Discrete Mathematics

Unit -I :- Mathematical logic :-

Introduction, Statements and Notations, Connectives, well found formula, Tautology, Duality law, equivalence, Implication, Normal forms, functionally complete set of connectives, Inference theory of statement calculus, predicate calculus, Inference theory of predicate calculus.

Discrete mathematics :-

Discrete mathematics is a branch of mathematics which deals with only those real numbers which are multiple of some basic unit.

Ex:- $f(n) = \langle 1,2,3,4 \ldots\ldots \rangle = \langle n \rangle$

$f(n) = \langle 4,8,12,16 \ldots\ldots \rangle = \langle 4n \rangle$

Uses:-

* Discrete Mathematics is widely used in studying and distributing the objects and problems in branches of computer science such as computer algorithms, programming languages, pictography, Automated theorem and software development etc.

Mathematical logic :-

Logic is a discipline which deals with methods of reasoning. Logical reasoning is used in computer science to verify the correctness of programs.

Statement proposition:-

A proposition or a statement is a declarative sentence that is either true or false but not both.

Ex:- $3+5=7$ is a statement & it is false.

$1-5=-4$ is a statement & it is true

$x+y>1$ is not a statement.

Note:-

* The truth values of a statement are proposition which are "True" or "False" which are respectively denoted by "T" and "F" (or) "1" and "0"

Logical Connectivenes & Compound Statements:-

The Three, most basic and fundamental connectivities are negation, conjunction and disjunction which are shown

English word	mathematical word	Symbol/Notation
NOT	Negation	~ or ¬
AND	Conjuction	$\wedge$
OR	Disjuction	$\vee$

→ Negations:-

For any preposition 'p' the negation of p can be formed by writing "It is not the case that" or "It is false that". Before p.

* It is denoted by ~p or ¬p.

Eg:- If p: The stars in the sky.

~p: The stars not in the sky.

(ii) If p: paris is in france.

~p: paris not in france.

Truthtable:-

p	~p
T	F
F	T

→ Conjunction:-

The conjuction of two statements p & q is a compound statement "p and q". It is denoted by "p∧q."

The compound statements $p \wedge q$ is true only when both p and q are true. Otherwise it is false.

Truth table:-

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Eg:- p: Ram is healthy.

q: He has blue eyes.

$p \wedge q$: Ram is healthy and he has blue eyes.

→ **Disjuction:-**

The disjuction of two statements p & q is a compound statement. "p or q". It is denoted by "$p \vee q$".

* The compound statement $p \vee q$ is false only when both p or q are false, otherwise it is true.

Truth Table:-

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Eg: p: It is cold.

q: It is raining.

$\sim p$: It is not cold.

$p \wedge q$: It is cold and raining.

$p \vee q$: It is cold or raining.

$\sim p \wedge q$: It is not cold and raining.

$p \wedge \sim q$: It is cold and not raining.

Conditional Statement / Implication:-

If p and q are two statements then the compound statement "if p then q" is called the conditional statement or Implication.

- It is denoted by "$p \Rightarrow q$".
- The conditional statement $p \Rightarrow q$ is false only if q is false and p is true, otherwise it is true.

Truth table:-

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

* Here the statement p is called "anticedant" or "Hypothesis" and statement q is called "consequent" or "conclusion".

Eg:- If Tomorrow is wednesday then today is tuesday.

Here p: Tomorrow is wednesday.

q: Today is Tuesday.

Note:- The conditional statement if p then q can also be read as

1) p implies q
2) p is sufficient for q.
3) p only of q.
4) q is necessary for p.
5) q if p.
6) q follows from p.
7) q is consequent of p.

Biconditional statement:-

If p and q are two statements then the compound statement "If p is if and only if q" is called the Biconditional statement.

* It is denoted by "$p \Leftrightarrow q$", "$p \longleftrightarrow q$", "$p \rightleftarrows q$".
* The Biconditional statement $p \Leftrightarrow q$ is true if and only if both q and p are true or false otherwise false.

Truth Table:-

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Eg:- p: A New car will be acquired.

q: Additional funding is available.

$p \Leftrightarrow q$: A New car will be acquired if and only if additional funding is available.

Logically equivalence:-

Two propositions $p(p_1, p_2, p_3 \ldots)$ and $q(q_1, q_2 \ldots)$ are said to be logically equivalent or simply equivalent if they have identical truth values.

* It is denoted by $p(p_1, p_2 \ldots) \equiv q(q_1, q_2 \ldots)$

→ Construct truth tables for each of following:-

(i) $p \wedge (\sim q \vee q)$

p	q	$\sim q$	$\sim q \vee q$	$p \wedge (\sim q \vee q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	T	F

(ii) $\sim(p \vee q) \vee (\sim p \wedge \sim q)$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim p \wedge \sim q$	$\sim(p \vee q)$	$\sim(p \vee q) \vee (\sim p \wedge \sim q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	F	T	T	F	T	T	T

→ Use truth table to show that $p \Rightarrow q \equiv \sim p \vee q$.

p	q	$\sim p$	$p \Rightarrow q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$\therefore$ $p \Rightarrow q \equiv \sim p \vee q$. are equivalent.

The truth values of $p \Rightarrow q$ and $\sim p \vee q$ are same.

Hence $p \Rightarrow q \equiv \sim p \vee q$.

→ Show that $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(\sim p \vee q) \wedge (\sim q \vee p)$ $(p \Rightarrow q) \wedge (q \Rightarrow p)$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

The truth values of $p \Leftrightarrow q$ and $(p \Rightarrow q) \wedge (q \Rightarrow p)$ are same.

$\therefore$ $p \Leftrightarrow q$ and $(p \Rightarrow q) \wedge (q \Rightarrow p)$ are equivalent.

Hence $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.

Laws of Algebra of propositions:

Idempotent law:-

* $p \vee p \equiv p$ And
* $p \wedge p \equiv p$

Associative law:-

* $p \vee (q \vee s) \equiv (p \vee q) \vee s$
* $p \wedge (q \wedge s) \equiv (p \wedge q) \wedge s$

Commutative law:-

* $\cancel{p \vee (q \vee s)}$ $p \wedge q \equiv q \wedge p$
* $p \vee q \equiv q \vee p$.

Complement law:-

* $p \vee \sim p \equiv T$
* $p \wedge \sim p \equiv F$

$\sim T \equiv F$; $\sim F \equiv T$.

Important

De morgan's law:-

* $\sim (p \vee q) \equiv \sim p \wedge \sim q$
* $\sim (p \wedge q) \equiv \sim p \vee \sim q$

Important

Distributive law:-

* $p \vee (q \wedge s) \equiv (p \vee q) \wedge (p \vee s)$
* $p \wedge (q \vee s) \equiv (p \wedge q) \vee (p \wedge s)$

Identity law:-

* $p \vee F \equiv p$
* $p \vee T \equiv T$
* $p \wedge F \equiv F$
* $p \wedge T \equiv p$.

Involution law:-

* $\sim(\sim p) \equiv p$.

→ Show that $p \Leftrightarrow q \equiv (p \vee q) \Rightarrow (p \wedge q)$ by using

a) Truth Table b) Algebra propostions.

a)

p	q	$p \Leftrightarrow q$	$p \vee q$	$p \wedge q$	$(p \vee q) \Rightarrow (p \wedge q)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T.

:- $p \Leftrightarrow q \equiv (p \vee q) \Rightarrow (p \wedge q)$

$p \Leftrightarrow q$ and $(p \vee q) \Rightarrow (p \wedge q)$ are equivalent.

Hence $p \Leftrightarrow q \equiv (p \vee q) \Rightarrow (p \wedge q)$ truth values are same.

b) By Algebra propositions:-

$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

$\equiv \underbrace{(\sim p \vee q)}_{a} \wedge (\underbrace{\sim q}_{b} \vee \underbrace{p}_{c})$

By distributive law

$\equiv ((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p)$

$\equiv ((\sim p \wedge \sim q) \vee (q \wedge \sim q)) \vee ((\sim p \wedge p) \vee (q \wedge p))$

By complement law

$\equiv ((\sim p \wedge \sim q) \vee F) \vee (F \vee (q \wedge p))$

By Idempotent law

$\equiv (\sim p \wedge \sim q) \vee (q \wedge p)$

By De morgan's & Commutative law

$\equiv \underbrace{\sim (p \vee q)}_{p} \vee \underbrace{(p \wedge q)}_{q}$

$\sim p \vee q \equiv p \Rightarrow q$

$\equiv (p \vee q) \Rightarrow (p \wedge q)$

$\therefore$ $p \Leftrightarrow q = ((p \vee q) \Rightarrow (p \wedge q))$

→ prove that $(\sim p \wedge (\sim q \wedge r)) \vee ((q \wedge r) \vee (p \wedge r)) \Leftrightarrow r$

$(\sim p \wedge (\sim q \wedge r)) \vee ((q \wedge r) \vee (p \wedge r)) \cancel{\Leftrightarrow} \equiv$

$\equiv ((\sim p \wedge \sim q) \wedge r) \vee ((q \vee p) \wedge r)$

By distributive law

$\equiv (\sim (p \vee q) \wedge r) \vee ((q \vee p) \wedge r)$

By Demorgan's law

$\equiv (\sim (p \vee q) \wedge r) \vee ((p \vee q) \wedge r)$

By distributive law

$\equiv (\sim (p \vee q) \vee (p \vee q)) \wedge r$ By complement law

$\equiv T \wedge r$ By Identity law

$\equiv r$

→ $p \Rightarrow (q \Rightarrow r) \Leftrightarrow p \Rightarrow (\sim q \vee r) \Leftrightarrow (p \wedge q) \Rightarrow r$

$p \Rightarrow (q \Rightarrow r) \equiv p \Rightarrow (\sim q \vee r)$ $\quad$ $[p \Rightarrow q \equiv (\sim p \vee q)]$

$\equiv \sim p \vee (\sim q \vee r)$ $\quad$ By associative law

$\equiv (\sim p \vee \sim q) \vee r$ $\quad$ By Demorgan's law

$\equiv \sim (p \wedge q) \vee r$

$\equiv (p \wedge q) \Rightarrow r$

→ $(p \Rightarrow q) \wedge (r \Rightarrow q) \Leftrightarrow ((p \vee r) \Rightarrow q)$

$= (\sim p \vee q) \wedge (\sim r \vee q)$ $\quad$ $[\because p \Rightarrow q \equiv (\sim p \vee q)]$

$\equiv (\sim p \wedge \sim r) \vee q$ $\quad$ By distributive law

$\equiv \sim (p \vee r) \vee q$ $\quad$ $[\because \sim p \vee q \equiv (p \Rightarrow q)]$

$\equiv ((p \vee r) \Rightarrow q)$

Tautology:-

→ A statement that is true for all possible values of its propositional variables is called a tautology.

Eg:- The proposition $p \vee \sim p$ is a tautology

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

* $\sim (p \wedge q) \vee q$

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim (p \wedge q) \vee q$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

* $p \Rightarrow (p \vee q)$

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

* x is a prime and x is an integer which is divisible by 1 and itself only.
* X is an ~~integer~~ prime and x is an even integer greater than 6. is not a tautology.

Contradiction:-

A compound proposition that is always falls for all possible values of its variables is called a contradiction.

Eg:- All men are good and All men are bad.

* The proposition $p \wedge (q \wedge \sim p)$

p	q	$\sim p$	$q \wedge \sim p$	$p \wedge (q \wedge \sim p)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

Contingency:-

A proposition that is neither tautology nor contradiction is called a contingency.

Eg:- The compound proposition $p \wedge (\sim p \vee q)$ is a contingency.

p	q	$\sim p$	$\sim p \vee q$	$p \wedge (\sim p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	F
F	F	T	T	F

Derived Connectives:-

1) NAND $\uparrow$:- Suppose (p & q) are two propositions NAND of p and q is a proposition which is false when both p and q are true otherwise it is true. It is denoted by $(p \uparrow q)$. i.e $\sim (p \wedge q)$ is the NAND of p and q.

$\boxed{p \uparrow q \equiv \sim(p \wedge q)}$

p	q	$p \uparrow q = p \wedge q$	$p \uparrow q$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

2) NOR ($\downarrow$):-

Suppose p and q are two propositions NOR of p and q is a proposition which is true only when both p and q are false otherwise it is false. It is denoted by $(p \downarrow q)$

i.e $\sim(p \vee q)$ is the NOR of p and q

$\boxed{p \downarrow q \equiv \sim(p \vee q)}$

p	q	$p \vee q$	$p \downarrow q$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

3) XOR:- [Exclusive OR $\oplus$]:-

Suppose p and q are two propositions the Exclusive OR (XOR) is a proposition, which is true when exactly one of p and q is true otherwise it is false. It is denoted by $(p \oplus q)$ or $(p \bar{\vee} q)$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Well formed formula (wffs):-

A statement formula is an expression which is a string consisting of variables, paranthesis and connective symbols. A grammatically correct expression is called a well-formed formula which is abbreviated as "wff" and is pronounced as woof.

Eg:- 1) $p \wedge (\sim p \rightarrow q)$ is a well formed formula.

2) $p \Rightarrow (q \wedge)$ is not a well formed formula.

3) $\sim p \vee q \Leftrightarrow (p \wedge \sim p \vee)$ is not a well formed formula.

Note:- If p and q are well formed formulas then $p \wedge q$, $p \vee q$, $p \Rightarrow q$, $p \Leftrightarrow q$ are also well formed formulas.

Duality law:-

Two formulas A and B are said to be duals of each other if either one can be obtained from the other by replacir $\wedge$ by $\vee$ (or) $\vee$ by $\wedge$.

Eg:- ① $(p \vee (p \wedge \sim q))$ and $(p \wedge (p \vee \sim q))$

② $p \vee (q \wedge R)$ and $p \wedge (q \vee R)$

Inverse, Converse and Contrapositive:-

If $p \Rightarrow q$ is an implication then

1) Inverse : $\sim p \Rightarrow \sim q$

2) Converse : $q \Rightarrow p$

3) Contrapositive : $\sim q \Rightarrow \sim p$

					Inverse	Converse	Contrapositive
p	q	$p \Rightarrow q$	$\sim p$	$\sim q$	$\sim p \Rightarrow \sim q$	$q \Rightarrow p$	$\sim q \Rightarrow \sim p$
T	T	T	F	F	T	T	T
T	F	F	F	T	T	T	F
F	T	T	T	F	F	F	T
F	F	T	T	T	T	T	T

Functionally complete set of connectives:-

A set of connectives in which every formula can be expressed in terms of an equivalent formula containing the connectives from the set (i.e, $\vee, \wedge, \sim$) is called the functionally complete set of connectives.

Note:-

In any formula we can replace all the biconditionals (i.e $\Leftrightarrow$) and then conditionals (i.e $\Rightarrow$) and finally all the conjunctions and disjunctions to obtain an equivalent formula containing either negation and disjunction [$(\sim, \vee)$ (or) $(\sim, \wedge)$] or negation and conjunction [$(\sim, \wedge)$] are minimal functionally complete set.

→ Show that $\{\uparrow\}$ and $\{\downarrow\}$ are functionally complete set of connectives.

Sol:- It is sufficiently prove that the sets $\{\sim, \wedge\}$ and $\{\sim, \vee\}$ can be expressed either in terms of $\{\uparrow\}$ (or) $\{\downarrow\}$

Eg:- * ~~$\sim p \Leftrightarrow (\sim p \vee \sim q)$~~ $\sim p = (\sim p \vee \sim q)$

$$= \sim (p \wedge q)$$

$$= p \uparrow q$$

× [* $p \wedge q = (p \wedge q) \vee (p \wedge q)$

$$= \sim(\sim(p \wedge q)) \vee \sim(p \wedge q)$$

$$= \sim (p \uparrow q) \vee (p \uparrow q)$$

$$= \sim ((p \uparrow q) \wedge (p \uparrow q))$$

$$= (p \uparrow q) \uparrow (p \uparrow q)]×$$

$\therefore p \uparrow q \equiv \sim (p \wedge q)$

$$* \; p \wedge q = (p \wedge q) \vee (p \wedge q)$$

$$= \sim(\underbrace{\sim(p \wedge q)}_{A} \wedge \underbrace{\sim(p \wedge q)}_{B})$$

$$= (\sim(p \wedge q)) \uparrow (\sim(p \wedge q))$$

$$= (p \uparrow q) \uparrow (p \uparrow q)$$

$\therefore \{\uparrow\}$ is a functionally complete set of connectives.

* $\sim p = \sim p \wedge \sim p$

$= \sim (p \vee p)$

$= p \downarrow p$

* $p \vee q = (p \vee q) \wedge (p \vee q)$

$= \sim(\sim(p \vee q) \vee \sim(p \vee q))$

$= (\sim(p \vee q)) \downarrow (\sim(p \vee q))$

$= (p \downarrow q) \downarrow (p \downarrow q)$

$\therefore$ $\{\downarrow\}$ is a functionally complete set of connectives

→ Write an equivalent expression for $(p \Rightarrow (q \wedge r)) \vee (r \Leftrightarrow s)$ which contains neither both biconditonal nor conditional.

$$(p \Rightarrow (q \wedge r)) \vee (r \Leftrightarrow s) \equiv (p \Rightarrow (q \wedge r)) \vee ((r \Rightarrow s) \wedge (s \Rightarrow r))$$
$$\equiv (\sim p \vee (q \wedge r)) \vee ((\sim r \vee s) \wedge (\sim s \vee r))$$

$\therefore$ $\left.\begin{aligned} p \Leftrightarrow q &\equiv (p \Rightarrow q) \wedge (q \Rightarrow p) \\ &\equiv (\sim p \vee q) \wedge (\sim q \vee p) \end{aligned}\right\}$ formula.

→ Express the following formulas ~~functions~~ into a functionally complete set of connectives.

$$\sim(p \Leftrightarrow (Q \Rightarrow (R \vee p)))$$
$$\equiv \sim(p \Rightarrow (Q \Rightarrow (R \vee p)) \wedge ((Q \Rightarrow (R \vee p)) \Rightarrow p)$$

$\therefore$ $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$

$$\equiv \sim(p \Rightarrow (\sim Q \vee (R \vee p)) \wedge (Q \Rightarrow (\sim(R \vee p)) \vee p))$$
$$\equiv \sim[(\sim p \vee (\sim Q \vee R \vee p)) \wedge (\sim Q \vee (\sim R \wedge \sim p \vee p))$$
$$\equiv \sim[(\sim p \vee \sim Q \vee R \vee p) \wedge (\sim Q \vee (\sim R \wedge \sim p \vee p)])$$
$$\equiv \sim(\sim p \vee \sim Q \vee R \vee p) \vee \sim(\sim Q \vee (\sim R \wedge \sim p \vee p)$$

$\rightarrow ((p \vee q) \wedge r) \Rightarrow (p \vee r)$

Normal form :-

Elementary product ($\wedge$):- A product of the variables and their negations in a formula is called an elementary product.

Eg:- If p and q are any two variables then the elementary products of p and q are $p \wedge \sim q, \sim p \wedge q, \sim p \wedge \sim$

Elementary Sums ($\vee$):- A sum of the variables and their negations is called the elementary sum.

Eg:- If p and q are two variables then the elementary sums of p and q are $p \vee \sim q, \sim p \vee q, \sim p \vee \sim q$.

Normal form :-

Converting the given statement formula into any one of the standard forms (elementary product or sums) is called the normal form or canonical form.

Normal forms are classified into two types.

(i) Disjunctive normal form (DNF) / Sum of Elementary products

(2) Conjunctive normal form (CNF) / product of Elementary Sums.

(i) DNF :-

A formula which is equivalent to the given formula and which consists of a sum of elementary products is called a disjuctive normal form.

Eg:- $(p \wedge Q) \vee (\sim p \wedge Q) \vee (\sim p \wedge \sim Q)$

$(p \wedge Q \wedge \sim R) \vee (\sim p \wedge Q \wedge R)$

$\rightarrow$ procedure to obtain DNF

* Remove all implication and Biimplications by equivalent expressions containing connectives $\wedge, \vee, \sim$

$(p \to Q) \Leftrightarrow (p \to Q) \wedge (Q \to p)$

$\Leftrightarrow (\sim p \vee Q) \wedge (\sim Q \vee p)$

$\Leftrightarrow (p \wedge Q) \vee (\sim p \wedge \sim Q)$

* Eleminate negation before sums and products by using double negation or De-morgan's law.

i.e $\sim(\sim p) \Leftrightarrow p$

$\sim(p \wedge Q) \Leftrightarrow \sim p \vee \sim Q$

$\sim(p \vee Q) \Leftrightarrow \sim p \wedge \sim Q$

* Apply the distribution law until a sum of Elementary products obtained.

$P \wedge (R \vee Q)$

$(P \wedge R) \vee (P \wedge Q)$

Note:-

Disjunctive normal form need not be unique.

Problems:-

→ Find the Disjunctive normal form of $P \wedge (p \to Q)$

$P \wedge (p \to Q) \Leftrightarrow P \wedge (\sim p \vee Q)$

$\Leftrightarrow (P \wedge \sim p) \vee (P \wedge Q)$

$P \wedge (p \to Q) \Leftrightarrow (P \wedge \sim p) \vee (p \wedge Q)$ (DNF)

→ Write an equivalent DNF for the equation

$P \vee (\sim p \to (Q \vee (Q \to \sim R)))$

$\equiv p \vee (\sim p \to (Q \vee (\sim Q \vee \sim R)))$

$\equiv p \vee (\sim p \to (Q \vee \sim Q \vee \sim R))$

$\equiv p \vee [\sim(\sim p) \vee (Q \vee \sim Q \vee \sim R))$

$\equiv p \vee (p \vee (Q \vee \sim Q \vee \sim R))$

$\equiv p \vee p \vee Q \vee \sim Q \vee \sim R$

→ $\sim(p \vee Q) \leftrightarrow (p \wedge Q)$

$\sim(p \vee Q) \leftrightarrow (p \wedge Q) \equiv (\sim(p \vee Q) \wedge (p \wedge Q)) \vee (\sim(\sim(p \vee Q)) \wedge (p \wedge Q))$

$\equiv ((\sim p \wedge \sim Q) \wedge (p \wedge Q)) \vee ((p \vee Q) \wedge (\sim p \vee \sim Q))$

$\equiv (\sim p \wedge \sim Q \wedge p \wedge Q) \vee (((p \vee Q) \wedge \sim p) \vee ((p \vee Q) \wedge \sim Q))$

$\equiv (\sim p \wedge p \wedge \sim Q \wedge Q) \vee ((p \wedge \sim p) \vee (Q \wedge \sim p) \vee ((p \wedge \sim Q) \vee (p \wedge \sim Q))$

$\equiv F \vee (F \vee (Q \wedge \sim p)) \vee ((p \wedge \sim Q) \vee F)$

$\equiv (Q \wedge \sim p) \vee (p \wedge \sim Q)$

→ $p \rightarrow (p \rightarrow Q) \wedge \sim(\sim Q \vee \sim p)$

$p \rightarrow (p \rightarrow Q) \wedge \sim(\sim Q \vee \sim p) \Leftrightarrow p \rightarrow (\sim p \vee Q) \wedge \sim(\sim Q \vee \sim p)$

$\equiv \sim p \vee ((\sim p \vee Q) \wedge \sim(\sim Q \vee \sim p))$

$\equiv \sim p \vee ((\sim p \vee Q) \wedge (Q \wedge p)$

$\equiv (\sim p \vee (\sim p \vee Q)) \wedge (\sim p \vee (Q \wedge p))$

$\equiv (\sim p \vee Q) \wedge ((\sim p \vee Q) \wedge (\sim p \vee p))$

$\equiv (\sim p \vee Q) \wedge (\sim p \vee Q) \wedge (\sim p \vee p)$

Conjunctive Normal form (CNF):-

A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a conjunctive Normal form of the given formula.

Eg:- * $p \wedge (p \vee q) \wedge (\sim p \vee q)$

* $(p \vee \sim Q \vee R) \wedge (p \vee \sim Q \vee \sim R)$

Problems:-

→ Find the CNF of $p \wedge (p \rightarrow Q) \Leftrightarrow p \wedge (\sim p \vee Q)$

↳ CNF

→ $(Q \vee (p \wedge R)) \wedge \sim((p \vee R) \wedge Q)$

$\equiv (Q \vee p) \wedge (Q \vee R) \wedge (\sim(p \vee R) \vee \sim Q)$

$\equiv (Q \vee p) \wedge (Q \vee R) \wedge ((\sim p \wedge \sim R) \vee \sim Q)$

$\equiv (Q \vee p) \wedge (Q \vee R) \wedge (\sim p \vee \sim Q) \wedge (\sim R \vee \sim Q)$

(iii) $\sim(p\vee q) \Leftrightarrow (p\wedge q)$ find equivalent CNF

$\sim(p\vee q) \Leftrightarrow (p\wedge q)$

$\equiv (\sim(\sim p\vee q))\vee(p\wedge q)\wedge(\sim(p\vee q)\vee\sim(p\wedge q))$

$\equiv (p\vee q)\vee(p\wedge q)\vee(\sim p\wedge\sim q)\vee(\sim p\vee\sim q)$

$\equiv ((p\wedge q)\vee(p\vee q))\wedge((\sim p\vee(\sim p\vee\sim q))\wedge(\sim q\vee(\sim p\vee\sim q)))$

$\equiv (p\vee(p\vee q))\wedge(q\vee(p\vee q))\wedge(\sim p\vee\sim q)\wedge(\sim p\vee\sim q)$

$\equiv (p\vee q)\wedge(q\vee p)\wedge(\sim p\vee\sim q)$

$\equiv (p\vee q)\wedge(\sim p\vee\sim q)$

→ Show that the formula $q\vee(p\wedge\sim q)\vee(\sim p\wedge\sim q)$ is a tautology

$\equiv q\vee(p\wedge\sim q)\vee(\sim p\wedge\sim q)$

$\equiv q\vee((p\vee\sim p)\wedge\sim q))$

$\equiv (q\vee(p\vee\sim p))\wedge(q\vee\sim q)$

$\equiv (q\vee p\vee\sim p)\wedge(q\vee\sim q)$

∴ Each of elementary sum is a tautology so the given formula is a tautology.

→ Principle Disjunctive Normal form:-

* Minterms:-

Let p, q be statement variables. Let us construct all possible formulas which consists of conjunction of p or its Negation and Conjunction of q or its negation which $p\wedge q$, $\sim p\wedge q$, $p\wedge\sim q$, $\sim p\wedge\sim q$, these formulas are called Minterms (or) Boolean conjunction of p and q.

Note:- (i) Minterms of 2 Variables are $2^2=4$

(2) Minterms of 3 variables p, q, r are $2^3=8$ which are $(p\wedge q\wedge r)$, $(\sim p\wedge q\wedge r)$, $(p\wedge\sim q\wedge r)$, $(p\wedge q\wedge\sim r)$, $(\sim p\wedge\sim q\wedge r)$, $(\sim p\wedge q\wedge\sim r)$, $(p\wedge\sim q\wedge\sim r)$, $(\sim p\wedge\sim q\wedge\sim r)$.

(3) Every minterm is an elementary product but every elementary product need not to be minterm.

* **Definition:-** An equivalent formula consisting of disjunction of Minterms only is called a principle Disjunctive Normal form.

Eg:- ① $(P\wedge Q)\vee(P\wedge\sim Q)\vee(\sim P\wedge Q)\vee(\sim P\wedge\sim Q)$ is a PDNF of 2 variables P and Q.

② $(P\wedge Q)\vee(P\wedge\sim Q\wedge R)$ is not a PDNF.

Note:-

→ principle Disjunctive Normal form is Unique.

→ Every PDNF is a DNF but converse need not to be true

→ There are two methods to obtain a PDNF which are
1) using truth table method 2) Replacement method.

Problems:-

→ Find PDNF of $P\rightarrow Q$

Truth Table for $P\rightarrow Q$ is as shown below

P	q	$P\rightarrow q$
T	T	T ✓
T	F	F
F	T	T ✓
F	F	T ✓

From the above table PDNF is $(P\wedge Q)\vee(\sim P\wedge Q)\vee(\sim P\wedge\sim Q)$

→ Find the PDNF of $P\vee Q$

Truth table for $P\vee Q$ is as shown below

P	q	$P\vee Q$
T	T	T ✓
T	F	F ✓
F	T	T ✓
F	F	F

PDNF is $(P\wedge Q)\vee(P\wedge\sim Q)\vee(\sim P\wedge Q)$

Replacement method:-

We need to follow the steps

* First replace the conditionals and Biconditionals by using equivalent formulas.

* The negations are applied to the variables by using De morgan's laws followed by the applications of distributi laws.

i.e; $\sim(P\wedge Q) \Leftrightarrow \sim P\vee\sim Q$

$\sim(P\vee Q) \Leftrightarrow \sim P\wedge\sim Q$

* Any elementary products which are contradictions to be dropped.

* Minterms are applied in the disjunctions by introducing missing factors.

Eg:- $P\vee(P\wedge\sim Q) \Leftrightarrow (P\wedge T)\vee(P\wedge\sim Q)$

$\Leftrightarrow (P\wedge(Q\vee\sim Q))\vee(P\wedge\sim Q)$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\sim Q)\vee(P\wedge\sim Q)$

* Identical minterms appearing in the disjunction are to be dropped.

Note :-

If two formulas are equivalent then both must have identical PDNF

Problems:-

$\longrightarrow$ Obtain the PDNF for the following formulas:-

① $P\rightarrow Q$

$P\rightarrow Q \Leftrightarrow \sim P\vee Q$

$\Leftrightarrow (\sim P\wedge T)\vee(Q\wedge T)$

$\Leftrightarrow (\sim P\wedge(Q\vee\sim Q))\vee(Q\wedge(P\vee\sim P))$

$\Leftrightarrow (\sim P\wedge Q)\vee(\sim P\wedge\sim Q)\vee(Q\wedge P)\vee(Q\wedge\sim P)$

$\Leftrightarrow (\sim P\wedge Q)\vee(\sim P\wedge\sim Q)\vee(Q\wedge P)$

② $\sim P \vee Q$ (same as ①)

③ $P \Leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

$\Leftrightarrow (\sim P \vee Q) \wedge (\sim Q \vee P)$ $\quad \because A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$

$\Leftrightarrow ((\sim P \vee Q) \wedge \sim Q) \vee ((\sim P \vee Q) \wedge P))$

$\Leftrightarrow (\sim P \wedge \sim Q) \vee (Q \wedge \sim Q) \vee (\sim P \wedge P) \vee (Q \wedge P)$

$\Leftrightarrow (\sim P \wedge \sim Q) \vee (Q \wedge P)$

④ $P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R)$

$(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R) \Leftrightarrow ((P \wedge Q) \wedge T) \vee ((\sim P \wedge R) \wedge T) \vee ((Q \wedge R) \wedge T)$

$\Leftrightarrow (P \wedge Q \wedge (R \vee \sim R)) \vee ((\sim P \wedge R) \wedge (Q \vee \sim Q)) \vee ((Q \wedge R) \wedge (P \vee \sim P))$

$\Leftrightarrow ((P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R)) \vee ((\sim P \wedge R \wedge Q) \vee (\sim P \wedge R \wedge \sim Q)) \vee ((Q \wedge R \wedge P) \vee (Q \wedge R \wedge \sim P))$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (\sim P \wedge R \wedge Q) \vee (\sim P \wedge R \wedge \sim Q)$.

→ Show that the following are equivalent formula's:

a) $P \vee (P \wedge Q) \Leftrightarrow P$ $\qquad$ b) $P \vee (\sim P \wedge Q) \Leftrightarrow P \vee Q$

Sol:- a) $P \vee (P \wedge Q) \Leftrightarrow P$

$P \vee (P \wedge Q) \equiv (P \wedge T) \vee (P \wedge Q)$

$\equiv (P \wedge (Q \vee \sim Q)) \vee (P \wedge Q)$

$\equiv (P \wedge Q) \vee (P \wedge \sim Q) \vee (P \wedge Q)$

$\equiv (P \wedge Q) \vee (P \wedge \sim Q)$

$\therefore$ PDNF of $P \vee (P \wedge Q)$ is $(P \wedge Q) \vee (P \wedge \sim Q) \to$ ①

$Q \Leftrightarrow P \wedge T$

$\equiv P \wedge (Q \vee \sim Q)$

$\equiv (P \wedge Q) \vee (P \wedge \sim Q)$

PDNF of Q is $(Q \wedge P) \vee (P \wedge \sim Q) \longrightarrow$ ②

from ① & ② the PDNF of $P \vee (P \wedge Q)$ and P are same.

$\therefore$ Hence $P \vee (P \wedge Q) \Leftrightarrow P$.

Maxterm:-

A maxterm consists of disjunctions in which each variable and its negation but not both appears only once.

Eg:- For two variables P and Q the number of maxte are $2^2 = 4$ which are $P \vee Q$, $P \vee \sim Q$, $\sim P \vee Q$, $\sim P \vee \sim Q$.

(2) For 3 variables P, Q and R the number of maxterms are $2^3 = 8$. which are

$(P \vee Q \vee R)$, $(\sim P \vee Q \vee R)$, $(P \vee \sim Q \vee R)$, $(P \vee Q \vee \sim R)$, $(\sim P \vee \sim Q \vee R)$, $(\sim P \vee Q \vee \sim R)$, $(P \vee \sim Q \vee \sim R)$, $(\sim P \vee \sim Q \vee \sim R)$

Note:-

The duals of minterms are called maxterms.

Principle Conjunctive Normal form (PCNF):-

Principle Conjunctive Normal form of a given formula can be defined as an equivalent formula consists of conjunction of maxterms only. This is also called product of sums canonical form.

Eg:- $(P \vee Q) \wedge (P \vee \sim Q) \vee (\sim P \vee Q)$

Note:-

1) The process for obtaining PCNF is similar to the process of PDNF.

2) The PCNF is unique.

3) Every compound proposition which is not a tautology have an equivalent PCNF.

4) If the compound proposition is a contradiction then its PCNF will contains all possible maxterms of its components.

Problems:-

→ The truth table for formula S is given in following. Determine its PDNF and PCNF.

P	Q	R	S
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

PDNF:-

$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R) \vee (P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge \sim Q \wedge R)$

PCNF :-

$(P \vee \sim Q \vee R) \wedge (\sim P \vee Q \vee R) \wedge (\sim P \vee Q \vee \sim R) \wedge (\sim P \vee \sim Q \vee \sim R)$

→ Find PCNF of $P \leftrightarrow Q$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

PCNF:-

~~$(\sim P \wedge Q \wedge R) \wedge$~~

$(P \vee \sim Q) \wedge (\sim P \vee Q)$

PDNF :- $(P \wedge Q) \vee (\sim P \wedge \sim Q)$.

→ Find PCNF of $P \leftrightarrow Q$ by replacement method.

$$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$$

$$\equiv (\sim P \vee Q) \wedge (\sim Q \wedge P).$$

Note:-

For obtaining PCNF of formula S, one can also construct the PDNF of ~S and then apply ~ (Negation).

→ Obtain the PCNF of a formula S $(\sim P \to R) \wedge (Q \Leftrightarrow P)$ also find the PDNF of S.

$$(\sim P \to R) \wedge (Q \Leftrightarrow P)$$

$$(\sim(\sim P) \vee R) \wedge ((Q \to P) \wedge (P \to Q))$$

$(p\vee R)\wedge(\sim Q\vee p)\wedge(\sim p\vee Q)$

$(p\vee R\vee F)\wedge((\sim Q\vee p)\vee F)\wedge((\sim p\vee Q)\vee F)$

$((p\vee R)\vee(Q\wedge\sim Q))\wedge[(\sim Q\vee p)\vee(R\wedge\sim R))\wedge((\sim p\vee Q)\vee(R\wedge\sim R)$

$(p\vee R\vee Q)\wedge(p\vee R\vee\sim Q)\wedge(\sim Q\vee p\vee R)\wedge(\sim Q\vee p\vee\sim R)\wedge(\sim p\vee Q\vee R)$
$\wedge(\sim p\vee Q\vee\sim R)$

$(p\vee Q\vee R)\wedge(p\vee\sim Q\vee R)\wedge(p\vee\sim Q\vee\sim R)\wedge(\sim p\vee Q\vee R)\wedge(\sim p\vee Q\vee\sim R$

which is the required PCNF.

Now the conjunctive Normal form "$\sim S$" can be obtained by writing the conjunction of remaining maxterms i.e.,

$$2^3=(8-5)=3$$

$(\sim p\vee\sim q\vee\sim r)\wedge(\sim p\vee\sim q\vee r)\wedge(p\vee q\vee\sim r)$ then considering the $\sim(\sim S$ we obtain PDNF of "S."

$\sim(\sim S)=\sim[(\sim p\vee\sim q\vee\sim r)\wedge(\sim p\vee\sim q\vee r)\wedge(p\vee q\vee\sim r)]$

$=\sim(\sim p\vee\sim q\vee\sim r)\vee\sim(\sim p\vee\sim q\vee r)\vee(p\vee q\vee\sim r)$

$=(p\wedge q\wedge r)\vee(p\wedge q\wedge\sim r)\vee(\sim p\wedge q\wedge r)$ which is required PCNF

→ Find PCNF of $(p\wedge q)\vee(\sim p\wedge r)$

$(p\wedge q)\vee(\sim p\wedge r)\Leftrightarrow[(p\wedge q)\wedge T)\vee((\sim p\wedge r)\wedge T)]$

$\Leftrightarrow[(p\wedge q)\wedge(r\vee\sim r)]\vee[(\sim p\wedge r)\wedge(q\vee\sim q)]$

$\Leftrightarrow[(p\wedge q\wedge r)\vee(p\wedge q\wedge\sim r)]\vee[(\sim p\wedge r\wedge q)\vee(\sim p\wedge q\wedge r)]$

$\Leftrightarrow(p\wedge q\wedge r)\vee(p\wedge q\wedge\sim r)\vee(\sim p\wedge q\wedge r)\vee(\sim p\wedge\sim q\wedge r)$

which are PCNF of 'S.'

The remaining Min terms of PDNF of S is

$\sim S=(p\wedge\sim q\wedge r)\vee(\sim p\wedge q\wedge\sim r)\vee(p\wedge\sim q\wedge\sim r)\vee(\sim p\wedge\sim q\wedge\sim r)$

PCNF. of 's' is $\sim(\sim S)\Leftrightarrow\sim[(p\wedge\sim q\wedge r)\vee(\sim p\wedge q\wedge\sim r)\vee(p\wedge\sim q\wedge\sim r)\vee$
$(\sim p\wedge\sim q\wedge$

$\Leftrightarrow\sim(p\wedge\sim q\wedge r)\wedge\sim(\sim p\wedge q\wedge\sim r)\wedge\sim(p\wedge\sim q\wedge\sim r)\wedge\sim(\sim p\wedge\sim q\wedge\sim r)$

$\Leftrightarrow(\sim p\vee q\vee\sim r)\wedge(p\vee\sim q\vee r)\wedge(\sim p\vee q\vee r)\wedge(p\vee q\vee r)$

→ Another method for $(p\wedge q)\vee(\sim p\wedge r)$

$\Leftrightarrow(p\vee(\sim p\wedge r))\wedge(q\vee(\sim p\wedge r))$

$\Leftrightarrow(p\vee\sim p)\wedge(p\vee r)\wedge(q\vee\sim p)\wedge(q\vee r)$

$\Leftrightarrow F \wedge (p \vee r) \wedge (q \vee \sim p) \wedge (q \vee r)$

$\Leftrightarrow (p \vee r) \wedge ((q \vee \sim p) \wedge (q \vee r))$

$\Leftrightarrow (p \vee r \vee F) \wedge (q \vee \sim p \vee F) \wedge (q \vee r \vee F)$

$\Leftrightarrow ((p \vee r) \vee (q \wedge \sim q)) \wedge ((q \vee \sim p) \vee (r \wedge \sim r)) \wedge ((q \vee r) \vee (p \wedge \sim p))$

$\Leftrightarrow (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \wedge (q \vee \sim p \vee r) \wedge (q \vee \sim p \vee \sim r)$

$\Leftrightarrow (p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

$\rightarrow$ verify $S \Leftrightarrow (p \wedge q) \vee (\sim p \wedge q \wedge r)$

$\Leftrightarrow ((p \wedge q) \wedge T) \vee (\sim p \wedge q \wedge r)$

$\Leftrightarrow (p \wedge q) \wedge (r \vee \sim r) \vee (\sim p \wedge q \wedge r)$

$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r)$

which is PDNF of 's'.

The remaining minterms of S are

$\sim S : (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

:- PCNF of "s" in $\sim(\sim S) \Leftrightarrow$

$\sim[(p \wedge \sim q \wedge r) \wedge (\sim p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)]$

$= \sim(p \wedge \sim q \wedge r) \wedge \sim(\sim p \wedge q \wedge \sim r) \wedge \sim(p \wedge \sim q \wedge r) \wedge \sim(\sim p \wedge \sim q \wedge r) \wedge \sim(\sim p \wedge \sim q \wedge \sim r)$

$= (\sim p \vee q \vee \sim r) \wedge (p \vee \sim q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$

which is required PCNF of "s".

Interference theory for Statement Calculus:-

The main function of logic is to provide rules of inferance, to inform a conclusion from a certain premices. The theory associated with rules of Inference is known as Inference theory

Deduction (or) formal proof:-

If a conclusion is derived from a set of premices by using the accepted rules of resigning then such a process of derivation is called a deduction or a formal proof and the argument is called a valid conclusion.

Definition:-

Let A and B be two statement formulas, we say that "B logically follows from A" or "B is a valid conclusion from the premice A" if and only if $A \rightarrow B$ is a tautology.

Note:-

Consider a set of premices $H_1, H_2, H_3 \cdots H_n$ and 'c' then Compound proposition of form $H_1 \wedge H_2 \wedge \cdots \wedge H_n \rightarrow c$ is called a Argument where $H_1, H_2, \ldots H_n$ are called premices or Assumptions (or) Hypothesis of the argument and c is called conclusion of Argument.

* To determine whether conclusion logically follows from the given premices we use the following two methods.

1) Truth table method.

2) Rules of Inference method.

→ Determine whether the conclusion c follows logically from hypothesis H_1 and H_2

① $H_1 : p \rightarrow q$, $H_2 : p$ $c : q$

p	q	$p\rightarrow q$	$H_1 \wedge H_2$	$H_1 \wedge H_2 \rightarrow c$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

∴ $H_1 \wedge H_2 \rightarrow c$ is a tautology

∴ c is a valid conclusion.

② $H_1 : p \rightarrow q$ $H_2 : \sim p$ $c : q$

p	q	$p\rightarrow q$	$\sim p$	$H_1 \wedge H_2$	$H_1 \wedge H_2 \rightarrow c$
T	T	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	F

∴ $H_1 \wedge H_2 \rightarrow c$ is not a tautology

∴ c is not a valid conclusion.

→ $H_1: \sim p$ $H_2: p \Leftrightarrow q$ $C: \sim(p \wedge q)$

p	q	$p \wedge q$	$\sim p$	$p \Leftrightarrow q$	$H_1 \wedge H_2$	$\sim(p \wedge q)$	$H_1 \wedge H_2 \to C$
T	T	T	F	T	F	F	T
T	F	F	F	F	F	T	T
F	T	F	T	F	F	T	T
F	F	F	T	T	T	T	T

→ $H_1: p \to q$, $H_2: \sim(p \wedge q)$, $C: \sim p$

→ $H_1: \sim q$, $H_2: p \to q$, $C: \sim p$

→ $H_1: p \to q$, $H_2: q \to r$, $C: p \to r$

→ $H_1: \sim p \vee q$, $H_2: \sim(q \wedge \sim r)$, $H_3: \sim r$, $C: \sim p$

p	q	r	$\sim p$	$\sim p \vee q$	$\sim r$	$q \wedge \sim r$	$\sim(q \wedge \sim r)$	$H_1 \wedge H_2 \wedge H_3$	$A \to C$
T	T	T	F	T	F	F	T	F	T
T	T	F	F	T	T	T	F	F	T
T	F	T	F	F	F	F	T	F	T
T	F	F	F	F	T	F	T	F	T
F	T	T	T	T	F	F	T	F	T
F	T	F	T	T	T	T	F	F	T
F	F	T	T	T	F	F	T	F	T
F	F	F	T	T	T	F	T	T	T

Rules of Inference :-

We know to describe a process by which one demonstra that a particular formula is valid consequence from a giver set of premices by using rules of Inference, sum implications and equivalences.

* Rules of Inferences are :-

Rule P :- A premice may be introduced at any point in the derivation.

Rule T :- A formula S may be introduced in deviation if S is a tautologically implied by one or more of preceeding formulas in derivation.

Implications:-

I_1 : $p \wedge q \Rightarrow p$

I_2 : $p \wedge q \Rightarrow q$

I_3 : $p \Rightarrow p \vee q$

I_4 : $q \Rightarrow p \vee q$

I_5 : $\sim p \Rightarrow p \rightarrow q$

I_6 : $q \Rightarrow p \rightarrow q$

I_7 : $\sim(p \rightarrow q) \Rightarrow p$

I_8 : $\sim(p \Rightarrow q \Rightarrow \sim q$

I_9 : $p, q \Rightarrow p \wedge q$

I_{10} : $\sim p, p \vee q \Rightarrow q$

I_{11} : $p, p \rightarrow q \Rightarrow q$

I_{12} : $\sim q, p \rightarrow q \Rightarrow \sim p$

I_{13} : $p \rightarrow q, q \rightarrow r \Rightarrow p \rightarrow r$

I_{14} : $p \vee q, p \rightarrow r, q \rightarrow r \Rightarrow r$.

Equivalences:-

E_1 : $\sim(\sim p) \Leftrightarrow p$

E_2 : $p \wedge q \Leftrightarrow q \wedge p$

E_3 : $p \vee q \Leftrightarrow q \vee p$

E_4 : $p \wedge (q \wedge r) \Leftrightarrow ((p \wedge q) \wedge r)$

E_5 : $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

E_6 : $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \vee r)$

E_7 : $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

E_8 : $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

E_9 : $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

E_{10} : $p \vee p \Leftrightarrow p$

E_{11} : $p \wedge p \Leftrightarrow p$

E_{12} : $R \vee (p \wedge \sim p) \Leftrightarrow R$

E_{13} : $r \wedge (p \vee \sim p) \Leftrightarrow r$

E_{14} : $R \vee (p \vee \sim p) \Leftrightarrow T$

E_{15} : $R \wedge (p \wedge \sim p) \Leftrightarrow F$

E_{16} : $p \rightarrow q \Leftrightarrow \sim p \vee q$

E_{17} : $\sim(p \rightarrow q) \Leftrightarrow p \wedge \sim q$

E_{18} : $p \rightarrow q \Leftrightarrow \sim q \rightarrow \sim p$

E_{19} : $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

E_{20} : $\sim(p \Leftrightarrow q) \Leftrightarrow (p \rightleftharpoons \sim q)$

E_{21} : $p \rightleftharpoons q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

E_{22} : $p \rightleftharpoons q \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$

→ Determine that R is a valid Inference from the premices $P \to Q$, $Q \to R$ and P.

$\{1\}$	(1)	P	Rule P
$\{2\}$	(2)	$P \to Q$	Rule P
$\{1,2\}$	(3)	Q	Rule T, (1),(2) & I_{11} $(P, P \to Q \Rightarrow Q)$
$\{4\}$	(4)	$Q \to R$	Rule P
$\{1,2,4\}$	(5)	R	Rule T (3),(4),(5) & I_{11} $(Q, Q \to R \Rightarrow R)$

∴ R is a valid conclusion.

(or)

$\{1\}$	(1)	$P \to Q$	P
$\{2\}$	(2)	$Q \to R$	P
$\{1,2\}$	(3)	$P \to R$	T, (1),(2) & I_{13}
$\{4\}$	(4)	P	P
$\{5\}$	(5)	R	T, (3),(4) & I_{11}

→ Show that $\sim P$ logically follows from the premices $\sim(P \wedge \sim Q)$, $\sim Q \vee R$, $\sim R$.

Sol:-	$\{1\}$	(1)	$\sim(P \wedge \sim Q)$	Rule P
	$\{1\}$	(2)	$\sim P \vee \sim(\sim Q)$	Rule T, (1) & E_8 $(\sim(P \wedge Q) \Rightarrow \sim P \vee \sim Q)$
	$\{1\}$	(3)	$\sim P \vee Q$	Rule T, (2) & E_1
	$\{1\}$	(4)	$P \to Q$	Rule T, (3) & E_{16} $(P \to Q \equiv \sim P \vee Q)$
	$\{5\}$	(5)	$\sim Q \vee R$	Rule P
	$\{5\}$	(6)	$Q \to R$	Rule T, (5) & E_{16}
	$\{1,5\}$	(7)	$P \to R$	Rule T, (4), (6) & I_{13}
	$\{1,5\}$	(8)	$\sim R \to \sim P$	Rule T, (7) & E_{18} $(P \to Q \equiv \sim Q \to \sim P)$
	$\{9\}$	(9)	$\sim R$	Rule P.
	$\{1,5,9\}$	(10)	$\sim P$	Rule T, (9) & I_{11}

∴ $\sim P$ is a valid conclusion.

→ Show that $R \vee S$ follows logically from the premices $C \vee D$, $(C \vee D) \rightarrow \sim H$, $\sim H \rightarrow (A \wedge \sim B)$ and $(A \wedge \sim B) \rightarrow R \vee S$

{1}	(1)	$C \vee D$	Rule P
{2}	(2)	$(C \vee D) \rightarrow \sim H$	Rule P
{1,2}	(3)	$\sim H$	Rule T, (1),(2) & I_{11}
{4}	(4)	$\sim H \rightarrow (A \wedge \sim B)$	Rule P
{1,2,4}	(5)	$A \wedge \sim B$	Rule T, (3), (4), & I_{11}
{6}	(6)	$(A \wedge \sim B) \rightarrow R \vee S$	Rule P
{1,2,4,6}	(7)	$R \vee S$	Rule T, (5),(6) & I_{11}

∴ $R \vee S$ logically follows from the given premices.

→ Show that $S \vee R$ is tautologically implied by $(P \vee Q)$ and $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$ to show that conjunctions of these i.e; $(P \vee Q) \wedge (P \rightarrow Q) \wedge (Q \rightarrow S) \Rightarrow S \vee R$ is a tautology. For this it is enough to prove that $S \vee R$ logically follows from the premices $(P \vee Q)$, $(P \rightarrow R)$ & $(Q \rightarrow S)$.

Sol:-

{1}	(1)	$P \vee Q$	Rule P
{1}	(2)	$\sim(\sim P) \vee Q$	Rule T, (1) & E_1
{1}	(3)	$\sim P \rightarrow Q$	Rule T, (2) & E_{16} $(P \rightarrow Q \equiv \sim P \vee Q)$
{4}	(4)	$Q \rightarrow S$	Rule P.
{1,4}	(5)	$\sim P \rightarrow S$	Rule T, (3), (4) & I_{13} $(P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R)$
{1,4}	(6)	$\sim S \rightarrow P$	Rule T, (5) & E_{18} $(P \rightarrow Q \equiv \sim Q \rightarrow \sim P)$
{7}	(7)	$P \rightarrow R$	Rule P
{1,4,7}	(8)	$\sim S \rightarrow R$	Rule T, (6), (7) & I_{13}
{1,4,7}	(9)	$\sim(\sim S) \vee R$	Rule T, (8) & E_{16}
{1,4,7}	(10)	$S \vee R$	Rule T, (9) & E_1

∴ $S \vee R$ is a valid conclusion.

→ Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premices $P \vee Q$, $Q \rightarrow R$, $P \rightarrow M$ and $\sim M$.

{1}	(1)	$P \rightarrow M$	Rule P
{2}	(2)	$\sim M$	Rule P
{1,2}	(3)	$\sim P$	Rule T, (1), (2) & I_{12} ($\sim Q, P \rightarrow Q \Rightarrow \sim P$)
{4}	(4)	$P \vee Q$	Rule P
{1,2,4}	(5)	Q	Rule T, (3), (4) & I_{10}
{6}	(6)	$Q \rightarrow R$	Rule P
{1,2,4,6}	(7)	R	Rule T, (5), (6) & I_{11}
{1,2,4,6}	(8)	$R \wedge (P \vee Q)$	Rule T, (4), (7) & I_9 ($P, Q \Rightarrow P \wedge Q$)

∴ $R \wedge (P \vee Q)$ is a valid conclusion.

→ Show that $\sim S$ is a valid argument from the premices $P \rightarrow Q$, $(\sim Q \vee R) \wedge (\sim R)$, $\sim(\sim P \wedge S)$

Rule of Conditional Prooof (Deduction Theorem):-

Rule CP:- If we can derive S from R and a set of premices then we can derive $R \rightarrow S$ from the set of premices alone. Rule CP follows from E_{19}

$$(P \wedge R) \rightarrow S \Leftrightarrow P \rightarrow (R \rightarrow S)$$

Let P be the conjunction of set of premices and R be any formula.

The above equivalence states that if R is included as additional premice and S is derived from P and R then $R \rightarrow S$ can be derived from the set of premices P alone.

Note:-

1) Rule CP is also called as the deduction theorem and is generally used if the conclusion is of the form $R \rightarrow S$.

2) In such cases R is taken as the additional premice and S is derived from the ~~set~~ given premices and R.

3) If we can prove $R \to S$ is a valid argument from the set of premices, it is enough to show that

$$R \wedge \begin{Bmatrix} \text{set of} \\ \text{premices} \end{Bmatrix} \to S$$

→ Show that $R \to S$ can be derived from the premices $P \to (Q \to S)$, $\sim R \vee P$, Q.

Now to prove $R \to S$ is a valid argument from the set of premices $P \to (Q \to S)$, $\sim R \vee P$, Q

{1}	(1)	R	Rule P (Assumed precise)
{2}	(2)	$\sim R \vee P$	Rule P
{2}	(3)	$R \to P$	Rule T, (2) & E_{16} ($P \to Q \Leftrightarrow \sim P \vee Q$)
{1,2}	(4)	P	Rule T, (1), (3) & I_{11} ($P, P \to Q \Rightarrow Q$)
{5}	(5)	$P \to (Q \to S)$	Rule P.
{1,2,5}	(6)	$Q \to S$	Rule T, (4), (5) & I_{11}
{7}	(7)	Q	Rule P
{1,2,5,7}	(8)	S	Rule T, (6), (7) & I_{11}
{1,2,5,7}	(9)	$R \to S$	Rule CP.

∴ $R \to S$ is a valid conclusion.

→ Derive the following using CP rule if necessary.

$\sim P \vee Q$, $\sim Q \vee R$, $R \to S \Rightarrow P \to S$

{1}	(1)	P	Rule P
{2}	(2)	$\sim P \vee Q$	Rule P
{2}	(3)	$P \to Q$	Rule T, (2) & E_{16}.
{1,2}	(4)	Q	Rule T, (1), (3) & I_{11}
{5}	(5)	$\sim Q \vee R$	Rule P
{5}	(6)	$Q \to R$	Rule T, (5) & E_{16}
{1,2,5}	(7)	R	Rule T, (4), (6), & I_{11}
{~~1,2~~ 8}	(8)	$R \to S$	Rule P
{1,2,5,8}	(9)	S	Rule T (7), (8) & I_{11}
{1,2,5,8}	(10)	$P \to S$	Rule CP.

∴ $P \to S$ is a valid conclusion

→ Show that $P \to Q \Rightarrow P \to (P \wedge Q)$

Now to prove $P \to (P \wedge Q)$ logically follows from the premice $P \to Q$.

{1}	(1)	P	Rule P (Assumed premice)
{2}	(2)	$P \to Q$	Rule P
{1,2}	(3)	Q	Rule T, (1), (2) & I_{11}
{1,2}	(4)	$P \wedge Q$	Rule T, (1), (3), & I_9
{1,2}	(5)	$P \to (P \wedge Q)$	Rule CP

→ Show that $(P \vee Q) \to R \Rightarrow (P \wedge Q) \to R$.

Now to prove $(P \wedge Q) \to R$ logically follows from the premice $(P \vee Q) \to R$.

{1}	(1)	$P \wedge Q$	Rule P (Assumed premice)
{2}	(2)	P	Rule T, (1) & I_1
{1}	(3)	$P \vee Q$	Rule T, (2) & I_3
{4}	(4)	$(P \vee Q) \to R$	Rule P
{1,4}	(5)	R	Rule T, (3), (4) & I_{11}
{1,4}	(6)	$(P \wedge Q) \to R$	Rule CP.

Consistency of premices and Indirect method of proof :-

Consistence :-

A set of premices $H_1, H_2, \ldots, H_n$ is said to be consistent if $H_1 \wedge H_2 \wedge \ldots \wedge H_n$ is a tautology. Otherwise the set of premices are said to be inconsistent.

i.e, $H_1 \wedge H_2 \wedge \ldots \wedge H_n$ is not a tautology (or) conjunction is false.

Eg:- Show that $\sim Q$, $P \to Q$, $P \vee R$ ~~tends to~~ → R.

Sol:-

{1}	(1)	$\sim Q$	Rule P
{2}	(2)	$P \to Q$	Rule P
{2}	(3)	$\sim Q \to \sim P$	Rule T, (2) & E_{18} $[P \to Q \Rightarrow (\sim Q \to \sim P)]$
{1,2}	(4)	$\sim P$	Rule T, (1), (3) & I_{11}

{5}	(5)	$P \vee R$	Rule P
{1,2,5}	(6)	R	Rule T, (4), (5), & I_{10} $(\sim P, P \vee Q \Rightarrow Q)$

∴ The set of given premices are consistent

Indirect method of proof :-

The notation of inconsistency is used in a procedure is called the indirect method of proof (or) method of contradicti

Note:-

To show that a conclusion C follows logically from the premic $H_1, H_2, \ldots H_n$. We assume that C is false. and $\sim C$ is an additional premice. If the new set of premices is inconsistent then they imply a contradiction. Therefore the assumption the $\sim C$ is true does not hold.

Problems:-

→ Show that $\sim(P \wedge Q)$ logically follows from $\sim P \wedge \sim Q$ by indirec method.

Sol:- To prove the statement by indirect method, we assume a new premice. $(\sim(\sim(P \wedge Q)))$

{1}	(1)	$\sim(\sim(P \wedge Q))$	Rule P (Assumed premice)
{1}	(2)	$\sim(\sim P \vee \sim Q)$	Rule T, (1) & E_8 $(\sim(P \wedge Q) \Leftrightarrow \sim P \vee \sim Q)$
{1}	(3)	$\sim(\sim P) \wedge \sim(\sim Q)$	Rule T, (2) & E_9
{1}	(4)	$P \wedge Q$	Rule T, (3) & E_1
{1}	(5)	P	Rule T, (4) & I_1
{6}	(6)	$\sim P \wedge \sim Q$	Rule P.
{6}	(7)	$\sim P$	Rule T, (6) & I_1
{1,6}	(8)	$P \wedge \sim P$	Rule T, (5), (7) & I_7

∴ $P \wedge \sim P \Rightarrow$ False.

∴ The assumption of $\sim(\sim(P \wedge Q))$ is false.

Hence $\sim(P \wedge Q)$ follows logically from $\sim P \wedge \sim Q$.

→ Using the indirect method (or) method of contradiction show that $P \to Q, Q \to R, \sim(P \wedge R), P \vee R \Rightarrow R$.

{1}	(1)	$\sim R$	Rule P (Assumed premice)
{2}	(2)	$P \to Q$	Rule P.
{3}	(3)	$Q \to R$	Rule P
{2,3}	(4)	$P \to R$	Rule T, (2), (3) & I_{13}
{2,3}	(5)	$\sim R \to \sim P$	Rule T, (4) & E_{18}
{1,2,3}	(6)	$\sim P$	Rule T, (1), (5) & I_{11}
{7}	(7)	$P \vee R$	Rule P
{1,2,3,7}	(8)	R	Rule T, (6), (7) & I_{10}.
{1,2,3,7}	(9)	$\sim R \wedge R$	Rule T, (1), (8) & $(P,Q \Rightarrow P \wedge Q)$

∴ $\sim R \wedge R \Rightarrow$ False

∴ The assumption is wrong.

Here R is a valid conclusion.

→ Show that the following premices are inconsistent.

$P \to Q, P \to R, Q \to \sim R, P$.

{1}	(1)	P	Rule P
{2}	(2)	$P \to Q$	Rule P
{1,2}	(3)	Q	Rule T, (1), (2) & I_{11}
{4}	(4)	$Q \to \sim R$	Rule P
{1,2,4}	(5)	$\sim R$	Rule T, (3), (4) & I_{11}
{6}	(6)	$P \to R$	Rule P
{6}	(7)	$\sim P \vee R$	Rule T, (6) & E_{16}
{6}	(8)	$R \vee \sim P$.	Rule T, (7) & E_3
{1,2,4,6}	(9)	$\sim P$	Rule T, (5); (8) & I_{10}.
{1,2,4,6,10}	(10)	$P \wedge \sim P$	Rule T, (1), (9) & $P,Q \to P \wedge Q$

∴ $P \wedge \sim P \Rightarrow$ False

∴ The given premises are inconsitent.

→ Show that the following statements are inconsistent:

Statement 1: Jack misses many classes through illness then he fails high school.

Statement 2: If Jack fails high school then he is uneducated.

Statement 3: If Jack reads a lot of books then he is not uneducated.

Statement 4: Jack misses many classes through illness and reads a lot of books.

Sol:- Let the statement.

P: Jack misses many classes through illness.
q: He fails high school.
r: He is uneducated.
S: Jack reads a lot of books.

Statement 1: $P \rightarrow Q$ Statement 3: $S \rightarrow \sim R$

Statement 2: $Q \rightarrow R$ Statement 4: $P \wedge S$

{1}	(1)	$P \rightarrow Q$	Rule P
{2}	(2)	$Q \rightarrow R$	Rule P
{1,2}	(3)	$P \rightarrow R$	Rule T, (1), (2) & I_{14}
{4}	(4)	$S \rightarrow \sim R$	Rule P.
{4}	(5)	$R \rightarrow \sim S$	Rule T, (4) & E_{18}
{1,2,4}	(6)	$P \rightarrow \sim S$	Rule T, (3), (5) & I_{13}
{1,2,4}	(7)	$\sim P \vee \sim S$	Rule T, (6) & E_{16} $(P \rightarrow Q \Leftrightarrow \sim P \vee Q)$
{1,2,4}	(8)	$\sim (P \wedge S)$	Rule T, (7) & E_8
{9}	(9)	$(P \wedge S)$	Rule P
{1,2,4,9}	(10)	$\sim (P \wedge S) \wedge (P \wedge S)$	Rule T, (8), (9) & I_9

$\therefore \sim (P \wedge S) \wedge (P \wedge S) \Rightarrow$ False

$\therefore$ The given statements are inconsistent.

Predicate Calculus:-

predicate describes something about one or more objects.

Eg:- Ramu is a good boy.

Here Ramu is an object and is a good boy is a predicate.

Note:-

→ Generally predicates are denoted by upper case letters A,B,C.... P,Q,R.....Z and objects are denoted by lower case letters a,b,c,....p,q,r,....z.

→ Any statement obtain of type "P is Q" where Q is a predicate and P is an object we represent $\boxed{Q(P)}$.

* $\underbrace{\text{Jack}}_{J}$ $\underbrace{\text{is taller than}}_{Q}$ $\underbrace{\text{Ramu}}_{r}$

The symbolic representation is $Q(j,r)$.

* The statement $\underbrace{\text{Naveen}}_{P}$ $\underbrace{\text{sits between}}_{S}$ $\underbrace{\text{Madhu}}_{q}$ and $\underbrace{\text{Madhan}}_{r}$.

The symbolic representation is $S(p,q,r)$

→ The order in which the names appear in the statement and the predicate should be taken in that order only.

→ If S is an n place predicate and $a_1, a_2, \ldots a_n$ are names of objects then $S(a_1, a_2, \ldots a_n)$ is a Statement and it is called an automic formula of predicate calculus.

Quantifiers:-

In predicate calculus each statement contains a word indicating quantity such as all, everyone, some and one. Such words are called Quantifiers.

Quantifiers are classified into two types:-

1) Universal Quantifiers

2) Existential Quantifiers.

1) Universal Quantifiers:-

It is used for the case of for all, for each and for every.

2) Existential Quantifiers:-

It is used for the case of for some and there exist.

Free and bounded Variables:-

Given a formula containing a part of the form $x\,p(x)$ (or) $(\exists x)\,p(x)$.

Such a part is called an x bound part of formula. Any occurance of an x in a x bound part of the formula is called bound occurance of x. By any occurance of x (or) of any variable that is not bound occurance is ix called a free occurance. The formula $p(x)$ in $(x)p(x)$ (or) $(\exists x)p(x)$ is described as the scope of the quantifier.

Ex:- $(x)\,p(x,y)$

Here, the occurance of variable x is bound occurance.

The occurance of variable y is free occurance.

The scope of the quantifier ix $p(x,y)$

→ Indicate the variables that are free and bounded also show the scope of the Quantifiers.

① $(x)(p(x) \wedge R(x)) \longrightarrow (x)\,p(x) \wedge Q(x)$

The scope of the first quantifier $p(x) \wedge R(x)$ and the occurance of the variable x is bound occurance.

The scope of the second quantifier (x) ix $p(x)$ and the occurance of the variable x is bound occurance. The occurance of the variable x in $Q(x)$ is the free occurance.

② $(x)(p(x) \wedge (\exists x)\,Q(x)) \vee (x)(p(x) \rightarrow Q(x))$

The scope of the first quantifier (x) is $p(x) \wedge (\exists x)\,Q(x)$.

The scope of the variable (x) is bound occurance. The occurance of the variable in $Q(x)$ ix bound occurance.

The scope of the second quantify (x) ix $p(x) \rightarrow Q(x)$

The occurance of the variable (x) is bound occurance.

③ $(x)(p(x) \rightleftharpoons Q(x) \wedge (\exists x) R(x)) \wedge S(x)$

The scope of the first quantifier (x) is $p(x) \rightleftharpoons Q(x)(\exists x)R(x)$

The occurance of the variable (x) is bound occurance.

The scope of the second quantify $(\exists x)$ is $R(x)$.

The occurance of the variable (x) ix bound occurance.

The occurance of the variable (x) ix $S(x)$ is free occurance.

Rewrite the prepositions symbolically

Statement :-

For each integer x there exist an integer y such that $x+y=0$.

for each integer of $x \Rightarrow (x)$

There exists an integer $\Rightarrow (\exists y)$

predicate $p(x,y) : x+y=0$.

The symbolic representation ix $(x)(\exists y) p(x,y)$

→ There exist an integer x such that $x+y=y$, for every integer y.

There exist an integer $\Rightarrow (\exists x)$

for every integer $y \Rightarrow (y)$

predicate $p(x,y) : x+y=y$

:- The symbolic representation ix $(\exists x)(y) p(x,y)$.

→ For all integers x and y such that $xy=yx$

for every integer $x \Rightarrow (x)$

for every integer $y \Rightarrow (y)$

predicate $p(x,y) : xy=yx$

:- The symbolic representation ix $(x)(y)p(x,y)$.

→ For all x "if x is a man than x is mortal".

for every integer $x \Rightarrow (x)$

x is a man $\Rightarrow$ $p(x)$

x is mortal $\Rightarrow$ $Q(x)$.

:- $(x)(p(x) \Rightarrow Q(x))$.

→ Any integer is positive (or) negative write the Atomic formula for the statement.

Given statement,

for all x if x is an integer then x is positive or x is negative.

for all $x \Rightarrow (x)$

x is an integer $\Rightarrow p(x)$

x is positive $\Rightarrow Q(x)$

x is negative $\Rightarrow R(x)$

:- $(x)(p(x) \rightarrow Q(x) \vee R(x))$

→ x is taller than y.

Given,

for all x, y, "If m is taller than y then y is not taller than x".

for all $x, y \Rightarrow (x)(y)$

x is taller than $y \Rightarrow p(x)$

y is not taller than $x \Rightarrow \sim p(y, x)$

$(x)(y)\,[p(x, y) \Rightarrow \sim p(y, x)]$

Inference theory for predicate Calculus:-

Rule US: [Universal specification]

$(x)\,p(x)$ one may conclude that $p(y)$.

Rule ES:- [Existential Specification]

$(\exists x)\,p(x)$ one may conclude that $p(y)$.

Rule UG:- [Universal Generalisation]

$p(y)$ one may conclude that $(x)\,p(x)$.

Rule EG:- [Existential Generalisation]

$p(y)$ one may conclude that $(\exists x)\,p(x)$.

Implications:-

I_{15} : $(x)(p(x) \vee Q(x)) \Rightarrow (x)\,p(x) \vee (x)\,Q(x)$

I_{16} : $(\exists x)(p(x) \wedge Q(x)) \Rightarrow (\exists x)\,p(x) \wedge (\exists x)\,Q(x)$

Equivalences:-

E_{23}: $(\exists x)(P(x) \vee Q(x)) \Leftrightarrow (\exists x)P(x) \vee (\exists x)Q(x)$.

E_{24}: $(x)(P(x) \wedge Q(x)) \Leftrightarrow (x)P(x) \wedge (x)Q(x)$

E_{25}: $\sim(\exists x)P(x) \Leftrightarrow (x)\sim P(x)$

E_{26}: $\sim(x)P(x) \Leftrightarrow (\exists x)\sim P(x)$

E_{27}: $(x)(A \vee B(x)) \Leftrightarrow A \vee (x)B(x)$

E_{28}: $(\exists x)(A \wedge (B(x))) \Leftrightarrow A \wedge (\exists x)B(x)$

E_{29}: $(x)A(x) \rightarrow B \Leftrightarrow (\exists x)(A(x) \rightarrow B)$

E_{30}: $(\exists x)A(x) \rightarrow B \Leftrightarrow (x)(A(x) \rightarrow B)$

E_{31}: $A \rightarrow (x)B(x) \Leftrightarrow (x)(A \rightarrow B(x))$

E_{32}: $A \rightarrow (\exists x)B(x) \Leftrightarrow (\exists x)(A \rightarrow B(x))$

→ Show that $(x)(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$.

Now to show that $M(s)$ logically follows from the premises

$(x)(H(x) \rightarrow M(x))$, $H(s) \Rightarrow M(s)$

{1}	(1)	$(x)(H(x) \rightarrow M(x))$	Rule P
{1}	(2)	$H(s) \rightarrow M(s)$	Rule US
{3}	(3)	$H(s)$	Rule P
{1,3}	(4)	$M(s)$	Rule (2), (3) & I_{11}

→ Show that $(x)(P(x) \rightarrow Q(x)) \wedge (x)(Q(x) \rightarrow R(x)) \rightarrow (x)(P(x) \rightarrow R(x))$

{1}	(1)	$(x)(P(x) \rightarrow Q(x))$	Rule P
{1}	(2)	$P(y) \rightarrow Q(y)$	Rule US, (1)
{3}	(3)	$(x)(Q(x) \rightarrow R(x))$	Rule P
{3}	(4)	$Q(y) \rightarrow R(y)$	Rule US, (3)
{1,3}	(5)	$P(y) \rightarrow R(y)$	Rule T, (2), (4) & I_{13}
{1,3}	(6)	$(x)(P(x) \rightarrow R(x))$	Rule UG, (5)

$\therefore$ $(x)(P(x) \rightarrow R(x))$ is a valid conclusion.

→ Show that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x) P(x) \wedge (\exists x) Q(x)$

$\{1\}$	(1)	$(\exists x)(P(x) \wedge Q(x))$	Rule P
$\{1\}$	(2)	$P(y) \wedge Q(y)$	Rule ES, (1)
$\{1\}$	(3)	$P(y)$	Rule T, (2) & I_1
$\{1\}$	(4)	$Q(y)$	Rule T, (2) & I_2
$\{1\}$	(5)	$(\exists x) P(x)$	Rule EG, (3)
$\{1\}$	(6)	$(\exists x) Q(x)$	Rule EG, (4)
$\{1\}$	(7)	$(\exists x) P(x) \wedge (\exists x) Q(x)$	Rule T, (5), (6) & I_9

$\therefore$ $(\exists x) \cdot P(x) \wedge (\exists x) Q(x)$ is a valid conclusion.

→ Show that from $(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$, $(\exists y)(M(y) \wedge \sim W(y))$ the conclusion $(x)(F(x) \rightarrow \sim S(x))$ follow

$\{1\}$	(1)	$(\exists y)(M(y) \wedge \sim W(y))$	Rule P
$\{1\}$	(2)	$M(z) \wedge \sim W(z)$	Rule ES, (1)
$\{1\}$	(3)	$\sim (M(z) \rightarrow W(z))$	Rule T, (2) & E_{17}
$\{1\}$	(4)	$(\exists x) \sim (M(x) \rightarrow W(x))$	Rule EG, (3)
$\{1\}$	(5)	$\sim (y)(M(y) \rightarrow W(y))$	E_{26}, (4)
$\{6\}$	(6)	$(\exists x)(F(x) \wedge S(x) \rightarrow (y)(M(y) \rightarrow W(y))$	Rule P.
$\{1,6\}$	(7)	$\sim (\exists x)(F(x) \wedge S(x))$	Rule T, (5), (6) & I_{12}
$\{1,6\}$	(8)	$(x) \sim (F(x) \wedge S(x))$	Rule T, (7) & E_{25}
$\{1,6\}$	(9)	$\sim (F(y) \wedge S(y))$	Rule US, (8)
$\{1,6\}$	(10)	$F(x) \rightarrow \sim S(x)$	Rule T, (9), E_{16}, E_{17}
$\{1,6\}$	(11)	$(y)(F(y) \rightarrow \sim S(y))$	Rule UG, (10).

$\therefore$ $y\,(F(y) \rightarrow \sim S(y))$ is a valid conclusion.