UNIT –IV
ISOPARAMETRIC FORMULATION

Definition:

The term ***isoparametric*** (same parameters) is derived from the use of the same shape (interpolation) functions ***N*** to define the element's ***geometric shape*** as are used to define the ***displacements*** within the element.

Alternatively:

The basic principle of isoparametric elements is that the interpolation functions for the displacements are also used to represent the geometry of the element.

$$u = \sum_{i=1}^{4} N_i u_i \quad , \quad v = \sum_{i=1}^{4} N_i v_i$$

$$x = \sum_{i=1}^{4} N_i x_i \quad , \quad y = \sum_{i=1}^{4} N_i y_i$$

Basic Principle of Isoparametric Elements

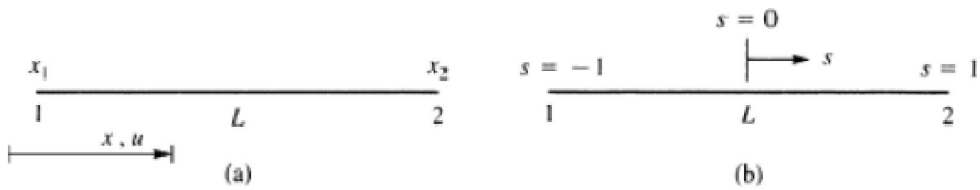
- In this formulation, displacements are expressed in terms of the ***natural*** (local) coordinates and then differentiated with respect to global coordinates. Accordingly, a transformation matrix [J], called ***Jacobian***, is produced.

- If the geometric interpolation functions are of lower order than the displacement shape functions, the element is called ***subparametric***. If the reverse is true, the element is referred to as ***superparametric***.

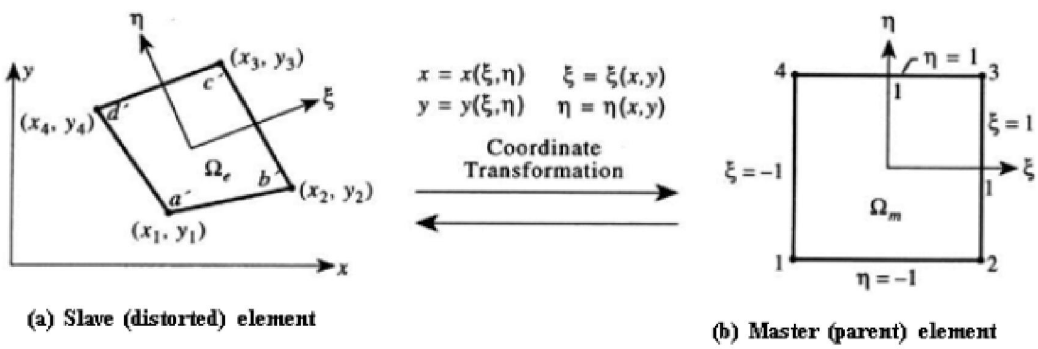
- The ***isoparametric formulation*** is generally applicable to 1-, 2- and 3- dimensional stress analysis. The isoparametric family includes elements for plane, solid, plate, and shell problems. Also, it is applicable for ***nonstructural*** problems.

- The isoparametric formulation makes it possible to generate elements that are **nonrectangular** and have **curved** sides. So it can facilitate an accurate representation of irregular elements.

- Numerous **commercial** computer programs have adopted this formulation for their various libraries of elements.



Two-Noded Bar Isoparametric Element

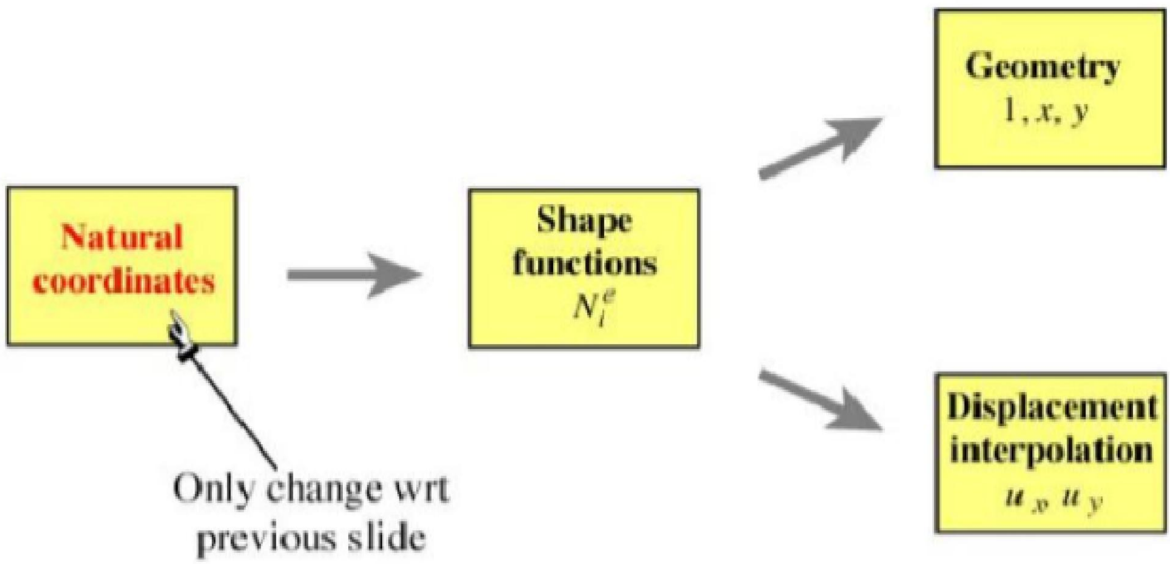


Isoparametric coordinate transformation.

As shown in the figure, the local (natural) coordinate system (ξ, η) for the two elements have their origins at the centroids of the elements, with (ξ, η) varying form –1 to 1. The natural coordinate system needs not to be orthogonal and neither has to be parallel to the x-y axes. The coordinate transformation will map the point (ξ, η) in the master element to $x(\xi, \eta)$ and $y(\xi, \eta)$ in the slave element.

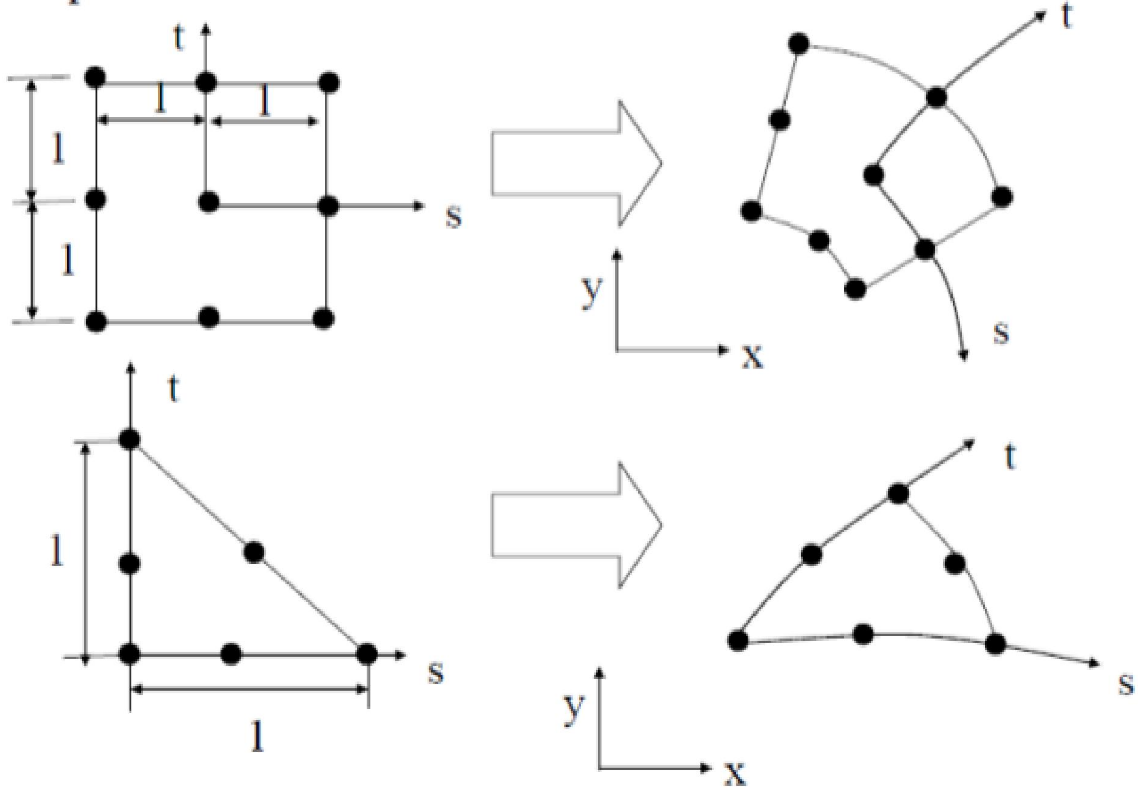
Finite Element Methods

Isoparametric Representation for *any* 2D Element



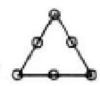
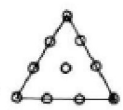
In 3D: $1,x,y$ **becomes** $1,x,y,z$ **etc**

Examples



Step 2: Select Displacement Functions

In other words, we look for shape functions that map the regular shape element in isoparametric coordinates to the quadrilateral in the x-y coordinates whose size and shape are determined by the eight nodal coordinates $x_1, y_1, x_2, y_2, \ldots, x_4, y_4$.

Terms in Pascal Triangle	*Polynomial Degree*	*Number of Terms Triangle*
1	0 (constant)	1
$x \quad y$	1 (linear)	3 CST 
$x^2 \quad xy \quad y^2$	2 (quadratic)	6 LST 
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3 (cubic)	10 QST 

$$x(\xi,\eta) = a_1 + a_2\,\xi + a_3\,\eta + a_4\,\xi\eta$$

$$y(\xi,\eta) = a_5 + a_6\,\xi + a_7\,\eta + a_8\,\xi\eta$$

$$\begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = \begin{Bmatrix} a_1 + a_2\,\xi + a_3\,\eta + a_4\,\xi\eta \\ a_5 + a_6\,\xi + a_7\,\eta + a_8\,\xi\eta \end{Bmatrix}$$

$$\begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} 1 & \xi & \eta & \xi\eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \xi & \eta & \xi\eta \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$x(\xi,\eta) = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$= \frac{1}{4}[(1-\xi)(1-\eta)x_1 + (1+\xi)(1-\eta)x_2 + (1+\xi)(1+\eta)x_3 + (1-\xi)(1+\eta)x_4]$$

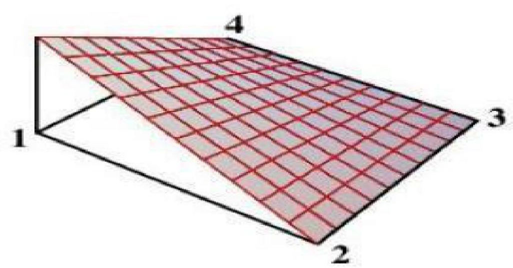
$$\begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{4} N_i\, x_i \\ \sum_{i=1}^{4} N_i\, y_i \end{Bmatrix}$$

Shape Function for 4-Nodes quadrilateral Elements

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \quad , \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) \quad , \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

These shape functions are seen to map the *(ξ, η)* coordinates of any point in the rectangular element in the above master element to ***x and y*** coordinates in the quadrilateral (slave) element.
For example, consider the coordinates of node 1, where:
$\xi=-1, \eta=-1$ using the above equation, we get $x=x_1$, $y=y_1$



$$N_1^e = \tfrac{1}{4}(1-\xi)(1-\eta)$$

Shape Function for 4-Nodes quadrilateral Elements

$$N_i = \begin{cases} 1 \; at \; node \, i \\ 0 \; at \; all \; other \; nodes \end{cases}$$

$$\sum_{i=1}^{n} N_i = 1 \quad , \qquad (i = 1, 2, , \cdots, n)$$

where n = the number of shape functions associated with number of nodes

$$\begin{Bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{4} N_i \, u_i \\ \sum_{i=1}^{4} N_i \, v_i \end{Bmatrix}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [N][d]$$

where *u and v* are displacements parallel to the global x and y coordinates

Step 3: Define the Strain/Displacement and Stress/Strain Relationships

Using Chain Rule

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

Can be computed $\Leftarrow$ $$\begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{Bmatrix}$$ $\Leftarrow$ We want to compute these for the B matrix

This is known as the **Jacobian** matrix (J) for the mapping $(\xi, \eta) \rightarrow (x, y)$

$$\begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

$$[J]^{-1} = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}$$

where

$$|J| = \frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \cdot \frac{\partial y}{\partial \xi}$$

Since:

$$\begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix}$$

$$\frac{\partial N}{\partial x} = \frac{1}{|J|} \left[\frac{\partial y}{\partial \eta} \frac{\partial N}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial N}{\partial \eta} \right]$$

$$\frac{\partial N}{\partial y} = \frac{1}{|J|} \left[\frac{\partial x}{\partial \xi} \frac{\partial N}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial N}{\partial \xi} \right]$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} & 0 \\ 0 & \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} \\ \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} & \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\{\varepsilon\} = [D'] \begin{Bmatrix} u \\ v \end{Bmatrix} = [D'][N][d]$$

$$\{\varepsilon\} = [D'][N][d]$$

$$\{\varepsilon\} = [B][d]$$

$$[D'] = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} & 0 \\ 0 & \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} \\ \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} & \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} \end{bmatrix}$$

$$\underset{(3\times 8)}{[B]} = \underset{(3\times 2)}{[D']} \; \underset{(2\times 8)}{[N]}$$

$$[k] = \iint_A [B]^T [D][B]\, t\, dx\, dy$$

$$\iint_A f(x,y)\, dx\, dy = \iint_A f(\xi,\eta) \left| J \right| d\xi\, d\eta$$

$$[k] = \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B]\, t \left|J\right| d\xi\, d\eta$$

The shape function are:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \quad , \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) \quad , \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Their derivatives:

$$\frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta) \quad , \frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1-\eta) \quad , \frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1+\eta) \quad , \frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(1+\eta)$$

and

$$\frac{\partial N_1}{\partial \eta} = -\frac{1}{4}(1-\xi) \quad , \frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(1+\xi) \quad , \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi) \quad , \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi)$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i \quad , \quad J_{12} = \frac{\partial y}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i \quad , \quad J_{22} = \frac{\partial y}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i$$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} N_{1,\xi} & N_{2,\xi} & N_{3,\xi} & N_{4,\xi} \\ N_{1,\eta} & N_{2,\eta} & N_{3,\eta} & N_{4,\eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

$$J_{22} = \frac{1}{4}\left[y_1(\xi-1) + y_2(-1-\xi) + y_3(1+\xi) + y_4(1-\xi) \right]$$

$$J_{12} = \frac{1}{4}\left[y_1(\eta-1) + y_2(1-\eta) + y_3(1+\eta) + y_4(-1-\eta) \right]$$

$$J_{11} = \frac{1}{4}\left[x_1(\eta-1) + x_2(1-\eta) + x_3(1+\eta) + x_4(-1-\eta) \right]$$

$$J_{21} = \frac{1}{4}\left[x_1(\xi-1) + x_2(-1-\xi) + x_3(1+\xi) + x_4(1-\xi) \right]$$

➢ *Derive the Element Stiffness Matrix and Equations*

$$[J]=\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$|J| = J_{11}\,J_{22} - J_{12}\,J_{21}$$

Explicit formulation for |J| for 4 node Element

$$|J| = \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix} = \frac{1}{8}\{x_1 \quad x_2 \quad x_3 \quad x_4\} \begin{bmatrix} 0 & 1-\eta & \eta-\xi & \xi-1 \\ \eta-1 & 0 & \xi+1 & -\xi-\eta \\ \xi-\eta & -\xi-1 & 0 & \eta+1 \\ 1-\xi & \xi+\eta & -\eta-1 & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

➢ *Derive the Element Stiffness Matrix and Equations*

$$[B]=[D'][N]$$

$$[B]=\frac{1}{|J|}\begin{bmatrix} J_{22}\dfrac{\partial()}{\partial\xi} - J_{12}\dfrac{\partial()}{\partial\eta} & 0 \\ 0 & J_{11}\dfrac{\partial()}{\partial\eta} - J_{21}\dfrac{\partial()}{\partial\xi} \\ J_{11}\dfrac{\partial()}{\partial\eta} - J_{21}\dfrac{\partial()}{\partial\xi} & J_{22}\dfrac{\partial()}{\partial\xi} - J_{12}\dfrac{\partial()}{\partial\eta} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$[B]=\frac{1}{|J|}[\,B_1 \quad B_2 \quad B_3 \quad B_4\,]$$

$$[B_i]=\begin{bmatrix} J_{22}\,N_{i,\xi} - J_{12}\,N_{i,\eta} & 0 \\ 0 & J_{11}\,N_{i,\eta} - J_{21}\,N_{i,\xi} \\ J_{11}\,N_{i,\eta} - J_{21}\,N_{i,\xi} & J_{22}\,N_{i,\xi} - J_{12}\,N_{i,\eta} \end{bmatrix}$$

➢ Derive the Element Stiffness Matrix and Equations

$$[B_i] = \begin{bmatrix} J_{22}\, N_{i,\xi} - J_{12}\, N_{i,\eta} & 0 \\ 0 & J_{11}\, N_{i,\eta} - J_{21}\, N_{i,\xi} \\ J_{11}\, N_{i,\eta} - J_{21}\, N_{i,\xi} & J_{22}\, N_{i,\xi} - J_{12}\, N_{i,\eta} \end{bmatrix}$$

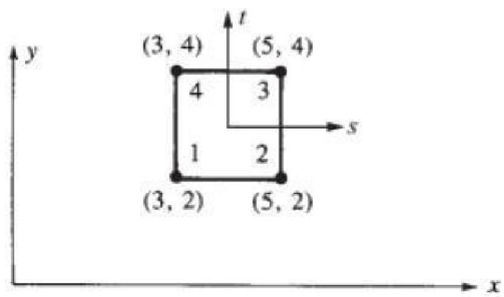
$$J_{22} = \frac{1}{4}\left[y_1(\xi-1) + y_2(-1-\xi) + y_3(1+\xi) + y_4(1-\xi)\right]$$

$$J_{12} = \frac{1}{4}\left[y_1(\eta-1) + y_2(1-\eta) + y_3(1+\eta) + y_4(-1-\eta)\right]$$

$$J_{11} = \frac{1}{4}\left[x_1(\eta-1) + x_2(1-\eta) + x_3(1+\eta) + x_4(-1-\eta)\right]$$

$$J_{21} = \frac{1}{4}\left[x_1(\xi-1) + x_2(-1-\xi) + x_3(1+\xi) + x_4(1-\xi)\right]$$

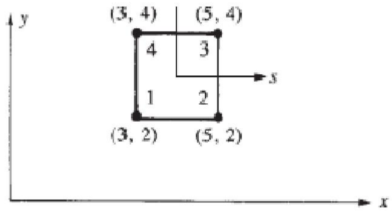
Evaluate the stiffness matrix for the quadrilateral element shown in Figure using the four-point Gaussian quadrature rule.
Let E = 30×10^6 psi, ν = 0.25 and h=1 in.



Solution

we evaluate the ***k*** matrix. Using the four-point rule, the four points are:

$$(\xi_1, \eta_1) = (-0.5773, -0.5773)$$
$$(\xi_2, \eta_2) = (-0.5773, 0.5773)$$
$$(\xi_3, \eta_3) = (0.5773, -0.5773) \qquad W_1 = W_2 = W_3 = W_4 = 1.0$$
$$(\xi_4, \eta_4) = (0.5773, 0.5773)$$



$$
\begin{aligned}
[k] = {} & [B(-0.5773,-0.5773)]^T[D][B(-0.5773,-0.5773)] \\
& |[J(-0.5773,-0.5773)]|(1)(1.000)(1.000) \\
& + [B(-0.5773,0.5773)]^T[D][B(-0.5773,0.5773)] \\
& |[J(-0.5773,0.5773)]|(1)(1.000)(1.000) \\
& + [B(0.5773,-0.5773)]^T[D][B(0.5773,-0.5773)] \\
& |[J(0.5773,-0.5773)]|(1)(1.000)(1.000) \\
& + [B(0.5773,0.5773)]^T[D][B(0.5773,0.5773)] \\
& |[J(0.5773,0.5773)]|(1)(1.000)(1.000)
\end{aligned}
$$

$$
|[J(-0.5773,-0.5773)]| = \frac{1}{8}[3 \quad 5 \quad 5 \quad 3]
\begin{bmatrix}
0 & 1-(-0.5773) & -0.5773-(-0.5773) & -0.5773-1 \\
-0.5773-1 & 0 & -0.5773+1 & -0.5773-(-0.5773) \\
-0.5773-(-0.5773) & -(-0.5773)-1 & 0 & -0.5773+1 \\
1-(-0.5773) & -0.5773+(-0.5773) & -0.5773-1 & 0
\end{bmatrix}
\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix} = 1.000
$$

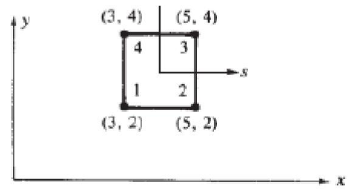
Similarly,

$$
|[J(-0.5773,0.5773)]| = 1.000
$$

$$
|[J(0.5773,-0.5773)]| = 1.000
$$

$$
|[J(0.5773,0.5773)]| = 1.000
$$

Even though $|[J]| = 1$ in this example, in general, $|[J]| \neq 1$ and varies in space.



$$
[B(-0.5773,-0.5773)] = \frac{1}{|[J(-0.5773,-0.5773)]|}[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]]
$$

$$
[B_1] = \begin{bmatrix}
J_{22}N_{1,\xi} - J_{12}N_{1,\eta} & 0 \\
0 & J_{11}N_{1,\eta} - J_{21}N_{1,\xi} \\
J_{11}N_{1,\eta} - J_{21}N_{1,\xi} & J_{22}N_{1,\xi} - J_{12}N_{1,\eta}
\end{bmatrix}
$$

$$
J_{22} = \frac{1}{4}\left[y_1(\xi-1) + y_2(-1-\xi) + y_3(1+\xi) + y_4(1-\xi)\right]
$$

$$
J_{22} = \frac{1}{4}\left[2(-0.5773-1) + 2(-1+0.5773) + 4(1-0.5773) + 4(1+0.5773)\right] = 1.0
$$

with similar computations used to obtain J_{12}, J_{11} and J_{21}. Also,

$$
N_{1,\xi} = -\frac{1}{4}(1-\eta) = -\frac{1}{4}(1+0.5773) = -0.3943
$$

$$
N_{1,\eta} = -\frac{1}{4}(1-\xi) = -\frac{1}{4}(1+0.5773) = -0.3943
$$

Similarly, $[B_2]$, $[B_3]$, and $[B_4]$ must be evaluated like $[B_1]$, at $(-0.5773, -0.5773)$. We then repeat the calculations to evaluate $[B]$ at the other Gauss points [Eq. (10.4.4a)].

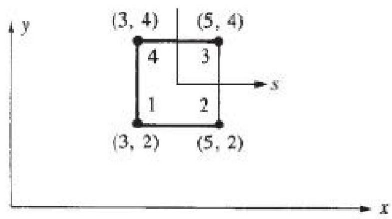
Using a computer program written specifically to evaluate $[B]$ at each Gauss point and then $[k]$, we obtain the final form of $[B(-0.5773, -0.5773)]$ as

$$[B(-0.5773, -0.5773)] =$$

$$\begin{bmatrix} -0.1057 & 0 & 0.1057 & 0 & 0 & -0.1057 & 0 & -0.3943 \\ -0.1057 & -0.1057 & -0.3943 & 0.1057 & 0.3943 & 0 & -0.3943 & 0 \\ 0 & 0.3943 & 0 & 0.1057 & 0.3943 & 0.3943 & 0.1057 & -0.3943 \end{bmatrix} \tag{10.4.4h}$$

with similar expressions for $[B(-0.5773, 0.5773)]$, and so on.

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12 \end{bmatrix} \quad 10^6 \text{ psi}$$



$$\begin{aligned} [k] = {} & [B(-0.5773, -0.5773)]^T[D][B(-0.5773, -0.5773)] \\ & |[J(-0.5773, -0.5773)]|(1)(1.000)(1.000) \\ & + [B(-0.5773, 0.5773)]^T[D][B(-0.5773, 0.5773)] \\ & |[J(-0.5773, 0.5773)]|(1)(1.000)(1.000) \\ & + [B(0.5773, -0.5773)]^T[D][B(0.5773, -0.5773)] \\ & |[J(0.5773, -0.5773)]|(1)(1.000)(1.000) \\ & + [B(0.5773, 0.5773)]^T[D][B(0.5773, 0.5773)] \\ & |[J(0.5773, 0.5773)]|(1)(1.000)(1.000) \end{aligned}$$

$$[k] = 10^4 \begin{bmatrix} 1466 & 500 & -866 & -99 & -733 & -500 & 133 & 99 \\ 500 & 1466 & 99 & 133 & -500 & -733 & -99 & -866 \\ -866 & 99 & 1466 & -500 & 133 & -99 & -733 & 500 \\ -99 & 133 & -500 & 1466 & 99 & -866 & 500 & -733 \\ -733 & -500 & 133 & 99 & 1466 & 500 & -866 & -99 \\ -500 & -733 & -99 & -866 & 500 & 1466 & 99 & 133 \\ 133 & -99 & -733 & 500 & -866 & 99 & 1466 & -500 \\ 99 & -866 & 500 & -733 & -99 & 133 & -500 & 1466 \end{bmatrix}$$

For the rectangular element shown previous Example, assume plane stress conditions

Let E = 30×10^6 psi, ν = 0.3 and displacements:

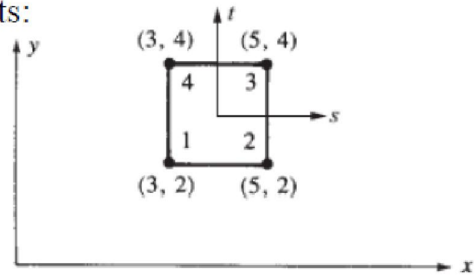
$u_1 = 0$, $v_1 = 0$

$u_2 = 0.001$, $v_2 = 0.0015$

$u_3 = 0.003$, $v_3 = 0.0016$

$u_4 = 0$, $v_4 = 0$

Evaluate the stresses at $s=0$, $t=0$



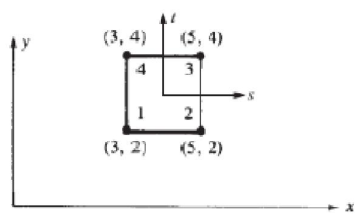
Solution

$$[B] = \frac{1}{|[J]|}[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]]$$

$$[B(0,0)] = \frac{1}{|[J(0,0)]|}[B_1(0,0)] \quad [B_2(0,0)] \quad [B_3(0,0)] \quad [B_4(0,0)]$$

Example 3

$$|[J(0,0)]| = \frac{1}{8}[3 \quad 5 \quad 5 \quad 3]\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix}$$



$$= \frac{1}{8}[-2 \quad -2 \quad 2 \quad 2]\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix}$$

$$|[J(0,0)]| = 1$$

$$[B_i] = \begin{bmatrix} J_{22}\,N_{i,\xi} - J_{12}\,N_{i,\eta} & 0 \\ 0 & J_{11}\,N_{i,\eta} - J_{21}\,N_{i,\xi} \\ J_{11}\,N_{i,\eta} - J_{21}\,N_{i,\xi} & J_{22}\,N_{i,\xi} - J_{12}\,N_{i,\eta} \end{bmatrix}$$

$$J_{22} = \frac{1}{4}\left[y_1(\xi - 1) + y_2(-1-\xi) + y_3(1+\xi) + y_4(1-\xi)\right]$$

$$J_{22} = \frac{1}{4}\left[2(0-1) + 2(-1-0) + 4(1+0) + 4(1-0)\right] = 1$$

Similiarly $J_{12} = 0, \quad J_{11} = 1, \quad J_{21} = 0$

$N_{1,\xi} = -\frac{1}{4}(1-\eta) = -\frac{1}{4}(1-0) = -\frac{1}{4}$ Similarly $N_{2,\xi} = \frac{1}{4}$, $N_{3,\xi} = \frac{1}{4}$ and $N_{4,\xi} = -\frac{1}{4}$

$N_{1,\eta} = -\frac{1}{4}(1-\xi) = -\frac{1}{4}(1-0) = -\frac{1}{4}$ Similarly $N_{2,\eta} = -\frac{1}{4}$, $N_{3,\eta} = \frac{1}{4}$ and $N_{4,\eta} = \frac{1}{4}$

$$[B_1] = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \qquad [B_2] = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad [B_3] = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad [B_4] = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$\{\sigma\} = [D][B]\{d\} =$$

$$= (30)\frac{10^6 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}}{1 - 0.09} \begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0.25 & 0 & -0.25 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.25 & 0 & 0.25 \\ -0.25 & -0.25 & -0.25 & 0.25 & 0.25 & 0.25 & 0.25 & -0.25 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.001 \\ 0.0015 \\ 0.003 \\ 0.0016 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 3.321 & 10^4 \\ 1.071 & 10^4 \\ 1.471 & 10^4 \end{Bmatrix} \text{psi}$$

Higher-Order Shape Functions

- In general, higher-order element shape functions can be developed by adding additional nodes to the sides of the linear element.
- These elements result in higher-order strain variations within each element, and convergence to the exact solution thus occurs at a faster rate using fewer elements.
- Another advantage of the use of higher-order elements is that curved boundaries of irregularly shaped bodies can be approximated more closely than by the use of simple straight-sided linear elements.

Shape function of a quadratic isoparametric element

$$N_1 = \frac{1}{4}(1-s)(1-t)(-s-t-1)$$

$$N_2 = \frac{1}{4}(1+s)(1-t)(s-t-1)$$

$$N_3 = \frac{1}{4}(1+s)(1+t)(s+t-1)$$

$$N_4 = \frac{1}{4}(1-s)(1+t)(-s+t-1)$$

or, in compact index notation, we express

$$N_i = \frac{1}{4}(1+ss_i)(1+tt_i)(ss_i+tt_i-1)$$

where ***i*** is the number of the shape function

$$s_i = -1, 1, 1, -1 \qquad (i = 1, 2, 3, 4)$$

$$t_i = -1, -1, 1, 1 \qquad (i = 1, 2, 3, 4)$$

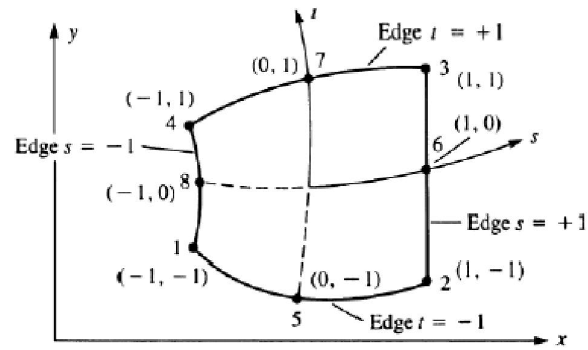


Figure 10–16 Quadratic isoparametric element

For the midside nodes ($i = 5, 6, 7, 8$),

$$N_5 = \frac{1}{2}(1-t)(1+s)(1-s)$$

$$N_6 = \frac{1}{2}(1+s)(1+t)(1-t)$$

$$N_7 = \frac{1}{2}(1+t)(1+s)(1-s)$$

$$N_8 = \frac{1}{2}(1-s)(1+t)(1-t)$$

Shape function of a cubic isoparametric element

For the corner nodes ($i = 1, 2, 3, 4$),

$$N_i = \frac{1}{32}(1+ss_i)(1+tt_i)[9(s^2+t^2)-10]$$

For the nodes on sides $s = \pm 1$ ($i = 7, 8, 11, 12$),

$$N_i = \frac{9}{32}(1+ss_i)(1+9tt_i)(1-t^2)$$

with $s_i = \pm 1$ and $t_i = \pm\frac{1}{3}$.

For the nodes on sides $t = \pm 1$ ($i = 5, 6, 9, 10$),

$$N_i = \frac{9}{32}(1+tt_i)(1+9ss_i)(1-s^2)$$

with $t_i = \pm 1$ and $s_i = \pm\frac{1}{3}$.

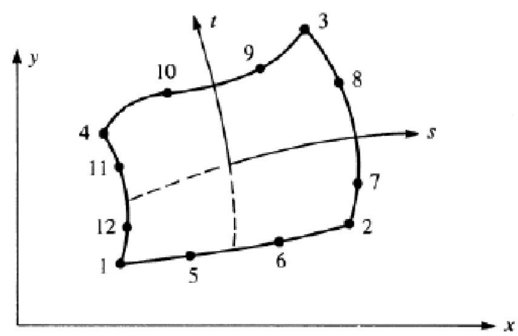


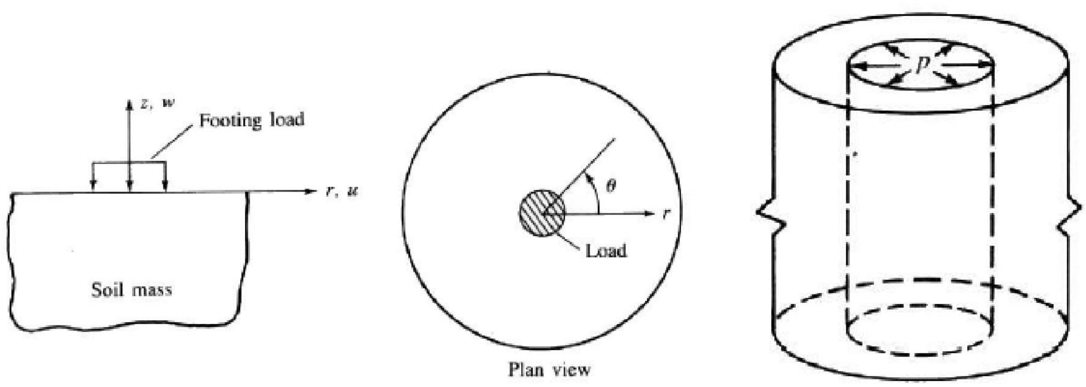
Figure 10–18 Cubic isoparametric element

Definition of an axisymmetric solid

- An axisymmetric solid (or a thick-walled body) of revolution is defined as a 3-D body that is generated by rotating a plane and is most easily described in cylindrical coordinates. Where z is called the axis of symmetry.
- If the geometry, support conditions, loads, and material properties are all axially symmetric (all are independent of θ), then the problem can be idealized as a two-dimensional one.

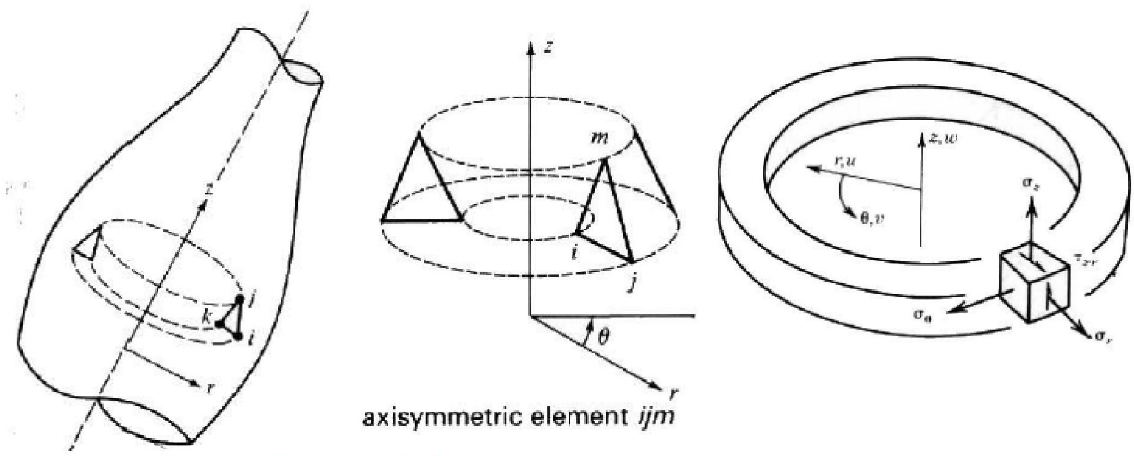
Examples of an axisymmetric solid

Problems such as soil masses subjected to circular footing loads, thick-walled pressure vessels, and a rocket nozzle subjected to thermal and pressure loading can often be analyzed using axisymmetric elements.



FE axisymmetric elements

axisymmetric problems can be analyzed by a finite element of revolution, called axisymmetric elements. Each element consists of a solid ring, the cross-section of which is the shape of the particular element chosen (triangular, rectangular, or quadrilateral elements). An axisymmetric element has nodal circles rather than nodal points



Equations of Equilibrium:

The three-dimensional elasticity equations in cylindrical coordinates can be summarized as follows

$$\left.\begin{aligned}
&\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}+X_r=0\\
&\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}+\frac{2}{r}\tau_{r\theta}+Y_\theta=0\\
&\frac{\partial \tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \sigma_z}{\partial z}+\frac{1}{r}\tau_{rz}+Z_b=0
\end{aligned}\right\}$$

The three-dimensional strain-displacement relationships of elasticity in cylindrical coordinates were ***u, v, w*** are the displacements in the ***r, θ,dz***, respectively, are:

$$
\left.
\begin{aligned}
\varepsilon_r &= \frac{\partial u}{\partial r} &,\quad \gamma_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \\
\varepsilon_\theta &= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} &,\quad \gamma_{rz} &= \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \\
\varepsilon_z &= \frac{\partial w}{\partial z} &,\quad \gamma_{\theta z} &= \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}
\end{aligned}
\right\}
$$

The three-dimensional stress-strain relationships for isotropic elasticity are:

$$
\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{r\theta} \\ \tau_{rz} \\ \tau_{\theta z} \end{Bmatrix}
= \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\
0 & 0 & 0 & & \frac{1-2\nu}{2} & 0 \\
0 & 0 & 0 & 0 & & \frac{1-2\nu}{2}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{r\theta} \\ \gamma_{rz} \\ \gamma_{\theta z} \end{Bmatrix}
$$

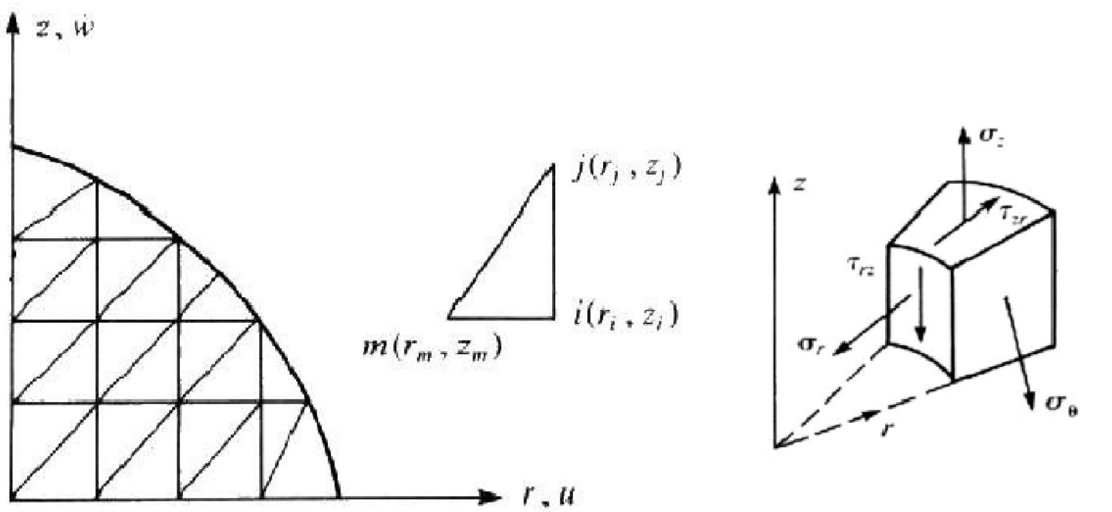
In axisymmetric problems, because of the symmetry about the z -axis, the stresses are independent of the θ coordinate. Therefore, all derivatives with respect to θ vanish and the circumferential (tangent to θ direction) displacement component is zero; therefore,

$$\gamma_{r\theta} = \gamma_{\theta z} = 0 \quad and \quad \tau_{r\theta} = \tau_{\theta z} = 0$$

$$\varepsilon_r = \frac{\partial u}{\partial r} \quad , \quad \varepsilon_\theta = \frac{u}{r} \quad , \quad \varepsilon_z = \frac{\partial w}{\partial z} \quad , \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

Derivation of the Stiffness Matrix and Equations

Step 1: Discretize and Select Element Type



Typical slice through an axisymmetric solid Discretized into triangular elements

$$\{d\} = \begin{Bmatrix} \underline{d}_i \\ \underline{d}_j \\ \underline{d}_m \end{Bmatrix} = \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

(u_i, w_i) → displacement components of ***node i*** in the *r* and *z* directions, respectively.

Step 2: Select Displacement Functions

$$u(r,z) = a_1 + a_2 r + a_3 z$$
$$w(r,z) = a_4 + a_5 r + a_6 z$$

$$\{\psi\} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{Bmatrix} a_1 + a_2 r + a_3 z \\ a_4 + a_5 r + a_6 z \end{Bmatrix} = \begin{bmatrix} 1 & r & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r & z \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$u_i = a_1 + a_2 r_i + a_3 z_i$$

$$u_j = a_1 + a_2 r_j + a_3 z_j$$

$$u_m = a_1 + a_2 r_m + a_3 z_m$$

$$w_i = a_4 + a_5 r_i + a_6 z_i$$

$$w_j = a_4 + a_5 r_j + a_6 z_j$$

$$w_m = a_4 + a_5 r_m + a_6 z_m$$

In Matrix Form

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix}^{-1} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} \qquad \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix}^{-1} \begin{Bmatrix} w_i \\ w_j \\ w_m \end{Bmatrix}$$

Solving for the a's

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} \qquad \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} w_i \\ w_j \\ w_m \end{Bmatrix}$$

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

$$\begin{aligned} 2A &= x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j) \\ &= \alpha_i + \alpha_j + \alpha_m \end{aligned}$$

A is the area of the triangle

$$\alpha_i = r_j z_m - z_j r_m \qquad \alpha_j = r_m z_i - z_m r_i \qquad \alpha_m = r_i z_j - z_i r_j$$

$$\beta_i = z_j - z_m \qquad \beta_j = z_m - z_i \qquad \beta_m = z_i - z_j$$

$$\gamma_i = r_m - r_j \qquad \gamma_j = r_i - r_m \qquad \gamma_m = r_j - r_i$$

$$\{u\} = \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\{u\} = \frac{1}{2A} \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

$$\{u\} = \frac{1}{2A} \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{Bmatrix} \alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m \end{Bmatrix}$$

$$u(r,z) = \frac{1}{2A} \left\{ (\alpha_i + \beta_i r + \gamma_i z) u_i + (\alpha_j + \beta_j r + \gamma_j z) u_j + (\alpha_m + \beta_m r + \gamma_m z) u_m \right\}$$

$$w(r,z) = \frac{1}{2A} \left\{ (\alpha_i + \beta_i r + \gamma_i z) w_i + (\alpha_j + \beta_j r + \gamma_j z) w_j + (\alpha_m + \beta_m r + \gamma_m z) w_m \right\}$$

$$\{\psi\} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

$$\{\psi\} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

$$N_i = \frac{1}{2A}\left(\alpha_i + \beta_i r + \gamma_i z\right)$$

$$N_j = \frac{1}{2A}\left(\alpha_j + \beta_j r + \gamma_j z\right)$$

$$N_m = \frac{1}{2A}\left(\alpha_m + \beta_m r + \gamma_m z\right)$$

$$\{\psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

$$\{\psi\} = [N]\{d\}$$

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

Step 3: Define the Strain/Displacement and Stress/Strain Relationships

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{u}{r} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} a_2 \\ a_6 \\ \dfrac{a_1}{r} + a_2 + \dfrac{a_3 z}{r} \\ a_3 + a_5 \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r} & 0 & \frac{\alpha_j}{r} + \beta_j + \frac{\gamma_j z}{r} & 0 & \frac{\alpha_m}{r} + \beta_m + \frac{\gamma_m z}{r} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{bmatrix} B_i & B_j & B_m \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix} \qquad B_i = \begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r} & 0 \\ \gamma_i & \beta_i \end{bmatrix}$$

$$\{\varepsilon\} = [B]\{d\}$$

B **is a function of** ***r*** **and** ***z***

Stress Strain Relationship

$$\{\sigma\} = [D][B]\{d\}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$

Step 4 :Derive the Element Stiffness Matrix and Equations

$$[k] = \iiint_V [B]^T [D][B]\, dV$$

$$[k] = 2\pi \iint_A [B]^T [D][B]\, r\, dr\, dz$$

1) Numerical integration (Gaussian quadrature)
2) Explicit multiplication and term-by-term integration.
3) Evaluate [B] at a centroidal point of the element

$$r = \bar{r} = \frac{r_i + r_j + r_m}{3} \qquad z = \bar{z} = \frac{z_i + z_j + z_m}{3}$$

$$[B(\bar{r}, \bar{z})] = [\bar{B}]$$

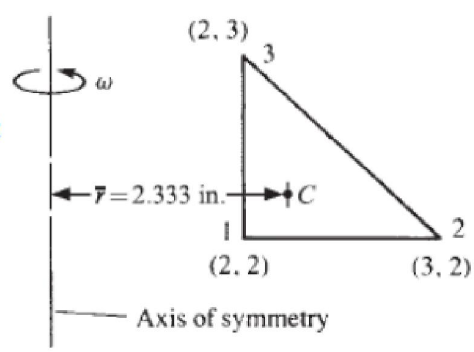
$$[k] = 2\pi \bar{r} A [\bar{B}]^T [D][\bar{B}]$$

Example 1

For the element of an axisymmetric body rotating with a constant angular velocity ***ω = 100 rev/min***.
Evaluate the approximate body force matrix, include the weight of the material, where the weight density ρ_w **is 0.283 *lb/in³***.
The coordinates of the element (in inches) are shown in the figure.

The body forces per unit volume evaluated at the centroid of the element are

Z_b =0.283 lb/in³



$$\bar{R}_b = \omega^2 \rho \bar{r} = \left[\; 100 \frac{\text{rev}}{\text{min}} \; \left(2\pi \frac{\text{rad}}{\text{rev}} \right) \left(\frac{1 \text{ min}}{60 \text{ s}} \right) \right]^2 \frac{(0.283 \text{ lb/in}^3)}{(32.2 \times 12) \text{ in./s}^2} (2.333 \text{ in.})$$

$$\bar{R}_b = 0.187 \text{ lb/in}^3$$

$$\frac{2\pi \bar{r} A}{3} = \frac{2\pi(2.333)(0.5)}{3} = 2.44 \text{ in}^3$$

$$f_{b1r} = (2.44)(0.187) = 0.457 \text{ lb}$$

$$f_{b1z} = -(2.44)(0.283) = -0.691 \text{ lb} \qquad \text{(downward)}$$

All r-directed and z-directed nodal body forces are equal

$$f_{b2r} = 0.457 \text{ lb} \qquad f_{b2z} = -0.691 \text{ lb}$$

$$f_{b3r} = 0.457 \text{ lb} \qquad f_{b3z} = -0.691 \text{ lb}$$