Fourier Transforms

→ Transformation is a mathematical device which converts one function into another function. Laplace transforms are used to solve ordinary differential equations and Fourier transforms are used to solve Partial differential Equations. (~~The L.T of function is defined by~~ ~~$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$~~ ~~The Fourier transform of the function $f(x)$ is defined by~~ ~~$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) dx$~~)

Fourier Integral theorem (statement only) :-

The Fourier Integral theorem states that

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos p(t-x) dt\, dp$$

Fourier sine and Cosine Integrals :-

We know that the Fourier integral states that

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos p(t-x) dt\, dp$$

$$= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) [\cos pt \cos px + \sin pt \sin px] dt\, dp$$

$$= \frac{1}{\pi} \int_0^\infty \cos px \int_{-\infty}^\infty \cos pt\, f(t) dt\, dp + \frac{1}{\pi} \int_0^\infty \sin px \int_{-\infty}^\infty \sin pt\, f(t) dt\, dp \quad \text{---(1)}$$

Case 1 :- If $f(t)$ is odd function then

$$\frac{1}{\pi} \int_0^\infty \cos px \int_{-\infty}^\infty \cos pt\, f(t) dt\, dp = 0$$

∴ Equ (1) becomes

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin px \int_0^\infty \sin pt\, f(t) dt\, dp$$

which is known as Fourier sine integral

Case 2 :- If $f(t)$ is even function then

$$\frac{1}{\pi} \int_0^\infty \sin px \int_{-\infty}^\infty \sin pt\, f(t) dt\, dp = 0$$

∴ Equ (1) becomes

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos px \int_0^\infty \cos pt\, f(t) dt\, dp$$

which is known as Fourier cosine integral

1. Use Fourier integral formula show that

$$e^{-x} \cos x = \frac{2}{\pi} \int_0^\infty \frac{\lambda^2 + 2}{\lambda^4 + 4} \cos \lambda x\, d\lambda.$$

Sol :- Given $f(x) = e^{-x} \cos x$

Since the integral contains cosine terms we have to use cosine integral formula

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos px \int_0^\infty \cos pt\, f(t) dt\, dp$$

Replacing p by λ

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty \cos \lambda t\, f(t) dt\, d\lambda$$

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty \cos\lambda t \; e^{t}\cos t \, dt \, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty e^{-t}\, 2\cos\lambda t \cos t \, dt \, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty e^{-t}\left[\cos(\lambda+1)t + \cos(\lambda-1)t\right] dt \, d\lambda \qquad \left(\because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}\right)$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x \left[\frac{1}{1+(\lambda+1)^2} + \frac{1}{1+(\lambda-1)^2}\right] d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x \left[\frac{1}{1+\lambda^2+2\lambda+1} + \frac{1}{1+\lambda^2-2\lambda+1}\right] d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x \left[\frac{1}{\lambda^2+2\lambda+2} + \frac{1}{\lambda^2-2\lambda+2}\right] d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x \left[\frac{\lambda^2-\cancel{2\lambda}+2+\lambda^2+\cancel{2\lambda}+2}{(\lambda^2+2\lambda+2)(\lambda^2-2\lambda+2)}\right]$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x \, \frac{2(\lambda^2+2)}{\lambda^4+4} \, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+4} \cos\lambda x \, d\lambda$$

2. Express the function $f(x) = \begin{cases} 1 & 0<x<\pi \\ 0 & x>\pi \end{cases}$ as a Fourier sine integral and Evaluate $\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x \, dx$.

Sol:- Given $f(x) = \begin{cases} 1 & 0<x<\pi \\ 0 & x>\pi \end{cases}$

The Fourier sine integral for the function $f(x)$ is

$$\frac{2}{\pi}\int_0^\infty \sin px \int_0^\infty \sin pt \, f(t)\,dt\,dp$$

replacing p by λ

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty \sin\lambda t \, f(t)\,dt\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[\int_0^\pi f(t)\sin\lambda t\,dt + \int_\pi^\infty f(t)\sin\lambda t\,dt\right] d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[\int_0^\pi f(t)\sin\lambda t\,dt + 0\right] d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[-\frac{\cos\lambda t}{\lambda}\right]_0^\pi d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[-\frac{\cos\lambda\pi}{\lambda} + \frac{\cos 0}{\lambda}\right] d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda} \sin\lambda x \, d\lambda$$

$$\Rightarrow \int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x \, d\lambda = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2} & \text{if } 0<x<\pi \\ 0 & \text{if } x>\pi \end{cases}$$

3. Show that $e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)} d\lambda$

Sol:- Given $f(x) = e^{-ax} - e^{-bx}$

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The Fourier sine integral for the function

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin px \int_0^\infty \sin pt\, f(t)\,dt\,dp$$

replacing p by λ

$$e^{-ax} - e^{-bx} = \frac{2}{\pi}\int_0^\infty (e^{-at} - e^{-bt}) \sin\lambda t\, dt\, d\lambda$$

$$e^{-ax} - e^{-bx} = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty \sin\lambda t (e^{-at} - e^{-bt})dt\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[\int_0^\infty e^{-at}\sin\lambda t\,dt - \int_0^\infty e^{-bt}\sin\lambda t\,dt\right]d\lambda$$

$$\left(\because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\right)$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[\frac{\lambda}{\lambda^2+a^2} - \frac{\lambda}{\lambda^2+b^2}\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \cdot \lambda\left(\frac{1}{\lambda^2+a^2} - \frac{1}{\lambda^2+b^2}\right)d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \lambda \sin\lambda x \left[\frac{\cancel{\lambda^2}+b^2-\cancel{\lambda^2}-a^2}{(\lambda^2+a^2)(\lambda^2+b^2)}\right]$$

$$= \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}d\lambda$$

4. Use Fourier integral formula show that

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\lambda x}{a^2+\lambda^2}d\lambda$$

Sol :- $f(x) = \frac{2}{\pi}\int_0^\infty \cos px \int_0^\infty \cos pt\, f(t)\,dt$

replacing p by λ

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty \cos\lambda t\; f(t)\,dt\,dp$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty e^{-at}\cos\lambda t\,dt\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\frac{a}{a^2+\lambda^2}\right]d\lambda$$

$$= \frac{2a}{\pi}\int_0^\infty \frac{\cos\lambda x}{a^2+\lambda^2}d\lambda$$

Fourier Transform and Inverse Fourier Transform

The complex form of Fourier integral of any function f(x) is in the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty e^{ip(t-x)} f(t)\,dt\,dp \quad \rightarrow (1)$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ipx}dp \int_{-\infty}^\infty e^{ipt}dt\, f(t)dt$$

Now write $F(p) = \int_{-\infty}^\infty e^{ipt}\, \cancel{dt}\, f(t)\,dt$

Then Eqn (1) becomes

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(p)e^{-ipx}dp \quad \rightarrow (2)$$

Here the function F(P) is called Fourier transform of f(x). ~~and denoted by F{f(t)} and~~ The function f(x) is called Inverse Fourier transform of F(P)

Definition:-

The Fourier transform of the function $f(x)$ is given by $F\{f(x)\} = F(P) = \int_{-\infty}^{\infty} f(x)\, e^{iPx}\, dx$ (Find)

The inverse Fourier transform of $F(P)$ is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(P)\, e^{-iPx}\, dP$$

Some Authors defines the Fourier transforms and inverse Fourier transforms as follows

$$F(P) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{iPx}\, dx$$

(Prove)

$$\text{and } f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(P)\, e^{-iPx}\, dP$$

Fourier Sine and Cosine transforms:-

Case 1:- The Fourier sine integral is

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin px \int_0^{\infty} f(t) \sin pt\, dt\, dp \longrightarrow (1)$$

define $F_s\{f(x)\} = F_s(P) = \int_0^{\infty} \sin pt\, f(t)\, dt \longrightarrow (2)$

then Equ (1) becomes

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin px\, F_s(P)\, dP \longrightarrow (3)$$

Equation (2) is called Fourier sine transform of the function $f(x)$ and Equ (3) is called Inverse Fourier sine transform of the function $F_s(P)$.

Case 2:-

The Four Cosine integral is

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos px \int_0^{\infty} f(t) \cos pt\, dt\, dp \longrightarrow (1)$$

define $F_c\{f(x)\} = F_c(P) = \int_0^{\infty} f(t) \cos pt\, dt \longrightarrow (2)$

then Equ (1) becomes

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos px\, F_c(P)\, dP \longrightarrow (3)$$

Equ (2) is called Fourier Cosine transform of $f(x)$ and Equ (3) is called Inverse Fourier Cosine transform of $F_c(P)$.

1. Show that $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$

Sol:- $\int_{-\infty}^{\infty} e^{-y^2} dy = 2\int_0^{\infty} e^{-y^2} dy$ ($\because$ the integrand is even)

let $y^2 = t$

$2y\, dy = dt \Rightarrow dy = \frac{dt}{2y} = \frac{dt}{2\sqrt{t}}$

$$\therefore \int_{-\infty}^{\infty} e^{-y^2} dy = \cancel{2} \int_0^{\infty} e^{-t} \frac{dt}{\cancel{2}\sqrt{t}}$$

$$= \int_0^{\infty} e^{-t}\, t^{-1/2}\, dt$$

$$\int_{-\infty}^{\infty} e^{-y^2} dy = \int_0^{\infty} e^{-t}\, t^{1/2 - 1}\, dt$$

This is of the form $\int_0^{\infty} e^{-x} x^{n-1} dx = \Gamma(n)$

which is a gamma function when $n = 1/2$

~~also $\Gamma(\frac{1}{2}) = \sqrt{\pi}$~~

$$\therefore \int_{-\infty}^{\infty} e^{-y^2} dy = \Gamma(\tfrac{1}{2}) \qquad (\because \Gamma(\tfrac{1}{2}) = \sqrt{\pi})$$

$$= \sqrt{\pi}$$

2. Show that the Fourier transform of $e^{-x^2/2}$ is self reciprocal.

Sol :- Given $f(x) = e^{-x^2/2}$

The Fourier transform of the function $f(x)$ is

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx$$

$$F\{e^{-x^2/2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{ipx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(x^2 - 2ipx)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(x-ip)^2} \cdot e^{-p^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-ip}{\sqrt{2}}\right)^2} \cdot e^{-p^2/2} dx$$

Let $\frac{x-ip}{\sqrt{2}} = y$

$x - ip = \sqrt{2}\, y$

$dx = \sqrt{2}\, dy$

$$\therefore F\{e^{-x^2/2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} \cdot e^{-p^2/2} \sqrt{2}\, dy.$$

$$= \frac{\sqrt{2}\, e^{-p^2/2}}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{e^{-p^2/2}}{\sqrt{\pi}} \sqrt{\pi} = e^{-p^2/2}$$

$\therefore$ The Fourier transform of the function is self reciprocal

3. Find the Fourier transform of the function $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$ and evaluate $\int_{-\infty}^{\infty} \frac{\sin ap \cos px}{p} dp$ also $\int_0^{\infty} \frac{\sin p}{p} dp$.

1st Part :-

Sol:- Given $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$

The Fourier transform of the given function is

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$

$$= \int_{-\infty}^{-a} e^{ipx} f(x)\, dx + \int_{-a}^{a} e^{ipx} f(x)\, dx + \int_{a}^{\infty} e^{ipx} f(x)\, dx$$

$$= 0 + \int_{-a}^{a} e^{ipx} f(x)\, dx + 0$$

$$= \int_{-a}^{a} e^{ipx}\, dx$$

$$= \left(\frac{e^{ipx}}{ip}\right)_{-a}^{a} = \frac{e^{ipa} - e^{-ipa}}{ip} = \frac{2i \sin pa}{ip} = \frac{2 \sin pa}{p}$$

2nd Part :-

The Inverse Fourier transform of the function $F(P)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx} F(P)\, dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2 \sin ap}{p} e^{-ipx}\, dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2 \sin ap}{p} \cos px\, dp - \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{2 \sin ap}{p} \sin px\, dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2 \sin ap}{p} \cos px\, dp - 0 \quad (\because \text{2nd integrand is odd})$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ap}{p} \cos px\, dp$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin ap \cos px}{p}\, dp = \pi f(x)$$

$$= \begin{cases} \pi & |x| < a \\ 0 & |x| > a \end{cases}$$

3rd Part :-

We have $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ap \cos px}{p}\, dp = \pi$

Put $x=0$, $a=1$

$$\int_{-\infty}^{\infty} \frac{\sin p}{p}\, dp = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin p}{p}\, dp = \pi$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin p}{p}\, dp = \frac{\pi}{2}$$

4. Find the Fourier transform of the function

$$f(x) = \begin{cases} \cos x & |x| \le a \\ 0 & |x| > a \end{cases}$$

Sol :- Given $f(x) = \begin{cases} \cos x & |x| \le a \\ 0 & |x| > a \end{cases}$

The Fourier transform of the given function is

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{ipx}\, dx$$

$$= \int_{-\infty}^{-a} f(x) e^{ipx}\, dx + \int_{-a}^{a} f(x) e^{ipx}\, dx + \int_{a}^{\infty} f(x) e^{ipx}\, dx$$

$$= 0 + \int_{-a}^{a} e^{ipx} \cos x\, dx + 0$$

$$= \left(\frac{e^{ipx}}{1 + i^2p^2}\left[ip \cos x + \sin x\right]\right)_{-a}^{a}$$

$$= \left[\frac{e^{ipa}}{1-p^2}(ip\cos a + \sin a) - \frac{e^{-ipa}}{1-p^2}(ip\cos a - \sin a)\right]$$

$$= \frac{1}{1-p^2}\left[(e^{ipa} - e^{-ipa})\, ip\cos a + (e^{ipa} + e^{-ipa}) \sin a\right]$$

$$= \frac{1}{1-p^2}\left[\rho i \sin \rho a \; i p \cos a + \rho \cos \rho a \sin a \right]$$

$$= \frac{2}{1-p^2}\left[\cos pa \sin a - \sin pa \cos a \right]$$

5. Show that the Fourier transform of $f(x) = \begin{cases} a-|x| & \text{for } |x|<a \\ 0 & |x|>a \end{cases}$ is $\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)$

Sol :- Given $f(x) = \begin{cases} a-|x| & |x|<a \\ 0 & |x|>a \end{cases}$

The Fourier transform of given function is

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} f(x) e^{isx} dx + \int_{-a}^{a} f(x) e^{isx} dx + \int_{a}^{\infty} f(x) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} a\, e^{isx} dx - \int_{-a}^{a} |x| e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[a\int_{-a}^{a} e^{isx} dx - \left[\int_{-a}^{0} |x| e^{isx} dx + \int_{0}^{a} |x| e^{isx} dx \right] \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[a\int_{-a}^{a} e^{isx} dx - \left[\int_{-a}^{0} -x e^{isx} dx + \int_{0}^{a} x e^{isx} dx \right] \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[a\int_{-a}^{a} e^{isx} dx + \int_{-a}^{0} x e^{isx} dx - \int_{0}^{a} x e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[I_1 + I_2 - I_3 \right] \longrightarrow (1)$$

Evaluation of I_1 :-

$$I_1 = a\int_{-a}^{a} e^{isx} dx = a\left[\frac{e^{isx}}{is} \right]_{-a}^{a}$$

$$= \frac{a}{is}\left[e^{isa} - e^{-isa} \right]$$

Evaluation of I_2 :-

$$I_2 = \int_{-a}^{0} x\, e^{isx}\, dx$$

$$= \left[x\frac{e^{isx}}{is} - (1)\frac{e^{isx}}{i^2 s^2} \right]_{-a}^{0}$$

$$= \left(x\frac{e^{isx}}{is} + \frac{e^{isx}}{s^2} \right)_{-a}^{0}$$

$$= 0 + \frac{1}{s^2} - \left(-a\frac{e^{-isa}}{is} + \frac{e^{-isa}}{s^2} \right)$$

$$= \frac{1}{s^2} + a\frac{e^{-ias}}{is} - \frac{e^{-ias}}{s^2}$$

$$= a\frac{e^{-ias}}{is} + \frac{1}{s^2}\left[1 - e^{-ias} \right]$$

Evaluation of I_3 :-

$$I_3 = \int_{0}^{a} x\, e^{isx}\, dx$$

$= \frac{1}{1-p^2}\left[q\,i \sin pa \; p\cos a + q\cos pa \sin a \right]$

$= \frac{q}{1-p^2}\left[\cos pa \sin a - \sin pa \cos a \right]$

5. Show that the Fourier transform of $f(x) = \begin{cases} a-|x| & \text{for } |x|<a \\ 0 & |x|>a \end{cases}$ is $\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)$

Sol :- Given $f(x) = \begin{cases} a-|x| & |x|<a \\ 0 & |x|>a \end{cases}$

The Fourier transform of given function is

$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} f(x) e^{isx} dx + \int_{-a}^{a} f(x) e^{isx} dx + \int_{a}^{\infty} f(x) e^{isx} dx \right]$

$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)\, e^{isx}\, dx$

$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} a\, e^{isx} dx - \int_{-a}^{a} |x| e^{isx} dx \right]$

$= \frac{1}{\sqrt{2\pi}} \left[a\int_{-a}^{a} e^{isx} dx - \left[\int_{-a}^{0} |x| e^{isx} dx + \int_{0}^{a} |x| e^{isx} dx \right] \right]$

$= \frac{1}{\sqrt{2\pi}} \left[a\int_{-a}^{a} e^{isx} dx - \left[\int_{-a}^{0} -x e^{isx} dx + \int_{0}^{a} x e^{isx} dx \right] \right]$

$= \frac{1}{\sqrt{2\pi}} \left[a\int_{-a}^{a} e^{isx} dx + \int_{-a}^{0} x e^{isx} dx - \int_{0}^{a} x e^{isx} dx \right]$

$= \frac{1}{\sqrt{2\pi}} \left[I_1 + I_2 - I_3 \right] \longrightarrow (1)$

Evaluation of I_1 :-

$I_1 = a\int_{-a}^{a} e^{isx} dx = a\left[\frac{e^{isx}}{is} \right]_{-a}^{a}$

$= \frac{a}{is}\left[e^{isa} - e^{-isa} \right]$

Evaluation of I_2 :-

$I_2 = \int_{-a}^{0} x\, e^{isx}\, dx$

$= \left[x \frac{e^{isx}}{is} - (1)\frac{e^{isx}}{i^2 s^2} \right]_{-a}^{0}$

$= \left(x\frac{e^{isx}}{is} + \frac{e^{isx}}{s^2} \right)_{-a}^{0}$

$= 0 + \frac{1}{s^2} - \left(-a\frac{e^{-isa}}{is} + \frac{e^{-isa}}{s^2} \right)$

$= \frac{1}{s^2} + a\frac{e^{-ias}}{is} - \frac{e^{-ias}}{s^2}$

$= a\frac{e^{-ias}}{is} + \frac{1}{s^2}\left[1 - e^{-ias} \right]$

Evaluation of I_3 :-

$I_3 = \int_{0}^{a} x\, e^{isx}\, dx$

$$= \left[x\frac{e^{isx}}{is} - (1)\frac{e^{isx}}{i^2 s^2} \right]_0^a$$

$$= \left[x\frac{e^{isx}}{is} + \frac{e^{isx}}{s^2} \right]_0^a = a\frac{e^{ias}}{is} + \frac{e^{ias}}{s^2} - \frac{1}{s^2}$$

$$= a\frac{e^{ias}}{is} + \frac{1}{s^2}\left[e^{ias} - 1\right]$$

Substitute I_1, I_2, I_3 in Equ ① we get

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\left[\cancel{\frac{a}{is}e^{-ias}} - \cancel{\frac{a}{is}e^{-ias}} + \cancel{a\frac{e^{-ias}}{is}} + \frac{1}{s^2} - \frac{e^{-ias}}{s^2} - \cancel{a\frac{e^{ias}}{is}} - \frac{1}{s^2}e^{ias} + \frac{1}{s^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{s^2} - \frac{1}{s^2}\left[e^{-ias} + e^{ias}\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{s^2} - 2\frac{\cos as}{s^2} \right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[\frac{1}{s^2} - \frac{\cos as}{s^2} \right] = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos as}{s^2}\right]$$

6. Find the Fourier transform of

$$f(x) = \begin{cases} 1-x^2 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

Sol:- $F\{f(x)\} = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$

$$= \int_{-\infty}^{-1} f(x)e^{ipx}dx + \int_{-1}^{1} f(x)e^{ipx}dx + \int_{1}^{\infty} f(x)e^{ipx}dx$$

$$= 0 + \int_{-1}^{1}(1-x^2)e^{ipx}dx + 0$$

$$= \left[(1-x^2)\frac{e^{ipx}}{ip} - (-2x)\frac{e^{ipx}}{i^2p^2} + (-2)\frac{e^{ipx}}{i^3p^3} \right]_{-1}^{1}$$

$$= \left[0 + \frac{2e^{ip}}{p^2} + \frac{2e^{ip}}{ip^3} - \left(0 + \frac{2e^{-ip}}{p^2} + 2\frac{e^{-ip}}{ip^3}\right)\right.$$

$$= -\frac{2e^{+ip}}{p^2} + \frac{2e^{ip}}{ip^3} - \frac{2e^{-ip}}{p^2} - \frac{2e^{-ip}}{ip^3}$$

$$= -\frac{2}{p^2}\left[e^{ip} + e^{-ip}\right] + \frac{2}{ip^3}\left[e^{ip} - e^{-ip}\right]$$

$$= -\frac{2}{p^2}(2\cos p) + \frac{2}{\cancel{i}p^3}(2\cancel{i}\sin p)$$

$$= -\frac{4}{p^2}\cos p + \frac{4}{p^3}\sin p$$

$$= \frac{4}{p^3}\left[\sin p - p\cos p\right]$$

7. S.T the Fourier transform of $f(x) = \begin{cases} a-|x| & |x|<a \\ 0 & |x|>a>0 \end{cases}$

is $\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos as}{s^2}\right]$ Hence deduce that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

Continuation of 5th problem

we have $F\{f(x)\} = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos as}{s^2}\right]$

Deduction:-

The inverse Fourier transform is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2/\pi} \left[\frac{1-\cos as}{s^2}\right](\cos sx - i\sin sx) \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \left[\int_{-\infty}^{\infty} \frac{1-\cos as}{s^2} \cos sx \, ds - i \int_{-\infty}^{\infty} \frac{1-\cos as}{s^2} \sin sx \, ds\right]$$

$$= \frac{1}{\pi} \left[\int_{-\infty}^{\infty} \frac{1-\cos as}{s^2} \cos sx \, ds\right] - 0 \quad (\because 2^{nd} \text{ integrand is } \cancel{\text{even}} \text{ odd})$$

$$\therefore \int_{-\infty}^{\infty} \frac{1-\cos as}{s^2} \cos sx \, ds = \pi f(x)$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{1-\cos as}{s^2} \cos sx \, ds = \pi f(x)$$

$$\Rightarrow \int_{0}^{\infty} \frac{1-\cos as}{s^2} \cos sx \, ds = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \frac{\pi}{2}(a-|x|) & |x|<a \\ 0 & |x|>a \end{cases}$$

Put $a=2$ and $x=0$

$$\therefore \int_{0}^{\infty} \frac{1-\cos 2s}{s^2} \cancel{\cos sx} \, ds = \frac{\pi}{2}(2-0)$$

$$\therefore \int_{0}^{\infty} \frac{2\sin^2 s}{s^2} \, ds = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = \pi \Rightarrow \int_{0}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{2}$$

(or) $$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

Q1. Find the Fourier sine and cosine transform of the function $\frac{e^{-ax}}{x}$ and deduce that $\int_{0}^{\infty} \frac{e^{-ax}-e^{-bx}}{x} \sin sx \, dx = \tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)$.

Sol:- Given $f(x) = \frac{e^{-ax}}{x}$

i) sine transform :- The Fourier sine transform of the function $f(x)$ is $F_s\{f(x)\} = \int_{0}^{\infty} f(t) \sin pt \, dt$

(replacing t by x)

$$F_s\left\{\frac{e^{-ax}}{x}\right\} = \int_{0}^{\infty} \frac{e^{-ax}}{x} \sin px \, dx$$

D.w.r.to p

$$\frac{d}{dp} F_s\left\{\frac{e^{-ax}}{x}\right\} = \int_{0}^{\infty} \frac{e^{-ax}}{\cancel{x}} \cos px \cdot \cancel{x} \, dx$$

$$= \frac{a}{a^2+p^2}$$

Integrate w.r.to p

$$F_s\left\{\frac{e^{-ax}}{x}\right\} = \int \frac{a}{a^2+p^2} \, dp = \tan^{-1}\frac{x}{a}$$

ii) cosine transform :-

$$F_c\{f(x)\} = \int_{0}^{\infty} f(t) \cos pt \, dt$$

$$F_c\left\{\frac{e^{-ax}}{x}\right\} = \int_{0}^{\infty} \frac{e^{-ax}}{x} \cos px \, dx$$

D. w.r.to p

$$\frac{d}{dp} F_c\left\{\frac{e^{-ax}}{x}\right\} = -\int_0^\infty \frac{e^{-ax}}{\cancel{x}} \sin px \cdot \cancel{x}\, dx$$

$$= -\frac{p}{a^2+p^2}$$

Integrate w.r.to p

$$F_c\left\{\frac{e^{-ax}}{x}\right\} = -\int \frac{p}{a^2+p^2}\,dp = -\frac{1}{2}\int \frac{2p}{a^2+p^2}$$

$$= -\frac{1}{2}\log(a^2+p^2)$$

Deduction :-

W.K.T the fourier sine transform of the function

$f(x)$ is

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin px\, dx$$

Let $f(x) = \frac{e^{-ax} - e^{-bx}}{x}$ replacing p by s

$$\therefore F_s\left\{\frac{e^{-ax}-e^{-bx}}{x}\right\} = \int_0^\infty \frac{e^{-ax}-e^{-bx}}{x} \sin sx\, dx$$

$$\Rightarrow \int_0^\infty \frac{e^{-ax}-e^{-bx}}{x} \sin sx\, dx = \int_0^\infty \frac{e^{-ax}}{x}\sin sx\, dx - \int_0^\infty \frac{e^{-bx}}{x}\sin sx\, dx$$

$$= \tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)$$

2. Find the inverse Fourier Cosine transform of

$$F_c\{p\} = \begin{cases} \frac{1}{2a}(a - p/2) & p \le 2a \\ 0 & p \ge 2a \end{cases}$$

Sol:- The Inverse Fourier Cosine transform of the function

$F_c(p)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(p)\cos px\, dp$$

$$= \frac{2}{\pi}\left[\int_0^{2a} F_c(p)\cos px\, dp + \int_{2a}^\infty F_c(p)\cos px\, dp\right]$$

$$= \frac{\cancel{2}}{\pi}\left[\int_0^{2a} \frac{1}{\cancel{2}a}\left(a - \frac{p}{2}\right)\cos px\, dp\right]$$

$$= \frac{1}{a\pi}\left[\left(a-\frac{p}{2}\right)\frac{\sin px}{x} - \left(-\frac{1}{2}\right)\left(-\frac{\cos px}{x^2}\right)\right]_0^{2a}$$

$$= \frac{1}{a\pi}\left[-\frac{1}{2}\frac{\cos px}{x^2}\right]_0^{2a} = -\frac{1}{2a\pi}\left[\frac{\cos 2ax}{x^2} - \frac{1}{x^2}\right]$$

$$= \frac{1}{2a\pi}\left[\frac{1-\cos 2ax}{x^2}\right]$$

$$= \frac{\cancel{2}\sin^2 ax}{\cancel{2}a\pi x^2} = \frac{\sin^2 ax}{a\pi x^2}$$

3. Find the Fourier transform of the function e^{-ax}, $x > 0$ and hence deduce the inversion formulae (or) deduce the integrals i) $\int_0^\infty \frac{\cos px}{a^2+p^2}\,dp$ ii) $\int_0^\infty \frac{p\sin px}{a^2+p^2}\,dp$

Sol :- Given $f(x) = e^{-ax}$

i) The Cosine transform :-

$$F_c\{f(x)\} = \int_0^\infty f(x) \cos px \, dx$$

$$F_c(p) = \int_0^\infty e^{-ax} \cos px \, dx$$

$$= \frac{a}{a^2+p^2}$$

ii) The sine transform :-

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin px \, dx$$

$$F_s(p) = \int_0^\infty e^{-ax} \sin px \, dx$$

$$= \frac{p}{a^2+p^2}$$

Deduction :-

i) The inverse fourier cosine transform is

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(p) \cos px \, dp$$

$$= \frac{2}{\pi} \int_0^\infty \frac{a}{a^2+p^2} \cos px \, dp$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{a}{a^2+p^2} \cos px \, dp$$

$$\Rightarrow \int_0^\infty \frac{\cos px}{a^2+p^2} dp = \frac{\pi}{2a} f(x) = \frac{\pi}{2a} e^{-ax}$$

ii) The Inverse fourier sine transform is

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(p) \sin px \, dp$$

$$= \frac{2}{\pi} \int_0^\infty \frac{p}{a^2+p^2} \sin px \, dp$$

$$\Rightarrow \int_0^\infty \frac{p \sin px}{a^2+p^2} dp = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-ax}$$

4. Find the Fourier sine and Cosine transform of $2e^{-5x} + 5e^{-2x}$.

Sol :- $f(x) = 2e^{-5x} + 5e^{-2x}$

i) The Fourier sine transform of the function $f(x)$ is

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin px \, dx = \int_0^\infty (2e^{-5x} + 5e^{-2x}) \sin px \, dx$$

$$= 2\int_0^\infty e^{-5x} \sin px \, dx + 5\int_0^\infty e^{-2x} \sin px \, dx$$

$$= \frac{2p}{p^2+25} + \frac{5p}{p^2+4}$$

ii) The Fourier Cosine transform of the function $f(x)$ is

$$F_c\{f(x)\} = \int_0^\infty f(x) \cos px \, dx = \int_0^\infty (2e^{-5x}+5e^{-2x}) \cos px \, dx$$

$$= 2\int_0^\infty e^{-5x} \cos px \, dx + 5\int_0^\infty e^{-2x} \cos px \, dx$$

$$= \frac{10}{p^2+25} + \frac{10}{p^2+4}$$

Properties :-

1. If $F(P)$ and $G(P)$ are the Fourier transforms of $f(x)$ & $g(x)$ then $F\{af(x)+bg(x)\} = a F\{P\} + b G(P)$

This is the Linearity property

2. Change of scale property :-

If $F(P)$ is the Complex Fourier transform of $f(x)$ then Fourier transform of $f(ax)$ is $\frac{1}{a} F\left(\frac{P}{a}\right)$

i.e., $F\{f(ax)\} = \frac{1}{a} F\left(\frac{P}{a}\right)$

If $F_s(P)$ and $F_c(P)$ are the Fourier sine & Cosine transforms of $f(x)$ respectively then

$F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{P}{a}\right)$

$F_c\{f(ax)\} = \frac{1}{a} F_c\left(\frac{P}{a}\right)$

3. Shifting Property :- If $F(P)$ is the Complex Fourier transform of $f(x)$ then

$F\{f(x-a)\} = e^{iPa} F(P)$

|||ly

$F_s\{f(x-a)\} = e^{iPa} F_s(P)$

$F_c\{f(x-a)\} = e^{iPa} F_c(P)$

Proof :-

we have $F(P) = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$

$\therefore F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{ipx} f(x-a)\,dx$ put $x-a=t$, $x = a+t$

$\therefore \quad = \int_{-\infty}^{\infty} e^{ip(a+t)} f(t)\,dt$

$= e^{ipa} \int_{-\infty}^{\infty} e^{ipt} f(t)\,dt$

$= e^{ipa} F(P)$

4. Modulation theorem :- If $F(P)$ is the Fourier transform of $f(x)$ then the Fourier transform of $f(x)\cos ax$ is $\frac{1}{2}[F(P+a)+F(P-a)]$

Proof :- we have $F(P) = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$

$\therefore F\{f(x)\cos ax\} = \int_{-\infty}^{\infty} e^{ipx} f(x) \cos ax\,dx$

$= \int_{-\infty}^{\infty} e^{ipx} f(x) \left[\frac{e^{iax}+e^{-iax}}{2}\right] dx$

$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(p+a)x} f(x)\,dx + \int_{-\infty}^{\infty} e^{i(p-a)x} f(x)\,dx\right]$

$= \frac{1}{2}[F(P+a) + F(P-a)]$

|||ly $F_s\{f(x)\cos ax\} = \frac{1}{2}[F_s(P+a)+F_s(P-a)]$

~~$F_c\{f(x)\sin ax\} = \frac{1}{2}[F_s(P+a) - F_s(P-a)]$~~

5. $F\{x^n f(x)\} = (-i)^n \frac{d^n}{dp^n} F(p)$

Sol:- By definition

$F(p) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{ipx} dx$

$\frac{d}{dp}[F(p)] = \frac{d}{dp}\int_{-\infty}^{\infty} f(x) e^{ipx} dx$

$= \int_{-\infty}^{\infty} f(x) \frac{d}{dp} e^{ipx} dx$

$= \int_{-\infty}^{\infty} f(x) \; e^{ipx} (ix) \; dx$

$= i \int_{-\infty}^{\infty} x \; f(x) \; e^{ipx} \; dx$

$= i \; F\{x f(x)\}$

$\Rightarrow F\{x f(x)\} = -i \frac{d}{dp}[F(p)$

$^{||}$ly $\frac{d^2}{dp^2} F(p) = \frac{d}{dp} \int_{-\infty}^{\infty} f(x)(ix) e^{ipx} dx$

$= i^2 \int_{-\infty}^{\infty} x^2 f(x) e^{ipx} dx$

$= i^2 \; F\{x^2 f(x)\}$

$\Rightarrow F\{x^2 f(x)\} = \frac{1}{i^2} \frac{d^2}{dp^2} F(p)$

$= (-i)^2 \frac{d^2}{dp^2} F(p)$

Problem

5. $F_s\{x f(x)\} = -\frac{d}{dp} F_c\{f(x)\}$

$F_c\{x \cdot f(x)\} = \frac{d}{dp} F_s\{f(x)\}$

Proof :-

i) W.K.T The fourier cosine transform of $f(x)$ is

$F_c(p)) = \int_0^{\infty} f(x) \cos px \, dx$

D. W. r. to p

$\frac{d}{dp} F_c(p) = \int_0^{\infty} f(x)(-x \sin px) \, dx$

$= -\int_0^{\infty} x \, f(x) \sin px \, dx = -F_s\{x f(x)\}$

$\Rightarrow F_s\{x f(x)\} = -\frac{d}{dp} F_c(p) = -\frac{d}{dp} F_c\{f(x)\}$

ii) W.K.T The Fourier sine transform of $f(x)$ is

$F_s(p) = \int_0^{\infty} f(x) \sin px \, dx$

$\frac{d}{dp} F_s(p) = \int_0^{\infty} x \, f(x) \cos px \; dx$

$\Rightarrow \frac{d}{dp} F_s(p) = F_c\{x f(x)\}$

$\Rightarrow F_c\{x f(x)\} = \frac{d}{dp} F_s(p) = \frac{d}{dp} F_s\{f(x)\}$

6. Find Fourier sine and Cosine transform of xe^{-ax}

Sol :- let $f(x) = e^{-ax}$

i) Four sine transform :-

W.K.T

$$F_s\{x f(x)\} = -\frac{d}{dp} F_c(p)$$

$$= -\frac{d}{dp} F_c\{f(x)\}$$

$$\therefore F_s\{xe^{-ax}\} = -\frac{d}{dp} F_c\{e^{-ax}\}$$

$$= -\frac{d}{dp}\int_0^{\infty} e^{-ax}\cos px\,dx$$

$$= -\frac{d}{dp}\left[\frac{a}{a^2+p^2}\right]$$

$$= -a\frac{d}{dp}\left[\frac{1}{a^2+p^2}\right]$$

$$= -a\left[\frac{a^2+p^2(0) - (1)(2p)}{(a^2+p^2)^2}\right]$$

$$= \frac{2ap}{(a^2+p^2)^2}$$

ii) Fourier Cosine transform :-

W.K.T

$$F_c\{x f(x)\} = \frac{d}{dp} F_s(p)$$

$$= \frac{d}{dp}\int_0^{\infty} f(x)\sin px\,dx$$

$$= \frac{d}{dp}\int_0^{\infty} e^{-ax}\sin px\,dx$$

$$= \frac{d}{dp}\left[\frac{p}{a^2+p^2}\right]$$

$$= \frac{(a^2+p^2)(1) - p(2p)}{(a^2+p^2)^2} = \frac{a^2+p^2-2p^2}{(a^2+p^2)^2} = \frac{a^2-p^2}{(a^2+p^2)^2}$$

Finite Fourier transforms :-

1. Sine transform :- The finite fourier sine transform of the function $f(x)$ in $0<x<l$ is given by

$$F_s\{f(x)\} = F_s(n) = \int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$ and

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{l}$$ is Called Inverse finite Fourier transform of $F_s(n)$.

2. Cosine transform :- The finite fourier cosine transform of the function $f(x)$ in $0<x<l$ is given by

$$F_c\{f(x)\} = F_c(n) = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$ and $$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{l}$$

~~$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{l}$~~ is Called Inverse ~~Cosine~~ finite fourier Cosine transform of $F_c(n)$.

1. Find the finite Fourier cosine transform of function $f(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}$ in $0 < x < \pi$

Sol :- Given $f(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}$

The finite Fourier cosine transform of the function $f(x)$ is

$$F_c(n) = \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx$$

here $l = \pi$

$$\therefore F_c(n) = \int_0^{\pi} \left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right) \cos nx \, dx$$

$$= \left[\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-1 + \frac{2x}{2\pi}\right)\left(\frac{-\cos nx}{n^2}\right) + \left(\frac{1}{\pi}\right)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

The 1^{st} and 3^{rd} ~~enter~~ terms of R.H.S vanishes at both upper and lower limits

$$F_c(n) = \left[\left(-1 + \frac{x}{\pi}\right)\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \left[0 - \left(-\frac{1}{n^2}\right)\right]$$

$$= \frac{1}{n^2}, \quad n > 0$$

2. Find the finite Fourier sine transform of the function $f(x) = x^2$, $0 < x < l$.

Sol :- The finite Fourier sine transform of the function $f(x)$ is

$$F_s(n) = \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \int_0^l x^2 \sin \frac{n\pi x}{l} \, dx$$

$$= \left[x^2 \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (2x)\left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (2)\frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$= \left(\frac{l^3}{n\pi}(-\cos n\pi) + \frac{2l^3}{n^3\pi^3}\cos n\pi\right) - \left(\frac{2l^3}{n^3\pi^3}\cos 0\right)$$

$$= \frac{l^3}{n\pi}\left(-(-1)^n\right) + \frac{2l^3}{n^3\pi^3}\left[(-1)^n - 1\right]$$

3. Find the Inverse finite sine ~~transform of~~ $F_s(n) = \frac{16(-1)^{n-1}}{n^3}$, $0 < x < 8$

Sol :- Given

$$F_s(n) = \frac{16(-1)^{n-1}}{n^3}, \quad 0 < x < 8$$

The Inverse finite sine transform of $F_s(n)$ is

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{l} \qquad (\because l = 8)$$

$$= \frac{2}{8}\sum_{n=1}^{\infty} \frac{16(-1)^{n-1}}{n^3} \sin \frac{n\pi x}{8}$$

$$= 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin \frac{n\pi x}{8}$$

$$= 4\left[\frac{1}{1^3}\sin\frac{\pi x}{8} - \frac{1}{2^3}\sin\frac{\pi x}{4} + \frac{1}{3^3}\sin\frac{3\pi x}{8} - \frac{1}{4^3}\sin\frac{\pi x}{2} + \cdots\right]$$

4. Find the Inverse finite cosine transform of

$F_c(n) = \dfrac{\cos\frac{2n\pi}{3}}{(2n+1)^2}$, $0<x<4$.

sol:- W.K.T the Inverse finite cosine transform of

$$\cancel{f(x) = \frac{2}{l} + \sum_{n=1}^{\infty} F_c(n) \cos\frac{n\pi x}{l}}$$

$$\cancel{= \frac{2}{4} + \sum_{n=1}^{\infty} \frac{\cos\frac{2n\pi}{3}}{(2n+1)^2} \cos\frac{n\pi x}{4}}$$

$$\cancel{= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n\pi}{3}\right)}{(2n+1)^2} \cos\frac{n\pi x}{4}}$$

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c(n) \cos\frac{n\pi x}{l}$$

$$= \frac{1}{4}(1) + \frac{2}{4} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n\pi}{3}\right)}{(2n+1)^2} \cos\frac{n\pi x}{4} \quad (\because l = 4)$$

$$= \frac{1}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2n\pi}{3}\right)}{(2n+1)^2} \cos\frac{n\pi x}{4}$$

$$= \frac{1}{4} + \frac{1}{2}\left[\frac{\cos\frac{2\pi}{3}}{9} \cos\frac{\pi x}{4} + \frac{\cos\frac{4\pi}{3}}{25} \cos\frac{2\pi x}{4} + \cdots\right]$$

5. Find the Finite Fourier sine and Cosine transforms of

$$f(x) = \begin{cases} 1 & \text{if } 0<x<\pi/2 \\ -1 & \text{if } \pi/2 < x < \pi \end{cases}$$

sol:- i) Finite fourier sine transform of $f(x)$ is given by

$$F_s(n) = \int_0^l f(x) \sin\frac{n\pi x}{l}\,dx = \int_0^{\pi} f(x) \sin nx\,dx \quad (\because l = \pi)$$

$$= \int_0^{\pi/2} 1 \cdot \sin nx\,dx + \int_{\pi/2}^{\pi} -1 \sin nx\,dx$$

$$= \left(-\frac{\cos nx}{n}\right)_0^{\pi/2} + \left(\frac{\cos nx}{n}\right)_{\pi/2}^{\pi}$$

$$= -\frac{\cos\frac{n\pi}{2}}{n} - \left(-\frac{\cos 0}{n}\right) + \frac{\cos n\pi}{n} - \frac{\cos\frac{n\pi}{2}}{n}$$

$$= -\frac{2\cos\frac{n\pi}{2}}{n} + \frac{1}{n} + \frac{\cos n\pi}{n}$$

$$= \frac{1}{n}\left[1 - 2\cos\frac{n\pi}{2} + \cos n\pi\right], \quad n = 1, 2, 3 \ldots$$

ii) Finite Fourier cosine transform of $f(x)$ is given by

$$F_c(n) = \int_0^l f(x) \cos\frac{n\pi x}{l}\,dx = \int_0^{\pi} f(x) \cos nx\,dx$$

$$= \int_0^{\pi/2} 1 \cos nx\,dx + \int_{\pi/2}^{\pi} -1 \cos nx\,dx$$

$$= \left(\frac{\sin nx}{n}\right)_0^{\pi/2} - \left(\frac{\sin nx}{n}\right)_{\pi/2}^{\pi}$$

$$= \frac{\sin\frac{n\pi}{2}}{n} - \frac{\sin 0}{n} - \left(\frac{\sin n\pi}{n} - \frac{\sin\frac{n\pi}{2}}{n}\right)$$

$$= \frac{\sin\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n} = \frac{1}{n}\left(2\sin\frac{n\pi}{2}\right), \quad n = 1, 2, 3, \ldots$$

6. Find the Finite Fourier sine & Cosine transform of e^{ax} in $(0, l)$

Parseval's Identity for Fourier-transforms :-

If $F(P)$ and $G(P)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively

i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(P)\,\overline{G(P)}\,dp = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx$

ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(P)|^2\,dp = \int_{-\infty}^{\infty} |f(x)|^2\,dx$

The Parseval's Identity for Fourier sine and cosine transforms are as follows

1. $\frac{2}{\pi}\int_0^{\infty} F_c(P)\,G_c(P)\,dp = \int_0^{\infty} f(x)\,g(x)\,dx$
2. $\frac{2}{\pi}\int_0^{\infty} |F_c(P)|^2\,dp = \int_0^{\infty} |f(x)|^2\,dx$
3. $\frac{2}{\pi}\int_0^{\infty} F_s(P)\,G_s(P)\,dp = \int_0^{\infty} f(x)\,g(x)\,dx$
4. $\frac{2}{\pi}\int_0^{\infty} |F_s(P)|^2\,dP = \int_0^{\infty} |f(x)|^2\,dx$

1. Find the Fourier transform of $f(x)=\begin{cases}1 & |x|<a\\ 0 & |x|>a\end{cases}$ and using Parseval's Identity prove that $\int_0^{\infty}\frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$.

Sol:- Given $f(x)=\begin{cases}1 & |x|<a\\ 0 & |x|>a\end{cases}$

The Fourier transform of $f(x)$ is given by

$$F\{f(x)\} = F(P) = \int_{-\infty}^{\infty} e^{iPx} f(x)\,dx$$

$$= \int_{-\infty}^{-a} f(x)e^{iPx}dx + \int_{-a}^{a} f(x)e^{iPx}dx + \int_a^{\infty} f(x)e^{iPx}dx$$

$$= 0 + \int_{-a}^{a} e^{iPx}dx + 0 = \left(\frac{e^{iPx}}{iP}\right)_{-a}^{a} = \frac{1}{iP}\left[e^{iPa} - e^{-iPa}\right]$$

$$= \frac{1}{\cancel{i}P}\,2\cancel{i}\sin aP$$

$$= \frac{2\sin aP}{P}$$

From the Parseval's Identity of Fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(P)|^2\,dP = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(2\,\frac{\sin aP}{P}\right)^2 dP = \int_{-a}^{a} 1\,dx$$

$$\frac{2}{2\pi}\int_0^{\infty}\left(2\,\frac{\sin aP}{P}\right)^2 dp = (x)_{-a}^{a}$$

$$\frac{4}{\pi}\int_0^{\infty}\frac{\sin^2 aP}{P^2}\,dp = 2a$$

$$\int_0^{\infty}\frac{\sin^2 aP}{P^2}\,dp = \cancel{2}a \times \frac{\pi}{\cancel{4}_2} = \frac{a\pi}{2}$$

Put $a=1$ and replacing P by t

$$\int_0^{\infty}\frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$$ //

2) Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & |x|>1 \end{cases}$

and using Parseval's Identity S.T $\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}$.

Sol:- Given $f(x) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & |x|>1 \end{cases}$

W.K.T the Fourier transform of the given function

is $F(P) = \frac{4}{p^3}(\sin p - p\cos p)$

From the Parseval's Identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(P)|^2 \, dp = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\frac{4}{p^3}(\sin p - p\cos p)\right]^2 dp = \int_{-\infty}^{-1} |f(x)|^2 dx + \int_{-1}^{1} |f(x)|^2 dx + \int_{1}^{\infty} |f(x)|^2 dx$$

$$\frac{16}{2\pi}\int_{0}^{\infty} \left[\frac{4}{p^3}(\sin p - p\cos p)\right]^2 dp = 0 + \int_{-1}^{1} (1-x^2)^2 dx + 0$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin p - p\cos p}{p^3}\right)^2 dp = \int_{-1}^{1} 1 + x^4 - 2x^2 \, dx$$

$$= \cancel{\left(1+\frac{x^5}{5} - \frac{2x^3}{3}\right)_{-1}^{1}}$$

$$= \cancel{1 + \frac{1}{5} - \frac{2}{3} - \left(1 - \frac{1}{5} + \frac{2}{3}\right)}$$

$$= \cancel{\frac{2}{5} - \frac{4}{3} = \frac{6-}{15}}$$

$$= \left(x + \frac{x^5}{5} - 2\frac{x^3}{3}\right)_{-1}^{1} = \left(1 + \frac{1}{5} - \frac{2}{3} - \left(-1 - \frac{1}{5} + \frac{2}{3}\right)\right)$$

$$= 1 + \frac{1}{5} - \frac{2}{3} + 1 + \frac{1}{5} - \frac{2}{3}$$

$$= 2 + \frac{2}{5} - \frac{4}{3} = \frac{16}{15}$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin p - p\cos p}{p^3}\right)^2 dp = \frac{16}{15} \times \frac{\pi}{16} = \frac{\pi}{15}$$

replacing p by x

$$\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}$$

3. Using Parseval's Identity Evaluate

i) $\int_0^\infty \frac{x^2}{(a^2+x^2)^2} dx$ ii) $\int_0^\infty \frac{dx}{(a^2+x^2)^2}$

Sol:- Let $f(x) = e^{-ax}$

The Fourier sine transform of the function $f(x)$ is

$$F_s\{f(x)\} = F_s(P) = \int_0^\infty f(x)\sin px \, dx$$

$$= \int_0^\infty e^{-ax}\sin px \, dx = \frac{p}{a^2+p^2}$$

The Fourier cosine transform of the function $f(x)$ is

$$F_c\{f(x)\} = F_c(P) = \int_0^\infty f(x)\cos px \, dx$$

$$= \int_0^\infty e^{-ax}\cos px \, dx = \frac{a}{a^2+p^2}$$

From the Parseval's Identity

i) $\frac{2}{\pi}\int_0^\infty |F_s(P)|^2 dp = \int_0^\infty |f(x)|^2 dx$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{p}{a^2+p^2}\right)^2 dp = \int_0^\infty (e^{-ax})^2 dx$$

$$= \int_0^\infty e^{-2ax}\,dx = \left(\frac{e^{-2ax}}{-2a}\right)_0^\infty$$

$$= 0 - \left(\frac{1}{-2a}\right) = \frac{1}{2a}$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{p^2}{(a^2+p^2)^2}\,dp = \frac{1}{2a}$$

$$\Rightarrow \int_0^\infty \frac{p^2}{(a^2+p^2)^2}\,dp = \frac{1}{2a}\times\frac{\pi}{2} = \frac{\pi}{4a}$$

replacing p by x

$$\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx = \frac{\pi}{4a}$$

ii) $$\frac{2}{\pi}\int_0^\infty |F_c(p)|^2\,dp = \int_0^\infty |f(x)|^2\,dx$$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{a}{a^2+p^2}\right)^2 dp = \int_0^\infty (e^{-ax})^2\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{a^2}{(a^2+p^2)^2}\,dp = \int_0^\infty e^{-2ax}\,dx$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{dp}{(a^2+p^2)^2} = \left(\frac{e^{-2ax}}{-2a}\right)_0^\infty = \frac{1}{2a}$$

replacing p by x

$$\int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{1}{2a}\times\frac{\pi}{2a^2} = \frac{\pi}{4a^3}$$

$$=$$

4. using Parseval's Identity S.T $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$

Sol:- Let $f(x) = e^{-ax}$ & $g(x) = e^{-bx}$

The Fourier Cosine transform of the function f(x) is

$$F_c\{f(x)\} = F_c(p) = \int_0^\infty f(x)\cos px\,dx$$

$$= \int_0^\infty e^{-ax}\cos px\,dx = \frac{a}{a^2+p^2}$$

The Fourier Cosine transform of the function g(x) is

$$F_c\{g(x)\} = G_c(p) = \int_0^\infty g(x)\,\cancel{\sin px}\,\cos px\,dx$$

$$F_c\{g(x)\} = G_c(p) = \int_0^\infty e^{-bx}\cos px\,dx = \frac{b}{b^2+p^2}$$

From the Parseval's Identity of cosine transform

$$\frac{2}{\pi}\int_0^\infty F_c(p)\,G_c(p)\,dp = \int_0^\infty f(x)g(x)\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{a}{a^2+p^2}\cdot\frac{b}{b^2+p^2}\,dp = \int_0^\infty e^{-ax}e^{-bx}\,dx$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{dp}{(a^2+p^2)(b^2+p^2)} = \int_0^\infty e^{-(a+b)x}\,dx$$

$$= \left(\frac{e^{-(a+b)x}}{-(a+b)}\right)_0^\infty = \frac{1}{a+b}$$

replacing p by x we have

$$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}$$

5. Find the Fourier transform of $f(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & |x|>1 \end{cases}$

and deduce that $\int_0^\infty \left(\frac{\sin x}{x}\right)^4 dx = \frac{\pi}{3}$.

Sol :- Given $f(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & |x|>1 \end{cases}$

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$$

$$= \int_{-\infty}^{-1} f(x)e^{ipx}dx + \int_{-1}^{1} f(x)e^{ipx}dx + \int_{1}^{\infty} f(x)e^{ipx}dx$$

$$= 0 + \int_{-1}^{1} 1-|x|\, e^{ipx}\,dx + 0$$

$$= \int_{-1}^{1} 1-|x|\,(\cos px + i\sin px)\,dx$$

$$= \int_{-1}^{1} 1-|x| \cos px\,dx + i\int_{-1}^{1} 1-|x| \sin px\,dx$$

$$= 2\int_0^1 1-|x| \cos px\,dx + 0 \qquad (\because 2^{nd} \text{ integrand is odd } 1^{st} \text{ is even})$$

$$= 2\left[\frac{1-x}{1}\frac{\sin px}{p} - (-1)\left(\frac{-\cos px}{p^2}\right)\right]_0^1$$

$$= 2\left[\left(0 - \frac{1}{p^2}\cos p\right) - \left(0 - \frac{1}{p^2}\right)\right]$$

$$F(p) = 2\left[-\frac{\cos p}{p^2} + \frac{1}{p^2}\right] = \frac{2}{p^2}(1-\cos p)$$

From the Parseval's identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(p)|^2 dp = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\frac{2}{p^2}(1-\cos p)\right]^2 dp = \int_{-\infty}^{\infty} (1-x)^2 dx$$

$$\frac{\cancel{2}}{2\pi}\int_0^\infty \frac{4}{p^4}(1-\cos p)^2 dp = \int_{-\infty}^{-1} (1-x)^2 dx + \int_{-1}^{1}(1-x)^2 dx + \int_1^\infty (1-x)^2 dx$$

$$= 0 + \int_{-1}^{1} (1-x)^2 dx + 0$$

$$\frac{4}{\pi}\int_0^\infty \frac{(1-\cos p)^2}{p^4} dp = 2\int_0^1 (1-x)^2 dx$$

$$\frac{4}{\pi}\int_0^\infty \frac{(2\sin^2 p/2)^2}{p^4} dp = 2\int_0^1 [1-2x+x^2]\,dx$$

$$\frac{16}{\pi}\int_0^\infty \frac{(\sin^2 p/2)^2}{p^4} dp = 2\left[x - \cancel{2}\frac{x^2}{\cancel{2}} + \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{3}\right] = \frac{2}{3}$$

$$\Rightarrow \int_0^\infty \frac{(\sin^2 p/2)^2}{p^4} dp = \frac{2}{3} \times \frac{\pi}{16\,8} = \frac{\pi}{24}$$

Let $\frac{p}{2} = x \Rightarrow p = 2x \Rightarrow dp = 2dx$

$$\therefore \int_0^\infty \frac{(\sin^2 x)^2}{(2x)^4} 2dx = \frac{\pi}{24}$$

$$\int_0^\infty \frac{(\sin^2 x)^2}{\cancel{16}_{8}\, x^4} \cancel{2}dx = \frac{\pi}{24}$$

$$\Rightarrow \int_0^\infty \frac{\sin^4 x}{x^4} dx = \frac{\pi}{24} \times 8 = \frac{\pi}{3}$$

6. using Parseval's Identity S.T $\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{2(a+b)}$

Sol :- Let $f(x) = e^{-ax}$ $g(x) = e^{-bx}$

The Fourier sine transform of $f(x)$ is

$$F_s\{f(x)\} = F_s(P) = \int_0^{\infty} f(x) \sin px \, dx$$

$$= \int_0^{\infty} e^{-ax} \sin px \, dx = \frac{p}{a^2+p^2}$$

The Fourier sine transform of $g(x)$ is

$$F_s\{g(x)\} = G_s(P) = \int_0^{\infty} g(x) \sin px \, dx$$

$$= \int_0^{\infty} e^{-bx} \sin px \, dx = \frac{p}{b^2+p^2}$$

From the Parseval's Identity of Fourier sine transform

$$\frac{2}{\pi}\int_0^{\infty} F_s(P)\, G_s(P)\, dp = \int_0^{\infty} f(x) g(x)\, dx$$

$$\frac{2}{\pi}\int_0^{\infty} \frac{p}{a^2+p^2} \cdot \frac{p}{b^2+p^2}\, dp = \int_0^{\infty} e^{-ax} e^{-bx}\, dx$$

$$\frac{2}{\pi}\int_0^{\infty} \frac{p^2}{(a^2+p^2)(b^2+p^2)}\, dp = \int_0^{\infty} e^{-(a+b)x}\, dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^{\infty} = \frac{1}{a+b}$$

replacing p by x

$$\int_0^{\infty} \frac{x^2}{(a^2+x^2)(b^2+x^2)}\, dx = \frac{\pi}{2(a+b)} \,//$$

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