Unit-1 Electrostatics

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→ Electrostatic fields:

→ The science of Electromagnetism deals with mutual interactions of Electric charges both at rest as well as in motion.

→ The branch of Electromagnetism which deals with the interaction of the charges at rest is called "Electrostatics".

→ An electric charge has its effect in a region or space around it. This region is called an Electricfield of that charge.

→ The electric field produced due to stationary electric charge does not vary with time. It is called Time invariant and called static electric field.

→ The study of time invariant electric fields in a space or vacuum, produced by various types of static charge distributions is called Electrostatics.

→ Examples are: most of the computer peripheral devices like Keyboards, Touch pads, liquid crystal display etc work on the principle of Electrostatics.

→ A variety of machines such as X-ray machine & medical instruments used for Electrocardiograms, scanning etc uses the principle of Electrostatics.

→ Properties of Lines of force:

1) The lines of Force begin with a positive charge and Terminate at a negative charge.
2) The lines of never intersect with Each other
3) Both the amount of charge and number of field lines are propotional.
4) These force lines will not be able to travel through a conductor.

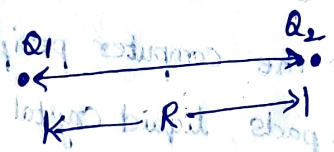
→ Collomb's law :-

Charles Augustein colddomb, French scientist, in 18th century performed a series of experiments to determined the force exerted between two objects having a static charge of Electricity.

Statement of coulomb's law :

The coulomb's law states that forces between two point charges Q_1 and Q_2

1) acts along the line joining the two point charges.
2) is directly proportional to the product $(Q_1 Q_2)$ of two charges.
3) is inversly proportional to the square of the distance between them.



Fig(1).

→ Consider two point charges Q_1 and Q_2 as shown in fig(1). Separated by distance R. The charge Q_1 exerts a force on Q_2, while Q_2 exerts a force on Q_1.

→ The force acts along the line joining Q_1 & Q_2.

The force exerted between them is repulsive if the ~~Sam~~ charges are of same polarity and it is attractive if the charges are of different polarity.

The force F between the charges can be expressed as

$$F \propto \frac{Q_1 Q_2}{R^2}$$

$$F = K \frac{Q_1 Q_2}{R^2}$$

where K = constant of proportionality

$$= \frac{1}{4\pi \varepsilon_0}$$

where ε_0 = permitivity of free space or vacuum

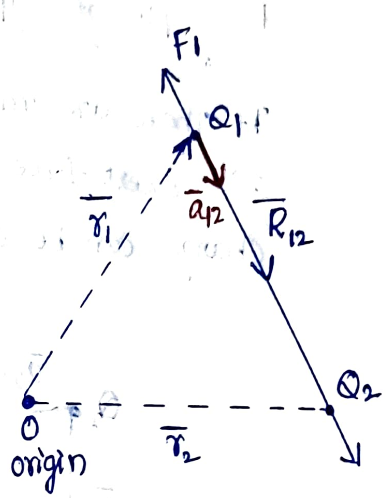
$= 8.854 \times 10^{-12}$ F/m.

where $Q_1 Q_2$ = product of two charges.

R = Distance b/w two charges.

Vector form of coulomb's law:

Consider two point charges Q_1 & Q_2 located at the points having position vectors $\bar{r}_1$ & $\bar{r}_2$ as shown in fig(2).



Fig(2)

Then the Force exerted by Q_1 on Q_2 acts along the direction $\bar{R}_{12}$, where $\bar{a}_{12}$ is the unit vector along $\bar{R}_{12}$.

Hence the force in vector form can be expressed as,

$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi \varepsilon_0 R_{12}^2} \bar{a}_{12}$$

where $\bar{a}_{12}$ = unit vector along $\bar{R}_{12}$ = $\frac{\text{Vector}}{\text{magnitude of Vector}}$

$$\bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{R}_{12}|} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|}$$

where $|R_{12}| = R =$ distance between the two charges.

$$\bar{a}_{21} = -\bar{a}_{12}$$

$$\bar{F}_1 = \frac{Q_1 Q_2}{4\pi \epsilon_0 R_{21}^2} (-\bar{a}_{21})$$

$$\bar{F}_1 = -\bar{F}_2$$

Hence the force exerted by two charges on each other is Equal but opposite in direction.

→ <u>Principle of Superposition</u>

If there are more than two point charges then Each will exert force on the other, then the net force on any charge can be obtained by the principle of Superposition.

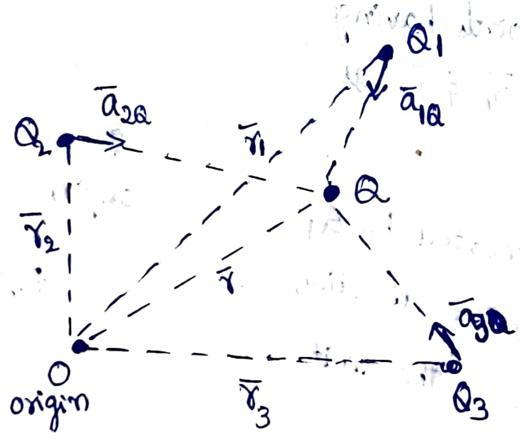


Fig (1).

consider a point charge Q surrounded by three other point charges Q_1, Q_2 & Q_3 as shown in fig

→ The Total force on Q is vector Sum of all the forces exerted on Q due to Each of other point charges Q_1, Q_2, & Q_3.

1) $$\bar{F}_{Q_1 Q} = \frac{Q_1 Q}{4\pi \epsilon_0 R_{1Q}^2} \bar{a}_{1Q} \quad , \quad \text{where } \bar{a}_{1Q} = \frac{\bar{r} - \bar{r}_1}{|\bar{r} - \bar{r}_1|}$$

2) Similarly force exerted due to Q_2 on Q is

$$F_{Q_2 Q} = \frac{Q_2 Q}{4\pi \epsilon_0 R_{2Q}^2} \bar{a}_{2Q} \quad , \quad \text{where } \bar{a}_{2Q} = \frac{\bar{r} - \bar{r}_2}{|\bar{r} - \bar{r}_2|}$$

3) Similarly force exerted due to Q_3 on Q is.

$$F_{Q_3 Q} = \frac{Q_3 Q}{4\pi \epsilon_0 R_{3Q}^2} \bar{a}_{3Q} \quad , \quad \text{where } \bar{a}_{3Q} = \frac{\bar{r} - \bar{r}_3}{|\bar{r} - \bar{r}_3|}$$

Hence Total force on Q is

$$\bar{F}_Q = \bar{F}_{Q_1 Q} + \bar{F}_{Q_2 Q} + \bar{F}_{Q_3 Q}.$$

If 'n' other charges present, then force exerted on Q due to all other charges is,

$$F_Q = \bar{F}_{Q_1 Q} + \bar{F}_{Q_2 Q} + \cdots + \bar{F}_{Q_n Q}$$

$$\Rightarrow \bar{F}_Q = \frac{Q}{4\pi \epsilon_0} \sum_{i=1}^{n} \frac{Q_i}{R_{iQ}^2} \frac{\bar{r} - \bar{r}_i}{|\bar{r} - \bar{r}_i|}$$

Problems

1) A charge $Q_1 = -20\ \mu C$ is located at $P(-6, 4, 6)$ and a charge $Q_2 = 50\ \mu C$ is located at $R(5, 8, -2)$ in a free space. find the force Exerted on Q_2 by Q_1 in vector form. The distances given are in meters.

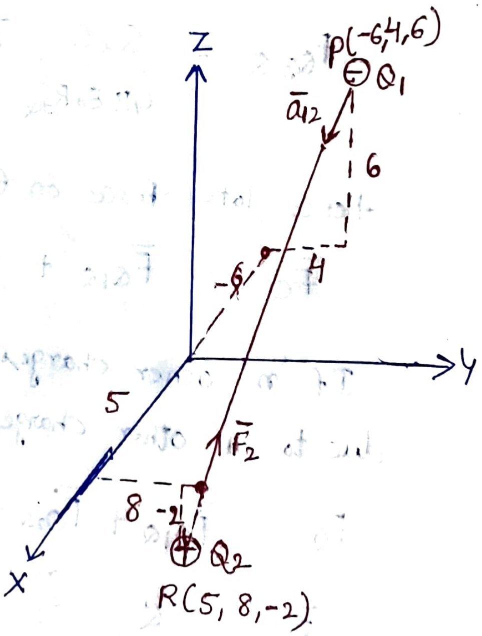
Sol: from the co ordinates of P & R, the respective position vectors are

$$\bar{P} = -6\,\bar{a}_x + 4\,\bar{a}_y + 6\,\bar{a}_z$$

$$\bar{R} = 5\,\bar{a}_x + 8\,\bar{a}_y - 2\,\bar{a}_z.$$

Force on Q_2 is given by

$$\Rightarrow \bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\,\bar{a}_{12}$$



$$\bar{R}_{12} = \bar{R}_{PR} = \bar{R} - \bar{P}$$

$$= [5-(-6)]\,\bar{a}_x + [8-4]\,\bar{a}_y + [-2-(6)]\,\bar{a}_z$$

$$\bar{R}_{12} = 11\,\bar{a}_x + 4\,\bar{a}_y - 8\,\bar{a}_z$$

$$|\bar{R}_{12}| = \sqrt{(11)^2 + (4)^2 + (-8)^2}$$

$$= 14.1774.$$

$$\bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{11\,\bar{a}_x + 4\,\bar{a}_y - 8\,\bar{a}_z}{14.1774}$$

$$\bar{a}_{12} = 0.7758\,\bar{a}_x + 0.2821\,\bar{a}_y - 0.5642\,\bar{a}_z$$

$$\Rightarrow \bar{F}_2 = \frac{-20\times10^{-6} \times 50\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(14.1774)^2}\,[\bar{a}_{12}]$$

$$= -0.0447\,[0.7758\,\bar{a}_x + 0.2821\,\bar{a}_y - 0.5642\,\bar{a}_z]$$

$\bar{F}_2 = -0.0346\,\bar{a}_x - 0.01261\,\bar{a}_y + 0.02522\,\bar{a}_z$ N

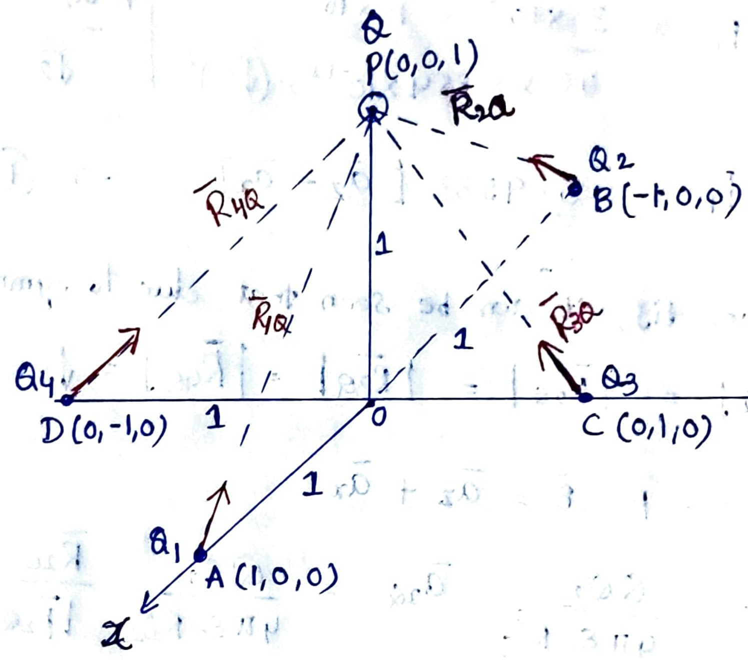
magnitude of force

$$|\bar{F}_2| = \sqrt{(-0.0346)^2 + (-0.01261)^2 + (0.02522)^2}$$

$$= 44.634 \text{ mN}$$

2) Four point charges each of 10 μC are placed in free space at the points (1,0,0), (-1,0,0), (0,1,0) & (0,-1,0) m respectively. Determine the force on a point charge of 30 μC located at a point (0,0,1) m.

Sol.



Use the principle of superposition as there are four charges exerting a force on the fifth charge Q.

→ The position vectors of four points at which the charges Q_1 to Q_4 are located can be obtained as,

A (1,0,0) ⇒ $\bar{A} = (1)\bar{a}_x + 0.\bar{a}_y + 0.\bar{a}_z \Rightarrow \bar{A} = \bar{a}_x$

B (-1,0,0) ⇒ $\bar{B} = (-1)\bar{a}_x + 0.\bar{a}_y + 0.\bar{a}_z \Rightarrow \bar{B} = -\bar{a}_x$

C (0,1,0) ⇒ $\bar{C} = 0.\bar{a}_x + 1.\bar{a}_y + 0.\bar{a}_z = \bar{C} = \bar{a}_y$

D (0,-1,0) ⇒ $\bar{D} = 0.\bar{a}_x + (-1).\bar{a}_y + 0.\bar{a}_z \Rightarrow \bar{D} = -\bar{a}_y$

Position vector of point P (0,0,1) is

$P(0,0,1) \Rightarrow \bar{P} = 0\cdot\bar{a}_x + 0\cdot\bar{a}_y + 1\cdot\bar{a}_z \Rightarrow \bar{P} = \bar{a}_z$

consider force on Q due to Q_1 alone is

$$\Rightarrow \bar{F}_1 = \frac{QQ_1}{4\pi\varepsilon_0 R_{1Q}^2}\,\bar{a}_{1Q} = \frac{QQ_1}{4\pi\varepsilon_0 R_{1Q}^2}\cdot\frac{\bar{R}_{1Q}}{|\bar{R}_{1Q}|}$$

$$\bar{R}_{1Q} = \bar{P} - \bar{A} = \bar{a}_z - \bar{a}_x$$

$$|R_{1Q}| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}.$$

$$\Rightarrow F_1 = \frac{30\times10^{-6} \times 10\times10^{-6}}{4\pi \times 8.854\times10^{-12} \times (\sqrt{2})^2}\left[\frac{\bar{a}_z - \bar{a}_x}{\sqrt{2}}\right]$$

$$F_1 = 0.9533\,[\bar{a}_z - \bar{a}_x]. \rightarrow (1)$$

from fig, it can be seen that due to symmetry,

$$|\bar{R}_{1Q}| = |\bar{R}_{2Q}| = |\bar{R}_{3Q}| = |\bar{R}_{4Q}| = \sqrt{2}$$

Now, $\bar{R}_{2Q} = \bar{P} - \bar{B} = \bar{a}_z + \bar{a}_x$

$$\Rightarrow F_2 = \frac{QQ_2}{4\pi\varepsilon_0 R_{2Q}^2}\,\bar{a}_{2Q} = \frac{QQ_2}{4\pi\varepsilon_0 R_{2Q}^2}\,\frac{\bar{R}_{2Q}}{|\bar{R}_{2Q}|}$$

$$F_2 = \frac{30\times10^{-6} \times 10\times10^{-6}}{4\pi \times 8.854\times10^{-12} \times(\sqrt{2})^2}\cdot\frac{\bar{a}_z + \bar{a}_x}{\sqrt{2}}$$

$$F_2 = 0.9533\,(\bar{a}_z + \bar{a}_x) \longrightarrow (2)$$

$$\bar{R}_{3Q} = \bar{P} - \bar{C} = \bar{a}_z - \bar{a}_y$$

$$\Rightarrow F_3 = \frac{QQ_3}{4\pi\varepsilon_0 R_{3Q}^2}\,\bar{a}_{3Q} = \frac{QQ_3}{4\pi\varepsilon_0 R_{3Q}^2}\,\frac{\bar{R}_{3Q}}{|\bar{R}_{3Q}|}$$

$$= \frac{30\times10^{-6} \times 10\times10^{-6}}{4\pi\times8.854\times10^{-12} \times(\sqrt{2})^2}\left[\frac{\bar{a}_z - \bar{a}_y}{\sqrt{2}}\right]$$

$$F_3 = 1.3481 \cdot \frac{\bar{a}_z - \bar{a}_y}{\sqrt{2}}$$

$$\Rightarrow F_3 = 0.9533 (\bar{a}_z - \bar{a}_y) \longrightarrow (3)$$

Now $\overline{R_{4Q}} = \bar{P} - \bar{D} = \bar{a}_z + \bar{a}_y$

$$\Rightarrow F_4 = \frac{Q Q_4}{4\pi \varepsilon_0 R_{4Q}^2} \bar{a}_{4Q} = \frac{Q Q_4}{4\pi \varepsilon_0 R_{4Q}^2} \cdot \frac{\overline{R_{4Q}}}{|\overline{R_{4Q}}|}$$

$$= \frac{30 \times 10^{-6} \times 10 \times 10^{-6}}{4\pi \times 8.854 \times 10^{-12} \times (\sqrt{2})^2} \left[\frac{\bar{a}_z + \bar{a}_y}{(\sqrt{2})} \right]$$

$$\Rightarrow F_4 = 0.9533 (\bar{a}_z + \bar{a}_y) \longrightarrow (4)$$

Hence Total force F_t, Exerted on Q due to all four charges is vector sum of all individual forces exerted on Q by the charges,

$$F_t = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$$

$$= 0.9533 [\cancel{\bar{a}_z} - \cancel{\bar{a}_x} + \bar{a}_z + \cancel{\bar{a}_x} + \bar{a}_z - \cancel{\bar{a}_y} + \bar{a}_z + \cancel{\bar{a}_y}]$$

$$= 0.9533 [4 \bar{a}_z]$$

$$\Rightarrow F_t = 3.813 \bar{a}_z \text{ Newton.}$$

→ Types of charge distribution :-

There are four types of continuous charge distribution which are :

1. point charge 2. Line charge 3. Surface charge

4. Volume charge.

1) Point charge : A point charge means that electric charge which is spreaded on a surface or space whose geometrical dimensions are very small compared to other dimensions.

→ The point charge has a position but not the dimensions.

It can be positive or negative.

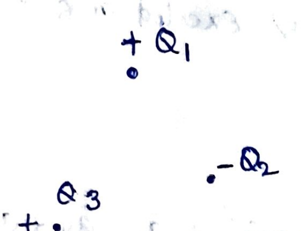
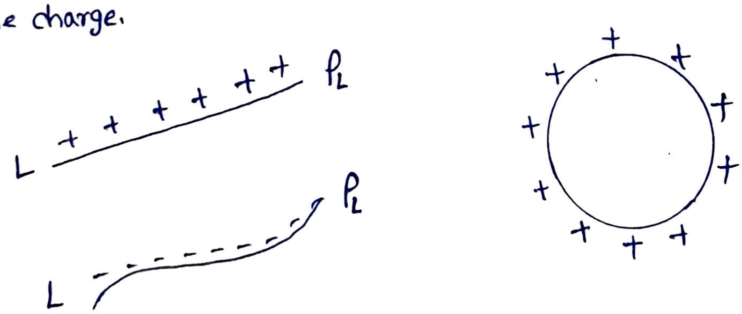


fig (1): point charge

2) Line charge : It is possible that the charge may be spreaded all along a line, which may be finite or infinite. such a charge uniformly distributed along a line is called a Line charge.



Fig(2): Line charges.

Charge density of line charge denoted as ρ_L & defined as charge per unit length.

$$\rho_L = \frac{\text{Total charge in coulomb}}{\text{Total length in meters}} \quad C/m$$

Method of finding Q from ρ_L :

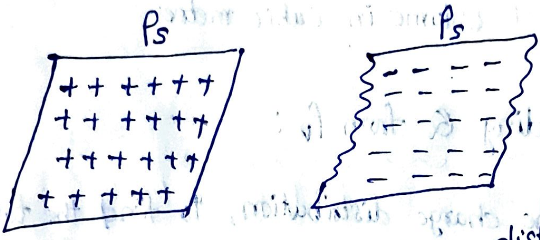
→ In line charge distribution, find the total charge Q by considering differential length dl of the line.

→ ~~By~~ $dQ = \rho_L \, dl$ = charge on differential length.

$$Q = \int_L dQ = \int_L \rho_L \, dl.$$

The integral is called line integral.

3) Surface charge : If the charge is distributed uniformly over a two dimensional surface, then it is called a surface charge or sheet of charge.



Fig(3) Surface charge distribution

Surface charge is denoted as ρ_S and defined as charge per unit surface area.

$$\rho_S = \frac{\text{Total charge in coulomb}}{\text{Total area in square metres}} \quad C/m^2.$$

Method of find Q from ρ_S :

→ In case of surface charge distribution, it is necessary to find the total charge Q by considering differential surface area ds.

$$dQ = \rho_S \, ds$$

$$Q = \int_S dQ = \int_S \rho_S \, ds.$$

The integral is called surface ~~charge~~ integral.

4) Volume charge: If the charge distributed uniformly in a volume then it is called volume charge.

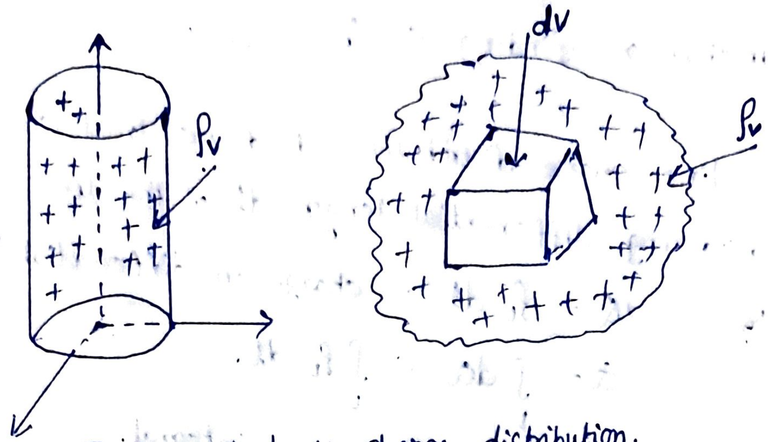


Fig (4): Volume charge distribution.

Volume charge density is denoted as ρ_v. and defined as the charge per unit volume.

$$\rho_v = \frac{\text{Total charge in Coulomb}}{\text{Total volume in cubic metres}} \; C/m^3.$$

method of finding Q from ρ_v:

In case of volume charge distribution, to find the total charge Q, consider the differential volume dv, as shown in fig 4.

$$dQ = \rho_v \, dv$$

$$Q = \int_{Vol} \rho_v \, dv$$

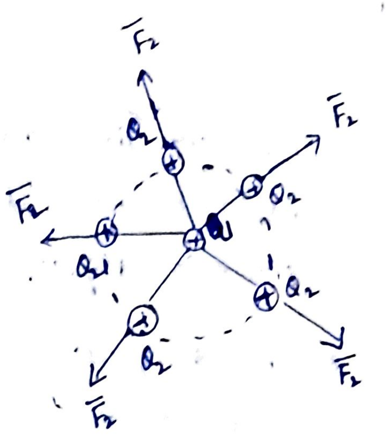
The Integral is called volume integral.

Electric Field Intensity (due to Point charge) :-

→ Consider a point charge Q_1 as shown in fig(1).

If any other similar charge Q_2 is brought near it, Q_2 experiences a force.

If Q_2 is moved around Q_1, still Q_2 experiences a force, as shown in fig(1).



Fig(1): Electric field

→ Thus there exist a region around a charge in which it exerts a force on any other charge. This region is called electric field of that charge.

→ The Force experienced by the charge Q_2 due to Q_1, by collomb's law as,

$$\bar{F_2} = \frac{Q_1 Q_2}{4\pi \varepsilon_0 R_{12}^2} \bar{a}_{12}$$

$$\frac{\bar{F_2}}{Q_2} = \frac{Q_1}{4\pi \varepsilon_0 R_{12}} \bar{a}_{12}$$

$$\bar{E_2} = \frac{Q_1}{4\pi \varepsilon_0 R_{12}} \bar{a}_{12} \quad \frac{\text{Newton}}{\text{Coloumb}} \text{ (or) } \frac{\text{volts}}{\text{metre}}$$

This force exerted per unit charge is called Electric field Intensity (or) Electric field strength.

It is a vector quantity, denoted as $\bar{E}$

$$\bar{E} = \frac{Q_1}{4\pi \varepsilon_0 R_{1P}} \bar{a}_{1P}$$

where p = position of any other charge around Q_1.

If Q_2 is placed at a distance R from Q_1, then

$$\bar{E} = \frac{Q_1}{4\pi \varepsilon_0 R^2} \bar{a}_R \quad V/m.$$

1) Electric field Intensity Due to Line charge :-

→ Consider a line charge distribution having a charge density ρ_L as shown in fig (1).

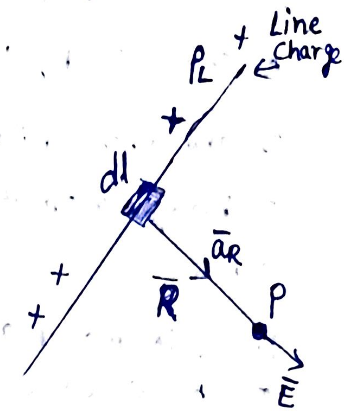


Fig (1)

→ The charge dQ on the differential length dl is

$$dQ = \rho_L \, dl$$

→ The differential Electric field $d\bar{E}$ at point P due to dQ is given by

$$d\bar{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\bar{a}_R = \frac{\rho_L \, dl}{4\pi\epsilon_0 R^2}\bar{a}_R$$

Hence the Total $\bar{E}$ at a point P due to line charge is

$$\bar{E} = \int_L \frac{\rho_L \, dl}{4\pi\epsilon_0 R^2}\bar{a}_R .$$

a) Electric field Intensity due to Infinite Line charge :-

→ Consider an infinitely long straight line carrying uniform line charge having density ρ_L C/m.

→ Let this line lies along z axis from $-\infty$ to ∞ & hence called Infinite Line charge.

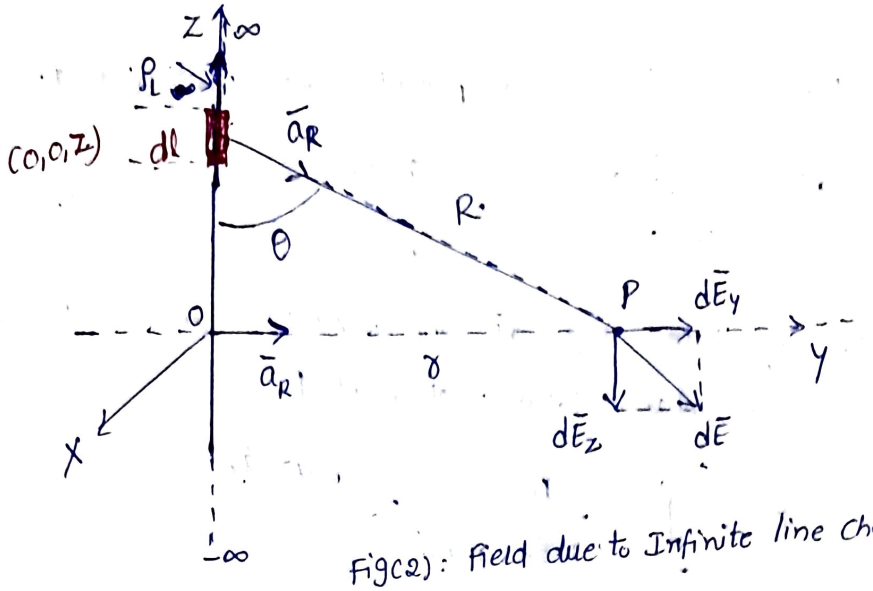
→ Let point P is on y axis at which electric field Intensity is to be determined. The distance of point P from the origin is 'r' as shown in fig(2).

→ Consider a small differential length dl carrying a charge dQ along the line as shown in fig (2).

It is along z axis hence $dl = dz$.

$$\therefore \quad dQ = \rho_L \, dl \quad \text{—— (1)}$$

$$dQ = \rho_L \, dz$$



Fig(2): Field due to Infinite line charge

The coordinates of dl are $(0, 0, z)$ & coordinates of Point P are $(0, r, 0)$.

$$\bar{r}_p = 0\cdot\bar{a}_x + r\cdot\bar{a}_y + 0\cdot\bar{a}_z = r\,\bar{a}_y$$

$$\bar{r}_{dl} = 0\cdot a_x + 0\,\bar{a}_y + z\cdot\bar{a}_z = z\cdot\bar{a}_z$$

Hence distance vector, $\bar{R} = \bar{r}_p - \bar{r}_{dl} = [r\,\bar{a}_y - z\,\bar{a}_z]$

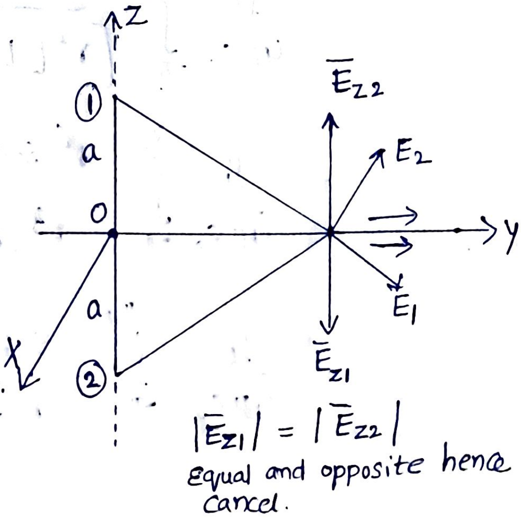
$$|\bar{R}| = \sqrt{r^2 + (-z)^2} = \sqrt{r^2+z^2}$$

$$\therefore \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{r\,\bar{a}_y - z\,\bar{a}_z}{\sqrt{r^2+z^2}} \longrightarrow (2)$$

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

$$= \frac{\rho_L\,dz}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2}\left[\frac{r\,\bar{a}_y - z\,\bar{a}_z}{\sqrt{r^2+z^2}}\right] \longrightarrow (3)$$

From fig(3), For every charge on positive Z axis, there is equal charge present on negative Z axis. Hence Z component of electric field intensities produced by such charges at point P will cancel each other. Hence no. Z component of $\bar{E}$ at P.



Fig(3)

By Eliminating $\bar{a}_z$ component, the equation of $d\bar{E}$ is

$$d\bar{E} = \frac{\rho_L\, dz}{4\pi\varepsilon_0 \left(\sqrt{r^2+z^2}\right)^2} \frac{r\,\bar{a}_y}{\sqrt{r^2+z^2}} \longrightarrow (4)$$

Now by integrating $d\bar{E}$ over the Z axis from $-\infty$ to ∞, we can obtain Total $\bar{E}$ at point P.

$$\therefore \bar{E} = \int_{-\infty}^{\infty} \frac{\rho_L}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} r\, dz\, \bar{a}_y$$

using $z = r\tan\theta \Rightarrow r = \frac{z}{\tan\theta}$.

$$\frac{dz}{d\theta} = r\sec^2\theta$$

$$\Rightarrow dz = r\sec^2\theta\, d\theta$$

For $z = -\infty$, $\theta = \tan^{-1}(-\infty) = -\frac{\pi}{2} = -90°$
$z = +\infty$ $\theta = \tan^{-1}(\infty) = \frac{\pi}{2} = +90°$ } changing the limits.

$$\bar{E} = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \frac{\rho_L}{4\pi\varepsilon_0 [r^2 + r^2\tan^2\theta]^{3/2}} r \times r\sec^2\theta\, d\theta\, \bar{a}_y.$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^2\sec^2\theta\, d\theta}{(r^2 + r^2\tan^2\theta)^{3/2}} \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int_{-\frac{\pi}{2}}^{\pi/2} \frac{r^2\sec^2\theta\, d\theta}{\left[r^2[1+\tan^2\theta]\right]^{3/2}} \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int_{-\frac{\pi}{2}}^{\pi/2} \frac{\cancel{r^2}\sec^2\theta\, d\theta}{r^{\cancel{3}}(\sec^3\theta)}$$ (since, $1+\tan^2\theta = \sec\theta$)

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int_{-\frac{\pi}{2}}^{\pi/2} \frac{\sec^{\cancel{2}}\theta\, d\theta}{r\cdot\sec^{\cancel{3}}\theta} \bar{a}_y$$

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0 r} \int_{-\pi/2}^{\pi/2} \frac{1}{\sec\theta}\, d\theta\ \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta\ \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} \left[\sin\theta\right]_{-\pi/2}^{\pi/2} \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} \left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right] \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} \left[1-(-1)\right] \bar{a}_y = \frac{\rho_L}{4\pi\varepsilon_0 r} \cdot 2\bar{a}_y$$

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_y \quad V/m. \longrightarrow (5)$$

The $\bar{a}_y$ unit vector along the distance r which is perpendicular distance of point P from the line charge. Thus in general,

$$\bar{a}_y = \bar{a}_r .$$

Hence, $$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \quad V/m \longrightarrow (6).$$

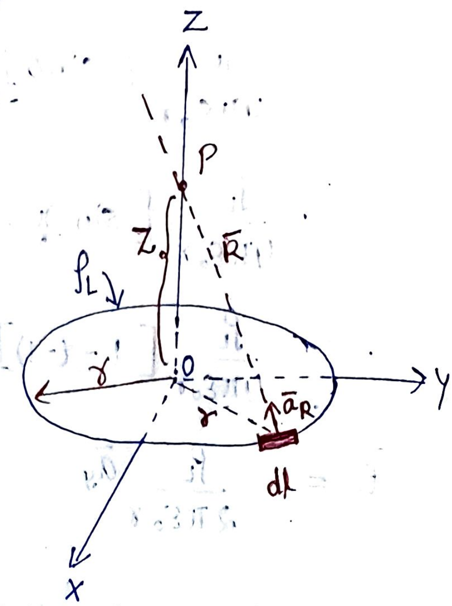
where r = perpendicular distance of point P from line charge

$\bar{a}_r$ = unit vector in the direction of perpendicular distance of point P from the line charge.

b) Electric field Intensity due to charged Circular Ring :-

→ Consider a charged circular ring of radius r placed in XY plane with the centre at origin, carrying a charge uniformly along its circumference. The charge density is ρ_L C/m.

→ The point P is at a perpendicular distance z from the ring as shown in fig(1).



Fig(1)

→ Consider a small differential length dl on this ring. The charge on it is dQ.

$$dQ = \rho_L \, dl.$$

$$d\bar{E} = \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2} \bar{a}_R \longrightarrow ①$$

where R = distance of point P from dl.

→ Consider the cylindrical coordinate system. For dl, we are moving in ϕ direction, then

$$dl = r \, d\phi \longrightarrow ②$$

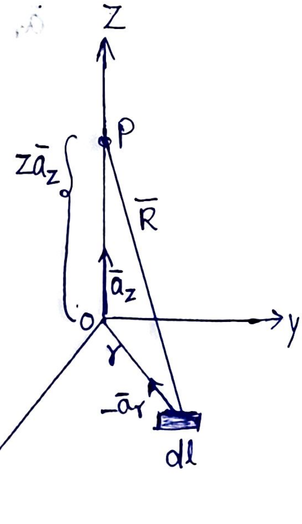
From fig(2),

$$R^2 = r^2 + z^2.$$

$\bar{R}$ can be obtained from two components in cylindrical system.

1) distance r in the direction of $-\bar{a}_r$, radially inwards.

i.e., $-r\,\bar{a}_r$.



Fig(2).

(2) distance z in the direction of $\bar{a}_z$ i.e., $z\,\bar{a}_z$.

$$\therefore \quad \bar{R} = -r\,\bar{a}_r + z\,\bar{a}_z \longrightarrow (3)$$

$$|\bar{R}| = \sqrt{(-r)^2 + z^2} = \sqrt{r^2+z^2} \longrightarrow (4)$$

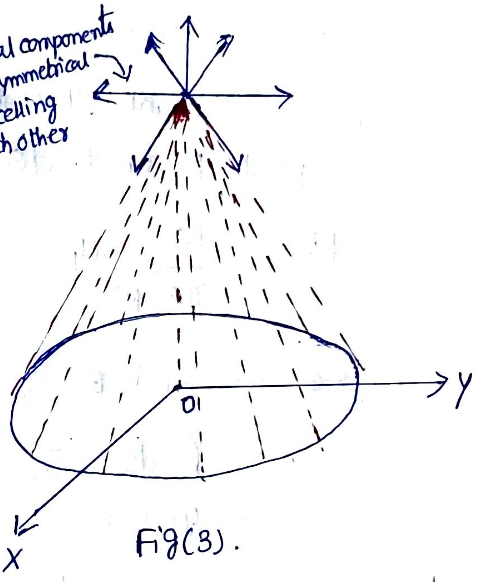
$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\,\bar{a}_r + z\,\bar{a}_z}{\sqrt{r^2+z^2}} \longrightarrow (5)$$

from Eq (1),

$$d\bar{E} = \frac{\rho_L\, dl}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2}\left[\frac{-r\,\bar{a}_r + z\,\bar{a}_z}{\sqrt{r^2+z^2}}\right] \longrightarrow$$

$$d\bar{E} = \frac{\rho_L\,(r\,d\phi)}{4\pi\varepsilon_0\,(r^2+z^2)^{3/2}}\left[-r\,a_r + z\,\bar{a}_z\right] \longrightarrow (6)$$

· Neglect $\bar{a}_r$ component, because the radial components of $\bar{E}$ at point P will be symmetrically placed in the plane parallel to xy plane & are going to cancel each other. This is shown in fig(3).



Fig(3).

→ Hence by neglecting $\bar{a}_r$ component from $d\bar{E}$ Eq (6), we get

$$d\bar{E} = \frac{\rho_L\, r\, d\phi}{4\pi\varepsilon_0\,(r^2+z^2)^{3/2}}\, z.\bar{a}_z \longrightarrow (7)$$

$$\therefore \bar{E} = \int_{\phi=0}^{2\pi} \frac{\rho_L\, r\, d\phi}{4\pi\varepsilon_0\,(r^2+z^2)^{3/2}}\, z.\bar{a}_z.$$

$$= \frac{\rho_L\, r}{4\pi\varepsilon_0\,(r^2+z^2)^{\frac{3}{2}}}\, z\,\bar{a}_z\,\left[\phi\right]_0^{2\pi} \qquad \left(\text{Since } \int 1.d\phi = \phi.\right)$$

$$\bar{E} = \frac{\rho_L\, r\, z\,\bar{a}_z}{4\pi\varepsilon_0\,(r^2+z^2)^{3/2}}\left[2\pi - 0\right]$$

$$\therefore \bar{E} = \rho_L\, r\, z\,\bar{a}_z \longrightarrow (8)$$

where r = radius of ring

z = Perpendicular distance of point P from ring along the axis of ring.

2) Electric-field Intensity due to Surface charge :

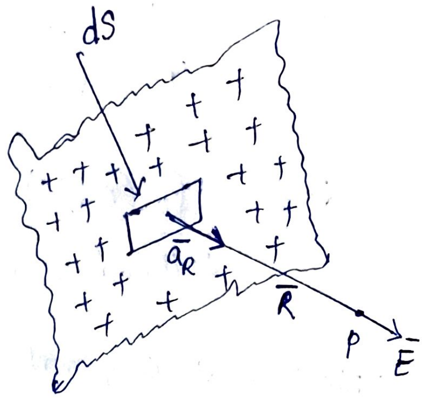
Consider a surface charge distribution having a charge density ρ_s as shown in fig(1).

The charge dQ on the differential surface area ds is

$$dQ = \rho_s \, ds.$$

The differential Electrical field $d\bar{E}$ at point P due to dQ is given by

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R = \frac{\rho_s \, ds \, \bar{a}_R}{4\pi\varepsilon_0 R^2}.$$



Fig(1).

Hence the total $\bar{E}$ is

$$\bar{E} = \int_S \frac{\rho_s \, ds}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

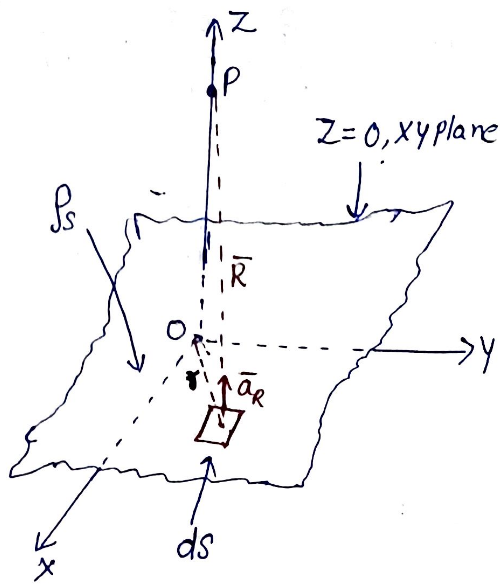
↳ Electric field due to Infinite Sheet of charge :

→ consider an infinite sheet of charge having uniform charge density ρ_s C/m² placed in xy plane as shown in fig(1).

→ Let us use cylindrical coordinates.

→ The point P at which $\bar{E}$ to be calculated is on z axis.

→ Consider a differential surface area ds carrying a charge dQ.



Fig(1)

The normal direction to ds is z direction hence ds normal to z direction is $r\,dr\,d\phi$.

$$dQ = \rho_s\, ds = \rho_s\, r\, dr\, d\phi. \longrightarrow (1).$$

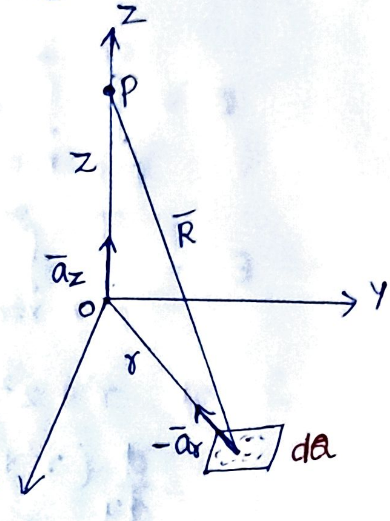


Fig (2)

So, $d\bar{E} = \dfrac{dQ}{4\pi \varepsilon_0 R^2}\, \bar{a}_R$.

$$= \frac{\rho_s\, r\, dr\, d\phi}{4\pi \varepsilon_0 R^2}\, \bar{a}_R \longrightarrow (2)$$

The distance vector $\bar{R}$ has two components as shown in fig (2).

(1) The radial component r along $-\bar{a}_r$. i.e., $-r\,\bar{a}_r$.

(2) The component z along $\bar{a}_z$. i.e., $z\,\bar{a}_z$.

$$\bar{R} = -r\,\bar{a}_r + z\,\bar{a}_z. \longrightarrow (3)$$

$$|\bar{R}| = \sqrt{(-r)^2 + z^2} = \sqrt{r^2+z^2} \longrightarrow (4)$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\,\bar{a}_r + z\,\bar{a}_z}{\sqrt{r^2+z^2}} \longrightarrow (5)$$

$$d\bar{E} = \frac{\rho_s\, r\, dr\, d\phi}{4\pi \varepsilon_0 \left(\sqrt{r^2+z^2}\right)^2} \cdot \left[\frac{-r\,\bar{a}_r + z\,\bar{a}_z}{\sqrt{r^2+z^2}}\right] \longrightarrow (6)$$

→ for infinite sheet in XY plane, r varies from 0 to ∞, while ϕ varies from 0 to 2π.

~~As~~ Neglecting $\bar{a}_r$ component, because, there is symmetry about Z axis from all radial direction, all $\bar{a}_r$ components of $\bar{E}$ are going to cancel each other.

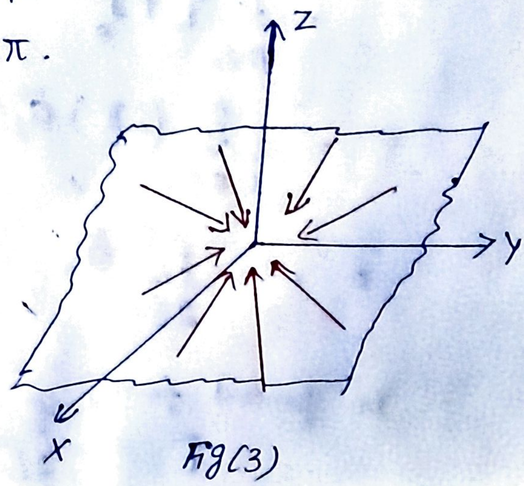


Fig (3)

By neglecting $\bar{a}_r$ component in $d\bar{E}$ Eq

$$d\bar{E} = \frac{\rho_s\, r\, dr\, d\phi}{4\pi\varepsilon_0 \left(\sqrt{r^2+z^2}\right)} \frac{z\,\bar{a}_z}{\sqrt{r^2+z^2}}.$$

$$E = \int_{\phi=0}^{2\pi} \int_{r=0}^{\infty} dE$$

$$E = \int_{\phi=0}^{2\pi} \int_{r=0}^{\infty} \frac{\rho_s\, r\, dr\, d\phi \quad z\,\bar{a}_z.}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}}$$

put $r^2+z^2 = u^2$

$$\frac{d}{du}(r^2+z^2) = \frac{d(u^2)}{du}$$

$$2r\frac{dr}{du} = 2u.$$

hence, $\cancel{2}r\,dr = \cancel{2}u\,du.$

$r\,dr = u\,du.$

for $r=0$, $u=z$ and $r=\infty$, $u=\infty$ ... changing limits.

$$\bar{E} = \int_0^{2\pi}\int_{u=z}^{\infty} \frac{\rho_s}{4\pi\varepsilon_0} \frac{u\,du}{(u^2)^{3/2}}\, d\phi\, z\,\bar{a}_z.$$

$$= \int_0^{2\pi}\int_{u=z}^{\infty} \frac{\rho_s}{4\pi\varepsilon_0} \frac{du}{u^2}\, d\phi\, (z\,\bar{a}_z)$$

$$= \int_0^{2\pi} \frac{\rho_s}{4\pi\varepsilon_0} d\phi\, z\,\bar{a}_z \left[-\frac{1}{u}\right]_z^{\infty} \quad \left(\text{since } \int \frac{1}{u^2} = -\frac{1}{u}.\right)$$

$$= \frac{\rho_s}{4\pi\varepsilon_0}[\phi]_0^{2\pi}\, z\,a_z \left[-\frac{1}{\infty} - \left(-\frac{1}{z}\right)\right]$$

$$= \frac{\rho_s (2\pi)\,\bar{a}_z}{4\pi\varepsilon_0}.$$

$$\bar{E} = \frac{\rho_s\,\bar{a}_z}{2\varepsilon_0} \quad V/m$$

In general form,

$$\bar{E} = \frac{\rho_s\,\bar{a}_n}{2\varepsilon_0} \quad V/m \qquad \to (7)$$

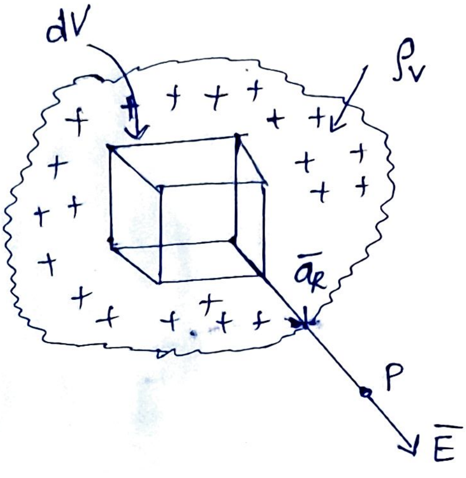
where $\bar{a}_n$ = Direction normal to the surface charge.

Thus for the points below XY plane, $\bar{a}_n = -\bar{a}_z$.

Hence $\bar{E} = -\dfrac{\rho_s}{2\varepsilon_0}\bar{a}_z$ V/m.

3) Electric field Intensity due to Volume charge:

→ Consider a volume charge distribution having a charge density ρ_v as shown in fig(1).



→ The charge dQ on the differential volume dv is

$dQ = \rho_v \, dv.$

→ Hence differential electric field $d\bar{E}$ at point P due to dQ is given by

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$$= \frac{\rho_v \, dv}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

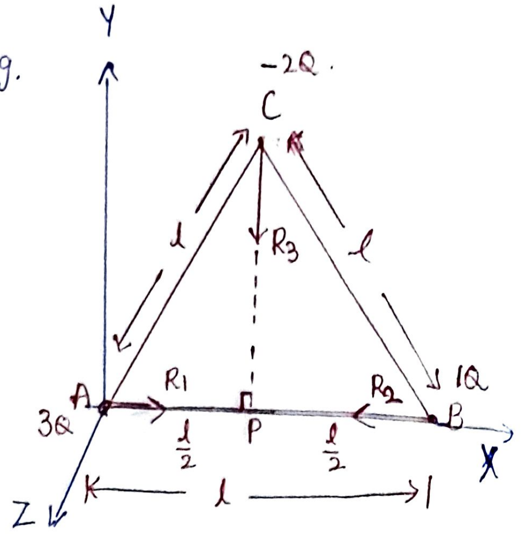
Hence Total $\bar{E}$ is given as

$$\bar{E} = \int_{Vol} \frac{\rho_v \, dv}{4\pi\varepsilon_0 R^2}\bar{a}_R .$$

Problems :

1) Point charges are located at each corner of an Equilateral triangle. If the charges are $3Q, -2Q$ & $1Q$. find $\bar{E}$ at midpoint of $3Q$ and $1Q$ side.

Sol: The arrangement is shown in fig.



Let $AB = BC = CA = l$.

So, $CP^2 = AC^2 - AP^2$

$= l^2 - \left(\frac{l}{2}\right)^2$

$CP^2 = \frac{3l^2}{2}$

$CP = \frac{\sqrt{3}\,l}{2}$

Co ordinates are

$A(0,0,0),\ B(l,0,0),\ C\left(\frac{l}{2}, \frac{\sqrt{3}l}{2}, 0\right)$ & $P\left(\frac{l}{2},0,0\right)$

$$E_1 = \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\,\bar{a}_{R1}$$

$$\bar{R}_1 = \bar{R}_P - \bar{R}_A$$

$$= \left(\frac{l}{2} - 0\right)\bar{a}_x + 0.\bar{a}_y + 0.\bar{a}_z$$

$$\bar{R}_1 = \frac{l}{2}\,\bar{a}_x = 0.5\,l\,\bar{a}_x$$

$$|\bar{R}_1| = 0.5\,l.$$

$$\bar{a}_{R_1} = \frac{\bar{R}_1}{|\bar{R}_1|} = \frac{0.5\,l\,\bar{a}_x}{0.5\,l} = \bar{a}_x$$

~~Eq 3Q~~ given $Q_1 = 3Q$.

$$\text{So, } E_1 = \frac{3Q}{4\pi\varepsilon_0 (0.5l)^2}\,\bar{a}_x$$

$$= \frac{1.078 \times 10^{11}\,Q\,\bar{a}_x}{l^2}$$

$$\Rightarrow \quad \bar{E}_2 = \frac{Q_2}{4\pi \varepsilon_0 R_2^2} \, \bar{a}_{R2}$$

$$\bar{R}_2 = \bar{R}_P - \bar{R}_B$$

$$\bar{R}_2 = \left(\frac{l}{2} - l\right)\bar{a}_x + 0 \cdot \bar{a}_y + 0 \cdot \bar{a}_z = -0.5 l \, \bar{a}_x$$

$$|\bar{R}_2| = \sqrt{(-0.5l)^2} = 0.5 l.$$

$$\bar{a}_{R2} = \frac{\bar{R}_2}{|\bar{R}_2|} = \frac{-0.5 l \, \bar{a}_x}{0.5 l} = -\bar{a}_x$$

Given $Q_2 = 1Q$.

so $$\bar{E}_2 = \frac{1Q}{4\pi\varepsilon_0 (-0.5l)^2} \cdot (-\bar{a}_x)$$

$$\Rightarrow \quad \bar{E}_2 = \frac{-3.595 \times 10^{10} \, Q \, \bar{a}_x}{l^2}$$

similarly $$\bar{E}_3 = \frac{Q_3}{4\pi\varepsilon_0 R_3^2} \cdot \bar{a}_{R3}$$

$$\bar{R}_3 = \bar{R}_P - \bar{R}_C$$

$$= \left(\frac{l}{2} - \frac{l}{2}\right)\bar{a}_x + \left(0 - \frac{\sqrt{3}}{2} l\right)\bar{a}_y + 0 \cdot \bar{a}_z.$$

$$\bar{R}_3 = -\frac{\sqrt{3}}{2} l \, \bar{a}_y = -0.866 \, l \, \bar{a}_y$$

$$|\bar{R}_3| = \sqrt{(-0.866 l)^2} = 0.866 l.$$

$$\bar{a}_{R3} = \frac{\bar{R}_3}{|\bar{R}_3|} = \frac{-0.866 l \, a_y}{0.866 l} = -\bar{a}_y.$$

given $Q_3 = -2Q$.

so, $$\bar{E}_3 = \frac{-2Q}{4\pi\varepsilon_0 (0.866 l)^2} \cdot (-\bar{a}_y).$$

$$= \frac{2.3968 \times 10^{10}}{l^2} \, \bar{a}_y.$$

Total $\bar{E}$ at Point P $\Rightarrow$ $\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3$

$$\bar{E} = \frac{1.078\times10^{11} Q\, \bar{a}_x}{l^2} + \frac{-3.595\times10^{10} Q\, \bar{a}_x}{l^2} + \frac{2.3968\times10^{10} Q\, \bar{a}_y}{l^2}$$

$$\bar{E} = \frac{Q}{l^2}\left[7.185\times10^{10}\, \bar{a}_x + 2.3968\times10^{10}\, \bar{a}_y\right] \; V/m.$$

2) A uniform line charge $\rho_L = 25$ nC/m lies on the line $x = -3$ and $z = 4$ in the free space. Find the Expression for $\bar{E}$ in the cartesian coordinates at the origin (0, 0, 0).

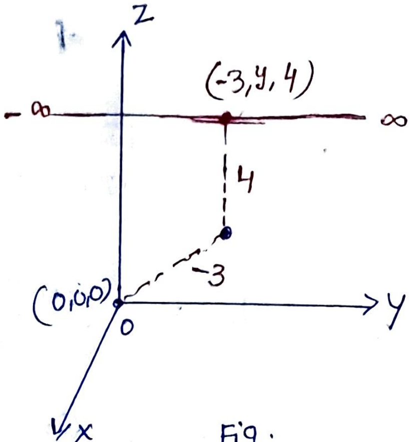


Fig:

Sol The line is shown in fig.

The line with $x = -3$ constant and $Z = 4$ constant is a line parallel to y axis as y can take any value.

$\bar{E}$ at origin is to be calculated.

$\bar{E}$ due to infinite line charge is given by,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\,\bar{a}_r.$$

where r = perpendicular distance of point from line.

→ To find r, consider point P on the line as $(-3, y, 4)$

→ Now the line charge is parallel to y axis. Hence $\bar{E}$ cannot have any $\bar{a}_y$ component

Hence point P $(-3, y, 4)$ & O $(0,0,0)$

$$\bar{r} = r_0 - r_p$$

$$= [0-(-3)]\,\bar{a}_x + (0-y)\,\bar{a}_y + (0-4)\,\bar{a}_z$$

$$= 3\bar{a}_x - y\,\bar{a}_y - 4\,\bar{a}_z$$

$$\bar{r} = \bar{r}_o - \bar{r}_p$$

$$= [0-(-3)]\,\bar{a}_x + [0-4]\,\bar{a}_z$$

$$\bar{r} = 3\bar{a}_x - 4\bar{a}_z.$$

$$|\bar{r}| = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = 5.$$

$$\bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{3\bar{a}_x - 4\bar{a}_z}{5}$$

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0(5)} \frac{[3\bar{a}_x - 4\bar{a}_z]}{5}$$

$$= \frac{25\times10^{-9}}{2\pi\times8.854\times10^{-12}(25)} [3\bar{a}_x - 4\bar{a}_z)$$

$$= 53.926\,\bar{a}_x - 71.902\,\bar{a}_z \quad V/m.$$

3) Two small identical conducting spheres have charges of 2 nC and -1 nC respectively. When they are separated by 4cm apart, find the magnitude of the force between them. If they are brought into contact and then again separated by 4cm, find the force between them.

Sol: Case 1 : Before the charges are brought into contact.

$$|\bar{F}| = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}$$

where $R_{12} = 4$ cm. $= 4\times10^{-2}$ m.

& given $Q_1 = 2$ nC

$Q_2 = -1$ nC.

$$|\bar{F}| = \frac{2\times10^{-9}\times(-1\times10^{-9})}{4\pi\times8.854\times10^{-12}\times(4\times10^{-12})}$$

$$|\bar{F}| = 11.234\ \mu N$$

Case 2: The charges are brought into contact & then separated:

→ When charges are brought into contact, the charge distribution takes place due to transfer of charge.

The transfer of charge continues till both the charges attain same value due to equal division of two charges.

Charges on each sphere $\Rightarrow \dfrac{Q_1 + Q_2}{2} = \dfrac{(2\times10^{-9}) + (-1\times10^{-9})}{2}$

$Q_1 = Q_2 \Rightarrow \dfrac{(2-1)\,10^{-9}}{2} = 0.5\, nC.$

$\therefore |F| = \dfrac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2} = \dfrac{0.5\times10^{-9} \times 0.5\times10^{-9}}{4\pi\times8.854\times10^{-12}\times(4\times10^{-2})^2}$

$|\bar{F}| = 1.404\ \mu N.$

4) A circular disc of 10 cm radius is charged uniformly with a total charge 100 μC. find $\bar{E}$ at a point 20 cm on its axis.

Sol:

$Q = 100\ \mu C$

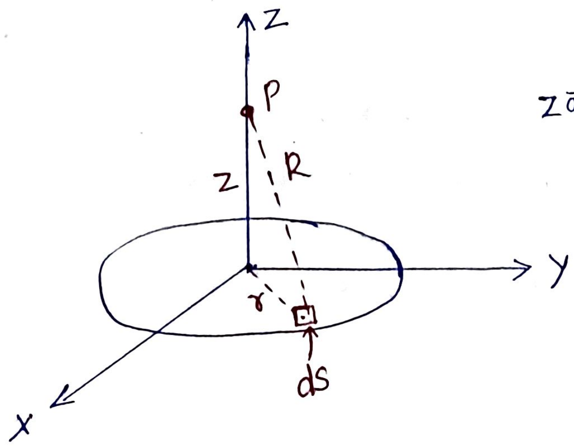
$r = 10\ cm = 0.1\ m$

Area of circle $= \pi r^2$

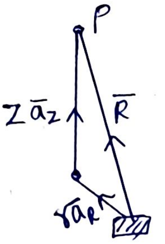
$= \pi (0.1)^2$

Area $= 0.03141\ m^2$

Surface charge, $\rho_s = \dfrac{Q}{Area} = \dfrac{100\times10^{-6}}{0.03141} = 3.1831\times10^{-3}\ C/m^2.$



fig(a)



fig(b)

$$dQ = \rho_s \, dS.$$

Consider differential surface area dS. Using cylindrical system,

$$dS = r \, dr \, d\phi .$$

$$\bar{R} = -r\bar{a}_r + z\bar{a}_z , \quad |\bar{R}| = \sqrt{r^2+z^2}$$

$$\bar{a}_R = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}}.$$

Neglect ar component, due to symmetry.

Ko $$\bar{E} = \int_S \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

$$= \int_S \frac{\rho_s \, dS}{4\pi\varepsilon_0 R^2}\,\bar{a}_R = \int_S \frac{\rho_s \, r\,dr\,d\phi}{4\pi\varepsilon_0 R^2}\,\bar{a}_R .$$

$$= \int_{\phi=0}^{2\pi}\int_{r=0}^{0.1} \frac{\rho_s\,[r\,dr\,d\phi]}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2} \cdot \frac{z\,\bar{a}_z}{\sqrt{r^2+z^2}}$$

$$\bar{E} = \frac{\rho_s z}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{r=0}^{0.1} \frac{r\,dr\,d\phi}{(r^2+z^2)^{3/2}} \cdot \bar{a}_z$$

use $r^2+z^2 = u^2$

$$\frac{d}{du}(r^2+z^2) = \frac{d}{du}(u^2)$$

$$2r\frac{dr}{du} + 0 = 2u$$

$$\cancel{2}r\,dr = \cancel{2}u\,du$$

$$\Rightarrow \quad r\,dr = u\,du.$$

Limits : $r=0$, $u_1 = z$ and $r = 0.1$, $u_2 = \sqrt{r^2+z^2} = \sqrt{0.1^2+z^2}$

$$\bar{E} = \frac{\rho_s z}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi}\int_{u_1}^{u_2} \frac{u\,du\,d\phi}{u^3}\,\bar{a}_z$$

$$= \frac{\rho_s z}{4\pi\varepsilon_0}\,[\phi]_0^{2\pi}\left[-\frac{1}{u}\right]_{u_1}^{u_2}\bar{a}_z .$$

$$\bar{E} = \frac{\rho_s z}{4\pi\varepsilon_0}(2\pi)\left(\frac{1}{u_1}-\frac{1}{u_2}\right)\bar{a}_z .$$

using $Z = 20\,cm, = 0.2\,m$

$u_1 = 0.2$

$u_2 = \sqrt{(0.1)^2 + (0.2)^2} = \sqrt{0.05} = 0.2236.$

$$\bar{E} = \frac{3.1831 \times 10^{-3} \times 0.2}{4\pi \times 8.854 \times 10^{-12}} \times 2\pi \times \left[\frac{1}{0.2} - \frac{1}{0.2236}\right] \bar{a}_z$$

$$\bar{E} = 18.9723\ \bar{a}_z.$$

5) Four like charges of 30 μC each are located at four courners of a square the diagonal of which measures 8m. Find the force on a 150 μC charge located at 3m above the centre of square.

$\boxed{\begin{matrix} l(AC) = 8\,m \\ l(BD) = 8\,m \end{matrix}}$

Sol: → The square is kept in X-Y plane with origin as one of its corners as shown in fig (1).

→ The diagonals AC = BD = 8m.

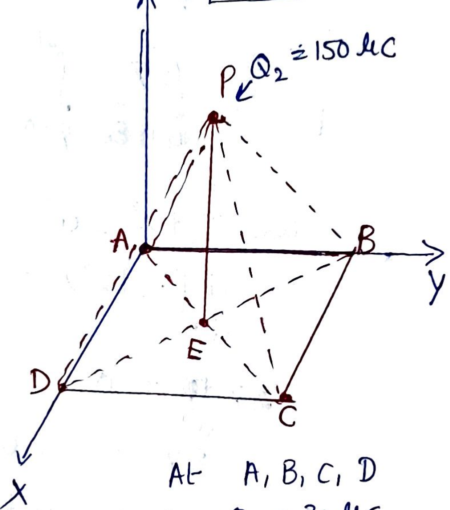
let AD = DC = BC = AB = l m

from fig, $AD^2 + DC^2 = AC^2$

$l^2 + l^2 = 8^2$

$2l^2 = 64$

$\therefore\ l = 5.656\,m$



At A, B, C, D

$Q_1 = 30\,\mu C$

Fig (1)

Hence the coordinates of various points are

A (0, 0, 0) B (0, 5.656, 0) C (5.656, 5.656, 0),

D (5.656, 0, 0).

The point E is centroid. hence E $(\frac{l}{2}, \frac{l}{2}, 0)$ = (2.828, 2.828, 0)

The point P is 3m above the centre E, hence the coordinates at P (2.828, 2.828, 3)

To find force on charge at P which is $Q_2 = 150\ \mu C$ due to charges at A, B, C & D of $Q_1 = 30\ \mu C$ each.

$$\therefore \bar{F}_P = \bar{F}_A + \bar{F}_B + \bar{F}_C + \bar{F}_D.$$

$$\bar{F}_A = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{AP}^2} \cdot \bar{a}_{AP} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{AP}^2} \frac{\bar{R}_{AP}}{|\bar{R}_{AP}|}$$

$$\bar{F}_B = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{BP}^2} \bar{a}_{BP} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{BP}^2} \cdot \frac{\bar{R}_{BP}}{|\bar{R}_{BP}|}$$

$$\bar{F}_C = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{CP}^2} \bar{a}_{CP} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{CP}^2} \frac{\bar{R}_{CP}}{|\bar{R}_{CP}|}$$

$$\bar{F}_D = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{DP}^2} \bar{a}_{DP} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{DP}^2} \frac{\bar{R}_{DP}}{|\bar{R}_{DP}|}$$

$$\bar{R}_{AP} = \bar{R}_P - \bar{R}_A = (2.828 - 0)\,\bar{a}_x + (2.828 - 0)\,\bar{a}_y + (3 - 0)\,\bar{a}_z$$
$$= 2.828\,\bar{a}_x + 2.828\,\bar{a}_y + 3\,\bar{a}_z$$

$$\bar{R}_{BP} = \bar{R}_P - \bar{R}_B = (2.828 - 0)\,\bar{a}_x + (2.828 - 5.656)\,\bar{a}_y + (3 - 0)\,\bar{a}_z$$
$$= 2.828\,\bar{a}_x - 2.828\,\bar{a}_y + 3\,\bar{a}_z.$$

$$\bar{R}_{CP} = \bar{R}_P - \bar{R}_C = (2.828 - 5.656)\,\bar{a}_x + (2.828 - 5.656)\,\bar{a}_y + (3 - 0)\,\bar{a}_z$$
$$= -2.828\,\bar{a}_x - 2.828\,\bar{a}_y + 3\,\bar{a}_z.$$

$$\bar{R}_{DP} = \bar{R}_P - \bar{R}_D = (2.828 - 5.656)\,\bar{a}_x + (2.828 - 0)\,\bar{a}_y + (3 - 0)\,\bar{a}_z.$$
$$= -2.828\,\bar{a}_x + 2.828\,\bar{a}_y + 3\,\bar{a}_z.$$

$$|\bar{R}_{AP}| = |\bar{R}_{BP}| = |\bar{R}_{CP}| = |\bar{R}_{DP}| = \sqrt{(2.828)^2 + (2.828)^2 + 3^2} = 5.$$

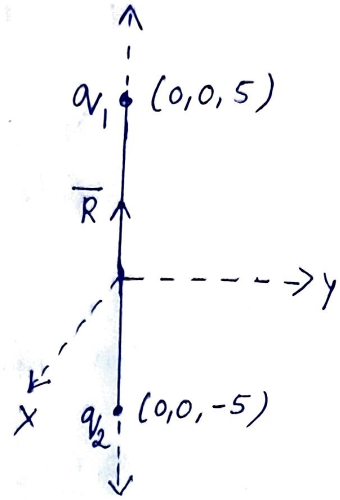
$$\therefore \bar{F}_P = \bar{F}_A + \bar{F}_B + \bar{F}_C + \bar{F}_D = \frac{Q_1 Q_2}{4\pi\varepsilon_0 (5)^3} [\bar{R}_{AP} + \bar{R}_{BP} + \bar{R}_{CP} + \bar{R}_{DP}]$$

$$\bar{F}_P = \frac{30 \times 10^{-6} \times 150 \times 10^{-6}}{4\pi \times 8.854 \times 10^{-12} \times (5)^3} [12\, a_z]$$

$$\therefore \bar{F}_P = 3.8827\,\bar{a}_z \quad N$$

6) Two point charges $q_1 = 250\,\mu C$ and $q_2 = 300\,\mu C$ are located at $(0,0,5)$ m and $(0,0,-5)$ m respectively. Find the force on q_1 ?.

Sol: The charges are shown in fig.



Fig

$$\bar{F}_1 = \frac{q_1 q_2}{4\pi \varepsilon_0 R^2}\,\bar{a}_R = \frac{q_1 q_2}{4\pi \varepsilon_0 R^2}\cdot\frac{\bar{R}}{|\bar{R}|}$$

$$\bar{R} = \bar{R}_{q_1} - \bar{R}_{q_2}$$

$$= \cancel{5-5}\; 0\cdot\bar{a}_x + 0\cdot\bar{a}_y + [5-(-5)]\,\bar{a}_z$$

$$\bar{R} = 10\,\bar{a}_z$$

$$|\bar{R}| = 10$$

$$\bar{F}_1 = \frac{250\times10^{-6}\times 300\times10^{-6}}{4\pi\times 8.854\times10^{-12}\times(10)^2}\left[\frac{10\,\bar{a}_z}{10}\right]$$

$$\bar{F}_1 = 6.7408\,\bar{a}_z \;\; N.$$

Two like charges repel each other hence the force on q_1 due to q_2 is in $+\bar{a}_z$ direction.

→ Work done in Moving a Point charge in Electrostatic Field :-

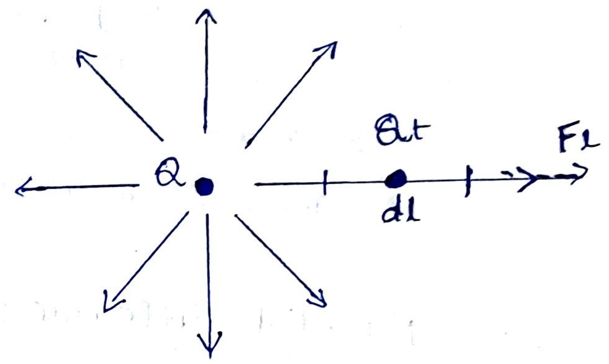
→ An Electric charge produces an electric field and if a test charge is brought to this region, it experiences a force.

→ When we think of force, it readily occurs some work (or) Energy is required to move the test charge from one point to another against the field.

→ Consider a positive charge Q and its Electric field $\bar{E}$

→ If a ~~test~~ positive test charge Q_t is placed in this field, it will move due to the force of repulsion.

→ Let the movement of charge Q_t is dl. The direction in which movement has taken place is denoted by unit vector $\bar{a}_L$ in the direction of dl. This is shown in fig (1).



According to Coulomb's law, the force exerted by field $\bar{E}$ is given by,

$$\bar{F} = Q_t \bar{E} \quad \text{Newton.}$$

The component of $\bar{F}$ in the direction of unit vector $\bar{a}_L$ is given by

$$\bar{F}_l = \bar{F} \cdot a_L = Q_t \bar{E} \cdot \bar{a}_L \quad \text{Newton}$$

This is the force responsible to move the charge Q_t through the distance dl in the direction of field.

→ To keep the charge in equilibrium, it is necessary to apply the force which is Equal & opposite to the force exerted by the field in the direction dl.

$$\therefore F_{applied} = -F_l = Q_t \bar{E} \cdot \bar{a}_L \quad \text{Newton}$$

In this case, the work is said to be done.

Hence mathematically, the differential workdone by an External source in moving the charge Q_t through the distance dl, against the direction of field $\bar{E}$ is given by,

$$dW = F_{applied} \times dl = -Q_t \bar{E} \cdot a_L \, dl$$

But $dl \, \bar{a}_L = d\bar{L}$ = distance vector.

$$\therefore \quad dW = -Q \bar{E} \cdot d\bar{L} \quad \text{Joule}$$

For finite distance,

$$W = \int_{initial}^{final} dW = \int_{initial}^{final} -Q \bar{E} \cdot d\bar{L}$$

$$\text{workdone}, \quad W = -Q \int_{initial}^{final} \bar{E} \cdot d\bar{L} \quad \text{Joule}$$

This workdone becomes the potential energy of the Test charge Q_t at the point at which it is moved.

→ Potential difference :-

work done in moving a point charge Q from B to A, in the electric field is given by,

$$W = -Q \int_B^A \bar{E} \cdot d\bar{L}$$

If the charge Q is selected as Unit Test charge, then

$$\text{workdone} \quad W = -\int_B^A \bar{E} \cdot d\bar{L}.$$

This workdone in moving unit charge from point B to A in Electric field $\bar{E}$ is called potential difference between the points B and A. It is denoted by V.

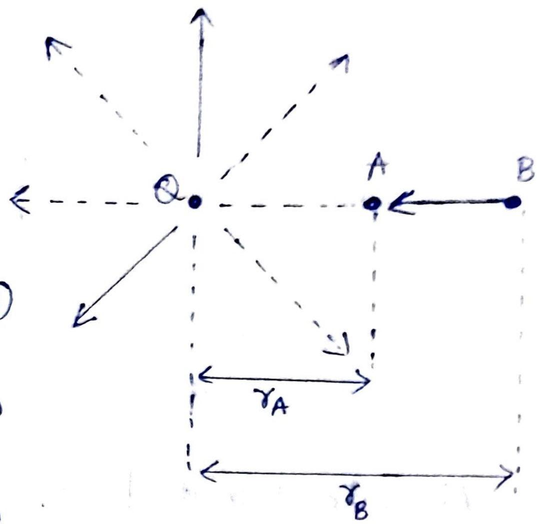
$$\therefore \text{Potential difference } V_{AB} = -\int_B^A E \cdot d\bar{L} \quad \text{Joule/Coulomb (or) Volt.}$$

Electric potential due to point charge :

1)

→ consider a point charge located at the origin of a spherical co ordinate system, producing $\bar{E}$ radially in all the directions as shown in fig(1).

Fig(1): Potential due to a point charge Q



→ $\bar{E}$ due to a point charge Q at a point having a radial distance r from origin is given by,

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r \longrightarrow ①$$

→ consider a unit charge which is placed at point B which is at radial distance of r_B from the origin.

→ It is moved against the direction of $\bar{E}$ from $\bar{B}$ to point A.

The point A is at radial distance of r_A from the origin.

The differential length in spherical system is,

$$d\bar{L} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi \longrightarrow ②$$

Hence the potential difference V_{AB} between points A & B is given by,

$$V_{AB} = -\int_B^A \bar{E}\cdot d\bar{L}, \quad \text{But } B \Rightarrow r_B \text{ and } A \Rightarrow r_A.$$

$$V_{AB} = -\int_{r_B}^{r_A}\left(\frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r\right)\cdot\left(dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi\right)$$

$$V_{AB} = -\int_{r_B}^{r_A}\frac{Q}{4\pi\varepsilon_0 r^2}dr \longrightarrow ③ \quad (\text{since } \bar{a}_r\cdot\bar{a}_r = 1)$$

$$V_{AB} = \frac{-Q}{4\pi\varepsilon_0}\int_{r_B}^{r_A} r^{-2}\,dr$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left[\frac{r^{-1}}{-1}\right]_{r_B}^{r_A}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{r_B}^{r_A}$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left[-\frac{1}{r_A} - \left(-\frac{1}{r_B}\right)\right]$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left[-\frac{1}{r_A} + \frac{1}{r_B}\right]$$

$$\therefore V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \quad \text{Volt}$$

This indicates the workdone by the External Source in moving unit charge from B to A.

→ **Absolute Potential :-**

The absolute potential at any point in an Electric field is defined as the workdone in moving a unit test charge from the infinity (or reference point at which potential is zero) to the point, against the direction of field.

→ Hence absolute potential at any point which is at a distance r from origin of a spherical system, where point charge Q is located is given by

$$V = \frac{Q}{4\pi\varepsilon_0 r}.$$

other way to define potential,

$$V = -\int_{\infty}^{r} \bar{E}\cdot d\bar{L} \quad \text{Volt}$$

where ∞ is selected as reference

Thus $V_A = -\int_{\infty}^{A} \bar{E}\cdot d\bar{L}$ Volt.

This is potential of point A with reference at infinity.

Note: Potential is a scalar quantity.

Potential due to point charge Not at origin :-

→ If the point charge Q is not located at the origin of a Spherical system then obtain the position vector r' of the point where Q is located.

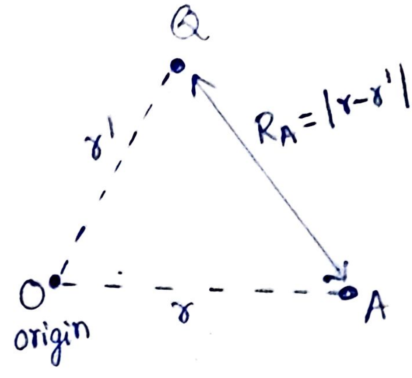


Fig.

→ Then the absolute potential at a point A located at a distance r from the origin is given by,

$$V(r) = V_A = \frac{Q}{4\pi\varepsilon_0 |r - r'|}$$

$$= \frac{Q}{4\pi\varepsilon_0 R_A}.$$

where $R_A = |r - r'|$ = Distance b/w point at which potential is to be calculated and the location of charge.

Potential due to Several point charges :

consider the various point charges $Q_1, Q_2 \ldots Q_n$ located at distance $r_1, r_2 \ldots r_n$ from origin. The potential due to all these point charges, at point A is to be determined. Use superposition principle.

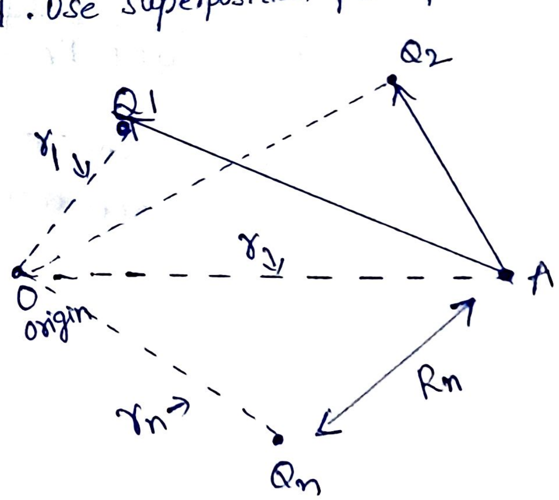


Fig: potential due to several point charges.

The potential V_{A1} due to Q_1 is

$$V_{A1} = \frac{Q}{4\pi\varepsilon_0 |r - r_1|} = \frac{Q}{4\pi\varepsilon_0 R_1}.$$

where $R_1 = |r - r_1|$
= distance b/w point A & position of Q_1.

The potential V_{A2} due to Q_2 is given by,

$$V_{A2} = \frac{Q_2}{4\pi\varepsilon_0 |r - r_2|} = \frac{Q_2}{4\pi\varepsilon_0 R_2} \text{ Volt}$$

Thus, the potential V_{An} due to Q_n is given by

$$V_{An} = \frac{Q_n}{4\pi\varepsilon_0 |r - r_n|} = \frac{Q_n}{4\pi\varepsilon_0 R_n} \text{ Volt}$$

The net potential at point A is

$$V(r) = V_A = V_{A1} + V_{A2} + \dots + V_{An}$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2} + \dots + \frac{Q_n}{4\pi\varepsilon_0 R_n}$$

$$\Rightarrow V_A = V(r) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0 |r - r_m|} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0 R_m} \text{ Volt.}$$

→ Potential calculation when Reference is other than Infinity :-

If any other point than infinity is selected as the reference then the potential at a point A due to point charge Q at the origin becomes,

$$V_A = \frac{Q}{4\pi\varepsilon_0 R_A} + C$$

where C = constant to be determined at choosen reference point where $V=0$.

Potential due to Line charge :

- consider a line charge having density ρ_L C/m as shown in fig
- consider differential length dL' at a distance r', then the differential charge on the length dL' is given by,

$$dQ = \rho_L(r')\, dL' \longrightarrow ①$$

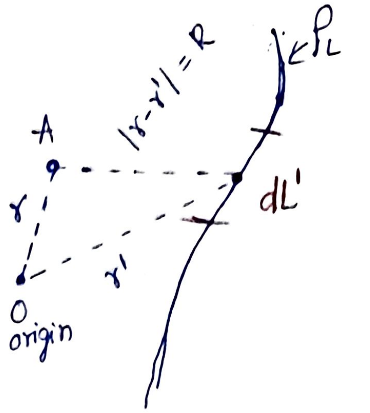


Fig : line charge

where : $\rho_L(r')$ = Line charge density at r'.

- Let the potential at A is to be determined, Then,

$$dV_A = \frac{dQ}{4\pi\varepsilon_0 |r-r'|} = \frac{dQ}{4\pi\varepsilon_0 R} \longrightarrow ②$$

$R = |r-r'|$ = distance of point A from differential charge.

- Potential V_A can be obtained by integrating dV_A over the length over which line charge is distributed

$$V_A = V(r) = \int_{Line} \frac{dQ}{4\pi\varepsilon_0 R}.$$

$$\Rightarrow \quad V_A = V(r) = \int_{Line} \frac{\rho_L(r')\, dL'}{4\pi\varepsilon_0 R} \quad \text{Volt}$$

Potential at a point on the axis of a ring of charge :

(or)

Potential V on Z axis at a distance z from origin when uniform line charge ρ_L in the form of a ring of radius a is placed in the $z=0$ plane :-

The arrangement is shown in fig.

- Point A $(0,0,z)$ is on z-axis, at a distance z from the origin while radius of ring is 'a'.

→ Consider differential length dL' at point P on the ring. The ring is in $z=0$ plane hence dL' in cylindrical system is,

$$dL' = r' d\phi = a\, d\phi$$

→ The distance of point A from the differential charge is $R = l(PA)$.

$$R = \sqrt{a^2+z^2}, \text{ from fig.}$$

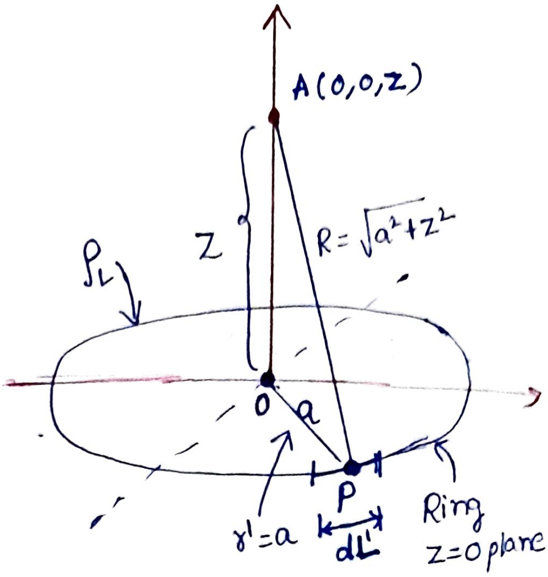


Fig.

The charge $dQ = \rho_L(r')\, dL'$

$$= \rho_L\, a\, d\phi.$$

Since $\rho_L(r') = \rho_L$ & $dL' = a\, d\phi$.

$$\therefore dV_A = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_L\, a\, d\phi}{4\pi\varepsilon_0 \sqrt{a^2+z^2}}$$

Hence the potential of A is to be obtained by integrating dV_A over the circular ring i.e., path with radius $r'=a$ and ϕ varies from 0 to 2π.

$$V_A = \int_{\phi=0}^{2\pi} \frac{\rho_L\, a\, d\phi}{4\pi\varepsilon_0\sqrt{a^2+z^2}} = \frac{\rho_L a}{4\pi\varepsilon_0\sqrt{a^2+z^2}} \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{\rho_L a}{4\pi\varepsilon_0\sqrt{a^2+z^2}} \left[\phi\right]_0^{2\pi} = \frac{\rho_L a}{4\pi\varepsilon_0\sqrt{a^2+z^2}} [2\pi - 0]$$

$$V_A = \frac{\rho_L a}{2\varepsilon_0\sqrt{a^2+z^2}} \text{ Volts}$$

Potential due to Volume charge :

→ Consider a uniform volume charge density ρ_v C/m^3 over the given volume as shown in fig.

→ Consider the differential volume dv' at point P where the charge density is $\rho_v(r')$.

differential charge,

$$dQ = \rho_v(r')\, dv' \longrightarrow (1)$$

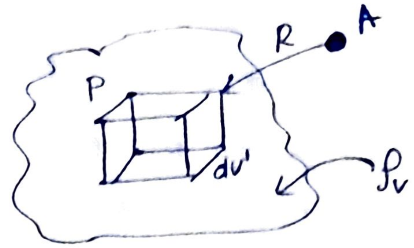


Fig: potential due to volume charge.

$$\therefore dV_A = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_v(r')\, dv'}{4\pi\varepsilon_0 R}.$$

where R = distance of point A from the differential charge.

→ The Total potential at A can be obtained by integrating dV_A over the given volume

$$V_A = \int_V \frac{\rho_v(r')\, dv'}{4\pi\varepsilon_0 R} \text{ Volt}$$

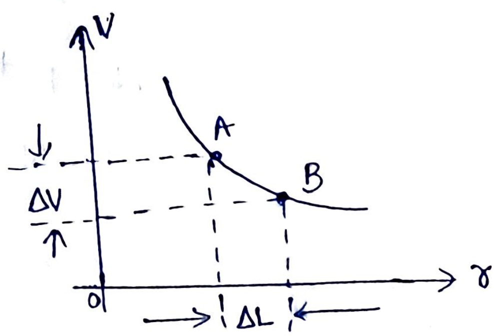
Note that for uniform volume charge distribution $\rho_v(r') = \rho_v$.

→ ## Potential Gradient (or) Relation between $\bar{E}$ and V :-

→ Consider a Electric field $\bar{E}$ due to positive charge placed at origin of a sphere, Then

$$V = -\int \bar{E}\cdot d\bar{L} = \frac{Q}{4\pi\varepsilon_0 r}.$$

The potential decreases as distance of point from the charge increases. This is shown in fig (1).



Fig(1): Potential gradient

→ For elementary length ΔL, we can write,

$$\therefore V_{AB} = \Delta V = -\bar{E}\cdot\Delta\bar{L}$$

The rate of change of potential with respect to the distance is called the Potential gradient.

$$\therefore \frac{dV}{dL} = \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \text{Potential Gradient}$$

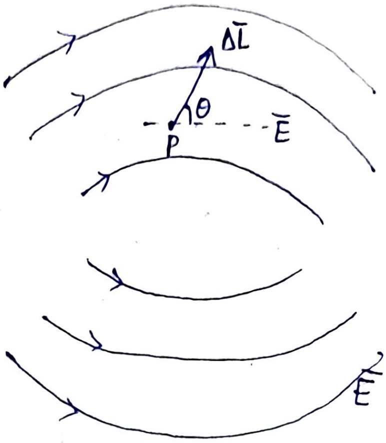
Potential gradient is nothing but the slope of graph of potential against distance at a point where ΔL is considered.

↳ Relation between $\bar{E}$ and V :

→ consider $\bar{E}$ due to a particular charge distribution in space.

→ The electric field $\bar{E}$ and potential V is changing from point to point in space.

→ consider a vector incremental length $\Delta \bar{L}$ making an angle θ with respect to direction of $\bar{E}$, as shown in fig.



Fig(2) : Incremental length at an angle θ.

→ To find incremental potential, we use,

$$\Delta V = - \bar{E} \cdot \Delta \bar{L} \longrightarrow (1)$$

$$\Delta \bar{L} = \Delta L \, \bar{a}_L \longrightarrow (2)$$

where $\bar{a}_L$ = unit vector in the direction of ΔL.

$$\bar{E} = E_L \, \bar{a}_L \longrightarrow (3)$$

$$\bar{E} \cdot \Delta \bar{L} = (E_L \bar{a}_L) \cdot (\Delta L \bar{a}_L)$$

$$\bar{E} \cdot \Delta \bar{L} = E_L \Delta L \longrightarrow (4) \quad (\text{since } a_L \cdot a_L = 1)$$

$$\Delta V = - E_L \Delta L \longrightarrow (5)$$

where E_L = component of $\bar{E}$ in the direction of $\bar{a}_L$.

In other words, dot product can be expressed as, in terms of $\cos\theta$ as,

$$\Delta V = - E \, \Delta L \cos\theta. \quad (\text{since } \bar{E} \cdot \Delta \bar{L} = |E||\Delta L| \cos\theta)$$

$$\frac{\Delta V}{\Delta L} = - E \cos\theta. \longrightarrow (6)$$

To find ΔV at a point, take $\lim \Delta L \to 0$.

$$\therefore \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = - E \cos\theta. \longrightarrow (7)$$

but $\lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \frac{dV}{dL}$ = potential gradient

$$\therefore \frac{dV}{dL} = -E\cos\theta \longrightarrow (8)$$

→ Hence Potential Gradient $\frac{dV}{dL}$ can be maximum only, when $\cos\theta = -1$, i.e., $\theta = +180°$

→ This indicates dL must be in the opposite direction to $\bar{E}$.

$$\therefore \left.\frac{dV}{dL}\right|_{max} = E. \longrightarrow (9)$$

→ Thus if $\bar{a}_n$ is the unit vector in the direction of increasing potential normal to Equipotential surface, then

$$\bar{E} = -\left.\frac{dV}{dL}\right|_{max} \bar{a}_n. \longrightarrow (10)$$

As $\bar{E}$ and potential gradient are in opposite direction, Eq (10) has a negative sign.

→ The maximum value of rate of charge of potential with distance $\frac{dV}{dL}$ is called gradient of V.

Gradient of V = grad V = ∇V.

∇V = grad V = $-\bar{E}$.

(or) $$\boxed{\bar{E} = -\nabla V = -(\text{grad } V)} \longrightarrow (11).$$

Eq (11) gives the relationship between $\bar{E}$ and V.

Note: grad V i.e., gradient of a scalar is a vector.

$$\nabla = \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z$$

→ Gauss Law

Karl Friedrich Gauss a German mathematician developed the divergence theorem popularly known by his name. Gauss law constitutes one of the fundamental laws of Electro-magnetism.

Statement of Gauss law:

"The total flux of the electric field over any closed surface is equal to the total charge enclosed in the surface."

Proof:

→ Consider any object of irregular shape as shown in fig.

→ The total charge enclosed by regular closed surface is Q.

→ Consider a small differential surface ds at point P. As the surface is irregular, the direction of $\bar{D}$ & its magnitude is going to change from point to point on the surface.

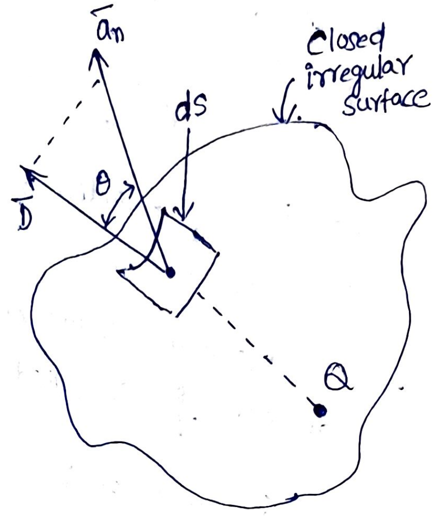


fig: flux through irregular closed surface

→ The flux $d\psi$ passing through the surface ds is the product of component of $\bar{D}$ in the direction normal to ds and $d\psi$.

$$d\psi = D_n \, ds.$$

where D_n = component of $\bar{D}$ in the direction of normal to surface ds.

From fig, $D_n = |\bar{D}| \cos\theta$.

$$d\psi = |\bar{D}| \cos\theta \, ds.$$

from the definition of dot product, $\bar{A} \cdot \bar{B} = |A| |B| \cos\theta_{AB}$.

$$|\bar{D}| ds \cos\theta = \bar{D} \cdot d\bar{s}$$

$$\therefore \quad d\psi = \bar{D} \cdot d\bar{s}.$$

Hence the total flux is obtained as

$$\psi = \int d\psi = \oint_S \bar{D} \cdot d\bar{s}.$$

$\oint$ sign indicates the integration over closed surface called closed surface integral.

→ Irrespective of shape of surface and charge distribution, total flux passing through the surface is the total charge enclosed by surface.

$$\psi = \oint_S \bar{D} \cdot d\bar{s} = Q = \text{charge enclosed.}$$

→ The common form used to represent Gauss law is

$$\psi = \oint_S \bar{D} \cdot d\bar{s} = \int_V \rho_v \, dv.$$

Application of Gauss's Law :-

→ Gauss law is the alternative statement of coulomb's law.

→ Gauss law provides an easy means of finding Electric field intensity $\bar{E}$ and electric flux density $\bar{D}$ for symmetrical charge distributions such as point charge, infinite line charge, infinite sheet charge and a spherical distribution of charge.

1) Point charge :

→ In spherical surface, let us consider a point charge Q. with centre as origin. The spherical surface is Gaussian surface.

→ $\bar{D}$ is always directed radially outwards along $\bar{a}_r$ which is normal to surface at any point P on the surface.

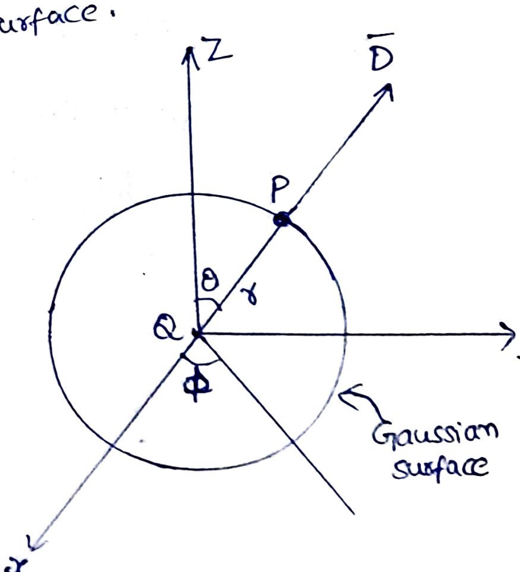


Fig: Gaussian surface about a point charge.

For a sphere of radius r, flux density $\bar{D}$ is in radial direction $\bar{a}_r$.

$$|\bar{D}| = D_r.$$

$$\bar{D} = D_r \, \bar{a}_r$$

For Gaussian surface i.e., sphere of radius r, ds normal to $\bar{a}_r$ is

$$d\bar{s} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$\bar{D} \cdot d\bar{s} = (D_r \bar{a}_r) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r).$$

$$\bar{D} \cdot d\bar{s} = D_r r^2 \sin\theta \, d\theta \, d\phi \qquad (\text{since, } \bar{a}_r \cdot \bar{a}_r = 1)$$

Now integrate over the surface of sphere of constant radius 'r'.

$$\oint_S \bar{D} \cdot d\bar{s} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r r^2 \sin\theta \, d\theta \, d\phi.$$

$$= D_r r^2 \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi.$$

$$= D_r r^2 \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= D_r r^2 \left[-(\cos\pi - \cos 0)\right] \left[2\pi - 0\right]$$

$$= D_r r^2 \left[-(-1-1)\right] \left[2\pi\right]$$

$$\oint_S \bar{D} \cdot d\bar{s} = 4\pi r^2 D_r.$$

But $\oint_S \bar{D} \cdot d\bar{s} = Q.$

$$\therefore Q = 4\pi r^2 D_r.$$

$$D_r = \frac{Q}{4\pi r^2}$$

Hence $\bar{D} = D_r \bar{a}_r = \dfrac{Q}{4\pi r^2} \bar{a}_r$

$$\bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 r^2} \bar{a}_r.$$

2) <u>Infinite Line charge</u> :-

→ Consider an infinite line charge of density ρ_L C/m lying along Z axis from $-\infty$ to ∞.

→ Consider the Gaussian surface as right circular cylinder with Z axis as its axis and radius r, as shown in fig.

→ $\bar{D}$ is directed radially outwards in the $\bar{a}_r$ direction, according to Cylindrical Co ordinate system.

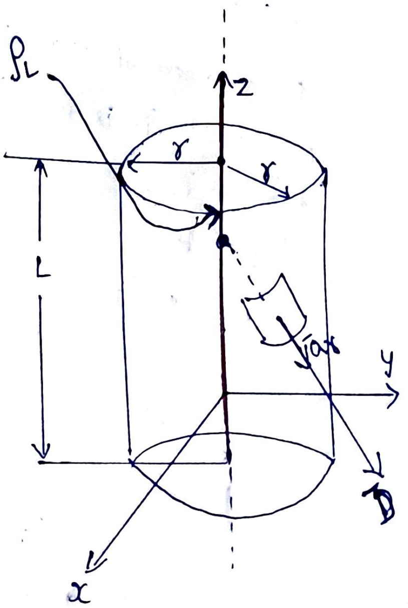


Fig: Infinite Line charge.

Consider a differential surface ds which is at a radial distance r from the line charge. The direction normal to ds is $\bar{a}_r$.

Now $Q = \oint_S \bar{D} \cdot d\bar{s}$.

The integration is to be evaluated for side surface, top surface and bottom surface.

$$Q = \oint_{side} \bar{D} \cdot d\bar{s} + \oint_{top} \bar{D} \cdot d\bar{s} + \oint_{bottom} \bar{D} \cdot d\bar{s}.$$

Now $\bar{D} = D_r a_r$

$$d\bar{s} = r\, d\phi\, dz\, \bar{a}_r$$

$$\therefore \bar{D} \cdot d\bar{s} = (D_r a_r) \cdot (r\, d\phi\, dz\, \bar{a}_r)$$

$$\bar{D} \cdot d\bar{s} = D_r r\, d\phi\, dz \qquad (\text{since } \bar{a}_r \cdot \bar{a}_r = 1).$$

As $\bar{D}$ has only radial component and no component along $\bar{a}_z$ and $-\bar{a}_z$ ∴ Hence integrations over top & bottom surfaces is zero.

$$\therefore \oint_{top} \bar{D} \cdot d\bar{s} = \oint_{bottom} \bar{D} \cdot d\bar{s} = 0.$$

$$\therefore Q = \oint_{side} \bar{D} \cdot d\bar{s} = \oint_{side} D_r r\, d\phi\, dz$$

$$= \int_{z=0}^{L} \int_{\phi=0}^{2\pi} D_r r\, d\phi\, dz$$

$$= r D_r \int_{z=0}^{L} dz \int_{\phi=0}^{2\pi} d\phi.$$

$$= r D_r [z]_0^L [\phi]_0^{2\pi}$$

$$= r D_r [L - 0][2\pi - 0].$$

$$Q = 2\pi r D_r L.$$

$$D_r = \frac{Q}{2\pi r L}.$$

$$\bar{D} = D_r \bar{a}_r = \frac{Q}{2\pi r L} \bar{a}_r$$

But $\frac{Q}{L} = \rho_L$, C/m.

$\therefore \bar{D} = \frac{\rho_L}{2\pi r} \bar{a}_r \quad C/m^2$.

$\bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{\rho_L}{2\pi \varepsilon_0 r} \bar{a}_r \quad V/m$

Note: The results are same as obtained from Coulomb's law

(3) Infinite sheet of charge :

→ Consider the infinite sheet of charge of uniform charge density ρ_s C/m², lying in z=0 plane i.e., xy plane as shown in fig.

→ Consider a Gaussian surface which is a rectangular box. and is cut by the sheet of charge to give ds = dx dy.

→ $\bar{D}$ acts normal to the plane. i.e., $\bar{a}_n = \bar{a}_z$ & $-\bar{a}_n = -\bar{a}_z$ direction.

Hence $\bar{D} = 0$ in x & y directions.

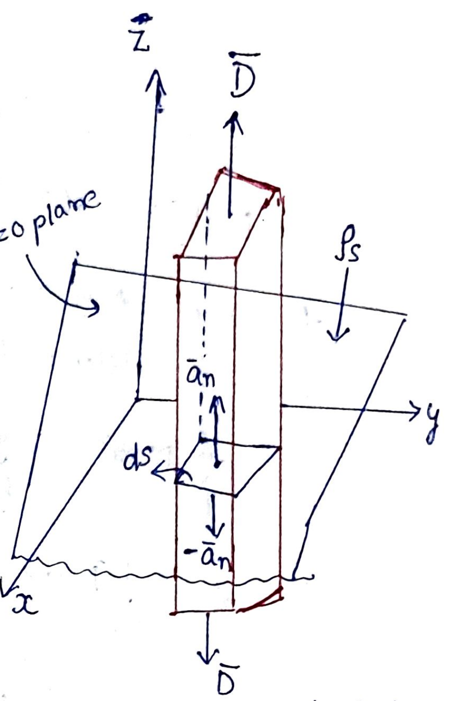


Fig: Infinite sheet of charge.

Hence the charge Enclosed can be written as;

$$Q = \oint_S \bar{D}\cdot d\bar{s} = \oint_{sides} \bar{D}\cdot d\bar{s} + \oint_{Top} \bar{D}\cdot d\bar{s} + \oint_{bottom} \bar{D}\cdot d\bar{s}.$$

But $\oint_{sides} \bar{D}\cdot d\bar{s} = 0$, as $\bar{D}$ has no component in x & y direction.

Now $\bar{D} = D_z \bar{a}_z$ for Top surface

$d\bar{s} = dx\, dy\, \bar{a}_z$

$\bar{D}\cdot d\bar{s} = D_z\, dx\, dy\, (\bar{a}_z \cdot \bar{a}_z)$

$= D_z\, dx\, dy.$

For bottom surface, $\bar{D} = D_z(-\bar{a}_z)$

$d\bar{S} = dx\,dy\,(-\bar{a}_z)$

$\bar{D}\cdot d\bar{S} = D_z(-\bar{a}_z)\cdot dx\,dy\,(-\bar{a}_z)$

$\bar{D}\cdot d\bar{S} = D_z\,dx\,dy$ (since $\bar{a}_z\cdot\bar{a}_z = 1$)

$$\therefore Q = \oint_{top} \bar{D}\cdot d\bar{S} + \oint_{bottom} \bar{D}\cdot d\bar{S}$$

$$Q = \oint_{top} D_z\,dx\,dy + \oint_{bottom} D_z\,dx\,dy.$$

Now $\oint_{top} dx\,dy = \oint_{bottom} dx\,dy = A$ = Area of surface

$$Q = 2D_z A.$$

But $Q = \rho_s \times A$ as ρ_s = Surface charge density.

$$\rho_s A = 2D_z A$$

$$\rho_s = 2D_z. \Rightarrow D_z = \frac{\rho_s}{2}.$$

$$\therefore \bar{D} = D_z\bar{a}_z = \frac{\rho_s\,\bar{a}_z}{2} \quad C/m^2.$$

$$\therefore \bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{\rho_s\,\bar{a}_z}{2\varepsilon_0} \quad V/m.$$

* The results are same as obtained by the Coulomb's law for infinite sheet of charge.

Maxwell's First Equation :-

The Maxwell first Equation is also called "point form of Gauss's law" (or) Gauss's law in differential form.

Statement of Gauss's law in point form is,

The divergence of Electric flux density in a medium at a point is equal to the volume charge density at the same point.

$$\operatorname{div}\bar{D} = \nabla\cdot\bar{D} = \rho_v .$$

Proof :

Divergence of Electric flux density $\bar{D}$ is

$$\operatorname{div}\bar{D} = \lim_{\Delta V \to 0} \frac{\oint_S \bar{D}\cdot d\bar{s}}{\Delta V} \longrightarrow (1)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} .$$

According to Gauss's law,

$$\psi = Q = \oint_S \bar{D}\cdot d\bar{s} . \longrightarrow (2)$$

Expressing Gauss's law per unit volume basis,

$$\frac{Q}{\Delta V} = \frac{\oint_S \bar{D}\cdot d\bar{s}}{\Delta V} \longrightarrow (3)$$

Taking lim $\Delta V \to 0$ i.e., volume shrinks to zero,

$$\lim_{\Delta V \to 0} \frac{Q}{\Delta V} = \lim_{\Delta V \to 0} \frac{\oint_S \bar{D}\cdot d\bar{s}}{\Delta V} \longrightarrow (4)$$

but $\lim_{\Delta V \to 0} \frac{Q}{\Delta V} = \rho_v$ at that point. $\longrightarrow (5)$

Substituting eq (5) & eq (1) in eq (4)

$$\rho_v = \operatorname{div}\bar{D}$$

$\text{div}\,\bar{D} = \rho_v$

$\rightarrow \boxed{\nabla\cdot\bar{D} = \rho_v} \longrightarrow (7)$

eq (7) is maxwell's first equation.

Problems :

1) Given $\bar{A} = 2xy\,\bar{a}_x + z\,\bar{a}_y + yz^2\,\bar{a}_z$ Find $\nabla\cdot\bar{A}$ at $P(2,-1,3)$

Sol: Given $\bar{A}$, $A_x = 2xy$, $A_y = z$, $A_z = yz^2$

$$\nabla\cdot\bar{A} = \text{div}\,\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial}{\partial x}[2xy] + \frac{\partial}{\partial y}[z] + \frac{\partial}{\partial z}[yz^2]$$

$$\nabla\cdot\bar{A} = 2y + 0 + 2zy = 2y + 2yz$$

At $P(2,-1,3)$, $x = 2$, $y = -1$, $z = 3$.

$$\nabla\cdot\bar{A} = 2(-1) + 2(-1)(3) = -8.$$

2) Find the divergence of $\bar{A}$ at $P(5, \frac{\pi}{2}, 1)$ where

$\bar{A} = rz\sin\phi\,\bar{a}_r + 3rz^2\cos\phi\,\bar{a}_\phi$.

Sol: Given $\bar{A}$ in cylindrical system,

$$\text{div}\,\bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}.$$

where $A_r = rz\sin\phi$, $A_\phi = 3rz^2\cos\phi$, $A_z = 0$.

$$\text{div}\,\bar{A} = \frac{1}{r}\frac{\partial}{\partial r}[r^2 z\sin\phi] + \frac{1}{r}\frac{\partial}{\partial\phi}[3rz^2\cos\phi] + 0.$$

$$= \frac{1}{r}z\sin\phi\,(2r) + \frac{1}{r}\cdot 3rz^2(-\sin\phi)$$

$$\nabla\cdot\bar{A} = 2z\sin\phi - 3z^2\sin\phi.$$

At point P, $r = 5$, $\phi = \frac{\pi}{2}$, $z = 1$.

$$\nabla\cdot\bar{A} = 2\times1\times\sin\frac{\pi}{2} - 3(1)^2\sin\frac{\pi}{2}.$$

$$= -1 \text{ at } P.$$

3) ~~The flux~~ Prove that the divergence of electric field and that of electric flux density in a charge free region is zero.

Sol:- from point form of Gauss's law, we can write,

$$\nabla \cdot \bar{D} = \text{div}\, \bar{D} = \rho_v \longrightarrow (1)$$

while $\bar{D} = \varepsilon_0 \bar{E}$.

$$\nabla \cdot (\varepsilon_0 \bar{E}) = \rho_v .$$

ε_0 is scalar and constant, so

$$\varepsilon_0 \nabla \cdot \bar{E} = \rho_v$$

$$\text{div}\, \bar{E} = \nabla \cdot \bar{E} = \frac{\rho_v}{\varepsilon_0} . \longrightarrow (2)$$

But in charge free region, $Q = 0$, hence there can not exist any charge density ρ_v.

$$\therefore \rho_v = 0 , \quad \text{for charge free region.} \rightarrow (3) .$$

Substitute Eqn (3) in (2) and (1) Eqn.

$$\nabla \cdot \bar{D} = \nabla \cdot \bar{E} = 0 , \quad \text{for charge free region} .$$

Laplace and Poisson's Equations :-

From the Gauss's law in the point form, Poisson's eq can be derived.

Consider the Gauss's law in the point form as,

$$\nabla \cdot \bar{D} = \rho_v \longrightarrow (1)$$

where $\bar{D}$ = flux density, ρ_v = volume charge density.

For homogeneous, isotropic and linear medium, flux density $\bar{D}$ and electric field intensity $\bar{E}$ are directly proportional.

Thus, $$\bar{D} = \varepsilon \bar{E} \longrightarrow (2)$$

$$\nabla \cdot \varepsilon \bar{E} = \rho_v \longrightarrow (3)$$

from the gradient relationship, $\bar{E} = -\nabla V \longrightarrow (4)$.

Substituting eq (4) in eq (3),

$$\nabla \cdot \varepsilon(-\nabla V) = \rho_v \longrightarrow (5)$$

Taking $-\varepsilon$ outside a constant,

$$-\varepsilon[\nabla \cdot \nabla V] = \rho_v$$

$$\therefore \quad \nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon} \longrightarrow (6)$$

Now $\nabla \cdot \nabla$ operation is called 'del squared' operation and denoted as ∇^2.

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}} \longrightarrow (7)$$

Eq (7) is called Poisson's Equation.

→ If in a certain region, volume charge density is zero ($\rho_v = 0$), which is true for di-electric medium, then the poisson's eq,

$$\boxed{\nabla^2 V = 0}, \text{ for charge free region} \longrightarrow (8)$$

→ This is special case of poisson's eq and is called "Laplace's Equation". ∇^2 operation is called Laplacian of V.

→ ∇^2 operation in different co ordinate system :

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z .$$

→ In cartesian co ordinate system,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 .$$

The above Eq is Laplace Eq in cartesian form.

→ In cylindrical co ordinate system,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0 .$$

The above Eq is Laplace Eq in cylindrical form.

→ In spherical coordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 .$$

The above Eq is Laplace Eq in spherical form.

→ Solution of Laplace Equation in one variable : (or)

Uniqueness Theorem :-

→ The solution of Laplace's Eq solved by uniqueness Theorem.

→ Assume that the Laplace's Eq. has two solutions say V_1 and V_2, both are function of the coordinates of system used. These solutions must satisfy Laplace's Eq. so we can write

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0 . \longrightarrow (1) .$$

Both the solutions must satisfy the boundary conditions as well. At the boundary, the potential at different points are same due to Equipotential surface, then, $\longrightarrow$ (2)

$$V_1 = V_2 .$$

Let the difference between two solutions is V_d.

$$V_d = V_2 - V_1 . \longrightarrow (3) .$$

Using Laplace's Eq for the difference V_d,

$$\nabla^2 V_d = \nabla^2 (V_2 - V_1) = 0 \longrightarrow (4)$$

$$\therefore \nabla^2 V_2 - \nabla^2 V_1 = 0 \longrightarrow (5)$$

on the boundary, $V_d = 0$, from Eq (2) & (3)

Now the divergence theorem states that,

$$\int_{vol} \nabla \cdot \bar{A} \, dV = \oint_S \bar{A} \cdot d\bar{s} \longrightarrow (6).$$

Let $\bar{A} = V_d \nabla V_d$ and from vector algebra,

$$\nabla \cdot (\alpha \bar{B}) = \alpha(\nabla \cdot \bar{B}) + \bar{B} \cdot (\nabla \alpha).$$

$\nabla \cdot (V_d \nabla V_d)$ with $\alpha = V_d$ and

$\therefore$ ~~∇~~. Now use this for

$$\nabla V_d = \bar{B}.$$

$$\therefore \nabla \cdot (V_d \nabla V_d) = V_d(\nabla \cdot \nabla V_d) + \nabla V_d \cdot (\nabla V_d).$$

But $\nabla \cdot \nabla = \nabla^2$ hence,

$$\therefore \nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d \cdot \nabla V_d. \longrightarrow (7)$$

using Eq (4),

$$\nabla \cdot (V_d \nabla V_d) = 0 + \nabla V_d \cdot \nabla V_d = \nabla V_d \cdot \nabla V_d \rightarrow (8)$$

~~Sub Eq (8) in Eq (6),~~

To use Eq (8) in Eq (6), let $\bar{A} = V_d \nabla V_d$, hence

$$\nabla \cdot V_d \nabla V_d = \nabla \cdot \bar{A} = \nabla V_d \cdot \nabla V_d.$$

$$\int_{Vol} \nabla V_d \cdot \nabla V_d \, dV = \oint_S V_d \nabla V_d \cdot d\bar{s} \longrightarrow (9).$$

But $V_d = 0$, on boundary, hence right hand side of Eq (9) is zero.

$$\therefore \int_{Vol} \nabla V_d \cdot \nabla V_d \, dV = 0 \longrightarrow (10)$$

It is known that, $\bar{C} \cdot \bar{C} = |\bar{C}|^2$,

$$\therefore \int_{Vol} |\nabla V_d|^2 dV = 0, \quad \text{as } \nabla V_d \text{ is Vector} \longrightarrow (11).$$

Now integration can be zero under two conditions,

i) The quantity under integral sign is zero,

ii) The quantity is positive in some regions and negative in other regions by equal amount and hence zero.

But square term cannot be negative in any region hence quantity under integral must be zero.

$$|\nabla V_d|^2 = 0.$$

$$\text{i.e., } \nabla V_d = 0.$$

As the gradient of $V_d = V_2 - V_1 = 0$, means

$V_2 - V_1 = \text{constant}$.

But considering boundary, it can be proved that;

$$V_2 - V_1 = \text{constant} = \text{Zero}$$

$$\therefore V_2 = V_1$$

This proves that both the solutions are equal and cannot be different.

Thus Uniqueness Theorem can be stated as :

If the solution of Laplace's Equation satisfies the boundary condition then that solution is unique, by whatever method it is obtained.

The solution of Laplace's Equation gives the field which is unique, satisfying the same boundary conditions, in a given region.

Problem :

Determine whether or not the following potential fields satisfy the Laplace's Equation.

a) $V = x^2 - y^2 + z^2$ b) $V = r\cos\phi + z$, c) $V = r\cos\theta + \phi$.

Laplace Equation $\nabla^2 V = 0$.

Sol:

a) $V = x^2 - y^2 + z^2$

In cartesian co ordinate system,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial}{\partial x^2}(x^2 - y^2 + z^2) + \frac{\partial}{\partial y^2}(x^2 - y^2 + z^2) + \frac{\partial}{\partial z^2}(x^2 - y^2 + z^2).$$

$$= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(2z).$$

$$\nabla^2 V = 2 - 2 + 2 = 2$$

So, $\nabla^2 V \neq 0$.

Hence field V does not satisfy Laplace's Equation.

b) $V = r\cos\phi + z$.

In cylindrical co ordinate system,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}.$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}(r\cos\phi + z) = \cos\phi.$$

$$\frac{\partial V}{\partial \phi} = \frac{\partial}{\partial \phi}(r\cos\phi + z) = -r\sin\phi.$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(r\cos\phi + z) = 1.$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}[r\cos\phi] = \frac{1}{r}\cos\phi.$$

$$\frac{1}{r^2}\left[\frac{\partial^2 V}{\partial \phi^2}\right] = \frac{1}{r^2}\left[\frac{\partial}{\partial \phi}(-r\sin\phi)\right] = -\frac{r\cos\phi}{r^2} = -\frac{\cos\phi}{r}.$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}(1) = 0.$$

$$\therefore \nabla^2 V = \frac{1}{r}\cos\phi - \frac{\cos\phi}{r} + 0 = 0.$$

$$\nabla^2 V = 0.$$

So this field satisfies Laplace's equation.

c) $V = r\cos\theta + \phi$.

In spherical system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$r^2 \frac{\partial V}{\partial r} = r^2 \frac{\partial}{\partial r}[r\cos\theta + \phi] = r^2 \cos\theta.$$

$$\sin\theta \frac{\partial V}{\partial \theta} = \sin\theta \frac{\partial}{\partial \theta}[r\cos\theta + \phi]$$

$$= \sin\theta[-r\sin\theta] = -r\sin^2\theta$$

$$\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}[r\cos\theta + \phi]$$

$$= \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}[1]$$

$$= 0.$$

$$\therefore \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\cos\theta] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(-r\sin^2\theta)$$

$$= \frac{1}{r^2} 2r\cos\theta + \frac{1}{r^2 \sin\theta}(-r\,2\sin\theta\cos\theta)$$

$$= \frac{2}{r}\cos\theta - \frac{2}{r}\cos\theta.$$

$$\nabla^2 V = 0.$$

So this field satisfies Laplace's Equation.

Electric Dipole :-

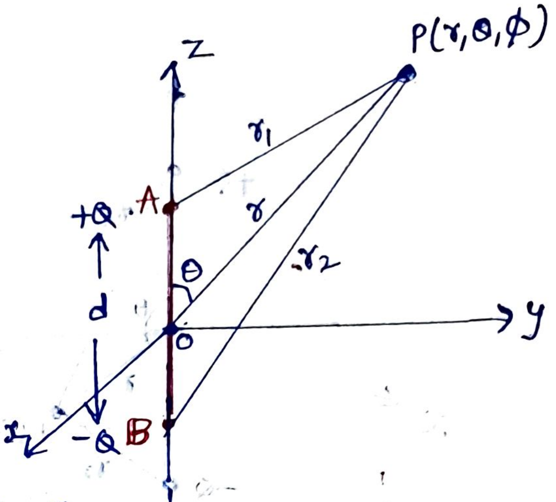
The two point charges of equal magnitude but opposite sign, separated by a very small distance give rise to an Electric Dipole.

Expression of $\bar{E}$ due to an Electric dipole :

Consider an electric dipole as shown in fig(1). The two point charges are $+Q$ & $-Q$, separated by a very small distance 'd'.

Consider a point $P(r, \theta, \phi)$ in spherical coordinate system.

It is required to find $\bar{E}$ due to an electric dipole at point P.



Fig(1): Field due to an Electric dipole

→ Let 'O' be the midpoint of AB.

The distance of point P from A is r_1.

while the distance of point P from B is r_2.

The distance of point O is r.

→ The coordinates of A are $(0, 0, +\frac{d}{2})$ and B are $(0, 0, -\frac{d}{2})$.

In spherical coordinates, the potential at point P due to charge $+Q$ is given by,

$$V_1 = \frac{+Q}{4\pi\varepsilon_0 r_1} \longrightarrow (1)$$

The potential at P due to charge $-Q$ is given by,

$$V_2 = \frac{-Q}{4\pi\varepsilon_0 r_2} \longrightarrow (2)$$

The total potential at point P is the algebraic sum of V_1 and V_2.

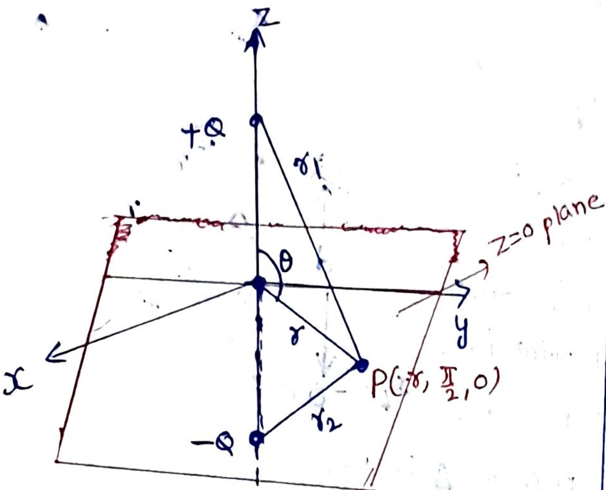
$$\therefore V = V_1 + V_2$$

$$= \frac{+Q}{4\pi\varepsilon_0 r_1} - \frac{Q}{4\pi\varepsilon_0 r_2}$$

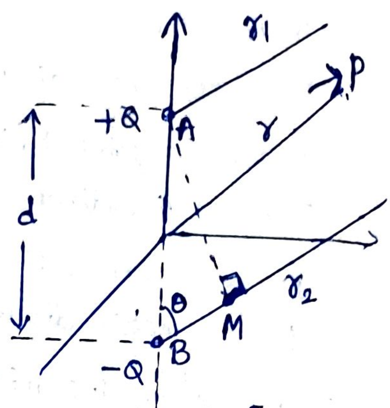
$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right].$$

$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]. \quad \text{——— (3)}$$

If now point P is located in the $Z=0$ plane as shown in fig(2), then $r_2 = r_1$. Hence we get $V=0$. Thus entire $Z=0$ plane i.e, xy plane is a zero potential surface.



Fig(2): point P in Z=0 plane.



Fig(3): Point P is too far away

Now consider that P is located far away from electric dipole, thus r_1, r_2 & r can be assumed to parallel to each other as shown in fig(3).

AM is drawn perpendicular from A on r_2. The angle made by r_1, r_2 and r with Z axis is θ, as all are parallel.

$$BM = AB\cos\theta = d\cos\theta \longrightarrow (4).$$

Now, $PB = PM + BM$.

$PA = PM$ as AM is perpendicular

and $PB = r_2$, $PA = r_1$.

$BM = PB - PM = r_2 - PM$

while, $PM = PA = r_1$.

$$\therefore BM = r_2 - r_1. \longrightarrow (5)$$

from eq (4),

$$\therefore r_2 - r_1 = d\cos\theta \longrightarrow (6).$$

As d is very small, $r_1 \approx r_2 \approx r$, hence $r_1 r_2 = r^2$.

from Eq (3),

$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{d\cos\theta}{r^2}\right] \longrightarrow (7).$$

Now $\bar{E} = -\nabla V$.

$$= -\left[\frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi\right]$$

$$\frac{\partial V}{\partial r} = \frac{Qd\cos\theta}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right] = \frac{Qd\cos\theta}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}(r^{-2})\right]$$

$$= \frac{Qd\cos\theta}{4\pi\varepsilon_0}\left[-2r^{3}\right) = -\frac{2Qd\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$\frac{\partial V}{\partial \theta} = \frac{Qd}{4\pi\varepsilon_0 r^2}\frac{\partial}{\partial\theta}(\cos\theta) = \frac{Qd}{4\pi\varepsilon_0 r^2}[-\sin\theta]$$

$$\frac{\partial V}{\partial \phi} = \frac{\partial}{\partial\phi}\left[\frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2}\right] = 0$$

$$\therefore \bar{E} = -\left[\frac{-2Qd\cos\theta}{4\pi\varepsilon_0 r^3}\bar{a}_r + \frac{1}{r}\left(\frac{-Qd\sin\theta}{4\pi\varepsilon_0 r^2}\right)\bar{a}_\theta + 0\right]$$

$$\bar{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right] \longrightarrow (8).$$

The above Eq (8) is Electric field $\bar{E}$ at point P due to electric dipole.

Dipole moment :-

The combination of two point charges of equal magnitude and opposite signs separated by a small distance d is referred as Electric dipole. and the product of charge and spacing (Qd) is called as Electric dipole moment.

$\boxed{\text{The dipole moment, } \bar{p} = Q\bar{d}.}$ Coulomb-metre (cm) let the vector length from $-Q$ to $+Q$ i.e, from B to A.

$\therefore \bar{d} = d\,\bar{a}_z$ $\longrightarrow$ (9)

Its component along $\bar{a}_r$ direction can be obtained as,

$$d_r = \bar{d}\cdot\bar{a}_r = d\,\bar{a}_z\cdot\bar{a}_r = d\cos\theta.$$

$\therefore \bar{d} = d\cos\theta\,\bar{a}_r$ $\longrightarrow$ (10)

Dipole moment $\bar{p} = Q\bar{d}$ $\longrightarrow$ (11).

$$\bar{p}\cdot\bar{a}_r = Q\bar{d}\cdot\bar{a}_r$$

$$= Q\,d\cos\theta\,\bar{a}_r\cdot\bar{a}_r$$

$$\bar{p}\cdot\bar{a}_r = Qd\cos\theta.$$

Hence the expression of potential V can be expressed as,

$$\Rightarrow \quad V = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\bar{p}\cdot\bar{a}_r}{4\pi\varepsilon_0 r^2} \text{ Volts} \longrightarrow (12)$$

If $p = |\bar{p}| = Q|\bar{d}| = Qd$, Then $\bar{E}$ due to dipole can be expressed interms of magnitude of dipole moment as,

$$\bar{E} = \frac{p}{4\pi\varepsilon_0 r^2}\left[2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right] \longrightarrow (13).$$

Torque on Electric dipole in an Electric Field :-

Consider an electric dipole in an uniform electric field $\bar{E}$, making an angle θ with respect to dipole as shown in fig(1)

The charges $+Q$ & $-Q$ experiences a force due to an $\bar{E}$ which is equal in magnitude but opposite in direction.

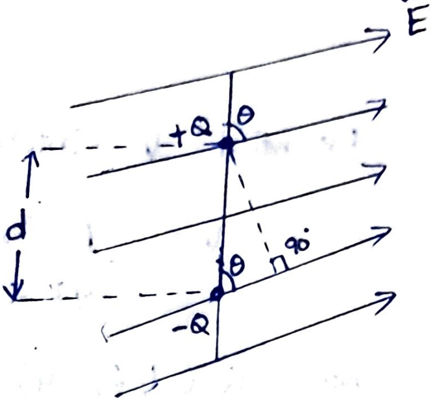


Fig: Electric dipole in uniform electric field.

Force on $+Q = \bar{F_1}$

Force on $-Q = \bar{F_2}$

$|\bar{F_1}| = |\bar{F_2}| = QE$

where E = magnitude of $\bar{E}$.

Torque = Force x perpendicular distance of separation

$T = F \times l$

$= (QE) \times l$

From fig, $\sin\theta = \frac{l}{d} \Rightarrow l = d\sin\theta$

$T = QE\,d\sin\theta$

$= (Qd)\,E\sin\theta$

But, Qd = dipole moment $= |\bar{P}|$

$E = |\bar{E}|$.

While $\sin\theta$ is the cross product between $|\bar{P}|$ & $|\bar{E}|$.

Hence in vector form, Torque on the dipole is expressed as

$$\boxed{\bar{T} = \bar{P} \times \bar{E}}$$

Problems :

1) A dipole having moment $\bar{p} = 3\bar{a}_x - 5\bar{a}_y + 10\bar{a}_z$ n cm is located at Q (1, 2, -4) in free space find V at P(2, 3, 4).

Sol : Potential V interms of dipole moment is,

$$V = \frac{\bar{p} \cdot \bar{a}_r}{4\pi \varepsilon_0 r^2}$$

Now Q (1, 2, -4) & P (2, 3, 4).

$$\bar{r} = \bar{P} - \bar{Q}$$

$$= (2-1)\bar{a}_x + (3-2)\bar{a}_y + (4-(-4))\bar{a}_z$$

$$= \bar{a}_x + \bar{a}_y + 8\bar{a}_z$$

$$|\bar{r}| = \sqrt{1^2+1^2+8^2} = \sqrt{1+1+64} = \sqrt{66}.$$

$$\bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{a}_x + \bar{a}_y + 8\bar{a}_z}{\sqrt{66}}$$

$$\bar{p} \cdot \bar{a}_r = (3\bar{a}_x - 5\bar{a}_y + 10\bar{a}_z) \cdot \left(\frac{\bar{a}_x + \bar{a}_y + 8\bar{a}_z}{\sqrt{66}}\right)$$

$$= \frac{3 - 5 + 80}{\sqrt{66}} = \frac{78}{\sqrt{66}} \times 10^{-9} \text{ as } \bar{p} \text{ in ncm.}$$

$$\therefore V = \frac{\bar{p} \cdot \bar{a}_r}{4\pi \varepsilon_0 r^2} = \frac{\frac{78}{\sqrt{66}} \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times (\sqrt{66})^2}$$

$$V = 1.3074 \text{ volt}$$

2) The Potential distribution is given by: $V = \dfrac{4}{x^2+y^2+z^2}$

find the Expression for $\bar{E}$.

Sol: $\bar{E} = -\nabla V = -\dfrac{\partial V}{\partial x}\bar{a}_x - \dfrac{\partial V}{\partial y}\bar{a}_y - \dfrac{\partial V}{\partial z}\bar{a}_z$.

$$\frac{\partial V}{\partial x} = 4\frac{\partial}{\partial x}\left[\frac{1}{x^2+y^2+z^2}\right] = 4\left[\frac{-2x}{(x^2+y^2+z^2)^2}\right].$$

$$\frac{\partial V}{\partial y} = 4\frac{\partial}{\partial y}\left[\frac{1}{x^2+y^2+z^2}\right] = 4\left[\frac{-2y}{(x^2+y^2+z^2)^2}\right]$$

$$\frac{\partial V}{\partial z} = 4\frac{\partial}{\partial z}\left[\frac{1}{x^2+y^2+z^2}\right] = 4\left[\frac{-2z}{(x^2+y^2+z^2)^2}\right].$$

$$\bar{E} = \frac{8}{(x^2+y^2+z^2)^2}\left[x\bar{a}_x + y\bar{a}_y + z\bar{a}_z\right]. \text{ Volt/m.}$$

3) find the Electric field at a point $(1, -2, 1)$ m if the given potential is $V = 3x^2y + 2yz^2 + 2xyz$.

Sol: Given $P(1, -2, 1)$, $V = 3x^2y + 2yz^2 + 2xyz$.

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right].$$

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left[3x^2y + 2yz^2 + 2xyz\right] = 6xy + 0 + 2yz.$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}\left[3x^2y + 2yz^2 + 2xyz\right] = 3x^2 + 2z^2 + 2xz.$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}\left[3x^2y + 2yz^2 + 2xyz\right] = 0 + 4y + 2xy$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(3x^2y + 2yz^2 + 2xyz$$

At point $P(\underset{x}{1}, \underset{y}{-2}, \underset{z}{1})$ m,

$$\frac{\partial V}{\partial x} = 6(1)(-2) + 2(-2)(1) = -12 + -4 = -16$$

$$\frac{\partial y}{\partial x} = 3(1)^2 + 2(1)^2 + 2(1)(1) = 3 + 2 + 2 = 7$$

$$\frac{\partial y}{\partial x} =$$

$$\frac{\partial V}{\partial z} = 4(-2) + 2(1)(-2) = -8-4 = -12$$

$$\therefore \bar{E} = -\left[-16\,\bar{a}_x + 7\,\bar{a}_y - 12\,\bar{a}_z\right]$$

$$\Rightarrow \bar{E} = 16\,\bar{a}_x - 7\,\bar{a}_y + 12\,\bar{a}_z$$

4) Find the stored Energy in a system of four identical charges $Q = 4\,nC$ at the corners of a square of 1 m on a side.

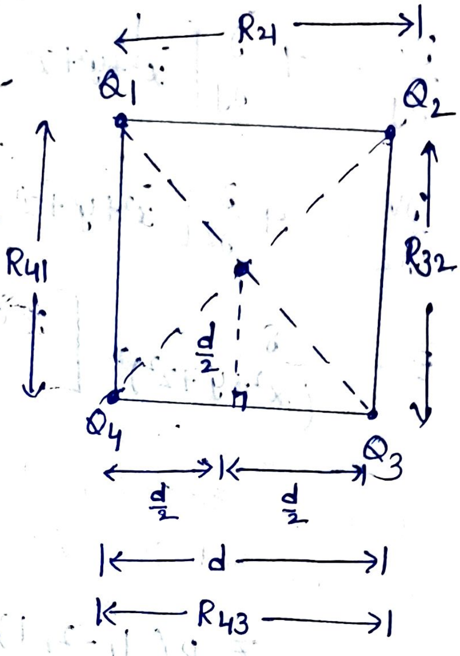


Fig.

Sol: The arrangement is shown in fig

Given $d = 1$

$$R_{31} = 2 \times \frac{\sqrt{2}\,d}{2}$$

$$R_{31} = \sqrt{2}\,d = \sqrt{2}(1)$$

$$\Rightarrow R_{31} = \sqrt{2}\ m = R_{42}$$

$$R_{21} = R_{32} = R_{43} = R_{41} = d = 1m.$$

let Q_1 is placed first, when all other charges are absent.

Hence $W_1 = 0$ Joules.

For Q_2, $W_2 = Q_2 V_{21} = Q_2\left[\frac{Q_1}{4\pi\varepsilon_0 R_{21}}\right]$

For Q_3, $W_3 = Q_3 V_{32} + Q_3 V_{31}$

$$W_3 = Q_3\left[\frac{Q_2}{4\pi\varepsilon_0 R_{32}}\right] + Q_3\left[\frac{Q_1}{4\pi\varepsilon_0 R_{31}}\right]$$

For Q_4, $W_4 = Q_4 V_{41} + Q_4 V_{42} + Q_4 V_{43}$

$$W_4 = Q_4\left[\frac{Q_1}{4\pi\varepsilon_0 R_{41}}\right] + Q_4\left[\frac{Q_2}{4\pi\varepsilon_0 R_{42}}\right] + Q_4\left[\frac{Q_3}{4\pi\varepsilon_0 R_{43}}\right]$$

But $Q_1 = Q_2 = Q_3 = Q_4 = Q = 4\ nC$, given

$$W = W_1 + W_2 + W_3 + W_4.$$

$$= 0 + \frac{Q^2}{4\pi\epsilon_0} + \frac{Q^2}{4\pi\epsilon_0} + \frac{Q^2}{4\pi\epsilon_0\sqrt{2}} + \frac{Q^2}{4\pi\epsilon_0} + \frac{Q^2}{4\pi\epsilon_0\sqrt{2}} + \frac{Q}{4\pi\epsilon_0}$$

$$= \frac{Q^2}{4\pi\epsilon_0}\left[4 + \frac{2}{\sqrt{2}}\right] = \frac{4\times10^{-9}}{4\pi\times8.854\times10^{-9}}\left[\frac{4\sqrt{2}+2}{\sqrt{2}}\right]$$

$$\Rightarrow W = 0.7785\ \mu J$$

5) The absalute potential (Electric) for some region is assumed to be $V = \frac{3000}{x} + \frac{2000}{x^2} + \frac{1000}{x^3}$ for all values of x, y, z where V is in volts and x is in metres. what is the electric field intensity at $x = 1\ m$?

Sol: $V = \frac{3000}{x} + \frac{2000}{x^2} + \frac{1000}{x^3}$

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left[\frac{3000}{x} + \frac{2000}{x^2} + \frac{1000}{x^3}\right]$$

$$= \left[-\frac{3000}{x^2} - \frac{4000}{x^3} - \frac{3000}{x^4}\right]$$

$$\frac{\partial V}{\partial y} = 0$$

$$\frac{\partial V}{\partial z} = 0$$

$$\therefore E = -\nabla V = -\left[-\frac{3000}{x^2} - \frac{4000}{x^3} - \frac{3000}{x^4}\right]\bar{a}_x$$

at $x = 1\ m$,

$$\bar{E} = -[-3000 - 4000 - 3000]\bar{a}_x$$

$$\Rightarrow \bar{E} = 10000\ \bar{a}_x\ V/m$$

6). Point charges of $+3\mu C$ and $-3\mu C$ are located at $(0,0,1)$ mm and $(0,0,-1)$ mm respectively, in free space.

i) Find dipole moment $\bar{p}$?

(ii) find $\bar{E}$ in spherical components at $P(r=2, \theta=40°, \phi=90°)$

Sol: i) The dipole moment $\bar{p} = Q\bar{d}$, where $\bar{d} = d\,\bar{a}_z$

Here d = distance between charges = 2 mm.

$Q = 3\times10^{-6}$ C.

$\therefore \bar{p} = (3\times10^{-6})(2\times10^{-3})\,\bar{a}_z$.

$\bar{p} = 6\,\bar{a}_z$ ncm

(ii) In spherical System,

$$\bar{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right]$$

$$= \frac{(3\times10^{-6})(2\times10^{-3})}{4\pi\times8.854\times10^{-12}(2)^3}\left[2\cos40°\,\bar{a}_r + \sin40°\,\bar{a}_\theta\right]$$

$\bar{E} = 10.3275\,\bar{a}_r + 4.333\,\bar{a}_\theta$. V/m.

Previous QP Problems

1) Two small identical conducting spheres have charge of 2 nC and -0.5 nC respectively separated by 4 cm. What is the force between them?

Sol: Given $Q_1 = 2nC$

$Q_2 = -0.5nC$.

distance $R = 4$ cm.

Force $F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}$

$= \frac{(2\times10^{-9})(-0.5\times10^{-9})}{4\pi\times 8.854\times10^{-12}\times(4)^2}$

$= \underline{5618.75 \text{ N}}$

2) Let $\rho_v = (3x + 4y + 2z)$ C/m², in the cubical region described by $0 \leq x, y, z \leq 3$. and the $\rho_v = 0$ outside the cube. Find the total charge contained within the cube.

Sol: Given Volume charge density $\rho_v = 3x + 4y + 2z$ C/m².

Total charge $Q = \int_V \rho_v \, dv$

The integral is a volume integral with the limits 0 and 3. and differential volume $dv = dx\,dy\,dz$.

$\therefore Q = \int_{x=0}^{3} \int_{y=0}^{3} \int_{z=0}^{3} \rho_v \, dx\,dy\,dz.$

$= \int_{y=0}^{3} \int_{z=0}^{3} \left[\int_{x=0}^{3} (3x+4y+2z)\,dx \right] dy\,dz.$

$= \int_{y=0}^{3} \int_{z=0}^{3} \left[\frac{3x^2}{2} + 4yx + 2zx \right]_0^3 dy\,dz$

$$Q = \int_{y=0}^{3}\int_{z=0}^{3}\left[\frac{3(3)^2}{2} + 4y(3) + 2z(3) - 0\right] dy\, dz$$

$$= \int_{z=0}^{3}\left[\int_{y=0}^{3}\left(\frac{27}{2} + 12y + 6z\right) dy\right] dz.$$

$$= \int_{z=0}^{3}\left[\frac{27y}{2} + \frac{12y^2}{2} + 6zy\right]_0^3 dz$$

$$= \int_{z=0}^{3} \frac{81}{2} + 54 + 18z \; dz$$

$$= \left[\frac{81z}{2} + 54z + 9z^2\right]_0^3$$

$$= \frac{81(3)}{2} + 54(3) + 9(3)^2 - 0$$

$\therefore$ $Q = 364.5$ Coulomb.

Four concentrated charges $Q_1 = 0.3\,\mu C$, $Q_2 = 0.2\,\mu C$, $Q_3 = 0.3\,\mu C$ and $Q_4 = 0.2\,\mu C$. are located at the vertices of a plane rectangle. The length of rectangle is 5 cm and breadth of rectangle is 2 cm. Find the magnitude and direction of resultant force on Q_1.

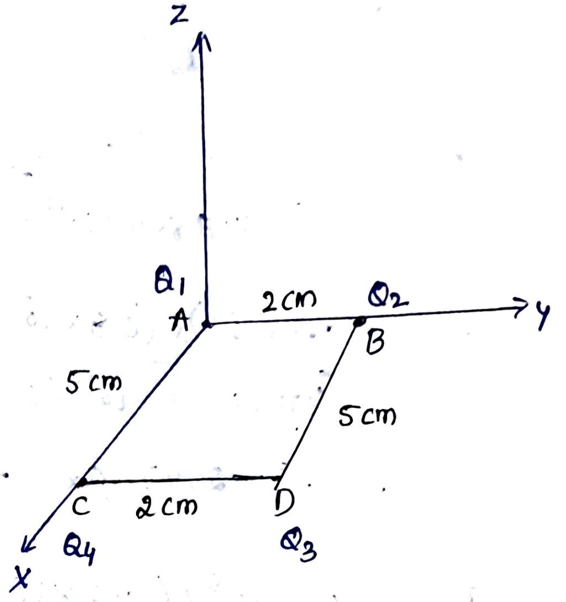


Fig.

Sol: The coordinates of 4 corners are $A(0,0,0)$ $B(0,0.02,0)$ $C(0.05,0.02,0)$ $D(0.05,0,0)$.

The Force on Q_1 is sum of the forces due to Q_2, Q_3 and Q_4.

$$\bar F_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2}\,\bar a_{21}$$

$$\bar F_{31} = \frac{Q_1 Q_3}{4\pi\varepsilon_0 R_{31}^2}\,\bar a_{31}$$

$$\bar F_{41} = \frac{Q_1 Q_4}{4\pi\varepsilon_0 R_{41}^2}\,\bar a_{41}$$

$\bar R_{21} = \cancel{-0.02\,\bar a_y}$, $|\bar R_{21}| = \cancel{0.02}$,

$$\bar R_{21} = (0-0)\,\bar a_x + (0-0.02)\,\bar a_y + (0-0)\,\bar a_z$$

$$\bar R_{21} = -0.02\,\bar a_y$$

$$|\bar R_{21}| = \sqrt{(-0.02)^2} = 0.02, \quad \bar a_{R21} = \frac{\bar R_{21}}{|\bar R_{21}|} = \frac{-0.02\,\bar a_y}{0.02} = -0.02\,\bar a_y$$

Similarly, $\bar R_{31} = (0-0.05)\,\bar a_x + (0-0.02)\,\bar a_y + (0-0)\,\bar a_z$

$$= -0.05\,\bar a_x - 0.02\,\bar a_y.$$

$$|\bar R_{31}| = \sqrt{(-0.05)^2 + (-0.02)^2} = 0.0538.$$

$$\bar a_{R31} = \frac{\bar R_{31}}{|\bar R_{31}|} = \frac{-0.05\,\bar a_x - 0.02\,\bar a_y}{0.0538}$$

$$\bar{R}_{41} = (0-0.05)\,\bar{a}_x + (0-0)\,\bar{a}_y + (0-0)\,\bar{a}_z$$

$$= -0.05\,\bar{a}_x$$

$$|\bar{R}_{41}| = 0.05$$

$$\bar{a}_{R41} = \frac{\bar{R}_{41}}{|\bar{R}_{41}|} = \frac{-0.05\,\bar{a}_x}{0.05} = -\bar{a}_x$$

$$\bar{F}_{21} = \frac{(0.3\times10^{-6})(0.2\times10^{-6})}{4\pi\times8.854\times10^{-12}\times(0.02)^2}\left[-a_y\right] = -1.348\,\bar{a}_y$$

$$\bar{F}_{31} = \frac{(0.3\times10^{-6})\times(0.3\times10^{-6})}{4\pi\times8.854\times10^{-12}(0.0538)^2}\left[\frac{-0.05\,\bar{a}_x-0.02\,\bar{a}_y}{0.0538}\right]$$

$$= -0.2597\,\bar{a}_x - 0.1038\,\bar{a}_y$$

$$F_{41} = \frac{(0.3\times10^{-6})(0.2\times10^{-6})}{4\pi\times8.854\times10^{-12}(0.05)^2}\left[-\bar{a}_x\right]$$

$$= -0.2157\,\bar{a}_x$$

$$\therefore\ \bar{F} = \bar{F}_{21} + \bar{F}_{31} + \bar{F}_{41}$$

$$\bar{F} = -0.4754\,\bar{a}_x - 1.4518\,\bar{a}_y \quad N$$

4) Find the workdone in moving a $3\mu C$ charge from origin $O(0,0,0)$ to point $A(1,-2,3)$ through the field $\bar{E} = 4xyz\,\bar{a}_x + 5xyz\,\bar{a}_y + 6x^2y\,\bar{a}_z$ V/m via the path $(0,0,0)$ to $(1,0,0)$ to $(1,-2,0)$ to $(1,-2,3)$.

Sol: Given Electric field $\bar{E} = 4xyz\,\bar{a}_x + 5xyz\,\bar{a}_y + 6x^2y\,\bar{a}_z$.

workdone in moving a charge $3\mu C$ from origin $O(0,0,0)$ to point $A(1,-2,3)$ through field $\bar{E}$.

$$W = -Q\int_{initial}^{final} \bar{E}\cdot d\bar{L} = -3\times10^{-6}\int_{0}^{A} \bar{E}\cdot d\bar{L}.$$

$$W = -3\times10^{-6}\int_{(0,0,0)}^{(1,-2,3)} \left(4xyz\,\bar{a}_x + 5xyz\,\bar{a}_y + 6x^2y\,\bar{a}_z\right)\cdot\left(dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z\right).$$

$$W = -3\times10^{-6}\int_{0,0,0}^{1,-2,3} 4xyz\,dx + 5xyz\,dy + 6x^2y\,dz \longrightarrow (1)$$

→ As per the data, the path of integration is $(0,0,0)$ to $(1,0,0)$ to $(1,-2,0)$ to $(1,-2,3)$. Between the first & second point, y and z coordinates are constant and x coordinate varies from $x=0$ to $x=1$. Hence in first integral substitute $y=0, z=0$.

→ Between second & third point x and z coordinates are constant and y coordinates varies from $y=0$ to $y=-2$. Hence in second integral substitute $x=1$ and $z=0$.

→ Between the third & fourth point, x & y coordinates are constant and z coordinates varies from $z=0$ to $z=3$. Hence in third integral substitute $x=1$ & $y=2$.

Hence Eq (1), becomes,

$$W = -3\times10^{-6}\int_0^1 4xyz\,dx + \int_0^{-2} 5xyz\,dy + \int_0^3 6x^2y\,dz.$$

$$W = -3\times10^{-6}\int_0^1 4(0)(0)\,dx + \int_0^{-2} 5(1)(0)\,dy + \int_0^3 6(1)^2(2)\,dz$$

$$W = -3\times10^{6}\left[0 + 0 + \int_0^3 12\, dz\right]$$

$$= -3\times10^{6}\times12\left[z\right]_0^3$$

$$= -36\times10^{6}\left[3\right]$$

$$W = -108\ \mu J$$

Electric Flux Density ($\overline{D}$):-

The net flux passing normal through the unit surface area is called Electric Flux density $\overline{D}$.

$$D = \frac{\psi}{S}$$

where ψ = total flux

S = Total Surface area of sphere

This is also called displacement flux density (or) displacement density.

The relation between $\overline{D}$ & $\overline{E}$ is

$$\overline{D} = \varepsilon \overline{E}$$

$$\overline{D} = \varepsilon_0 \varepsilon_r \overline{E}$$