UNIT- 2

Time Response Analysis.

* **Time Response of Control system:-** If the output of control system for an input varies with respect to time, then it is called the "time response of the control system"

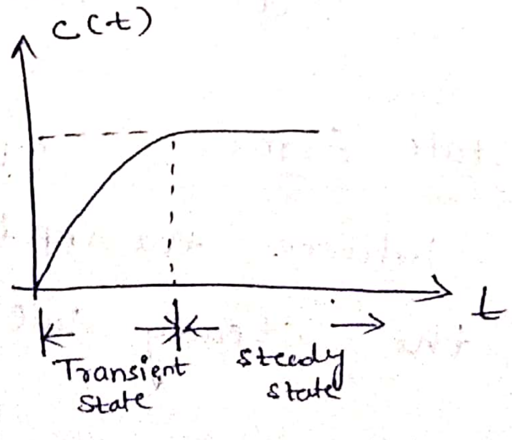
→ It is represented by c(t).

→ The response (or) output of control system is in time domain

The time Response of a control system consists of two parts

1. Transient Response
2. Steady state Response

1. **Transient Response :-**



→ After applying the input to the control system, output takes certain time to reach steady state.

∴ the response of the control system during the transient state is known as the transient Response.

→ Transient response means, response goes from initial state to the final state.

2. **Steady state response** :- The response during the steady state is called steady state response

→ In which the system output 't' approaches infinity.

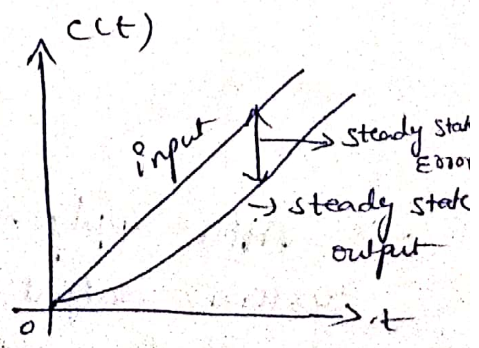
∴ thus the system response c(t) may be written as

$$\boxed{c(t) = c_{tr}(t) + c_{ss}(t)}$$

where $c_{tr}(t)$ = Transient Response

$c_{ss}(t)$ = Steady state response.

Steady State Error :- It is defined as the difference between the input and the output reached the steady state.



The steady state error will depend on the type of the input (i.e ramp, step, parabolic etc) and as well as the system type.

* **Step Response:-** The response of a system to the unit step input is called the step response

* **Impulse Response:-** The response of a system to the impulse signal input is called impulse response.

→ Impulse function is used to check the stability of a system.

* **Order of a system:-** The order of the system is given by the highest order of the differential equation representing the system.

→ If the system is represented by n^{th} order differential equation, then the system is called n^{th} order system.

Ex.. $a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \text{------}$

In case of transfer function, the order of the system is given by the maximum power of 's' in the denominator

$$T(s) = \frac{b_0 s^m + b_1 s^{m-1} + \text{----}}{a_0 s^n + a_1 s^{n-1} + \text{----}} = \frac{C(s)}{R(s)}$$

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In case of transfer function, the order of the system is given by the maximum power of 's' in the denominator.

$$T(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots}{a_0 s^n + a_1 s^{n-1} + \cdots} = \frac{C(s)}{R(s)}$$

where

$$R(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots$$

Here 'n' is the order of the system.

when $n = 0$, zero order system

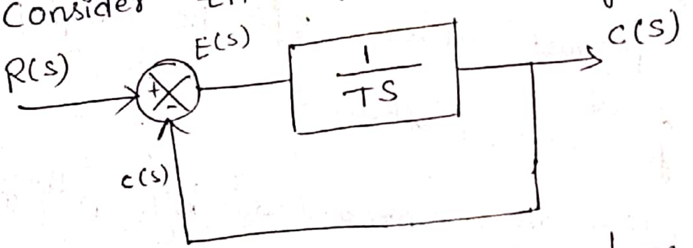
$n = 1$, first order system

$n = 2$, Second order system.

* **Characteristic Equation :-** the characteristic equation is nothing but setting the denominator of the closed loop transfer function to zero.

* **Time Response of first order Systems :-**

Consider the first order system



The transfer function of above block diagram is

$$C(s) = E(s) \cdot \frac{1}{TS} \longrightarrow ①$$

$$E(s) = R(s) - C(s) \longrightarrow ②$$

sub ② in ①

$$C(s) = \frac{R(s) - C(s)}{TS}$$

$$C(s)TS = R(s) - C(s)$$

$$C(s)TS + C(s) = R(s)$$

$$(1 + TS)\, C(s) = R(s)$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{1}{1+TS}}$$

∴ The simplified block diagram is

$$R(s) \longrightarrow \boxed{\frac{1}{1+TS}} \longrightarrow C(s)$$

∴ The input - output relation is given by

$$\boxed{\frac{C(s)}{R(s)} = \frac{1}{1+TS}}$$

Unit Step response of First order System:-

If the input is unit step then $r(t) = 1$

Then laplace transform of $r(t)$ is $\boxed{R(s) = 1/s} \longrightarrow (1)$

we know the time response of first order system is

$$\frac{C(s)}{R(s)} = \frac{1}{1+TS} \Rightarrow C(s) = R(s) \cdot \frac{1}{1+TS}$$

from eq (1)

$$C(s) = \frac{1}{S} \cdot \frac{1}{1+TS} \longrightarrow (2)$$

Unit step response can be obtained by applying the inverse lapdace transfarm to the equation (2)

$$\mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{S(1+TS)}\right]$$

Apply partial fractions

$$C(s) = \frac{1/T}{s(s+1/T)} \Rightarrow \frac{A}{S} + \frac{B}{s+1/T}$$

$$\frac{1}{T} = A(s+1/T) + BS$$

If $s = 0 \Rightarrow \frac{1}{T} = \frac{A}{T} + 0$ ∴ $\boxed{A = 1}$

If $s = -\frac{1}{T} \Rightarrow \frac{1}{\cancel{T}} = 0 - \frac{B}{\cancel{T}}$ ∴ $\boxed{B = -1}$

$$\therefore \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1/T}\right]$$

$$c(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1/T}\right]$$

$$\boxed{c(t) = 1 - e^{-t/\tau}}$$

when $t=0$

$c(t) = 1 - e^{0} = 0$

when $t = 2T$

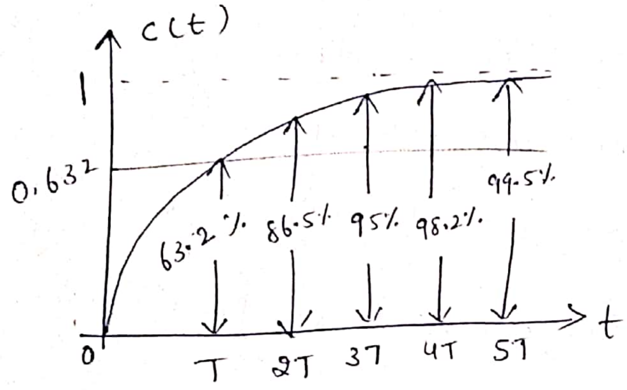
$c(t) = 1 - e^{-2} = 0.865$

when $t = 3T$

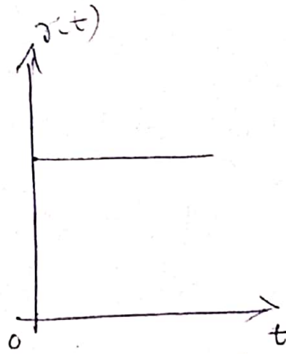
$c(t) = 1 - e^{-3} = 0.95$

at $t = T$

$$c(T) = 1 - e^{-T/T} = 1 - e^{-1} = 0.632$$



Unit step response of first order system



unit step input

Unit ramp response of first order system :-

If the input is unit ramp, then $r(t) = t$

The laplace transform of ramp input is $\boxed{R(s) = \frac{1}{s^2}}$ → (1)

we know the transfer function of first order system is $\frac{C(s)}{R(s)} = \frac{1}{1+TS}$

$$\boxed{C(s) = R(s) \cdot \frac{1}{1+TS}} \rightarrow (2)$$

sub (1) in (2), $C(s) = \frac{1}{s^2} \frac{1}{(1+TS)}$

$$C(s) = \frac{1\,(1/T)}{s^2\,(s+1/T)}$$

Apply partial fractions

(7)

$$c(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 1/T}$$

$$\therefore \frac{1}{T} = As\left(s+\frac{1}{T}\right) + B\left(s+\frac{1}{T}\right) + Cs^2$$

If $s = 0$

$$\frac{1}{T} = 0 + \frac{B}{T} + 0$$

$$\boxed{\therefore B = 1}$$

If $s = -1/T$

$$\frac{1}{T} = 0 + 0 + \frac{C}{T^2}$$

$$1 = \frac{C}{T}$$

$$\boxed{C = T}$$

On comparing the coefficients of s^2 $\quad A + C = 0$

$$A + T = 0$$

$$\boxed{A = -T}$$

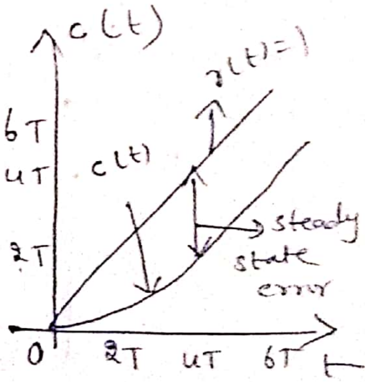
$$\therefore c(s) = \frac{-T}{s} + \frac{1}{s^2} + \frac{T}{s + 1/T}$$

Apply the inverse laplace transform to the above equation

$$\mathcal{L}^{-1}[c(s)] = \mathcal{L}^{-1}\left[\frac{-T}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + T\,\mathcal{L}^{-1}\left[\frac{1}{s+1/T}\right]$$

$$c(t) = -T + t + Te^{-t/T} \quad , \quad t \geq 0$$

$$\boxed{c(t) = t + T\left(e^{-t/T} - 1\right)}$$



Steady state error

$$\boxed{e(t) = r(t) - c(t)}$$

Unit impulse response of first order system. ⑧

If the input is impulse then $\boxed{R(s) = 1} \rightarrow (1)$

we know the transfer function of first order system is

$$\frac{C(s)}{R(s)} = \frac{1}{1+TS} \rightarrow (2)$$

sub eq (1) in (2)

$$C(s) = (1) \cdot \frac{1}{1+TS}$$

$$C(s) = \frac{1}{1+TS} \Rightarrow \frac{1/T}{(s+1/T)}$$

Apply inverse laplace transform to the above equation

$$L^{-1}[C(s)] = \frac{1}{T} L^{-1}\left[\frac{1}{s+1/T}\right]$$

$$c(t) = \frac{1}{T} e^{-t/T}$$

∴ The time response of unit impulse is

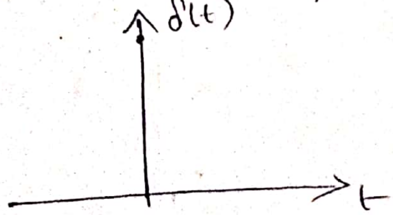
$$\boxed{c(t) = \frac{1}{T} e^{-t/T}}$$

when $t = 0$,

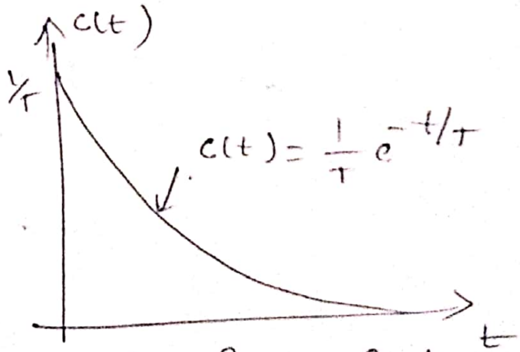
$c(t) = \frac{1}{T} e^{0} = 1/T$

when $t = 1T$

$c(t) = \frac{1}{T} e^{-T/T} = \frac{1}{T} e^{-1}$



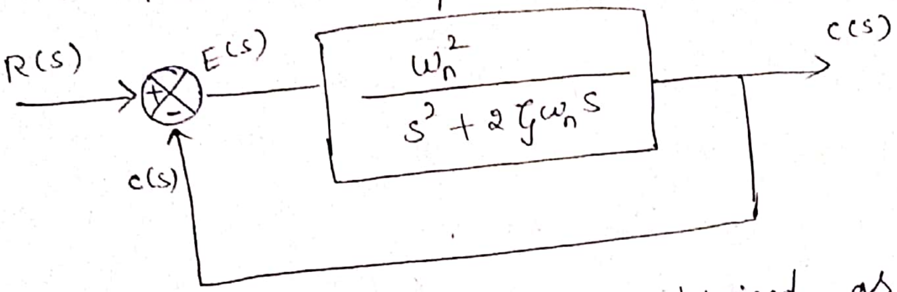
Input impulse signal



output: Impulse Response of first order system

* Second Order System :-

The closed loop second order system is given by



The transfer function can be obtained as

$$C(s) = E(s) \cdot \frac{\omega_n^2}{s^2+2\zeta\omega_n s} \longrightarrow ①$$

$$E(s) = R(s) - C(s) \longrightarrow ②$$

sub ② in ①

$$C(s) = \left[R(s) - C(s)\right] \frac{\omega_n^2}{s^2+2\zeta\omega_n s}$$

$$C(s) + C(s)\frac{\omega_n^2}{s^2+2\zeta\omega_n s} = \frac{R(s)\,\omega_n^2}{s^2+2\zeta\omega_n s}$$

$$C(s)\left[\frac{s^2+2\zeta\omega_n s+\omega_n^2}{\cancel{s^2+2\zeta\omega_n s}}\right] = \frac{R(s)\,\omega_n^2}{\cancel{s^2+2\zeta\omega_n s}}$$

$$\boxed{\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

$\therefore$ The standard form of closed loop transfer function of second order system is given by

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

where ω_n = undamped natural frequency, rad/sec

ζ = Damping ratio.

Damping Ratio :- The damping ratio is defined as the ratio of actual damping to the critical damping

→ The response of c(t) of the second order system depends on the value of damping ratio.

Depending on the damping ratio, the system can be classified into the following four cases.

Case 1 : Undamped System, $\zeta = 0$

Case 2 : Underdamped system, $0 < \zeta < 1$

Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Overdamped System, $\zeta > 1$

The characterstic equation of the second order system is given by

$$\boxed{s^2 + 2\zeta\omega_n s + \omega_n^2 = 0}$$

Roots of the above equation are given by

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

⇒ when $\zeta = 0$, $\boxed{s_1, s_2 = \pm j\omega_n}$ { Roots are purely imaginary and the system is undamped

⇒ when $\zeta = 1$, $\boxed{s_1, s_2 = -\omega_n}$; { Roots are real and equal and the system is critically damped when $\zeta > 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$ { Roots are real and unequal and the system is overdamped (11)

when $0 < \zeta < 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

$$= -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{(-1)}\sqrt{1-\zeta^2}$$

$$\boxed{s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}}$$

So, the roots are complex conjugate and the system is underdamped.

$$\boxed{\therefore\ s_1, s_2 = -\zeta\omega_n \pm j\omega_d}$$ where $$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

ω_d = damped frequency of oscillation of the system

* <u>Response of undamped Second order system for unit step unit</u>

for undamped System $\boxed{\zeta = 0} \longrightarrow (1)$

we know the transfer function of second order System is

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \longrightarrow (2)$$

Sub (1) in (2),

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2} \longrightarrow (3)$$

when the input is unit step, $r(t) = 1$

Laplace transform of unit step is, $\boxed{R(s) = 1/s}$ → (4)

from equation (3)

$$C(s) = R(s) \cdot \frac{\omega_n^2}{s^2+\omega_n^2}$$

from eq (4)

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2+\omega_n^2}$$

Apply the partial fractions

$$C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+\omega_n^2}$$

$$\therefore \omega_n^2 = A(s^2+\omega_n^2) + (Bs+C)(s)$$

on coparing the coefficients of ω_n^2 $A\omega_n^2 = \omega_n^2$

$$\boxed{A = 1}$$

co-efficients of $s^2 \Rightarrow A + B = 0$

$$1 + B = 0$$

$$\boxed{B = -1}$$

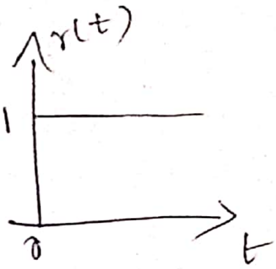
Co-efficients of $s \Rightarrow \boxed{C = 0}$

$$\therefore C(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$$

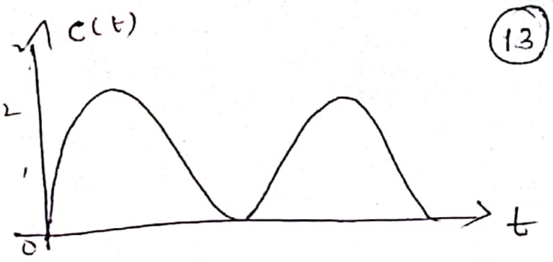
apply the inverse laplace transfam to the above equation

$$\mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+\omega_n^2}\right]$$

$\boxed{c(t) = 1 - \cos\omega_n t}$ this is the time domain response of second order system for the unit step input.



Unit step input



Time domain response.

Response of underdamped Second-order system for Unit step input

For underdamped system, $0 < \zeta < 1$ and the roots of the denominator are complex conjugate.

Since, $\zeta < 1$, ζ^2 is also < 1 and $1 - \zeta^2$ is always +ve

$$\therefore \quad s = -\zeta\omega_n \pm \omega_n \sqrt{(-1)(1-\zeta^2)}$$

$$\boxed{s = -\zeta\omega_n \pm j\omega_n \sqrt{1-\zeta^2}}$$

The damped frequency of Oscillation is $\boxed{\omega_d = \omega_n \sqrt{1-\zeta^2}}$

$$\boxed{\therefore \; s = -\zeta\omega_n \pm j\omega_d}$$

We know the ~~th~~ transfer function of second order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

When the input is unit step input, then $R(s) = 1/s$

$$\therefore \; C(s) = R(s) \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} \, \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

By partial fractions

$$C(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = A(s^2 + 2\zeta\omega_n s + \omega_n^2) + Bs + C(s)$$

$$\omega_n^2 = As^2 + 2\zeta\omega_n s A + A\omega_n^2 + Bs^2 + Cs$$

on comparing the co-efficients

$$A + B = 0, \quad 2\zeta\omega_n + C = 0, \quad A\cancel{\omega_n^2} = \cancel{\omega_n^2}$$

$$\boxed{A = 1}$$

$$1 + B = 0$$

$$\boxed{B = -1}$$

$$\boxed{C = -2\zeta\omega_n}$$

$$\therefore C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

adding & substracting the $\zeta^2\omega_n^2$ to the denominator of second term.

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2(1 - \zeta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Multiply & divide by ω_d in the third term of equation

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d} \frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

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apply the inverse laplace transform to the above equation.

$$\mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right] - \frac{\xi\omega_n}{\omega_d}\mathcal{L}^{-1}\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right]$$

we Know that

$$\mathcal{L}^{-1}\left[\frac{\omega}{(s+a)^2+\omega^2}\right] = e^{-at}\sin\omega t$$

$$\mathcal{L}^{-1}\left[\frac{s+a}{(s+a)^2+\omega^2}\right] = e^{-at}\cos\omega t$$

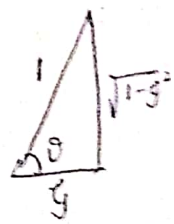
$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}e^{-\xi\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi\omega_n}{\omega_d}\sin\omega_d t\right]$$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\sin\omega_d t\right]$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right]$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t\right]$$



$\sin\theta = \sqrt{1-\xi^2}$

$\cos\theta = \xi$

$\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$

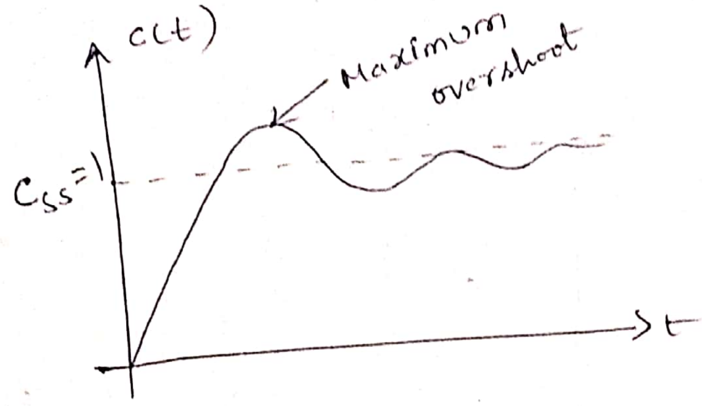
$$\boxed{c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin(\theta+\omega_d t)\right]}$$

where $\boxed{\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}$

when $t = 0$, $c(0) = 0$

$t = \infty$, $c(\infty) = 1$

$c(\infty) = C_{ss}$ = steady state value



Response of Critically damped second order system for unit step unit.

The transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

for critical damping $\xi = 1$

then

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} \Rightarrow \frac{\omega_n^2}{(s+\omega_n)^2}$$

when the input is unit step then $R(s) = 1/s$

$$\therefore C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{(s+\omega_n)^2}$$

$$\therefore C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$\omega_n^2 = A(s+\omega_n)^2 + Bs(s+\omega_n) + Cs$$

$$\omega_n^2 = As^2 + A\omega_n^2 + A\,2\omega_n s + Bs^2 + B\omega_n s + Cs.$$

on Comparing the co efficients

$A + B = 0.$ $\quad A\omega_n^2 = \omega_n^2$ $\quad 2A\omega_n + B\omega_n + C = 0$

$\boxed{A = 1}$ $\quad 2\omega_n - \omega_n + C = 0$

$1 + B = 0$ $\quad \omega_n + C = 0$

$\boxed{B = -1}$ $\quad \boxed{C = -\omega_n}$

$$\therefore\; C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

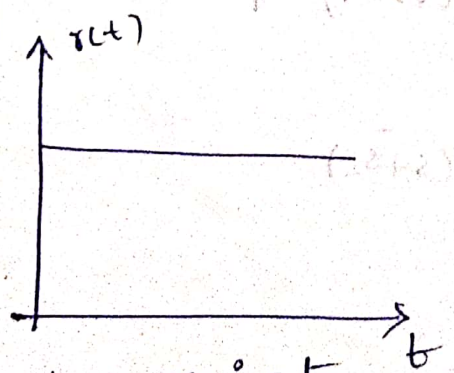
Apply the inverse laplace transform to the above equation

$$\mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+\omega_n}\right] - \omega_n \mathcal{L}^{-1}\left[\frac{1}{(s+\omega_n)^2}\right]$$

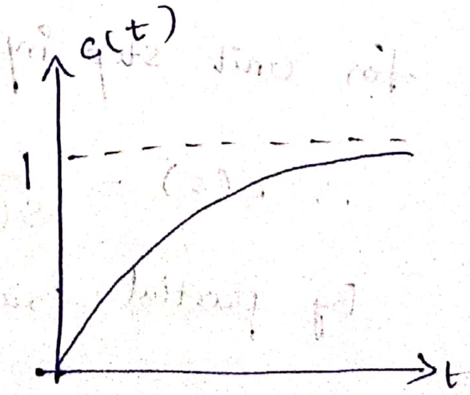
$$c(t) = 1 - e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t}$$

$$\boxed{c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)}$$

this is the time response of the ~~second~~ critically damped second order system for unit step input



unit step input



Response

Response of overdamped Second order system for unit step input

(18)

The transfer function of the second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

for overdamped system, $\xi > 1$. so the roots of the denominator are real and distinct.

Let the roots of the denominator are S_a & S_b

$$S_a, S_b = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$
$$= -\left[\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}\right]$$

Let $S_1 = -S_a$ and $S_2 = -S_b$

$$\therefore S_1 = \xi\omega_n - \omega_n\sqrt{\xi^2-1}$$
$$S_2 = \xi\omega_n + \omega_n\sqrt{\xi^2-1}$$

The closed loop transfer function can be written in terms of S_1 and S_2 as shown below

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+S_1)(s+S_2)}$$

for unit step input $r(t) = 1$, & $R(s) = 1/s$

$$\therefore C(s) = \frac{\omega_n^2}{s(s+S_1)(s+S_2)}$$

By partial fractions

$$C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

$$\cancel{C(s)}\ \omega_n^2 = A(s+s_1)(s+s_2) + B\,s(s+s_2) + C\,s(s+s_1)$$

when $s = 0$

$\Rightarrow \omega_n^2 = A(s_1)(s_2) + 0 + 0$, sub s_1 & s_2

$$A = \frac{\omega_n^2}{s_1 s_2} = \frac{\omega_n^2}{[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}][\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}]}$$

$$= \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$\boxed{\therefore A = 1}$$

when $s = -s_1$

$$\omega_n^2 = A(0) + B(-s_1)(-s_1+s_2) + 0$$

$$B = \frac{\omega_n^2}{(-s_1)(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}]}$$

$$B = \frac{-\omega_n^{\cancel{2}}}{s_1([2\omega_n\sqrt{\zeta^2-1}])}$$

$$\boxed{\therefore B = \frac{-\omega_n}{2\sqrt{\zeta^2-1}} \cdot \frac{1}{s_1}}$$

when $s = -s_2$

$$\omega_n^2 = 0 + 0 + C(-s_2)(-s_2+s_1)$$

$$C = \frac{\omega_n^2}{(-s_2)(-s_2+s_1)}$$

$$C = \frac{\omega_n^2}{(-s_2)\left[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]}$$

$$C = \frac{\omega_n^{2}}{s_2\left[2\omega_n\sqrt{\zeta^2-1}\right]}$$

$$\boxed{C = \frac{\omega_n}{\left[2\sqrt{\zeta^2-1}\right]s_2}}$$

Substitute A, B, C in c(s)

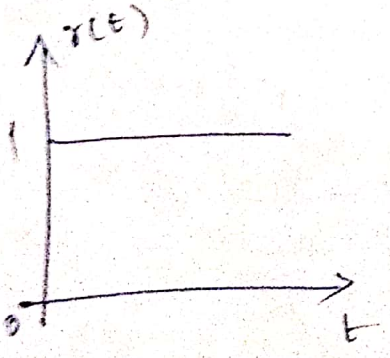
$$\therefore c(s) = \frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_1}\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_2}\frac{1}{(s+s_2)}$$

Apply the inverse laplace transform to the above equation

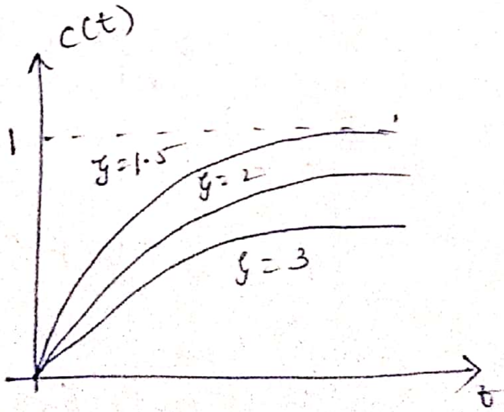
$$\mathcal{L}^{-1}[c(s)] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_1}\mathcal{L}^{-1}\left[\frac{1}{s+s_1}\right] + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_2}\mathcal{L}^{-1}\left[\frac{1}{s+s_2}\right]$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_1}e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{s_2}e^{-s_2 t}$$

$$\boxed{c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right]}$$



Unit step input

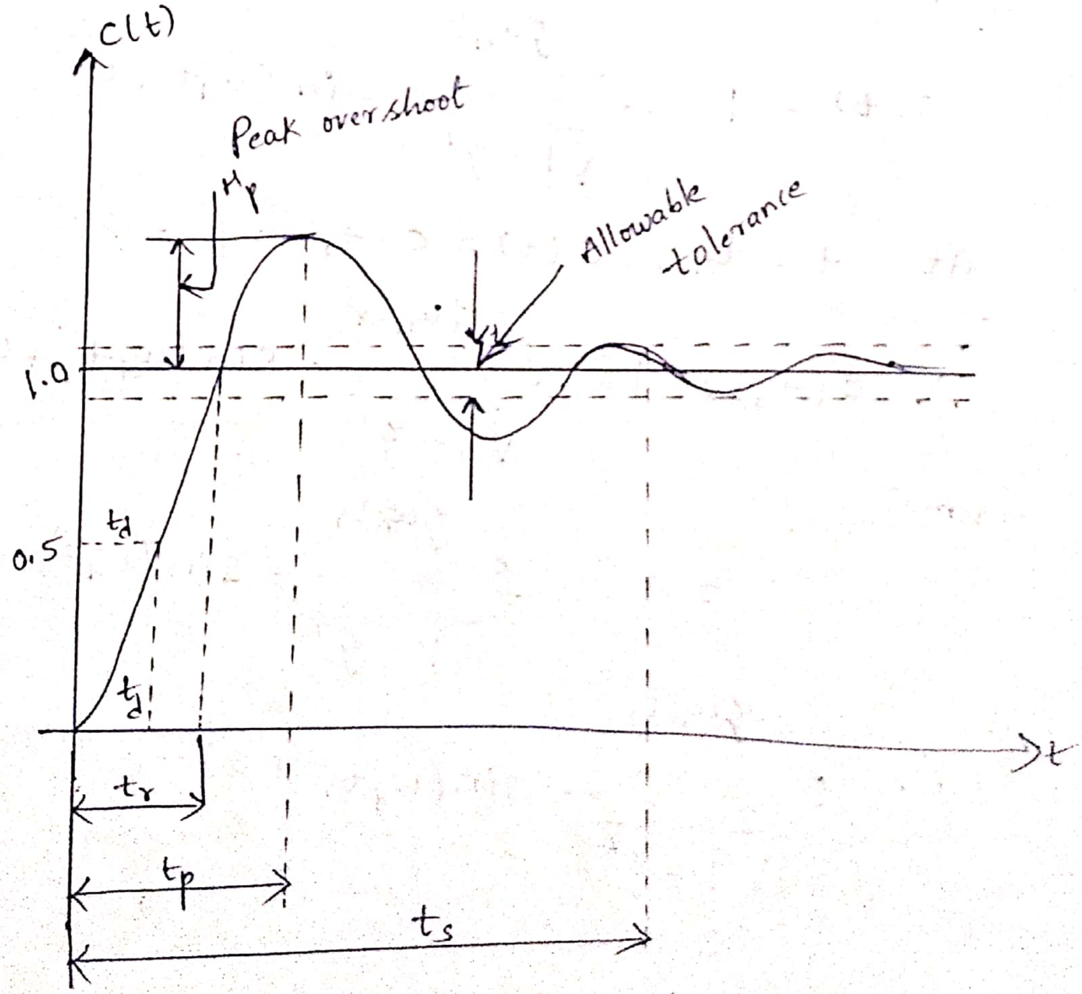


Time domain Specifications (or) Time Response Specifications

The desired performance characteristics of Control system are specified in terms of time domain specifications.

Time domain Specifications are

1. Delay time t_d
2. Rise time t_r
3. Peak time t_p
4. Peak overshoot M_p
5. Settling time t_s
6. Steady State Error e_{ss}



Time domain Specifications

1. **Delay time t_d :-** It is the time required for the response to reach 50% of the final value in first attempt.

2. **Rise time t_r :-** It is the time required for the response to rise from 10% to 90% of the final value for overdamped systems and 0 to 100% of the final value for underdamped systems

Expression for the Rise time t_r :-

The unit step response of second order system for underdamped case is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1 \longrightarrow (1)$

$$\therefore c(t_r) = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta)$$

from (1)

$$1 = 1 - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta)$$

$$\therefore \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = 0$$

Since $e^{-\zeta \omega_n t_r} \neq 0$

so, $\sin(\omega_d t_r + \theta) = 0$

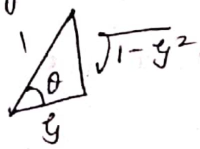
$$\therefore \omega_d t_r + \theta = \sin^{-1}(0)$$

$$\omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

$$\boxed{\therefore \text{Rise time}, \ t_r = \frac{\pi - \theta}{\omega_d}} \rightarrow (2)$$

On constructing the right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get



$$\therefore \tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Here $\left.\begin{array}{l} \theta = \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right) \text{ and} \\ \text{Damped frequency of oscillation, } \omega_d = \omega_n\sqrt{1-\zeta^2} \end{array}\right\} \rightarrow (3)$

Substitute Equations (3) in eq (3)

$$\therefore \text{Rise time} \ \boxed{t_r = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}{\omega_n\sqrt{1-\zeta^2}}}$$

(3) **Peak time** :- It is the time required for the response to reach the peak of time response

Expression for peak time :- To find the expression for peak time, t_p, differentiate $c(t)$ with respect to t and equating to zero

i.e $\left.\dfrac{d\,c(t)}{dt}\right|_{t=t_p} = 0$

The unit step response of under damped second order system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

Differentiating $c(t)$ w.r. to t

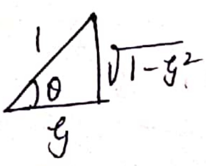
$$\frac{d}{dt} c(t) = \frac{-e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} (-\xi\omega_n) \sin(\omega_d t + \theta) + \left(\frac{-e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\right) \cos(\omega_d t + \theta)\, \omega_d$$

and we know $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\therefore \frac{d}{dt} c(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} (\xi\omega_n) \sin(\omega_d t + \theta) - \frac{\omega_n\sqrt{1-\xi^2}}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cos(\omega_d t + \theta)$$

$$= \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[\xi \sin(\omega_d t + \theta) - \sqrt{1-\xi^2} \cos(\omega_d t + \theta)\right]$$

from



$\Rightarrow$ $\sin\theta = \sqrt{1-\xi^2}$

$\cos\theta = \xi$

$$\therefore \frac{d}{dt} c(t) = \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[\cos\theta \sin(\omega_d t + \theta) - \sin\theta \cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[\sin(\omega_d t + \theta) \cos\theta - \cos(\omega_d t + \theta) \sin\theta\right]$$

$$\frac{dc(t)}{dt} = \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[\sin(\omega_d t + \theta - \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_d t)$$

at $t = t_p$, $\frac{d\, c(t)}{dt} = 0$

$$\therefore \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t_p} \sin(\omega_d t_p) = 0.$$

Since $e^{-\xi\omega_n t_p} \neq 0$, then $\sin(\omega_d t_p) = 0$

$$\therefore \omega_d t_p = \sin^{-1}(0)$$

$$\omega_d t_p = \pi$$

$$\boxed{\therefore \text{Peak time } t_p = \frac{\pi}{\omega_d}}$$

we know, the damped frequency of oscillation

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

$$\boxed{\therefore \text{peak time } t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

4. Peak overshoot :- It indicates the normalized difference between the time response peak and the steady output and is defined as

$$\text{Peak percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

where $c(t_p)$ = Peak response at $t = t_p$

$c(\infty)$ = Final steady state value

The unit step response of second order system is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

At $t = \infty$, $c(t) = c(\infty) = 1 - \frac{e^{-\infty}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$

$$= 1 - 0$$

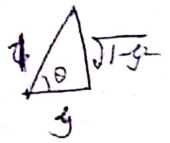
At $t = t_p$, $c(t) = c(t_p) = 1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \theta)$

we already know $t_p = \frac{\pi}{\omega_d}$ & $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\therefore\ c(t_p) = 1 - \frac{e^{-\frac{\xi\cancel{\omega_n}\pi}{\cancel{\omega_n}\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin\left(\cancel{\omega_d}\frac{\pi}{\cancel{\omega_d}} + \theta\right)$$

$$c(t_p) = 1 - \frac{e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin(\pi + \theta)$$

$$c(t_p) = 1 + \frac{e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin\theta$$



$\sin\theta = \sqrt{1-\xi^2}$

$$c(t_p) = 1 + \frac{e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}{\cancel{\sqrt{1-\xi^2}}} (\cancel{\sqrt{1-\xi^2}})$$

$$\boxed{c(t_p) = 1 + e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}}$$

$\therefore$ % Peak overshoot, $\%\, M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$

$$\% M_p = \frac{1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} - 1}{1} \times 100$$

$$\boxed{\% M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100}$$

5. Settling time (t_s) :- It is the time required for the response to reach and stay with in a specified tolerance band (usually 2% (or) 5%) of its final value.

the response of second order system has two components. They are

1. Decaying exponential component, $\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$
2. Sinusoidal component, $\sin(\omega_d t + \theta)$

→ Among the above two components decaying exponential term reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by the exponential component.

→ the settling time can be found out by equating exponential component to % tolerance errors

for 2% tolerance error band, at $t = t_s$

$$\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$$

for least values of t_s,

$$e^{-\xi\omega_n t_s} = 0.02$$

$$-\xi\omega_n t_s = \ln(0.02)$$

$$-\xi\omega_n t_s = -4$$

$$\boxed{t_s = \frac{4}{\xi\omega_n}}$$

$\therefore$ Settling time, $\boxed{t_s = \frac{4}{\xi\omega_n}}$ for 2% error

for 5% tolerance error.

$$-\xi\omega_n t_s = \ln(0.05)$$

$$-\xi\omega_n t_s = -3$$

$$\boxed{t_s = \frac{+3}{\xi\omega_n}}$$

$\therefore$ Settling time, $\boxed{t_s = \frac{3}{\xi\omega_n}}$ for 5% error

6. Steady State error:- It indicates the error between the actual output and desired output as t tends to ∞ i.e

$$e_{ss} = \lim_{t\to\infty} [r(t) - c(t)]$$

Type Number of control System:-

The number of poles of the loop transfer function lying at the origin decides the type number of the system. The type number is specified for loop transfer function $G(s)H(s)$.

→ the loop transfer function can be expressed as a ratio of two polynomials in s.

$$G(s)H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)(s+z_2)(s+z_3)\cdots}{s^N(s+P_1)(s+P_2)(s+P_3)\cdots}$$

where $z_1, z_2, z_3 \cdots$ are zeros of transfer function

$P_1, P_2, P_3 \cdots$ are poles of transfer function

K = Constant

N = Number of poles at the origin.

The value of 'N' in the denominator of loop transfer function represents the type number of the System.

If $N=0$, system is type-0 system

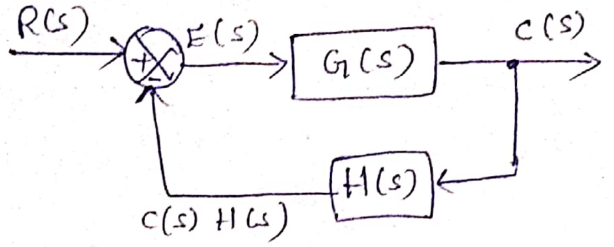
$N=1$, System is type-1 system

$N=2$, System is type-2 system and so on

Steady State Error:-

The steady state error is the value of error signal $e(t)$, when t tends to ∞. These errors arise from the nature of inputs, type of system and from non-linearity of system components.

Consider a closed loop system



the error signal $E(s) = R(s) - C(s)\,H(s)$. → ①

The output signal $C(s) = E(s)\,G(s)$. → ②

sub eq ② in eq ①

$$E(s) = R(s) - [E(s)\,G(s)]\,H(s)$$

$$E(s) + E(s)\,G(s)\,H(s) = R(s)$$

$$E(s)\,[1 + G(s)\,H(s)] = R(s)$$

$$\therefore E(s) = \frac{R(s)}{1 + G(s)\,H(s)} \rightarrow ③$$

Let $e(t)$ = error signal in time domain

The steady state error is defined as the value of $e(t)$ when t tends to ∞

$$\therefore e_{ss} = \lim_{t\to\infty} e(t)$$

The final value theorem of laplace transform states that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$$

So, the steady state error can be written as

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s)$$

from eq ③

$$\boxed{\therefore e_{ss} = \lim_{s\to 0} s \cdot \frac{R(s)}{1 + G(s)\,H(s)}}$$

Static Error Constants

The value of steady state error depends on the type number and the input signal static error constants are 3 types.

Type-0 static error constants

1. Positional error constant $K_p = \lim_{s \to 0} G(s)H(s)$
2. Velocity error constant $K_v = \lim_{s \to 0} s\,G(s)H(s)$
3. Acceleration error constant $K_a = \lim_{s \to 0} s^2 G(s)H(s)$

1. Type-0 system will have a constant steady state error when the input is step signal
2. Type-1 system will have a constant steady state error when the input is ramp signal
3. Type-2 system will have a constant steady state error when the input is parabolic signal

Steady State error when the input is unit step signal

Steady state error, $e_{ss} = \lim_{s \to 0} \frac{s \cdot R(s)}{1 + G(s)H(s)}$

when the input is unit step, $R(s) = 1/s$.

$$\therefore\ e_{ss} = \lim_{s \to 0} \frac{\cancel{s} \cdot 1/s}{1 + G(s)H(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{1 + G(s)H(s)}$$

$$e_{ss} = \frac{1}{1 + \lim_{s \to 0} G(s)H(s)}$$

$$e_{ss} = \frac{1}{1 + K_p}$$ where K_p = positional error constant

Type - 0 System

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$= \lim_{s \to 0} K \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$K_p = K \frac{z_1 z_2 z_3 \ldots}{P_1 P_2 P_3 \ldots} = \text{Constant}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{Constant}$$

Hence in type - 0 systems when the input is unit step there will be a constant steady state error.

Type - I System

$$K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} K \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{s(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$K_p = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0.$$

In systems with type number 1 and above for unit step input the value of K_p is ∞ and steady state error is zero.

Steady state error when the input is unit Ramp signal

$$\text{Steady state error } e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1+G(s)H(s)}$$

when the input is unit Ramp, $R(s) = \frac{1}{s^2}$

$$\therefore e_{ss} = \lim_{s \to 0} \frac{\cancel{s} \cdot 1/s^{\cancel{2}}}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{1}{s + sG(s)H(s)}$$

$$e_{ss} = \frac{1}{\lim_{s \to 0} sG(s)H(s)} = \frac{1}{K_v}$$

where $K_v = \lim_{s \to 0} sG(s)H(s)$

the constant 'K_v' is called velocity error constant

Type - 0 System

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} s.K \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$= 0$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

Hence in Type-0 systems when the input is unit ramp, the steady state error is infinity

Type-1 System

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \cancel{s}K \frac{(s+z_1)(s+z_2)\ldots}{\cancel{s}(s+P_1)(s+P_2)\ldots}$$

$$K_v = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{\text{Constant}} \cancel{= \infty} = \text{constant}$$

Hence in type-1 systems when the input is unit ramp there will be a constant steady state error

Type-2 System

$$K_v = \lim_{s \to 0} s\,G(s)H(s)$$

$$= \lim_{s \to 0} sK \frac{(s+z_1)(s+z_2)\ldots}{s^2(s+p_1)(s+p_2)\ldots} = \infty$$

$$\boxed{K_v = \infty}$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

In systems with type number 2 and above for unit ramp unit, the value of K_v is ∞ so the steady state error is zero

Steady State error when the input is unit parabolic Signal

Steady state error, $e_{ss} = \lim_{s \to 0} s \frac{R(s)}{1+G(s)R(s)}$

when the input is unit parabolic $R(s) = 1/s^3$

$$\therefore \cancel{e_{ss} = \lim_{s \to 0} s \frac{R(s)H(s)}{1+G(s)H(s)}}$$

$$\therefore e_{ss} = \lim_{s \to 0} s \frac{1/s^3}{1+G(s)H(s)} = \lim_{s \to 0} \frac{1}{s^2+s^2G(s)H(s)}$$

$$e_{ss} = \frac{1}{\lim_{s \to 0} s^2 G(s)H(s)} = \frac{1}{K_a}$$

where $K_a = \lim_{s \to 0} s^2 G(s)H(s)$ and it is accelaration error constant

Type-0 System

$$K_a = \lt_{s\to 0} s^2\, G(s)\, H(s) = \lt_{s\to 0} s^2 K \frac{(s+z_1)(s+z_2)\ldots}{(s+P_1)(s+P_2)\ldots}$$

$$\boxed{K_a = 0}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity

Type-I System

$$K_a = \lt_{s\to 0} s^2 G(s) H(s) = \lt_{s\to 0} \cancel{s^2} K \frac{(s+z_1)(s+z_2)\ldots}{\cancel{s}(s+P_1)(s+P_2)\ldots}$$

$$\boxed{K_a = 0}$$

~~Hence~~ $\therefore e_{ss} = \frac{1}{0} = \infty$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity

Type-2 System

$$K_a = \lt_{s\to 0} s^2 G(s) H(s) = \lt_{s\to 0} \cancel{s^2} K \frac{(s+z_1)(s+z_2)\ldots}{\cancel{s^2}(s+P_1)(s+P_2)\ldots}$$

$$\boxed{K_a = \text{Constant}}$$

$$\therefore e_{ss} = \frac{1}{\text{Constant}} = \text{Constant}$$

Hence in type-2 system for unit parabolic input, the steady state error is constant

Type-3 system

$$K_a = \lt_{s\to 0} s^2 G(s) H(s) = \lt_{s\to 0} \cancel{s^2} K \frac{(s+z_1)(s+z_2)\ldots}{s^{\cancel{3}}(s+P_1)(s+P_2)\ldots}$$

$$\boxed{K_a = \infty}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

Hence in type-3 system for unit parabolic input, the steady state error is zero and K_a is infinity

Static Error Constants for various type number of system

Error Constant	Type number of system			
	0	1	2	3
K_p	Constant	∞	∞	∞
K_v	0	Constant	∞	∞
K_a	0	0	Constant	∞

Steady State error for various types of inputs

Input Signal	Type number of Systems			
	0	1	2	3
Unit step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0