COURSE MATERIAL	
SUBJECT	NUMERICAL METHODS AND PROBABILITY THEORY (20A54402)
UNIT	1
COURSE	B. TECH
SEMESTER	2 - 2
DEPARTMENT	HUMANITIES & SCIENCE
PREPARED BY (Faculty Name/s)	Department of Mathematics

8. LECTURE NOTES

Solutions of Algebraic and Transcendental equations:

1) **Polynomial function:** A function $f(x)$ is said to be a polynomial function

if $f(x)$ is a polynomial in x.

ie, $f(x) = a_0x^n + a_1x^{n-1} + \cdots \ldots \ldots \ldots .. + a_{n-1}x + a_n$

where $a_0 \neq 0$, the co-efficients $a_0, a_1 \ldots\ldots\ldots . a_n$ are real constants and n is a

non-negative integer.

2) **Algebraic function:** A function which is a sum (or) difference (or) product of

two polynomials is called an algebraic function. Otherwise, the function is called a transcendental (or) non-algebraic function.

Eg: (i) $f(x) = c_1e^x + c_2e^{-x} = 0$ (ii) $f(x) = e^{5x} - \frac{x^3}{2} + 3 = 0$

3) **Root of an equation:** A number α is called a root of an equation $f(x) = 0$ if

$f(\alpha) = 0$. We also say that α is a zero of the function.

Note: The roots of an equation are the abscissae of the points where the graph $y = f(x)$ cuts the x-axis.

Methods to find the roots of f (x) = 0

Direct method:

We know the solution of the polynomial equations such as linear equation $ax + b$ =0, and quadratic equation $ax^2 + bx + c = 0$,using direct methods or analytical methods. Analytical methods for the solution of cubic and quadratic equations are also available.

1.1. **Bisection method:**

Bisection method is a simple iteration method to solve an equation. This method is also known as Bolzano method of successive bisection. Sometimes it is referred to as half-interval method.

(i) Suppose we know an equation of the form $f(x)=0$ has exactly one real root between two real numbers x_0, x_1 .The number is chosen such that $f(x_0)$ and $f(x_1)$ will have opposite sign.

(ii) Let us bisect the interval $[x_0, x_1]$ into two half intervals and find the mid point $x_2 = \frac{x_0 + x_1}{2}$. If $f(x_2)=0$ then x_2 is a root.

(iii) If $f(x_1)$ and $f(x_2)$ have same sign then the root lies between x_0 and x_2.

(iv) The interval is taken as $[x_0, x_2]$. Otherwise the root lies in the interval $[x_2, x_1]$.

(v) Next calculate x 3 , x4, x5---,until two consecutive iterations are equal. Then we stop the process after getting desired accuracy.

This method is known as Bisection Method

PROBLEMS

1). Find a root of the equation $x^3 - 5x + 1 = 0$ using the bisection method in 5 – stages

Sol Let $f(x) = x^3 - 5x + 1$. We note that $\begin{matrix} f(0) > 0 \\ f(1) < 0 \end{matrix}$ $\quad and$

$\therefore$ One root lies between 0 and 1

Consider $x_0 = 0 \;\; and \;\; x_1 = 1$

By Bisection method the next approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1}{2}(0+1) = 0.5$$
$$\Rightarrow f(x_2) = f(0:5) = -1.375 < 0 \;\; and \;\; f(0) > 0$$

We have the root lies between 0 and 0.5

Now $x_3 = \frac{0+0.5}{2} = 0.25$

We find $f(x_3) = -0.234375 < 0 \quad and \quad f(0) > 0$

Since $f(0) > 0$, we conclude that root lies between $x_0 \; and \; x_3$

The third approximation of the root is

$x_4 = \frac{x_0+x_3}{2} = \frac{1}{2}(0+0.25) = 0.125$

We have $f(x_4) = 0.37495 > 0$

Since $f(x_4) > 0 \; and \; f(x_3) < 0$, the root lies between

$x_4 = 0.125 \; and \; x_3 = 0.25$

Considering the 4th approximation of the roots

$$x_5 = \frac{x_3+x_4}{2} = \frac{1}{2}(0.125+0.25) = 0.1875$$

$f(x_5) = 0.06910 > 0$, since $f(x_5) > 0 \; and \; f(x_3) < 0$ the root must lie between

$$x_5 = 0.18758 \; and \; x_3 = 0.25$$

Here the fifth approximation of the root is

$$\begin{aligned} x_6 &= \frac{1}{2}(x_5+x_3) \\ &= \frac{1}{2}(0.1875+0.25) \\ &= 0.21875 \end{aligned}$$

We are asked to do up to 5 stages

We stop here 0.21875 is taken as an approximate value of the root

and it lies between 0 and 1

2) Find a root of the equation $x^3 - 4x - 9 = 0$ using bisection method in four stages

Sol Let $f(x) = x^3 - 4x - 9$

We note that $f(2) < 0$ and $f(3) > 0$

$\therefore$ One root lies between 2 and 3

Consider $x_0 = 2 \ \ and \ \ x_1 = 3$

By Bisection method $x_2 = \dfrac{x_0 + x_1}{2} = 2.5$

Calculating $f(x_2) = f(2.5) = -3.375 < 0$

$\therefore$ The root lies between $x_2 \ and \ x_1$

The second approximation is $x_3 = \frac{1}{2}(x_1 + x_2) = \frac{2.5+3}{2} = 2.75$

Now $f(x_3) = f(2.75) = 0.7969 > 0$

$\therefore$ The root lies between $x_2 \ \ and \ \ x_3$

Thus the third approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.625$$

Again $f(x_4) = f(2.625) = -1.421 < 0$

$\therefore$ The root lies between $x_3 \ \ and \ \ x_4$

Fourth approximation is $x_5 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(2.75 + 2.625) = 2.6875$

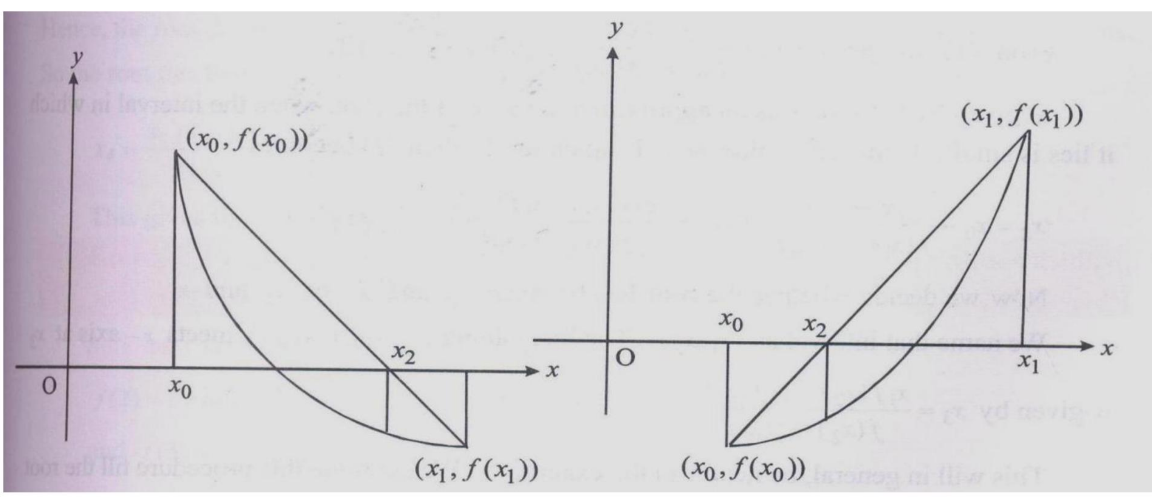
1.2. False Position Method (Regula – Falsi Method)

In the false position method we will find the root of the equation $f(x)=0$ Consider two initial approximate values x_0 and x_1 near the required root so that $f(x_0)$ and $f(x_1)$ have different signs. This implies that a root lies between x_0 and x_1. The curve $f(x)$ crosses x- axis only once at the Point x_2 lying between the points x_0 $and x_1$. Consider the point $A=(x_0, f(x_0))$ and $B=(x_1, f(x_1))$ on the graph and suppose they are connected by a straight line. Suppose this line cuts x-axis atx_2. We calculate the value of $f(x_2)$ at the point. If $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2 and value x_1 is replaced by x_2

Otherwise the root lies between x_2 and x_1 and the value of x_0 is replaced by x_2.Another line is drawn by connecting the newly obtained pair of values. Again the point here cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points

x_2, x_3, x_4 ,...obtained converge to the expected root of the equation $y=f(x)$

The below graph shows how to execute Regula Falsi Method



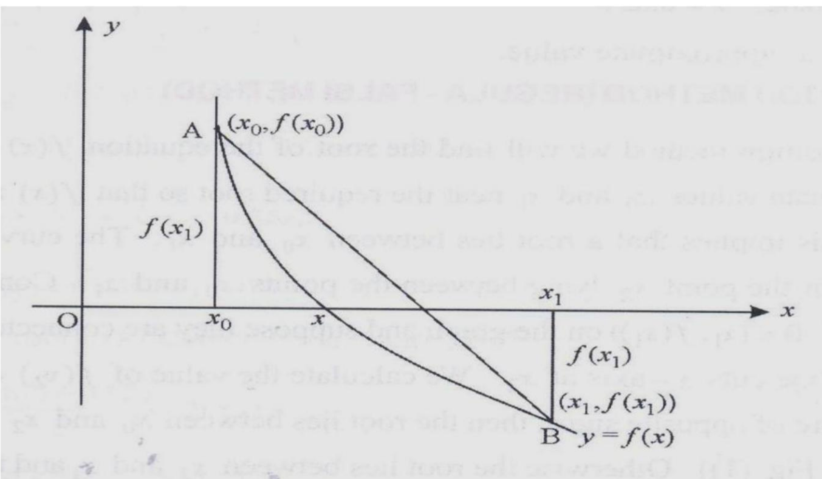
To Obtain the equation to find the next approximation to the root

Let $A=\left(x_0, f\left(x_0\right)\right)$ *and* $B=\left(x_1, f\left(x_1\right)\right)$ be the points on the curve $y=f(x)$ Then the equation to the chord AB is $\frac{y-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0}------(1)$

At the point C where the line AB crosses the x – axis, where $f(x)=0$ $ie,$ $y=0$

From (1), we get $x=x_0-\frac{x_1-x_0}{f\left(x_1\right)-f\left(x_0\right)} f\left(x_0\right)-$ --------------(2)

x is given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new value of x is taken as x_2 then (2) becomes



$$
\begin{aligned}
x_2 &= x_0-\frac{\left(x_1-x_0\right)}{f\left(x_1\right)-f\left(x_0\right)} f\left(x_0\right) \\
&= \frac{x_0 f\left(x_1\right)-x_1 f\left(x_0\right)}{f\left(x_1\right)-f\left(x_0\right)} \rightarrow (3)
\end{aligned}
$$
-------------(3)

Now we decide whether the root lies between

$$x_0 \;\; and \;\; x_2 \;\; (or) x_2 \;\; and \;\; x_1$$

We name that interval as $\left(x_1, x_2\right)$ The line joining$(x_1, y_1),(x_2, y_2)$ meets x – axis at x_3

is given by $x_3=\frac{x_1 f\left(x_2\right)-x_2 f\left(x_1\right)}{f\left(x_2\right)-f\left(x_1\right)}$

This will in general, be nearest to the exact root. We continue this procedure till the root is found to the desired accuracy

The iteration process based on (3) is known as the method of false position

The successive intervals where the root lies, in the above procedure are named as

$(x_0, x_1), (x_1, x_2), (x_2, x_3)$ etc

Where $x_i < x_{i+1}$ and $f(x_0), f(x_{i+1})$ are of opposite signs.

Also $x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

PROBLEMS:

1. By using Regula - Falsi method, find an approximate root of the equation $x^4 - x - 10 = 0$ that lies between 1.8 and 2. Carry out three approximations

Sol. Let us take $f(x) = x^4 - x - 10$ and $x_0 = 1.8, x_1 = 2$

Then $f(x_0) = f(1.8) = -1.3 < 0$ and $f(x_1) = f(2) = 4 > 0$

Since $f(x_0)$ and $f(x_1)$are of opposite signs, the equation $f(x) = 0$ has a root between x_0 *and* x_1

The first order approximation of this root is

$$\begin{aligned} x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 1.8 - \frac{2 - 1.8}{4 + 1.3} \times (-1.3) \\ &= 1.849 \end{aligned}$$

We find that $f(x_2) = -0.161$ so that $f(x_2)$ *and* $f(x_1)$ are of opposite signs. Hence the root lies between x_2 *and* x_1 and the second order approximation of the root is

$$x_3 = x_2 - \left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right].f(x_2)$$

$$= 1.8490 - \left[\frac{2 - 1.849}{0.159}\right] \times (-0.159)$$

$$= 1.8548$$

We find that $f(x_3) = f(1.8548)$

$$= -0.019$$

So that $f(x_3)$ *and* $f(x_2)$ are of the same sign. Hence, the root does not lie between x_2 *and* x_3.But $f(x_3)$ *and* $f(x_1)$ are of opposite signs. So the root lies between x_3 *and* x_1 and the third order approximate value of the root is $x_4 = x_3 - \left[\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right] f(x_3)$

$$= 1.8548 - \frac{2 - 1.8548}{4 + 0.019} \times (-0.019)$$

$$= 1.8557$$

This gives the approximate value of x.

2. Find out the roots of the equation $x^3 - x - 4 = 0$ using False position method

Sol. Let $f(x) = x^3 - x - 4 = 0$

Then $f(0) = -4, f(1) = -4, f(2) = 2$

Since $f(1)$ *and* $f(2)$ have opposite signs the root lies between 1 and 2

By False position method $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$x_2 = \frac{(1 \times 2) - 2(-4)}{2 - (-4)}$$

$$= \frac{2 + 8}{6} = \frac{10}{6} = 1.666$$

$$f(1.666) = (1.666)^3 - 1.666 - 4$$
$$= -1.042$$

Now, the root lies between 1.666 and 2

$$x_3 = \frac{1.666 \times 2 - 2 \times (-1.042)}{2 - (-1.042)} = 1.780$$
$$f(1.780) = (1.780)^3 - 1.780 - 4$$
$$= -0.1402$$

Now, the root lies between 1.780 and 2

$$x_4 = \frac{1.780 \times 2 - 2 \times (-0.1402)}{2 - (-0.1402)} = 1.794$$
$$f(1.794) = (1.794)^3 - 1.794 - 4$$
$$= -0.0201$$

Now, the root lies between 1.794 and 2

$$x_5 = \frac{1.794 \times 2 - 2 \times (-0.0201)}{2 - (-0.0201)} = 1.796$$
$$f(1.796) = (1.796)^3 - 1.796 - 4 = -0.0027$$

Now, the root lies between 1.796 and 2

$$x_6 = \frac{1.796 \times 2 - 2 \times (-0.0027)}{2 - (-0.0027)} = 1.796$$

The root is 1.796

1.3. Newton- Raphson Method:-

The Newton- Raphson method is a powerful and elegant method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let x_0 be an approximate root of $f(x)=0$ and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1)=0$. We use Taylor's theorem and expand $f(x_1) = f(x_0+h) = 0$

$$\Rightarrow f(x_0) + hf^1(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f^1(x_0)}$$

Substituting this in x_1 ,we get

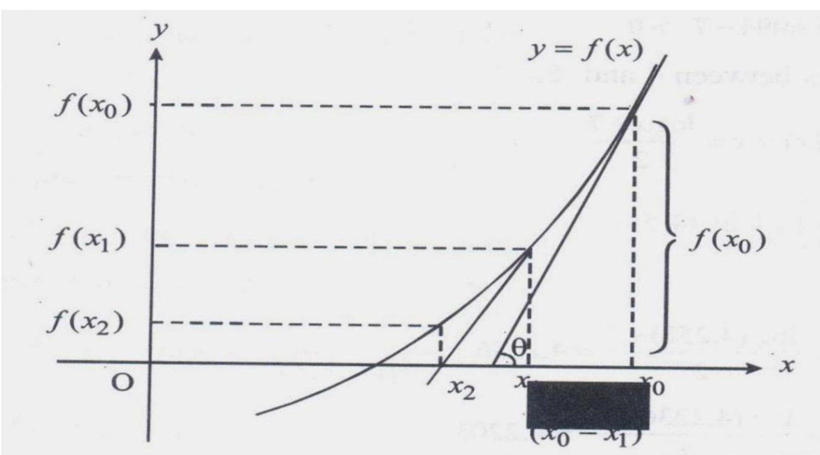
$$x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$\therefore x_1$ is a better approximation than x_0

Successive approximations are given by

$x_2, x_3 \dots\dots\dots.. x_{n+1}$ Where $x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$



PROBLEMS:

1. Apply Newton – Rapson method to find an approximate root, correct to three decimal places, of the equation $x^3 - 3x - 5 = 0$, which lies near $x = 2$

Sol:- Here $f(x) = x^3 - 3x - 5 = 0$ *and* $f^1(x) = 3(x^2 - 1)$

$\therefore$ The Newton – Raphson iterative formula

$$x_{i+1} = x_i - \frac{x_i^3 - 3x_i - 5}{3\left(x_i^2 - 1\right)} = \frac{2x_i^3 + 5}{3\left(x_i^2 - 1\right)}, i = 0,1,2....(1)$$

To find the root near $x = 2$, we take $x_0 = 2$ then (1) gives

$$x_1 = \frac{2x_0^3 + 5}{3\left(x_0^2 - 1\right)} = \frac{16+5}{3(4-1)} = \frac{21}{9} = 2.3333$$

$$x_2 = \frac{2x_1^3 + 5}{3\left(x_1^2 - 1\right)} = \frac{2\times(2.3333)^3 + 5}{3\left[(2.3333)^2 - 1\right]} = 2.2806$$

$$x_3 = \frac{2x_2^3 + 5}{3(x_2^3 - 1)} = \frac{2 \times (2.2806)^3 + 5}{3[(2.2806)^2 - 1]} = 2.2790$$

$$x_4 = \frac{2 \times (2.2790)^3 + 5}{3[(2.2790)^2 - 1]} = 2.2790$$

Since x_3 and x_4 are identical up to 3 places of decimal, we take $x_4 = 2.279$ as the required root, correct to three places of the decimal

2. Using Newton – Raphson method

a) Find square root of a number

b) Find reciprocal of a number

Sol. a) **Square root:-**

Let $f(x) = x^2 - N = 0$, where N is the number whose square root is to be found. The solution to $f(x)$ is then $x = \sqrt{N}$

Here $f'(x) = 2x$

By Newton-Raphson technique

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} = x_i - \frac{x_i^2 - N}{2x_i}$$

$$\Rightarrow x_{i+1} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

Using the above iteration formula the square root of any number N can be found to any desired accuracy. For example, we will find the square root of $N = 24$.

Let the initial approximation be $x_0 = 4.8$

$$x_1 = \frac{1}{2}\left(4.8 + \frac{24}{4.8}\right) = \frac{1}{2}\left(\frac{23.04 + 24}{4.8}\right) = \frac{47.04}{9.6} = 4.9$$

$$x_2 = \frac{1}{2}\left(4.9 + \frac{24}{4.9}\right) = \frac{1}{2}\left(\frac{24.01 + 24}{4.9}\right) = \frac{48.01}{9.8} = 4.898$$

$$x_3 = \frac{1}{2}\left(4.898 + \frac{24}{4.898}\right) = \frac{1}{2}\left(\frac{23.9904 + 24}{4.898}\right) = \frac{47.9904}{9.796} = 4.898$$

Since $x_2 = x_3$, therefore the solution to $f(x) = x^2 - 24 = 0$ is 4.898. That means,

The square root of 24 is 4.898

b) Reciprocal:-

Let $f(x) = \frac{1}{x} - N = 0$ where N is the number whose reciprocal is to be found

The solution to $f(x)$ is then $x = \frac{1}{N}$. Also, $f^1(x) = \frac{-1}{x^2}$

To find the solution for $f(x) = 0$, apply Newton – Raphson method

$$x_{i+1} = x_i - \frac{\left(\frac{1}{x_i} - N\right)}{-1/x_i^2} = x_i(2 - x_i N)$$

For example, the calculation of reciprocal of 22 is as follows

Assume the initial approximation be $x_0 = 0.045$

$$\therefore x_1 = 0.045(2 - 0.045 \times 22)$$
$$= 0.045(2 - 0.99)$$
$$= 0.0454(1.01) = 0.0454$$
$$x_2 = 0.0454(2 - 0.0454 \times 22)$$
$$= 0.0454(2 - 0.9988)$$
$$= 0.0454(1.0012) = 0.04545$$
$$x_3 = 0.04545(2 - 0.04545 \times 22)$$
$$= 0.04545(1.0001) = 0.04545$$

$$x_4 = 0.04545(2 - 0.04545 \times 22)$$
$$= 0.04545(2 - 0.99998)$$
$$= 0.04545(1.00002)$$
$$= 0.0454509$$

$\therefore$ The reciprocal of 22 is 0.04545

3. Find by Newton's method, the real root of the equation $xe^x - 2 = 0$ correct to three decimal places.

Sol. Let $f(x) = xe^x - 2 \to (1)$

Then $f(0) = -2$ and $f(1) = e - 2 = 0.7183$

So root of $f(x)$ lies between 0 and 1

It is near to 1. So we take $x_0 = 1$ and $f^1(x) = xe^x + e^x$ *and* $f^1(1) = e + e = 5.4366$

$\therefore$ By Newton's Rule

First approximation $x_1 = x_0 - \dfrac{f(x_0)}{f^1(x_0)}$

$$= 1 - \frac{0.7183}{5.4366} = 0.8679$$

$\therefore f(x_1) = 0.0672 \qquad f^1(x_1) = 4.4491$

The second approximation $x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$

$$= 0.8679 - \frac{0.0672}{4.4491}$$
$$= 0.8528$$

$\therefore$ Required root is 0.853 correct to 3 decimal places.

1.4.GAUSS JORDAN METHOD:

SYSTEM OF NON HOMOGENEOUS LINEAR EQUATIONS

An equation of the form $a_1x_1 + a_2x_2 + ...+ a_nx_n = b$ where $x_1, x_2, ...,x_n$ are unknowns and $a_1, a_2, ...,a_n$, b are constants is called linear equation in n unknowns .

Definition: Consider the system of m linear equations in n unknowns $x_1, x_2, ...,x_n$ as given below:

$$a_{11}x_1 + a_{12}x_2 + ...+ a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ...+ a_{2n}x_n = b_2$$

....

$$a_{i1}x_1 + a_{i2}x_2 + ...+ a_{in}x_n = b_i$$

....

$$a_{m1}x_1 + a_{m2}x_2 + ...+ a_{mn}x_n = b_m$$

The number a_{ij}'s are known as coefficient and $b_1, b_2, ...b_m$ are constants. An ordered n-tuple $(x_1, x_2, ...,x_n)$ satisfying all the equations simultaneously is called a solution of system.

Non-Homogeneous system:

If all $b_i \neq 0$ i.e.at least one $b_i \neq 0$.

Matrix Representation:

The above system of linear non Homogeneous equations can be written in Matrix form as AX=B

$$\begin{bmatrix} a_{11} & a_{12}\dots\dots\dots & a_{1n} \\ a_{21} & a_{22}\dots\dots\dots & a_{2n} \\ a_{m1} & a_{m2}\dots\dots\dots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_m \end{bmatrix}$$

Augmented Matrix:

It is denoted by [A/B] or [A B] is obtained by Augmenting A by the column B.

$$\therefore [A/B] = \begin{bmatrix} a_{11} & a_{12}\dots\dots\dots & a_{1n} & b_1 \\ a_{21} & a_{22}\dots\dots\dots & a_{2n} & b_2 \\ a_{m1} & a_{m2}\dots\dots\dots & a_{mn} & b_m \end{bmatrix}$$

By reducing [A /B] into its row echelon form the existence and uniqueness of solution

AX = B exists.

NOTE:

Given a system, we do not know in general whether it has a solution or not .If there is at least one solution , then the system is said to be consistent .If does not have any solution then the system is inconsistent.

CONSISTENT**:** A system is said to be consistent if it has at least one solution

NOTE: Here rank is denoted by ρ

Gauss Jordan Method: In Gauss Jordan method augmented matrix [A/B] can be reduced to identity matrix and column matrix by elementary row operations. Finally last column gives solutions of given linear system.

The Augmented matrix [A/B] can be reduced as follows by elementary row operations

$$[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\alpha \\ 0 & 1 & 0\beta \\ 0 & 0 & 1\gamma \end{bmatrix}$$

Then last column is the solution set of given linear system

For Non Homogeneous System, The system AX = B is consistent i.e it has a solution.

The system is inconsistent i.e. it has no solution.

NOTE: Find the rank A and rank [A /B] by reducing the augmented matrix [A /B] to Echelon form by elementary row operations. Then the matrix A will be reduced to Echelon form.

This procedure is illustrated through the following examples.

Example 1: Find whether the following equations are consistent, if so solve them

By Gauss Jordan method x + y + 2z= 4; 2x – y + 3z = 9; 3x – y - z = 2.

Solution: The given equations can be written in the matrix form as $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

i.e. .AX = B Use Gauss Jordan method

The Augmented matrix $[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

Applying $R_3 \to 3R3 - 4R_2$

$$[A / B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$$

Since Rank of A = 3 & Rank of [A / B] = 3

Since the number of non- zero rows of matrix A is 3

Since the number of non- zero rows of matrix [A /B] is 3

$\therefore$ Rank of A = Rank of [A B]

i.e. $\rho(A) = \rho(AB)$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

$\Rightarrow$ R3← R3/(-17)

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$\Rightarrow$ R1← R1-2 R3 and R2→R2+R3

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Next perform R2/(-3) and R1→R1-R2

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Then solution set X= $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

∴ x = 1, y = -1 , z = 2 is the solution.

Example 2: Using Gauss Jordan method solve linear equations given below

x + 2y+ 2z = 2; 3x - 2y - z = 5: 2x - 5y + 3z = -4; x + 4y + 6z = 0.

Solution: The given equations can be written in the matrix form as AX = B

i.e. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$

The Augmented matrix [A/ B] = $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$

Use Gauss Jordan method

Applying $R_2 \rightarrow R_2 - 3R_1$; $R_3 \rightarrow R_3 - 2R_1$; $R_4 \rightarrow R_4 - R_1$

[A /B] = $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$

Applying $R_3 \rightarrow 8R_3 - 9R_2$;; $R_4 \rightarrow 4R_4 + R_2$,we get

[A/B] = $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 55 & -55 \\ 0 & 0 & 9 & -9 \end{bmatrix}$

Applying $R_3 \rightarrow R_3/55$; $R_4 \rightarrow R_4/9$

$$[A/B] \approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying $R_4 \rightarrow R_4 - R_3$

$$[A/B] \approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since Rank of A = 3 & Rank of [A/ B] = 3

$\therefore$ Rank of A = Rank of [A /B]

i.e. $\rho(A) = \rho(AB)$

The given system is consistent, so it has a solution.

We have $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

$\Rightarrow$ Apply R1-2R3,R2+7R3

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ -1 \\ 0 \end{bmatrix}$$

Next R2/(-8) and R1-2R2 then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$\therefore x = 2$, $y = 1$, $z = -1$ is the solution.

1.5. Gauss Siedel Method:

Algorithm: Consider the linear system of equations as

(i) $a_1 x+b_1y+c_1z=d_1$

$a_2x+b_2y+c_2z=d_2$

$a_3x+b_3y+c_3z=d_3$

(ii) If a_1,b_2, c_3 are large as compared with other coefficients , then solve them for x,y,z respectively .

The system can be written in the below form

$$X = \frac{1}{a1}(d_1-b_1y-c_1z)$$

$$Y = \frac{1}{b2}(d_2-a_2x-c_2z)$$

$$Z= \frac{1}{c3}(d_3-a_3x-b_3y)$$

(iii) First iteration: We can calculate first iteration values in the following equations

$$X^1 = \frac{1}{a1}(d_1-b_1y^0+c_1z^0)$$

$$Y^1 = \frac{1}{b2}(d_2-a_2x^1-c_2z^0)$$

$$Z^1= \frac{1}{c3}(d_3-a_3x^1-b_3y^1)$$

(iv) Second iteration: Formulas for second iteration

$$X^2 = \frac{1}{a1}(d_1-b_1y^1+c_1z^1)$$

$$Y^2= \frac{1}{b2}(d_2-a_2x^2-c_2z^1)$$

$$Z^2= \frac{1}{c3}(d_3-a_3x^2-b_3y^2)$$

First take initial values zeroes as new approximation for an un known value found, it is immediately used in next step. We continued these processes up to two successive iterations are approximately equal. This procedure is called as Gauss Siedal iteration method.

PROBLEMS:

1. Solve by Gauss Siedal method 10x+y+z=12 , 2x+10y+z=13 , 2x+2y+10z=14

Sol. Given equations are 10x+y+z=12----(1)

2x+10y+z=13----(2)

2x+2y+10z=14---(3)

From (1) $x=\frac{1}{10}(12-y-z)$

From (2) $y=\frac{1}{10}(13-2x-z)$

From (3) $z=\frac{1}{10}(14-2x-2y)$

First iteration: $X^1=\frac{1}{10}(12-y^0-z^0)=\frac{1}{10}(12-0-0)=1.2$

$Y^1=\frac{1}{10}(13-2x^1-z^0)=\frac{1}{10}(13-2(1.2)-0)=1.06$

$Z^1=\frac{1}{10}(14-2x^1-2y^1)=\frac{1}{10}(14-2(1.2)-2(1.06))=0.948$

Second iteration: $X^2=\frac{1}{10}(12-y^1-z^1)=\frac{1}{10}(12-1.2-1.06)=0.999$

$Y^2=\frac{1}{10}(13-2x^2-z^1)=\frac{1}{10}(13-2(0.999)-0.948)=1.005$

$Z^2=\frac{1}{10}(14-2x^2-2y^2)=\frac{1}{10}(14-2(0.999)-2(1.005))=0.999$

Third iteration: $X^3=\frac{1}{10}(12-y^2-z^2)=\frac{1}{10}(12-1.005-0.999)=1$

$Y^3=\frac{1}{10}(13-2x^3-z^2)=\frac{1}{10}(13-2(1)-0.999)=1$

$Z^3=\frac{1}{10}(14-2x^3-2y^3)=\frac{1}{10}(14-2(1)-2(1))=1$

Fourth iteration: $X^4=\frac{1}{10}(12-y^3-z^3)=\frac{1}{10}(12-1-1)=1$

$Y^4=\frac{1}{10}(13-2x^4-z^3)=\frac{1}{10}(13-2(1)-1)=1$

$Z^4=\frac{1}{10}(14-2x^4-2y^4)=\frac{1}{10}(14-2(1)-2(1))=1$

Since third and fourth iterations are equal then desired set of solutions are

X=1, y=1 , z=1

2. Using Gauss Siedal method solve the linear system 20x+y-2z=17,3x+20y-z=-18,2x-3y+20z=25
3. Solve 6x+y+z=105,4x+8y+3z=155,5x+4y-10z=65 by Gauss Siedal method.

9. Practice Quiz

1. Newton's iterative formula for finding the Cube root of a number N is x_{n+1} = [b]

a) $\frac{1}{3}\left[2x_n + \frac{N}{x_n^{\,2}}\right]$

b) $\frac{1}{3}\left[2x_n + \frac{N}{x_n^{\,3}}\right]$

c) $\frac{1}{3}\left[2x_n - \frac{N}{x_n^{\,2}}\right]$

d) $\frac{1}{3}\left[2x_n - \frac{N}{x_n^{\,3}}\right]$

2. Iteration formula in Newton-Raphson method is [b]

a) $x_{n+1} = x_n + \frac{f(x_n)}{f^1(x_n)}$

b) $x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$

c) $x_{n+1} = x_n + \frac{f^1(x_n)}{f(x_n)}$

d) $x_{n+1} = x_n - \frac{f^1(x_n)}{f(x_n)}$

3. Which of the following is an algebraic equation ……….. [b]

a) $x^2 - \log x - 1.2 = 0$

b) $x^3 + 2x^2 + x + 1 = 0$

c) $\cos x = xe^x$

d) $xe^x - 1 = 0$

4. Which of the following is a transcendental equation… [a]

a) $x^2 - \log x = 1.2$

b) $x^3 + 2x^2 + x + 1 = 0$

c) $x^3 - 3x - 5 = 0$

d) $x^3 - 5x + 1 = 0$

5. Using the false position method, the formula for the approximate root of the equation $f(x) = 0$ is………… [a]

a) $x = \frac{af(b) - bf(a)}{f(b) - f(a)}$

b) $x = \frac{bf(b) - af(a)}{f(b) - f(a)}$

c) $x = \frac{af(b) + bf(a)}{f(b) + f(a)}$

d) $x = \frac{bf(b) + af(a)}{f(b) + f(a)}$

6. If the root of the equation $x^3 - 6x + 4 = 0$ lies between 0 & 1, then the first approximation of the required root using Newton-Raphson method is……. [c]

a)0.55555
b)0.4444
c)0.77777
d)0.66666

7. The n^th order difference of a polynomial of n^th degree is______ [a]

a) Constant