COMPLEX VARIABLES & TRANSFORMS

UNIT-1

COMPLEX VARIABLE : DIFFERENTIATION

Functions of Complex Variable

A number of the form $x+iy$ where x and y are real numbers is called the complex number and it is denoted by 'z'

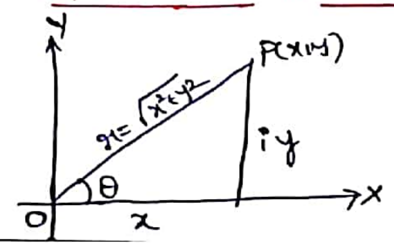
The Part x is called real part and y is called imaginary part, $i=\sqrt{-1}$

The form $z=x+iy$ is the Cartesian form of complex number and can be written as $z(x,y)$ (or) $z=P(x,y)$

Note:-

1. Two complex numbers are said to be equal iff their real & imaginary parts are equal
2. The conjugate of a complex number is defined as
$\bar{z} = x-iy$

Graphical representation of a complex number



The plane XOY is called complex plane where x denotes real axis and Y denotes imaginary axis

Note:-

1. OP is called magnitude of z and can be written as $|z|$, it is denoted by 'r' defined as the distance b/w origin and complex number in a z plane

$$|z| = r = \sqrt{x^2+y^2}$$

2. θ is called amplitude (or) Argument of z and defined as $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Complex plane :- If we represent any complex number in xy plane, it is called complex plane.

Complex function :-

The function $w = f(z) = u(x,y) + i\,v(x,y)$ where u, v are two real valued functions is called complex function.

Polar form of a complex number :-

WKT $z = x+iy$

Put $x = r\cos\theta$ $y = r\sin\theta$

$\therefore z = r\cos\theta + i r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$\therefore z = re^{i\theta}$

The polar form of a complex number is denoted by $z(r,\theta)$.

Continuity of f(z) :-

The function f(z) is said to be Continuous at $z=z_0$ if it satisfies the condition

$\lim_{z \to z_0} f(z) = f(z_0)$ exist uniquely & finitely

Differentiability of f(z) :-

The function f(z) is said to be differentiable at point z_0 if it satisfies the condition

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Case i) :- if $z = z_0 + h$

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

Case ii) :- if $h = \Delta z$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Analytic of f(z) :-

The function f(z) is said to be analytic at the point z_0 if

i) $f'(z_0)$ exist

ii) $f'(z)$ exist for all z in the neighbourhood of z_0

(or)

$f'(z)$ exist for all z such that $|z - z_0| < \epsilon$ where $\epsilon > 0$

~~Theorem~~ ~~Nee~~

Cauchy - Reimann Equations :-

C-R Eqns are used to determined whether the complex function is analytic (or) not.

suppose $w = f(z) = u(x,y) + iv(x,y)$

then

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

The Equations $u_x = v_y$ and $u_y = -v_x$ are called C-R Equations.

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Theorem :-

Necessary and Sufficient Conditions for C-R Equations :-

The complex function $f(z) = u(x,y) + iv(x,y)$ is analytic (or) differentiable at the point z iff 'u' and 'v' are differentiable and satisfies C-R Equations.

Proof :- Necessary condition

If $f(z)$ is differentiable at point z then

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

W.K.T $f(z) = u(x,y) + iv(x,y)$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - (u(x,y) + iv(x,y))}{\Delta x + i\Delta y} \to (1)$$

Case i :- let $\Delta z \to 0$ along x-axis

$\therefore \Delta z = \Delta x$ & $\Delta y = 0$

$\therefore$ from (1)

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - [u(x,y) + iv(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \to (2)$$

Case ii) :- let $\Delta z \to 0$ along y axis then $\Delta z = i\Delta y$ & $\Delta x = 0$

$\therefore$ from (1)

$$f'(z) = \lim_{\Delta y \to 0} \frac{u[x, y+\Delta y] + i\, v[x, y+\Delta y] - [u(x,y) + iv(x,y)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \cancel{i} \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\cancel{i}\, \Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \to (3)$$

from Equ (2) & (3)

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Comparing real & img parts

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$\therefore$ C-R Equs are satisfied

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Sufficient Condition:-

If u and v are differentiable and satisfies C-R Equations

$u_x = v_y$ & $u_y = -v_x$

By Taylor's series for the function of two variables x and y we have

$$f(x+h, y+k) = f(x,y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x \partial y}\right] + \cdots$$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

$$= \left[u(x,y) + \Delta x\frac{\partial u}{\partial x} + \Delta y\frac{\partial u}{\partial y} + \cdots\right] + i\left[v(x,y) + \Delta x\frac{\partial v}{\partial x} + \Delta y\frac{\partial v}{\partial y}\right] + \cdots$$

taking first powers of Expansion and neglecting higher order terms we get

$$f(z+\Delta z) = [u(x,y) + iv(x,y)] + \Delta x\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + \Delta y\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$f(z+\Delta z) - f(z) = \Delta x\left[\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}\right] + \Delta y\left[-\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right]$$

(from C-R Eqns)

$$= \Delta x\left[\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}\right] + i\Delta y\left[\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}\right]$$

$$= (\Delta x + i\Delta y)\left(\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}\right]$$

$$= (\Delta z)\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] \quad \left(\because \text{C-R Eqn } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right)$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = u_x + iv_x \text{ exist uniquely \& finitely}$$

$$= f'(z)$$

$\therefore f(z)$ is differential at the point z

$\therefore f(z)$ is analytic

C-R Equations in Polar-form:-

The C-R Eqns in Polar form are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

where $f(z) = f(r,\theta) = u(r,\theta) + iv(r,\theta)$

Proof :- Given $f(z) = u(r,\theta) + iv(r,\theta) \rightarrow (1)$

let $x = r\cos\theta$ $y = r\sin\theta$

$z = x+iy = r\cos\theta + i\sin\theta$

$z = re^{i\theta}$

$f(z) = f(re^{i\theta}) \rightarrow (2)$

from (1) & (2)

$u(r,\theta) + iv(r,\theta) = f(re^{i\theta}) \rightarrow (3)$

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diff Eqn ③ Partially w.r.to 'r' then

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})e^{i\theta} \rightarrow ④$$

diff Eqn ③ Partially w.r.to 'θ' then

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, ire^{i\theta} \rightarrow ⑤$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \quad \text{from ④}$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

Equating real and imaginary parts

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

2. Use C-R Equations, Examine which of the following are differentiable.

i) $f(z) = x^2+y^2$ ii) $f(z) = e^y(\cos x + i\sin x)$

iii) $f(z) = z \cdot \text{real}\, z$

Sol:- i) $f(z) = x^2+y^2$

here $u = x^2+y^2$, $v=0$

$\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = 2y$

$\frac{\partial v}{\partial x} = 0$ $\frac{\partial v}{\partial y} = 0$

$u_x \neq v_y$, $u_y \neq -v_x$

C-R Eqns are not satisfied

Hence the given function is not differentiable

ii) $f(z) = e^y(\cos x + i\sin x)$

$u = e^y\cos x$ $v = e^y \sin x$

$\frac{\partial u}{\partial x} = e^y(-\sin x)$ $\frac{\partial u}{\partial y} = e^y\cos x$

$\frac{\partial v}{\partial x} = e^y\cos x$ $\frac{\partial v}{\partial y} = e^y\sin x$

$u_x \neq v_y$ & $u_y \neq -v_x$

∴ C-R Eqns are not satisfied

Hence the given function is not differentiable

iii) $f(z) = z\, \text{real}\, z$

$f(z) = (x+iy)x$

$u = x^2$, $v = xy$

$\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = y$ $\frac{\partial v}{\partial y} = x$

∴ $u_x \neq v_y$ & $u_y \neq -v_x$

∴ C-R Equations are not satisfied

Hence the given function is not differentiable

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2. Show that $f(z) = \frac{1}{x+iy}$ is differentiable

Sol:- $f(z) = \frac{1}{x+iy}$

$$f(z) = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

$$u = \frac{x}{x^2+y^2} \qquad v = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(0) + y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\therefore u_x = v_y$ & $u_y = -v_x$

C-R Eqns are satisfied

The given function is differentiable

3. P.T z^n is analytic everywhere if n is a positive integer

Sol:- WKT $z = re^{i\theta}$

$$z = r[\cos\theta + i\sin\theta]$$

$$z^n = r^n(\cos\theta + i\sin\theta)^n$$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$u = r^n\cos n\theta \qquad v = r^n \sin n\theta$$

$$\frac{\partial u}{\partial r} = n r^{n-1}\cos n\theta \qquad \frac{\partial v}{\partial r} = n r^{n-1}\sin n\theta$$

$$\frac{\partial u}{\partial \theta} = r^n(-\sin n\theta)n \qquad \frac{\partial v}{\partial \theta} = r^n(\cos n\theta)n$$

$$\frac{\partial u}{\partial r} = n r^{n-1}\cos n\theta$$

$$= \frac{n r^n}{r}\cos n\theta$$

$$\therefore \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

$$\frac{\partial u}{\partial \theta} = -n r^n \sin n\theta$$

$$= (-r)\, n r^{n-1} \sin n\theta = -r\frac{\partial v}{\partial r}$$

$$\therefore \boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

$\therefore$ C-R Eqns are satisfied

Hence the given function is analytic

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4. Verify $f(z) = 2xy + i(x^2-y^2)$ is analytic or not.

Sol :- $f(z) = \cancel{2x}2xy + i(x^2-y^2)$

$u = 2xy \qquad v = x^2-y^2$

$\frac{\partial u}{\partial x} = 2y \,,\ \frac{\partial u}{\partial y} = 2x$

$\frac{\partial v}{\partial x} = \cancel{2y}\,2x \qquad \frac{\partial v}{\partial y} = -2y$

$u_x \neq v_y$ & $u_y \neq -v_x$

$\therefore$ The function is not analytic

5. ^{HW} show that $f(z) = z + 2\bar{z}$ is not analytic everywhere in complex plane.

Theorem :- If $f''(z)$ exist then show that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{and}$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

Proof :- W.K.T C-R Eqns in polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \rightarrow (1)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \rightarrow (2)$$

diff (1) Partially w.r. to 'r'

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \cdot \partial r} \rightarrow (3)$$

multiply eqn (1) with $\frac{1}{r}$

$$\frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{r^2}\frac{\partial v}{\partial \theta} \rightarrow (4)$$

Diff eqn (2) Partially w.r. to θ

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta \partial r} \rightarrow (5)$$

multiply (5) with $\frac{1}{r^2}$

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} \rightarrow (6)$$

adding (3), (4), (6)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\cancel{\frac{\partial v}{\partial \theta}} + \frac{1}{r}\cancel{\frac{\partial^2 v}{\partial \theta \cdot \partial r}} - \frac{1}{r}\cancel{\frac{\partial^2 v}{\partial \theta \cdot \partial r}} + \frac{1}{r^2}\cancel{\frac{\partial v}{\partial \theta}}$$

$$= 0$$

$$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

My $\quad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$

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6. P.T $f(z) = \frac{x^3-y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}$ satisfies C-R Equations at origin but not differentiable there.

Sol :- $f(z) = \frac{x^3-y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}$

here $u = \frac{x^3-y^3}{x^2+y^2}$ $\quad v = \frac{x^3+y^3}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

at the origin $(x,y) = (0,0)$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - 0}{\Delta x} = \frac{(\Delta x)^3/(\Delta x)^2}{\Delta x} = \frac{\Delta x}{\Delta x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(\Delta y)}{\Delta y} = \frac{-(\Delta y)^3/(\Delta y)^2}{\Delta y} = -\frac{\Delta y}{\Delta y} = -1$$

Ily $\frac{\partial v}{\partial x} = \frac{(\Delta x)^3/(\Delta x)^2}{\Delta x} = 1$

$\frac{\partial v}{\partial y} = \frac{(\Delta y)^3/(\Delta y)^2}{\Delta y} = 1$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

C-R Equations are satisfied at the origin

W.K.T

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \frac{f(z) - f(z_0)}{z - z_0}$$

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{f(z)}{z}$$

Case i) :- Along x axis, $y = 0$

$$f'(0) = \lim_{z \to 0} \frac{\frac{x^3}{x^2} + i\,\frac{x^3}{x^2}}{x} = 1 + i$$

Case ii) :- Along y axis $x = 0$

$$f'(0) = \lim_{z \to 0} \frac{-y^3/y^2 + i\,y^3/y^2}{iy} = -\frac{1+i}{i} = i+1$$

Case iii) :- Along $y = mx$

$$f'(0) = \lim_{z \to 0} \frac{\frac{x^3 - m^3x^3}{x^2+m^2x^2} + i\,\frac{x^3+m^3x^3}{x^2+m^2x^2}}{x + imx}$$

$$f'(0) = \frac{(1-m^3) + i(1+m^3)}{(1+m^2)(1+im)}$$

It depends upon the value of m and is not unique

$\therefore$ $f'(0)$ does not exist

hence $f(z)$ is not differentiable at origin although C-R Equs are satisfied

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2. If $f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{x^2+y^4}, & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

P.T $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any axis except at $x = my^2$.

Sol:- $f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{x^2+y^4}, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}$

$$f(z) = \frac{x^2y^2 + i xy^3}{x^2+y^4}$$

$$u = \frac{x^2y^2}{x^2+y^4} \qquad v = \frac{xy^3}{x^2+y^4}$$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

at the origin $(x,y) \to (0,0)$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(\Delta x)}{\Delta x} = 0$$

lly $\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are also zero

$\therefore u_x = v_y$ and $u_y = -v_x$

there CR Eqns are satisfied at origin

w.k.t $f'(z_0) = \lim_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$

$\Rightarrow f'(0) = \lim_{z \to 0} \dfrac{f(z)}{z}$

Along $y = mx$:-

$$f'(0) = \lim_{z \to 0} \frac{\dfrac{x^4m^2 + i x^4 m^3}{x^2 + m^4x^4}}{x + imx}$$

$$= \lim_{x \to 0} \frac{\dfrac{x^4[m^2 + im^3]}{x^2[1+m^4x^2]}}{x[1+im]} = \lim_{x \to 0} \frac{x(m^2+im^3)}{(1+m^4x^2)(1+im)} = 0$$

$f'(0) = 0$ at $y = mx$

i.e., $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$

lly along $x = my^2$:-

$$f'(0) = \lim_{y \to 0} \frac{\dfrac{m^2y^4y^2 + i m y^2 y^3}{m^2y^4 + y^4}}{my^2 + iy}$$

$$= \lim_{y \to 0} \frac{\dfrac{m^2y^6 + imy^5}{y^4(m^2+1)}}{y(my+i)}$$

$$= \lim_{y \to 0} \frac{\dfrac{y^5(m^2y + im)}{y^4(m^2+1)}}{y(my+i)} = \lim_{y \to 0} \frac{m^2y + im}{(m^2+1)(my+i)} = \frac{m}{(m^2+1)}$$ not unique

$\therefore f'(0)$ does not exist at $x = my^2$

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3. P.T $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ is continous and C-R Equs are not satisfied at origin yet $f'(0)$ does not exist.

Sol :- $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

$= \dfrac{x^3 + ix^3 - y^3 + iy^3}{x^2+y^2} = \dfrac{x^3 - y^3 + i(x^3+y^3)}{x^2+y^2}$

$u = \dfrac{x^3 - y^3}{x^2+y^2}$ $\quad v = \dfrac{x^3+y^3}{x^2+y^2}$

$\dfrac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \dfrac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$

at the origin $(x,y) = (0,0)$

$\dfrac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \dfrac{u(\Delta x)}{\Delta x} = \dfrac{(\Delta x)^3/(\Delta x)^2}{\Delta x} = 1$

$\dfrac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \dfrac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$

$= \lim_{\Delta y \to 0} \dfrac{u(\Delta y)}{\Delta y} = -1$

lly $\dfrac{\partial v}{\partial x} = 1$, $\dfrac{\partial v}{\partial y} = 1$

$\therefore u_x = v_y$ & $u_y = -v_x$

Hence C-R Equs are satisfied

$\lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \dfrac{x^3 - y^3}{x^2+y^2} + i\,\dfrac{x^3+y^3}{x^2+y^2}$

$= \lim_{y \to 0} \dfrac{-y^3}{y^2} + i\,\dfrac{y^3}{y^2} = \lim_{y \to 0} [-y + iy] = 0$

$\therefore f(z)$ is continous at $z \to 0$

W.K.T $\quad f'(z_0) = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$

$f'(0) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z} = \lim_{z \to 0} \dfrac{f(z)}{z}$

$= \lim_{z \to 0} \dfrac{\dfrac{x^3 - y^3}{x^2+y^2} + i\,\dfrac{x^3+y^3}{x^2+y^2}}{x+iy}$

Case i) :- Along $y = mx$

$f'(0) = \lim_{x \to 0} \dfrac{x^3 - m^3x^3 + i(x^3 + m^3x^3)}{(x^2 + m^2x^2)(x + imx)}$

$= \lim_{x \to 0} \dfrac{x^3(1-m^3) + i\,x^3(1+m^3)}{x^2(1+m^2)\,x(1+im)}$

$= \dfrac{(1-m^3) + i(1+m^3)}{(1+m^2)(1+im)}$

It depends up on the value of m, not unique

$\therefore f(z)$ is not differentiable at origin

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4. If $f(z) = \begin{cases} \frac{x^3 y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ P.T $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector except at the curve $y = ax^3$

Sol:- $f(z) = \frac{x^3 y(y-ix)}{x^6+y^2} = \frac{x^3y^2 - ix^4y}{x^6+y^2}$

$u = \frac{x^3y^2}{x^6+y^2}$ $v = \frac{-x^4y}{x^6+y^2}$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

at the origin $(x,y) = (0,0)$

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(\Delta x)}{\Delta x} = 0$$

Illy $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ are equal to zero

$\therefore u_x = v_y$ & $u_y = -v_x$

C-R Equations are satisfies. we have $f'(0) = \lim_{z \to 0} \frac{f(z)}{z}$

i) Along the path $y = mx$:-

$$f'(0) = \lim_{z \to 0} \frac{x^3[(mx)^2 - ix^4(mx)]}{x^6 + (mx)^2 (x+imx)}$$

$$= \lim_{x \to 0} \frac{m^2x^5 - mix^5}{(x^6 + (mx)^2)(x+imx)}$$

$$= \lim_{x \to 0} \frac{x^5[m^2 - im]}{x^2[x^4+m^2]\,x(1+im)} = \lim_{x \to 0} \frac{x^2(m^2-im)}{(x^4+m^2)(1+im)} = 0$$

Along $y = ax^3$:-

$$f'(0) = \lim_{x \to 0} \frac{x^3(ax^3)^2 - ix^4ax^3}{(x^6 + (ax^3)^2)(x+iax^3)}$$

$$= \lim_{x \to 0} \frac{x^9a^2 - iax^7}{(x^6+ax^6)(x+iax^3)}$$

$$= \lim_{x \to 0} \frac{\cancel{x^4}[x^2a^2 - ia]}{\cancel{x^6}(1+a)\,x(1+iax^2)} = \lim_{x \to 0} \frac{x^2a^2 - ia}{(1+a)(1+iax^2)}$$

$= \frac{-ia}{1+a}$ depends upon the value of a

$\therefore$ It is not unique

Hence the function is not differentiable at $y = ax^3$

5. P.T $f(z) = \sqrt{|xy|}$ satisfies C-R equations at the origin but not differentiable there.

Sol:- $f(z) = \sqrt{|xy|}$

$u = \sqrt{|xy|}$

$\frac{\partial u}{\partial x} = 0$ $\frac{\partial u}{\partial y} = 0$ $\frac{\partial v}{\partial x} = 0$ $\frac{\partial v}{\partial y} = 0$

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$u_x = v_y$ & $u_y = -v_x$

Along $y = mx$:-

$$f'(0) = \lim_{z \to 0} \frac{f(z)}{z}$$

$$= \lim_{x \to 0} \frac{\sqrt{x\,mx}}{x + imx} = \lim_{x \to 0} \frac{x\sqrt{m}}{x(1+im)} = \frac{\sqrt{m}}{1+im}$$

It is not unique

$\therefore$ It is not differentiable at origin

Laplace Equation :-

The Laplace Equation for the function $u(x,y)$ is given by

$$u_{xx} + u_{yy} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Ily for $v(x,y)$ is given by

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Harmonic function :-

The function $f(z) = u(x,y) + i\,v(x,y)$ is said to be harmonic if u and v satisfies Laplace Equations.

Conjugate harmonic function:-

The function $f(z) = u(x,y) + i\,v(x,y)$ is said to be conjugate harmonic if u and v are harmonic & satisfy C-R Equations

Note:- If $f(z) = u + iv$ is analytic, u and v satisfies Laplace Equations then the part 'u' is refered as harmonic and the part 'v' is referred as conjugate harmonic of u.

1. Show that the both real & imaginary parts of analytic function are harmonic

Sol:- Let $f(z) = u + iv$ is analytic

$\therefore f(z)$ satisfies C-R Equations

$$\underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}_{(1)} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow (2)$$

Real Part :- Diff Eqn (1) Partially w.r.to x we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \rightarrow (3)$$

Diff Eqn (2) Partially w.r.to y we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \rightarrow (4)$$

$$(3) + (4) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{\frac{\partial^2 v}{\partial x \partial y}} - \cancel{\frac{\partial^2 v}{\partial x \partial y}} = 0$$

$\therefore u$ is harmonic

Imaginary part :-

Diff Eqn (1) w.r.to y Partially

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} \rightarrow (5)$$

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diff Eqn ② partially w.r.to x

$$-\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x \partial y} \rightarrow (6)$$

$$(5)-(6) \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \cancel{\frac{\partial^2 u}{\partial x \partial y}} - \cancel{\frac{\partial^2 u}{\partial x \partial y}} = 0$$

$\therefore$ v is harmonic

hence the result

2. Every analytic function $f(z) = u+iv$ defines two family of curves $u(x,y)=k_1$ and $v(x,y)=k_2$ forming an orthogonal system.

Sol:- Let $f(z)$ be analytic function

$\therefore u_x = v_y$ & $u_y = -v_x$

Given $u(x,y) = k_1 \rightarrow (1)$ $\quad v(x,y) = k_2 \rightarrow (2)$

diff ① Partially w.r.to 'x' using chainrule

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = m_1 \text{ (say)}$$

diff ② partially w.r.to x using chainrule

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} \Big/ \frac{\partial u}{\partial x} = m_2 \text{ (say)}$$

Now $m_1 \cdot m_2 = \left(-\frac{\partial u}{\partial x} / \frac{\partial u}{\partial y}\right) \cdot \left(\frac{\partial u}{\partial y} / \frac{\partial u}{\partial x}\right) = -1$

$\therefore$ The given family of curves forming an orthogonal system.

3. Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2 = 2|f'(z)|^2$ where $w = f(z)$ is analytic.

Sol:- Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function

Diff w.r.to x

$f'(z) = u_x + iv_x$

$|f'(z)| = \sqrt{u_x^2 + v_x^2} \Rightarrow |f'(z)|^2 = u_x^2 + v_x^2$

$= u_x^2 + (-u_y)^2$

$= u_x^2 + u_y^2$

$$\frac{\partial}{\partial x}(u^2) = 2u \frac{\partial u}{\partial x}$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2\left[u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] \rightarrow (1)$$

lly $\frac{\partial^2}{\partial y^2}(u^2) = 2\left[u \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right] \rightarrow (2)$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\left[u(0) + u_x^2 + u_y^2\right]$$

$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2 = 2|f'(z)|^2$

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4. Determine value of P such that $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{Px}{y}\right)$ be an analytic function

Sol: Given $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{Px}{y}\right)$

$u = \frac{1}{2}\log(x^2+y^2) \qquad v = \tan^{-1}\left(\frac{Px}{y}\right)$

$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2} \rightarrow (1)$

lly $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \rightarrow (2)$

$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{Px}{y}\right)^2}\left(\frac{P}{y}\right) = \frac{y^{\cancel{2}}}{y^2+P^2x^2}\left(\frac{P}{\cancel{y}}\right) = \frac{Py}{y^2+P^2x^2} \rightarrow (3)$

$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{Px}{y}\right)^2}\left(-\frac{Px}{y^2}\right) = \frac{\cancel{y^2}}{y^2+P^2x^2}\frac{(-Px)}{\cancel{y^2}} = \frac{-Px}{y^2+P^2x^2} \rightarrow (4)$

fam (1) & (4)

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ when $P = -1$

fam (2) & (3)

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ when $P = -1$

$\therefore$ The CR Equs of given functions satisfies if $P=-1$

Hence the given function is analytic when $P=1$.

5. Find the value of K such that $f(x,y) = x^3+3Kxy^2$ is harmonic

Sol:- $f(x,y) = x^3+3Kxy^2$

$\frac{\partial f}{\partial x} = 3x^2+3Ky^2 \qquad \frac{\partial f}{\partial y} = 6Kxy$

$\frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial^2 f}{\partial y^2} = 6Kx$

the given function is harmonic if

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

$6x + 6Kx = 0$

$6x(1+K) = 0$

$1+K = 0 \Rightarrow K = -1$

6. P.T the function $u = x^2-y^2$ $v = \frac{-y}{x^2+y^2}$ are harmonic but $f(z) = u+iv$ is not analytic.

Sol:- $u = x^2-y^2$

$\frac{\partial u}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = -2y$

$\frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial^2 u}{\partial y^2} = -2$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \therefore u$ is harmonic

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$$v = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(0) + y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(2y) - 2xy\,2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2)\left[(x^2+y^2) - 4x^2\right]}{(x^2+y^2)^4} = \frac{2y[-3x^2+y^2]}{(x^2+y^2)^4}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(2y) - (y^2-x^2)\,2(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2)\left[x^2+y^2 - 2y^2 + 2x^2\right]}{(x^2+y^2)^4}$$

$$= \frac{2y[3x^2-y^2]}{(x^2+y^2)^4} = \frac{-2y(-3x^2+y^2)}{(x^2+y^2)^4}$$

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore$ v is harmonic

here $u_x \neq v_y$ & $u_y \neq v_x$

so $f(z)$ does not satisfy C-R Equations

Hence $f(z) = u+iv$ is not analytic

Milne Thomson's Method :-

This method is used to find the function $f(z)$ when real (or) imaginary part is given.

By this method we can find $f(z)$ directly without finding real (or) imaginary parts.

1. If the real part 'u' is given, we can find $f(z)$ and imaginary part 'v' by using the formula

$$f(z) = \int [u_x(z,0) - i u_y(z,0)]\, dz + C$$

2. If the imaginary part 'v' is given, we can find $f(z)$ and real part 'u' by using the formula

$$f(z) = \int [v_y(z,0) + i v_x(z,0)]\, dz + C$$

Proof:- Let $f(z) = u(x,y) + iv(x,y)$

Now $f'(z) = u_x(x,y) + i v_x(x,y)$

we have $z = x+iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x \Rightarrow x = \frac{z+\bar{z}}{2}$ $\quad \left(\because u_x = v_y \;\&\; u_y = -v_x\right)$

$z - \bar{z} = 2iy \Rightarrow y = \frac{z-\bar{z}}{2i}$

$$f'(z) = u_x\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right] - i u_y\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right]$$

Put $\bar{z} = z$

$f'(z) = u_x(z,0) - i u_y(z,0)$

$\Rightarrow f(z) = \int [u_x(z,0) - i u_y(z,0)]\, dz + C$

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Ily

$f(z) = \int [V_y(z,0) + iV_x(z,0)]\, dz + C$

Note :- By Milne Thomson method $f'(z)$ can be expressed interms of $(z,0)$ i.e., $f'(z)$ can be obtained by replacing 'x' by z and 'y' by zero.

1. Show that the following functions are harmonic, find $f(z)$ & imaginary parts

i) $u = x^2 - y^2$

Sol :- $\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$

$\cancel{\frac{\partial v}{\partial x} = 0} \qquad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial^2 u}{\partial y^2} = -2$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ u is harmonic function

To find $f(z)$:-

By Milne thomson method

$f(z) = \int u_x(z,0) - iu_y(z,0)\, dz + C$

$u_x(x,y) = 2x \quad | \quad u_y(x,y) = -2y$

$u_x(z,0) = 2z \quad | \quad u_y(z,0) = 0$

$\therefore f(z) = \int 2z\, dz + C = \frac{\cancel{2}z^2}{\cancel{2}} + C$

To find imaginary part :-

$f(z) = z^2 + C$

$= (x+iy)^2 + C$

$= x^2 - y^2 + 2ixy + C$

$\therefore v = 2xy + c$ where c is complex constant

ii) $u = 4xy - 3x + 2$

$\frac{\partial u}{\partial x} = 4y - 3 \qquad \frac{\partial u}{\partial y} = 4x$

$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ u is harmonic function.

To find $f(z)$:-

$u_x(x,y) = 4y - 3 \quad | \quad u_y(x,y) = 4x$

$u_x(z,0) = -3 \quad | \quad u_y(z,0) = 4z$

$f(z) = \int u_x(z,0) - iu_y(z,0)\, dz$

$= \int -3 - i(4z)\, dz + C$

$= -3z - i4\frac{z^2}{2} + C$

$= -3z - i2z^2 + C$

To find imaginary part :-

$f(z) = -3(x+iy) - i2(x+iy)^2 + C$

$= -3x - 3iy - 2ix^2 + 2iy^2 + 4xy + C$

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$f(z) = 4xy - 3x + c + i[2y^2 - 2x^2 - 3y]$

$\therefore v = 2y^2 - 2x^2 - 3y$ ($\because$ a/c to problem $u = 4xy - 3x + 2$ $c = 2$ where c is constant)

2. Show that $v(x,y) = 2xy$ is conjugate harmonic and find $f(z)$, $u(x,y)$.

Sol:- Given $v(x,y) = 2xy$

$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2x$

$\frac{\partial^2 v}{\partial x^2} = 0 \qquad \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore$ v is harmonic

To find $f(z)$:-

$f(z) = \int v_y(z,0) + i v_x(z,0)\, dz + c$

$v_x(x,y) = 2y \quad | \quad v_y(x,y) = 2x$

$v_x(z,0) = 0 \quad | \quad v_y(z,0) = 2z$

$f(z) = \int 2z + i(0)\, dz + c$

$= \cancel{2}\frac{z^2}{\cancel{2}} + c$

$f(z) = z^2 + c$

To find real part :-

$f(z) = z^2 + c$

$= (x+iy)^2 + c$

$= x^2 - y^2 + 2ixy + c$

$\therefore u = x^2 - y^2 + c$

$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$

$u_x = v_y$ & $u_y = -v_x$

$\therefore$ C-R Equations are satisfied

Hence $v = 2xy$ is conjugate harmonic of u.

3. Find conjugate harmonic of $u = e^{x^2-y^2}\cos 2xy$ and find $f(z)$ in terms of 'z'.

Sol:- $u = e^{x^2-y^2}\cos 2xy$

$\frac{\partial u}{\partial x} = e^{x^2-y^2}(-2y \sin 2xy) + \cos 2xy\, e^{x^2-y^2}(2x)$

$u_x(x,y) = 2e^{x^2-y^2}[-y \sin 2xy + x\cos 2xy]$

$\frac{\partial u}{\partial y} = e^{x^2-y^2}[-\cancel{2x} \sin 2xy] + \cos 2xy\, e^{x^2-y^2}(-2y)$

$u_y(x,y) = -2e^{x^2-y^2}[+x \sin 2xy + y\cos 2xy]$

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$u_x[z,0] = 2e^{z^2}[0+z] = 2ze^{z^2}$

$u_y(z,0) = -2e^{z^2}[0] = 0$

By Milne Thomson method,

$f(z) = \int [u_x(z,0) - iu_y(z,0)]dz + c$

$= \int 2ze^{z^2} dz + c$ Put $z^2 = t$, $2z\,dz = dt$

$= \int e^t \, dt + c = e^t + c = e^{z^2} + c$

To find Imaginary Part :-

$f(z) = e^{z^2} + c$

$= e^{(x+iy)^2} + c$

$= e^{x^2-y^2} \cdot e^{2ixy} + c$

$= e^{x^2-y^2}[\cos 2xy + i\sin 2xy] + c$

$= e^{x^2-y^2}[\cos 2xy] + i[e^{x^2-y^2}\sin 2xy + c]$

$\therefore v = e^{x^2-y^2}\sin 2xy + c$ where c is complex constant

4. If $f(z)$ is analytic, $u-v = (x-y)(x^2+4xy+y^2)$ find $f(z)$ in terms of z.

Sol :- Let $f(z) = u+iv \rightarrow (1)$

then $if(z) = iu - v \rightarrow (2)$

$f(z) + if(z) = u + iv + iu - v$

$(1+i)f(z) = (u-v) + i(u+v) = U + iV \rightarrow (3)$

where $U = u-v$
$V = u+v$

diff (3) partially w.r.to x then

$(1+i)f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$

$= \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$ (By C-R Equations $U_y = -V_x \Rightarrow V_x = -U_y$)

$= \frac{\partial}{\partial x}[(x-y)(x^2+4xy+y^2)] - i\frac{\partial}{\partial y}[(x-y)(x^2+4xy+y^2)]$

$(1+i)f'(z) = [(x-y)(2x+4y) + (x^2+4xy+y^2)(1)] - i[(x-y)(4x+2y) - 1[x^2+4xy+y^2]]$

$= (2x^2+4xy-2xy-4y^2+x^2+4x^2+y^2) - i(4x^2+2xy-4xy-2y^2-x^2-4xy-y^2)$

$= [3x^2-3y^2+6xy] - i[3x^2-3y^2-6xy]$

By Milne-thomson method,
$f'(z)$ can be expressed in terms of z by replacing x by z and y by zero

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$(1+i) f'(z) = (3z^2) - i(3z^2) = (1-i) 3z^2$

$f'(z) = \frac{1-i}{1+i} 3z^2$

$\int f'(z)\, dz = \frac{1-i}{1+i} \int 3z^2\, dz$

$f(z) = \frac{1-i}{1+i} \cancel{3} \frac{z^3}{\cancel{3}} + C$

$= \frac{(1-i)}{(1+i)} \times \frac{(1-i)}{(1-i)} z^3 + C$

$= \frac{(1-i)^2}{1+1} z^3 + C$

$= \frac{1+i^2-2i}{2} z^3 + C = \frac{\cancel{1}-\cancel{1}-2i}{2} z^3 + C$

$= -iz^3 + C$

$\therefore f(z) = -iz^3 + C$

5. show that $v = x^2 - y^2 + \frac{x}{x^2+y^2}$ is imaginary part of analytic function, Determine real part and f(z) interms of z.

Sol:- $v = x^2 - y^2 + \frac{x}{x^2+y^2}$

$\frac{\partial v}{\partial x} = 2x + \frac{x^2+y^2(1) - x(2x)}{(x^2+y^2)^2}$

$v_x(x,y) = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$

$v_x(z,0) = 2z + \frac{-z^2}{z^4} = 2z - \frac{1}{z^2}$

$\frac{\partial v}{\partial y} = -2y + \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2}$

$v_y(x,y) = -2y - \frac{2xy}{(x^2+y^2)^2}$

$v_y(z,0) = 0$

By Milne-thomson method

$f(z) = \int v_y(z,0) + i v_x(z,0)\, dz + C$

$= \int 0 + i\left(2z - \frac{1}{z^2}\right) dz + C$

$= i\left[\left[2\frac{z^2}{2}\right] - \int \frac{1}{z^2} dz \cancel{+C}\right] + C$

$f(z) = i\left[z^2 + \frac{1}{z}\right] + C$

<u>To find real part</u>:-

$f(z) = i\left[(x+iy)^2 + \frac{1}{x+iy}\right] + C$

$= i\left[x^2 - y^2 + 2ixy + \frac{x-iy}{x^2+y^2}\right] + C$

$= ix^2 - iy^2 - 2xy + \frac{ix}{x^2+y^2} + \frac{y}{x^2+y^2} + C$

$= \left[-2xy + \frac{y}{x^2+y^2} + C\right] + i\left[x^2 - y^2 + \frac{x}{x^2+y^2}\right]$

$\therefore u = -2xy + \frac{y}{x^2+y^2} + C$

$v = x^2 - y^2 + \frac{x}{x^2+y^2}$ is imaginary part

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6 show that $v = e^{x}[x\sin y + y\cos y]$ is imaginary part and find real part & $f(z)$.

Sol:-

$$v = e^{x} x \sin y + e^{x} y \cos y$$

$$\frac{\partial v}{\partial x} = \sin y\,[xe^{x} + e^{x}] + y\cos y\, e^{x}$$

$$\frac{\partial v}{\partial y} = e^{x} x \cos y + e^{x}[-y \sin y + \cos y]$$

$$= xe^{x}\cos y + e^{x}(\cos y - y\sin y)$$

$$v_x(x,y) = e^{x}[x\sin y + \sin y + y\cos y]$$

$$v_y(x,y) = e^{x}[x\cos y + \cos y - y\sin y]$$

$$v_x(z,0) = e^{z}[0] = 0$$

$$v_y(z,0) = e^{z}[z+1-0] = ze^{z} + e^{z}$$

$$\therefore f(z) = \int v_y(z,0) + i v_x(z,0)\, dz + C$$

$$= \int ze^{z} + e^{z}\, dz + c$$

$$= ze^{z} - \int e^{z}\, dz + e^{z} + C$$

$$= ze^{z} - \cancel{e^{z}} + \cancel{e^{z}} + C = ze^{z} + C$$

$$f(z) = ze^{z} + c$$

To find real part :-

$$f(z) = ze^{z} + C$$

$$= (x+iy)\, e^{x+iy} + c$$

$$= xe^{x}e^{iy} + iy e^{x} e^{iy} + c$$

$$= xe^{x}[\cos y + i\sin y] + iy e^{x}[\cos y + i\sin y] + c$$

$$= [e^{x}(x\cos y - y\sin y) + c] + ie^{x}[x \sin y + y \cos y]$$

$$u = e^{x}[x\cos y - y\sin y] + c$$

$$v = e^{x}[x\sin y + y\cos y]$$

Conformal Mapping

If $y = f(x)$ is a real valued function of x then it gives relationship between the points in x-axis & y-axis.

We represent this relationship by drawing a curve in xy plane.

If $w = f(z)$ is a complex valued function of complex variable 'z' then there will be no such graphical representation because z and w requires two planes to represent them.

However the relation between z-plane & w-plane can establish a graphical correspondence.

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Mapping :-

The graphical correspondence defined by $w = f(z)$ b/w z-plane & w-plane is called mapping (or) Transformation from z-plane to w-plane.

The set of points in w-plane are called the images of the points z in the z-plane and there is a one to one correspondence b/w z & w planes.

Conformal mapping :- $w = f(z)$ is said to be conformal if the angle b/w the two curves c_1 & c_2 intersecting at the point $P(x,y)$ in z plane is equal to magnitude and sense to the angle b/w the curves c_1' & c_2' at the point $P'(u,v)$ in the w-plane.

If the angle b/w curves preserve same magnitude but not sense then it is said to be an "Isogonal mapping".

Theorem :-

Condition to $w = f(z)$ is a conformal mapping

statement :- If $f(z)$ is analytic function in domain 'D' and $f'(z) \neq 0$ in 'D' then $w = f(z)$ is a conformal mapping at all points of 'z'.

Proof :-

Let $P(z)$ be any point in the region 'R' of the z-plane

$P'(w)$ be any point in the region 'R' of the w-plane

Let c be the curve on z plane &

c' be the curve on w plane

Let $Q(z+\Delta z)$ be the neighbourhood point of the curve c

& $Q'(w+\Delta w)$ be the neighbourhood point on w plane

Let $\overline{PQ} = \Delta z$ & $\overline{P'Q'} = \Delta w$

Thus Δz is a complex number with modulus of length r and 'θ' be the amplitude that makes angle made by PQ with x-axis

$\therefore \Delta z = r e^{i\theta}$

$|||^{ly}$ $\Delta w = r' e^{i\theta'}$

then $\dfrac{\Delta w}{\Delta z} = \dfrac{r'}{r} e^{i(\theta' - \theta)}$

We know that $w = f(z)$ & $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\cancel{w} + \Delta w - \cancel{w}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{r'}{r} e^{i(\theta' - \theta)} \longrightarrow (1)$$

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suppose the tangents of the curves c & c' makes the angles α and α' we have

$\theta' = \alpha'$ & $\theta = \alpha$

since $f'(z) \neq 0$ let $f'(z) = \rho e^{i\phi} \rightarrow (2)$

where ρ is modulus

ϕ is amplitude

from (1) & (2)

$$\rho = \lim_{\Delta z \to 0} \frac{\rho'}{\rho}$$

$$\phi = \theta' - \theta = \alpha' - \alpha \rightarrow (3)$$

let c_1 be the another curve on z and c_1' be the another curve on 'w'. suppose the tangents to the curves c_1 and c_1' makes angles β and β' then proceeding as above we get

$$\phi = \beta' - \beta \rightarrow (4)$$

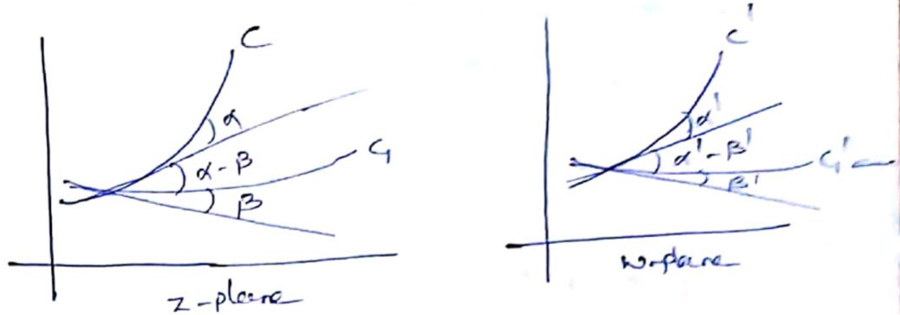
from (3) & (4)

$$\alpha' - \alpha = \beta' - \beta$$

$$\Rightarrow \beta' - \alpha' = \beta - \alpha \Rightarrow \alpha' - \beta' = \alpha - \beta$$

$\therefore$ The angle b/w the two curves c and c_1 in z plane equal to magnitude & sense to the angle b/w c' and c_1' in the w plane

$\therefore w = f(z)$ is conformal



Standard Transformations :-

1. Translation :- The transformation of the form $w = z + c$ where c is the complex constant is called translation.

Suppose $z = x + iy$ $c = c_1 + ic_2$ & $w = u + iv$ then we get

$$w = z + c$$

i.e., $u + iv = x + iy + c_1 + ic_2$

Comparing real & imaginary parts

$u = x + c_1$ $v = y + c_2$

$\therefore$ A point $P(x, y)$ on z plane maps to the point $P'(x + c_1, y + c_2)$ on w-plane

under these transformation, the figure on z plane maps to w-plane with same size and shape.

2. Expansion, Contraction & Rotation :-

Consider the transformation $w = cz$

let $w = Re^{i\phi}$
$z = re^{i\theta}$
$c = \beta e^{i\alpha}$

$w = cz$

$\Rightarrow Re^{i\phi} = \beta e^{i\alpha} . re^{i\theta}$

$R = \beta r \qquad \phi = \theta + \alpha$

$\therefore$ The point $P(r, \theta)$ on z-plane maps to the point $P'(\beta r, \theta+\alpha)$ in the w-plane.

Thus the transformation effects expansion when $|c| > 1$ and contraction when $0 < c < 1$, rotation through the angle $c = \alpha$.

3. Inversion $w = \frac{1}{z}$:-

The transformation $w = \frac{1}{z}$ is called inversion

let $w = Re^{i\phi} \quad z = re^{i\theta}$

$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$

$R = \frac{1}{r} \qquad \phi = -\theta$

$\therefore$ The point $P(r, \theta)$ in z-plane maps to the point $P'(\frac{1}{r}, -\theta)$ in w-plane

under these transformation, a straightline (or) circle in z plane maps to the circle (or) a straight line in w-plane.

Some special Transformations :-

1. $w = e^z$:-

Let $w = Re^{i\phi} \quad z = x + iy$

$w = e^z$

$\Rightarrow Re^{i\phi} = e^{x+iy} = e^x e^{iy}$

$\therefore R = e^x \quad \phi = y$

Thus the point $P(x, y)$ on z-plane maps to the point $P'(e^x, y)$ in w-plane

2. $w = \sin z$:-

Let $w = u + iv$, $z = x + iy$

$w = \sin z$

$\Rightarrow u + iv = \sin(x + iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + \cos x \, i \sinh y$

$\therefore u = \sin x \cosh y \quad v = \cos x \sinh y$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

$\cos ix = \frac{e^{-x} + e^{x}}{2} = \cosh x$

$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$\sin ix = \frac{e^{-x} - e^{x}}{2i}$

$= -i \left[\frac{e^{-x} - e^{x}}{2}\right]$

$= i \left[\frac{e^{x} - e^{-x}}{2}\right]$

$= i \sinh x$

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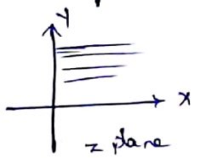
$\sin x = \frac{u}{\cosh y}$ $\cos x = \frac{v}{\sinh y}$

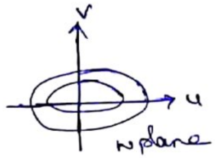
$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$

If $y = c$ is a constant then

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$$

Thus the line parallel to x axis z plane maps to family of ellipse in w-plane



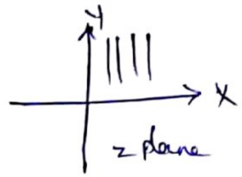


Now $\cosh y = \frac{u}{\sin x}$ $\sinh y = \frac{v}{\cos x}$

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$

let $x = c$ is a constant then $\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$

Thus the lines parallel to y axis on z plane maps to family of hyperbola on w plane



3. $w = \cos z$:- This transformation is same as $w = \sin z$

4. $w = \ln(z) = \log z$:-

let $\omega = u + iv$ $\log z = \log(x + iy)$

$u + iv = \log(x + iy)$

$e^u e^{iv} = x + iy$

$e^u [\cos v + i \sin v] = x + iy$

$\Rightarrow x = e^u \cos v$ $y = e^u \sin v$

under these transformation the lines on z plane maps to curves on w plane

5. $w = z^2$:-

$w = u + iv$ $z = x + iy$

$u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$

$u = x^2 - y^2$ $v = 2xy$

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Case i) :- If $u = a$ & $v = b$

$a = x^2 - y^2 \quad b = 2xy$

$\frac{x^2}{a} - \frac{y^2}{a} = 1 \quad xy = \frac{b}{2}$

which are the pair of hyperbolas

$\therefore$ The pair of lines in w-plane maps to pair of hyperbolas in z plane

Case ii) :- let $x = c$

$u = c^2 - y^2 \quad v = 2cy$

$y^2 = c^2 - u \quad \Rightarrow y = \frac{v}{2c}$

$\frac{v^2}{4c^2} = c^2 - u$

$v^2 = 4c^2(c^2 - u)$ which is ~~hyperbola~~ Parabola in w-plane

$\therefore$ The pair of lines in z-plane maps to pair of parabolas in w-plane.

1. Under the transformation $w = \frac{1}{z}$ find the image of $|z-2i| = 2$.

Sol :- $w = \frac{1}{z}$

$\Rightarrow z = \frac{1}{w}$

$x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2} \quad y = \frac{-v}{u^2+v^2}$

$|z-2i| = 2$

$|x+iy-2i| = 2$

$|x+i(y-2)| = 2$

$\sqrt{x^2+(y-2)^2} = 2$

$x^2+y^2-4y+4 = 4$

$x^2+y^2-4y = 0$

$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 4\frac{v}{u^2+v^2} = 0$

$\frac{1}{u^2+v^2} + \frac{4v}{u^2+v^2} = 0$

$1+4v = 0$

which is a st. line in w-plane

2. S.T the transformation $w = \frac{1}{z}$ maps a ~~circle st~~ circle ~~line in z-plane to a circle in w plane~~ to a circle or st line in w plane.

Sol :- $w = \frac{1}{z}$

$z = \frac{1}{w}$

$x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2} \quad y = \frac{-v}{u^2+v^2}$

The General Eqn of circle is

$a(x^2+y^2)+bx+cy+d = 0$

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$$a\left[\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}\right]+b\frac{u}{u^2+v^2}-\frac{cv}{u^2+v^2}+d=0$$

$$\frac{a}{u^2+v^2}+\frac{bu}{u^2+v^2}-\frac{cv}{u^2+v^2}+d=0$$

$$a+bu-cv+d(u^2+v^2)=0$$

$$d(u^2+v^2)+bu-cv+a=0$$

which is a circle in w-plane

3. show that the image of hyperbola $x^2-y^2=1$ under $w=\frac{1}{z}$ is $\rho^2=\cos 2\phi$

Sol:- Given transformation $w=\frac{1}{z}$

$$\Rightarrow z=\frac{1}{w}$$

$$re^{i\theta}=\frac{1}{\rho e^{i\phi}}$$

$$\rho e^{i\phi}=\frac{1}{r}e^{-i\phi}$$

$$\rho=\frac{1}{r}\qquad \phi=-\theta$$

Given hyperbola $x^2-y^2=1$ $\quad\left(\because x=r\cos\theta,\ y=r\sin\theta\right.$

$$r^2(\cos^2\theta-\sin^2\theta]=1$$

$$r^2(\cos 2\theta)=1$$

$$\cos 2\theta=1/r^2$$

$$\cos 2(-\phi)=\frac{1}{r^2}$$

$$\cos 2\phi=\rho^2$$

4. P.T the transformation $w=\sin z$ maps the family of lines $x=a$ & $y=b$ into a family of confocal central conics (hyperbola)

Sol:- $w=\sin z$

$$u+iv=\sin(x+iy)$$

$$u+iv=\sin x\cos iy+\cos x\sin iy$$

$$u+iv=\sin x\cosh y+i\cos x\sinh y$$

$$u=\sin x\cosh y\qquad v=\cos x\sinh y$$

$$\frac{u}{\sin x}=\cosh y\qquad \frac{v}{\cos x}=\sinh y\ \Rightarrow\ \frac{u^2}{\sin^2 x}-\frac{v^2}{\cos^2 x}=1$$

If $x=a$

$$\frac{u^2}{\sin^2 a}-\frac{v^2}{\cos^2 a}=1$$

which is hyperbola

Hly the line $x=b$ in z plane maps to hyperbola in w plane

$\therefore$ The pair of lines $x=a$ & $x=b$ in z plane maps a family of hyperbolas in w plane under the transformation $w=\sin z$

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5. Find the Image of infinite strip bounded by $x=0$ & $x=\frac{\pi}{4}$ under $w=\cos z$.

Sol:- Given $w=\cos z$

$u+iv=\cos(x+iy)$

$=\cos x \cos iy - \sin x \sin iy$

$=\cos x \cosh y - i \sin x \sinh y$

$u=\cos x \cosh y \qquad v=-\sin x \sinh y$

$\frac{u}{\cos x}=\cosh y \qquad \frac{-v}{\sin x}=+\sinh y$

$\frac{u^2}{\cos^2 x}-\frac{v^2}{\sin^2 x}=\cosh^2 y-\sinh^2 y=1$

If $x=0$ then $v=0$ and $-\infty<\cosh y<\infty$

$\therefore$ The value of y are infinite

If $x=\pi/4$

$\frac{u^2}{\cos^2\frac{\pi}{4}}-\frac{v^2}{\sin^2\frac{\pi}{4}}=1$

which is a hyperbola in wplane

$\therefore$ The line $x=0$ maps to the line $v=0$ & $-\infty<u<\infty$ and The line $x=\pi/4$ maps to hyperbola in wplane.

6. Find the image of z plane bw the lines $y=0$ and $y=\pi/2$ under the transformation $w=e^z$

Sol:- $w=e^z$

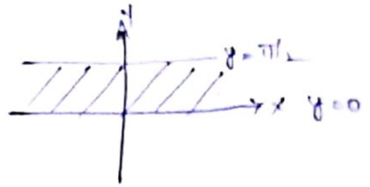
$\rho e^{i\phi}=e^{(x+iy)}$

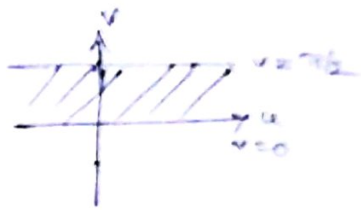
$\rho=e^x \quad \phi=y$

If $y=0$ then $\phi=0$

$y=\pi/2$ then $\phi=\pi/2$

The line $y=0$ & $y=\pi/2$ in z plane maps to $\phi=0$ & $\phi=\pi/2$ in w plane





7. S.T the transformation $w=z^2$ maps to the circle $|z-1|=1$ into $\rho=2(1+\cos 2\phi)$ where $w=\rho e^{i\phi}$ in wplane

Sol:- Given $w=z^2$

$\rho e^{i\phi}=(re^{i\theta})^2$

$\rho e^{i\phi}=r^2 e^{i2\theta}$

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$\rho = r^2 \quad \phi = 2\theta$

Given $|z-1| = 1$

$|x+iy-1| = 1$

$\sqrt{(x-1)^2+y^2} = 1$

$(x-1)^2 + y^2 = 1$

$x^2 - 2x + \cancel{1} + y^2 = \cancel{1}$

$x^2 + y^2 - 2x = 0$

$r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta = 0$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

$r^2 = 4\cos^2\theta = 4\left(\frac{1+\cos 2\theta}{2}\right) = 2(1+\cos 2\theta)$

$\Rightarrow \rho = 2(1+\cos 2\theta) = 2(1+\cos\phi)$

8. Find the image of triangle with vertices $i, 1+i, 1-i$ in z plane under the transformation $w = 3z+4-2i$

Sol:- Given $w = 3z+4-2i$

$u+iv = 3x+3iy+4-2i = (3x+4) + i(3y-2)$

$u = 3x+4 \quad v = 3y-2$

at the point P :- $(0,1)$:-

$\therefore u = 4 \quad v = 1$

at the point $(1+i)$:-

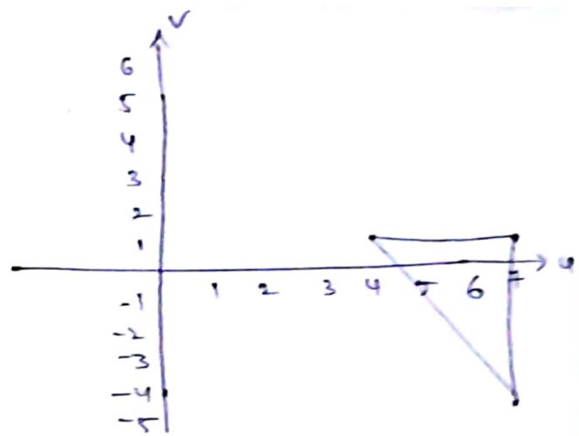
$x=1 \quad y=1$

$\therefore u = 7 \quad v = 1$

at the point $(1-i)$:-

$x=1 \quad y=-1$

$u=7 \quad v=-5$



$\therefore$ The image of triangle in z plane maps to another triangle in w-plane

9. Find the region bounded by $x=0$ $y=0$, $x=2$ $y=1$ under the transformation $w = z+(2+3i)$

Sol:- $w = z+(2+3i) = x+iy+2+3i = (x+2)+i(y+3)$

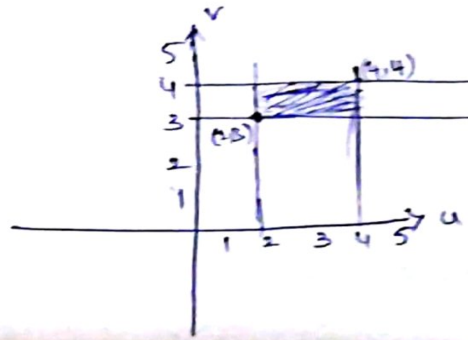
$\therefore u = x+2 \quad v = y+3$

at $x=0$ $y=0$

$\Rightarrow u=2 \quad v=3$

at $x=2$ $y=1$

$\Rightarrow u=4 \quad v=4$



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10. Find the image of finite strip $0 < y < \frac{1}{2}$ under the transformation $w = 1/z$.

Sol :- $w = \frac{1}{z}$

$\Rightarrow z = \frac{1}{w}$

$x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$

If $y = 0 \Rightarrow v = 0$

If $y = \frac{1}{2}$ then $\frac{1}{2} = \frac{-v}{u^2+v^2}$

$\Rightarrow u^2+v^2+2v = 0$

$\Rightarrow u^2+(v+1)^2 = 1$

which is a circle with centre $(0,-1)$ and radius 1

Thus a line $y=0$ in z plane maps to the line $v=0$ in w plane & the line $y = 1/2$ maps to circle in w plane.

11. Find the image of line $x=4$ in z plane under the transformation $w = z^2$

Sol :- $w = z^2$

$u+iv = (x+iy)^2 = x^2-y^2+2ixy$

$u = x^2-y^2 \qquad v = 2xy$

$y = v/2x = \frac{v}{8} \; (\because x=4)$

$u = x^2-y^2 = 16 - \frac{v^2}{64}$

$u-16 = -\frac{v^2}{64}$

$v^2 = -64(u-16)$

$= 64(16-u)$

$v^2 = 4(4^2)(4^2-u)$

This is in the form $4c^2(c^2-u)$ which represents parabola in w plane

12. Under the transformation $w = \frac{z-i}{1-iz}$ find the image of unit circle i) $|w| = 1$ ii) $|z| = 1$

Sol i) Given $w = \frac{z-i}{1-iz}$

$|w| = 1$

$\Rightarrow \left|\frac{z-i}{1-iz}\right| = 1$

$|z-i| = |1-iz|$

$|x+iy-i| = |1-i(x+iy)|$

$|x+i(y-1)| = |(1+y)-ix|$

$\sqrt{x^2+(y-1)^2} = \sqrt{(1+y)^2+x^2}$

Squaring on both sides

$x^2+(y-1)^2 = x^2+(1+y)^2$

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$$\cancel{x^2}+\cancel{y^2}-2y+\cancel{1} = \cancel{x^2}+\cancel{y^2}+\cancel{1}+2y$$

$$4y = 0 \Rightarrow y = 0$$

which is a real axis in z plane

ii) $w = \frac{z-i}{1-iz}$

$$w(1-iz) = z-i$$

$$w - wiz = z-i$$

$$w+i - wiz - z = 0$$

$$(w+i) - z(1+iw) = 0$$

$$z = \frac{i+w}{1+iw}$$

Given $|z| = 1$

$$\left|\frac{w+i}{1+iw}\right| = 1$$

$$|w+i| = |1+iw|$$

$$|u+iv+i| = |1+i(u+iv)|$$

$$|u+i(1+v)| = |(1-v)+u|$$

$$\sqrt{u^2+(1+v)^2} = \sqrt{(1-v)^2+u^2}$$

squaring on both sides

$$u^2+v^2+2v+1 = 1+v^2-2v+u^2$$

$$4v = 0 \Rightarrow v = 0$$

which is a real axis in w plane

13. S.T the transformation $w = z + 1/z$ converts $\arg z = a$ into a branch of hyperbola of eccentricity $\sec a$?

Sol:- Given $w = z + \frac{1}{z}$

Let $w = u+iv$ $\quad z = \cancel{re^{i\theta}}\ re^{i\theta}$

$$u+iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$u+iv = r[\cos\theta + i\sin\theta] + \frac{1}{r}[\cos\theta - i\sin\theta]$$

Comparing real & imaginary parts

$$u = \left(r+\frac{1}{r}\right)\cos\theta \qquad v = \left(r-\frac{1}{r}\right)\sin\theta$$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2$$

$$= \left(r^2+2\cdot\cancel{r}\cdot\frac{1}{\cancel{r}}+\frac{1}{r^2}\right) - \left(r^2 - 2\cancel{r}\frac{1}{\cancel{r}}+\frac{1}{r^2}\right)$$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4$$

$$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1$$

Put $\theta = \arg z = a$

$$\frac{u^2}{4\cos^2 a} - \frac{v^2}{4\sin^2 a} = 1$$

which represents hyperbola

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here $a^2 = 4\cos^2 a$ $\quad b^2 = 4\sin^2 a$

Eccentricity $e = \sqrt{1+\frac{b^2}{a^2}}$

$$e = \sqrt{1+\frac{\cancel{4}\sin^2 a}{\cancel{4}\cos^2 a}}$$

$$= \sqrt{\frac{\cos^2 a+\sin^2 a}{\cos^2 a}}$$

$$= \sqrt{\frac{1}{\cos^2 a}} = \sqrt{\sec^2 a}$$

$$\boxed{\therefore e = \sqrt{\sec^2 a}}$$

Bilinear Transformation :-

The transformation of the form

$N = \frac{az+b}{cz+d}$. is called Bilinear transformation (or) Mobius transformation (or) linear fractional transformation

The condition $ad-bc \neq 0$ is ensure that the transformation is conformal for

$$\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2}$$

If $ad-bc = 0$ then every point on z is critical point.

Also $z = \frac{-dw+b}{cw-a}$ is inversal mapping also a conformal mapping.

Invarient (or) Fixed Points :-

For the mapping $w = f(z)$ the points that are mapped on themselves are called Invarient (or) fixed points

let $w = \frac{az+b}{cz+d}$ thus for all the points of z

$$z = \frac{az+b}{cz+d}$$

$$cz^2 + dz = az+b$$

$$cz^2 + (d-a)z - b = 0 \rightarrow (1)$$

This is a quadratic in z and roots of this equation are called "Invarient or fixed points"

To find Bilinear transformation whose fixed points are α, β :-

The quadratic eqn in z with α, β as roots is

$$z^2 - (\alpha+\beta)z + \alpha\beta = 0$$

$$z^2 - (\alpha+\beta)z + \alpha\beta - \gamma z + \gamma z = 0$$

where γ = any complex constant

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$$z^2 - z[(\alpha+\beta) - \gamma] = \gamma z - \alpha\beta$$

$$z[z - ((\alpha+\beta) - \gamma)] = \gamma z - \alpha\beta$$

$$z = \frac{\gamma z - \alpha\beta}{z - (\alpha+\beta) - \gamma} = \frac{\gamma z - \alpha\beta}{z - [(\alpha+\beta)] + \gamma}$$

Then Bilinear transformation when α, β are fixed points

is $w = \frac{\gamma z - \alpha\beta}{z - ((\alpha+\beta) + \gamma)}$ $\quad w = \frac{\gamma z - \alpha\beta}{z - (\alpha+\beta) + \gamma}$

Bilinear transformation preserves cross ratio:-

If t_1, t_2, t_3, t_4 are four points then

$\frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_4)(t_3 - t_2)}$ is said to be cross ratio

If z_1, z_2, z_3, z_4 on z plane maps to w_1, w_2, w_3, w_4 in w-plane then $\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$

→ If z_1, z_2, z_3 are the points maps to w_1, w_2, w_3

then $\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$.

Note:- If the point of infinity introduced as one of the prescribed points in either z-plane (or) w plane

i.e., $\frac{(z - z_1)(z_2 - \infty)}{(z - \infty)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - \infty)}{(w - \infty)(w_2 - w_1)}$

take $\frac{z_2 - \infty}{z - \infty} = 1$, $\frac{w_2 - \infty}{w - \infty} = 1$

1. Find the bilinear transformation which maps the points $z = 1, i, -1$ to the points $w = i, 0, -i$ Hence find the invarience points of this transformation.

Sol:- Given $z = 1, i, -1$ $\quad w = i, 0, -i$

~~w.k.t~~ We know that the Bilinear transformation preserves the cross ratio.

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

here $z_1 = 1 \quad z_2 = i \quad z_3 = -1$
$\quad w_1 = i \quad w_2 = 0 \quad w_3 = -i$

$$\frac{(w - i)(0 + i)}{(w + i)(0 - i)} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)}$$

$$-\frac{(w - i)\cancel{i}}{(w + i)\cancel{i}} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)} = -\frac{(z - 1)(1 + i)}{(z + 1)(1 - i)}$$

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By componendo & dividendo rule

if $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

$$\frac{(w-1)+(w+i)}{(w-1)-(w+i)} = \frac{(z-1)(1+i)+(z+1)(1-i)}{(z-1)(1+i)-(z+1)(1-i)}$$

$$\frac{w-1+w+i}{w-1-w-i} = \frac{z+iz-1-i+z-iz+1-i}{z+iz-1-i-(z-iz+1-i)}$$

$$\frac{2w}{-2i} = \frac{2z-2i}{2iz-2}$$

$$\frac{w}{-i} = \frac{z-i}{iz-1}$$

$$w = \frac{-iz-1}{iz-1} = \frac{-(1+iz)}{-(1-iz)} = \frac{1+iz}{1-iz}$$

Put $w=z$ in the above transformation we get invariant points

$$\therefore z = \frac{1+iz}{1-iz}$$

$$z - iz^2 = 1+iz$$

$$z - iz^2 - 1 - iz = 0$$

$$-iz^2 + z(1-i) - 1 = 0$$

$$iz^2 - z(1-i) + 1 = 0$$

$$z = \frac{(1-i) \pm \sqrt{(1-i)^2 - 4 \times 1 \times i}}{2i} = \frac{(1-i) \pm \sqrt{1-1-2i-4i}}{2i}$$

$$= \frac{(1-i) \pm \sqrt{-6i}}{2i} = \frac{1-i+\sqrt{6i}}{2i}, \frac{1-i-\sqrt{6i}}{2i}$$

2. Find the Bilinear Transformation that maps $z_1=0, z_2=1$, $z_3=\infty$ on to $w_1=-1, w_2=-i, w_3=1$

Sol:- Given $z_1=0, z_2=1, z_3=\infty$

$w_1=-1, w_2=-i, w_3=1$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w+1)(-i-1)}{(w-1)(-i+1)} = \frac{(z-0)(1-\infty)}{(z-\infty)(1-0)}$$

$$\frac{-(w+1)(1+i)}{(w-1)(1-i)} = \frac{z}{1}$$

$$-\frac{(w+1)}{w-1} = z\frac{(1-i)}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{z(1-i)^2}{1-i^2} = \frac{z(1-2i-1)}{2} = \frac{-2zi}{2}$$

$$\frac{-(w+1)}{w-1} = -iz$$

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$$\frac{w+1}{w-1} = iz$$

$$iz(w-1) = w+1$$

$$wiz - iz = w+1$$

$$wiz - iz - w - 1 = 0$$

$$w(iz-1) = iz+1$$

$$w = \frac{iz-1}{iz-1}$$

3. Find the Bilinear transformation that maps $z_1=-1, z_2=0, z_3=1$ & $w_1=-1, w_2=-i, w_3=1$

Sol:-

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{(z+1)(0-1)}{(z-1)(0+1)} = \frac{(w+1)(-i-1)}{(w-1)(-i+1)}$$

$$\cancel{-}\frac{(z+1)}{(z-1)} = \cancel{-}\frac{(w+1)(1+i)}{(w-1)(1-i)}$$

$$\frac{z+\cancel{1}+z-\cancel{1}}{z+1-\cancel{z}+1} = \frac{(w+1)(1+i)+(w-1)(1-i)}{(w+1)(1+i)-(w-1)(1-i)}$$

$$\frac{2z}{2} = \frac{w+i\cancel{w}+\cancel{1}+i+w-i\cancel{w}-\cancel{1}+i}{w+iw+1+\cancel{i}-[w-iw-1+\cancel{i}]}$$

$$z = \frac{2w+2i}{2iw+2}$$

$$z = \frac{\cancel{2}(w+i)}{\cancel{2}(iw+1)}$$

$$iwz+z = w+i$$

$$iwz+z-w-i = 0$$

$$w(iz-1)+z-i = 0$$

$$w = \frac{i-z}{1-iz}$$

4. Find the Bilinear transformation that maps $(0,1,\infty)$ in z plane on to the points $(-1,-2,-i)$ in w plane

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{(z-0)(1-\infty)}{(z-\infty)(1-0)} = \frac{(w+1)(-2+i)}{(w+i)(-2+1)}$$

$$\frac{z}{1} = \frac{(w+1)(i-2)}{(w+i)(-1)}$$

$$-wz - i\cancel{z} = wi - 2w + i - 2$$

$$-wz - i\cancel{z} - wi + 2w - i + 2 = 0$$

$$w[-z-i+2] = iz-2+i$$

$$w = \frac{iz-2+i}{-z-i+2}$$ //

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5. Find the Bilinear transformation $(\infty, i, 0)$ on to the point $(0, i, \infty)$ in the w-plane.

Sol:- Given $z_1 = \infty \quad z_2 = i \quad z_3 = 0$

$w_1 = 0 \quad w_2 = i \quad w_3 = \infty$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-\infty)}{(w-\infty)(i-0)} = \frac{(z-\infty)(i-0)}{(z-0)(i-\infty)}$$

$$\frac{w}{i} = \frac{i}{z}$$

$$w = i^2/z = -1/z$$

6. Find the Bilinear transformation whose fixed points are $i, -i$

Sol:- The Bilinear transformation whose fixed points α, β is

$$w = \frac{\gamma z - \alpha\beta}{z-(\alpha+\beta)+\gamma} = \frac{\gamma z - (-i^2)}{z-(0)+\gamma} = \frac{\gamma z - 1}{z+\gamma}$$

If $\gamma = 0$, $\quad w = \frac{-1}{z}$

$\gamma = 1$, $\quad w = \frac{z-1}{z+1}$

$\gamma = 2$, $\quad w = \frac{2z-1}{z+2}$

7. Find the Bilinear transformation whose fixed points are $1, i$ and maps from 0 to -1.

Sol:- The Bilinear transformation whose fixed points are α, β is

$$w = \frac{\gamma z - \alpha\beta}{z-(\alpha+\beta)+\gamma}$$

$$w = \frac{\gamma z - i}{z-(1+i)+\gamma} \quad \longrightarrow (1)$$

since $z = 0$ maps to $w = -1$ we get

$$-1 = \frac{-i}{-(1+i)+\gamma}$$

$$-1 = \frac{-i}{-1-i+\gamma}$$

$$1 + i - \gamma = -i$$

$$\gamma = 1 + 2i$$

$\therefore$ from (1)

$$w = \frac{(1+2i)z - i}{z-(1+i)+(1+2i)}$$

8. Find the fixed points of $w = \frac{2i-6z}{iz-3}$

Sol:- $w = \frac{2i-6z}{iz-3}$

Put $w = z$

$$z = \frac{2i-6z}{iz-3}$$

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$iz^2 - 3z - 2i + 6z = 0$

$iz^2 + 3z - 2i = 0$

$$z = \frac{-3 \pm \sqrt{9 - 4i\cdot(-2i)}}{2i} = \frac{-3 \pm \sqrt{9-8}}{2i} = \frac{-3 \pm \sqrt{1}}{2i}$$

$$z = \frac{-3+1}{2i}, \quad \frac{-3-1}{2i}$$

$$z = \frac{-\cancel{2}}{\cancel{2}i}, \quad \frac{-\cancel{4}^{2}}{\cancel{2}i} = -\frac{1}{i}, \quad -\frac{2}{i}$$

$$z = i, 2i$$

~~9. Find the Bilinear transformation~~ that maps

$z_1 = -1$, $z_2 = 0$, $z_3 = 1$ & ~~$w_1 = -1$, $w_2 = -i$~~, $W_3 = 1$

Sol:- Given $z_1 = -1$, $z_2 = 0$, $z_3 = 1$

$w_1 = -1$, $w_2 = -i$, $w_3 = 1$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(\cancel{z}-z_3)(z_2-z_1)}$$

$$\frac{(w+1)(-i-1)}{(w-1)(-i+1)} = z+1$$

9. Find the Bilinear transformation that maps $(1+i)$, $-i$, $2i$ of triangle of t of z plane on to the point $0, 1, i$ in w-plane

Sol:- Given $z_1 = 1+i$, $z_2 = -i$, $z_3 = 2i$

$w_1 = 0$, $w_2 = 1$, $w_3 = i$

The required Bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(1-i)}{(w-i)(1-0)} = \frac{(z-1-i)(-i-2+i)}{(z-2+i)(-i-1-i)}$$

$$\frac{w(1-i)}{w-i} = \frac{(z-1-i)(-2)}{(z-2+i)(-2i-1)}$$

$$\frac{w(1-i)}{w-i} = \frac{2[1+i-z]}{(2-i-z)(1+2i)}$$

$$\frac{w-i}{w(1-i)} = \frac{(1+2i)(2-i-z)}{2(1+i-z)}$$

$$\frac{w-i}{w} = \frac{(1+2i)(1-i)(2-i-z)}{2(1+i-z)}$$

$$= \frac{(1+2i-i+2)(2-i-z)}{2(1+i-z)}$$

$$= \frac{(3+i)(2-i-z)}{2(1+i-z)}$$

$$1 - \frac{i}{w} = \frac{6-3i-3z+2i+1-iz}{2(1+i-z)}$$

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$$1-\frac{i}{W} = \frac{7-3z-i-iz}{2(1+i-z)}$$

$$\frac{i}{W} = 1 - \frac{7-3z-i-iz}{2(1+i-z)}$$

$$\frac{i}{W} = \frac{2+2i-2z-7+3z+i+iz}{2(1+i-z)}$$

$$\frac{i}{W} = \frac{-5+3i+z+iz}{2(1+i-z)} = \frac{(-5+3i)+z(1+i)}{2(1+i-z)}$$

$$\Rightarrow \frac{W}{i} = \frac{2(1+i-z)}{(-5+3i)+z(1+i)}$$

$$\therefore W = \frac{2i(1+i-z)}{(3i-5)+z(1+i)}$$

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