Unit - 4 - Multiple integrals

Double Integrals :- Double integral of $f(x,y)$ over the region R and is denoted by the symbol $\iint_R f(x,y)\,dR$ or $\iint_R f(x,y)\,dx\,dy$

Note :- 1. $\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\right]dx$

2. $\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{f_1(y)}^{f_2(y)} f(x,y)\,dx\right]dy$

Problems :-

1) Solve $\int_0^2 \int_0^3 xy\,dx\,dy$

Sol) Given $\int_0^2\int_0^3 xy\,dx\,dy = \int_{y=0}^{2}\int_{x=0}^{3} xy\,dx\,dy$

$$= \int_{y=0}^{2}\left[\int_{x=0}^{3} x\,dx\right] y\,dy$$

$$= \int_{y=0}^{2}\left(\frac{x^2}{2}\right)_0^3 y.dy$$

$$= \int_{y=0}^{2} \frac{9}{2}.y.dy$$

$$= \frac{9}{2}\int_{y=0}^{2} y.dy$$

$$= \frac{9}{2}\left(\frac{y^2}{2}\right)_0^2$$

$$= \frac{9}{2}\left[\frac{4}{2}-0\right]$$

$$= \frac{9}{\cancel{2}}\times\cancel{2}$$

$$= 9 \,//$$

2) $\int_0^3\int_0^2 (4-y)^2\,dy\,dx$

Sol) $\int_0^3\int_0^2 (4-y)^2\,dy\,dx = \int_{x=0}^{3}\left[\int_{y=0}^{2}(4-y)^2\,dy\right]dx$

(x, y)
by the
y
] dx
dy

$$= \int_{x=0}^{3}\left[\int_0^2 \{16+y^2-8y\}\,dy\right]dx \qquad \int x^2\,dx = \frac{x^3}{3}$$

$$= \int_{x=0}^{3}\left[16y+\frac{y^3}{3}-8\frac{y^2}{2}\right]_0^2 dx \qquad \int k\,dx = k\int(1)\,dx = kx$$

$$= \int_{x=0}^{3}\left[\left\{16(2)+\frac{2^3}{3}-8\cdot\frac{2^2}{2}\right\}-0\right]dx$$

$$= \int_{x=0}^{3}\left[32+\frac{8}{3}-16\right]dx$$

$$= \int_{x=0}^{3}\left(\frac{56}{3}\right)dx$$

$$= \frac{56}{3}\int_{x=0}^{3}dx$$

$$= \frac{56}{3}\,[x]_0^3$$

$$= \frac{56}{\cancel{3}}\,[\cancel{3}]$$

$$= 56_{\,//}$$

3) $\int_0^3\int_1^2 xy(x+y)\,dx\,dy$

Sol) Let $\int_0^3\int_1^2 xy(x+y)\,dx\,dy \Rightarrow \int_{x=0}^{3}\int_{y=1}^{2} xy(x+y)\,dx\,dy$

$$\Rightarrow \int_{x=0}^{3}\int_{y=1}^{2}(x^2y+xy^2)\,dx\,dy$$

$$\Rightarrow \int_{x=0}^{3}\left[\int_{y=1}^{2}(x^2y+xy^2)\,dy\right]dx$$

$$\Rightarrow \int_{x=0}^{3}\left[x^2\frac{y^2}{2}+x\frac{y^3}{3}\right]_1^2 dx$$

$$\Rightarrow \int_{x=0}^{3}\left[x^2\cdot\frac{4}{\cancel{2}}^{2}+x\cdot\frac{8}{3}\right]-\left[x^2\frac{1}{2}+x\,\frac{1}{3}\right]dx$$

$$\Rightarrow \int_{x=0}^{3}\left[2x^2+\frac{8x}{3}-\frac{x^2}{2}-\frac{x}{3}\right]dx$$

$$\Rightarrow \int_{x=0}^{3}\left[\frac{3x^2}{2}+\frac{7x}{3}\right]dx$$

$$\Rightarrow \left[\frac{3}{2}\left[\frac{x^3}{3}\right]+\frac{7}{3}\left(\frac{x^2}{2}\right)\right]$$

$$= \frac{3}{2}\left[\frac{27}{3}\right] + \frac{7}{3}\cdot\frac{9}{2}\Big]^{3} - 0$$

$$= \frac{27}{2} + \frac{21}{2}$$

$$= \frac{48}{2}$$

4] $\int_0^3 \int_1^2 xy(1+x+y)\,dy\,dx$

Sol) $\int_0^3 \int_1^2 xy(1+x+y)\,dy\,dx \Rightarrow \int_{x=0}^{3} \int_{y=1}^{2} xy(1+x+y)\,dy\,dx$

$$\Rightarrow \int_{x=0}^{3}\left[\int_{y=1}^{2} xy(1+x+y)\,dy\right]dx$$

$$\Rightarrow \int_{x=0}^{3}\left[\int_{y=1}^{2} (xy + x^2y + xy^2)\,dy\right]dx$$

$$\Rightarrow \int_{x=0}^{3}\left[x\frac{y^2}{2} + x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_1^2 dx$$

$$\Rightarrow \int_{x=0}^{3}\left[x\cdot\frac{4}{2} + x^2\frac{4}{2} + x\cdot\frac{8}{3}\right] - \left[x\cdot\frac{1}{2} + x^2\frac{1}{2} + x\cdot\frac{1}{3}\right]dx$$

$$\Rightarrow \int_{x=0}^{3}\left[2x + 2x^2 + \frac{8x}{3} - \frac{x}{2} - \frac{x^2}{2} - \frac{x}{3}\right]dx$$

$$\Rightarrow \int_{x=0}^{3}\left[\frac{3x^2}{2} + \frac{23x}{6}\right]dx$$

$$\Rightarrow \left[\frac{3}{2}\frac{x^3}{3} + \frac{23}{6}\frac{x^2}{2}\right]_0^3$$

$$= \frac{3}{2}\left[\frac{27}{3}\right] + \frac{23}{6}\left[\frac{9}{2}\right] - 0$$

$$= \frac{135}{3} + \frac{87}{4}$$

$$= \frac{81}{6} + \frac{69}{4}$$

$$= \frac{123}{4}$$

5] $\int_0^2 \int_0^x y\,dy\,dx$

Sol) Let $\int_0^2 \int_0^x y\,dy$

6) $\int_0^1 \int_x^1 (x$

Sol) $\int_0^1 \int_x^1 (x$

5] $\int_0^2 \int_0^x y \, dy \, dx$

Sol) Let $\int_0^2 \int_0^x y \, dy \, dx \Rightarrow \int_{x=0}^{2} \int_{y=0}^{x} y \, dy \, dx$

$$\Rightarrow \int_{x=0}^{2} \left[\int_{y=0}^{x} y \, dy \right] dx$$

$$\Rightarrow \int_{x=0}^{2} \left[\frac{y^2}{2} \right]_0^x dx$$

$$\Rightarrow \int_{x=0}^{2} \left(\frac{x^2}{2} - 0 \right) dx$$

$$\Rightarrow \int_{x=0}^{2} \left[\frac{x^2}{2} \right] dx$$

$$\Rightarrow \left[\frac{1}{2} \cdot \frac{x^3}{3} \right]_0^2$$

$$\Rightarrow \left[\frac{1}{2} \cdot \frac{8}{3} - 0 \right]$$

$$= \frac{\cancel{8}\,4}{\cancel{6}\,3}$$

$$= \frac{4}{3}$$

$\int^1 \int^1 (x^2 + y^2) \, dy \, dx = \frac{1}{3}$

Sol: $\int_{x=0}^{1}\left[x^2 y + \frac{y^3}{3}\right]_{x}^{1} dx$

$$\Rightarrow \int_{x=0}^{1} x^2(1) + \frac{1}{3} - x^2(x) + \frac{x^3}{3}\, dx$$

$$= \int_{x=0}^{1} \left[x^2 + \frac{1}{3} - \frac{4x^3}{3}\right] dx$$

$$\left[\frac{1}{3} + \frac{x^3}{3} - \frac{4}{3}\cdot\frac{x^4}{4}\right]_0^1$$

$$\left(\frac{1}{3} + \frac{x^3}{3} - \frac{4x^4}{4\times 3}\right)_0^1$$

$$\frac{1}{3} + \frac{1}{3} - \frac{1}{3} - 0$$

$$\Rightarrow \frac{1}{3}$$

7) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy.$

Sol) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\left[\int_{y=x}^{\sqrt{x}} (x^2+y^2)\,dy\right] dx$

$$= \int_{x=0}^{1} \left[x^2(y) + \frac{y^3}{3}\right]_{y=x}^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} \left[x^2\sqrt{x} + \frac{(\sqrt{x})^3}{3} - \left(x^2 x + \frac{x^3}{3}\right)\right] dx$$

$$= \int_{x=0}^{1} \left[x^{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{3} - x^3 - \frac{x^3}{3}\right] dx$$

$$= \int_{x=0}^{1} \left[x^{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{3} - \frac{4x^3}{3}\right] dx$$

$$= \int_{x=0}^{1} \left[\frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{x^{\frac{5}{2}}}{3\left[\frac{5}{2}\right]} - \frac{\cancel{4}}{3}\frac{x^4}{\cancel{4}}\right]_0^1$$

$$= \left[\frac{2x^{\frac{7}{2}}}{7} + \frac{3x^{\frac{5}{2}}}{5} - \frac{x^4}{3}\right]_0^1$$

$$= \frac{2(1)^{\frac{7}{2}}}{7} + \frac{3(1)^{\frac{5}{2}}}{5} - \frac{1^4}{3} - 0\,\Big]$$

$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3}$

$= \frac{3}{35}$ //

8] $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

Sol) $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_{x=0}^{1} \int_{y=0}^{1} \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

$= \int_{x=0}^{1} \left[\int_{y=0}^{1} \frac{1}{\sqrt{(1-y^2)}} \right] dy \, \frac{1}{\sqrt{(1-x^2)}} \, dx$

$= \int_{x=0}^{1} \left[\sin^{-1} y \right]_0^1 \frac{dx}{\sqrt{1-x^2}}$

$= \int_{x=0}^{1} \left[\sin^{-1}(1) - \sin^{-1} 0 \right] \frac{dx}{\sqrt{1-x^2}}$

$= \int_{x=0}^{1} \left[\frac{\pi}{2} - 0 \right] \frac{dx}{\sqrt{1-x^2}}$

$= \frac{\pi}{2} \int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}} \, dx$

$= \frac{\pi}{2} \left[\sin^{-1} x \right]_0^1$

$= \frac{\pi}{2} \left[\sin^{-1}(1) - \sin^{-1}(0) \right]$

$= \frac{\pi}{2} \left[\frac{\pi}{2} - 0 \right]$

$= \frac{\pi^2}{4}$ //

9) Evaluate i) $\int_0^2 \int_0^x e^{x+y} \, dy \, dx$

ii) $\int_0^1 \int_0^x e^{x+y} \, dy \, dx$

Sol) i) $\int_0^2 \int_0^x e^{x+y} \, dy \, dx \Rightarrow \int_{x=0}^{2} \int_{y=0}^{x} e^x . e^y \, dy \, dx$

$\Rightarrow \int_{x=0}^{2} \left[\int_{y=0}^{x} e^y \right] dy \; e^x \, dx$

$\Rightarrow \int_{x=0}^{2} [e^y]_0^x$

$\Rightarrow \int_{x=0}^{2} [e^x -$

$\Rightarrow \int_{x=0}^{2} [e^x$

$$\Rightarrow \int_{x=0}^{2} [e^{y}]_{0}^{x} e^{x} dx$$

$$\Rightarrow \int_{x=0}^{2} [e^{x} - e^{0}] e^{x} dx$$

$$\Rightarrow \int_{x=0}^{2} [e^{x} - 1] e^{x} dx$$

$$\Rightarrow \int_{x=0}^{2} [e^{2x} - e^{x}] dx$$

$$\Rightarrow \left[\frac{e^{2x}}{2} - e^{x}\right]_{0}^{2x}$$

$$\Rightarrow \frac{e^{4}}{2} - e^{2} - \frac{e^{0}}{2} + e^{0}$$

$$\Rightarrow \frac{e^{4}}{2} - e^{2} - \frac{1}{2} + 1$$

$$\Rightarrow \frac{e^{4}}{2} - e^{2} + \frac{1}{2}$$

ii) $\int_{0}^{1} \int_{0}^{x} e^{x+y} \, dy \, dx$

Sol) $\Rightarrow \int_{x=0}^{1} \int_{y=0}^{x} e^{x} \cdot e^{y} \, dy \, dx \Rightarrow \int_{x=0}^{1} \left[\int_{y=0}^{x} e^{y}\right] dy \; e^{x} dx$

$$\Rightarrow \int_{x=0}^{1} [e^{y}]_{0}^{x} e^{x} dx$$

$$\Rightarrow \int_{x=0}^{1} [e^{x} - 1] e^{x} dx$$

$$= \int_{x=0}^{1} [e^{x} - 1] e^{x} dx$$

$$= \int_{x=0}^{1} [e^{2x} - e^{x}] dx$$

$$= \left[\frac{e^{2x}}{2} - e^{x}\right]_{0}^{1}$$

$$= \frac{e^{2}}{2} - e^{1} - \frac{e^{0}}{2} + e^{0}$$

$$= \frac{e^{2}}{2} - e^{1} - \frac{1}{2} + 1$$

$$= \frac{e^{2}}{2} - e^{1} + \frac{1}{2}$$

10) Evaluate i) $\int_0^4 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx$

ii) $\int_0^1 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx$

Sol) i) $\int_0^4 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx \Rightarrow \int_{x=0}^4 \left[\int_{y=0}^{x^2} e^{\frac{y}{x}}\right] dy\, dx$

$\Rightarrow \int_{x=0}^4 \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}}\right]_0^{x^2} dx$

$\Rightarrow \int_{x=0}^4 \left[x\cdot e^{\frac{y}{x}}\right]_0^{x^2} dx$

$\Rightarrow \int_{x=0}^4 \left[x\cdot e^{\frac{x^2}{x}} - x\cdot e^{\frac{0}{x}}\right] dx$

$\Rightarrow \int_{x=0}^4 \left[x\cdot e^x - x(1)\right] dx$

$\Rightarrow \int_{x=0}^4 \left[x\cdot e^x - x\right] dx$

$\Rightarrow \left[e^x(x-1) - \frac{x^2}{2}\right]_0^4$

$\Rightarrow \left[e^4(4-1) - \frac{4^2}{2} - e^0(0-1) + \frac{0}{2}\right]$

$= e^4(3) - 8 + 1 + 0$

$= 3e^4 - 7$ //

ii) $\int_0^1 \int_0^{x^2} e^{\frac{y}{x}}\, dy\, dx \Rightarrow \int_{x=0}^1 \left[\int_{y=0}^{x^2} e^{\frac{y}{x}} dy\right] dx$

$\Rightarrow \int_0^1 \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}}\right]_0^{x^2} dx$

$\Rightarrow \int_0^1 \left[x\cdot e^{\frac{y}{x}}\right]_0^{x^2} dx.$

$\Rightarrow \int_0^1 \left[x\cdot e^{\frac{x^2}{x}} - x\cdot e^{\frac{0}{x}}\right] dx$

$\Rightarrow \int_0^1 \left[x\cdot e^x - x(1)\, dx\right]$

$= \int_0^1 \left[x e^x - x\right] dx$

$= \left[e^x(x-1) - \frac{x^2}{2}\right.$

$\Rightarrow e^1(1-1) - \frac{1}{2}$

$\Rightarrow e^1(0) - \frac{1}{2}$

$\Rightarrow$

11) $\int\int_R y\, d$

** the parab

Sol) Given parabola

Dra

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and P

To fin

two parabolas

Let $y^2 = 4x$

$x^2 = 4y$

from ② $y =$

from ① $\left(\frac{x^2}{4}\right.$

$\Rightarrow \left[e^{x}(x-1) - \frac{x^2}{2}\right]_0^1$

$\Rightarrow e^1(1-1) - \frac{1}{2} - e^0(0-1) + \frac{0}{2}$

$\Rightarrow e^1(0) - \frac{1}{2} + 1 + 0$

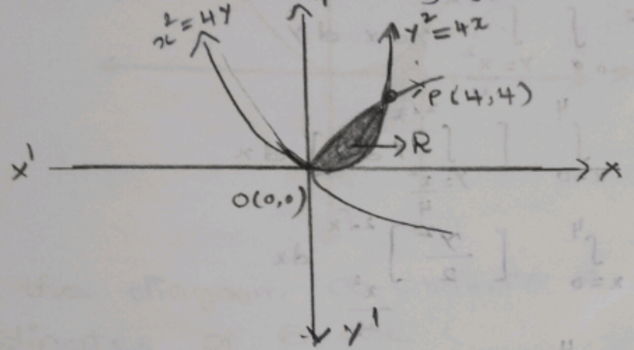
$\Rightarrow \frac{1}{2}$ //

11) $\iint_R y\,dx\,dy$ where R is the region bounded by

*** the parabola $y^2=4x$, $x^2=4y$

Sol) Given parabolas $y^2=4x$, $x^2=4y$

Draw the parabolas in xy plane.



from the diagram R is the region bounded by the two parabolas intersecting at two points O and P.

To find Quadinates of O and P solve the given two parabolas.

Let $y^2 = 4x \longrightarrow (1)$

$x^2 = 4y \longrightarrow (2)$

from (2) $y = \frac{x^2}{4}$

from (1) $\left(\frac{x^2}{4}\right)^2 = 4x$

$\frac{x^4}{16} = 4x$

$x^4 = 64x$

$x^4 - 64x = 0$

$x(x^3-64) = 0$

$x=0$ or $x^3-64=0$

$x^3 = 4^3$ $\therefore x = 4$ //

from ① If $x=0$ then $y^2=4(0) \Rightarrow y=0 \Rightarrow O(0,0)$

If $y=4$ then $y^2=4(4) \Rightarrow y^2=16 \Rightarrow y=\pm 4 \Rightarrow P(4,4)$

Limits:- from the diagram, $x \longrightarrow 0$ to 4

$y \longrightarrow \frac{x^2}{4}$ to $2\sqrt{x}$

(or)

from the diagram, $y \longrightarrow 0$ to 4

$x \longrightarrow 2\sqrt{y}$ to $\frac{y^2}{4}$,

$$\int_R\int y\,dx = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} y\,dx\,dy$$

$$= \int_{x=0}^{4} \int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\,dx\,dy$$

$$= \int_{x=0}^{4} \left[\int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\right] dx$$

$$= \int_{x=0}^{4} \left[\frac{y^2}{2}\right]_{\frac{x^2}{4}}^{2\sqrt{x}} dx$$

$$= \int_{x=0}^{4} \left[\frac{(2\sqrt{x})^2}{2} - \frac{\left(\frac{x^2}{4}\right)^2}{2}\right] dx$$

$$= \int_{x=0}^{4} \left[\frac{4x}{2} - \frac{\frac{x^4}{16}}{2}\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{4} \left[4x - \frac{x^4}{16}\right] dx$$

$$= \frac{1}{2}\left[\cancel{4}^{2}\frac{x^2}{\cancel{2}} - \frac{x^5}{5} \times \frac{1}{16}\right]_0^4$$

$$= \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4$$

$$= \frac{1}{2}\left[2(16) - \frac{4^5}{80} - 0\right]$$

$$= \frac{1}{2}\left[32 - \frac{1024}{80}\right]$$

$$= \frac{1}{2}\left(\frac{2560-1024}{80}\right)$$

$$= \frac{1}{2}\left[\frac{1536}{80}\right]$$

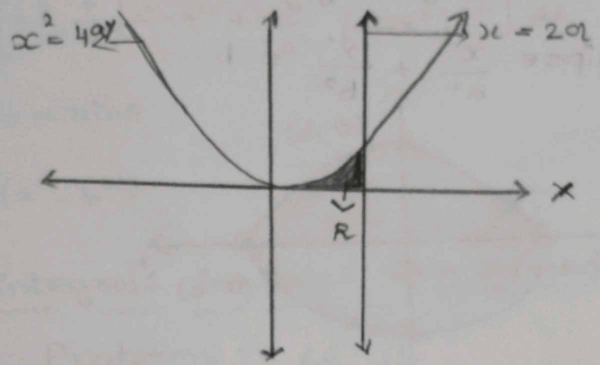
$$= \frac{768}{80}$$

$$= \frac{48}{5}$$

12) Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

Sol) Given,

Draw the line $x = 2a$ and $x^2 = 4ay$ in xy plane



from the diagram co-ordinate A are $(2a, 0)$ and co-ordinates of B are (

Put $x = 2a$ in $x^2 = 4ay \Rightarrow (2a)^2 = 4ay$

$$\Rightarrow \cancel{4}a^{\cancel{2}} = \cancel{4}\cancel{a}y$$

$$\Rightarrow \boxed{y = a}$$

Consider $B(2a, a)$

Limits :- $x \rightarrow 0$ to $2a$

$y \rightarrow 0$ to $\frac{x^2}{4a}$

$$\iint_R xy\,dx\,dy \Rightarrow \int_0^{2a}\left[\int_0^{x^2/4a} y\,dy\right] x\,dx$$

$$\Rightarrow \int_0^{2a} \left[\frac{y^2}{2}\right]_0^{x^2/4a} x\,dx$$

$$\Rightarrow \frac{1}{2}\int_0^{2a} \left(\frac{x^2}{4a}\right)^2 - 0 \; x\,dx$$

$$\Rightarrow \frac{1}{2}\int_0^{2a} \frac{x^4}{16a^2}\, x\,dx$$

$$\Rightarrow \frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

$$\Rightarrow \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

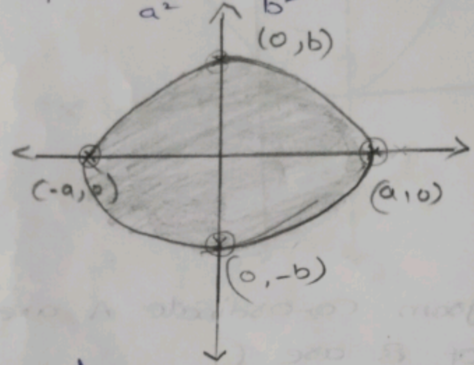
$$\Rightarrow \frac{1}{192a^2}\left[(2a)^6 - 0\right]$$

$$\Rightarrow \frac{1}{192a^2} \times 64a^6$$

$$\Rightarrow \frac{a^4}{3}$$

13) Evaluate $\iint_R x^2+y^2 \, dx\, dy$ over the area bounded
*** by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol) Given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



from the diagram,

x-limits are $\longrightarrow$ $-a$ to a

y-limits are $\longrightarrow$ let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$= \pm b\sqrt{\frac{a^2-x^2}{a^2}}$$

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

Let $\iint x^2+y^2 \, dx\, dy$

$$\int_{x=-a}^{a}\left[\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\, dy\right] dx$$

$$\Rightarrow \int_{x=-a}^{a} \left[x^2 y + \frac{y^3}{3} \right]_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{x^2+y^2}}$$

$\therefore \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$$\Rightarrow 2\int_{x=-a}^{a} \left[\int_{0}^{b/a\sqrt{a^2-x^2}} (x^2+y^2)\,dy \right] dx$$

$f(-x) = f(x) \longrightarrow f(x)$ is even

$f(-x) = -f(x) \longrightarrow f(x)$ is odd

$$\Rightarrow 2\int_{-a}^{a} \left[x^2 y + \frac{y^3}{3} \right]_{0}^{b/a\sqrt{a^2-x^2}}$$

$$\Rightarrow 2\int_{-a}^{a} \left[x^2 \frac{b}{a}\sqrt{a^2-x^2} + \frac{\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^3}{3} \right] dx$$

$$\Rightarrow 2\int_{-a}^{a} \left[x^2 \frac{b}{a}\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^3)^{3/2} \right] dx$$

Put $x = a\sin\theta$

$$\Rightarrow \frac{\pi ab}{4}(a^2-b^2)$$

Double integral's in polar co-ordinates :-

Problems :-

1) $\displaystyle\int_{0}^{\pi}\int_{0}^{a\sin\theta} r\,dr\,d\theta$

Sol) $\displaystyle\int_{0}^{\pi}\int_{0}^{a\sin\theta} r\,d\theta\,dr \Rightarrow \int_{\theta=0}^{\pi}\left[\int_{r=0}^{a\sin\theta} r\,dr\right] d\theta$

$$\Rightarrow \int_{\theta=0}^{\pi} \left[\frac{r^2}{2}\right]_{0}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi} a^2\sin^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\pi} \sin^2\theta\,d\theta$$

$\therefore \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= \frac{a^2}{2}\left[\int_{\theta=0}^{\pi} \frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \frac{a^2}{4}\int_{0}^{\pi} (1-\cos 2\theta)\,d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_{0}^{\pi}$$

$$= \frac{a^2}{4}\left[\left[\pi - \frac{\sin 2\pi}{2}\right] - \left[0 - \frac{\sin 0}{2}\right]\right]$$

$$= \frac{a^2}{4}[\pi - 0] - 0$$

$$= \frac{\pi a^2}{4}$$

2) Evaluate $\int_0^\infty \int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr$

Sol) $\int_0^\infty \int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr \Rightarrow \int_{r=0}^\infty \int_{\theta=0}^{\pi/2} e^{-r^2} r\,d\theta\,dr$

$$\Rightarrow \int_{r=0}^\infty \left[\int_{\theta=0}^{\pi/2} d\theta\right] e^{-r^2} r\,dr$$

$$\Rightarrow \int_{r=0}^\infty [\theta]_0^{\pi/2} e^{-r^2} r\,dr$$

$$\Rightarrow \int_{r=0}^\infty [\pi/2 - 0]\, e^{-r^2} r\,dr$$

$$\Rightarrow \frac{\pi}{2} \int_{r=0}^\infty e^{-r^2} r\,dr$$

Put $r^2 = t$

$2r \cdot dr = dt$

$r \cdot dr = \frac{dt}{2}$

If $r=0$ then $t = 0^2 = 0$

$r=\infty$ then $t = \infty^2 = \infty$

$$\Rightarrow \frac{\pi}{2} \int_{t=0}^\infty e^{-t} \frac{dt}{2}$$

$$\Rightarrow \frac{\pi}{4} \int_0^\infty e^{-t}\,dt$$

$\because e^\infty = \infty$

$\therefore e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$

$$= \frac{\pi}{4}\left[-e^{-t}\right]_0^\infty$$

$$= -\frac{\pi}{4}\left[e^{-\infty} - e^0\right]$$

$$= \frac{-\pi}{4}[0 - 1]$$

$$= \pi/4$$

3) Evaluate $\int_0^\pi$

Sol) $\int_0^\pi \int_0^{a\cos\theta} r\sin\theta$ d

Put

If

change

Change of

$\iint_R F(x,$

$\iint F($

3) Evaluate $\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta \, dr \, d\theta$

Sol) $\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta \, dr \, d\theta \Rightarrow \int_{\theta=0}^{\pi}\left[\int_{r=0}^{a\cos\theta} r\,dr\right]\sin\theta\,d\theta$

$$\Rightarrow \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a\cos\theta}\sin\theta\,d\theta$$

$$\Rightarrow \frac{1}{2}\int_{\theta=0}^{\pi}\left[a^2\cos^2\theta-0\right]\sin\theta\,d\theta$$

$$\Rightarrow \frac{a^2}{2}\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta$$

Put $\cos\theta = t$

$-\sin\theta\,d\theta = dt$

if $\theta=0$ then $t=\cos 0 = 1$

$\theta=\pi$ then $t=\cos\pi=-1$

$$\Rightarrow \frac{a^2}{2}\int_1^{-1} t^2(-dt)$$

$$\Rightarrow -\frac{a^2}{2}\left[\frac{t^3}{3}\right]_1^{-1}$$

$$\Rightarrow -\frac{a^2}{6}\left[(-1)^3-(1)^3\right]$$

$$\Rightarrow -\frac{a^2}{6}\left[-1-1\right]$$

$$= \frac{-a^2}{\cancel{6}_3}(-\cancel{2})$$

$$= \frac{a^2}{3}$$

Change of variables in double integrals:-

Change of variables from Cartigion to Polar Co-ordinates:-

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

and

$$\iint F(r,\theta)\,dA = \int_{\theta=\theta_1}^{\theta_2}\int_{r=f_1(\theta)}^{f_2(\theta)} F(r,\theta)\, r\,dr\,d\theta$$

Problems :-

1) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$ by changing to Polar co-ordinates and hence show that integral $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Sol) $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$

Here, x and y limits are from $0-\infty$, the region of integration is the 1^{st} Quadrant of the xy plane.

changing to polar co-ordinates then put

$x = r\cos\theta$, $y = r\sin\theta$

then we have $dx\, dy = r\, dr\, d\theta$

then we get $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(1)$

$= r^2$

In the region of integration r limits are 0 to ∞ and θ varies are 0 to $\pi/2$.

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r\, dr\, d\theta$$

$$= \int_{r=0}^{\infty} \left[\int_{\theta=0}^{\pi/2} d\theta \right] e^{-r^2} r\, dr$$

$$= \int_{r=0}^{\infty} [\theta]_0^{\pi/2} e^{-r^2} r\, dr$$

$$= \int_{r=0}^{\infty} [\pi/2 - 0]\, e^{-r^2} r\, dr$$

$$= \frac{\pi}{2} \int_{r=0}^{\infty} e^{-r^2} r\, dr$$

Put $r^2 = t$

$2r \cdot dr = dt$

$r\, dr = \frac{dt}{2}$

If $r=0$ then $t=0$

$r=\infty$ then $t=\infty$

$= \pi/2 \int_{t=0}^{\infty} e^{-t} \frac{dt}{2}$

$= \pi/2\cdot 4 \int_{t=0}^{\infty} e^{-t}\, dt$

$= \frac{\pi}{4} [-e^{-t}]_0^{\infty}$

$= \frac{-\pi}{4} [e^{-\infty} - e^{0}]$

$= \frac{-\pi}{4} [-1]$

$= \pi/4$ "

Let $\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$ we have to prove

Let $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\,dy = \int_0^{\infty}\int_0^{\infty} e^{-x^2} e^{-y^2}\, dx\,dy$

$= \int_0^{\infty} e^{-x^2} dx \times \int_0^{\infty} e^{-y^2} dy$

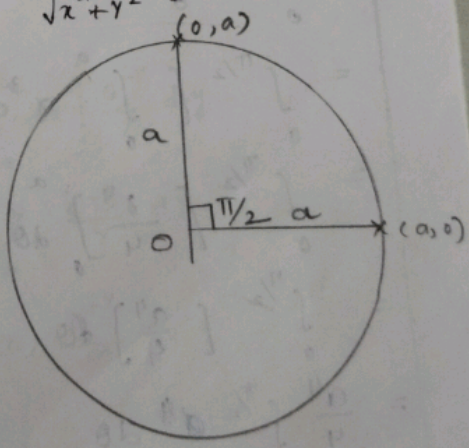
$= \int_0^{\infty} e^{-x^2} dx \times \int_0^{\infty} e^{-x^2} dy$

$= \left[\int_0^{\infty} e^{-x^2} dx\right]^2$

$\pi/4 = \left[\int_0^{\infty} e^{-x^2} dx\right]^2$

$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ "

2] Transform the integral into polar co-ordinates and hence evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy$



Sol) Here, $x^2+y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

and $r \longrightarrow 0$ to a

$\theta \longrightarrow 0$ to $\pi/2$

$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy$

$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} \sqrt{r^2}\; r\,dr\, d\theta$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} (a^3 - 0)\, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/2} d\theta$$

$$= \frac{a^3}{3}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{a^3}{3}\left[\pi/2 - 0\right]$$

$$= \frac{\pi a^3}{6} \;\; //$$

HW 3] Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dx\, dy$ by changing into polar co-ordinates.

Sol) Here $x^2 + y^2 = r^2$

$dx\, dy = r\, dr\, d\theta$

and $r \longrightarrow 0$ to a

$\theta \longrightarrow 0$ to $\pi/2$

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dx\, dy$$

$$= \int_{y=0}^a \int_{x=0}^{\sqrt{a^2-y^2}} r^2\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^a r^2\, r\, d\theta\, dr$$

$$= \int_0^{\pi/2} \left[\int_0^a r^3\, dr\right] d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \int_0^{\pi/2} \left[\frac{a^4}{4}\right] d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2} \cancel{a^4}\, d\theta$$

$$= \frac{a^4}{4}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{a^4}{4}\left[\pi/2 - 0\right]$$

$$= \frac{\pi a^4}{8}$$

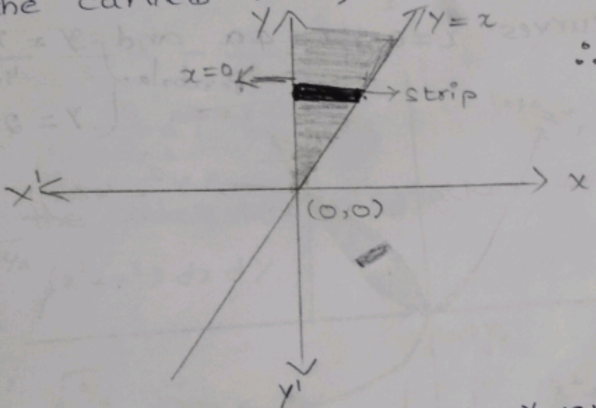
Evaluation of integrals by change of order of integration :-

1. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

Sol) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_{x=0}^\infty \int_{y=x}^\infty \frac{e^{-y}}{y}\, dy\, dx$

Here x limits are fixed and y varies from x to ∞.

Draw the curves $y = x$, $y = \infty$, $x = 0$, $x = \infty$



$\therefore x = 0 \Rightarrow$ y-axis

$x = k \Rightarrow \parallel$ to y-axis

$y = 0 \Rightarrow$ x-axis

$y = k \Rightarrow \parallel$ to x-axis

Here x limits are fixed and y varies from x to ∞. So that draw a strip which is $\parallel^{lel}$ to x-axis in the region

from the diagram changed limits are

$y \rightarrow 0$ to ∞ [fixed]

$x \rightarrow 0$ to y [varies

$$\int_{x=0}^{\infty} \int_{y=x}^{y=\infty} \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^{\infty} \left[\int_{x=0}^{y} dx\right] \frac{e^{-y}}{y}\, dy$$

$$= \int_{y=0}^{\infty} [x]_0^y \frac{e^{-y}}{y}\, dy$$

$$= \int_{y=0}^{\infty} (\cancel{y} - 0) \frac{e^{-y}}{\cancel{y}}\, dy$$

$$= \int_{y=0}^{\infty} e^{-y}\, dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= -(e^{-\infty} - e^{0})$$

$$= -(0-1)$$

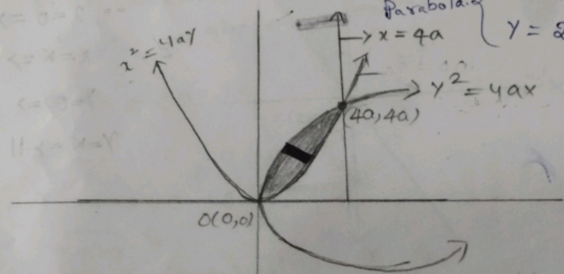
$$= 1$$

2) $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

Sol) Given :,

$$\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx \Rightarrow \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$$

Draw the curves $x=0$, $x=4a$ and Parabola $\begin{cases} y=\frac{x^2}{4a} \Rightarrow x^2=4ay \\ y=2\sqrt{ax} \Rightarrow y^2=4ax \end{cases}$



Here x limits are fixed and y limits are varies. so, draw a strip Parallel to x-axis
from the diagram :,
changed the limits are

$$y \longrightarrow 0 \text{ to } 4a$$

$$x \longrightarrow \frac{y^2}{4a} \text{ to } 2\sqrt{ay}$$

$$\therefore \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_{y=0}^{4a} \left[\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right] dy$$

$$= \int_{y=0}^{4a} [x]_{y^2/4a}^{2\sqrt{ay}}\, dy$$

$$= \int_{y=0}^{4a} \left[2\sqrt{ay} - \frac{y^2}{4a}\right] dy$$

$$= \int_0^{4a} \left[2\sqrt{a}\, y^{1/2} - \frac{y^2}{4a}\right] dy$$

$$= \int_0^{4a} \left[2\sqrt{a} \cdot \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{y^3}{12a}\right]_0^{4a}$$

$$= \left[2\sqrt{a}\, \frac{2}{3} y^{3/2} - \frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3} (4a)^{3/2} - \frac{(4a)^3}{12a} - 0$$

$$= \frac{4\sqrt{a}}{3} \left[(2^2) a\right]^{3/2} - \frac{4^3 a^3}{12a}$$

$$= \frac{32}{3} \sqrt{a} \cdot a^{3/2} - \frac{\overset{16}{\cancel{64}}}{\underset{3}{\cancel{12}}} \cdot a^2$$

$$= \frac{32}{3} a^2 - \frac{16}{3} a^2$$

$$= \frac{16}{3} a^2 \quad //$$

3) Change the order of integration and Evaluate

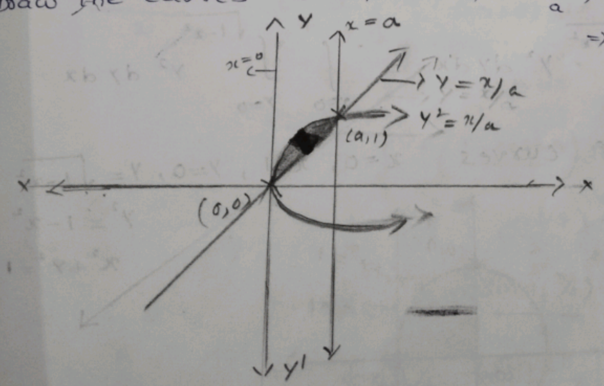
$$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\, dx\, dy$$

A) Given ;

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\, dx\, dy = \int_{x=0}^{a} \int_{y=x/a}^{\sqrt{x/a}} (x^2+y^2)\, dx\, dy$$

Draw the curves $x=0$, $x=a$, $y=\frac{x}{a}$, $y=\sqrt{\frac{x}{a}}$ S.O.B.S

$\Rightarrow y^2 = \frac{x}{a} \rightarrow$ Parabola



Here x limits are fixed and y limits are varies.
So, draw a strip parallel to x-axis.

$y \longrightarrow 0$ to 1 (fixed)

$x \longrightarrow ay^2$ to ay (y)

$$\Rightarrow \int_{y=0}^{1} \left[\int_{x=ay^2}^{ay} (x^2+y^2)\,dx \right] dy$$

$$\Rightarrow \int_{y=0}^{1} \left[\frac{x^3}{3} + xy^2 \right]_{ay^2}^{ay} dy$$

$$\Rightarrow \int_{y=0}^{1} \left[\frac{a^3y^3}{3} + \frac{ay^3}{1} - \frac{a^3y^6}{3} - \frac{ay^4}{1} \right] dy$$

$$\Rightarrow \left[\frac{a^3}{3}\cdot\frac{y^4}{4} + a\cdot\frac{y^4}{4} - \frac{a^3}{3}\cdot\frac{y^7}{7} - a\cdot\frac{y^5}{5} \right]_0^1$$

$$\Rightarrow \frac{7a^3-4a^3}{84} + \frac{5a-4a}{20} \qquad \frac{a^3}{12} - \frac{a^3}{21}$$

$$\Rightarrow \frac{3a^3}{84} + \frac{a}{20}$$

$$\Rightarrow \frac{a^3}{28} + \frac{a}{20} \;//$$

4) Change the order of integration and evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} xy^2\,dy\,dx$$

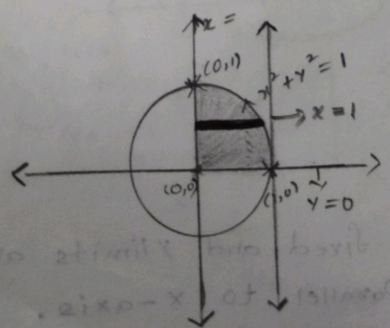
A) Given

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y^2\,dy\,dx$$

Draw the curves $x=0,\ x=1,\ y=0,\ y=\sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2=1$



from the diagram limits are

$x \longrightarrow 0$ to $\sqrt{1-y^2}$

$y \longrightarrow 0$ to 1 (fixed)

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx = \int_{y=0}^{y=1} \left[\int_{x=0}^{x=\sqrt{1-y^2}} dx \right] y^2\,dy$$

$$= \int_{y=0}^{y=1} \left[x \right]_0^{\sqrt{1-y^2}} y^2\,dy$$

$$= \int_{y=0}^{y=1} \left(\sqrt{1-y^2} - 0 \right) y^2\,dy$$

$$= \int_{y=0}^{y=1} \sqrt{1-y^2}\; y^2\,dy$$

Put $y = \sin\theta$, $dy = \cos\theta\,d\theta$

Limits:

If $y=0$ then $\sin\theta = 0$, $\theta = 0$

$y=1$ then $\sin 1 = \sin$, $\theta = \pi/2$

$$= \int_{y=0}^{\pi/2} \sqrt{1-\sin^2\theta}\; \sin^2\theta \cos\theta\,d\theta$$

$$= \int_{y=0}^{\pi/2} \sqrt{\cos^2\theta}\; \sin^2\theta \cos\theta\,d\theta$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= \int_0^{\pi/2} \sin^2\theta \cos^2\theta\,d\theta$$

$\sin^2 2\theta = \frac{1-\cos 2(2\theta)}{2}$

$= \frac{1-\cos 4\theta}{2}$

$$= \frac{1}{4} \int_0^{\pi/2} (2\sin\theta\cos\theta)^2\,d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} (\sin 2\theta)^2\,d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} \sin^2 2\theta\,d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{2} \right) d\theta$$

$$= \frac{1}{8} \left(\theta - \frac{\sin 4\theta}{2} \right)_0^{\pi/2}$$

$$= \frac{1}{8} \left(\pi/2 - \frac{\sin 4(\pi/2)}{2} - 0 - \frac{\sin(0)}{2} \right)$$

$$= \frac{1}{8} \left[\pi/2 - \frac{\sin 2\pi}{2} \right]$$

$$= \frac{\pi}{16}$$

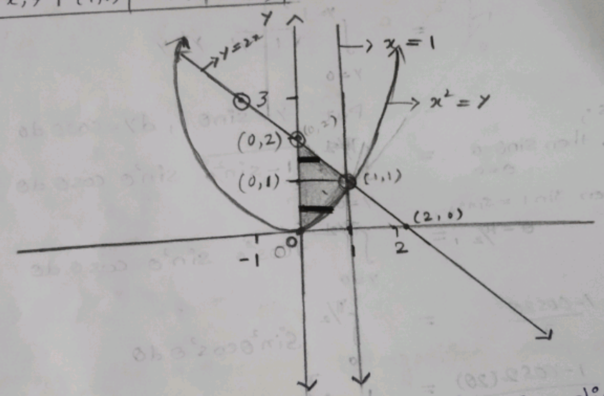
5) Change the order of integration in $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the double integral?

Sol) $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy \Rightarrow \int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dx\,dy$

Draw the curves $x=0,\ x=1,\ y=x^2,\ y=2-x \Rightarrow x+y=2$

$y = 2-x$

x	1	0	-1
y	1	2	3
x, y	(1,1)	(0,2)	(-1,3)



from the diagram, integral OABCO is divided into two regions OACO and ABCA

OACO :- $x \longrightarrow 0$ to $\sqrt{y}$ (varies)

$y \longrightarrow 0$ to 1 (fixed)

ABCA :- $x \longrightarrow 0$ to $2-y$ (varies)

$y \longrightarrow 1$ to 2 (fixed)

$$= \iint_{OACO} + \iint_{ABCA} \Rightarrow \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$\Rightarrow \int_{y=0}^{1}\left[\int_{x=0}^{\sqrt{y}} x\,dx\right] y\,dy + \int_{y=1}^{2}\left[\int_{x=0}^{2-y} x\,dx\right] y\,dy$$

$$\Rightarrow \int_{y=0}^{1} \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} y\,dy + \int_{y=1}^{2} \left[\frac{x^2}{2}\right]_0^{2-y} y\,dy$$

$$\Rightarrow \int_{y=0}^{1} \left[\frac{y}{2} - 0\right] y\,dy + \int_{y=1}^{2} \frac{(2-y)^2}{2}\, y\,dy$$

$$\Rightarrow \frac{1}{2}\int_{y=0}^{1} y^2\,dy + \frac{1}{2}\int_{y=1}^{1} (4+y^2-4y)\,y\,dy$$

$$\Rightarrow \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y+y^3-4y^2)\,dy$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{3}-0\right] + \frac{1}{2}\left[4\left[\frac{y^2}{2}\right] + \frac{y^4}{4} - 4\left(\frac{y^3}{3}\right)\right]_1^2$$

$$\Rightarrow \frac{1}{6} + \frac{1}{2}\left[2y^2 + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2$$

$$\Rightarrow \frac{1}{6} + \frac{1}{2}\left[2(4) + \frac{16}{4} - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$\Rightarrow \frac{1}{6} + \frac{1}{2}\left[8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$\Rightarrow \frac{1}{6} + \frac{1}{2}\left[12 - \frac{32}{3} - \frac{4\cdot 3}{12}\right]$$

$$\Rightarrow \frac{1}{6} + \frac{1}{2}\left[10 - \frac{32}{3} - \frac{1}{4} + \frac{4}{3}\right]$$

$$\Rightarrow \frac{1}{6} + \frac{1}{2}\left[\cancel{\frac{31}{4}}\right]\left[\frac{5}{12}\right]$$

$$\Rightarrow \frac{1}{6} \cancel{+\frac{31}{8}}\ \frac{1}{6} + \frac{5}{24}$$

$$\Rightarrow \frac{3}{8}$$

Triple integrals:-

* The standard form of triple integral is given by.

1. $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz$

2. $\int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right]dy\,dx$

Problem's:-

1) Evaluate $\int_0^1\int_0^1\int_0^1 dx\,dy\,dz$

Sol) $\int_0^1\int_0^1\int_0^1 dx\,dy\,dz \Rightarrow \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} dx\,dy\,dz$

$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1}\left[\int_{z=0}^{1} dz\right]dy\,dx$

$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1} [z]_0^1\,dy\,dx$

$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1} (1-0)\,dy\,dx$

$\Rightarrow \int_{x=0}^{1}\left[\int_{y=0}^{1} dy\right]dx$

$\Rightarrow \int_{x=0}^{1} [y]_0^1\,dx$

$\Rightarrow \int_{x=0}^{1} (1-0)\,dx$

$\Rightarrow \int_{x=0}^{1} dx$

$\Rightarrow (x)_0^1$

$= 1$

given

2) Evaluate $\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c}(x^2+y^2+z^2)\,dx\,dy\,dz$

Sol) $\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c}(x^2+y^2+z^2)\,dx\,dy\,dz$

$$\Rightarrow \int_{x=-a}^{a}\int_{y=-b}^{b}\left(\int_{z=-c}^{c}(x^2+y^2+z^2)\,dz\right)dy\,dx$$

$$\Rightarrow \int_{x=-a}^{a}\int_{y=-b}^{b}\left[x^2z+y^2z+\frac{z^3}{3}\right]_{-c}^{c}dy\,dx$$

$$\Rightarrow \int_{x=-a}^{a}\int_{y=-b}^{b}\left[x^2c+y^2c+\frac{c^3}{3}-\left(x^2(-c)+y^2(-c)+\frac{(-c)^3}{3}\right)\right]dy\,dx$$

$$\Rightarrow \int_{x=-a}^{a}\int_{y=-b}^{b}\left[2x^2c+2y^2c+\frac{2c^3}{3}\right]dy\,dx$$

$$\Rightarrow \int_{x=-a}^{a}\left[2x^2cy+2c\frac{y^3}{3}+\frac{2c^3}{3}y\right]_{-b}^{b}dx$$

$$\Rightarrow \int_{x=-a}^{a}\left[2x^2cb+2c\frac{b^3}{3}+\frac{2c^3}{3}b-\left[2x^2c(-b)+2c\frac{(-b)^3}{3}+\frac{2c^3}{3}(-b)\right]\right)dx$$

$$\Rightarrow \int_{x=-a}^{a}\left[4x^2cb+4c\frac{b^3}{3}+\frac{4c^3}{3}b\right]dx$$

$$\Rightarrow \left[4cb\frac{x^3}{3}+4c\frac{b^3}{3}(x)+\frac{4c^3}{3}bx\right]_{-a}^{a}$$

$$\Rightarrow 4cb\frac{a^3}{3}+4c\frac{b^3}{3}(a)+\frac{4c^3b}{3}(a)+\left(\frac{4cb(-a)^3}{3}+\frac{4cb^3}{3}(-a)+\frac{4c^3b}{3}(-a)\right)$$

$$\Rightarrow \frac{8a^3bc}{3}+\frac{8ab^3c}{3}+\frac{8abc^3}{3}$$

$$\Rightarrow \frac{8abc}{3}\left[a^2+b^2+c^2\right]$$

3) Evaluate $\int_0^1\int_0^1\int_0^1(x^2+y^2+z^2)\,dz\,dy\,dx$

Sol) $\int_0^1\int_0^1\int_0^1(x^2+y^2+z^2)\,dz\,dy\,dx$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1}\left[\int_{z=0}^{1}(x^2+y^2+z^2)\,dz\right]dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1}\left(x^2z + y^2z + \frac{z^3}{3}\right)_0^1 dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1}\left[x^2 + y^2 + \frac{1}{3}\right] dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\left[x^2y + \frac{y^3}{3} + \frac{1}{3}(y)\right]_0^1 dx$$

$$\Rightarrow \int_{x=0}^{1}\left[x^2 + \frac{1}{3} + \frac{1}{3}\right] dx$$

$$\Rightarrow \int_{x=0}^{1}\left[x^2 + \frac{2}{3}\right] dx$$

$$\Rightarrow \left[\frac{x^3}{3} + \frac{2}{3}\cdot x\right]_0^1$$

$$\Rightarrow \frac{1}{3} + \frac{2}{3} = \frac{\cancel{3}}{\cancel{3}} = 1 \,/\!/$$

4) $\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} dx\,dy\,dz = 48$

5) $\int_{0}^{3}\int_{0}^{2}\int_{0}^{1}(x+y+z)\,dx\,dy\,dz$

6) Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$

Sol) Given, $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1}\left[\int_{z=0}^{1} e^{x+y+z}\,dz\right] dx\,dy$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} e^x\cdot e^y\cdot e^z\,dz\,dx\,dy$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1}\left[\int_{z=0}^{1} e^z\,dz\right] e^y\cdot e^x\cdot dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1}\left[e^z\right]_0^1 e^y\cdot e^x\,dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{1}\left[e^1 - e^0\right] e^y\cdot e^x\,dy\,dx$$

$$\Rightarrow e-1 \int_{x=0}^{1}\left[\int_{y=0}^{1} e^{y}\,dy\right] e^{x}\,dx$$

$$\Rightarrow e-1 \int_{x=0}^{1} \left[e^{y}\right]_0^1 e^{x}\,dx$$

$$\Rightarrow e-1 \int_{x=0}^{1} (e^{1}-e^{0})\, e^{x}\,dx$$

$$\Rightarrow (e-1)^2 \int_{x=0}^{1} e^{x}\,dx$$

$$\Rightarrow (e-1)^2 \left(e^{x}\right)_0^1$$

$$\Rightarrow (e-1)^2 \left(e^{1}-e^{0}\right)$$

$$\Rightarrow (e-1)^3 \,//$$

4) Evaluate $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$ where V is the region of space bounded by $x=0$, $x=1$, $y=0$, $y=2$, $z=0$, $z=3$. $33/2$

Sol) Given, $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{2}\left[\int_{z=0}^{3}(xy+yz+zx)\,dz\right]dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{2}\left[xyz+y\frac{z^2}{2}+\frac{z^2}{2}x\right]_0^3 dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{2}\left[3xy+\frac{9y}{2}+\frac{9x}{2}\right]dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\left[3x\frac{y^2}{2}+\frac{9}{2}\cdot\frac{y^2}{2}+\frac{9x}{2}y\right]_0^2 dx$$

$$\Rightarrow \int_{x=0}^{1}\left[3x\frac{4}{2}+\frac{9}{2}\cdot\frac{4}{2}+\frac{9x}{2}(2)\right]dx$$

$$\Rightarrow \int_{x=0}^{1}\left[6x+9+9x\right]dx$$

$$\Rightarrow \int_{x=0}^{1}\left[15x+9\right]dx$$

$$\Rightarrow \left[15\frac{x^2}{2}+9x\right]_0^1$$

$15\left[\frac{1}{2}\right]+9=0$

$\frac{15}{2}+9=\frac{33}{2}$ //

4) $\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} dx\,dy\,dz$

Sol) Given; $\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} dx\,dy\,dz$

$\Rightarrow \int_{x=-1}^{1}\int_{y=-2}^{2}\left[\int_{z=-3}^{3} dz\right] dy\,dx$

$\Rightarrow \int_{x=-1}^{1}\int_{y=-2}^{2}\left[z\right]_{-3}^{3} dy\,dx$

$\Rightarrow \int_{x=-1}^{1}\int_{y=-2}^{2}(3+3)\,dy\,dx$

$\Rightarrow \int_{x=-1}^{1}\int_{y=-2}^{2}(6)\,dy\,dx$

$\Rightarrow 6\int_{x=-1}^{+1}\left[\int_{y=-2}^{2} dy\right] dx$

$\Rightarrow 6\int_{x=-1}^{1}(y)_{-2}^{2}\,dx$

$\Rightarrow 6\int_{x=-1}^{1}(2+2)\,dx$

$\Rightarrow 24\int_{x=-1}^{1} dx$

$\frac{24\times 2}{48}$

$\Rightarrow 24\,(x)_{-1}^{1}$

$\Rightarrow 24\,[1+1]$

$= 24(2)$

$= 48$ //

5) $\int_{0}^{3}\int_{0}^{2}$

Sol) Given

8) e

Sol)

5) $\int_0^3 \int_0^2 \int_0^1 (x+y+z)\, dx\, dy\, dz$

Sol) Given $\int_0^3 \int_0^2 \int_0^1 (x+y+z)\, dx\, dy\, dz$

$\Rightarrow \int_{x=0}^{3} \int_{y=0}^{2} \int_{z=0}^{1} (x+y+z)\, dx\, dy\, dz$

$\Rightarrow \int_{x=0}^{3} \int_{y=0}^{2} \left[\int_{z=0}^{1} (x+y+z)\, dz \right] dy\, dx$

$\Rightarrow \int_{x=0}^{3} \int_{y=0}^{2} \left[xz + yz + \frac{z^2}{2} \right]_0^1 dy\, dx$

$\Rightarrow \int_{x=0}^{3} \int_{y=0}^{2} \left[x + y + \frac{1}{2} \right] dy\, dx$

$\Rightarrow \int_{x=0}^{3} \cancel{\frac{1}{2}} \int_{y=0}^{2} \left[x+y+\cancel{\tfrac{1}{2}} \right] dy\, dx$

$\Rightarrow \cancel{\frac{1}{2}} \int_{x=0}^{3} \int_{y=0}^{2} \left[xy + \frac{y^2}{2} + \cancel{\tfrac{y}{2}} \right]_0^2 dy\, dx$

$\Rightarrow \cancel{\frac{1}{2}} \int_{x=0}^{3} (2x + 2)\, dx$

$\Rightarrow \cancel{\frac{1}{2}} \int_{x=0}^{3} (2x+2)\, dx$

$\Rightarrow \int_{x=0}^{3} 2\left[\frac{x^2}{2} + x \right]_0^3 dx$

$\Rightarrow \left[\cancel{2}\left[\frac{x^2}{\cancel{2}} \right] + 3x \right]_0^3$

$\Rightarrow \left[x^2 + 3x \right]_0^3$

$= 9 + 9 = 18$

8) Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy$

Sol) Given $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy$

$\int_{y=1}^{e} \int_{x=1}^{\log y} \left[\int_{z=1}^{e^x} \log z\, dz \right] dx\, dy$

$$\Rightarrow \int_{y=1}^{e} \int_{x=1}^{\log y} [z \log z - z]_1^{e^x} \, dx \, dy$$

$$\Rightarrow \int_{y=1}^{e} \int_{x=1}^{\log y} \left(e^x \log_e(e^x) - e^x\right] - \{1 (\log 1) - 1\} \, dx \, dy$$

$$\Rightarrow \int_{y=1}^{e} \int_{x=1}^{\log y} (e^x . x - e^x + 1) \, dx \, dy \qquad \therefore \int x e^x dx = e^x(x-1)$$

$$\Rightarrow \int_{y=1}^{e} \left(x e^x - e^x - e^x + x\right]_1^{\log y} dy$$

$$= \int_{y=1}^{e} \left[\log y \, e^{\log_e y} - e^{\log_e y} - e^{\log_e y} + \log y - e^1 - e^1 - e^1 + 1\right] dy$$

$$= \int_{y=1}^{e} \left(\log_e y \, e^{\log_e y} - 2e^{\log_e y} + \log y_1 + 2e + e^1 + 1\right] dy$$

$$= \int_{y=1}^{e} [\log_e y \;\; y - 2y + \log_e y + e - 1] \, dy$$

$$= \int_{y=1}^{e} [\log_e y \, (y+1) - 2y + e - 1] \, dy$$

$$= \int_{y=1}^{e} \log_e y(y+1) - 2\int_{y=1}^{e} y \, dy + e - 1 \int_{y=1}^{e} dy$$

$$\therefore \int uv \, dx = u \int v \, dx - \int (u' \int v \, dx) \, dx$$

$$= \log y \int (y+1) dy - \int\left(\frac{1}{y} \int (y+1) dy\right) dy - 2\left(\frac{y^2}{2}\right)_1^e + e - 1 \, (y)_1^e$$

$$= \log y\left[\frac{y^2}{2} + y\right] - \int\left(\frac{1}{y}\left[\frac{y^2}{2} + y\right]\right) dy - e^2 + 1 + (e-1)(e-1)$$

$$= \log y\left(\frac{y^2}{2} + y\right] - \int\left(\frac{y}{2} + 1\right) dy - e^2 + 1 + (e-1)^2$$

$$= \left[\log y \left(\frac{y^2}{2} + y\right] - \left(\frac{y^2}{4} + y\right)\right]_1^e - e^2 + 1 + (e-1)^2$$

$$= \log_e e\left(\frac{e^2}{2} + e\right] - \frac{e^2}{4} + e - \left[\log(1)\left(\frac{1}{2} + 1\right) - \frac{1}{4} + 1\right] - e^2 + 1 + (e-1)^2$$

$$= \left[1 \left[\frac{e^2}{2} + \cancel{e}\right] - \frac{e^2}{4} + \cancel{e}\right] - \left[0 - \frac{5}{4}\right] - e^2 + 1 + (e-1)^2$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{5}{4} - \cancel{e^2} + 1 + \cancel{e^2} + 1 - 2e$$

$$= \frac{2e^2 - e^4}{4} +$$

$$= \frac{e^2 - 8e + 1}{4}$$

9) Evalueate

10) evaluate

3d) Given.

19] Eval
region
$z = 0$

$$= \frac{2e^2-e^4}{4} + \frac{13}{4} - 2e$$

$$= \frac{e^2-8e+13}{4} \text{ //}$$

9) Evalueate $\int_1^e \int_0^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy$

10) evaluate $\int_0^1 \int_y^1 \int_0^{1-x} x\, dz\, dx\, dy$

Sol) Given :, $\int_0^1 \int_y^1 \int_0^{1-x} x\, dz\, dx\, dy$

$$\Rightarrow \int_{y=0}^{1} \int_{x=y}^{1} \left[\int_{z=0}^{1-x} dz \right] x\, dx\, dy$$

$$\Rightarrow \int_{y=0}^{1} \int_{x=y}^{1} [z]_0^{1-x} x\, dx\, dy$$

$$\Rightarrow \int_{y=0}^{1} \int_{x=y}^{1} [1-x]\, x\, dx\, dy$$

$$\Rightarrow \int_{y=0}^{1} \int_{x=y}^{1} [x - x^2]\, dx\, dy$$

$$\Rightarrow \int_{y=0}^{1} \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_y^1 dy$$

$$\Rightarrow \int_{y=0}^{1} \left(\frac{1}{2} - \frac{1}{3} - \frac{y^2}{2} + \frac{y^3}{3}\right) dy$$

$$\Rightarrow \int_{y=0}^{1} \left[\frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3}\right] dy$$

$$\Rightarrow \left[\frac{y}{6} - \frac{y^3}{6} + \frac{y^4}{12}\right]_0^1$$

$$= \cancel{\frac{1}{6}} - \cancel{\frac{1}{6}} + \frac{1}{12}$$

$$= \frac{1}{12} \text{ //}$$

11) Evaluate $\iiint dx\, dy\, dz$ where V is the finite region of space formed by the planes $x=0, y=0$ $z=0$ and $2x+3y+4z=12$

z limits:-

Sol) Let $2x+3y+4z=12$

$$4z = 12-2x-3y$$

$$z = \frac{1}{4}[12-2x-3y]$$

$z \to 0$ to $\frac{1}{4}[12-2x-3y]$

y limits

Let $2x+3y+4z=12$

Let $z=0 \Rightarrow 2x+3Y+0=12$

$$\Rightarrow 3Y = 12-2x$$

$$Y = \frac{1}{3}(12-2x)$$

$Y \to 0$ to $\frac{1}{3}(12-2x)$

x limits:-

Let $2x+3Y+4z=12$

Let $Y=0 \quad z=0$

$$2x = 12$$

$$x = 6$$

$\therefore x \to 0$ to 6