# Fourier Series :-

(2) Suppose that a given function $f(x)$ defined in $[-\pi, \pi]$ or $[0, 2\pi]$ or in any other interval can be expressed as a trignometric series as

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

$$= \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots (a_n \cos nx + b_n \sin nx) + \dots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $n = 1, 2, 3, \dots$ & $a, b$ are constants.

Such series is called as Fourier series.

(1) Fourier series is an infinite series representation of a periodic function in terms of sine and cosines. Fourier series is possible for continuous functions, periodic functions and function discontinuous in their and derivatives. Fourier series is useful to solve ordinary and partial D.E's particularly with periodic functions appearing as non-homogeneous terms.

## Periodic function :-

A function $f(x)$ is said to be of ~~periodic~~ periodic with period $T > 0$ if for all real $x$, $f(x+T) = f(x)$ and $T$ is the least of such values.

Ex: $\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) = \dots$

The function $\sin x$ is periodic with period $2\pi$,

→ In similarly $\cos x$ is periodic with period $2\pi$,

→ The period of $\tan x$ is '$\pi$'

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

<u>By Euler's Formula</u>:- The Fourier series for the function $f(x)$ in the interval $c \le x \le c+2\pi$ is given by ~~where $a_0 =$~~

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

If function $f(x)$ of Fourier series in interval $[-\pi, \pi]$

Then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

If the interval $\{0, 2\pi\}$, then

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

## Dirichlet conditions for Fourier Expansion:-

Dirichlet has formulated certain conditions known as Dirichlet conditions under .... certain functions posses valid Fourier expansions.

A function $f(x)$ has a valid Fourier series expansion of the form $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

i) $f(x)$ is well defined, periodic, single valued and finite.

ii) $f(x)$ has a finite number of finite discontinuities in any one period.

iii) $f(x)$ has at most a finite number of maxima and minima in the in the interval of defined.

Note: The above conditions are sufficient but not necessary

→ $\sin n\pi = 0$, $\sin 2n\pi = 0$, for $n \in Z$ where '$Z$' is the set of all integers

→ $\cos n\pi = (-1)^n$ , $\cos 2n\pi = 1$ , $n \in Z$

→ $\sin (n+\frac{1}{2})\pi = (-1)^n$, $n \in Z$

→ $\cos (n+\frac{1}{2})\pi = 0$, $n \in Z$

→ $\sin \frac{n\pi}{2} = \begin{cases} (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$

→ $\cos \frac{n\pi}{2} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2}, & \text{if } n \text{ is even} \end{cases}$

1) Find the Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$

<u>sol</u>

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

then $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \Rightarrow \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$

$$\Rightarrow \frac{1}{3\pi}(8\pi^{\cancel{3}\,2}) \Rightarrow \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$$

$$\Rightarrow \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$u = x^2$, $v = \cos nx$

$u' = 2x$, $v_1 = \frac{\sin nx}{n}$

$u'' = 2$, $v_2 = -\frac{\cos nx}{n^2}$

$v_3 = -\frac{\sin nx}{n^3}$

$$\Rightarrow \frac{1}{\pi}\left( \cancel{\cos nx}\, x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$\Rightarrow \frac{1}{\pi}\left[\left(4\pi^2 \frac{\sin n 4\pi}{n} + 4\pi \frac{\cos n 2\pi}{n^2} + 2\frac{\sin n 2\pi}{n^3}\right) - (0)\right]$$

$$\Rightarrow \frac{1}{\cancel{\pi}}\left[\frac{4\cancel{\pi} \cos 2n\pi}{n^2}\right] = \frac{4}{n^2} \quad [\because \cos 2n\pi = 1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) + 2x\left(\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\frac{-4\pi^2 \cos n2\pi}{n} + \frac{2\pi \sin n}{n^2} + \frac{2\cos n 2\pi}{n^3} - \frac{2}{n^3}\right\}$$

$$\Rightarrow \frac{1}{\cancel{\pi}}\left(\frac{-4\pi^{\cancel{2}}}{n} + \cancel{\frac{2}{n^3}} - \cancel{\frac{2}{n^3}}\right) \Rightarrow -4\pi/n$$

$$(1) \Rightarrow \quad x^2 = \frac{4\cancel{8}\pi^2}{\cancel{2}\times 3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$(\because \frac{a_0}{2})$

(1) Obtain the fourier series for $f(x) = x - x^2$ in the interval $[-\pi, \pi]$ Hence show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

Sol The fourier series of $f(x) = x - x^2$ in $[-\pi, \pi]$ is given by

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

Now, we determine the values of $a_0, a_n, b_n$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\,dx - \int_{-\pi}^{\pi} x^2\,dx\right]$$

$$= \frac{1}{\pi}\left[0 - \int_{-\pi}^{\pi} x^2\,dx\right]$$

$$\Rightarrow \frac{1}{\pi}\left(-\frac{x^3}{3}\right]_{-\pi}^{\pi} \Rightarrow \frac{1}{\pi}(-)\left(\frac{\pi^3}{3} + \frac{\pi^3}{3}\right) \Rightarrow -\frac{2\pi^3}{3\pi} \Rightarrow -\frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos nx\,dx$$

$$\left( \begin{array}{l} (u, v) = uv_1 - u'v_2 + u''v_3 - \\ u, u', u'', \ldots \text{ are Differentiation values} \\ v_1, v_2, v_3 \ldots \text{ are integrations} \end{array} \right.$$

$$\Rightarrow \frac{1}{\pi}\left[(x - x^2)\left(+\frac{\sin nx}{n}\right) - (1 - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$\cancel{\Rightarrow \frac{1}{\pi}\left[-\frac{(1+2x)(-)\cos nx}{n^2}\right]_{-\pi}^{\pi} \Rightarrow \frac{1}{\pi}\left[\frac{1 - 2\pi \cos n\pi}{n^2} + \frac{1 + 2\pi}{}\right.}$$

$$\Rightarrow \frac{1}{\pi}\left[(-1 + 2x)\frac{\cos nx}{n^2}\right]_{-\pi}^{\pi} \Rightarrow \frac{1}{\pi}\left[\frac{-1 + 2\pi \cos n\pi}{n^2} - \frac{(-1 - 2\pi)\cos n\pi}{n^2}\right] \left[\because \cos(-\theta) = \cos\theta\right.$$

$$\Rightarrow \frac{1}{\pi}\left(\frac{-1 + 2\pi \cos n\pi + 1 + 2\pi \cos n\pi}{n^2}\right) \Rightarrow \frac{1}{\pi}\left(\frac{4\pi \cos n\pi}{n^2}\right)$$

$$a_n = \frac{-4\cos n\pi}{n^2} = \frac{-4(-1)^n}{n^2} \quad n \neq 0$$

$$\Rightarrow \cancel{-4(-1)^n}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x-x^2)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos nx}{n}\right) - (1-2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$\cancel{= \frac{1}{\pi}\left((-2)\frac{\cos nx}{n^3}\right)_{-\pi}^{\pi} \Rightarrow \frac{1}{\pi}\left(\frac{(-2)\cos n\pi}{n^3} - \frac{(-2)\cos n(-\pi)}{n^3}\right)}$$

$$\cancel{\Rightarrow \frac{1}{\pi}\left[\frac{-2\cos n\pi}{n^3}\right.}$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{-\cos nx}{n} - 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\frac{-2\pi\cos n\pi}{n}\right] \Rightarrow \frac{-2\cos n\pi}{n} = \frac{-2(-1)^n}{n}$$

substitute the values of $a_0, a_n, b_n$ in eqn (1), we get

$$x - x^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{-4(-1)^n}{n^2}\cos nx + \frac{(-2)(-1)^n \sin nx}{n}\right]$$

$$= \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n^2}\cos nx + \frac{2(-1)^{n+1}\sin nx}{n}\right]$$

$$(x-x^2) = \frac{-\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right] \rightarrow (2)$$

<u>Deduction:</u> $x=0$ is a point of continuity of $f(x)$. Hence the Fourier Series of $f(x)$ at $x=0$ converges to $f(0)$

Putting $x=0$ in (2) we get.

$$0 = \frac{-\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \dots\right]$$

$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots\cdots = \frac{\pi^2}{12}$

(or) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

(2) Expand $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 \le x < 2\pi$ in a fourier series

~~obtain~~ the fourier series to ~~represent~~

$f(x) = \frac{1}{\cdot}$ $\qquad a_0 = \frac{\pi^2}{6}$, $a_n = \frac{1}{n^2}$, $b_n = 0$.

Sol: given $f(x) = \left(\frac{\pi - x}{2}\right)^2 \longrightarrow (1)$, $0 \le x \le 2\pi$

the solution of given function. i.e, fourier series of $f(x)$ is

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \rightarrow (2)$$

determine $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4}\,dx \Rightarrow \frac{1}{4\pi}\int_0^{2\pi} \pi^2 + x^2 - 2\pi x\,dx$$

$$= \frac{1}{4\pi}\left[x\pi^2 + \frac{x^3}{3} - \pi x^2\right]_0^{2\pi}$$

$$\cancel{= \frac{1}{4\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} \Rightarrow \frac{1}{4\pi}\left(\frac{8\pi^3}{3}\right) \Rightarrow \frac{8\pi^2}{12}}$$

$$= \frac{1}{4\pi}\left(2\pi(\pi^2) + \frac{8\pi^3}{3} - 4\pi^3\right) \Rightarrow \frac{1}{4\pi}\left[\frac{8\pi^3}{3} - 2\pi^3\right]$$

$$= \frac{1}{4\pi}\left[\frac{8\pi^3 - 6\pi^3}{3}\right] \Rightarrow \frac{2\pi^3}{12\pi} \Rightarrow \frac{\pi^2}{6}$$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$u = (\pi - x)^2$ | $v = \cos nx$

$u' = -2(\pi - x)$ | $v_1 = \frac{\sin nx}{n}$

$u'' = -2(-1) \Rightarrow 2$ | $v_2 = -\frac{\cos nx}{n^2}$

$v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{1}{4\pi}\int_0^{2\pi} \underset{u}{(\pi - x)^2}\cdot \underset{v}{\cos nx}\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(0 + \frac{2\pi \cos 2n\pi}{n^2} + 0\right) - \left(0 - \frac{2\pi \cos 0}{n^2} + 0\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] \quad \left[\because \cos 2n\pi = \cos 0 = 1\right]$$

$$a_n = \frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right] = \frac{1}{n^2}$$

$$b_n = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi - x)(-1)\left[\frac{-\sin nx}{n^2}\right] + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left[-\frac{\pi^2 \cos 2n\pi}{n} - 0 + 2\frac{\cos 2n\pi}{n^3}\right] - \left(\frac{-\pi^2}{n} - 0 + \frac{2}{n^3}\right)\right]$$

$$= \frac{1}{4\pi}\left[\left(\frac{-\pi^2}{n} + \frac{2}{n^3}\right) - \left(\frac{-\pi^2}{n} + \frac{2}{n^3}\right)\right] \quad \because \cos 2n\pi = 1$$

$$b_n = 0$$

Substitute the values of $a_0, a_n, b_n$ in eq (1)

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots$$

3) Find a Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$

Sol: $x^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty} \frac{4\pi}{n}\sin nx$

where; $a_n = \frac{4}{n^2}$, $b_n = \frac{-4\pi}{n}$, $a_0 = \frac{8}{3}\pi^2$

Fourier series:

Q) Expand $f(x) = x \sin x \quad 0 < x < 2\pi$ as a Fourier series

$\therefore \quad x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$u = x, \quad v = \sin x$

$u' = 1, \quad v_1 = -\cos x$

$v_2 = -\sin x$

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x\, dx$

$= \frac{1}{\pi}\left[x(-\cos x) + (1)\sin x\right]_0^{2\pi}$

$= \frac{1}{\pi}\left[-2\pi \cos 2\pi + \sin 2\pi - (0(-\cos 0) + \sin 0)\right]$

$a_0 = -2$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx \Rightarrow \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\, dx$

$\Rightarrow \frac{2}{\pi}\int_0^{2\pi} x[\sin(1+n)x + \sin(1-n)x]$

$\Rightarrow \frac{2}{2\pi}\left[\int_0^{2\pi} x \sin(1+n)x + \int_0^{2\pi} x \sin(1-n)x\, dx\right]$

$\Rightarrow \frac{2}{2\pi}\left[\frac{x(-\cos(1+n)x)}{1+n} + \frac{\sin(1+n)x}{(1+n)^2} + \frac{x(-\cos(1-n)x)}{(1-n)} + \frac{\sin(1-n)x}{(1-n)^2}\right]_0^{2\pi}$

$\Rightarrow \frac{12}{2\pi}\left[\left(\frac{-2\pi\cos(1+n)2\pi}{1+n} - \frac{2\pi\cos(1-n)2\pi}{1-n}\right) - 0\right]$

$\Rightarrow \frac{12}{2\pi}\left(\frac{-\cancel{4}^{2}\pi}{1^2-n^2}\right) \Rightarrow \frac{-2}{1^2-n^2} \Rightarrow \frac{2}{n^2-1} \quad [n \neq 1]$

If $n=1$, then we have

$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos(1)x\, dx \Rightarrow \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x\, dx$

$\Rightarrow \frac{1}{2\pi}\left[\frac{x(-\cos 2x)}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi} \Rightarrow \frac{1}{2\pi}\left[\left(\frac{(2\pi)(-\cos 2(2\pi))}{2} + \frac{\sin 4\pi}{4} - (0)\right]\right.$

$\Rightarrow \frac{1}{2\pi}\left[\frac{2\pi(-1)}{2}\right] = -\frac{1}{2}$

Now $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, \sin nx\, dx \Rightarrow \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx\, dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(1-n)x - \cos(1+n)x]\, dx$

$\Rightarrow \frac{1}{2\pi}\left[\int_0^{2\pi} x \cos(1-n)x\, dx - \int_0^{2\pi} x \cos(1+n)\, dx\right]$

$$= \frac{1}{2\pi}\left[\frac{x\sin(1-n)x}{1-n} + \frac{\cos(1-n)x}{(1-n)^2} + \frac{x\sin(1+n)x}{1+n} + \frac{\cos(1+n)x}{(1+n)^2}\right]_0^{2\pi}$$

$$\Rightarrow \frac{1}{2\pi}\left[\frac{\cos(1-n)2\pi}{(1-n)^2} + \frac{\cos(1+n)2\pi}{(1+n)^2} - \frac{1}{(1-n)^2} - \frac{1}{(1+n)^2}\right]$$

$$\Rightarrow \frac{1}{2\pi}\left[\frac{1}{(1-n)^2} + \frac{1}{(1+n)^2} - \frac{1}{(1-n)^2} - \frac{1}{(1+n)^2}\right] \quad \text{if } (n \neq 1)$$

then, $b_n = 0$

If $n=1$ then we get

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x\, 2\sin^2 x\, dx \qquad \left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx \Rightarrow \frac{1}{2\pi}\left[\int_0^{2\pi} x\,dx - \int_0^{2\pi} x\cos 2x\,dx\right]$$

$$\Rightarrow \frac{1}{2\pi}\left(\frac{x^2}{2} - \frac{x\sin 2x}{2} - \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$\Rightarrow \frac{1}{2\pi}\left(\frac{4\pi^2}{2} - \frac{2\pi\sin 4\pi}{2} - \frac{\cos 4\pi}{4} + \frac{1}{4}\right) \Rightarrow \frac{1}{\cancel{2\pi}}\left(\frac{\cancel{4}\pi^{\cancel{2}}}{\cancel{2}} - \frac{1}{4} + \frac{1}{4}\right) = \boxed{\pi = b_1}$$

Substituting the values of $a_0, a_n, b_n$ in eqn ①, we get,

$$x\sin x = \frac{-2}{2} + \left(-\frac{1}{2}\right)\cos(1)x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + \pi\sin x; \quad [\because b_n = 0]$$

$$\Rightarrow -1 + \pi\sin x - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}$$

Q) $f(x) = e^x$ for $[-\pi, \pi]$; Hence derive a series for $\frac{\pi}{\sinh\pi}$

Sol.

$$a_0 = \frac{2}{\pi}\left(\frac{e^{\pi}-e^{-\pi}}{2}\right) = \frac{2}{\pi}\sinh\pi \qquad \left[\because \frac{e^{\theta}-e^{-\theta}}{2} = \sinh\theta\right]$$

$$a_n = \frac{\cos n\pi\,(e^{\pi}-e^{-\pi})}{\pi(1+n^2)} \Rightarrow \frac{2(-1)^n\sinh\pi}{\pi(1+n^2)}$$

$$b_n = \frac{n\cos n\pi\,(e^{-\pi}-e^{\pi})}{\pi(1+n^2)} \Rightarrow \frac{n(-1)^n(e^{-\pi}-e^{\pi})}{\pi(1+n^2)} \Rightarrow \frac{n(-1)^{n+1}(e^{\pi}-e^{-\pi})}{\pi(1+n^2)} \Rightarrow \frac{2n(-1)^{n+1}\sinh\pi}{\pi(1+n^2)}$$

Q) $f(x) = x\cos x$, $[0, 2\pi]$

Ans: $f(x) = \pi\cos x - \frac{1}{2}\sin x - 2\sum_{n=2}^{\infty}\frac{n}{n^2-1}\sin nx$

## Even and odd functions:-

A function $f(x)$ is said to be even function if $f(-x) = f(x)$ and odd function if $f(-x) = -f(x)$

Ex: $x^2$, $x^4+x^2+1$, $e^x+e^{-x}$, $\cos x$, $\sec x$ are all even function of $x$, and $x$, $x^3$, $x^5+2x^3+3$, $\sin x$, $\operatorname{cosec} x$, $\tan x$ are all odd functions.

Fourier series for Even & odd functions:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

(i) when $f(x)$ is an even function.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

Since '$\cos x$' is an even function, '$f(x)\cos nx$' is also an even function.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

again '$\sin nx$' is odd function, $f(x)\sin nx$ is an odd func

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

Note:- If a function "$f(x)$" is even in $(-\pi, \pi)$ its fourier series expansion contains only "cosine terms".

i.e; $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx \quad , \; n = 0, 1, 2, \ldots$$

Case ii) If a function $f(x)$ defined in $(-\pi, \pi)$ is "odd", it's fourier Expansion contains only "Sine terms"

i.e; $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

(1) Sine terms are odd function

$\cos x$ terms are even function.

$$\sin ax = \frac{e^{ax} - e^{-ax}}{2} \quad , \quad \cos ax = \frac{e^{ax} + e^{-ax}}{2}$$

$\int_0^{2\pi}$ $\int_{-\pi}^{\pi}$

(1) Expand the function $f(x) = x^2$ as a fourier series in $[-\pi, \pi]$ and prove $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos n\theta}{n^2}$

**Sol**

$f(x) = x^2$

$f(-x) = (-x^2) = x^2$

So the given function is even function. Hence in its Fourier series expansion, the sine terms are absent

$$\therefore x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \Rightarrow \frac{2}{\pi}\int_{0}^{\pi} x^2\,dx \quad \cancel{\Rightarrow \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}}$$

$[-\pi,\pi]$

$\int_{-\pi}^{\pi}$

$$\cancel{\Rightarrow \frac{1}{\pi}\left[2\pi - 2\cdot\pi\right] \Rightarrow \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]} \Rightarrow$$

$2\int_0^{\pi}$

$$\Rightarrow \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \Rightarrow \frac{2}{\pi}\left(\frac{\pi^3}{3}\right) \Rightarrow \frac{2\pi^2}{3}$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} \underset{u}{x^2} \underset{v}{\cos nx}\,dx$

$u = x^2$, $u' = 2x$, $u'' = 2$ | $v = \cos nx$, $v_1 = \frac{-\sin nx}{n}$, $v_2 = \frac{-\cos nx}{n^2}$, $v_3 = \frac{\sin nx}{n^3}$

$$= \frac{2}{\pi}\left\{x^2\left(\frac{-\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + (2)\frac{\sin nx}{n^3}\right\}$$

$$= \frac{2}{\pi}\left[2x\frac{\cos nx}{n^2}\right]_0^{\pi} \Rightarrow \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2} - \frac{2(0)\cos n(0)}{n^2}\right]$$

$$= \frac{4\pi\cos n\pi}{\pi n^2} \Rightarrow \frac{4\cos n\pi}{n^2} \Rightarrow \frac{4(-1)^n}{n^2}$$

Substitute the values in eqn (1)

$$x^2 = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

$$\Rightarrow \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \cos nx \Rightarrow \frac{\pi^2}{3} + 4\left[\sum (-1)^{n+1} \cos nx\right]$$

$$x^2 = \frac{\pi^2}{3} + 4\left[\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right.$$

Deduction put $x = 0$

$$0 = \frac{\pi^2}{3} + 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right)$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

put $x = \pi$ then

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

add the values $x=0$ & $x=\pi$ then we get,

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

(2) Find the fourier Series to represent the function $f(x) = |\sin x|$, $-\pi < x < \pi$

Sol Since $|\sin x|$ is an even function

$\therefore b_n = 0$ for all $n$

let $f(x) = |\sin x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow (1)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x|\,dx \Rightarrow \frac{2}{\pi}\int_0^{\pi} \sin x\,dx \quad [\because |\sin x| = \sin x]$$

$$\Rightarrow \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} \Rightarrow \frac{2}{\pi}\left[-\cos\pi + \cos(0)\right]$$

$$a_0 \Rightarrow \frac{2}{\pi}\left[-(-1)+1\right] \Rightarrow \frac{4}{\pi} \quad \left[\left[\because \cos\pi = -1\right] \quad \cos n\pi = (-1)^n\right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \Rightarrow \frac{2}{\pi}\int_0^{\pi} \sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(1+n)x + \sin(1-n)x\right]dx$$

$[\because 2\sin a\cos b = \sin(a+b) + \sin(a-b)]$

$[2\sin x\sin nx = \cos(1-n)x - \cos(1+n)]$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1+n)\pi}{(1-n)} - \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{-1}{\pi}\left[\frac{(-1)^{n+1}-1}{1+n} + \frac{(-1)^{n+1}-1}{1-n}\right]$$

$$= \frac{-1}{\pi}\left\{(-1)^{n+1}\left[\frac{1}{1+n} - \frac{1}{1+n} + \frac{1}{1-n} - \frac{1}{1-n}\right]\right\}$$

$$= \frac{-1}{\pi}\left\{(-1)^{n+1}\left\{\frac{1}{1+n}+\frac{1}{1-n}\right\} - \left\{\frac{1}{1+n}+\frac{1}{1-n}\right\}\right]$$

$$= \frac{-1}{\pi}\left[(-1)^{n+1}\left\{\frac{2}{1-n^2}\right\} - \left\{\frac{2}{1-n^2}\right\}\right]$$

$$= \frac{+2}{\pi(1-n^2)}\left[(-1)^{n+1}-1\right]$$

$$\Rightarrow \frac{-2}{\pi(n^2+1)}\left[1+(-1)^n\right]$$

$$a_n = \begin{cases} 0 & \text{if 'n' is odd } n \neq 1 \\ \frac{-4}{\pi(n^2-1)} & \text{if 'n' is even} \end{cases} \quad \Rightarrow n=1 \text{ i.e. } a_n = 0$$

consider n is even

Substitute the values in eqn (1)

$$|\sin x| = \frac{2}{\pi} + \sum_{n=2,4,6}^{\infty} \frac{-4}{\pi(n^2-1)} \cos nx .$$

$$= \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=2,4,6} \frac{\cos nx}{n^2-1}$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \cdots\right]$$

(3) $f(x) = |\cos x|;\ [-\pi, \pi]$

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \cdots\right)$$

$a^m \cdot a^n = a^{m+n}$

$a^m \div a^n = a^m$

(3) $f(x) = x^3, [-\pi, \pi]$ ; $x^3 = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}(6\pi - \pi^2 n^2)\sin nx$

1) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

2) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx + b\cos bx)$

3) $2\cos a \cos b = \cos(a+b) + \cos(a-b)$

4) $2\cos a \sin b = \sin(a+b) + \sin(a-b)$

5) $2\sin a \cos b \Rightarrow \cos(a-b) - \cos a+b$

6) $2\sin a \sin b = \cos(a-b) - \cos(a+b)$

4) $f(x) = \cosh ax \quad (-\pi, \pi)$ $\quad \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$

is even function so that we have find, $a_0, a_n$

Let, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{a\pi}\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right) = \frac{2}{a\pi} \sinh a\pi$

$a_n = \frac{2a(-1)^n}{\pi(n^2+a^2)}\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right) \Rightarrow \frac{2(a)(-1)^n \sinh a\pi}{\pi(n^2+a^2)}$

$\therefore f(x) = \frac{2a}{\pi}\sinh a\pi \left\{\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2+a^2}\right\}$

i.e; $= \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2a(-1)^n \sinh a\pi}{\pi(n^2+a^2)} \cos nx$ //

## Function Having the points Discontinuity:-

1) Find the Fourier series of the periodic function defined as $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots = \frac{\pi^2}{8}$

**Sol:** let, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \rightarrow (1)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \Rightarrow \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$\Rightarrow \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_{0}^{\pi} x\,dx\right] \Rightarrow \frac{1}{\pi}\left[\left[-\pi x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right]$$

$$\Rightarrow \frac{1}{\pi}\left\{\left[(-\pi(0)) - (-\pi)(-\pi)\right] + \left(\frac{\pi^2}{2} - \frac{0^2}{2}\right)\right\}$$

$$\Rightarrow \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] \Rightarrow \frac{1}{\pi}\left[\frac{-2\pi^2+\pi^2}{2}\right] \Rightarrow \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \Rightarrow \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\}$$

$$\Rightarrow \frac{1}{\pi}\left\{\left[(-\pi)\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[x\frac{\sin nx}{n} - \frac{-\cos nx}{n^2}\right]_{0}^{\pi}\right\}$$

$$\Rightarrow \frac{1}{\pi}\left\{\left(0 - \frac{(-\pi)\sin n(-\pi)}{n}\right) + \left(\frac{\pi \sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \left(\frac{0\,\sin n(0)}{n} + \frac{\cos n(0)}{n^2}\right)\right)\right\}$$

$$\Rightarrow \frac{1}{\pi}\left\{\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right\} \Rightarrow \frac{1}{\pi}\left\{\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right\} \Rightarrow \frac{1}{\pi}\left\{\frac{(-1)^n - 1}{n^2}\right\}$$

$$\therefore a_1 = \frac{-2}{1^2\pi},\ a_2 = 0,\ a_3 = \frac{-2}{3^2\pi},\ a_4 = 0,\ a_5 = \frac{-2}{5^2\pi}, \ldots$$

and $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \Rightarrow \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\}$$

$$\Rightarrow \frac{1}{\pi}\left\{\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-x\cdot\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_{0}^{\pi}\right\}$$

$$\Rightarrow \frac{1}{\pi}\left\{\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}(\cos n\pi)\right\} \Rightarrow \frac{1}{n}(1 - 2\cos n\pi) \Rightarrow \frac{1}{n}\{1 - 2(-1)^n\}$$

$\therefore b_1 = 3,\ b_2 = \frac{-1}{2},\ b_3 = 1,\ b_4 = \frac{-1}{4}$ and so on.

Substituting the values of $a_0, a_n, b_n$ in eqn ①,

we get,

$$f(x) = \frac{-\pi}{4} + \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right) + \left(3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{4\sin 4x}{4} + \dots\right) \rightarrow ②$$

put $x = 0$ in eqn ②, we obtain

$$0 = -\frac{\pi}{4} + \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$\frac{\pi}{4} = \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Q) $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x^2, & 0 \le x \le \pi \end{cases}$

$a_0 = \frac{\pi^2}{3}$, $a_n = \frac{2}{n^2}(-1)^n$ for $n = 1, 2, 3 \dots$,

$$b_n = \frac{-\pi}{n}(-1)^n + \frac{2}{\pi n^3}\left[(-1)^n - 1\right]$$

$$f(x) = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos nx + \sum_{n=1}^{\infty}\left[\frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}(-1)^n - 1\right]\sin nx$$

~~Notes~~:

# Half-Range Fourier series:

**Sine series:-** $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

**Cosine series:-** $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$

① Expand $f(x) = \cos x$, $0 \leq x < \pi$ in half range sine series.

**Ans** let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$. $\longrightarrow$ ①

then $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2 \sin nx \cos x$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x + \sin(n-1)x] \, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^n}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right) \Rightarrow \frac{1}{\pi}\left[\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} + \left\{\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[\left\{(-1)^n + 1\right\}\left[\frac{1}{n+1} + \frac{1}{n-1}\right]\right]$$

$$= \frac{1}{\pi}\left[\left\{(-1)^n + 1\right\}\frac{2n}{n^2-1}\right] \Rightarrow \frac{2n}{\pi}\left[\frac{1+(-1)^n}{n^2-1}\right]$$

If 'n' is odd then $b_n = 0$ $\quad n \neq 1$

If 'n' is even then $b_n = \frac{4n}{\pi(n^2-1)}$

∴ If 'n' is even $b_n = \frac{4n}{\pi(n^2-1)}$.

Substitute the value in ①

$$f(x) = \sum_{n=2,4,6\ldots}^{\infty} \frac{4n}{\pi(n^2-1)} \sin nx$$

$$= \frac{4}{\pi} \sum_{n=2,4,\ldots}^{\infty} \frac{n}{(n^2-1)} \sin nx$$

$$= \cancel{\frac{4}{\pi}\left[\frac{2}{3}\sin 2x +\right.}$$

If $n$ is even then replace '$n$' is '$2n$'

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{2n}{4n^2-1} \sin 2nx$$

$$= \frac{8}{\pi}\left[\frac{1}{3}\sin 2x + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \cdots\right.$$

② If $f(x) = \begin{cases} kx & 0 \le x < \frac{\pi}{2} \\ k(\pi - x) & \frac{\pi}{2} < x < \pi \end{cases}$ Find the half range sine series

Sol The half range sine series of given $f(x)$ in interval $(0,\pi)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow ①$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} f(x) \sin nx \, dx + \int_{\pi/2}^{\pi} f(x) \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} kx \cdot \sin nx \, dx + \int_{\pi/2}^{\pi} k(\pi - x) \sin nx \, dx\right]$$

$$= \frac{2k}{\pi}\left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx\right]$$

$$= \frac{2k}{\pi}\left[\left\{x\cdot\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right\}_0^{\pi/2} + \left\{(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\frac{-\sin nx}{n^2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2k}{\pi}\left[\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi/2} + \left[\frac{-(\pi - x)\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi}\right]$$

$$\pi - \frac{\pi}{2}$$

$$= \frac{2k}{\pi}\left[\left\{-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{\sin n\pi}{2n^2}\right\} - \left\{0 + \frac{\sin n0}{n^2}\right\} - \left\{\cancel{\pi - \frac{\pi}{2}} - \frac{\pi}{2}\cos\frac{n\pi}{2} - \frac{\sin n\pi}{2n^2}\right\}\right]$$

$$= \frac{2k}{\pi}\left(-\frac{\pi}{2n}\cancel{\cos\frac{n\pi}{2}} + \frac{\sin n\pi}{2n^2} + \cancel{\frac{\pi}{2}\cos\frac{n\pi}{2}} + \frac{\sin n\pi}{2n^2}\right)$$

$$\Rightarrow \frac{2k}{\pi}\cdot\frac{2\sin n\pi}{2n^2} = \cancel{\frac{4k}{\pi}\frac{2\cos n\pi}{n^2}}$$

$$\Rightarrow \frac{4k}{\pi}\frac{\sin n\pi}{2n^2} \Rightarrow \frac{4k}{\pi n^2}\cdot\sin\frac{n\pi}{2}$$

where $b_n = 0$ for $n$ is even. i.e; $b_2 = b_4 = b_6 = \dots = 0$

~~Bout~~ for $n = 1, 3, 5, \dots$ then $b_n \neq 0$

Hence $$f(x) = \frac{4k}{\pi}\sum_{n=1,3,5\dots}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx$$

$$= \frac{4k}{\pi}\left[\frac{1}{1^2}\sin\frac{\pi}{2}\cdot\sin x + \frac{1}{3^2}\sin\frac{3\pi}{2}\sin 3x + \frac{1}{5^2}\sin\frac{5\pi}{2}\sin 5x + \dots\right.$$

Q) Find Half Range sine series for $f(x) = x(\pi - x), [0, \pi]$

Hence show that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$

Ans $b_n = \frac{4}{\pi n^3}(1-(-1)^n)$ and put $x = \frac{\pi}{2}$ for get above result.

Q) Obtain the Fourier ~~series~~ cosine series $f(x) = x \sin x \ [0, \pi]$

show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \cdots = \frac{\pi - 2}{4}$

Ans $a_0 = 2$, $a_n = \frac{2(-1)^{n+1}}{n^2-1}$ ; $n \neq 1$

$a_2 = \frac{-2}{1.3}$, $a_3 = \frac{2}{2.4}$, $a_4 = \frac{-2}{3.5}$, $a_5 = \frac{2}{4.6}$ ; ....

and $a_1 = -1/2$ & put $x = \pi/2$ then we get above result.

$\sin(1+n)\pi/2$ $\quad (-1)^{1+\frac{n-1}{2}} \Rightarrow$

## Intervals other than $[-\pi,\pi]$ and $[0,2\pi]$

Suppose the intervals $[0,2l]$, $[-l,l]$

Let $f(x)$ be a periodic function with period $2l$. defined in the interval $c < x < c+2l$ then Fourier Expansion for $f(x)$ is as follows

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$$

If Interval $[0,2l]$

put $c=0$ in the beside formulas

where $$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x).\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x).\sin\frac{n\pi x}{l}\,dx$$

Fourier series for Even and odd functions $[-l,l]$

Case(i):- Fourier cosine series

Let $f(x)$ be even function $[-l,l]$. then.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx \quad [\because f(x) \text{ is even}]$$

$$a_n = \frac{2}{l}\int_0^{l} f(x).\cos\frac{n\pi x}{l}\,dx$$

case (ii) Fourier sine series:-

Let $f(x)$ be an odd function in $[-l, l]$

Then, $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

where $b_n = \frac{2}{l}\int_0^l f(x) . \sin \frac{n\pi x}{l} \, dx$.

① Express $f(x) = x^2$ as a Fourier series in $[-l, l]$

Sol Since $f(-x) = (-x)^2 = x^2 = f(x)$.

$\therefore$ $f(x)$ is even function. then Fourier series of given function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n . \cos \frac{n\pi x}{l} \longrightarrow ①$$

from ① $a_0 = \frac{2}{l}\int_0^l f(x)\,dx \Rightarrow \frac{2}{l}\int_0^l x^2\,dx$

$$= \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l \Rightarrow \frac{2}{\cancel{l}}\left[\frac{l^{\cancel{3}2}}{3}\right] = \frac{2l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x) . \cos \frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x^2 . \cos \frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x^2 . \left(\frac{\sin \frac{n\pi x}{l}}{n\pi / l}\right) - 2x\left(\frac{-\cos \frac{n\pi x}{l}}{n^2\pi^2 / l^2}\right) + 2\left(\frac{-\sin \frac{n\pi x}{l}}{n^3\pi^3 / l^3}\right)\right]_0^l$$

$\therefore$ $\sin n\pi = 0$ so in above eqn 'sin' series is vanish

$$= \frac{2}{l}\left[2x \frac{\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right]_0^l$$

$$= \frac{2}{l}\left[\left\{2l \frac{\cos n\pi l/l}{n^2\pi^2/l^2}\right\} - \{0\}\right]$$

$$= \frac{2}{\cancel{l}}\left[2\cancel{l} \frac{\cos n\pi}{n^2\pi^2/l^2}\right] \Rightarrow \frac{4 l^2 \cos n\pi}{n^2\pi^2}$$

$$a_n = \frac{4l^2(-1)^n}{n^2\pi^2}$$

substitute the values of $a_0$, $a_n$ in eq ① then we get

$$x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2} \cos\frac{n\pi x}{l}.$$

$$= \frac{l^2}{3} + \frac{(-)4l^2}{1^2\pi^2}\cos\frac{\pi x}{l} + \frac{4l^2}{2^2\pi^2}\frac{\cos 2\pi x}{l} - \frac{4l^2}{3^2\pi^2}\cos\frac{3\pi x}{l} + \dots$$

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos\frac{n\pi}{l}}{1^2} - \frac{\cos\frac{2\pi x}{l}}{2^2} + \frac{\cos\frac{3\pi x}{l}}{3^2} \dots\dots\right]$$

$\therefore$ which is our required fourier Expansion of given function.

② Find fourier series to $f(x) = x^2 - 2$ when $-2 \le x \le 2$.

Sol. $f(x) = x^2 - 2$ is even function. Here $l = 2$

then fourier series of given f(x) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}. \longrightarrow ①$$

find the values of $a_0$, $a_n$ as follows

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \Rightarrow \frac{2}{2}\int_0^2 (x^2-2)\,dx$$

$$= \left[\frac{x^3}{3} - 2x\right]_0^2 \Rightarrow \left[\left\{\frac{(2)^3}{3} - 2(2)\right\} - \{0\}\right]$$

$$= \left[\frac{8}{3} - 4\right] = -\frac{4}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx \Rightarrow \frac{2}{2}\int_0^2 (x^2-2)\cos\frac{n\pi x}{2}\,dx \quad [\because l=2]$$

$$= \int_0^2 (x^2-2)\cos n\pi x/2$$

$$= \left[(x^2-2)\left(\frac{\sin n\pi x/2}{n\pi/2}\right) - (2x)\left(\frac{-\cos n\pi x/2}{n^2\pi^2/4}\right) + (2)\left(\frac{-\sin n\pi x/2}{n^3\pi^3/8}\right)\right]_0^2$$

$$= \left[2x\,\frac{\cos n\pi x/2}{n^2\pi^2/4}\right]_0^2 \Rightarrow \left[\left\{2(2)\left(\frac{\cos n\pi 2/2}{n^2\pi^2/4}\right)\right\} - \{0\}\right]$$

$$= 4\left(\frac{\cos \pi n}{n^2\pi^2/4}\right) \Rightarrow \cancel{\frac{4}{\pi}(-1)^n}$$

$$a_n = \frac{16}{n^2\pi^2}\left(\cos\frac{n\pi 2}{2}\right) = \frac{16}{n^2\pi^2}(\cos n\pi) = \frac{16}{n^2\pi^2}(-1)^n$$

Substitute the values of $a_0$, $a_n$ in eqn (1), we get,

$$x^2-2 = \frac{-4}{3\times 2} + \sum_{n=1}^{\infty}\frac{16}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{2} \quad \because l=2.$$

$$= -\frac{4}{6} + \frac{16}{\pi^2}\left[\frac{\cos\pi x}{2} - \frac{\cos\pi x}{2^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} - \frac{\cos\frac{4\pi x}{2}}{4^2} + \cdots\right]$$

$\therefore$ which is our required Fourier Expansion of f(x).

$[0, 2l]$

Q) $f(x) = 2x - x^2$ ; $0 < x < 3$ Find Fourier series of periodicity '3' i.e; $\boxed{2l = 3} \Rightarrow \boxed{l = 3/2}$

Sol. The F.S of given $f(x)$ is as follows:

$$2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{3}\right) + b_n \sin\left(\frac{2n\pi x}{3}\right) \quad \because l = 3/2$$

$\boxed{a_0 = 0}$, ie; $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx \Rightarrow \frac{2}{3}\int_0^3 (2x - x^2)\,dx = 0$ $\boxed{\because 2l = 3}$

$$a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\frac{n\pi x}{l}\,dx \Rightarrow \frac{2}{3}\int_0^3 (2x - x^2)\cos\left(\frac{2\pi n x}{3}\right)dx$$

$$\boxed{a_n = \frac{9}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right)dx \Rightarrow \frac{2}{3}\int_0^3 (2x - x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$\boxed{b_n = \frac{3}{n\pi}}$$

$$\therefore 2x - x^2 = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{\pi n}\sin\left(\frac{2n\pi x}{3}\right)\right]$$

Q) Find Fourier series for $f(x) = 2lx - x^2$ $[0, 2l]$ and show that $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

Sol $a_0 = 4l^2$, $a_n = \frac{-4l^2}{n^2\pi^2}$, $b_n = 0$

and put $x = 0$ then we get above result.

Q) Find Fourier series expansion for $\underline{[-2, 2]}$

$$f(x) = \begin{cases} 2, & \text{if } -2 \le x \le 0 \\ x, & \text{if } 0 \le x \le 2 \end{cases}$$

Sol here $l = 2$, $a_0 = 3$, $a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$

$b_n = \frac{-2}{n\pi}$

if 'n' is even, then $a_n = 0$

if 'n' is odd, then $a_n = \frac{-4}{n^2\pi^2}$

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\sum_{n=1,3,5..}^{\infty} \frac{1}{n^2}\cos\left(\frac{n\pi x}{2}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{2}\right)$$

Q) Find Fourier Series expansion for $\underline{[0, 2]}$ i.e; $[0, 2l]$

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2, \end{cases}$$

sol: here $2l = 2, \Rightarrow l = 1$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$$

Q) Find the Fourier Series for $f(x) = x - x^2$; $(-1, 1)$

Sol: here $\boxed{l = 1}$

$$f(x) = \frac{-1}{3} + \frac{4}{\pi^2}\left(\frac{\cos\pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \cdots\right) + \frac{2}{\pi}\left(\sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x - \cdots\right)$$

Q) ~~$f(x) = \begin{cases} \pi x; & 0 < x < 1 \\ \pi(2-x); & 1 < x < 2 \end{cases}$~~

## Half Range Expansion:- $(0, l)$

### Half-Range Sine series:- $(0, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

### Half-Range Cosine series: $(0, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\, dx$

$a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

1) Find the Half-Range sine series for $f(x) = ax+b$ ; $0<x<1$

Sol Half-Range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \longrightarrow (1)$$

here $l=1$ [$\because$ the intervals are $(0,1)$ as $(0,l)$]

$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \Rightarrow 2\int_0^1 (ax+b) \sin n\pi x\, dx$

$\Rightarrow 2\left\{(ax+b)(-\cos n\pi x) - (a)^{-\sin n\pi x}\right\}_0^1$

$\Rightarrow 2\left\{[(a+b)(-\cos n\pi) + a \sin n\pi] - [(a(0)+b)(-\cos n\pi(0)) + a \sin n\pi(0)]\right\}$

$b_n \Rightarrow 2\{(a+b)(-1)^{n+1} + b\}$

$b_n = \left\{\frac{-2a}{n\pi}\right.$ , when 'n' is even

$b_n = \left\{\frac{2}{n\pi}(a+2b)\right.$ , if 'n' is odd.

Substitute the value $b_n$ in eqn ①

$$ax+b = \frac{2}{\pi}(a+2b)\sin \pi x - \frac{2a}{2\pi}\sin 2\pi x + \frac{2}{3\pi}(a+2b)\sin 3\pi x - \cdots$$

Q) Find Fourier Half-Range cosine series

$f(x) = x(2-x)$, in $0 \le x \le 2$ and hence show that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Sol: $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

here $l = 2$, $\because$ the interval $[0,2]$ of $[0,l]$

$$a_0 = \frac{2}{2}\int_0^2 (2x - x^2)\,dx = 4/3$$

$$a_n = \frac{2}{l}\int_0^2 f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$\Rightarrow \frac{2}{2}\int_0^2 (2x-x^2)\cos\left(\frac{n\pi x}{2}\right)dx \Rightarrow \int_0^2 (2x-x^2)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow \left[(2x-x^2)\frac{\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)} + (2-2x)\frac{\cos\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^2} + (2)\frac{\sin\left(\frac{n\pi x}{2}\right)}{(n\pi/2)^3}\right]_0^2$$

$$\Rightarrow \frac{-8}{n^2\pi^2}\cos n\pi - \frac{8}{n^2\pi^2} \Rightarrow \frac{-8}{n^2\pi^2}\left[1+(-1)^n\right]$$

$$\therefore a_n = \begin{cases} \frac{-16}{n^2\pi^2}, & \text{if 'n' is even} \\ 0, & \text{if n is odd} \end{cases}$$

consider 'n' is even then $a_n = \frac{-16}{n^2\pi^2}$

substitute the value of $a_0, a_n$ in eqn ①

$$2x - x^2 = \frac{2}{3} - \frac{16}{\pi^2}\sum_{n=2,4,6\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2}$$

$$2x - x^2 = \frac{2}{3} - \frac{16}{\pi^2}\left\{\frac{1}{2^2}\cos\pi x + \frac{1}{4^2}\cos 2\pi x + \frac{1}{6^2}\cos 3\pi x + \cdots\right\}$$

put $x=1$ in above series

$$2-1 = \frac{2}{3} - \frac{16}{\pi^2}\left[\frac{1}{2^2}\cos\pi x + \frac{1}{4^2}\cos 2\pi + \frac{1}{6^2}\cos 3\pi + \dots\right]$$

$$1 = \frac{2}{3} - \frac{16}{\pi^2}\left[-\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots\right]$$

$$1 = \frac{2}{3} - \frac{\cancel{16}^{4}}{\cancel{4}\pi^2}\left[-1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$1 = \frac{2}{3} + \frac{4}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right]$$

$$1 - \frac{2}{3} = \frac{4}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right]$$

$$\frac{1}{3} \times \frac{\pi^2}{4} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

3) Half-Range cosine series $f(x) = (x-1)^2$ $0 < x < 1$,
show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$, here $l = 1$,

Sol. $a_0 = \frac{2}{3}$, $a_n = \frac{4}{n^2\pi^2}$

If $x=0$ in then we get $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$
If $x=1$, then $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{5^2} + \dots = \frac{\pi^2}{12}$
adding the series of $x=0$ & $x=1$, Then we get, $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

4) Half Range sine series $f(x) = e^x$ in $0 < x < 1$

$$b_n = \frac{2n\pi}{n^2\pi^2 + 1}\left[1 - e(-1)^n\right]$$

5) H.R. cosine series $f(x) = \sin\left(\frac{\pi x}{l}\right)$; $(0, l)$

$$a_0 = \frac{4}{\pi}, \quad a_n = \begin{cases} 0, & \text{if } 'n' \text{ is odd} \\ \frac{-4}{\pi(n+1)(n-1)} & \text{if } 'n' \text{ is even} \end{cases}$$

$$f(x) = |\cos x| \quad [-\pi, \pi]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \;\&\; b_n = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \Rightarrow \frac{2}{\pi}\left\{\int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right\}$$

$$\Rightarrow \frac{2}{\pi}\left\{\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx\right\}$$

$$\Rightarrow \frac{2}{\pi}\left[\left[\sin x\right]_0^{\pi/2} + \left[\sin x\right]_{\pi/2}^{\pi}\right]$$

$$\Rightarrow \frac{2}{\pi}\left\{\sin(\pi/2) - \sin(0) + (-\sin(\pi)) - (-\sin(\pi/2))\right\}$$

$$\Rightarrow \frac{2}{\pi}\{1+1\} \Rightarrow 4/\pi$$

$$a_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} \cos x \cos nx\,dx - \int_{\pi/2}^{\pi} \cos x \cos nx\,dx\right\}$$

$$\Rightarrow \frac{1}{\pi}\left\{\int_0^{\pi/2} (\cos(1+n)x + \cos(1-n)x)\,dx - \int_{\pi/2}^{\pi} (\cos(1+n)x + \cos(1-n)x)\,dx\right.$$

$$\Rightarrow \frac{1}{\pi}\left\{\left(\frac{\sin(1+n)x}{1+n} + \frac{\sin(1-n)x}{1-n}\right)_0^{\pi/2} - \left(\frac{\sin(1+n)x}{1+n} + \frac{\sin(1-n)x}{1-n}\right)_{\pi/2}^{\pi}\right\}$$

$$\Rightarrow \frac{1}{\pi}\left\{\frac{\sin(1+n)\pi/2}{1+n} + \frac{\sin(1-n)\pi/2}{1-n} + \frac{\sin(1+n)\pi/2}{1+n} + \frac{\sin(1-n)\pi/2}{1-n}\right.$$

$$\Rightarrow \frac{1}{\pi}\left\{\frac{2\sin(1+n)\pi/2}{1+n} + \frac{2\sin(1-n)\pi/2}{1-n}\right\}$$

$$\Rightarrow \cancel{\frac{1}{\pi}\left(2\sin(1+n)\frac{\pi}{2}\left(\frac{1}{1+n}+\frac{1}{1-n}\right)\right)} = \frac{2}{\pi}\left\{\frac{\sin(1+n)\pi/2}{1+n} + \frac{\sin(1-n)\pi/2}{1-n}\right\}$$

$$\cancel{\frac{1}{\pi}\left\{2\sin\left(\frac{2}{1-n^2}\right)\right\} = \frac{4}{\pi(1-n^2)}} \Rightarrow \frac{2}{\pi}\left\{\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\left(\frac{n\pi}{2}\right)}{n-1}\right\}$$

$$\cancel{\frac{2}{\pi}\left\{\frac{\cos(1+n)\pi/2}{1+n} + \cos(1-n)\pi/2\right.}$$

$$\Rightarrow \frac{-4\cos n\pi/2}{\pi(n^2-1)} \quad \text{if } n \neq 1$$

$$\therefore |\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left\{\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \dots\right\}$$

## Parsevals Formula:-

To prove that $\int_{-l}^{l} [f(x)]^2 dx = l \left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right)$

provided that the Fourier series for $f(x)$ converges uniformly in the interval $(-l, l)$

proof: Fourier series for $f(x)$ in $(-l, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right] \rightarrow ①$$

multiplying both sides of ① by $f(x)$ and integrating term by term from $-l$ to $l$, which is justified as the series ① is ~~con~~ uniformly convergent. we get,

$$\int_{-l}^{l} [f(x)]^2 dx = \frac{a_0}{2} \int_{-l}^{l} f(x) dx + \sum_{n=1}^{\infty} \left[ a_n \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx + b_n \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx \right] \rightarrow ②$$

we know, $a_0 = \frac{1}{l} \int_{-l}^{l} f(x) dx$ i.e; $\int_{-l}^{l} f(x) dx = l a_0$

$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$, then $\int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx = l a_n$

$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$, then $\int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx = l b_n$

then from ② substitute above values in eqn ② then

$$② \Rightarrow \int_{-l}^{l} [f(x)]^2 dx = l \left[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

which is the desired parsevals formula.

Case (i): If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$ in $(0, ~~2\pi~~)$, then
$(0, 2l)$

$$\int_0^{2l} [f(x)]^2 dx = l\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$$

Case (ii):- If the half-range cosine series in $(0, l)$ for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \text{ then}$$

$$\int_0^{l} [f(x)]^2 = \frac{l}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \dots \infty\right]$$

Case (iii):- If the half-range sine series in $(0, l)$ for $f(x)$ is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \text{ then}$$

$$\int_0^{l} [f(x)]^2 = \frac{l}{2}\left\{b_1^2 + b_2^2 + b_3^2 + \dots \infty\right\}$$