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UNIT- II Partial differentiation.

Definition: A differential eqn is said to be partial differential equation, if it contains atleast two independt. variables and one dependent variable.

EX: ① $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

② $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$

Order of P.D.E: The order of the partial differential eqn. is the order of highest order partial derivative in the equation.

Degree of P.D.E: The degree of an equation is the degree of the highest order derivative occuring in the eqn.

EX: 1. $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$ first order and first degree

2. $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+\frac{\partial^2 z}{\partial x\partial y}=0$ is of second order and first degree

The general solution of a partial differential eqn is a function of independent variables which satisfies the given differential equation.

whenever we consider the case of two independent variables, we take 'x' and 'y' and take 'z' be dependent variable.

Let $\frac{\partial z}{\partial x}=p$ $\frac{\partial z}{\partial y}=q$ $\frac{\partial^2 z}{\partial x^2}=r$ $\frac{\partial^2 z}{\partial x\partial y}=s$ $\frac{\partial^2 z}{\partial y^2}=t$

NOTE: If the number of constants to be eliminated is equal to the number of independent variables, the partial differential eqns obtained are of first order.

In case the number of constants to be eliminated is more than the number of independent variables, we obtain the P.D.E of second and higher order.

2)

① Form the partial diff equation by eliminating the arbitrary constants 'a' and 'b' from

(a). $z = ax + by + a^2 + b^2$

(b). $z = ax + by + \frac{a}{b} - b$.

Soln: (a). The given function $z = ax + by + a^2 + b^2 \rightarrow ①$

Diff ① partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = a(1) + 0 + 0 + 0$$

$$\boxed{p = a} \rightarrow ② \quad \left|\ \because \frac{\partial z}{\partial x} = p\right.$$

again diff ① partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = 0 + b(1) + 0 + 0$$

$$\boxed{q = b} \rightarrow ③ \quad \left|\ \frac{\partial z}{\partial y} = q\right.$$

Substitute ② & ③ in ① we get

$$z = ax + by + a^2 + b^2$$

$$\boxed{z = px + qy + p^2 + q^2}$$ is the required P.D.E.

(b). The given function $z = ax + by + \frac{a}{b} - b \rightarrow ①$

To eliminate constants a, b diff ① partially w.r.t 'x' and w.r.t 'y'.

Diff ① partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = a(1) + 0 + 0 - 0$$

$$\boxed{p = a} \rightarrow ②$$

Diff ① partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = 0 + b(1) + 0 - 0$$

$$\boxed{q = b} \rightarrow ③$$

Substitute ② & ③ in eqn ① we get

$$z = ax + by + \frac{a}{b} - b$$

$$\boxed{z = px + qy + \frac{p}{q} - q}$$ is required P.D.E.

② . Eliminate h, K from $(x-h)^2 + (y-K)^2 + z^2 = a^2$

Soln: the given function $(x-h)^2+(y-K)^2+z^2=a^2 \longrightarrow ①$

To eliminate consts h, K, Diff partially w.r.t 'x' and 'y'.

Diff ① w.r.t 'x' partially

$$2(x-h) + 0 + 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$2[(x-h) + z \cdot p] = 0 \qquad \frac{\partial z}{\partial x} = p$$

$$(x-h) + zp = 0$$

$$\therefore \boxed{(x-h) = -zp} \longrightarrow ②$$

again Diff ① partially w.r.t 'y' then

$$0 + 2(y-K) + 2z \cdot \frac{\partial z}{\partial y} = 0$$

$$2[(y-K) + z \cdot q] = 0 \qquad \frac{\partial z}{\partial y} = q$$

$$(y-K) + z \cdot q = 0$$

$$\therefore \boxed{(y-K) = -zq} \longrightarrow ③$$

Substitute ② & ③ in given eqn ①

$$(x-h)^2 + (y-K)^2 + z^2 = a^2$$

$$(-zp)^2 + (-zq)^2 + z^2 = a^2$$

$$z^2p^2 + z^2q^2 + z^2 = a^2$$

$$\boxed{z^2(p^2+q^2+1) = a^2}$$ is required P.D.E.

③ Form the p.d.e by eliminating constants from $z=(x^2+a)(y^2+b)$

Soln: the given function $z = (x^2+a)(y^2+b) \longrightarrow ①$

Diff ① partially w.r.t 'x' (y = const)

$$\frac{\partial z}{\partial x} = p = (y^2+b)(2x+0)$$

$$\therefore y^2+b = \frac{p}{2x} \longrightarrow ②$$

Diff ① partially w.r.t 'y', $\frac{\partial z}{\partial y} = (x^2+a)(2y+0)$

$$q = (x^2+a)(2y)$$

$$(x^2+a) = \frac{q}{2y} \longrightarrow (3)$$

Substitute (2) & (3) in eqn (1)

$$z = (x^2+a)(y^2+b)$$

$$z = \frac{q}{2y} \cdot \frac{p}{2x}$$

$$z = \frac{pq}{4xy}$$

$\therefore \boxed{pq = 4xyz}$ is required P.D.E.

(4). Form the partial differential eqn by eliminating Constants a, b from $z = a \cdot \log\left[\frac{b(y-1)}{1-x}\right]$

Soln: The given function $z = a \cdot \log\left[\frac{b(y-1)}{1-x}\right]$

$\log\left(\frac{m}{n}\right) = \log m - \log n$

$\log(m \cdot n) = \log m + \log n$

$$z = a \cdot \left[\log[b(y-1)] - \log(1-x)\right]$$

$$z = a \cdot \left[\log b + \log(y-1) - \log(1-x)\right] \longrightarrow (1)$$

To eliminate Constant a, b diff eqn (1) partially w.r.t 'x' and 'y'

Diff (1) w.r.t 'x' partially,

$$\frac{\partial z}{\partial x} = a\left[0 + 0 - \frac{1}{(1-x)}(0-1)\right]$$

$\frac{\partial z}{\partial x} = p$

$$p = \frac{a}{1-x}$$

$$\therefore a = p(1-x) \rightarrow (2)$$

Diff (1) partially w.r.t 'y' then

$$\frac{\partial z}{\partial y} = a\left[0 + \frac{1}{y-1} - 0\right]$$

$$q = \frac{a}{y-1}$$

$\frac{\partial z}{\partial y} = q$

$$a = q(y-1) \longrightarrow (3)$$

Substitute (2) & (3) in eqn (1) then 'b' not eliminated

then from relation from (2) & (3)

$p(1-x) = q(y-1)$ (or) $\boxed{px + qy = p + q}$

is required p.d.e.

5) Form p.d.e by eliminating the constants from $(x-a)^2+(y-b)^2 = z^2\cot^2\alpha$

Soln: The given function $(x-a)^2+(y-b)^2 = z^2\cot^2\alpha \rightarrow (1)$

To eliminate constants a, b.

Diff (1) partially w.r.t 'x' we get

$$\cancel{2}(x-a)+0 = \cancel{2}z\cdot\frac{\partial z}{\partial x}\cdot\cot^2\alpha.$$

$$(x-a) = zp\cdot\cot^2\alpha \rightarrow (2) \quad \Big| \ \frac{\partial z}{\partial x}=p$$

Diff (1) partially w.r.t 'y' we get

$$0+\cancel{2}(y-b) = \cancel{2}z\cdot\frac{\partial z}{\partial y}\cdot\cot^2\alpha$$

$$y-b = zq\cdot\cot^2\alpha \rightarrow (3) \quad \Big| \ \frac{\partial z}{\partial y}=q$$

(2) & (3) substitute in eqn (1) we get

$$(x-a)^2+(y-b)^2 = z^2\cot^2\alpha$$

$$(zp\cdot\cot^2\alpha)^2+(zq\cot^2\alpha)^2 = z^2\cot^2\alpha$$

$$z^2p^2\cot^4\alpha+z^2q^2\cot^4\alpha = z^2\cot^2\alpha$$

$$\cancel{z^2\cot^2\alpha}\left[p^2\cot^2\alpha+q^2\cot^2\alpha\right] = \cancel{z^2\cot^2\alpha}$$

$$p^2\cot^2\alpha+q^2\cot^2\alpha = 1$$

$$(p^2+q^2)\cot^2\alpha = 1$$

$$p^2+q^2 = \frac{1}{\cot^2\alpha} = \tan^2\alpha$$

$\therefore$ $p^2+q^2 = \tan^2\alpha$ is the required

NOTE: If the number of constants are greater than the number of independent variables then we need to go second and higher order partial differentiation.

⑥. Form the partial differential eqn by eliminating the arbitrary constants a, b, c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ⑥

Soln: The given function $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \rightarrow ①$

Diff ① partially w.r.t 'x' both sides

$$\frac{1}{a^2}(2x)+0+\frac{1}{c^2}\left(2z\cdot\frac{\partial z}{\partial x}\right)=0$$

$$2\left[\frac{x}{a^2}+\frac{1}{c^2}\cancel{z}\cdot p\right]=0 \qquad \frac{\partial z}{\partial x}=p$$

$$\frac{x}{a^2}+\frac{zp}{c^2}=0 \rightarrow ②$$

Diff ① partially w.r.t 'y' both sides

$$0+\frac{1}{b^2}(2y)+\frac{1}{c^2}\left(2z\cdot\frac{\partial z}{\partial y}\right)=0$$

$$2\left[\frac{y}{b^2}+\frac{zq}{c^2}\right]=0 \qquad \frac{\partial z}{\partial y}=q$$

$$\therefore \frac{y}{b^2}+\frac{zq}{c^2}=0 \rightarrow ③$$

Here eqns ② & ③ not enough to eliminate a, b, c because the no of constants are greater than the independt variables. [constants = a, b, c Variables = x, y]

again diff ② partially w.r.t 'x' both sides

$$\frac{\partial}{\partial x}\left[\frac{x}{a^2}+\frac{zp}{c^2}\right]=0$$

$$\frac{1}{a^2}(1)+\frac{1}{c^2}\left(z\cdot\frac{\partial p}{\partial x}+p\cdot\frac{\partial z}{\partial x}\right)=0 \qquad \frac{\partial z}{\partial x}=p$$

$$\frac{1}{a^2}+\frac{1}{c^2}\left(z\cdot r+p^2\right)=0 \rightarrow ④$$

$$\frac{\partial p}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=r$$

Multiply eqn ④ × x and subtract eqn ②

$$\left[\cancel{\frac{x}{a^2}}+\frac{x}{c^2}(zr+p^2)\right]-\left[\cancel{\frac{x}{a^2}}+\frac{zp}{c^2}\right]=0$$

$$\frac{1}{c^2}\left[xzr+xp^2-pz\right]=0$$

$$xzr+xp^2-pz=0$$

$$\therefore \boxed{pz=xzr+xp^2}$$ is required P.D.E.

⑦ Form the p.d.e by eliminating the arbitrary constants a, b from $2z = (x+a)^{1/2} + (y-a)^{1/2} + b$

Soln: The given function $2z = (x+a)^{1/2} + (y-a)^{1/2} + b$

To eliminate contants a, b $\quad 2z = \sqrt{x+a} + \sqrt{y-a} + b \rightarrow (1)$

Diff (1) partially w.r.t 'x' both sides

$$2 \cdot \frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x+a}}(1+0) + 0 + 0$$

$$2p = \frac{1}{2\sqrt{x+a}} \qquad \frac{\partial z}{\partial x} = p$$

$$\sqrt{x+a} = \frac{1}{4p} \rightarrow (2)$$

again diff (1) partially w.r.t 'y' both sides

$$2 \cdot \frac{\partial z}{\partial y} = 0 + \frac{1}{2\sqrt{y-a}}(1-0) + 0$$

$$2q = \frac{1}{2\sqrt{y-a}} \qquad \frac{\partial z}{\partial y} = q$$

$$\sqrt{y-a} = \frac{1}{4q} \rightarrow (3)$$

Substitute (2) & (3) in eqn (1) the const 'b' not eliminates then form the relation between (2) & (3)

Squaring (2) $\quad x + \cancel{a} = \frac{1}{16p^2}$

Squaring (3) $\quad y - \cancel{a} = \frac{1}{16q^2}$

(+)

$$x + y = \frac{1}{16p^2} + \frac{1}{16q^2}$$

$$x + y = \frac{1}{16}\left(\frac{1}{p^2} + \frac{1}{q^2}\right)$$

(or) $\quad \boxed{\frac{1}{p^2} + \frac{1}{q^2} = 16(x+y)}$ is required P.D.E.

(or) $\quad \boxed{q^2 + p^2 = 16p^2q^2(x+y)}$

⑧ Form the diff eqn by eliminating constants a, b from

$$\log(az-1) = x + ay + b$$

<u>Soln:</u> Given function $\log(az-1) = x+ay+b \rightarrow (1)$

<u>To eliminate constants a, b</u>

Diff (1) partially w.r.t 'x' both sides

$$\frac{1}{az-1}\left(a\cdot\frac{\partial z}{\partial x} - 0\right) = 1+0+0 \qquad ⑥$$

$$\frac{1}{az-1}(ap) = 1 \qquad \frac{\partial z}{\partial x} = p$$

$$\therefore\ ap = az-1 \rightarrow (2)$$

Diff (1) partially w.r.t 'y' both sides

$$\frac{1}{az-1}\left(a\cdot\frac{\partial z}{\partial y} - 0\right) = 0 + a(1) + 0$$

$$\frac{1}{az-1}(\cancel{a}q) = \cancel{a} \qquad \frac{\partial z}{\partial y} = q$$

$$q = az-1 \rightarrow (3)$$

from (2) & (3) $\quad ap = az-1 = q$

$$\therefore\ \boxed{ap = q} \qquad \boxed{a = \frac{q}{p}}$$

Substitute in eqn (2) we get

$$ap = az-1$$

$$q = \left(\frac{q}{p}\right)z - 1$$

$$q = \frac{qz-p}{p}$$

$$qp = qz - p$$

$$qp + p = qz$$

$\boxed{(q+1)p = qz}$ is the required P.D.E.

(9) Form the partial diff eqn by eliminating the arbitrary constants a, b and c from $z = a(x+y) + b(x-y) + abt + c$.

Soln: The given function $z = a(x+y) + b(x-y) + abt + c \rightarrow (1)$

To eliminate consts a, b, c:

Diff (1) partially w.r.t 'x' bothsides

$$\frac{\partial z}{\partial x} = a(1+0) + b(1-0) + 0 + 0$$

$$z_x = a+b \rightarrow (2) \qquad \frac{\partial z}{\partial x} = p = z_x$$

Diff (1) partially w.r.t 'y' both sides

$$\frac{\partial z}{\partial y} = a(0+1) + b(0-1) + 0 + 0$$

$$z_y = a-b \rightarrow (3) \qquad \frac{\partial z}{\partial y} = q = z_y$$

again diff (1) partially w.r.t 't' both sides

$$\frac{\partial z}{\partial t} = 0 + 0 + ab(1) + 0$$

$$z_t = ab \rightarrow (4)$$

Now make relation from (2), (3) and (4)

$$z_x^2 - z_y^2 = (a+b)^2 - (a-b)^2 = 4ab = 4z_t$$

$\therefore \boxed{z_x^2 - z_y^2 = 4z_t}$ is required P.D.E.

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problems: Exercise: Form p.d.e by eliminating consts from.

(1) $z = ax^3 + by^3$ (a, b consts)

(2) Form the p.d.e of all spheres whose centres lie on z-axis with a given radius 'r'. $(x^2+y^2+(z-c)^2 = r^2)$ (c, r consts)

(3) Form the d.eqn of all planes passing through the origin $(ax+by+cz=0)$ (a, b, c consts)

(4) Find the diff eqn of all planes having equal intercepts on x and y axes $\left(\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1\right)$ (a, c consts)

9 Form the Partial diff equation by eliminating the arbitrary Constants a, b and c from $z = a(x+y)+b(x-y)+abt+c$

Soln: The given function $z = a(x+y)+b(x-y)+abt+c \rightarrow (1)$

To eliminate Consts a, b, c:

Diff (1) Partially w.r.t x, $z_x = \frac{\partial z}{\partial x} = a(1+0)+b(1-0)+0+0$

Formation of P.D.E by eliminating the arbitrary functions:

Let $u = u(x,y,z)$, $v = v(x,y,z)$ be independent functions of the variables x, y, z. Let $\phi(u,v) = 0 \rightarrow (i)$

We shall obtain a partial differential equation by eliminating the functions 'u' and 'v'. We treat 'z' as dependent variable and x and y as independent variables.

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = 0 \quad (10)$$

Problems

(1) Form a partial differential equation by eliminating the arbitrary functions from

(i). $z = f(x^2+y^2)$

(ii). $z = f(x) + e^y \cdot g(x)$

Soln: (i). The given function $z = f(x^2+y^2) \longrightarrow (1)$

Here we have to eliminate the arbitrary function 'f'

Diff (1) partially w.r.t 'x' both sides

$$\frac{\partial z}{\partial x} = f'(x^2+y^2)\cdot(2x+0)$$

$$p = 2x \cdot f'(x^2+y^2) \longrightarrow (2) \quad \frac{\partial z}{\partial x} = p$$

Diff (1) partially w.r.t 'y' bothsides

$$\frac{\partial z}{\partial y} = f'(x^2+y^2)\cdot(0+2y)$$

$$q = 2y\, f'(x^2+y^2) \longrightarrow (3) \quad \frac{\partial z}{\partial y} = q$$

(2) ÷ (3)

$$\frac{p}{q} = \frac{\cancel{2}x \cdot \cancel{f'(x^2+y^2)}}{\cancel{2}y\, \cancel{f'(x^2+y^2)}}$$

$$\frac{p}{q} = \frac{x}{y} \Rightarrow \boxed{py - qx = 0}$$ is required partial diff^{al} equation.

Sol^n: (ii). we have $z = f(x) + e^{y} \cdot g(x) \longrightarrow (1)$

we note that two arbitrary functions 'f' and 'g' are given. Therefore after elimination of these two functions we may get second (or) higher order partial differential eqn.

Diff (1) Partially w.r.t 'x' bothsides

$$\frac{\partial z}{\partial x} = f'(x) + e^{y} \cdot g'(x) \longrightarrow (2)$$

(11)

Diff (1) Partially w.r.t 'y' bothsides

$$\frac{\partial z}{\partial y} = 0 + e^{y} \cdot g(x). \qquad \frac{\partial}{\partial y}(e^{y}) = e^{y}$$

$$\frac{\partial z}{\partial y} = e^{y} \cdot g(x) \longrightarrow (3)$$

From (2) & (3) it is not Possible to eliminate 'f' and 'g'

again diff (3) w.r.t 'y' both sides

$$\frac{\partial^2 z}{\partial y^2} = e^{y} \cdot g(x) \longrightarrow (4) \qquad t = \frac{\partial^2 z}{\partial y^2}$$

From (3) & (4) $\boxed{\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}}$ (or) $\boxed{t - q = 0}$

(2). Form the P.D.E by eliminating arbitrary function from $z = f\left(\frac{y}{x}\right)$.

Sol^n: Given $z = f\left(\frac{y}{x}\right) \longrightarrow (1)$

Diff (1) Partially w.r.t 'x' bothsides

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right) \cdot y\left(-\frac{1}{x^2}\right)$$

$$p = -\frac{y}{x^2} f'\left(\frac{y}{x}\right) \longrightarrow (2)$$

Diff (1) w.r.t 'y' Partially bothsides

$$\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}(1)$$

$$q = \frac{1}{x} \cdot f'\left(\frac{y}{x}\right) \longrightarrow (3)$$

$$(2) \div (3) \quad \frac{p}{q} = \frac{-\frac{y}{x^2} \cancel{f'\left(\frac{y}{x}\right)}}{\frac{1}{x} \cancel{f'\left(\frac{y}{x}\right)}} = \frac{-y}{x^2} \times \frac{x}{1} = -\frac{y}{x}$$

$\boxed{px + qy = 0}$ is required P.D.E

③ Form a P.D.E by eliminating arbitrary functions $f(x)$ and $g(y)$ from $z = y\cdot f(x) + x\cdot g(y)$.

Soln: Given $z = y\cdot f(x) + x\cdot g(y) \longrightarrow ①$

To eliminate the functions 'f' and 'g' diff partially w.r.t 'x' and 'y'

Diff ① partially w.r.t 'x' both sides

$$\frac{\partial z}{\partial x} = y\cdot f'(x) + g(y)(1) \qquad \Big|\ \frac{\partial z}{\partial x} = p$$

$$p = y\cdot f'(x) + g(y) \longrightarrow ②$$

⑫

Diff ① partially w.r.t 'y' both sides

$$\frac{\partial z}{\partial y} = f(x)\cdot(1) + x\cdot g'(y) \qquad \Big|\ \frac{\partial z}{\partial y} = q$$

$$q = f(x) + x\cdot g'(y) \longrightarrow ③$$

Since relations ①,②,③ are not sufficient to eliminate f, g, f' and g'. So we go for second order partial derivatives

Diff ② partially w.r.t 'x' $\quad \frac{\partial^2 z}{\partial x^2} = y\cdot f''(x) \longrightarrow ④$

Diff ③ partially w.r.t 'x' $\quad \frac{\partial^2 z}{\partial x\,\partial y} = f'(x) + g'(y)(1) \longrightarrow ⑤$

Diff ③ partially w.r.t 'y' $\quad \frac{\partial^2 z}{\partial y^2} = 0 + x\cdot g''(y) \longrightarrow ⑥$

From ② & ③ find $f'(x), g'(y)$ and substitute in ⑤

from ② $p = y\cdot f'(x) + g(y)$

$\frac{1}{y}[p - g(y)] = f'(x)$

from ③ $q = f(x) + x\cdot g'(y)$

$\frac{1}{x}[q - f(x)] = g'(y)$

∴ from ⑤

$$\frac{\partial^2 z}{\partial x\,\partial y} = f'(x) + g'(y)$$

$$s = \frac{1}{y}[p - g(y)] + \frac{1}{x}[q - f(x)]$$

$$s\cdot xy = x[p - g(y)] + y[q - f(x)]$$

$$= px + qy - [y\cdot f(x) + x\cdot g(y)]$$

from ①

$$\boxed{xys = px + qy - z}$$

is the required partial diff equation.

④ Form the differential equation by eliminating the arbitrary function 'f' from $xyz = f(x^2+y^2+z^2)$

Soln: Given $xyz = f(x^2+y^2+z^2) \longrightarrow ①$

Diff ① partially w.r.t 'x' both sides $(y, z = \text{Const})$

$$y\,\frac{\partial}{\partial x}(xz) = f'(x^2+y^2+z^2)\left(2x+0+2z\cdot\frac{\partial z}{\partial x}\right)$$

$$y\left[x\cdot\frac{\partial z}{\partial x}+z(1)\right] = f'(x^2+y^2+z^2)\cdot 2(x+zp)$$

$$yxp + yz = 2(x+zp)\cdot f'(x^2+y^2+z^2) \longrightarrow ②$$

Diff ① partially w.r.t 'y' both sides (13)

$$x\cdot\frac{\partial}{\partial y}(yz) = f'(x^2+y^2+z^2)\cdot\left(0+2y+2z\cdot\frac{\partial z}{\partial y}\right)$$

$$x\cdot\left[y\cdot\frac{\partial z}{\partial y}+z(1)\right] = f'(x^2+y^2+z^2)\cdot 2(y+zq)$$

$$xyq + xz = 2(y+zq)\cdot f'(x^2+y^2+z^2) \longrightarrow ③$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

② ÷ ③

$$\frac{yxp+yz}{xyq+xz} = \frac{\cancel{2}(x+zp)\cdot \cancel{f'(x^2+y^2+z^2)}}{\cancel{2}(y+zq)\,\cancel{f'(x^2+y^2+z^2)}}$$

$$(yxp+yz)(y+zq) = (x+zp)(xyq+xz)$$

$$y^2xp + y^2z + \cancel{xyzpq} + z^2yq - x^2yq - x^2z - \cancel{xyzpq} + xpz^2 = 0$$

$$\boxed{z(y^2-x^2) + z^2(qy+px) + xy(yp-xq) = 0}$$

(or) $$\boxed{z(x^2-y^2) + z^2(px-qy) + xy(xq-yp) = 0}$$

is the required P.D.E

(5). Eliminate the arbitrary functions ϕ_1 and ϕ_2 from
$z = \phi_1(x+iy) + \phi_2(x-iy)$

Soln: Given $z = \phi_1(x+iy) + \phi_2(x-iy)$

take $\boxed{\begin{matrix} x+iy = u \\ x-iy = v \end{matrix}}$ $\Rightarrow \frac{\partial u}{\partial x} = 1$, $\Rightarrow \frac{\partial v}{\partial x} = 1$

$\therefore z = \phi_1(u) + \phi_2(v) \longrightarrow (1)$

$\frac{\partial u}{\partial y} = i$

$\frac{\partial v}{\partial y} = -i$

Diff (1) partially w.r.t 'x' we get

$$\frac{\partial z}{\partial x} = \phi_1'(u) \cdot \frac{\partial u}{\partial x} + \phi_2'(v) \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \phi_1'(u) + \phi_2'(v) \longrightarrow (2)$$

and diff (1) partially w.r.t 'y' both sides (14)

$$\frac{\partial z}{\partial y} = \phi_1'(u) \cdot \frac{\partial u}{\partial y} + \phi_2'(v) \cdot \frac{\partial v}{\partial y}$$

$$= \phi_1'(u)(i) + \phi_2'(v) \cdot (-i)$$

$$\frac{\partial z}{\partial y} = i\phi_1'(u) - i\phi_2'(v) \longrightarrow (3)$$

Eqns (2) & (3) are not enough to eliminate ϕ_1 and ϕ_2

again diff (2) partially w.r.t 'x' then

$$\frac{\partial^2 z}{\partial x^2} = \phi_1''(u) \cdot \frac{\partial u}{\partial x} + \phi_2''(v) \cdot \frac{\partial v}{\partial x} = \phi_1''(u) + \phi_2''(v) \longrightarrow (4)$$

again diff (3) partially w.r.t 'y' then

$$\frac{\partial^2 z}{\partial y^2} = i \cdot \phi_1''(u) \cdot \frac{\partial u}{\partial y} - i\phi_2''(v) \cdot \frac{\partial v}{\partial y}$$

$$= i\phi_1''(u) \cdot (i) - i\phi_2''(v)(-i)$$

$$= i^2\phi_1''(u) + i^2\phi_2''(v) \qquad i^2 = -1$$

$$= -\left[\phi_1''(u) + \phi_2''(v)\right]$$

$$-\frac{\partial^2 z}{\partial y^2} = \phi_1''(u) + \phi_2''(v) \longrightarrow (5)$$

from (4) & (5) $\frac{\partial^2 z}{\partial x^2} = -\frac{\partial^2 z}{\partial y^2} \Rightarrow \boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0}$

(or) $\boxed{r + t = 0}$

(6) Form the partial diff^al equation by eliminating the arbitrary function 'f' from $f(x^2+y^2, x^2-z^2)=0$

(5)

Soln: Given $f(x^2+y^2, x^2-z^2) = f(u,v) = 0 \longrightarrow (1)$

$\because u = x^2+y^2$

$v = x^2 - z^2$

Here we eliminate 'f' such that find $\frac{\partial u}{\partial x}$; $\frac{\partial u}{\partial y}$; $\frac{\partial v}{\partial x}$. $\frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial z}$; $\frac{\partial v}{\partial z}$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = 0 \longrightarrow (2)$$

$p = \frac{\partial z}{\partial x}$

$u = x^2+y^2$

$\frac{\partial u}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = 2y$

$\frac{\partial u}{\partial z} = 0$

$v = x^2 - z^2$

$\frac{\partial v}{\partial x} = 2x - 2z \cdot \frac{\partial z}{\partial x} = 2(x - zp)$

$\frac{\partial v}{\partial y} = 0$

$\frac{\partial v}{\partial z} = 0 - 2z = -2z$

$\therefore$ from (2)
$$\begin{vmatrix} 2x + p(0) & 2(x-zp) + p(-2z) \\ 2y + q(0) & 0 + q(-2z) \end{vmatrix} = 0$$

$$= \begin{vmatrix} 2x & 2(x-2pz) \\ 2y & -2qz \end{vmatrix} = 0$$

$$= (2)(2) \begin{vmatrix} x & x-2pz \\ y & -qz \end{vmatrix} = 0$$

$= 4[(-xqz) - y(x-2pz)] = 0$

$= 4[-xqz - yx + 2ypz] = 0$

$-xqz - yx + 2ypz = 0$

$\boxed{xy = z(2yp - xq)}$ is required P.D.E.

⑦ Form the partial differential equation by eliminating the arbitrary function 'ϕ' from $\phi(x^2+y^2+z^2,\ z^2-2xy)=0$

Soln: Given $\phi(x^2+y^2+z^2,\ z^2-2xy) = \phi(u,v) = 0 \longrightarrow (1)$ (16)

To eliminate the arbitrary function 'ϕ' such that

$$\begin{vmatrix} \dfrac{\partial u}{\partial x}+p\dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial x}+p\dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial y}+q\dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial y}+q\dfrac{\partial v}{\partial z} \end{vmatrix} = 0 \longrightarrow (2)$$

$u = x^2+y^2+z^2$

$\dfrac{\partial u}{\partial x} = 2x+0+2z\cdot\dfrac{\partial z}{\partial x} = 2(x+zp)$

$\dfrac{\partial u}{\partial y} = 0+2y+2z\cdot\dfrac{\partial z}{\partial y} = 2(y+zq)$

$\dfrac{\partial u}{\partial z} = 0+0+2z = 2z$

$v = z^2-2xy$

$\dfrac{\partial v}{\partial x} = 2z\cdot\dfrac{\partial z}{\partial x} - 2y(1) = 2(zp-y)$

$\dfrac{\partial v}{\partial y} = 2z\cdot\dfrac{\partial z}{\partial y} - 2x(1) = 2(zq-x)$

$\dfrac{\partial v}{\partial z} = 2z-0 = 2z$

from (2)
$$\begin{vmatrix} 2(x+zp)+p(2z) & 2(zp-y)+p(2z) \\ 2(y+zq)+q(2z) & 2(zq-x)+q(2z) \end{vmatrix} = 0$$

$$= \begin{vmatrix} 2(x+2zp) & 2(2zp-y) \\ 2(y+2zq) & 2(2zq-x) \end{vmatrix} = 0$$

$$= (2)(2)\begin{vmatrix} x+2zp & 2zp-y \\ y+2zq & 2zq-x \end{vmatrix} = 0$$

$$[(x+2zp)(2zq-x)] - [(2zp-y)(y+2zq)] = 0$$

$$(2xzq+\cancel{4z^2pq}-x^2-2zpx) - (2zpy+\cancel{4z^2pq}-y^2-2zyq) = 0$$

$$2xzq - x^2 - 2zpx - 2zpy + y^2 + 2zyq = 0$$

$$2zq(x+y) - 2pz(x+y) + y^2 - x^2 = 0$$

$$2(zq-pz)(x+y) + [(y+x)(y-x)] = 0$$

$$(x+y)[2z(q-p)+(y-x)] = 0$$

$$2z(q-p) = -(y-x)$$

$$\boxed{2z(q-p) = x-y}$$

is required P.D.E.

(8). Form the Partial differential equation $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Soln: Given $z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \rightarrow (1)$

Eliminate the arbitrary function 'f' such that (17)

Diff (1) Partially w.r.t 'x' both sides

$$\frac{\partial z}{\partial x} = 0 + 2f'\left(\frac{1}{x} + \log y\right) \cdot \left(\frac{-1}{x^2} + 0\right)$$

$$p = -\frac{2}{x^2} f'\left(\frac{1}{x} + \log y\right) \rightarrow (2)$$

Diff (1) partially w.r.t 'y' bothsides

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \left(0 + \frac{1}{y}\right)$$

$$q = 2y + \frac{2}{y} f'\left(\frac{1}{x} + \log y\right)$$

$$q - 2y = \frac{2}{y} f'\left(\frac{1}{x} + \log y\right) \rightarrow (3)$$

(2) ÷ (3)
$$\frac{p}{q-2y} = \frac{-\frac{2}{x^2} f'\cancel{\left(\frac{1}{x} + \log y\right)}}{\frac{2}{y} f'\cancel{\left(\frac{1}{x} + \log y\right)}} = -\frac{\cancel{2}}{x^2} \times \frac{y}{\cancel{2}}$$

$$\frac{p}{q-2y} = -\frac{y}{x^2}$$

$$px^2 = -y(q - 2y)$$

$$\boxed{px^2 + qy = 2y^2}$$ is required P.D.E.

(9) obtain Partial diffal eqn from $z = f(\sin x + \cos y)$

Soln: Given $z = f(\sin x + \cos y) \rightarrow (1)$,

To eliminate the arbitrary function 'f' Diff (1) w.r.t x, y.

$$\frac{\partial z}{\partial x} = f'(\sin x + \cos y) \cdot (\cos x + 0)$$

$$p = \cos x \cdot f'(\sin x + \cos y) \rightarrow (2)$$

$$\frac{\partial z}{\partial y} = f'(\sin x + \cos y) \cdot [0 + (-\sin y)]$$

$$q = -\sin y \, f'(\sin x + \cos y) \rightarrow (3)$$

(2) ÷ (3)
$$\frac{p}{q} = \frac{\cos x \cdot f'\cancel{(\sin x + \cos y)}}{-\sin y \cdot f'\cancel{(\sin x + \cos y)}} \Rightarrow -p\sin y = q\cos x$$

$$\boxed{p\sin y + q\cos x = 0}$$

is required Partial diff eqn.

(10). Form the partial differential equation by eliminating the arbitrary functions from

(a). $z = (x+y) \cdot f(x^2-y^2)$

(b). $\phi\left(\frac{y}{x}, x^2+y^2+z^2\right) = 0$ 18

(c). $z = f(x^2+y^2+z^2)$

(d). $z = f\left(\frac{xy}{z}\right)$

(e). $z = f(x+iy) + g(x-iy)$ (solved in examples)

Solutions of first order equations using Lagrange's method:

Lagrange's Linear equation:

An equation of the form $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$ is called Lagrange's Linear equation.

It's general solution is given by $f(u,v)=0$ where 'f' is arbitrary function and u, v are functions of x,y,z.

Solution of Lagrange's Linear equation:

① Write the given diff^al equation in the standard form

$$P(x,y,z).p + Q(x,y,z).q = R(x,y,z) \quad \text{(or)} \quad Pp + Qq = R$$

② Form the Lagrange's auxillary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

③ Solve the simultaneous equations to obtain two independ^t solutions as $u = c_1$, $v = c_2$.

④ Write the general solution of the given equation as $f(u,v) = 0$ (or) $u = \phi(v)$.

Example ① Solve : $yzp - xzq = xy$

Soln: Write the given fun in standard form $Pp + Qq = R$.

i.e $yzp - xzq = xy \rightarrow ①$

where $\boxed{\begin{array}{l} P = yz \\ Q = -xz \\ R = xy \end{array}}$

the auxillary eqn $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$

Taking first two functions

$$\frac{dx}{y\cancel{z}} = \frac{dy}{-x\cancel{z}}$$

$$x\,dx + y\,dy = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$$x^2 + y^2 = 2c = c_1 \rightarrow ②$$

Taking Last two functions

$$\frac{dy}{-\cancel{x}z} = \frac{dz}{\cancel{x}y}$$

$$y\,dy + z\,dz = 0$$

$$\frac{y^2}{2} + \frac{z^2}{2} = c'$$

$$y^2 + z^2 = 2c' = c_2 \rightarrow ③$$

the general solution $f(u,v) = 0 \Rightarrow f(c_1, c_2) = 0$

i.e $f(x^2+y^2,\ y^2+z^2) = 0$

(2) Solve : $z(z^2+xy)(px-qy) = x^4$

Soln: we have $z(z^2+xy)(px-qy) = x^4 \longrightarrow (1)$

i.e $z(z^2+xy)px - z(z^2+xy)qy = x^4$

$$[zx(z^2+xy)]\cdot p + [-zy(z^2+xy)]\cdot q = x^4$$

It is of the form $P(x,y,z)\cdot p + Q(x,y,z)\cdot q = R(x,y,z)$

the Lagrange's auxillary eqn $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\therefore \frac{dx}{zx(z^2+xy)} = \frac{dy}{-zy(z^2+xy)} = \frac{dz}{x^4}$$

Take first two functions

$$\frac{dx}{\cancel{z}x\cancel{(z^2+xy)}} = \frac{dy}{-\cancel{z}y\cancel{(z^2+xy)}}$$

$$\frac{dx}{x} = \frac{dy}{-y}$$

Integrating $\int \frac{1}{x}dx + \int \frac{1}{y}dy = c \Rightarrow \log x + \log y = \log c_1$

$\log(xy) = \log c_1$

$\boxed{xy = c_1}$

Take first ~~Last~~ (two) functions

$$\frac{dx}{\cancel{z}x(z^2+xy)} = \frac{dz}{x^{\cancel{4}3}}$$

$x^3\,dx = \cancel{z}(z^2+xy)\,dz \quad \because xy = c_1$

$$x^3\,dx = (z^3 + c_1 z)\,dz$$

Integrating $\int x^3\,dx = \int (z^3 + c_1 z)\,dz$

$$\frac{x^4}{4} = \frac{z^4}{4} + c_1\frac{z^2}{2} + c_2$$

$$\frac{x^4}{4} - \frac{z^4}{4} - c_1\cdot\frac{z^2}{2} = c_2$$

$$x^4 - z^4 - 2c_1 z^2 = 4c_2$$

(or) $x^4 - z^4 - 2xyz^2 = c_2 \quad c_1 = xy$

$\therefore$ General solution $f(c_1, c_2) = 0 \Rightarrow \boxed{f(xy,\ x^4 - z^4 - 2xyz^2) = 0}$

(3) solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Soln: The given equation is Lagrange's Method Linear eqn

$P.p + Q.q = R$ Given $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

$P = \sqrt{x}$, $Q = \sqrt{y}$, $R = \sqrt{z}$

the auxillary eqn $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Take First, second terms

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$$

Integrating $\int \frac{1}{\sqrt{x}} dx = \int \frac{1}{\sqrt{y}} dy + C_1$

$$2\sqrt{x} = 2\sqrt{y} + C_1$$

$$2\sqrt{x} - 2\sqrt{y} = C_1$$

$$\sqrt{x} - \sqrt{y} = \frac{C_1}{2} = \bar{C}_1$$

Take second, third terms

$$\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Integrating $\int \frac{1}{\sqrt{y}} dy = \int \frac{1}{\sqrt{z}} dz + C$

$$2\sqrt{y} = 2\sqrt{z} + C_2$$

$$2\sqrt{y} - 2\sqrt{z} = C_2$$

$$\sqrt{y} - \sqrt{z} = \frac{C_2}{2} = \bar{C}_2$$

∴ the general solution $\phi(u, v) = 0$ (or) $\phi(\bar{C}_1, \bar{C}_2) = 0$

$$\boxed{\phi(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0}$$

(4) Solve: $p \tan x + q \tan y = \tan z$

Soln: Given equation $p \tan x + q \cdot \tan y = \tan z$ is of the form $P.p + Q.q = R$ $P = \tan x$, $Q = \tan y$, $R = \tan z$

The auxillary eqn $\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$

$$\cot x \, dx = \cot y \, dy = \cot z \, dz$$

Consider $\cot x \, dx = \cot y \, dy$

Integrating $\int \cot x \, dx = \int \cot y \, dy + \cancel{\log c}$ C_1

$$\log(\sin x) = \log(\sin y) + \log C_1$$

$$\log(\sin x) - \log(\sin y) = \log C_1$$

$$\log\left(\frac{\sin x}{\sin y}\right) = \log C_1$$

$$\frac{\sin x}{\sin y} = C_1$$

$\cot y \, dy = \cot z \, dz$

$$\int \cot y \, dy = \int \cot z \, dz + C$$

$$\log(\sin y) = \log(\sin z) + \log C$$

$$\log\left(\frac{\sin y}{\sin z}\right) = \log C_2$$

$$\frac{\sin y}{\sin z} = C_2$$

General Solution $\phi(C_1, C_2) = 0 \Rightarrow \boxed{\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0}$

5. Solve the partial diff^{al} eqn. $z(x-y) = px^2 - qy^2$

Soln: Given eqn is $z(x-y) = px^2 - qy^2 \rightarrow (1)$

It is of the form, Lagrange's method $P.p + Q.q = R$

$$P = x^2, \; Q = -y^2 \; ; \; R = z(x-y)$$

The Lagrange's auxillary eqn $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e $\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)} \rightarrow (2)$

Taking first '2' function

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

Integrating $\int \frac{1}{x^2} dx = \int -\frac{1}{y^2} dy$

$$-\frac{1}{x} = -\left(-\frac{1}{y}\right) + c_1$$

$$\frac{1}{x} + \frac{1}{y} = c_1 \rightarrow (3)$$

Taking Last two terms with first term.

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)}$$

$$\cancel{\frac{x^2 dy + y^2 dx}{x^2 + y^2}} = \frac{dz}{z(x-y)}$$

$$\frac{dx + dy}{x^2 - y^2}$$

$$\frac{dx + dy}{(x+y)\cancel{(x-y)}} = \frac{dz}{z\cancel{(x-y)}}$$

Integrating $\int \frac{1}{x+y} d(x+y) = \int \frac{dz}{z}$

$$\log(x+y) = \log z + \log c_2$$

$$\log(x+y) - \log z = \log c_2$$

$$\log\left(\frac{x+y}{z}\right) = \log c_2$$

$$c_2 = \frac{x+y}{z} \rightarrow (4)$$

from (3) & (4)

General solution $f(c_1, c_2) = 0$

$$\therefore \boxed{f\left(\frac{1}{x} + \frac{1}{y}, \frac{x+y}{z}\right) = 0}$$

where 'f' is arbitrary function

Note: (1) Method of grouping $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(2) Method of Multipliers

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$ l, m, n are

Constants such that $lP + mQ + nR = 0$, then $l\,dx + m\,dy + n\,dz = 0$

To solve $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(i) Method of grouping:

In some problems it is possible that two of the functions $\frac{dx}{P} = \frac{dy}{Q}$ (or) $\frac{dy}{Q} = \frac{dz}{R}$ (or) $\frac{dx}{P} = \frac{dz}{R}$ are directly solvable to get general solution $u(x,y) = C_1$, $v(x,y) = C_2$ such that $f(u,v) = 0$.

(ii). In some problems one of the equations $\frac{dx}{P} = \frac{dy}{Q}$ can be directly solved to get $u(x,y) = C_1$, using this we may express 'y' as a function of 'x' and substitute in $\frac{dy}{Q} = \frac{dz}{R}$ to get $v(x,y) = C_2$ so that Complete Solution is $f(u,v) = 0$

(iii). Method of Multipliers:

Write $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$

where l, m, n are called Multipliers

Choose l, m, n such that $\boxed{l \cdot P + m \cdot Q + n \cdot R = 0}$

then $l\,dx + m \cdot dy + n \cdot dz = 0$, integrating this we get

$u(x,y) = C_1$

Similarly we can find another set of multipliers to get $v(x,y) = C_2$.

(6) Solve: $x^2(y-z) \cdot p + y^2(z-x) \cdot q = z^2(x-y)$

Soln: The given Lagrange's p.d.eqn $x^2(y-z) \cdot p + y^2(z-x) \cdot q = z^2(x-y) \rightarrow (1)$

It is of the form $P \cdot p + Q \cdot q = R$

where $P = x^2(y-z)$; $Q = y^2(z-x)$, $R = z^2(x-y)$

The Laglange's auxillary eqn $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ (24)

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \longrightarrow (2)$$

Choose Multipliers such that $\boxed{l \cdot P + m \cdot Q + n \cdot R = 0}$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l \cdot dx + m \cdot dy + n \cdot dz}{l \cdot P + m \cdot Q + n \cdot R} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

Case (1)
from (2)

$$\frac{\frac{dx}{x^2}}{y-z} = \frac{\frac{dy}{y^2}}{z-x} = \frac{\frac{dz}{z^2}}{x-y}$$

$$\frac{\frac{dx}{x^2}}{\frac{1}{x^2}\left[x^2(y-z)\right]} = \frac{\frac{dy}{y^2}}{\frac{1}{y^2}\left[y^2(z-x)\right]} = \frac{\frac{dz}{z^2}}{\frac{1}{z^2}\left[z^2(x-y)\right]}$$

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{(y-z) + (z-x) + (x-y)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

Integrating $\int \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$

$$\left(\frac{-1}{x}\right) + \left(\frac{-1}{y}\right) + \left(\frac{-1}{z}\right) = c_1$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1$$

Case (2) from (2)

$$\frac{\frac{dx}{x}}{x(y-z)} = \frac{\frac{dy}{y}}{y(z-x)} = \frac{\frac{dz}{z}}{z(x-y)}$$

Integrating $\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2.$$

$\therefore$ General Soln, $f(u,v) = 0 \Rightarrow f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$

7. Solve : $x(y-z)p + y(z-x)q = z(x-y)$

Soln: The given P.D.E $x(y-z)p + y(z-x)q = z(x-y) \rightarrow (1)$

is the Lagrange's P.D.E is of the form $P.p + Q.q = R$

Here $P = x(y-z)$, $Q = y(z-x)$, $R = z(x-y)$

The Auxillary eqn $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \rightarrow (2)$$

By Method of Multipliers,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + n \cdot R}$$

Choose l, m, n such that $lP + mQ + nR = 0$

case(i): from (2) $\frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{dx+dy+dz}{0}$

$\because x(y-z)+y(z-x)+z(x-y) = 0$

$\therefore$ we get $dx + dy + dz = 0$

$$\int dx + \int dy + \int dz = c$$

$$x+y+z = c_1 \rightarrow (3)$$

case(ii): Also $\frac{\frac{dx}{x}}{(y-z)} = \frac{\frac{dy}{y}}{z-x} = \frac{\frac{dz}{z}}{x-y}$

since $(y-z) + (z-x) + (x-y) = 0$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = c$

$$\log x + \log y + \log z = \log c$$

$$\log(x \cdot y \cdot z) = \log c$$

$$xyz = c_2$$

$\therefore$ General Solution $f(u,v) = 0 \Rightarrow \boxed{f(x+y+z, xyz) = 0}$

8. Solve: $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

Soln: The given P.D.E $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2) \rightarrow (1)$

It is of the form $P.p + Q.q = R$

Here $P = x(y^2+z)$, $Q = -y(x^2+z)$; $R = z(x^2-y^2)$

The Auxillary eqn in Lagrange's method $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \rightarrow (2)$

Take multipliers such that the sum of denominators equal to zero.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

Case(i) take multipliers $l = \frac{1}{x}$, $m = \frac{1}{y}$, $n = \frac{1}{z}$

from (2) $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$

$$\frac{\frac{1}{x}dx}{(y^2+z)} = \frac{\frac{1}{y}dy}{-(x^2+z)} = \frac{\frac{1}{z}dz}{(x^2-y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y^2+z) - (x^2+z) + (x^2-y^2)}$$

$\because (y^2+z) - (x^2+z) + (x^2-y^2) = 0$

$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = c$

$\log x + \log y + \log z = \log c$

$\log(xyz) = \log c$

$\boxed{xyz = c_1} \rightarrow (3)$

Case(ii) Take Multipliers $l = x$, $m = y$, $n = -1$

from (2) $\frac{x\,dx + y\,dy - (1)dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} = \frac{x\,dx + y\,dy - dz}{0}$

$\therefore x\,dx + y\,dy - dz = 0$

$\int x\,dx + \int y\,dy - \int dz = c \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} - z = c$

$x^2 + y^2 - 2z = 2c = c_2$

$\therefore$ General Solution $f(u,v) = 0 \Rightarrow \boxed{f(xyz, x^2+y^2-2z) = 0}$

9. Solve: $z - xp - yq = a\sqrt{x^2+y^2+z^2}$

Soln: Given P.D.E $z - xp - yq = a\sqrt{x^2+y^2+z^2}$.

write Lagrange's P.D.E form $P \cdot p + Q \cdot q = R$

$$xp + yq = z - a\sqrt{x^2+y^2+z^2}$$

Here $P = x$, $Q = y$, $R = z - a\sqrt{x^2+y^2+z^2}$.

The Auxillary eqn $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2+y^2+z^2}} \rightarrow (2)$$

By Multipliers

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}}$$

Let $x^2+y^2+z^2 = u^2$

$2x\,dx + 2y\,dy + 2z\,dz = 2u\,du$

$\therefore x\,dx + y\,dy + z\,dz = u\,du$

$\because \sqrt{x^2+y^2+z^2} = u$

from (2) $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - au}$

take $\boxed{l = x, m = y, n = z}$

$$\frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z(z - au)} = \frac{u\,du}{(x^2+y^2+z^2) - azu}$$

$$= \frac{u\,du}{u^2 - azu} = \frac{du}{u - az}$$

$$\therefore \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - au} = \frac{du}{u - az} \rightarrow (3)$$

$\therefore \frac{dx}{x} = \frac{dy}{y}$

$\int \frac{1}{x}dx - \int \frac{1}{y}dy = c$

$\log x - \log y = \log c$

$\log\left(\frac{x}{y}\right) = \log c$

$\frac{x}{y} = c_1$

$\frac{dz}{z - au} = \frac{du}{u - az}$

$\frac{dz + du}{(z - au) + (u - az)} = \frac{dz + du}{z(1-a) + u(1-a)}$

$= \frac{dz + du}{(1-a)[u+z]}$

$\therefore \frac{dx}{x} = \frac{dz + du}{(1-a)[u+z]}$

$(1-a)\int \frac{1}{x}dx = \int \frac{1}{u+z}dz + \int \frac{1}{u+z}du + c$

$(1-a)\log x = \log(u+z) + \log(u+z) + \log$

$\log x^{(1-a)} = \log(u+z)^2 + \log c$

$\log\left(\frac{x^{(1-a)}}{(u+z)^2}\right) = \log c_2$

$c_2 = \frac{x^{(1-a)}}{(u+z)^2}$

$$\therefore G.S. = f\left(\frac{x}{y}, \frac{x^{(1-a)}}{[\sqrt{x^2+y^2+z^2} + z]^2}\right) = 0$$