510 ~~507~~ Numerical Solution of ordinary differential Equations.

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516
04150 Many problems in science & Engineering can be formulated into ordinary differential eqns. and then we use Analytical Methods to solve these d. Eqns. Some times equations appearing in physical problems do not belonging to any of these familiar types for such problems we use numerical methods for solving.

Solution of a differential equation:-

The solution of an ordinary d.E in which 'x' is independent Variable and y depends on 'x' (f(x)). Here we use the following Neumerical methods (Techniques) to solve the ordinary differential eqns.

1. Taylor's series Method ✓
2. Euler's Method ✓
3. Modified Euler Method
4. Picard's method of successive approximation ✓
5. Runge - Kutta Method ✓
6. predictor correctar Methods
   (a). Milne predictor corrector formula
   (b). Adams Moulton method.

For the solution of ordinary differential eqns. Consider the general first order d. eqn.

$$\frac{dy}{dx} = f(x, y), \text{ with the initial Condition } y(x_0) = y_0$$

Initial & boundary conditions ;

An ordinary differential equation of n^{th} order is of the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right) = 0$

It's general solution will contain 'n' arbitrary constants. ~~and~~ ~~if~~ i.e, $f(x, y, c_1, c_2 \cdots c_n) = 0$

To obtain It's particular solution n-conditions must be given so that the constants $c_1, c_2 \cdots c_n$ can be determined. The problems in which $y, y', y'', \cdots y^{(n-1)}$ are all specified at some value of $x = x_0$ are called initial-value problems. If the conditions on y are prescribed at n distinct points then the problems are called boundary-value problems.

In this chapter we discuss numerical methods to solve initial value problems.

① . Taylor-series Method :-

To find the numerical solution of d. eqn

$$\frac{dy}{dx} = f(x, y), \quad \text{initial condition } y(x_0) = y_0$$

$y(x)$, can be expanded about the point $x = x_0$ (or) $(x - x_0)$ in taylor's series (powers of $(x - x_0)$)

$$y(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \cdots + \frac{(x - x_0)^n}{n!} y^n(x_0) + \cdots$$

where $y^i(x_0)$ is i^{th} derivative of $y(x)$ at $x = x_0$.

To evaluate $y(x)$ first we should know the values of derivatives

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{d}{dx}\left[f(x, y)\right]$$

$$= \frac{\partial}{\partial x}\left[f(x, y)\right] + \frac{\partial}{\partial y}\left[f(x, y)\right] \cdot \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot f \qquad \left(\because f_x = \frac{\partial f}{\partial x},\ f_y = \frac{\partial f}{\partial y},\ f = \frac{dy}{dx} \right)$$

$$= f_x + f \cdot f_y$$

$$y''' = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right]$$

$$= \frac{d}{dx}\left[f_x + f \cdot f_y\right]$$

$$= \frac{\partial}{\partial x}\left[f_x + f \cdot f_y\right] + \frac{\partial}{\partial y}\left[f_x + f \cdot f_y\right] \cdot \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + f \cdot f_y^2 + f_y \cdot f_y$$

$$y(x) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

$$y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \cdots$$

Using Taylor's series Method solve $y' = xy + y^2$, $y(0) = 1$ at $x = 0.1, 0.2, 0.3$

Given that

$$y' = xy + y^2 \quad \text{and} \quad y(0) = 1 \quad \left(\because y' = \frac{dy}{dx}\right)$$

$$y(x_0) = y_0$$

$x_0 = 0$, $y_0 = 1$

$$y'_0 = x_0 y_0 + y_0^2 = 0(1) + 1^2 = 1$$

$$y''_0 = xy' + y(1) + 2y \cdot y'$$

$$= x_0(y'_0) + y_0 + 2y_0 \cdot y'_0$$

$$= 0 + 1 + 2 = 3$$

$$y'''_0 = xy'' + y'(1) + y' + 2[y_0 y''_0 + y'_0 y'_0]$$

$$= xy'' + 2y' + 2y_0'^2 + 2y_0 y''_0$$

$$= 0 + 2 + 2 + 2(3) = 10$$

$$y^{iv}_0 = xy''' + y''(1) + 2y'' + 4y' \cdot y'' + 2[y_0 y'''_0 + y''_0 y'_0]$$

$$= xy''' + 3y'' + 4y'y'' + 2y'y'' + 2yy'''$$

$$y_0^{iv} = 0+3+6+12+6+20 = 47$$

(i). For $x=0.1$:

By Taylor's series we have

$$y(x) = y_0 + h\,y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots$$

where $h = 0.1$

$$y(0.1) = 1 + (0.1)(1) + \frac{(0.01)}{2}(3) + \frac{0.001}{6}(10) + \frac{0.0001}{24}(47) + \cdots$$

$$= 1.1 + 0.015 + 0.00166 + 0.0001958$$

$$\boxed{y(0.1) = 1.11686}$$

For $x = 0.2$ $\quad x_1 = 0.1, \quad y_1 = 1.11686$

$$y(0.2) = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1'''' + \cdots$$

where $x_1 = 0.1, \quad y_1 = 1.11686$

$$y_1' = x_1 y_1 + y_1^2 = (0.1)(1.11686) + (1.11686)^2$$

$$= 0.111686 + 1.2474$$

$$y_1' = 1.359086$$

$$y_1'' = x_1 y_1' + y_1 + 2 y_1 y_1'$$

$$= (0.1)(1.359086) + (1.11686) + 2(1.11686)(1.359086)$$

$$= 4.2859$$

$$y_1''' = x y_1'' + 2 y_1' + 2 y_1'^2 + 2 y y_1''$$

$$= (0.1)(4.2885) + 2(1.3590) + 2(1.3590)^2 + 2(1.3590)(4.2855)$$

$$= 16.42026$$

$$y_1^{iv} = x_1 y_1''' + y_1'' + 2y_1'' + 4y_1' y_1'' + 2 y_1' y_1'' + 2 y y'''$$

$$= 86.15743$$

$$\therefore y(0.2) = 1.1168 + (0.1)(1.3590) + \frac{(0.1)^2}{2}(4.2885) + \frac{(0.1)^3}{6}(16.4202) + \frac{(0.1)^4}{24}(86.1574)$$

$$y(0.2) = 1.1168 + (0.1)(13590) + \frac{(0.01)}{2} \times (4.2885) + \frac{(0.001)}{6}(16.42 + \frac{(0.0001)}{24}(86.1574)$$

* Solve: $y' = x+y$ given $y(1) = 0$. Find $y(1.1)$ & $y(1.2)$ by using Taylor's series method.

* Find $y(0.1)$ and $y(0.2)$ using Taylor's series method given that $y' = y^2 + x$ and $y(0) = 1$

* Solve the equation $\frac{dy}{dx} = x - y^2$ with the conditions $y(0) = 1$ and $y'(0) = 1$. Find $y(0.2)$ and $y(0.4)$ using Taylor's series method.

Soln. Given that, $\frac{dy}{dx} = x - y^2$

$y(x_0) = y_0 \Rightarrow y(0) = 1$

$y'(x_0) = y_0' \Rightarrow y'(0) = 1$

$y' = x - y^2 \rightarrow ①$

diff ① w.r.t x both sides.

$x_0 = 0 \Rightarrow y_0 = 1$

$x_0 = 0 \Rightarrow y_0' = 1$

$$y'' = 1 - 2yy' \rightarrow ②$$

$$y''' = 0 - 2[yy'' + y'y'] \rightarrow ③$$

$$= -2yy'' - 2(y')^2$$

$$y'''' = -2[yy''' + y''y'] - 2 \cdot 2y' \cdot y''$$

$$= -4y' \cdot y'' - 2y'y'' - 2yy''' \rightarrow ④$$

According to Taylor's series expansion

$$f(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0'''' + \dots$$

$$y_0' = x_0 - y_0^2 = 0 - (1)^2 = -1$$

$$y_0'' = 1 - 2y_0 \cdot y_0' \Rightarrow 1 - 2(1) \times (1)^2 = -1$$

$y_0''' = -2(y_0')^2 - 2y_0 y_0''$

$= -2(1)^2 - 2(1)(-1)$

$= -2+2 = 0$

$y_0'''' = -4(y_0')y_0'' - 2y_0' y_0'' - 2y_0 y_0'''$

$= -4(1).(1) - 2(1)(-1) - 2(1)(0)$

$= 4+2 = 6$

$f(x)$, $x = 0.2$

$y(0.2) = 1 + (0.2)(1) + \frac{(0.04)}{2} \times (-1) + \frac{0.008}{6} \times (0) + \frac{0.0016}{24} \times (6)$

$= 1+0.2 - 0.02 + 0.0004$

$\boxed{y(0.2) = 1.1804}$

$x_1 = 0.2 \qquad y_1 = 1.1804$

$y(0.4) = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1'''' + \cdots$

$y_1' = x_1 - y_1^2 \rightarrow y_1' = 0.2 - (1.1804)^2 =$

$y_1'' = 1 - 2y_1 y_1' \Rightarrow 1 - 2(1.1804)(\text{''})$

$y_1''' = -2(y_1')^2 - 2y_1 y_1''$

$y_1'''' = -4(y_1')y_1'' - 2y_1' y_1'' - 2y_1 y_1'''$

Taylor's series Method for simultaneous first order differential equations.

The eqns of the type $\frac{dy}{dx} = f(x,y,z)$, $\frac{dz}{dx} = g(x,y,z)$ with initial conditions $y(x_0) = y_0$, $z(x_0) = z_0$

* Find $y(0.1)$, $y(0.2)$, $z(0.1)$, $z(0.2)$ given $\frac{dy}{dx} = x+z$, $\frac{dz}{dx} = x - y^2$ with $y(0) = 2$, $z(0) = 1$ by using Taylor's series method.

Soln: Given that $y' = x+z$

Take $y(0) = 2 \rightarrow y(x_0) = y_0 \qquad h = 0.1$

$x_0 = 0, \; y_0 = 2$

Now we have to find $y(x_1) = y(0.1)$ and $y(x_2) = y(0.2)$

write the derivates, given $y' = x+z$

$$y'' = 1+z'$$
$$y''' = 0+z''$$
$$y^{iv} = z'''$$

and $z' = x-y^2$ given $z(0)=1 \Rightarrow x_0 = 0,\ z_0 = 1$

$h = 0.1$

we have to find $z(x_1) = z(0.1)$

$z(x_2) = z(0.2)$

Now $z' = x-y^2$

$$z'' = 1-2.y.y'$$
$$z''' = 0-2[y'.y'+y.y''] = -2(y')^2 - 2yy''.$$
$$z'''' = -2.2y'.y'' - 2[y''.y'+y.y''']$$
$$= -4y'.y'' - 2y'y'' \cancel{+} 2y.y'''$$
$$= -6y'y'' - 2yy'''$$

By Taylor series for $y(x_1)$ & $z(x_1)$ we have

$$y(x_1) = y(0.1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv}$$
$$z(x_1) = z(0.1) = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots$$

we have

$$y_0' = x_0 + z_0 = 0+1 = 1$$
$$y_0'' = 1+z_0' = 1+(x_0-y_0^2) = 1+(0-2^2) = 1-4 = -3$$
$$y_0''' = z_0'' = 1-2y_0y_0' = 1-2(2).(1) = 1-4 = -3$$
$$y_0^{iv} = z_0''' = -2(y_0')^2 - 2y_0y_0'' = -2(1)-2(2)(-3)$$
$$= -2+12 = 10$$

and $z_0 = 1,\ z_0' = x_0 - y_0^2 = 0-2^2 = -4$

$$z_0'' = 1-2y_0.y_0' \Rightarrow 1-2(2)(1) = 1-4 = -3$$
$$z_0''' = -2(y_0')^2 - 2y_0y_0'' = -2(1)^2 - 2(2)(-3)$$
$$= -2+12 = 10$$

substituting

$y(x_1) = y(0.1) = 2 + (0.1)(1) + \frac{0.01}{2}(-3) + \frac{0.001}{6}(-3) + \cdots$

$= 2 + 0.1 - 0.015 - 0.0005 + \cdots$

$= 2.0845$

Taylor's series $y(x_2) = y(0.2)$ & $z(x_2) = z(0.2)$:-

we have.

$x_1 = 0.1$ & $h = 0.1$, $y_1 = 2.0845$

$y_1' = x_1 + z_1 = 0.1 + 0.5867 = 0.6867$

$y_1'' = 1 + z_1' = 1 + x_1 - y_1^2$

$= 1 + 0.1 - (2.0845)^2$

$= -3.2451$

$y_1''' = z_1'' = 1 - 2y_1 \cdot y_1'$

$= 1 - 2(2.0845)(0.6867)$

$= -1.8628$

and $z_1 = 0.5867$

$z_1' = x_1 - y_1^2$

$= 0.1 - (2.0845)^2$

$= -4.2451$

$z_1''' = -2[y_1 \cdot y_1'' + (y_1')^2]$

$= -2[(2.0845) \cdot (-3.2451) + (0.6867)^2]$

$= -2[-6.7644 + 0.4716]$

$= -2[-6.2928]$

$= 12.5856$

substituting

$y(0.2) = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$

$= 2.0845 + (0.1)(0.6867) + \frac{0.01}{2}(-3.2451) + \frac{0.001}{6}(-1.8628) + \cdots$

Evaluate the values of $y(1.1)$ and $y(1.2)$ from

$y'' + y^2 y' = x^3$

$y(1) = 1$

$y'(1) = 1$

Using Taylor's Series Method

$= 2.0845 + 0.06867 - 0.0162 - 0.0003104 + \cdots$

$= 2.1367$ (correction 4 decimals)

$z(0.2) = 0.5867 + (0.1)(-4.2451) + \frac{0.01}{2}(-1.8628)$
$+ 0.001(12.5856)$

$= 0.5867 - 0.42451 - 0.009314 + 0.0020976 + \cdots$

$= \underline{0.15497}$

PICARD's Method of successive approximations

Consider the differential eqn $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$. → (1)

To get the solution, $dy = f(x,y)\,dx$

Integrating in the interval $[x_0, x]$ we get

$$\int_{x=x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$y\Big]_{x_0}^{x} = \int_{x_0}^{x} f(x,y)\,dx$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\,dx$$

$$y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)\,dx$$

(or) $\boxed{y(x) = y_0 + \int_{x_0}^{x} f(x,y)\,dx}$ → (2)

It can be solved by successive approximations

To get first approximation, $y^{(1)}(x)$, put $y = y_0$ in the integrand of (2) we get

$$y^{(1)}(x) = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx$$

$$y^{(2)}(x) = y_0 + \int_{x_0}^{x} f(x,y_1)\,dx$$

$\vdots$

Proceding in this way we get the n^{th} approximation $y^{(n)}(x)$ for y is,

$$\boxed{y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx} \rightarrow (3)$$

$n = 1, 2, 3, \ldots$

this is called picard's Iterative (successive approxi^n) formula. Repeat this process until two successive approximations $y^{(i)}, y^{(i-1)}$ are sufficiently close.

* Find the value of y for $x = 0.4$ by picard's method given that $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 0$.

Given that $\frac{dy}{dx} = f(x,y) = x^2 + y^2$, $y(x_0) = y_0 \Rightarrow y(0) = 0$

i.e $x_0 = 0, y_0 = 0$

By picard's method,

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{n-1})\,dx$$

First app: $y^{(1)} = 0 + \int_0^x f(x^2 + y_0^2)\,dx$

$$= \int_0^x (x^2 + y_0^2)\,dx \quad (\because y_0 = 0)$$

$$= \int_0^x (x^2 + 0)\,dx = \left.\frac{x^3}{3}\right]_0^x = \frac{x^3}{3}$$

Second approximation:

$$y^{(2)}(x) = y_0 + \int_{x_0}^{x} (x^2 + y_1^2)\,dx$$

$$= 0 + \int_0^x \left[x^2 + \left(\frac{x^3}{3}\right)^2\right] dx$$

$$= \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx$$

$$= \left.\frac{x^3}{3} + \frac{x^7}{7 \times 9}\right]_0^x = \frac{x^3}{3} + \frac{x^7}{63}$$

calculating $y^3(x)$ is tedious take upto $y^{(2)}(x)$

For $x = 0.4 \Rightarrow$ $y = \frac{(0.4)^3}{3} + \frac{(0.4)^7}{63}$

$= 0.021333 + 0.00026$

$= 0.0213663$

$= 0.0214$

* Find the value of y for $x = 0.1$ by Picard's Method given that $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$

(or) Find $y(0.1)$ given $y' = \frac{y-x}{y+x}$, $y(0) = 1$ by Picard's Method.

Soln: Given that $y' = f(x,y) = \frac{y-x}{y+x}$, $y(x_0) = y_0$

$y(0) = 1$

i.e $x_0 = 0, y_0 = 1$

By Picard's Method

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

$$= y_0 + \int_{x_0}^{x} \left(\frac{y-x}{y+x}\right) dx$$

First approximation.

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0) =$$

$$= y_0 + \int_{x_0}^{x} \left(\frac{y_0 - x}{y_0 + x}\right) dx$$

$$= 1 + \int_{0}^{x} \left(\frac{1-x}{1+x}\right) dx$$

$$= 1 + \int_{0}^{x} \left(-1 + \frac{2}{1+x}\right) dx$$

$$= 1 + \left[\int_{0}^{x} (-1)\,dx + 2\int_{0}^{x} \frac{1}{1+x}\,dx\right] = 1 + \left[-x + 2\log(1+x)\right]_0^x$$

$$= 1 + [-x + 2\log(1+x)] - [0 + 2\log(1+0)]$$

$$= 1 - x + 2\log(1+x)$$

For the second approximation,

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= y_0 + \int_{x_0}^{x} \left(\frac{y_1 - x}{y_1 + x}\right) dx$$

$$= 1 + \int_0^x \frac{[1 - x + 2\log(1+x)] - x}{[1 - x + 2\log(1+x)] + x}\, dx$$

$$= 1 + \int_0^x \left[\frac{1 - 2x + 2\log(1+x)}{1 + 2\log(1+x)}\right] dx$$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1 + 2\log(1+x)}\right] dx$$

$$= 1 + \int_0^x dx - 2\int_0^x \left[\frac{x}{1 + 2\log(1+x)}\right] dx$$

$$= 1 + x - 2\int_0^x \frac{x}{1 + 2\log(1+x)}\, dx$$

which is very difficult to integrate.

Hence write first approximation $y_1^{(1)}$ is the value of y

$$\therefore y(x) = y^{(1)} = 1 - x + 2\log(1+x)$$

put $x = 0.1$

$$y(0.1) = 1 - (0.1) + 2\log(1 + 0.1)$$

$$= 1 - 0.1 + 2\log(1.1)$$

$$= 1 - 0.1 + 0.1906$$

$$= 1.0906 \quad \text{(correcting 4 decimals)}$$

* Solve: $y' = y - x^2$, $y(0) = 1$, by picard's method upto Fourth approximations. Hence find $y(0.1)$ & $y(0.2)$

Soln: $f(x,y) = y - x^2$ $\quad y(x_0) = y_0$ $\quad y(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$ → (1)

$y(0) = 1$

By picard's method, First approximation.

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\,dx$$

$$= 1 + \int_0^x (y_0 - x^2)\,dx$$

$$= 1 + \int_0^x (1 - x^2)\,dx \Rightarrow 1 + \int_0^x dx + \int_0^x -x^2\,dx$$

$$= 1 + x\Big]_0^x - \frac{x^3}{3}\Big]_0^x$$

$$= 1 + x - \frac{x^3}{3}$$

Second approximation,

$$y^{(2)} = y_1 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

$$= y_1 + \int_{x_0}^{x} (y^{(1)} - x^2)\,dx$$

$$= 1 + \int_0^x \left(1 + x - \frac{x^3}{3} - x^2\right)dx$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3}$$

$a^2(1 + \cos^2\theta + 2\cos\theta)$
$+ a^2\sin^2\theta$

$a^2 + a^2\cos^2\theta + 2a^2\cos\theta$
$+ a^2\sin^2\theta$

$\& (\cos^2\theta, \sin^2\theta)$

Third approximation,

$$y^{(3)} = y_1 + \int_{x_0}^{x} f(x, y^{(2)})\,dx$$

$$= y_1 + \int_{x_0}^{x} (y^{(2)} - x^2)\,dx$$

$$= 1 + \int_0^x \left(1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3} - x^2\right)dx$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^5}{60} - \frac{x^4}{12} - \frac{x^3}{3}$$

Fourth approximation

$$y^{(4)} = 1 + \int_0^x \left[\left(1+x+\frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{60}\right) - x^2\right] dx$$

$$= 1 + \int_0^x \left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{60}\right) dx$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} - \frac{x^5}{60} - \frac{x^6}{360} \rightarrow (2)$$

Calculating $y^{(5)}$ is tedious stop. take $y^{(4)}$ as the value of 'y'.

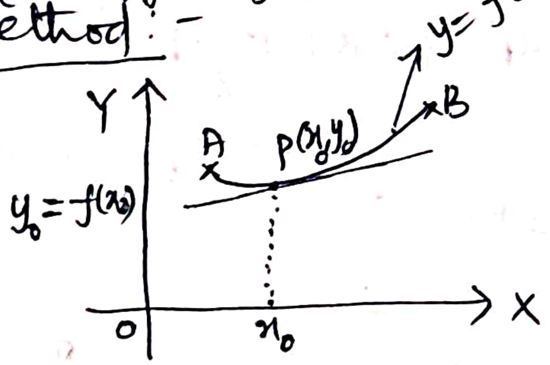
Putting $x = 0.1$ in the above (2)

$$y^{(4)}(x) = y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} - \frac{(0.1)^4}{24} - \frac{(0.1)^5}{60} - \frac{(0.1)^6}{360}$$

$$y^{(4)}(x) = y(0.2) = 1 + (0.2) + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{6} - \frac{(0.2)^4}{24} - \frac{(0.2)^5}{60} - \frac{(0.2)^6}{360}$$

* Find the value of y at $x = 0.4$ by Picard's Method given that $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 0$.

Euler's Method :-



Suppose we wish to solve the equation $\frac{dy}{dx} = f(x,y)$ subject to the condition that $y(x_0) = y_0$

i.e the curve $y = f(x)$ passes through the point (x_0, y_0) and slope is $m = \frac{dy}{dx} = f(x,y)_{at (x_0,y_0)}$

$\therefore$ The equation of Tangent at (x_0, y_0) is
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad x_1 \; y_1$

$$y - y_0 = \cancel{y_0} \; m \cdot (x - x_0)$$

$$y - y_0 = \left.\frac{dy}{dx}\right)_{at (x_0,y_0)} \cdot (x - x_0)$$

$$\boxed{\because \frac{dy}{dx} = f(x,y)}$$

i.e $y - y_0 = f(x_0, y_0)(x - x_0)$

$y = y_0 + (x - x_0) f(x_0, y_0)$ $\qquad x_0 + h = x_1$, $h = x_1 - x_0$

$(\because h = b - a = x - x_0)$

$y = y_0 + h\, f(x_0, y_0)$

This gives the approximate value of y at $x = x_1$

we shall denote this by y_1.

similarly $y_2 = y_1 + h f(x_1, y_1)$

$\vdots$

In general; the recursive relation

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

* solve by Euler's Method $\frac{dy}{dx} = x + y$, $y(0) = 1$ and find $y(0.3)$ taking step size $h = 0.1$. Compare the result obtained by this method with result obtained by Analytical Method.

Soln: Given $f(x, y) = \frac{dy}{dx} = x + y$, $\quad y(x_0) = y_0$, $y(0) = 1$

i.e $x_0 = 0$, $y_0 = 1$, $h = 0.1$

Euler's algorithm,

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_1 = y_0 + h f(x_0, y_0) \quad (\because n = 0)$$

$y(0.1) = 1 + (0.1)(x_0 + y_0)$

$= 1 + (0.1)(0 + 1) = \boxed{1.1}$

$y_2 = y_1 + h f(x_1, y_1)$

$= y_1 + h(x_1 + y_1)$

$= (1.1) + (0.1)[0.1 + 1.1] = 1.1 + (0.1)(1.2)$

$= 1.1 + 0.12 = \boxed{1.22}$

$$y_3 = y_2 + h\, f(x_2, y_2)$$
$$= y_2 + h\,(x_2 + y_2)$$
$$= (1.22) + (0.1)[0.2 + 0.22]$$
$$= 1.22 + 0.142$$
$$\boxed{y(0.3) = 1.362}$$

To Compare with Analytical Solution:-

Given that $\frac{dy}{dx} = x + y$

$$\frac{dy}{dx} - y = x$$

which is linear equation in y, $\frac{dy}{dx} + Py = Q$

$P = -1$, $Q = x$

$$I.F = e^{\int P dx} = e^{-\int dx} = e^{-x}$$

$$G.S = y \times I.F = \int Q.\, I.F\, dx + C$$
$$= y \times e^{-x} = \int x \cdot e^{-x}\, dx + C$$
$$y e^{-x} = -(x+1)e^{-x} + C$$

$$y = -(x+1) + ce^{x}$$

$= x\int e^{-x}dx + e^{-x}\int x\,dx$

$x(-e^{-x}) + e^{-x}$

$u = x,\ dv = e^{-x}$, $v = -e^{-x}$

$\int u\,dv = uv - \int v\,du$

$= x(-e^{-x}) - \int(-e^{-x})(1)$

$= -xe^{-x} + -e^{-x}$

$= -(x+1)e^{-x}$

$x = 0 \Rightarrow y = 1$

$$1 = -1 + ce^{0}$$

$$c = 2$$

$\therefore$ ~~P.I~~ $\therefore G.S = \quad y = ce^{x} - (x+1)$

$$y = 2e^{x} - (x+1)$$

$$y(0.1) = 2e^{(0.1)} - (0.1 + 1)$$
$$= 2(1.10517) - 1.1$$
$$= 1.11034$$

$y' = x^2 - y$, $y(0) = 1$
find correct upto 4 decimal places the value of $y(0.1)$ Euler's method

$y(0.2) = 1.3428$, $y(0.3) = 1.5997$

x:	0	0.1	0.2	0.3
Euler y:	1	1.1	1.22	1.362
Exact y	1	1.11034	1.3428	1.5997

Modified Euler's Method :-

$\frac{dy}{dx} = f(x,y)$ given that $y = y_0$ at $x = x_0$

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$$

where $y_1^{(0)} = y_0 + h f(x_0, y_0)$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]$$

$$y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(k)}) \right]$$

If two successive values of $y_1^{(k)}$, $y_1^{(k+1)}$ are sufficiently close to one another, we will take common value as y_1.

Now we have $\frac{dy}{dx} = f(x,y)$ with $y = y_1$ at $x = x_1$.

To get $y_2 = y(x_2) = y(x_1 + h)$ continue........

* Using modified Euler method find $y(0.2)$ and $y(0.4)$ given $y' = y + e^x$, $y(0) = 0$

Soln: $f(x,y) = y + e^x$, $y(x_0) = y_0$, $y(0) = 0$, $h = 0.2$

Using Euler's formula, $y_1^{(0)} = y_0 + h\, f(x_0, y_0)$

$= 0 + (0.2) f(0,0)$

$= 0 + (0.2)\left[0 + e^0\right]$

$= 0.2$

$$y_1^{(1)} = y(0.2) = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$$

$= 0 + \frac{0.2}{2}\left[1 + (0.2 + e^{0.2})\right]$

$= 0.1\left[1 + 1.4214\right]$

$= 0.24214$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 0 + \frac{0.2}{2}\left[1 + f(0.2, 0.24214)\right]$$

$$= 0.1\left[1 + (0.24214) + e^{0.2}\right]$$

$$= 0.2463$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 0 + \frac{0.2}{2}\left[1 + f(0.2, 0.2463)\right]$$

$$= 0 + 0.1\left[1 + (0.2463 + e^{0.2})\right]$$

$$= 0.2468$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 0 + \frac{0.2}{2}\left[1 + (0.2, 0.2468)\right]$$

$$= (0.1)\left[1 + (0.2468 + 1.2214)\right]$$

$$= 0.2468$$

$\therefore$ $y_1^{(3)}$, $y_1^{(4)}$ are equal we take, $\boxed{y_1 = y(0.2) = 0.2468}$

To find $y_2 \Rightarrow y(0.2)$:

$x_1 = 0.2$, $y_1 = 0.2468$, $x_2 = 0.4$, $h = 0.2$

$$f(x_1, y_1) = f(0.2, 0.2468) = 0.2468 + e^{0.2} = 0.2468 + 1.2214$$
$$= 1.4682$$

First approximation,

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(0)})\right]$$

But $y_2^{(0)} = y_1 + h\, f(x_1, y_1)$

$$= 0.2468 + (0.2)[1.4682]$$

$$= 0.5404$$

$$\Rightarrow 0.2468+(0.1)[1.4682+f(0.4,0.5404)]$$
$$=0.2468+(0.1)[1.4682+[0.5404+e^{(0.4)}]]$$
$$=0.2468+(0.1)[1.4682+(0.5404+1.4918)]$$
$$=\underline{0.5968}$$

$$y_2^{(2)}=y_1+\frac{h}{2}[f(x_1,y_1)+f(x_2,y_2^{(1)})]$$
$$=0.2468+(0.1)[1.4682+f(0.4,0.5968)]$$
$$=0.2468+(0.1)[1.4682+(0.5968+1.4918)]$$
$$=0.6025$$

$$y_2^{(3)}=y_1+\frac{h}{2}[f(x_1,y_1)+f(x_2,y_2^{(2)})]$$
$$=0.2468+(0.1)[1.4682+f(0.4,0.6025)]$$
$$=0.603$$

$$y_2^{(4)}=y_2^{(4)}=y_1+\frac{h}{2}[f(x_1,y_1)+f(x_2,y_2^{(3)})]$$
$$=0.2468+(0.1)(1.4682+f(0.4,0.603))$$
$$=0.2468+(0.1)(1.4682+(0.603+1.4918))$$
$$=0.6031$$

$$y_2^{(5)}=y_1+\frac{h}{2}[f(x_1,y_1)+f(x_2,y_2^{(4)})]$$
$$=0.6031$$

$$\therefore \boxed{y_{x=0.2}=0.2468}$$

$$\boxed{y_{x=0.4}=0.6031}$$

* Given $y' = x + \sin y$ $y(0) = 1$ Compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ Using Euler's Modified Method.

Here $f(x,y) = x + \sin y = \frac{dy}{dx}$, $y(x_0) = y_0$

$y(0) = 1$

i.e $x_0 = 0$, $y_0 = 1$, $h = 0.2$

<u>To find</u> $y_1^{(1)}$ <u>i.e $y(0.2)$</u>

$$y_1^1 = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$$

where $f(x_0,y_0) = x_0 + \sin y_0 = 0 + \sin 1 = \sin 1$

$f(x_1, y_1^{(0)}) = f(0.2, 1.163) = 0.2 + \sin(1.163)$

$= 1.12$

$$y_1^{(0)} = y_0 + h f(x_0,y_0)$$
$$= 1 + (0.2)[x_0 + \sin y_0]$$
$$= 1 + (0.2)[0 + \sin 1]$$
$$= 1.163$$

$$y_1^{(1)} = 1 + \frac{0.2}{2}\left[\sin 1 + 1.12\right]$$
$$= 1.1961$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 1 + \frac{0.2}{2}\left[\sin 1 + [0.2 + \sin(1.1961)]\right]$$
$$= 1.2038$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x, y_1^{(2)})\right]$$
$$= 1 + \frac{0.2}{2}\left[0.8414 + x_0 + \sin(1.2038)\right]$$
$$= 1 + 0.1[0.8414 + 1.2098]$$
$$= 1.20482.$$

$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(3)})]$

$= 1 + \frac{0.2}{2}[0.8414 + 1.2045\cancel{2}]$

$= 1.2046$

$y_1^{(5)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(4)})] = 1.2046$

$\because y_1^{(4)} = y_1^{(5)} = 1.2046$, therefore $\boxed{y_1 = y(0.2) = 1.2046}$

To find y_2, i.e $y(0.4)$:-

take $x_1 = 0.2$, $y_1 = 1.2046$, $x_2 = 0.4$ & $h = 0.2$

$\therefore f(x_1, y_1) = f(0.2, 1.2046) = 0.2 + \sin(1.2046)$
$= 1.1337$

First approx

$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1,y_1) + f(x_2, y_2^{(0)})]$

$y_2^{(0)} = y_1 + h f(x_1, y_1)$ By Euler's formula.

$= 1.2046 + (0.2)[1.1337]$

$= 1.4313$

$y_2^{(1)} \Rightarrow 1.2046 + \frac{0.2}{2}[1.1337 + [x_2 + \sin y_2^{(0)}]]$

$= 1.2046 + \frac{0.2}{2}[1.1337 + 0.4 + \sin(1.4313)]$

$= 1.2046 + (0.1)[1.1337 + 1.4313]$

$= 1.4611$

$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1,y_1) + f(x_2, y_2^{(1)})]$

$= 1.4641$

$y_2^{(3)} = 1.4644$

$y_2^{(4)} = 1.4644$

$\therefore y_2^{(3)} = y_2^{(4)} = 1.4644$, therefore $\boxed{y_2 = y(0.4) = 1.4644}$

The above results write in table:

x	New y
0.2	1.163
0.2	1.1961
0.2	1.2038
0.2	1.20452
0.2	1.2046
0.2	1.2046

x	New y
0.4	1.4313
0.4	1.4611
0.4	1.4641
0.4	1.4644
0.4	1.4644
0.4	1.4644

Runge-Kutta Methods

Runge-Kutta Methods mainly discussed based on their order. Euler's & Modified Euler's Method are the Runge-Kutta methods of first and second orders respectively.

The merits of Runge-Kutta is, the self-starting feature and the demerit is it is required to evaluate $f(x,y)$ for several slightly different values of x and y in every step of the function. which gives less efficiency.

First order Runge-Kutta method :- (Euler's Method)

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= y_0 + h y_0' \qquad (\because y' = f(x,y),\ y_0' = f(x_0,y_0))$$

and also Taylor's series, $y_1 = y(x_0+h) = \underbrace{y_0 + h y_0'} + \frac{h^2}{2!} y_0'' + \dots$

i.e Euler's Method agree with Taylor's series solution upto the term h.

Hence, Euler's Method is, R-K method of First order.

Second order R-K Method :-

The Modified Euler's method is,

$$y_1^{(0)} = y_0 + h f(x_0, y_0) = y_0 + h f_0$$
$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(0)})]$$
$$= y_0 + \frac{h}{2}[f_0 + f(x_0+h, y_0+hf_0)] \qquad hf_0 = K_1,\ (h f(x_0+h, y_0+K_1) = K_2$$
$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

$$\therefore \boxed{y_1 = y_0 + \frac{1}{2}(K_1 + K_2)}$$

where $K_1 = h f(x_0, y_0)$ K_1

$K_2 = h f(x_0 + h, y_0 + h f_0)$

Third-order R-K Method:

$$\boxed{y_1 = y_0 + \frac{1}{6}(K_1 + 4K_2 + K_3)}$$

where $K_1 = h f(x_0, y_0)$

$K_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

and $K_3 = h f(x_0 + h, y_0 + 2K_2 - K_1)$

Fourth-order R-K Method:-

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$K_1 = h f(x_0, y_0)$

$K_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$K_3 = h f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_2)$

$K_4 = h f(x_0 + h, y_0 + K_3)$

Solve $\frac{dy}{dx} = xy$ using R.K method for $x = 0.2$ taking $h = 0.2$

given $y(0) = 1$, $y'(0) = 0$

Given that $\frac{dy}{dx} = f(x, y) = xy$ with $y(x_0) = y_0$, $y'(x)$; $y(0) = 1$, $y'(0) = 0$

$x_0 = 0$, $y_0 = 1$, $y_0' = 0$

Now we have to find $y(0.2)$

Applying Runge-Kutta Method

$K_1 = h f(x_0, y_0)$

$= (0.2) f(0, 1)$

$= (0.2)(0 \times 1) = 0$

$K_2 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$= (0.2) \times f\left(0 + \frac{0.2}{2}, 1 + \frac{0}{2}\right)$

$= (0.2)\, f(0.1, 1)$

$= (0.2)(0.1) = 0.02$

$K_3 = h\, f\{x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$

$= (0.2)\, f[0 + (0.1), 1 + \frac{0.02}{2})$

$= (0.2)\, f(0.1, 1.01)$

$= (0.2)\, (0.1) \times (1.01)$

$= 0.0202$

$K_4 = h\, [f(x_0 + h), y_0 + K_3]$

$= (0.2)\, \cancel{f}f(0 + 0.2), 1 + 0.0202]$

$= (0.2)\, [(0.2) \times 1.0202]$

$= 0.0408$

$\therefore y_1 = y(0.2) = y_0 + \frac{1}{6}[K_1 + 2(K_2 + K_3) + K_4]$

$= 1 + \frac{1}{6}[0 + 0.04 + 0.0404 + 0.0408]$

$= 1.0202$

Now we have $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$y_1 = 1.0202$

$K_1 = h\, f(x_1, y_1)$ $\quad (\because f(x,y) = xy)$

$= (0.2)[x_1 \times y_1]$

$= (0.2)[(0.2) \times (1.0202)]$

$= 0.040808$

$K_2 = h\, f(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2})$

$= (0.2)\ f[(0.2+0.1), (1.0202+0.020404)]$

$= (0.2) f[0.3, 1.040604]$

$= 0.2\ (0.3 \times 1.040604)$

$= 0.0624$

$K_3 = h\ f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$

$= (0.2).\ f[(0.2+0.1); (1.0202+0.0312)]$

$= (0.2)\ [0.3 \times 1.0514]$

$= 0.063084$

$K_4 = h\ f[x_1+h,\ y_1+K_3]$

$= (0.2) \times f[0.4, (1.0202+0.06308)]$

$= (0.2)\ [0.4 \times 1.08328]$

$= 0.0866624$

$\therefore\ y_2 = y(0.2) = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1.0202 + \frac{1}{6}[0.040808 + 0.1248 + 0.126168 + 0.0866624]$

$\boxed{y_2 = 1.08326}$

Apply the Fourth order R-K Method to find $y(0.1)$ and $y(0.2)$ given that $y' = xy + y^2$, $y(0) = 1$

Solve the following using R-K Fourth method $y' = y - x$, $y(0) = 2$, $h = 0.2$. Find $y(0.2)$

Given equation $y' = f(x,y) = y - x$

$x_0 = 0,\ y_0 = 2,\ h = 0.2$

$K_1 = h\ f(x_0, y_0)$

$= (0.2)\ f(0, 2)$

$= (0.2)\ [2-0]$

$= 0.4$

$$K_2 = h\, f\left[x_0 + \frac{h}{2},\ y_0 + K_1/2\right]$$
$$= (0.2)\ [0.1,\ 2+0.2]$$
$$= (0.2)\ [2.2 - 0.1]$$
$$= 0.42$$

$$K_3 = h \times f\left[x_0 + \frac{h}{2},\ y_0 + K_2/2\right]$$
$$= (0.2)\left[(0.1),\ 2 + 0.21\right]$$
$$= (0.2)\ [2.11]$$
$$= 0.422$$

$$K_4 = h \times f\left[x_0 + \frac{h}{2},\ y_0 + K_3/2\right]$$
$$= (0.2)\left[0.1,\ 2 + 0.211\right]$$
$$= (0.2)\ (2.111)$$
$$K_4 = 0.4222$$

$\therefore$ By R-K 4^{th} Method

$$y_1 = y(0.2) = y_0 + \frac{1}{6}\left(K_1 + 2(K_2 + K_3) + K_4\right)$$
$$= 2 + \frac{1}{6}\left[(0.4) + 2\left[0.42 + 0.422\right) + 0.4222\right]$$

$$\boxed{y(0.2) = 2.4177}$$