The Rules of sum and product.

In many Situations of computational work, we employ two basic rules of counting, called the Sum Rule and the product Rule. These rules are stated and illustrated in the Following Paragraphs.

The Sum rule.

Suppose two tasks T_1 and T_2 are to be performed. if the task T_1 can be performed in m different ways and the task T_2 can be performed in n different ways and if these two tasks cannot be performed simultaneously, then one of the two tasks (T_1 or T_2) can be performed in $m+n$ ways.

More generally, if $T_1, T_2, T_3, ---- T_k$ are k tasks such that no two of these tasks can be performed at the same time and if the task T_i can be performed in n_i different ways, then one of the k tasks (namely T_1 or T_2 or T_3 ----, or T_k) can be performed in $n_1+n_2+----+n_k$ ways.

Example 1 :-

Suppose there are 16 boys and 18 girls in a class and we wish to select one of these Students (either a boy or a girl) as a class representative. the number of ways of selecting a boy is 16 and the number of ways of selecting a girl is 18. Therefore the number of ways of selecting a student (boy or girl) is $16+18 = 34$.

Example 2 :-

Suppose a hostel Library has 12 books on Mathematics, 10 books on physics, 16 books on computer Science and 11 books on Electronics. suppose a student wishes to choose one of these books for study. the number of ways in which he can choose book is

$12+10+16+11 = 49$.

Example 3 :-

Suppose T1 is the task of selecting a prime number less than 10 and Task T2 is of Selecting a even number less than 10. then T1 can be performed in 4 ways (- by selecting 2 or 3 or 5 or 7), and Task T2 can be performed in 4 ways (-by selecting 2 or 4 or 6 or 8). But, since 2 is both a prime and an even number less than 10, the task T1 or T2 can be performed in

4 + 4 - 1 = 7 ways.

The product rule:-

Suppose that two tasks T1 and T2 are to be performed one after the other. if T1 can be performed in n_1 different ways, & for each of these ways T2 can be performed in n_2 different ways, then both of the tasks can be performed in $n_1 n_2$ different ways.

More generally, suppose that k tasks $T_1, T_2, T_3, \ldots T_k$ are to be performed in a Sequence. if T1 can be performed in n_1 different ways and for each of these ways T2 can be performed in n_2 different ways, and for each of $n_1 n_2$ different ways and for each of $n_1 n_2$ different ways of performing T1 & T2 in that order, T3 can be performed in n_3 different ways, and so on, then the sequence of tasks $T_1, T_2, T_3, \ldots T_k$ can be performed in $n_1, n_2, n_3, \ldots n_k$ different ways.

Example 4 :-

Suppose a person has 3 shirts and 5 ties. Then he has 3×5 = 15 different ways of choosing a shirt and a tie.

Example ~~4~~ 5 :-

Suppose we wish to construct sequences of four symbols in which the first 2 are English letters and the next 2 are single digit numbers. if no letter or digit can be repeated, then the number of different sequences that we can construct is 26 × 25 × 10 × 9 = 58500. * if repetition of letters & digits is allowed then the number of different sequences that we can construct is

$$26 \times 26 \times 10 \times 10 = 67600.$$

Example 6 :-

Suppose a restaurant sells 6 South indian dishes, 4 North indian dishes, 3 hot beverages & 2 cold bevarages. For breakfast a student wishes to buy 1 south indian dish & 1 hot bevarage, or 1 North indian dish & 1 cold beverage. Then he can have the first choice in $6 \times 3 = 18$ ways. & he can have the second choice in $4 \times 2 = 8$ ways. The total number of ways he can buy his breakfast items is $18 + 8 = 26$

The following are some more illustrative Examples:-

Example 7 :-

There are 20 married couple in a party. Find the numbe of ways of choosing one woman & one man from the party such that the two are not married to each other.

Sol :- From the party, a women can be choosen in 20 ways. Among the 20 men, in the party, one is her husband. Out of the 19 other men, one can be choosen in 19 ways. Therefore the required number is $20 \times 19 = 380$.

Example 8 :-

A License plate consists of two English letters followed by four digits. If repetitions are allowed, how many of the plates have only vowels (A, E, I, O, U) and even digits?

Soln :- Each of the first two positions in a plate can be filled in 5 ways (with vowels) and each of the remaining four places can be filled in 5 ways (with digits 0, 2, 4, 6, 8). Therefore, the number of possible license plates of the given type is $(5 \times 5) \times (5 \times 5 \times 5 \times 5) = 5^6 = 15,625$.

Example 9 :-

There are four bus routes between the places A & B and three bus routes between the places B and C. Find the number of ways a person can make a round trip from A to A via B. If he does not use a route more than once

Soln :- The person can travel from A to B in four ways and from B to C in three ways, but only in two ways from C to B. & only in three ways from B to A. if he does not use a route more than once. therefore, the number of ways he can made the round trip under the given condition is

$$4 \times 3 \times 2 \times 3 = 72$$

example 10:-

Let A be a set with n elements. How many different sequences, each of length r, can be formed using the elements from A. if the elements in the sequence may be repeated?

Soln :- Since repetition is allowed., each place in the sequence can be filled in n different ways. thus, in a sequence of length r, there are n^r ways of filling the r places in the sequence. This means that there are n^r possible sequences (of the required type).

Example 11:-

a) Find the number of binary sequences of length n.

b) Find the number of binary sequences of length n that contain an even number of 1's.

a) soln :- A binary sequence of length n contains n positions. Each of these positions can be filled in two ways (with 0 or 1) therefore, the number of ways of filling n positions in 2^n. This is precisely the number of binary sequences of length n.

b) soln :- In binary sequence of length $n-1$ has an even number of 1's we append the digit 0 to it to obtain a binary sequence of length n which contains an even number of 1's. if a binary sequence of length $n-1$ contains an odd number of 1's we append the digit 1 to it to obtain a sequence of length n which contains an even number of 1's. As such, the number of binary sequences of length n with an even number of 1's is equal to the number of binary sequences of length $n-1$, which is ~~2^n~~ 2^{n-1}.

Example 12:-

A bit is either 0 or 1. A byte is a Sequence of 8 bits. Find (i) The number of bytes,
(ii) The number of bytes that begin with 11 & End with 11,
(iii) The number of bytes that begin with 11 & do not end with 11,
(iv) The number of bytes that began with 11 or end with 11.

Soli:- (i) Since Each byte contains 8 bits & each bit is a 0 or 1 (two choices), the number of bytes is $2^8 = 256$.

(ii) In a byte begining & Ending with 11, therefore there occur 4 open positions. These can be Filled in $2^4 = 16$ ways. Therefore, there are 16 bytes which began & End with 11.

(iii) There occur six open positions in a byte beginning with 11. These positions can be Filled in $2^6 = 64$ ways. Thus, there are 64 bytes that begin with 11. Since there are 16 bytes that began & End with 11.

The number of bytes that began with 11 but not end with 11 is $64 - 16 = 48$.

(iv) As in (iii), the number of bytes that end with 11 is 64. Also the number of bytes that began & end with 11 is 16. Therefore, the number of bytes that begin or end with 11 is $64 + 64 - 16 = 112$.

Example 13:-

A telegraph can transmit two different signals: a dot and a dash. What length of these Symbols is needed to encode 26 Letters of the English alphabet and the ten digits 0, 1, 2, ----- 9?

Soli:- Since there are two Choices for each signal, the number of different Sequences of length k of these signals is 2^k.

Therefore, the number of nontrivial sequences of length n or less is

$$2+2^2+2^3+2^4+\cdots+2^n = 2^{n+1}-2.$$

To Encode 26 letters and 10 digits, we require atleast $26+10=36$ sequences of the above type; that is

$$2^{n+1}-2 \geq 36.$$

The Least value of n (positive integer) for which this inequality holds is $n=5$. Hence, the length of the Symbols needed to encode 26 alphabets & 10 digits is at least 5.

Example 14:-

Find the number of 3-digit even numbers with no repeated digits.

Soln:- Here we consider numbers of the form xyz, where each of x, y, z represents a digit under the given restrictions. Since x, y, z has to be even, z has to be 0, 2, 4, 6 or 8. if z is 0, then x has 9 choices and if z is 2, 4, 6 or 8 (4 choice) then x has 8 choices. (Note that x cannot be zero) Therefore z, and x can be choosen in $(1\times9)+(4\times8)=41$ ways. For each of these ways, y can be choosen in 8 ways. Hence, the desired number is $41\times8=328$.

Example 15:-

How many among the first 100,000 positive integers contain exactly one 3, one 4 & one 5 in their decimal representation?

Soln:- The number of 100,000 does not contain 3 or 4 or 5 Therfore we have to consider all possible positive intege with 5 places that meet the given conditions. In a 5-plac integer the digit 3 can be in any one of the 5 places. Subsequently, the digit 4 can be in any one of the 4 remaining places. Then the digit 5 can be in any one of the 3 remaining places. there are 2 places left & either of these may be filled by 7 digits(-digits from 0 to 9 other than 3,4,5)

Thus there are $5 \times 4 \times 3 \times 7 \times 7 = 2940$ integers of the required type.

Example 16:-

Find the number of proper divisors of 441000.

Soln :- we note that $441000 = 2^3 \times 3^2 \times 5^3 \times 7^2$. There fore, every divisor of 441000 must be of the form $d = 2^p \times 3^q \times 5^r \times 7^s$ where $0 \le p \le 3$, or $0 \le q \le 2$, $0 \le r \le 3$, $0 \le s \le 2$. Thus for a divisor d, p can be choosen in 4 ways, q in 3 ways, r in 4 ways & s in 3 ways. Accordingly, the number of possible d's is $4 \times 3 \times 4 \times 3 = 144$. of these, two are not proper divisors. There fore the number of proper divisors is $144 - 2 = 142$.

Permutations:-

Suppose that we are given n distinct objects and wish to arrange r of these objects in a line. since there are n ways of choosing the first object, & after this is done, n-1 ways or choosing the second object, ------ and finally n-r+1 ways of choosing the rth object, it follows by the product rule of counting (stated in the preceeding section) that the number of different arrange-ments, or permutations (as they are commonly called) is $n(n-1)(n-2) \text{-----} (n-r+1)$. We denote this number by $P(n,r)$ & is referred to as the number of permutations of size r of n objects Thus (by definition)

$$P(n,r) = n(n-1)(n-2)(n-3) \text{-----} (n-r+1)$$

using the factorial notation defined by

$$k! = k(k-1)(k-2) \text{----} 2\cdot1$$

For any positive integer k, & $0! = 1$, we find that.

$$P(n,r) = n(n-1)(n-2)(n-3)\cdots(n-r+1)$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 2\cdot 1}{(n-r)(n-r-1)\cdots 2\cdot 1}$$

$$= \frac{n!}{(n-r)!}$$

As a particular case of this, we get,

$$P(n,n) = n!$$

That is, the number of different arrangements (permutations) of n distinct objects, taken all at a time, is $n!$. This is simply called the number of permutations of n distinct objects.

In the above analysis, we have considered the situation where all the objects that are to be arranged are distinct.

Suppose it is required to find the number of permutations that can be formed from a collection of n objects of which n_1 are of one type, n_2 are of a second type, --- n_k are of kth type, with $n_1+n_2+\cdots+n_k = n$. Then the number of permutations of the n objects is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

PROOF:-

There are $n!$ permutations when all the n objects are different. We must therfore divide $n!$ by $n_1!$ to account for the fact that the n_1 objects which are alike will identify $n_1!$ of these permutations. (for any given set of positions of the n_1 objects in the permutation) Similarly, we must divide $n!$ by $n_2!, n_3!, \cdots, n_k!$, which are the numbers of permutations of the corresponding alike objects. Thus, $n!$ divided by all of

Example 1 :-

How many different strings (sequences) of length 4 can be formed using the letters of the word Flower?

Sol :- The given word has 6 letters all of which are distinct. Therefore, the required number of strings is.

$$P(6,4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6\times5\times4\times3\times\cancel{2\times1}}{\cancel{2\times1}} = 360.$$

Example 2 :-

Find the number of permutations of the letters of the word Success.

Sol :- The given word has 7 letters, of which 3 are s; 2 are c and 1 each are U & E. Therefore, the required number of permutations is.

$$\frac{7!}{3!\cdot2!\cdot1!\cdot1!} = \frac{7\times6\times5\times4\times3\times2\times1}{(3\times2)\times(2\times1)\times1\times1} = 420.$$

Example 3 :-

How many 9 letter "words" can be formed by using the letters of the word DIFFicult?

Sol The given word contains 9 letters of which there are 2F's, 2I's and 1each of D, C, U, L, T.

The number of permutations of these letters is the required number of "words" II. This number is

$$\frac{9!}{2!\cdot2!\cdot1!\cdot1!\cdot1!\cdot1!\cdot1!} = 90720.$$

Example 4 :-

Find the number of permutations of the letters of the word MASSASAUGA. In how many of these, all four A's are together? How many of them begin with s?

Sol :- The given word has 10 letters of which 4 are A, 3 are s and 1 each are M, U & G.

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25,200.$$

if, in a permutation, all A's are to be together, we treat all of A's as one single letter. Then the letters are to be ~~together~~ permuted read (AAAA), S,S, S, M, U, G. and the number of permutations is

$$\frac{7!}{1!\,3!\,1!\,1!\,1!} = 840.$$

For permutations begining with S, there occur nine open positions to fill, where two are S, four are A, and one each are M, U, G. The number of such permutations is

$$\frac{9!}{2!\,4!\,1!\,1!\,1!} = 7560$$

Example 5:-

How many positive numbers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

Sol:- Here n must be of the form.

$$n = x_1 x_2 x_3 x_4 x_5 x_6 x_7$$

where $x_1, x_2, ---- x_7$ are the given digits with $x_1 = 5, 6$ or:

Suppose we take $x_1 = 5$ then $x_2 x_3 x_4 x_5 x_6 x_7$ is arrangement of the remaining 6 digits which contains two 4's & one each of 3, 5, 6, 7. The number of such arrangement is

$$\frac{6!}{2!\,1!\,1!\,1!\,1!} = 360.$$

Next suppose we take $x_1 = 6$. Then, $x_2 x_3 x_4 x_5 x_6 x_7$ is an arrangement of 6 digits which contains two each of 4 and 5 & one each of 3 & 7. The number of such arrangement is

$$\frac{6!}{1!\cdot 2!\,2!\cdot 1!} = 180.$$

Similarly, if we take $x_1 = 7$, the number of arrangements is

$$\frac{6!}{1!\,2!\,2!\,1!} = 180$$

Accordingly, by the sum rule, the number of n's of the desired type is

$360 + 180 + 180 = 720.$

Example 6:-

In how many ways can n persons be seated at a round table if arrangements are considered the same when one can be obtained from the other by rotation?

Soli:-

Let one of them be seated anywhere. Then the remaining $n-1$ persons can be seated in $(n-1)!$ ways. This is the total number of ways of arranging the n persons in a circle.

Example 7:-

It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrange ments are possible?

Soli:-

The 5 men may be seated in odd places in $5!$ ways and the 4 women may be seated in even places in $4!$ ways, & correspondi-ng to each arrangement of the men there is an arrange-ment of the women. Therefore, the total number of arrangement of the desired type is

$5! \times 4! = 120 \times 24 = 2880.$

Example 8:-

In how many ways can 6 men & 6 women be seated in a row.

(i) if any person may sit next to any other?

(ii) if men & women must occupy alternate seats?

Soli:- (i) if any person may sit next to any other, no distinc-ion need to be made between men & women in their seating. Accordingly, since there are 12 persons in all, the number of ways they can be seated is

$12! = 479,001,600.$

(ii) when men & women are to occupy alternate seats, the six men can be seated in $6!$ ways in odd places & the six women can be seated in $6!$ ways. in even places, & corresponding to each arrangement of the men there is an arrangement of the women.

Therefore, the number of ways in which the men occupy the odd places and the women the even places is

$$6! \times 6! = 720 \times 720 = 518400.$$

Similarly, the number of ways in which the women occupy the odd places & the men the even places. is 518400. Accordingly, the total number of ways is

$$518400 + 518400 = 1,036,800.$$

Example 9:-

Four different mathematics books, five different computer science books. & two different control theory books are to be arranged in a shelf. How many different arrangements are possible if (a) the books in each particular subject must all be together? (b) only the mathematics books must be together?

Sol:- a) The mathematics books can be arranged among themselves in $4!$ different ways, the computer science books in $5!$ ways, the control theory books in $2!$ ways, & the three groups in $3!$ ways. Therefore the number of possible arrangement is

$$4! \times 5! \times 2! \times 3! = 24 \times 120 \times 2 \times 6 = 34560.$$

b) Consider the four mathematics books as one single book. Then we have 8 books which can be arranged in $8!$ ways. in all ~~#~~ of these ways the mathematics books are together, But the mathematics books can be arranged among themselves in $4!$ ways. Hence, the number of arrangements is

$8! \times 4! = 40320 \times 24$
$= 967,680.$

Example 10:-

Find the total number of positive integers that can be formed from the digits 1,2,3,4 if no digit is repeated in any one integer.

Sol:- We first note that no integer of the required type can obtain more than 4 digits. Let s_1, s_2, s_3, s_4 denote the number of integers of the required type containing one, two, three, four digits respectively.

Since there are four digits, there are four integers containing exactly one digit (i.e., $s_1 = 4$), there are $4 \times 3 = 12$ integers containing exactly two digits (i.e., $s_2 = 12$), there are $4 \times 3 \times 2 = 24$ integers containing exactly three digits (i.e., $s_3 = 24$) Therefore, the required number is

$$s_1 + s_2 + s_3 + s_4 = 64.$$

Example 11:-

How many 8-digit telephone numbers have one or more repeated digits?

Sol:- The number of 8-digit numbers in which repetitions of integers is allowed is 10^8. Of these, $P(10,8)$ numbers do not contain repetitions. Therefore the required number is $10^8 - P(10,8)$

Example 12:-

Find the value of n so that $2P(n,2) + 50 = P(2n,2)$.

Soln:- From the given condition, we have

$$2 \times \frac{n!}{(n-2)!} + 50 = \frac{(2n)!}{(2n-2)!}$$

i.e., $2n(n-1) + 50 = 2n(2n-1)$ or $n^2 = 25$.

This gives $n = 5$ or -5 since n cannot be negative. the value of n should be 5.

Example 13:-

Prove that for all integers $n, r \geq 0$, if $n+1 > r$ then

$$P(n+1, r) = \left(\frac{n+1}{n+1-r}\right) P(n, r)$$

Sol:- we have

$$P(n, r) = \frac{n!}{n-r!} \quad \& \quad P(n+1, r) = \frac{(n+1)!}{(n+1-r)!}$$

therefore,

$$\frac{P(n+1, r)}{P(n, r)} = \frac{(n+1)!}{(n+1-r)!} \cdot \frac{(n-r)!}{n!}$$

$$= \frac{(n+1)!}{n!} \cdot \frac{(n-r)!}{(n+1-r)!} = \frac{n+1}{n+1-r}$$

$$P(n+1, r) = \left(\frac{n+1}{n+1-r}\right) P(n, r).$$

Example 14:- if k is a positive integer, & $n = 2k$, prove that $\frac{n!}{2^k}$ is a positive integer.

Sol:- consider the symbols $x_1, x_1, x_2, x_2, x_3, x_3, ---- x_k$ in which each of $x_1, x_2 ----- x_k$ is 2 in number. Evidently, the number of these symbols is $2k$. Therefore, the number of permutations of these $2k$ symbols is

$$\frac{(2k)!}{\underbrace{2! \cdot 2! --- 2!}_{k\text{-Factors}}} = \frac{n!}{2^k}.$$

This number has to be a Positive number.

Example 15:-

Prove that $(n!)!$ is divisible by $(n!)^{(n-1)!}$

Sol: Let us set $n! = N$ so that $(n-1)! = \frac{n!}{n} = \frac{N}{n} = M$ (say)

consider a collection of N objects of M types with n objects in each type. The number of permutations of these N objects is

$$\frac{N!}{\underbrace{(n!)(n!) ---- (n!)}_{M\text{Factors}}} = \frac{N!}{(n!)^M}$$

this has to be a Positive integer: ..... This means $N!$ is divisible by $(n!)^M$. This Proves

Combinations:

Suppose we are interested in Selecting (choosing) a set of r objects from a set of $n \geq r$ objects without regard to order. The set of r objects being Selected is traditionally Called a Combination of r objects.

The total number of combinations of r different objects that can be selected from n different objects can be obtained by proceeding in the following way. Suppose this number is equal to c, say; that is, Suppose there is a total of c number of Combinations of r different objects chosen from n different objects. Take any one of the combinations. The r objects in this Combination can be arranged in $r!$ different ways. Since there are C combinations, the total number of permutations is $(C \cdot r!)$. But this is equal to $P(n,r)$. Thus,

$$C \cdot r! = P(n,r), \quad \text{or} \quad C = \frac{P(n,r)}{r!}$$

Thus, the total number of Combinations of r different objects that can be selected from n different objects is equal to $\frac{P(n,r)}{r!}$. This number is denoted by $C(n,r)$ or $\binom{n}{r}$**. Thus,

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!} \quad \text{for } 0 \leq r \leq n.$$

Replacing r by $n-r$ in this expression, we get.

$C(n, n-r) = \binom{n}{n-r} = \frac{n!}{r!(n-r)!} = C(n,r) = \binom{n}{r}$ for $0 \leq r \leq n$.

Consequently, we have.

$C(n,n) = \binom{n}{n} = C(n,0) = \binom{n}{0} = 1$ and $C(n,1) = \binom{n}{1} = C(n, n-1) = \binom{n}{n-1} = n$.

For $r > n$, $C(n,r)$ is defined to be equal to zero.

Example-1:-

1. How many committees of five with a given chairperson can be selected from 12 persons?

Sol:- The chairperson can be choosen in 12 ways, and, following this, the other four on the Committee can be choosen in $C(1,4)$ ways. Therefore, the possible number of such Committees is.

$$12 \times C(11,4) = 12 \times \frac{11!}{4!\,7!} = 12 \times 330 = 3960.$$

Example-2:-

2. Find the number of committees of 5 that can be selected from 7 men and 5 women if the Committee is to consist of at least 1 man and at least 1 woman.

Sol:- From the given 12 persons the number of Committees of 15 that can be formed is $C(12,5)$. Among these possible Committees, there are $C(7,5)$ committees consisting of 5 men and $1 = C(5,5)$ committee consisting of 5 women. Accordingly, the number of Committees containing atleas one man and one woman is

$$C(12,5) - C(7,5) - 1 = \frac{12!}{7!\,5!} - \frac{7!}{5!\,2!} - 1 = 792 - 21 - 1 = 77$$

Example-3

3\. At a certain college hostel, the housing office has decided to appoint, for each floor, one male and one female residential advisor. How many different pairs of advisors can be selected for a seven-floor building from 12 male and 15 female candidates?

Sol:- From 12 male candidates, 7 candidates can be selected in $C(12,7)$ ways. From 15 female candidates, 7 candidates can be selected in $C(15,7)$ ways. Therefore, the total number of possible pairs of advisors of the required type is.

$$C(12,7) \times C(15,7) = \frac{12!}{7!\,5!} \times \frac{15!}{7!\,8!} = 792 \times 6435 = 5,096,520.$$

Example-4:-

4\. A certain question paper contains two parts A and B each containing 4 questions. How many different ways a student can answer 5 questions by selecting at least 2 questions from each part?

Sol:- The different ways a student can select his 5 questions are

(I) 3 questions from part A and 2 questions from part B. This can be done in $C(4,3) \times C(4,2) = 4 \times 6 = 24$ ways.

(II) 2 questions from part 4 and 2 questions from part B. This can be done in $C(4,2) \times C(4,3) = 24$ ways.

Therefore, the total number of ways a student can answer 5 questions under the given restrictions is $24 + 24 = 48$.

Example-5:-

5. A Certain question paper contains three parts A, B, C with four questions in part A, five questions in part B and six questions in part C. It is required to answer Seven questions Selecting at least two questions from each part. In how many different ways can a Student Select his seven questions for answering?

Sol:- The different possible ways in which a student can make a Selection are.

(I) 2 questions from part A, 2 from part B and 3 from part C.

(II) 2 questions from part A, 3 from part B and 2 from part C.

(III) 3 questions from part A, 2 from part B and 2 from part C.

Now, Selection (I) can be made in

$C(4,2) \times C(5,2) \times C(6,3) = 6 \times 10 \times 20 = 1200$ ways, the

Selection (II) can be made in

$C(4,2) \times C(5,3) \times C(6,2) = 6 \times 10 \times 15 = 900$ ways, and the

Selection (III) can be made in.

$C(4,3) \times C(5,2) \times C(6,2) = 4 \times 10 \times 15 = 600$ ways.

Consequently, the total number of possible selections is

$1200 + 900 + 600 = 2700.$

Example-6:-

6. A woman has 11 close relatives and she wishes to invite 5 of them to dinner. In how many ways can she invite them in the following situations.

(i) There is no restriction on the choice.

(ii) Two particular persons will not attend Separately.

(iii) Two particular persons will not attend together.

Sol:-

(i) Since there is no restriction on the choice of invitees five out of 11 Can be invited in $C(11,5) = \frac{11!}{6!5!} = 462$ ways.

(ii) Since two particular persons will not attend Separately, they should both be invited or not invited. If both of them are invited, then three more invitees are to be Selected from the remaining 9 relatives. This Can be done in

$$C(9,3) = \frac{9!}{6!\,3!} = 84 \text{ ways.}$$

If both of them are not invited, then five invitees are to be selected from 9 relatives. This Can be done in

$$C(9,5) = \frac{9!}{5!\,4!} = 126 \text{ ways.}$$

(iii) Since two particular persons (say A and B) will not attend together, only one of them can be invited or none of them can be invited. The number of ways of choosing the invitees with A invited is.

$$C(9,4) = \frac{9!}{5!\,4!} = 126.$$

Similarly the number of ways of choosing the invitees with B invited is 126. If both A and B are not invited, the number of ways of choosing the invitees is

$$C(9,5) = 126.$$

Thus, the total number of ways in which the invitees Can be selected in this Case is $126+126+126 = 378$.

Example-7:-

7. Find the number of 5-digit positive integers such that in each of them every digit is greater than the digit to the right.

Sol:- A set of 5 distinct digits can be selected in $C(10,5)$ ways. Once these digits are chosen, there is only one way of arranging them in a decreasing order from left to right. So, the number of 5-digit positive integers of the required type is $1 \times C(10,5) = C(10,5)$.

Example-8:-

8. From seven consonants and five vowels, how many sets containing of four different consonants and three different vowels can be formed?

Sol:- The four different consonants can be selected in $C(7,4)$ different ways and three different vowels can be selected in $C(5,3)$ ways, and the resulting seven different letters (four consonants and three vowels) can then be arranged among themselves in $7!$ ways. Therefore, the number of possible sets is.

$$C(7,4) \times C(5,3) \times 7! = \frac{7!}{4!\,3!} \times \frac{5!}{3!\,2!} \times 7! = 35 \times 10 \times 5040$$
$$= 1,764,000.$$

Example-9:-

Find the number of arrangements of the letters in TALLAHASSEE which have no adjacent A's.

Here the number of letters is 11 of which 3 are A's, 2 each are L's, S's, E's and 1 each are T and H. First, let us disregard the A's. The remaining 8 letters can be arranged in $\frac{8!}{2!2!2!1!1!} = 5040$ ways.

In each of these arrangements, there are 9 possible locations for the three A's. These locations can be chosen in $C(9,3)$ ways. Therefore, by the product rule, the required number of arrangements is

$$5040 \times C(9,3) = 5040 \times \frac{9!}{3!6!} = 5040 \times 84 = 423,360.$$

Example-10:-

10. Find the number of ways of seating r out of n persons around a circular table, and the others around another circular table.

Sol:- First, choose a set of r persons for the first table - this can be done in $C(n,r)$ ways. these r persons can be seated around the first table in $(r-1)!$ ways. The remaining $(n-r)$ persons can be seated around the second table in $(n-r-1)!$ ways. So, the required number is

$$C(n,r) \times (r-1)! \times (n-r-1)!$$

Example-11:-

11. A party is attended by n persons. If each person in the party shakes hands with all the others in the party, find the number of handshakes.

Sol:- Each handshake is determined by exactly two persons. Therefore, if each person shakes hands with all the other persons, the total number of handshakes is equal to the number of combinations of two persons that can be selected from the n persons. This number is

$$C(n,2) = n!/(n-2)!2! = \frac{1}{2}n(n-1)$$

Example-12:-

12. There are n married Couples attending a party. Each person shakes hands with every person other than his or her spouse. Find the total number of handshakes.

Sol:- The number of persons at the party is $2n$. these $2n$ persons fail into $C(2n,2)$ pairs out of which n pairs are married couples. Thus, the number of pairs who are not married Couples in

$$C(2n,2)-n = \frac{(2n!)}{(2n-2)!\,2!}-n = \frac{1}{2}\cdot 2n(2n-1)-n = 2n(n-1)$$

This number is identical with the number of handshakes

Example-13:-

13. (a) How many diagonals are there in a regular polygon with n sides?

(b) which regular polygon has the same number of diagonals are sides?

Sol:- (a) A regular polygon of n sides has n vertices. Any two vertices determine either a side or a diagonal. Thus, the number of sides plus the number of diagonals is $C(n,2)$. consequently, the number of diagonals is.

$$C(n,2)-n = \frac{n!}{(n-2)!\,2!}-n = \frac{1}{2}n(n-1)-n = \frac{1}{2}n(n-3).$$

(b) If the number of diagonals is the same as the number of sides, we should have.

$$\frac{1}{2}n(n-3)=n, \text{ or } n^2-5n=0, \text{ or } n(n-5)=0.$$

Since $n>0$, we should have $n=5$. Thus, the regular polygon which has the same number of diagonals as sides num must have 5 sides; that is, it must be a pentagon.

Example-14:-

14. A String of length n is a sequence of the form $x_1 x_2 x_3 \ldots x_n$, where each x_i is a digit. The sum $x_1+x_2+x_3 \ldots + x_n$ is called the weight of the string. If each x_i can be one of 0, 1 or 2, find the number of Strings of Length $n=10$. Of those, find the number of Strings whose weight is an even number.

Sol:- There are 10 positions in a string of length 10, and each of these positions can be filled in 3 ways (with 0, 1, 2). Therefore, the number of ways of filling the 10 positions of a string of length 10 is 3^{10}. This means that there are 3^{10} number of strings of length 10 (with 0, 1 or 2 as its digits).

Since each digits in the strings being Considered here is 0, 1, or 2, the weight of a string is even only when it contains an even number of 1's. Thus, strings of even weight have zero, two, four, six, eight or ten number of 1's.

If a string has no 1's, then all its places are filled by 0's and 2's. The number of such strings is 2^{10}.

If a String has two 1's, it can have two 1's in $C(10,2)$ number of locations. For each of these locations, the remaining eight locations are filled by 0's and 2's.

Therefore, the number of strings having two 1's is $C(10,2) \times 2^8$.

Similarly, the numbers of strings having four 1's, six 1's and eight 1's are $C(10,4) \times 2^6$, $C(10,6) \times 2^4$ and $C(10,8) \times 2^2$ respectively.

Lastly, the number of strings having ten 1's is evidently only one.

Accordingly, the number of strings that have been even weight is.

$2^{10} + C(10,2) \times 2^8 + C(10,4) \times 2^6 + C(10,6) \times 2^4 + C(10,8) \times 2^2 + 1.$

Example-15:-

15. prove the following identities:

(i) $C(n+1,r) = C(n,r-1) + C(n,r)$ (ii) $C(m+n,2) - C(n,2) = mn$.

Sol:- we have

(i) $$C(n,r-1) + C(n,r) = \frac{n!}{(r-1)!\,(n-r+1)!} + \frac{n!}{r!\,(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left\{\frac{1}{n-r+1} + \frac{1}{r}\right\}.$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \cdot \frac{n+1}{r(n-r+1)}$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!} = C(n+1,r).$$

(ii) $$C(m,2) + C(n,2) + mn = \frac{m!}{(m-2)!\cdot 2} + \frac{n!}{(n-2)!\cdot 2} + mn$$

$$= \frac{1}{2}\{m(m-1) + n(n-1)\} + mn$$

$$= \frac{1}{2}(m+n)(m+n-1) = \frac{(m+n)!}{2(m+n-2)!}$$

$$= C(m+n, 2).$$

Example -16 :-

16. prove the identity

$C(n,r) \cdot C(r,k) = C(n,k) \cdot C(n-k, r-k)$, for $n \geq r \geq k$

Deduce that, if n is a prime number, then $C(n,r)$ is divisible by n.

Sol :- we have,

$$C(n,r) \cdot C(r,k) = \frac{n!}{r!\,(n-r)!} \cdot \frac{r!}{k!\,(r-k)!}$$

$$= \frac{n!}{(n-k)!\,k!} \cdot \frac{(n-k)!}{(n-r)!\,(r-k)!}$$

$$= \frac{n!}{(n-k)!\,k!} \cdot \frac{(n-k)!}{\{(n-k)-(r-k)\}!\,(r-k)!}$$

$$= C(n,k) \cdot C(n-k, r-k).$$

For $k=1$, this identity becomes

$$r \cdot C(n,r) = n \cdot C(n-1, r-1).$$

Since n divides the R.H.S., it must divide the L.H.S. Thus, n must divide r or $C(n,r)$. But n cannot divide r (because $r \leq n$). Hence n must divide $C(n,r)$

Binomial and Multinomial Theorems

One of the basic properties of $C(n,r) = \binom{n}{r}$ is that it is the coefficient of $x^r y^{n-r}$ in the expansion of the expression $(x+y)^n$, where x and y are any real numbers. In other words,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

This result is known as the Binomial Theorem for a positive integral index.

The numbers $\binom{n}{r}$ for $r = 0, 1, 2, \ldots n$ in the above result are known as the binomial coefficients.

The student is already familiar with the proof by mathematical induction of the above mentioned binomial theorem.

The following is a generalization of the binomial theorem, known as the Multinomial Theorem.

Theorem : For positive integers n and t, the coefficients of $x_1^{n_1}, x_2^{n_2}, x_3^{n_3} \ldots x_t^{n_t}$ in the expansion of $(x_1+x_2+\cdots+x_t)^n$ is

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!}$$

where each n_i is a nonnegative integer $\leq n$, and $n_1+n_2+n_3+\cdots+n_t = n$.

Proof : We note that in the expansion of $(x_1+x_2+\cdots+x_t)^n$ the coefficients of $x_1^{n_1} x_2^{n_2} \ldots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n-n_1$ remaining factors, x_3 from n_3 of the $n-n_1-n_2$ remaining factors, and so on. Therefore this coefficient is, by the product rule,

$$C(n, n_1)\cdot C(n-n_1, n_2)\cdot C(n-n_1-n_2, n_3)\cdots C(n-n_1-n_2-\cdots-n_{t-1}, n_t)$$

$$= \frac{n!}{n_1!\,(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\cdot\frac{(n-n_1-n_2)!}{n_3!\,(n-n_1-n_2-n_3)!}\cdots$$

$$\cdots\frac{(n-n_1-n_2-\cdots-n_{t-1})!}{n_t!\,(n-n_1-n_2-\cdots-n_{t-1}-n_t)!}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!}$$

This proves the required result.

Note : Another way of stating the Multinomial theorem is :

The general term in the expansion of

$(x_1+x_2+x_3+\cdots x_t)^n$ is $\dfrac{n!}{n_1!\,n_2!\cdots n_t!}\, x_1^{n_1}\, x_2^{n_2}\cdots x_t^{n_t}$,

where $n_1, n_2, \ldots n_t$ are non negative integers not exceeding n

and $n_1+n_2+n_3+\cdots+n_t = n$.

The Expression $\dfrac{n!}{n_1!\,n_2!\cdots n_t!}$ is also written as $\binom{n}{n_1, n_2, n_3, \ldots n_t}$ and

is called a multinomial coefficient.

Example-1 : Prove the following identities for a positive integer n:

(i) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n} = 2^n$ (ii) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\cdots+(-1)^n\binom{n}{n} = 0$

→ The Binomial Theorem for a positive integral index n reads

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r} x^r y^{n-r}.$$

when $x=y=1$, this becomes

$$2^n = \sum_{r=0}^{n}\binom{n}{r} 1^r\cdot 1^{n-r} = \sum_{r=0}^{n}\binom{n}{r} = \binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$$

and when $x=-1$ and $y=1$, we get

$$0 = \sum_{r=0}^{n}\binom{n}{r}(-1)^r = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\cdots+(-1)^n\binom{n}{n}.$$

Example-2: Find the coefficient of x^9y^3 in the expansion of $(2x-3y)^{12}$.

→ We have, by the binomial theorem,

$$(2x-3y)^{12} = \sum_{r=0}^{12} \binom{12}{r} \cdot (2x)^r (-3y)^{12-r}$$

$$= \sum_{r=0}^{12} \binom{12}{r} \cdot 2^r (-3)^{12-r} x^r y^{12-r}.$$

In this expansion, the coefficient of x^9y^3 (which corresponds to $r=9$):

$$\binom{12}{9} 2^9 (-3)^3 = -2^9 \times 3^3 \times \frac{12!}{9!\,3!} = -2^9 \times 3^3 \times \frac{12 \times 11 \times 10}{6}$$

$$= -2^{10} \times 3^3 \times 11 \times 10 = 1946.$$

Example-3: Evaluate : $\binom{12}{5,3,2,2}$

→ We have $\binom{12}{5,3,2,2} = \frac{12!}{5!\,3!\,2!\,2!} = 166320$

Example-4: Find the term which contains x^{11} and y^4 in the expansion of $(2x^3-3xy^2+z^2)^6$.

→ By the multinomial theorem, the general term in the given expansion is

$$\binom{6}{n_1,n_2,n_3}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3} = \binom{6}{n_1,n_2,n_3} 2^{n_1}(-3)^{n_2} x^{3n_1+n_2} y^{2n_2} z^{2n_3}$$

Thus, for the term containing x^{11} and y^4 we should have $3n_1+n_2=11$ and $2n_2=4$, so that $n_1=3$ and $n_2=2$. Since $n_1+n_2+n_3=6$, we should then have $n_3=1$. Accordingly, the term containing x^{11} and y^4 is

$$\binom{6}{3,2,1} 2^3(-3)^2 x^{11}y^4z^2 = \left\{\frac{6!}{3!\,2!\,1!} \times 8 \times 9\right\} x^{11}y^4z^2 = 4320\,x^{11}y^4z^2.$$

Example-5: Determine the coefficient of

(i) xyz^2 in the expansion of $(2x-y-z)^4$, and

(ii) $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$.

i) By the multinomial theorem, we note that the general term in the expansion of $(2x-y-z)^4$ is

$$\binom{4}{n_1,n_2,n_3}(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}.$$

For $n_1=1$, $n_2=1$ and $n_3=2$, this reads

$$\binom{4}{1,1,2}(2x)(-y)(-z)^2 = \binom{4}{1,1,2} \times 2 \times (-1) \times (-1)^2 \times x y z^2$$

$$= -2 \times \binom{4}{1,1,2} xyz^2$$

$$= -2 \times \frac{4!}{1!\,1!\,2!} xyz^2 = -24\,xyz^2$$

Thus, the required coefficient is -24.

ii) By the multinomial theorem, we note that the general term in the expansion of $(a+2b-3c+2d+5)^{16}$ is

$$\binom{16}{n_1,n_2,n_3,n_4,n_5}(a)^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}.$$

For $n_1=2$, $n_2=3$, $n_3=2$, $n_4=5$ and $n_5=16-(2+3+2+5)=4$, this becomes

$$\binom{16}{2,3,2,5,4} a^2(2b)^3(-3c)^2(2d)^5 5^4 = \binom{16}{2,3,2,5,4} \times 2^3 \times (-3)^2 \times 2^5 \times 5^4 \times a^2 b^3 c^2 d^5$$

$$= 2^8 \times 3^2 \times 5^4 \times \frac{16!}{2!\,3!\,2!\,5!\,4!} a^2 b^3 c^2 d^5$$

$$= 3 \times 2^5 \times 5^3 \times \frac{16!}{(4!)^2} a^2 b^3 c^2 d^5$$

Thus, the required coefficient is $\dfrac{16! \times 2^5 \times 5^3 \times 3}{(4!)^2}$

Combinations with Repetitions

Suppose we wish to select, with repetition, a combination of r objects from a set of n distinct objects. The number of such selections is given by

$$C(n+r-1, r) \equiv \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!\,(n-1)!}$$

$$= \binom{r+n-1}{n-1} \equiv C(r+n-1, n-1)$$

In other words, $C(n+r-1, r) = C(r+n-1, n-1)$ represents the number of combinations of n distinct objects, taken r at a time, with repetitions allowed.

The following are other interpretations of this number:

(1) $C(n+r-1, r) = C(r+n-1, n-1)$ represents the number of ways in whic r identical objects can be distributed among n distinct containers.

(2) $C(n+r-1, r) = C(r+n-1, n-1)$ represents the number of nonnegative integer solutions of the equation

$$x_1 + x_2 + \dots + x_n = r.$$

Example-1:-

A bag contains coins of seven different denominations, with atleast one dozen coins in each denomination. In how many ways can we select a dozen coins from the bag?

→ The selection consists in choosing with repetitions, $r=12$ coins of $n=7$ distinct denominations. The number of ways of making this selection is $C(7+12-1, 12) = C(18, 12) = \frac{18!}{12!\,6!} = 18,564.$

Example-2:-

In how many ways can we distribute 10 identical marbles among 6 distinct containers?

→ The required number is

$$C(6+10-1, 10) = C(15, 10) = \frac{15!}{10!\,5!} = 3003.$$

Example-3: Find the number of nonnegative integer solutions of the Equation $x_1+x_2+x_3+x_4+x_5 = 8$.

→ The required number is

$$c(5+8-1, 8) = c(12, 8) = 495.$$

Example-4: Find the number of distinct terms in the expansion of

$$(x_1+x_2+x_3+x_4+x_5)^{16}.$$

→ Every term in the expansion is of the form (by multinomial theorem

$$\binom{16}{n_1, n_2, n_3, n_4, n_5} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5}$$

where each n_i is a nonnegative integer, and these n_i's sum to 16. Therefore, the number of distinct terms in the expansion is precisely equal to the number of nonnegative integer solutions of the equation

$$n_1+n_2+n_3+n_4+n_5 = 16.$$

This number is $c(5+16-1, 16) = c(20, 16) = 4845.$

Example-5: Find the number of nonnegative integer solutions of the inequality $x_1+x_2+x_3+\dots+x_6 < 10$.

→ We have to find the number of nonnegative integer solutions of the equation $x_1+x_2+x_3+\dots+x_6 = 9-x_7$ where $9-x_7 \le 9$ so that x_7 is a nonnegative integer. Thus, the required number is the number of nonnegative solutions of the Equation $x_1+x_2+x_3+\dots+x_7 = 9$.

This number is

$$c(7+9-1, 9) = c(15, 9)$$

$$= \frac{15!}{9!\,6!} = 5005$$

Example-6: Find the number of positive integer solutions of the equation $x_1+x_2+x_3=17$.

→ Here, we require $x_1 \geq 1$, $x_2 \geq 1$, $x_3 \geq 1$. Let us set $y_1 = x_1-1$, $y_2 = x_2-1$, $y_3 = x_3-1$. Then y_1, y_2, y_3 are all non negative integers.

When written in terms of y's, the given equation reads

$$(y_1+1)+(y_2+1)+(y_3+1)=17, \text{ or } y_1+y_2+y_3=14.$$

The number of non negative integer solutions of this equation is the required number. This number is

$$c(3+14-1, 14) = c(16,14) = \frac{16!}{14!\,2!} = \frac{16\times 15}{2} = 120.$$

Example-7: Find the number of integer solutions of

$$x_1+x_2+x_3+x_4+x_5=30$$

where $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 4$, $x_4 \geq 2$, $x_5 \geq 0$.

→ Let us set $y_1 = x_1-2$, $y_2 = x_2-3$, $y_3 = x_3-4$, $y_4 = x_4-2$, $y_5 = x_5$.

Then $y_1, y_2, \ldots y_5$ are all non negative integers

when written in terms of y's, the given equation reads

$$(y_1+2)+(y_2+3)+(y_3+4)+(y_4+2)+y_5=30, \text{ (or) } y_1+y_2+y_3+y_4+y_5=19.$$

The number of nonnegative integer solutions of this equation is the required number, and the number is

$$c(5+19-1, 19) = c(23,19) = \frac{23!}{19!\,4!} = 8855$$

Example-8: In how many ways can we distribute 12 identicals pencils to 5 children so that every child gets atleast 1pencil?

→ First we distribute, one pencil to each child. Then there remain 7 pencils to be distributed. The number of ways of distibuting these 7 pencils to 5 children is the required number. This number is

$$c(5+7-1, 7) = c(11,7) = \frac{11!}{7!\,4!} = 330$$

Example-9 :

A total amount of Rs.1500 is to be distributed to 3 poor students A, B, C of a class. In how ways the distribution can be made in multiples of Rs.100

(i) If Everyone of these must get atleast Rs.300?

(ii) If A must get atleast Rs.500, and B and C must get atleast Rs.400 each?

→ Taking Rs.100 as a unit, there are 15 units for distribution.
In case (i), each of three students must get atleast 3 units. let us first distribute 3 units to each of the 3 students. Then there remain 6 units for distribution. The number of ways of distributing these 6 units to A, B, C is the required number (in this case). This number is $C(3+6-1,6) = C(8,6) = 28$

In Case (ii) A must get atleast 5 units, B and C must get atleast 4 units each. Let us distribute 5 units to A and 4 units to each of B and C. Then there remain 2 units for distribution. Accordingly, the number of ways of making the distribution in this case is $C(3+2-1,2) = C(4,2) = 6$.

Example-10 :

In how many ways can we distribute 7 apples and 6 oranges among 4 children so that each child gets atleast 1 apple?

→ Suppose we first give 1 apple to each child. This exhausts 4 apples. The remaining 3 apples can be distributed among the 4 children in $C(4+3-1,3) = C(6,3)$ ways. Also, 6 oranges can be distributed among the 4 children in $C(4+6-1,6) = C(9,6)$ ways. Therefore, by the product rule, the number of ways of distributing the

the given fruits under the given condition is

$$C(6,3) \times C(9,6) = \frac{6!}{3!\,3!} \times \frac{9!}{6!\,3!} = 20 \times 84 = 1680.$$

Example-11:

Find the number of ways of giving 10 identical gift boxes to 6 persons. A, B, C, D, E, F in such a way that the total number of boxes given to A and B together does not exceed 4.

→ Of the 10 boxes, suppose r boxes are given to A and B together. Then $0 \le r \le 4$. The number of ways of giving r boxes to A and B is

$$C(2+r-1, r) = C(r+1, r) = r+1.$$

The number of ways in which the remaining $(10-r)$ boxes can be given to C, D, E, F is $C(4+(10-r)-1, (10-r)) = C(13-r, 10-r)$

$$= C(13-r, 3).$$

Consequently, the number of ways in which r boxes may be given to A and B and $10-r$ boxes to C, D, E, F, is by the product rule

$$(r+1) \times C(13-r, 3).$$

Since $0 \le r \le 4$, the total number of ways in which the boxes may be given is, by the sum rule,

$$\sum_{r=0}^{4} (r+1) \times C(13-r, 3).$$

Example-12:

A message is madeup of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 blank spaces between the symbols, with atleast three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

→ The 12 symbols can be arranged in 12! ways. For each of these arrangements, there are 11 positions between the 12 symbols. Since there must be atleast three spaces between successive symbols, 33 of the 45 spaces will be used up. The remaining 12 spaces are to be accomodated in 11 positions. This can be done in $C(11+12-1, 12) = C(22, 12)$ ways. Consequently, by the product rule, the required number is

$$12! \times C(22, 12) = \frac{22!}{10!} = 3.097445 \times 10^{14}.$$

Example-13:

Show that $C(n-1+r, r)$ represents the number of binary numbers that contains $(n-1)$ 1's and r 0's.

→ A binary number that contains $(n-1)$ 1's and r 0's, has $n-1+r$ positions and is determined by r positions of 0's. The number of such binary numbers is therefore $C(n-1+r, r)$.

Example-14:

Given positive integers m, n with $m \geq n$, show that the number of ways to distribute m identical objects into n distinct containers such that each container gets atleast r objects, where $r \leq (m/n)$ is $C(m-1+(1-r)n, n-1)$.

→ Suppose we place r of the m identical objects into each of the n distinct containers. Then, there remain $(m-nr)$ identical objects to be distributed into n distinct containers. The number of ways of doing this is the required number. This number is

$$C(n+(m-nr)-1, m-nr) = C(n+(m-nr)-1, n-1)$$
$$= C(m-1+(1-r)n, n-1)$$

The principle of Inclusion - Exclusion :-

Consider a finite set S containing p number of elements. Here, the number p is called the order, size or the Cardinality of the set S and is denoted by $o(s)$, or $n(s)$, or $|S|$.

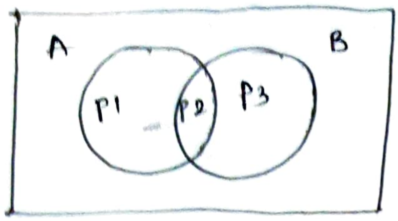
For example, if $A = \{1,2,6\}$ and $B = \{a,b,c,d\}$, then $o(A) = |A| = 3$ and $o(B) = |B| = 4$. It is obvious that $|\phi| = 0$, and $|S| \geq 1$ for every non-empty finite set S. Further, for any two finite sets A and B, if $A \subseteq B$ then $|A| \leq |B|$ and if $A \subset B$ then $|A| < |B|$.

If A is a subset of a finite universal set U, then the number of elements in the complement $\bar{A}$ (of A in U is given by $|\bar{A}| = |U| - |A|$.

Suppose we consider the union of two finite set A and B and wish to determine the number of element in $A \cup B$. Since the elements of $A \cup B$ consist of all elements which are in A or B or both A and B, the number of elements in $A \cup B$ is equal to the number of elements in A plus the number of elements in B minus the number of elements (if any) that are common to A and B. That is,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

A more explicit (visual) way of obtaining this result is through the use of a Venn diagram.



Consider the Venn diagram shown above. In this diagram, the set A is made up of two parts P_1 and P_2, and the set B is made up of two parts P_2 and P_3, where $P_2 = A \cap B$, and $A \cup B$ is made up of parts P_1, P_2 & P_3. Therefore,

$|A| =$ Number of elements in P_1 + Number of elements in P_2

$$= |P_1| + |P_2|.$$

Similarly, $|B| = |P_2| + |P_3|$, $|A \cap B| = |P_2|$ and

$$|A \cup B| = |P_1| + |P_2| + |P_3|.$$

From these we get.

$$|A \cup B| = |P_1| + |P_2| + |P_3| = (|P_1| + |P_2|) + (|P_2| + |P_3|) - |P_2|$$

$$= |A| + |B| - |A \cap B|.$$

Thus, for determing the number of elements in $A \cup B$, we first include all elements in A and all elements B, and then exclude all elements that are common to A and B.

If U is a finite universal set of which A and B are subsets, then, by virtue of a De'Morgan law and the expression (1) above, we have

$$|\bar{A} \cap \bar{B}| = |\overline{A \cup B}| = |U| - |A \cup B|$$

with the use of formula (2) above, this becomes

$$|\bar{A} \cap \bar{B}| = |U| - \{|A| + |B| - |A \cap B|\}$$
$$= |U| - |A| - |B| + |A \cap B|.$$

Expression (2) and (3) are equivalent to one another. Either of these is referred to as the Addition principle (Rule) or the Principle of inclusion - exclusion for two sets.

In the particular case where A and B are disjoint Sets so that $A \cap B = \phi$, the addition rule (2) becomes.

$$|A \cup B| = |A| + |B| - |\phi| = |A| + |B|.$$

this is known as the principle of disjunctive Counting for two sets.

Example - 1 :-

1. A Computer Company requires 30 programmers to handle systems programming jobs and 40 programmers for applications programming. If the Company appoints 55 programmers to carry out these jobs, how many of these perform jobs of both types? How many handle only system programming jobs? How many handle only applications programming?

Sol :- Let A denote the Set of programmers who handle Systems programming job and B the set of programmers who handle applications programming. Then $A \cup B$ is the Set of programmers.

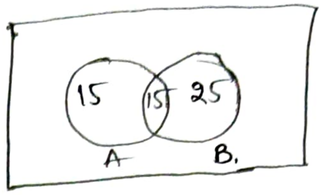
$$|A| = 30, \; |B| = 40, \; |A \cup B| = 55.$$

Therefore, the addition rule $|A \cup B| = |A| + |B| - |A \cap B|$ gives

$|A \cap B| = |A| + |B| - |A \cup B| = 30 + 40 - 55 = 15.$

Similarly, the number of programmers who handle only applications programming is

$|B - A| = |B| - |A \cap B| = 40 - 15 = 25.$



These results are illustrated in the following Venn diagram

Example-2 :-

2. In a class of 52 students, 30 are studying C++, 28 are studying Pascal and 13 are studying both Languages. How many in this class are studying at least one of these languages? How many are studying neither of these languages?

Sol :- Let U denote the set of all students in the class, A denote the set of students in the class who are Studying C++, and B is the set of students in the class who are studying pascal.

Then, the set of students in the class, A denote the set of students in the class who are studying both languages is $A \cap B$, the set of are

nei

Then, the set of students in the class who are studying both languages is $A \cap B$, the set of students who are studying at least one of two languages is $A \cup B$ and the set of students who are studying neither of these languages is $(\overline{A \cup B})$.

From what is given, we have

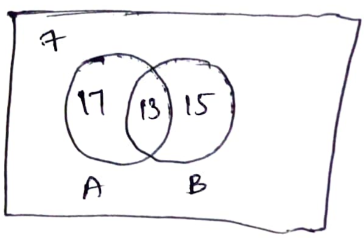
$$|U| = 52, \; |A| = 30, \; |B| = 28, \; |A \cap B| = 13.$$

∴ By the addition principle,

$$|A \cup B| = |A| + |B| - |A \cap B| = 30 + 28 - 13 = 45.$$

Also, $|(\overline{A \cup B})| = |U| - |A \cup B| = 52 - 45 = 7.$

Thus, 45 students of the class study at least one of two languages indicated and 7 students of the class study neither of these languages. The result is illustrated in the fig



Principle of Inclusion-Exclusion for n sets :- The principle of inclusion-exclusion as given by expression (2) can be extended to n sets, $n > 2$.

Let U be a finite universal set and $A_1, A_2, \ldots, A_n$ be subsets of U. then the principle of Inclusion-Exclusion for $A_1, A_2, \ldots A_n$ states that.

$$|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| = \Sigma |A_i| - \Sigma |A_i \cap A_j| + \Sigma |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n| \to$$

Proof :-

Take any $x \in A_1 \cup A_2 \cup \cdots \cup A_n$. Then x is in m of the sets $A_1, A_2, \ldots A_n$ where $1 \le m \le n$. without loss of generality, Let us assume that $x \in A_i$ for $1 \le i \le m$ and $x \notin A_i$ for $i > m$. Then x and will be Counted once in each of the terms $|A_i|, i = 1, 2 \ldots, m$. Thus, x will be Counted m times in $\Sigma |A_i|$.

We note that there are $C(m,2)$ pairs of Sets A_i, A_j where x is in both A_i and A_j. As such, x will be ~~Conduc~~ Counted $C(m,2)$ times in $\Sigma |A_i \cap A_j|$.

Similarly, x will be Counted $C(m,3)$ times in $\Sigma |A_i \cap A_j \cap A_k|$ and So on.

* In expression (5) and in what follows, the following notation is used:

$\Sigma |A_i|$ = Sum of the Cardinalities of sets, $A_1, A_2, \ldots A_n$,

$\Sigma |A_i \cap A_j|$ = Sum of the Cardinalities of intersections of $A_1, A_2 \ldots A_n$, taken 2 at a time,

$\Sigma |A_i \cap A_j \cap A_k|$ = Sum of the cardinalities of intersections of $A_1, A_2 \ldots A_n$ taken 3 at a time, and So on.

we note that:

the sum $\Sigma |A_i|$ Contains $C(n,2) \equiv \binom{n}{2}$ terms,

the sum $\Sigma |A_i|$ Contains n terms,

the sum $\Sigma |A_i \cap A_j \cap A_k|$ Contains $C(n,3) \equiv \binom{n}{3}$ terms,

and So on.

Continuing in this way, we see that, in the right hand side of expression(5), x is counted

$$m - C(m,2) + C(m,3) + \cdots + (-1)^{m-1} C(m,m).$$

number of times. (Bear in mind that $C(m,n) = 0$ for $n > m$).

we note that,

$$m - C(m,2) + C(m,3) + \cdots (-1)^{m-1} C(m,m).$$

$$= 1 - \{1 - m + C(m,2) - C(m,3) + \cdots + (-1)^m C(m,m)\}.$$

$$= 1 - (1 + (-1))^m \text{, by binomial theorem.}$$

$$= 1.$$

thus, on the right hand side of expression (5) every element x of $A_1 \cup A_2 \cup \cdots \cup A_n$ is counted exactly once. This means that the number of elements in $A_1 \cup A_2 \cup \cdots \cup A_1$ is equal to the right hand side of expression (5). This completes the proof of expression (5).

Corollary :- By Virtue of a De' Morgan Law, we have

$$\overline{(A_1 \cup A_2 \cup A_3 \cdots \cup A_n)} = \bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n.$$

Since $|\bar{A}| = |U| - |A|$ for any Subset A of U, this yields.

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n| = |\overline{(A_1 \cup A_2 \cup A_3 \cup \cdots A_n)}|$$

$$= |U| - |(A_1 \cup A_2 \cup \cdots \cup A_n)|.$$

Using expression (5), this becomes

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \cap \bar{A}_n| = |U| - \Sigma |A_i| + \Sigma |A_i \cap A_j| - \Sigma |A_i \cap A_j \cap A$$

$$+ \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n| \rightarrow (6).$$

This is an equivalent version of the principle of inclusion-exclusion, given by (5). Note that, for $n=2$, expressions (5) and (6) reduce to expressions (2) and (3) respectively.

Alternative Versions :-

We may rewrite expressions (5) and (6) in other forms as well.

Suppose A_1 represents the set of all those elements of U with satisfy a certain Condition C_1, A_2 represents the set of all those elements of U which satisfy a certain Condition C_2, and so on. Let us put.

$$N=|U|, \quad N(c_i)=|A_i|, \quad N(\bar{c}_i)=|\bar{A}_i|,$$

$$N(c_ic_j)=|A_i\cap A_j|, \quad N(\bar{c}_i\bar{c}_j)=|\bar{A}_i\cap\bar{A}_j|,$$

$$N(c_ic_jc_k)=|A_i\cap A_j\cap A_k|, \quad N(\bar{c}_i\bar{c}_j\bar{c}_k)=|\bar{A}_i\cap\bar{A}_j\cap\bar{A}_k|,$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$N(c_1c_2c_3\cdots c_n)=|A_1\cap A_2\cap A_3\cap\cdots A_n|,$$

$$\bar{N}=N(\bar{c}_1\bar{c}_2\bar{c}_3\cdots\bar{c}_n)=|\bar{A}_1\cap\bar{A}_2\cap\bar{A}_3\cap\cdots\cap\bar{A}_n|, \text{ and}$$

$$N(c_i \text{ or } c_j)=|A_i\cup A_j|, \quad N(c_i \text{ or } c_j \text{ or } c_k)=|A_i\cup A_j\cup A_k|,$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$N(c_1 \text{ or } c_2 \text{ or } c_3 \text{ or} \cdots \text{or } c_n)=|A_1\cup A_2\cup\cdots A_n|.$$

Then, expression (5) reads

$$N(c_1 \text{ or } c_2 \text{ or } c_3 \text{ or} \cdots \text{or } c_n)=\sum N(c_i)-\sum N(c_ic_j)+\sum N(c_ic_jc_k)\cdots+(-1)^{n-1}N(c_1c_2c_3\cdots c_n)$$

and expression (6) reads.

$$\bar{N} = N - \Sigma N(c_i) + \Sigma N(c_i c_j) - \Sigma N(c_i c_j c_k) + \cdots$$
$$(-1)^n N(c_1 c_2 c_3 \cdots c_n) \rightarrow (8).$$

Using (8), expression (7) can be written as

$$N(c_1 \text{ or } c_2 \text{ or } c_3 \text{ or } \cdots \text{ or } c_n) = N - \bar{N}.$$

putting

$S_0 = N = |U|$, $S_1 = \Sigma N(c_i) = \Sigma |A_i|$, $S_2 = \Sigma N(c_i c_j) = \Sigma |A_i \cap A$

$S_3 = \Sigma N(c_i c_j c_k) = \Sigma |A_i \cap A_j \cap A_k|$,.

and so on, expression (7) and (8) can be rewritten respectively as follows.

$$N(c_1 \text{ or } c_2 \text{ or } c_3 \text{ or } \cdots \text{ or } c_n) = S_1 - S_2 + S_3 - \cdots + (-1)^{n-1} S_n \rightarrow (10)$$

$$\bar{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n \rightarrow (11).$$

Generalization :-

The principle of inclusion-exclusion as given by (11) gives the number of elements in U that satisfy none of the conditions $c_1, c_2, \cdots c_n$. The following expression determines the number of elements in U that statisfy exactly m of the n Conditions $(0 \le m \le n)$.

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{n-m} \binom{n}{n-m} S_n \rightarrow (1$$

for $m=0$, this Exception Expression reduces to expression (11).

Further, the following expression determines the number of elements in U that satisfy as least m of the n conditions $(1 \le m \le n)$;

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} + \cdots + (-1)^{n-m} \binom{n-1}{m-1} S_n \quad (13)$$

For $m=1$, this expression reduces to expression (10).

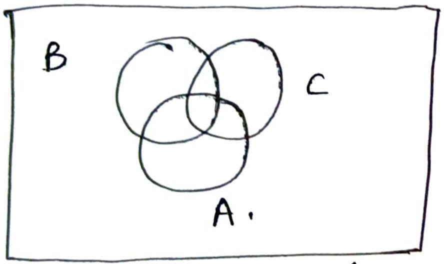
The proofs of expressions (12) and (13) are comitted.

Example-3:

3. If A, B, C are finite sets, prove that.

$$|A-B-C| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|.$$

Sol. We first note that $A-B-C$ is the set of elements that belong to A, but not to B or C.



$A-B-C$ (shaded)

Therefore,

$$|A-B-C| = |A \cup B \cup C| - |B \cup C| \text{; see fig}$$

$$= (|A|+|B|+|C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|) -$$

$(|B|+|C| - |B \cap C|)$ on using addition principle.

$$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|.$$

Example-4:- In a Sample of 100 logic chips, 23 have a defect D_1, 26 have a defect D_2, 30 have a defect D_3, 7 have defects D_1 and D_2, 8 have effects D_1 and D_3, 10 have defects D_2 and D_3 and 3 have all the three defects. Find the number of chips having (i) atleast one defect, (ii) no defect.

Sol:- Let U denote the set of all chips (being considered), and A, B, C denote the sets of chips having defects D_1, D_2, D_3 respectively. Then, from what is given, we have.

$$|U| = 100, \ |A| = 23, \ |B| = 26, \ |C| = 30,$$

$$|A \cap B| = 7, \ |A \cap C| = 8, \ |B \cap C| = 10, \ |A \cap B \cap C| = 3.$$

therefore, the set of chips having atleast one defect is $A \cup B \cup C$, and the number of such chips is.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 23 + 26 + 30 - 7 - 8 - 10 + 3 = 57.$$

The set of chips having no defect is $(\overline{A \cup B \cup C})$ and the number of such chips is

$$(\overline{A \cup B \cup C}) = |U| - |A \cup B \cup C| = 100 - 57 = 43.$$

Example-5:-

A Survey of 500 television viewers of a sports channel produced the following information: 285 watch cricket 195 watch hockey, 115 watch football, 45 watch cricket and football, 70 watch cricket and hockey

50 watch hockey and football and 50 do not watch any of the three kinds of games.

(a) How many viewers in the survey watch all three kinds of games?

(b) How many viewers watch exactly one of the sports?

Sol:- Let U denote the set of all viewers included in the survey, A denote the set of viewers who watch cricket, B denote the set of viewers who watch hockey, and C denote the set of viewers who watch football, Then from what is given, we have

$$|U| = 500, \quad |A| = 285, \quad |B| = 195, \quad |C| = 115,$$

$$|A \cap C| = 45, \quad |A \cap B| = 70, \quad |B \cap C| = 50,$$

$$|\overline{A \cup B \cup C}| = 50, \quad |A \cup B \cup C| = 500 - 50 = 450.$$

Using the addition principle for 3 sets namely,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

we find that

$$|A \cap B \cap C| = |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |A \cap C|$$

$$= 450 - 285 - 195 - 115 + 70 + 50 + 45 = 20.$$

Thus, the number of viewers who watch all three kinds of games is 20.

~~(Let $A_1 = A - B - C$, and by virtue of the result proved in Example, and~~

Let A_1 denote the set of viewers who watch only cricket, B_1 denote the set of viewers who watch only football.

Then, $A_1 = A - B - C$, and by virtue of the result proved in Example 3, we have.

$$|A_1| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|.$$

Accordingly, the number of viewers who watch only cricket is

$$|A_1| = 285 - 70 - 45 + 20 = 190.$$

Similarly, the number of viewers who watch only hockey is

$$|B_1| = |B| - |B \cap A| - |B \cap C| + |B \cap C \cap A|$$

$$= 195 - 70 - 50 + 20 = 95.$$

and the number of viewers who watch only fotball is

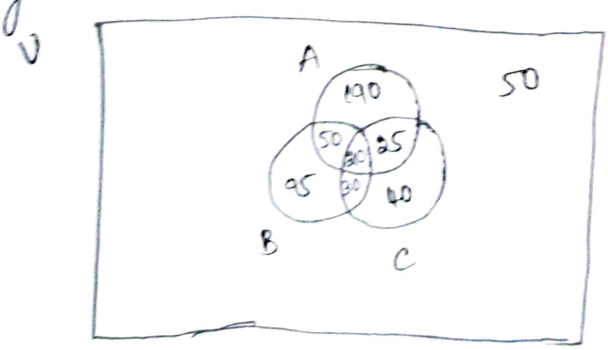
$$|C_1| = |C| - |C \cap A| - |C \cap B| + |C \cap A \cap B|$$

$$= 115 - 45 - 50 + 20 = 40.$$

From these, we find that the number of viewers who watch exactly one of the sports is

$$|A_1| + |B_1| + |C_1| = 190 + 95 + 40 = 325.$$

The results of this example are depicted in the following Venn diagram.



Example-6:

6. A Survey of a Sample of 25 new cars being sold by an auto-dealer was conducted to see which of the three popular options: air-conditioning, 12 had radio, 11 had power windows, 5 had air-conditioning and power windows, 9 had air-conditioning and ratio, 4 had radio and power windows, and 3 had all three options. Find the number of cars that had;

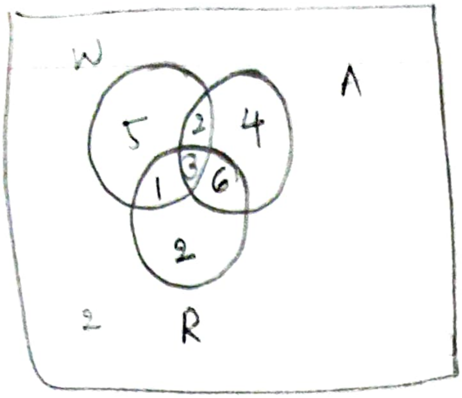
(i) only power windows,
(ii) only air-conditioning,
(iii) only radio,
(iv) only one of the options,
(v) at least one option.
(vi) None of the options.

Sol:- Let A, R and W be the sets of cars included in the Sample that had air-conditioning, radio and power windows, respectively. Also, Let U denote the set of all cars in the Sample. Then from what is given we have the following:

$$|U| = 25, \ |A| = 15, \ |R| = 12, \ |W| = 11, \ |A \cap W| = 5,$$

$$|A \cap R| = 9, \ |R \cap W| = 4, \ |A \cap R \cap W| = 3.$$

Now, Let W_1, A_1 and R_1 respectively denote the



Example-7:-

7. Thirty cars are assembled into a factory. The options available are a music system, an air conditioner and power windows. It is known that 15 of the cars have music systems, 8 have air conditioners and 6 have power windows. Further, 3 have all options. Determine atleast how many cars do not have any option at all.

Sol:- Let U denote the set of all cars being considered, and A, B, C respectively denote the sets of cars having music system, air conditioner and power windows respectively. Then, from what is given, we have

$|U| = 30$, $|A| = 15$, $|B| = 8$, $|C| = 6$, $|A \cap B \cap C| = 3$.

We note that, $A \cup B \cup C$ denotes the set of cars that have atleast one of the options, so that $\overline{A \cup B \cup C}$ is the set of cars that do not have any option.

By the addition rule, we have

Sets of cars that had only power windows, only air-conditioning, and only radio. Then we find that

$$|W_1| = |W-A-R| = |W| - |W\cap A| - |W\cap R| + |W\cap A\cap R|.$$

$$= 11-5-4+3=5.$$

Thus, the number of cars that had only power windows is 5.

Similarly, the number of cars which had only air-Conditioning is

$$|A_1| = |A| - |A\cap W| - |A\cap R| + |A\cap W\cap R|$$

$$= 15-5-9+3=4,$$

and the number of cars which had only radio is

$$|R_1| = |R| - |R\cap W| - |R\cap A| + |R\cap W\cap A|$$

$$= 12-4+9+3=2.$$

Consequently, the number of cars that had only One of the Options is.

$$|W_1| + |A_1| + |R_1| = 5+4+2=11.$$

Next, the number of cars which had atleast one option is.

$$|W\cup A\cup R| = |W| + |A| + |R| - |W\cap A| - |W\cap R| - |A\cap R| + |W\cap A\cap R|.$$

$$= 11+15+12-5-9-4+3 = 23.$$

The results of this Example are depicted in the following Venn diagram.

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|B\cap C|-|C\cap A|+|A\cap B\cap C| \quad (ii)$$

Since $A\cap B\cap C$ is a subset of $A\cap B$, $B\cap C$ and $C\cap A$, we have

$$|A\cap B| \geq |A\cap B\cap C|,\ |B\cap C| \geq |A\cap B\cap C|,\ |C\cap A| \geq |A\cap B\cap C|$$

Using these in (ii), we get

$$|A\cup B\cup C| \leq |A|+|B|+|C|-|A\cap B\cap C|-|A\cap B\cap C|-|A\cap B\cap C|+|A\cap B\cap C|.$$

$$= |A|+|B|+|C|-2|A\cap B\cap C| = 15+8+6-6 = 23.$$

Consequently,

$$|\overline{(A\cup B\cup C)}| = |U|-|A\cup B\cup C| \geq 30-23 = 7.$$

Example-8:-

8. A student visits a sports club every day from Monday to Friday after school hours and plays one of the three games: Cricket, Tennis, Football. In how many ways can he play each of the three games at least once during a week (from Monday to Friday)?

Sol:- On each day, the student has three choices of games. Therefore, the total number of choices of games in a 5-day period is 3^5. thus, if U is the set of all choice of games in a 5-day period, we have $|U| = 3^5$.

Let A denote the set of all choices of games which excludes cricket. Then, the number of choices of games in a 5-day period which excludes Cricket is $|A| = 2^5$. Similarly, if B is the set of all choices of games which excludes Tennis in a 5-day period and C is the set of all choices of games which excludes Football in a 5-day period, we have $|B| = 2^5$ and $|C| = 2^5$.

Consequently, $A \cap B$ is the set of all choices of games in a 5-day period which excludes Cricket and tennis, and $|A \cap B| = 1^5$. Similarly, $|B \cap C| = 1^5$, $|A \cap C| = 1^5$. Also, $A \cap B \cap C$ is the set of all choices of games which excludes all of the three games in the 5-day period, and this set is the null set. Therefore, $|A \cap B \cap C| = 0$.

Further, $A \cup B \cup C$ is the set of all choices of games which excludes atleast one of the three games in the 5-day period, and $|A \cup B \cup C|$ is given by.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$= 2^5 + 2^5 + 2^5 - 1^5 - 1^5 - 1^5 + 0 = 3 \times 2^5 - 3 = 93.$

Therefore, the number of choices of games in the 5-day period which does not exclude any game is

$|(\overline{A \cup B \cup C})| = |U| - |A \cup B \cup C| = 3^5 - 93 = 243 - 93 = 150.$

Thus, there are 150 ways for the student to select ~~this~~ his daily games so that he plays every games at least once during a week (from Monday to friday).

Example-9:-

9. Let X be the set of all three-digit integers; that is, $X = \{x \text{ is an integer} \mid 100 \le x \le 999\}$. If A_i is the set of numbers in X whose ith digit is i, compute the cardinality of the set $A_1 \cup A_2 \cup A_3$.

Sol:- We first observe the following:

$A_1 = \{100, 101, 102, \ldots 199\}$, so that $|A_1| = 100$,

$A_2 = \{120, 121, 122, \ldots 129, 220, 221, 229, 320, 321, \ldots 329, \ldots 920, 921, 922, \ldots 929\}$. So that $|A_2| = 90$, and

$A_3 = \{103, 113, 123, \ldots 193, 203, 213, 223, \ldots 293, 303, 313, 323, \ldots 393, \ldots 903, 913, 923, \ldots 993\}$.

So that $|A_3| = 90$.

Further, we find that

$A_1 \cap A_2 = \{120, 121, 122, \ldots 129\}$ so that $|A_1 \cap A_2| = 10$;

$A_1 \cap A_2 = \{103, 113, 123, \ldots 193\}$ so that $|A_1 \cap A_3| = 10$,

$A_2 \cap A_3 = \{123, 223, 323, \ldots 923\}$ so that $|A_2 \cap A_3| = 9$.

$A_1 \cap A_2 \cap A_3 = \{123\}$ so that $|A_1 \cap A_2 \cap A_3| = 1$.

Therefore,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A$$

$$+ |A_1 \cap A_2 \cap A_3|.$$

$$= 100 + 90 + 90 - 10 - 10 - 9 + 1 = 252.$$