BINOMIAL DISTRIBUTION

Definition:

A random variable 'x' is said to follow a binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$P(x) = {}^nC_x \, p^x q^{n-x}$$

where n = number of trials / random experiment

x = number of success

p = probability of success

q = probability of failure

Note:

1. n and p are called parameters of the binomial distribution.

2. A random variable 'x' which follow a binomial distribution which parameters n, p is denoted by $x \sim B(n, p)$.

3. $$\sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} {}^nC_x \, p^x q^{n-x}$$

$$= {}^nC_0 \, p^0 q^{n-0} + {}^nC_1 \, p^1 q^{n-1} + {}^nC_2 \, p^2 q^{n-2} + \dots + {}^nC_n \, p^n q^{n-n}$$

Here ${}^nC_0 = 1$

${}^nC_n = 1$

$p^0 = 1$

$q^0 = 1$

$$= q^n + {}^nC_1 \, p^1 q^{n-1} + {}^nC_2 \, p^2 q^{n-2} + \dots + p^n$$

$$= (p+q)^n$$

$$= (1)^n$$

$$\sum_{x=0}^{n} P(x) = 1$$

Properties :

1. Arithmetic mean (or) mean :

μ_1' - non central

$$\mu_1' = E(x) = \sum_{x=0}^{n} x \cdot p(x)$$

$$\mu_1' = \sum_{x=0}^{n} x \cdot {}^nC_x \, p^x q^{n-x}$$

$$\mu_1' = \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$\mu_1' = \sum_{x=1}^{n} x \cdot \frac{n(n-1)!}{x(x-1)!\,[(n-1)-(x-1)]!} \cdot \frac{p}{p} p^x q^{n-x}$$

$$\mu_1' = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} \frac{1}{p} p^x q^{n-x}$$

$$\mu_1' = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{-1} p^x q^{n-x}$$

$$\mu_1' = np \sum_{x=1}^{n} {}^{(n-1)}C_{(x-1)} \, p^{x-1} q^{n-x}$$

where

$$\sum_{x=1}^{n} {}^{n-1}C_{x-1} \, p^{x-1} q^{n-x} = (q+p)^{n-1}$$

$$= (p+q)^{n-1}$$

$$= (1)^{n-1}$$

$$= 1$$

$$\mu_1' = np(1)$$

$$\boxed{\mu_1' = np}$$

2. Variance :

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_2' = E(x^2) = \sum_{x=0}^{n} x^2 p(x)$$

$$\mu_2' = \sum_{x=0}^{n} [x(x-1)+x]\, p(x)$$

$$\mu_2' = \sum_{x=0}^{n} x(x-1)\,p(x) + \sum_{x=0}^{\infty} x\,p(x)$$

$$\mu_2' = \sum_{x=0}^{n} x(x-1)\,{}^nC_x\, p^x q^{n-x} + \sum_{x=0}^{n} x\cdot {}^nC_x\, p^x q^{n-x}$$

$$\mu_2' = \sum_{x=0}^{n} x(x-1)\frac{n!}{x!\,(n-x)!}\, p^x q^{n-x} + np$$

$$\mu_2' = \sum_{x=2}^{n} \cancel{x}(\cancel{x-1})\cdot \frac{n(n-1)(n-2)!}{\cancel{x}(\cancel{x-1})(x-2)!\,(n-x)!}\,\frac{p^2}{p^2}\, p^x q^{n-x} + np$$

$$\mu_2' = n(n-1)\,p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!}\,\frac{1}{p^2}\, p^x q^{n-x} + np$$

$$\mu_2' = n(n-1)\,p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!}\, p^{-2}\, p^x q^{n-x} + np$$

$$\mu_2' = n(n-1)\,p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{n-x} + np$$

$$\mu_2' = n(n-1)\,p^2\,(1) + np$$

$$\mu_2' = n(n-1)p^2 + np$$

$$\mu_2' = (n^2-n)p^2 + np$$

$$\mu_2' = n^2p^2 - np^2 + np$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\boxed{\mu_2 = np - np^2}$$

$$\mu_2 = np(1-p)$$

$$\boxed{\mu_2 = npq}$$

Note:

Standard deviation $= \sqrt{npq}$

A, B, M

20. Tens coins are thrown simultaneously find the probability of getting atleast 7 heads.

Sol: No. of trails $= n = 10$

Let $p =$ the probability of getting a head

i.e., $p = \frac{1}{2}$

Let $q =$ the probability of getting not a head or getting tail.

i.e., $q = \frac{1}{2}$

probability mass function of binomial distribution is $p(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$

Here $n = 10$ $p = \frac{1}{2}$ $q = \frac{1}{2}$

$x = 0, 1, 2, 3, \ldots, 10.$

$$P(x) = {}^{10}C_{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{10-x}$$

$$= {}^{10}C_{x} \left(\frac{1}{2}\right)^{x+10-x}$$

$$P(x) = {}^{10}C_{x} \left(\frac{1}{2}\right)^{10}$$

$$P(x) = \frac{1}{2^{10}} \times {}^{10}C_{x}$$

probability of getting atleast 7 heads is given by

$$P(x) \geq 7$$

$$p(x \geq 7) = p(x) = 7 + p(x=8) + p(x=9) + p(x=10)$$

$$p(x \geq 7) = \frac{1}{2^{10}}\, {}^{10}C_{x} + \frac{1}{2^{10}}\, {}^{10}C_{x} + \frac{1}{2^{10}}\, {}^{10}C_{x} + \frac{1}{2}\, {}^{10}C_{x}$$

$$p(x \geq 7) = \frac{1}{2^{10}}\, {}^{10}C_{7} + \frac{1}{2^{10}}\, {}^{10}C_{8} + \frac{1}{2^{10}}\, {}^{10}C_{9} + \frac{1}{2^{10}}\, {}^{10}C_{10}$$

$$p(x \geq 7) = \frac{1}{2^{10}} \left({}^{10}C_{7} + {}^{10}C_{8} + {}^{10}C_{9} + {}^{10}C_{10}\right)$$

$$p(x \geq 7) = \frac{11}{64} = 0.1718$$

B, A, M

2. If 3 of 20 tyres are defective and 4 of them are randomly choosen for inspection. What is the probability that only one of defective tyre will be included.

Sol: Let p = probability of a defective tyre (success) $= \frac{3}{20}$

$$p = \frac{3}{20}$$

q = probability of a non defective tyre

$$q = 1 - p$$

$$q = 1 - \frac{3}{20}$$

$$q = \frac{17}{20}$$

Given that,

$$n = 4.$$

The probability that exactly only one tyre will be defective $p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

where $n = 20,\ p = \frac{3}{20},\ q = \frac{17}{20},\ x = 1$

$$P(x) = {}^{20}C_1 \left(\frac{3}{20}\right)^{x} \left(\frac{17}{20}\right)^{20-1}$$

$$P(x) = {}^{20}C_1 \left(\frac{3}{20}\right)^{1} \left(\frac{17}{20}\right)^{19}$$

$$p(x) = 0.13679$$

3. The incidence of an occupational disease in an industry is such that the workers have a 20% chance of suffering from it. what is the probability that out of 6 workers (n), choosen at random 4 or more will suffer from the disease.

Sol: No of workers = n = 6

Let p = the probability that a worker suffering from a disease.

$$p = 20\% = \frac{20}{100}$$

$$p = 0.2$$

Let q = probability that a worker not suffering from a disease

$$q = 1 - 0.2$$

$$q = 0.8 = \frac{80}{100}$$

The probability that 4 or more workers will suffer from a disease

$$P(x \geq 4) = P(x=4) + P(x=5) + P(x=6)$$

$$P(x \geq 4) = {}^{n}C_x\, p^x q^{n-x} + {}^{n}C_x\, p^x q^{n-x} + {}^{n}C_x\, p^x q^{n-x}$$

$$P(x \geq 4) = {}^{6}C_4 (0.2)^4 (0.8)^{6-4} + {}^{6}C_5 (0.2)^5 (0.8)^{6-5} + {}^{6}C_6 (0.2)^6 (0.8)^{6}$$

$$P(x \geq 4) = 15 \cdot \frac{1}{625} \cdot \frac{16}{25} + 6 \cdot \frac{1}{3125}\left(\frac{4}{5}\right) + 1 \frac{1}{15625} 1$$

$$P(x \geq 4) = \frac{53}{3125}$$

$$P(x \geq 4) = 0.01696$$

4. Determine the binomial distribution for which the mean is 4 and variance is 3. A, B, M

Sol: Given that

In binomial distribution,

$$\text{mean} = np = 4$$

$$\text{variance} = npq = 3$$

$$\frac{npq}{np} = \frac{3}{4}$$

$$\boxed{q = \frac{3}{4}} \qquad p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\boxed{p = \frac{1}{4}}$$

$\Rightarrow n \cdot p = 4$

$n\left(\frac{1}{4}\right) = 4$

$$\boxed{n = 16}$$

B, M, A

5\. In 256 sets of 12 tosses of a coin in how many cases one can expect 8 heads and 4 tails.

Sol: Let p = probability of getting a head

$p = \frac{1}{2}$

q = probability of getting a tail

$q = \frac{1}{2}$

NO. of trails $= n = 12$

NO. of sets $= N = 256$

The probability of getting 8 heads and 4 tails in 12 trails

$$P(x) = n_{C_x} p^x q^{n-x}$$

$$P(x) = 12_{C_8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{12-8}$$

$$= 495 \frac{1}{256} \cdot \frac{1}{16}$$

$$= \frac{495}{4096}$$

$$P(x) = 0.12084$$

$\therefore$ The expected number of such cases in 256 sets

$= 256 \times P(x=8)$

$= 256 \times 0.12084$

$= 30.93504$

≈ 31

6. Out of 800 families with 5 children each, how many would you expect to have 1. 3 boys
2. 5 girls
3. either 2 or 3 boys.

Assume equal probabilities for boys and girls.

Sol: Let x = number of boys in each family

Let p = the probability of getting a boy

$p = \frac{1}{2}$

q = the probability of getting a girl

$q = \frac{1}{2}$

Number of children = $n = 5$ (boys)

probability mass fuction of BD is

$$P(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

$$P(x) = 5_{C_x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{5-x}$$

$$p(x) = 5_{C_x}\left(\frac{1}{2}\right)^{5} \text{ per family}$$

1. 3 boys :

$P[3 \text{ boys}] = P(x=3)$

$= 5_{C_3} \cdot \frac{1}{2^5}$

$= \frac{5}{16}$

$P[3\text{boys}] = 0.3125$

∴ Thus for 800 families, the probability of no. of families having 3 bo

$= 800 \times 0.3125 = 250$ fami

2. 5 girls :

$P[5 \text{ girls}] = P(x=5) = P(\text{no boys}) = P(x=0)$

$= 5_{C_0}\left(\frac{1}{2}\right)^{5}$

$= 1 \cdot \frac{1}{32}$

$= 0.03125$

Thus for 800 families, the probability of no. of families having 5 girls $= 800 \times 0.03125$

$= 25$ families.

3. either 2 or 3 boys :

$p(\text{either 2 or 3 boys}) = p(2) + p(3) = p(x=2) + p(x=3)$

$= {}^5C_2\left(\frac{1}{2}\right)^5 + {}^5C_3\left(\frac{1}{2}\right)^5$

$= 0.3125 + 0.3125$

$= 0.625$

$= \frac{5}{8}$

Thus for 800 families, the probability of no. of families with either 2 or 3 boys $= 800 \times (p(x=2) + p(x=3))$

$= 800 \times 0.625$

$= 500$ families.

7. The mean and variance of a binomial distribution are 4 and $\frac{4}{3}$ respectively. Find $p(x \geq 1)$.

B, A

Sol: In binomial distribution,

Mean $= np = 4$

variance $= npq = \frac{4}{3}$

$$\frac{npq}{np} = \frac{\frac{4}{3}}{4}$$

$$\boxed{q = \frac{1}{3}}$$

$p = 1 - q$

$p = 1 - \frac{1}{3}$

$$\boxed{p = \frac{2}{3}}$$

$np = 4$

$n \cdot \frac{2}{3} = 4 \Rightarrow n = \cancel{4}^{2} \times \frac{3}{\cancel{2}}$ $\boxed{n = 6}$

$P[x \geq 1] = P[x=1] + P[x=2] + P[x=3] + P[x=4] + P[x=5] + \ldots$

(or)

$P[x \geq 1] + P[x < 1] = 1$

$P[x \geq 1] = 1 - P[x < 1]$

$= 1 - P[x = 0]$

$= 1 - {}^{n}C_{x}\, p^{x} q^{n-x}$

$= 1 - {}^{6}C_{0} \left(\frac{2}{3}\right)^{0} \left(\frac{1}{3}\right)^{6-0}$

$= 1 - 1(1)\frac{1}{729}$

$= 1 - \frac{1}{729}$

$P[x \geq 1] = 0.99862$

8. The mean & variance of a binomial variable 'x' with parameters n and p are 16 and 8. Find $P[x \geq 1]$ and $P[x > 2]$

Sol: Given that,

$\text{mean} = np = 16$

$\text{variance} = npq = 8$

$\frac{\text{variance}}{\text{mean}} = \frac{\cancel{np}q}{\cancel{np}} = \frac{\cancel{16}\,8}{8\,16} = \frac{1}{2}$

$q = 1/2 = 0.5$

$p = 1 - q$

$p = 1 - 0.5$

$p = \frac{1}{2} = 0.5$

$np = 16$

$n\frac{1}{2} = 16$

$\boxed{n = 32}$

$$P[x \geq 1] + P[x < 1] = 1$$

$$P[x \geq 1] = 1 - P(x<1)$$

$$= 1 - P[x=0]$$

$$= 1 - 32C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{32-0}$$

$$= 1 - 32C_0 \left(\frac{1}{2}\right)^{32}$$

$$= 1 - 1\left(\frac{1}{2^{32}}\right)$$

$$= 0.9999$$

$$P(x>2] + P[x \leq 2] = 1$$

$$P(x>2] = 1 - P[x \leq 2]$$

$$= 1 - P[x=0] + P[x=1] + P[x=2]$$

$$= 1 - \left\{\frac{1}{2^{32}} + 32C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{31} + 32C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{30}\right\}$$

$$= 1 - \left\{\frac{1}{2^{32}} + \frac{32}{2^{32}} + \frac{496}{4 \times 2^{30}}\right\}$$

$$= 1.0000$$

9. In 8 throws of a die 5 (or) 6 is considered a success. Find the mean no. of success and the standard deviation. A, B, M

Sol: No. of throws $= n = 8$

Let p = probability of success

$$P[5 \text{ or } 6] = P[x=5] + P[x=6]$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{2}{6}$$

$$\boxed{P = \frac{1}{3}}$$

q = Probability of failure

$$q = 1 - \frac{1}{3}$$

$$\boxed{q = \frac{2}{3}}$$

Mean $= np$

$= 8 \times \frac{1}{3}$

Mean $= \frac{8}{3}$

Variance $= npq$

$= 8 \times \frac{1}{3} \times \frac{2}{3}$

$= \frac{16}{9}$

Variance $= \cancel{\frac{8}{3}}\ \frac{16}{9}$

Standard deviation $= \sqrt{npq}$

$= \sqrt{\frac{16}{9}}$

Standard deviation $= \frac{4}{3}$

A, M

10. In a binomial distribution consisting of 5 independent trails, probabilities of 1 and 2 success are 0.4096 and 0.2048 respectively. Find the parameter p of the distribution.

2m

Sol: No. of trials $= n = 5$

Given that

$P(1) = P[x=1] = 0.4096$

$P(2) = P[x=2] = 0.2048$

$$\frac{P(x=1)}{P(x=2)} = \frac{0.4096}{0.2048} = 2$$

$$\frac{{}^{n}C_{x}\, p^{x} q^{n-x}}{{}^{n}C_{x}\, p^{x} q^{n-x}} = \frac{2}{1}$$

$$\frac{{}^{5}C_{1}\, p^{1} q^{5-1}}{{}^{5}C_{2}\, p^{2} q^{5-2}} = \frac{2}{1}$$

$$\frac{7}{16_{2}} \frac{p\, q^4}{p^{2}\, q^{3}} = \frac{2}{1}$$

$$\frac{q}{2p} = \frac{2}{1}$$

$$q = 4p$$

$$1-p = 4p$$

$$1 = 5p$$

$$\boxed{p = \frac{1}{5}} = 0.2$$

$$q = 1 - \frac{1}{5}$$

$$q = \frac{4}{5} = 0.8$$

B, M

11. 20% of items produced from a factory ~~or~~ are deffective. (P). Find the probability that in a sample of 5. Choosen at random. find 1. none is defective
2. one is defective
3. $P[1<x<4]$

Sol: Let p = probability of items defective

$$p = \frac{20}{100} = \frac{1}{5} = 0.2$$

q = probability of items non defective

$$q = 1 - 0.2$$

$$q = 0.8 = \frac{4}{5}$$

Given $n = 5$ $\Rightarrow P(x) = 5c_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{5-x}$

1. none is defective

$$P(x=0) = 5c_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^5$$

$$= 1 \cdot 1 \cdot \frac{1024}{3125}$$

$$= 0.32768$$

2. 1 is defective

$P(x=1) = 5C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^4$

$= 5C_1 \frac{1}{5} \cdot \frac{4^4}{5^4}$

$= \cancel{5} \cdot \frac{1}{\cancel{5}} \frac{256}{625}$

$= 0.4096$

3. $P[1<x<4] = P(x=2) + P(x=3)$

$= 5C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^3 + 5C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^2$

$= 10 \times \frac{1}{25} \times \frac{64}{125} + 10 \times \frac{1}{125} \times \frac{16}{25}$

$= \frac{32}{125} = 0.256$

12. Find the maximum 'n' such that the probability of getting no head in tossing a coin, n times is greater than 0.1.

Sol: Let p = probability of getting head

$\boxed{p = \frac{1}{2}}$

q = Probability of getting tail

$q = 1 - p$

$q = 1 - \frac{1}{2}$

$\boxed{q = \frac{1}{2}}$

The probability of getting no head in tossing 'n' times is greater than 0.1.

i.e., $P(x=0) > 0.1$

$nC_x \, p^x \, q^{n-x} > 0.1$

$nC_0 \, p^0 \, q^{n-0} > 0.1$

$q^n > 0.1$

$\left(\frac{1}{2}\right)^n > 0.1$

put $n=1$

$$\left(\frac{1}{2}\right)^1 > 0.1$$

$$0.5 > 0.1$$

put $n=2$

$$\left(\frac{1}{2}\right)^2 > 0.1$$

$$0.25 > 0.1$$

put $n=3$

$$\left(\frac{1}{2}\right)^3 > 0.1$$

$$\frac{1}{8} > 0.1$$

$$0.125 > 0.1$$

put $n=4$

$$\left(\frac{1}{2}\right)^4 > 0.1$$

$$0.0625 > 0.1 \text{ (wrong)}.$$

The minimum value of the ~~prob~~ required value of n is 3

13. Assume that B, M, A 50% of all engineering students are good in mathematics. Determine the probability that among 18 engineering students.

1. Exactly 10
2. atleast 10
3. atmost 8
4. atleast 2 and atmost 9 are good in mathematics.

Sol: No. of students $= n = 18$

$p=$ probability of students good in mathematics

$$p = 50\% = \frac{50}{100} = 0.5$$

$q=$ Probability of students not good in mathematics

$$q = \frac{1}{2} = 0.5$$

let x = the number of engineering
are good in mathematics.

Binomial distribution $= {}^{n}C_{x}\, p^{x}\, q^{n-x}$

$= {}^{18}C_{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{18-x}$

$p(x) = {}^{18}C_{x}\left(\frac{1}{2}\right)^{18}$

1. Exactly 10 :

$p(x=10) = {}^{18}C_{10}\,\frac{1}{2^{18}}$

$= \frac{43758}{262144}$

$= 0.166923$

2. Atleast 10 :

$p(\text{atleast } 10) = p(x \geq 10)$

$= \sum_{x=10}^{18}\left(\frac{1}{2}\right)^{18}$

$= \frac{1}{2^{18}}\left[{}^{18}C_{10} + {}^{18}C_{11} + {}^{18}C_{12} + \cdots + {}^{18}C_{18}\right]$

$= 0.407264$

3. Atmost 8 :

$P(\text{atmost } 8) = \sum_{x=0}^{8} {}^{18}C_{x}\,\frac{1}{2^{18}}$

$=$

$= \frac{1}{2^{18}} \sum_{x=0}^{8} {}^{18}C_{x}.$

$= 0.407264$

4. atleast 2 & almost 9

$$p(2 \le x \le 9) = \sum_{x=2}^{9} {}^{18}C_x \frac{1}{2^{18}}$$

$$= \frac{1}{2^{18}} \sum_{x=2}^{9} {}^{18}C_x$$

$$= 0.592662$$

14. The probability of a man hitting a target $= \frac{1}{3}$

1. If he fires 5 times, what is the probability of his hitting the target atleast twice.

2. How many times must he ~~twice~~ fire so that the probability of hitting his target atleast once is more than 90%.

Sol: Given, No. of trails = 5

P = probability of man hitting a target

$p = \frac{1}{3}$

q = probability of man not hitting a target

$q = 1 - p = 1 - \frac{1}{3}$

$q = \frac{2}{3}$

1. $p(\text{atleast twice}) = p(x \ge 2)$

$$= p(x=2) + p(x=3) + p(x=4) + p(x=5)$$

$$= 1 - p(x > 2)$$

$$= 1 - \{p(x=0) + p(x=1)\}$$

$$= 1 - \left\{ {}^5C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^5 + {}^5C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^4 \right\}$$

$$= 1 - \left\{ 1 \cdot 1 \cdot \frac{32}{243} + 5 \frac{1}{3} \frac{16}{81} \right\}$$

$$= 1 - \frac{112}{243}$$

$$= \frac{131}{243}$$

$$= 0.539095$$

2. Given that = p(at least once) > 90% $= \frac{90}{100} = 0.9$

i.e., $p(x \geq 1) > 0.9$

$= \{1 - p(x=0)\} > 0.9$

$= \{1 - {}^nC_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^n\} > 0.9$

$= 1 - \left(\frac{2}{3}\right)^n > 0.9$

let $n = 1$,

$1 - \left(\frac{2}{3}\right)^1 > 0.9$

$0.33333 > 0.9$ (x)

let $n = 2$,

$1 - \frac{4}{9} > 0.9$

$0.5555 > 0.9$ (x)

let $n = 3$

$1 - \frac{8}{27} > 0.9$

$0.70370 > 0.9$ (x)

let $n = 4$

$1 - \frac{16}{81} > 0.9$

$0.80246 > 0.9$ (x)

let $n = 5$

$1 - \left(\frac{2}{3}\right)^5 > 0.9$

$0.86831 > 0.9$ (x)

let $n = 6$

$1 - \left(\frac{2}{3}\right)^6 > 0.9$

$0.91222 > 0.9$ (✓)

$\therefore$ Hence the value of $n = 6$

This is satisfy for $n=6$,

because $1-\left(\frac{2}{3}\right)^6 = 0.91220 > 0.9$

15. Determine the probabity of getting a sum of 9 exactly twice in 3 throws with a pair of fair dice.

Sol: Total possible outcomes $= 6^n = 6^2$

$= 36$

Let E = the two dice a sum of $9 = \{(5,4)(4,5)(3,6)(6,3)\}$

No. of favourable cases $= m = 4$

p = probability of a sum is 9

$p = \frac{4}{36} = \frac{1}{9}$

q = probability of a sum is not 9

$q = 1-\frac{1}{9}$

$q = \frac{8}{9}$

Here,

∴ The probability of getting a sum of 9, exactly twice in three throws,

here $x=2$, $n=3$, $p=\frac{1}{9}$, $q=\frac{8}{9}$

$$P(x) = {}^nC_x\, p^x\, q^{n-x}$$

$$= {}^3C_2 \left(\frac{1}{9}\right)^2 \left(\frac{8}{9}\right)^{3-2}$$

$$= 3 \cdot \frac{1}{81} \cdot \frac{8}{9}$$

$$= \frac{8}{243}$$

$$= 0.032921$$

16. The probability that John hits a [illegible]
He fires 6 times. find the probability that hits the target.

1. exactly two times
2. more than 4 times
3. atleast once.

Sol: let

p = probability that john hits the target

$p = \frac{1}{2}$

q = probability that john miss the target

$q = 1 - \frac{1}{2}$

$q = \frac{1}{2}$

NO. of times he fires $n = 6$

$$p(x) = {}^6C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{6-x}$$

$$= {}^6C_x \left(\frac{1}{2}\right)^6$$

1. exactly 2 times

$$P(x=2) = {}^6C_2 \left(\frac{1}{2}\right)^6$$

$$= \frac{1}{64} \cdot 15$$

$$= 0.234375$$

2. more than 4 times

$$P(x>4) = p(x=5) + p(x=6)$$

$$= {}^6C_5 \left(\frac{1}{2}\right)^6 + {}^6C_6 \left(\frac{1}{2}\right)^6$$

$$= \left(\frac{1}{2}\right)^6 \left({}^6C_5 + {}^6C_6\right)$$

$$= \frac{1}{64} \cdot \frac{7}{1} = 0.109375$$

3 Atleast once:

$$P(x \geq 1) = P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5) + P(x=6)$$

$$= \frac{1}{2^6}\left(6c_1 + 6c_2 + 6c_3 + 6c_4 + 6c_5 + 6c_6\right)$$

$$= \frac{1}{2^6} \sum_{x=1}^{6} 6c_x$$

$$= \frac{63}{64}$$

$$= 0.984375$$

17 The mean of B.D is 3 & variance is 9/4. find

1. value of n

2. $P(x \geq 7)$

3. $P(1 \leq x < 6)$

Sol: Given

1. Mean $= np = 3$

Variance $= npq = \frac{9}{4}$

$$\frac{npq}{np} = \frac{9/4}{3}$$

$$q = \frac{3}{4}$$

$$p = 1 - q$$

$$p = 1 - \frac{3}{4}$$

$$p = \frac{1}{4}$$

$$npq = 9/4$$

$$n \times \frac{1}{4} \times \frac{3}{4} = \frac{9}{4}$$

$$n = \frac{9}{4} \times \frac{4}{3} \times 4 \Rightarrow \boxed{n = 12}$$

2. $P(x \geq 7) = 1 - [P(x=0) + P(x=1) + P[x=2] + P(x=3] + P(x=4] + P[x=5] + P(x=6]]$

$$= 1 - 0.8554$$

$$P(x \geq 7) = 0.1446$$

3. $P(1 \le x < 6) = P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5)$

$$= {}^{12}C_1 \frac{1}{4}\left(\frac{3}{4}\right)^{11} + {}^{12}C_2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{10} + {}^{12}C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^9 + {}^{12}C_4 \left(\frac{1}{4}\right)^4$$

$$+ {}^{12}C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^7 + \cancel{{}^{12}C_6 \left(\frac{1}{4}\right)^6 \left(\frac{3}{4}\right)^6} + 1$$

$$= 0.82$$

A

18. The probability that the life of a bulb is 100 days is 0.05. find the probability that 1 of 6 bulbs.

1. Atleast one $P[x \ge 1]$
2. greater than 4 $P[x > 4]$
3. None., will be having a life of 100 days.

p = probability that life of bulb

$p = 0.05$

q = probability that ~~not~~ life of bulb

$q = 1 - 0.05$

$q = 0.95$

NO of bulbs - 6

$$P(E) = {}^6C_x (0.05)^x (0.95)^{6-x}$$

$$P(x=1) = {}^6C_1 (0.05)^1 (0.95)^5$$

$$= 6(0.05)(0.77378)$$

$$= 0.232134$$

1. $P[x \ge 1] + P[x=0] = 1$

$$P[x \ge 1] = 1 - P[x=0]$$

$$= 1 - {}^6C_0 (0.05)^0 (0.95)^6$$

$$= 1 - (6)(1)(0.73509)$$

$$=$$

2. $P(x > 4] = P(x=5) + P(x=6)$

$$= {}^6C_5 (0.05)^5 (0.95)^1 + {}^6C_6 (0.05)^6 (1)$$

$$= 1.79687 \times 10^{-6}$$

3. None - $P[x=0]$

$= {}^6C_0 (0.05)^0 (0.95)^6$

$= 4.41054$

A, B

19. 2 dice are thrown 5 times. If getting a doublet is a success. Find the probability that getting the success atleast once.

Sol: Total possible outcomes $n = 6^n$

$= 6^2$

$n = 36$

Getting a doublet numbers = $\{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (5,5)\ (6,6)\}$

favourable cases = $m = 6$

p = the probability that getting doublet

$p = \frac{6}{36} = \frac{1}{6}$

q = the probability that not getting doublet

$q = 1 - \frac{1}{6}$

$q = \frac{5}{6}$

No. of trails = 5

The probability getting a success atleast once.

i.e., $P[x \geq 1]$

$P[x \geq 1] + P[x < 1] = 1$

$P[x \geq 1] = 1 - P[x < 1]$

$= 1 - P[x = 0]$

$= 1 - {}^nC_x\, p^x\, q^{n-x}$

$= 1 - {}^5C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5$

$= 1 - 1 \cdot 1 \cdot \frac{3125}{7776} = 0.59812$

20. 2 dice are thrown 5 times . find ... of getting 7 as sum. A, B

1. atleast once
2. two times
3. $P[1<x<5]$.

Sol: Total probability outcomes $= 6^2$
$= 36$

No. of trails $= 5$

$E =$ Event of getting 7 as sum $= \{(1,6)(6,1)(2,5)$
$(3,4)\ (4,3)\}$

$m = 6$

$P =$ probability of sum (9) ~~7~~

$p = \frac{6}{36} = \frac{1}{6}$

$q = \frac{5}{6}$

$$P(x) = {}^5C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{5-x}.$$

1. $p(\text{atleast once}) = p(x \geq 1)$

$$p(x \geq 1) + p(x<1) = 1$$

$$p(x \geq 1) = 1 - P[x=0]$$
$$= 1 - {}^5C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5$$
$$= 1 - 1 \cdot \frac{3125}{7776}$$
$$= 0.598122$$

2. $P(\text{two times}) = P(x=2)$

$$= {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3$$
$$= 10 \times \frac{1}{36} \times \frac{125}{216}$$
$$= 0.16075$$

3. $P[1<x<5] = P[x=2] + P[x=3] + P[x=4]$

$$= {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 + {}^5C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 + {}^5C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^1$$
$$= \cancel{0.16075} + 0.03215 + \frac{25}{7776} + \frac{625}{3888} + \frac{125}{3888}$$
$$= \frac{1525}{7776}$$

21. In a family of 5 children find the probability that there are
1. 2 boys
2. atleast 1 boy
3. all are boys
4. no boys.

B A

Sol: Given

Number of children $= 5$

$p =$ probability of getting a boy $= \frac{1}{2}$

$q = 1 - p$

$q = 1 - \frac{1}{2}$

$q = \frac{1}{2}$

$$P(x) = {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}$$

$$P(x) = {}^5C_x \left(\frac{1}{2}\right)^5$$

1. $P(2 \text{ boys}) = {}^5C_2 \left(\frac{1}{2}\right)^5$

$= \frac{10}{32}$

$= 0.3125$

2. $p(\text{atleast one boy}) = p(x \geq 1) = 1 - p(x=0)$

$= 1 - {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{5-0} = 1 - \frac{1}{32} = \frac{31}{32}$

3. $p(\text{all are boys}) = p(x=5) = {}^5C_5 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

4. $p(\text{no boy}) = p(x=0) = {}^5C_0 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

22. A box contains 9 cards. Numb... drawn with replacement. What is the probabi... that none is one.

Sol: No. of trails $= n = 4$

Total possible outcomes $= 9$

Let p = the probability of getting one

$p = \frac{1}{9}$

q = the probability of not getting one

$q = 1 - \frac{1}{9}$

$q = \frac{8}{9}$

Hence the probability of getting none is one.

$$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$P[x=0] = {}^{4}C_{0} \left(\frac{1}{9}\right)^{0} \left(\frac{8}{9}\right)^{4}$$

$$= 1 \cdot 1 \cdot \frac{4096}{6561}$$

$$P[x=0] = 0.62429$$

Random variable:

B.P

* A random variable x is defined as the sum o... the number on the faces when two dice are thrown. Find the mean of 'x'.

Sol: Let x = the sum of the numbers on the faces when 2 dice are thrown.

Total possible outcomes $= 6^{n}$

$= 6^{2}$

$= 36$

$(1,1) = 1+1 = 2$ (1)

$(1,2)(2,1) = 1+2 = 3$ (2)

$(1,3)(2,2)(3,1) = 3+1 = 4$ (3)

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(1,4) (4,1) (3,2) (2,3) = 2+3=5 (4)

(1,5) (5,1) (3,3) (2,4) (4,2) = 4+2=6(5)

$$\text{mean} = \Sigma x\, p(x)$$

$$= 2\times\frac{1}{36} + \frac{3}{36}\times 2 + 4\times\frac{3}{36} + 5\times\frac{4}{36} + 6\times\frac{5}{36} + 7\times\frac{6}{36} + 8\times\frac{5}{36}$$

$$+ 9\times\frac{4}{36} + 10\times\frac{3}{36} + 11\times\frac{2}{36} + 12\times\frac{1}{36}$$

$$\mu = 7$$

Binomial distribution:

B, M A direct method.

23. Fit a Binomial distribution to the following data.

x	0	1	2	3	4	5
$f(x)$	2	14	20	34	22	8

Sol: Given

x	f	fx
0	2	0
1	14	14
2	20	40
3	34	102
4	22	88
5	8	40
	$\Sigma f=100$	$\Sigma fx=284$

$n = 5$

$N = \Sigma f = \text{Total frequency}$

$N = 100$

$\Sigma fx = 284$

$$\text{Mean} = \bar{x} = \frac{\Sigma fx}{N}$$

$$= \frac{284}{100}$$

$$= \frac{71}{25}$$

$\text{Mean} = \bar{x} = 2.84$

$\bar{x} = np$

$5p = 2.84$

$$p = \frac{2.84}{5}$$

$p = 0.568$

$q = 1 - p$

$q = 0.432$

Expected frequencies :

Note:

Multiplying both sides by N

$$N\, P(x) = N\, {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

$$f(x) = N \cdot {}^{n}C_{x} \cdot p^{x}\, q^{n-x}$$

put $x = 0$

$$f(x) = 100 \cdot {}^{5}C_{0}\,(0.568)^{0}\,(0.432)^{5}$$

$$= 100 \cdot 1 \cdot 1 \cdot 0.015045$$

$f(x) = 1.504591 = 2$ (approximately)

put $x = 1$

$f(x) = 100 \cdot {}^5C_1 \cdot (0.568)^1 (0.432)^4$

$f(x) = 9.89129$

put $x = 2$

$f(x) = 100 \; 5C_2 \; (0.568)^2 (0.432)^3$

$f(x) = 26.01045$

put $x = 3$

$f(x) = 34.19892$

put $x = 4$

$f(x) = 22.48262$

put $x = 5$

$f(x) = 100 \cdot 5C_5 \; (0.568)^5$

$f(x) = 5.91209 \approx 6$ B, M A

Q4. Four coins are tossed 160 times, a number of times heads occur is given below (Recurrance method)

x	0	1	2	3	4	
No. of times	8	34	69	43	6	

Fit a B.D to this data on the hypothesis that coins are unbiased

H_0 - null hypothesis

Sol: H_0 : The coins are unbiased

p = Probability of head

$p = \frac{1}{2}$

q = Probability of tail

$q = \frac{1}{2}$

$n = 4$

$N = \Sigma f$ = Total frequency

$N = 160$

Let f = no. of times

$N = 160$

$P(x) = N \cdot {}^{n}C_{x}\, p^{x} q^{n-x}$

$f(x) = N \cdot {}^{n}C_{x}\, p^{x} q^{n-x}.$

$= 160 \cdot {}^{4}C_{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{4-x}$

$= 16$

put $x = 0$

$f(x) = 160 \cdot {}^{4}C_{0} \left(\frac{1}{2}\right)^{4}$

$f(x) = 10 \Rightarrow f(0) = 10$

put $x = 1$

$f(x) = 160 \cdot {}^{4}C_{1} \left(\frac{1}{2}\right)^{4}$

$f(x) = 40$]

we have the recurrance relation.

$$f(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} f(x)$$

put $x = 0$

$$f(1) = \frac{4-0}{0+1} \cdot \frac{\frac{1}{2}}{\frac{1}{2}} f(0)$$

$= 4 \times 10$

$f(1) = 40$

put $x = 1$

$$f(1+1) = \frac{4-1}{1+1} \cdot 1 \cdot (40)$$

$f(2) = \frac{3}{2} \cdot \cancel{40}\,20$

$f(2) = 60$

put $x = 2$

$$f(2+1) = \frac{4-2}{2+1} \times 60$$

$$= \frac{2}{3} \times 60^{20}$$

$$f(3) = 40$$

put $x = 3$

$$f(3+1) = \frac{4-3}{3+1} \times 60$$

$$= \frac{1}{4_2} \times 60^{30}$$

$$f(4) = 15$$

POISSON DISTRIBUTION

Definition:

A discrete random variable x is said to follow Poisson distribution, if it assumes only non negative values and its probability mass function is given by

$$P(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

where $x = 0, 1, 2, 3 \ldots \infty$

1. Show that the poisson distribution is a limiting form of a binomial distribution.

proof:

Under the following conditions the binomial distribution tends to poission distribution.

1. $n \to \infty$
2. $p \to 0$
3. $np \to \lambda$

   $p = \frac{\lambda}{n}$

The probability mass function of binomial distribution is given by

$$P(x) = n_{C_x}\, p^x q^{n-x}$$

$$P(x) = n_{C_x}\, p^x (1-p)^{n-x}$$

$$\lambda = np \Rightarrow \boxed{p = \frac{\lambda}{n}}$$

$$P(x) = n_{C_x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$P(x) = n_{C_x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n} \left(1 - \frac{\lambda}{n}\right)^{-x}$$

$$p(x) = {}^{n}C_{x} \left(\frac{\lambda}{n}\right)^{x} \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$p(x) = \frac{n!}{x!(n-x)!} \cdot \frac{\lambda^{x}}{n^{x}} \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$p(x) = \frac{n(n-1)(n-2)\cdots(n-x+1)\cancel{(n-x)!}}{x!\,\cancel{(n-x)!}} \cdot \frac{\lambda^{x}}{n^{x}} \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$p(x) = \frac{n(n-1)(n-2)\cdots(n-x+1)}{n^{x}} \cdot \frac{\lambda^{x}}{x!} \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$p(x) = \frac{n(n-1)(n-2)\cdots(n-x+1)}{n\cdot n\cdot n\cdots n\ (x\text{ times})} \cdot \frac{\lambda^{x}}{x!} \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$p(x) = \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{(n-x+1)}{n} \frac{\lambda^{x}}{x!} \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$p(x) = 1\left(\frac{n}{n}-\frac{1}{n}\right)\left(\frac{n}{n}-\frac{2}{n}\right)\cdots\left(\frac{n}{n}-\frac{(x-1)}{n}\right) \frac{\lambda^{x}}{x!} \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$p(x) = 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right) \cdot \frac{\lambda^{x}}{x!} \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{x}}$$

$$\lim_{n\to\infty} p(x) = 1\cdot 1\cdot 1\cdots 1\cdot \frac{\lambda^{x} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}}{x!\,\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{x}}$$

Here $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n} = e^{-\lambda}$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{x} = e^{0} = 1$$

$$\lim_{n\to\infty} p(x) = \frac{\lambda^{x}}{x!}\frac{e^{-\lambda}}{1}$$

$$\boxed{\lim_{n\to\infty} p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}}$$

Properties :

1. Arithmetic mean :

$$\mu_1' = E(x) = \sum_{x=0}^{\infty} x\, p(x)$$

$$\mu_1' = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$\mu_1' = e^{-\lambda} \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x}{x!}$$

$$\mu_1' = e^{-\lambda} \sum_{x=1}^{\infty} x \cdot \frac{\lambda^x}{x(x-1)!} \cdot \frac{\lambda}{\lambda}$$

$$\mu_1' = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \cdot \frac{1}{\lambda}$$

$$\mu_1' = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \cdot \lambda^{-1}$$

$$\mu_1' = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$\mu_1' = \lambda e^{-\lambda} \left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots \right]$$

where

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\mu_1' = \lambda e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right]$$

$$\mu_1' = \lambda e^{-\lambda} e^{\lambda}$$

$$\mu_1' = \lambda e^{-\lambda+\lambda}$$

$$\mu_1' = \lambda e^0$$

$$\boxed{\mu_1' = \lambda}$$

variance:

$$\mu_2' = \mu_2' - (\mu_1')^2$$

$$\mu_2' = E(x^2) = \sum_{x=0}^{\infty} x^2 p(x)$$

$$\mu_2' = \sum_{x=0}^{\infty} [x(x-1) + x] p(x)$$

$$\mu_2' = \sum_{x=0}^{\infty} x(x-1)\, p(x) + \sum_{x=0}^{\infty} x\, p(x)$$

$$\mu_2' = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\mu_2' = e^{-\lambda} \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x}{x!} + \lambda$$

$$\mu_2' = e^{-\lambda} \sum_{x=2}^{\infty} x(x-1) \frac{\lambda^x}{x(x-1)(x-2)!} \cdot \frac{\lambda^2}{\lambda^2} + \lambda$$

$$\mu_2' = e^{-\lambda} \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} \frac{\cancel{\lambda^2}}{\lambda^2} + \lambda$$

$$\mu_2' = e^{-\lambda} \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} \frac{1}{\lambda^2} + \lambda$$

$$\mu_2' = e^{-\lambda} \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$\mu_2' = e^{-\lambda} \lambda^2 \left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots \right] + \lambda$$

where $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$$\mu_2' = e^{-\lambda} \lambda^2 \left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right] + \lambda$$

$$\mu_2' = e^{-\lambda} \lambda^2 e^{\lambda} + \lambda$$

$$\mu_2' = e^{-\lambda+\lambda} \lambda^2 + \lambda$$

$$\mu_2' = \lambda^2 e^0 + \lambda$$

$$\mu_2' = \lambda^2 + \lambda$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_2 = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$\boxed{\mu_2 = \lambda}$$

In poission distribution mean = variance

Normal Distribution

Definition :

A random variable 'x' is said to have normal distribution with parameters mean 'μ' and varianc σ^2 if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; \quad \begin{array}{l} -\infty \le x \le \infty \\ -\infty \le \mu \le \infty \\ \sigma > 0 \end{array}$$

Note:

1. A random variable 'x' which follows a Normal Distribution with parameters μ, σ^2 is denoted by $x \sim N(\mu, \sigma^2)$.

2. If $X \sim N(\mu, \sigma^2)$; $z = \frac{X-\mu}{\sigma}$ is called Standard normal variate. μ is a normal variate with mean '0' and variance unity is called a standard Normal variate.

   The distribution of 'z' is called standard normal distribution.

3. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

4. The probability density function of a standard normal distribution is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} \quad , \quad -\infty \le z \le \infty$$

$$\int_{-\infty}^{\infty} \phi(z)\,dz = 1$$

$$\int_{-\infty}^{\infty} \phi(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}\,dz = 1$$

$$2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}\,dz = 1$$

$$\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}\,dz = \frac{1}{2}$$

Properties:

moment generating function:

By definition $M_x(t) = E[e^{tx}]$

$$M_x(t) = \int_{-\infty}^{\infty} e^{+tx} f(x)\,dx$$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

let $z = \dfrac{x-\mu}{\sigma} \Rightarrow \sigma z = x - \mu$

$\sigma z + \mu = x$

$$z = \frac{x-\mu}{\sigma}$$

$$dz = \frac{dx}{\sigma} \Rightarrow dx = \sigma\,dz$$

$$M_x(t) = \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-z^2/2} \cancel{\sigma}\,dz$$

$$M_x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\mu + t\sigma z}\, e^{-z^2/2}\,dz$$

$$M_x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t}\, e^{t\sigma z}\, e^{-z^2/2}\,dz$$

$$M_x(t) = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z}\, e^{-z^2/2}\,dz$$

$$\mu_x(t) = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z^2 - 2\sigma z t)} \, dz$$

$$\mu_x(t) = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z^2 - 2\sigma z t + \sigma^2 t^2)} \cdot e^{\frac{\sigma^2 t^2}{2}} \, dz$$

$$\mu_x(t) = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z - \sigma t)^2} \cdot e^{\frac{+\sigma^2 t^2}{2}} \, dz$$

$$\mu_x(t) = \frac{e^{\mu t}}{\sqrt{2\pi}} e^{\frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z - \sigma t)^2} \, dz$$

let $y = z - \sigma t$

$dy = dz$

$$\mu_x(t) = \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} \, dy$$

Here $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} \, dy = 1$

$$\mu_x(t) = e^{\mu t + \frac{t^2 \sigma^2}{2}} \quad (1)$$

$$\mu_x(t) = e^{\mu t + \frac{t^2 \sigma^2}{2}}$$

Moments of normal distribution:

Mean:

$$\mu_r' = \frac{d^r}{dt^r} [\mu_x(t)]_{t=0}$$

put $r = 1$

$$\mu_1' = \frac{d}{dt} [\mu_x(t)]_{t=0}$$

$$\mu_1' = \frac{d}{dt} \left[e^{t\mu + \frac{t^2 \sigma^2}{2}} \right]_{t=0}$$

$$\mu_1' = \left[e^{t\mu + \frac{t^2\sigma^2}{2}} \left(\mu + \frac{\cancel{2}t\sigma^2}{\cancel{2}}\right) \right]_{t=0}$$

$$\mu_1' = e^{0\mu + \frac{0\sigma^2}{2}} (\mu + (0)\sigma^2)$$

$$\mu_1' = e^0 (\mu + 0) = \mu$$

$$\boxed{\mu_1' = \mu}$$

Variance:

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_2' = \frac{d^2}{dt^2} \left[M_x(t) \right]_{t=0}$$

$$\mu_2' = \frac{d}{dt} \left[\frac{d}{dt} M_x(t) \right]_{t=0}$$

$$\mu_2' = \frac{d}{dt} \left[(\mu + t\sigma^2) \cdot e^{t\mu + \frac{t^2\sigma^2}{2}} \right]_{t=0}$$

$$\mu_2' = \left[(\mu + t\sigma^2) e^{t\mu + \frac{t^2\sigma^2}{2}} \cdot (\mu + t\sigma^2) + e^{t\mu + \frac{t^2\sigma^2}{2}} (\sigma^2) \right]_{t=0}$$

$$\mu_2' = \left[(\mu + t\sigma^2)^2 e^{t\mu + \frac{t^2\sigma^2}{2}} + \sigma^2 e^{t\mu + \frac{t^2\sigma^2}{2}} \right]_{t=0}$$

$$\mu_2' = \left[(\mu + 0\sigma^2)^2 e^{0(\mu) + \frac{0^2\sigma^2}{2}} + \sigma^2 e^{0\mu + \frac{0^2\sigma^2}{2}} \right]$$

$$\mu_2' = (\mu + 0)^2 e^0 + \sigma^2 e^0$$

$$\mu_2' = \mu^2 + \sigma^2$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_2 = \mu^2 + \sigma^2 - \mu^2$$

$$\therefore \boxed{\mu_2 = \sigma^2}$$

Mode :

Mode is the value of 'x' for which $f(x)$ is maximum.

It is obtained by solution of the equation $f'(x)=0$ and $f''(x)<0$

The probability density function of the normal distribution is

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Taking log on both sides,

$$\log f(x) = \log\left[\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right]$$

$$\log f(x) = \log\frac{1}{\sigma\sqrt{2\pi}} + \log e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\log f(x) = \log\frac{1}{\sigma\sqrt{2\pi}} - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \log_e e$$

$$\log f(x) = \log\frac{1}{\sigma\sqrt{2\pi}} - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 (1)$$

differentiate on both sides wrt 'x'

$$\frac{1}{f(x)} f'(x) = 0 - \frac{1}{2}\cdot 2\left(\frac{x-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right)$$

$$\frac{f'(x)}{f(x)} = \frac{-(x-\mu)}{\sigma^2}$$

$$f'(x) = \frac{-(x-\mu)}{\sigma^2} f(x)$$

put $f'(x)=0$

$$-\frac{(x-\mu)}{\sigma^2} f(x) = 0$$

$$-(x-\mu) = 0$$

$$-x+\mu = 0$$

$$+x = +\mu$$

$$\boxed{x = \mu}$$

$$f'(x) = \frac{-1}{\sigma^2}\left[(x-\mu) f(x)\right]$$

Again diff $f'(x)$ wrt 'x'

$$f''(x) = \frac{-1}{\sigma^2}\left[(x-\mu) f'(x) + f(x)\right]$$

put $x = \mu$

$$f''(x) = \frac{-1}{\sigma^2}\left[(\mu-\mu) f'(\mu) + f(\mu)\right]$$

$$f''(x) = \frac{-1}{\sigma^2} f(\mu)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

put $x = \mu$

$$f(\mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{\mu-\mu}{\sigma}\right)^2}$$

$$f(\mu) = \frac{1}{\sigma\sqrt{2\pi}} e^0 = \frac{1}{\sigma\sqrt{2\pi}}$$

$$f''(x) = -\frac{1}{\sigma^2} \frac{1}{\sigma\sqrt{2\pi}}$$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} < 0$$

$$f''(x) < 0$$

$$\text{Mode} = \mu$$

Median :

If 'M' is median

then $\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2}$

$$\int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2}$$

According to normal distribution table

$$\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2}$$

$$\int_{\mu}^{M} f(x)\,dx = \frac{1}{2} - \frac{1}{2} = 0$$ [If integral is '0' the limits are equal]

$$[x]_{\mu}^{M} = 0$$

$$M - \mu = 0$$

$$\boxed{M = \mu}$$

$$\boxed{\text{Mean} = \text{Median}}$$

Mean deviation about mean:

$$M \cdot D_{\bar{x}} = E\left[|X - E(X)|\right]$$

Here $E(x) = \mu$

$$M \cdot D_{\bar{x}} = E\left[|x - \mu|\right]$$

$$M \cdot D_{\bar{x}} = \int_{-\infty}^{\infty} |x - \mu| \, f(x)\,dx.$$

$$M \cdot D_{\bar{x}} = \int_{-\infty}^{\infty} |x - \mu| \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

let $z = \frac{x - \mu}{\sigma}$

$$\sigma z = x - \mu$$

$$z = \frac{x - \mu}{\sigma}$$

$$dz = \frac{dx}{\sigma} \Rightarrow \sigma dz = dx$$

$$M \cdot D_{\bar{x}} = \int_{-\infty}^{\infty} |\sigma z| \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-z^2/2} \cancel{\sigma} dz$$

$$M \cdot D_{\bar{x}} = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| \, e^{-\frac{z^2}{2}} dz$$

$$M \cdot D_{\bar{x}} = \frac{\sigma}{\sqrt{2\pi}} \, 2\int_0^\infty |z| \, e^{-z^2/2} \, dz$$

$$M \cdot D_{\bar{x}} = \frac{2\sigma}{\sqrt{2\pi}} \int_0^\infty z e^{-z^2/2} \, dz$$

let $t = \frac{z^2}{2} \Rightarrow dt = \frac{\cancel{2} z \, dz}{\cancel{2}}$

$$dt = z\,dz$$

$$M \cdot D_{\bar{x}} = \frac{2\sigma}{\sqrt{2\pi}} \int_0^\infty e^{-t} \, dt$$

$$M \cdot D_{\bar{x}} = \frac{2\sigma}{\sqrt{2\pi}} \left[\frac{e^{-t}}{-1} \right]_0^\infty$$

$$M \cdot D_{\bar{x}} = \frac{-2\sigma}{\sqrt{2\pi}} \left[e^{-\infty} - e^{0} \right]$$

$$M \cdot D_{\bar{x}} = -\sigma \sqrt{\frac{2}{\pi}} \, [0 - 1]$$

$$M \cdot D_{\bar{x}} = \sigma \sqrt{\frac{2}{\pi}} = \sigma \sqrt{\frac{2}{22/7}}$$

$$M \cdot D_{\bar{x}} = \sigma \sqrt{\frac{14}{22}}$$

$$M \cdot D_{\bar{x}} \doteq \frac{4\sigma}{5} \text{ (approximately)}$$

Chief characteristics of Normal Distribution:

The normal distribution with the probability density function is given by

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \, dx$$ has the following properties.

1. The curve is bell shaped and is symmetrical about the line $x = \mu$

2. Mean, median and mode coincides

3. As 'x' increases numerically, $f(x)$ decreases. The maximum probability is at the point $x=\mu$ and its value is equal to $\frac{1}{\sigma\sqrt{2\pi}}$

4. $\beta_1 = 0$ and $\beta_2 = 3$

5. $\mu_{2n+1} = 0$ and $\mu_{2n} = 1 \cdot 3 \cdot 5 \cdots (2n-1)\,\sigma^{2n}$

6. Linear Combination of independent normal variates is also a normal variate.

7. Mean deviation about mean is equal to $\frac{4\sigma}{5}$ approximately

Q.D : M.D : S.D :: 10 : 12 : 15

8. $P[\mu-\sigma \le X \le \mu+\sigma] = 0.6826$

$P[\mu-2\sigma \le X \le \mu+2\sigma] = 0.9544$

$P[\mu-3\sigma \le X \le \mu+3\sigma] = 0.9974$

Importance of Normal Distribution:

The Normal Distribution plays a vital role in statistics because of following reasons.

1. Most of the distribution ofor example binomial, poisson, hyper geometric distribution etc tends to normal distribution is the limit for further many of the sampling distribution. Such as χ^2, t, F tends to normality for large samples.

2. Even if a variable is not normally distributed it can be brought to normal form by simple transformation.

3. If $X \sim N(\mu, \sigma^2)$

$P[\mu-3\sigma \le X \le \mu+3\sigma] = 0.9974$

$P[-3 \leq Z \leq 3] = 0.9974$

$P[|Z|>3] = 1-0.9974$

$= 0.0026$

This property of the normal distribution is the basis of the entire large sample theory.

4. Many of the distribution of the sample statistic follow normal distribution when the sample size in large.

5. Normal distribution has wide application in industrial statistics such as statistical quality control.

Derive Normal Distribution from Binomial Distribution. Show that the binomial distribution tends to normal distribution as $n \to \infty$.

<u>proof</u>:

Let $x \sim B(n, p)$

$E(x) = np = \mu$.

$V(x) = npq = \sigma^2$

$M_x(t) = (q+pe^t)^n$

$M_x(t/\sigma) = (q+pe^{t/\sigma})^n$.

$M_x(t/\sigma) = [q+pe^{t/\sqrt{npq}}]^n$

Now the standard binomial variate can be defined as follows

$$z = \frac{x-\mu}{\sigma}$$

$$M_z(t) = E[e^{tz}]$$

$$M_z(t) = E\left[e^{t\left(\frac{x-\mu}{\sigma}\right)}\right]$$

$$M_z(t) = E\left[e^{\frac{tx}{\sigma} - \frac{t\mu}{\sigma}}\right]$$

$$M_z(t) = E\left[e^{tx/\sigma} \cdot e^{-t\mu/\sigma}\right]$$

$$M_z(t) = e^{-t\mu/\sigma} E\left[e^{tx/\sigma}\right]$$

$$M_z(t) = e^{-t\mu/\sigma} M_x(t/\sigma)$$

$$= e^{-t\mu/\sigma}\left[q + p e^{t/\sqrt{npq}}\right]^n$$

$$= e^{-t\mu/\sigma}\left[q + p e^{t/\sqrt{npq}}\right]^n$$

$$= e^{\frac{-t \cdot np}{\sqrt{npq}}}\left[q + p e^{t/\sqrt{npq}}\right]^n$$

$$= \left(e^{\frac{-tp}{\sqrt{npq}}}\left[\left(q + p e^{t/\sqrt{npq}}\right)\right]\right)^n$$

$$= \left[q\, e^{\frac{-tp}{\sqrt{npq}}} + p\, e^{\frac{t}{\sqrt{npq}}} \cdot e^{\frac{-tp}{\sqrt{npq}}}\right]^n$$

$$= \left[q\, e^{\frac{-tp}{\sqrt{npq}}} + p\, e^{\frac{t}{\sqrt{npq}}(1-p)}\right]^n$$

$$= \left[q\, e^{\frac{-tp}{\sqrt{npq}}} + p\, e^{\frac{tq}{\sqrt{npq}}}\right]^n$$

$$= \left\{q\left[1 - \frac{tp}{\sqrt{npq}} + \frac{t^2p^2}{2!\,npq} - o'(n^{-3/2})\right] + p\left[1 + \frac{tq}{\sqrt{npq}} + \frac{t}{2} \quad + o''(n^{-3/2})\right]\right\}$$

$$= \left\{q - \frac{tpq}{\sqrt{npq}} + \frac{t^2p^2q}{2!\,npq} - o'(n^{-3/2}) + p + \frac{tpq}{\sqrt{npq}} + \frac{t}{2!} \quad + o''(n^{-3/2})\right\}^n$$

$$M_2(t) = \left\{(p+q) + \frac{t^2 pq}{2npq}(p+q) + o(n^{-3/2})\right\}^n$$
(1)

$$= \left\{1 + \frac{t^2 pq}{2npq} + o(n^{-3/2})\right\}^n$$

$$= \left\{1 + \frac{t^2}{2n} + o(n^{-3/2})\right\}^n$$

Taking log on both sides

$$\log M_2(t) = n \log \left[\underbrace{1 + \frac{t^2}{2n} + o(n^{-3/2})}_{z}\right]$$

$$\log M_2(t) = n\left\{\frac{t^2}{2n} + o(n^{-3/2}) - \frac{1}{2}\left(\frac{t^2}{2n} + o(n^{-3/2})\right)^2 + \frac{1}{3}\left(\frac{t^2}{2n} + o(n^{-3/2})\right)^3 - \frac{1}{4}\left(\frac{t^2}{2n} + o(n^{-3/2})\right)^4\right\}$$

$$\log M_2(t) = \frac{t^2}{2} + o(n^{-3/2}) - n(0)$$

$$\lim_{n\to\infty} \log M_2(t) = \frac{t^2}{2}$$

$$\lim_{n\to\infty} M_2(t) = e^{\frac{t^2}{2}}$$

$$= \frac{\Gamma(m-1) \cdot \Gamma(n+1)}{\Gamma m \cdot \Gamma n}$$

$$= \frac{\cancel{\Gamma(m-1)} \cdot n\cancel{\Gamma n}}{(m-1)\cancel{\Gamma(m-1)}\, \cancel{\Gamma n}}$$

$$\frac{1}{H} = \frac{n}{m-1}$$

$$\therefore H = \frac{m-1}{n}$$

$$\boxed{\text{Harmonic mean} = \frac{m-1}{n}}$$

Exponential Distribution

(iv) Definition:- A continuous random variable x is said to follow exponential distribution if it assumes only non-negative values and its probability density function is given by

$$f(x) = \theta \cdot e^{-\theta x} \; ; \; 0 \le x \le \infty$$

Here θ is the parameter of this distribution.

(14). Moment Generating Function of Exponential distribution and hence mean and variance

Let 'x' denotes a continuous random variable and $f(x) = \theta \cdot e^{-\theta x}$; $0 \le x \le \infty$.

The moment generating function of x is given by

$$M_x(t) = E(e^{tx})$$

$$= \int_0^{\infty} e^{tx} \cdot f(x) \cdot dx$$

$$= \int_0^\infty e^{tx} \, \theta e^{-\theta x} \, dx$$

$$= \theta \int_0^\infty e^{tx - \theta x} \, dx$$

$$= \theta \int_0^\infty e^{(t-\theta)x} \, dx.$$

$$= \left[\frac{\theta e^{(t-\theta)x}}{(t-\theta)} \right]_0^\infty$$

$$= \frac{\theta (e^{\infty} - e^{0})}{(t-\theta)} \qquad \left(\because e^{\infty} = 0 \atop \because e^{0} = 1 \right)$$

$$= \frac{\theta (0-1)}{t-\theta}$$

$$M_x(t) = \frac{-\theta}{t-\theta}$$

$$= \frac{\theta}{\theta - t}$$

$$\boxed{M_x(t) = \theta(\theta - t)^{-1}}$$

Mean:- $\frac{d}{}$

$$\mu_1' = \frac{d}{dt} (M_x(t)) \Big]_{t=0}$$

$$= \frac{d}{dt} \theta (\theta - t)^{-1}$$

$$= \theta \frac{d}{dt} (\theta - t)^{-1}$$

$$= \theta (-1)(\theta - t)^{-2} (-1)$$

$$\mu_1' = \theta(\theta - t)^{-2}$$

put $t = 0$

$$\mu_1' = \theta(\theta)^{-2}$$

$$= \theta^{-1}$$

$$= \frac{1}{\theta}$$

$$\boxed{\therefore \text{Mean} = \frac{1}{\theta}}$$

$$\mu_2' = \frac{d^2}{dt^2} M_X(t) \Big]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{d}{dt} M_X(t)\right]$$

$$\mu_2' = \frac{d}{dt}\left(\theta(\theta-t)^{-2}\right)$$

$$= \theta \frac{d}{dt}(\theta-t)^{-2}$$

$$= \theta(-2)(\theta-t)^{-3}(-1)$$

$$\mu_2' = 2\theta(\theta-t)^{-3}$$

put $t=0$

$$= 2\theta(\theta)^{-3}$$

$$\mu_2' = 2\theta^{-2}$$

$$\mu_2' = \frac{2}{\theta^2}$$

$\therefore$ variance : $V(X) = \mu_2' - \mu_1'^2$

$$= \frac{2}{\theta^2} - \frac{1}{\theta^2}$$

$$= \frac{1}{\theta^2}$$

$$\boxed{\therefore \text{ variance} = V(X) = \frac{1}{\theta^2}}$$

Distribution function of exponential distributio

Distribution function is given by

$$F(x) = P(X \le x)$$

$$= \int_0^x f(x)\,dx$$

$$= \int_0^x \theta e^{-\theta x}\,dx$$

$$= \theta \int_0^x e^{-\theta x}\,dx.$$

$$= \theta \int_a^{x+a} e^{-\theta x}\, dx$$

$$= \cancel{\theta}\left[\frac{e^{-\theta x}}{-\cancel{\theta}}\right]_a^{x+a}$$

$$= -\left[e^{-\theta(x+a)} - e^{-\theta a}\right]$$

$$= -e^{-\theta x - \theta a} + e^{-\theta a}$$

$$= -\cancel{e}^{-\theta x - \theta a - e^{-\theta a}}$$

$$= -e^{-\theta x} \cdot e^{-\theta a} + e^{-\theta a}$$

$$= e^{-\theta a}\left[1 - e^{-\theta x}\right]$$

$$P(\cancel{y}\, Y \le x \cap X \ge a) = e^{-\theta a}(1 - e^{-\theta x})$$

Again from ①

$$P(X \ge a) = \int_a^{\infty} f(x)\, dx$$

$$= \int_a^{\infty} \theta e^{-\theta x}\, dx$$

$$= \theta \int_a^{\infty} e^{-\theta x}\, dx$$

$$= \cancel{\theta}\left[\frac{e^{-\theta x}}{-\cancel{\theta}}\right]_a^{\infty}$$

$$= \left(-e^{-\theta x}\right)_a^{\infty}$$

$$= -\left[e^{-\theta\infty} - e^{-\theta a}\right]$$

$$= -e^{-\infty} + e^{-\theta a}$$

$$\boxed{P(X \ge a) = e^{-\theta a}}$$

$$\therefore P\{Y \le x / X \le a\} = \frac{\cancel{e^{-\theta a}}(1 - e^{-\theta x})}{\cancel{e^{-\theta a}}}$$

$$= 1 - e^{-\theta x}$$

$$= \theta\left(\frac{e^{-\theta x}}{-\theta}\right)_0^x$$

$$= -\left[e^{-\theta x} - e^{-0}\right]$$

$$= -e^{-\theta x} + e^0$$

$$F(x) = 1 - e^{-\theta x}$$

$$\therefore \boxed{F(x) = 1 - e^{-\theta x}}$$

- Show that ~~Geometric~~ Exponential distribution lacks the memory?

Statement:- If 'x' is an Exponential variate, ~~a~~ great $a \geq 0$, then exponential distribution lacks the memory in the following sense

i.e., $P\{y \leq x / x \geq a\} = P(x \leq x)$ when $y = x - a$

Proof:- We know that distribution function

i.e., $F(x) = 1 - e^{-\theta x} = P(x \leq x)$

$$P\{y \leq x / x \geq a\} = \frac{P(y \leq x \cap x \geq a)}{P(x \geq a)} \longrightarrow (1)$$

from (1) $P(y \leq x \cap x \geq a\} = P\{y \leq x \text{ and } x \geq a\}$

$$= P\{y \leq x, x \geq a\}$$

$$= P\{x - a \leq x, x \geq a\}$$

$$= P\{x \leq x + a, a \leq x\}$$

$$= P\{a \leq x \leq x + a\}$$

$$= \int_a^{x+a} f(x)\, dx.$$

$$= \int_a^{x+a} \theta e^{-\theta x}\, dx$$

$= P(X \le x)$

$\therefore P\{Y \le x / X \le a\} = P(X \le x)$

$\therefore$ Exponential distribution lacks the memory.

imp 12 <u>Normal Distribution</u>

(or)

<u>Gaussian Distribution</u>

23-12-04

<u>Definition</u>:- A continuous random variable 'x' is said to follow a normal distribution if it assumes only non-negative values and its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; -\infty \le x \le \infty$$
$$\mu, \sigma^2 > 0$$

Here μ, σ^2 are the parameters of the normal distribution.

Symbolically $X \sim N(\mu, \sigma^2)$

<u>Standard Normal variate and distribution</u>:-

If 'x' is a normal variate then x follows

$X \sim N(\mu, \sigma^2)$

The standard normal variate is defined as follows and

standard normal variate $= Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ and standard normal distribution is called standard normal distribution

with mean '0' and variance '1'.

$$E(x^2) = \int_a^b x^2 f(x)\, dx$$

$$= \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\int_a^b x^2 dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right)$$

$$= \frac{\cancel{(b-a)}\,(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$E(x^2) = \frac{b^2+ab+a^2}{3}$$

$$\text{Variance} = V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{b^2+ab+a^2}{3} - \left[\frac{a+b}{2}\right]^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{(a^2+b^2+2ab)}{4}$$

$$= \frac{4b^2+4a^2+4ab-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$= \frac{(a-b)^2}{12}$$

$$\therefore \boxed{\text{variance} = \frac{(a-b)^2}{12}}$$

<u>Moment Generating Function of Rectangular Distribution:-</u>

Let x denotes a continuous random variable

and $f(x) = \frac{1}{b-a}$; $a<x<b$

Moment generating function of x is given by

16-12-09

Continuous Distributions

1. Rectangular Distribution (or) Uniform Distribution

17) Def:- A continuous Random variable 'x' is said to follow rectangular distribution over an interval (a, b) and its assumes only non-negative values and the probability density function of x is given by

$$f(x) = \frac{1}{b-a} \; ; \; a < x < b$$

Mean and Variance of Rectangular Distributions:-

Let X denotes a continuous random variable and $f(x) = \frac{1}{b-a} \; ; \; a < x < b$

$$\text{Mean} = E(X) = \int_a^b x \cdot f(x)\, dx.$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b x\, dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a} \left[\frac{b^2}{2} - \frac{a^2}{2}\right]$$

$$= \frac{1}{b-a} \left[\frac{b^2 - a^2}{2}\right]$$

$$= \frac{(b+a)(b-a)}{(b-a)2}$$

$$\boxed{\text{Mean} = \frac{a+b}{2}}$$

$$M_x(t) = E(e^{tx})$$

$$= \int_a^b e^{tx} f(x)\, dx$$

$$= \int_a^b e^{tx} \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx}\, dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b$$

$$= \frac{1}{b-a} \left[\frac{e^{tb} - e^{at}}{t} \right]$$

$$\therefore \boxed{M_x(t) = \frac{e^{bt} - e^{at}}{t(b-a)}}$$

Characteristic Function of Rectangular Distribution:-

Let x denotes a continuous random variable and $f(x) = \frac{1}{b-a}$; $a<x<b$

Characteristic function of x is given by

$$\phi_x(t) = E(e^{itx})$$

$$= \int_a^b e^{itx} f(x)\, dx$$

$$= \int_a^b e^{itx} \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b e^{itx}\, dx$$

$$= \frac{1}{b-a} \left[\frac{e^{itx}}{it} \right]_a^b$$

$$\boxed{\therefore \phi_x(t) = \frac{e^{bit} - e^{ait}}{it(b-a)}}$$

The distribution function of rectangular distribution:-

Let x denotes a continuous random variable

and $f(x) = \frac{1}{b-a}$; $a<x<b$

The distribution function of rectangular distribution is given by

$$F(x) = P(X \leq x)$$

$$= \int_a^x f(x)\,dx$$

$$= \int_a^x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\int_a^x dx.$$

$$= \frac{1}{b-a}\left[x\right]_a^x$$

$$\boxed{F(x) = \frac{x-a}{b-a}}$$

17) ## The mean deviation about mean of rectangular distribution:-

Let x denotes a continuous random variable

and $f(x) = \frac{1}{b-a}$; $a<x<b$

The mean deviation about mean is given by

$$M.D = E\left|x - \text{mean}\right|$$

$$= E\left|x - \frac{a+b}{2}\right| \qquad \left(\because \text{Mean} = \frac{a+b}{2}\right)$$

$$= \int_a^b \left|x - \frac{(a+b)}{2}\right| f(x)\,dx.$$

$$= \int_a^b \left|x - \frac{a+b}{2}\right| \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| dx$$

Let $t = x - \frac{a+b}{2}$

$dt = dx$

If $x = a$ then $t = a - \frac{a+b}{2}$

$$= \frac{2a-a+b}{2}$$

$$= \frac{a-b}{2} = \frac{-(b-a)}{2}$$

If $x = b$ then $t = b - \frac{a+b}{2}$

$$= \frac{2b-a-b}{2}$$

$$= \frac{b-a}{2}$$

$$\therefore \quad MD = \frac{1}{b-a}\int_{\frac{-(b-a)}{2}}^{\frac{b-a}{2}} |t|\, dt$$

$$= \frac{1}{b-a}\, 2\int_0^{\frac{b-a}{2}} t\, dt$$

$$= \frac{1}{b-a}\, 2\left[\frac{t^2}{2}\right]_0^{\frac{b-a}{2}}$$

$$= \frac{1}{\cancel{2}(b-a)}\cancel{2}\left[\left(\frac{b-a}{2}\right)^2\right]$$

$$= \frac{1}{\cancel{2}(b-a)}\left[\frac{(b-a)^{\cancel{2}}}{4}\right]$$

$$M \cdot D = \frac{b-a}{4}$$

$$\boxed{M \cdot D = \frac{b-a}{4}}$$