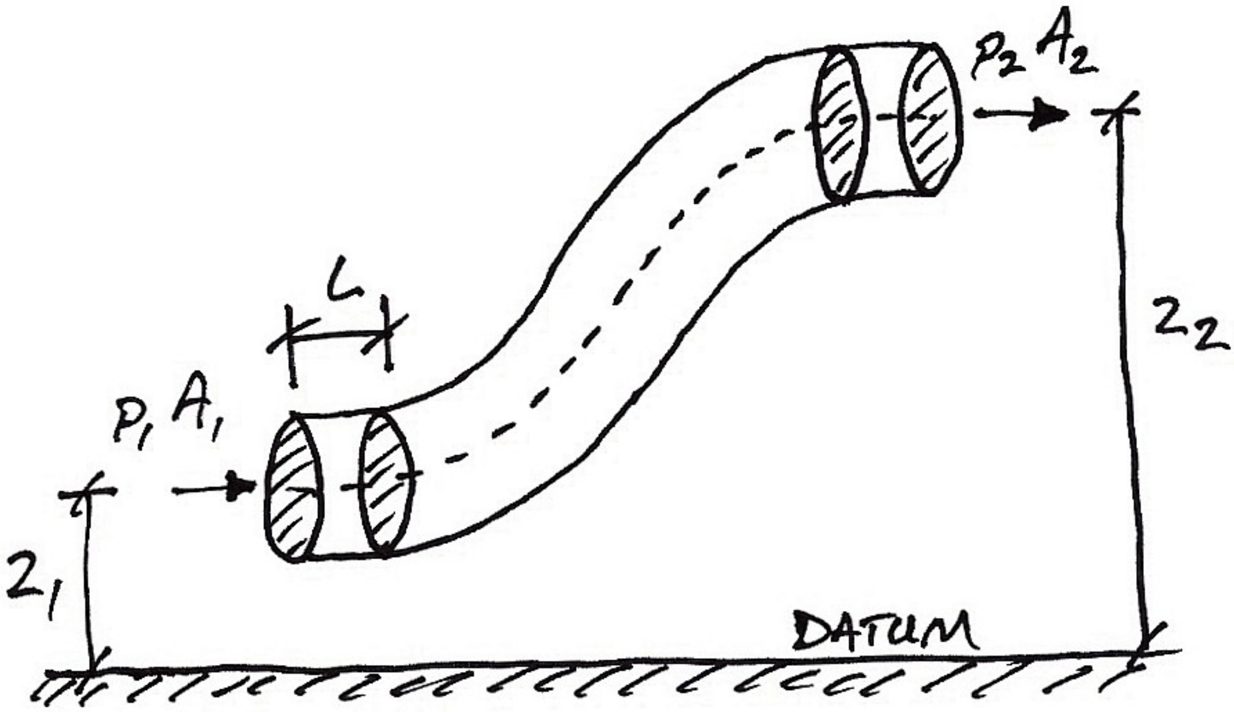
And so it is likely to be a 450 mm ∅ pipe.

Note that in a problem such as this the individual calculations do not pose a problem. It is the strategy used to solve it that is key. In this problem, we started from some knowns and one equation. Even though we couldn't see all the way to the end from Step 1, with each new calculation another possibility opened up. This is the 'art of problem solving' and it can only be learned by practice!

4.3 The Energy Equation

Development

We apply the Law of Conservation of Energy to a control volume. To do so, we must identify the forms of energy in the control volume. Consider the following system:



The forms of energy in this system are:

- Pressure energy:
  The pressure in a fluid also does work by generating force on a cross section which then moves through a distance. This is energy since work is energy.
- Kinetic energy:
  This is due to the motion of the mass of fluid.
- Potential energy:
  This is due to the height above an arbitrary datum.

Pressure Energy

The combination of flow and pressure gives us work. The pressure results in a force on the cross section which moves through a distance L in time δt. Hence the pressure energy is the work done on a mass of fluid entering the system, which is:

$$m = \rho_1 A_1 L$$

Be careful to distinguish the density ρ and the pressure *p*.

And so the pressure energy at the entry is:

$$\text{PrE} = pAL = p_1 A_1 L$$

Kinetic Energy

From classical physics, the kinetic energy of the mass entering is:

$$\text{KE} = \frac{1}{2} m v^2 = \frac{1}{2} \rho_1 A_1 L v_1^2$$

Potential Energy

The potential energy of the mass entering, due to the height above the datum is:

$$\text{PE} = mgz = \rho_1 A_1 L g z_1$$

Total Energy

The total energy at the entry to the system is just the sum:

$$H_1^* = p_1 A_1 L + \frac{1}{2} \rho_1 A_1 L v_1^2 + \rho_1 A_1 L g z_1$$

Final Form

It is more usual to consider the energy per unit weight, and so we divide through by $mg = \rho_1 g A_1 L$:

$$
\begin{aligned}
H_1 &= \frac{H_1^*}{\rho_1 g A_1 L} \\
&= \frac{p_1 A_1 L}{\rho_1 g A_1 L} + \frac{1}{2}\frac{\rho_1 A_1 L v_1^2}{\rho_1 g A_1 L} + \frac{\rho_1 A_1 L g z_1}{\rho_1 g A_1 L} \\
&= \frac{p_1}{\rho_1 g} + \frac{v_1^2}{2g} + z_1
\end{aligned}
$$

Similarly, the energy per unit weight leaving the system is:

$$H_2 = \frac{p_2}{\rho_2 g} + \frac{v_2^2}{2g} + z_2$$

Also, the energy entering must equal the energy leaving as we assume the energy cannot change. Also, assuming incompressibility, the density does not change:

$$
\begin{gathered}
H_1 = H_2 \\
\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2
\end{gathered}
$$

And so we have Bernoulli's Equation:

$$\boxed{\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 = H = \text{constant}}$$

Comments

From Bernoulli's Equation we note several important aspects:

1. It is assumed that there is no energy taken from or given to the fluid between the entry and exit. This implies the fluid is frictionless as friction generates heat energy which would be an energy loss. It also implies that there is no energy added, say by a pump for example.

2. Each term of the equation has dimensions of length, L, and units of metres. Therefore each term is known as a head:
   - Pressure head: $\frac{p}{\rho g}$;
   - Kinetic or velocity head: $\frac{v^2}{2g}$;
   - Potential or elevation head: z.

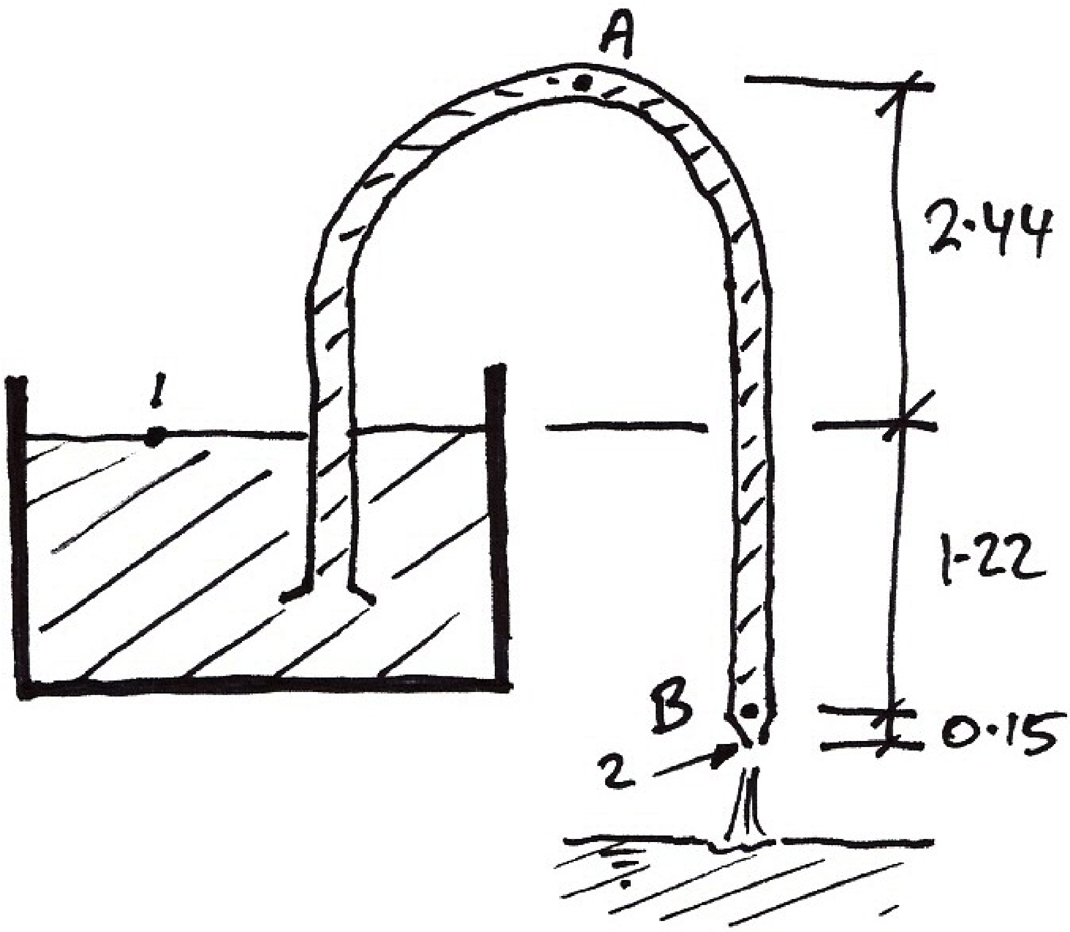
3. The streamtube must have very small dimensions compared to the heights above the datum. Otherwise the height to the top of a cross-section would be different to the height to the bottom of a cross-section. Therefore, Bernoulli's Equation strictly only applies to streamlines.

We have derived the equation from energy considerations. It can also be derived by force considerations upon an elemental piece of fluid.

Energy Equation – Example

Problem

For the siphon shown, determine the discharge and pressure heads at *A* and *B* given that the pipe diameter is 200 mm and the nozzle diameter is 150 mm. You may neglect friction in the pipe.



Solution

To find the discharge (or flow) apply Bernoulli's Equation along the streamline connecting points 1 and 2. To do this note:

- Both p_1 and p_2 are at atmospheric pressure and are taken to be zero;
- v_1 is essentially zero.

$$\left(\frac{p_1}{\rho g}\right)_{=0} + \left(\frac{v_1^2}{2g}\right)_{=0} + z_1 = \left(\frac{p_2}{\rho g}\right)_{=0} + \frac{v_2^2}{2g} + z_2$$

$$z_1 - z_2 = \frac{v_2^2}{2g}$$

Hence, from the figure:

$$1.22 + 0.15 = \frac{v_2^2}{2 \times 9.81}$$

$$v_2 = 5.18 \text{ m/s}$$

And using continuity:

$$\begin{aligned} Q_2 &= A_2 v_2 \\ &= \frac{\pi (0.15)^2}{4} \cdot 5.18 \\ &= 0.092 \text{ m}^3 / s \end{aligned}$$

For the pressure head at A, apply Bernoulli's equation from point 1 to A:

$$\left(\frac{p_1}{\rho g}\right)_{=0} + \left(\frac{v_1^2}{2g}\right)_{=0} + z_1 = \frac{p_A}{\rho g} + \frac{v_A^2}{2g} + z_A$$

Hence:

$$\frac{p_A}{\rho g} = \left(z_1 - z_A\right) - \frac{v_A^2}{2g}$$

Again using continuity between point 2 and *A* and the diameter of the pipe at *A*:

$$\begin{aligned} Q_A &= Q_2 \\ A_A v_A &= 0.092 \\ \frac{\pi\left(0.2\right)^2}{4} \cdot v_A &= 0.092 \\ v_A &= 2.93 \text{ m/s} \end{aligned}$$

Hence the kinetic head at *A* is just $\frac{v_A^2}{2g} = 0.44 \text{ m}$, and so:

$$\begin{aligned} \frac{p_A}{\rho g} &= -2.44 - 0.44 \\ &= -2.88 \text{ m} \end{aligned}$$

This is negative gauge pressure indicating suction. However, it is still a positive absolute pressure.

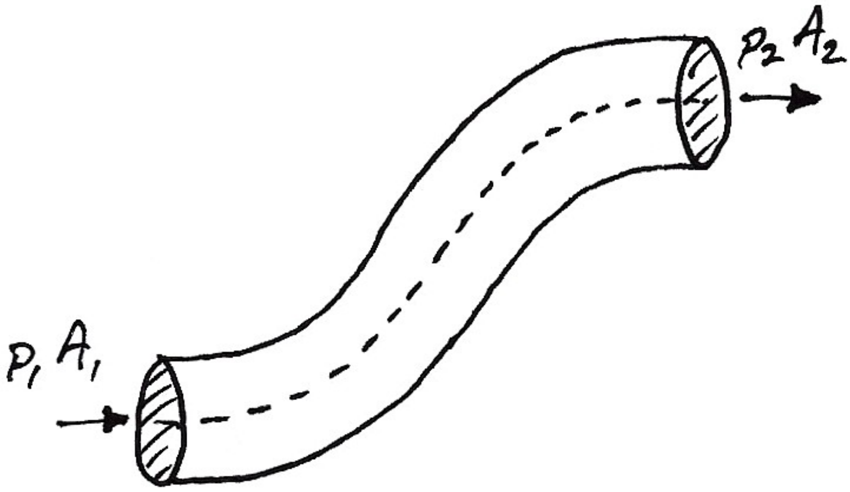
Similarly to *A*, at *B* we have, $v_B = v_A$ and $z_1 - z_B = 1.22 \text{ m}$ and so:

$$\begin{aligned} \frac{p_B}{\rho g} &= \left(z_1 - z_B\right) - \frac{v_B^2}{2g} \\ &= 1.22 - 0.44 \\ &= 0.78 \text{ m} \end{aligned}$$

4.4 The Momentum Equation

Development

We consider again a general streamtube:



In a given time interval, δt, we have:

$$\text{momentum entering} = \rho Q_1 \, \delta t \, v_1$$
$$\text{momentum leaving} = \rho Q_2 \, \delta t \, v_2$$

From continuity we know $Q = Q_1 = Q_2$. Thus the force required giving the change in momentum between the entry and exit is, from Newton's Second Law:

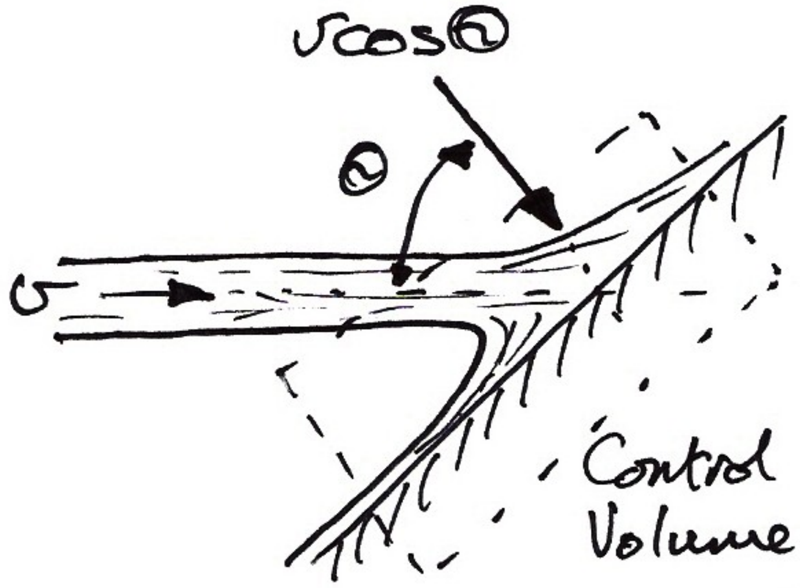
$$F = \frac{d(mv)}{dt}$$

$$F = \frac{\rho Q \, \delta t (v_2 - v_1)}{\delta t}$$
$$= \rho Q (v_2 - v_1)$$

This is the force acting on a fluid element in the direction of motion. The fluid exerts an equal but opposite reaction to its surroundings.

Application – Fluid Striking a Flat Surface

Consider the jet of fluid striking the surface as shown:



The velocity of the fluid normal to the surface is:

$$v_{normal} = v\cos\theta$$

This must be zero since there is no relative motion at the surface. This then is also the change in velocity that occurs normal to the surface. Also, the mass flow entering the control volume is:

$$\rho Q = \rho A v$$

Hence:

$$F = \frac{d(mv)}{dt}$$
$$= (\rho A v)(v \cos\theta)$$
$$= \rho A v^2 \cos\theta$$

And if the plate is perpendicular to the flow then:

$$F = \rho A v^2$$

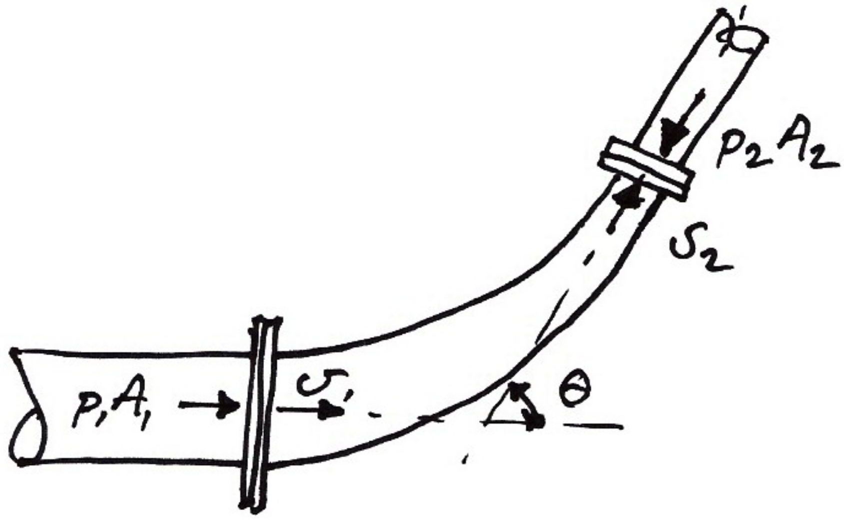
Notice that the force exerted by the fluid on the surface is proportional to the velocity squared. This is important for wind loading on buildings. For example, the old wind loading code CP3: Chapter V gives as the pressure exerted by wind as:

$$q = 0.613 v_s^2 \qquad (\text{N/m}^2)$$

In which v_s is the design wind speed read from maps and modified to take account of relevant factors such as location and surroundings.

Application – Flow around a bend in a pipe

Consider the flow around the bend shown below. We neglect changes in elevation and consider the control volume as the fluid between the two pipe joins.



The net external force on the control volume fluid in the *x*-direction is:

$$p_1 A_1 - p_2 A_2 \cos\theta + F_x$$

In which F_x is the force on the fluid by the pipe bend (making it 'go around the corner'). The above net force must be equal to the change in momentum, which is:

$$\rho Q\left(v_2 \cos\theta - v_1\right)$$

Hence:

$$\begin{aligned} p_1 A_1 - p_2 A_2 \cos\theta + F_x &= \rho Q\left(v_2 \cos\theta - v_1\right) \\ F_x &= \rho Q\left(v_2 \cos\theta - v_1\right) - p_1 A_1 + p_2 A_2 \cos\theta \\ &= \left(\rho Q v_2 + p_2 A_2\right)\cos\theta - \left(\rho Q v_1 + p_1 A_1\right) \end{aligned}$$

Similarly, for the *y*-direction we have:

$$\begin{aligned} -p_2 A_2 \sin\theta + F_y &= \rho Q\left(v_2 \sin\theta - 0\right) \\ F_y &= \rho Q\left(v_2 \sin\theta - 0\right) + p_2 A_2 \sin\theta \\ &= \left(\rho Q v_2 + p_2 A_2\right)\sin\theta \end{aligned}$$

The resultant is:

$$F = \sqrt{F_x^2 + F_y^2}$$

And which acts at an angle of:

$$\theta = \tan^{-1}\frac{F_y}{F_x}$$

This is the force and direction of the bend on the fluid. The bend itself must then be supported for this force. In practice a manhole is built at a bend, or else a thrust block is used to support the pipe bend.

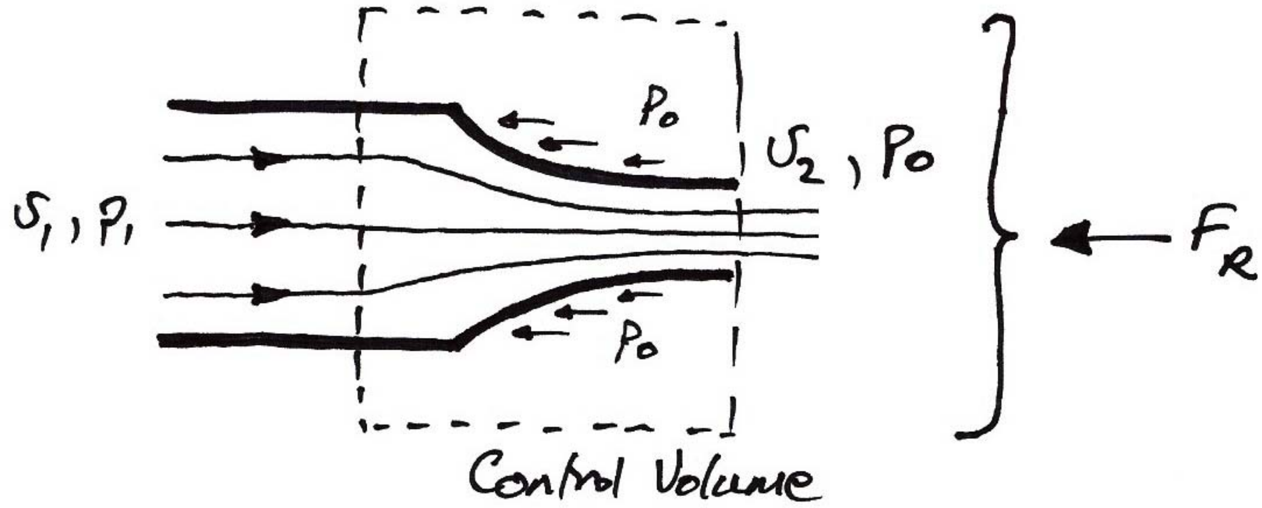
Application – Force exerted by a firehose

Problem

A firehose discharges 5 l/s. The nozzle inlet and outlet diameters are 75 and 25 mm respectively. Calculate the force required to hold the hose in place.

Solution

The control volume is taken as shown:



There are three forces in the *x*-direction:

- The reaction force F_R provided by the fireman;
- Pressure forces F_P: p_1A_1 at the left side and p_0A_0 at the right hand side;
- The momentum force F_M caused by the change in velocity.

So we have:

$$F_M = F_P + F_R$$

The momentum force is:

$$F_M = \rho Q\left(v_2 - v_1\right)$$

Therefore, we need to establish the velocities from continuity:

$$\begin{aligned} v_1 &= \frac{Q}{A_1} = \frac{5\times10^{-3}}{\pi\left(0.075\right)^2/4} \\ &= 1.13 \text{ m/s} \end{aligned}$$

And

$$\begin{aligned} v_2 &= \frac{5\times10^{-3}}{\pi\left(0.025\right)^2/4} \\ &= 10.19 \text{ m/s} \end{aligned}$$

Hence:

$$\begin{aligned} F_M &= \rho Q\left(v_2 - v_1\right) \\ &= 10^3\left(5\times10^{-3}\right)\left(10.19 - 1.13\right) \\ &= 45 \text{ N} \end{aligned}$$

The pressure force is:

$$F_P = p_1 A_1 - p_0 A_0$$

If we consider gauge pressure only, the $p_0 = 0$ and we must only find p_1. Using Bernoulli's Equation between the left and right side of the control volume:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \left(\frac{p_0}{\rho g}\right)_{=0} + \frac{v_0^2}{2g}$$

Thus:

$$\begin{aligned} p_1 &= \left(\frac{\rho}{2}\right)\left(v_1^2 - v_0^2\right) \\ &= \left(\frac{10^3}{2}\right)\left(10.19^2 - 1.13^2\right) \\ &= 51.28 \text{ kN/m}^2 \end{aligned}$$

Hence

$$\begin{aligned} F_P &= p_1 A_1 - p_0 A_0 \\ &= \left(51.28 \times 10^3\right)\left(\frac{\pi\left(0.075\right)^2}{4}\right) - 0 \\ &= 226 \text{ N} \end{aligned}$$

Hence the reaction force is:

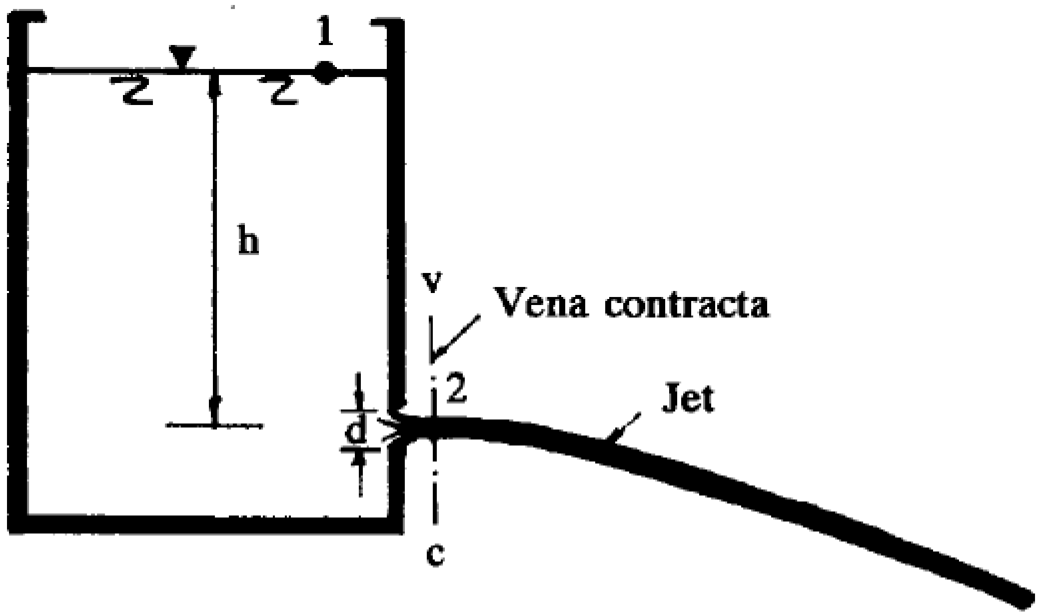
$$\begin{aligned} F_R &= F_M - F_P \\ &= 45 - 226 \\ &= -181 \text{ N} \end{aligned}$$

This is about a fifth of an average body weight – not inconsequential.

4.5 Modifications to the Basic Equations

Flow Measurement – Small Orifices

Consider the following tank discharge through a small opening below its surface:



If the head is practically constant across the diameter of the orifice ($h > d$) then, using the energy equation:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + h = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + 0$$

With both pressures atmospheric and taking $v_1 = 0$ we have:

$$h = \frac{v_2^2}{2g}$$

And so the velocity through the orifice is:

$$v_2 = \sqrt{2gh}$$

This is Torricelli's Theorem and represents the theoretical velocity through the orifice. Measured velocities never quite match this theoretical velocity and so we introduce a *coefficient of velocity*, C_v, to get:

$$v_{actual} = C_v \sqrt{2gh}$$

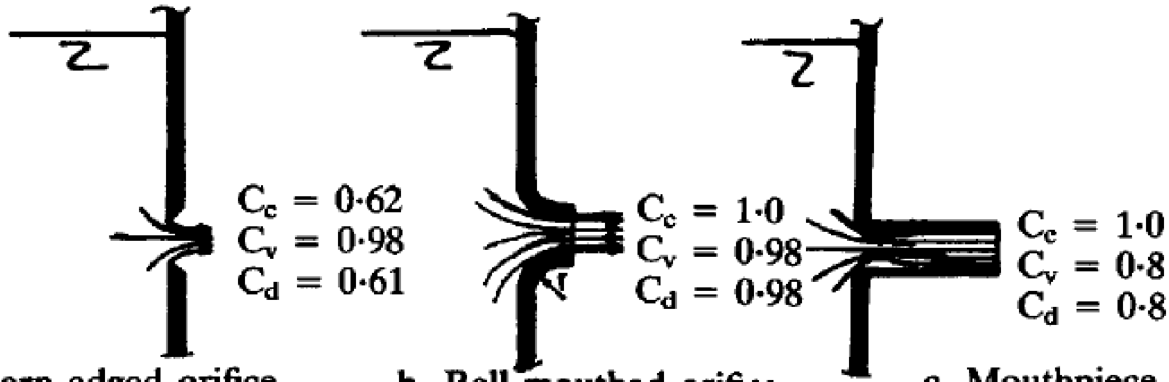
Also, due to viscosity the area of the jet may not be the same as that of the orifice and so we introduce a *coefficient of contraction*, C_c:

$$C_c = \frac{\text{Area of jet}}{\text{Area of orifice}}$$

Lastly, the discharge through the orifice is then:

$$\begin{aligned} Q &= Av \\ &= \left(C_c a\right)\left(C_v \sqrt{2gh}\right) \\ &= C_d a \sqrt{2gh} \end{aligned}$$

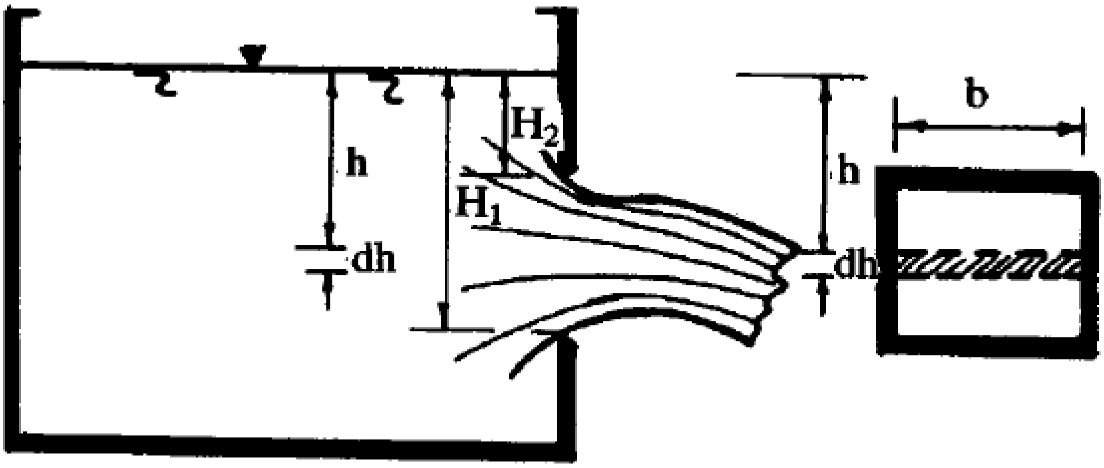
In which C_d is the coefficient of discharge and is equal to $C_c C_v$. For some typical orifices and mouthpieces values of the coefficient are:



a. Sharp edged orifice b. Bell mouthed orifice c. Mouthpiece

Flow Measurement – Large Orifices

When studying small orifices we assumed that the head was effectively constant across the orifice. With large openings this assumption is not valid. Consider the following opening:



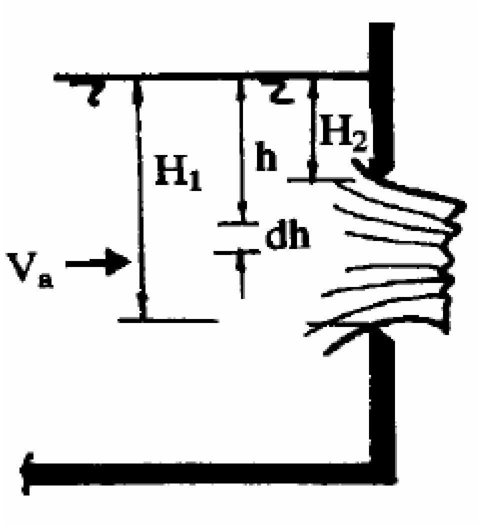
To proceed, we consider the infinitesimal rectangular strip of area $b \cdot dh$ at depth h. The velocity through this area is $\sqrt{2gh}$ and the infinitesimal discharge through it is:

$$dq = C_d \, b \, dh \sqrt{2gh}$$

Thus the total discharge through the opening is the sum of all such infinitesimal discharges:

$$\begin{aligned} Q &= \int dq \\ &= C_d b \sqrt{2g} \int_{H_2}^{H_1} \sqrt{h} \, dh \\ &= \frac{2}{3} C_d b \sqrt{2g} \left(H_1^{3/2} - H_2^{3/2} \right) \end{aligned}$$

Large openings are common in civil engineering hydraulics, for example in weirs. But in such cases the fluid has a velocity (V_a) approaching the large orifice:



Using the energy equation:

$$\frac{V_a^2}{2g}+h=\frac{v_{jet}^2}{2g}$$

Hence:

$$\begin{aligned} v_{jet} &= \sqrt{2gh+V_a^2} \\ &= \sqrt{2g\left(h+V_a^2/2g\right)} \end{aligned}$$

In which each term in the brackets is a head. Given the velocity we can find the discharge through the strip to be:

$$dq = C_d\ b\ dh\sqrt{2g\left(h+V_a^2/2g\right)}$$

And so the total discharge is:

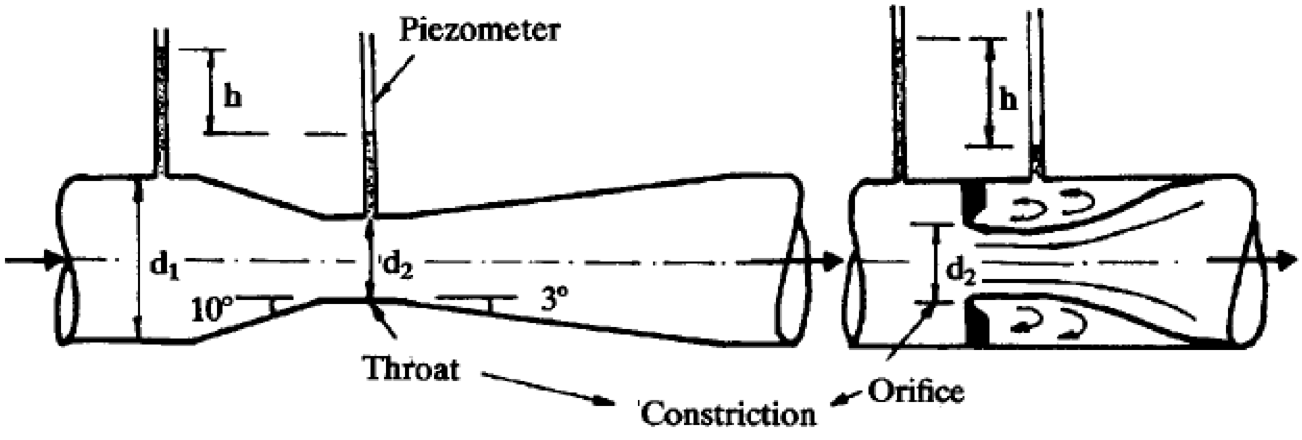
$$Q = \int dq$$

$$= C_d b \sqrt{2g} \int_{H_2}^{H_1} \sqrt{h + V_a^2/2g}\ dh$$

$$= \frac{2}{3} C_d b \sqrt{2g} \left[\left(H_1 + V_a^2/2g \right)^{3/2} - \left(H_2 + V_a^2/2g \right)^{3/2} \right]$$

Discharge Measurement in Pipelines

We consider two kinds of meters based on constricting the flow: the Venturi meter and the Orifice meter, as shown.



a. Venturi meter b. Orifice meter

The constriction in these meters causes a difference in pressure between points 1 and 2, and it is this pressure difference that enables the discharge to be measured. Applying the energy equation between the inlet and the constriction:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g}$$

Thus the difference in height in the piezometer is:

$$h = \frac{p_1 - p_2}{\rho g} = \frac{v_1^2 - v_2^2}{2g}$$

And from continuity $Q = A_1 v_1 = A_2 v_2$, and using $k = A_1 / A_2$ we get:

$$v_2^2 = \left(A_1^2 / A_2^2\right) v_1^2 = k^2 v_1^2$$

And also:

$$v_1^2 = 2gh + v_2^2$$

Which after substituting for v_2^2 and rearranging gives:

$$v_1 = \sqrt{\frac{2gh}{k^2 - 1}}$$

Hence the discharge is:

$$\begin{aligned} Q &= A_1 v_1 \\ &= A_1 \sqrt{\frac{2gh}{k^2 - 1}} \end{aligned}$$

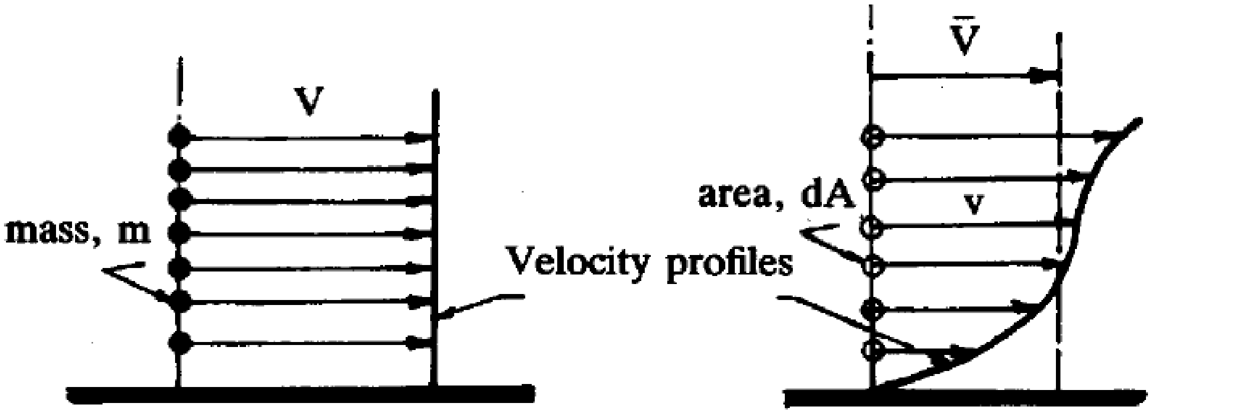
This equation neglects all losses. The actual discharge requires the introduction of the coefficient of discharge, C_d :

$$Q_{actual} = C_d A_1 \sqrt{\frac{2gh}{k^2 - 1}}$$

For properly designed Venturi meters, C_d is about 0.97 to 0.99 but for the Orifice meter it is much lower at about 0.65.

Velocity and Momentum Factors

The velocity and momentum terms in the energy and momentum equations (and in any resultant developments) have assumed uniform flow and thus require modification due to velocity variations:



a. Uniform distribution b. Nonuniform distribution

We require a factor that accounts for the real velocity profile and so we equate kinetic energies. For the true profile the mass passing through a small area is $\rho v\, dA$. Hence the kinetic energy passing through this small area is $\frac{1}{2}(\rho v\, dA)v^2$ and so the total energy is:

$$\frac{1}{2}\rho\int v^3 dA$$

With an imaginary uniform flow of the average velocity $\bar{V}$, the total energy is $\alpha\frac{1}{2}\rho\bar{V}^3 A$ in which α is the velocity correction factor. Hence:

$$\alpha\frac{1}{2}\rho\bar{V}^3 A = \frac{1}{2}\rho\int v^3 dA$$

And so

$$\alpha = \frac{1}{A}\int \left(\frac{v}{\overline{V}}\right)^3 dA$$

Usual values for α are 1.03 to 1.3 for turbulent flows and 2 for laminar flows.

The momentum correction factor follows a similar idea and is:

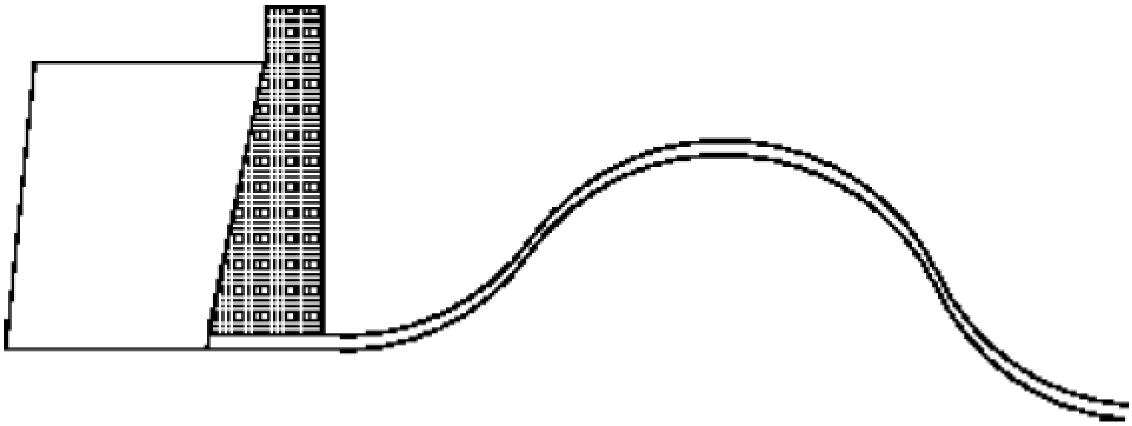
$$\beta = \frac{1}{A}\int \left(\frac{v}{\overline{V}}\right)^2 dA$$

Its values are lower than α.

Both correction factors are usually close to unity and are usually ignored, but this is not always the case.

Accounting for Energy Losses

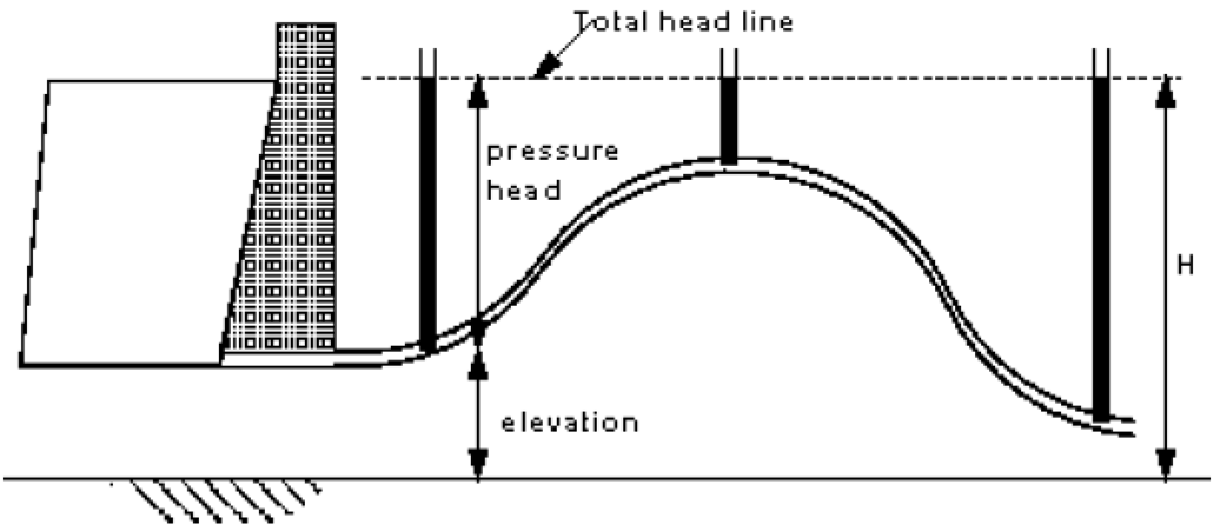
Consider the following reservoir and pipe system:



The energy equation gives us:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 = H = \text{constant}$$

Taking there to be zero velocity everywhere, we can draw this total head on the diagram:

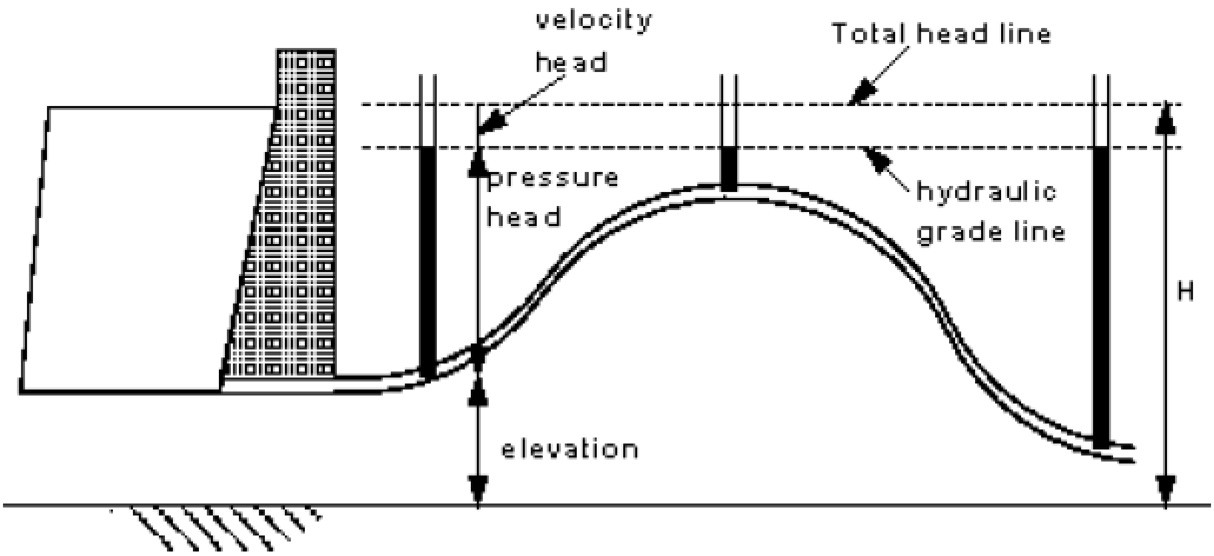


Hence at each point we have an exchange between pressure head and static head:

$$\frac{p}{\rho g} + z = H$$

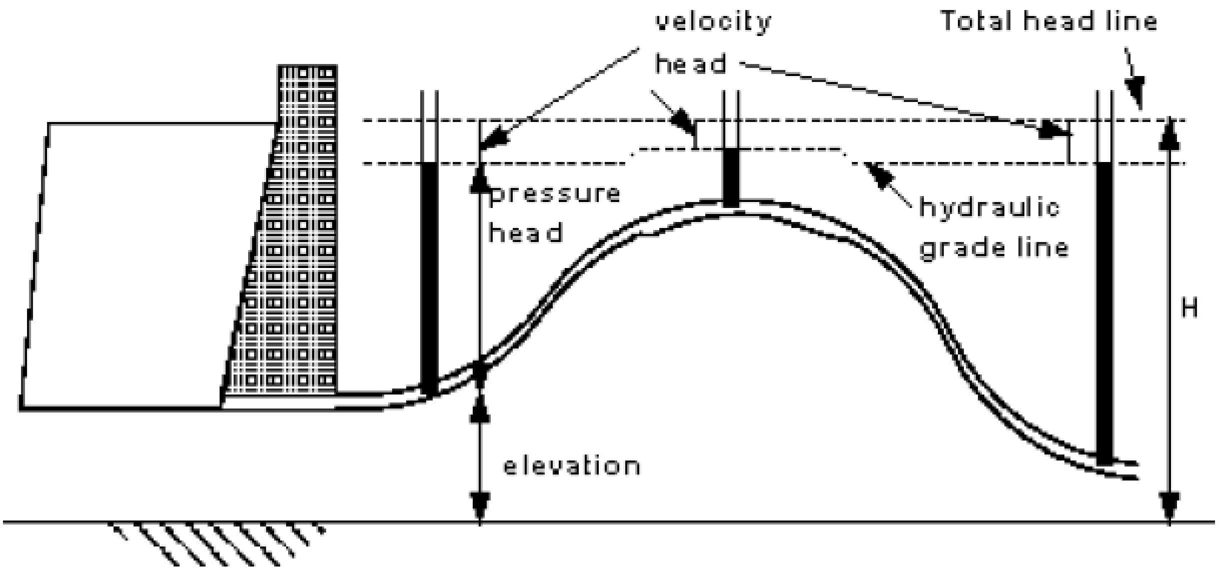
If we introduce the effect of velocity into the diagram we know that the pressure must fall by an amount $\frac{v^2}{2g}$ since we now have

$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = H$$

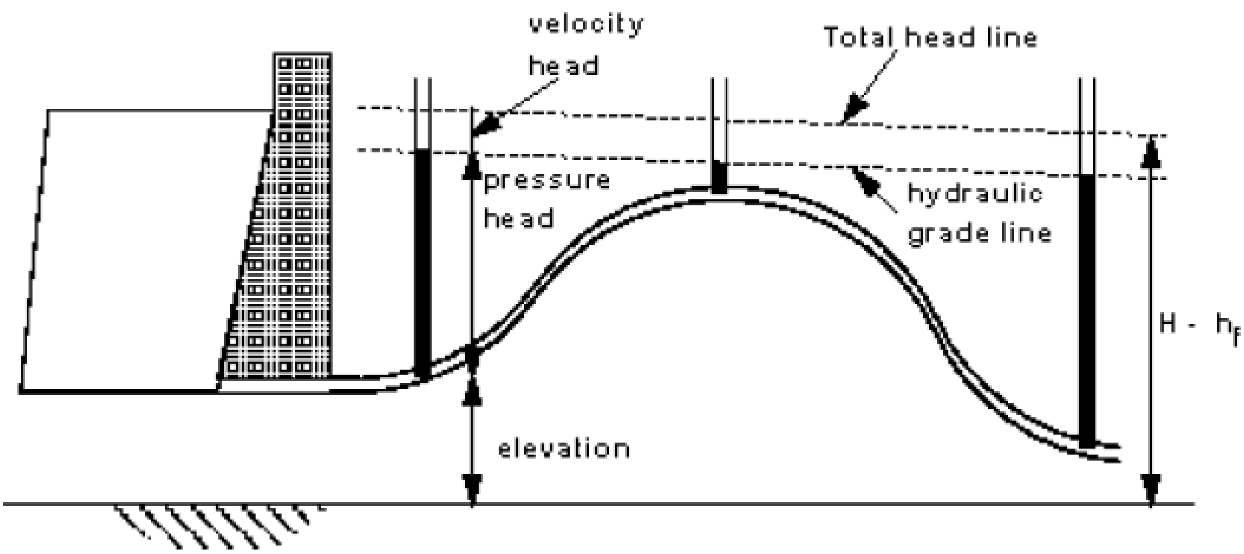


The *hydraulic grade line* is the line showing the pressure and static heads only.

If the velocity varies over the length of the pipe due to changes in diameter, say, we now have:



Note that the hydraulic grade line rises at the larger pipe section since the velocity is less in the larger pipe ($Q = Av$). If we now consider energy to be lost at every point along the length of the pipe, the total head will reduce linearly:



Thus denoting h_f as the *friction head loss*, we modify the energy equation to take account of friction losses between two points:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_f$$

Problems – Energy Losses and Flow Measurement

1. Estimate the energy head lost along a short length of pipe suddenly enlarging from a diameter of 350mm to 700 mm which discharges 700 litres per second of water. If the pressure at the entrance of the flow is 10^5 N/m^2, find the pressure at the exit.

   (Ans. 0.28 m, 1.02 × 10^5 N/m^2)

2. A Venturi meter is introduced in a 300 mm diameter horizontal pipeline carrying water under a pressure of 150 kN/m^2. The throat diameter of the meter is 100 mm and the pressure at the throat is 400 mm of mercury below atmosphere. If 3% of the differential pressure is lost between the inlet and outlet throat, determine the flow rate in the pipe.

   (Ans. 157 l/s)

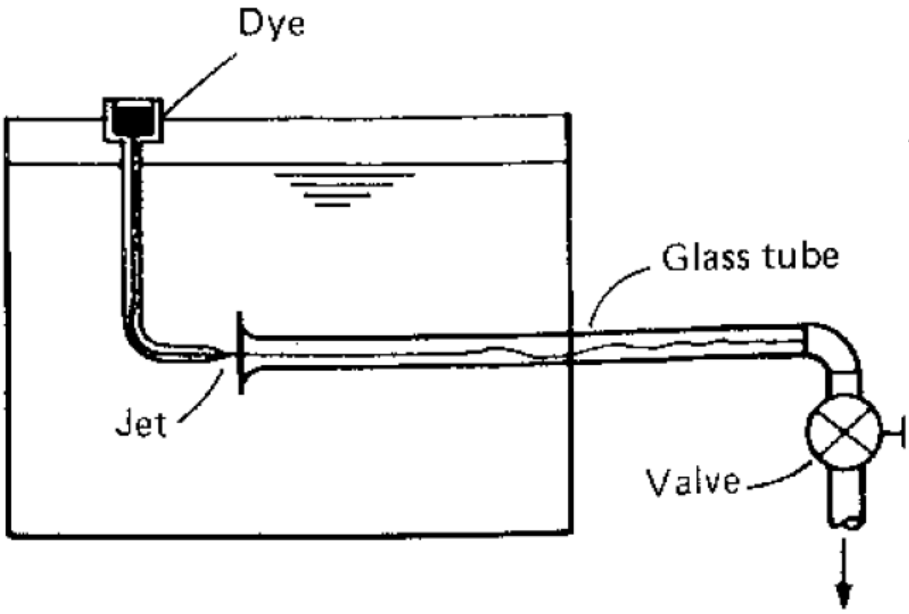
3. A 50 mm inlet/25 mm throat Venturi meter with a coefficient of discharge of 0.98 is to be replaced by an orifice meter having a coefficient of discharge of 0.6. If both meters are to give the same differential mercury manometer reading for a discharge of 10 l/s, determine the diameter of the orifice.

   (Ans. 31.2 mm)

5. Hydrodynamics: Flow in Pipes

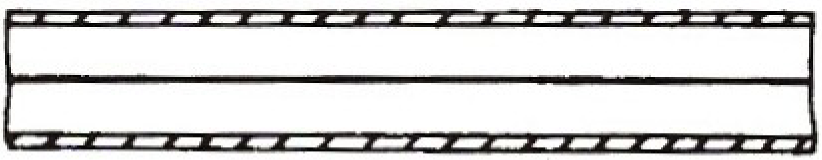
5.1 General Concepts

The real behaviour of fluids flowing is well described by an experiment carried out by Reynolds in 1883. He set up the following apparatus:

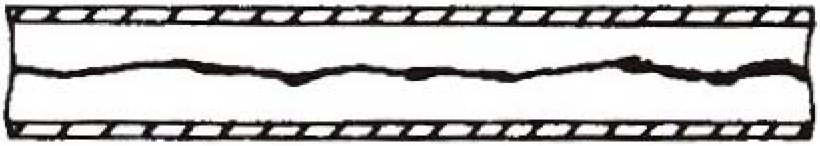


The discharge is controlled by the valve and the small 'filament' of dye (practically a streamline) indicates the behaviour of the flow. By changing the flow Reynolds noticed:

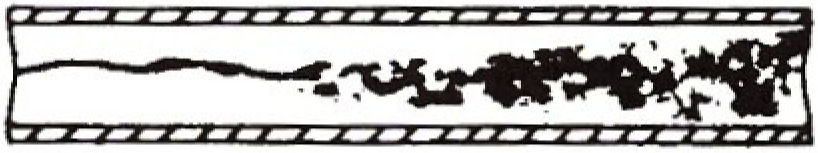
- At low flows/velocities the filament remained intact and almost straight. This type of flow is known as **laminar flow**, and the experiment looks like this:



- At higher flows the filament began to oscillate. This is called **transitional flow** and the experiment looks like:



- Lastly, for even higher flows again, the filament is found to break up completely and gets diffused over the full cross-section. This is known as **turbulent flow**:



Reynolds experimented with different fluids, pipes and velocities. Eventually he found that the following expression predicted which type of flow was found:

$$\mathrm{Re} = \frac{\rho \bar{v} l}{\mu}$$

In which Re is called the Reynolds Number; ρ is the fluid density; $\bar{v}$ is the average velocity; l is the characteristic length of the system (just the diameter for pipes), and; μ is the fluid viscosity. The Reynolds Number is a ration of forces and hence has no units.

Flows in pipes normally conform to the following:

- $\mathrm{Re} < 2000$: gives laminar flow;

- $2000 < \text{Re} < 4000$: transitional flow;
- $\text{Re} > 4000$: turbulent flow.

These values are only a rough guide however. Laminar flows have been found at Reynolds Numbers far beyond even 4000.

For example, if we consider a garden hose of 15 mm diameter then the limiting average velocity for laminar flow is:

$$\text{Re} = \frac{\rho \bar{v} l}{\mu}$$

$$2000 = \frac{\left(10^3\right)\bar{v}\left(0.015\right)}{0.55 \times 10^{-3}}$$

$$\bar{v} = 0.073 \text{ m/s}$$

This is a very low flow and hence we can see that in most applications we deal with turbulent flow.

The velocity below which there is no turbulence is called the **critical velocity**.

Characteristics of Flow Types

For laminar flow:

- Re < 2000;
- ‘low’ velocity;
- Dye does not mix with water;
- Fluid particles move in straight lines;
- Simple mathematical analysis possible;
- Rare in practical water systems.

Transitional flow

- 2000 < Re < 4000
- ‘medium’ velocity
- Filament oscillates and mixes slightly.

Turbulent flow

- Re > 4000;
- ‘high’ velocity;
- Dye mixes rapidly and completely;
- Particle paths completely irregular;
- Average motion is in the direction of the flow;
- Mathematical analysis very difficult - experimental measures are used;
- Most common type of flow.

Background to Pipe Flow Theory

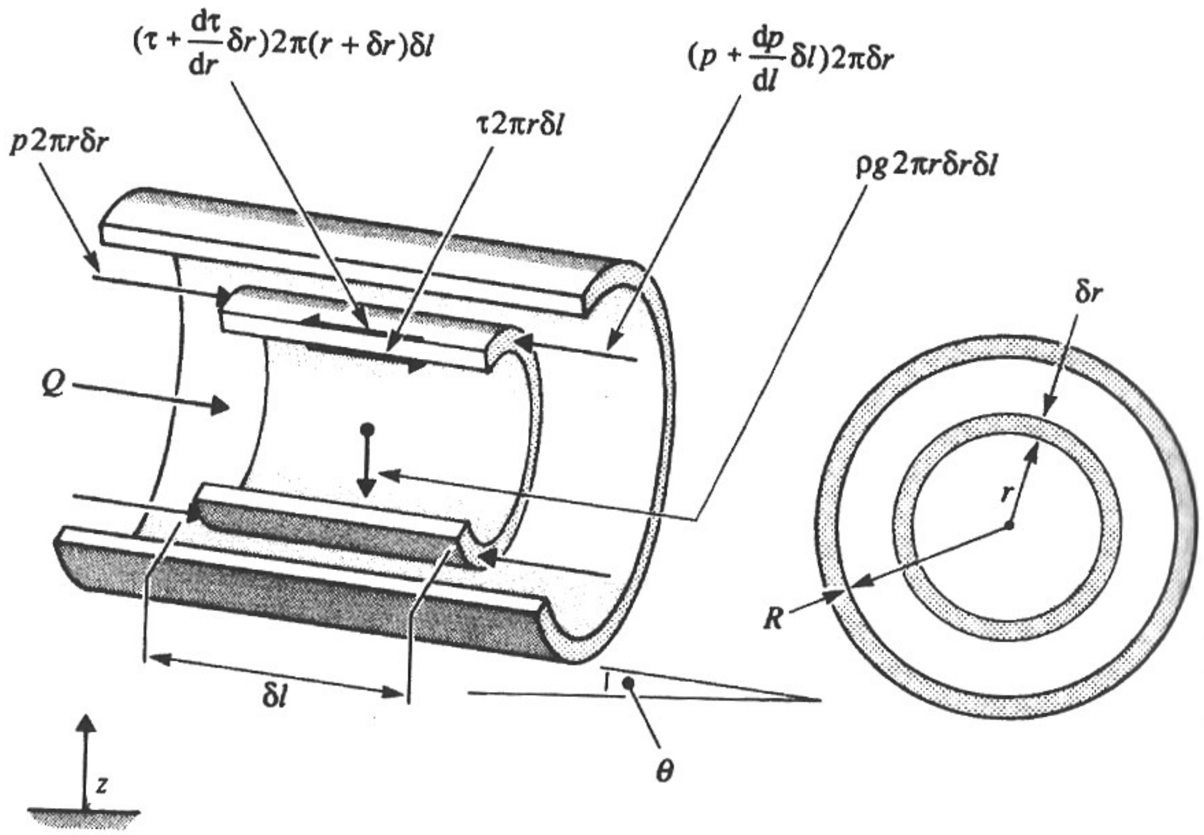
To explain the various pipe flow theories we will follow the historical development of the subject:

Date	Name	Contribution
~1840	Hagen and Poiseuille	Laminar flow equation
1850	Darcy and Weisbach	Turbulent flow equation
1883	Reynolds	Distinction between laminar and turbulent flow
1913	Blasius	Friction factor equation or smooth pipes
1914	Stanton and Pannell	Experimental values of friction factor for smooth pipes
1930	Nikuradse	Experimental values of friction factor for artificially rough pipes
1930s	Prandtl and von Karman	Equations for rough and smooth friction factors
1937	Colebrook and White	Experimental values of the friction factor for commercial pipes and the transition formula
1944	Moody	The Moody diagram for commercial pipes
1958	Ackers	Hydraulics Research Station charts and tables for the design of pipes and channels
1975	Barr	Solution of the Colebrook-White equation

5.2 Laminar Flow

Steady Uniform Flow in a Pipe: Momentum Equation

The development that follows forms the basis of the flow theories applied to laminar flows. We remember from before that at the boundary of the pipe, the fluid velocity is zero, and the maximum velocity occurs at the centre of the pipe. This is because of the effect of viscosity. Therefore, at a given radius from the centre of the pipe the velocity is the same and so we consider an elemental annulus of fluid:



In the figure we have the following:

- δr – thickness of the annulus;
- δl – length of pipe considered;
- R – radius of pipe;
- θ – Angle of pipe to the horizontal.

The forces acting on the annulus are:

- The pressure forces:
  - Pushing the fluid: $p2\pi\,\delta r$
  - Resisting $\left(p+\dfrac{dp}{dl}\delta l\right)2\pi\,\delta r$

- The shear forces (due to viscosity):
  - Inside the annulus: $\tau 2\pi r\,\delta l$
  - Outside the annulus $\left(\tau+\dfrac{d\tau}{dr}\delta r\right)2\pi\left(r+\delta r\right)\delta l$

- The weight of the fluid (due to the angle θ):

$$\rho g 2\pi\delta l\delta r\sin\theta$$

The sum of the forces acting is equal to the change in momentum. However, the change in momentum is zero since the flow is steady and uniform. Thus:

$$p2\pi\,\delta r-\left(p+\frac{dp}{dl}\delta l\right)2\pi\,\delta r+\tau 2\pi r\,\delta l-\left(\tau+\frac{d\tau}{dr}\delta r\right)2\pi\left(r+\delta r\right)\delta l+\rho g 2\pi\delta l\delta r\sin\theta=0$$

Using $\sin\theta=-dz/dl$, and dividing by $2\pi r\delta l\delta r$ gives:

$$-\frac{dp}{dl}-\frac{d\tau}{dr}-\frac{\tau}{r}-\rho g\frac{dz}{dl}=0$$

In which second order terms have been ignored. We introduce the term $p^* = p + \rho g z$ which is the piezometric pressure measured from the datum $z = 0$ to give:

$$-\frac{dp^*}{dl} - \left(\frac{\tau}{r} + \frac{d\tau}{dr}\right) = 0$$

Examining the term in brackets, we see:

$$\frac{d\tau}{dr} + \frac{\tau}{r} = \frac{1}{r}\left(r\frac{d\tau}{dr} + \tau\right) = \frac{1}{r}\frac{d}{dr}(\tau r)$$

Hence:

$$-\frac{dp^*}{dl} - \frac{1}{r}\frac{d}{dr}(\tau r) = 0$$

$$\frac{d}{dr}(\tau r) = -r\frac{dp^*}{dl}$$

Integrating both sides:

$$\tau r = -\frac{r^2}{2}\frac{dp^*}{dl} + C$$

But at the centreline, $r = 0$ and thus the constant of integration $C = 0$. Thus:

$$\boxed{\tau = -\frac{r}{2}\frac{dp^*}{dl}}$$

Thus the shear stress at any radius is known in terms of the piezometric pressure.

Hagen-Poiseuille Equation for Laminar Flow

We can use the knowledge of the shear stress at any distance from the centre of the pipe in conjunction with our knowledge of viscosity as follows:

$$\tau = \mu \frac{dv}{dy} = -\mu \frac{dv_r}{dr}$$
$$= -\frac{r}{2}\frac{dp^*}{dl}$$

Hence:

$$\frac{dv_r}{dr} = \frac{r}{2\mu}\frac{dp^*}{dl}$$

Integrating:

$$v_r = \frac{r^2}{4\mu}\frac{dp^*}{dl} + C$$

At the pipe boundary, $v_r = 0$ and $r = R$, Hence we can solve for the constant as:

$$C = -\frac{R^2}{4\mu}\frac{dp^*}{dl}$$

And so:

$$\boxed{v_r = -\frac{1}{4\mu}\frac{dp^*}{dl}\left(R^2 - r^2\right)}$$

Thus the velocity distribution is parabolic (i.e. a quadratic in *r*). The total discharge can now be evaluated:

$$\delta Q = \left(2\pi r \delta r\right) v_r$$

Introducing the equation for the velocity at radius *r* and integrating gives:

$$\begin{aligned} Q &= 2\pi \int_0^R r v_r \, dr \\ &= -\frac{2\pi}{4\mu}\frac{dp^*}{dl}\int_0^R r\left(R^2 - r^2\right) dr \\ &= -\frac{\pi}{8\mu}\frac{dp^*}{dl} R^4 \end{aligned}$$

The mean velocity, $\bar{v}$ is obtained from *Q* as:

$$\begin{aligned} \bar{v} &= \frac{Q}{A} \\ &= -\frac{\pi}{8\mu}\frac{dp^*}{dl} R^4 \frac{1}{2\pi R^2} \\ &= -\frac{1}{8\mu}\frac{dp^*}{dl} R^2 \end{aligned}$$

At this point we introduce the allowance for the frictional head loss, which represents the change in pressure head occurring over the length of pipe examined, i.e.:

$$h_f = -\frac{\Delta p^*}{\rho g}$$

Therefore, introducing this and the relation for the pipe diameter $R^2 = D^2/4$, the equation for the mean velocity becomes:

$$\bar{v} = \frac{1}{8\mu}\frac{h_f}{L}\rho g \frac{D^2}{4}$$

And rearranging for the head loss that occurs, gives the Hagen-Poiseuille Equation:

$$\boxed{h_f = \frac{32\mu L\bar{v}}{\rho g D^2}}$$

Example: Laminar Flow in Pipe

Problem

Oil flows through a 25 mm diameter pipe with mean velocity of 0.3 m/s. Given that the viscosity $\mu = 4.8\times10^{-2}$ kg/ms and the density $\rho = 800$ kg/m^3, calculate: (a) the friction head loss and resultant pressure drop in a 45 m length of pipe, and; (b) the maximum velocity, and the velocity 5 mm from the pipe wall.

Solution

Firstly check that the laminar flow equations developed apply, that is, Re < 2000:

$$\begin{aligned}\mathrm{Re} &= \frac{\rho D\bar{v}}{\mu} \text{ for pipe flow}\\ &= \frac{(800)(0.025)(0.3)}{4.8\times10^{-2}}\\ &= 125\\ &< 2000 \text{ thus laminar equations apply}\end{aligned}$$

1. To find the friction head loss, we apply the Hagen-Poiseuille Equation:

$$\begin{aligned}h_f &= \frac{32\mu L\bar{v}}{\rho g D^2}\\ &= \frac{32\left(4.8\times10^{-2}\right)(45)(0.3)}{(800)(9.81)(0.025)^2}\\ &= 4.228 \text{ m of oil}\end{aligned}$$

The associated pressure drop is:

$$\begin{aligned}\Delta p &= -\rho g h_f \\ &= -(800)(9.81)(4.228) \\ &= -33.18 \text{ kN/m}^2\end{aligned}$$

The negative sign is used to enforce the idea that it is a pressure drop.

2. To find the velocities, use the equation for velocities at a radius:

$$v_r = -\frac{1}{4\mu}\frac{dp^*}{dl}\left(R^2 - r^2\right)$$

The maximum velocity occurs furthest from the pipe walls, i.e. at the centre of the pipe where $r = 0$, hence:

$$\begin{aligned}v_{\max} &= -\frac{1}{4\left(4.8\times10^{-2}\right)}\frac{\left(-33.18\times10^3\right)}{45}\left(\left(\frac{0.025}{2}\right)^2 - 0\right) \\ &= 0.6 \text{ m/s}\end{aligned}$$

Note that the maximum velocity is twice the mean velocity. This can be confirmed for all pipes algebraically. The velocity at 5 mm from the wall is:

$$\begin{aligned}v_{\max} &= -\frac{1}{4\left(4.8\times10^{-2}\right)}\frac{\left(-33.18\times10^3\right)}{45}\left(\left(\frac{0.025}{2}\right)^2 - \left(0.0075\right)^2\right) \\ &= 0.384 \text{ m/s}\end{aligned}$$

In which it must be remembered that at 5 mm from the wall, $r = \frac{25}{2} - 5 = 7.5 \text{ mm}$.

5.3 Turbulent Flow

Description

Since the shearing action in laminar flows is well understood, equations describing the flow were easily determined. In turbulent flows there is no simple description of the shear forces that act in the fluid. Therefore the solutions of problems involving turbulent flows usually involve experimental results.

In his work, Reynolds clarified two previous results found experimentally:

- Hagen and Poiseuille found that friction head loss is proportional to the mean velocity:

$$h_f \propto \bar{v}$$

  Reynolds found that this only applies to laminar flows, as we have seen.

- Darcy and Weisbach found that friction head loss is proportional to the mean velocity squared:

$$h_f \propto \bar{v}^2$$

  Reynolds found that this applies to turbulent flows.

Empirical Head Loss in Turbulent Flow

Starting with the momentum equation previously developed, and considering only the shear stress at the pipe wall, τ_0, we have:

$$\tau_0 = -\frac{R}{2}\frac{dp^*}{dl}$$

We also know from the Hagen-Poiseuille equation that:

$$-\frac{dp^*}{dl} = \frac{h_f \rho g}{L}$$

Hence:

$$\tau_0 = \frac{h_f}{L}\rho g \frac{R}{2}$$

Using he experimental evidence that $h_f \propto \bar{v}^2$, we introduce $h_f = K_1\bar{v}^2$:

$$\begin{aligned}\tau_0 &= \frac{K_1\bar{v}^2}{L}\rho g \frac{R}{2} \\ &= K_2\bar{v}^2\end{aligned}$$

Hence, from previous

$$K_2\bar{v}^2 = \tau_0 = \frac{h_f}{L}\rho g \frac{R}{2}$$

And rearranging for the friction head loss:

$$h_f = \frac{4K_2 L\bar{v}^2}{\rho g D}$$

If we substitute in for some of the constants $\lambda = 8K_2/\rho$ we get:

$$\boxed{h_f = \frac{\lambda L\bar{v}^2}{2gD}}$$

This is known as the **Darcy-Weisbach Equation**.

In this equation, λ is known as the pipe friction factor and is sometimes referred to as f in American practice. It is a dimensionless number and is used in many design charts. It was once though to be constant but is now known to change depending on the Reynolds number and the 'roughness' of the pipe surface.

5.4 Pipe Friction Factor

Many experiments have been performed to determine the pipe friction factor for many different arrangements of pipes and flows.

Laminar Flow

We can just equate the Hagen-Poiseuille and the Darcy-Weisbach Equations:

$$\frac{32\mu L\overline{v}}{\rho g D^2} = \frac{\lambda L \overline{v}^2}{2gD}$$

Hence, for laminar flow we have:

$$\lambda = \frac{64\mu}{\rho D\overline{v}} = \frac{64}{\mathrm{Re}}$$

Smooth Pipes – Blasius Equation

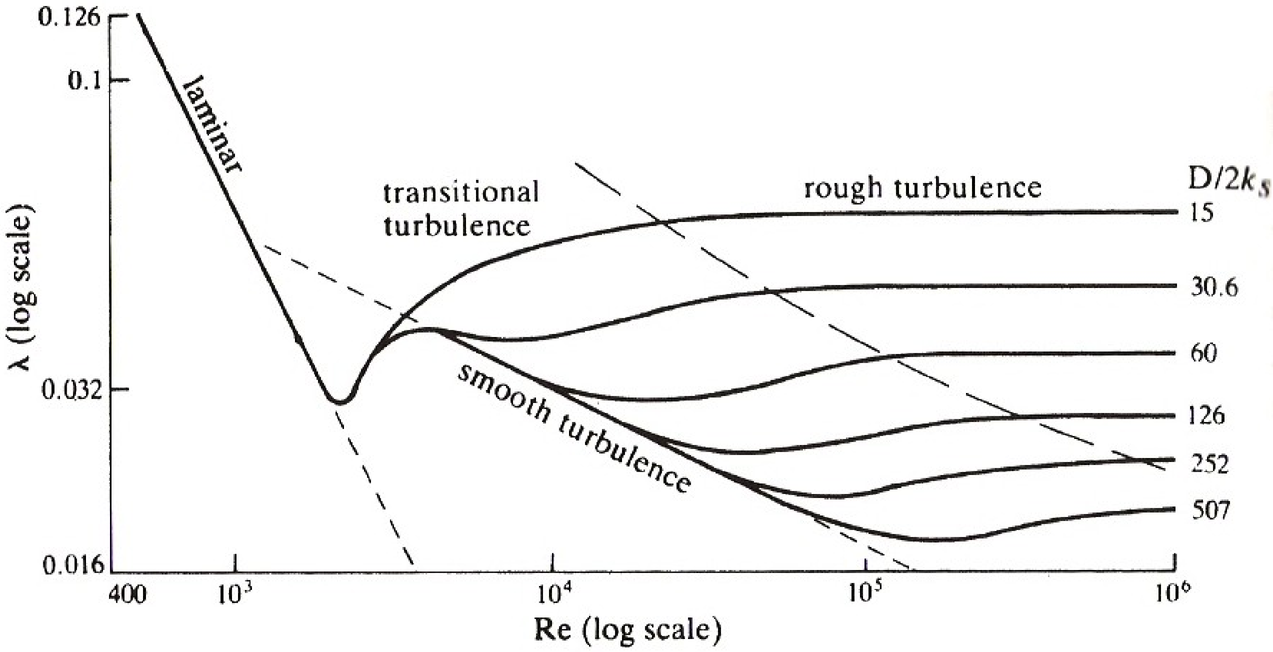
Blasius determined the following equation from experiments on ‘smooth’ pipes:

$$\lambda = \frac{0.316}{\mathrm{Re}^{0.25}}$$

Stanton and Pannell confirmed that this equation is valid for $\mathrm{Re} < 10^5$. Hence it is for ‘smooth’ pipes.

Nikuradse's Experiments

Nikuradse carried out many experiments up to $\mathrm{Re} = 3\times10^6$. In the experiments, he artificially roughened pipes by sticking uniform sand grains to smooth pipes. He defined the relative roughness (k_s/D) as the ration of the sand grain size to the pipe diameter. He plotted his results as $\log\lambda$ against $\log\mathrm{Re}$ for each k_s/D, shown below.



There are 5 regions of flow in the diagram:

1. Laminar Flow – as before;
2. Transitional flow – as before, but no clear λ;
3. Smooth turbulence – a limiting line of turbulence as Re decreases for all k_s/D;
4. Transitional turbulence – λ varies both with Re and k_s/D, most pipe flows are in this region;
5. Rough turbulence - λ is constant for a given k_s/D and is independent of Re.

The von Karman and Prandlt Laws

von Karman and Prandlt used Nikuradse's experimental results to supplement their own theoretical results which were not yet accurate. They found semi-empirical laws:

- Smooth pipes:

$$\frac{1}{\sqrt{\lambda}} = 2\log\frac{\mathrm{Re}\sqrt{\lambda}}{2.51}$$

- Rough pipes:

$$\frac{1}{\sqrt{\lambda}} = 2\log\frac{3.7}{k_s/D}$$

The von Karman and Prandlt Law for smooth pipes better fits the experimental data than the Blasius Equation.

The Colebrook-White Transition Formula

The friction factors thus far are the result of experiments on artificially roughened pipes. Commercial pipes have roughnesses that are uneven in both size and spacing. Colebrook and White did two things:

1. They carried out experiments and matched commercial pipes up to Nikuradse's results by finding an 'effective roughness' for the commercial pipes:

Pipe/Material	k_s (mm)
Brass, copper, glass, Perspex	0.003
Wrought iron	0.06
Galvanized iron	0.15
Plastic	0.03
Concrete	6.0

2. They combined the von Karman and Prandlt laws for smooth and rough pipes:

$$\boxed{\frac{1}{\sqrt{\lambda}} = -2\log\left(\frac{k_s}{3.7D} + \frac{2.51}{\mathrm{Re}\sqrt{\lambda}}\right)}$$

This equation is known as the **Colebrook-White transition formula** and it gives results very close to experimental values for transitional behaviour when using effective roughnesses for commercial pipes.

The transition formula must be solved by trial and error and is not expressed in terms of the preferred variables of diameter, discharge and hydraulic gradient. Hence it was not used much initially.

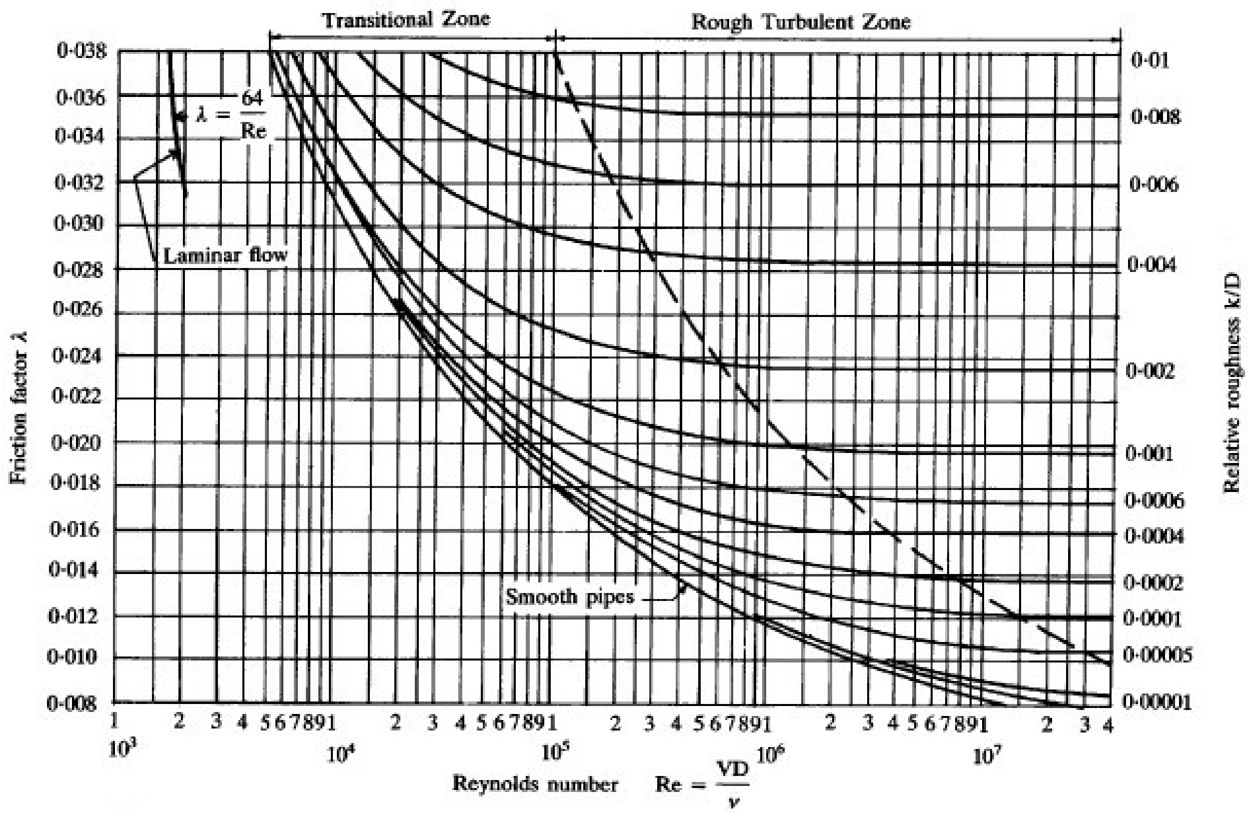
Moody

Moody recognized the problems with the Colebrok-White transition formula and did two things to remove objections to its use:

1. He presented an approximation to the Colebrook-White formula:

$$\lambda = 0.0055\left[1+\left(\frac{20\times 10^{3}k_{s}}{D}+\frac{10^{6}}{\mathrm{Re}}\right)^{1/3}\right]$$

2. He plotted λ against $\log \mathrm{Re}$ for commercial pipes, this is now known as the Moody diagram:



Barr

One last approximation to the Colebrook-White formula is that by Barr, who substituted the following approximation for the smooth law component:

$$\frac{5.1286}{\mathrm{Re}^{0.89}} \cong \frac{2.51}{\mathrm{Re}\sqrt{\lambda}}$$

To get:

$$\frac{1}{\sqrt{\lambda}} = -2\log\left(\frac{k_s}{3.7D} + \frac{5.1286}{\mathrm{Re}^{0.89}}\right)$$

This formula provides an accuracy of $\pm 1\%$ for $\mathrm{Re} > 10^5$.

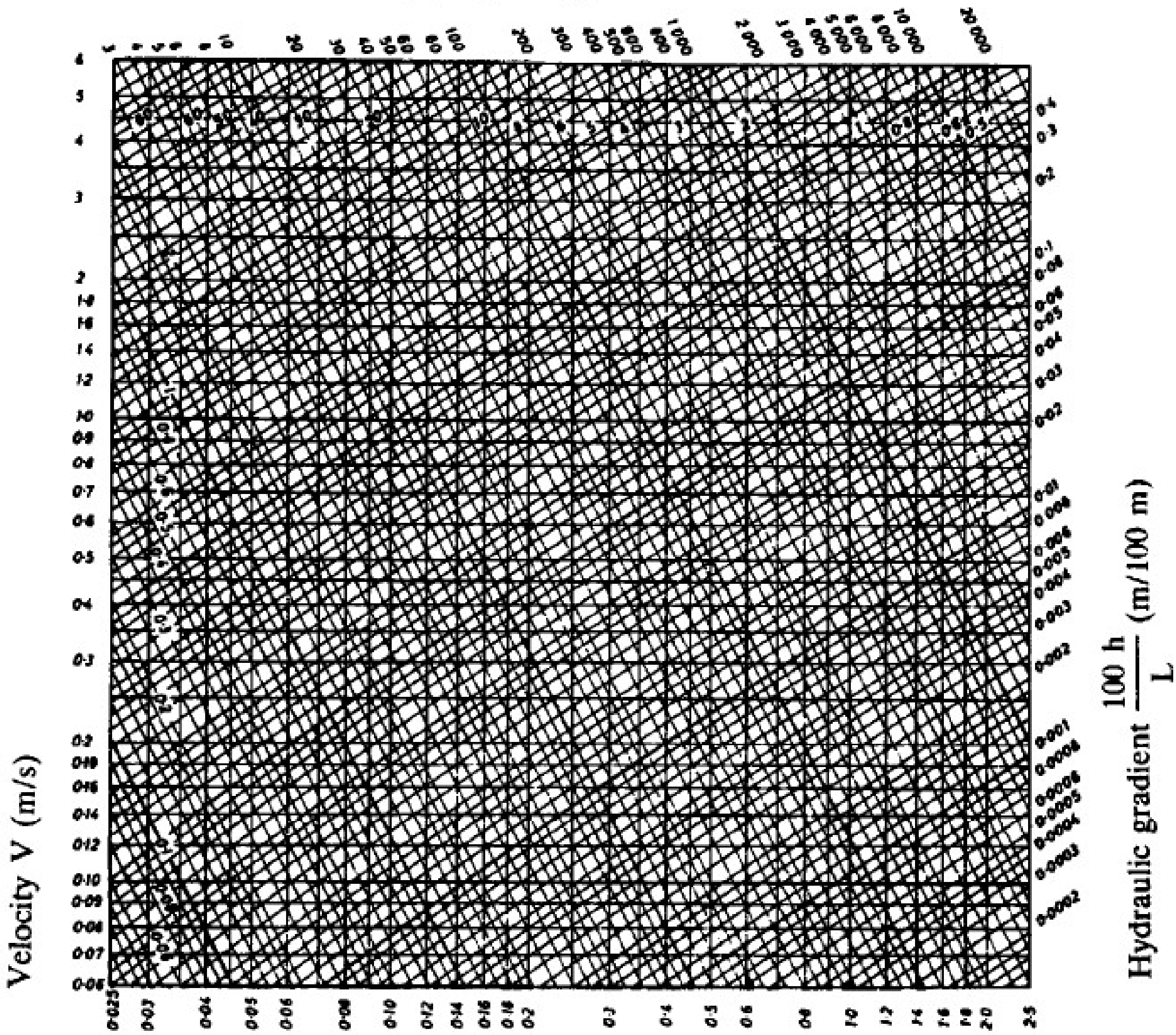
Hydraulics Research Station Charts

To derive charts suitable for design, the Colebrook-White and Darcy-Weisbach formulas were combined to give:

$$\bar{v} = -2\sqrt{2gDS_f}\,\log\left[\frac{k_s}{3.7D} + \frac{2.51\nu}{D\sqrt{2gDS_f}}\right]$$

In which $\nu = \mu/\rho$ and is known as the kinematic viscosity and S_f is the hydraulic gradient, i.e $S_f = h_f/L$. A sample chart is:

Discharge Q (l/s) *for pipes flowing full*



Diameter D (m)

Water at 15°C

k_s = 0·03 mm

Example

Problem

A plastic pipe, 10 km long and 300 mm diameter, conveys water from a reservoir (water level 850 m above datum) to a water treatment plant (inlet level 700 m above datum). Assuming the reservoir remains full, estimate the discharge using the following methods:

1. the Colebrook-White formula;
2. the Moody diagram;
3. the HRS charts.

Take the kinematic viscosity to be $1.13\times10^{-6}\ \text{m}^2/\text{s}$.

Solution

1. Using the combined Colebrook-White and Darcy-Weisbach formula:

$$\bar{v} = -2\sqrt{2gDS_f}\,\log\left[\frac{k_s}{3.7D} + \frac{2.51\nu}{D\sqrt{2gDS_f}}\right]$$

We have the following input variables:

1. $D = 0.3\ \text{m}$;
2. from the table for effective roughness, $k_s = 0.03\ \text{mm}$;
3. the hydraulic gradient is:

$$S_f = \frac{850-700}{10000} = 0.015$$

$$\bar{v} = -2\sqrt{2g(0.3)(0.015)}\,\log\left[\frac{0.03\times10^{-3}}{3.7\times0.3} + \frac{2.51\left(1.13\times10^{-6}\right)}{0.3\sqrt{2g(0.3)(0.015)}}\right]$$

$$= 2.514\ \text{m/s}$$

Hence the discharge is:

$$Q = A\bar{v} = 2.514\left(\frac{\pi 0.3^2}{4}\right) = 0.178 \text{ m}^3$$

2. To use the Moody chart proceed as:
    1. calculate k_s/D;
    2. assume a value for $\bar{v}$;
    3. calculate Re;
    4. estimate λ from the Moody chart;
    5. calculate h_f;
    6. compare h_f with the available head, H;
    7. if $h_f \neq H$ then repeat from step 2.

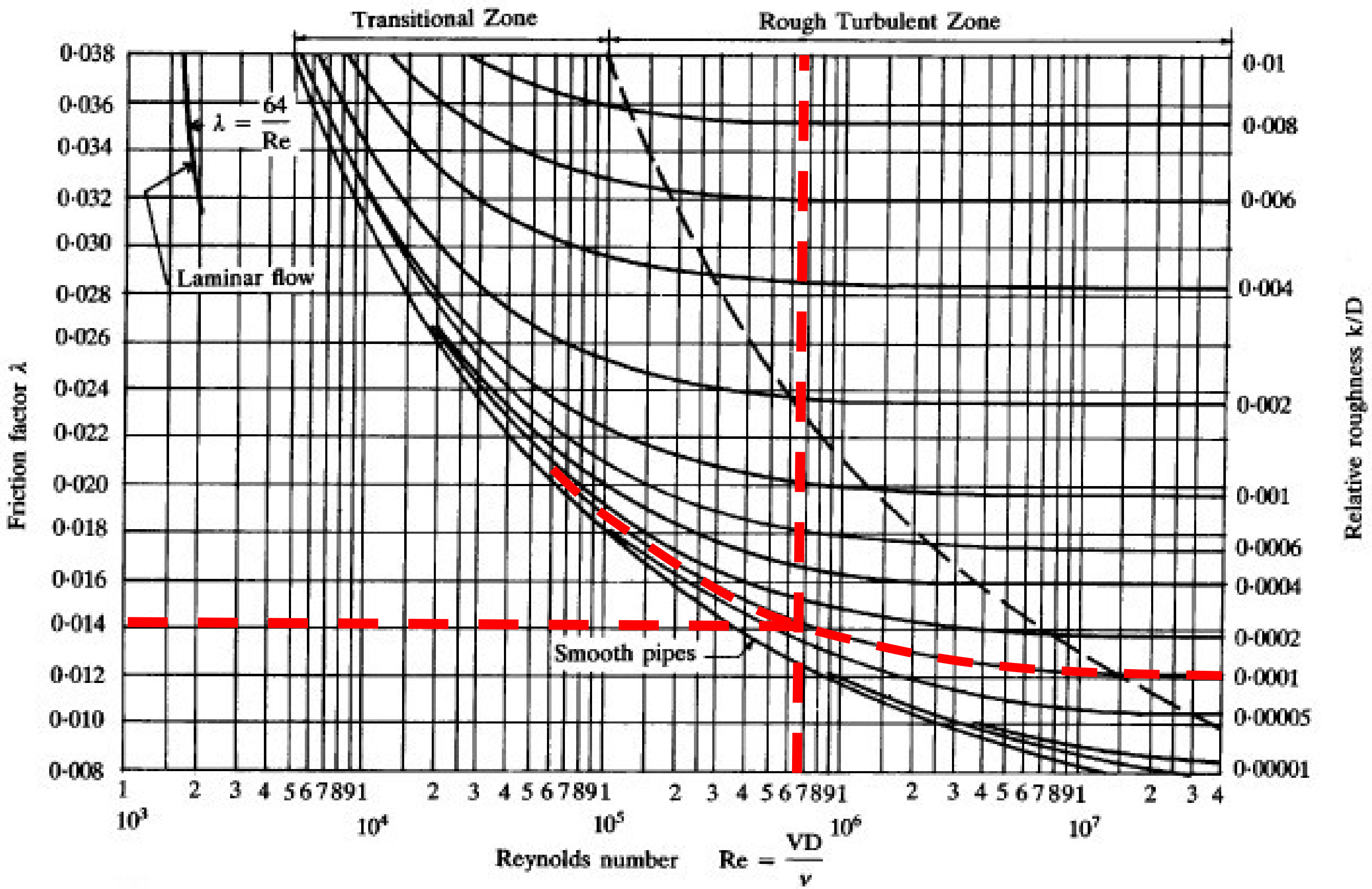
This is obviously tedious and is the reason the HRS charts were produced. The steps are:

1. $k_s/D = 0.03\times10^{-3}/0.3 = 0.0001$;
2. We'll take $\bar{v}$ to be close to the known result from part 1 of the question to expedite the solution: $\bar{v} = 2.5 \text{ m/s}$;
3. The Reynolds number:

$$\begin{aligned}\text{Re} &= \frac{\rho\bar{v}l}{\mu} = \frac{D\bar{v}}{\nu} \text{ for a pipe} \\ &= \frac{0.3\times2.5}{1.13\times10^{-6}} = 0.664\times10^{6}\end{aligned}$$

4. Referring to the Moody chart, we see that the flow is in the turbulent region. Follow the k_s/D curve until it intersects the Re value to get:

$$\lambda \simeq 0.014$$


Transitional Zone
Rough Turbulent Zone
$\lambda = \frac{64}{Re}$
Laminar flow
Smooth pipes
Friction factor λ
Relative roughness k/D
Reynolds number Re = $\frac{VD}{\nu}$
0·038
0·036
0·034
0·032
0·030
0·028
0·026
0·024
0·022
0·020
0·018
0·016
0·014
0·012
0·010
0·008
0·01
0·008
0·006
0·004
0·002
0·001
0·0006
0·0004
0·0002
0·0001
0·00005
0·00001
10^3
10^4
10^5
10^6
10^7

5. The Darcy-Wesibach equation then gives:

$$h_f = \frac{\lambda L \bar{v}^2}{2gD}$$
$$= \frac{0.014\left(10\times10^3\right)\left(2.5\right)^2}{2g\left(0.3\right)}$$
$$= 148.7 \text{ m}$$

6. The available head is $H = 850 - 700 = 150 \approx 148.7 \text{ m}$ so the result is quite close – but this is because we assumed almost the correct answer at the start.

Having confirmed the velocity using the Moody chart approach, the discharge is evaluated as before.

3. Using the HRS chart, the solution of the combined Colebrook-White and Darcy-Weisbach formula lies at the intersection of the hydraulic gradient line (sloping downwards, left to right) with the diameter (vertical) and reading off the discharge (line sloping downwards left to right):

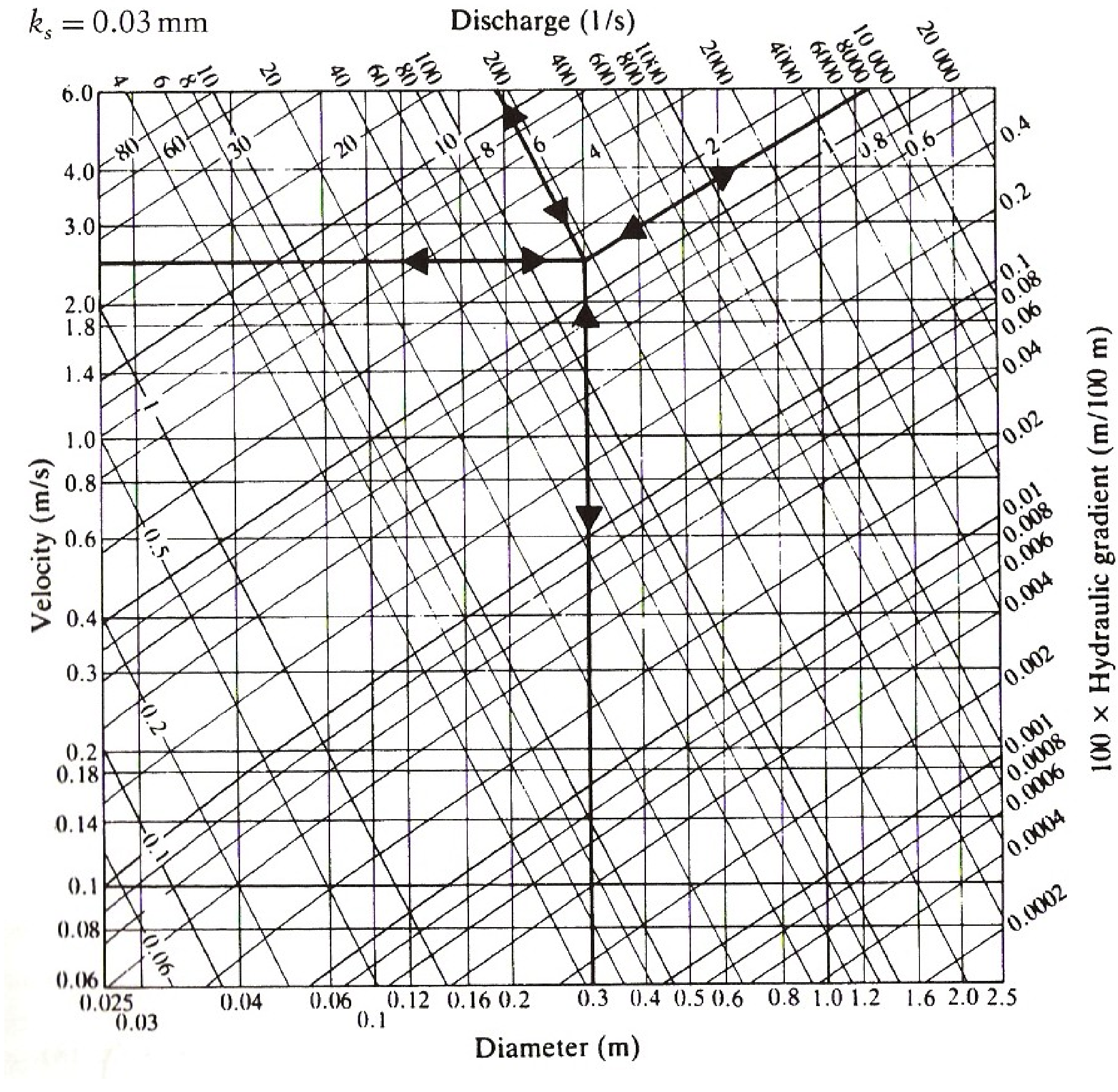
The inputs are:

- $S_f = 0.015$ and so $100S_f = 1.5$;
- $D = 300 \text{ mm}$.

Hence, as can be seen from the attached, we get:

$$Q = 180 \text{ l/s}$$
$$= 0.18 \text{ m}^3\text{/s}$$

Which is very similar to the exact result calculated previously.


$k_s = 0.03$ mm
Discharge (l/s)
Velocity (m/s)
100 × Hydraulic gradient (m/100 m)
Diameter (m)
0.025 0.03 0.04 0.06 0.1 0.12 0.16 0.2 0.3 0.4 0.5 0.6 0.8 1.0 1.2 1.6 2.0 2.5
6.0 4.0 3.0 2.0 1.8 1.4 1.0 0.8 0.6 0.4 0.3 0.2 0.18 0.14 0.1 0.08 0.06
4 6 8 10 20 40 60 80 100 200 400 600 800 1000 2000 4000 6000 8000 10 000 20 000
0.4 0.2 0.1 0.08 0.06 0.04 0.02 0.01 0.008 0.006 0.004 0.002 0.001 0.0008 0.0006 0.0004 0.0002

Problems – Pipe Flows

1. Determine the head loss per kilometre of a 100 mm diameter horizontal pipeline that transports oil of specific density 0.925 and viscosity 0.065 Ns/m^2 at a rate of 10 l/s. Determine also the shear stress at the pipe wall.

   (Ans. 29.2 m/km, 6.62 N/m^2)

2. A discharge of 400 l/s is to be conveyed from a reservoir at 1050 m AOD to a treatment plant at 1000 m AOD. The length of the pipeline is 5 km. Estimate the required diameter of the pipe taking $k_s = 0.03$ mm.

   (Ans. 450 mm)

3. The known outflow from a distribution system is 30 l/s. The pipe diameter is 150 mm, it is 500 m long and has effective roughness of 0.03 mm. Find the head loss in the pipe using:
   a. the Moody formula;
   b. the Barr formula;
   c. check these value against the Colebrook-White formula.

   (Ans. 0.0182, 8.94 m)

4. A plunger of 0.08m diameter and length 0.13m has four small holes of diameter 5/1600 m drilled through in the direction of its length. The plunger is a close fit inside a cylinder, containing oil, such that no oil is assumed to pass between the plunger and the cylinder. If the plunger is subjected to a vertical downward force of 45N (including its own weight) and it is assumed that the upward flow through the four small holes is laminar, determine the speed of the fall of the plunger. The coefficient of viscosity of the oil is 0.2 kg/ms.

   (Ans. 0.00064 m/s)

5.5 Pipe Design

Local Head Losses

In practice pipes have fittings such as bends, junctions, valves etc. Such features incur additional losses, termed local losses. Once again the approach to these losses is empirical, and it is found that the following is reasonably accurate:

$$h_L = k_L \frac{\overline{v}^2}{2g}$$

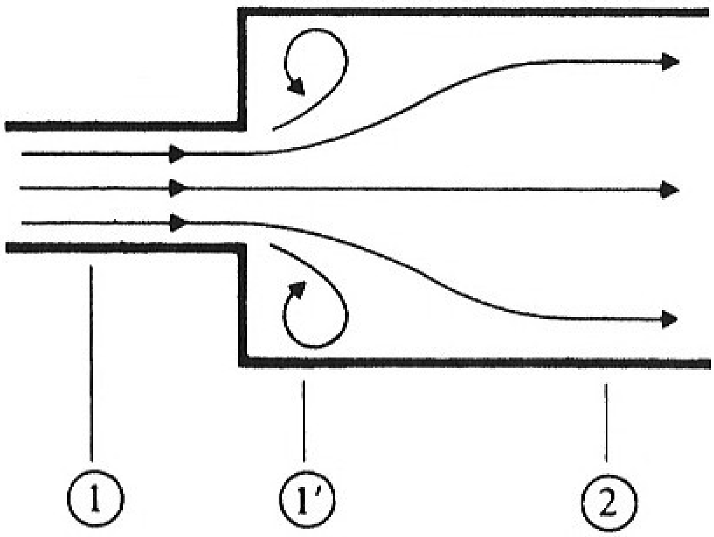
In which h_L is the local head loss and k_L is a constant for a particular fitting.

Typical values are:

Fitting	Local Head Loss Coefficient, k_L	
	Theoretical/Experimental	Design Practice
Bellmouth entrance	0.05	0.10
Bellmouth exit	0.2	0.5
90˚ bend	0.4	0.5
90˚ tees:		
- in-line flow	0.35	0.4
- branch to line	1.20	1.5
- gate valve (open)	0.12	0.25

Sudden Enlargement

Sudden enlargements (such as a pipe exiting to a tank) can be looked at theoretically:



From points 1 to 2 the velocity decreases and so the pressure increases. At 1' turbulent eddies are formed. We will assume that the pressure at 1 is the same as the pressure at 1'. Apply the momentum equation between 1 and 2:

$$p_1 A_1 - p_2 A_2 = \rho Q\left(\overline{v}_2 - \overline{v}_1\right)$$

Using continuity, $Q = A_2 \overline{v}_2$ and so:

$$\frac{p_2 - p_1}{\rho g} = \frac{\overline{v}_2}{g}\left(\overline{v}_1 - \overline{v}_2\right)$$

Now apply the energy equation from 1 to 2:

$$\frac{p_1}{\rho g} + \frac{\overline{v}_1^2}{2g} = \frac{p_2}{\rho g} + \frac{\overline{v}_2^2}{2g} + h_L$$

And so

$$h_L = \frac{\overline{v}_1^2 - \overline{v}_2^2}{2g} - \frac{p_1 - p_2}{\rho g}$$

Substituting for $\frac{p_2 - p_1}{\rho g}$ from above:

$$h_L = \frac{\overline{v}_1^2 - \overline{v}_2^2}{2g} - \frac{\overline{v}_2}{g}\left(\overline{v}_1 - \overline{v}_2\right)$$

Multiplying out and rearranging:

$$h_L = \frac{\left(\overline{v}_1 - \overline{v}_2\right)^2}{2g}$$

Using continuity again, $\overline{v}_2 = \overline{v}_1\left(A_1/A_2\right)$ and so:

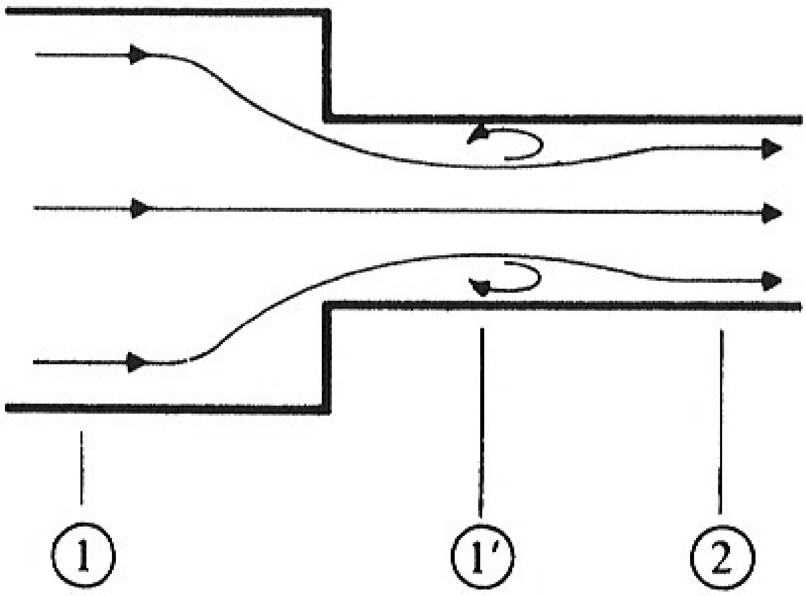
$$h_L = \frac{\left(\overline{v}_1 - \overline{v}_1\dfrac{A_1}{A_2}\right)^2}{2g}$$
$$= \left(1 - \frac{A_1}{A_2}\right)^2 \frac{\overline{v}_1^2}{2g}$$

Therefore in the case of sudden contraction, the local head loss is given by:

$$k_L = \left(1 - \frac{A_1}{A_2}\right)^2$$

Sudden Contraction

We use the same approach as for sudden enlargement but need to incorporate the experimental information that the area of flow at point 1' is roughly 60% of that at point 2.



Hence:

$$A_{1'} \simeq 0.6A_2$$

$$\begin{aligned} h_L &= \left(1 - \frac{0.6A_2}{A_2}\right)^2 \frac{\left(\overline{v}_2/0.6\right)^2}{2g} \\ &= 0.44\frac{\overline{v}_2^2}{2g} \end{aligned}$$

And so:

$$k_L = 0.44$$

Example – Pipe flow incorporating local head losses

Problem

For the previous example of the 10 km pipe, allow for the local head losses caused by the following items:

- 20 90˚ bends;
- 2 gate valves;
- 1 bellmouth entry;
- 1 bellmouth exit.

Solution

The available static head of 150 m is dissipated by the friction and local losses:

$$H = h_f + h_L$$

Using the table of loss coefficients, we have:

$$\begin{aligned} h_L &= \left[(20\times 0.5)+(2\times 0.25)+0.1+0.5\right]\frac{\bar{v}^2}{2g} \\ &= 11.1\frac{\bar{v}^2}{2g} \end{aligned}$$

To use the Colebrook-White formula (modified by Darcy's equation) we need to iterate as follows:

1. Assume $h_f \simeq H$ (i.e. ignore the local losses for now);
2. calculate $\bar{v}$ and thus h_L;
3. calculate $h_f + h_L$ and compare to H;
4. If $H \neq h_f + h_L$ then set $h_f = H - h_L$ and repeat from 2.

From the last example, we will take $\bar{v} = 2.514\text{ m/s}$. Thus:

$$h_L = 11.1\frac{2.514^2}{2g} = 3.58\text{ m}$$

Adjust h_f:

$$h_f = 150 - 3.58 = 146.42\text{ m}$$

Hence:

$$S_f = \frac{146.42}{10000} = 0.01464$$

Substitute into the Colebrook-White equation:

$$\bar{v} = -2\sqrt{2g(0.3)(0.01464)}\log\left[\frac{0.03\times10^{-3}}{3.7\times0.3} + \frac{2.51\left(1.13\times10^{-6}\right)}{0.3\sqrt{2g(0.3)(0.01464)}}\right]$$
$$= 2.386\text{ m/s}$$

Recalculate h_L:

$$h_L = 11.1\frac{2.386^2}{2g} = 3.22\text{ m}$$

Check against *H*:

$$h_f + h_L = 146.42 + 3.22 = 149.64 \approx 150\text{ m}$$

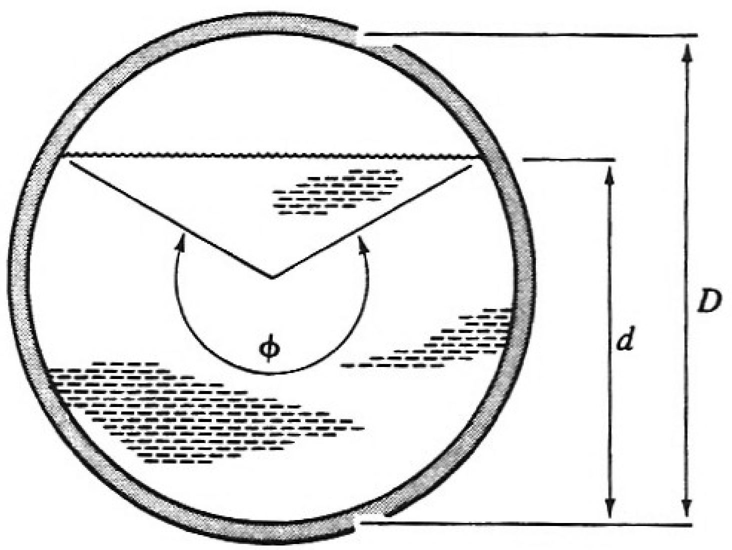
This is sufficiently accurate and gives $Q = 0.17\text{ m}^3\text{/s}$. Note that ignoring the local losses gives $Q = 0.18\text{ m}^3\text{/s}$, as previous.

Partially Full Pipes

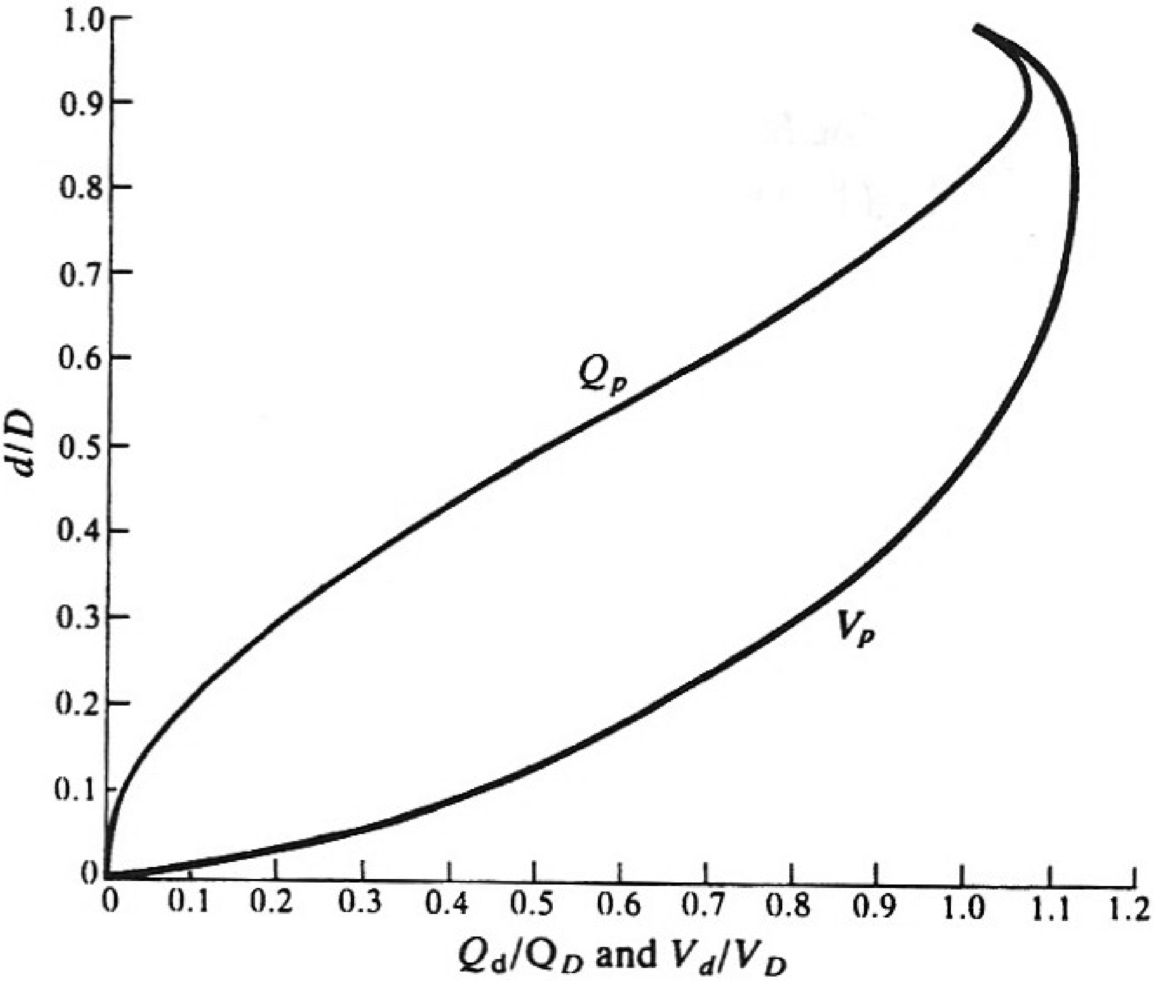
Surface water and sewage pipes are designed to flow full, but not under pressure. Water mains are designed to flow full and under pressure. When a pipe is not under pressure, the water surface will be parallel to the pipe invert (the bottom of the pipe). In this case the hydraulic gradient will equal the pipe gradient, S_0:

$$S_0 = \frac{h_f}{L}$$

In these non-pressurized pipes, they often do not run full and so an estimate of the velocity and discharge is required for the partially full case. This enables checking of the **self-cleansing velocity** (that required to keep suspended solids in motion to avoid blocking the pipe).



Depending on the proportional depth of flow, the velocity and discharge will vary as shown in the following chart:



This chart uses the subscripts: *p* for proportion; *d* for partially full, and; *D* for full.

Note that it is possible to have a higher velocity and flow when the pipe is not full due to reduced friction, but this is usually ignored in design.

Example

Problem

A sewerage pipe is to be laid at a gradient of 1 in 300. The design maximum discharge is 75 l/s and the design minimum flow is 10 l/s. Determine the required pipe diameter to carry the maximum discharge and maintain a self-cleansing velocity of 0.75 m/s at the minimum discharge.

Solution

(Note: a sewerage pipe will normally be concrete but we'll assume it's plastic here so we can use the chart for $k_s = 0.03$ mm)

$$Q = 75 \text{ l/s}$$
$$\frac{100h_f}{L} = \frac{100}{300} = 0.333$$

Using the HRS chart for $k_s = 0.03$ mm, we get:

$$D = 300 \text{ mm}$$
$$\overline{v} = 1.06 \text{ m/s}$$

Check the velocity for the minimum flow of 10 l/s:

$$Q_p = \frac{10}{75} = 0.133$$

Hence from the proportional flow and discharge graph:

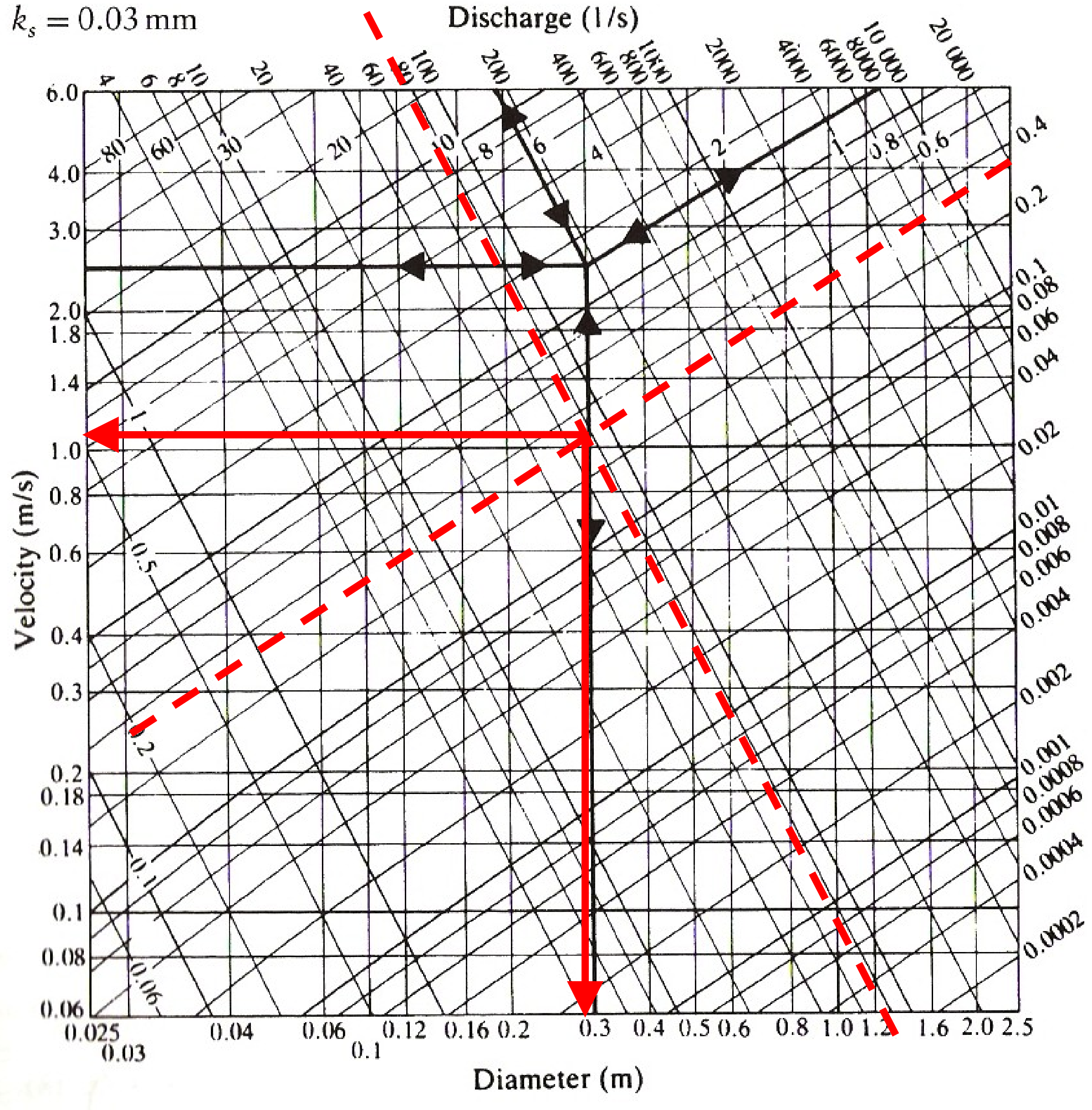
$$\frac{d}{D} = 0.25 \text{ and } v_p = 0.72$$

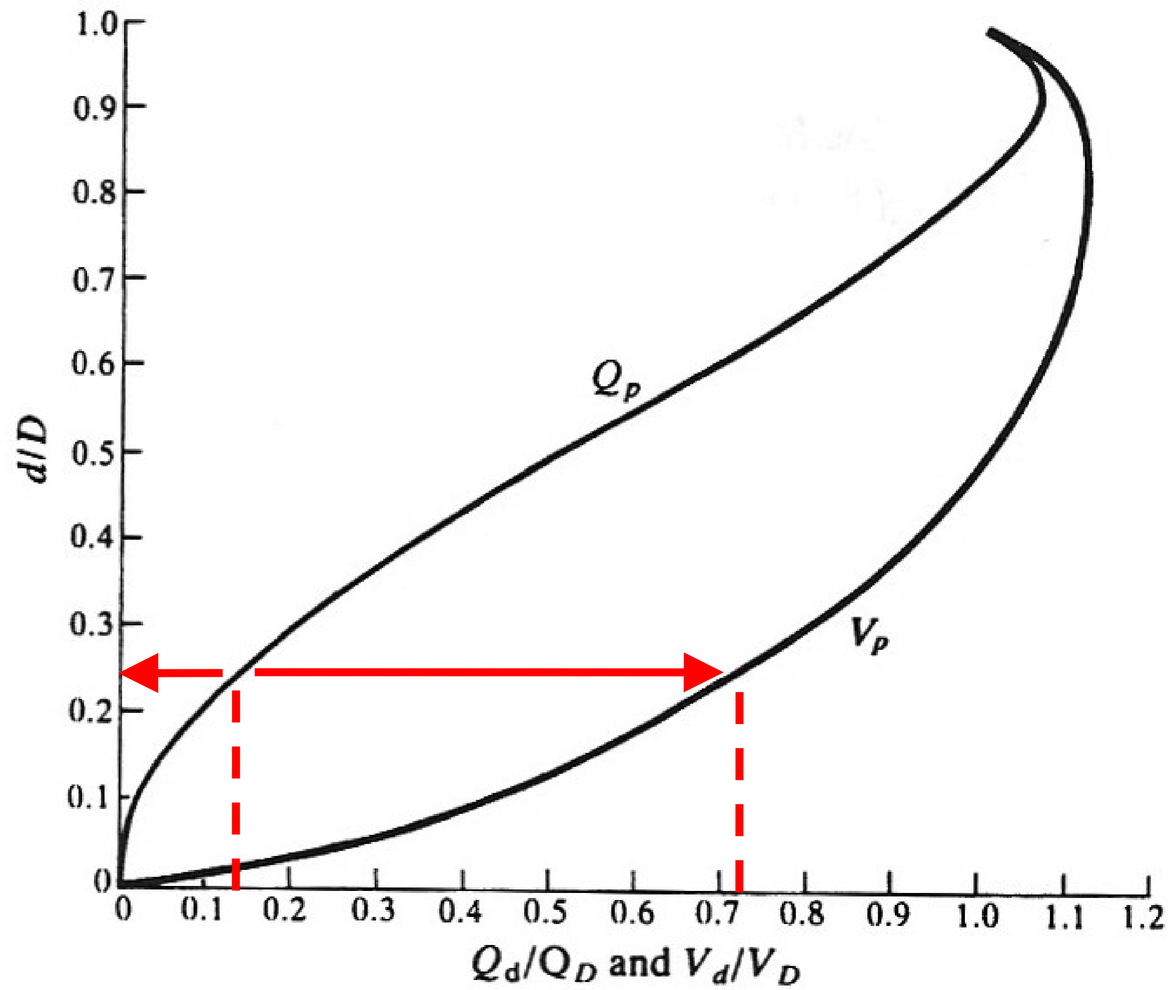
Thus:

$$\bar{v}_d = 0.72 \times 1.06 = 0.76 \text{ m/s}$$

This is greater then the minimum cleaning velocity required of 0.75 m/s and hence the 300 mm pipe is satisfactory.

The lookups are:




1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
d/D
Q_p
V_p
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
Q_d/Q_D and V_d/V_D

Problems – Pipe Design

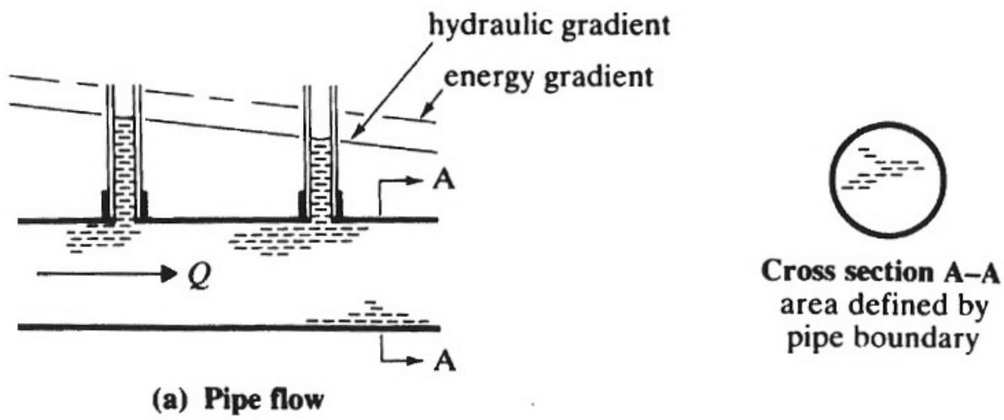
1. A uniform pipeline, 5000 m long, 200 mm in diameter and of effective roughness 0.03 mm, conveys water between two reservoirs whose surfaces are kept at a constant 50 m difference in elevation. There is an entry loss of 0.5 times the velocity head and a valve produces a loss of 10 times the velocity head. Determine the steady-state discharge using the HRS charts and confirm using the Colebrook-White transitional formula.

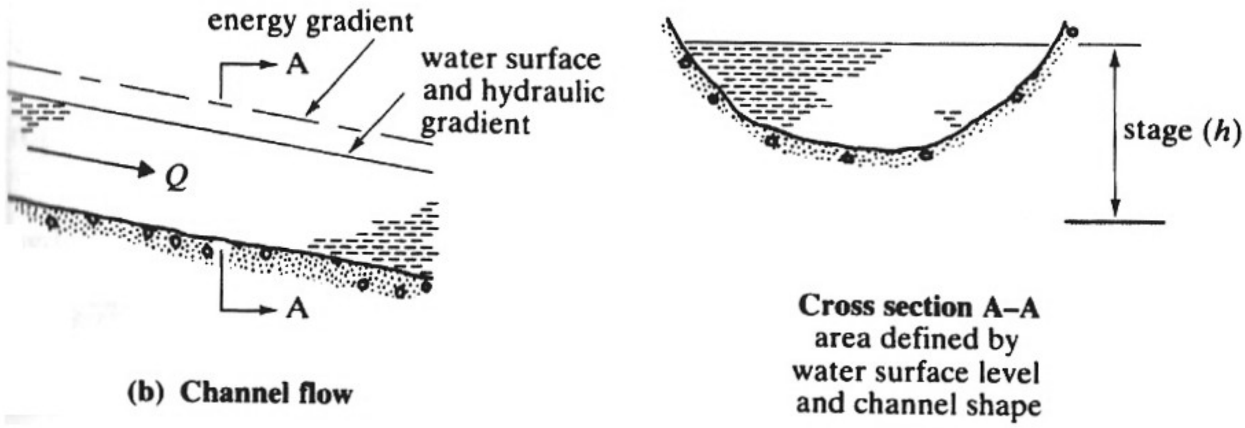
   (Ans. 1.54 m/s, 48.4 l/s)

6. Hydrodynamics: Flow in Open Channels

6.1 Description

The main difference between what we have studied so far and open channels is the existence of the free surface. It has great effect as can be seen from the following comparison:





In general, the analysis of channel flow is more difficult than that of pipe flow as there are many more variables. Some approximate analyses are possible.

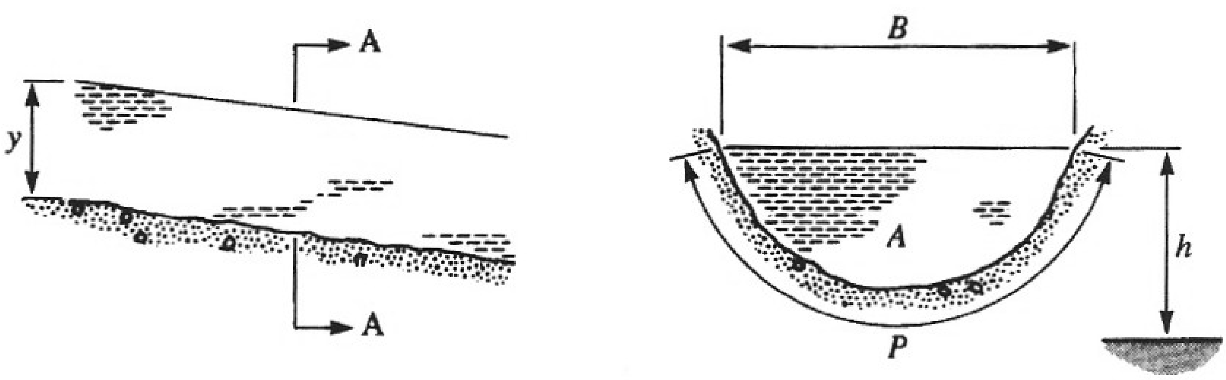
Natural channels (mainly rivers) are the most variable whilst man-made channels are more regular and thus hydrological theories are more accurate.

Properties

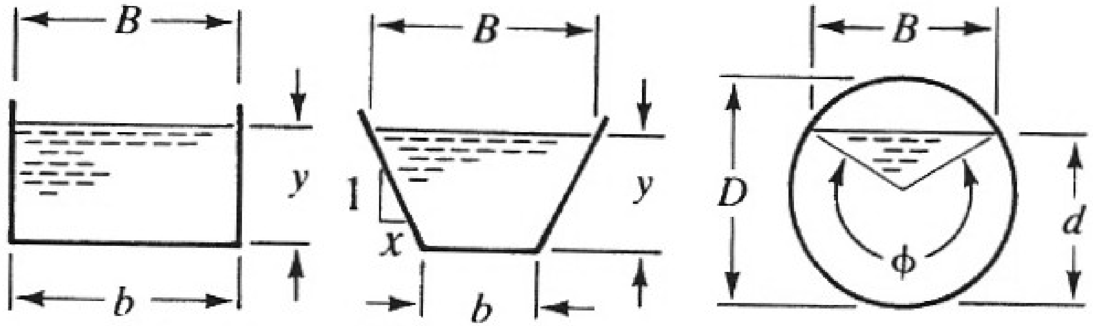
Properties used are:

- **Depth** (y): the vertical distance from the lowest point of the channel to the free surface;
- **Stage** (h): the vertical distance from an arbitrary datum to the free surface;
- **Area** (A): the cross sectional area of flow normal to the flow direction;
- **Wetted perimeter** (P): the length of the wetted surface measured normal to the flow;
- **Surface width** (B): the width fo the channel at the free surface;
- **Hydraulic radius** (R): the ration of area to wetted perimeter (A/P);
- **Hydraulic mean depth**: the ratio of area to surface width (A/B).

The properties on a general channel are thus:



For various shapes these properties are:



	Rectangle	Trapezoid	Circle
area, A	by	$(b+xy)y$	$\frac{1}{8}(\phi-\sin\phi)D^2$
wetted perimeter, P	$b+2y$	$b+2y\sqrt{1+x^2}$	$\frac{1}{2}\phi D$
top width, B	b	$b+2xy$	$\left(\sin\frac{\phi}{2}\right)D$
hydraulic radius, R	$\frac{by}{b+2y}$	$\frac{(b+xy)y}{b+2y\sqrt{1+x^2}}$	$\frac{1}{4}\left(1-\frac{\sin\phi}{\phi}\right)D$
hydraulic mean depth, D_m	y	$\frac{(b+xy)y}{b+2xy}$	$\frac{1}{8}\left(\frac{\phi-\sin\phi}{\sin(1/2\phi)}\right)D$

6.2 Basics of Channel Flow

Laminar and Turbulent Flow

For a pipe we saw that the Reynolds Number indicates the type of flow:

$$\mathrm{Re} = \frac{\rho D \bar{v}}{\mu}$$

For laminar flow, $\mathrm{Re} < 2000$ and for turbulent flow, $\mathrm{Re} > 4000$. These results can be applied to channels using the equivalent property of the hydraulic radius:

$$\mathrm{Re}_{\mathrm{Channel}} = \frac{\rho R \bar{v}}{\mu}$$

For a pipe flowing full, $R = D/4$, hence:

$$\mathrm{Re}_{\mathrm{Channel}} = \mathrm{Re}_{\mathrm{Pipe}}/4$$

Hence:

- Laminar channel flow: $\mathrm{Re}_{\mathrm{Channel}} < 500$
- Turbulent channel flow $\mathrm{Re}_{\mathrm{Channel}} > 1000$

Moody Diagrams for Channel Flow

Using the Darcy-Weisbach equation:

$$h_f = \frac{\lambda L \bar{v}^2}{2gD}$$

And substituting for channel properties: $R = D/4$ and $h_f / L = S_0$ where S_0 is the bed slope of the channel, we have:

$$S_0 = \frac{\lambda \bar{v}^2}{8gR}$$

Hence, for a channel

$$\lambda = \frac{8gRS_0}{\bar{v}^2}$$

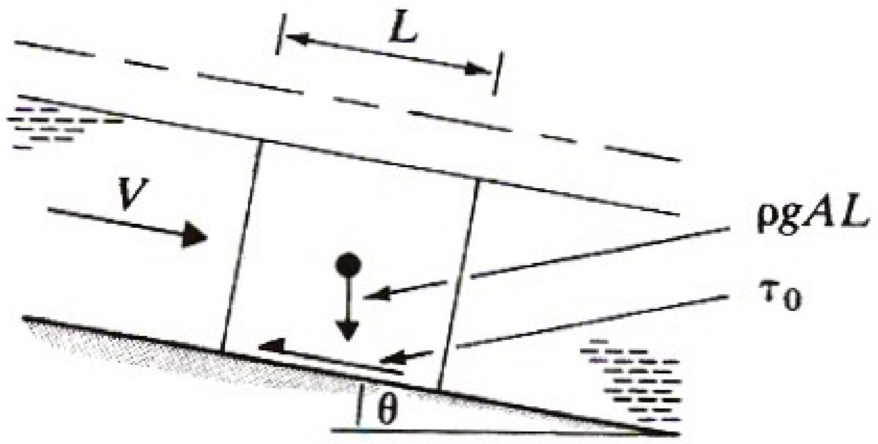
The $\lambda - \mathrm{Re}$ relationship for pipes is given by the Colebrook-White equation and so substituting $R = D/4$ and combining with Darcy's equation for channels gives:

$$\bar{v} = -2\sqrt{8gRS_0}\log\left(\frac{k_s}{14.8R} + \frac{0.6275\nu}{R\sqrt{8gRS_0}}\right)$$

A diagram, similar to that for pipes, can be drawn based on this equation to give channel velocities. This is not as straightforward though, since *R* varies along the length of a channel and the frictional resistance is far from uniform.

Friction Formula for Channels

For uniform flow, the gravity forces exactly balance those of the friction forces at the boundary, as shown in the diagram:



The gravity force in the direction of the flow is $\rho g A L \sin\theta$ and the shear force in the direction of the flow is $\tau_0 P L$, where τ_0 is the mean boundary shear stress. Hence:

$$\tau_0 P L = \rho g A L \sin\theta$$

Considering small slopes, $\sin\theta \approx \tan\theta \approx S_0$, and so:

$$\tau_0 = \frac{\rho g A S_0}{P} = \rho g R S_0$$

To estimate τ_0 further, we again take it that for turbulent flow:

$$\tau_0 \propto \overline{v}^2 \qquad \text{or} \qquad \tau_0 = K\overline{v}^2$$

Hence we have:

$$\overline{v} = \sqrt{\frac{\rho g}{K} R S_0}$$

Or taking out the constants gives the **Chézy Eqaution**:

$$\overline{v} = C\sqrt{RS_0}$$

In which C is known as the Chézy coefficient which is not entirely constant as it depends on the Reynolds Number and the boundary roughness.

From the Darcy equation for a channel we see:

$$C = \sqrt{\frac{8g}{\lambda}}$$

An Irish engineer, Robert Manning, presented a formula to give C, known as **Manning's Equation**:

$$C = \frac{R^{1/6}}{n}$$

In which n is a constant known as Manning's n. Using Manning;s Equation in the Chézy Equation gives:

$$\overline{v} = \frac{R^{2/3}\sqrt{S_0}}{n}$$

And the associated discharge is:

$$Q = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}\sqrt{S_0}$$

Manning's Equation is known to be both simple and reasonably accurate and is often used.

Evaluating Manning's *n*

This is essentially a roughness coefficient which determines the frictional resistance of the channel. Typical values for *n* are:

Channel type	Surface material and alignment	
river	earth, straight	0.02–0.025
	earth, meandering	0.03–0.05
	gravel (75–150 mm), straight	0.03–0.04
	gravel (75–150 mm), winding or braided	0.04–0.08
unlined canals	earth, straight	0.018–0.025
	rock, straight	0.025–0.045
lined canals	concrete	0.012–0.017
models	mortar	0.011–0.013
	Perspex	0.009

Example –Trapezoidal Channel

Problem

A concrete lined has base width of 5 m and the sides have slopes of 1:2. Manning's *n* is 0.015 and the bed slope is 1:1000:

1. Determine the discharge, mean velocity and the Reynolds Number when the depth of flow is 2 m;
2. Determine the depth of flow when the discharge is 30 cumecs.

Solution

The channel properties are:

$$A = (5+2y)y \qquad P = 5+2y\sqrt{1+2^2}$$

$$R = \frac{(5+4)2}{5+\left(2\times 2\sqrt{5}\right)} = 1.29 \text{ m}$$

1. Using the equation for discharge and for $y = 2$ m, we have:

$$\begin{aligned} Q &= \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}\sqrt{S_0} \\ &= \frac{1}{0.015}\frac{\left[(5+4)2\right]^{5/3}}{\left[5+\left(2\times 2\sqrt{5}\right)\right]^{2/3}}\sqrt{0.001} \\ &= 45 \text{ m}^3/\text{s} \end{aligned}$$

For the mean velocity, using the continuity equation:

$$\bar{v} = \frac{Q}{A} = \frac{45}{(5+4)2} = 2.5 \text{ m/s}$$

And for the Reynolds number we have:

$$\begin{aligned}\mathrm{Re} &= \frac{\rho R\bar{v}}{\mu} \\ &= \frac{10^3 \times 1.29 \times 2.5}{1.14 \times 10^{-3}} \\ &= 2.83 \times 10^6\end{aligned}$$

2. We have the following relationship between flow and depth:

$$\begin{aligned}Q &= \frac{1}{0.015}\frac{\left[(5+2y)y\right]^{5/3}}{\left[5+2\sqrt{5}y\right]^{2/3}}\sqrt{0.001} \\ &= 2.108\frac{\left[(5+2y)y\right]^{5/3}}{\left[5+2\sqrt{5}y\right]^{2/3}}\end{aligned}$$

This is a difficult equation to solve and a trial an error solution is best. Since $y = 2$ m gives us 45 cumecs, try $y = 1.7$ m:

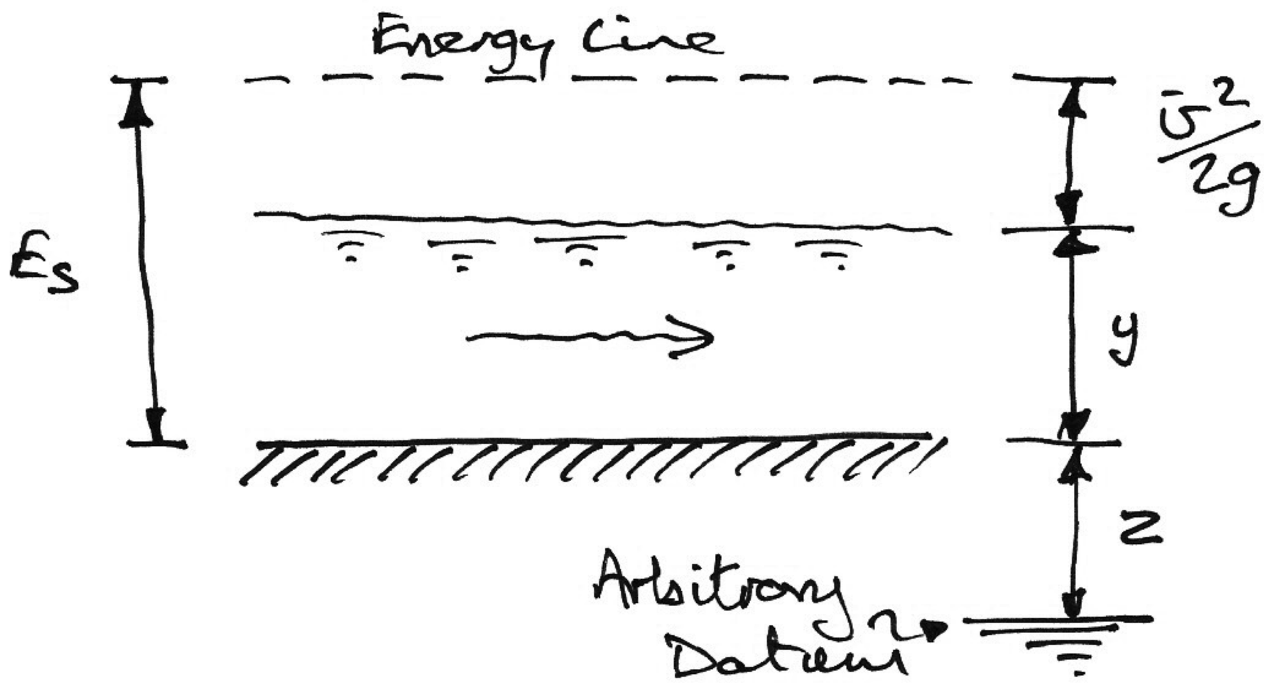
$$Q = 2.108\frac{\left[(5+2(1.7))1.7\right]^{5/3}}{\left[5+2\sqrt{5}(1.7)\right]^{2/3}} = 32.7 \text{ m}^3/\text{s}$$

Try $y = 1.6$ m to get $Q = 29.1 \text{ m}^3/\text{s}$. Using linear interpolation, the answer should be around $y = 1.63$ m for which $Q = 30.1 \text{ m}^3/\text{s}$ which is close enough. Hence for $Q = 30 \text{ m}^3/\text{s}$, $y = 1.63$ m.

6.3 Varying Flow in Open Channels

The Energy Equation

Assuming that the channel bed is has a very small slope, the energy lines are:



Hence Bernoulli's Equation is:

$$H = y + \frac{\overline{v}^2}{2g} + z$$

To avoid the arbitrary datum, we use a quantity called the **specific energy**, E_s:

$$E_s = y + \frac{\overline{v}^2}{2g}$$

For steady flow we can write this as:

$$E_s = y + \frac{(Q/A)^2}{2g}$$

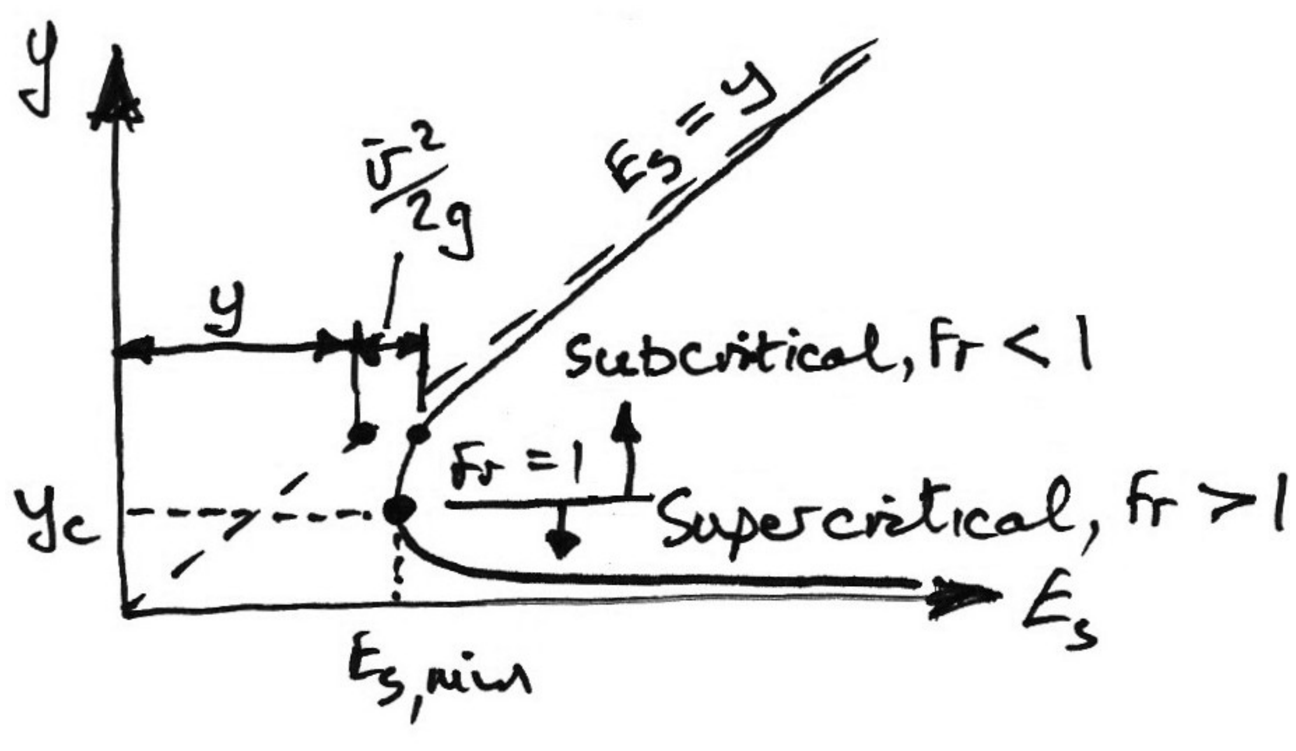
And if we consider a rectangular channel:

$$\frac{Q}{A} = \frac{bq}{by} = \frac{q}{y}$$

In which *q* is the mean flow per metre width of channel. Hence we have:

$$E_s = y + \frac{(q/y)^2}{2g}$$
$$(E_s - y)\,y^2 = \frac{q^2}{2g} = \text{constant}$$

This is a cubic equation in *y* for a given *q*:



Flow Characteristics

In this graph we have also identified the **Froude Number**, Fr:

$$\text{Fr} = \frac{\bar{v}}{\sqrt{gL}}$$

In which L is the characteristic length of the system. The different types of flows associated with Fr are:

- $\text{Fr} < 1$: Subcritical or tranquil flow;
- $\text{Fr} = 1$: critical flow;
- $\text{Fr} > 1$: Supercritical or rapid flow.

The Froude Number for liquids is analogous to Mach number for the speed of sound in air. In subcritical flow, a disturbance (waves) can travel up and down stream (from the point of view of a static observer). In supercritical flow, the flow is faster than the speed that waves travel at and so no disturbance travels upstream.

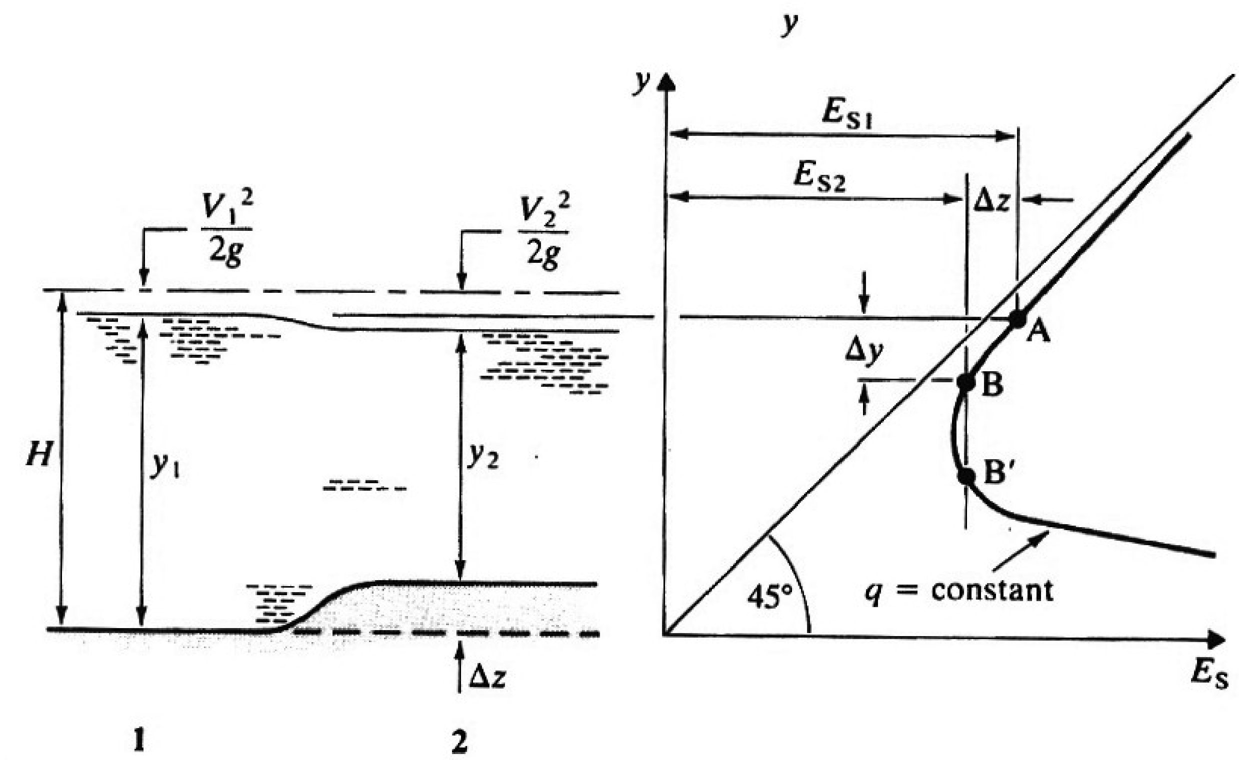
Associated with the critical flow, as shown on the graph, we have the critical depth:

$$y_c = \frac{Q^2}{gA^2}$$

A change in flow from subcritical to supercritical is termed a **hydraulic jump** and happens suddenly.

Flow Transition

Consider the situation shown where a steady uniform flow is interrupted by the presence of a hump in the streambed. The upstream depth and discharge are known; it remains to find the downstream depth at section 2.



Applying the energy equation, we have:

$$y_1 + \frac{\overline{v}_1^2}{2g} = y_2 + \frac{\overline{v}_2^2}{2g} + \Delta z$$

In addition we also have the continuity equation:

$$\overline{v}_1 y_1 = \overline{v}_2 y_2 = q$$

Combining we get:

$$y_1 + \frac{q^2}{2gy_1^2} = y_2 + \frac{q^2}{2gy_2^2} + \Delta z$$

Which gives:

$$2gy_2^3 + y_2^2\left(2g\Delta z - 2gy_1 - \frac{q^2}{y_1^2}\right) + q^2 = 0$$

Which is a cubic equation in y_2 which mathematically has three solutions, only one of which is physically admissible.

At this point refer to the specific energy curve. We see:

$$E_{s1} = E_{s2} + \Delta z$$

Also we see:

- point A on the graph represents conditions at section 1 of the channel;
- Section 2 must lie on either point B or B' on the graph;
- All points between 1 and 2 lie on the E_s graph between A and B or B';
- To get to B' the river would need to jump higher than Δz (since $E_{s1} - E_{s2} > \Delta z$ between B and B'). This is physically impossible (rivers jumping?!) and so section 2 corresponds to point B.

Example – Open Channel Flow Transition

Problem

The discharge in a rectangular channel of width 5 m and maximum depth 2 m is 10 cumecs. The normal depth of flow is 1.25 m. Determine the depth of flow downstream of a section in which the river bed rises by 0.2 m over 1.0 m length.

Solution

Flow properties:

$$\bar{v} = \frac{10}{5 \times 1.25} = 1.6 \text{ m/s}$$

Using

$$E_{s1} = E_{s2} + \Delta z$$

We have:

$$E_{s1} = y_1 + \frac{\bar{v}^2}{2g} = 1.25 + \frac{1.6^2}{2g} = 1.38 \text{ m}$$

$$\mathrm{E}_{s2} = y_2 + \frac{\left[10/(5y_2)\right]^2}{2g} = y_2 + \frac{2^2}{2gy_2^2}$$

$$\Delta \mathrm{z} = 0.2 \text{ m}$$

Hence:

$$1.38 = y_2 + \frac{2^2}{2gy_2^2} + 0.2$$
$$1.18 = y_2 + \frac{2}{gy_2^2}$$

Looking at the specific energy curve, point B must have a depth of flow less than $1.25 - 0.2 = 1.05$ m. Using a trial and error approach:

- $y_2 = 0.9$ m: Hence $E_{s2} = y_2 + \frac{2}{gy_2^2} = 0.9 + \frac{2}{g0.9^2} = 1.15 \neq 1.18$ m;
- $y_2 = 1.0$ m: Hence $E_{s2} = 1.0 + \frac{2}{g1.0^2} = 1.2 \neq 1.18$ m;
- $y_2 = 0.96$ m: Hence $E_{s2} = 0.96 + \frac{2}{g0.96^2} = 1.18$ m;

Hence the depth of flow below the hump is less than that above it due to the acceleration of the water caused by the need to maintain continuity.

Problems – Open Channel Flow

1. Measurements carried out on the uniform flow of water in a long rectangular channel 3 m wide and with a bed slope of 0.001, revealed that at a depth of flow of 0.8 m the discharge of water was 3.6 cumecs. Estimate the discharge of water using (a) the Manning equation and (b) the Darcy equation.

(Ans. 8.6 m^3/s, 8.44 m^3/s)

2. A concrete-lined trapezoidal channel has a bed width of 3.5 m, side slopes at 45° to the horizontal, a bed slope of 1 in 1000 and Manning roughness coefficient of 0.015. Calculate the depth of uniform flow when the discharge is 20 cumecs.

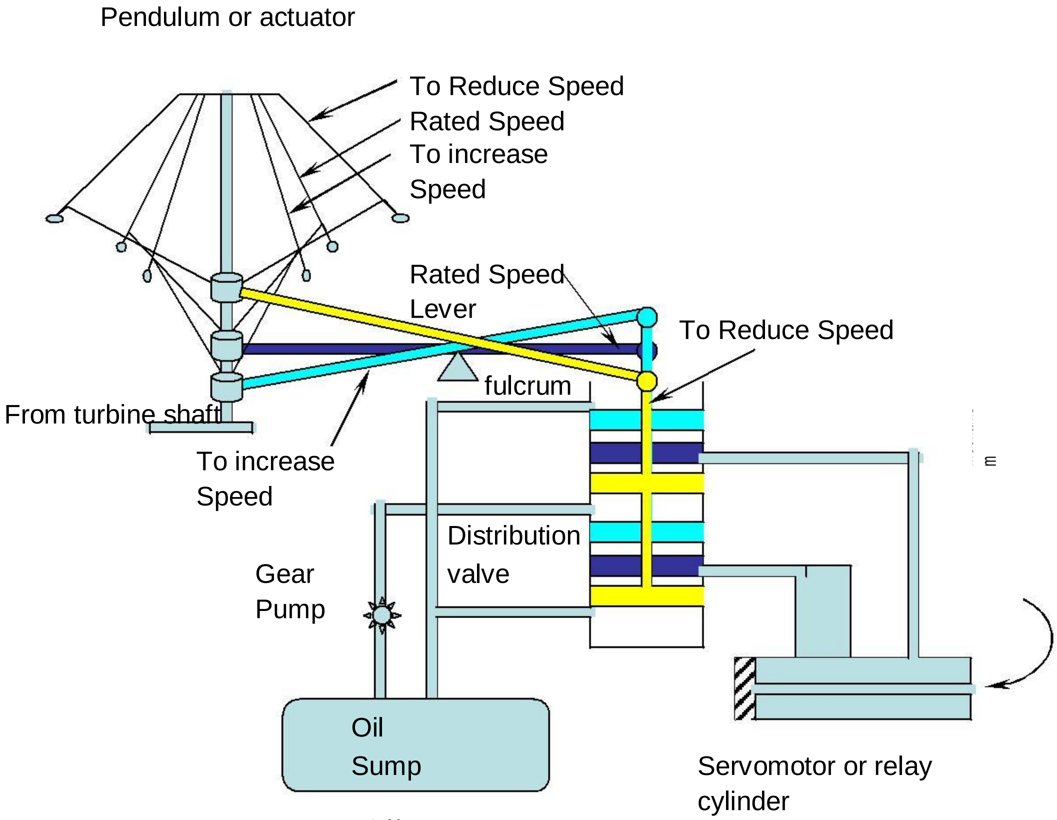
(Ans. 1.73 m)

LECTURE NOTES ON

Hydraulics and Hydraulic Machines

Department of Civil Engineering

Governors for Turbines



Oil pressure governor

A Governor is a mechanism to regulate the speed of the shaft of a turbine. The turbine is coupled to the shaft of the generator, which is generating power/electricity. The power generated should have uniform rating of current and frequency which in turn depends on the speed of the shaft of the turbine. Fig shows the oil pressure governor for a turbine. The main component parts of the governor are:

1. The servomotor or Relay cylinder
2. The distribution valve or control valve
3. Actuator or Pendulum
4. Oil Sump
5. Gear pump which runs by tapping power from the power shaft by belt drive
6. A pipe system communicating with the control valve, servomotor and the sump

When the turbine is subjected to its normal load, it runs at the normal speed *N*. When the load on the turbine increases or decreases the speed of the turbine also will accordingly decrease ot increase.

The oil pressure governor will restore the speed to the normal value. The normal position of the governor at the normal speed is shown in fig.

As the load on the turbine increases, the speed decreases in turn reducing the speed of the vertical bar of the governor. The fly balls of the centrifugal governor are brought to a lower level, thereby bringing the displacement lever downward. This through the fulcrum lifts the piston of the control valve and thereby opens the valve A and closes the valve B. Oil is pumped through valve A and into the servomotor, thereby pushing the piston of the servomotor backwards. This in turn increases the inlet area of the discharge into the turbine, thereby increasing the speed.

Similarly, with decrease in load on the turbine, the fly balls move farther away from the vertical shaft of the governor, thereby lifting the displacement lever upwards. This through the fulcrum lowers the piston of the control valve and thereby opens the valve B and closes the valve A. Oil is pumped through valve B and into the servomotor, thereby pushing the piston of the servomotor forwards. This in turn decreases the inlet area of the discharge into the turbine, thereby decreasing the speed.

In both the cases mentioned above, the process continues until the normal position is reached.

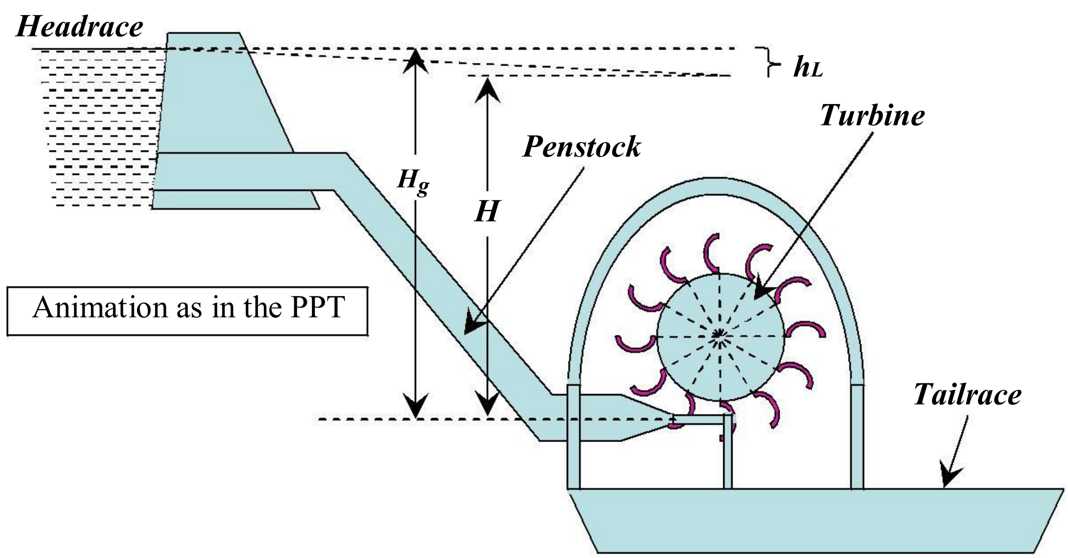
HYDRAULIC TURBINES

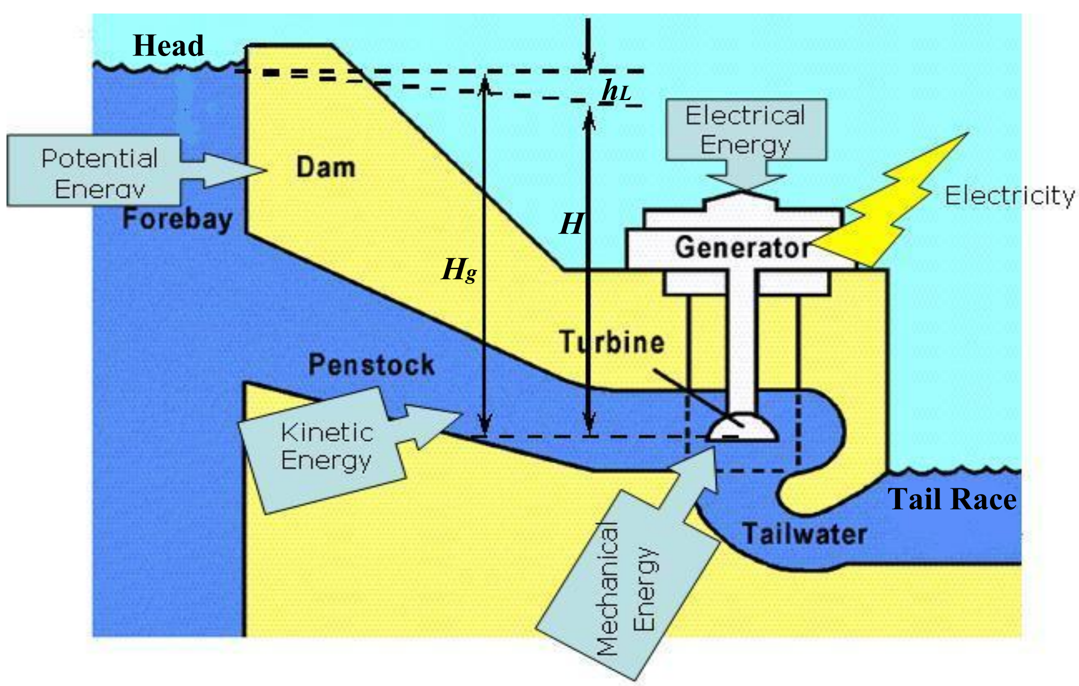
Introduction:

The device which converts h ydraulic energy into mechanical energy or vice versa is known as ***Hydraulic Machines* .** The h ydraulic machines which convert h ydraulic energy into mechanical energy are known as

Turbines and that convert mechanical energy into h ydraulic energy is known as ***Pumps* .**

Fig . shows a general layout of a h ydroelectric plant .





It consists of the following:

1 . A ***Dam*** constructed across a river or a channel to store water. The reservoir is also

known as ***Headrace.***

2 . Pipes of large diameter called ***Penstocks*** which carry water under pressure from storage reservoir to the turbines . These pipes are usuall y made of steel or reinforced concrete.

3 . ***Turbines*** having different t ypes of vanes or buckets or blades mounted on a wheel called runner.

4 . Tailrace which is a channel carrying water away from the turbine after the water has worked on the turbines . The water surface in the tailrace is also referred to as tailrace .

Important Terms:

Gross Head (H_g): It is the vertical difference between headrace and tailrace.

Net Head:(H): Net head or effective head is the actual head available at the inlet of the to work on the turbine .

$$H = H_g - h_L$$

Where h_L is the total head loss during the transit of water from the headrace to tailrace which is m ainl y head loss due to friction, and is given b y

$$h_f = \frac{4 f L V^2}{2 g d}$$

Where ***f*** is the coefficient of friction of penstock depending on the type of material of penstock

L is the total length of penstock

V is the mean flow velocit y of water through the p enstock

D is the diameter of penstock and

g is the acceleration due to gravit y

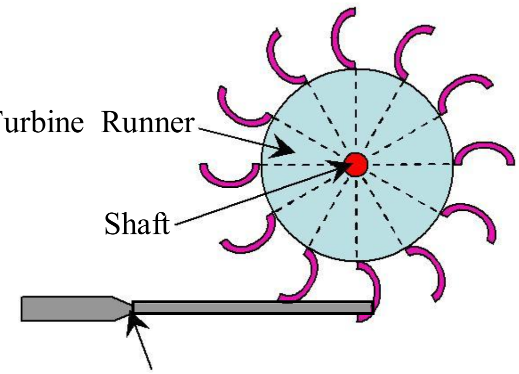
TYPES OF EFFICIENCIES

Depending on the considerations of input and output, the efficiencies can be classified as

(i) H ydraulic Efficiency

(ii) Mechanical Efficiency

(iii) Overall efficienc y



(i) H ydraulic Efficiency: (η_h)

It is the ratio of the power developed b y the runner of a turbine to the power supplied at the inlet of a turbine. Since the power supplied is hydraulic, and the probable loss is between the striking jet and vane it is rightly called hydraulic efficiency.

If R.P. is the Runner Power and W.P. is the Water Power

$$\eta_h = \frac{\text{R.P.}}{\text{W.P.}} \qquad (01)$$

7. Mechanical Efficiency: (ηm)

It is the ratio of the power available at the shaft to the power developed by the runner of a turbine. This depends on the slips and other mechanical problems that will create a loss of energy between the runner in the annular area between the nozzle and spear, the amount of water reduces as the spear is pushed forward and vice -versa .

and shaft which is purel y mechanical and hence mechanical efficiency.

If S . P . is the Shaft Power

$$\eta_m = \frac{\text{S.P.}}{\text{R.P.}} \qquad (02)$$

(iii) Overall Efficiency: (η)

It is the ratio of the power available at the shaft to the power supplied at the inlet of a turbine . As this covers overall problems of losses in energy, it is known as overall efficienc y. This depends on both the h ydraulic losses and the slips and other mechanical problems that will create a loss of energy between the jet power supplied and the power generated at the shaft available for coupling of the generator.

$$\eta = \frac{\text{S.P.}}{\text{W.P.}} \qquad (03)$$

From Eqs 1,2 and 3, we have

$\eta = \eta_h \text{ x } \eta_m$

Classification of Turbines

The h ydraulic turbines can be classified based on t ype of energy at the inlet, direction of flow through the vanes, head available at the inlet, discharge through the vanes and specific speed . They can b e arranged as per the following table:

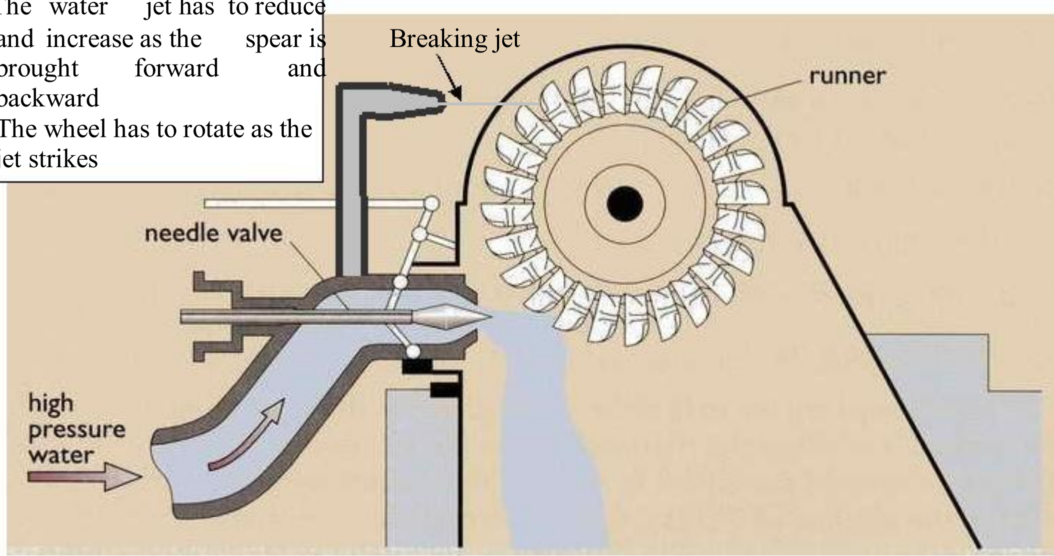
Turbine		Type of energy	Head	Discharge	Direction of flow	Specific Speed
Name	Type					
Pelton Wheel	Impulse	Kinetic	High Head > 250m to 1000m	Low	Tangential to runner	Low <35 Single jet 35 – 60 Multiple jet
Francis Turbine	Reaction Turbine	Kinetic + Pressure	Medium 60 m to 150 m	Medium	Radial flow Mixed Flow	Medium 60 to 300
Kaplan Turbine			Low < 30 m	High	Axial Flow	High 300 to 1000

As can be seen from the above table, an y specific t ype can be explained b y suitable construction of sentences b y selecting the other items in the table along the row .

PELTON WHEEL OR TURBINE

Pelton wheel, named after an eminent engineer, is an impulse turbine wherein the flow is tangential to the runner and the available energy at the entrance is completel y kinetic energy. Further, it is preferred at a very high head and low discharges with low specific speeds . The pressure available at the inlet and the outlet is atmospheric .

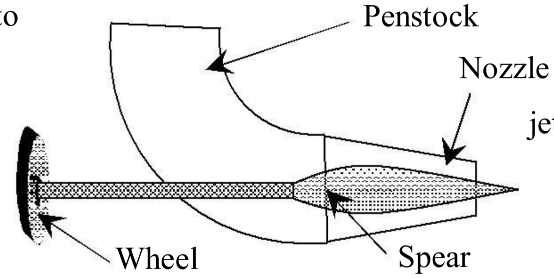
Animation:
(i) The water jet has to reduce and increase as the spear is brought forward and backward
(ii) The wheel has to rotate as the jet strikes



The main components of a Pelton turbine are:

(i) *Nozzle and flow regulating arrangement:*

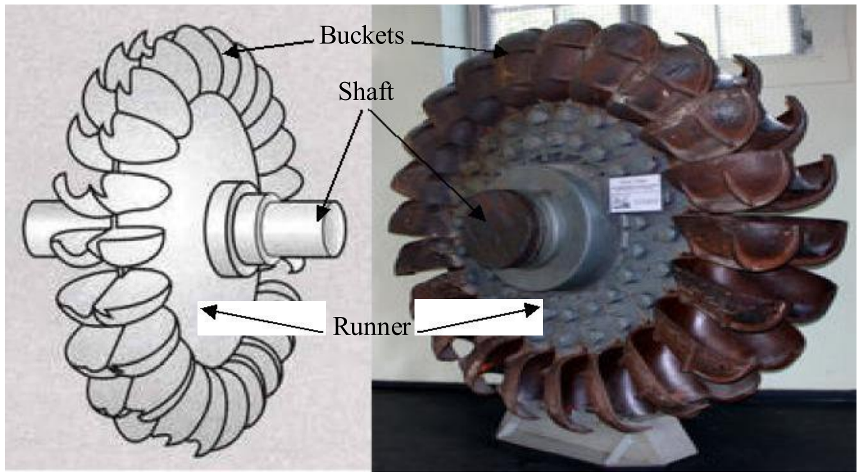
Water is brought to the h ydroelectric plant site through large penstocks at the end of which there will be a nozzle, which converts the pressure energy completel y into kinetic energy. This will convert the liquid flow into a high -speed jet, which strikes the buckets or vanes mounted on the runner, which in -turn rotates the runner of the turbine. The amount of water striking the vanes is controlled b y the forward and backward motion of the spear . As the water is flowing in the annular area between the annular area between the



nozzle opening and the spear, the flow gets reduced as the spear moves forward and vice - versa.

(ii) *Runner with buckets:*

Runner is a circular disk mounted on a shaft on the periphery of



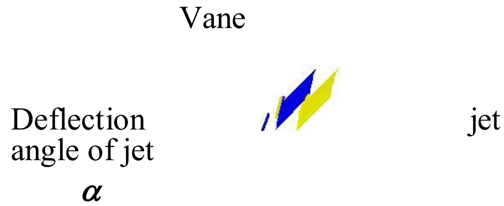
which a number of buckets are fixed equall y spaced as shown in Fig . The buckets are made of cast -iron cast -steel, bronze or stainless steel depending upon the head at the inlet of the turbine. The water jet strikes the bucket on the splitter of the bucket and gets deflected through (α) 160 - 170 ^{0} .

(iii) *Casing:*

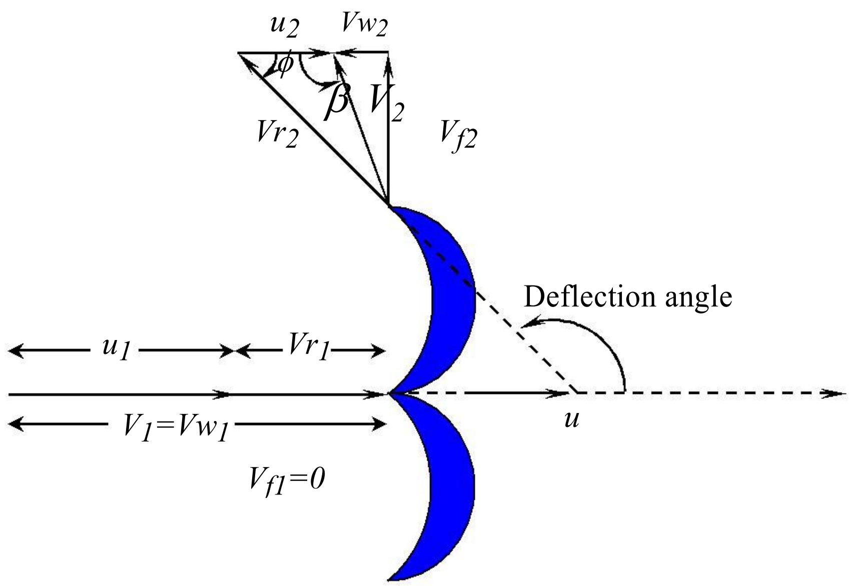
It is made of cast - iron or fabricated steel plates . The main function of the casing is to prevent splashing of water and to discharge the water into tailrace .

(iv) *Breaking jet:*

Even after the amount of water striking the buckets is comple tel y stopped, the runner goes on rotating for a very long time due to inertia. To stop the runner in a short time, a small nozzle is provided which directs the jet of water on the back of bucket with which the rotation of the runner is reversed . This jet i s called as breaking jet .



3 D Picture of a jet striking the splitter and getting split in to two parts and deviating.



Velocit y triangles for the jet striking the bucket

From the impulse -momentum theorem, the force with which the jet strikes tthe bucket along the direction of vane is given b y

F_x = rate of change of momentum of the jet along the direction of vane motion

F_x = (Mass of water / second) x change in velocit y along the x direction

$$=\rho a V_1 \left[V_{w1} - (-V_{w2})\right]$$

$$=\rho a V_1 \left[V_{w1} + V_{w2}\right]$$

Work done per second b y the jet on the vane is given b y the product of Force exerted on the vane and the distance moved b y the vane in one second

W . D . /S = F_x x u

$$=\rho a V_1 \left[V_{w1} + V_{w2}\right] u$$

Input to the jet per second = Kinetic energy of the jet per second

$$=\frac{1}{2} \rho a V_1^3$$

Efficiency of the jet $= \dfrac{Output / \sec ond}{Input / \sec ond} = \dfrac{Workdone / \sec ond}{Input / \sec ond}$

$$\eta = \frac{\rho a V_1 \left[V_{w1} + V_{w2}\right] u}{\frac{1}{2} \rho a V_1^3}$$

$$\eta = \frac{2u\left[V_{w1} + V_{w2}\right]}{V_1^2}$$

From inlet velocit y triangle, $V_{w1} = V_1$

Assuming no shock and ignoring frictional losses through the vane, we have $V_{r1} = V_{r2} = (V_1 - u_1)$

In case of Pelton wheel, the inlet and outlet are located at the same radial distance from the centre of runner and hence $u_1 = u_2 = u$

From outlet velocit y triangle, we have $V_{w2} = V_{r2}\ Cos\ \phi - u_2$

$$= (V_1 - u) Cos\ \phi - u$$

$$F_x = \rho a V_1 \left[V_1 + (V_1 - u) Cos\phi - u\right]$$

$$F_x = \rho a V_1 (V_1 - u)\left[1 + Cos\phi\right]$$

Substituting these values in the above equation for efficiency, we have

$$\eta = \frac{2u\left[V_1 + (V_1 - u)\cos\phi - u\right]}{V_1^2}$$

$$\eta = \frac{2u}{V_1^2}\left[(V_1 - u) + (V_1 - u)\cos\phi\right]$$

$$\eta = \frac{2u}{V_1^2}(V_1 - u)[1 + \cos\phi]$$

The above equation gives the efficiency of the jet striking the vane in case of Pelton wheel .

To obtain the maximum efficiency for a given jet velocit y and vane angle, from maxima -minima, we have

$$\frac{d\eta}{du} = 0$$

$$\Rightarrow \frac{d\eta}{du} = \frac{2}{V_1^2}[1 + \cos\phi]\frac{d}{du}\left(uV_1 - u^2\right) = 0$$

$$V_1 - 2u = 0$$

or $$u = \frac{V_1}{2}$$

i . e . When the bucket speed is maintained at half the velocity of the jet, the efficiency of a Pelton wheel will be maximum . Substituting we get,

$$\eta_{\max} = \frac{2u}{(2u)^2}(2u - u)[1 + \cos\phi]$$

$$\eta_{\max} = \frac{1}{2}[1 + \cos\phi]$$

From the above it can be seen that more the value of cos ϕ, more will be the efficiency. Form maximum efficiency, the value of cos ϕ should be 1 and the value of ϕ should be 0^0 . This condition makes the jet to completel y deviate by 180^0 and this, forces the jet striking the bucket to strike the successive bucket on the back of it acting like a breaking jet . Hence to avoid this situation, at least a small angle of $\phi = 5^0$ should be provided .

Dec -06/Jan07

6 a. i)Sketch the layout of a PELTON wheel turbine showing the details of nozzle, buckets and wheel when the turbine axis is horizontal(04) ii) Obtain an expression for maximum - efficiency of an impulse turbine. (06)

July 06

6 (a) With a neat sketch explain the l ayout of a h ydro -electric plant (06)

(b) With a neat sketch explain the parts of an Impulse turbine. (06)

Jan 06

6 (a) What Is specific speed of turbine and state Its significance. (04)

(b) Draw a neat sketch of a h ydroelectric plant and mention the function of each component . (08)

Jan 05

6 (a) Classify the turbines based on head, specific speed and h ydraulic actions . Give examples for each . (06)

(b) What is meant b y Governing of turbines? Explain with a neat sketch the governing of an impulse turbine (06)

July 04

5 (a) Explain the classification of turbines . (08)

The head at the base of the nozzle of a Pelton wheel is 640 m . The outlet vane angle of the bucket is 15 ^{o} . The relative velocit y at the outlet is reduced b y 15% due to friction along the vanes . If the discharge at outlet is without whirl find the ratio of bucket speed to the jet speed . If the jet diameter is 100 mm while the wheel diameter is 1 . 2 m, find the spe ed of the turbine in rpm, the force exerted b y the jet on the wheel, the Power developed and the h ydraulic efficiency. Take C_v =0.97.

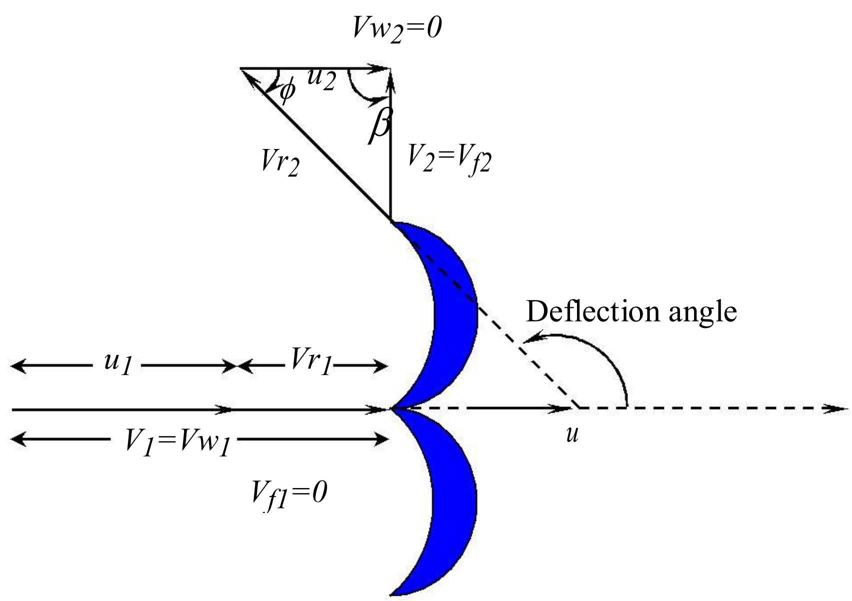
Solution:

H = 640 m; ϕ=15 ^{o} ; V_{r1} = 0 . 85 V_{r2} ; V_{w2} = 0; d = 100 mm; D = 1 . 2 m;

C_v = 0 . 97; K_u = ?; N = ?; F_x = ?; P = ?; η_h = ?

We know that the absolute velocit y of jet is given b y

$V = C_v\sqrt{2\,g\,H} = 0.97\sqrt{2\times 10\times 640}$ =109.74 m/s



Let the bucket speed be u

Relative velocit y at inlet = $V_{r1} = V_1 - u$ = 109 . 74 - u

Relative velocit y at outlet = V_{r2} = (1 -0 . 15)V_{r1} = 0 . 85(109 . 74 - u)

But $V_{r2}\cos\phi = u \Rightarrow$ 0 . 85(109 . 74 -u)cos15

Hence u = 49 . 48 m/s

But $u = \dfrac{\pi D N}{60}$ and hence

$N = \frac{60\,u}{\pi D} = \frac{60 \times 49.48}{\pi \times 1.2} = 787.5$ rpm (Ans)

Jet ratio = $m = \frac{u}{V} = \frac{49.48}{109.74} = 0.45$

Weight of water supplied = $\gamma Q = 10 \times 1000 \times \frac{\pi}{4} \times 0.1^2 \times 109.74^2 = 8.62$ kN/s

Force exerted = $F_x = \rho a V_1 \left(V_{w1} - V_{w2}\right)$

But $V_{w1} = V_1$ and $V_{w2} = 0$ and hence

$F_x = 1000 \times \frac{\pi}{4} \times 0.1^2 \left(109.74\right)^2 = 94.58$ kN

Work done/second = F_x x u = 94 . 58 x 49 . 48 = 4679 . 82 kN/s

Kinetic Energy/second = $\frac{1}{2} \rho a V_1^3 = \frac{1}{2} \times 1000 \times \frac{\pi}{4} \times 0.1^2 \times 109.74^3 = 5189.85$ kN/s

H ydraulic Efficiency = $\eta_h = \frac{\text{Work done/s}}{\text{Kinetic Energy/s}} = \frac{4679.82}{5189.85} \times 100 = 90.17\%$

Dec 06 -Jan 07

A PE LTON wheel turbine is having a mean runner diameter of 1 . 0 m and is running at 1000 rpm . The net head is 100 . 0 m . If the side clearance is 20° and discharge is 0 . 1 m^3/s, find the power available at the nozzle and h ydraulic efficiency of the turbine . (10)

Solution:

D = 1 . 0 m; N = 1000 rpm; H = 100 . 0 m; $\phi = 20^o$; Q = 0 . 1 m^3/s; WD/s = ? and η_h = ?

Assume $C_v = 0.98$

We know that the velocit y of the jet is given b y

$V = C_v \sqrt{2\,g\,H} = 0.98\sqrt{2 \times 10 \times 1000} = 43.83$ m/s

The absolute velocity of the vane is given b y

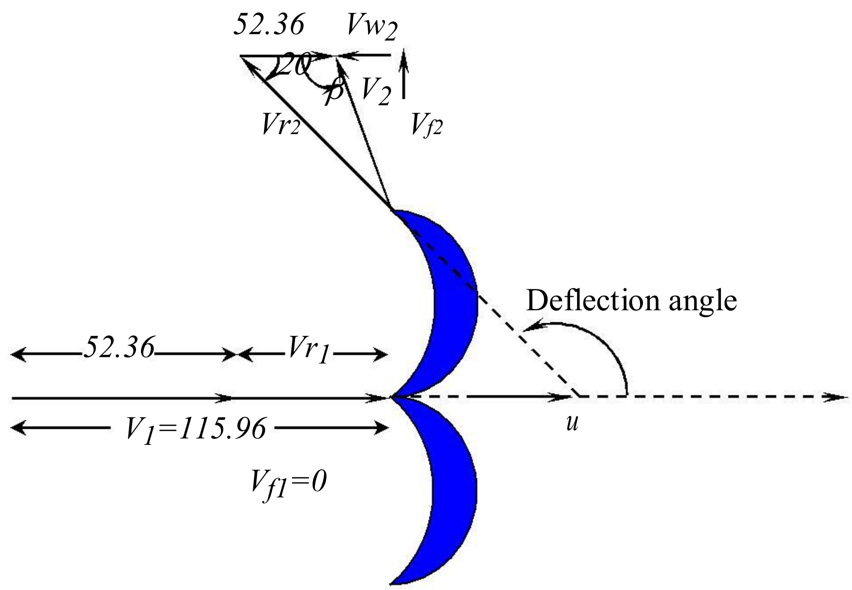
$u = \frac{\pi D N}{60} = \frac{\pi \times 1 \times 1000}{60} = 52.36$ m/s

This situation is impracticable and hence the data has to be modified . Clearl y state the assumption as follows:

Assume H = 700 m (Because it is assumed that the t yping and seeing error as 100 for 700)

Absolute velocit y of the jet is given b y

$$V = C_v\sqrt{2gH} = 0.98\sqrt{2 \times 10 \times 700} = 115.96 \text{ m/s}$$



Power available at the n ozzle is the given b y work done per second

WD/second = $\gamma Q H = \rho g Q H$ = 1000x10x0 . 1x700 = 700 kW

H ydraulic Efficiency is given b y

$$\eta_h = \frac{2u}{V_1^2}(V - u)[1 + \cos\phi] = \frac{2 \times 52.36}{115.96^2}(115.96 - 52.36)(1 + \cos 20) = 96.07\%$$

July 06

A Pelton wheel has a mean bucket speed of 10 m/s with a jet of water flowing at the rate of 700 lps under a head of 30 m . The buckets deflect the jet through an angle of 160° . Calculate the power given b y water to the runner and the h ydraulic efficiency of the turbine . Assume the coefficient of nozzle as 0 . 98 . (08)

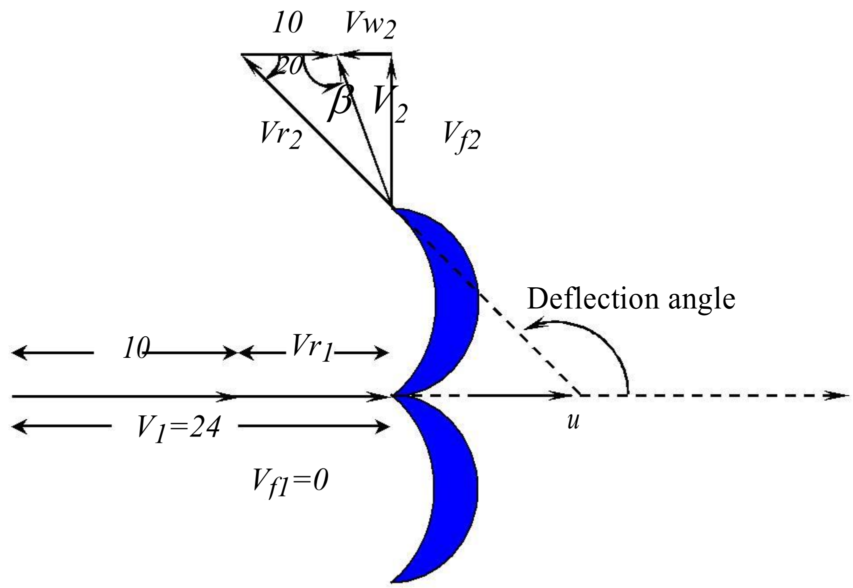
Solution:

u = 10 m/s; Q = 0 . 7 m^3/s; ϕ = 180 -160 = 20^o; H = 30 m; C_v = 0 . 98;

WD/s = ? and η_h = ?

Assume g = 10m/s^2

$V = C_v \sqrt{2\,g\,H} = 0.98\ \sqrt{2 \times 10 \times 30} = 24$ m/s



$V_{r1} = V_1 - u = 24 - 10 = 14$ m/s

Assuming no shock and frictional losses we have $V_{r1} = V_{r2} = 14$ m/s

$V_{w2} = V_{r2}\ Cos\ \phi - u = 14 \times Cos\ 20 - 10 = 3.16$ m/s

We know that the Work done b y the jet on the vane is given by WD/s $= \rho$ $aV_1 \left[V_{w1} + V_{w2}\right] u = \rho\, Q\, u \left[V_{w1} + V_{w2}\right]$ as $Q = aV_1$

$= 1000 \times 0.7 \times 10 \left[24 + 3.16\right] = 190.12$ kN -m/s (Ans)

IP/s = KE/s = $\frac{1}{2} \rho a V_1^3 = \frac{1}{2} \rho Q V_1^2 = \frac{1}{2} \times 1000 \times 0.7 \times 24^2 = 201.6$ kN -m/s

H ydraulic Efficiency = Output/ Input = 190 . 12/201 . 6 = 94 . 305% It can also be directly calculated b y the derived equation as

$\eta_h = \frac{2u}{V_1^2}\left(V_1 - u\right)\left[1 + \cos\phi\right] = \frac{2 \times 10}{24^2}\left(24 - 10\right)\left[1 + \cos 20\right] = 94.29\%$ (Ans)

Jan 06

A Pelton wheel has to develop 13230 kW under a net head of 800 m while running at a speed of 600 rpm . If the coefficient of Jet $C_y = 0.97$, speed ratio ϕ=0 . 46and the ratiooftheJetdiameteris 1 /16 of wheel diameter . Calculate

i) Pitch circle diameter ii) the diameter of jet

iii) the quantit y of water supplied to the wheel

iv) the number of Jets required .

Assume over all efficiency as 85%. (08)

Solution:

P = 13239 kW; H = 800 m; N = 600 rpm; C_v = 0 . 97; ϕ = 0 . 46 (Speed ratio) d/D = 1/16; η_o = 0 . 85; D = ?; d = ?; n = ?;

Assume g = 10 m/s^2 and ρ = 1000 kg/m^3

We know that the overall efficiency is given b y

$$\eta_o = \frac{Output}{Input} = \frac{P}{\gamma Q H} = \frac{13239 \times 10^3}{10 \times 1000 \times Q \times 800} = 0.85$$

Hence Q = 1 . 947 m^3/s (Ans)

Absolute velocit y of jet is given b y

$$V = C_v\sqrt{2\,g\,H} = 0.97\sqrt{2 \times 10 \times 800} = 122.696 \text{ m/s}$$

Absolute velocit y of vane is given b y

$$u = \phi\sqrt{2\,g\,H} = 0.46\sqrt{2\times10\times800} = 58.186 \text{ m/s}$$

The absolute velocity of vane is also given b y

$$u = \frac{\pi D N}{60} \text{ and hence}$$

$$D = \frac{60\,u}{\pi N} = \frac{60\times58.186}{\pi\times600} = 1.85\ m \text{ (Ans)}$$

$$d = \frac{1.85}{16} = 115\ .\ 625 \text{ mm (Ans)}$$

Discharge per jet = $q = \frac{\pi}{4} d^2 \times V = \frac{\pi}{4} \times 0.115625^2 \times 122.696 = 1.288 \text{ m}^3/\text{s}$

No . of jets = $n = \frac{Q}{q} = \frac{1.947}{1.288} \approx 2$ (Ans)

July 05

Design a Pelton wheel for a head of 80m . and speed of 300 RPM . The Pelton wheel develops 110 kW . Take co - eficient of velocit y= 0 . 98, speed ratio= 0 . 48 and overall efficiency = 80%. (10)

Solution:

H = 80 m; N = 300 rpm; P = 110 kW; C_v = 0 . 98, K_u =0 . 48; η_o = *0 . 80*

Assume $g = 10$ m/s ^{2} and $\rho = 1000$ kg/m ^{3}

We know that the overall efficiency is given b y

$$\eta_o = \frac{Output}{Input} = \frac{P}{\gamma Q H} = \frac{110 \times 10^3}{10 \times 1000 \times Q \times 80} = 0.8$$

Hence $Q = 0.171875$ m ^{3} /s

Absolute velocit y of jet is given b y

$V = C_v \sqrt{2 g H} = 0.98 \sqrt{2 \times 10 \times 80} = 39.2$ m/s

Absolute velocit y of vane is given b y $u = \phi$

$\sqrt{2 g H} = 0.48 \sqrt{2 \times 10 \times 80} = 19.2$ m/s

The absolute velocity of vane is also given b y

$u = \frac{\pi D N}{60}$ and hence

$D = \frac{60 u}{\pi N} = \frac{60 \times 19.2}{\pi \times 300} = 1.22\ m$ (Ans)

Single jet Pelton turbine is assumed

The diameter of jet is given b y the discharge continuit y equation

$Q = \frac{\pi}{4} d^2 \times V = \frac{\pi}{4} \times d^2 \times 39.2 \Rightarrow 0.171875$

Hence $d = 74.7$ mm

The design parameters are

Single jet

Pitch Diameter = 1 . 22 m

Jet diameter = 74 . 7 mm

Jet Ratio = $m = \frac{D}{d} = \frac{1.22}{0.0747} = 16.32$

No . of Buckets = 0 . 5x m + 15 = 24

Jan 05

It is desired to generate 1000 kW of power and survey reveals that 450 m of static head and a minimum flow of 0 . 3 m ^{3} /s are available. Comment whether the task can be accomplished b y installing a Pelton wheel run at 1000 rpm and having an overall efficiency of 80% .

Further, design the Pelton wheel assuming suitable data for coefficient of velocit y and coefficient of drag. (08)

Solution:

P = 1000 kW; H = 450 m; Q = 0 . 3 m^3/s; N = 1000 rpm; η_o = 0 . 8

Assume C_v = 0 . 98; K_u =0 . 45; ρ = 1000 kg/m^3 ; g = 10 m/s^2

$$\eta_o = \frac{Output}{Input} = \frac{P}{\gamma Q H} = \frac{1000 \times 10^3}{10 \times 1000 \times 0.3 \times 450} = 0.74$$

For the given conditions of P, Q and H , it is not possible to achieve the desired efficiency of 80%.

To decide whether the task can be accomplished b y a Pelton turbine compute the specific sp eed N_s

$$N_s = \frac{N\sqrt{P}}{H^{5/4}};$$

where N is the speed of runner, P is the power developed in kW and H is the head available at the inlet .

$$N_s = \frac{1000\sqrt{1000}}{450^{5/4}} = 15.25 < 35$$

Hence the installation of single jet Pelton wheel is justified . Absolute velocit y of jet is given b y

$$V = C_v\sqrt{2gH} = 0.98\sqrt{2 \times 10 \times 450} = 92.97 \text{ m/s}$$

Absolute velocit y of vane is given b y $u = \phi$

$$\sqrt{2gH} = 0.48\sqrt{2\times10\times80} = 19.2 \text{ m/s}$$

The absolute velocity of vane is also given b y

$$u = \frac{\pi D N}{60} \text{ and hence}$$

$$D = \frac{60u}{\pi N} = \frac{60\times19.2}{\pi\times300} = 1.22\ m \text{ (Ans)}$$

Single jet Pelton turbine is assumed

The diameter of jet is given b y the discharge continuit y equation

$Q = \frac{\pi}{4} d^2 \times V = \frac{\pi}{4} \times d^2 \times 39.2 \Rightarrow 0.171875$

Hence $d = 74.7$ mm

The design parameters are

Single jet

Pitch Diameter = 1 . 22 m

Jet diameter = 74 . 7 mm

Jet Ratio = $m = \frac{D}{d} = \frac{1.22}{0.0747} = 16.32$

No . of Buckets = 0 . 5x m + 15 = 24

July 04

A double jet Pelton wheel develops 895 MKW with an overall efficienc y of 82% under a head of 60m . The speed ratio = 0 . 46, jet ratio = 12 and the nozzle coefficient = 0 . 97 . Find the jet diameter, wheel diameter and wheel speed in RPM . (12)

Solution:

No . of jets = $n = 2$; $P = 895$ kW; $\eta_o = 0.82$; $H = 60$ m; $K_u = 0.46$; $m = 12$;

$C_v = 0.97$; $D = ?$; $d = ?$; $N = ?$

We know that the absolute velocit y of jet is given b y

$V = C_v \sqrt{2 g H} = 0.97\sqrt{2 \times 10 \times 60} = 33.6$ m/s

The absolute velocity of vane is given b y

$u = K_u \sqrt{2 g H} = 0.46\sqrt{2 \times 10 \times 60} = 15.93$ m/s

Overall efficiency is given b y

$\eta_o = \frac{P}{\gamma Q H}$ and hence $Q = \frac{P}{\gamma \eta H} = \frac{895 \times 10^3}{10 \times 10^3 \times 0.82 \times 60} = 1.819$ m^3/s

Discharge per jet = $q = \frac{Q}{n} = \frac{1.819}{2} = 0.9095$ m^3/s

From discharge continuit y equation, discharge per jet is also given b y

$q = \frac{\pi d^2}{4} V = \frac{\pi d^2}{4} \times 33.6 \Rightarrow 0.9095$

$d = 0.186\ m$

Further, the jet ratio $m = 12 = \frac{D}{d}$

Hence $D = 2.232$ m

Also $u = \frac{\pi D N}{60}$ and hence $N = \frac{60\,u}{\pi D} = \frac{60 \times 15.93}{\pi \times 2.232} = 136$ rpm

Note: Design a Pelton wheel: Width of bucket = 5 *d* and depth of bucket is 1 . 2 *d*

The following data is related to a Pelton wheel:

Head at the base of the nozzle = 80m; Diameter of the jet = 100 mm; Discharge of the nozzle = 0 . 3m^3/s; Power at the shaft = 206 kW; Power absorbed in mechanical resistance = 4 . 5 kW . Determine (i) Power lost in the nozzle and (ii) Power lost due to h ydraulic resistanc e in the runner .

Solution:

H = 80 m; d = 0 . 1m; $a = \frac{1}{4}\pi d^2$ = 0.007854 m^2; Q = 0.3 m^3/s; SP = 206 kW; Power absorbed in mechanical resistance = 4.5 kW.

From discharge continuity equation, we have, $Q = a$ x

V = 0.007854 x $V \Rightarrow 0.3$

V = 38.197 m/s

Power at the base of the nozzle = $\rho g Q H$

= 1000 x 10 x 0.3 x 80 = 240 kW Power

corresponding to the kinetic energy of the jet = ½ $\rho a V^3$

= 218.85 kW

(i) Power at the base of the nozzle = Power of the jet + Power lost in the nozzle

Power lost in the nozzle = 240 — 218.85 = 21.15 kW (Ans)

(ii) Power at the base of the nozzle = Power at the shaft + Power lost in the (nozzle + runner + due to mechanical resistance)

Power lost in the runner = 240 – (206 + 21.15 + 4.5) = 5.35 kW (Ans)

The water available for a Pelton wheel is 4 m^3/s and the total head from reservoir to the nozzle is 250 m. The turbine has two runners with two jets per runner. All the four jets have the same diameters. The pipeline is 3000 m long. The efficiency if power transmission through the pipeline and the nozzle is 91% and efficiency of each runner is 90%. The velocity coefficient of each nozzle is 0.975 and coefficient of friction *4f* for the pipe is 0.0045. Determine:

(i) The power developed by the turbine; (ii) The diameter of the jet and (iii) The diameter of the pipeline.

Solution:

$Q = 4$ m^3/s; $H_g = 250$ m; No. of jets = $n = 2 \times 2 = 4$; Length of pipe = $l = 3000$ m;

Efficiency of the pipeline and the nozzle = 0.91 and Efficiency of the runner =

$\eta_h = 0.9$; $C_v = 0.975$; $4f = 0.0045$

Efficiency of power transmission through pipelines and nozzle =

$$\eta = \frac{H_g - h_f}{H_g} \Rightarrow 0.91 = \frac{250 - h_f}{250}$$

Hence $h_f = 22.5$ m

Net head on the turbine = $H = H_g - h_f = 227.5$ m

Velocity of jet = $V_1 = C_v . \sqrt{2\,g\,H} = 0.975\sqrt{2 \times 10 \times 227.5} = 65.77$ m/s

(i) Power at inlet of the turbine = WP = Kinetic energy/second = ½ $\rho\, a\, V^3$

WP = ½ x 4 x 65.77^2 = 8651.39 kW

$$\eta_h = \frac{\text{Power developed by turbine}}{\text{WP}} = \frac{\text{Power developed by turbine}}{8651.39} \Rightarrow 0.9$$

Hence power developed by turbine = 0.9 x 8651.39 = 7786.25 kW (Ans)

(ii) Discharge per jet $= q = \dfrac{\text{Total discharge}}{\text{No. of jets}} = \dfrac{4.0}{4} = 1.0$ m^3/s

But $q = \dfrac{\pi}{4} d^2 \times V_1 \Rightarrow 1.0 = \dfrac{\pi}{4} d^2 \times 65.77$

Diameter of jet = d = 0.14 m (Ans)

(iii) If D is the diameter of the pipeline, then the head loss through the pipe is given by =

h_f

$$h_f = \frac{4fLV^2}{2gD} = \frac{fLQ^2}{3D^5} \qquad \text{(From } Q=aV\text{)}$$

$$h_f = \frac{0.0045 \times 3000 \times 4^2}{3D^5} \Rightarrow 22.5$$

Hence D = 0.956 m (Ans)

The three jet Pelton wheel is required to generate 10,000 kW under a net head of 400 m. The blade at outlet is 15^{o} and the reduction in the relative velocity while passing over the blade is 5%. If the overall efficiency of the wheel is 80%, C_v = 0.98 and the speed ratio = 0.46, then find: (i) the diameter of the jet, (ii) total flow (iii) the force exerted by a jet on the buckets (iv) The speed of the runner.

Solution:

No of jets = 3; Total Power P = 10,000 kW; Net head H = 400 m; Blade angle = $\phi = 15^{o}$; $V_{r2} = 0.95\ V_{r1}$; Overall efficiency = $\eta_o = 0.8$; $C_v = 0.98$; Speed ratio = K_u = 0.45; Frequency = f = 50 Hz/s.

We know that $\eta_o = \dfrac{P}{\rho g Q H} \Rightarrow 0.8 = \dfrac{10{,}000 \times 10^3}{1000 \times 10 \times Q \times 400}$

$Q = 3.125\ m^3/s$ (Ans)

Discharge through one nozzle = $q = \dfrac{Q}{n} = \dfrac{3.125}{3} = 1.042\ m^3/s$

Velocity of the jet = $V_1 = C_v \sqrt{2gH} = 0.98\sqrt{2 \times 10 \times 400} = 87.65\ m^3/s$

But $q = \dfrac{\pi}{4} d^2 \times V_1 \Rightarrow 1.042 = \dfrac{\pi}{4} d^2 \times 87.65$

d = 123 mm (Ans)

Velocity of the Vane = $u = K_u \sqrt{2gH} = 0.46\sqrt{2 \times 10 \times 400} = 41.14\ m^3/s$

$V_{r1} = (V_1 - u_1) = 87.65 - 41.14 = 46.51$ m/s

$V_{r2} = 0.95\ V_{r1} = 0.95 \times 46.51 = 44.18$ m/s

$V_{w1} = V_1 = 87.65$ m/s

$V_{w2} = V_{r2} \cos\phi - u_2 = 44.18 \cos 15 - 41.14 = 1.53$ m/s

Force exerted by the jet on the buckets = $F_x = \rho q (V_{w1} + V_{w2})$

F_x = 1000 x 1.042 (87.65+1.53) = 92.926 kN (Ans)

Jet ratio = $m = \frac{D}{d} \Rightarrow 10$ (Assumed)

D = 1.23 m

$$u = \frac{\pi D N}{60}$$

Hence $N = \frac{60\,u}{\pi D} = \frac{60 \times 41.14}{\pi \times 1.23}$ =638.8 rpm (Ans)

Reaction Turbines

Reaction turbines are those turbines which operate under hydraulic pressure energy and part of kinetic energy. In this case, the water reacts with the vanes as it moves through the vanes and transfers its pressure energy to the vanes so that the vanes move in turn rotating the runner on which they are mounted.

The main types of reaction turbines are

8. ***Radially outward flow reaction turbine:*** This reaction turbine consist a cylindrical disc mounted on a shaft and provided with vanes around the perimeter. At inlet the water flows into the wheel at the centre and then glides through radially provided fixed guide vanes and then flows over the moving vanes. The function of the guide vanes is to direct or guide the water into the moving vanes in the correct direction and also regulate the amount of water striking the vanes. The water as it flows along the moving vanes will exert a thrust and hence a torque on the wheel thereby rotating the wheel. The water leaves the moving vanes at the outer edge. The wheel is enclosed by a water-tight casing. The water is then taken to draft tube.

9. ***Radially inward flow reaction turbine:*** The constitutional details of this turbine are similar to the outward flow turbine but for the fact that the guide vanes surround the moving vanes. This is preferred to the outward flow turbine as this turbine does not develop racing. The centrifugal force on the inward moving body of water decreases the relative velocity and thus the speed of the turbine can be

   controlled easily.

The main component parts of a reaction turbine are:

(1) Casing, (2) Guide vanes (3) Runner with vanes (4) Draft tube

Casing: This is a tube of decreasing cross-sectional area with the axis of the tube being of geometric shape of volute or a spiral. The water first fills the casing and then enters the guide vanes from all sides radially inwards. The decreasing cross-sectional area helps the velocity of the entering water from all sides being kept equal. The geometric shape helps the entering water avoiding or preventing the creation of eddies..

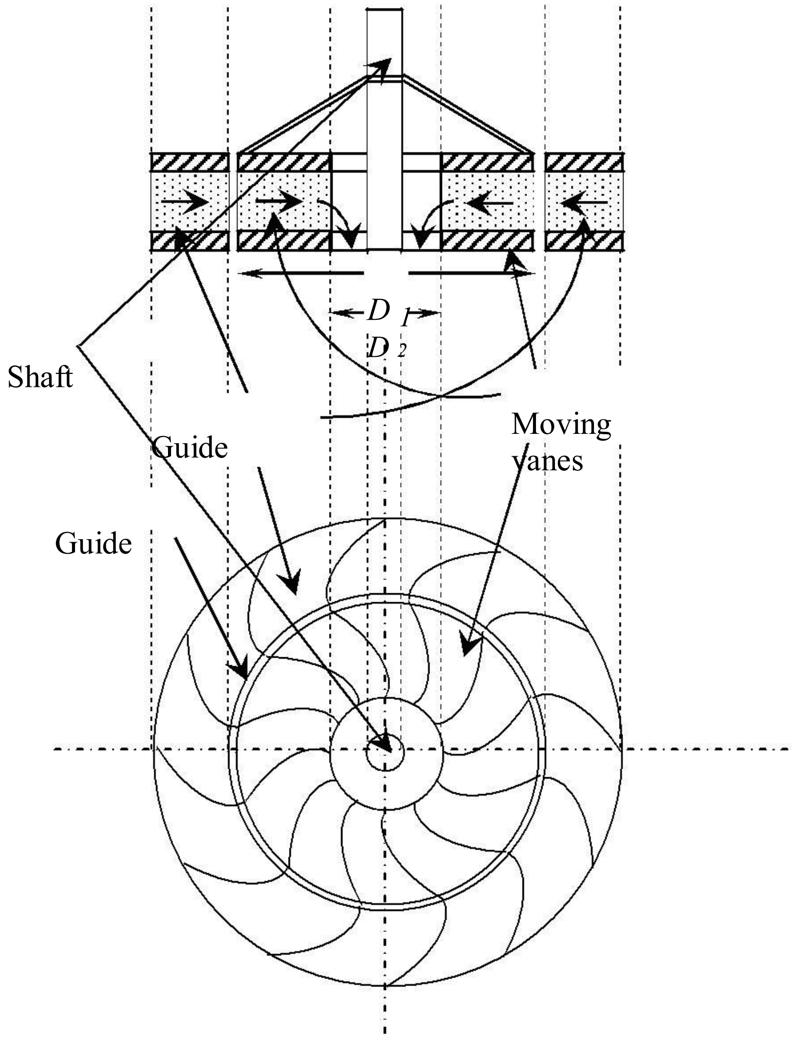
Guide vanes: Already mentioned in the above sections.

Runner with vanes: The runner is mounted on a shaft and the blades are fixed on the runner at equal distances. The vanes are so shaped that the water reacting with them will pass through them thereby passing their pressure energy to make it rotate the runner.

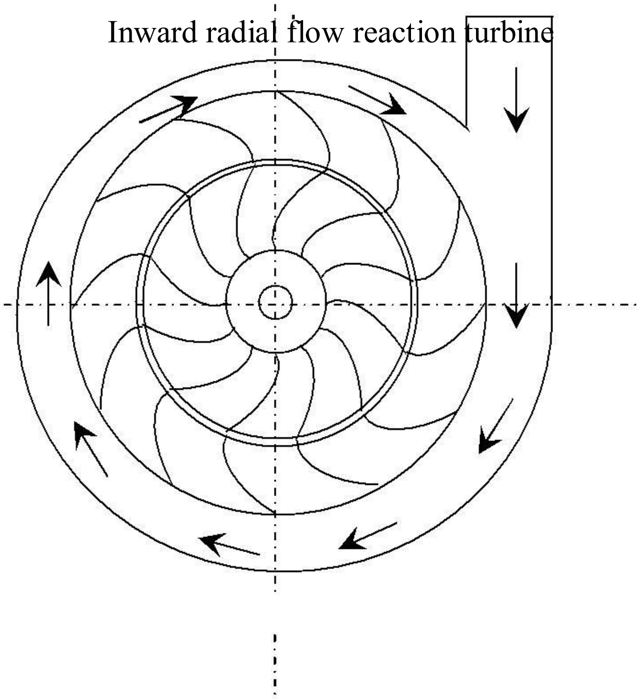
Draft tube: This is a divergent tube fixed at the end of the outlet of the turbine and the other end is submerged under the water level in the tail race. The water after working on the turbine, transfers the pressure energy there by losing all its pressure and falling below atmospheric pressure. The draft tube accepts this water at the upper end and increases its pressure as the water flows through the tube and increases more than atmospheric pressure before it reaches the tailrace.

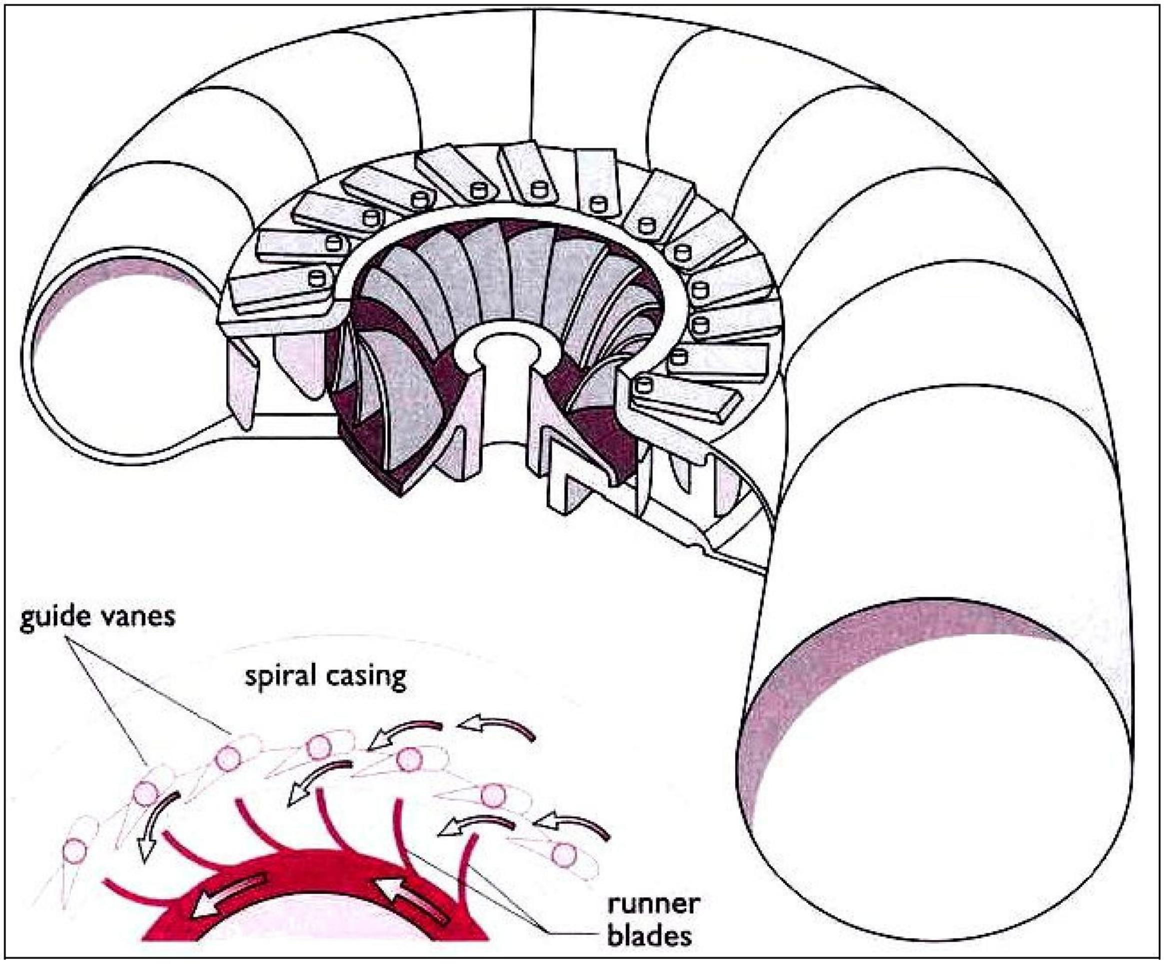
(iv)***Mixed flow reaction turbine***: This is a turbine wherein it is similar to inward flow reaction turbine except that when it leaves the moving vane, the direction of water is turned from radial at entry to axial at outlet. The rest of the parts and functioning is same as that of the inward flow reaction turbines.

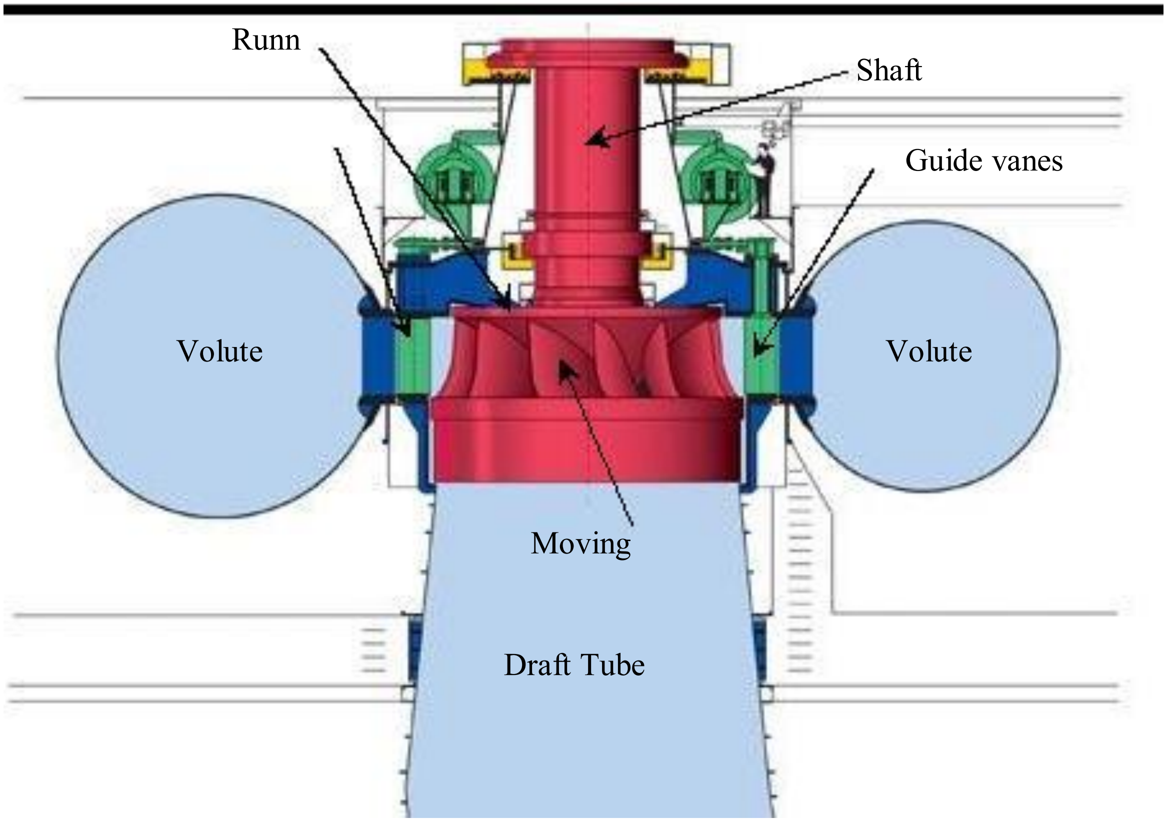
(v) ***Axial flow reaction turbine***: This is a reaction turbine in which the water flows parallel to the axis of rotation. The shaft of the turbine may be either vertical or horizontal. The lower end of the shaft is made larger to form the ***boss*** or the ***hub***. A number of vanes are fixed to the boss. When the vanes are composite with the boss the turbine is called ***propeller turbine***. When the vanes are adjustable the turbine is called a ***Kaplan turbine***.



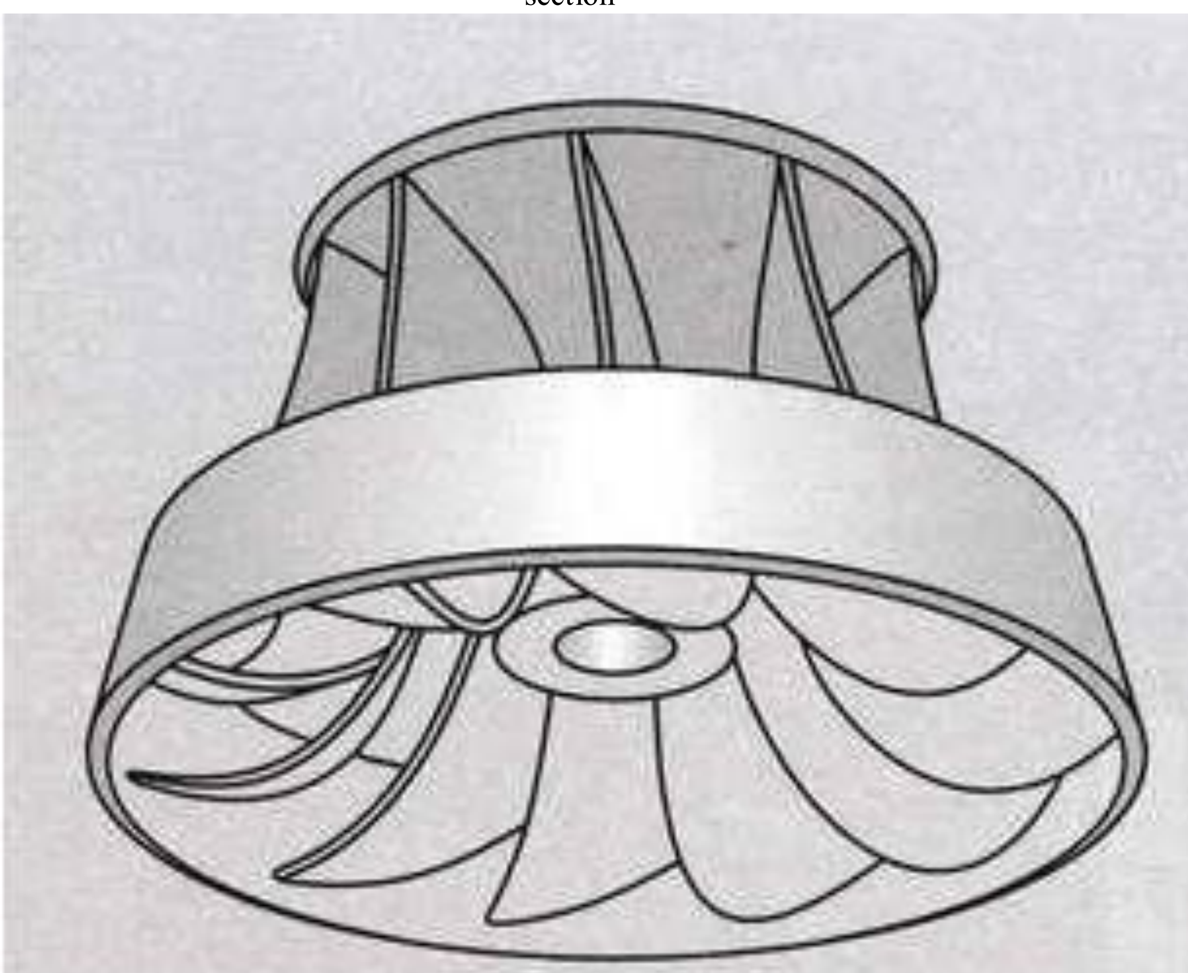
Inward radial flow reaction turbine

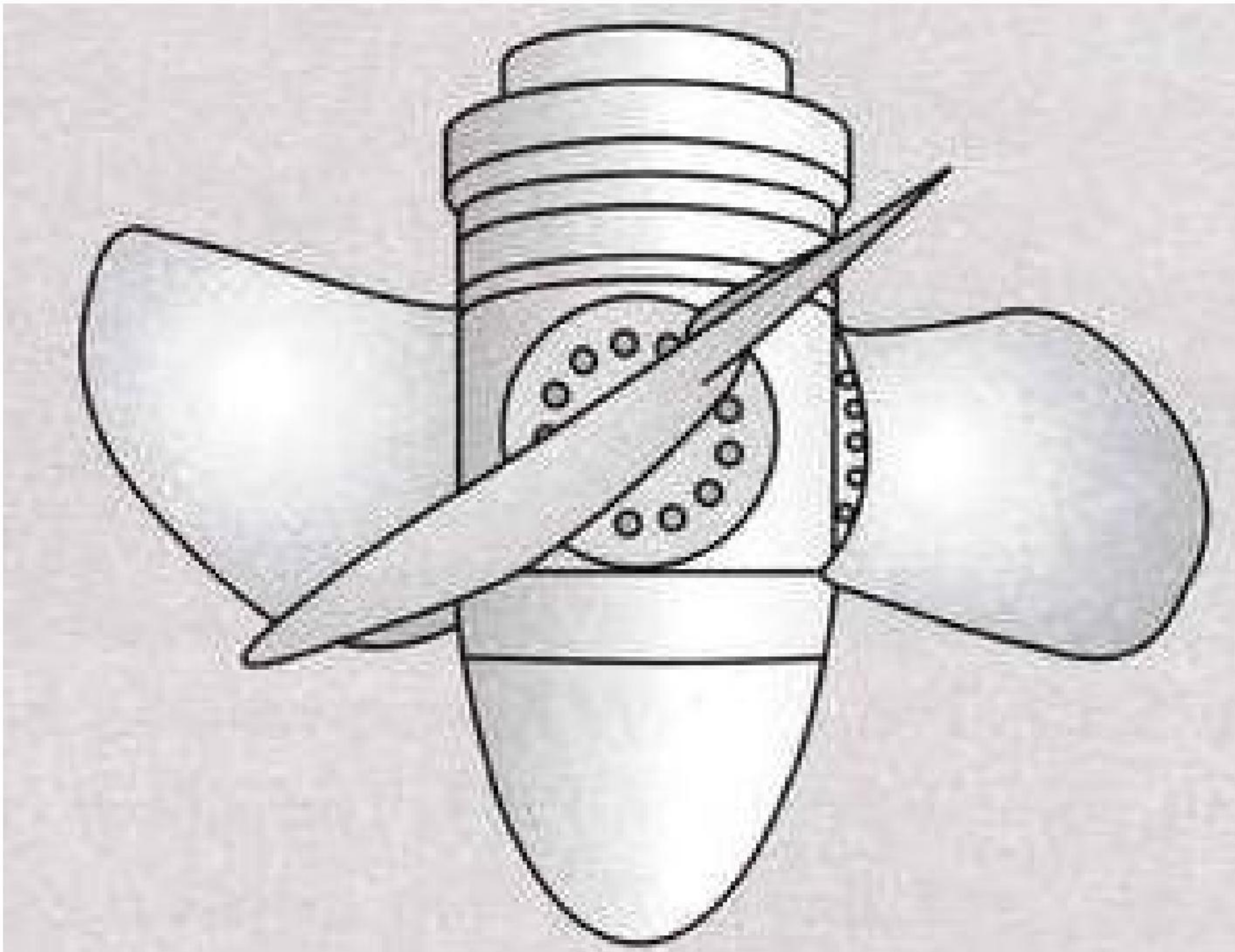


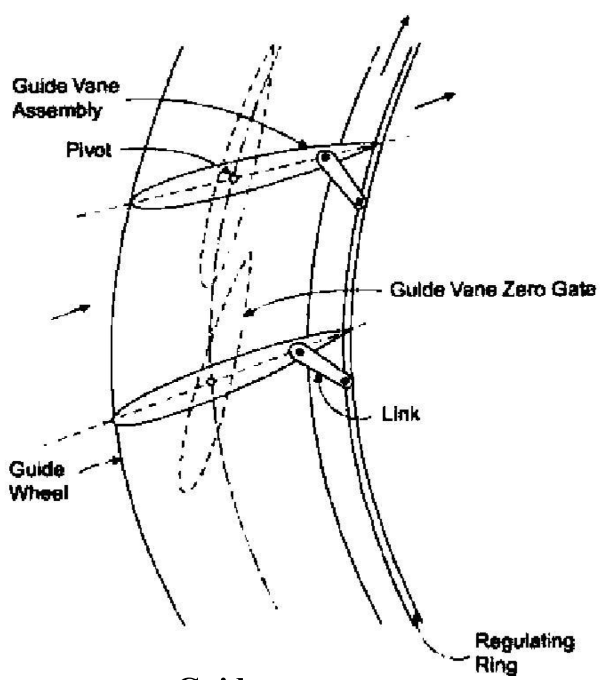

guide vanes
spiral casing
runner
blades



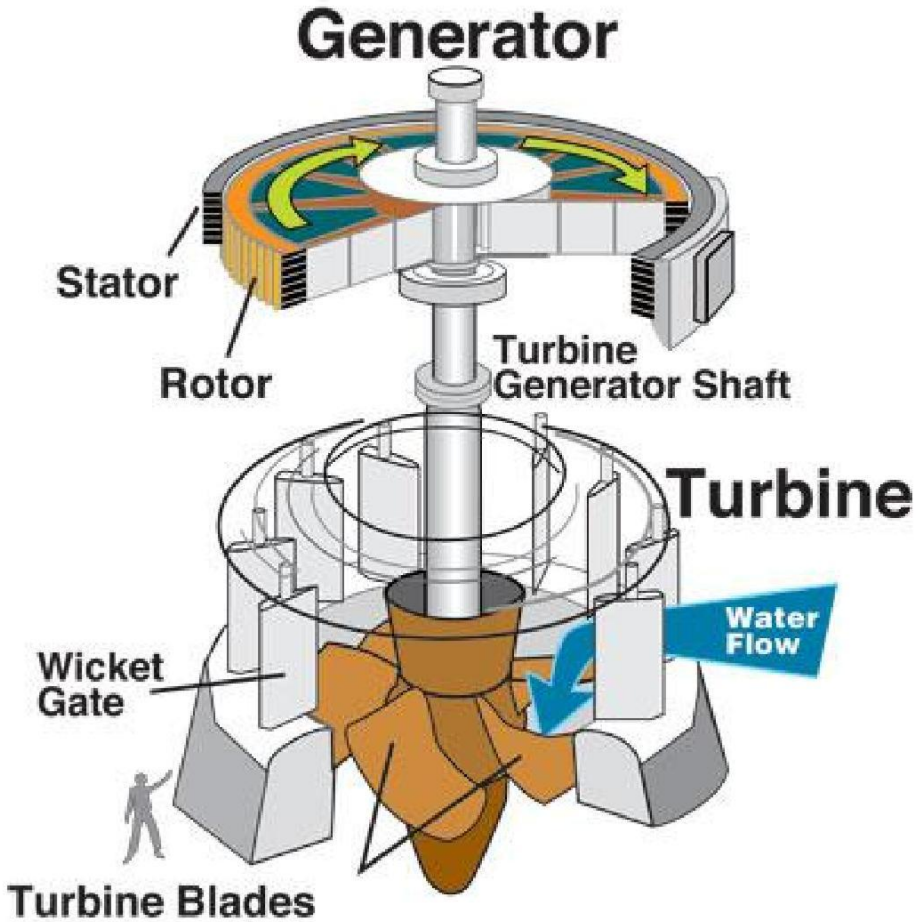
Francis Turbine Cross - section

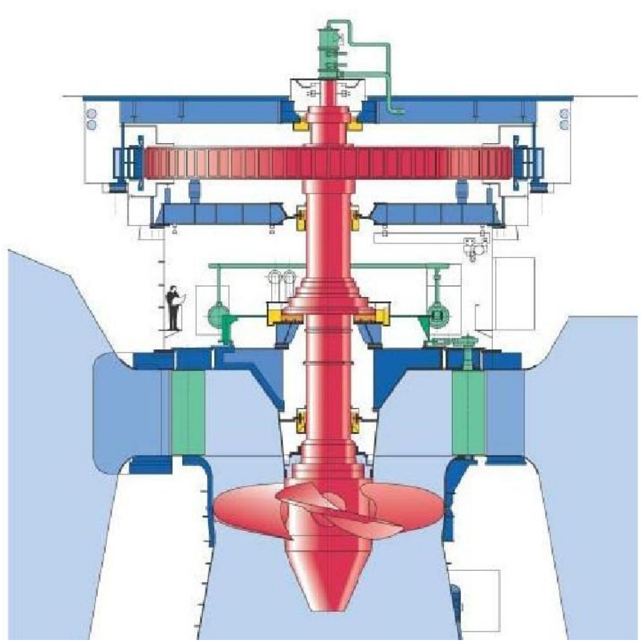




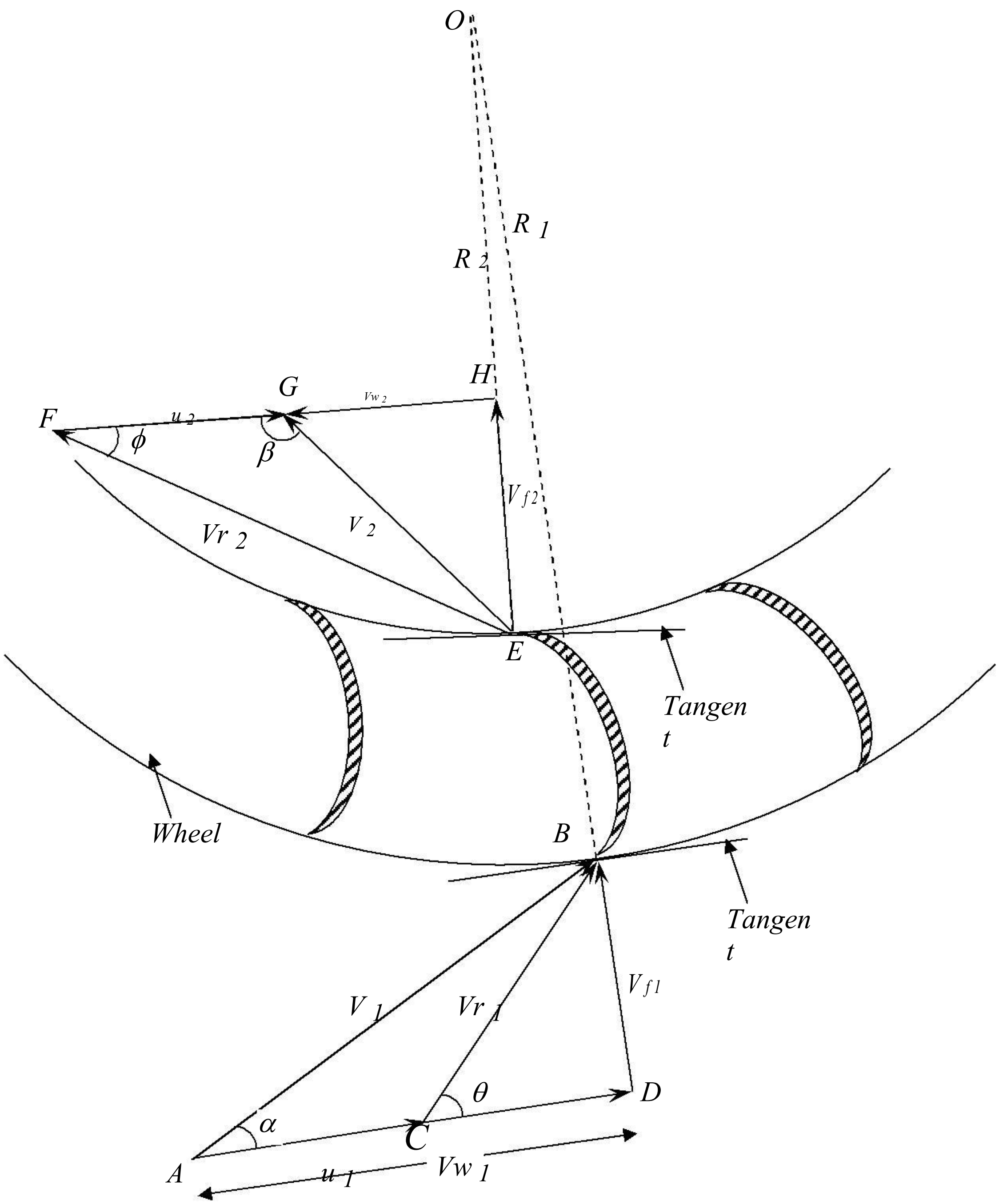

Guide Vane
Assembly
Pivot
Guide Vane Zero Gate
Link
Guide
Wheel
Regulating
Ring

Guide vane


Generator
Stator
Rotor
Turbine
Generator Shaft
Turbine
Water
Flow
Wicket
Gate
Turbine Blades



Derivation of the efficiency of a reaction turbine



Let

R_1 = Radius of wheel at inlet of the v ane

R_2 = Radius of wheel at outlet of the vane

ω = Angular speed of the wheel

Tangential speed of the vane at inlet = $u_1 \quad = \omega R_1$

Tangential speed of the vane at outlet = $u_2 = \omega R_2$

The velocit y triangles at inlet and outlet are drawn as shown in Fig .

α and β are the angles between the absolute velocities of jet and vane at inlet and outlet respectivel y

θ and ϕ are vane angles at inlet and outlet respectivel y

The mass of water striking a series of vanes per second = $\rho a V_1$

where a is the area of jet or flow a nd V_1 is the velocit y of flow at inlet . The momentum of water striking a series of vanes per second at inlet is given b y the product of mass of water striking per second and the component of velocity of flow at inlet

= $\rho a V_1$ x V_{w1} (V_{w1} is the velocit y component of flow at inlet along tangential direction)

Similarl y momentum of water striking a series of vanes per second at outlet is given b y

= $\rho a V_1$ x $(-V_{w2})$ (V_{w2} is the velocit y component of flow at outlet along

component is acting in the opposite direction)

Now angular momentum per second at inlet is given b y the product of momentum of water at inlet and its radial distance = $\rho a V_1$ x V_{w1} x R_1

And angular momentum per second at inlet is given b y = $-\rho a V_1$ x V_{w2} x R_2

Torque exerted by water on the wheel is given by impulse momentum theorem as the rate of change of angular momentum

$T = \rho a V_1$ x V_{w1} x $R_1 - -\rho a V_1$ x V_{w2} x R_2

$T = \rho a V_1 (V_{w1} R_1 + V_{w2} R_2)$

Workdone per second on the wheel = Torque x Angular velocity = T x ω

WD/s = $\rho a V_1 (V_{w1} R_1 + V_{w2} R_2)$ x ω

= $\rho a V_1 (V_{w1} R_1$ x $\omega + V_{w2} R_2$ x $\omega)$

As $u_1 = \omega R_1$ and $u_2 = \omega R_2$, we can simplify the above equation as

WD/s = $\rho a V_1 (V_{w1} u_1 + V_{w2} u_2)$

In the above case, always the velocit y of whirl at outlet is given b y both magnitude and direction as $V_{w2} = (Vr_2 \cos\phi - u_2)$

If the discharge is radial at outlet, then $V_{w2} = 0$ and hence the equation reduces to

WD/s = $\rho\, a\, u_1\, V_1\, V_{w1}$

KE/s = ½ $\rho\, a\, V_1^{\,3}$

Efficiency of the reaction turbine is given b y

$$(\iota\iota\iota) = \frac{\text{Workdone/second}}{\text{Kinetic Energy/second}} = \frac{\rho a V_1 \left(V_{w1} u_1 + V_{w2} u_2\right)}{\frac{1}{2}\rho a V_1^3}$$

$$(\iota\varpi) = \frac{2\left(V_{w1} u_1 + V_{w2} u_2\right)}{V_1^2}$$

Note: The value of the velocit y of whirl at outlet is to be substituted as along with its sign .

$V_{w2} = (Vr_2 \cos\phi - u_2)$

Summary

(i) Speed ratio = $\dfrac{u_1}{\sqrt{2gH}}$ where H is the Head on turbine

(ii) Flow ratio = $\dfrac{V_{f1}}{\sqrt{2gH}}$ where V_{f1} is the velocit y of flow at inlet

(iii) Discharge flowing through the reaction turbine is given b y

$$Q = \pi D_1 B_1 V_{f1} = \pi D_2 B_2 V_{f2}$$

Where D_1 and D_2 are the diameters of runner at inlet and exit

B_1 and B_2 are the widths of runner at inlet and exit

V_{f1} and V_{f2} are the Velocit y of flow at inlet and exit

If the thickness (t) of the vane is to be considered, then the area through which flow takes place is given b y ($\pi D_1 - nt$) where n is the number of vanes mounted on the runner.

Discharge flowing through the reactio n turbine is given b y

$$Q = (\pi D_1 - nt) B_1 V_{f1} = (\pi D_2 - nt) B_2 V_{f2}$$

(iv) The head (H) on the turbine is given b y $H = \dfrac{p_1}{\rho g} + \dfrac{V_1^2}{2g}$

Where p_1 is the pressure at inlet .

(v) Work done per second on the runner = $\rho a V_1 (Vw_1 u_1 \pm Vw_2 u_2) = \rho Q (Vw_1 u_1 \pm Vw_2 u_2)$

(vi) $u_1 = \dfrac{\pi D_1 N}{60}$

(vii) $u_2 = \dfrac{\pi D_2 N}{60}$

(viii) Work done per unit weight $= \dfrac{\text{Work done per second}}{\text{Weight of water striking per second}}$

$$= \frac{\rho Q (V_{w1} u_1 \pm V_{w2} u_2)}{\rho Q g} = \frac{1}{g}(V_{w1} u_1 \pm V_{w2} u_2)$$

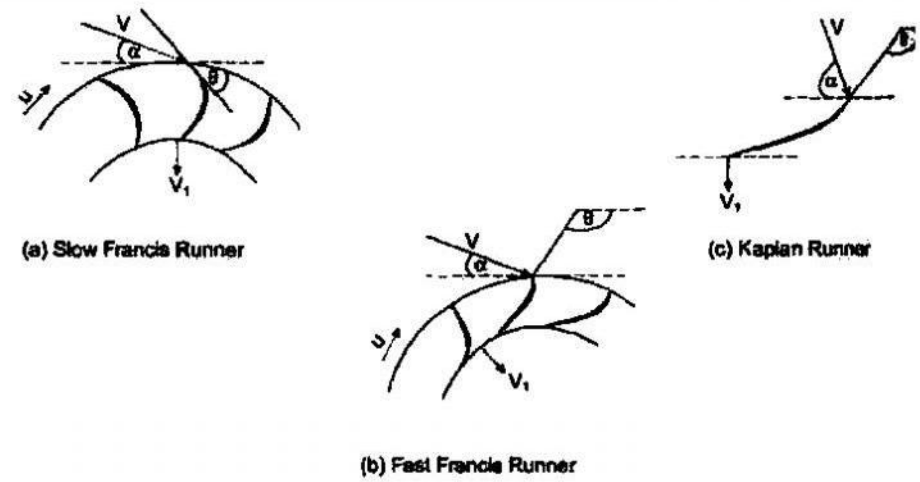
If the discharge at the exit is radial, then $Vw_2 = 0$ and hence

Work done per unit weight $= \dfrac{1}{g}(V_{w1} u_1)$

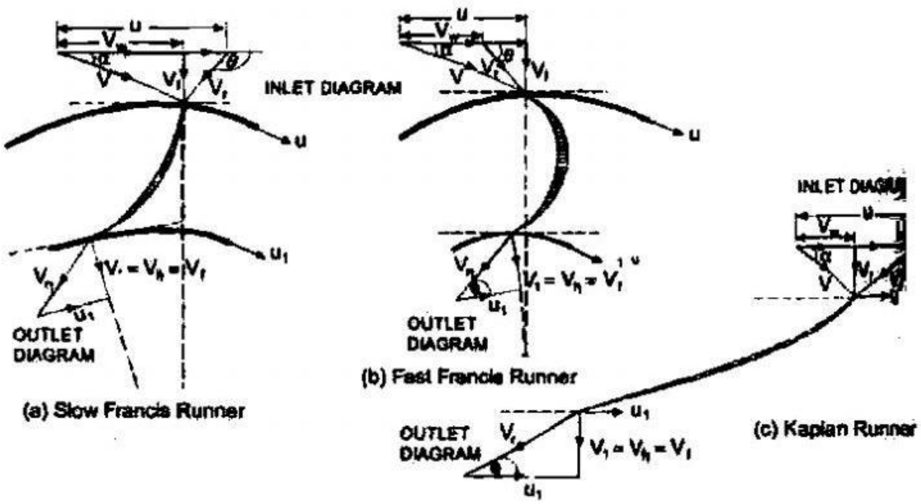
(ix) H ydraulic efficiency = $\dfrac{R.P.}{W.P.} = \dfrac{\rho Q (V_{w1} u_1 \pm V_{w2} u_2)}{\rho g Q H} = \dfrac{1}{g H}(V_{w1} u_1 \pm V_{w2} u_2)$

If the discharge at the exit is radial, then $Vw_2 = 0$ and hence

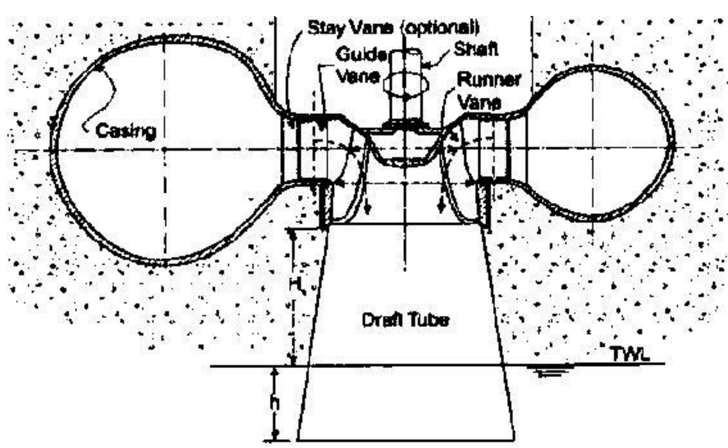
H ydraulic efficiency = $\dfrac{1}{g H}(V_{w1} u_1)$



Blade

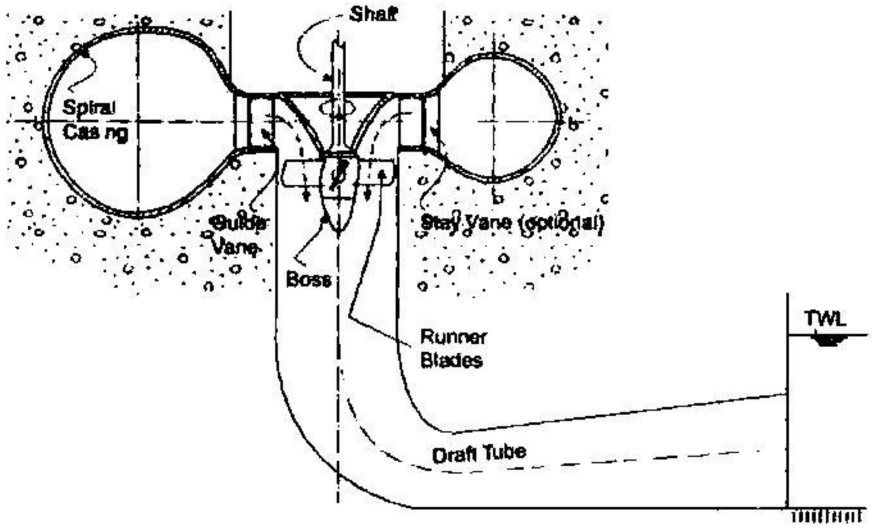


Velocity



Francis Turbine installation with straight

WORKING OF A KAPLAN TURBINE



Kaplan Turbine installation with an Elbow

The reaction turbine developed b y Victor Kaplan (1815 - 1892) is an improved version of the older propeller turbine . It is particularl y suitable for generating h ydropower in locations where large quantities of water are available under a relativel y low head . Consequentl y the specific speed of these turbines is high, viz . , 300 to 1000 . As in the case of a Francis turbine, the Kaplan turbine is provided with a spiral casing, guide vane assembl y and a draft tube. The blades of a Kaplan turbine, three to eight in number are pivoted around the central hub or boss, thus permitting adjustment of their orientation for changes in load and head . This arrangement is generall y carried out b y the governor which also moves the guide vane suitabl y. For this reason, while a fixed blade propeller turbine gives the best performance under the desi gn load conditions, a Kaplan turbine gives a consistentl y high efficiency over a larger range of heads, discharges and loads . The facilit y for adjustment of blade angles ensures shock -less flow even under non - design conditions of operation .

Water entering radiall y from the spiral casing is imparted a substantial whirl component b y the wicket gates . Subsequentl y, the curvature of the housing makes the flow become axial to some extent and finall y then relative flow as it enters the runner, is tangential to th e leading edge of the blade as shown in Fig 1(c), Energy transfer from fluid to runner depends essentiall y on the extent to which the blade is capable of extinguishing the whirl component of fluid . In most Kaplan runners as in Francis runners, water leaves the wheel axiall y with almost zero whirl or tangential component . The velocit y triangles shown in Fig 1(c) are at the inlet and outlet tips of the runner vane at mid radius, i . e. , midway between boss periphery and runner periphery.

Com parison between Reaction and Im pulse Turbines

SN	***Reaction turbine***	***Impulse turbine***
1	Only a fraction of the available hydraulic energy is converted into kinetic energy before the fluid enters the runner.	All the available hydraulic energy is converted into kinetic energy by a nozzle and it is the jet so produced which strikes the runner blades.
2.	Both pressure and velocity change as the fluid passes through the runner. Pressure at inlet is much higher than at the outlet.	It is the velocity of jet which changes, the pressure throughout remaining atmospheric.
3	The runner must be enclosed within a watertight casing (scroll casing).	Water-tight casing is not necessary. Casing has no hydraulic function to perform. It only serves to prevent splashing and guide water to the tail race
4.	Water is admitted over the entire circumference of the runner	Water is admitted only in the form of jets. . There may be one or more jets striking equal number of buckets simultaneously.
5.	Water completely fills at the passages between the blades and while flowing between inlet and outlet sections does work on the blades	The turbine does not run full and air has a free access to the buckets
6.	The turbine is connected to the tail race through a draft tube which is a gradually expanding passage. It may be installed above or below the tail race	The turbine is always installed above the tail race and there is no draft tube used
7.	The flow regulation is carried out by means of a guide-vane assembly. Other component parts are scroll casing, stay ring, runner and the draft tube	Flow regulation is done by means of a needle valve fitted into the nozzle.

KAPLAN TURBINE - SUMMARY

1 . Peripheral velocities at inlet and outlet are same and given b y

$$u_1 = u_2 = \frac{\pi D_o N}{60}$$

where D_o is the outer diameter of the runner

2 . Flow velocities at inlet and outlet are same. i . e. $V_{f1} = V_{f2}$ 3 .
Area of flow at inlet is same as area of flow at outlet

$$Q = \frac{\pi}{4}\left(D_o^2 - D_b^2\right)$$

where D_b is the diameter of the boss .

The external diameter of an inward flow reaction turbine is 0 . 5 m . The width of the wheel at inlet is 150 mm and the velocit y of flow at inlet is 1 . 5 m/s . Find the rate of flow passing through the turbine.

Solution:

$D_1 = 0.5$ m, $B_1 = 0.15$ m, $V_{f1} = 1.5$ m/s, $Q = ?$

Discharge through the turbine = $Q = \pi D_1 B_1 V_{f1} = \pi \times 0.5 \times 0.15 \times 1.5$

$Q = 0.353$ m^3/s (Ans)

The external and internal diameters of an inward flow reaction turbine are 600 mm and 200 mm respectivel y and the breadth at inlet is 150 mm . If the velocit y of flow through the runner is constant at 1 . 35 m^3/s, find the discharge through turbine and the width of wheel at outlet .

Solution:

$D_1 = 0.6$ m, $D_2 = 0.2$ m, $B_1 = 0.15$ m, $V_{f1} = V_{f2} = 1.35$ m/s, $Q = ?$, $B_2 = ?$

Discharge through the turbine = $Q = \pi D_1 B_1 V_{f1} = \pi \times 0.6 \times 0.15 \times 1.35$

$Q = 0.382$ m^3/s (Ans)

Also discharge is given b y $Q = \pi D_2 B_2 V_{f2} = \pi \times 0.2 \times B_2 \times 1.35 \Rightarrow 0.382$

$B_2 = 0.45$ m/s (Ans)

An inward flow reaction turbine running at 500 rpm has an external diameter is 700 mm and a width of 180 mm . If the gu ide vanes are at 20º to the wheel tangent and the absolute velocit y of water at inlet is 25 m/s, find (a) discharge through the turbine (b) inlet vane angle.

Solution:

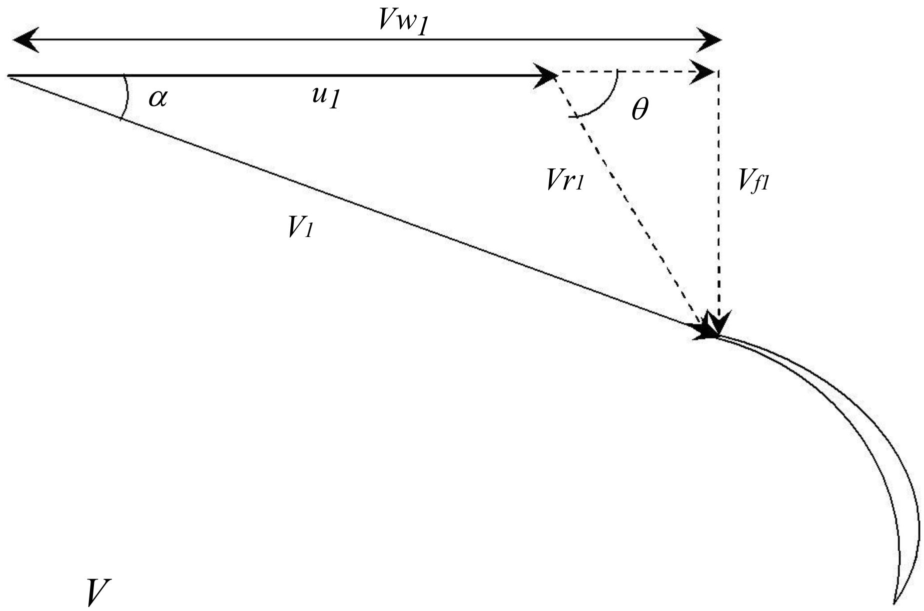
$N = 500$ rpm, $D_1 = 0.7$ m, $B_1 = 0.18$ m, $a = 20°$, $V_1 = 25$ m/s, $Q = ?$, $\theta = ?$ We know that the peripheral velocit y is given b y

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 0.7 \times 500}{60} = 18.33\ m/s$$

From inlet velocit y triangle, we have

$V_{f1} = V_1 \sin\alpha = 25 \times \sin 20 = 8.55$ m/s

$Vw_1 = V_1 \cos\alpha = 25 \times \cos 20 = 23.49$ m/s



$$\tan\theta = \frac{V_{f1}}{V_{w1} - u_1} = \frac{8.55}{23.49 - 18.33} = 1.657$$

$\theta = 58.89^\circ$ (Ans)

$Q = \pi D_1 B_1 V_{f1} = \pi \times 0.7 \times 0.18 \times 8.55 = 3.384$ m^3/s (Ans)

A reaction turbine works at 450 rpm under a head of 120 m . Its diameter at inlet is 1 . 2 m and the flow area is 0 . 4 m^2 . The angle made b y the absolute and relative velocities at inlet are 20º and 60º respectivel y with the tangential velocit y. Determine (i) the discharge through the turbine (ii) power developed (iii) efficiency. Assume radi al discharge at outlet .

Solution:

$N = 450$ rpm, $H = 120$ m, $D_1 = 1.2$ m, $a_1 = 0.4$ m^2 , $\alpha = 20^\circ$ and $\theta = 60^\circ$

$Q = ?$, $\eta = ?$, $Vw_2 = 0$

We know that the peripheral velocit y is given b y

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 1.2 \times 450}{60} = 28.27\ m/s$$

$$\text{Tan}\,\theta = \frac{V_{f1}}{V_{w1} - u_1}$$

$$\text{Tan}\,60 = \frac{V_{f1}}{V_{w1} - 28.27}$$

Hence $V_{f1} = (V_{w1} - 28.27)\ \text{Tan}\ 60$ (01)

Further $\text{Tan}\,\alpha = \frac{V_{f1}}{V_{w1}} = \text{Tan}\ 20$

Hence $V_{f1} = (V_{w1})\ \text{Tan}\ 20$ (02)

From equations 1 and 2, we get

$(V_{w1} - 28.27)\ \text{Tan}\ 60 = V_{w1}\ \text{Tan}\ 20$

Hence $V_{w1} = 35.79$ m/s

$V_{f1} = 35.79 \times \text{Tan}\ 20 = 13.03$ m/s

Discharge $Q = \pi D_1 B_1 V_{f1} = a_1 V_{f1} = 0.4 \times 13.03 = 5.212\ \text{m}^3/\text{s}$ (Ans)

Work done per unit weight of water =

$$\frac{1}{g}(V_{w1} u_1) = \frac{1}{10}(35.79 \times 28.27) = 101.178\ kN - m / N$$

Water Power or input per unit weight = H = 120 kN -m/N

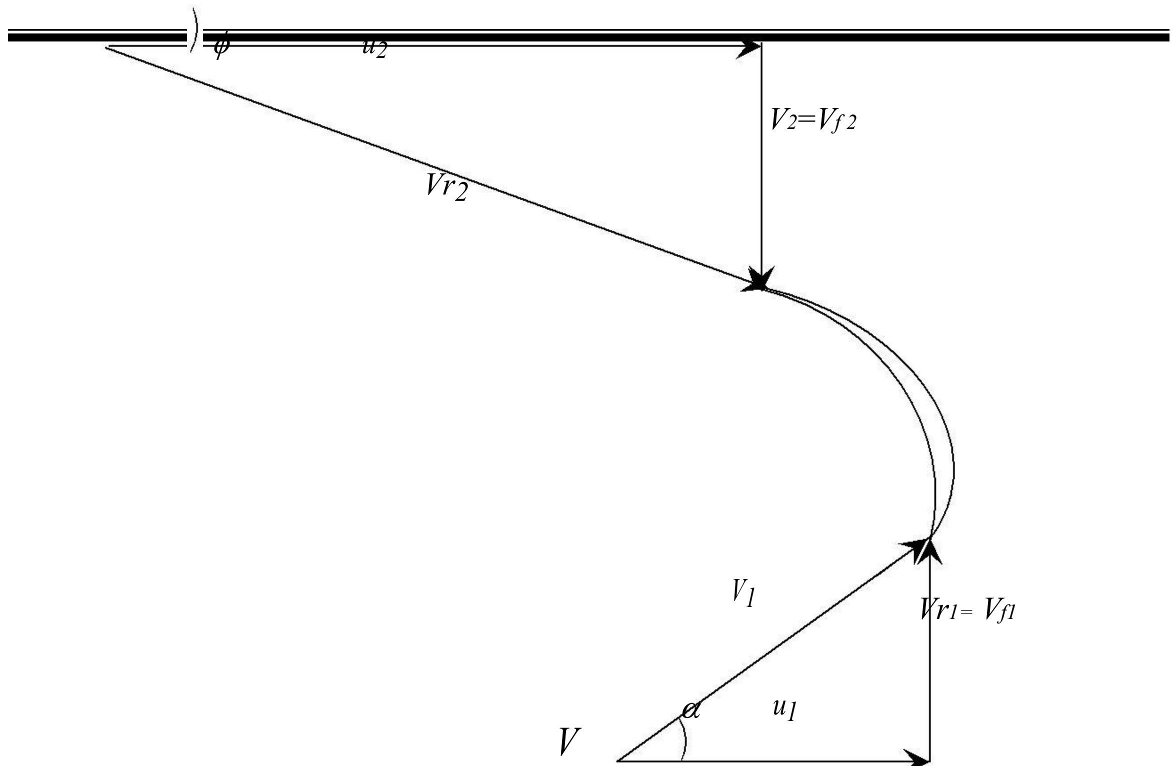
Hydraulic efficiency $\eta = \frac{101.178}{120} = 84.31\%$

The peripheral velocity at inlet of an outward flow reaction turbine is 12 m/s. The internal diameter is 0.8 times the external diameter. The vanes are radial at entrance and the vane angle at outlet is 20º. The velocity of flow through the runner at inlet is 4 m/s. If the final discharge is radial and the turbine is situated 1 m below tail water level, determine:

1. The guide blade angle
2. The absolute velocity of water leaving the guides
3. The head on the turbine
4. The hydraulic efficiency

Solution:

$u_1 = 12$ m/s, $D_1 = 0.8 D_2$, $\theta = 90º$, $\phi = 20º$, $V_{f1} = 4$ m/s, $Vw_2 = 0$, Pressure head at outlet = 1m, $\alpha = ?$, $V_1 = ?$, $H = ?$, $\eta_h = ?$



From inlet velocit y triangle, Tan $\alpha = \frac{V_{f1}}{u_1} = \frac{4}{12}$, Hence $\alpha = 18.44^{o}$

Absolute velocit y of water leaving guide vanes is

$$V_1 = \sqrt{u_1^2 + V_{f1}^2} = \sqrt{12^2 + 4^2} = 12.65 \text{ m/s}$$

$$u_1 = \frac{\pi D_1 N}{60} \text{ and } u_2 = \frac{\pi D_2 N}{60}$$

Comparing the above 2 equations, we have $\frac{60\,u_1}{\pi D_1} = \frac{60\,u_2}{\pi D_2}$ and hence $\frac{u_1}{D_1} = \frac{u_2}{D_2}$

Hence $u_2 = \frac{D_2}{D1} u_1 = \frac{12}{0.8} = 15$ m/s

From outlet velocit y triangle, $V_2 = V_{f2} = u_2 \tan 20 = 15 \tan 20 = 5.46$ m/s As $Vw_2 = 0$

Work done per unit weight of water = $\frac{Vw_1 u_1}{g} = \frac{12 \times 12}{10} = 14.4$ kN − m/N

Head on turbine *H*

Energy Head at outlet = WD per unit weight + losses

$$H = \left(1 + \frac{V_2^2}{2g}\right) + \left(\frac{Vw_1 u_1}{g}\right) \text{ and hence}$$

$$10. \quad = \left(1 + \frac{5.46^2}{2g}\right) + 14.4 = 16.89 \text{ m}$$

Hydraulic efficiency = $\eta_h = \frac{Vw_1 u_1}{g H} = \frac{12 \times 12}{10 \times 16.89} \times 100 = 85.26\ \%$

Jan/Feb 2006

An inward flow water turbine has blades the inner and outer radii of which are 300 mm and 50 mm respectively. Water enters the blades at the outer periphery with a velocity of 45 m/s making an angle of 25º with the tangent to the wheel at the inlet tip. Water leaves the blade with a flow velocity of 8 m/s. If the blade angles at inlet and outlet are 35º and 25º respectively, determine

(i) Speed of the turbine wheel
(ii) Work done per N of water (08)

Solution:

$D_1 = 0.6$ m; $D_2 = 0.1$ m, $V_1 = 45$ m/s, $\alpha = 25°$, $V_2 = 8$ m/s, $\theta = 35°$, $\phi = 25°$, $N = ?$, $WD/N = ?$

$$Sin\alpha = \frac{V_{f1}}{V_1} = Sin25 = 0.423$$

Hence $V_{f1} = 0.423 \times 45 = 19.035$ m/s

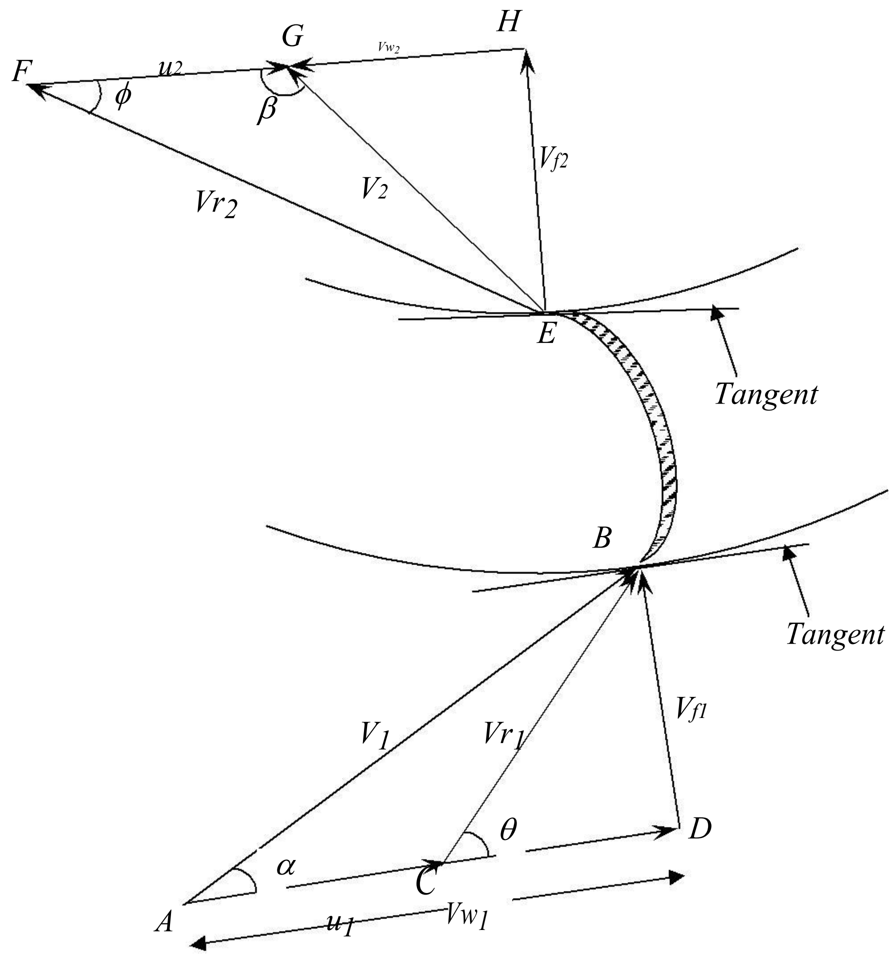
$$Tan\alpha = \frac{V_{f1}}{V_{w1}} = \tan 25 = 0.466$$

Hence $Vw_1 = 40.848$ m/s

$$Tan\theta = \frac{V_{f1}}{V_{w1} - u_1} = \tan 35 \Rightarrow 0.7 = \frac{10.035}{40.848 - u_1}$$

$u_1 = 13.655$ m/s

$$u_1 = \frac{\pi D_1 N}{60} \text{ and hence } N = \frac{60\, u_1}{\pi D_1} = \frac{60 \times 13.655}{\pi \times 0.6} = 434.65 \text{ RPM (Ans)}$$



$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 0.1 \times 869.3}{60} = 4.552 \text{ m/s}$$

Ignoring shock losses, $V_{r2} = V_{r1} = \frac{V_{f1}}{\sin\theta} = \frac{19.035}{\sin 35} = 33.187$ m/s

$Vw_2 = V_{r2}\cos\phi - u_2 = 33.187\cos 25 - 4.552 = 25.526$ m/s

Work done per unit weight of water = $\frac{1}{g}\left(V_{w1}u_1 + V_{w2}u_2\right)$

$$WD/N = \frac{1}{g}\left(40.848 \times 13.655 + 25.526 \times 4.552\right) = 67.4 \text{ m/s (Ans)}$$

July/Aug 2005

A reaction turbine 0 . 5 m dia develops 200 kW while running at 650 rpm and requires a discharge of 2700 m^3/hour; The pressure head at entrance to the turbine is 28 m, the elevation of the turbine casing above the tail water level is 1 . 8 m and the water enters the turbine with a velocit y of 3 . 5 m/s . Calculate (a) The effective head and efficiency, (b) The speed, discharge and power if the same machine is made to operate under a head of 65 m

Solution:

$D = 0.5$ m, $P = 200$ kW, $N = 650$ rpm, $Q = 2700/60^2 = 0.75\ m^3/s$,

$V_1 = 3.5$ m/s, $\frac{p_1}{\rho g} = 28\ m$

The effective head = H =Head at entry to runner –Kinetic energy in tail race + elevation of turbine above tailrace

$$H = \frac{p_1}{\rho g} - \frac{V_2^2}{2g} = 28 - \frac{3.5^2}{2\times10} + 1.8 = 29.1875 \text{ m (Ans)}$$

$$\text{H ydraulic efficiency} = \eta_0 = \frac{P}{\rho g Q H} = \frac{200\times10^3}{1000\times10\times0.75\times29.1875}\times100 = 91.36\ \%$$

Further unit quantities are given b y

$$\text{Unit speed} = N_u = \frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}}$$

$$\text{Unit Discharge} = Q_u = \frac{Q_1}{\sqrt{H_1}} = \frac{Q_2}{\sqrt{H_2}}$$

$$\text{Unit Power} = P_u = \frac{P_1}{H_1^{3/2}} = \frac{P_2}{H_2^{3/2}}$$

$$N_u = \frac{650}{\sqrt{29.1875}} = \frac{N_2}{\sqrt{65}} = 120.31$$

$N_2 = 969.97$ rpm (Ans)

$$Q_u = \frac{0.75}{\sqrt{29.1875}} = \frac{Q_2}{\sqrt{65}} = 0.1388$$

$Q_2 = 1.119\ m^3/s$ (Ans)

$$P_u = \frac{200}{29.1875^{3/2}} = \frac{P_2}{65^{3/2}} = 1.268$$

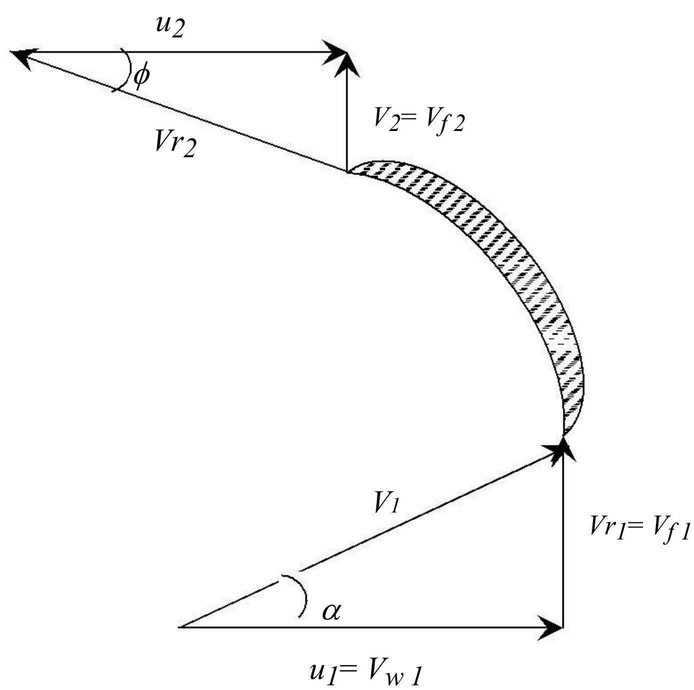
$P_2 = 664.49$ kW (Ans)

July/Aug 2005

A Francis turbine has inlet wheel diameter of 2 m and outlet diameter of 1 . 2 m . The runner runs at 250 rpm and water flows at 8 cumecs . The blades have a constant width of 200 mm . If the vanes are radial at inlet and the discharge is radiall y outwards at exit, make calculations for the angle of guide vane at inlet and blade angle at outlet (10)

Solution:

$D_1 = 2$ m, $D_2 = 1.2$ m, $N = 250$ rpm, $Q = 8$ m^3/s, $b = 0.2$ m, $Vw_1 = u_1$,
$Vw_2 = 0$, $\alpha = ?$, $\phi = ?$



$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 2 \times 250}{60} = 26.18 \text{ m/s}$$

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times 1.2 \times 250}{60} = 15.71 \text{m/s}$$

$Q = \pi D_1 b V_{f1} = \pi D_2 b V_{f2}$

$8 = \pi \times 2 \times 0.2 \times V_{f1}$

Hence $V_{f1} = 6.366$ m/s

Similarl y $8 = \pi \times 1.2 \times 0.2 \times V_{f2}$

$V_{f2} = 10.61$ m/s

$\tan \alpha = \frac{V_{f1}}{u_1} = \frac{6.366}{26.18}$

$\alpha = 13.67°$(Ans)

$\tan \phi = \frac{V_{f2}}{u_2} = \frac{10.61}{15.71}$

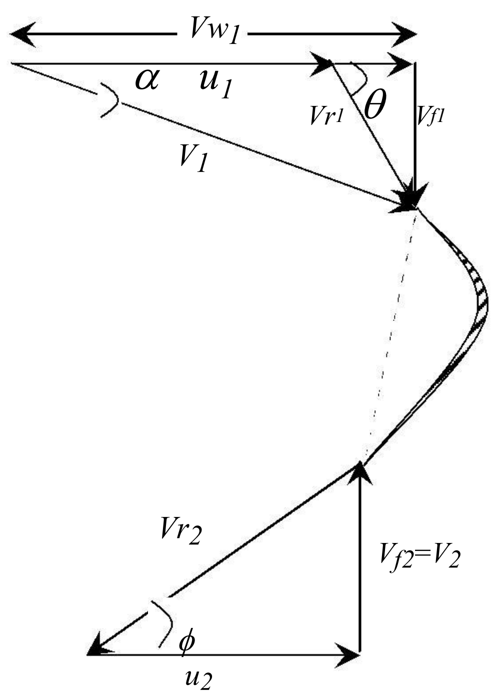
$\phi = 34.03°$ (Ans)

Determine the overall and h ydraulic efficiencies of an inward flow reaction turbine using the following data. Output Power = 2500 kW, effective head = 45 m, diameter of runner = 1 . 5 m, width of runner = 200 mm, guide vane angle = 20 °, runner vane angle at inlet = 60 ° and specific speed = 100 .

Solution:

$P = 2500$ kW, $H = 45$ m, $D_1 = 1.5$ m, $b_1 = 0.2$ m, $\alpha = 20°$, $\theta = 60°$,

$N_s = 110$, $\eta_o = ?$, $\eta_h = ?$



We know that specific speed is given b y

$$N_s = \frac{N\sqrt{P}}{H^{5/4}} \quad \text{and hence } N = \frac{N_s H^{5/4}}{\sqrt{P}} = \frac{100 \times 45^{5/4}}{\sqrt{2500}} = 233\ rpm$$

$$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 1.5 \times 233}{60} = 18.3\ m/s$$

But from inlet velocity triangle, we have

$$u_1 = \frac{V_{f1}}{\tan\alpha} - \frac{V_{f1}}{\tan\theta}$$

$$18.3 = \frac{V_{f1}}{\tan 20} - \frac{V_{f1}}{\tan 60}$$ and hence V_{f1} = 8 . 43 m/s

$$V_{w1} = \frac{V_{f1}}{\tan\alpha} = \frac{8.43}{\tan 20} = 23.16 \text{ m/s}$$

V_{w2} = 0 and hence

$$\eta_h = \frac{V_{w1} u_1}{g H} = \frac{23.16 \times 18.3}{10 \times 45} \times 100 = 94.18\ \%\ \text{(Ans)}$$

$Q = \pi D_1\ b_1\ V_{f1} = \pi$ x 1 . 5 x 0 . 2 x 8 . 43 = 7 . 945 m^3/s

$$\eta_o = \frac{P}{\rho g Q H} = \frac{2500 \times 10^3}{1000 \times 10 \times 7.945 \times 45} \times 100 = 69.93\ \%\ \text{(Ans)}$$

Determine the output Power, speed, specific speed and vane angle at exit of a Francis runner using the following data . Head = 75 m, H ydraulic efficiency = 92%, overall efficiency = 86 %, runner diameters = 1 m and 0 . 5 m, width = 150 mm and guide blade angle = 18 °. Assume that the runner vanes are set normal to the periphery at inlet .

Solution:

Data: H = 75 m, η_h = 0 . 92, η_o = 0 . 86, D_1 = 1 m, D_2 = 0 . 5 m, α = 18 °,

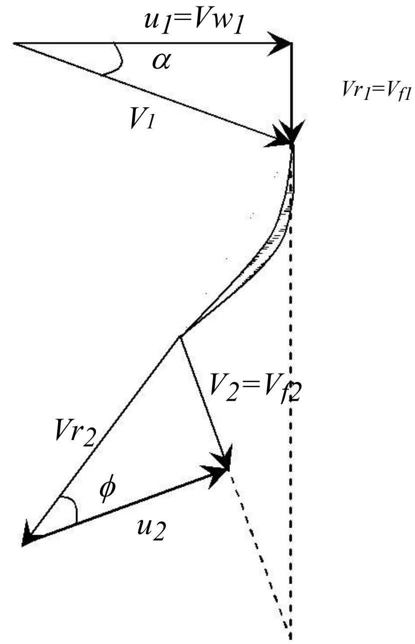
$Vw_1 = u_1$, P = ?, N = ?, ϕ = ?

$$\eta_h = \frac{Vw_1 u_1}{g H} = \frac{u_1^2}{g H}$$

u_1^2 = 0.92 x 10 x 75 =

690 u_1 = 26.27 m/s

$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi \times 1.0 \times N}{60} = 26.27\ m/s$



$N = 501.7$ RPM

$V_{f1} = u_1 \tan\alpha = 26.27 \text{ x } \tan 18 = 8.54$ m/s

$Q = \pi D_1 b_1 V_{f1} =$

$= \pi \text{ x } 1.0 \text{ x } 0.15 \text{ x } 8.54 = 4.02\ m^3/s$

$\frac{u_1}{D_1} = \frac{u_2}{D_2}$ and hence $u_2 = 0.5 \text{ x } u_1 = 13.135$ m/s

Assuming $V_{f1} = V_{f2}$

From outlet velocit y triangle, we have

$\tan\phi = \frac{V_{f2}}{u_2} = \frac{8.54}{13.135} = 0.65$

Hence $\phi = 33°$

$\eta_o = \frac{P}{\rho g Q H} = \frac{P}{1000 \times 10 \times 4.02 \times 75} = 0.86$

Hence $P = 2592.9$ kW (Ans)

Specific speed = $N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{501.7\sqrt{2592.9}}{75^{5/4}} = 115.75$ RPM

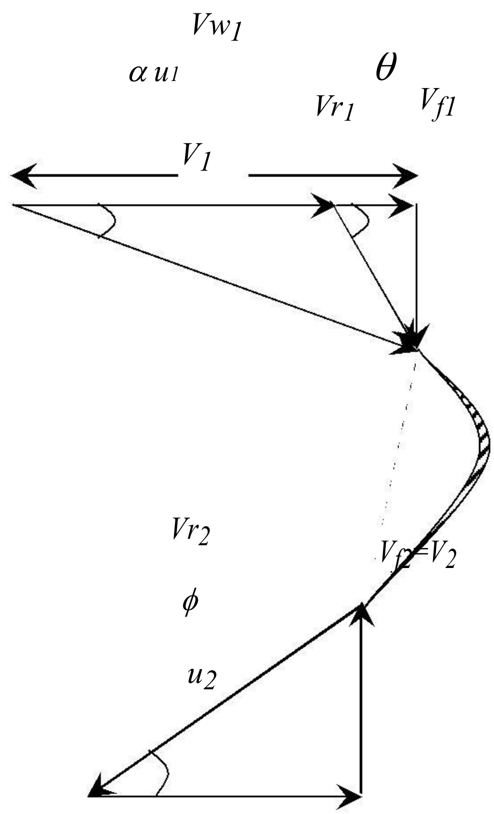
The following data is given for a Francis turbine . Net Head = 60 m; speed N = 700 rpm; Shaft power = 294 . 3 kW; η_o = 84%; η_h = 93%; flow ratio = 0 . 2; breadth ratio n = 0 . 1; Outer diameter of the runner = 2 x inner diameter of the runner . The thickness of the vanes occupies 5% circumferential area of the runner, velocity of flow is constant at inlet and outlet and discharge is radial at outlet . Determine:

(i) Guide blade angle
(ii) Runner vane angles at inlet and outlet
(iii) Diameters of runner at inlet and outlet
(iv) Width of wheel at inlet

Solution

$H = 60$ m; $N = 700$ rpm; $P = 294.3$ kW; $\eta_o = 84\%$; $\eta_h = 93\%$;

flow ratio = $\frac{V_{f1}}{\sqrt{2gH}} = 0.2$



$V_{f1} = 0.2\sqrt{2\times10\times60} = 6.928$ m/s

Breadth ratio $\frac{B_1}{D_1} = 0.1$

$D_1 = 2 \times D_2$

$V_{f1} = V_{f2} = 6.928$ m/s

Thickness of vanes =

5% of circumferential area of runner

$\therefore$ Actual area of flow = $0.95\,\pi D_1 B_1$

Discharge at outlet = Radial and hence

$V_{w2} = 0$ and $V_{f2} = V_2$

We know that the overall efficiency is given by

$\eta_0 = \frac{P}{\rho g Q H}$;$0.84 = \frac{294.3\times10^3}{1000\times10\times Q\times60}$

$Q = 0.584$ m^3/s

$Q = 0.95\,\pi D_1 B_1 V_{f1} = 0.95\,\pi D_1 \times (0.1\,D_1) \times 6.928 = 0.584$

Hence $D_1 = 0.531$ m (Ans)

$\frac{B_1}{D_1} = 0.1$ and $B_1 = 53.1$ mm (Ans)

$u_1 = \frac{\pi D_1 N}{60} = \frac{\pi\times0.531\times700}{60} = 19.46$ m/s

Hydraulic efficiency $\eta_h = \frac{V_{w1}u_1}{gH}$;$0.93 = \frac{V_{w1}\times19.46}{10\times60}$

$V_{w1} = 28.67$ m/s

From Inlet velocity triangle $\tan\alpha = \frac{V_{f1}}{V_{w1}} = \frac{6.928}{28.67} = 0.242$

Hence Guide blade angle = $\alpha = 13.58^\circ$ (Ans)

$$\tan\theta = \frac{V_{f1}}{V_{w1} - u_1} = \frac{6.928}{28.67 - 19.46} = 0.752$$

Vane angle at inlet = $\theta = 37^\circ$ (Ans)

$$u_2 = \frac{\pi D_2 N}{60} = \frac{\pi \times \left(0.531/2\right) \times 700}{60} = 9.73 \text{ m/s}$$

From outlet velocit y triangle, we have

$$\tan\phi = \frac{V_{f2}}{u_2} = \frac{6.928}{9.73} = 0.712$$

$\phi = 35.45^\circ$(Ans)

Diameters at inlet and outlet are $D_1 = 0.531$m and $D_2 = 0.2655$ m

A Kaplan turbine develops 9000 kW under a net head of 7 . 5 m . Overall efficiency of the wheel is 86% The speed ratio based on outer diameter is 2 . 2 and the flow ratio is 0 . 66 . Diameter of the boss is 0 . 35 times the external diameter of the wheel . Determine the diameter of the runner and the specific speed of the runner.

Solution:

$P = 9000$ kW; $H = 7.5$ m; $\eta_o = 0.86$; Speed ratio = 2 . 2; flow ratio = 0 . 66;

$D_b = 0.35 D_o$;

$$\frac{u_1}{\sqrt{2gH}} = 2.2$$

$$u_1 = 2.2\sqrt{2 \times 10 \times 7.5} = 26.94 \text{ m/s}$$

$$\frac{V_{f1}}{\sqrt{2gH}} = 0.66$$

$$V_{f1} = 0.66\sqrt{2 \times 10 \times 7.5} = 8.08 \text{ m/s}$$

$$\eta_0 = \frac{P}{\rho g Q H}; 0.86 = \frac{9000 \times 10^3}{1000 \times 10 \times Q \times 7.5}$$

$Q = 139.5\ \text{m}^3/\text{s}$

$$Q = \frac{\pi}{4}\left(D_o^2 - D_b^2\right)\times V_{f1} \quad \Rightarrow \frac{\pi}{4}\left(D_o^2 - \left[0.35D_o\right]^2\right)\times 8.08 = 139.5$$

D_o = 5 . 005 m (Ans)

$$u = \frac{\pi D_o N}{60} = \frac{\pi\times 5.005\times N}{60} = 26.94 \text{ m/s}$$

N =102 . 8 rpm (Ans)

$$N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{102.8\sqrt{9000}}{7.5^{5/4}} = 785.76 \text{ rpm (Ans)}$$

A Kaplan turbine working under a head of 25 m develops 16,000 kW shaft power. The outer diameter of the runner is 4 m and hub diameter is 2 m . The guide blade angle is 35° . The hydraulic and overall efficiency are
90% and 85% respectivel y. If the velocit y of whirl is zero at outlet, determine runner vane angles at inlet and outlet and speed of turbine .

Solution

H = 25 m; P = 16,000 kW; D_b = 2 m; D_o = 4 m; α = 35°; η_h = 0 . 9;

η_o = 0 . 85; V_{w2} = 0; θ = ?; ϕ = ?; N = ?

$$\eta_0 = \frac{P}{\rho g Q H}; \; 0.85 = \frac{16000\times 10^3}{1000\times 10\times Q\times 25}$$

Q = 75 . 29 m^3/s

$$Q = \frac{\pi}{4}\left(D_o^2 - D_b^2\right)\times V_{f1} \Rightarrow \frac{\pi}{4}\left(4^2 - 2^2\right)\times V_{f1} = 75.29$$

V_{f1} = 7 . 99 m/s

From inlet velocit y triangle,

$$\tan\alpha = \frac{V_{f1}}{V_{w1}}$$

$$V_{w1} = \frac{7.99}{\tan 35} = 11.41 \text{m/s}$$

From H ydraulic efficiency

$$\eta_h = \frac{V_{w1} u_1}{g H}$$

$$0.9 = \frac{11.41 \times u_1}{10 \times 25}$$

$u_1 = 19.72$ m/s

$$\tan\theta = \frac{V_{f1}}{u_1 - V_{w1}} = \frac{7.99}{19.72 - 11.41} = 0.9614$$

$\theta = 43.88^\circ$ (Ans)

For Kaplan turbine, $u_1 = u_2 = 19.72$ m/s and $V_{f1} = V_{f2} = 7.99$ m/s

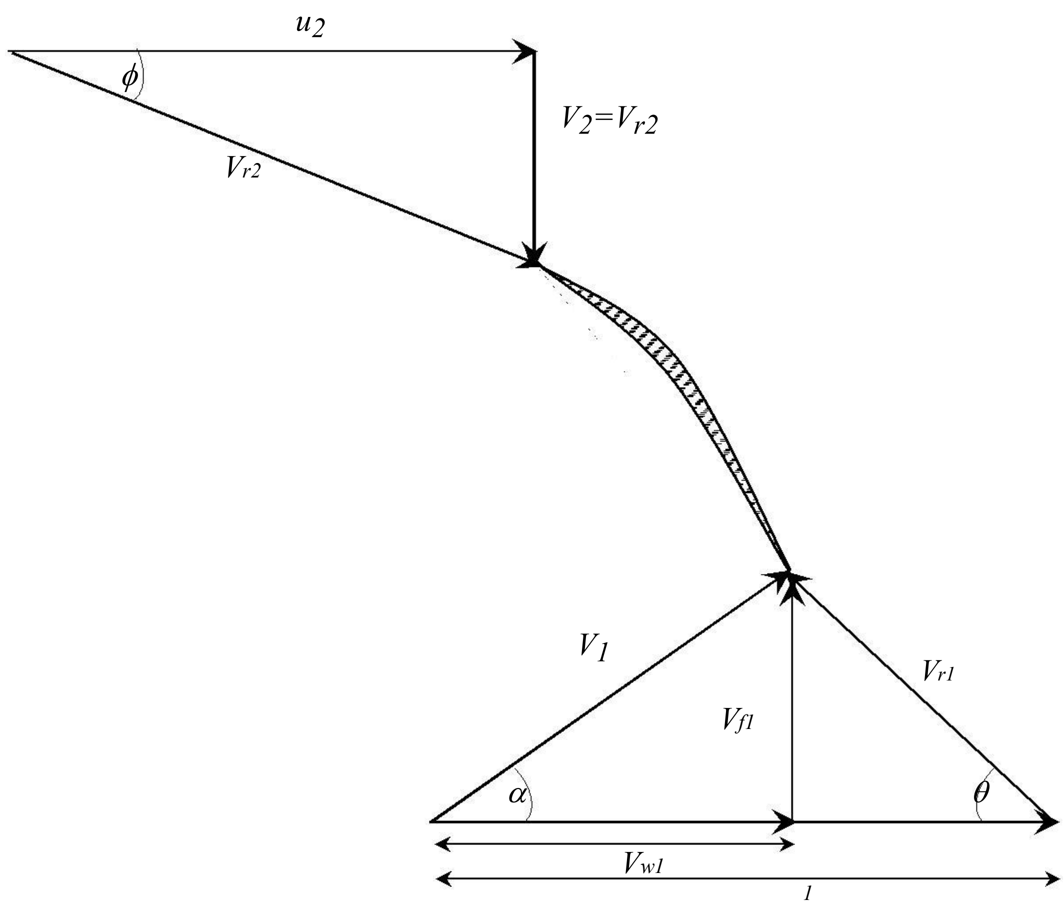
From outlet velocit y triangle

$$\tan\phi = \frac{V_{f2}}{u_2} = \frac{7.99}{19.72} = 0.4052$$

$\phi = 22.06^\circ$ (Ans)

$$u_1 = u_2 = \frac{\pi D_o N}{60} = \frac{\pi \times 4 \times N}{60} = 19.72 \text{ m/s}$$

$N = 94.16$ rpm (Ans)

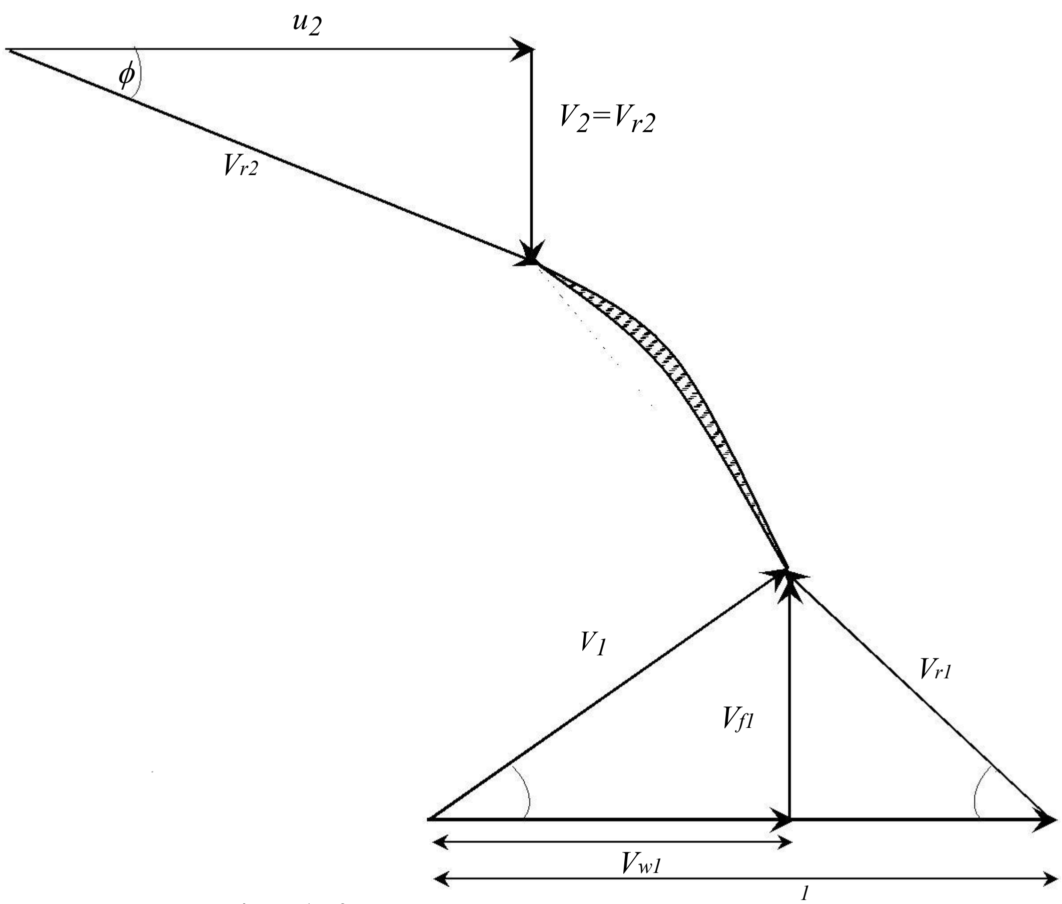


A Kaplan turbine works under a head of 22 m and runs at 150 rpm . The diameters of the runner and the boss are 4 . 5 m and 12 m respectivel y. The flow ratio is 0 . 43 . The inlet vane angle at the extreme edge of the runner is 163°19′. If the turbine discharges radially at outlet, determine the discharge, the h ydraulic efficiency, the guide blade angle at the extreme edge of the runner and the outlet vane angle at the extreme edge of the manner.

Solution:

$H = 22$ m; $N = 150$ rpm; $D_o = 4.5$ m; $D_b = 2$ m; $\theta = 163°19'$; $V_{\omega 2} = 0$; $\frac{V_{f1}}{\sqrt{2gH}} = 0.43$

$V_2 = V_{f2} = V_{f1}$; $Q = ?$; $\eta_h = ?$; $\alpha = ?$; $\phi = ?$;



$$u_1 = u_2 = \frac{\pi D_o N}{60} = \frac{\pi \times 4.5 \times 150}{60} = 35.34 \text{ m/s}$$

$V_{f1} = 0.43\sqrt{2\times10\times22} = 9.02$ m/s

$$\tan(180-\theta)=\frac{V_{f1}}{(u_1 - V_{w1})}$$

$$\tan(180-163°19')=\frac{9.02}{(35.34-V_{w1})}=0.2997$$

$V_{w1} = 5.24$ m/s

Hydraulic efficiency is given by

$$\eta_h = \frac{V_{w1}u_1}{gH} = \frac{5.24\times35.34}{10\times22} = 84.17\%$$

$$\tan\alpha = \frac{V_{f1}}{V_{w1}} = \frac{9.02}{5.24} = 1.72$$

$\alpha = 59.85°$ (Ans)

$$\tan\phi = \frac{V_{f2}}{u_2} = \frac{9.02}{35.34} = 0.2552$$

$\phi = 14.32°$ (Ans)

A kaplan turbine is to be designed to develop 7,350 kW . The net available head is 5 . 5 m . Assume that the speed ratio as 0 . 68 and the overall efficiency as 60%. The diameter of the boss is ⅓ rd of the diameter of the runner. Find the diameter of the runner, its speed and its specific speed .

Solution:

$P = 7350$ kW, $H = 5.5$ m

$\frac{V_{f1}}{\sqrt{2gH}} = 0.68$ and hence $V_{f1} = 0.68\sqrt{2\times10\times5.5} = 7.13$ m/s

$\frac{u_1}{\sqrt{2gH}} = 2.09$ and hence $u_1 = 2.2\sqrt{2\times10\times5.5} = 23.07$ m/s

$$\eta_0 = \frac{P}{\rho g Q H}; \; 0.6 = \frac{7350\times10^3}{1000\times10\times Q\times5.5}$$

$Q = 222.72 \text{ m}^3/\text{s}$

$$Q = \frac{\pi}{4}\left(D_o^2 - D_b^2\right) \times V_{f1} \Rightarrow \frac{\pi}{4}\left(D_o^2 - \left[\frac{D}{3}\right]^2\right) \times 7.13 = 222.72$$

$D_o = 6.69 \text{ m}$ (Ans)

$$u = \frac{\pi D_o N}{60} = \frac{\pi \times 6.69 \times N}{60} = 23.07 \text{ m/s}$$

$N = 65.86 \text{ rpm}$ (Ans)

$$N_s = \frac{N\sqrt{P}}{H^{5/4}} = \frac{65.86\sqrt{7350}}{5.5^{5/4}} = 670.37 \text{ rpm (Ans)}$$

MOMENTUM EQUATION FOR FLUIDS

Session – II

Net force experienced by fluid along x-direction.

$$\Sigma F_x = \frac{m}{t}(V_x - U_x)$$

11. $F_x = m(V_x - U_x)$

12. $F_y = m(V_y - U_y)$

Where m is the mass flow

rate $m = \rho Q$

V = Final velocity of fluid along the direction.

U = Initial velocity of fluid along the direction.

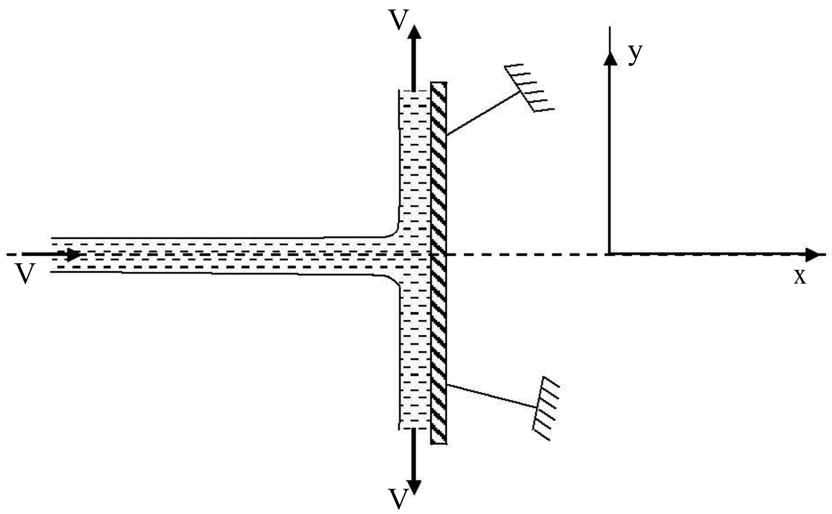
(ϖι) Capability **of Momentum and Energy Equations**

	Momentum Equation	**Energy Equation**
Applicable	To any fluid flow	To steady flow where energy changes are zero or known
Information required	Velocity distribution at one end of control volume. Total forces on the boundaries of control distribution at the other end.	Velocity and pressure at one point on the stream line with independent knowledge of energy changes Pressure variation or velocity variation along stream line
Solution gives	Average final velocity of stream or total force	Velocity variation or pressure variation along stream lime
Solution will not give	Actual velocity distribution or pressure distribution	Tangential forces due to friction
Best application	When energy changes are unknown and only overall, knowledge of flow is required. Eg: Total force, Mean velocity	When energy changes are known and detailed information of flow is required. Eg: Velocity and Pressure distribution.

(ਙ) Force Exerted by Jet on Plates

Case-I

To compute the impact of field jet on stationary flat plate held normal to the jet.



V – Velocity of jet striking the plate

a – Area of cross section of jet.

$\boxed{\therefore\ m = \rho a V}$

Force exerted by plate on fluid jet along x – direction = $F_x = m\,[V_x - U_x]$

Force exerted by the jet on the plate along x – direction will be equal and opposite to that of force exerted by plate on the jet.

$\therefore$ Force exerted by jet on plate along x – direction = $F_x = m\,(U_x - V_x)$ $F_x = \rho av\,[V - 0]$

$\boxed{F_x = \rho av^2}$

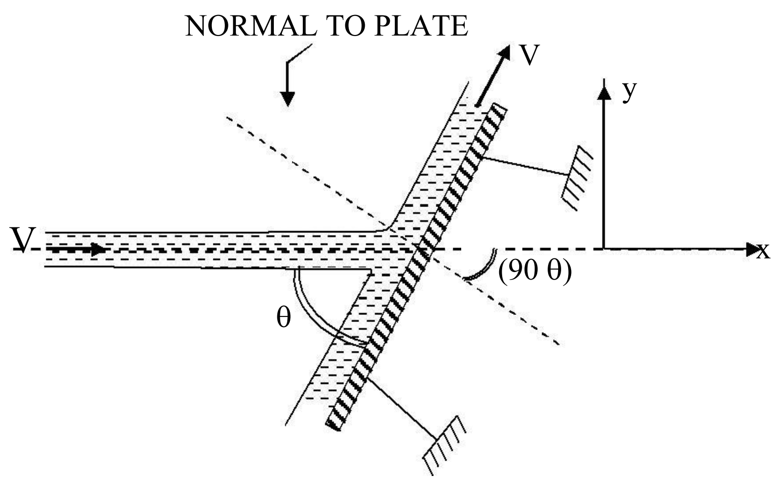
Work done by the jet = Force x Velocity of plate

(ii) Force x 0

(iii) 0

Case-II

To compute the impact of jet on a stationary flat plate held inclined to the direction of jet.

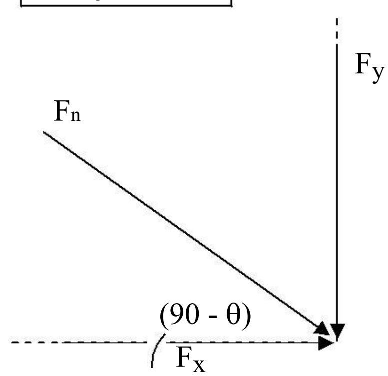


Force exerted by jet on vane along normal direction.

(iii) $F_n = m\,[U_n - V_n]$

$F_n = \rho a V\,[(V\sin\theta) - O]$

$\boxed{\mathbf{F_n = \rho a V^2 \sin\theta}}$



$F_x = F_n \cos(90 - \theta)$

$F_x = [\rho a V^2 \sin\theta]\sin\theta$

$\boxed{\mathbf{F_x = \rho a V^2 \sin^2\theta}}$

$F_y = F_n \sin(90 - \theta)$

$F_y = [\rho a V^2 \sin\theta]\cos\theta$

$\boxed{\mathbf{F_y = \rho a V^2 \sin\theta \cos\theta}}$

Work done by the jet on vane

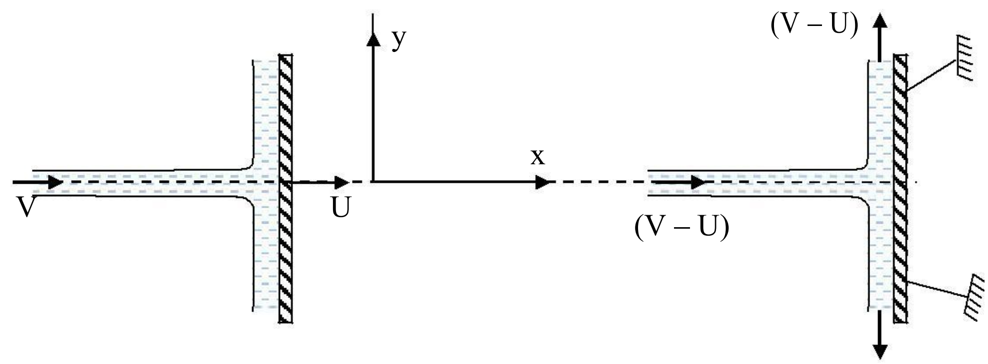
(v) Force x Velocity of vane

(vi) Force x 0

(vii) 0

Case-III

To compute the impact of jet on a moving flat plate held normal to the jet.



V = Velocity of jet striking the plate

U = Velocity of vane along the direction of vane.

Adopting the concept of relative velocity, the system can be considered to be a stationary plate, the jet striking the vane with a relative velocity (V – U).

= $m = \rho Q$

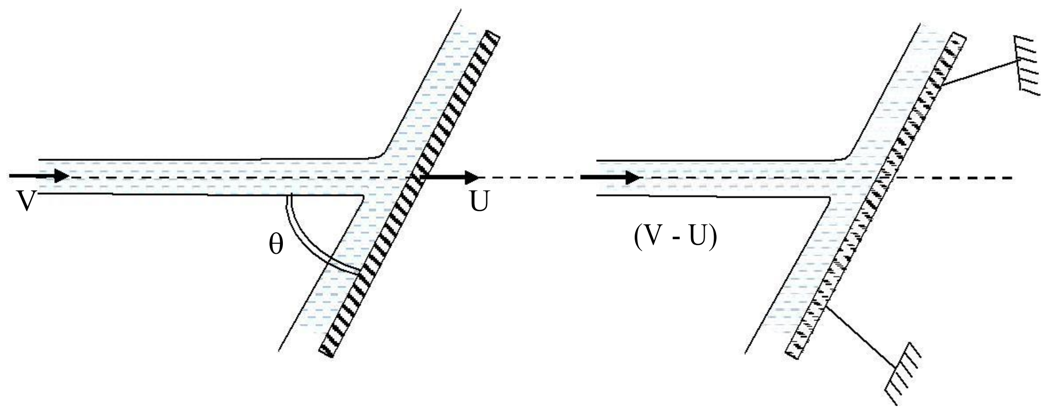
$\rho a(V - U)$

$F_x = \rho a(V - U)^2$

Work done by the jet on plate = Force x Velocity of plate = $\rho a\,(V - U)^2 \times U$

Case-IV

To compute the impact of jet on a moving flat plate held inclined to the direction of jet.



V = Velocity of jet

U = Velocity of plate along the direction of jet.

Adopting the concept of relative velocity, the above case can be considered to be fixed vane with a jet velocity of (V – U).

$$\therefore F_n = \rho a\,(V - U)^2$$
$$F_x = \rho a\,(V - U)^2 \sin^2\theta$$
$$F_y = \rho a\,(V - U)^2 \sin\theta \cos\theta$$

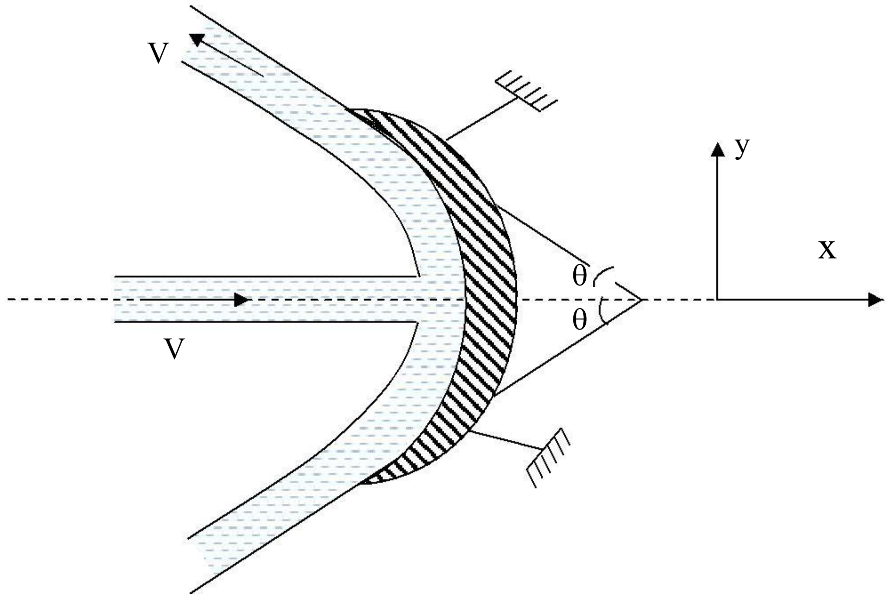
Work done by the jet on vane plate along x – direction

= F_x x Velocity of plate along x – direction

= $\rho a\,(V - U)^2 \sin^2\theta\, U$

Case-V

To compute the impact of jet on a stationery symmetrical curved plate, the jet striking the plate at its centre.



$\boxed{m = \rho a V}$

$= F_x = m\,[U_x - V_x]$

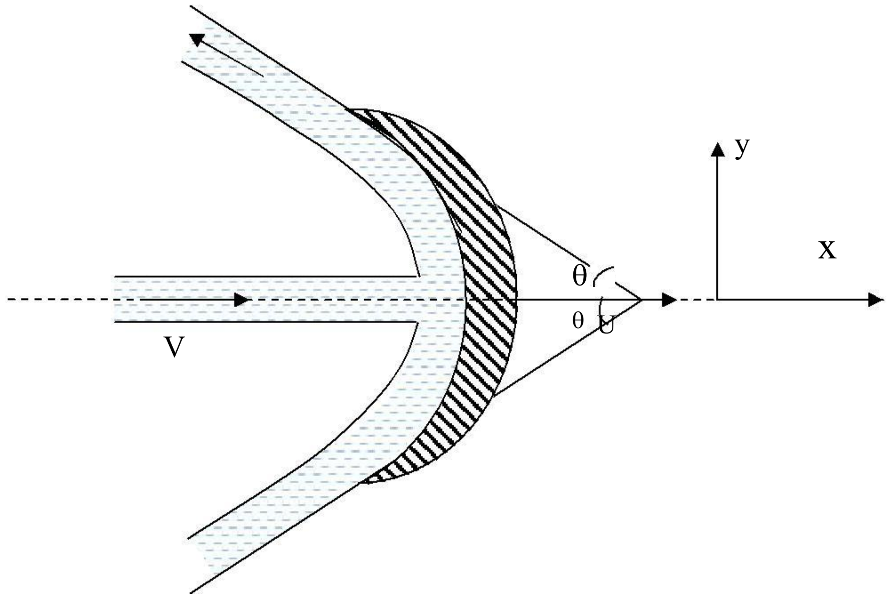
$F_x = (\rho aV)\,[V - (-V\cos\theta)]$

$F_x = \rho aV^2\,(1 + \cos\theta]$

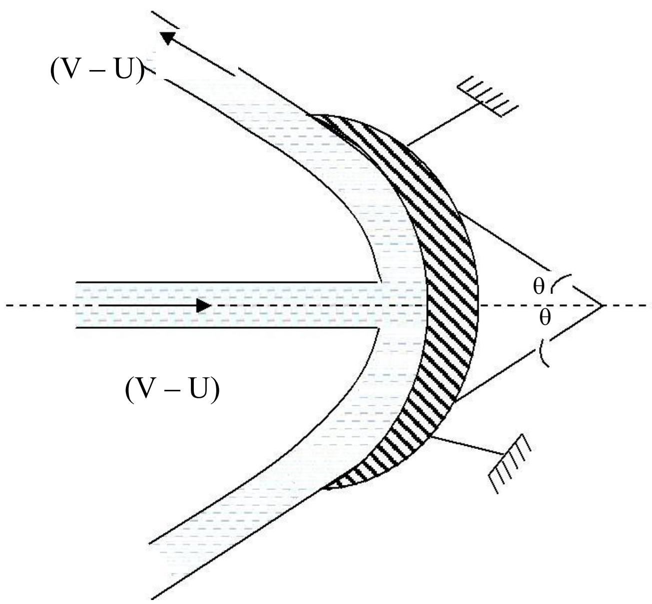
Work done by the jet on plate is zero since the plate is stationery.

Case-VI

To compute the impact of jet on a moving symmetrical curved plate, the jet striking the plate at its centre.



Adopting relative velocity concept, the system can be considered to be a jet of relative velocity (V – U) striking a fixed plate.



$$\boxed{m = \rho a\,(V - U)}$$

$F_x = m\,[U_x - V_x]$

$F_x = \rho a\,(V - U)\,[(V - U) - (-V - U)\cos\theta]$

$F_x = \rho a\,(V - U)^2\,(1 + \cos\theta]$

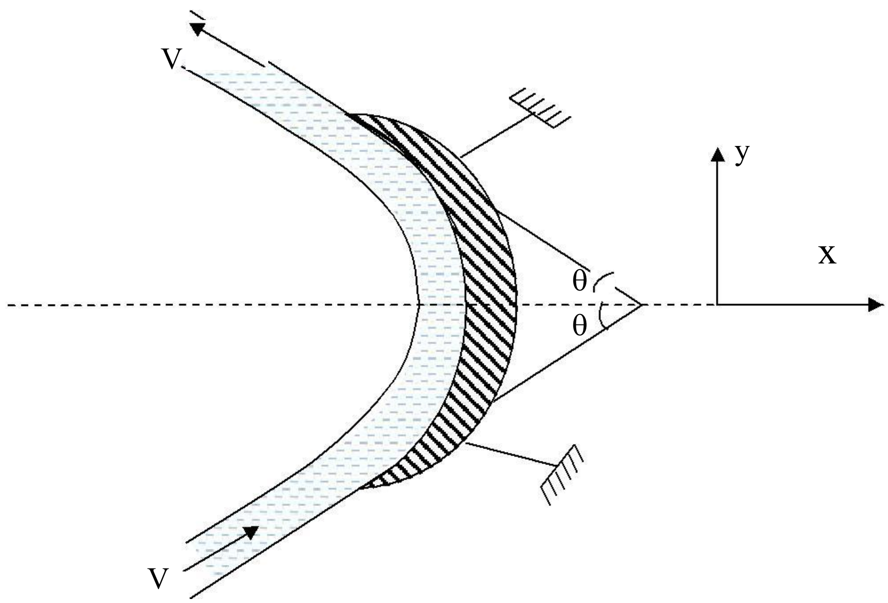
Work done by the jet on plate = Force x Velocity of date

$= F_x \; U$

$= \rho a(V - U)^2 \, (1 + \cos\theta] \, U$

Case-VII

To compute the impact of jet on a stationery symmetrical curved plate, the jet striking the plate at one of the tips tangentially.



$\boxed{m = \rho a V}$

$F_x = m \, [U_x - V_x]$

$F_x = \rho a V \, [V\cos\theta - (-V\cos\theta)]$

$F_x = \rho a V^2 \, (1 + \cos\theta]$

$F_y = m \, [U_y - V_y]$

$F_x = \rho a V \, [V\sin\theta - V\sin\theta]$

$F_x = 0$

Work done by the jet on plate is zero since the plate is stationery.

MOMENTUM EQUATION FOR FLUIDS

Session – II

Net force experienced by fluid along x-direction.

$$\sum F_x = \frac{m}{t}(V_x - U_x)$$

13. $F_x = m(V_x - U_x)$

14. $F_y = m(V_y - U_y)$

Where m is the mass flow

rate $m = \rho Q$

V = Final velocity of fluid along the direction.

U = Initial velocity of fluid along the direction.

(ϖιι) Capability **of Momentum and Energy Equations**

	Momentum Equation	**Energy Equation**
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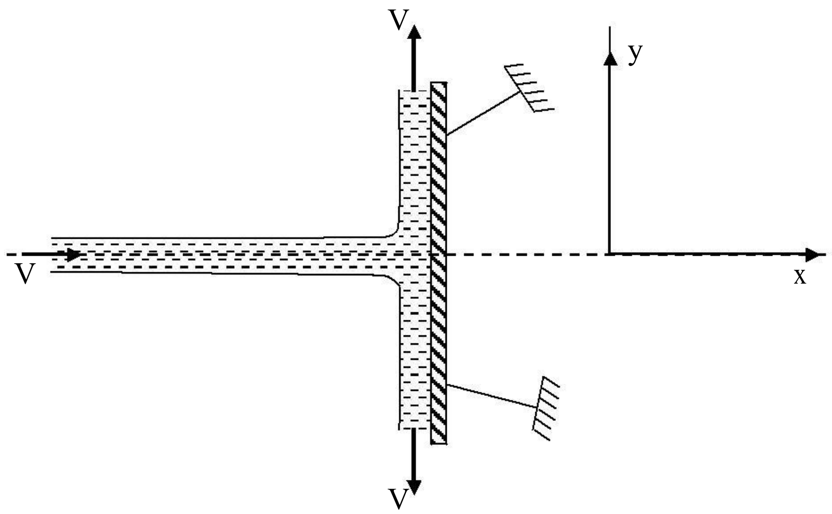
(ııı) Force Exerted by Jet on Plates

Case-I

To compute the impact of field jet on stationary flat plate held normal to the jet.



V – Velocity of jet striking the plate

a – Area of cross section of jet.

$\boxed{\therefore\ m = \rho a V}$

Force exerted by plate on fluid jet along x – direction = $F_x = m\,[V_x - U_x]$

Force exerted by the jet on the plate along x – direction will be equal and opposite to that of force exerted by plate on the jet.

$\therefore$ Force exerted by jet on plate along x – direction = $F_x = m\,(U_x - V_x)$ $F_x = \rho a v\,[V - 0]$

$\boxed{F_x = \rho a v^2}$

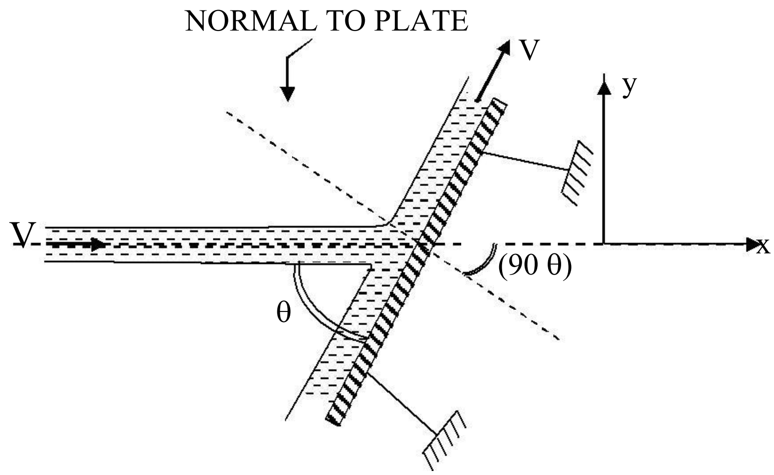
Work done by the jet = Force x Velocity of plate

(iv) Force x 0

(v) 0

Case-II

To compute the impact of jet on a stationary flat plate held inclined to the direction of jet.

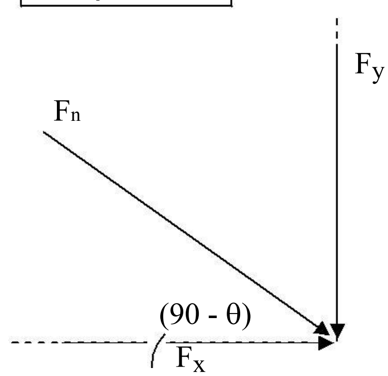


Force exerted by jet on vane along normal direction.

(iv) $F_n = m\,[U_n - V_n]$

$F_n = \rho aV\,[(V\sin\theta) - O]$

$\boxed{\mathbf{F_n = \rho a V^2 \sin\theta}}$



$F_x = F_n \cos(90 - \theta)$

$F_x = [\rho aV^2 \sin\theta]\sin\theta$

$\boxed{\mathbf{F_x = \rho a V^2 \sin^2\theta}}$

$F_y = F_n \sin(90 - \theta)$

$F_y = [\rho aV^2 \sin\theta]\cos\theta$

$\boxed{\mathbf{F_y = \rho a V^2 \sin\theta \cos\theta}}$

Work done by the jet on vane

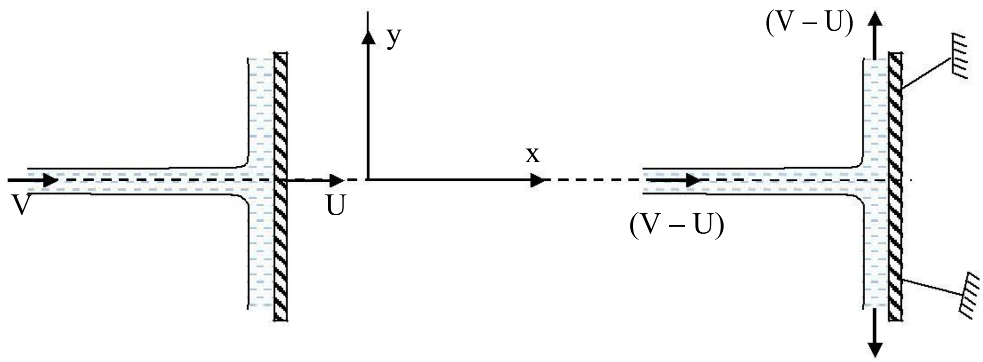
(viii) Force x Velocity of vane

(ix) Force x 0

(x) 0

Case-III

To compute the impact of jet on a moving flat plate held normal to the jet.



V = Velocity of jet striking the plate

U = Velocity of vane along the direction of vane.

Adopting the concept of relative velocity, the system can be considered to be a stationary plate, the jet striking the vane with a relative velocity (V – U).

$= \quad m = \rho Q$

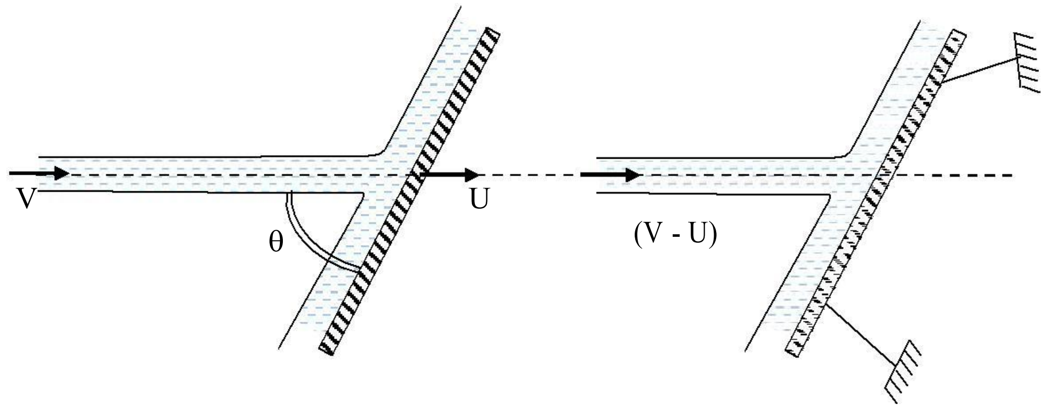
$\rho a(V - U)$

$F_x = \rho a(V - U)^2$

Work done by the jet on plate = Force x Velocity of plate = $\rho a\,(V - U)^2 \times U$

Case-IV

To compute the impact of jet on a moving flat plate held inclined to the direction of jet.



V = Velocity of jet

U = Velocity of plate along the direction of jet.

Adopting the concept of relative velocity, the above case can be considered to be fixed vane with a jet velocity of (V – U).

$$\therefore F_n = \rho a\,(V - U)^2$$
$$F_x = \rho a\,(V - U)^2 \sin^2\theta$$
$$F_y = \rho a\,(V - U)^2 \sin\theta \cos\theta$$

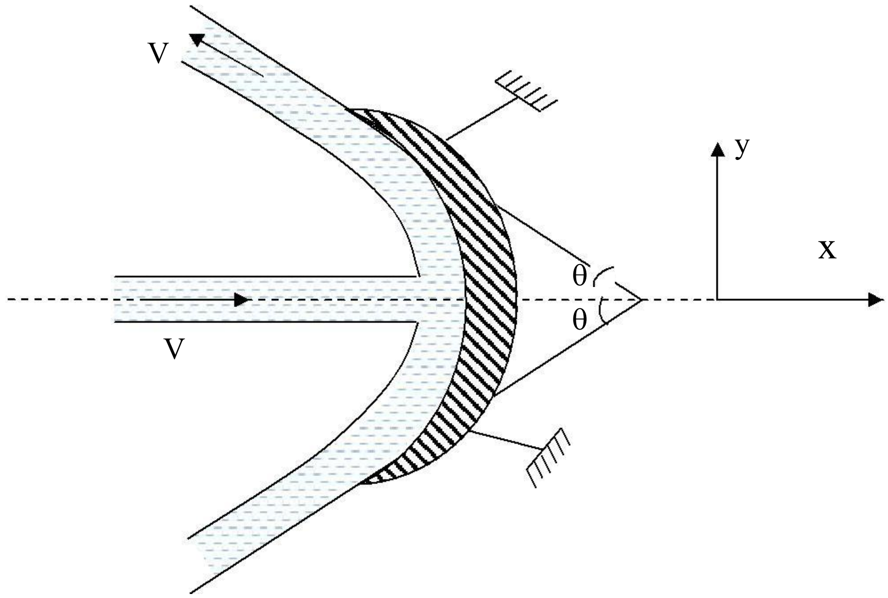
Work done by the jet on vane plate along x – direction

$= F_x$ x Velocity of plate along x – direction

$= \rho a\,(V - U)^2 \sin^2\theta\, U$

Case-V

To compute the impact of jet on a stationery symmetrical curved plate, the jet striking the plate at its centre.



$\boxed{m = \rho a V}$

$= F_x = m\,[U_x - V_x]$

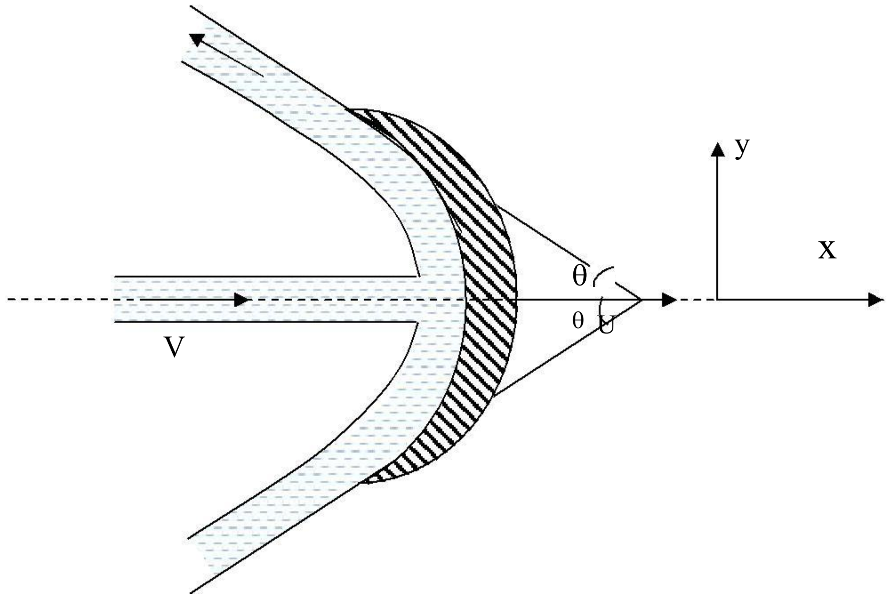
$F_x = (\rho a V)\,[V - (-V\cos\theta)]$

$F_x = \rho a V^2\,(1 + \cos\theta]$

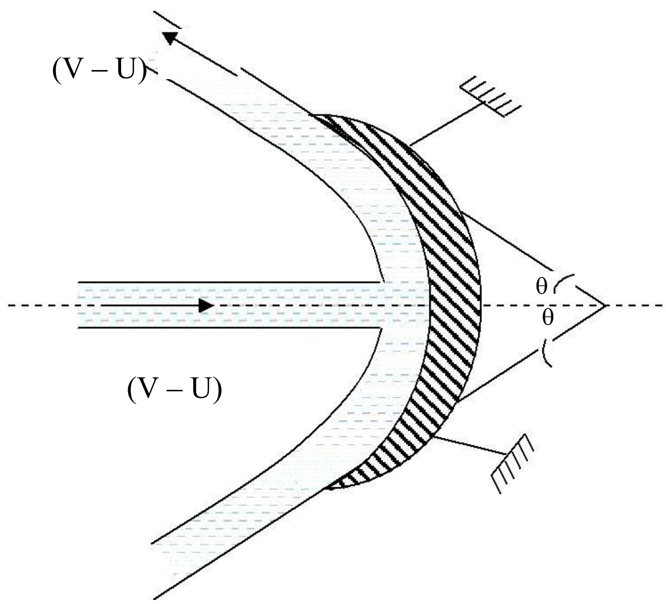
Work done by the jet on plate is zero since the plate is stationery.

Case-VI

To compute the impact of jet on a moving symmetrical curved plate, the jet striking the plate at its centre.



Adopting relative velocity concept, the system can be considered to be a jet of relative velocity (V – U) striking a fixed plate.



$$\boxed{m = \rho a\,(V - U)}$$

$F_x = m\,[U_x - V_x]$

$F_x = \rho a\,(V - U)\,[(V - U) - (-V - U)\cos\theta]$

$F_x = \rho a\,(V - U)^2\,(1 + \cos\theta]$

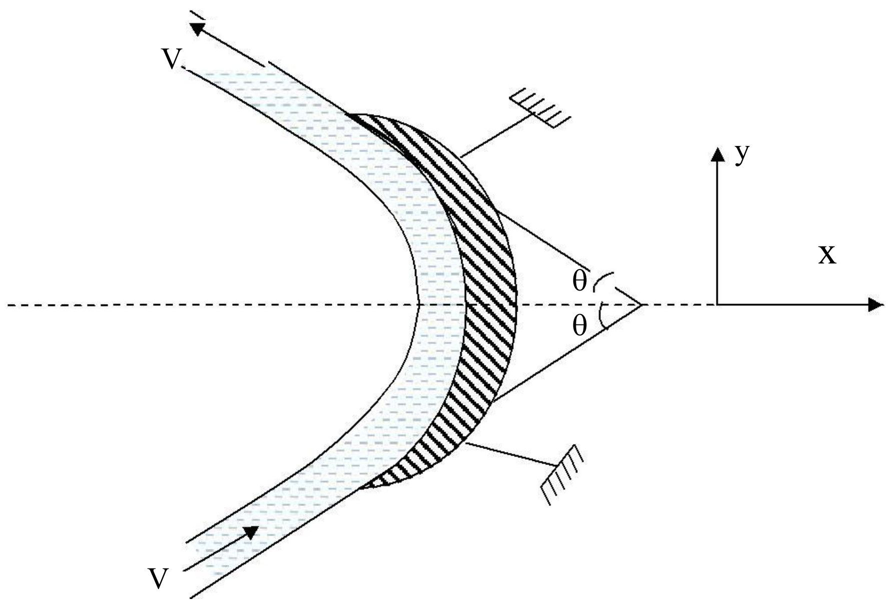
Work done by the jet on plate = Force x Velocity of date

$= F_x \; U$

$= \rho a(V - U)^2 (1 + \cos\theta] \, U$

Case-VII

To compute the impact of jet on a stationery symmetrical curved plate, the jet striking the plate at one of the tips tangentially.



$$\boxed{m = \rho a V}$$

$F_x = m\,[U_x - V_x]$

$F_x = \rho aV\,[V\cos\theta - (-V\cos\theta)]$

$F_x = \rho aV^2 (1 + \cos\theta]$

$F_y = m\,[U_y - V_y]$

$F_x = \rho aV\,[V\sin\theta - V\sin\theta]$

$F_x = 0$

Work done by the jet on plate is zero since the plate is stationery.

IMPACT OF JET ON VANES

Session – III

15. Problems - 1

A jet of water 50 mm diameter strikes a flat plate held normal to the direction of jet. Estimate the force exerted and work done by the jet if.

(viii) The plate is stationary

(ix) The plate is moving with a velocity of 1 m/s away from the jet along the line of jet.

(x) When the plate is moving with a velocity of 1 m/s towards the jet along the same line.

The discharge through the nozzle is 76 lps.

(ϖιι) Solution:

$d = 50 \text{ mm} = 50 \times 10^{-3} \text{ m}$

$a = \frac{\Pi}{4} \times (50 \times 10^{-3})^2$

$\mathbf{a = 1.9635 \times 10^{-3} m^2}$

$Q = aV$

$76 \times 10^{-3} = 1.9635 \times 10^{-3} \times V$

V = 38.70 m/s

Case a) When the plate is stationary

$F_x = \rho a V^2$

$F_x = 1000 \times (1.9635 \times 10^{-3}) \times (38.70)^2$

$\mathbf{F_x = 2940.71\ N}$

Work done/s = $F_x \times U$

Work done/s = $F_x \times 0$

Work done/s = 0

Case b) V = 38.70 m/s (→)

U = 1 m/s (→)

$F_x = \rho a (V - U)^2$

$F_x = 1000 \times 1.9635 \times 10^{-3} \times (38.7 - 1)^2$

$\mathbf{F_x = 2790\ TN}$

Work done/s = F_x x U

Work done/s = 2790.7 x 1

Work done/s = 2790.7 Nm/s or J/s or W

Case c) V = 38.70 m/s (→)

U = 1 m/s (←)

$F_x = \rho a (V - U)^2$

$F_x = 1000 \times 1.9635 \times 10^{-3} \times (38.7 + 1)^2$

$\mathbf{F_x = 3094.65\ N}$

Work done/s = F_x x U

Work done/s = 3094.65 x 1

Work done/s = 3094.65 Nm/s

(ϖι) Problems - 2

A jet of water 50 mm diameter exerts a force of 3 kN on a flat vane held perpendicular to the direction of jet. Find the mass flow rate.

(ϖ) Solution:

$d = 50\ mm = 50 \times 10^{-3}\ m$

$a = \frac{\Pi}{4} \times (50 \times 10^{-3})^2$

$\mathbf{a = 1.9635 \times 10^{-3} m^2}$

$F_x = \rho a V^2$

$3000 = 1000 \times 1.9635 \times 10^{-3} \times V^2$

$\mathbf{V = 39.09\ m/s}$

m = ρQ

m = ρaV

m = 1000 x 1.9635 x 10^{-3} x 39.09

m = 76.75 kg/s

(ξι)Problems - 3

A jet of data 75 mm diameter has a velocity of 30 m/s. It strikes a flat plate inclined at 45^o to the axis of jet. Find the force on the plate when.

- The plate is stationary
- The plate is moving with a velocity of 15 m/s along and away from the jet.

Also find power and efficiency in case (b)

- **Solution:**

d = 75 x 10^{-3} m

$a = \frac{\Pi}{4}$ x (75 x $10^{-3})^2$

V = 30 m/s

a = 4.418 x 10^{-3} m^2

θ = 45^o

Case a) When the plate is stationary

$F_X = \rho a V^2 \sin^2\theta$

 1000 x 4.418 x 10^{-3} x 30^2 x Sin^2 45

 1988.10 N

Case b) When the plate is moving

V = 30 m/s (→)

U = 15 m/s (→)

$F_X = \rho a\ (V - U)^2 \sin^2\theta$

F_X = 1000 x 4.418 x 10^{-3} x $(30 - 15)^2$ $\sin^2 45$

F_X = 497.03 TN

Output power = Work done/s

Output power = F_x x U

Output power = 497.03 x 15

Output power = 7455.38 W

Input power = Kinetic energy of jet/s

Input power = $\frac{1}{2}$ x m x V^2

Input power = $\frac{1}{2}$ x (ρaV) x V^2

Input power = $\frac{1}{2}$ x1000 x 4.418 x10^{-3} x 30^3

Input power = 59643 W

Efficiency of the system = $\frac{O/P}{I/P}$ x100

Efficiency of the system = $\frac{7455.38}{59643}$ x100

Efficiency of the system = 12.5%

η **Problem – 4**

A 75 mm diameter jet having a velocity of 12 m/s impinges a smooth flat plate, the normal of which is inclined at 60^o to the axis of jet. Find the impact of jet on the plate at right angles to the plate when the plate is stationery.

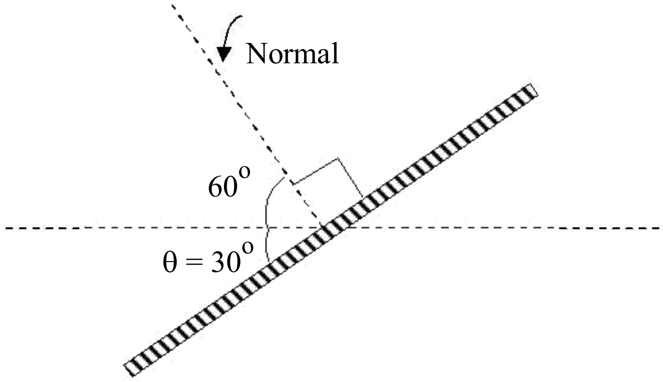
What will be the impact if the plate moves with a velocity of 6 m/s in the direction of jet and away from it. What will be the force if the plate moves towards the plate.

η **Solution**

d = 75 x 10^{-3} m

a = $\frac{\pi}{4}$ x $(75 \text{ x}10^{-3})^2$

a = 4.418 x 10^{-3} m^2



When the plate is stationery

$F_n = \rho a V^2 \sin\theta$

$F_n = 1000 \times (4.418 \times 10^{-3})\ 12^2 \sin 30$

$\mathbf{F_n = 318.10\ N}$

When the plate is moving away from the jet

$F_n = \rho a\ (V - U)^2 \sin\theta$

$F_n = 1000 \times 4.418 \times 10^{-3}\ (12 - 6)^2 \sin 30$

$\mathbf{F_n = 79.52\ N}$

When the plate is moving towards the jet

$F_n = \rho a\ (V + U)^2 \sin\theta$

$F_n = 1000 \times 4.418 \times 10^{-3}\ (12 + 6)^2 \sin 30$

$\mathbf{F_n = 715.72\ N}$

Ω **Problem – 5**

A vertical flat plate is hinged at its top. A jet of water strikes at the centre of the plate. Due to the impact of jet, the plate attains equilibrium at an angle 'θ' with the vertical. Show that $\sin\theta = \frac{\rho a V^2}{W}$ where

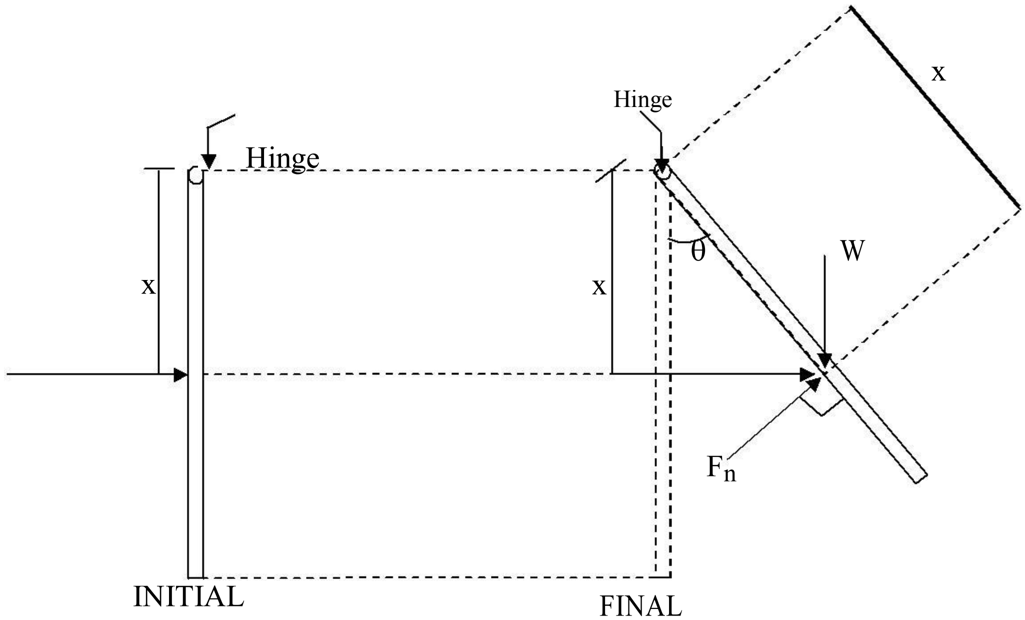
ρ – Mass density of fluid

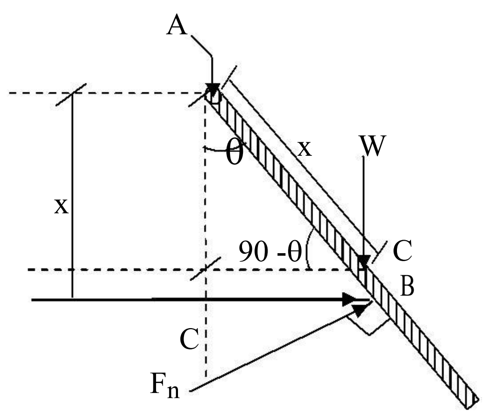
a – area of cross section of jet.

V – Velocity of jet

W – Weight of the plate

- **Solution**





$\Sigma M_{Hinge} = 0$

$-F_n \times AB + W \times CA \sin\theta = 0$

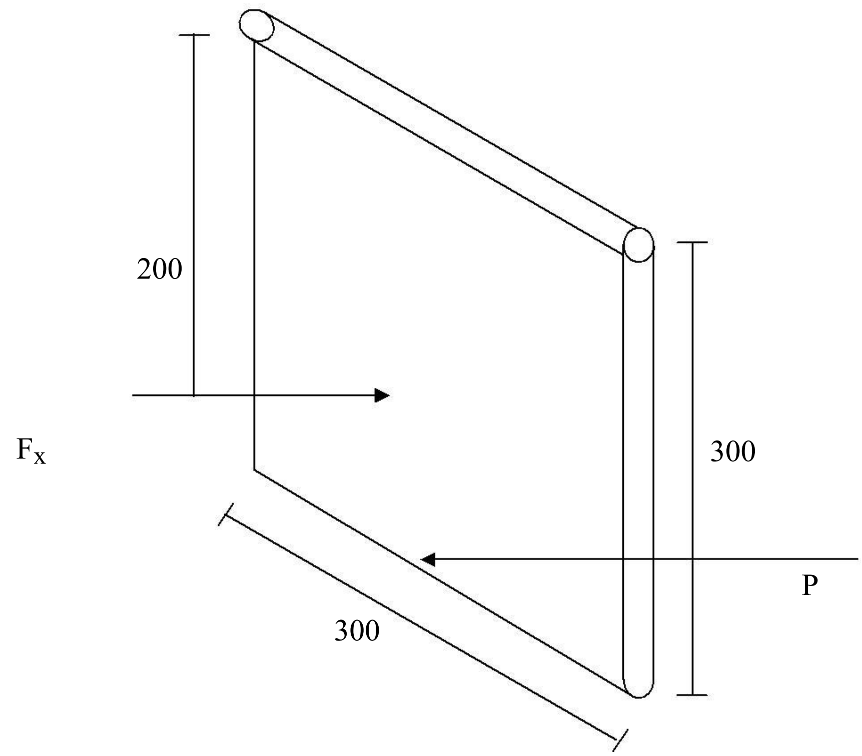
$-\rho a V^2 \sin(90 - \theta) \cdot \dfrac{x}{\cos\theta} + W \cdot x \sin\theta = 0$

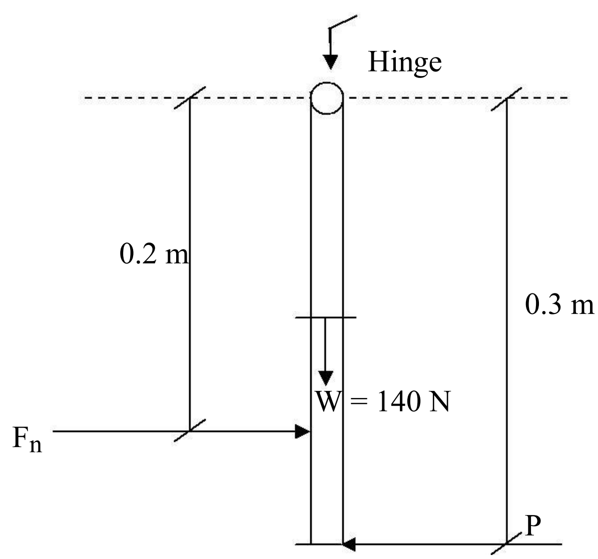
$\rho a V^2 = W \sin\theta$

$$\mathbf{\sin\theta = \frac{\rho a V^2}{W}}$$

- **Problem – 6**

A square plate weighing 140 N has an edge of 300 mm. The thickness of the plate is uniform. It is hung so that it can swing freely about the upper horizontal edge. A horizontal jet of 20 mm diameter having 15 m/s velocity impinges on the plate. The centre line of jet is 200 mm below. The centre line of jet is 200 mm below the upper edge of plate. Find what force must be applied at the lower edge of plate in order to keep it vertical.





$$a = \frac{\pi}{4} \times \left[20 \times 10^{-3}\right]^2$$

$$\mathbf{a = 314.16 \times 10^{-6} m^2}$$

$V = 15$ m/s

$\Sigma M_{Hinge} = 0$

$F_x \times 0.2 + P \times 0.3 = 0$

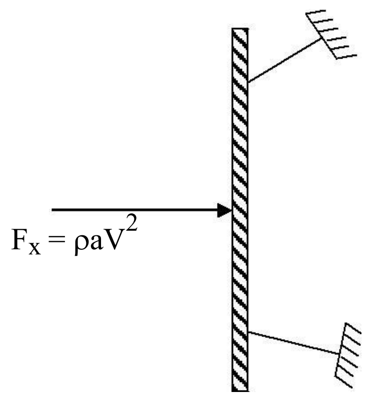
$\rho a V^2 = 0.2 = P \times 0.3$

$1000 \times 314.16 \times 10^{-6} \times 0.2 \times 15^2 = P \times 0.3$

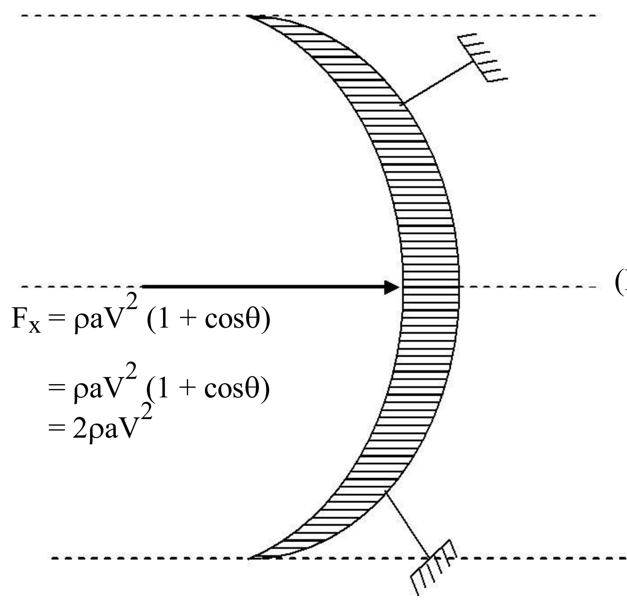
$\mathbf{P = 47.12\ N}$

- **Problem – 7**

Show that the force exerted by a jet on a hemispherical stationery vane is twice the force exerted by the same jet on flat stationery normal vane.



(DERIVE)



(DERIVE)

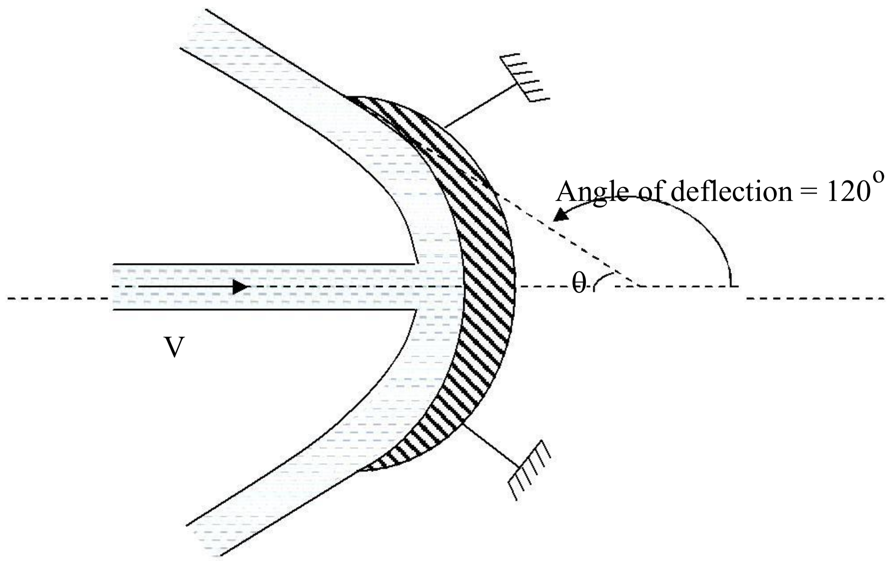
$$\frac{F_{x2}}{F_{x1}} = \frac{2\rho aV^2}{\rho aV^2}$$

$$\therefore F_{x2} = 2F_{x1}$$

- **Problem – 8**

A jet of water of diameter 50 mm strikes a stationary, symmetrical curved plate with a velocity of 40 m/s. Find the force extended by the jet at the centre of plate along its axis if the jet is deflected through 120^o at the outlet of the curved plate.

Solution:



$d = 50 \times 10^{-3} m$

$a = \frac{\pi}{4} \times (50 \times 10^{-3})^2$

$\mathbf{a = 1.963 \times 10^{-3}\ m^2}$

$V = 40$ m/s

$\theta = 60^o$

$F_x = \rho aV^2 (1 + \cos\theta)$

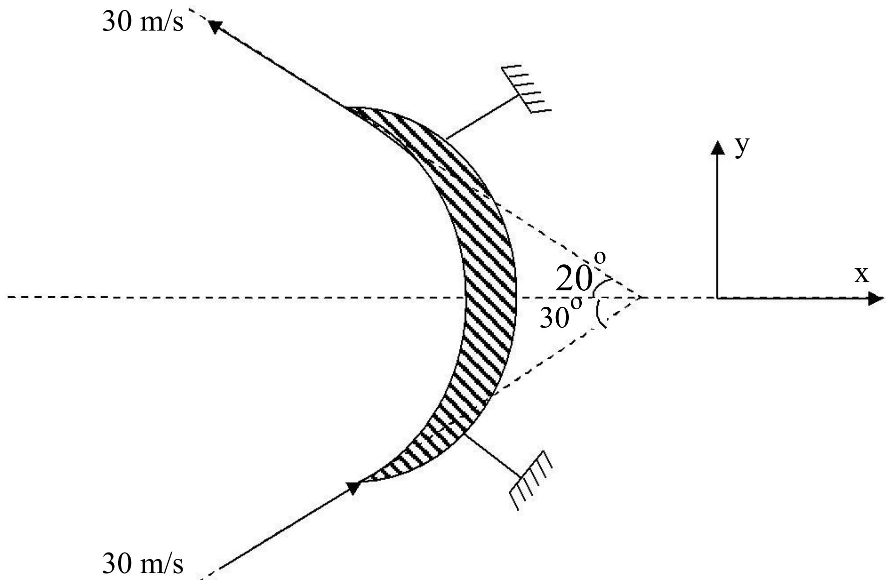
$F_x = 1000 \times 1.963 \times 10^{-3} \times 40^2 (1 + \cos 60)$

$\mathbf{F_x = 4711\text{-}2\ N}$

- **Problem – 9**

A jet of water strikes a stationery curved plate tangentially at one end at an angle of 30^o. The jet of 75 mm diameter has a velocity of 30 m/s. The jet leaves at the other end at angle of 20^o to the horizontal. Determine the magnitude of force exerted along 'x' and 'y' directions.

Solution:



$d = 50 \times 10^{-3} m$

$a = \frac{\pi}{4} \times (75 \times 10^{-3})^2$

$\mathbf{a = 4.418 \times 10^{-3}\ m^2}$

$F_X = m [U_X - V_X]$

$F_X = \rho aV [30 \cos 30 – (-30 \cos 20)]$

$F_X = 1000 \times 4.418 \times 10^{-3} \times 30 (30 \cos 30 + 30\cos20)$

$\mathbf{F_X = 7179.90\ N}$

- **Problem – 10**

A jet of water of diameter 75 mm strikes a curved plate at its centre with a velocity of 25 m/s. The curved plate is moving with a velocity of 10 m/s along the direction of jet. If the jet gets deflected through 165^o in the smooth vane, compute.

a) Force exerted by the jet.

b) Power of jet.

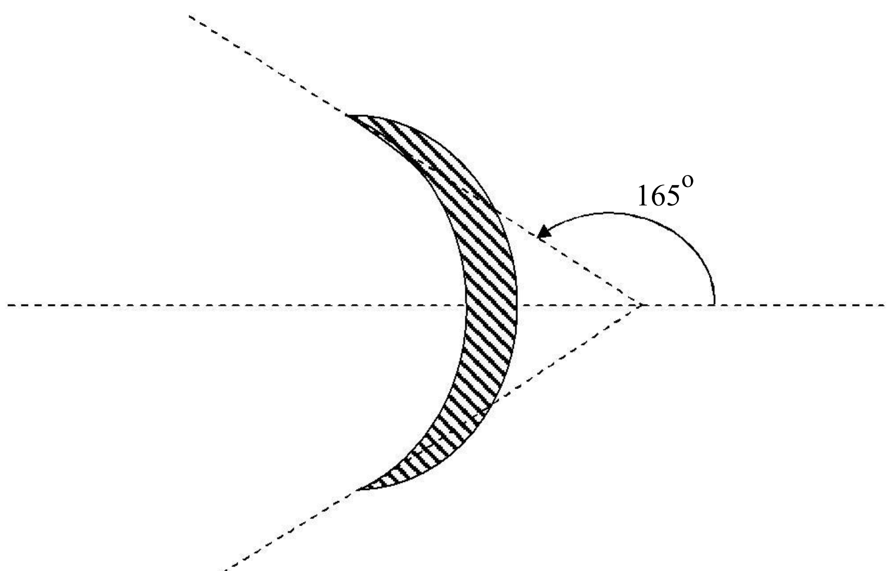
c) Efficiency of jet.

$d = 75 \text{ mm} = 75 \times 10^{-3} \text{m}$

$a = \frac{\pi}{4} \times \left(75 \times 10^{-3}\right)^2$

$a = 4.418 \times 10^{-3} \text{ m}^2$

Solution:



$\theta = 15^o$

$V = 25 \text{ m/s}$

$U = 10 \text{ m/s}$

$F_X = \rho a (V - U)^2 (1 + \cos\theta)$

$F_X = 1000 \times 4.418 \times 10^{-3} [25 - 10]^2 \times (1 + \cos 15)$

$F_X = 1954.23$ N

Power of jet = Work done/s

Power of jet = F_x x U

Power of jet = 1954.23 x 10

Power of jet = 19542.3 W

Kinetic energy of jet/s = $\frac{1}{2} mV^2$

Kinetic energy of jet/s = $\frac{1}{2}\left[1000 \text{ x } 4.418 \text{ x}10^3 \text{ x } 25\right]25^2$

Kinetic energy of jet/s = 34515.63 W

$\eta = \frac{\text{Out put}}{\text{In put}}$

$\eta = \frac{19542.3}{34515.63}$

η **= 56.4 %**

12

IMPACT OF JET ON VANES

Session – IV

16. Problems - 11

A symmetrical curved vane is moving with a velocity of ‘U’ and a jet of velocity ‘V’ strikes at the centre along the direction of motion.

Derive expressions for

(xi) Force exerted along the direction of motion.

(xii) Work done/s.

(xiii) Efficiency of the system.

(xiv) Maximum efficiency.

(ϖιιι) Solution

$$F_x = \rho a\,(V - U)^2\,(1 + \cos\theta)$$

$$\text{Work done/s} = \rho a\,(V - U)^2\,(1 + \cos\theta)\,U$$

$$\text{Efficiency} = \frac{\text{Work done / s}}{\text{K.E.supplied / s}}$$

$$\eta = \frac{\rho a(V - U)^2\,(1+\cos\theta)U}{\frac{1}{2}}$$

$$\eta = \frac{\rho a(V - U)^2\,(1+\cos\theta)U}{\frac{1}{2} \times \rho a V \,.\, V^2}$$

$$\eta = \frac{2(V - U)^2\,(1 + \cos\theta)U}{V^3}$$

$$\eta = \frac{2\left(V^2U - 2U^2V + U^3\right)(1 + \cos\theta)U}{V^3}$$

For maximum efficiency

$$\frac{d\eta}{dU} = 0$$

$$\frac{d\eta}{dU} = 0 = \frac{2\left(V^2 - 4UV + 3U^2\right)(1+\cos\theta)}{V^3}$$

$$V^2 - 4UV + 3U^2 = 0$$

$$(V - 3U)\,(V - U) = 0$$

$V - 3U = 0$ or $V - U = 0$

$(V - U) \neq 0$

$\therefore\ V = 3U$ or $U = \dfrac{V}{3}$

For maximum efficiency velocity of vane should be $\frac{1}{3}$ x velocity of jet

$$\eta_{max} = \frac{2(V-U)^2(1+\cos\theta)U}{V^3}$$

Substituting $U = \dfrac{V}{3}$

$$\eta_{max} = \frac{16}{27}\cos^2\left[\frac{\theta}{2}\right]$$

if the vane is hemispherical $\theta = 0$

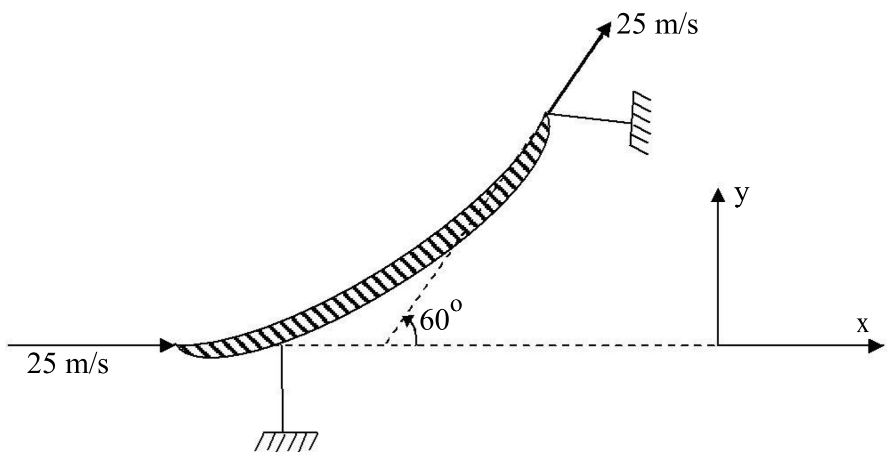
$$\eta_{max} = \frac{16}{27}$$

$$\eta_{max} = 59.25\%$$

(ϖιι) Problems - 12

A jet of water from a nozzle is deflected through 60^o from its direction by a curved plate to which water enters tangentially without shock with a velocity of 30m/s and leaver with a velocity of 25 m/s as shown in figure. If the discharge from the nozzle is 0.8 kg/s, calculate the magnitude and direction of resultant force on the vane.

(ϖι) Solution



$m = 0.8$ kg/s

$F_x = m\ (U_x - V_x)$

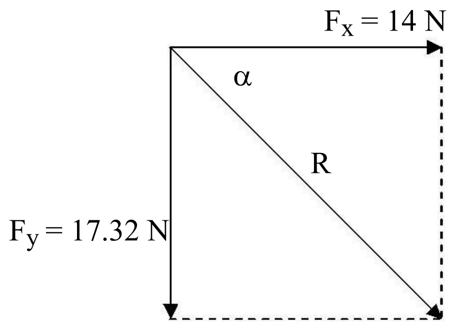
$F_x = 0.8\ (30 - 25\cos 60)$

$F_x = 14$ N (→)

$F_y = m\ (U_y - V_y)$

$F_y = 0.8\ (0 - 25\sin 60)$

$F_y = 17.32$ N (↓)



$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

R = 22.27 N

$$\alpha = \tan^{-1}\left(\frac{\Sigma F_y}{\Sigma F_x}\right)$$

$\alpha = 51.05$

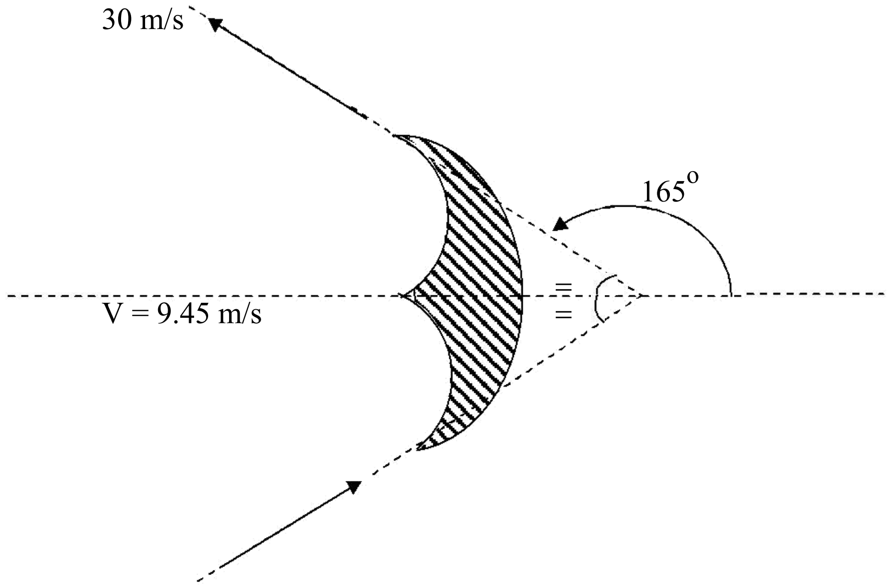
(ξιι) Problems - 13

A jet of water 50 mm in diameter impinges on a fixed cup which deflects the jet by 165° as shown in figure. If the reaction of the cup was found to be 26.5 N when the discharge was 980 N/minute compute the ratio of.

Actual force to theoretical force of jet.

Velocity of outlet to velocity at inlet.

= **Solution**



$d = 50 \times 10^{-3} m$

$= = \frac{\pi}{4} (50 \times 10^{-3})^2$

$a = 1.963 \times 10^{-3} m^2$

$\theta = 15^o$

W =m g

$\frac{980}{60}$ m x 9.81

m = 1.665 kg/s

$F_x = m (U_x – V_x)$

F_x = 1.665 (9.45 + 9.45 cos 15) F_x = 30.93 N

$\frac{F_{act}}{F_{th}} = \frac{26.5}{30.93}$

$\frac{F_{act}}{F_{th}} = 0.857$

Let, V_1 be the actual velocity at out $F_{x\ act} = m (U_x – V_x)$

$26.5 = 1.665\,(9.45 + V_1 \cos 15)$

$V_1 = 6.964$ m/s

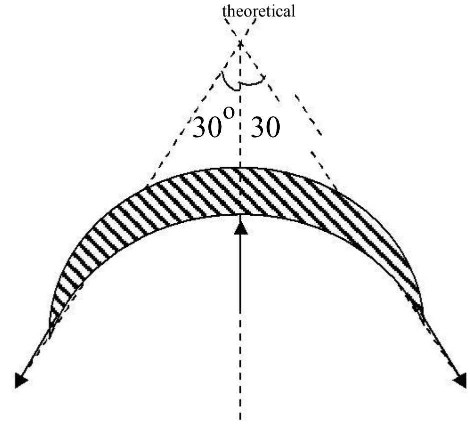
$\dfrac{\text{Velocity at outlet}}{\text{Velocity at inlet}} = \dfrac{6.694}{6.45}$

$\dfrac{\text{Velocity at outlet}}{\text{Velocity at inlet}} = 0.708$

= **Problem – 14**

In a laboratory setup, a symmetrical curved vane which deflects the centerline of jet by 30^o support a total mass of 180 grams. Determine the discharge that must be supplied for balancing the vane. The jet diameter is 5 mm and vane co-efficient is 0.75.

Note: Co-efficient of impact = $C_i = \dfrac{F_{actual}}{F_{theoretical}}$



$d = 5 \times 10^{-3} m^2$

$d = \dfrac{\pi}{4} x \left(5x10^{-3}\right)^2$

$\mathbf{a = 1.963 \times 10^{-5} m^2}$

W = mg

W = 0.180 x 9.81

W = 1.7658 N

$Ci = \dfrac{F_{act}}{F_{th}}$

5

$$0.75 = \frac{1.7658}{F_{theoretical}}$$

$$\therefore T_{heoretical} = \frac{1.7658}{0.75}$$

$$\mathbf{= 2.3544\ N}$$

$F = \rho a V^2 (1 + \cos\theta)$

$2.354 = 1000 \times 1.963 \times 10^{-3} \times V^2 (1 + \cos 30)$

$V^2 = 0.6426$

$\mathbf{V = 0.8016\ m/s}$

$Q = aV$

$Q = 1.963 \times 10^{-5} \times 0.8016$

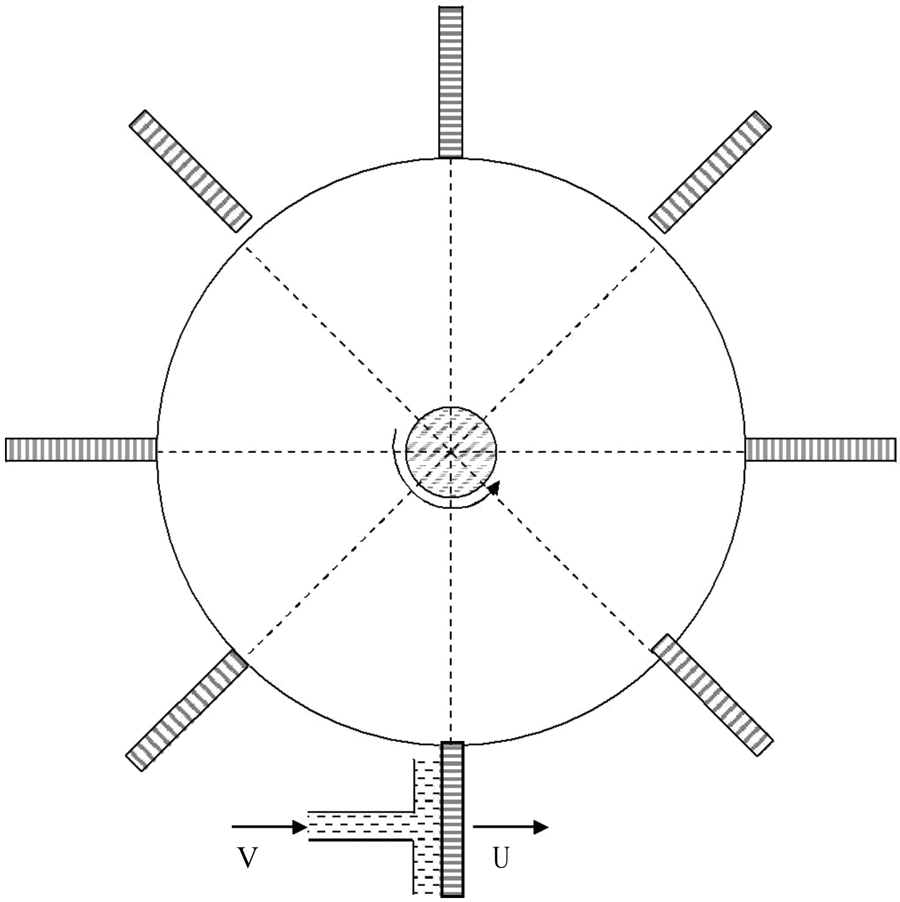
$\mathbf{Q = 1.5735 \times 10^{-5}\ m^3/s}$

Case-VIII

To derive expressions for the force exerted, work done and efficiency of impact of jet on a series of flat vanes mounted radially on the periphery of a circular wheel.

OR

To show that efficiency of impact of jet on radially mounted flat vanes is 50% when the jet strikes normally on the vane.



Let us consider flat vanes mounted radially on the periphery of a circular wheel. V is the velocity of jet and 'U' is the velocity of vane. The inpact of jet on vanes will be continuous since vanes occupy one after another continuously.

$$F_x = m\,(U_x - V_x)$$

$$F_x = \rho a V\,[(V - U) - 0]$$

$$\mathbf{F_x = \rho a V\,(V - U)}$$

Work done/s or Power = $F_x \cdot U$

$$\mathbf{Power = \rho a V\,(V - U)\,U}$$

$$\text{Efficiency} = \quad \eta = \frac{O / P}{I / P}$$

$$\eta = \frac{\rho a V (V - U) U}{\frac{1}{2}(\rho a V) V^2}$$

7

$$\eta = \frac{2(V-U)U}{V^2}$$

Conduction for maximum efficiency:

$$\eta = \frac{2}{V^2}\left(VU - U^2\right)$$

For maximum efficiency

$$\frac{d\eta}{dU} = 0$$

$$\frac{d\eta}{dU} = 0 = \frac{2}{V^2}\left(V - 2U\right)$$

$$\therefore U = \frac{V}{2}$$

$\therefore$ Efficiency is maximum when the vane velocity is 50% of velocity of jet.

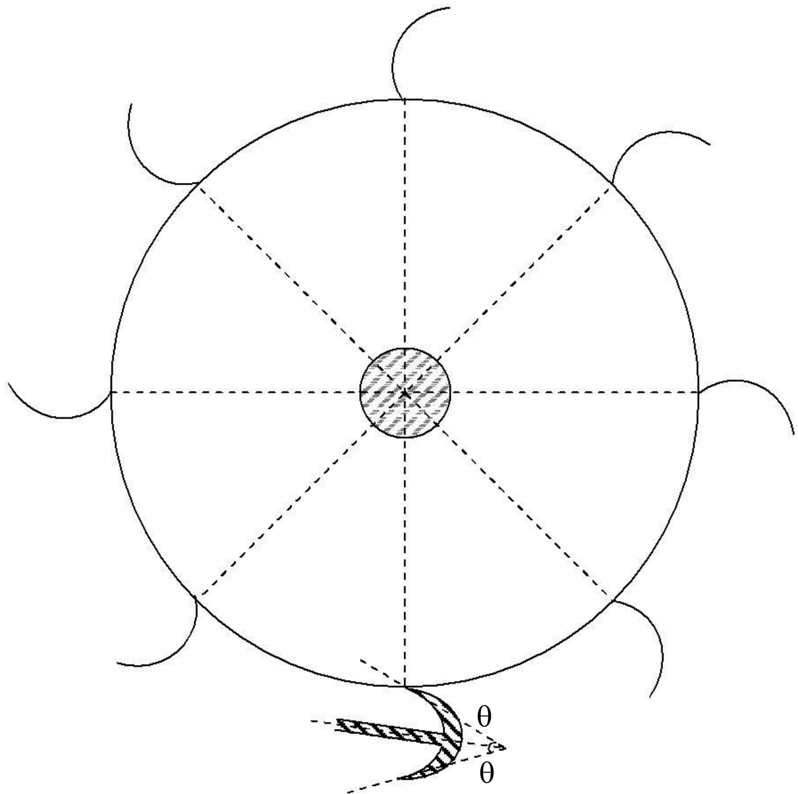
$$\eta_{max} = \frac{2\left(V - \frac{V}{2}\right)\frac{V}{2}}{V^2}$$

$$= \frac{1}{2}$$

$$= \mathbf{50\%}$$

Case-IX

To derive expression for the force exerted, power and efficiency of impact of jet on a series of symmetrical curved vanes mounted on the periphery of a wheel.



$F_x = (\rho aV)\ (U_x – V_x)$

$F_x = (\rho aV)\ [(V – U) + (V – U) \cos\theta]$

$\mathbf{F_x = \rho aV\ [(V – U)\ (1 + \cos\theta)}$

Power = work done/s

Power = F_x x U

Power = $\rho aV(V – U)\ (1 + \cos\theta)\ U$

Input = $\frac{1}{2} m V^2$

Input = $\frac{1}{2} \rho a V^3$

$\eta = \frac{O/P}{I/P}$

$$\eta = \frac{\rho a V (V - U)(1 + \cos\theta) U}{\frac{1}{2}\rho a V^3}$$

$$\eta = \frac{2(V - U)(1 + \cos\theta) U}{\frac{1}{2}\rho a V^3}$$

Condition for maximum efficiency

$$\eta = \frac{2(VU - U^2)(1 + \cos\theta) U}{V^2}$$

For maximum efficiency

$$\frac{d\eta}{dU} = 0 = \frac{2}{V^2}(V - 2U)(1 + \cos\theta)$$

$$V - 2U = 0$$

$$\mathbf{U = \frac{V}{2}}$$

Vane velocity = $\frac{1}{2}$ x velocity of jet

$$\eta_{max} = \frac{2\left(V - \frac{V}{2}\right)(1 + \cos\theta)\frac{V}{2}}{V^2}$$

$$\eta_{max} = \frac{(1 + \cos\theta)}{2}$$

If the vanes are hemispherical θ = 0

$$\eta_{max} = \frac{H \cos\theta}{2}$$

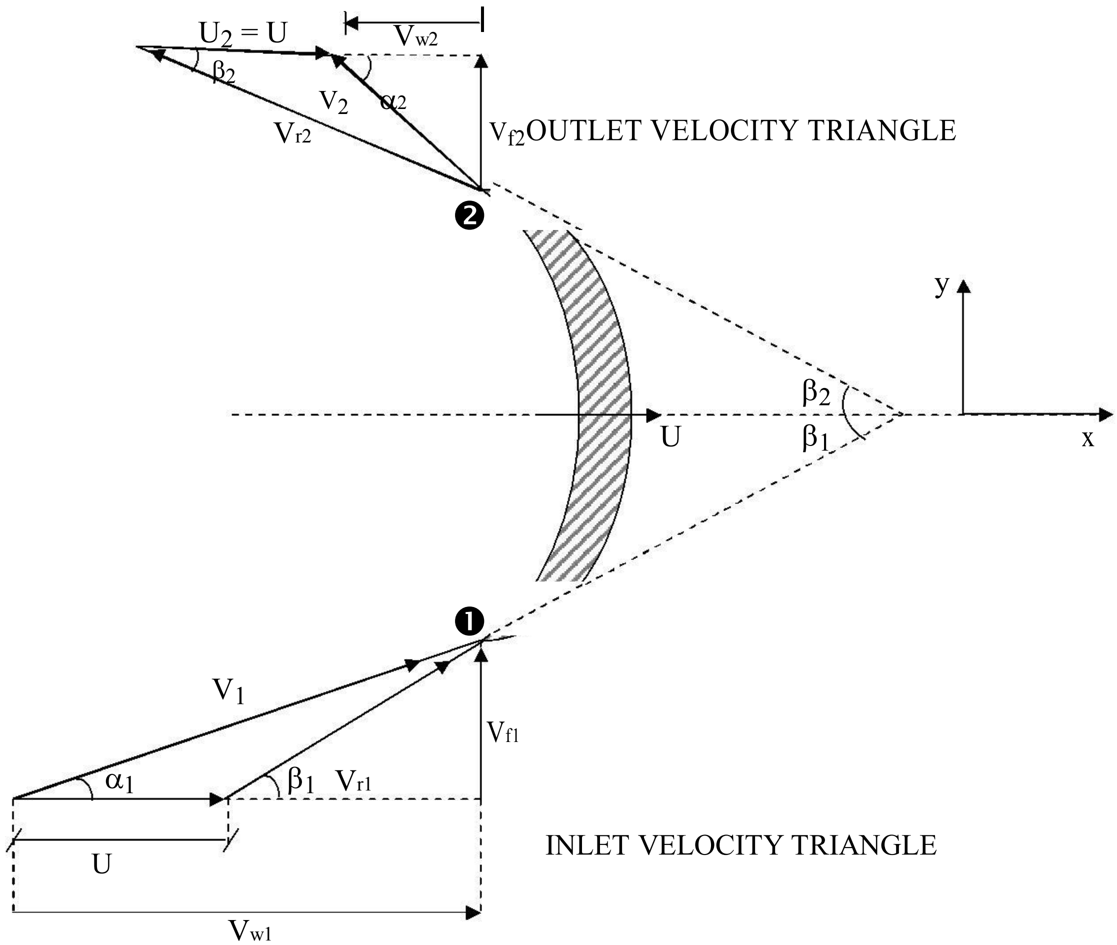
$\eta_{max} = 1$ or 100%

10

IMPACT OF JET ON VANES

Session – V

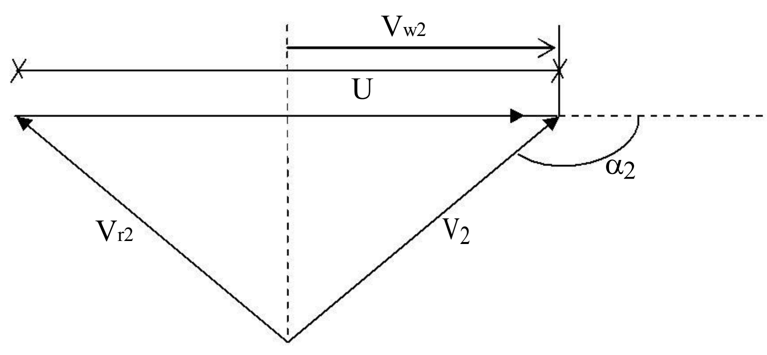
Case X: **Force exerted by a jet of water on an asymmetrical curved vane when the jet strikes tangentially at one of the tips:**



$$F_x = m\,[U_x - V_x]$$

$$F_x = \rho a V_{r1}\,[(V_{w1} - U) - (-(V_{w2} + U))]$$

$$\mathbf{F_x = \rho a V_{r1}\,[V_{w1} + V_{w2}]}$$



1

$F_x = \rho a V_{r1}\ [(V_{w1} - U) - (-(U - V_{w2}))]$

$\mathbf{F_x = \rho a V_{r1}\ [V_{w1} - V_{w2}]}$

$\therefore \mathbf{F_x = \rho a V_{r1}\ [V_{w1} \pm V_{w2}]}$

Work done/s $= F_x \times U$

Work done/s = $\rho a V_{r1}\ [V_{w1} \pm V_{w2}]\ U$

Work done for unit mass flow rate = $[V_{w1} \pm V_{w2}]\ U$

Work done for unit weight flow rate = $[V_{w1} \pm V_{w2}]\ \dfrac{U}{g}$

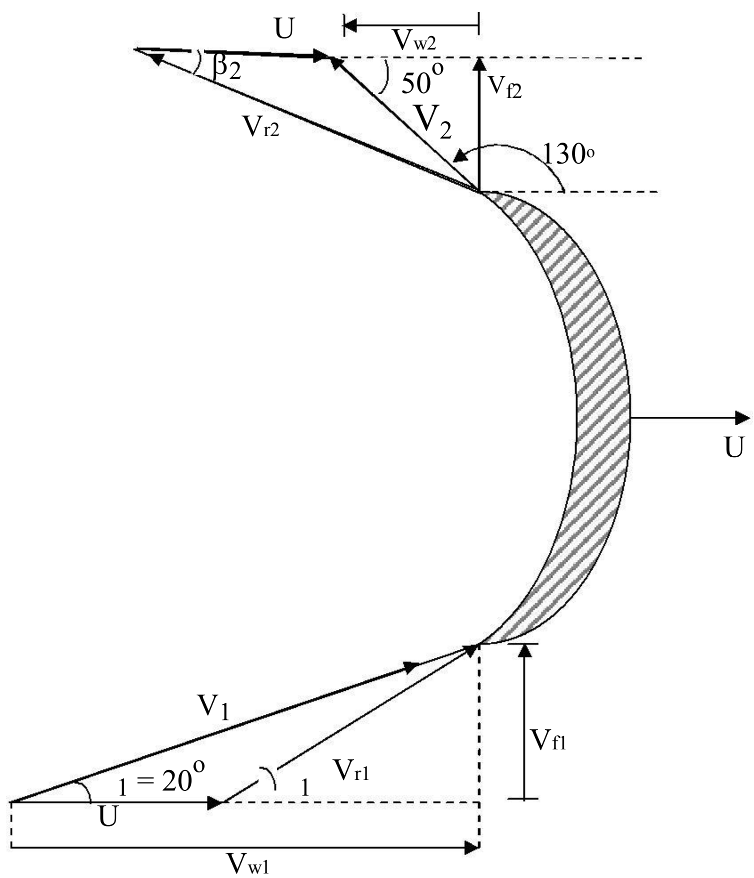
17. Problem:

A jet of water impinges a curved plate with a velocity of 20 m/s making an angle of 20^o with the direction of motion of vane at inlet and leaves at 130^o to the direction of motion at outlet. The vane is moving with a velocity of 10 m/s. Compute.

(xv) Vane angles, so that water enters and leaves without shock.

(xvi) Work done/s

(ιξ)Solution:



$V_1 = 20$ m/s

$U_1 = U_2 = 10$ m/s

Assuming number loss $V_{r1} = V_{r2}$

$V_{w1} = 20 \cos 20 = 18.79$ m/s

$V_{f1} = 20 \sin 20 = 6.84$ m/s

$$\tan \beta_1 = \frac{V_{f1}}{(V_{w1} - U)}$$

$$\tan \beta_1 = \frac{6.84}{(18.79 - 10)}$$

$$\mathbf{\tan \beta_1 = 37.88^o}$$

$$\sin \beta_1 = \frac{V_{f1}}{V_{r1}}$$

$$\sin 37.88 = \frac{6.84}{V_{r1}}$$

$V_{r1} = 11.14$ m/s

$$\mathbf{V_{r2} = V_{r1} = 11.14 \text{ m/s}}$$

$$\frac{V_{r2}}{\sin 130} = \frac{U}{\sin(180 - 130 - \beta_2)}$$

$$\sin(150 - \beta_2) = \frac{10 \sin 130}{11.14}$$

$$\sin(50 - \beta_2) = 0.6877$$

$$\mathbf{\beta_2 = 6.55^o}$$

Work done per unit mass flow rate

(viii) $(V_{w1} + V_{w2})\, U$

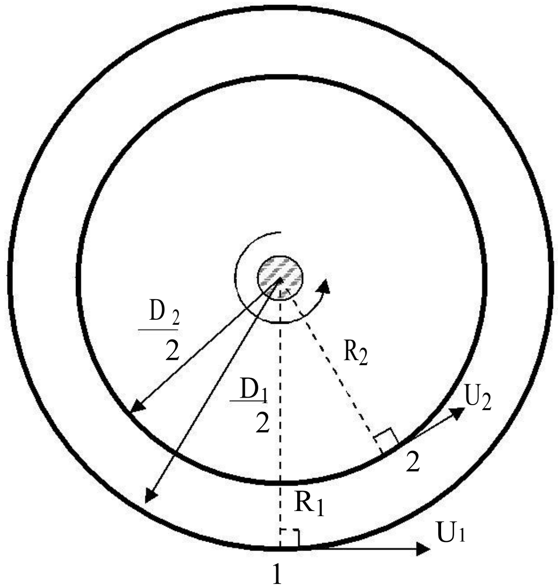
(ix) $[18.79 + (V_{r2} \cos \beta_2 - U)]$

(x) $[18.79 + 11.14 \cos 6.55 - 10]\, 10$

(xi) 198.57 W/kg

Case XI: Work done by water striking the vanes of a reaction turbine.

Note:



$$U_1 = \frac{\pi D_1 N}{60} \qquad U_1 = R_1\omega$$

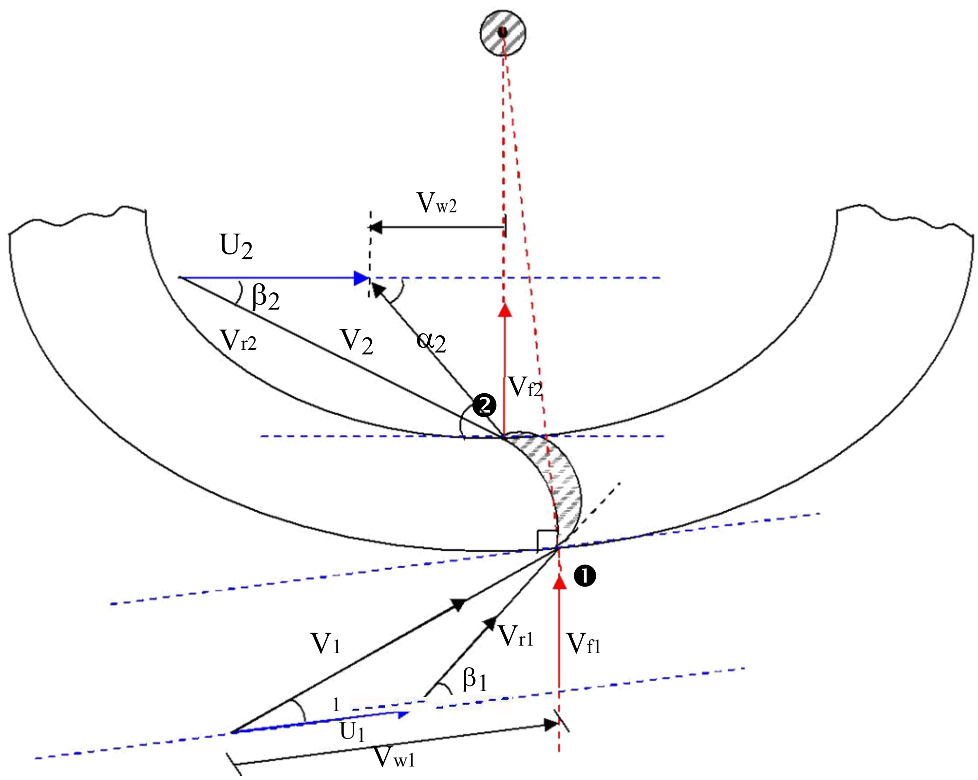
$$U_2 = \frac{\pi D_2 N}{60} \qquad U_2 = R_2\omega$$

2. Angular Momentum Principle:

Torque = Rate of change of angular momentum

$T = (\rho Q) \cdot [V_{w1} R_1 - V_{w2} R_2]$

4



☎❖♓♓☽ - Inlet tip

☎⊠♓♓♓☽ - Outlet tip

U_1 - Tangential velocity of wheel at inlet = $\frac{\pi D_1 N}{60}$

U_2 - Tangential velocity of wheel at outlet = $\frac{\pi D_2 N}{60}$

V_1 - Absolute velocity of fluid at inlet

V_2 - Absolute velocity of fluid at outlet

V_{w1} - Tangential component of absolute velocity at inlet – velocity of wheel at inlet = $V_1 \cos \alpha_1$.

V_{w2} - Tangential component of absolute velocity at outlet – velocity of wheel at outlet = $V_2 \cos \alpha_2$.

V_{f1} - Absolute velocity of flow at inlet

V_{f2} - Absolute velocity of flow at outlet

V_{r1} - Relative velocity at inlet

V_{r2} - Relative velocity at outlet

α_1 - Guide angle or guide vane angle at inlet

β_1 - Vane angle at inlet

β_2 - Vane angle at outlet

By angular momentum equation

$$T = m\,[V_{w1} R_1 - (-V_{w2} R_2)]$$

$$\mathbf{T = m\,[V_{w1} R_1 + V_{w2} R_2]}$$

$$\mathbf{T = m\,[V_{w1} R_1 \pm V_{w2} R_2]} \qquad \textbf{----- (1)}$$

Work done/s or power = T x Angle velocity

Work done/s or power = $T \cdot \omega$

Work done/s or power = $m\,[V_{w1} R_1 \pm V_{w2} R_2]\,\omega$

Work done/s or power = $m\,[V_{w1} (\omega R_1) \pm V_{w2} (\omega R_2)]$

Work done/s or power = $m\,[V_{w1} U_1 \pm V_{w2} U_2]$

Work done per unit mass flow rate = $[V_{w1} U_1 \pm V_{w2} U_2]$

Work done per unit weight flow rate $= \frac{1}{g} (V_{w1} U_1 \pm V_{w2} U_2)$

Efficiency of the system $= \eta = \dfrac{m\left[V_{w1} U_1 \pm V_{w2} U_2\right]}{\frac{1}{2} m V_1^2}$

$$\eta = \frac{2\left[V_{w1} U_1 \pm V_{w2} U_2\right]}{V_1^2}$$

= **Problem:**

A jet of water having a velocity of 35 m/s strikes a series of radial curved vanes mounted on a wheel. The wheel has 200 rpm. The jet makes 20^o with the tangent to wheel at inlet and leaves the wheel with a velocity of 5 m/s at 130^o to tangent to the wheel at outlet. The diameters of wheel are 1 m and 0.5 m. Find

 Vane angles at inlet and outlet for radially outward flow turbine.

 Work done

 Efficiency of the system

η **Solution**

V_1 - 35 m/s

N - 200 rpm

α_1 - 20^o

α_2 - $180 - 130 = 50^o$

V_2 - 5 m/s

D_1 - 1 m

D_2 - 0.5 m

$$U_1 = \frac{\pi D_1 N}{60}$$

$$U_1 = \frac{\pi x 1 x\ 200}{60}$$

$$\mathbf{U_1 = 10.47\ m/s}$$

$$U_2 = \frac{\pi D_2 N}{60}$$

$$U_2 = \frac{\pi x\ 0.5\ x\ 200}{60}$$

$$\mathbf{U_2 = 5.236\ m/s}$$

$V_{w1} = V_1 \cos \alpha_1$

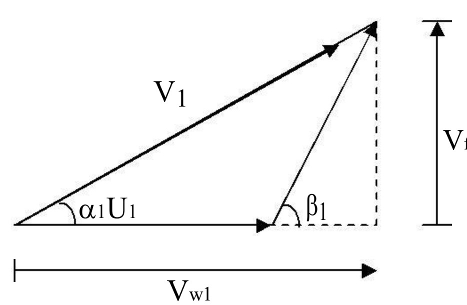
$V_{w1} = 35 \cos 20$

$\mathbf{V_{w1} = 32.89\ m/s}$

$V_{f1} = V_1 \sin \alpha_1$

$\mathbf{V_{f1} = 11.97\ m/s}$

V_{r1}

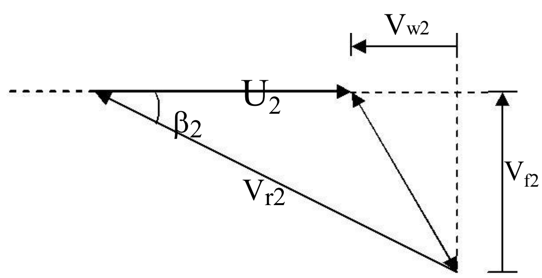


7

$$\tan \beta_1 = \frac{V_{f1}}{(V_{w1} - U_1)}$$

$$\tan \beta_1 = \frac{11.97}{(32.89 - 10.47)}$$

$$\boldsymbol{\beta_1 = 28.10^o}$$



$V_{f2} = 5 \sin 50$

$\mathbf{V_{f2} = 3.83\ m/s}$

$V_{w2} = 5 \cos 50$

$\mathbf{V_{w2} = 3.214\ m/s}$

$$\tan \beta_2 = \frac{V_{f2}}{U_2 + V_{w2}}$$

$$\tan \beta_2 = \frac{3.83}{5.236 + 3.214}$$

$$\boldsymbol{\tan \beta_2 = 24.38^o}$$

Work done per unit mass flow rate = $[V_{w1}\, U_1 + V_{w2}\, U_2]$

Work done per unit mass flow rate = [32.89 x 10.47 + 3.214 x 5.236] Work done per unit mass flow rate = 362.13 W/kg

$$\text{Efficiency} = \eta = \frac{2[V_{\omega 1}\, U_1 + V_{\omega 2}\, U_2]}{V_1^2}$$

$$\text{Efficiency} = \eta = \frac{2[32.89 \times 10.47 + 3.214 \times 5.236]}{35^2}$$

Efficiency = η = 0.5896 or 58.96 %

Session – VI

It is a mathematical technique which makes use of study of dynamics as an art to the solution of engineering problems.

18. Fundamental Dimensions

All physical quantities are measured by comparison which is made with respect to a fixed value.

Length, Mass and Time are three fixed dimensions which are of importance in fluid mechanics and fluid machinery. In compressible flow problems, temperature is also considered as a fundamental dimensions.

(ξϖιι) Secondary Quantities or Derived Quantities

Secondary quantities are derived quantities or quantities which can be expressed in terms of two or more fundamental quantities.

(ξ) Dimensional Homogeneity

In an equation if each and every term or unit has same dimensions, then it is said to have Dimensional Homogeneity.

$$V = u + at$$

$$m/s \quad m/s \quad m/s^2 \cdot s$$

$$LT^{-1} = (LT^{-1}) + (LT^{-2})\,(T)$$

(ξι) **Uses of Dimensional Analysis**

It is used to test the dimensional homogeneity of any derived equation.

It is used to derive equation.

Dimensional analysis helps in planning model tests.

(ξιι) **Dimensions of quantities**

1. Length LM^oT^o
2. Mass L^oMT^o
3. Time L^oM^oT
4. Area $L^2M^oT^o$
5. Volume $L^3M^oT^o$

(iii) Velocity	LM^oT^{-1}
(iv) Acceleration	LM^oT^{-2}
(v) Momentum	LMT^{-1}
(vi) Force	LMT^{-2}
(vii) Moment or Torque	L^2MT^{-2}
(viii) Weight	LMT^{-2}
(ix) Mass density	$L^{-3}MT^o$
(x) Weight density	$L^{-2}MT^{-2}$
(xi) Specific gravity	$L^oM^oT^o$
(xii) Specific volume	$L^3M^{-1}T^o$
(xiii) Volume flow rate	$L^3M^oT^{-1}$
(xiv) Mass flow rate	L^oMT^{-1}
(xv) Weight flow rate	LMT^{-3}
(xvi) Work done	L^2MT^{-2}
(xvii) Energy	L^2MT^{-2}
(xviii) Power	L^2MT^{-3}
(xix) Surface tension	L^oMT^{-2}
(xx) Dynamic viscosity	$L^{-1}M^{+1}T^{-1}$
(xxi) Kinematic viscosity	$L^2M^oT^{-1}$
(xxii) Frequency	$L^oM^oT^{-1}$
(xxiii) Pressure	$L^{-1}MT^{-2}$
(xxiv) Stress	$L^{-1}MT^{-2}$
(xxv) E, C, K	$L^{-1}MT^{-2}$
(xxvi) Compressibility	$LM^{-1}T^2$
(xxvii) Efficiency	$L^oM^oT^o$
(xxviii) Angular velocity	$L^oM^oT^{-1}$
(xxix) Thrust	LMT^{-2}
(xxx) Energy head (Energy/unit mass)	$L^2M^oT^{-2}$
(xxxi) Energy head (Energy/unit weight)	LM^oT^o

2

DIMENSIONAL ANALYSIS

Session – VII

19. **Methods of Dimensional Analysis**

There are two methods of dimensional analysis.

Rayleigh's method

Buckingham's (Π – theorem) method

 Rayleigh's method

Rayleigh's method of analysis is adopted when number of parameters or variables are less (3 or 4 or 5).

Methodology

X_1 is a function of

$X_2, x_3, X_4, \ldots\ldots, X_n$ then it can be written as

$X_1 = f(X_2, x_3, X_4, \ldots\ldots, X_n)$

$X_1 = K\,(X_2^{a}, x_3^{b}, X_4^{c}, \ldots\ldots)$

Taking dimensions for all the quantities

$$[X_1] = [X_2]^a\,[X_3]^b\,[X_4]^c \ldots\ldots$$

Dimensions for quantities on left hand side as well as on the right hand side are written and using the concept of Dimensional Homogeneity a, b, c …. can be determined.

Then,

$$X_1 = K \cdot X_2^{a} \cdot X_3^{b} \cdot X_4^{c} \cdot \ldots\ldots$$

(ϖ) **Problems 1:** Velocity of sound in air varies as bulk modulus of electricity K, Mass density ρ. Derive an expression for velocity in form $C = \sqrt{\dfrac{K}{\rho}}$

- **Solution:**

$$C = f\,(K, \rho)$$

$$\mathbf{C = M \cdot K^{a} \cdot \rho^{b}}$$

M – Constant of proportionality

$[C] = [K]^a \cdot [\rho]^b$

$[LM^oT^{-1}] = [L^{-1}MT^{-2}]^a \ [L^3MT^o]^b$

C – Velocity	– LM^oT^{-1}
K – Bulk modulus	– $L^{-1}MT^{-2}$
ρ - Mass density	– $L^{-3}MT^o$

$[LM^oT^{-1}] = [L^{-a+(-36)} M^{a+b} T^{-2a}]$

$-a - 3b = 1$

$a + b = 0$

$-2b = 1$

$b = -\frac{1}{2}$

$\mathbf{a = \frac{1}{2}}$

$C = MK^{1/2} \rho^{-1/2}$

$C = M\sqrt{\frac{K}{\rho}}$

If, M = 1, $C = \sqrt{\frac{\mathbf{K}}{\boldsymbol{\rho}}}$

= **Problem 2:** Find the equation for the power developed by a pump if it depends on bead H discharge Q and specific weight γ of the fluid.

= **Solution**:

$P = f(H, Q, \gamma)$

$P = K \cdot H^a \cdot Q^b \cdot \gamma^c$

$[P] = [H]^a \cdot [Q]^b \cdot [\gamma]^c$

$[L^2MT^{-3}] = [LM^oT^o]^a \cdot [L^{-2}MT^{-2}]^b \cdot [L^{-2}MT^{-2}]^c$

Power	$= L^2MT^{-3}$
Head	$= LM^oT^o$
Discharge	$= L^3M^oT^{-1}$
Specific Weight	$= L^{-2}MT^{-2}$

$2 = a + 3b - 2c$

$1 = c$

$-3 = -b - 2$

$-3 = -b - 2$

$b = -2 + 3$

$\mathbf{b = 1}$

$2 = a + 3 - 2$

$\mathbf{a = 1}$

$$P = K \cdot H^1 \cdot Q^1 \cdot \gamma^1$$

$$P = K \cdot H \cdot Q \cdot \gamma$$

When, $K = 1$

$$\mathbf{P = H \cdot Q \cdot \gamma}$$

= **Problem 3:** Find an expression for drag force R on a smooth sphere of diameter D moving with uniform velocity V in a fluid of density ρ and dynamic viscosity μ..

= **Solution:**

$$R = f(D, V, \rho, \mu)$$

$$R = K \cdot D^a \cdot V^b \cdot \rho^c, \mu^d$$

$$[R] = [D]^a \cdot [V]^b \cdot [\rho]^c \cdot [\mu]^d$$

$$[LMT^{-2}] = [LM^oT^o]^a \cdot [LM^oT^{-1}]^b \cdot [L^{-3}MT^o]^c \cdot [L^{-1}MT^{-1}]^d$$

Force	$= LMT^{-2}$
Diameter	$= LM^oT^o$
Velocity	$= LM^oT^{-1}$
Mass density	$= L^3MT^o$

$$c + d = 1$$

$$\mathbf{c = 1 - d}$$

$$-b - d = -2$$

$$\mathbf{b = 2 - d}$$

$$1 = a + b - 3c - d$$

$$1 = a + 2 - d - 3(1 - d) - d$$

$$1 = a + 2 - d - 3 + 3d - d$$

$$\mathbf{a = 2 - d}$$

$$R = K \cdot D^{2-d} \cdot V^{2-d} \cdot \rho^{1-d}, \mu^d$$

$$R = K\ \frac{D^2}{D^d} \cdot \frac{V^2}{V^d} \cdot \frac{\rho}{\rho^d} \cdot \mu^d$$

$$R = K \cdot \rho\ V^2 D^2 \left[\frac{\mu}{\rho VD}\right]^d$$

$$R = \rho V^2 D^2\ \phi \left[\frac{\mu}{\rho VD}\right]$$

$$R = \rho V^2 D^2 \phi \left[\frac{\rho V D}{\mu} \right]$$

$$\mathbf{R = \rho V^2 D^2 \phi [N_{Re}]}$$

η **Problem 4:** The efficiency of a fan depends on the density ρ dynamic viscosity μ, angular velocity ω, diameter D, discharge Q. Express efficiency in terms of dimensionless parameters using Rayleigh's Method.

η - $L^oM^oT^o$
ρ - $L^{-3}MT^o$
μ - $L^{-1}MT^{-1}$
ω - $L^oM^oT^{-1}$
D - LM^oT^o

η **Solution:**

$$\Xi = f(\rho, \mu, \omega, D, Q)$$

$$\Psi = k \cdot \rho^a \cdot \mu^b \cdot \omega^c \cdot D^d \cdot Q^e$$

$$[\eta] = [\rho]^a \cdot [\mu]^b \cdot [\omega]^c \cdot [D]^d \cdot [Q]^e$$

$$[L^oM^oT^o] = [L^{-3}MT^o]^a \cdot [L^{-1}MT^{-1}]^b \cdot [L^oM^oT^{-1}]^c \cdot [LM^oT^o]^d \cdot$$

$$[L^3M^oT^{-1}]^{-1} \cdot \; [L^oM^oT^o] \cdot \; [L^{-3a-b+d+3e} \cdot M^{a+b} \cdot T^{-b-c-e}]$$

$a + b = 0$

a = – b

$– b – c – e = 0$

c = – b – e

$– 3a – b + d + 3e = o$

$+ 3b – b + d + 3e = 0$

d = – 2b – 3e

$$\therefore \eta = K \cdot \rho^{-b} \cdot \mu^b \cdot \omega^{-b-e} \cdot D^{-2b-3e} \cdot Q^e$$

$$\eta = K \cdot \frac{1}{\rho^b} \cdot \mu^b \, \frac{1}{\omega^b \cdot \omega^e} \cdot \frac{1}{(D^2)^b \cdot (D^3)^e} \, Q^2$$

$$\eta = K \left(\frac{\mu}{\rho\omega D^2}\right)^b \cdot \left(\frac{Q}{\omega D^3}\right)^e$$

$$\boldsymbol{\eta = Q\left[\frac{\mu}{\rho\omega D^2}, \frac{Q}{\omega D^3}\right]}$$

- **Problem 5:** The capillary rise H of a fluid in a tube depends on its specific weight γ and surface tension σ and radius of the tube R prove that $\frac{\mathbf{H}}{\mathbf{R}} = \phi\left[\frac{\boldsymbol{\sigma}}{\boldsymbol{\gamma R}^2}\right]$.

- **Solution:**

$H = f(\gamma, \sigma, R)$

$H = K \cdot \gamma^a \cdot \sigma^b \cdot R^c$

$[H] = [\gamma]^a \cdot [\sigma]^b \cdot [R]^c$

$[LM^oT^o] = [L^{-2}MT^{-2}]^a \cdot [L^oMT^{-2}]^b \cdot [LM^oT^o]^c$

$[LM^oT^o] = \cdot [L^{-2a+c} \cdot M^{a+b} \cdot T^{-2a-2b}]$

$-2a + c =$

$1\ a + b = 0$

$\mathbf{-2a - 2b = 0}$

$H - LM^oT^o$
$= - L^{-2}MT^{-2}$
$- L^oMT^{-2}$
$R - LM^oT^o$

$= \left(\frac{c-1}{2}\right) a$

$\frac{c-1}{2} + b = 0$

$\left(\frac{1-c}{2}\right) \mathbf{b =}$

$H = K \cdot \gamma^{\left(\frac{c-1}{2}\right)} \cdot \sigma^{\frac{1-c}{2}} \cdot R^c$

$H = K \cdot \frac{\gamma^{c/2}}{\gamma^{1/2}} \cdot \frac{\sigma^{1/2}}{\sigma^{c/2}} \cdot R^c$

$\frac{H}{R} = K \cdot \frac{\gamma^{c/2}}{\gamma^{1/2}} \cdot \frac{\sigma^{1/2}}{\sigma^{c/2}} \cdot \frac{R^c}{R}$

$\frac{H}{R} = K \cdot \left(\frac{\gamma^{1/2} \cdot R}{\sigma^{1/2}}\right) \cdot \frac{\sigma^{1/2}}{\gamma^{c/2}} \cdot 1$

$H = K \cdot \gamma^{-b} \cdot \sigma^b \cdot R^{1-2b}$

$\frac{\sigma^b}{\ } \quad \frac{R}{\ }$

$H = K \quad \gamma^b \cdot (R^2)^b$

$$H = K\left[\frac{\sigma}{\gamma R^2}\right]^b$$

$$\frac{H}{R} = \phi\left[\frac{\sigma}{\gamma R^2}\right]$$

2. *Buckingham's Π Method*

This method of analysis is used when number of variables are more.

Buckingham's Π Theorem

If there are n – variables in a physical phenomenon and those n-variables contain 'm' dimensions, then the variables can be arranged into (n-m) dimensionless groups called Π terms.

Explanation:

If $f(X_1, X_2, X_3, \ldots\ldots\ldots X_n) = 0$ and variables can be expressed using m dimensions then.

$$f(\Pi_1, \Pi_2, \Pi_3, \ldots\ldots\ldots \Pi_{n-m}) = 0$$

Where, $\Pi_1, \Pi_2, \Pi_3, \ldots\ldots\ldots$ are dimensionless groups.

Each Π term contains (m + 1) variables out of which m are of repeating type and one is of non-repeating type.

Each Π term being dimensionless, the dimensional homogeneity can be used to get each Π term.

- **Selecting Repeating Variables**
  1. Avoid taking the quantity required as the repeating variable.
  2. Repeating variables put together should not form dimensionless group.
  3. No two repeating variables should have same dimensions.
  4. Repeating variables can be selected from each of the following properties.
     a. Geometric property → Length, height, width, area
     b. Flow property → Velocity, Acceleration, Discharge
     c. Fluid property → Mass density, Viscosity, Surface tension

DIMENSION ANALYSIS

Session – VIII

Problem 1: Find an expression for drag force R on a smooth sphere of diameter D moving with uniform velocity V in a fluid of density ρ and dynamic viscosity μ..

20. **Solution:**

$R = LMT^{-2}$
$D = LM^oT^o$
$V = LT^{-1}$
$\rho = ML^{-3}$
$\mu = L^{-1}MT^{-1}$

$f(R, D, V, \rho, \mu) = 0$

Here, $n = 5, m = 3$

(ξϖιιι) Number of Π terms = (n – m) = 5 – 3 = 2

(ξιξ) $f(\Pi_1, \Pi_2, \Pi_3) = 0$

Let D, V, ρ be the repeating variables.

$\Pi_1 = D^{a_1} \cdot V^{b_1} \cdot \rho^{c_1} \cdot R$

$[L^oM^oT^o] = [L]^{a_1} [LT^{-1}]^{b_1} [ML^{-3}]^{c_1} [LMT^{-2}]$

$L^oM^oT^o = [L]^{a_1 + b_1 - 3c_1 + 1} [M]^{c_1+1} [T]^{-b_1 - 2}$

(xi) b

$_1 = 2$

$\mathbf{b_1 = -2}$

$c_1 + 1 = 0$

$\mathbf{c_1 = -1}$

$a_1 + b_1 - 3c_1 + 1 = 0$

$a_1 + 2 + 3 + 1 = 0$

$\mathbf{a_1 = -2}$

$\Pi_1 = D^{-2} \cdot V^{-2} \cdot \rho^{-1} \cdot R$

$\Pi_1 = \dfrac{\mathbf{R}}{\mathbf{D^2 V^2 \rho}}$

$\Pi_2 = D^{a_2} \cdot V^{b_2} \cdot \rho^{c_2} \cdot \mu$

$[L^o M^o T^o] = [L]^{a_2} [LT^{-1}]^{b_2} [ML^{-3}]^{c_2} [L^{-1}MT^{-1}]$

$[L^oM^oT^o] = [L]^{a_2 + b_2 - 3c_2 - 1} [M]^{c_2+1} [T]^{-b_2 - 1}$

$-b_2 - 1 = 0$

$\mathbf{b_2 = -1}$

$\mathbf{c_2 = -1}$

$a_2 + b_2 - 3c_2 - 1 =$

$0 \; a_2 - 1 + 3 - 1 = 0$

$\mathbf{a_2 = -1}$

$\Pi_2 = D^{-1} \cdot V^{-1} \cdot \rho^{-1} \cdot \mu$

$$\boldsymbol{\Pi_2 = \frac{\mu}{\rho VD}}$$

$f(\Pi_1, \; \Pi_2) = 0$

$$f\left(\frac{R}{D^2 V^2 \rho}, \frac{\mu}{\rho VD}\right) = 0$$

$$\frac{R}{D^2 V \rho} = \phi\left(\frac{\mu}{\rho VD}\right)$$

$$R = \rho V^2 D^2 \phi\left(\frac{\mu}{\rho VD}\right)$$

(ξιϖ) **Problem 2:** The efficiency of a fan depends on density η, dynamic viscosity μ, angular velocity ω, diameter D and discharge Q. Express efficiency in terms of dimensionless parameters.

(ξϖ) **Solution:**

$f(\eta, \rho, \mu\ \omega, D, Q) = 0$

Here, $n = 6, m = 3$

(ϖιι) Number of Π terms = 3

(ιξ) $f(\Pi_1, \Pi_2, \Pi_3) = 0$

$\eta = L^oM^oT^o$
$\rho = ML^{-3}$
$\mu = L^{-1}MT^{-1}$
$\omega = T^{-1}$
$D = L$
$Q = L^3T^{-1}$

Let, D, ω, ρ be the repeating variables.

$\Pi_1 = D^{a_1} \cdot \omega^{b_1} \cdot \rho^{c_1} \cdot \mu$

$[L^oM^oT^o] = [L]^{a_1} [T^{-1}]^{b_1} [ML^{-3}]^{c_1} [L^{-1}MT^{-1}]$

$[L^oM^oT^o] = [L]^{a_1 - 3c_1 - 1} [M]^{c_1 + 1} [T]^{-b_1 - 1}$

$\mathbf{b_1 = -1}$

$c_1 = -1$

$a_1 - 3c_1 - 1 =$

$0 \; a_1 = -2$

$\Pi_1 = D^{-2} \cdot \omega^{-1} \cdot \rho^{-1} \cdot \mu$

$$\Pi_1 = \frac{\mu}{D^2 . \omega . \rho}$$

$\Pi_2 = D^{a_2} \cdot \omega^{b_2} \cdot \rho^{c_2} \cdot Q$

$[L^o M^o T^o] = [L]^{a_2} [T^{-1}]^{b_2} [ML^{-3}]^{c_2} [L^3 T^{-1}]$

$[L^o M^o T^o] = [L]^{a_2+3-3C_2} [M]^{c_2} [T]^{-b_2-1}$

$c_2 = 0$

$-b_2 - 1 =$

$0 \; b_2 = -1$

$a_2 + 3 - 3c_2 = 0$

$a_2 + 3 = 0$

$a_2 = -3$

$\Pi_2 = D^{-3} \cdot \omega^{-1} \cdot \rho^{o} \cdot Q$

$$\Pi_2 = \frac{Q}{\omega D^3}$$

$\Pi_3 = D^{a_3} \cdot \omega^{b_3} \cdot \rho^{c_3} \cdot \eta$

$[L^o M^o T^o] = [L]^{a_3} [T^{-1}]^{b_3} [ML^{-3}]^{c_3} [M^o L^o T^o]$

$[L^o M^o T^o] = [L]^{a_{3-3}c_3} [M]^{c_3} [T]^{-b_3}$

$b_3 = 0$

$c_3 = 0$

$a_3 = 0$

$\Pi_3 = \eta$

$f(\Pi_1, \Pi_2, \Pi_3) = 0$

$$f\left(\frac{\mu}{D^2 \cdot \omega \cdot \rho}, \frac{\mu}{\omega D^3}, \eta\right) = 0$$

$$\eta = \phi\left(\frac{\mu}{D^2 \omega\rho}, \frac{\mu}{\omega D^3}\right)$$

3

(ξιϖ) **Problem 3:** The resisting force of a supersonic plane during flight can be considered as dependent on the length of the aircraft L, velocity V, viscosity μ, mass density ρ, Bulk modulus K. Express the fundamental relationship between resisting force and these variables.

(ξϖ) **Solution:**

$f(R, L, K, \mu, \rho, V) = 0$

$n = 6$

$=$ Number of Π terms $= 6 - 3 = 3$

$= f(\Pi_1, \Pi_2, \Pi_3) = 0$

Let, L, V, ρ be the repeating variables.

$\Pi_1 = L^{a_1} \cdot V^{b_1} \cdot \rho^{c_1} \cdot K$

$L^o M^o T^o = [L]^{a_1} [LT^{-1}]^{b_1} [ML^{-3}]^{c_1} [L^{-1}MT^{-2}]$

$L^o M^o T^o = [L]^{a_1 + b_1 - 3c_1 - 1} [M]^{c_1 + 1} [T]^{-b_1 - 2}$

$\mathbf{b_1 = -2}$

$\mathbf{c_1 = -1}$

$a_1 + b_1 - 3c_1 - 1 =$

$0 \; a_1 - 2 + 3 - 1 = 0$

$a_1 = 0$

$\Pi_1 = L^o \cdot V^{-1} \cdot \rho^{-1} \cdot K$

$\Pi_1 = \dfrac{K}{V^2 \cdot \rho}$

$\Pi_2 = L^{a_2} \cdot V^{b_2} \cdot \rho^{c_2} \cdot R$

$L^o M^o T^o = [L]^{a_2} [LT^{-1}]^{b_2} [ML^{-3}]^{c_2} [LMT^{-2}]$

$L^o M^o T^o = [L]^{a_2 + b_2 - 3c_2 + 1} [M]^{c_2 + 1} [T]^{-b_2 - 2}$

$-b_2 - 2 =$

$0 \; \mathbf{b_2 = -2}$

$\mathbf{c_2 = -1}$

4

$a_2 + b_2 - 3c_2 + 1 =$

$0\ a_1 + 2 + 3 + 1 = 0$

$a_2 = -2$

$\Pi_2 = L^{-2} \cdot V^{-2} \cdot \rho^{-1} \cdot R$

$$\Pi_2 = \frac{R}{L^2 V^2 \rho}$$

$\Pi_3 = L^{a_3} \cdot V^{b_3} \cdot \rho^{c_3} \cdot \mu$

$L^o M^o T^o = [L]^{a_3} [LT^{-1}]^{b_3} [ML^{-3}]^{c_3} [L^{-1}MT^{-1}]$

$L^o M^o T^o = [L]^{a_3 + b_3 - 3c_3 - 1} [M]^{c_3 + 1} [T]^{-b_3 - 1}$

$-b_3 - 1 =$

0 $\mathbf{b_3 = -1}$

$\mathbf{c_3 + 1 = 0}$

$\mathbf{c_3 = -1}$

$a_3 + b_3 - 3c_3 - 1 =$

$0\ a_1 - 1 + 3 - 1 = 0$

$a_3 = -1$

$\Pi_3 = L^{-1} \cdot V^{-1} \cdot \rho^{-1} \cdot \mu$

$$\Pi_2 = \frac{\mu}{LV\rho}$$

$$f\left(\frac{K}{V^2\rho}, \frac{R}{L^2V^2\rho}, \frac{\mu}{LV\rho}\right) = 0$$

$$\frac{R}{L^2V^2\rho} = \phi\left(\frac{K}{V^2\rho}, \frac{\mu}{LV\rho}\right)$$

$$R = L^2V^2\rho\,\phi\left(\frac{K}{V^2\rho}, \frac{\mu}{LV\rho}\right)$$

= **Problem 4:** Using Buckingham Π - theorem, show that velocity of fluid through a circular orifice is given by $V = \sqrt{2gh}\,\phi\left(\frac{D}{H}, \frac{\mu}{\rho VH}\right)$.

η **Solution:**

We have, f (V, D, H, μ, ρ, g) = 0

$V = LT^{-1}$, D = L, H = L, $\mu = L^{-1}MT^{-1}$, $\rho = ML^{-3}$, g = LT^{-2} n = 6

m = 3

∴ Number of Π terms = (6 – 3) =

3 Let H, g and ρ be repeating variables

$\Pi_1 = H^{a1} \cdot g^{b_1} \cdot \rho^{c_1} \cdot V$

$[L^oM^oT^o] = [L]^{a1} [LT^{-2}]^{b1} [ML^{-3}]^{c1} [LT^{-1}]$

$[L^oM^oT^o] = [L]^{a_1+b_1-3c_1+1} [M]^{c_1} [T]^{-2b_1-1}$

$-2b_1 = 1$

$b_1 = -\frac{1}{2}$

$c_1 = 0$

$a_1 + b_1 - 3c_1 + 1 = 0$

$a_1 - \frac{1}{2} - 0 + 1 = 0$

$a_1 = -\frac{1}{2}$

$\Pi_1 = H^{-1/2} \cdot g^{-1/2} \cdot \rho^o \cdot V$

$\Pi_1 = \frac{V}{\sqrt{gH}}$

$\Pi_2 = H^{a2} \cdot g^{b2} \cdot \rho^{c2} \cdot D$

$[L^oM^oT^o] = [L]^{a_2} [LT^{-2}]^{b_2} [ML^{-3}]^{c_2} [L]$

$[L^oM^oT^o] = [L]^{a_2+b_2-3c_2+1} [M]^{c_2} [T]^{-2b_2}$

$-2b_2 = 0$

$b_2 = 0$

$c_2 = 0$

$a_2 + b_2 - 3c_2 + 1 =$
$0\ a_2 = -1$

$\Pi_2 = H^{-1} \cdot g^o \cdot \rho^o \cdot D$

$\Pi_2 = \frac{H}{D}$

$\Pi_3 = H^{a_3} \cdot g^{b_3} \cdot \rho^{c_3} \cdot \mu$

$[M^o L^o T^o] = [L]^{a_3} [LT^{-2}]^{b_3} [ML^{-3}]^{c_3} [L^{-1} MT^{-1}]$

$[M^o L^o T^o] = [M]^{c_3+1} [L]^{a_3+b_3-3c_3-1} [T]^{-2b_3-1}$

$c_3 = -1$

$b_3 = -\frac{1}{2}$

$a_3 - \frac{1}{2} + 3 - 1 = 0$

$c_3 = -\frac{3}{2}$

$\Pi_3 = H^{-3/2} \cdot g^{-1/2} \cdot \rho^{-1} \cdot \mu$

$\Pi_3 = \frac{\mu}{\rho\sqrt{gH^3}}$

$\Pi_3 = \frac{\mu}{\rho\sqrt{gH.H}}$

$f(\Pi_1, \Pi_2, \Pi_3) = 0$

$$f\left(\frac{V}{\sqrt{gh}}, \frac{H}{D}, \frac{\mu}{\rho\sqrt{gH \cdot H}}\right) = 0$$

$$\frac{V}{\sqrt{gH}} = \phi\left(\frac{H}{D}, \frac{\mu}{\rho\sqrt{gH \cdot H}}\right)$$

$$v = \sqrt{2gH}\,\phi\left(\frac{D}{H}, \frac{\mu}{\rho VH}\right)$$

7

η **Problem 5**: Using dimensional analysis, derive an expression for thrust P developed by a propeller assuming that it depends on angular velocity ω, speed of advance V, diameter D, dynamic viscosity μ, mass density ρ, elasticity of the fluid medium which can be denoted by speed of sound in the medium C.

η **Solution:**

$f(P, \omega, V, D, \mu, \rho, C) =$

$0 \; n = 7$

$m = 3$

Taking D, V and ρ as repeating variables.

$P = LMT^{-2}$

$\omega = L^oM^oT^{-1}$

$V = LM^oT^{-1}$

$D = L$

$\mu = L^{-1}MT^{-1}$

$\rho = ML^{-3}$

$C = LT^{-1}$

$\Pi_1 = D^{a_1} \cdot V^{b_1} \cdot \rho^{c_1} \cdot P$

$[L^oM^oT^o] = [L]^{a_1} [LT^{-1}]^{b_1} [ML^{-3}]^{c_1} [LMT^{-2}]$

$[L^oM^oT^o] = [L]^{a_1+b_1-3c_1+1} [M]^{c_1+1} [T]^{-b_1-2}$

$\mathbf{-b_1 - 2 =}$

$\mathbf{0 \; b_1 = -2}$

$\mathbf{c_1 + 1 = 0}$

$\mathbf{c_1 = -1}$

$a_1 + b_1 - 3c_1 + 1 =$

$0 \; a_1 - 2 + 3 + 1 = 0$

$\mathbf{a_1 = -2}$

$\Pi_1 = D^{-2} \cdot V^{-2} \cdot \rho^{-1} \cdot P$

$\Pi_1 = \dfrac{P}{D^2 V^2 \rho}$

$\Pi_2 = D^{a_2} \cdot V^{b_2} \cdot \rho^{c_2} \cdot \omega$

$[L^oM^oT^o] = [L]^{a_2} [LT^{-2}]^{b_2} [ML^{-3}]^{c_2} [T^{-1}]$

$[L^oM^oT^o] = [L]^{a_2+b_2-3c_2} [M]^{c_2} [T]^{-b_2-1}$

$\mathbf{-b_2 - 1 =}$

$\mathbf{0 \; b_2 = -1}$

$\mathbf{c_2 = 0}$

8

$a_2 - 1 + 0 =$

$0\ a_2 = -1$

$\Pi_2 = D^1 \cdot V^{-1} \cdot \rho^o \cdot \omega$

$\Pi_2 = \frac{D\omega}{V}$

$\Pi_3 = D^{a_3} \cdot V^{b_3} \cdot \rho^{c_3} \cdot \mu$

$[M^oL^oT^o] = [L]^{a_3}\,[LT^{-1}]^{b_3}\,[ML^{-3}]^{c_3}\,[L^{-1}MT^{-1}]$

$[M^oL^oT^o] = [L]^{a_3+b_3-3c_3-1}\,[M]^{c_3+1}\,[T]^{-b_3-1}$

$b_3 = -1$

$c_3 - 1 + 3 - 1 =$

$0\ c_3 = -1$

$\Pi_3 = D^{-1} \cdot V^{-1} \cdot \rho^{-1} \cdot \mu$

$\Pi_3 = \frac{\mu}{\rho VD}$

$\Pi_4 = D^{a_4} \cdot V^{b_4} \cdot \rho^{c_4} \cdot C$

$M^oL^oT^o = [L]^{a_4}\,[LT^{-1}]^{b_4}\,[ML^{-3}]^{c_4}\,[LT^{-1}]$

$M^oL^oT^o = [L]^{a_4-b_4-3c_4+1}\,[M]^{c_4}[T]^{-b_4-1}$

$b_4 = -$

$1\ c_4 = 0$

$a_4 - 1 + 0 + 1 =$

$0\ a_4 = 0$

$\Pi_4 = D^o \cdot V^{-1} \cdot \rho^o \cdot C$

$\Pi_4 = \frac{C}{V}$

$f(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = 0$

$$f\left(\frac{P}{D^2V^2\rho}, \frac{D\omega}{V}, \frac{\mu C}{\rho VD\ V}\right) = 0$$

$$P = D^2V^2\rho\,\phi\left(\frac{D\omega}{V}, \frac{\mu}{\rho VD}, \frac{C}{V}\right)$$

9

Z **Problem 6**: The pressure drop ΔP in a pipe depends on mean velocity of flow V, length of pipe l, viscosity of the fluid μ, diameter D, height of roughness projection K and mass density of the liquid ρ. Using Buckingham's method obtain an expression for ΔP.

AA **Solution:**

$\Delta P = L^{-1}MT^{-2}$

$V = LM^{o}T^{-1}$

$l = LM^{o}T^{o}$

$\mu = L^{-1}MT^{-1}$

$K = LM^{o}T^{o}$

$\rho = ML^{-3}$

f (ΔP, V, l, μ, D, K, ρ) = 0

n = 7

∴ number of Π terms = 7 – 3 = 4

Let D, V, ρ be the repeating variables

$\Pi_1 = D^{a1} \cdot V^{b1} \cdot \rho^{c1} \cdot \Delta P$

$[L^{o}M^{o}T^{o}] = [L]^{a1} [LT^{-1}]^{b1} [ML^{-3}]^{c1} [L^{-1}MT^{-2}]$

$[L^{o}M^{o}T^{o}] = [L]^{a1+b1-3c1-1} [M^{c1+1}] [T^{-b1-2}]$

$-b_1 – 2 =$

$0 \; b_1 = - 2$

$c_1 = - 1$

$a_1 + b_1 – 3c_1 – 1 =$

$0 \; a_1 – 2 + 3 – 1 =$

$0 \; a_1 = 0$

$\Pi_1 = D^{o} \cdot V^{-2} \cdot \rho^{-1}$

$\cdot \Delta P \; \Pi_1 = \dfrac{\Delta P}{V^2 \cdot \rho}$

$\Pi_2 = D^{a2} \cdot V^{b2} \cdot \rho^{c2} \cdot l$

$[L^{o}M^{o}T^{o}] = [L]^{a2} [LT^{-1}]^{b2} [ML^{-3}]^{c2} [L]$

$[L^{o}M^{o}T^{o}] = [L]^{a2+b2-3c2+1} [M]^{c2} [T]^{-b2}$

$b_2 = 0$

$c_2 = 0$

$a_2 + 0 + 0 + 1 =$

$0 \; a_2 = – 1$

$\Pi_2 = D^{-1} \cdot V^{o} \cdot \rho^{o} \cdot$

$L\ \Pi_2 = \frac{l}{D}$

$\Pi_3 = D^{a3} \cdot V^{b3} \cdot \rho^{c3} \cdot K$

$[L^{o}M^{o}T^{o}] = [L]^{a3}\ [LT^{-1}]^{b3}\ [ML^{-3}]^{c3}\ [L]$

$[L^{o}M^{o}T^{o}] = [L]^{a3+b3-3c3+1}\ [M]^{c3}\ [T]^{-b3}$

$b_3 = 0$

$c_3 = 0$

$a_3 + b_3 - 3c_3 + 1 =$

$0\ a_3 = -1$

$\Pi_3 = \frac{K}{D}$

$\Pi_4 = D^{a4} \cdot V^{b4} \cdot \rho^{c4} \cdot \mu$

$[L^{o}M^{o}T^{o}] = [L]^{a4}\ [LT^{-1}]^{b4}\ [ML^{-3}]^{c4}\ [L^{-1}MT^{-1}]$

$[L^{o}M^{o}T^{o}] = [L]^{a4+b4-3c4-1}\ [M]^{c4+1}\ [T]^{-b4-1}$

$b_4 = -1$

$c_4 = -1$

$a_4 + b_4 - 3c_4 - 1 =$

$0\ a_4 - 1 + 3 - 1 =$

$0\ a_4 = -1$

$\Pi_4 = D^{-1} \cdot V^{-1} \cdot \rho^{-1} \cdot \mu$

$\Pi_4 = \frac{\mu}{\rho VD}$

$$f\left(\frac{\Delta P}{\rho v^2}, \frac{l}{D}, \frac{K}{D}, \frac{\mu}{\rho VD}\right) = 0$$

$$\Delta P = \rho V^2\ \phi\left(\frac{L}{D}, \frac{K}{D}, \frac{\mu}{\rho VD}\right)$$

11

MODEL ANALYSIS

Session – IX

Before constructing or manufacturing hydraulics structures or hydraulics machines tests are performed on their models to obtain desired information about their performance. Models are small scale replica of actual structure or machine. The actual structure is called prototype.

21. **Similitude / Similarity**

It is defined as the similarity between the prototype and it's model.

(ξξ) **Types of Similarity**

There are three types of similarity.

- Geometric similarity
- Kinematic similarity
- Dynamic similarity

(ξι) **Geometrical Similarity**

Geometric similarity is said to exist between the model and prototype if the ratio of corresponding linear dimensions between model and prototype are equal.

i.e. $\frac{L_p}{L_m} = \frac{h_p}{h_m} = \frac{H_p}{H_m} \ldots\ldots\ldots\ldots = L_r$

L_r → scale ratio / linear ratio

$$\frac{A_p}{A_m} = (L_r)^2 \quad \frac{V_p}{V_m} = (L_r)^3$$

- **Kinematic Similarity**

Kinematic similarity exists between prototype and model if quantities such at velocity and acceleration at corresponding points on model and prototype are same.

$$\frac{(V_1)_p}{(V_1)_m} = \frac{(V_2)_p}{(V_2)_m} = \frac{(V_3)_p}{(V_3)_m} \ldots\ldots\ldots\ldots = V_r$$

V_r → Velocity ratio

(ξξξι) **Dynamic Similarity**

Dynamic similarity is said to exist between model and prototype if ratio of forces at corresponding points of model and prototype is constant.

$$\frac{(F_1)_p}{(F_1)_m} = \frac{(F_2)_p}{(F_2)_m} = \frac{(F_3)_p}{(F_3)_m} \ldots\ldots\ldots\ldots = F_R$$

F_R→ Force ratio

- **Dimensionless Numbers**

  Following dimensionless numbers are used in fluid mechanics.

  1. Reynold's number
  2. Froude's number
  3. Euler's number
  4. Weber's number
  5. Mach number

1. Reynold's number

It is defined as the ratio of inertia force of the fluid to viscous force.

$$\therefore N_{Re} = \frac{F_i}{F_v}$$

Expression for N_{Re}

F_i = Mass x Acceleration

$F_i = \rho$ x Volume x Acceleration

$$F_i = \rho \times \text{Volume} \times \frac{\text{Change in velocity}}{\text{Time}}$$

$F_i = \rho \times Q \times V$

$F_i = \rho A V^2$

$F_V \rightarrow$ Viscous force

$F_V = \tau \times A$

$$F_V = \mu \frac{V}{y} A$$

$$F_V = \mu \frac{V}{L} A$$

$$N_{Re} = \frac{\rho A V^2}{\mu \frac{V}{L} A}$$

$$N_{Re} = \frac{\rho V L}{\mu}$$

In case of pipeline diameter is the linear dimension.

$$N_{Re} = \frac{\rho V D}{\mu}$$

2. Froude's Number (F_r)

It is defined as the ratio of square root of inertia force to gravity force.

$$F_r = \sqrt{\frac{F_i}{F_g}}$$

$F_i = m \text{ x } a$

$F_i = \rho \text{ x Volume x Acceleration}$

$F_i = \rho AV^2$

$F_g = m \text{ x } g$

$F_g = \rho \text{ x Volume x } g$

$F_g = \rho \text{ x A x L x g}$

$$F_\gamma = \sqrt{\frac{\rho AV^2}{\rho \text{ x A x L x g}}}$$

$$F_\gamma = \sqrt{\frac{V^2}{Lg}}$$

$$F_\gamma = \frac{V}{\sqrt{Lg}}$$

3. Euler's Number (ε_u)

It is defined as the square root of ratio of inertia force to pressure force.

$$\varepsilon_u = \sqrt{\frac{F_i}{F_p}}$$

$F_i = \text{Mass x Acceleration}$

$$F_i = \rho \text{ x Volume x } \frac{\text{Velocity}}{\text{Time}}$$

$F_i = \rho \text{ x Q x}$

$V\ F_i = \rho AV^2$

$F_p = p \text{ x A}$

$$\varepsilon_u = \sqrt{\frac{\rho AV^2}{pA}} = V\sqrt{\frac{\rho}{p}}$$

$$\varepsilon_u = \frac{V}{\sqrt{\frac{p}{\rho}}}$$

4. Weber's Number (W_b)

It is defined as the square root of ratio of inertia force to surface tensile force.

$$W_b = \sqrt{\frac{F_i}{F_p}}$$

$$F_b = \rho A V^2$$

$$F_s = \sigma \text{ x } L$$

$$W_b = \sqrt{\frac{\rho A V^2}{\sigma L}} = V\sqrt{\frac{\rho L}{\sigma}}$$

$$\mathbf{W_b} = \frac{V}{\sqrt{\frac{\sigma}{\rho L}}}$$

5. Mach Number (M)

It is defined as the square root of ratio of inertia force to elastic force.

$$M = \sqrt{\frac{F_i}{F_e}}$$

$$F_i = \rho A V^2$$

$$F_e = K \text{ x } A$$

K → Bulk modulus of elasticity

A → Area

$$M = \sqrt{\frac{\rho A V^2}{KA}}$$

$$M = \frac{V}{\sqrt{K / \rho}}$$

$$M = \frac{V}{C}$$

C → Velocity of sound in fluid.

MODEL LAWS (SIMILARITY LAWS)

1. Reynold's Model Law

For the flows where in addition to inertia force, similarity of flow in model and predominant force, similarity of flow in model and prototype can be established if Re is same for both the system.

This is known as Reynold's Model Law.

Re for model = Re for prototype

$(N_{Re})_m = (N_{Re})_p$

$$\left(\frac{\rho VD}{\mu}\right)_m = \left(\frac{\rho VD}{\mu}\right)_p$$

$$\frac{\dfrac{\rho_m \cdot V_m \cdot D_m}{\rho_p \cdot V_p \cdot D_p}}{\dfrac{\mu_m}{\mu_p}} = 1$$

$$\frac{\rho_r \cdot V_r \cdot D_r}{\mu_r} = 1$$

Applications:

i) In flow of incompressible fluids in closed pipes.

ii) Motion of submarine completely under water.

iii) Motion of air-planes.

2. Froude's Model Law

When the force of gravity is predominant in addition to inertia force then similarity can be established by Froude's number. This is known as Froude's model law.

$(F_r)_m = (F_r)_p$

$$\left(\frac{V}{\sqrt{gL}}\right)_m = \left(\frac{V}{\sqrt{gL}}\right)_p$$

$$\left(\frac{V}{\sqrt{gL}}\right)_r = 1$$

Applications:

- Flow over spillways.
- Channels, rivers (free surface flows).
- Waves on surface.
- Flow of different density fluids one above the other.

5

MODEL ANALYSIS

Session – X

3. Euler's Model Law

When pressure force is predominant in addition to inertia force, similarity can be established by equating Euler number of model and prototype. This is called Euler's model law.

$$(\varepsilon_u)_m = (\varepsilon_u)_p$$

$$\left(\frac{V}{\sqrt{p_m / \rho_m}} \right) = \left(\frac{V_p}{\sqrt{p_p / \rho_p}} \right)$$

Application: Turbulent flow in pipeline where viscous force and surface tensile forces are entirely absent.

4. Mach Model Law

In places where elastic forces are significant in addition to inertia, similarity can be achieved by equating Mach numbers for both the system.

This is known as Mach model law.

$$\mathbf{M_m = M_\gamma}$$

$$\left(\frac{V_m}{\sqrt{K_m/\rho_m}}\right) = \left(\frac{V_p}{\sqrt{K_p/\rho_p}}\right) \qquad \left(\frac{V_\gamma}{\sqrt{K_\gamma/\rho_\gamma}}\right) = 1$$

<u>Applications:</u>

22. Aerodynamic testing where velocity exceeds speed of sound.

    Eg: Flow of airplane at supersonic speed.

23. Water hammer problems.

5. Weber's Model Law:

If surface tension forces are predominant with inertia force, similarity can be established by equating Weber number of model and prototype.

$W_m = W_\gamma$

$$\left(\frac{V}{\sqrt{\sigma/\rho L}}\right)_m = \left(\frac{V}{\sqrt{\sigma/\rho L}}\right)_p$$

$$\left(\frac{V}{\sqrt{\omega/\rho L}}\right)_r = 1$$

Applications:

(xxi) Flow over wires with low heads.

(xxii) Flow of very thin sheet of liquid over a surface.

(xxiii) Capillary flows.

- **Problem 1**: A pipe of diameter 1.5 m is required to transmit an oil of S = 0.9 and viscosity 3 x 10^{-2} poise at 3000 lps. Tests were conducted on 15 cm diameter pipe using water at 20^{o}C. Find velocity and rate of flow of model if μ water at 20^{o}C is 0.01 poise.

- **Solution**

$D_p = 1.5$ m
$S_p = 0.9$
$\mu_p = 3 \times 10^{-2}$ poise $= 3 \times 10^{-3}$ Ns/m^2
$Q_p = 3000$ lps $= 3000 \times 10^{-3}$ m^3/s $= 3$ m^3/s
$D_m = 0.15$
m $S_m = 1$
$V_m = ?$
$Q_m = ?$
$A_p V_p = Q_p$
$V_p = 1.698$ m/s
$\mu_m = 0.01$ poise
$= 0.001$ poise
$\rho_m = 1000$ kg/m^3
$\rho_p = 0.9 \times 1000 = 900$ kg/m^3

$(R_e)_m = (R_e)_p$

$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_p V_p D_p}{\mu_p}$$

$$\frac{1000 \times V_m \times 0.15}{0.001} = \frac{900 \times 1.698 \times 1.5}{3 \times 10^{-3}}$$

$V_m = 5.094$ m/s

$Q = A_m V_m$

$$Q = \frac{\pi}{4}(0.15)^2(5.094)$$

$Q = 0.09\ m^3/s$

Q = 90 lps.

$D_p = 1.5$ m
$S_p = 0.9$
$\mu_p = 3 \times 10^{-2}$ poise $= 3 \times 10^{-3}\ Ns/m^2$
$Q_p = 3000$ lps $= 3000 \times 10^{-3}\ m^3/s = 3\ m^3/s$
$D_m = 0.15$ m
$S_m = 1$
$V_m = ?$
$Q_m = ?$
$A_p V_p = Q_p$
$V_p = 1.698$ m/s
$\mu_m = 0.01$ poise
$= 0.001$ poise
$\rho_m = 1000\ kg/m^3$
$\rho_p = 0.9 \times 1000 = 900\ kg/m^3$

- **Problem 2**: In a 1 in 40 model of spillway velocity and discharge are 2 m/s and 2.5 m^3/s. Find the corresponding velocity and discharge in prototype.

- **Solution**

Since it is a spillway problem,
Froude's law of similarity is used.

$$(F_\gamma)_m = (F_\gamma)_p$$

$$\left(\frac{V}{\sqrt{gL}}\right)_m = \left(\frac{V}{\sqrt{gL}}\right)_p$$

$$\frac{2}{\sqrt{9.81 \times 1}} = \frac{V_p}{\sqrt{9.81 \times 40}}$$

$$V_p = 12.65 \text{ m/s}$$

$$L_\gamma = \frac{L_m}{L_p} = \frac{1}{40}$$

$V_m = 2$ m/s

$Q_m = 2.5\ m^3/s$

For a spillway,

$$Q \alpha L^{2.5}$$

$$\frac{Q_p}{Q_m} = \frac{L_p^{2.5}}{L_m^{2.5}}$$

$$\frac{Q_p}{2.5} = (40)^{2.5}$$

$$Q_p = 25298.22 \text{ m}^3/\text{s}$$

- **Problem 3**: Experiments area to be conducted on a model ball which is twice as large as actual golf ball. For dynamic similarity, find ratio of initial velocity of model to that of actual ball. Take fluid in both cases as air at STP.

It is a case of motion of fully submerged body.

∴ Reynolds's number of flow determines dynamic similarity.

- **Solution**

$\therefore (R_e)_m = (R_e)_p$

$\rho_m = \rho_p$

$\mu_m = \mu_p$

$$\left(\frac{\rho V d}{\mu}\right)_m = \left(\frac{\rho V d}{\mu}\right)_p$$

$$\frac{d_m}{d_p} = 2$$

$V_m \cdot d_m = V_p \cdot d_p$

$$\frac{V_m}{V_p} = \frac{d_p}{d_m}$$

$$\frac{V_m}{V_p} = \frac{1}{2}$$

$V_m = 0.5\ V_p$

- **Problem 4**: Water at $15^{o}C$ flows at 4 m/s in a 150 mm diameter pipe. At what velocity oil at $30^{o}C$ must flow in a 75 mm diameter pipe for the flows to be dynamically similar? Take kinematic viscosity of water at $15^{o}C$ as 1.145×10^{-6} m^2/s and that for oil at $30^{o}C$ as 3×10^{-6} m^2/s.

- **Solution**

$V_p = 4$ m/s

$d_p = 0.15$ m

$V_m = ?$

$D_m = 0.075$ m

$$\left(\frac{\mu}{\rho}\right)_p = 1.145 \times 10^{-6}\ m^2/s$$

$$\left(\frac{\mu}{\rho}\right)_m = 3 \times 10^{-6}\ m^2/s$$

$$\left(\frac{\rho V d}{\mu}\right)_m = \left(\frac{\rho V d}{\mu}\right)_p$$

$$\frac{V_m \times 0.075}{3 \times 10^{-6}} = \frac{4 \times 0.15}{1.145 \times 10^{-6}}$$

$V_m = 20.96$ m/s

- **Problem 5**: A model with linear scale ratio (model to prototype) x, of a mach 2 supersonic aircraft is tested in a wind tunnel where in pressure is y times the atmospheric pressure. Determine the speed of model in tunnel given that velocity of sound m atmospheric air is Z.

- **Solution**

$\frac{L_m}{L_p} = X$

$M = 2$

$P_m = y\, p_{atm}$

$\rho_m = y\, \rho_{atm}$

$C = Z$

$\frac{V}{C} = 2$

$\frac{V}{Z} = 2$

$V_p = 2Z$

Dynamic similarity in this case is established by Reynold's Model law.

$$(R_e)_m = (R_e)_p$$

$$\left(\frac{\rho VL}{\mu}\right)_m = \left(\frac{\rho VL}{\mu}\right)_p$$

$$y\ \frac{\rho_{atm} \times V_m \times L_m}{\mu_m} = \frac{\rho_p \times V_p \times L_p}{\mu_p}$$

$$y\ \frac{\rho_{atm}}{\mu_m} \cdot \frac{V_m}{1} \cdot x = \rho_{atm} \cdot 2Z$$

$$V_m = \frac{2Z}{xy}$$

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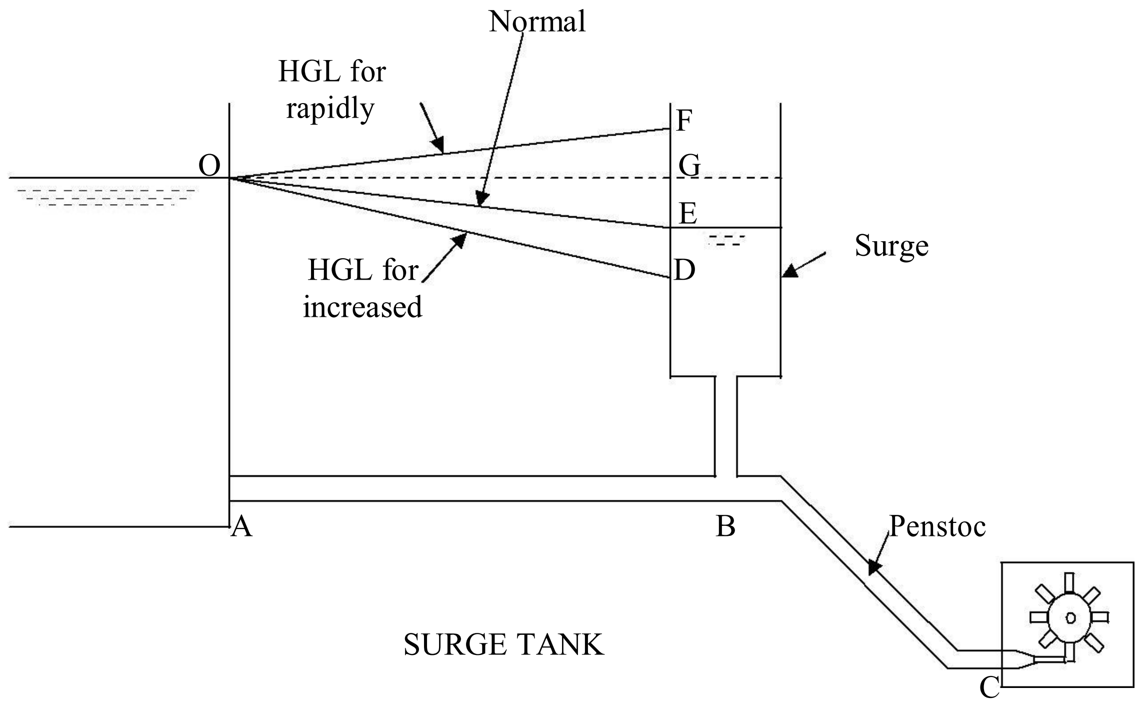
Surge Tank

Surge is the instantaneous rise in pressure due to sudden partial or complete closure of valve on the downstream end of a long pipeline. Surge tanks are generally built as a part of hydroelectric plant. In a long pipeline (Penstock), conveying water from a reservoir to the turbines, there will be sudden fluctuations in the discharge at the outlet of the pipeline with the varying load on the generator coupled to the turbine. There will be need for the generator speed to be cut down suddenly due to decrease in load which in turn decreases the discharge. This affects over a long pipeline instantaneously increasing the pressure at the outlet, thereby bursting the pipe. If there is an open tank whose level is kept well above the supply reservoir, located closer to the outlet, it can temporarily accommodate the additional supply of water coming from both the reservoir and the backwater from the control valve.

Similarly, there will be need for the generator speed to be increased suddenly due to increase in the load which in turn increases the discharge. This additional supply of discharge which has to be obtained from the reservoir in turn immediately increases the flow velocity in the pipeline thereby decreasing the pressure. This results in crushing of t he pipe as the external pressure is far more than the internal pressure. The surge tank if provided can augment the supply of water due to sudden increase in discharge temporarily and prevent the damage to the pipeline.

The surge tank is an open topped large chamber provided so as to communicate freely with the pipe line bringing water from the reservoir. The upper lip of the surge tank is situated at a suitable height above the maximum water level in the reservoir. When the turbine is working under steady load and the flow through the pipe is uniform there will be a normal pressure gradient *OE*. The water level in the surge tank will be lower than that in the reservoir by *GE* which represents the loss of head in the pipe line due to friction . If now the rate of flow in the pipe line is suddenly decreased, there will be a sudden pressure rise and this will result in **a** sudden rise in the water level in the surge tank so that the hydraulic gradient is now along OF . In this situation, the water level in the surge tank will be higher than that in the reservoir. This condition prevails only for a short duration. The surge tank acts as an auxiliary storage reservoir



SURGE TANK

to collect the flow down the pipe when the flow through the pipe is reduced or stopped . The excess water is accumulated in the surge tank. This arrangement eliminates the instantaneous expansion of the pipe line and thus prevents pipe bursting. Similarly, due to increased flow condition during excess of water requirement will result in **a** sudden decrease in the water level in the surge tank so that the hydraulic gradient is now along

OD .

Other Types of Surge Tanks, Besides the simple cylindrical surge tank, other types are also adopted.

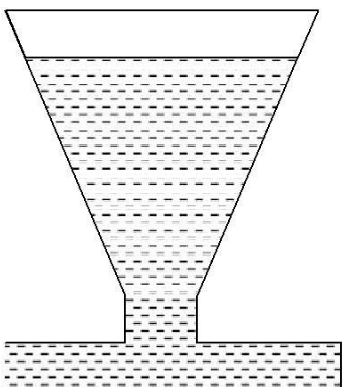
(i) Conical surge tank *(ii)* Surge tank with internal bell -mouthed spillway
(iii) Differential surge tank

Fig. a shows a conical surge tank which is similar to the simple surge tank described earlier, except in this case the tank has a conical shape.

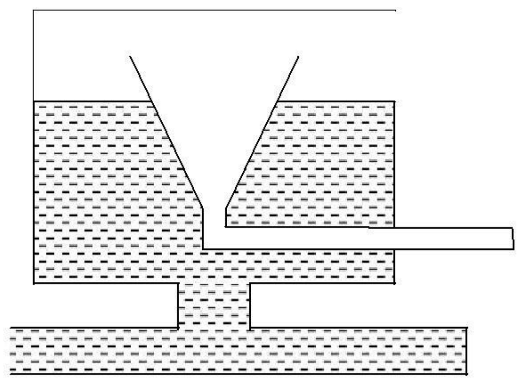
2

Fig. b shows a surge tank provided with internal bell -mouthed spill way. This arrangement allows the overflow to be conveniently disposed of.

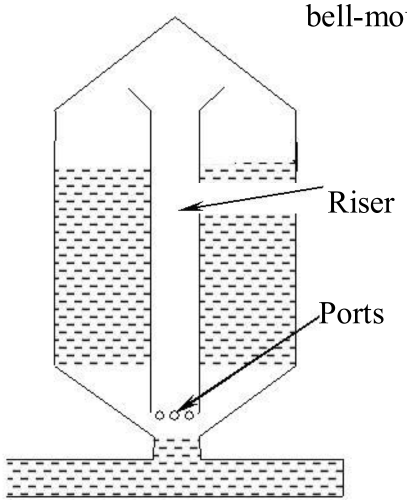
Fig. c shows a differential surge tank. This has the advantage that for the same stabilising effect its size can be very much less than that of the ordinary surge tank . Inside the surge tank there is a riser pipe provided with ports at its bottom. When there is an increase in pressure in the pipe, some small quantity of water enters the surge tank through these ports but the major bulk of the incoming flow mounts to the top of the riser and th en spills over into the tank . Thus this provides a substantial retarding head while in the ordinary surge tank the head only builds up gradually as the tank gets filled - It may further be realized that the water is not allowed to waste in the differential tank



(a) Conical Surge tank



(b) Surge tank with internal bell-mouthed spillway



(c) Differential Surge tank

Unit Quantities

In the studies of comparison of the performances of turbines of different output, speeds and different heads, it is convenient to determine the output, the speed and the discharge, when the head on the turbine is reduced to unit y, i . e . 1 m . The conditions of the turbine under unit head are such that the efficiency of the turbine remains unaffected . Thus the velocit y triangles under working conditions and under unit head are geometricall y similar.

Given a turbine every velocit y vector (V_1 , U_1 , Vw_1 , V_{f1}) is a function of H where H is the head on the turbine. With this basic concept, we can determine the speed, discharge and power under unit head .

***Unit Speed* (N_u):**

This is the speed of a turbine working under a unit head

Let N be the speed of turbine, H be the head on the turbine and u be the peripheral velocit y.

We know that the peripheral velocit y u is given b y

$u = \frac{\pi D N}{60}$ where D is the mean diameter of the runner which is treated as constant and N is the speed of the runner, Hence

$u \propto N$

But $u = K_u \sqrt{2 \gamma H}$ and hence $u \, \alpha \, \sqrt{H}$

Hence $N \, \alpha \, \sqrt{H}$

i . e . $N = K_1 \sqrt{H}$, where K_1 is the proportionalit y constant .

From definition of Unit speed, it is the speed of a turbine when working under unit head . Hence at $H=1$, $N=N_u$. Substituting, we get

$N_u = K_1 \sqrt{1}$

$$N = Nu \sqrt{H} \text{ or } Nu = \frac{N}{\sqrt{H}} \qquad (01)$$

***Unit Discharge* (Q_u) :**

This is the discharge through the turbine working under a unit head

Consider the Q as the discharge through a turbine. From discharge continuit y equation, $Q = a$ x V, where a is the cross -sectional area of flow and V is the mean flow velocit y.

For a given turbine, the cross - sectional area is constant and hence $Q \propto V$

But $V = C_v \sqrt{2gH}$ and hence $V \alpha \sqrt{H}$ and hence $Q \alpha \sqrt{H}$

i . e . , $Q = K_2 \sqrt{H}$ where K_2 is the proportionalit y constant .

From definition of Unit discharge, it is the discharge through the turbine when working under unit head . Hence at $H=1$, $Q=Q_u$. Substituting, we get

$$Q_u = K_2 \sqrt{1}$$

$$Q = Qu \sqrt{H} \quad \text{or} \quad Qu = \frac{Q}{\sqrt{H}} \qquad (02)$$

Unit Power (P_u) :

This is the Power developed b y the turbine working under a unit head Consider the P as the power developed by the turbine.

We know that the efficiency of turbine is given b y

$$\eta = \frac{P}{\gamma Q H}$$

Where γ is the weight densit y of the fluid/water passing through the turbine, Q is the discharge through the turbine and H is the head under which the turbine is working . But efficiency of a turbine and weight densit y of water are constants and hence, we can write

$$P \propto Q H$$

From discharge continuit y equation, $Q = a$ x V , where a is the cross - sectional area of flow and V is the mean flow velocit y.

For a given turbine, the cross - sectional area is constant and hence $Q \propto V$

But $V = C_v \sqrt{2gH}$ and hence $V \alpha \sqrt{H}$ and hence $Q \alpha \sqrt{H}$

Substituting, we get

$$P \alpha H\sqrt{H} \quad \text{or} \quad P = K_3 H \sqrt{H}$$

where K_3 is the proportionalit y constant .

From definition of Unit Power, it is the power developed b y the turbine when working under unit head . Hence at $H=1$, $P=P_u$. Substituting, we get

$$P_u = K_3 \, 1\sqrt{1}$$

$$P = P_u H \sqrt{H} \quad \text{or} \quad P_u = \frac{P}{H\sqrt{H}} = \frac{P}{H^{3/2}} \qquad (03)$$

Unit Speed, Unit discharge and Unit Power is definite characteristics of a turbine.

If for a given turbine under heads $H_1, H_2, H_3, ...$ the corresponding speeds are $N_1, N_2, N_3, ...,$ the corresponding discharges are $Q_1, Q_2, Q_3, ...$ and the powers developed are $P_1, P_2, P_3, ...$ Then

Unit speed = $N_u = \frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}} = \frac{N_3}{\sqrt{H_3}}$

Unit Discharge = $Q_u = \frac{Q_1}{\sqrt{H_1}} = \frac{Q_2}{\sqrt{H_2}} = \frac{Q_3}{\sqrt{H_3}}$

Unit Power = $P_u = \frac{P_1}{H_1\sqrt{H_1}} = \frac{P_2}{H_2\sqrt{H_2}} = \frac{P_3}{H_3\sqrt{H_3}}$ or $P_u = \frac{P_1}{H_1^{3/2}} = \frac{P_2}{H_2^{3/2}} = \frac{P_3}{H_3^{3/2}}$

Thus if speed, discharge and power developed b y a turbine under a certain head are known, the corresponding quantities for an y other head can be determined .

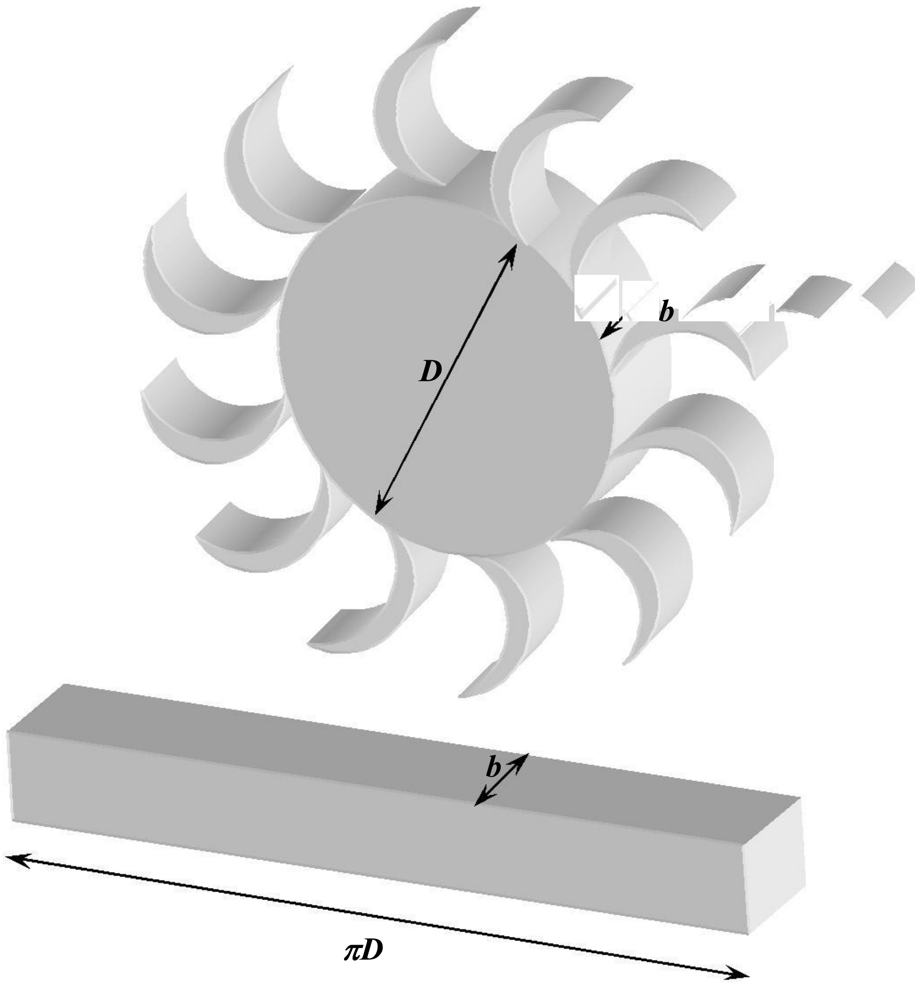
Specific Speed of a Turbine **(N_s)**

The specific speed of a turbine is the speed at which the turbine will run when developing unit power under a unit head . This is the t ype characteristics of a turbine . For a set of geometricall y similar turbines the specific speed will have the same value.

Consider the P as the power developed by the turbine. We know that the efficiency of turbine is given b y

$$\eta = \frac{P}{\gamma Q H}$$

Where γ is the weight densit y of the fluid/water passing through the turbine, Q is the discharge through the turbine and H is the head under which the turbine is working . But efficiency of a turbine and weight densit y of water are constants and hence, we can write



$P \propto Q\,H$

Discharge is given by the product of cross sectional area of flow and the flow velocit y. Cross sectional area is $\pi\, d\, b$ and hence

$Q = \pi\, D\, b\, V_f$

But $V = C_v \sqrt{2\, g\, H}$ and hence $V\ \alpha\ \sqrt{H}$

And $u = K_u \sqrt{2\, \gamma H}$ and hence $u\ \alpha\ \sqrt{H}$

Further, the peripheral velocit y is given by

$u = \dfrac{\pi D\, N}{60}$ where D is the mean diameter of the runner .

From the above two equations of u , we can write that

$DN \,\alpha\, \sqrt{H}$

or $D\alpha \sqrt{\frac{H}{N}}$

But in turbines the width of flow area b is proportional to the diameter D . Hence $D \propto b,$ with which

$b\alpha \sqrt{\frac{H}{N}}$

Hence $P \,\alpha \sqrt{\frac{H}{N}} \sqrt{\frac{H}{N}} H$

or $P\alpha \frac{H^{5/2}}{N^2}$

or $N^2 \,\alpha \frac{H^{5/2}}{P} = K \frac{H^{5/2}}{P}$

Simplifying further

$N = K \frac{H^{5/4}}{\sqrt{P}}$

But from the definition of specific speed, it is the speed of a turbine when it is working under unit head developing unit power. Hence when $H = 1$ and $P = 1$. $N = N_S$. Hence

$N_s = K \frac{1^{5/4}}{\sqrt{1}}$ and $K = N_s$

Substituting we get

$N = N_s \frac{H^{5/4}}{\sqrt{P}}$

or $N_S = \frac{N\sqrt{P}}{H^{5/4}}$

Examination questions

Dec/Jan 07

Define unit power, unit speed, unit discharge and specific speed with reference to h ydraulic turbines . Derive expressions for these terms . (06)

July 06

Derive expressions for specific speed of a turbine (06)

A turbine is to operate under a head of 25 m at 200 rpm . The discharge is 9 m^3/s . If the efficiency is 90 %, determine :

i) Power generated ii) Speed and Power at a head of 20 m . (07)

Jan 06

What is specific speed of a turbine and ex plain its significance (04)

Jan 05

Define and derive an expression for specific speed of a turbine, indicating its significance. (08)

Define the terms unit power, unit speed and unit discharge with reference to a h ydraulic turbine. Also give the expressions for these terms (06)

July 04

Define specific speed of a h ydraulic turbine . Derive an equation for specific speed in terms of operating speed, power and head (08)

Dec/Jan 07

Suggest a suitable type of a turbine to develop 7000 kW power under a head of 20 m, while operating at 220 rpm . What are the considerations fo r your suggestion? (04)

Solution:

P = 7000 kW; H = 20 m; N = 220 rpm .

We know that speci fic speed of a turbine is given b y

$$N_s = \frac{N\sqrt{P}}{H^{5/4}}$$

Substituting, we get

$$N_s = \frac{220\sqrt{7000}}{20^{5/4}} = 435 \text{ rpm}$$

As the specific speed is between 300 and 1000, Kaplan turbine can be suggested .

Jan 06

A turbine is to operate under a head of 25 m at 200 rpm . The disc harge is 9 m^3/s . If the efficiency is 90%, determine the performance of the turbin e under a head of 20 m (08)

Solution:

H_1 = 25 m; N_1 = 200 rpm; Q = 9 m^3/s; η = 0 . 90, H_2 = 20 m; N_2 = ?; Q_2 = ?

P_1 = ?, P_2 = ?

We know that

$$\eta = \frac{P}{\gamma Q H}$$

$$0.9 = \frac{(P)}{10 \times 1000 \times 9 \times 25}$$

P_1 = 2025 kW

Further unit quantities are given b y

Unit speed = $N_u = \frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}}$

Unit Discharge = $Q_u = \frac{Q_1}{\sqrt{H_1}} = \frac{Q_2}{\sqrt{H_2}}$

Unit Power = $P_u = \frac{P_1}{H_1^{3/2}} = \frac{P_2}{H_2^{3/2}}$

$N_u = \frac{200}{\sqrt{25}} = \frac{N_2}{\sqrt{20}} = 40$

$N_2 = 178.9$ rpm(Ans)

$Q_u = \frac{9}{\sqrt{25}} = \frac{Q_2}{\sqrt{20}} = 1.8$

$Q_2 = 8.05$ m^3/s (Ans)

$P_u = \frac{2025}{25^{3/2}} = \frac{P_2}{20^{3/2}} = 16.2$

$P_2 = 1449$ kW (Ans)

July 04

Suggest a suitable type of turbine to develop 7500 kW of power under a head of 25 m, while operating at 220 rpm . If the same turbine has to work under a head of 10 m, what power would devel op? What would be the new speed? (12)

Solution:

$P_1 = 7500$ kW; $H_1 = 25$ m; $N_1 = 220$ rpm; $H_2 = 10$; $P_2 = ?$; $N_2 = ?$ We know that specific speed of a turbine is given b y

$$N_s = \frac{N\sqrt{P}}{H^{5/4}}$$

Substituting, we get

$$N_s = \frac{220\sqrt{7500}}{25^{5/4}} = 340.8 \text{ rpm}$$

As the specific speed is between 300 and 1000, Kaplan turbine can be suggested .

Further unit quantities are given b y

Unit speed = $N_u = \frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}}$

Unit Power = $P_u = \frac{P_1}{H_1^{3/2}} = \frac{P_2}{H_2^{3/2}}$

$$N_u = \frac{220}{\sqrt{25}} = \frac{N_2}{\sqrt{10}} = 44$$

N_2 = 139 . 14 (Ans)

$$P_u = \frac{7500}{25^{3/2}} = \frac{P_2}{10^{3/2}} = 60$$

P_2 = 1897 . 4 kW (Ans)