COMPLEX VARIABLES & TRANSFORMS

UNIT-2

COMPLEX VARIABLE : INTEGRATION

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Complex Integration

Consider $F(t) = u(t) + iv(t)$ be a complex function where

F, u, v are functions of t in the interval (a, b)

define

$$\int_a^b F(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

$\therefore$ Real part of $\int_a^b F(t)\,dt$ is $\int_a^b u(t)\,dt$ and

Img part of $\int_a^b F(t)\,dt$ is $\int_a^b v(t)\,dt$

Note:-

1. $\int_a^b F(t)\,dt = -\int_b^a F(t)\,dt$

2. $\int_a^b F(t)\,dt = \int_a^c F(t)\,dt + \int_c^b F(t)\,dt \quad , \quad a \le c \le b$

3. $\int_a^b k\,F(t)\,dt = k\int_a^b F(t)\,dt$ where k is complex constant

4. $\left| \int_a^b F(t)\,dt \right| \le \int_a^b |F(t)|\,dt$

Line integral (or) complex integral :-

If $f(z)$ is any complex function and c is a curve joining from $z=a$ to $z=b$ then

$\int_c f(z)\,dz$ is called line integral (or) complex integral

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Real & Img Parts of Complex Integrals :-

W.K.T $z = x + iy$

$dz = dx + i\,dy$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

1. Find the value of $\int_C (x+y)\,dx + x^2y\,dy$ along the parabola $y = x^2$ having end points $(0,0)$ & $(3,9)$

Sol :- Given parabola $y = x^2$

$dy = 2x\,dx$

$$\therefore \int_C (x+y)\,dx + x^2y\,dy = \int_{x=0}^{3} (x+x^2)\,dx + x^4\,2x\,dx$$

$$= \int_0^3 (x + x^2 + 2x^5)\,dx$$

$$= \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{2x^6}{6} \right)_0^3$$

$$= \frac{9}{2} - 0 + \frac{27}{3} + \frac{1}{3}(729)$$

$$= \frac{513}{2}$$

2. Evalue $\int_0^{1+i} (x^2 - iy)\,dz$ along $y = x$ & $y = x^2$

Sol:- $z = x + iy$

$dz = dx + i\,dy$

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if $z=0 \Rightarrow x=0, y=0$

$z=1+i \Rightarrow x=1, y=1$

i) Along $y=x$:-

$$\int_0^{1+i}(x^2-iy)\,dz = \int_0^1 (x^2-ix)(dx+idy)$$

$y=x \Rightarrow dy=dx$

$$\int_0^{1+i}(x^2-iy)\,dz = \int_0^1 (x^2-ix)(dx+idx)$$

$$= \int_0^1 x^2\,dx + ix^2\,dx - ix\,dx + x\,dx$$

$$= \left[\frac{x^3}{3} + i\frac{x^3}{3} - i\frac{x^2}{2} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{3} + \frac{i}{3} + \frac{1}{2} - \frac{i}{2}$$

$$= \frac{5-i}{6}$$

ii) Along $y=x^2$

$y=x^2 \Rightarrow dy = 2x\,dx$

$$\int_0^{1+i}(x^2-iy)(dx+idy) = \int_0^1 (x^2-ix^2)(dx+i2x\,dx)$$

$$= \int_0^1 x^2\,dx + i2x^3\,dx - ix^2\,dx + 2x^3\,dx$$

$$= \left[\frac{x^3}{3} + i\frac{2x^4}{4} - \frac{ix^3}{3} + \frac{2x^4}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{i}{2} - \frac{i}{3} + \frac{1}{2} = \frac{5+i}{6}$$

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3. Evaluate $\int_C (y^2+z^2)dx + (x^2+z^2)dy + (x^2+y^2)dz$ from

$(0,0,0)$ to $(1,1,1)$ where c is a curve such that

$x=t$, $y=t^2$, $z=t^3$

Sol:- at $x=0$ $y=0$ $z=0$ $\Rightarrow$ $t=0$

$x=1$ $y=1$ $z=1$ $\Rightarrow$ $t=1$

also $x=t \Rightarrow dx = dt$

$y=t^2 \Rightarrow dy = 2t\,dt$

$z=t^3 \Rightarrow dz = 3t^2\,dt$

$$\therefore \int_C (y^2+z^2)dx + (x^2+z^2)dy + (x^2+y^2)dz$$

$$= \int_0^1 (t^4+t^6)dt + (t^2+t^6)2t\,dt + (t^2+t^4)3t^2dt$$

$$= \int_0^1 (t^4+t^6 + 2t^3+2t^7+3t^4+3t^6)dt$$

$$= \left(\frac{t^5}{5}+\frac{t^7}{7}+2\frac{t^4}{4}+2\frac{t^8}{8}+3\frac{t^5}{5}+3\frac{t^7}{7}\right)_0^1$$

$$= \frac{1}{5}+\frac{1}{7}+\frac{1}{2}+\frac{1}{4}+\frac{3}{5}+\frac{3}{7} = \frac{297}{140}$$

4. Evaluate $\int_C f(z)dz = \int_C (x^2+ixy)\,dz$ from $A(1,1)$ to $B(2,8)$ along i) st line AB

ii) Curve c : $x=t$, $y=t^3$

Sol :- $z=x+iy \Rightarrow dz = dx+i\,dy$

i) Along st. line AB:- The eqn of line passing through

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A(1,1) to B(2,8) is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\Rightarrow \frac{y-1}{8-1} = \frac{x-1}{2-1} \Rightarrow \frac{y-1}{7} = \frac{x-1}{1}$$

$$\Rightarrow y-1 = 7x - \cancel{7}$$

$$\Rightarrow y = 7x-6$$

$$\Rightarrow dy = 7dx$$

$$\therefore \int_C (x^2+ixy)\,dz = \int_C (x^2+ixy)(dx+i\,dy)$$

$$= \int_C (x^2+ix(7x-6)(dx+i7dx)$$

$$= \int_1^2 (x^2+i7x^2 - i6x + i7x^2 - 49x^2 + 42x)\,dx$$

$$= \int_1^2 -48x^2 dx + i\,14x^2 dx - i6x\,dx + 42x\,dx$$

$$= \left[-48\left(\frac{x^3}{3}\right) + i\,14\frac{x^3}{3} - i6\frac{x^2}{2} + 42\left(\frac{x^2}{2}\right)\right]_1^2$$

$$= -48\left[\frac{8}{3}-\frac{1}{3}\right] + i14\left[\frac{8}{3}-\frac{1}{3}\right] - i6\left[\frac{4}{2}-\frac{1}{2}\right] + 42\left[\frac{4}{2}-\frac{1}{2}\right]$$

$$= -48\left(\frac{7}{3}\right) + i14\left(\frac{7}{3}\right) - i6\left(\frac{3}{2}\right) + \overset{21}{\cancel{42}}\left(\frac{3}{\cancel{2}}\right)$$

$$= \left[-\frac{48}{3}+63\right] + i\left[\frac{98}{3}-\frac{18}{2}\right]$$

ii) Along the curve $x=t$ $y=t^3$

at $x=1$ $y=1 \Rightarrow t=1$

$x=2$ $y=8 \Rightarrow t=2$

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also $x=t \Rightarrow dx=dt$

$y=t^3 \Rightarrow dy = 3t^2dt$

$$\int_C (x^2+ixy)(dx+idy) = \int_1^2 (t^2+it^4)(dt+i3t^2dt)$$

$$= \int_1^2 (t^2 + i3t^4 + it^4 - 3t^6)dt$$

$$\cancel{= \int \frac{t^3}{3} + i3t^5}$$

$$= \int_1^2 t^2 dt + i\,4t^4dt - 3t^6 dt$$

$$= \left[\frac{t^3}{3} + i4\frac{t^5}{5} - \frac{3t^7}{7}\right]_1^2$$

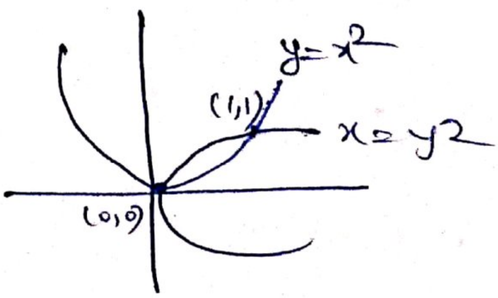
$$= \left(\frac{8}{3}-\frac{1}{3}\right) + i4\left(\frac{32}{5}-\frac{1}{5}\right) - \left[\frac{384}{7}-\frac{3}{7}\right]$$

$$= \frac{7}{3} + \frac{124i}{5} - \frac{381}{7} = \frac{-1094}{21} + \frac{124i}{5}$$

5. Evaluate $\int_C (y^2+2xy)dx + (x^2-2xy)dy$ where C is boundary of region along $y=x^2$ & $x=y^2$

Sol:- Given curves $y=x^2$ and $x=y^2$

The two curves are intersecting at $(0,0)$ $(1,1)$



Along $y=x^2$:-

$dy = 2x\,dx$

$$\int_C (y^2+2xy)dx + (x^2-2xy)dy$$

$$= \int_0^1 x^4+2x^3 + (x^2-2x^3)2x\,dx$$

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$= \int_0^1 3x^4 + 4x^3 \, dx$

$= \left(-3\frac{x^5}{5} + 4\frac{x^4}{4}\right)_0^1 = \left(\frac{-3}{5} + 1\right) = \frac{2}{5}$

Along $x = y^2$:-

$dx = 2y \, dy$

$\int_C (y^2 + 2xy) dx + (x^2 - 2xy) dy = \int_1^0 (y^2 + 2y^3) 2y \, dy + (y^4 - 2y^3) dy$

$= \int_1^0 2y^3 + 4y^4 + y^4 - 2y^3 \, dy$

$= \int_1^0 5y^4 \, dy = 5\left(\frac{y^5}{5}\right)_1^0 = -1$

6. Evaluate $\int_0^{1+i} (x - y^2 + ix^3) \, dz$ along real axis from $z = 0$ to $z = 1$.

Sol :- $z = x + iy$

Along real axis $y = 0$

$\therefore z = x$

$\Rightarrow dz = dx$

Also If $z = 0 \rightarrow x = 0 \quad y = 0$

$z = 1 + i \Rightarrow x = 1 \quad y = 1$

$\therefore \int_0^{1+i} (x - y^2 + ix^3) \, dz = \int_0^1 (x + ix^3) \, dx$

$= \left(\frac{x^2}{2} + i\frac{x^4}{4}\right)_0^1 = \frac{1}{2} + \frac{i}{4} = \frac{2 + i}{4}$

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7. Evaluate $\int_{1-i}^{2+i}(2x+iy+1)\,dz$ along st line joining

$(1-i)$ & $(2+i)$

Sol :- If $z = 1-i \Rightarrow x=1 \; y=-1$

If $z = 2+i \Rightarrow x=2 \; y=1$

The Eqn of st line joining $(1-i), (2+i)$ is

$$\frac{y+1}{2} = \frac{x-1}{1} \Rightarrow y+1 = 2x-2$$

$\Rightarrow y = 2x-3$

$dy = 2dx$

$$\int_{1-i}^{2+i}(2x+iy+1)\,dz = \int_{1}^{2}(2x+i(2x-3)+1)(dx+i2dx)$$

$$= \int_{1}^{2}(2x+2ix-i3+1dx + 4ix\,dx - 4x+6+2i)dx$$

$$= \int_{1}^{2} -2x+6ix-i+7\,dx$$

$$= \left[-\cancel{x}\frac{x^2}{\cancel{x}} + \overset{3}{\cancel{6}}i\frac{x^2}{\cancel{2}} - ix + 7x\right]_1^2$$

$= -4+12i-2i+14-(-1+3i-i+7)$

$= -4+12i+14+1-3i+i-7$

$= -4+8i+8$

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Cauchy Integral Theorem :-

statement :- If $f(z)$ is analytic in the region R of simple closed curve C and $f'(z)$ is continous in the simple closed curve then $\int_C f(z)\,dz = 0$

(or)

~~Proof~~ :-

If $f(z)$ is analytic function and $f'(z)$ is continous at each point with in a simple closed curve C then $\int_C f(z)\,dz = 0$

Proof :- Let R be the region bounded by the curve 'C'

and let $f(z) = u(x,y) + iv(x,y)$

then $\int_C f(z)\,dz = \int_C (u+iv)\,dx+idy$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy \rightarrow (1)$$

since $f'(z)$ is continous, the partial derivativatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continous

By Greens theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\therefore \int_C u\,dx - v\,dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy \rightarrow (2)$$

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Ily $\int_C v\,dx + u\,dy = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \rightarrow (3)$

substitute (2) & (3) in (1)

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

since $f(z)$ is analytic at each point in the closed curve 'C' we have

$u_x = v_y$ & $u_y = -v_x$

$$\therefore \int_C f(z)\,dz = \iint_R \left(\cancel{\frac{\partial u}{\partial y}} - \cancel{\frac{\partial u}{\partial y}}\right) dx\,dy + i \iint_R \left(\cancel{\frac{\partial v}{\partial y}} - \cancel{\frac{\partial v}{\partial y}}\right) dx\,dy$$

$$\therefore \int_C f(z)\,dz = 0$$

Hence proved

This theorem is also known as Cauchy's - Goursat theorem.

Cauchy's Integral theorem for multiple Connected regions (Generalised Cauchy's Integral ~~formula~~ Theorem)

Statement:- If $f(z)$ is analytic in region between the two curves C_1 & C_2 then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

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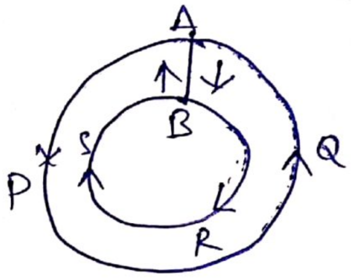
(or)

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

Proof:-

If $f(z)$ is analytic in the region R then by Cauchy's Integral theorem we have $\int_C f(z)\,dz = 0$

make a cross cut AB ~~thus making~~ to connect the curves C_1 & C_2 ~~a double~~ connected in counter clockwise direction ~~region~~ into ~~simple connected one~~



let the simple connected region as APQ ABRSBA

Apply Cauchy's integral them then

$$\int_{APQABRSBA} f(z)\,dz = 0 \rightarrow (1)$$

$$\int_{APQA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BRSB} f(z)\,dz + \int_{BA} f(z)\,dz = 0$$

$$\therefore \int_{C_1} f(z)\,dz + \cancel{\int_{AB} f(z)\,dz} + \int_{C_2} f(z)\,dz - \cancel{\int_{AB} f(z)\,dz} = 0$$

$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0 \rightarrow (2)$$

Since the two curves traversed in opposite direction we have $\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$

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$$\boxed{\therefore \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz}$$

Also from (1) & (2)

$$\boxed{\int_{C} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz}$$

The above result has capability of extension if there is a closed curve C has the intersection of curves $C_1\ C_2 \dots C_n$

i.e., $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \dots + \int_{C_n} f(z)\,dz$

Morera's theorem :-

Statement :- If $f(z) = u+iv$ is continous in a simple connected region 'R' and u, v has continous partial derivatives and $\int_C f(z)\,dz = 0$ along closed curve 'C' in the region 'R' then f(z) is analytic.

Proof :- Let R be the region bounded by the curve 'C'

Consider $F(z) = A + iB = \int_C f(z)\,dz$

w.k.T $f(z) = u+iv$

then $\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i \int_C v\,dx + u\,dy$

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$\therefore A = \int_C u\,dx - v\,dy \qquad B = \int_V v\,dx + u\,dy$

$\frac{\partial A}{\partial x} = u \quad \frac{\partial A}{\partial y} = -v \quad \frac{\partial B}{\partial x} = v \quad \frac{\partial B}{\partial y} = u$

$A_x = B_y$ & $A_y = -B_x$

C-R Eqns are satisfied

$\therefore F(z) = A + iB$ is analytic

$\Rightarrow F'(z) = A_x + iB_x = u + iv = f(z)$ is also analytic

$\Rightarrow f(z)$ is analytic

hence the theorem

Cauchy Integral Formula:-

Statement:- If $f(z)$ is analytic in a closed curve 'c' and a be the point within 'c' then

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a}\,dz$$

Proof:- Consider the function $\phi(z) = \frac{f(z)}{z-a}$ which is analytic at all points except at $z=a$.

Draw a small circle C_1 lying inside c with centre 'a' and radius 'r'

The function $\phi(z)$ is analytic in the region enclosed by c and C_1

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then by Cauchy Integral theorem for multiple connected region

$$\int_C \phi(z)\,dz = \int_{C_1} \phi(z)\,dz \rightarrow (1)$$

Also C_1 is a circle ~~with centre~~ $|z-a| = r$

let $z-a = re^{i\theta}$

$z = a + re^{i\theta}$ where ~~θ lies~~ $0 < \theta < 2\pi$

$dz = ire^{i\theta}d\theta$.

$$\int_C \phi(z)\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz$$

$$\int_{C_1} \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{r\cancel{e^{i\theta}}}\, ir\cancel{e^{i\theta}}\,d\theta$$

as $r \rightarrow 0$ we have

$$\int_{C_1} \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} i f(a)\,d\theta$$

$$= i f(a)\,(\theta)_0^{2\pi}$$

$$= i f(a)\,(2\pi)$$

$$= 2\pi i f(a)$$

$$\therefore \int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \quad \left(\text{from (1) } \int_{C_1} = \int_C\right)$$

$$\Rightarrow f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\,dz$$

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Generalised Cauchy Integral formula :-

If $f(z)$ is analytic function in a simple closed curve C and 'a' is any point within 'C' then

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Proof :- From Cauchy Integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz \rightarrow (1)$$

differentiate Eqn (1) w.r.to a

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{-f(z)}{(z-a)^2}(-1)\,dz$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

again diff w.r.to a

$$f''(a) = \frac{1}{2\pi i}\int_C \frac{-2f(z)}{(z-a)^3}(-1)\,dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

$$\text{Ily}\quad \boxed{f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

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1. Evaluate $\int_C \frac{z^2-z+1}{z-1}\,dz$ where c is a circle $|z|=1$

Sol:- W.K.T C.I.F

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$\therefore$ $f(z) = z^2 - z + 1$ and $a = 1$

$\Rightarrow f(1) = 1$

$$\therefore \int_C \frac{z^2-z+1}{z-1}\,dz = 2\pi i\, f(1)$$

$$= 2\pi i$$

2. Prove that $\int_C \frac{dz}{z-a} = 2\pi i$ where c is circle $|z-a| = r$

Sol:- Given $|z-a| = r$ is a circle with centre 'a' and radius 'r'

let $z - a = re^{i\theta}$, $\theta \in (0, 2\pi)$

$$z = a + re^{i\theta}$$

$$dz = ire^{i\theta}\,d\theta$$

$$\therefore \int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{i\,\cancel{re^{i\theta}}}{\cancel{re^{i\theta}}}\,d\theta = i\,(2\pi) = 2\pi i$$

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3. Evaluate $\oint_C \frac{z^2+4}{z-3}\,dz$ where c is circle i) $|z|=5$
ii) $|z|=2$

Sol:- Given $f(z) = z^2+4$ & $a=3$

i) $|z|=5$ is a circle with centre 0 and radius 5

By C.I.F

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{z^2+4}{z-3}\,dz = 2\pi i\, f(3)$$

$$\int_C \frac{z^2+4}{z-3}\,dz = 2\pi i\,(13) = 26\pi i$$

ii) $|z|=2$ is a circle with centre 0 and radius 2 centre

here $a=3$ lies outside the circle

$$\therefore \int_C \frac{z^2+4}{z-3}\,dz = 0$$

4. Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$ where c is the circle $|z|=3$

Sol:- $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

by solving $A=-1$ $B=1$

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$$\frac{1}{(z-2)(z-1)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = \int_C -\frac{e^{+2z}}{(z-1)}\,dz + \int_C \frac{e^{2z}}{(z-2)}\,dz$$

$$= \int_C \left(\frac{e^{2z}}{(z-2)} - \frac{e^{+2z}}{z-1}\right) dz \rightarrow (1)$$

Now $\int_C \frac{e^{2z}}{(z-2)}\,dz = 2\pi i\, f(2)$ [By C.I.F]

$f(z) = e^{2z} \Rightarrow f(2) = e^4$

$\therefore \int_C \frac{e^{2z}}{(z-2)}\,dz = 2\pi i\, e^4 \rightarrow (2)$

Now $\int_C \frac{e^{+2z}}{z-1}\,dz = 2\pi i\, f(1)$

$f(z) = e^{+2z} \Rightarrow f(1) = e^2$

$\int_C \frac{e^{2z}}{z-1}\,dz = 2\pi i\, e^2 \rightarrow (3)$

substitute (2) & (3) in eqn (1)

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = 2\pi i\,[e^4 - e^2]$$ //

5. Evaluate $\int_C \frac{e^{2z}}{(z+1)^2}\,dz$ where C is circle $|z|=2$

Sol:- Given $f(z) = e^{2z}$, $a = -1$

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also $|z|=2$ is circle with centre O and radius 2 units

$\therefore a=-1$ lies inside the circle

By Generalise C.I.F

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

$$\int_C \frac{e^{2z}}{(z+1)^2} dz = \frac{2\pi i}{n!} f'(a)$$

$$= \frac{2\pi i}{1!} f'(-1)$$

$f(z) = e^{2z}$

$\Rightarrow f'(z) = 2e^{2z}$

$\Rightarrow f'(-1) = 2e^{-2}$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^2} dz = 2\pi i\,(2e^{-2}) = 4\pi i e^{-2}$$

6. If $F(a) = \int_C \frac{3z^2+7z+1}{z-a} dz$ where C is circle $|z|=2$

find the values of $F(1)$, $F(3)$, $F'(1-i)$ $F''(1-i)$.

Sol:- $F(a) = \int_C \frac{3z^2+7z+1}{(z-a)} dz$

by C.I.F $\int_C \frac{3z^2+7z+1}{z-a} dz = 2\pi i\, f(a)$

$F(a) =$

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$$F(a) = \int_C \frac{3z^2+7z+1}{(z-a)} dz = 2\pi i\, f(3a^2+7a+1)$$

$$F(1) = 2\pi i\,(11) = 22\pi i$$

$$F(3) = 2\pi i\,(0) = 0 \quad (\because a=3 \text{ lies outside } |z|=2)$$

WKT $f(z) = 3z^2+7z+1$ $\Rightarrow$ $f'(a) = 6a+7$

$f'(z) = 6z+7$ $\Rightarrow$ $f''(a) = 6$

$f''(z) = 6$

$$F'(a) = 2\pi i\, f'(a) = 2\pi i\,(6a+7)$$

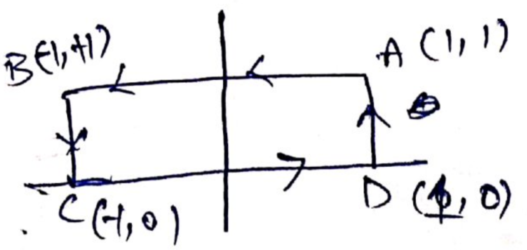
$$\therefore F'(1-i) = 2\pi i\,(6(1-i)+7) = 2\pi i\,[6-6i+7]$$
$$= 26\pi i + 12\pi$$

$$F''(a) = 2\pi i\, f''(a)$$

$$F''(1-i) = 2\pi i\,(6) = 12\pi i$$

7. verify Cauchy's theorem for integral of z^3 taken over the boundary of rectangle with vertices $1+i, -1+i, -1, 1$.

Sol:- Given $f(z) = z^3$



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The boundary of rectangle consists of four curves C_1, C_2, C_3, C_4

$$\int_C z^3\,dz = \int_{C_1} z^3 dz + \int_{C_2} z^3\,dz + \int_{C_3} z^3 dz + \int_{C_4} z^3 dz$$

Along C_1 :- $A(1,1)$ $B(-1,1)$

$y=1$, x varies from 1 to -1

$dy=0$

$dz = dx + i\,dy = dx$

$$\int_{C_1} z^3 dz = \int_1^{-1} (x+iy)^3\,dx = \int_{+1}^{-1} (x+i)^3 dx$$

$$= \int_{+1}^{-1} (x^3+3x^2 i + 3x i^2 + i^3)\,dx = \int_{+1}^{-1} (x^3+3x^2 i - 3x - i)\,dx$$

$$= \left(\frac{x^4}{4} + \cancel{3} i \frac{x^3}{\cancel{3}} - 3\frac{x^2}{2} - ix \right)_1^{-1} = \left[\frac{1}{4} - i - \frac{3}{2} + i\right] - \left(\frac{1}{4} + i - \frac{3}{2} - i\right)$$

$$= 0$$

Along C_2 :- $B(-1,+1)$, $C(-1,0)$

$\therefore x = -1$ y varies from 1 to 0

$dx = 0$

$\therefore dz = i\,dy$

$$\int_{C_2} z^3 dz = \int_1^0 (x+iy)^3\, i\,dy$$

$$= \int_1^0 (-1+iy)^3\, i\,dy$$

$$= i\int_1^0 \left[(-1)^3 + (iy)^3 + 3iy + 3(-1)(iy)^2\right] dy$$

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$$= i\int_1^0 (-1 - iy^3 + 3iy + 3y^2)\, dy$$

$$= i\left[-y - i\frac{y^4}{4} + 3i\frac{y^2}{2} + \frac{3y^3}{3}\right]_1^0$$

$$= i\left[0 - (\cancel{1} - \frac{i}{4} + \frac{3i}{2} + \cancel{1})\right] = -\frac{1}{4} + \frac{3}{2} = 5/4$$

Along C_3 :- $C(-1,0)$ $D(1,0)$

$y = 0$ x varies from -1 to 1

$dy = 0$

$\Rightarrow dz = dx$

$$\int_{C_3} z^3\, dx = \int_{-1}^{1} (x+iy)^3\, dx = \int_{-1}^{1} x^3\, dx = \left(\frac{x^4}{4}\right)_{-1}^{1} = 0$$

Along C_4 :- $D(1,0)$ $A(1,1)$

$x = 1$ $y \to 0$ to 1

$dx = 0$

$\Rightarrow dz = i\, dy$

$$\int_{C_4} z^3\, dz = \int_0^1 (x+iy)^3\, i\, dy$$

$$= i\int_0^1 (1+iy)^3\, dy = i\int_0^1 \left[(1 - iy^3 + 3iy - 3y^2)\right] dy$$

$$= i\left[y - i\frac{y^4}{4} + 3i\frac{y^2}{2} - \cancel{3}\frac{y^3}{\cancel{3}}\right]_0^1$$

$$= i\left[\cancel{1} - \frac{i}{4} + \frac{3i}{2} - \cancel{1}\right] = \frac{1}{4} - \frac{3}{2} = -5/4$$

$\therefore$ from ① $\int_C z^3\, dz = 0 + \cancel{5/4} + 0 - \cancel{5/4} = 0$

$\therefore$ The Cauchy's Integral theorem is verified

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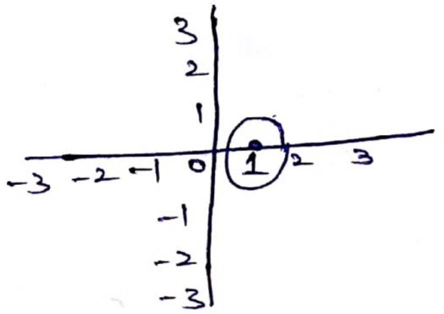
8. Evaluate $\int_C \frac{z^3 e^{-z}}{(z-1)^3}\, dz$ where c is circle $|z-1| = 1/2$

Sol :- $f(z) = z^3 e^{-z}$, $a=1$, $n=2$

By generalised Cauchy's Integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{n}(a)$$



$$\int_C \frac{z^3 e^{-z}}{(z-1)^3}\, dz = \frac{\cancel{2}\pi i}{\cancel{2}!} f''(1)$$

$$= \pi i f''(1)$$

$$f(z) = z^3 e^{-z}$$

$$f'(z) = -z^3 e^{-z} + 3z^2 e^{-z}$$

$$f''(z) = -3z^2 e^{-z} + z^3 e^{-z} + 6ze^{-z} - 3z^2 e^{-z}$$

$$= -6z^2 e^{-z} + z^3 e^{-z} + 6ze^{-z}$$

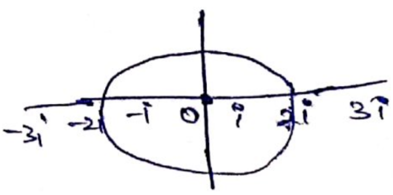
$$f''(1) = -6\cancel{e^{-1}} + \cancel{6e^{-1}} + e^{-1} = e^{-1}$$

$$\therefore \int_C \frac{z^3 e^{-z}}{(z-1)^3}\, dz = e^{-1}\pi i \;//$$

9. Evaluate $\int_C \frac{\cos z - \sin z}{(z+i)^3}\, dz$ wher c is circle $|z| = 2$.

Sol :- $f(z) = \cos z - \sin z$ $\quad a = -i$

$a = i$ which lies inside the circle



also $n=2$

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By Generalised Cauchy's Integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(a)$$

$$\int_C \frac{\cos z - \sin z}{(z+i)^3}\,dz = \frac{2\pi i}{2!} f''(-i)$$

$f(z) = \cos z - \sin z$

$f'(z) = -\sin z - \cos z$

$f''(z) = -\cos z + \sin z$

$f''(-i) = -\cos(-i) + \sin(-i)$

$= -\cos i \mp \sin i$

$$\therefore \int_C \frac{\cos z - \sin z}{(z+i)^3}\,dz = -\pi i(\cos i + \sin i)$$

10. Evaluate $\int_C \frac{z^3 - \sin 3z}{(z-\frac{\pi}{2})^3}\,dz$ where c is circle $|z-\frac{\pi}{2}| = 2$

Sol :- here $f(z) = z^3 - \sin 3z$ $\quad a = \pi/2 \quad n = 2$

By Generalised Cauchy's Integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(a)$$

$$\int_C \frac{z^3 - \sin 3z}{(z-\frac{\pi}{2})^3}\,dz = \frac{2\pi i}{2!} f''(\tfrac{\pi}{2})$$

$f(z) = z^3 - \sin 3z$

$f'(z) = 3z^2 - 3\cos 3z$

$f''(z) = 6z + 9\sin 3z$

$f''(\frac{\pi}{2}) = 6\frac{\pi}{2} + 9\sin\frac{3\pi}{2} = 3\pi + 9\sin(\frac{\pi}{2} + \pi)$

$= 3\pi + 9\cos\pi = 3\pi - 9$

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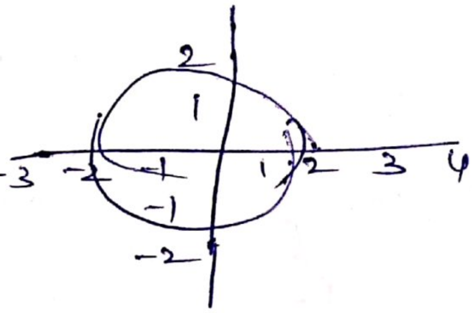
$$\int_C \frac{z^3 \sin 3z}{(z-\pi/2)^3}\, dz = \pi i (3\pi - 9)$$
$$= 3\pi i (\pi - 3)$$

11. Evaluate $\int_C \frac{dz}{z^3(z+4)}$ where C is circle $|z|=2$.

Sol :- Given function has two singular points i.e., $z=0$ & $z=-4$. here $z=0$ lies inside c and $z=-4$ lies outside c.



$$\int_C \frac{dz}{z^3(z+4)} = \int_C \frac{\frac{1}{z+4}}{z^3}\, dz$$

$f(z) = \frac{1}{z+4}$, $a=0$, $n=2$

By Generalised CIF

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^n(a)$$

$$\int_C \frac{\frac{1}{z+4}}{z^3}\, dz = \int_C \frac{1}{z^3(z+4)}\, dz = \frac{\cancel{2}\pi i}{\cancel{2}!} f''(0)$$

$$f(z) = \frac{1}{z+4}$$

$$f'(z) = \frac{-1}{(z+4)^2}$$

$$f''(z) = \frac{2}{(z+4)^3} \Rightarrow f''(0) = \frac{2}{64} = \frac{1}{32}$$

$$\therefore \int_C \frac{1}{z^3(z+4)}\, dz = \frac{\pi i}{32}$$

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12. Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where c is circle

i) $|z|=1$ ii) $|z+1-i|=2$ iii) $|z+1+i|=2$

Sol:- $\int_C \frac{z+4}{z^2+2z+5}\,dz = \int_C \frac{(z+4)}{(z+1)^2-(2i)^2}\,dz = \int_C \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz$

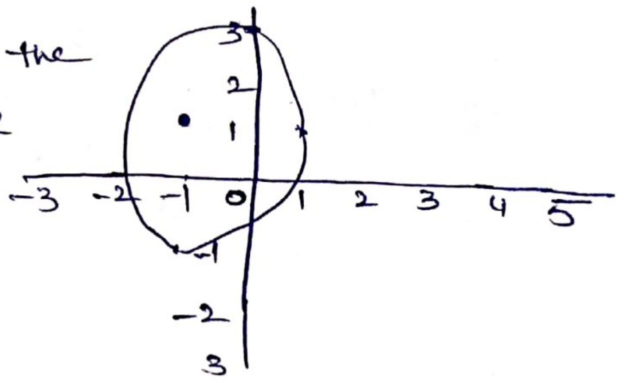
The function has two singular point $z=-1-2i, -1+2i$

i) c is a circle $|z|=1$

clearly the point $(-1,-2i)$ & $(-1+2i)$ lies outside the circle $\therefore \int_C \frac{z+4}{z^2+2z+5}\,dz = 0$

ii) c is a circle with centre $-1+i$ and radius 2 units

Here the point $-1-2i$ lies outside the circle & $-1+2i$ lies inside the circle



$\therefore \int_C \frac{z+4}{z^2+2z+5}\,dz = \int_C \frac{\frac{z+4}{z+1+2i}}{z+1-2i}\,dz$

$f(z) = \frac{z+4}{z+1+2i}$ $a = -1+2i$

$\therefore$ By C.I.F

$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$

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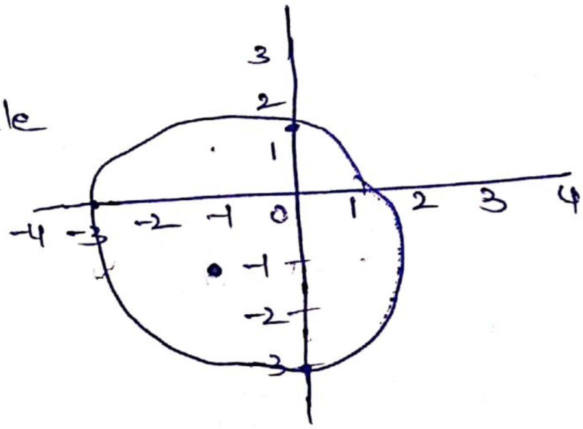
$$f(a) = \frac{a+4}{a+1+2i} \Rightarrow f(-1+2i) = \frac{-1+2i+4}{\cancel{-1}+2i+\cancel{1}+2i} = \frac{3+2i}{4i}$$

$$\therefore \int_C \frac{z+4}{z^2+2z+5} dz = 2\pi i \, f(a)$$

$$= \cancel{2}\pi \cancel{i} \left[\frac{3+2i}{\cancel{4}_2 \cancel{i}} \right] = \frac{\pi}{2}(3+2i)$$

iii) $|z+1+i| = 2$ is a circle with centre $(-1-i)$ and radius 2 units

$\therefore$ The points $-1-2i$ lies inside the circle & $-1+2i$ lies outside the circle



$\therefore$ the function can be written as

$$\int_C \frac{\frac{z+4}{z+1-2i}}{z+1+2i} dz$$

here $f(z) = \frac{z+4}{z+1-2i}$ $\quad a = -1-2i$

$$f(-1-2i) = \frac{-1-2i+4}{\cancel{-1}-2i+1-2i} = \frac{3-2i}{-4i}$$

By C.I.F

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i \, f(a)$$

$$\int_C \frac{z+4}{z^2+2z+5} dz = 2\pi i \, f(-1-2i)$$

$$= 2\pi \cancel{i} \left(\frac{+3-2i}{-4\cancel{i}} \right) = \frac{-\pi}{2}(3-2i)$$

$$= \frac{\pi}{2}(2i-3)$$

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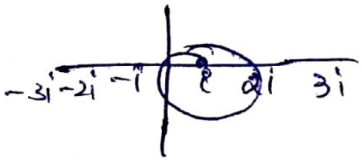
13. Evaluate $\int_C \frac{z^2-1}{z^2+1} dz$ $\quad C : |z-i|=1$

Sol :- $\int_C \frac{z^2-1}{z^2+1} dz = \int_C \frac{z^2-1}{(z+i)(z-i)} dz$

the function has two singular points $z=-i, i$

$z=-i$ lies outside the circle

$z=i$ lies inside the circle



$$\int_C \frac{z^2+1}{(z+i)(z-i)} dz = \int_C \frac{\frac{z^2-1}{(z+i)}}{z-i} dz$$

$f(z) = \frac{z^2-1}{z+i}$, $a=i$

$f(i) = \frac{i^2-1}{i+i} = \frac{-2}{2i} = i$

By C.I.F $\quad \int_C \frac{f(z)}{z-a} dz = 2\pi i f(a) = 2\pi i (i) = -2\pi$

14. Evaluate $\int_C \frac{\log z}{(z-1)^3} dz$ where C is $|z-1| = 1/2$

using CIF.

Sol :- By Generalised CIF

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

$f(z) = \log z \quad a=1$

$f'(z) = \frac{1}{z}$

$f''(z) = \frac{-1}{z^2} \Rightarrow f''(1) = -1$

$$\therefore \int_C \frac{\log z}{(z-1)^3} dz = \frac{\cancel{2}\pi i}{\cancel{2}!} f''(1) = \pi i (-1) = -\pi i$$

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Complex Power Series

Let $\Sigma z_n = (a_1+ib_1)+(a_2+ib_2)+\dots+(a_n+ib_n)+\dots$ be an infinite series where $a_1, a_2 \dots a_n$ are real numbers series and $b_1, b_2 \dots b_n$ are series of imaginary numbers.

The series Σz_n is convergent if the both real and imaginary are convergent and Σz_n is absolutely convergent if $\Sigma |z_n|$ is convergent.

Power series :- A series of the form $\Sigma a_n z^n$ is called power series.

If $\Sigma a_n z^n$ converges at $z = z_1$ then it converges absolutely for all z such that $|z| < |z_1|$.

Region of Convergence :- It is the set of all points for which the series convergent.

The power series converges for all z such that $|z| < R$ and diverges at $|z| > R$, where R is the radius of convergence.

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Taylor's theorem :-

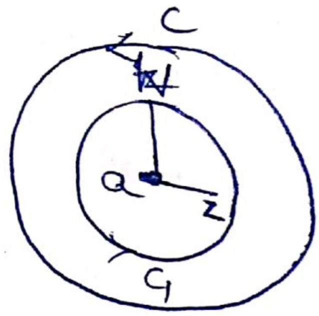
If $f(z)$ is analytic inside the circle 'c' whose centre is 'a' then for all 'z'

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a) + \cdots\cdots$$

This is called the Taylor's series about the point $z=a$

Proof:- Let z be any point inside the circle 'c' with centre 'a'

draw a small circle C_1 with 'a' as centre and let w be any point on the circle C_1.



then $|z-a| < |w-a|$

Consider $\frac{1}{w-z} = \frac{1}{w-a+a-z}$

$$= \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{(w-a)\left[1-\frac{z-a}{w-a}\right]}$$

$$= \frac{1}{w-a}\left[1-\left(\frac{z-a}{w-a}\right)\right]^{-1}$$

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^n+\cdots\right]$$

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Now integrating over C_1 and multiply with $\frac{f(w)}{2\pi i}$ on both sides we get

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-a}\,dw + \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^2}(z-a)\,d$$

$$+ \cdots + \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}}(z-a)^n \longrightarrow (1)$$

$\because$ $f(w)$ is analytic on circle C_1 by Cauchy's integral formula we have

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-a}\,dw = f(a)$$

$$f'(a) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^2}\,dw$$

$$f''(a) = \frac{2!}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^3}\,dw$$

$$\text{lly } f^n(a) = \frac{n!}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw$$

Substitute all these values in Eqn (1)

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a) + \cdots$$

$=$

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1. Expand Taylor's series for the function $f(z) = \cos z$ at $z = \pi/4$.

Sol:- Given $f(z) = \cos z$

$$a = \pi/4$$

wkt Taylor's series about the point $z=a$ is

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots\cdots$$

$$\cos z = f(\pi/4) + (z-\pi/4) f'(\pi/4) + \frac{(z-\pi/4)^2}{2!} f''(\pi/4) + \cdots\cdots$$

$$f(z) = \cos z \Rightarrow f(\pi/4) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'(z) = -\sin z \Rightarrow f'(\pi/4) = -\sin(\pi/4) = -1/\sqrt{2}$$

$$f''(z) = -\cos z \Rightarrow f''(\pi/4) = -\cos(\pi/4) = -1/\sqrt{2}$$

$$\therefore \cos z = \frac{1}{\sqrt{2}} + (z-\pi/4)\left(\frac{-1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \cdots\cdots$$

$$\cos z = \frac{1}{\sqrt{2}}\left[1 - \left(z-\frac{\pi}{4}\right) - \frac{(z-\pi/4)^2}{2!} + \cdots\cdots\right]$$

2. obtain the Taylor's series for $f(z) = \frac{1}{(z+1)^2}$ with centre at $-i$

Sol:- $f(z) = \frac{1}{(z+1)^2}$

$a = -i$

If $|z| < 1$ then

1) $(1-z)^{-1} = 1+z+z^2+\cdots\cdots$

2) $(1+z)^{-1} = 1-z+z^2-\cdots$

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W.K.T Taylor series about $z=a$ is

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots \rightarrow (1)$$

$$f(z) = \frac{1}{(z+1)^2} \Rightarrow f(-i) = \frac{1}{(1-i)^2}$$

$$f'(z) = \frac{-2}{(z+1)^3} \Rightarrow f'(-i) = \frac{-2}{(1-i)^3}$$

$$f''(z) = \frac{6}{(z+1)^4} \Rightarrow f''(-i) = \frac{6}{(1-i)^4}$$

$$\therefore \frac{1}{(z+1)^2} = \frac{1}{(1-i)^2} + (z+i)\left(\frac{-2}{(1-i)^3}\right) + \frac{(z+i)^2}{2!}\left(\frac{6}{(1-i)^4}\right) + \cdots$$

$$\frac{1}{(z+1)^2} = \frac{1}{(1-i)^2}\left[1 - \frac{2(z+i)}{1-i} + \frac{3(z+i)^2}{(1-i)^2} + \cdots\right]$$

Special Case :- If we put $a=0$ in Taylor's series we get the Maclaurin's series about the point $z=0$ as

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \cdots$$

3. obtain Taylor's series for $f(z) = \frac{z-1}{z+1}$ at $z=0$.

Sol :- Given $f(z) = \frac{z-1}{z+1}$

W.K.T the Taylor's series about $z=a$ is

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

$$\frac{z-1}{z+1} = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \cdots$$

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$f(z) = \frac{z-1}{z+1} \Rightarrow f(0) = -1$

$$f'(a) = \frac{(z+1)(1) - (z-1)(1)}{(z+1)^2} \Rightarrow f'(0) = 2$$

$$f''(a) = \frac{-4}{(z+1)^3} \Rightarrow f''(0) = -4$$

$$\therefore \frac{z-1}{z+1} = (-1) + z(2) + \frac{z^2}{2!}(-4) + \dots$$

$$= -1 + 2z - 2z^2 + \dots$$

$=$

4. obtain Taylor's series for $f(z) = \frac{1}{(z-1)(z-2)}$ in powers of z

Sol :- $f(z) = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{2(\frac{z}{2}-1)} - \frac{1}{-(1-z)}$$

$$= \frac{1}{2(\frac{z}{2}-1)} + \frac{1}{1-z}$$

$$= -\frac{1}{2}(1-\frac{z}{2})^{-1} + (1-z)^{-1}$$

$$= -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \dots\right] + \left[1 + z + z^2 + \dots\right]$$

$$\therefore f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{2} + \frac{3z}{4} + \frac{7}{8}z^2 + \dots$$

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5. Obtain expansion of $\frac{1}{(z-1)(z-3)}$ in Taylor's series in powers of $(z-4)$

Sol:- Given $f(z) = \frac{1}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$

$$= \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(z-4)+1} - \frac{1}{(z-4)+3}\right]$$

$$= \frac{1}{2}\left[(1+(z-4))^{-1} - \frac{1}{\left[3\left(\frac{z-4}{3}+1\right)\right]}\right]$$

$$= \frac{1}{2}(1+(z-4))^{-1} - \frac{1}{6}\left(1+\left(\frac{z-4}{3}\right)\right)^{-1}$$

$$= \frac{1}{2}\left[1-(z-4)+(z-4)^2+\cdots\right] \cancel{\frac{1}{6}\left[1-\frac{(z-4)}{3}+\frac{(z-4)^3}{4}\right]}$$
$$-\frac{1}{6}\left[1-\left(\frac{z-4}{3}\right)+\left(\frac{z-4}{3}\right)^2-\cdots\right]$$

6. Obtain the Taylor's series for

1) $f(z) = e^{1+z}$ at $z=1$

2) $f(z) = \log(1-z)$ at $z=0$

3) $f(z) = \frac{1}{z}$ at $z=1$

Sol:- i) $f(z) = e^{1+z}$, $a=1$

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots\cdots$$

$$e^{1+z} = f(1) + (z-1)f'(1) + \frac{(z-1)^2}{2!}f''(1) + \cdots\cdots$$

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$$f(z) = e^{1+z} \Rightarrow f(1) = e^2$$

$$f'(z) = e^{1+z}(1) \Rightarrow f'(1) = e^2$$

$$f''(z) = e^{1+z}(1) \Rightarrow f''(1) = e^2$$

$$e^{1+z} = e^2 + (z-1)e^2 + \frac{(z-1)^2}{2!}e^2 + \cdots\cdots$$

$$e^{1+z} = e^2\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\right]$$

ii) $f(z) = \log(1-z)$, $a = 0$

$$f(z) = \log(1-z) \Rightarrow f(0) = \log 1 = 0$$

$$f'(z) = \frac{1}{1-z}(-1) \Rightarrow f'(0) = -1$$

$$f''(z) = \frac{-1}{(1-z)^2}(-1) \Rightarrow f''(0) = +1$$

$$\therefore f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots\cdots$$

$$\log(1-z) = f(0) + z f'(0) + \frac{z^2}{2!}f''(0) + \cdots\cdots$$

$$\log(1-z) = 0 + z(-1) + \frac{z^2}{2!}(+1) + \cdots\cdots$$

$$\log(1-z) = -\left[z - \frac{z^2}{2!} + \cdots\cdots\right]$$

iii) $f(z) = \frac{1}{z}$, $a = 1$

$$f(z) = \frac{1}{z} \Rightarrow f(1) = 1$$

$$f'(z) = -\frac{1}{z^2} \Rightarrow f''(1) = 1$$

$$f''(z) = \frac{2}{z^3} \Rightarrow f''(1) = 2$$

$$\therefore \frac{1}{z} = 1 + (z-1)(-1) + \frac{(z-1)^2}{2!}(2) + \cdots\cdots$$

$$= 1 - (z-1) + 2\frac{(z-1)^2}{2!} + \cdots\cdots$$

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Note:- If $f(z)$ is analytic inside the region bounded by two curves C_1 & C_2 and a is any point then $f(a) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-a}dz$

This is known as Cauchy's Integral formula for double connected regions.

State & prove Laurent's theorem:-

If $f(z)$ is analytic inside and on the boundary of region R, the boundary is connected of two circles C_1 & C_2 of radii r_1 and r_2 where $r_1 > r_2$ respectively having centre 'a' for all z in the region R

then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$

where $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw \qquad n=0,1,2\ldots\ldots$

$a_{-n} = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw \qquad n=1,2,\ldots\ldots$

Proof:- Let z be any point in the region R bounded by two circles C_1 & C_2 then

By Cauchy integral formula for double connected regions we have

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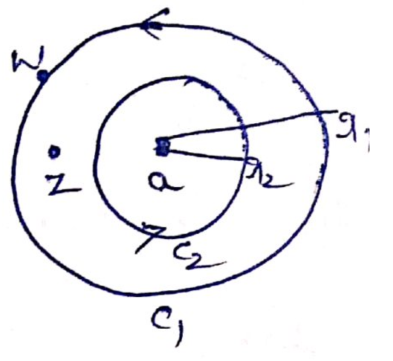
$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w-z}\, dw \rightarrow (1)$$

Case 1 :-

Let w be the point lies on circle C_1



$|z-a| < |w-a| \Rightarrow \left|\frac{z-a}{w-a}\right| < 1$

Consider $\frac{1}{w-z} = \frac{1}{w-a+a-z}$

$$= \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left[1 - \left(\frac{z-a}{w-a}\right)\right]^{-1}$$

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1 + \left(\frac{z-a}{w-a}\right) + \left(\frac{z-a}{w-a}\right)^2 + \cdots\right] \rightarrow (2)$$

integrating eqn (2) over C_1 and multiplied by $\frac{f(w)}{2\pi i}$

$$\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-a}\, dw + \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^2} (z-a)\, dw + \cdots$$

$$= a_0 + a_1 (z-a) + a_2 (z-a)^2 + \cdots$$

$$\frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z}\, dw = \sum_{n=0}^{\infty} a_n (z-a)^n \rightarrow (3)$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}}\, dw \quad n = 0, 1, 2, \ldots$

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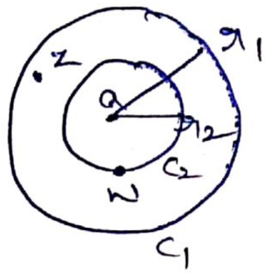
Case 2:- Let w be the point lies on circle C_2

$$|w-a| < |z-a| \Rightarrow \left|\frac{w-a}{z-a}\right| < 1$$



Consider $\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{(w-a)-(z-a)}$

$$= \frac{1}{z-a\left[\frac{w-a}{z-a} - 1\right]} = \frac{-1}{z-a}\left[1-\left(\frac{w-a}{z-a}\right)\right]^{-1}$$

$$\frac{1}{w-z} = \frac{-1}{z-a}\left[1 + \left(\frac{w-a}{z-a}\right) + \left(\frac{w-a}{z-a}\right)^2 + \cdots\right] \longrightarrow (4)$$

Integrating ~~multiply~~ Eqn (4) over C_2 & multiply by $\frac{f(w)}{2\pi i}$ we get

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{z-a}\,dw + \frac{1}{2\pi i}\int_{C_2}\frac{f(w)\,(w-a)\,dw}{(z-a)^2}$$

$$+ \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(z-a)^3}(w-a)^2\,dw + \cdots$$

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)\,dw}{z-a} + \frac{1}{2\pi i}\int_{C_2} f(w)\,(w-a)(z-a)^{-2}\,dw$$

$$+ \frac{1}{2\pi i}\int_{C_2} f(w)\,(w-a)^2\,(z-a)^{-3}\,dw + \cdots$$

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \longrightarrow (5)$$

where $a_{-n} = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\,dw \quad n = 1, 2, 3, \ldots$

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Substitute ③ & ⑤ in Eqn ①

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw$

$$a_{-n} = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw$$

1. Find the Laurent series expansion of $f(z) = \frac{1}{z^2-4z+3}$ for $1<|z|<3$

$1<1$

Sol :-

$$f(z) = \frac{1}{z^2-4z+3} = \frac{1}{z^2-3z-z+3} = \frac{1}{z(z-3)-1(z-3)}$$

$$= \frac{1}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$1 = Az-3A+Bz-B$$

$$f(z) = -\frac{1}{2(z-1)} + \frac{1}{2(z-3)} = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

Given $1<|z|<3$

$\frac{1}{z}<1$, $\frac{z}{3}<1$

$$= \frac{1}{2}\left[\frac{1}{3(\frac{z}{3}-1)} - \frac{1}{z(1-\frac{1}{z})}\right]$$

$1<|z|$

$|\frac{1}{z}|<1$

$$= -\frac{1}{6}\left[1-\frac{z}{3}\right]^{-1} - \frac{1}{2z}\left[1-\frac{1}{z}\right]^{-1}$$

$$f(z) = -\frac{1}{6}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right] - \frac{1}{2z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

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2. Expand $f(z) = \frac{7z-2}{z(z+1)(z-2)}$ in Powers of $(z+1)$ in the range $1<|z+1|<3$.

Sol:- $f(z) = \frac{7z-2}{z(z+1)(z-2)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-2}$.

$7z-2 = A(z+1)(z-2) + B(z)(z-2) + Cz(z+1)$

$7z-2 = Az^2 - Az - 2A + Bz^2 - 2Bz + Cz^2 + Cz$

$A+B+C=0$
$-2A-2B+C=7$
$(+)$

$A+B+C=0$
$-A-2B+C=7$
$-2A=-2 \Rightarrow A=1$

$A+B+C=0$
$-A-2B+C=7$
$(+)\ (+)\ (-)\ (-)$
$2A+3B=-7 \Rightarrow 2+3B=-7 \Rightarrow 3B=-9 \Rightarrow B=-3$

$A+B+C=0 \Rightarrow 1-3+C=0 \Rightarrow C=2$

$\therefore f(z) = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$

$= \frac{-3}{z+1} + \frac{2}{z+1-1-2} + \frac{1}{z+1-1}$

$= \frac{-3}{z+1} + \frac{2}{z+1-3} + \frac{1}{z+1-1}$

$= \frac{-3}{z+1} + \frac{2}{3\left(\frac{z+1}{3}-1\right)} + \frac{1}{(z+1)\left(1-\frac{1}{z+1}\right)}$

$= \frac{-3}{z+1} - \frac{2}{3}\left[1-\frac{z+1}{3}\right]^{-1} + \frac{1}{z+1}\left[1-\frac{1}{z+1}\right]^{-1}$

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$$f(z) = -\frac{3}{z+1} - \frac{2}{3}\left[1 + \left(\frac{z+1}{3}\right) + \frac{(z+1)^2}{9} + \cdots\right] + \frac{1}{z+1}\left[1 + \frac{1}{z+1} + \left(\frac{1}{z+1}\right)^2 + \cdots\right]$$

3. Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ as a Laurent series about singular point $z=1$ also find region of convergence.

Sol:- Given $f(z) = \frac{e^{2z}}{(z-1)^3}$

Put $z-1 = w \Rightarrow z = w+1$

$$f(z) = \frac{e^{2(w+1)}}{w^3} = e^2 \frac{1}{w^3}\left[1 + 2w + \frac{(2w)^2}{2!} + \cdots\right]$$

$$= e^2 \frac{1}{w^3} \sum_{n=0}^{\infty} \frac{(2w)^n}{n!}, \quad w \neq 0$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} w^{n-3}, \quad w \neq 0$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3}, \quad (z-1) \neq 0 \quad \cancel{z-1 \neq 0 \ \& \ |z-1|<1}$$

i.e., $\cancel{0 < |z-1| < 1}$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3} \text{ if } |z-1| > 0 \text{ and } |z-1| < 1$$

$$0 < |z-1| < 1$$

4. Find the Laurent series of $f(z) = \frac{1}{z^2(1-z)}$ in two different ways and specify the regions in which these expansions are valid.

Sol:- Case i):- $f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}(1-z)^{-1}$, $z \neq 0$ and $|z|<1$

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$= \frac{1}{z^2}\left[1+z+z^2+\cdots\right]$, if $|z|>0$ & $|z|<1$

$= \frac{1}{z^2}\left[1+z+z^2+\cdots\right]$ if $0<|z|<1$ (Since the function has two singular point $z=0$ &

ii) $f(z) = \frac{1}{z^2(1-z)} = \cancel{\frac{1}{z^2 \cdot z(1-\frac{1}{z})}}$

$= \frac{1}{z^2 z(\frac{1}{z}-1)} = \frac{-1}{z^3}\left(1-\frac{1}{z}\right)^{-1}$

$= \frac{-1}{z^3}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$ if $z\neq 0$ and $\left|\frac{1}{z}\right|<1$.

$= -\left(\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}+\cdots\right)$ if $|z|>1$

5. Find the Laurent series expansion of $f(z) = \frac{z}{(z+1)(z+2)}$ about $z=-2$.

Sol:- $f(z) = \frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{(z+2)}$

$z = Az+2A+Bz+2B$

$A+B=1$

$2A+B=0$

$-A=1 \Rightarrow A=-1 \Rightarrow B=2$

$f(z) = \frac{-1}{z+1} + \frac{2}{z+2}$

$= \frac{-1}{z+2-2+1} + \frac{2}{z+2}$

$= \frac{-1}{z+2-1} + \frac{2}{z+2}$

$= \frac{-1}{-(1-(z+2))} + \frac{2}{z+2}$

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$$= \frac{2}{z+2} + [1-(z+2)]^{-1}$$

$$= \frac{2}{z+2} + [1+(z+2)+(z+2)^2+\cdots]$$

6. Find the Laurent's series expansion for $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$

Put $z+2 = t$

Sol:- $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \frac{A}{z-1} + \frac{B}{z-3} + \frac{C}{z+2}$

$$f(z) \frac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \frac{1}{z-1} - \frac{1}{z-3} + \frac{1}{z+2}$$

$$= \frac{1}{z+2-2-1} - \frac{1}{z+2-2-3} + \frac{1}{z+2}$$

$3 < |z+2| < 5$

$\frac{3}{z+2} < 1, \quad \frac{z+2}{5} < 1$

$$= \frac{1}{z+2-3} - \frac{1}{z+2-5} + \frac{1}{z+2}$$

$$= \frac{1}{z+2\left[1-\frac{3}{z+2}\right]} - \frac{1}{5\left[\frac{z+2}{5}-1\right]} + \frac{1}{z+2}$$

$$= \frac{1}{z+2}\left[1-\frac{3}{z+2}\right]^{-1} + \frac{1}{5}\left[1-\frac{z+2}{5}\right]^{-1} + \frac{1}{z+2}$$

$$= \frac{1}{z+2}\left(1+\frac{3}{z+2}+\left(\frac{3}{z+2}\right)^2+\cdots\right] + \frac{1}{5}\left[1+\frac{z+2}{5}+\left(\frac{z+2}{5}\right)^2+\cdots\right] + \frac{1}{z+2}$$

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Zeros and Singularities :-

The point $z=a$ is called the zero of the function $f(z)$ if $f(a)=0$

Zero of order m :- If an analytic function is expressed in the form $f(z) = (z-a)^m \phi(z)$ and $\phi(a) \neq 0$ then $z=a$ is called zero of order 'm'

Singular point :- The singular point of $f(z)$ is the point at which the function $f(z)$ is not analytic

1. Isolated singular point :- The point $z=a$ is called Isolated singular point of $f(z)$ if

i) $f(z)$ is not analytic at $z=a$

ii) $f(z)$ is analytic in neighbourhood of $z=a$

i.e., it contains no ~~other~~ other singularities ~~other than~~ of $z=a$

Ex:- $\frac{e^{2z}}{z^2+1}$

The function has only ~~one~~ two singular point $z=-i, i$

$\therefore z=-i, i$ is isolated singular points.

poles of an analytic function :- If $z=a$ is isolated singular point and by Laurent series $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$

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here the series of negative powers namely $\sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$ is called principal part of Laurent series

If the principal part of Laurent series contains finite no. of terms (m say) then $z=a$ is called the pole of order m. If $m>1$ then it is called multiple pole. If $m=1$ then $z=a$ is called simple pole. (or)

If $\lim_{z \to a} (z-a)^m f(z) = A \neq 0$ then $z=a$ is pole of order m

Ex:- $\frac{z^2}{(z+1)(z-3)^2}$

here $z=-1$ is called simple pole

$z=3$ is called pole of order 2

Essential singular point:-

If the principle part of Laurent's series contains infinite no. of terms then $z=a$ is called essential singular point.

Removable singular point:-

If the Laurent series has no ~~principle part~~ negative power series i.e., $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ then $z=a$ is called removable singular point.

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Calculus of Residues

Residue :- The coefficient of $\frac{1}{z-a}$ in the Laurent series expansion of $f(z)$ about the isolated singular point $z=a$ is called residue of $f(z)$ and it is denoted by Res $[f(z)]$ at $z=a$

We know that the Laurent series about the point $z=a$ is
$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + a_{-1}\frac{1}{z-a} + a_{-2}\frac{1}{(z-a)^2} + \dots$$

here the coefficient of $\frac{1}{z-a}$ is a_{-1}

$\therefore$ a_{-1} is called residue of $f(z)$ and is defined by
$$a_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz$$
$$\therefore \int_C f(z)\,dz = 2\pi i\,(a_{-1})$$

(from Laurent's theorem
$$a_{-n} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{-n+1}}dz$$
$$a_{-1} = \frac{1}{2\pi i}\int_C f(z)dz$$

Residue of Simple pole :-

Case 1 :- If a is the simple pole of $f(z)$ then
$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{a_{-1}}{(z-a)}$$
multiply by $(z-a)$

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$$f(z)(z-a) = \lim_{z \to a} \sum_{n=0}^{\infty} a_n (z-a)^{n+1} + a_{-1}$$

taking $\lim_{z \to a}$ on both sides

$$\lim_{z \to a} f(z)(z-a) = \lim_{z \to a} \sum_{n=0}^{\infty} a_n (z-a)^{n+1} + a_{-1}$$

$$\therefore \boxed{a_{-1} = \lim_{z \to a} f(z)(z-a)}$$

Case 2:- Let $f(z) = \dfrac{\phi(z)}{\psi(z)}$

where $\psi(z)$ is simple zero at a i.e., $\psi(a)=0$

$$a_{-1} = \left[\operatorname{Res}_{z=a} f(z)\right] = \lim_{z \to a} (z-a) f(z)$$

$$= \lim_{z \to a} (z-a) \frac{\phi(z)}{\psi(z)}$$

$$= \lim_{z \to a} (z-a) \frac{\phi(z)}{\psi(a) + (z-a)\psi'(a) + \dots}$$

(by talors series)

$$= \lim_{z \to a} (z-a) \frac{\phi(z)}{\psi(a) + (z-a)\psi'(a)}$$

$$= \lim_{z \to a} \cancel{(z-a)} \frac{\phi(z)}{0 + \cancel{(z-a)}\psi'(a)}$$

$$a_{-1} = \frac{\phi(a)}{\psi'(a)}$$

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Residue of multiple pole

If $f(z)$ is analytic and has a pole of order m at $z=a$ then residue at $z=a$ is given by

$$a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$

Proof:- Given that $f(z)$ has a pole of order m

then $\lim_{z \to a} f(z)(z-a)^m = \phi(a)$ where $\phi(a) \neq 0$

let $f(z) = \dfrac{\phi(a)}{(z-a)^m}$

$$a_{-1} = \frac{1}{2\pi i} \int_c f(z)\, dz$$

$$= \frac{1}{2\pi i} \int_c \frac{\phi(a)}{(z-a)^m} dz$$

$$= \frac{1}{(m-1)!} \frac{(m-1)!}{2\pi i} \int_c \frac{\phi(a)}{(z-a)^m} dz$$

$$= \frac{1}{(m-1)!} \cancel{[\phi(a)]^{m-1}} \phi^{m-1}(a)$$

$$= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \phi(a)$$

$$\boxed{a_{-1} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)}$$

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Problems :-

1. Find the poles and residues of $f(z) = \frac{1}{(z+1)(z+3)}$

Sol :- $f(z) = \frac{1}{(z+1)(z+3)}$

The function has two simple poles $z=-1$ & $z=-3$

Residue at $z=-1$:-

$$a_{-1} = \lim_{z \to a} (z-a) f(z)$$

$$= \lim_{z \to -1} \cancel{(z+1)} \frac{1}{\cancel{(z+1)}(z+3)} = \frac{1}{2}$$

Residue of $f(z)$ at $z=-3$:-

$$a_{-1} = \lim_{z \to a} (z-a) f(z)$$

$$= \lim_{z \to -3} \cancel{(z+3)} \frac{1}{(z+1)\cancel{(z+3)}} = \frac{-1}{2}$$

2. Find the poles of $f(z) = \frac{z^2}{(z-1)(z-2)^2}$ and also find residues at these poles.

Sol :- $f(z) = \frac{z^2}{(z-1)(z-2)^2}$

It has the simple pole at $z=1$ and pole of order 2 at $z=2$

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$\left[\text{Res } f(z)\right]_{z=1}$:-

$$a_{-1} = \lim_{z\to a}(z-a)f(z) = \lim_{z\to 1}\cancel{(z-1)}\frac{z^2}{\cancel{(z-1)}(z-2)^2} = \frac{1}{(1-2)^2} = 1$$

Res of $f(z)$ at $z=2$:-

$$a_{-1} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}} f(z)(z-a)^m$$

here $m=2$

$$\therefore a_{-1} = \frac{1}{1!}\lim_{z\to 2}\frac{d}{dz}\left[\frac{z^2}{(z-1)\cancel{(z-2)^2}}\cancel{(z-2)^2}\right]$$

$$= \lim_{z\to 2}\frac{(z-1)(2z) - z^2(1)}{(z-1)^2} = \lim_{z\to 2}\frac{2z^2-2z-z^2}{(z-1)^2}$$

$$= \lim_{z\to 2}\frac{z^2-2z}{(z-1)^2} = 0$$

3. Find the residue of $f(x) = \frac{7-3z}{z^2-z}$

Sol :- $f(z) = \frac{7-3z}{z^2-z} = \frac{7-3z}{z(z-1)}$

The given function has two simple pole $z=0$ & $z=1$

also $f(z) = \frac{7-3z}{z^2-z} = \frac{\phi(z)}{\psi(z)}$ where $\psi(z) = z^2-z$
$\psi'(z) = 2z-1$

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Res of $f(z)$ at $z=a$ is

$$a_{-1} = \frac{\phi(a)}{\psi'(a)}$$

$$\therefore \left[\text{Res } f(z)\right]_{z=0} = a_{-1} = \frac{\phi(0)}{\psi'(0)} = \frac{7}{-1} = -7$$

$$\left[\text{Res } f(z)\right]_{z=1} = a_{-1} = \frac{\phi(1)}{\psi'(1)} = \frac{7-3}{1} = 4$$

4. Find the residue of $\dfrac{ze^z}{(z-1)^3}$.

Sol :- $f(z) = \dfrac{ze^z}{(z-1)^3}$

The function $f(z)$ has $z=1$ is the pole of order 3

$$\therefore a_{-1} = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$$

here $m=3$, $a=1$

$$a_{-1} = \frac{1}{2!} \lim_{z\to 1} \frac{d^2}{dz^2} \cancel{(z-1)^3} \frac{ze^z}{\cancel{(z-1)^3}}$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d}{dz} \left[ze^z + e^z\right]$$

$$= \frac{1}{2} \lim_{z\to 1} \left[ze^z + e^z + e^z\right]$$

$$= \frac{1}{2}\left[3e^1\right] = \frac{3e^1}{2}$$

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5. Find the residue of $f(z) = \frac{1+e^z}{\sin z + z\cos z}$ at $z=0$.

Sol:- Given $z=0$ is the pole of $f(z) = \frac{1+e^z}{\sin z + z\cos z}$

W.K.T

$$a_{-1} = \lim_{z \to a} (z-a) f(z)$$

$$= \lim_{z \to 0} z \frac{1+e^z}{\sin z + z\cos z} = \lim_{z \to 0} \frac{\cancel{z}(1+e^z)}{\cancel{z}\left[\cos z + \frac{\sin z}{z}\right]}$$

~~using L hospital rule~~

$$= \frac{1+e^0}{\cos 0 + \sin 0} = 2$$

(or)

using L hospital rule

$$= \lim_{z \to 0} \frac{z(e^z) + (1+e^z)(1)}{\cos z + \cos z - z\sin z} = \frac{0+(1+1)}{1+1-0} = \frac{2}{2} = 1$$

6. Find the poles and residues of $f(z) = \frac{z^2}{z^4-1}$

Sol:- Given $f(z) = \frac{z^2}{z^4-1} = \frac{z^2}{(z^2+1)(z^2-1)} = \frac{z^2}{(z+1)(z-1)(z+i)(z-i)}$

$\therefore$ The function has four simple poles $z=1, z=-1, z=i, z=-i$

also $f(z) = \frac{z^2}{z^4-1} = \frac{\phi(z)}{\psi(z)}$

where $\psi(z) = z^4-1 \Rightarrow \psi'(z) = 4z^3$

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$\left[\text{Res } f(z)\right]_{z=1}$:-

$$a_{-1} = \frac{\phi(a)}{\psi'(a)} = \frac{\phi(1)}{\psi'(1)} = \frac{1}{4}$$

$\left[\text{Res } f(z)\right]_{z=-1}$:-

$$a_{-1} = \frac{\phi(a)}{\psi'(a)} = \frac{\phi(-1)}{\psi'(-1)} = -\frac{1}{4}$$

$\left[\text{Res } f(z)\right]_{z=i}$:-

$$a_{-1} = \frac{\phi(i)}{\psi'(i)} = \frac{i^2}{4i^3} = \frac{1}{4i} = \frac{-i}{4}$$

$\left[\text{Res } f(z)\right]_{z=-i}$:-

$$a_{-1} = \frac{\phi(-i)}{\psi'(-i)} = \frac{i^2}{4(-i)^3} = \frac{-1}{4i} = \frac{i}{4}$$

7. Find the poles and residues of $\frac{e^{iz}}{(z^2+a^2)}$

Sol:- $f(z) = \frac{e^{iz}}{z^2+a^2} = \frac{e^{iz}}{(z+ai)(z-ai)}$ and $f(z) = \frac{e^{iz}}{z^2+a^2} = \frac{\phi(z)}{\psi(z)}$

The function has two simple poles $z=ai$, $z=-ai$ | $\psi(z) = z^2+a^2$, $\psi'(z) = 2z$

$\left[\text{Res } f(z)\right]_{z=ai}$:-

$$a_{-1} = \frac{\phi(a)}{\psi'(a)} = \frac{\phi(ai)}{\psi'(ai)} = \frac{e^{i(ai)}}{2ai} = \frac{e^{-a}}{2ai} = -\frac{ie^{-a}}{2a}$$

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$[\text{Res } f(z)]_{z=-ai}$:- $a_{-1} = \dfrac{\phi(-ai)}{\psi'(-ai)} = \dfrac{e^{i(-ai)}}{2(-ai)} = -\dfrac{e^{a}}{2ai} = \dfrac{ie^{a}}{2a}$

Cauchy - Residue Theorem :-

If $f(z)$ is analytic inside the simple closed curve 'c'

Except at finite no. of isolated singular points $a_1, a_2, \dots a_n$

then $\int_c f(z)\,dz = 2\pi i$ [sum of residues]

Proof :- Draw a small circles $c_1, c_2, \dots c_n$ with centres

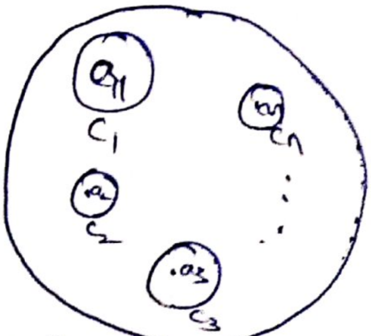
$a_1, a_2 \dots a_n$ lies inside the curve 'c'.

Since $f(z)$ is analytic function

By the cauchy's theorem for multiple connected regions

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \dots + \int_{c_n} f(z)\,dz \rightarrow (1)$$

By the definition of residue



$$[\text{Res } f(z)]_{z=a} = \frac{1}{2\pi i}\int_{c} f(z)\,dz$$

$$\therefore [\text{Res } f(z)]_{z=a_1} = \frac{1}{2\pi i}\int_{c_1} f(z)\,dz \Rightarrow \int_{c_1} f(z)\,dz = 2\pi i\,[\text{Res } f(z)]_{z=a_1}$$

$$[\text{Res } f(z)]_{z=a_2} = \frac{1}{2\pi i}\int_{c_2} f(z)\,dz \Rightarrow \int_{c_2} f(z)\,dz = 2\pi i\,[\text{Res } f(z)]_{z=a_2}$$

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$$[\text{Res } f(z)]_{z=a_n} = \frac{1}{2\pi i}\int_{C_n} f(z)\,dz \Rightarrow \int_{C_n} f(z)\,dz = 2\pi i\,[\text{Res } f(z)]_{z=a_n}$$

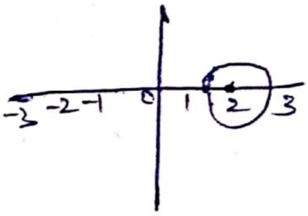
Substitute the above residues in Eqn ①

$$\int_C f(z)\,dz = 2\pi i\left[[\text{Res } f(z)]_{z=z_1} + [\text{Res } f(z)]_{z=z_2} + \cdots + [\text{Res } f(z)]_{z=z_n}\right]$$

$$\int_C f(z)\,dz = 2\pi i\,[\text{sum of residues}]$$

1. Evaluate $\int_C \frac{z}{(z-1)(z-2)^2}\,dz$ where C is circle $|z-2| = \frac{1}{2}$ using Cauchy's residue theorem.

Sol:- Given $\int_C f(z)\,dz = \int_C \frac{z}{(z-1)(z-2)^2}\,dz$



here the function has two poles

$z=1$ lies outside circle

$z=2$ lies inside circle $|z-2| = \frac{1}{2}$

$\therefore [\text{Res } f(z)]_{z=2}$ is $a_{-1} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)$

$$a_{-1} = \frac{1}{1!}\lim_{z\to 2}\frac{d}{dz}\cancel{(z-2)^2}\frac{z}{(z-1)\cancel{(z-2)^2}}$$

$$= \lim_{z\to 2}\frac{(z-1)(1) - z(1)}{(z-1)^2} = \lim_{z\to 2}\frac{\cancel{z}-1-\cancel{z}}{(z-1)^2} = -1$$

$\therefore$ By Cauchy's Residue theorem

$$\int_C \frac{z}{(z-1)(z-2)^2}\,dz = 2\pi i\,(\text{sum of residues})$$
$$= 2\pi i\,(-1) = -2\pi i$$

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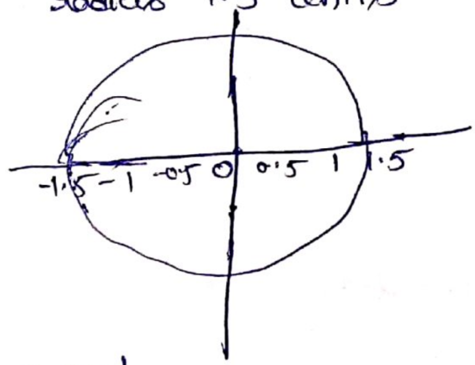
2. Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)} dz$ where c is circle $|z| = \frac{3}{2}$.

Sol:- $f(z) = \frac{4-3z}{z(z-1)(z-2)}$; c is circle with centre 0 and radius 1.5 units



The function has 3 simple poles in which

$z=0$, $z=1$ lies inside circle and

$z=2$ lies outside circle

∴ we have to find residues at $z=0$ & $z=1$

∴ $\left[\text{Res } f(z)\right]_{z=0}$ is

$$a_{-1} = \lim_{z\to a} (z-a) f(z) = \lim_{z\to 0} \cancel{z} \frac{4-3z}{\cancel{z}(z-1)(z-2)} = \frac{4}{2} = 2$$

$\left[\text{Res } f(z)\right]_{z=1}$:-

$$a_{-1} = \lim_{z\to 1} \cancel{(z-1)} \frac{4-3z}{z\cancel{(z-1)}(z-2)} = \frac{1}{-1} = -1$$

∴ By Cauchy's Residue theorem

$$\int_C f(z) dz = 2\pi i (\text{sum of residues})$$

$$= 2\pi i (2-1) = \underline{\underline{2\pi i}}$$

3. Evaluate $\int_C \frac{z^3}{z^4-1} dz$ where c is circle $|z|=2$ by Cauchy residue theorem

Sol:- $f(z) = \frac{z^3}{z^4-1} = \frac{z^3}{(z^2+1)(z^2-1)} = \frac{z^3}{(z+i)(z-i)(z+1)(z-1)}$

The function has four poles $z=1$, $z=-1$, $z=i$, $z=-i$ which
point (1,0) (-1,0) (0,1) (0,-1)

lies inside the circle $|z|=2$

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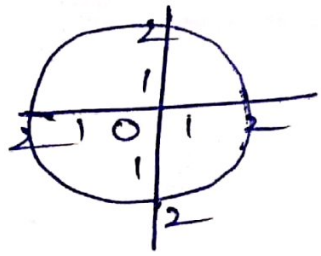
$\left[\operatorname{Res} f(z)\right]_{z=1}$:-

$$a_{-1} = \cancel{\lim_{z\to 1} (z-1)} \frac{\phi(a)}{\psi'(a)} = \frac{\phi(1)}{\psi'(1)} = \frac{1}{4}$$



$\phi(z) = z^3$
$\psi(z) = z^4 - 1$
$\psi'(z) = 4z^3$

$\left[\operatorname{Res} f(z)\right]_{z=-1}$:-

$$a_{-1} = \frac{\phi(a)}{\psi'(a)} = \frac{\phi(-1)}{\psi'(-1)} = \frac{-1}{-4} = \frac{1}{4}$$

$\left[\operatorname{Res} f(z)\right]_{z=i}$:-

$$a_{-1} = \frac{\phi(i)}{\psi'(i)} = \frac{i^3}{4i^3} = \frac{1}{4}$$

$\left[\operatorname{Res} f(z)\right]_{z=-i}$:-

$$a_{-1} = \frac{\phi(-i)}{\psi'(-i)} = \frac{-i^3}{-4i^3} = \frac{1}{4}.$$

∴ By Cauchy's Residue theorem :-

$$\int_C f(z)\,dz = 2\pi i\,[S.O.R] = 2\pi i\left[\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right] = 2\pi i(1) = 2\pi i$$

4. Evaluate $\int_C \frac{e^z}{(z-1)^n}\,dz$ where c is circle $|z|=2$.

Sol:- Given $\int_C \frac{e^z}{(z-1)^n}\,dz$

$$f(z) = \frac{e^z}{(z-1)^n}$$

here $z=1$ is the pole of order n.

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$[\text{Res } f(z)]_{z=1}$:-

$$a_{-1} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$$

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \to 1} \frac{d^{n-1}}{dz^{n-1}} \cancel{(z-a)^n} \frac{e^z}{\cancel{(z-1)^n}}$$

$$= \frac{1}{(n-1)!} \lim_{z \to 1} e^z = \frac{e^1}{(n-1)!}$$

By Cauchy's Residue theorem

$$\int_C \frac{e^z}{(z-1)^n} dz = 2\pi i \left[\frac{e^1}{(n-1)!}\right] = \frac{2\pi i e}{(n-1)!}$$

5. Evaluate $\int_C \frac{2e^z}{z(z-3)} dz$ $C : |z| = 2$

Sol :- $f(z) = \frac{2e^z}{z(z-3)}$ $|z| = 2$

here $z = 0$ is simple pole lies inside the circle

$z = 3$ lies outside the circle

$$[\text{Res } f(z)]_{z=0} = \lim_{z \to 0} \cancel{(z-0)} \frac{2e^z}{\cancel{z}(z-3)}$$

$$= -\frac{2}{3}$$

By Cauchy residue theorem

$$\int_C f(z)\, dz = 2\pi i [\text{sum of residues}]$$

$$= 2\pi i \left[-\frac{2}{3}\right] = \frac{-4\pi i}{3}$$

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Evaluation of definite integrals by Contour Integration

In this section, we consider certain types of definite integrals. To evaluate these integrals we apply Cauchy's Residue theorem which is simpler than usual method.

The process of evaluating definite integrals by making path of integration about the contour (closed curve) in a complex plane is called contour integration.

Type 1:- Integration around the unit circle
(or)
Integration of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$
where F is real.

To evaluate these types of integrals put $z = e^{i\theta}$

$$dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

also $$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2} = \frac{z^2-1}{2iz}$$

Now reduce $\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$ into $\int_C f(z)\,dz$

By Cauchy residue theorem $\int_C f(z)\,dz = 2\pi i\,[\text{sum of residues}]$

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1. show that $\int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \frac{2\pi}{\sqrt{3}}$.

Sol :- Put $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta \Rightarrow d\theta = \frac{dz}{iz}$

also $\sin\theta = \frac{z^2-1}{2iz}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \int_C \frac{\frac{dz}{iz}}{2-\left(\frac{z^2-1}{2iz}\right)} = \int_C \frac{\frac{dz}{\cancel{iz}}}{\frac{4iz-z^2+1}{2\cancel{iz}}}$$

$$= 2\int_C \frac{dz}{4iz-z^2+1} = -2\int_C \frac{dz}{z^2-4iz-1} = \int_C \frac{-2}{z^2-4iz-1}\, dz$$

where $f(z) = \frac{-2}{z^2-4iz-1}$

Now the poles of $f(z)$ are the roots of $z^2-4iz-1$

$$z = \frac{4i \pm \sqrt{-16+4}}{2} = \frac{4i \pm i\sqrt{12}}{2} = \frac{4i \pm i\sqrt{3}}{2} = 2i \pm i\sqrt{3}$$

$\therefore z = 2i + i\sqrt{3}\,,\ 2i - i\sqrt{3}$

let $\alpha = 2i + i\sqrt{3}$ $\quad \beta = 2i - i\sqrt{3}$.

$$\therefore f(z) = \frac{-2}{z^2-4iz-1} = \frac{-2}{(z-(2i+i\sqrt{3}))(z-(2i-i\sqrt{3}))} = \frac{-2}{(z-\alpha)(z-\beta)}$$

$\therefore \alpha, \beta$ are the poles of $f(z)$ in which α lies outside and β lies inside the circle.

$\alpha = 2i + i\sqrt{3}$

$\therefore$ The point is $(0, 2+\sqrt{3}) = 2 + 1.73 = 3.73$ outside the unit circle

$\beta = 2i - i\sqrt{3}$ $\quad (0, 2-\sqrt{3}) = 2 - 1.73 = 0.27$ inside the unit circle

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Now

$$[\text{Res } f(z)]_{z=\beta} = \lim_{z\to\beta} (z-\beta)\, f(z)$$

$$= \lim_{z\to\beta} \cancel{(z-\beta)} \frac{-2}{(z-\alpha)\cancel{(z-\beta)}}$$

$$= \frac{-2}{\beta-\alpha} = \frac{-2}{\cancel{2i}-i\sqrt{3}-\cancel{2i}-i\sqrt{3}} = \frac{-\cancel{2}}{-\cancel{2}i\sqrt{3}} = \frac{1}{i\sqrt{3}}$$

By Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i\,[\text{Sum of residues}]$$

$$\int_C \frac{-2}{z^2-4iz-1}\,dz = 2\pi \cancel{i} \frac{1}{\cancel{i}\sqrt{3}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2-\sin\theta} = \frac{2\pi}{\sqrt{3}}$$

Q. Show that $\displaystyle\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$, $a>b>0$

If $f(2\pi-\theta) = f(\theta)$

then $\displaystyle\int_0^{2\pi} f(\theta)d\theta = 2\int_0^{\pi} f(\theta)d\theta$

$\Rightarrow \displaystyle\int_0^{\pi} f(\theta)d\theta = \frac{1}{2}\int_0^{2\pi} f(\theta)d\theta$

Sol :- Put $z = e^{i\theta} \Rightarrow d\theta = \dfrac{dz}{iz}$

$$\cos\theta = \frac{z^2+1}{2z}$$

here $$\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} \rightarrow (1)$$

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Now $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{\frac{dz}{iz}}{a+b\left(\frac{z^2+1}{2z}\right)} = \int_C \frac{\frac{dz}{iz}}{\frac{2az+bz^2+b}{2z}} = \int_C \frac{dz}{iz} \times \frac{2z}{2az+bz^2+b}$

$= \frac{2}{i}\int_C \frac{dz}{b\left(\frac{2az}{b}+z^2+1\right)} dz = \frac{2}{ib}\int_C \frac{dz}{z^2+\frac{2ab}{z}+1}$

$= \frac{1}{ib}\int_C f(z)\,dz$

where $f(z) = \frac{2}{z^2+\frac{2az}{b}+1}$

The poles of $f(z)$ are the roots of $z^2+\frac{2a}{b}z+1=0$

$z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2} = \frac{-\frac{2a}{b} \pm 2\frac{\sqrt{a^2-b^2}}{b}}{2}$

$= \frac{-2a \pm 2\sqrt{a^2-b^2}}{2b}$

$= -\frac{a}{b} \pm \frac{\sqrt{a^2-b^2}}{b}$

$\therefore \alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

$\therefore a > b > 0$
let $a=2$ $b=1$
$\alpha = \frac{-2+\sqrt{3}}{2} = -0.26$
lies inside
$\beta = \frac{-2-\sqrt{3}}{2} = -1.86$
lies outside

$\therefore f(z) = \frac{2}{(z-\alpha)(z-\beta)}$

in which $z=\alpha$ lies inside C and $z=\beta$ lies outside C

$\therefore [\text{Res } f(z)]_{z=\alpha} = \lim_{z\to\alpha} \cancel{(z-\alpha)}\frac{2}{\cancel{(z-\alpha)}(z-\beta)} = \frac{2}{\alpha-\beta}$

$= \frac{2}{\cancel{-\frac{a}{b}}+\frac{\sqrt{a^2-b^2}}{b}\cancel{+\frac{a}{b}}+\frac{\sqrt{a^2-b^2}}{b}} = \frac{\cancel{2}b}{\cancel{2}\sqrt{a^2-b^2}} = \frac{b}{\sqrt{a^2-b^2}}$

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∴ By Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i\,[S.O.R]$$

$$\int_C \frac{2}{z^2+\frac{2az}{b}+1}\,dz = 2\pi i \frac{b}{\sqrt{a^2-b^2}}$$

$$\frac{1}{ib}\int_C \frac{2}{z^2+\frac{2az}{b}+1}\,dz = \frac{1}{\cancel{ib}}\,2\pi \cancel{i}\,\frac{\cancel{b}}{\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

from (1)

$$\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{\cancel{2}}\frac{\cancel{2}\pi}{\sqrt{a^2-b^2}} = \frac{\pi}{\sqrt{a^2-b^2}}$$

3. Show that $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right], a>b>0$

Sol:- $$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \int_0^{2\pi} \frac{1-\cos2\theta}{2(a+b\cos\theta)}\,d\theta = \text{Real part}\int_0^{2\pi} \frac{1-e^{2i\theta}}{2(a+b\cos\theta)}\,d\theta$$

Now put $z=e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$ also $\cos\theta = \frac{z^2+1}{2z}$

$$\Rightarrow \text{Real part}\int_0^{2\pi} \frac{1-e^{2i\theta}}{2(a+b\cos\theta)}\,d\theta = \int_C \frac{1-z^2}{2\left[a+b\left(\frac{z^2+1}{2z}\right)\right]}\frac{dz}{iz}$$

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$$= \int_C \frac{(1-z^2)\cancel{z}}{2az+bz^2+b}\frac{dz}{i\cancel{z}} = \frac{1}{ib}\int_C \frac{(1-z^2)}{z^2+\frac{2a}{b}z+1}dz = \frac{1}{ib}\int_C f(z)\,dz$$

where $f(z) = \dfrac{1-z^2}{z^2+\frac{2a}{b}z+1}$

The poles of $f(z)$ are the roots of $z^2+\frac{2a}{b}z+1=0$

$$\therefore \alpha = \frac{-a+\sqrt{a^2-b^2}}{b} \qquad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

$$\therefore f(z) = \frac{1-z^2}{(z-\alpha)(z-\beta)}$$

here $z=\alpha$ lies inside the unit circle and $z=\beta$ lies outside the unit circle

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=\alpha} = \lim_{z\to\alpha}\cancel{(z-\alpha)}\frac{(1-z^2)}{\cancel{(z-\alpha)}(z-\beta)} = \frac{1-\alpha^2}{\alpha-\beta}$$

$$= \frac{\alpha\left[\frac{1}{\alpha}-\alpha\right]}{(\alpha-\beta)} \qquad (\because \alpha\beta=1 \Rightarrow \beta=1/\alpha)$$

$$= \frac{\alpha(\beta-\alpha)}{\alpha-\beta} = \frac{-\alpha\cancel{(\alpha-\beta)}}{\cancel{(\alpha-\beta)}}$$

$$= -\alpha$$

$$= \frac{a-\sqrt{a^2-b^2}}{b}$$

Now by Cauchy's residue theorem

$$\int_C \frac{1-z^2}{z^2+\frac{2a}{b}z+1}dz = 2\pi i\left[\frac{a-\sqrt{a^2-b^2}}{b}\right]$$

$$\frac{1}{ib}\int_C \frac{1-z^2}{z^2+\frac{2a}{b}z+1}dz = \frac{1}{\cancel{i}b}2\pi\cancel{i}\left[\frac{a-\sqrt{a^2-b^2}}{b}\right] = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$$

$$\therefore \int_0^{2\pi}\frac{\sin^2\theta}{a+b\cos\theta}d\theta = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$$

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4. show that $\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$, $a>b>0$

Sol:- $\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} \longrightarrow ①$

Consider $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \int_C \frac{\frac{dz}{iz}}{\left[a+b\left(\frac{z^2+1}{2z}\right)\right]^2}$

$$= \int_C \frac{\frac{dz}{iz}}{\frac{(2az+bz^2+b)^2}{(2z)^2}} = \int_C \frac{dz}{iz} \times \frac{4z^2}{(2az+bz^2+b)^2}$$

$$= \frac{4}{ib^2}\int_C \frac{z}{\left(z^2+\frac{2a}{b}z+1\right)^2} dz = \frac{4}{ib^2}\int_C f(z)\, dz$$

Now the poles of $f(z)$ are the roots of $z^2+\frac{2a}{b}z+1=0$

$$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b} \qquad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

$$\therefore f(z) = \frac{z}{\left(z^2+\frac{2a}{b}z+1\right)^2} = \frac{z}{[(z-\alpha)(z-\beta)]^2} = \frac{z}{(z-\alpha)^2(z-\beta)^2}$$

The poles of $f(z)$ are $z=\alpha$ and $z=\beta$ in which $z=\beta$ lies outside circle. $z=\alpha$ lies inside circle and

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=\alpha} = \frac{1}{(m-1)!} \lim_{z\to\alpha} \frac{d^{m-1}}{dz^{m-1}} (z-\alpha)^m f(z)$$

$$= \frac{1}{1!} \lim_{z\to\alpha} \frac{d}{dz} \cancel{(z-\alpha)^2} \frac{z}{\cancel{(z-\alpha)^2}(z-\beta)^2}$$

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$$= \lim_{z\to\alpha} \frac{(z-\beta)^2 - z\,2(z-\beta)}{(z-\beta)^4} = \lim_{z\to\alpha} \frac{z-\beta(z-\beta-2z)}{(z-\beta)^4} = \lim_{z\to\alpha} \frac{-z-\beta}{(z-\beta)^3}$$

$$= \frac{-\alpha-\beta}{(\alpha-\beta)^3} = \frac{\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b} + \frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b}}{\left(\cancel{-\frac{a}{b}} + \frac{\sqrt{a^2-b^2}}{b} + \cancel{\frac{a}{b}} + \frac{\sqrt{a^2-b^2}}{b}\right)^3} = \frac{\frac{2a}{b}}{\left(2\frac{\sqrt{a^2-b^2}}{b}\right)^3}$$

$$= \frac{2a}{b} \times \frac{b^3}{8\left(\sqrt{a^2-b^2}\right)^3} = \frac{ab^2}{4(a^2-b^2)^{3/2}}$$

By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,[S.O.R]$$

$$= 2\pi i\left[\frac{ab^2}{4(a^2-b^2)^{3/2}}\right] = \frac{\pi i a b^2}{2(a^2-b^2)^{3/2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{\pi i a b^2}{(a^2-b^2)^{3/2}}$$

$\therefore$ from ①

$$\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{1}{2}\pi i a b^2$$

$$\therefore \frac{4}{ib^2}\int_c \frac{z}{(z^2+\frac{2a}{b}z+1)^2}\,dz = \frac{\cancel{4}^{2}}{\cancel{ib^2}} \frac{\pi \cancel{i} a \cancel{b^2}}{\cancel{2}(a^2-b^2)^{3/2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2a\pi}{(a^2-b^2)^{3/2}}$$

$$\therefore \text{from ① } \int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}a\pi}{(a^2-b^2)^{3/2}}\right] = \frac{\pi a}{(a^2-b^2)^{3/2}}$$

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5. Show that $\int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta} = \frac{\pi}{a\sqrt{1+a^2}}$ for $a>0$

Sol:- $\int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta} = \int_0^{\pi} \frac{d\theta}{a^2+\frac{1-\cos 2\theta}{2}} = \int_0^{\pi} \frac{2\,d\theta}{2a^2+1-\cos 2\theta} \rightarrow (1)$

Put $2\theta = \phi \Rightarrow 2\,d\theta = d\phi$ ~~$\Rightarrow d\theta = d\phi$~~

If $\theta = 0 \Rightarrow \phi = 0$

$\theta = \pi \Rightarrow \phi = 2\pi$

and Let $z = e^{i\phi} \Rightarrow dz = ie^{i\phi}\,d\phi \Rightarrow d\phi = \frac{dz}{iz}$

also $\cos\phi = \frac{z^2+1}{2z}$

$\therefore \int_0^{\pi} \frac{2\,d\theta}{2a^2+1-\cos 2\theta} = \int_0^{2\pi} \frac{d\phi}{2a^2+1-\cos\phi} \rightarrow (2)$

Now $\int_0^{2\pi} \frac{d\phi}{2a^2+1-\cos\phi} = \int_C \frac{dz/iz}{2a^2+1-\left(\frac{z^2+1}{2z}\right)} = \int_C \frac{dz}{iz} \times \frac{2z}{4a^2z+2z-z^2-1}$

$= \frac{1}{i}\int_C \frac{2\,dz}{4a^2z+2z-z^2-1} = \frac{1}{i}\int_C \frac{2\,dz}{-z^2+z(4a^2+2)-1}$

$= \frac{1}{i}\int_C \frac{-2\,dz}{z^2-z(4a^2+2)+1} = \frac{1}{i}\int_C f(z)\,dz$

The poles of $f(z)$ are the roots of $z^2-(4a^2+2)z+1=0$

$$z = \frac{(4a^2+2) \pm \sqrt{(4a^2+2)^2-4}}{2}$$

$$= \frac{(4a^2+2) \pm \sqrt{(4a^2+2)^2-2^2}}{2} = \frac{(4a^2+2) \pm \sqrt{(4a^2+2+2)(4a^2+2-2)}}{2}$$

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$$= \frac{(4a^2+2) \pm \sqrt{16a^4+16a^2}}{2} = \frac{(4a^2+2) \pm 4a\sqrt{a^2+1}}{2}$$

$$= (2a^2+1) \pm 2a\sqrt{a^2+1}$$

$\therefore \alpha = (2a^2+1) + 2a\sqrt{a^2+1} \qquad \beta = (2a^2+1) - 2a\sqrt{a^2+1}$

Poles of $f(z)$ are $z = \alpha$ and $z = \beta$ in which $z = \beta$ lies inside the ~~circle~~ unit circle

Given $a > 0$
let $a = 1$
$\alpha = 3 + 2\sqrt{2}$
$= 5.82$
$\beta = 3 - 2\sqrt{2}$
$= 0.1416$
lies inside c

$$[\operatorname{Res} f(z)]_{z=\beta} = \lim_{z \to \beta} \cancel{z-\beta} \frac{-2}{(z-\alpha)\cancel{(z-\beta)}} = \frac{-2}{\beta - \alpha}$$

$$= \frac{-2}{\cancel{(2a^2+1)} - 2a\sqrt{a^2+1} - (\cancel{2a^2}+1) - 2a\sqrt{a^2+1}}$$

$$= \frac{-2}{-4a\sqrt{a^2+1}} = \frac{1}{2a\sqrt{a^2+1}}$$

By Cauchy Residue theorem

$$\int_C \frac{2\,dz}{z^2 - z(4a^2+2)+1} = \cancel{2}\pi i \left[\frac{1}{\cancel{2}a\sqrt{a^2+1}}\right]$$

$$\frac{1}{i} \int_C \frac{2\,dz}{z^2 - z(4a^2+2)+1} = \frac{1}{\cancel{i}} \left[\frac{\pi \cancel{i}}{a\sqrt{a^2+1}}\right] = \frac{\pi}{a\sqrt{a^2+1}}$$

$$\therefore \int_0^{2\pi} \frac{d\phi}{2a^2+1-\cos\phi} = \frac{\pi}{a\sqrt{a^2+1}}$$

$$2\theta \therefore \text{from (2)} \int_0^{\pi} \frac{2\,d\theta}{2a^2+1-\cos 2\theta} = \frac{\pi}{a\sqrt{a^2+1}}$$

$$\therefore \text{from (1)} \int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta} = \frac{\pi}{a\sqrt{a^2+1}}$$

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6. Evaluate $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$

Sol:- Put $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_C \frac{\frac{dz}{iz}}{5-3\left(\frac{z^2+1}{2z}\right)} = \int_C \frac{dz/iz}{\frac{10z-3z^2-3}{2z}}$$

$$= \int_C \frac{dz}{i\cancel{z}} \times \frac{2\cancel{z}}{10z-3z^2-3} = \frac{1}{i}\int_C \frac{-2\,dz}{3z^2-10z+3} = \frac{1}{i}\int_C f(z)dz$$

Now the poles of $f(z)$ are $3z^2-10z+3=0$

$$z = \frac{10 \pm \sqrt{100-36}}{6} = \frac{10 \pm 8}{6} = 3, \frac{1}{3}$$

$\alpha = 3, \quad \beta = \frac{1}{3}$

in which $\alpha = 3$ lies outside & $\beta = \frac{1}{3}$ lies inside unit circle

$$\left[\text{Res } f(z)\right]_{z=\frac{1}{3}} = \lim_{z \to \frac{1}{3}} \cancel{\left(z-\frac{1}{3}\right)}\left[\frac{-2}{\cancel{\left(z-\frac{1}{3}\right)}(z-3)}\right]$$

$$= \frac{-2}{\frac{1}{3}-3} = \frac{-2}{\frac{1-9}{3}} = \frac{-6}{-8} = \frac{\cancel{+}3}{\cancel{-}4} = \frac{3}{4}$$

By Cauchy Residue theorem

$$\int_C f(z)dz = 2\pi i\,[S.O.R]$$

$$\Rightarrow \int_C \frac{-2}{3z^2-10z+3}dz = 2\pi i\left[\frac{3}{4}\right] = \frac{3\pi i}{2}$$

$$\therefore \frac{1}{i}\int_C \frac{-2}{3z^2-10z+3}dz = \frac{1}{\cancel{i}}\left[\frac{3\pi \cancel{i}}{2}\right] = \frac{3\pi}{2}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \frac{3\pi}{2}$$

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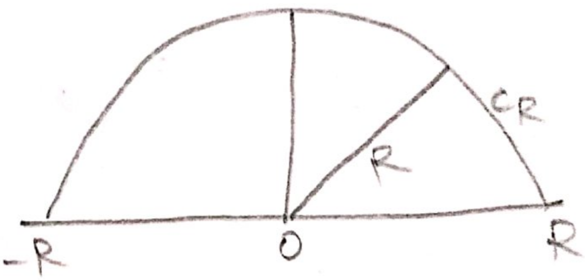
Type 2 :- Integral of the form $\int_{-\infty}^{\infty} f(x)\,dx$

(or)

Integration ~~of the form~~ around semicircle.

To evaluate these types of integrals, consider $\int_C f(z)dz$ where C is the closed contour consisting of semicircle C_R with centre at origin and radius R along with real axis $-R$ to R.



Here all the singularities are in the upper half plane.

By Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i\,[\text{sum of residues}]$$

$$\int_{C_R} f(z)dz + \int_{-R}^{R} f(z)dz = 2\pi i\,[\text{sum of residues}] \quad \text{--- (1)}$$

as $z \to \infty$ $\int_{C_R} f(z)dz = 0$ and

along the real axis $z = x$ we have love $R \to \infty$ as $z \to \infty$

from (1)

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\,[\text{sum of residues}]$$

Note:- In this case $f(z)$ has no poles on the real axis

$\therefore$ poles of $f(z)$ are imaginary.

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1. By contour integration Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx$

Sol:- $\int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx \rightarrow (1)$ ($\because$ Integrand is even)

Now $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx$

Put $x = z \Rightarrow dx = dz$

$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx = \int_C \frac{z^2}{(z^2+1)^2} dz$

where C is the contour consisting of semicircle C_R together with the real axis $[-R, R]$

$\int_C \frac{z^2}{(z^2+1)^2} dz = \int_C \frac{z^2}{(z+i)^2(z-i)^2} dz = \int_C f(z) dz$

The function $f(z)$ has two poles $z=i$ $z=-i$ in which $z=i$ pole of order 2 lies inside C_R and $z=-i$, pole of order 2 lies out side C_R.

$\therefore [\text{Res} f(z)]_{z=i} = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$

$= \frac{1}{1!} \lim_{z\to i} \frac{d}{dz} \cancel{(z-i)^2} \frac{z^2}{(z+i)^2 \cancel{(z-i)^2}}$

$= \lim_{z\to i} \frac{(z+i)^2(2z) - z^2 2(z+i)}{(z+i)^4}$

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$$= \lim_{z\to i} \frac{(z+i)\left[2z(z+i)-2z^2\right]}{(z+i)^4} = \lim_{z\to i} \frac{\cancel{(z+i)}\left[\cancel{2z^2}+2iz-\cancel{2z^2}\right]}{(z+i)^3}$$

$$= \frac{-2}{8i^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

∴ By Cauchy residue theorem

$$\int_C f(z)\,dz = 2\pi i\,[S.O.R]$$

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi i\left[\frac{1}{4i}\right]$$

$\Rightarrow \int_C$ as $z \to \infty$ $\int_{C_R} f(z)\,dz = 0$ and along real axis $z = x$

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{2}$

$R \to \infty$ as $z \to \infty$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx = \frac{\pi}{2}$$

from ①

$$\int_0^{\infty} \frac{x^2}{(x^2+1)^2}\,dx = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

2. Evaluate $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$

Sol:- put $x = z \Rightarrow dx = dz$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \int_C \frac{z^2\,dz}{(z^2+1)(z^2+4)}\,dz = \int_C f(z)\,dz$$

where $f(z) = \frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)}$

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$\therefore$ The poles of $f(z)$ are $i, -i, 2i, -2i$ of which

$z = i, 2i$ lies inside the circle

$$[\text{Res } f(z)]_{z=i} = \underset{z \to i}{\text{lt}} \cancel{(z-i)} \frac{z^2}{\cancel{(z-i)}(z+i)(z-2i)(z+2i)} = \frac{i^2}{(2i)(-i)(3i)} = \frac{-1}{6i}$$

$$[\text{Res } f(z)]_{z=2i} = \underset{z \to 2i}{\text{lt}} \cancel{(z-2i)} \frac{z^2}{(z-i)(z+i)\cancel{(z-2i)}(z+2i)} = \frac{4i^2}{(i)(3i)(4i)} = \frac{-4}{-12i} = \frac{1}{3i}$$

By Cauchy residue theorem

$$\int_C f(z)\,dz = 2\pi i\,[\text{S.O. R}]$$

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi i\left[\frac{-1}{6i} + \frac{1}{3i}\right]$$

as $z \to \infty$ $\int_{C_R} f(z)dz = 0$ and along real axis $z = x$ $R \to \infty$ as $z \to \infty$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi \cancel{i}}{6\cancel{i}}[-1+2] = \frac{\pi}{3}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{3}$$

3. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ by Cauchy residue theorem.

Sol:- $$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$$

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Put $x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \int_C \frac{dz}{(z^2+a^2)^2} = \int_C f(z)\,dz$$

where $f(z) = \frac{1}{(z^2+a^2)^2} = \frac{1}{(z+ai)^2(z-ai)^2}$

poles of $f(z)$ are $z = ai$ and $z = -ai$ which are of order 2

in which $z = ai$ lies inside C_R

$$[\text{Res } f(z)]_{z=ai} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$$

$$= \lim_{z \to ai} \frac{d}{dz} \cancel{(z-ai)^2} \frac{1}{(z+ai)^2 \cancel{(z-ai)^2}}$$

$$= \lim_{z \to ai} \frac{-2}{(z+ai)^3} = \frac{-2}{(2ai)^3} = \frac{\cancel{2}i^2}{\cancel{8}_4 a^3 i^3} = \frac{1}{4a^3 i}$$

By Cauchy residue theorem

$$\int_C f(z)\,dz = 2\pi i\,[S.O.R]$$

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi \cancel{i} \left[\frac{1}{4a^3 \cancel{i}}\right] = \frac{2\pi}{4a^3}$$

as $z \to \infty$ $\int_{C_R} f(z)\,dz \to 0$ and along real axis $z = x$ $R \to \infty$ as $z \to \infty$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi}{4a^3}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{2\pi}{4a^3}$$

from (1) $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\left(\frac{2\pi}{4a^3}\right) = \frac{\pi}{4a^3}$

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4. Evaluate by contour integration $\int_{-\infty}^{\infty} \frac{1}{1+x^6} dx$

Sol :- Put $x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{1+x^6} dx = \int_C \frac{1}{1+z^6} dz$$

where c is the contour consisting of semicircle C_R together with the real axis $-R$ to R

$$f(z) = \frac{1}{1+z^6}$$

The poles of $f(z)$ are the roots of $1+z^6=0$

$$z^6 = -1$$

$$\Rightarrow z = (-1)^{1/6}$$

$$= (\cos\pi + i\sin\pi)^{1/6}$$

$$= (\cos n\pi + i\sin n\pi)^{1/6} \quad \text{where } n = \pm 1, \pm 3, \pm 5$$

$$= (e^{in\pi})^{1/6} = e^{in\pi/6}$$

$\therefore$ The pole of $f(z)$ are $z_1 = e^{i\pi/6}$, $z_2 = e^{-i\pi/6}$, $z_3 = e^{3i\pi/6} = e^{i\pi/2}$

$z_4 = e^{-3i\pi/6} = e^{-\pi i/2}$, $z_5 = e^{i5\pi/6}$, $z_6 = e^{-i5\pi/6}$

$\therefore$ z_1, z_3, z_5 lies inside c and z_2, z_4, z_6 lies outside c

$$f(z) = \frac{1}{1+z^6} = \frac{\phi(z)}{\psi(z)} \quad \text{where } \phi(z) = 1 = -z^6 \quad (\because 1+z^6 = 0,\ z^6 = -1,\ -1 = -z^6)$$

$$\psi(z) = 1+z^6$$

$$\psi'(z) = 6z^5$$

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$$\left[\text{Res} f(z)\right]_{z=e^{i\pi/6}} = \frac{\phi(z)}{\psi'(z)} = \frac{-z^6}{6z^5} = \frac{-z}{6} = \frac{e^{-i\pi/6}}{6}$$

$$\left[\text{Res} f(z)\right]_{z=e^{i\pi/2}} = \frac{\phi(z)}{\psi'(z)} =$$

$$\left[\text{Res} f(z)\right]_{\text{at } z=z_1} = \frac{\phi(z_1)}{\psi'(z_1)} = \frac{-z_1^6}{6z_1^5} = \frac{-z_1}{6} = \frac{-e^{i\pi/6}}{6}$$

$$\left[\text{Res} f(z)\right]_{z=z_3} = \frac{\phi(z_3)}{\psi'(z_3)} = \frac{-z_3}{6} = -\frac{e^{i\pi/2}}{6}$$

$$\left[\text{Res} f(z)\right]_{z=z_5} = \frac{\phi(z_5)}{\psi'(z_5)} = \frac{-z_5}{6} = -\frac{e^{i5\pi/6}}{6}$$

By Cauchy Residue theorem

$$\int_C f(z)\,dz = 2\pi i\,[\text{S.O.R}]$$

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi i\left[-\frac{e^{i\pi/6}}{6} - \frac{e^{i\pi/2}}{6} - \frac{e^{i5\pi/6}}{6}\right]$$

$$= \frac{-2\pi i}{6}\left[e^{i\pi/6} + e^{i\pi/2} + e^{i5\pi/6}\right]$$

$$= \frac{-\pi i}{3}\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} + \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} + \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$$

$$= \frac{-\pi i}{3}\left[\cancel{\frac{\sqrt{3}}{2}} + \frac{i}{2} + 0 + i + \cancel{\left(-\frac{\sqrt{3}}{2}\right)} + \frac{i}{2}\right]$$

$$= \frac{-\pi i}{3}\left[i + i\right] = \frac{-2i^2\pi}{3} = \frac{2\pi}{3}$$

as $z \to \infty$ $\int_{C_R} f(z)\,dz = 0$ along $z = x$, $R \to \infty$ as $x \to \infty$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi}{3} \Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx = \frac{2\pi}{3}$$

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5. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$ by Cauchy residue theorem.

Sol:- $\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \int_C \frac{dz}{1+z^4} = \int_C f(z)\,dz$

The poles of $f(z)$ are the roots of

$1+z^4 = 0$

$z^4 = -1 \Rightarrow z = (-1)^{\frac{1}{4}}$

$= (\cos\pi + i\sin\pi)^{\frac{1}{4}}$

$= (\cos n\pi + i\sin n\pi)^{\frac{1}{4}}$ where $n = \pm 1, \pm 3$

$= (e^{in\pi})^{\frac{1}{4}} = e^{in\pi/4}$

$\therefore$ The poles of $f(z)$ are

$z_1 = e^{i\pi/4}$, $z_2 = e^{-i\pi/4}$, $z_3 = e^{3i\pi/4}$, $z_4 = e^{-3\pi i/4}$

In which z_1, z_3 lies in side C and z_2, z_4 lies outside C

also $f(z) = \frac{1}{1+z^4} = \frac{-z^4}{1+z^4} = \frac{\phi(z)}{\psi(z)}$

where $\phi(z) = -z^4$

$\psi(z) = 1+z^4 \Rightarrow \psi'(z) = 4z^3$

$[\text{Res } f(z)]_{z=z_1} = \frac{\phi(z_1)}{\psi'(z_1)} = \frac{-z_1^4}{4z_1^3} = \frac{-z_1}{4} = \frac{-e^{i\pi/4}}{4}$

$[\text{Res } f(z)]_{z=z_3} = \frac{\phi(z_3)}{\psi'(z_3)} = \frac{-z_3^4}{4z_3^3} = \frac{-z_3}{4} = -\frac{e^{3i\pi/4}}{4}$

By Cauchy Residue theorem

$\int_C f(z)\,dz = 2\pi i\,[S.O.R]$

$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi i\left[\frac{-e^{i\pi/4} - e^{3i\pi/4}}{4}\right]$

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$$= -\frac{\pi i}{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} + \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$

$$= -\frac{\pi i}{2}\left[\cancel{\frac{1}{\sqrt2}} + i\frac{1}{\sqrt2} - \cancel{\frac{1}{\sqrt2}} + i\frac{1}{\sqrt2}\right] = -\frac{\pi i}{\cancel{2}}\left(\frac{\cancel{2}i}{\sqrt2}\right) = \frac{\pi}{\sqrt2}$$

as $z \to \infty$, $\int_{C_R} f(z)dz = 0$ & along $z = x$, $R \to \infty$ as $z \to \infty$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sqrt2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \frac{\pi}{\sqrt2}$$

$=$

Note:- here we have to take only positive 'i' inside C because for suppose $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i = (0,1)$ lies inside semicircle and $e^{-i\pi/2} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = 0 - i = (0,-1)$ lies out side C. This is the reson to take the positives as inside C

6. Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)} dx$

Sol:- $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)} dx = \int_C \frac{dz}{z^2+1} = \int_C f(z)\,dz$

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

The poles of $f(z)$ are $z = i$, $z = -i$ in which $z = i$ lies inside C

$$[\text{Res } f(z)]_{z=i} = \lim_{z \to i} \cancel{(z-i)} \frac{1}{(z+i)\cancel{(z-i)}} = \frac{1}{2i}$$

$$\int_C f(z)\,dz = 2\pi i\,[\text{S.O.R}]$$

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = \cancel{2}\pi \cancel{i}\left(\frac{1}{\cancel{2i}}\right) = \pi$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \pi$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)}\,dx = \pi$$

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Type 3:- Integration of the form $\int_{-\infty}^{\infty} e^{imx}\phi(x)\,dx$ (or) $\int_{-\infty}^{\infty} \cos mx\,\phi(x)\,dx$ (or) $\int_{-\infty}^{\infty} \sin mx\,\phi(x)\,dx$

To evaluate these types of Integrals

Consider $\int_{-\infty}^{\infty} e^{imx}\phi(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = \int_{C} f(z)\,dz$

where C is the contour consisting of semicircle C_R together with real axis $-R$ to R.

By Cauchy residue theorem $\int_C f(z)dz = 2\pi i\,[S.O.R]$

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi i\,[S.O.R]$$

as $z \to \infty$, $f(z) \to 0$ and along real axis $z = x$

$R \to \infty$ as $z \to \infty$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} e^{imx}\phi(x)\,dx = 2\pi i\,[S.O.R]$$

Note:- 1. $\int_{-\infty}^{\infty} x\cos ax\,dx = R.P\int_{-\infty}^{\infty} xe^{iax}\,dx$

2. $\int_{-\infty}^{\infty} x\sin ax\,dx = I.P\int_{-\infty}^{\infty} xe^{iax}\,dx$

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1. Evaluate $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} dx$

Sol:- $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} dx$ = Real part $\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx$

Put $x = z \Rightarrow dx = dz$

$$= R.P \int_{-\infty}^{\infty} \frac{e^{iaz}}{z^2+1} dz \quad = R.P \int_C f(z)\, dz$$

where $f(z) = \frac{e^{iaz}}{z^2+1} = \frac{e^{iaz}}{(z+i)(z-i)}$

The poles of $f(z)$ are $z = i, z = -i$ in which $z = i$ lies inside C and $z = -i$ lies outside C

$$\therefore [\text{Res } f(z)]_{z=i} = \lim_{z \to i} \cancel{(z-i)} \frac{e^{iaz}}{(z+i)\cancel{(z-i)}} = \frac{e^{iai}}{2i} = \frac{e^{-a}}{2i}$$

By Cauchy residue theorem $\int_C f(z)\,dz = 2\pi i\, [S.O.R]$

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi i \left[\frac{e^{-a}}{2i}\right] = \pi e^{-a}$$

as $z \to \infty$ $\int_{C_R} f(z)\,dz = 0$ and along real axis $z = x$

$R \to \infty$ as $z \to \infty$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \pi e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx = \pi e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{\cos ax + i \sin ax}{(x^2+1)} dx = \pi e^{-a}$$

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Equating real parts

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}$$

2. Show that $\displaystyle\int_0^{\infty} \frac{\cos mx}{(a^2+x^2)^2}\,dx = \frac{\pi}{4a^3}(1+am)e^{-am}$

Sol:- $\displaystyle\int_0^{\infty} \frac{\cos mx}{(a^2+x^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos mx}{(a^2+x^2)^2}\,dx \longrightarrow (1)$

Consider $\displaystyle\int_{-\infty}^{\infty} \frac{\cos mx}{(a^2+x^2)^2}\,dx = R.P \int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)^2}\,dx$

$$= R.P \int_c \frac{e^{imz}}{(z^2+a^2)^2}\,dz = R.P \int_c f(z)\,dz$$

$$f(z) = \frac{e^{imz}}{(z^2+a^2)^2} = \frac{e^{imz}}{(z+ai)^2(z-ai)^2}$$

The poles of $f(z)$ are $z=ai$, $z=-ai$ in which

$z=ai$ is pole of order 2 which lies inside C

$$[\text{Res}\, f(z)]_{z=ai} = \frac{1}{(m-1)!} \lim_{z\to ai} \frac{d^{m-1}}{dz^{m-1}} (z-ai)^2 \frac{e^{imz}}{(z-ai)^2(z+ai)^2}$$

here $m=2$

$$= \frac{1}{1!} \lim_{z\to ai} \frac{d}{dz}\left[\frac{e^{imz}}{(z+ai)^2}\right]$$

$$= \lim_{z\to ai}\left[\frac{(z+ai)^2 e^{imz}(im) - e^{imz}\,2(z+ai)}{(z+ai)^4}\right]$$

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$$= \lim_{z \to ai} \frac{(z+ai)\left[(z+ai)\, im\, e^{imz} - 2e^{imz}\right]}{(z+ai)^4}$$

$$= \lim_{z \to ai} \frac{(z+ai)\, im\, e^{imz} - 2e^{imz}}{(z+ai)^3}$$

$$= \frac{(2ai)(im)\, e^{-am} - 2e^{-am}}{(2ai)^3}$$

$$= \frac{-2am\, e^{-am} - 2e^{-am}}{-8a^3 i} = \frac{2am e^{-am} + 2e^{-am}}{8a^3 i}$$

$$= \frac{2e^{-am}[am+1]}{8a^3 i} = \frac{e^{-am}(am+1)}{4a^3 i}$$

By Cauchy's residue theorem

$$\int_C f(z)\, dz = 2\pi i\, [S.O.R]$$

$$\int_{C_R} f(z)\, dz + \int_{-R}^{R} f(z)\, dz = \frac{2\pi \cancel{i}}{4a^3 \cancel{i}} \left[e^{-am}(am+1)\right]$$

as $z \to \infty$ $\int_{C_R} f(z) dz = 0$ along $z = x$, $R \to \infty$ as $z \to \infty$

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi}{2a^3}\left(e^{-am}(am+1)\right)$$

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)^2}\, dx = \frac{\pi}{2a^3}\left[e^{-am}(am+1)\right]$$

Comparing real parts

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+a^2)^2}\, dx = \frac{\pi}{2a^3}\left[e^{-am}(1+am)\right]$$

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$\therefore$ from ①

$$\int_0^\infty \frac{\cos mx}{(x^2+a^2)^2}\,dx = \frac{1}{2}\left[\frac{\pi}{2a^3} e^{-am}(1+am)\right] = \frac{\pi}{4a^3}\left[e^{-am}(1+am)\right]$$

3. using Cauchy's residue theorem Evaluate $\int_0^\infty \frac{x\sin x}{(x^2+a^2)}\,dx$

Sol:- $\int_0^\infty \frac{x\sin x}{(x^2+a^2)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin x}{(x^2+a^2)}\,dx \rightarrow$ ①

Consider $\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+a^2}\,dx =$ Imaginary Part of $\int_{-\infty}^{\infty}\frac{x e^{ix}}{x^2+a^2}\,dx$

$$= I.P\int_C \frac{z e^{iz}}{z^2+a^2}\,dz = I.P\int_C f(z)\,dz$$

where $f(z) = \frac{z e^{iz}}{z^2+a^2} = \frac{z e^{iz}}{(z+ai)(z-ai)}$

The poles of $f(z)$ are $z=ai$, $z=-ai$ in which $z=ai$ lies inside C

$$\left[\text{Res } f(z)\right]_{z=ai} = \lim_{z\to ai}\cancel{(z-ai)}\frac{z e^{iz}}{(z+ai)\cancel{(z-ai)}}$$

$$= \frac{ai\, e^{i(ai)}}{2ai} = \frac{ai\, e^{-a}}{2ai} = \frac{e^{-a}}{2}$$

By Cauchy residue theorem

$$\int_C f(z)\,dz = 2\pi i\,[S.O.R]$$

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \cancel{2}\pi i\left(\frac{e^{-a}}{\cancel{2}}\right) = \pi i e^{-a}$$

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as $z \to \infty$ $\int_{C_R} f(z) \to 0$, along $z = x$ $R \to \infty$ as $z \to \infty$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \pi i\, e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+a^2}\,dx = \pi i e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{x(\cos x + i\sin x)}{x^2+a^2}\,dx = \pi i e^{-a}$$

Equating imaginary parts

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\,dx = \pi e^{-a}$$

$$\therefore \text{from } ① \quad \int_{0}^{\infty} \frac{x \sin x}{x^2+a^2}\,dx = \frac{1}{2}(\pi e^{-a}) = \frac{\pi e^{-a}}{2}$$

4. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx$ by Cauchy residue theorem.

Sol:- $$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \text{R.P of} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx$$

$$= \text{R.P of} \int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz = \text{R.P} \int_C f(z)\,dz$$

where $$f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} = \frac{e^{iz}}{(z+ai)(z-ai)(z+bi)(z-bi)}$$

in which $z = ai$, $z = bi$ lies inside C

$$\therefore [\text{Res} f(z)]_{z=ai} = \lim_{z \to ai} (z-ai) \frac{e^{iz}}{(z+ai)(z-ai)(z+bi)(z-bi)}$$

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$= \dfrac{e^{i ai}}{(2ai)(a-b)i(a+b)i} = \dfrac{e^{-a}}{2ai^3(a^2-b^2)} = \dfrac{e^{-a}}{-2ai(a^2-b^2)} = \dfrac{e^{-a}}{2ai(b^2-a^2)}$

$[\text{Res } f(z)]_{z=bi} = \lim_{z \to bi} \dfrac{\cancel{(z-bi)}\, e^{iz}}{(z-ai)(z+ai)(z+bi)\cancel{(z-bi)}}$

$= \dfrac{e^{i(bi)}}{(b-a)i(b+a)i(2bi)} = \dfrac{e^{-b}}{2bi^3(b^2-a^2)} = \dfrac{e^{-b}}{-2bi(b^2-a^2)}$

$= \dfrac{e^{-b}}{2bi(a^2-b^2)}$

By Cauchy residue theorem

$\int_C f(z)\,dz = 2\pi i\,[S.O.R]$

$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz = 2\pi i \left[\dfrac{e^{-a}}{2ai(b^2-a^2)} + \dfrac{e^{-b}}{2bi(a^2-b^2)} \right]$

$= \dfrac{\cancel{2}\pi \cancel{i}}{\cancel{2i}(b^2-a^2)} \left[\dfrac{e^{-a}}{a} - \dfrac{e^{-b}}{b} \right]$

$= \dfrac{\pi}{b^2-a^2} \left[\dfrac{e^{-a}}{a} - \dfrac{e^{-b}}{b} \right]$

as $z \to \infty$ $\int_{C_R} f(z)\,dz = 0$ and along $z = x$, $R \to \infty$ as $z \to \infty$

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \dfrac{\pi}{b^2-a^2} \left[\dfrac{e^{-a}}{a} - \dfrac{e^{-b}}{b} \right]$

$\int_{-\infty}^{\infty} \dfrac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx = \dfrac{\pi}{b^2-a^2} \left[\dfrac{e^{-a}}{a} - \dfrac{e^{-b}}{b} \right]$

Equating real parts

$\int_{-\infty}^{\infty} \dfrac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \dfrac{\pi}{b^2-a^2} \left[\dfrac{e^{-a}}{a} - \dfrac{e^{-b}}{b} \right]$

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