DISCRETE TIME SIGNALS AND SYSTEMS

1.1 INTRODUCTION TO DSP

Digital signal processing is an area of science that has rapid development with the significant advances in digital computer technology and integrated circuit fabrication. The conventional microprocessors are used for off-line scientific computations and business applications. The specialized digital signal processor hardware can be used for processing real time signals like ECG, EEG, EMG, Radar and weather signals, earthquake signals etc.

The signals in real world are analog in nature. Such signals may be processed directly by analog filters, frequency multipliers and amplifiers etc. Both the input signal and output signal are analog in nature as shown in Fig. 1.1.

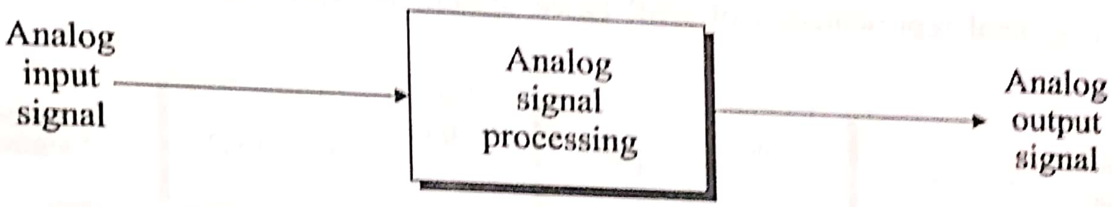


Fig. 1.1 General Representation of Analog signal processing

The Digital signal processing is an alternative to process the given signals. The digital representation of original signal is obtained by using Analog to digital conversion and the digital output is given to the digital signal processor for further analysis of signal as shown in Fig. 1.2.

The applications of real time signal processing is enormous but the constraint in the implementation of signal processing applications lies in the design of accurate analog to digital converter for extremely wide band signal bandwidths. But due to advancement in VLSI technology and digital computer technology the above said limitation had been overcome in recent years.

1.2 DEFINITION OF DIGITAL SIGNAL PROCESSING

Digital signal processing (DSP) is concerned with the digital representation of signals and the use of digital processors to analyze, modify, or extract information from signals. The term DSP is defined as:

★ **Digital:** Operating by the use of discrete signals to represent data in the form of binary numbers.

★ **Signal:** A variable parameter by which information is conveyed through an electronic circuit.

★ **Processing:** To perform operations on data according to programmed instructions.

It is used in Speech (Telephony, Radio, Everyday communication), Biomedical signals (EEG brain signals), Sound and music, Video and image and Radar signals (range and bearing).

The general representation of DSP based system is represented by:

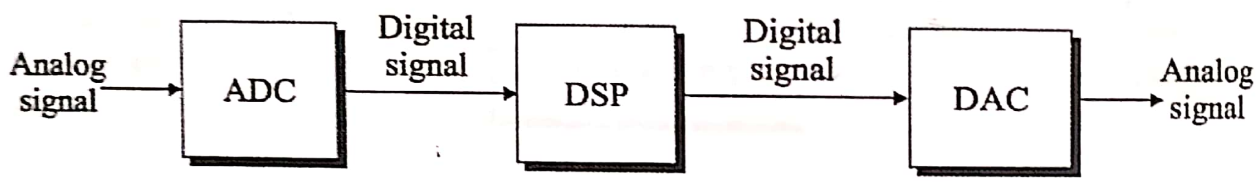


Fig. 1.2 General representation of DSP based systems

Analog to Digital Conversion (ADC) represents signals by a sequence of numbers by using sampling or analog-to-digital conversions techniques. The processing of digital representation of data is performed by using DSP processor. The analog representation of reconstructed digital signal is done by using Digital to Analog Conversion (DAC).

1.3 MERITS AND DEMERITS OF DIGITAL SIGNAL PROCESSING

With the advancement in digital computer technology and in very large scale integrated electronic circuits the advantages of signal processing applications is very high when compared to its disadvantages.

1. Merits of DSP

The advantages of Digital Signal Processing are given by:

1. Accuracy:

Accuracy can be controlled by choosing word length of the signal processors.

2. Flexibility:

Digital programmable systems allow flexibility in reconfiguring the DSP operations by changing simply the program.

3. Easy storage:

Digital signals can be easily stored in magnetic media without loss of signal fidelity. Transmission of signals from one place to another is easy by using higher end DSP processors.

4. Low cost:

With the advancement in VLSI technology the digital implementation of signal processing system is cheaper and cost effective.

2. Demerits of DSP

The limitations of DSP are that the conversion speed of ADC and the process speed of signal processors should be very high to perform real time processing. Signals of high bandwidth require fast sampling rate ADC's and high speed processors. Quantization and round-off errors limit the performance of DSP processors.

1.4 REVIEW OF SIGNAL AND SYSTEMS

The applications of signals and systems play an important role in the field of many areas of Science and Technology. The concepts are very extensively applied in the field of circuit analysis and design, long distance communications, power system generation and distribution, electrical machines, Biomedical engineering, aeronautics, process control, consumer electronics, speech and image processing are to mention a few.

Representation of Discrete – time Signal.

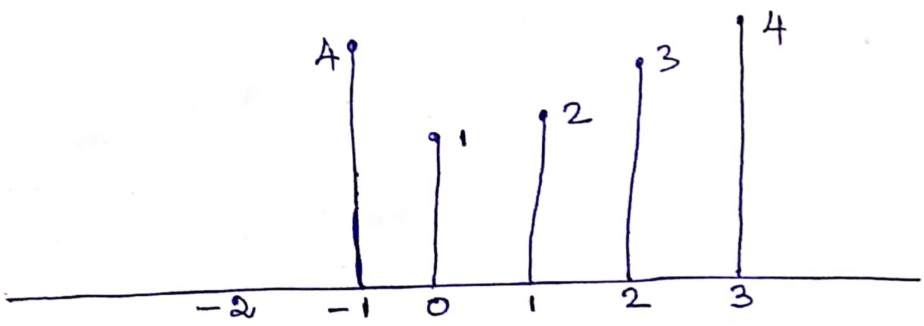
* The discrete time signals are represented by

1. Graphical Representation.
2. Functional Representation.
3. Tabular Representation.
4. Sequence Representation.

Graphical Representation:-

Consider a signal $x(n)$ with values $x(-1) = 4$, $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, $x(3) = 4$.

The Discrete Time Signal is represented graphically as.



Functional Representation:

$$x(n) = \begin{cases} 4, & n = -1 \\ 1, & n = 0 \\ 2, & n = 1 \\ 3, & n = 2 \\ 4 & n = 3. \end{cases}$$

Tabular Representation

n	-1	0	1	2	3
$x(n)$	4	1	2	3	4

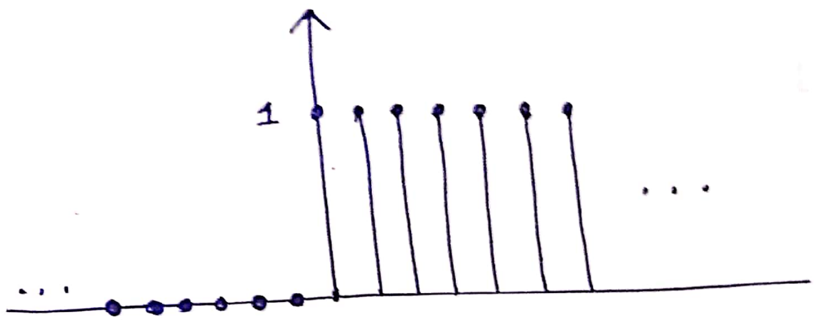
Sequence Representation.

A finite duration sequence $x(n)$ with time origin $n=0$ indicated as $\uparrow$ is given by

$$x(n) = \{4, \underset{\uparrow}{1}, 2, 3, 4\}$$

Discrete Time Unit Step function or Unit Sequence function.

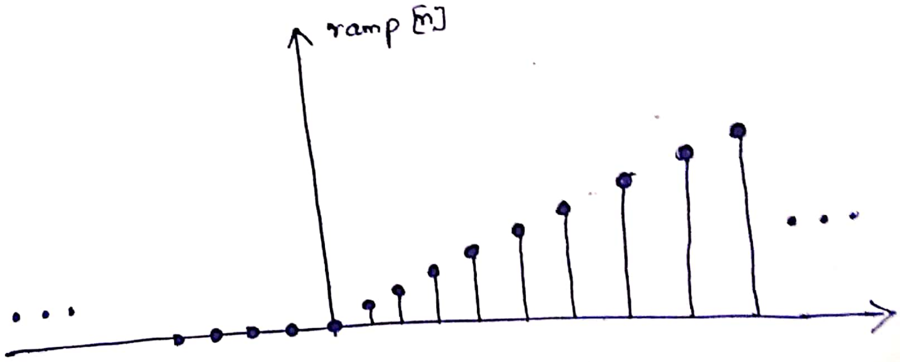
$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Discrete Time Unit Ramp function.

The basic unit ramp sequence is denoted by $r(n)$ and is

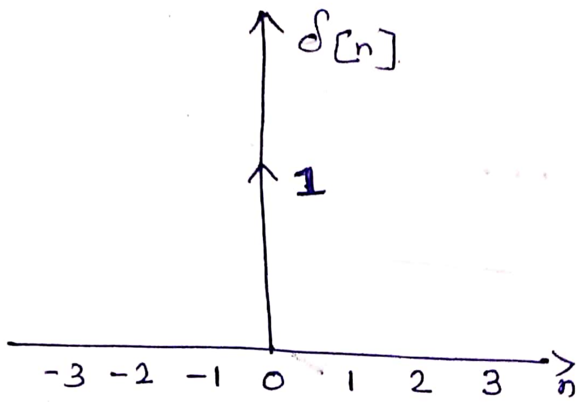
$$r(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Discrete Time Unit Impulse Function or Unit pulse seq

The unit impulse sequence is defined as.

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Operation of Signals.

* A given mathematical function may completly describe a signal and its properties.

* Different operations are required for different purposes of arbitrary signals.

Let $x(n)$ be the given signal and the operations performed on the signal are.

1. Time shifting
   (a) Time Delaying
   (b) Time advancing.
2. Time Scaling or rate changing.
3. Time reversing or Time Folding.
4. Signal multiplication.
5. Signal addition.
6. Signal scaling.

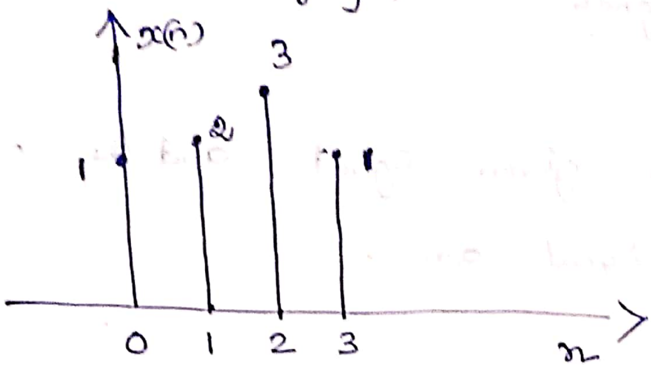
Time Shifting:-

The original signal $x(n)$ is shifted by an amount n_0

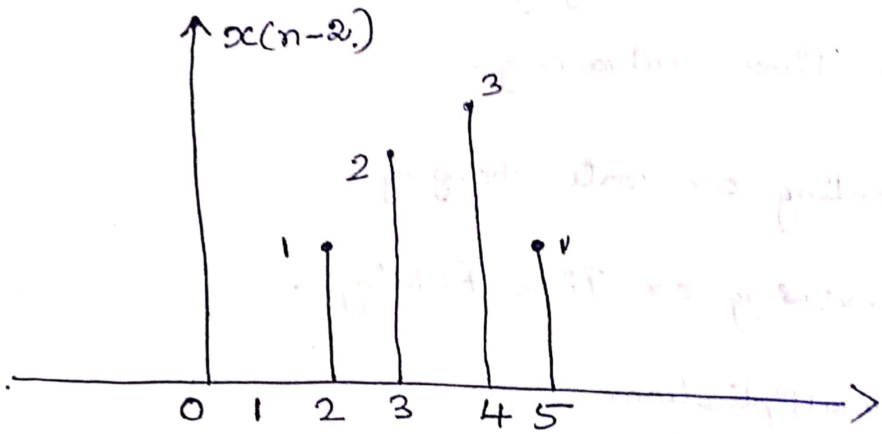
$$y(n) = x(n-n_0)$$

* If $x(n)$ is the given signal and shifting the signal by n_0 by two units depends on the shifting parameter.

* If n_0 is positive the signal is shifted to the right. This results in time delaying.

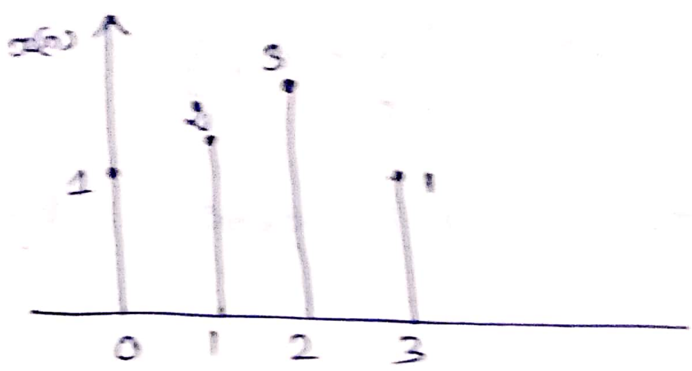


Original signal.

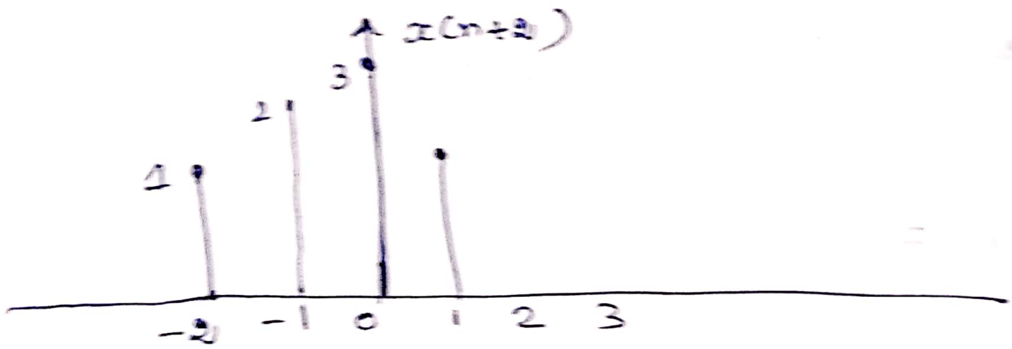


original signal shifted to the right by 2 units (delayed version)

* If n_0 is negative the signal is shifted to the left. This gives time advancing of the signal.



The original Signal.



original Signal shifted to the left by two unit (advanced version)

Time Scaling :-

Let x(n) be the original signal.

x(an) is the time scaled version of x(n).

For a > 1, the period of function x(n) reduces and function speeds up. Graph of the function shrinks similar to compression of signals.

For a < 1, the period of the x(n) increases and the function slows down. Graph of the function expands. This is similar to expansion of signals.

Consider a sequence $x(n) = \{1, 2, 3, 4\}$

The scaled version of the signal is obtained by.

$y(n) = x(an)$

if $a = 2$

$y(n) = x(2n)$

for $n = 0$

$y(0) = x(0) = 1$

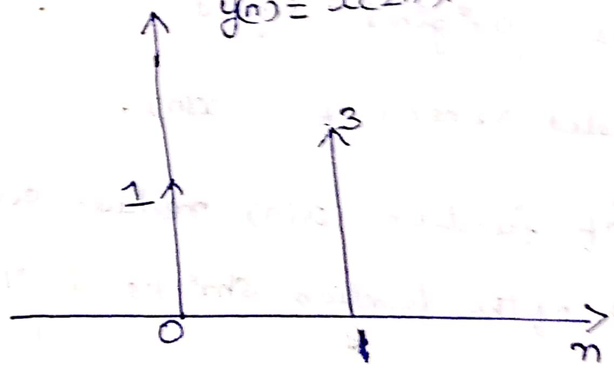
for $n = 1$

$y(1) = x(2) = 3$

for $n = 2$

$y(2) = x(4) =$

$y(n) = x(2n)$.



The signal $x(2n)$ is obtained by reducing the sampling rate by a factor of 2.

This process of reducing sampling rate is defined as down sampling or decimation.

Consider another type of Sequence.

$y(n) = x\left(\frac{n}{N}\right)$ where $N = 2$.

when $n = 0$

$y(0) = x\left(\frac{0}{2}\right)$

$y(0) = x(0)$

$y(0) = 1$

when $n = 1$

$y(1) = x\left(\frac{1}{2}\right)$

$= x(0.5)$

when $n = 2$

$y(2) = x\left(\frac{2}{2}\right)$

of sampo

$= x(1)$

$y(2) = 2$

when $n = 3$

$y(3) = x\left(\frac{3}{2}\right)$

$= x(\quad)$

$=$

when $n = 4$

$y(4) = x\left(\frac{4}{2}\right)$

$= x(2)$

$= 3$

when $n = 5$

$y(5) = x\left(\frac{5}{2}\right)$

$=$

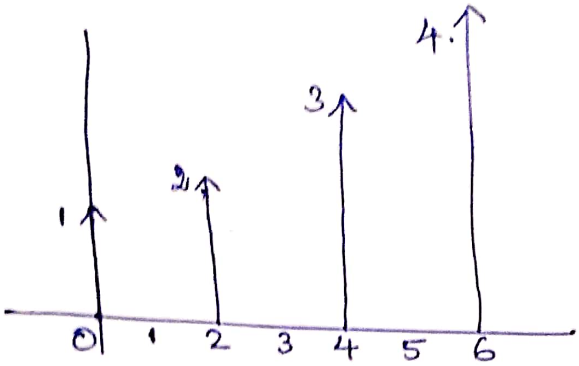
when $n = 6$

$y(6) = x\left(\frac{6}{2}\right)$

$= x(3)$

$= 4$

The new sequence obtained is not defined for odd values of n.

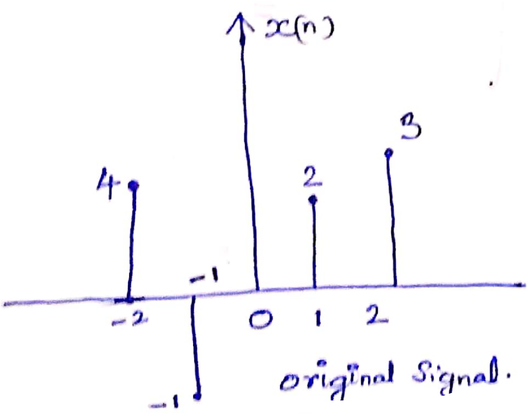


In the above sequence $y(n) = 0$ for odd values of n.

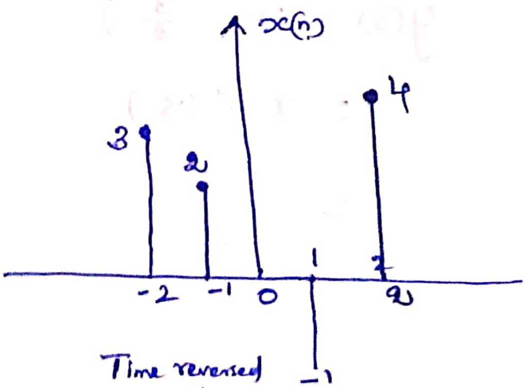
In this sequence $x(n)$, $N-1$ zeros are inserted between each of the sequence values in $x(n)$. This is called upsampling or expansion of signals.

Time Reversing or Folding.

* Time reversal is also called time folding.
* In time reversal the signal is reversed with respect to time. i.e $y(n) = x(-n)$ is obtained for the given function.
* The time reversal of the sequence is obtained by folding the signal about $n=0$



original Signal.

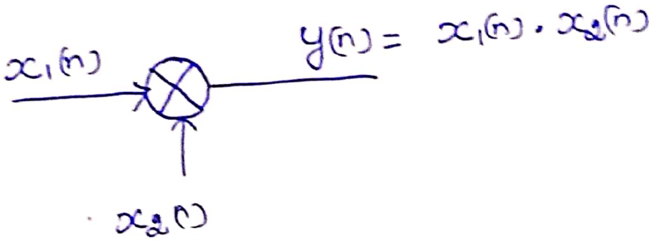


Time reversed Signal.

Signal multiplication

Two signals $x_1(n)$ and $x_2(n)$ can be multiplied by a multiplier to obtain a new sequence

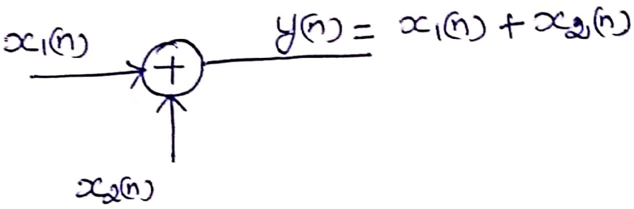
$$y(n) = x_1(n) \cdot x_2(n)$$



Signal addition:-

Two signals $x_1(n)$ and $x_2(n)$ can be added by an adder to obtain a new sequence.

$$y(n) = x_1(n) + x_2(n)$$



Signal Scaling

The signal $x(n)$ is multiplied by scale factor 'a' to obtain new signal.

$$x(n) \xrightarrow{\quad a \quad} y(n) = a\,x(n)$$

Classification of Signals.

Generally the signals are classified into

1. Deterministic and Non Deterministic signals.
2. Periodic and Aperiodic Signals.
3. Even and odd Signals.
4. Energy and Power Signals.
5. Causal and Non causal Signals.

Deterministic and Non Deterministic Signals.

Deterministic Signals.

* For deterministic signals the behaviour of these signals is predictable. w.r.t time.
* There is no uncertainty with respect to its value at any time.
* These signals can be expressed mathematically.

(eg) $x(t) = \sin(3t)$ is deterministic signal.

2. Non Deterministic or Random Signals

* The behavior of non-deterministic signals or random signals cannot be predicted w.r.t time.
* There is an uncertainty with respect to its value at any time.
* These signals can't be expressed mathematically.

(eg) Thermal noise generated in communication systems is non-deterministic signal.

Periodic and Non-periodic Signals

If $x(t)$ is a continuous time signal then the signal $x(t)$ is periodic if and only if $x(t) = x(t+T_0)$ for any T_0

The fundamental period is given by $T = \frac{1}{f}$

$\omega = 2\pi f$

where T is the fundamental period.

The smallest value of T that satisfies the above condition is known as fundamental period.

For non-periodic signals

$$x(t) \neq x(t+T)$$

A non-periodic signal is assumed to have a period $T = \infty$.

Example of non periodic signal is an exponential signal. It does not have fundamental time period.

discrete time signal $x(n)$ is said to periodic with period N if and only if,

$$x(n) = x(N+n) \quad \text{for all } N.$$

The smallest value of N in the above equation is called as fundamental period.

The discrete time signal $x(n)$ is said to be aperiodic

$$\text{if} \quad x(n) \neq x(N+n)$$

Find whether the following signals are periodic or not. Determine the fundamental period of the signal, if it is periodic.

(i) $x(n) = e^{j6\pi n}$ (ii) $x(n) = e^{j\frac{2}{5}(n+\frac{1}{2})}$ (iii) $x(n) = \sin^2 \frac{\pi}{4} n$

(iv) $x(n) = \sin\left(\frac{n}{2} - \pi\right)$ (v) $x(n) = \cos \frac{2\pi}{3} n + \cos \frac{3\pi}{4} n$

Solution:-

(i) $x(n) = e^{j6\pi n}$

The exponential signal in general are represented by

$$x(n) = e^{j\omega_0 n} \quad \text{——— (1)}$$

The given signal is

$$x(n) = e^{j6\pi n}. \quad \text{——— (2)}$$

comparing (1) and (2)

$\omega_0 = 6\pi.$

The fundamental frequency is multiple of π.

The given signal is periodic.

To find the fundamental period

$$N = 2\pi\left[\frac{m}{\omega_0}\right]$$

$$N = 2\pi\left[\frac{m}{6\pi}\right]$$

$$N = \frac{m}{3}$$

The minimum value of m for which N is an integer is 3.

$$N = 2\pi \frac{3}{6\pi} \qquad \boxed{N = 1}$$

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Fundamental period, $N = 1$

$\therefore x(n)$ is periodic.

(ii) $x(n) = e^{j\frac{2}{5}(n+\frac{1}{2})}$

Solution

The exponential signal in general are represented by

$x(n) = e^{j\omega_0 n}$ —①

The given signal is

$x(n) = e^{j\frac{2}{5}(n+\frac{1}{2})}$ —②

Comparing ① and ②

$\omega_0 = \frac{2}{5}$ which is not a multiple of π.

The given signal is aperiodic or non-periodic.

(iii) $x(n) = \sin^2 \frac{\pi}{4} n$

Solution:-

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$x(n) = \sin^2 \frac{\pi}{4} n$

$x(n) = \frac{1 - \cos 2\times\frac{\pi}{4} n}{2}$

$x(n) = \frac{1}{2} - \frac{1}{2}\cos\frac{\pi}{2} n$

$x(n) = x_1(n) + x_2(n)$

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$$x_1(n) = \frac{1}{2}$$

$$x_1(n) = \frac{1}{2}(1)^n$$

$N_1 = 1$

$x_1(n)$ is periodic.

$$x_2(n) = -\frac{1}{2} \cos \frac{\pi}{2} n \quad \text{—— (1)}$$

The general representation of cosine signal is given

$$x(n) = a \cos \omega_0 n \quad \text{—— (2)}$$

comparing (1) and (2)

$$\omega_0 = \frac{\pi}{2}.$$

$$N_2 = \frac{2\pi}{\omega_0} m$$

$$= \frac{2\pi}{\frac{\pi}{2}} \times m$$

$$N_2 = 4 \times m$$

if $m = 1$

$$\boxed{N_2 = 4}$$

To find the sum of two signals as periodic.

$$\frac{N_1}{N_2} = \frac{1}{4}.$$

$$4N_1 = N_2 = N$$

$\therefore N = 4$

The given signal is periodic.

(iv) $x(n) = \sin\left(\frac{n}{8} - \pi\right)$

Solution:-

The general representation of Sine signal is given by

$$x(n) = \sin w_0 n \quad \text{——} \ (1)$$

The given signal is

$$x(n) = \sin\left(\frac{n}{8} - \pi\right) \quad \text{——} \ (2)$$

comparing (1) and (2)

$$w_0 = \frac{1}{8}$$

$$N = \frac{2\pi}{w_0} m$$

$$N = \frac{2\pi}{\frac{1}{8}} m$$

$$N = 16\pi m.$$

For any integer value of m, N is not an integer.

The Signal is aperiodic.

(v) $x(n) = \cos\frac{2\pi}{3}n + \cos\frac{3\pi}{4}n.$

Solution:.

$$x(n) = \cos\frac{2\pi}{3}n + \cos\frac{3\pi}{4}n$$

$$x(n) = x_1(n) + x_2(n)$$

$$x_1(n) = \cos\frac{2\pi}{3}n \quad \text{——} \ (1)$$

The general representation of cosine signal is

$$x_1(n) = \cos w_0 n \quad \text{——} \ (2)$$

comparing (1) and (2)

$\omega_0 = \frac{2\pi}{3}$

$N_1 = \frac{2\pi}{\omega_0} m$

$N_1 = \frac{2\pi}{\frac{2\pi}{3}} \times m$

$N_1 = 3m$

for $m = 1$

$\boxed{N_1 = 3}$

$x_2(n) = \cos \frac{3\pi}{4} n$ — (3)

The general representation of cosine signal is

$x(n) = \cos \omega_0 n$ — (4)

$\omega_0 = \frac{3\pi}{4}$

$N_2 = \frac{2\pi m}{\omega_0}$

$N_2 = \frac{2\pi}{\frac{3\pi}{4}} m$

$N_2 = \frac{8}{3} m$

if $m = 3$

$\boxed{N_2 = 8}$

To find sum of two signals as periodic.

$$\frac{N_1}{N_2} = \frac{3}{8}$$

$$8N_1 = 3N_2.$$

$\therefore$ The fundamental period is $N=24$.
The given signal is periodic.

Determine the fundamental period of the following signals, if they are periodic.

(i) $x(n) = \sin\left(\frac{\pi}{4}\right)n.$

Solution:-

The general representation of sinusoidal signal is

$$x(n) = \sin \omega_0 n \qquad \text{——— (1)}$$

The given signal is

$$x(n) = \sin \frac{\pi}{4} n \qquad \text{——— (2)}$$

comparing (1) and (2)

$$\omega_0 = \frac{\pi}{4}$$

The given signal is periodic.

$$N = \frac{2\pi}{\omega_0} m$$

$$N = \frac{2\pi}{\frac{\pi}{4}} \times m$$

$$N = \frac{8\pi}{\pi} \times m$$

$$N = 8m$$

if $m=1$

$$\boxed{N = 8}$$

Fundamental Period $N=8$

$x(n) = e^{j6n}$

Solution:-

The general representation of the exponential signal is

$x(n) = e^{j\omega_0 n}$ —— ①

The given signal is

$x(n) = e^{j6n}.$ —— ②

comparing ① and ②

$\omega_0 = 6$

The signal is aperiodic.

(iii) $x(n) = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$

Solution:-

$x(n) = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}.$

$x(n) = x_1(n) + x_2(n)$

$x_1(n) = e^{j\frac{2\pi}{3}n}$ —— ①

$x_1(n) = e^{j\omega_0 n}$ —— ②

comparing ① & ②

$\omega_0 = \frac{2\pi}{3}$

$N_1 = \frac{2\pi}{\omega_0} m$

$N_1 = \frac{2\pi m}{\frac{2\pi}{3}}$

$N_1 = 3m$

if $m = 1$

$$\boxed{N_1 = 3}$$

$$x_2(n) = e^{j\frac{3\pi}{4}n} \quad \text{------} \quad (3)$$

$$x_2(n) = e^{j w_0 n} \quad \text{------} \quad (4)$$

comparing (3) & (4)

$$w_0 = \frac{3\pi}{4}$$

$$N_2 = \frac{2\pi m}{w_0}$$

$$N_2 = \frac{2\pi m}{\frac{3\pi}{4}}$$

$$N_2 = \frac{8m}{3}$$

if $m = 3$

$$\boxed{N_2 = 8}$$

To find the sum of two signals as periodic.

$$\frac{N_1}{N_2} = \frac{3}{8}$$

$$8N_1 = 3N_2$$

$$\boxed{N = 24}$$

The signal is periodic.

$x(n) = e^{j\pi n}$

Solution:-

$x(n) = e^{j\pi n}$ —— ①

The general representation of exponential signal is

$x(n) = e^{j\omega_0 n}$ —— ②

Comparing ① and ②

$\boxed{\omega_0 = \pi}$

$N = \frac{2\pi m}{\omega_0}$

$N = \frac{2\pi m}{\pi}$

$N = 2m$ if $m = 1$

$\boxed{N = 2}$

The signal is periodic.

Find whether the following are periodic or not.

(i) $\cos(0.1\pi n)$ (ii) $\sin\left(\frac{6\pi}{7}n + 1\right)$ (iii) $12\cos(20n)$

(iv) $\cos\left(\frac{1}{4}n\right)$

Solution

(i) $x(n) = \cos(0.1\pi n)$ —— ①

The general representation of cosine signal is

$x(n) = \cos(\omega_0 n)$ —— ②

comparing ① and ②

$\omega_0 = 0.1\pi$.

$$N = \frac{2\pi}{\omega_0} m$$

$$N = \frac{2\pi}{0.1\pi} m$$

$$N = 20m$$

if $m = 1$

$$\boxed{N = 20}$$

The sequence is periodic.

(ii) $\sin\left(\frac{6\pi}{7}n + 1\right)$

$$x(n) = \sin\left(\frac{6\pi}{7}n + 1\right) \longrightarrow ①$$

The general representation of sin signal is

$$x(n) = \sin \omega_0 n \longrightarrow ②$$

comparing ① and ②

$$\omega_0 = \frac{6\pi}{7}$$

$$N = \frac{2\pi}{\omega_0} m$$

$$N = \frac{2\pi}{\frac{6\pi}{7}} m$$

$$N = \frac{7}{3} m$$

if $m = 3$

$$\boxed{N = 7}$$

The given signal is periodic.

$x(n) = 12\cos(20n)$

The general representation of cosine signal is

$x(n) = \cos \omega_0 n$ ——— ①

The given signal $x(n) = 12\cos(20n)$ ——— ②

Comparing ① and ②

$\omega_0 = 20$

$N = \dfrac{2\pi m}{\omega_0}$

$N = \dfrac{2\pi m}{20}$

$N = 0.1\pi m$

For any value of m N is not an integer.

Therefore the given signal $12\cos(20n)$ is a aperiodic sequence.

(iv) $x(n) = \cos\left(\frac{n}{4}\right)$

The general representation of cosine signal is

$x(n) = \cos \omega_0 n$ ——— ①

The given signal is

$x(n) = \cos \frac{n}{4}$ ——— ②

Comparing ① and ②

$\omega_0 = \frac{1}{4}$

$$N = \frac{2\pi m}{\omega_0}$$

$$N = \frac{2\pi m}{\frac{1}{4}}$$

$$N = 8\pi m$$

For any value m, N is not an integer.

Therefore the Signal. $\cos\left(\frac{n}{4}\right)$ is a aperiodic signal.

Even and odd Signal:-

- A discrete-time Signal $x(n)$ is even if it Satisfies the condition $x(-n) = x(n)$ for all n.

* The even Signal is Symmetric with respect to origin.

* A discrete time Signal $x(n)$ is said to be odd if $x(n) = -x(-n)$ and is anti symmetric with respect to origin.

The even part or even component of the Signal is

$$x_e(n) = \frac{x(n) + x(-n)}{2}.$$

The odd component of the Signal is

$$x_o(n) = \frac{x(n) - (x-n)}{2}.$$

Determine whether the following Signal is odd or even.

(i) $x(n) = \sin 2\pi n$. (ii) $x(n) = \cos 2\pi n$.

Solution:-

$x(n) = \sin 2\pi n$. $\sin(-\theta) = -\sin\theta$

$x(-n) = \sin(-2\pi n)$

$x(-n) = -\sin(2\pi n)$ $-x(-n) = \sin 2\pi n$

$-x(-n) = x(n)$

$x(n) = -x(n)$

The given signal is odd signal.

(ii) $x(n) = \cos 2\pi n.$

$x(-n) = \cos(-2\pi n)$

$x(-n) = \cos 2\pi n.$

$\boxed{x(n) = x(-n)}$

The given signal is even.

Find the odd and even components of Discrete signal.

$x(n) = \{-3, 2, \underset{\uparrow}{1}, 5, 3\}$

$x(-n) = \{3, 5, \underset{\uparrow}{1}, 2, -3$

$x_e = \frac{1}{2}[x(n) + x(-n)]$

$= \frac{1}{2}\left[\{-3, 2, \underset{\uparrow}{1}, 5, 3\} + \{3, 5, \underset{\uparrow}{1}, 2, -3\}\right]$

$= \frac{1}{2}\left[-3+3, 2+5, \underset{\uparrow}{1+1}, 5+2, 3-3\right]$

$= \frac{1}{2}[0, 7, \underset{\uparrow}{2}, 7, 0]$

$x_e = \{0, 3.5, \underset{\uparrow}{1}, 3.5, 0\}$

$$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

$$= \frac{1}{2}\left[-3-3,\ 2-5,\ 1-1,\ 5-2,\ 3+3\right]$$

$$= \frac{1}{2}\left[-6, -3, 0, 3, 6\right]$$

$$x_o = \left[-3, 1.5, 0, 1.5, 3\right]$$

Find the even and odd components of the following signals.

$$x(n) = \cos n + \sin n + \cos n \sin n$$

$$x(-n) = \cos(-n) + \sin(-n) + \cos(-n)\sin(-n)$$

$$x(-n) = \cos n - \sin(n) \cancel{+} \cos(n)\sin(n)$$

$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

$$= \frac{1}{2}\left[\cos n + \cancel{\sin n} + \cos n \cancel{\sin n} + \cos n - \cancel{\sin n} - \cos n \cancel{\sin n}\right]$$

$$= \frac{1}{2} 2 \cos n$$

$$\boxed{x_e(n) = \cos n}$$

$$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

$$= \frac{1}{2}\left[\cancel{\cos n} + \sin n + \cos n \sin n - \cancel{\cos n} + \sin n + \cos n \sin n\right]$$

$$x_o(n) = \frac{1}{2}\left[2\sin n + 2\sin n \cos n\right]$$

$$\boxed{x_o(n) = \sin n + \sin n \cos n}$$

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Causal and Non-causal Signals.

* A signal $x(n)$ is said to be causal if its value is zero for $n < 0$.

* A signal is non causal if its value is zero for $n \geq 0$.

Example of causal signals.

$$x_1(n) = a^n u(n)$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, -1\}$$

Example of Non-causal Signals.

$$x_1(n) = a^n u(-n+1)$$

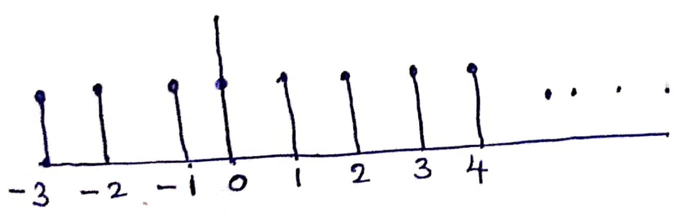
$$x_2(n) = \{1, 0, \underset{\uparrow}{-1}, 2, 3\}$$

Find which of the following signals are causal or non causal.

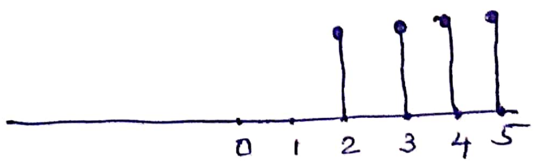
(i) $x_1(n) = u(n+3) - u(n-2)$

$u(n)$

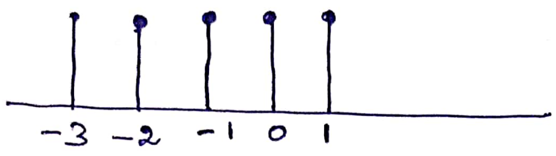
$u(n+3)$



$u(n-2)$



$x_1(n) = u(n+3) - u(n-2)$



The Signal $x_1(n)$ is defined for values $n<0$. Therefore $x_1(n)$ is a non-causal Signal.

(ii) $x_2(n) = \left(\frac{1}{2}\right)^n u(n+2)$

$u(n+2)$ means it is having values for $n<0$

The given Signal is a Non-Causal Signal.

Energy and Power for Discrete time Signal

A discrete time signal with finite energy and zero power is called Energy Signal.

$$0 < E < \infty \quad \text{and} \quad P = 0$$

The Signal energy for a discrete time signal $x[n]$ is

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

The average signal power of a discrete time power signal $x[n]$ is

$$P_x = \underset{N \to \infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

For a periodic signal $x[n]$ the average signal power is

$$P_x = \frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2$$

$N \to$ Length of the signal.

Determine the values of power and energy of the following signals, find whether the signals are power, energy or neither energy or power signals.

(i) $x(n) = \left(\frac{1}{2}\right)^n u(n)$ (ii) $x(n) = e^{j\left(\frac{\pi}{2} n + \frac{\pi}{8}\right)}$ (iii) $x(n) = \left(\frac{1}{3}\right)^n u(n)$

(iv) $x(n) = u(n)$ (v) $x(n) = e^{2n} u(n)$

Solution:-

$x(n) = \left(\frac{1}{2}\right)^n u(n)$

$u(n) = 1 \quad n \geq 0$
$= 0 \quad n < 0$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^n\right]^2$$

$$= \sum_{n=0}^{\infty} \left(\left(\frac{1}{2}\right)^2\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$E = \frac{1}{1-\frac{1}{4}}$$

$$E = \frac{1}{\frac{3}{4}}$$

$$\boxed{E = \frac{4}{3}}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left[\left(\frac{1}{2}\right)^n\right]^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left[\frac{1}{4}\right]^n.$$

$$\lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{1-\left(\frac{1}{4}\right)^{N+1}}{1-\frac{1}{4}}\right]$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{1-\left(\frac{1}{4}\right)^{\infty}}{\frac{3}{4}}\right]$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{1-0}{\frac{3}{4}}\right]$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{4}{3}\right]$$

$$= \frac{1}{\frac{1}{0}}$$

$$= \frac{0}{1} \left[\frac{4}{3}\right]$$

$$= 0 \left[\frac{4}{3}\right]$$

$$\boxed{P = 0}$$

(ii) $x(n) = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{8}\right)}$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} \left| e^{j\left(\frac{\pi}{2}n + \frac{\pi}{8}\right)} \right|^2$$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} 1$$

$$E = \lim_{N\to\infty} (2N+1)$$

$$\boxed{E = \infty}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j\left(\frac{\pi}{2}n + \frac{\pi}{8}\right)} \right|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$\left| e^{j(\omega+\theta)} \right| = 1$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \times (2N+1)$$

$$\sum_{n=-N}^{N} 1 = 2N+1$$

$$P = \lim_{N\to\infty} 1$$

$$\boxed{P = 1}$$

The energy is infinite and the power is finite. Therefore $x_2(n)$ is a power signal.

$$x(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2$$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} \left[\left(\frac{1}{3}\right)^n\right]^2 u(n)$$

Since $u(n) = 1$ for $n \geq 0$
The limit is changed
to 0 to N

$$E = \lim_{N\to\infty} \sum_{n=0}^{N} \left[\left(\frac{1}{3}\right)^2\right]^n$$

$$E = \lim_{N\to\infty} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^n$$

$$E = \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n$$

$$E = \frac{1}{1-\frac{1}{9}}$$

$$E = \frac{1}{\frac{8}{9}}$$

$$\boxed{E = \frac{9}{8}}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left[\left(\frac{1}{3}\right)^n\right]^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\left(\frac{1}{3}\right)^2\right)^n$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^n$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \left[\frac{1 - \left(\frac{1}{9}\right)^{N+1}}{1-9} \right]$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \left[\frac{1 - \left(\frac{1}{9}\right)^{\infty}}{1-9} \right]$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \left[\frac{1-0}{1-9} \right]$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \left[\frac{1}{-8} \right]$$

$$P = \frac{1}{\infty} \left[-\frac{1}{8} \right]$$

$$P = \frac{1}{\frac{1}{0}} \left[-\frac{1}{8} \right]$$

$$P = \frac{0}{1} \left[-\frac{1}{8} \right]$$

$$\boxed{P = 0}$$

$$x(n) = u(n)$$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E = \sum_{n=0}^{\infty} |u(n)|^2$$

$$E = \sum_{n=0}^{\infty} 1$$

$$\boxed{E = \infty}$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} |u(n)|^2$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} 1$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} (N+1)$$

$$P = \underset{N\to\infty}{Lt} \frac{(N+1)}{2N+1}$$

Taking N outside from the numerator and denominator.

$$P = \underset{N\to\infty}{Lt} \frac{\cancel{N}\left(1+\frac{1}{N}\right)}{\cancel{N}\left(2+\frac{1}{N}\right)}$$

$$P = \frac{\left(1+\frac{1}{\infty}\right)}{\left(2+\frac{1}{\infty}\right)}$$

$$P = \frac{(1+0)}{(2+0)}$$

$$\boxed{P = \frac{1}{2}}$$

(v) $x(n) = e^{2n} \, u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E = \sum_{n=0}^{\infty} |e^{2n}|^2$$

$$E = \sum_{n=0}^{\infty} e^{4n}$$

$$= 1 + e^4 + e^8 + e^{12} + \dots + \infty$$

$$\boxed{E = \infty}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} |e^{2n}|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} e^{4n}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \left[1 + e^4 + e^8 + e^{12} + \dots + N\right]$$

But $N \to \infty$

$$P = \frac{1}{2N+1} \cdot \left[1+e^{4}+e^{8}+e^{12}+\dots+\infty\right]$$

$$P = \frac{1}{\infty+1} \cdot \infty$$

$$\boxed{P = \infty}$$

Since energy and power are infinite the given signal is neither energy nor power signal.

Find whether the signals are power, energy or neither energy or power signals.

(i) $x(n) = \cos\frac{\pi}{2}n$ (ii) $x(n) = A\,\delta(n)$ (iii) $x(n) = 2e^{j3\pi n}$.

Solution:-

$$x(n) = \cos\frac{\pi}{2}n.$$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$E = \sum_{n=-\infty}^{\infty} \left|\cos\frac{\pi}{2}n\right|^2$$

$$E = \sum_{n=-\infty}^{\infty} \cos^2\frac{\pi}{2}n.$$

$$E = \sum_{n=-\infty}^{\infty} \frac{1+\cos 2\frac{\pi}{2}n}{2}$$

$$E = \sum_{n=-\infty}^{\infty} \frac{1+\cos\pi n}{2}$$

$$E = \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} 1 + \sum_{n=-\infty}^{\infty} \cos \pi n\right]$$

$$E = \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} 1 + 0\right]$$

$$E = \frac{1}{2}[\infty]$$

$$\boxed{E = \infty}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos^2 \frac{\pi}{2} n$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left[\frac{1 + \cos \frac{2\pi n}{2}}{2}\right]$$

$$= \frac{1}{2}\left[\lim_{N\to\infty} \frac{1}{2N+1}\left[\sum_{-N}^{N} 1 + \sum_{-N}^{N} \cos \pi n\right]\right]$$

$$= \frac{1}{2}\left[\lim_{N\to\infty} \frac{1}{2N+1}\left[[2N+1] + 0\right]\right]$$

$$P = \frac{1}{2}\left[\lim_{N\to\infty} \frac{1}{\cancel{2N+1}} \times \cancel{2N+1}\right]$$

$$\boxed{P = \frac{1}{2}}$$

$E = \infty, \quad P = \frac{1}{2}.$

So the given signal is a power signal.

(ii) $x(n) = A\,\delta(n)$

$\delta(n) = 1, \; n = 0$

$\delta(n) = 0; \; n \neq 0$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E = \sum_{n=0}^{0} |A\,\delta(n)|^2$$

$$E = \sum_{n=0}^{0} A^2$$

$$\boxed{E = A^2}$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |A\,\delta(n)|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{0} |A|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} A^2$$

$$P = \frac{1}{\frac{1}{0}}$$

$$P = \frac{1}{\infty} A^2$$

$$P = \frac{1}{\frac{1}{0}} \times A^2$$

$$\boxed{P = 0}$$

The given signal is an energy signal.

$$x(n) = 2e^{j3\pi n}.$$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} |2e^{j3\pi n}|^2$$

$$= \lim_{N\to\infty} \sum_{n=-N}^{N} 4|e^{j3\pi n}|^2$$

$$e^{j3\pi} = \cos 3\pi + j\sin 3\pi$$

$$|\cos 3\pi + j\sin 3\pi| = \sqrt{\cos^2 3\pi + \sin^2 3\pi}$$

$$= 1$$

$$= 4 \lim_{N\to\infty} \sum_{-N}^{N} 1$$

$$= 4 \lim_{N\to\infty} (2N+1)$$

$$\boxed{E = \infty}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |2e^{j3\pi n}|^2$$

$$= 4 \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |e^{j3\pi n}|^2$$

$$= 4 \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$P = 4 \lim_{N \to \infty} \frac{1}{\cancel{(2N+1)}} \cancel{(2N+1)}$$

$$\boxed{P = 4}$$

The power is finite & Energy is infinite

so the given signal is a power signal.

$$x(n) = 8(0.5)^n u(n).$$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2$$

$$E = \sum_{n=0}^{\infty} |8(0.5)^n u(n)|^2$$

$$E = \sum_{n=0}^{\infty} 8^2 ((0.5)^2)^n.$$

$$E = \sum_{n=0}^{\infty} 64 (0.25)^n$$

$$E = 64 \sum_{n=0}^{\infty} (0.25)^n$$

$$E = 64 \times \frac{1}{1-0.25}$$

$$\boxed{E = 85.33}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \times 64 \sum_{n=0}^{\cancel{N}} (0.25)^n$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \times 85.33$$

$$P = \frac{1}{\infty} \times 85.33$$

$$P = \frac{1}{\frac{1}{0}} \times 85.33$$

$$\boxed{P = 0}$$

The given signal is a Energy signal.

Classify the following signals as energy signals, power signals or neither.

(a) $x(n) = 2^n u(-n)$ (b) $x(n) = \cos n\pi$ (c) $x(n) = e^{jn\pi}$

(d) $x(n) = (j)^n + (j)^{-n}$.

Solution

$u(-n) = 1$ for $n \le 0$

(i) $x(n) = 2^n u(-n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E = \sum_{n=-\infty}^{0} |2^n u(-n)|^2$$

$$E = \sum_{n=-\infty}^{0} (2^2)^n$$

$$E = \sum_{n=-\infty}^{0} (4)^n$$

put $n = -n$.

$$E = \sum_{n=0}^{\infty} (4)^{-n}$$

$$E = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

$$= \frac{1}{1-\frac{1}{4}}$$

$$\boxed{E = \frac{4}{3}}$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{0} |2^n u(-n)|^2$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{0} (2^{2n})$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{0} (4)^n$$

put $n = -n$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{\infty} 4^{-n}$$

$$\boxed{\sum_{n=0}^{N} a^n = \frac{1}{1-a}}$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \times \left(\frac{1}{1-\frac{1}{4}}\right)$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \times \frac{4}{3}$$

$$P = \frac{1}{\infty} \times \frac{4}{3}$$

$$\boxed{P = 0}$$

The given signal is a Energy Signal.

$x(n) = \cos n\pi$.

General form of cosine sequence is

$x(n) = \cos n\omega_0$

$$\boxed{\omega_0 = \pi}$$

Fundamental period $N = \frac{2\pi}{\omega_0} m$

$$N = \frac{2\pi}{\pi} \times m$$

$$\boxed{N = 2} \quad \text{when } m = 1$$

For periodic signal the expression for Period is

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

$$P = \frac{1}{2} \sum_{n=0}^{2-1} (\cos n\pi)^2$$

$$P = \frac{1}{2} \sum_{n=0}^{1} (\cos n\pi)^2$$

$$P = \frac{1}{2} \left[\cos^2 0 + \cos^2 \pi \right]$$

$$P = \frac{1}{2} [1 + 1]$$

$$P = \frac{1}{2} \times 2.$$

$$\boxed{P = 1}$$

(iii) $x(n) = e^{jn\pi}$

$x(n) = \cos n\pi + j \sin n\pi$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2$$

$$\sqrt{\cos^2\pi + \sin^2\pi}$$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} |\cos n\pi + j \sin n\pi|^{2}$$

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} 1$$

$$E = \lim_{N\to\infty} 2N+1$$

$$\boxed{E = \infty}$$

(iii) $$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{-N}^{N} |x(n)|^2$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{-N}^{N} 1$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \times (2N+1)$$

$$\boxed{P = 1}$$

The given signal is power signal

$$x(n) = (j)^n + (j)^{-n}$$

$$e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2}$$

$$= 0 + j\,1$$

$$\boxed{e^{j\frac{\pi}{2}} = j}$$

$$\boxed{e^{-j\frac{\pi}{2}} = -j}$$

$$x(n) = \left(e^{j\frac{\pi}{2}}\right)^n + \left(e^{j\frac{\pi}{2}}\right)^{-n}$$

$$x(n) = e^{jn\frac{\pi}{2}} + e^{-jn\frac{\pi}{2}}$$

$$x(n) = 2.\ \cos\frac{n\pi}{2}$$

$$\frac{e^{jx} + e^{-jx}}{2} = \cos x$$

$$\omega_0 = \frac{\pi}{2}$$

$$N = \frac{2\pi}{\omega_0} m$$

$$N = \frac{2\pi}{\frac{\pi}{2}} \times m$$

$$N = \frac{4\pi}{\pi} \times m$$

$$\boxed{N = 4}$$

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

$$P = \frac{1}{4} \sum_{n=0}^{3} \left|2 \cos\frac{n\pi}{2}\right|^2$$

$$P = \frac{4}{4} \sum_{n=0}^{3} \left|\cos\frac{n\pi}{2}\right|^2$$

$$P = 1 + \cos^2\frac{\pi}{2} + \cos^2\pi + \cos^2\frac{3\pi}{2}$$

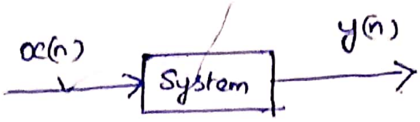
$$P = 1 + 0 + 1 + 0$$

$\boxed{P = 2}$

The given signal is a power signal.

System

* A System is defined as a physical device that generates a response or output signal, for a given input Signal.



Mathematically $y(n) = T[x(n)]$

The input $x(n)$ is transformed to $y(n)$

A discrete-time system is one which operates on a discrete time input signal and produces a discrete-time output signal.

Types of Systems.

The systems are classified into

1. Causal and Non-Causal System.
2. Linear and Non linear system.
3. Time Variant and Time-invariant system.
4. Stable and Unstable System.
5. Static and Dynamic system.
6. Invertible and Inverse Systems.

1. Causal and Non- Causal Systems.

Causal System:-

A system is said to be causal if the present value of the output signal depends only on the present and/or past values of the input signal.

(Eg) $y(n) = x(n) + \frac{1}{2} x(n-1)$.

Anticausal or Non- Causal System:-

A system is said to be anticausal if the present value of the output signal depends only on the future values of the input signal.

(Eg) $y(n) = x(n^2)$

Problems:-

Determine whether the following systems are causal or not.

(i) $y(n) = x(n) + x(n-1) + x(n-2)$ (ii) $y(n) = x(2n+2)$

(iii) $y(n-1) = x(n)$ (iv) $y(n) = \cos[x(n)]$ (v) $y(n) = \sum_{k=-\infty}^{n+10}$

Solution:-

$y(n) = x(n) + x(n-1) + x(n-2)$

put $n=0$

$y(0) = x(0) + x(-1) + x(-2)$

put $n=1$

$y(1) = x(1) + x(0) + x(-1)$

The output depends on the present Value and the past value

So the System is Causal.

(ii) $y(n) = x(2n+2)$

Solution:-

$y(n) = x(2n+2)$

put $n=0$

$y(0) = x(2)$

The output depends on the future Values of the input.

The System is Non-Causal.

(iii) $y(n-1) = x(n)$

Solution:-

$y(n-1) = x(n)$

put $n=0$

$y(-1) = x(0)$

put $n=1$

$y(0) = x(1)$

The output depends on the future input. The System is Non-causal.

(iv) $y(n) = \cos x(n)$

Solution

$y(n) = \cos\boxed{x(n)}$

put $n=0$

$y(0) = \cos x(0)$

The output depends on the present input.

So the system is causal.

(v) $$y(n) = \sum_{k=-\infty}^{n+10} x(k)$$

Solution

$$= x[-\infty] + x[-\infty+1] + \dots + x(-1) + x(0) + x(1) + x(2)$$

The output depends on past, present and future inputs.

Hence the system is Non causal.

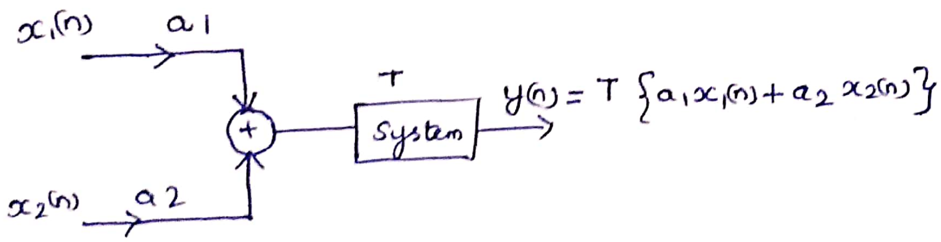
Linear and Non Linear Systems.

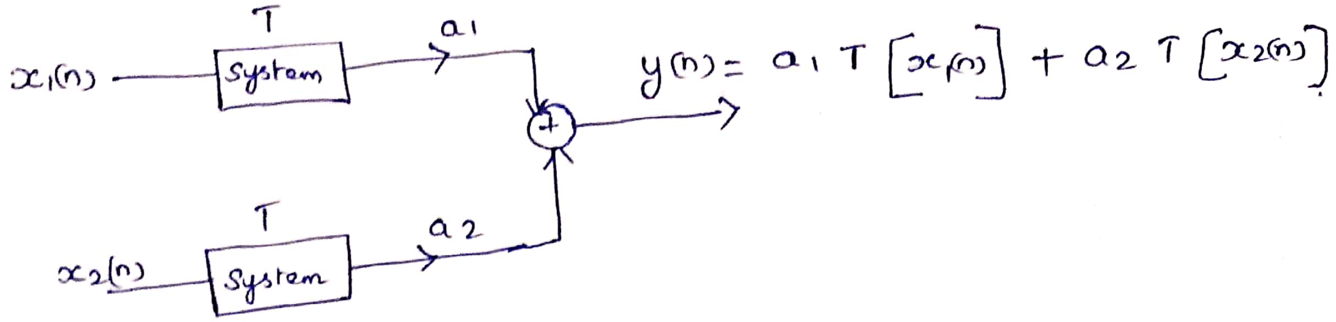
* A system is said to be linear if it satisfies the principle of superposition.

* Let $x_1(n)$ and $x_2(n)$ be the two input sequences. Then the system is said to be linear if and only if.

$$T\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 T[x_1(n)] + a_2 T[x_2(n)].$$

Here a_1 and a_2 are arbitrary constansts.





Check whether the following systems are linear or not.

(i) $y(n) = x^2(n)$ (ii) $y(n) = x(n^2)$ (iii) $y(n) = x(4n+1)$

(iv) $y(n) = x(n) + \dfrac{1}{x(n-1)}$ (v) $y(n) = x(n) + n\,x(n+1)$

(vi) $y(n) = n\,x(n)$

Solution

(i) $y(n) = x^2(n)$

The response due to linear combination of input is

$$a_1 y_1(n) = a_1 x_1^2(n)$$

$$a_2 y_2(n) = a_2 x_2^2(n)$$

The weighted sum of output $y_3(n)$ is

$$y_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

L.H.S. $y_3(n) = a_1 x_1^2(n) + a_2 x_2^2(n)$ —— (1)

R.H.S The output due to the weighted sum of input is

$$y_3(n) = [a_1 x_1(n) + a_2 x_2(n)]^2.$$

$$y_3(n) = a_1^2 x_1^2(n) + a_2^2 x_2^2(n) + 2a_1a_2 x_1(n) x_2(n) \quad \text{—— (2)}$$

$① \neq ②$

The System is non-linear.

(ii) $y(n) = x(n^2)$

The response due to linear combination of inputs is

$$a_1 y_1(n) = a_1 x_1(n^2)$$
$$a_2 y_2(n) = a_2 x_2(n^2)$$

L.H.S

The weighted sum of output $y_3(n)$ is

$$y_3(n) = a_1 y_1(n) + a_2 y_2(n)$$
$$y_3(n) = a_1 x_1(n^2) + a_2 x_2(n^2) \quad — ①$$

R.H.S

The output $y_3(n)$ due to weighted sum of input is

$$y_3(n) = a_1 x_1(n^2) + a_2 x_2(n^2) \quad — ②$$

$$① = ②$$

The System is Linear.

(iii) $y(n) = x(4n+1)$

The response due to linear combination of inputs is

$$a_1 y_1(n) = a_1 x_1(4n+1)$$
$$a_2 y_2(n) = a_2 x_2(4n+1)$$

L.H.S

The weighted. Sum of output $y_3(n)$ is

$y_3(n) = a_1 y_1(n) + a_2 y_2(n)$

$y_3(n) = a_1 x_1(4n+1) + a_2 x_2(4n+1)$ — ①

R.H.S

The output due to weighted sum of Input is

$y_3(n) = a_1 x_1(4n+1) + a_2 x_2(4n+1)$ — ②

① = ②

The system is linear.

(iv) $y(n) = x(n) + \dfrac{1}{x(n-1)}$

The response due to linear combination of inputs is

$a_1 y_1(n) = a_1 \left[x_1(n) + \dfrac{1}{x_1(n-1)} \right]$

$a_2 y_2(n) = a_2 \left[x_2(n) + \dfrac{1}{x_2(n-1)} \right]$

L.H.S

The weighted Sum of output $y_3(n)$ is

$y_3(n) = a_1 y_1(n) + a_2 y_2(n)$

$y_3(n) = a_1 \left[x_1(n) + \dfrac{1}{x_1(n-1)} \right] + a_2 \left[x_2(n) + \dfrac{1}{x_2(n-1)} \right]$ — ①

R.H.S

$$y_3(n) = a_1 x_1(n) + a_2 x_2(n)$$

$$= a_1 \quad x_1(n) + \frac{1}{a_1 x_1(n-1)} + a_2 \quad x_2(n) + \frac{1}{a_2 x_2(n-1)}$$

$$y_3(n) = a_1 x_1(n) + a_2 x_2(n) + \frac{1}{a_1 x_1(n-1)} + \frac{1}{a_2 x_2(n-1)}$$

L.H.S $\neq$ R.H.S

The system is non-linear.

(v) $y(n) = x(n) + n\, x(n+1)$

The response due to linear combination of inputs is

$$a_1 y_1(n) = a_1 \left[x_1(n) + n \; x_1(n+1)\right]$$

$$a_2 y_2(n) = a_2 \left[x_2(n) + n \; x_2(n+1)\right]$$

L.H.S

The weighted sum of the output is

$$y_3(n) = a_1 \; y_1(n) + a_2 y_2(n)$$

$$y_3(n) = a_1 \left[x_1(n) + n\, x_1(n+1)\right] + a_2 \left[x_2(n) + n\, x_2(n+1)\right]$$

R.H.S. The output due to weighted sum of input is

$$y_3(n) = a_1 x_1(n) + a_2 x_2(n)$$

$$y_3(n) = a_1 \left[x_1(n) + n\, x_1(n+1)\right) + a_2 \left[x_2(n) + n\, x_2(n+1)\right)$$

L.H.S = R.H.S

The System is linear.

(vi) $y(n) = n x(n)$

$y_1(n) = a_1 n x_1(n)$

$y_2(n) = a_2 n x_2(n)$

$y_3(n) = y_1(n) + y_2(n)$

L.H.S

$y_3(n) = a_1 n x_1(n) + a_2 n x_2(n)$

R.H.S =

$y_3(n) = a_1 x_1(n) + a_2 x_2(n)$

$y_3(n) = a_1 n x_1(n) + a_2 n x_2 n$

L.H.S = R.H.S

The System is linear.

Time Invariant and Time Variant Systems.

* A system is said to be time invariant if a time delay or time advance of the input signal leads to a identical time shift in the output signal.

Steps to find the Time invariance.

(i) For the delayed input $x(n-k)$ find the output $y(n,k)$

(ii) Determine the delayed output $y(n-k)$ by substituting $n = n-k$ in $y(n)$.

(iii) If $y(n,k) = y(n-k)$, the system is Time invariant. otherwise the system is time variant.

Determine whether the following systems are time invariant / Variant systems.

(i) $y(n) = x(-n)$ (ii) $y(n) = x(n) + x(n-1)$ (iii) $y(n) = n\,x(n)$

(iv) $y(n) = \cos[x(n)]$ (v) $y(n) = x(2n)$ (vi) $y(n) = x(n)\,x(n-1)$

Solution:-

(i) $y(n) = x(-n)$

To prove $y(n,k) = y(n-k)$

$$y(n) = T[x(n)].$$

If the input is delayed k units in time

$$y(n,k) = T[x(n-k)]$$

$$= T[x(-n-k)]$$

$$y(n,k) = x(-n-k)$$

If the output is delayed by K units.

$$y(n-k) = x(-(n-k))$$

$$y(n-k) = x(-n+k)$$

$$y(n,k) \neq y(n-k).$$

The system is time variant.

(ii) $y(n) = n\,x(n)$

To prove $y(n,k) = y(n-k)$

If input is delayed by K units of time

$$y(n,k) = n\,(x(n-k)) \quad \text{— ①}$$

If the output is delayed by K units of time.

$$y(n-k) = (n-k)\;x(n-k). \quad \text{— ②}$$

① ≠ ②

The System is time variant.

iii) $y(n) = x(n) + x(n-1)$

To prove $y(n,k) = y(n-k)$.

If the input is delayed by K units of time.

$$y(n,k) = x(n-k) + x(n-1-k) \quad \text{— ①}$$

If the output is delayed by K units of time.

$$y(n-k) = x(n-k) + x(n-k-1) \quad \text{— ②}$$

① = ②

The system is time invariant.

(iv) $y(n) = \cos[x(n)]$

If the input is delayed by k units of time.

$y(n,k) = \cos[x(n-k)]$ —— ①

If the output is delayed by k units of time.

(v) $y(n) = x(2n)$

If the input is delayed by k units of time

$y(n,k) = x(2n-k)$ —— ①

If the output is delayed by k units of time

$y(n-k) = x(2(n-k))$.

$y(n-k) = x(2n-2k)$ —— ②

① ≠ ②

The System is time variant.

(vi) $y(n) = x(n)\ x(n-1)$

If input is delayed by k units of time.

$y(n,k) = x(n-k)\ x(n-k-1)$ —— ①

If output is delayed by k units of time.

$y(n-k) = x(n-k)\ x(n-k-1)$ —— ②

① = ②

The System is time invariant.

Stable and Unstable systems.

* When the every bounded input produces a bounded output, then the system is called Bounded Input Bounded Output Stable (BIBO).

* The input $x(n)$ is said to be bounded if there exists some finite number M_x such that

$$|x(n)| \leq M_x < \infty$$

Similarly output $y(n)$ is bounded if there exists some finite number M_y such that

$$|y(n)| \leq M_y < \infty$$

* If the output is unbounded for any bounded input, then the system is unstable.

Static and Dynamic Systems.

* A static system is memoryless system.
* It has no storage devices.
* Its output signal depends on present values of the input signal.

(eg) $i(t) = \frac{1}{R} V(t)$

* A dynamic system posses memory.
* It has storage devices.
* A system is said to possess memory if its output signal depends on past values of the input signal.

Invertible and Inverse System.

A system is said to be invertible if the input to the system may be uniquely determined from the output.
In order to have a system to be invertible, it is necessary that distinct input produces distinct outputs.
If a system is invertible then an inverse system exists.

Causal and Non causal System:

A system is said tobe causal, if the present value of the output signal depends only on the present value or past value or combination of present and past values of the input signal.

If the present value of the output signal depends on future value of the input signal then system is non-causal.

Time Variant and Time Invariant Systems.

* A system is time-invariant if a time shift in the input signal produces an identical time shift in the output signal.

* A system is time variant if a time shift in the input signal produces different time shift in the output signal.

Linear and Non linear system :-

* A System is linear if it follows the principle of Superposition.

$$T\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 T\{x_1(n)\} + a_2 T\{x_2(n)\}$$

Determine whether the following systems are static / dynamic.

(i) $y(n) = x(n) - x(n-1)$ (ii) $y(n) = \cos[x(n)]$ (iii) $y(n) = x[3n]$

(iv) $y(n) = x(n)\, x(n-1)$ (v) $y(n) = x^3(n) + x(n)$

Solution:-

(i) $y(n) = x(n) - x(n-1)$

put $n=0$

$y(0) = x(0) - x(-1)$

put $n=1$

$y(1) = x(1) - x(0)$

The output depends on the past input which requires memory.

So the system is dynamic.

(ii) $y(n) = \cos[x(n)]$

put $n=0$

$y(0) = \cos x(0)$

put $n=1$

$y(1) = \cos x(1)$

The output depends on the present input only

So the system is static.

(iii) $y(n) = x[3n]$

put $n=0$

$y(0) = x[0]$

put $n=1$

$y(1) = x[3]$

put $n=-1$

$y(-1) = x[-3]$

The output depends on the future input and the past input so the system is dynamic.

(iv) $y(n) = x(n)\, x(n-1)$

put $n=0$

$y(0) = x(0)\, x(-1)$

put $n=1$

$y(1) = x(1)\, x(0)$

put $n=-1$

$y(-1) = x(-1)\, x(-2)$

The output depends on the past input

The system is dynamic.

(v) $y(n) = x^3(n) + x(n)$

put $n=0$

$y(0) = x^3(0) + x(0)$

put $n=0$

$y(1) = x^3(1) + x(1)$

put $n=-1$

$y(-1) = x^3(-1) + x(-1)$

The output depends on the present input only.

The system is static.

Necessary and Sufficient condition for stability of a System

Let $x(n)$ be a bounded input sequence, $h(n)$ be the impulse response of the system and $y(n)$ be the output sequence.

The output of the system $y(n)$ can be found by using convolution sum.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

or

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k).$$

The magnitude of the output is given by.

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \right| \quad \text{—— (1)}$$

we know that the magnitude of the sum of terms is less than or equal to the sum of the magnitudes. hence.

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)|\, |x(n-k)| \quad \text{—— (2)}$$

Let the bounded value of the input is equal to M, the (2) can be written as.

$$|y(n)| \leq M \sum_{k=-\infty}^{\infty} h(k)$$

The above condition will be satisfied when

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

So, the necessary and sufficient condition for stability is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Test the stability of the systems given below.

(i) $y(n) = e^{x(n)}$ (ii) $h(n) = \left(\frac{1}{2}\right)^n u(n)$ (iii) $y(n) = \cos x(n)$

(iv) $y(n) = \sum_{k=0}^{n+1} x(k)$ (v) $h(n) = 3^n u(n+3)$ (vi) $h(n) = 3^n u(n-3)$

(vii) $h(n) = 2^n u(-n)$ (viii) $y(n) = x(n-1) + x(n) + x(n+1)$.

Solution:-

(i) $y(n) = e^{x(n)}$

The necessary and sufficient condition for stability is

$$\sum_{n=-\infty}^{\infty} y(n) < \infty \text{ for given value of input.}$$

For $x(n)$ bounded, $e^{x(n)}$ is bounded for all values of n

and the system is stable.

(ii) $h(n) = \left(\frac{1}{2}\right)^n u(n)$

The condition for stability is $\sum_{n=-\infty}^{\infty} h(n) < \infty$

$$\sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n)$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{1-\frac{1}{2}}$$

$$= \frac{1}{\frac{1}{2}}$$

$$= 2$$

$2 < \infty$

The system is stable.

(iii) $y(n) = \cos x(n)$

If $x(n)$ is bounded then $\cos x(n)$ is also bounded.

The system is stable.

(iv) $y(n) = \sum_{k=0}^{n+1} x(k)$

when n is finite, the system is stable.

when n is infinite, the output $y(n)$ is also infinite and the output is unbounded. The system is unstable

(v) $h(n) = 3^n u(n+3)$

$$h(n) = \sum_{n=-\infty}^{\infty} 3^n u(n+3)$$

$$= \sum_{n=-3}^{\infty} 3^n$$

$$= 3^{-3} + 3^{-2} + 3^{-1} + 3^{0} + 3^{1} + 3^{2} + 3^{3} + \dots + 3^{\infty}$$

$$h(n) = \infty$$

The output is unbounded

The system is unstable.

(vi) $h(n) = 3^n u(n-3)$

$$= \sum_{n=-\infty}^{\infty} 3^n u(n-3)$$

$$= \sum_{3}^{\infty} 3^n$$

$$= 3^3 + 3^4 + \dots + 3^{\infty}$$

$$h(n) = \infty$$

The output is unbounded so the system is unstable.

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(vii) $h(n) = 2^n u(-n)$

$$= \sum_{n=-\infty}^{\infty} 2^n u(-n)$$

$$= \sum_{n=-\infty}^{0} 2^n$$ put $n = -n$

$$= \sum_{n=0}^{\infty} 2^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{1-\frac{1}{2}}$$

$$= \frac{1}{\frac{1}{2}}$$

$$= 2$$

$2 < \infty$

The system is stable.

(viii) $y(n) = x(n-1) + x(n) + x(n+1)$

consider $x(n)$ as impulse input.

$y(n) = \delta(n-1) + \delta(n) + \delta(n+1)$

when $n = 0$

$y(0) = \delta(-1) + \delta(0) + \delta(1)$

$y(0) = 0 + 1 + 0$

$\boxed{y(0) = 1}$

when $n = 1$

$y(1) = \delta(0) + \delta(1) + \delta(2)$

$= 1 + 0 + 0$

$\boxed{y(1) = 1}$

when $n = 2$

$y(2) = \delta(1) + \delta(2) + \delta(3)$

$\boxed{y(2) = 0}$

when $n = -1$

$y(-1) = \delta(-2) + \delta(-1) + \delta(0)$

$= 0 + 0 + 1$

$\boxed{y(-1) = 1}$

For all the possible values of input conditions.

$$y(n) = \sum_{K=-\infty}^{\infty} |h(K)| < \infty$$

$$y(n) = 1+1+1+0 \cdots$$

$$y(n) < \infty$$

The system is stable.

Determine whether the system $y(n) = x(-n-2)$ is

(i) Causal (ii) Linear (iii) Dynamic (iv) Time invariant.

(v) Stable.

Solution

(i) To check whether $y(n) = x(-n-2)$ is Causal or anticausal

$$y(n) = x(-n-2)$$

For $n=0$

$$y(0) = x(-2)$$

For $n=1$

$$y(1) = x(-1-2)$$

$$y(1) = x(-3)$$

For $n=-1$

$$y(-1) = x(-(-1)-2)$$

$$= x(1-2)$$

$$y(-1) = x(-1)$$

For $n=-2$

$$y(-2) = x(-(-2)-2)$$

$$= x(2-2)$$

$$y(-2) = x(0)$$

For $n=-3$

$$y(-3) = x(-(-3)-2)$$

$$y(-3) = x(3-2)$$

$y(-3) = x(1)$

Since the output depends upon past and future inputs the System is non causal.

(ii) To check whether the Signal is Linear or non Linear.

The System is linear if and only if

$$T\left[a_1 x_1(n) + a_2 x_2(n)\right] = a_1 T\left[x_1(n)\right] + a_2 T\left[x_2(n)\right]$$

$$y(n) = x(-n-2)$$

The response due to linear combination input is

$$a_1 y_1(n) = a_1 x_1(-n-2)$$

$$a_2 y_2(n) = a_2 x_2(-n-2)$$

L.H.S

The weighted Sum of output $y_3(n)$ is

$$y_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

$$y_3(n) = a_1 x_1(-n-2) + a_2 x_2(-n-2) \quad \text{— (1)}$$

R.H.S

The output $y_3(n)$ due To weighted Sum of input is

$$y_3(n) = a_1 x_1(-n-2) + a_2 x_2(-n-2) \quad \text{— (2)}$$

$(1) = (2)$

The given system is linear.

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(iii) To check whether the system is Static or Dynamic.

$$y(n) = x(-n-2)$$

For $n=0$

$$y(0) = x(-2)$$

For $n=1$

$$y(1) = x(-1-2)$$

$$y(1) = x(-3)$$

Since the present output depends on past inputs
The system is Dynamic.

(iv) To check whether the system is Time Invariant or Time Variant.

A system is said to be Time-Invariant if and only if $y(n,k) = y(n-k)$

$$y(n) = x(-n-2)$$

If the input is delayed by k-Samples.

$$y(n,k) = x(-n-k-2) \quad \text{--- (1)}$$

If the output is delayed by k-Samples.

$$y(n-k) = x(-(n-k)-2)$$

$$y(n-k) = x(-n+k-2) \quad \text{--- (2)}$$

$① \neq ②$.

$y(n,k) \neq y(n-k)$

The system is Time Variant.

(V) <u>To check whether the system is stable or not.</u>

The condition for stability is $y(n) < \infty$

$$y(n) = x(-n-2)$$

If we substitute any finite value for n the output $y(n)$ will be also finite.

So the system is Stable.

Determine whether the following signals are Energy or Power Signals.

(i) $x(n) = \left(\frac{1}{4}\right)^n u(n)$ (ii) $x(n) = \sin\left(\frac{n\pi}{3}\right)$

<u>Solution:-</u>

(i) $x(n) = \left(\frac{1}{4}\right)^n u(n)$

The Energy of the Signal $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$.

$$E = \sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{4}\right)^n u(n)\right|^2 \qquad u(n) = 1 \text{ for } n \geq 0$$

$$E = \sum_{n=0}^{\infty} \left|\left(\frac{1}{4}\right)^n\right|^2$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$E = \sum_{n=0}^{\infty} \left(\left(\frac{1}{4}\right)^2\right)^n$$

$$E = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n$$

$$E = \frac{1}{1-\frac{1}{16}}$$

$$E = \frac{1}{\frac{15}{16}}$$

$$\boxed{E = \frac{16}{15}}$$

power of the signal $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\left(\frac{1}{4}\right)^n u(n)\right|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left|\left(\frac{1}{4}\right)^n\right|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \cdot \sum_{n=0}^{N} \left(\left(\frac{1}{4}\right)^2\right)^n$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{16}\right)^n$$

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}\left[\frac{1-\left(\frac{1}{16}\right)^{N+1}}{1-\left(\frac{1}{16}\right)}\right]$$

$$= \frac{1}{2\times\infty+1}\left[\frac{1-\left(\frac{1}{16}\right)^{\infty}}{1-\left(\frac{1}{16}\right)}\right]$$

$$= \frac{1}{\infty}\left[\frac{1-0}{\frac{15}{16}}\right]$$

$$= \frac{1}{\infty} \times \frac{16}{15}$$

$$= \frac{1}{\frac{1}{0}} \times \frac{16}{15}$$

$$= 0 \times \frac{16}{15}$$

$$\boxed{P = 0}$$

The Energy is finite and power is zero.

The given Signal is an Energy Signal.

(ii) $x(n) = \sin\left(\frac{n\pi}{3}\right)$

The Energy of the Signal $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$$E = \sum_{n=-\infty}^{\infty} \sin^2\left(\frac{n\pi}{3}\right)$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$E = \sum_{n=-\infty}^{\infty} \left[\frac{1 - \cos \frac{2\pi}{3} n}{2} \right]$$

$$E = \frac{1}{2} \left[\sum_{n=-\infty}^{\infty} 1 - \sum_{n=-\infty}^{\infty} \cos \frac{2\pi}{3} n \right]$$

$$E = \frac{1}{2} \left[\sum_{n=-\infty}^{\infty} 1 - 0 \right]$$

$$E = \frac{1}{2} \times \infty$$

$$\boxed{E = \infty}$$

power of the Signal

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| \sin^2 \frac{\pi}{3} n \right|$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1 - \cos\left(\frac{2\pi}{3} n\right)}{2}$$

$$P = \frac{1}{2} \lim_{N \to \infty} \frac{1}{2N+1} \left[\sum_{n=-N}^{N} 1 - \sum_{n=-N}^{N} \cos\left(\frac{2\pi}{3} n\right) \right]$$

$$P = \frac{1}{2} \lim_{N \to \infty} \frac{1}{2N+1} \left[\sum_{n=-N}^{N} 1 - 0 \right]$$

$$P = \frac{1}{2} \lim_{N \to \infty} \frac{1}{\cancel{(2N+1)}} \times \cancel{(2N+1)}$$

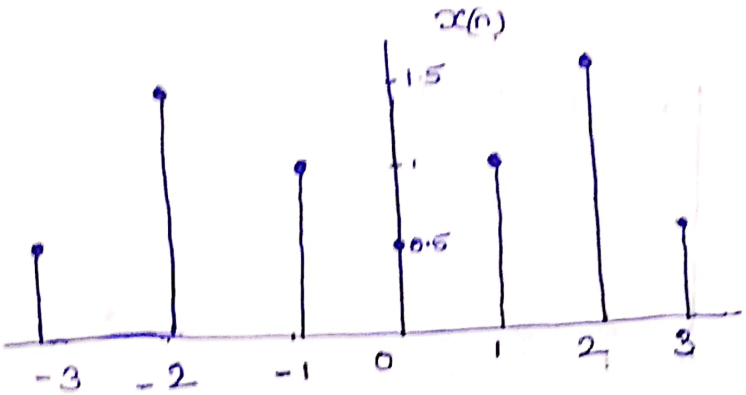
$$\boxed{P = \frac{1}{2}}$$

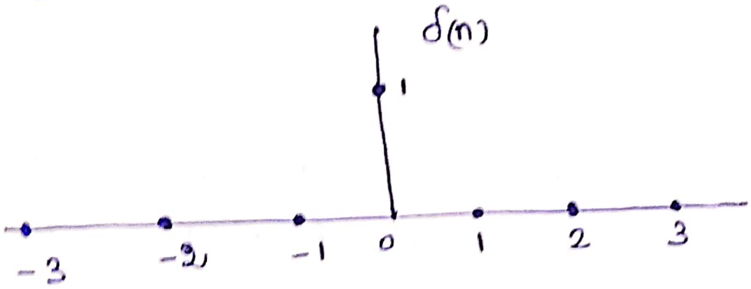
The Energy is infinite and power is finite.
Hence the Signal is a Power Signal.

Representation of an Arbitrary Sequence.

* Any arbitrary sequence $x(n)$ can be represented in terms of delayed and scaled impulse sequence $\delta(n)$.

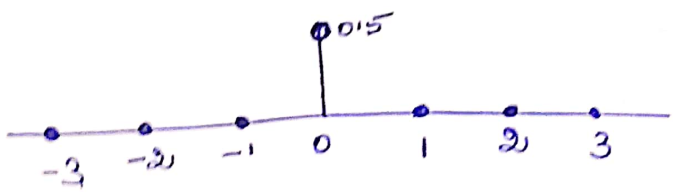
Let $x(n)$ be an infinite sequence as shown in figure.





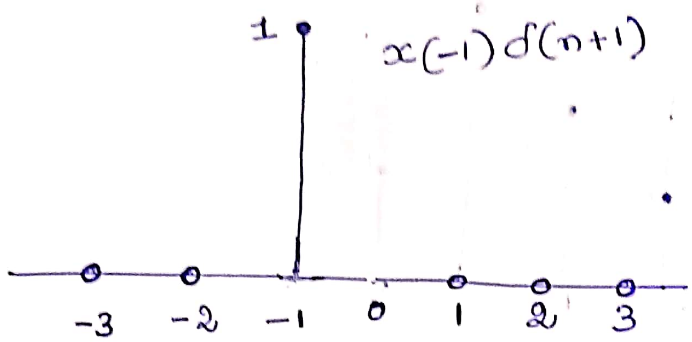
The sample $x(0)$ can be obtained by multiplying $x(0)$, the magnitude, with unit impulse $\delta(n)$

ie $x(0)\,\delta(n) = \begin{cases} x(0) & \text{for } n=0 \\ 0, & \text{for } n\neq 0 \end{cases}$



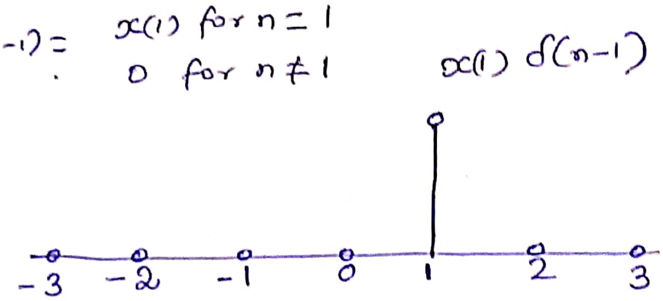
Similarly, the sample $x(-1)$ can be obtained by multiplying $x(-1)$ & the magnitude, with one sample advanced unit impulse $\delta(n+1)$

i.e $$x(-1)\,\delta(n+1) = \begin{cases} x(-1) & \text{for } n=-1 \\ 0 & \text{for } n \neq -1 \end{cases}$$



The sample $x(1)$ can be obtained by multiplying $x(1)$ the magnitude, with one sample delayed unit impulse $\delta(n-1)$

i.e $x(1)\,\delta(n-1) =$ $x(1)$ for $n=1$
0 for $n \neq 1$



The sum of the sequences are given by

$$x(-3)\,\delta(n+3) + x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + x(3)\,\delta(n-3)$$

$$x(n) = \ldots x(-3)\,\delta(n+3) + x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + x(3)\,\delta(n-3) + \ldots$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

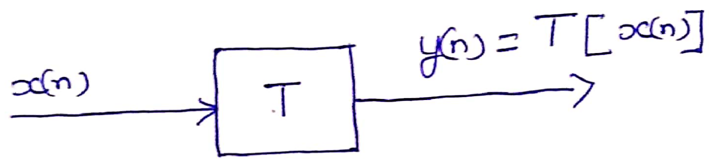
where $\delta(n-k)$ is unity for $n=k$ and zero for all other terms.

Impulse Response and Convolution Sum.

A discrete-time system performs an operation on an input signal based on a predefined criteria to produce a modified output signal.

The input signal $x(n)$ is the system excitation, and $y(n)$ is the system response.

This transform operation is shown figure below.



If the input to the system is unit impulse ie $x(n) = \delta(n)$, then the output of the system is known as impulse response denoted by $h(n)$ where

$$h(n) = T[\delta(n)]$$

we know that any arbitrary sequence $x(n)$ can be represented as a weighted sum of discrete impulses.

Now the system response is given by

$$y(n) = T[x(n)]$$

$$= T\left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right]$$

For linear System.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, T[\delta(n-k)] \quad \text{— (1)}$$

The response to the shifted impulse sequence can be denoted by $h(n,k)$ defined as

$$h(n,k) = T[\delta(n-k)]$$

For a time invariant system

$$h(n,k) = h(n-k)$$

$$T[\delta(n-k)] = h(n-k) \quad \text{— (2)}$$

sub (2) in (1)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

For a linear time invariant system, if the input sequence $x(n)$ and impulse response $h(n)$ are given, we can find the output $y(n)$ by using the equation.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

which is known as the convolution sum and can be represented as

$$y(n) = x(n) * h(n)$$

where $*$ denotes the convolution operation.

Convolution is an important property for analysis of system output for the given input since convolution of signals in time domain is equivalent to multiplication of two signals in frequency domain

The convolution sum of two Sequence. can be found by using the following steps.

Step 1:

Choose an initial value of n, the starting time for evaluating the output sequence $y(n)$.

If $x(n)$ starts at $n=n_1$ and $h(n)$ starts at $n=n_2$ then $n=n_1+n_2$ is a good choice.

Step 2:- Express both sequences in terms of the index k.

Step 3:- Folding

Fold $h(k)$ about $k=0$ to obtain $h(-k)$.

Step 4:- Shifting

Shift $h(-k)$ by n to the right if n is positive and left if n is negative To obtain $h(n-k)$

Step 5:- Multiplication.

Multiply $x(k)$ and $h(n-k)$ element by element.

Step 6 :- Summation.

Sum all the values of the product sequence. To get $y(n)$

Step 7 :- Increment the index n, shift the sequence $h(n-k)$ to right by one sample and do step 5 & 6

Step 8 :- Repeat step 7 until the sum of products is zero for all the remaining values of n.

Properties of Convolution

(i) Commutative law.

$$x(n) * h(n) = h(n) * x(n)$$

(ii) Associative law

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

(iii) Distributive law.

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

Find the Convolution Sum of the two Sequences $x_1(n)$ and $x_2(n)$ given below

$$x_1(n) = \{1, 2, 3\}$$
$$x_2(n) = \{2, 1, 4\}$$

Solution:-

Here we have to convolute two finite duration sequences.

$$x_1(n) = 1 \cdot \delta(n) + 2\,\delta(n-1) + 3\,\delta(n-2)$$
$$x_2(n) = 2 \cdot \delta(n) + 1 \cdot \delta(n-1) + 4\,\delta(n-2)$$

$$y(n) = x_1(n) * x_2(n)$$

$$= \left[\delta(n) + 2\,\delta(n-1) + 3\,\delta(n-2)\right] * \left[2\delta(n) + \delta(n-1) + 4\delta(n-2)\right]$$

$$= 2\delta(n) * \delta(n) + 4\,\delta(n-1) * \delta(n) + 6\,\delta(n-2) * \delta(n) + \delta(n-1) * \delta(n) +$$
$$2\,\delta(n-1) * \delta(n-1) + 3\,\delta(n-2) * \delta(n-1) + 4\,\delta(n-2) * \delta(n) +$$
$$8\,\delta(n-2) * \delta(n-1) + 12\,\delta(n-2) * \delta(n-2)$$

$$= 2\,\delta(n) + 4\,\delta(n-1) + 6\,\delta(n-2) + \delta(n-1) + 4\,\delta(n-2) + 8\,\delta(n-3)$$
$$+ 2\delta(n-2) + 3\,\delta(n-3) + 12\,\delta(n-4)$$

$$= 2\,\delta(n) + 5\,\delta(n-1) + 12\,\delta(n-2) + 11\,\delta(n-3) + 12\,\delta(n-4)$$

or

$$y(n) = \{\underset{\uparrow}{2}, 5, 12, 11, 12\}$$

Compute the convolution of two sequences $x_1(n)$ and $x_2(n)$ given below.

$$x_1(n) = (\underset{\uparrow}{1}, 2, 3) \qquad x_2(n) = (1, 2, \underset{\uparrow}{3}, 4)$$

$$x_1(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2)$$

$$x_2(n) = 1\cdot\delta(n+2) + 2\delta(n+1) + 3\delta(n) + 4\delta(n-1)$$

$$y(n) = x_1(n) * x_2(n)$$

$$y(n) = \left[\delta(n) + 2\delta(n-1) + 3\delta(n-2)\right]\left[\delta(n+2) + 2\delta(n+1) + 3\delta(n) + 4\delta(n-1)\right]$$

$$= \delta(n+2)*\delta(n) + 2\delta(n+2)*\delta(n-1) + 3\delta(n-2)*\delta(n+2) +$$
$$2\delta(n+1)*\delta(n) + 4\delta(n+1)*\delta(n-1) + 6\delta(n-2)*\delta(n+1) + 3\delta(n)*\delta(n)$$
$$+ 6\delta(n-1)*\delta(n) + 9\delta(n-2)*\delta(n) + 4\delta(n-1)*\delta(n) + 8\delta(n-1)*\delta(n)$$
$$+ 12\delta(n-2)*\delta(n-1)$$

$$= \delta(n+2) + 2\delta(n+1) + 3\delta(n) + 2\delta(n+1) + 4\delta(n) + 6\delta(n-1)$$
$$+ 3\delta(n) + 6\delta(n-1) + 9\delta(n-2) + 4\delta(n-1) + 8\delta(n-2) +$$
$$12\delta(n-3)$$

$$y(n) = \delta(n+2) + 4\delta(n+1) + 10\delta(n) + 16\delta(n-1) + 17\delta(n-2) + 12\delta(n-3)$$

$$y(n) = (1, 4, \underset{\uparrow}{10}, 16, 17, 12)$$

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Alternate Method.

Compute the convolution of two sequences $x_1(n)$ and $x_2(n)$ given below.

$$x_1(n) = (\underset{\uparrow}{1}, 2, 3) \quad \text{and} \quad x_2(n) = (1, 2, \underset{\uparrow}{3}, 4)$$

Solution:-

$$y(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

The sequence $x_1(n)$ starts at $n=0$, and $x_2(n)$ starts at $n=-2$.

The starting time of evaluating The output sequence $y(n) = n_1 + n_2$

$$= 0 - 2$$

$$= -2$$

$$y(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

for $n=0$

$$y(0) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(-k)$$

$x_1(k)$:		$\downarrow$ 1	2	3
$x_2(\ )$:	4	3	2	1

$1\times3 + 2\times2 + 3\times1$

$3+4+3$

$$\boxed{y(0) = 10}$$

For $n = 1$

$$y(1) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(1-k)$$

$x_1(k)$	:	1	2	3	
$x_2(1-k)$	:	4	3	2	1

$y(1)$ $\quad 1\times4 + 2\times3 + 3\times2$

$4 + 6 + 6$

$$\boxed{y(1) = 16}$$

For $n = 2$

$$y(2) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(2-k)$$

$x_1(k)$	:	1	2	3		
$x_2(2-k)$	:		4	3	2	1

$y(2)$ $\quad 2\times4 + 3\times3$

$y(2) = 8 + 9$

$$\boxed{y(2) = 17}$$

For $n = 3$

$$y(3) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(3-k)$$

$\downarrow$

$x_1(k)$:	1	2	3		
$x_2(3-k)$:			4	3	2	1
$y(3)$			3×4		

$$\boxed{y(3) = 12}$$

For $n \geq 4$ the Sequence $x_1(k)$ and $x_2(n-k)$ do not overlap

Hence $y(n) = 0$

For $n = -1$

$$y(-1) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(-1-k)$$

$x_1(k)$:			1	2	3
$x_2(-1-k)$:	4	3	2	1	
$y(-1)$			$1\times2 + 2\times1$		

$2+2$

$$\boxed{y(-1) = 4}$$

$n = -2$

$$y(-2) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(-2-k)$$

$x_1(k)$:				$\downarrow$ 1	2	3
$x(-2-k)$	4	3	2	1		
$y(-2)$				1×1		

$$\boxed{y(-2) = 1}$$

For $n \leq -3$ the sequence $x_1(k)$ and $x_2(n-k)$ do not overlap and hence $y(n) = 0$.

Therefore $y(n) = \{1, 4, \underset{\uparrow}{10}, 16, 17, 12\}$

Determine the convolution sum of two Sequences.

$x(n) = \{3, 2, 1, 2\}$ $h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$

Solution:-

$x(n) = \underset{\uparrow}{3}, 2, 1, 2$

$h(n) = 1, \underset{\uparrow}{2}, 1, 2$

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

For $n=0$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k)$$

$x(k)$:			$\downarrow$ 3	2	1	2
$h(-k)$	2	1	2	1		

$y(0)$ $\quad 3\times2 + 2\times1$

$6+2$

$$\boxed{y(0) = 8}$$

For $n=1$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k)$$

$x(k)$:		$\downarrow$ 3	2	1	2
$h(1-k)$	2	1	2	1	

$y(1)$ $\quad 3\times1 + 2\times2 + 1\times1$

$= 3+4+1$

$$\boxed{y(1) = 8}$$

for $n=2$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)\, h(2-k)$$

$$\begin{array}{llllll} x(k): & \overset{\downarrow}{3} & 2 & 1 & 2 & \\ h(2-k) & 2 & 1 & 2 & 1 & \\ \hline y(2) & \multicolumn{5}{l}{3\times 2 + 2\times 1 + 1\times 2 + 2\times 1} \end{array}$$

$6+2+2+2$

$\boxed{y(2) = 12}$

For $n = 3$

$$y(3) = \sum_{n=-\infty}^{\infty} x(k)\, h(3-k)$$

$$\begin{array}{llllll} x(k): & \overset{\downarrow}{3} & 2 & 1 & 2 & \\ h(3-k) & & 2 & 1 & 2 & 1 \\ \hline y(3) & \multicolumn{5}{l}{2\times 2 + 1\times 1 + 2\times 2} \end{array}$$

$4+1+4$

$\boxed{y(3) = 9}$

For $n=4$

$$y(4) = \sum_{n=-\infty}^{\infty} x(k)\, h(4-k)$$

$$\begin{array}{llllll} x(k): & \overset{\downarrow}{3} & 2 & 1 & 2 & \\ h(4-k) & & & 2 & 1 & 2 \;\; 1 \\ \hline y(4) & \multicolumn{5}{l}{1\times 2 + 2\times 1} \end{array}$$

$2+2$

$\boxed{y(4) = 4}$

for $n=5$

$$y(5)=\sum_{k=-\infty}^{\infty} x(k)\, h(5-k)$$

$x(k)$:	$\downarrow$ 3	2	1	2			
$h(5-k)$				2	1	2	1
$y(5)$				2×2			

$$\boxed{y(5)=4}$$

for $n\geq 6$ $x(k)$ and $h(n-k)$ do not over lap and hence

$y(n)=0$

for $n=-1$

$$y(-1)=\sum_{n=-\infty}^{\infty} x(k)\cdot h(-1-k)$$

$x(k)$:				$\downarrow$ 3	2	1	2
$h(-1-k)$	2	1	2	1			
$y(-1)$				3×1			

$$\boxed{y(-1)=3}$$

$y(n)=\{y(-1), y(0), y(1), y(2), y(3), y(4), y(5)\}$

$y(n)=\{3, 8, 8, 12, 9, 4, 4\}$

Determine the convolution sum of two sequences

$x(n) = \{3, 2, 1, 2\}$ $\quad h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$

k		-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
x(k)								3	2	1	2					
h(k)							1	2	1	2						
n=0	h(0-k)					2	1	2	1							
n=1	h(1-k)						2	1	2	1						
n=2	h(2-k)							2	1	2	1					
n=3	h(3-k)								2	1	2	1				
n=4	h(4-k)									2	1	2	1			
n=5	h(5-k)										2	1	2	1		
n=-1	h(-1-k)				2	1	2	1								

$y(0) = 3 \times 2 + 2 \times 1$

$= 6 + 2$

$y(0) = 8$

$y(1) = 3 \times 1 + 2 \times 2 + 1 \times 1$

$= 3 + 4 + 1$

$\boxed{y(1) = 8}$

$y(2) = 3 \times 2 + 2 \times 1 + 1 \times 2 + 2 \times 1$

$= 6 + 2 + 2 + 2$

$\boxed{y(2) = 12}$

$y(3) = 2 \times 2 + 1 \times 1 + 2 \times 2$

$= 4 + 1 + 4$

$\boxed{y(3) = 9}$

$y(4) = 1 \times 2 + 2 \times 1$

$= 2 + 2$

$\boxed{y(4) = 4}$

$y(5) = 2 \times 2$

$\boxed{y(5) = 4}$

$y(-1) = 3 \times 1$

$\boxed{y(-1) = 3}$

$y(n) = \{ y(-1), y(0), y(1), y(2), y(3), y(5) \}$

$y(n) = \{ 3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4 \}$

Determine the convolution sum of two sequences.

$x(n) = \{3, 2, 1, 2\}$ $\quad h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$

Alternate Method.

Step 1:

write down the sequence $x(n)$ and $h(n)$ as shown.

$$\begin{array}{c|cccc} & & x(n) & & \\ & 3 & 2 & 1 & 2 \\ \hline 1 & & & & \\ h(n) \quad 2 & & & & \\ 1 & & & & \\ 2 & & & & \end{array}$$

Step 2:-

Multiply each and every sample in h(n) by the samples of x(n) and tabulate the values.

Step 3: Divide the elements in the table by drawing diagonal lines

	x(n): 3	2	1	2
1	3	2	1	2
2	6	4	2	4
1	3	2	1	2
2	6	4	2	4

Step 4:-

Starting from the left, sum all the elements in each strip and write down in the same order.

$$= 3,\ (6+2),\ (3+4+1),\ (6+2+2+2),\ (4+1+4),\ (2+2),\ 4$$

$$\boxed{y(n) = 3, 8, 8, 12, 9, 4, 4.}$$

Step 5:-

The starting value $n = n_1 + n_2$

$n_1 = 0, \quad n = -1$

$n = 0 + -1$

$$\boxed{y(n) \quad 3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4}$$

Determine the output response $y(n)$ if $x(n) = \{1,2,3,2\}$

$h(n) = \{1,2,2\}$.

Solution:-

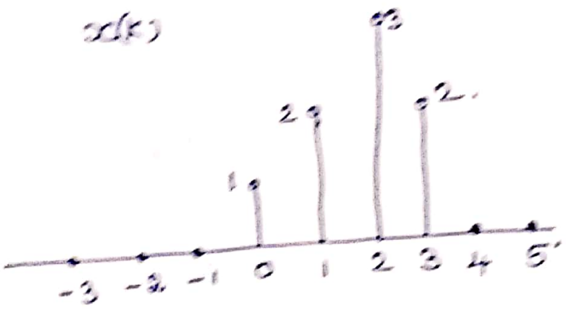
$x(n) = \{1,2,3,2\}$ $\quad$ $h(n) = \{1,2,2\}$

Both the sequences starts at $n=0$

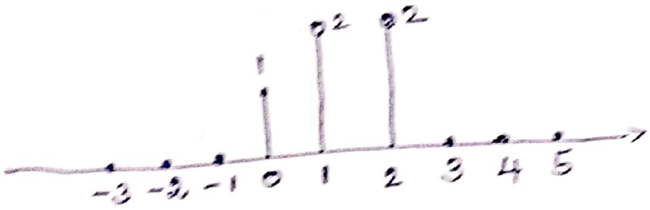
Therefore the output $y(n)$ also starts at $n=0$.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$x(k)$

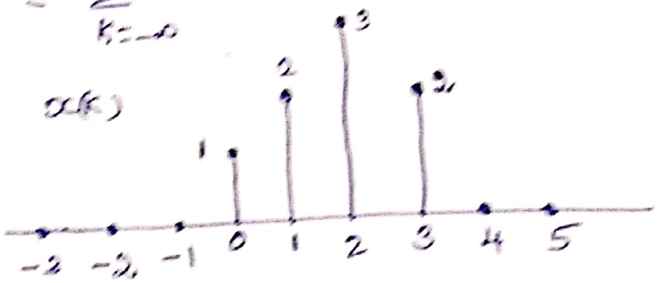


$h(k)$

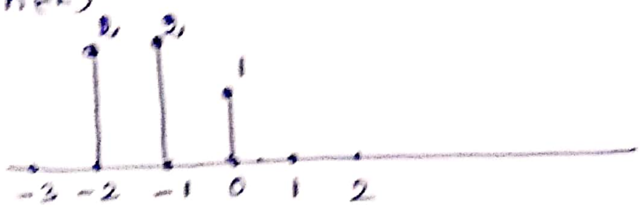


$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k)$$

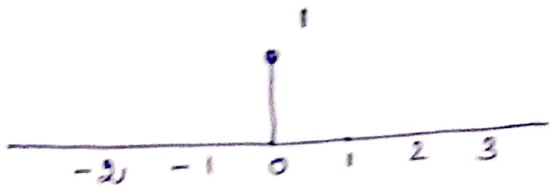
$x(k)$



$h(-k)$



$V_0(k)$

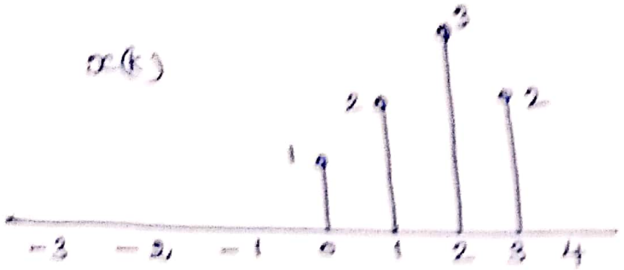


$y(0) = \sum V_0(k)$

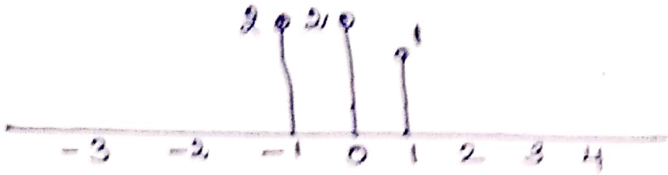
$\boxed{y(0) = 1}$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k)$$

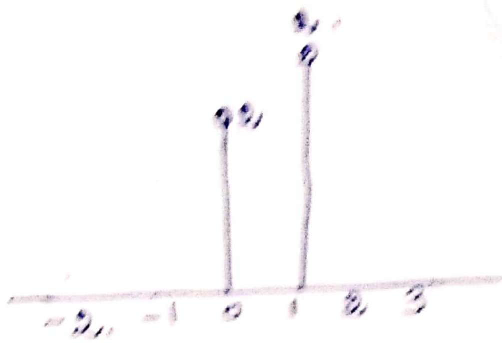
$x(k)$



$h(1-k)$



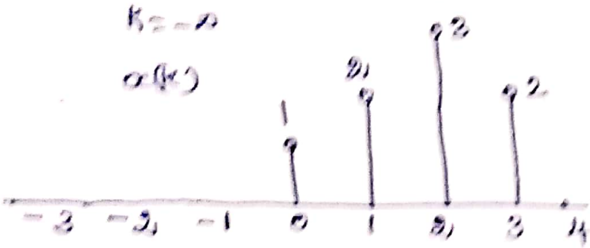
$V_1(k)$



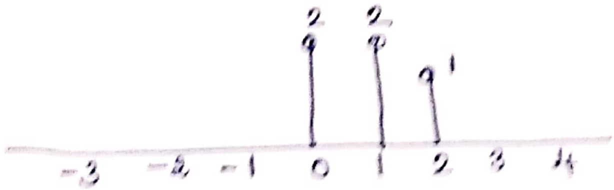
$$y(1) = \sum V(k)$$
$$= 2 + 2$$
$$\boxed{y(1) = 4}$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)\, h(2-k)$$

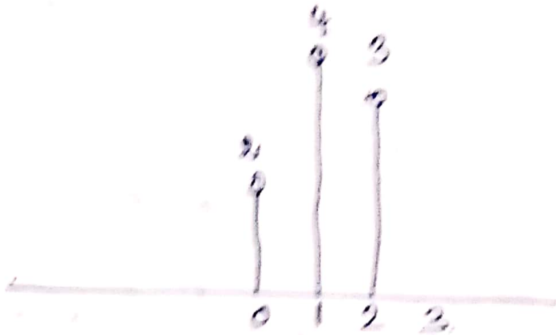
$x(k)$



$h(2-k)$



$V_2(k)$



$$y(2) = \sum V_2(k)$$
$$= 2 + 4 + 3$$
$$= 9.$$

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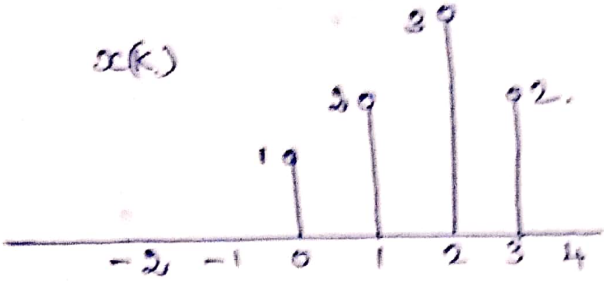
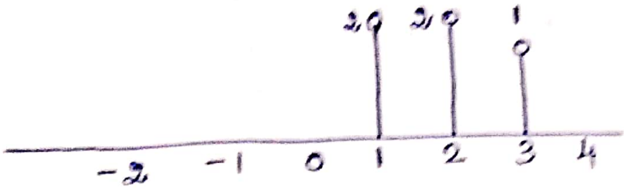
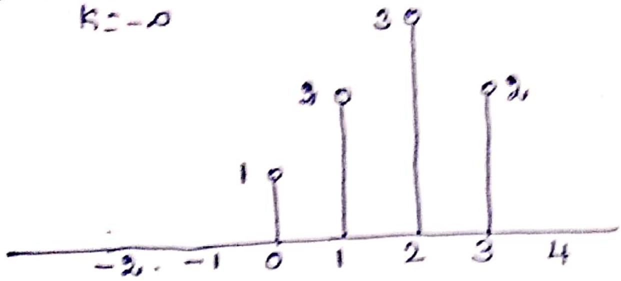
$$y(3) = \sum_{k=-\infty}^{\infty} x(k)\, h(3-k)$$



h(3-k)

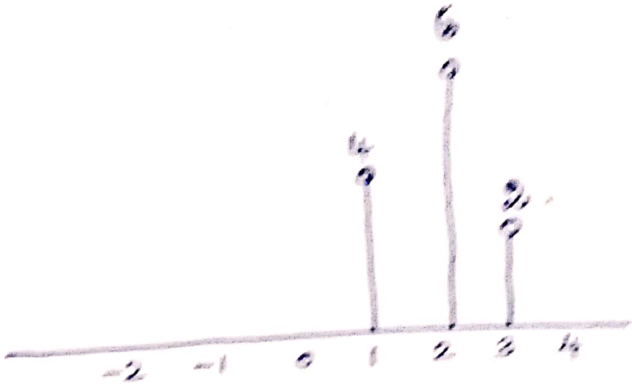


$$y(4) = \sum_{k=-\infty}^{\infty} x(k)\, h(4-k)$$



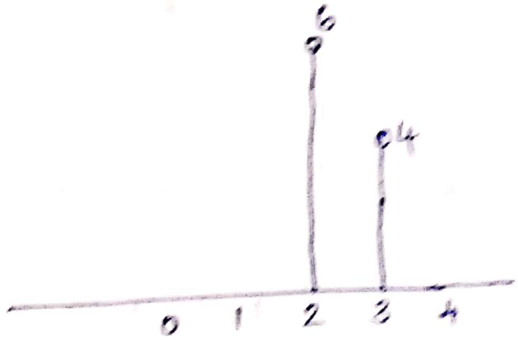
h(4-k)

$v_3(k)$



$$y(3) = \sum v_3(k)$$
$$= 4+6+2$$
$$\boxed{y(3) = 12}$$

$v_4(k)$

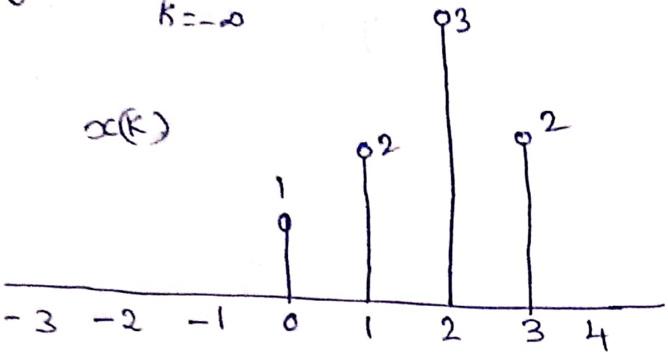


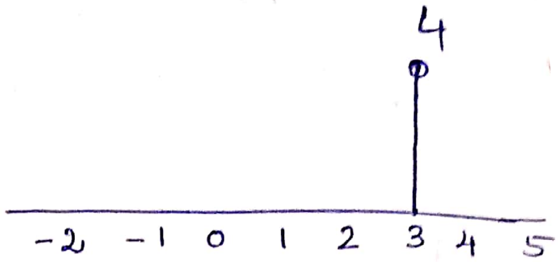
$$y(4) = \sum v_4(k)$$
$$= 6+4$$
$$\boxed{y(4) = 10}$$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k)\, h(5-k)$$

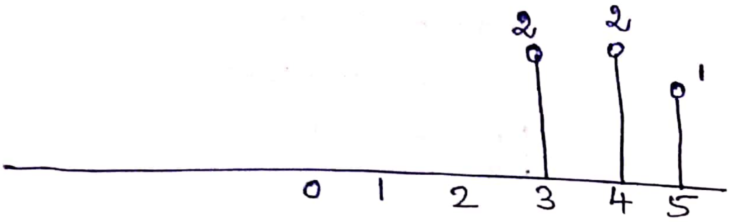
$V_5(k)$

$x(k)$





$h(5-k)$



$$y(5) = \sum V_5(k)$$

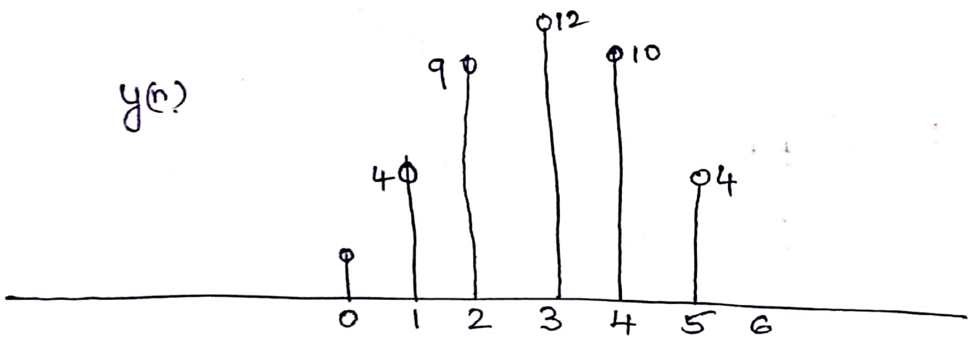
$$\boxed{y(5) = 4}$$

for $n \geq 6$ the $x(k)\, h(n-k)$ will not operlap so $y(n) = 0$

$$y(n) = \{ y(0), y(1), y(2), y(3), y(4), y(5) \}$$

$$= \{ 1, 4, 9, 12, 10, 4 \}$$

$y(n)$



Determine the convolution sum of two sequences.

$x(n) = \{1, \underset{\uparrow}{4}, 3, 2\}$: $h(n) = \{1, 3, 2, 1\}$.

Solution:-

$x(n) = \{1, \underset{\uparrow}{4}, 3, 2\}$ $\quad h(n) = \{1, 3, 2, 1\}$.

The sequence $x(n)$ starts at $n_1 = -1$

The sequence $h(n)$ starts at $n_2 = 0$

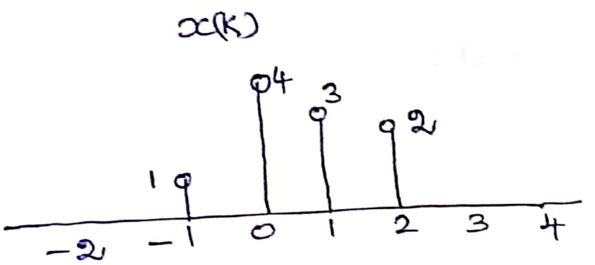
The starting time for evaluating the output sequence $y(n)$ is

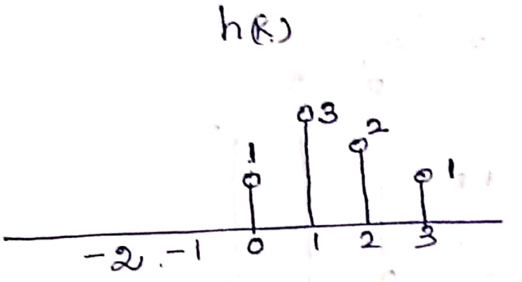
$n = n_1 + n_2$

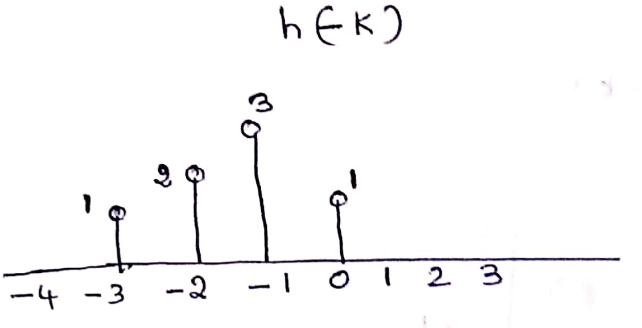
$= -1 + 0$

$n = -1$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

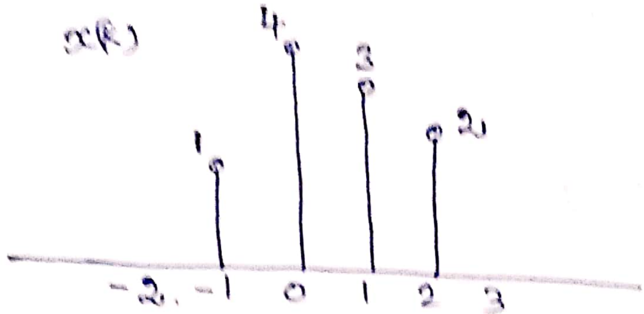




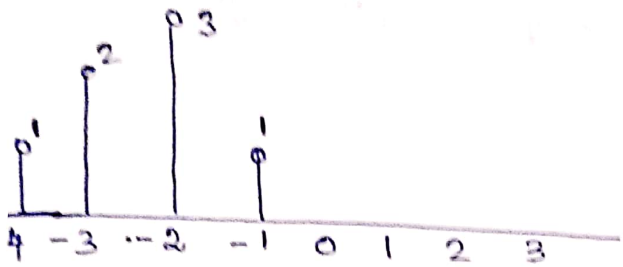


$$y(-1) = \sum_{n=-\infty}^{\infty} x(k)\, h(-1-k)$$

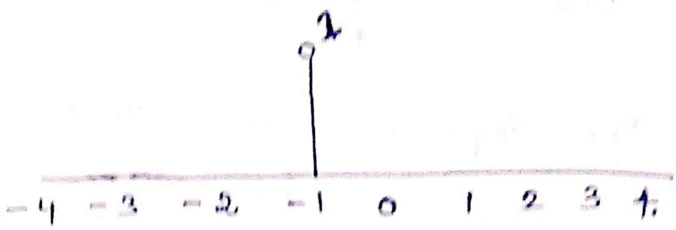
$x(k)$



$h(-1-k)$



$V_{-1}(k)$

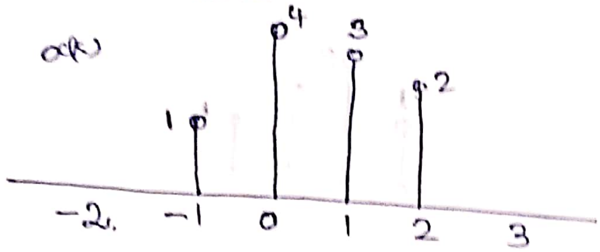


$$y(-1) = \sum V_{-1}(k)$$

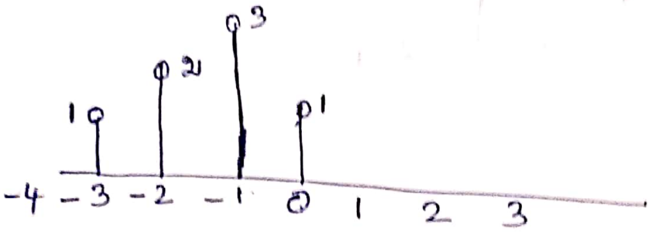
$$\boxed{y(-1) = 1}$$

$$y(0) = \sum_{n=-\infty}^{\infty} x(k)\, h(-k)$$

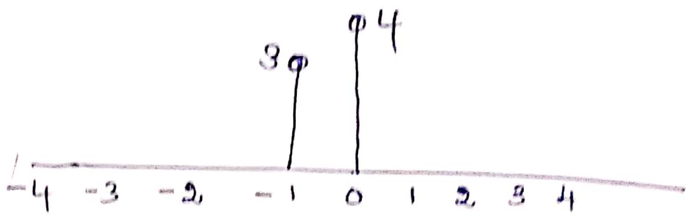
$x(k)$



$h(-k)$



$V_0(k)$

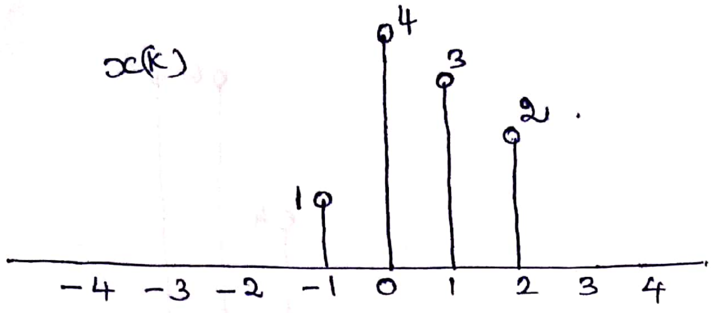


$$y(0) = 3+4$$

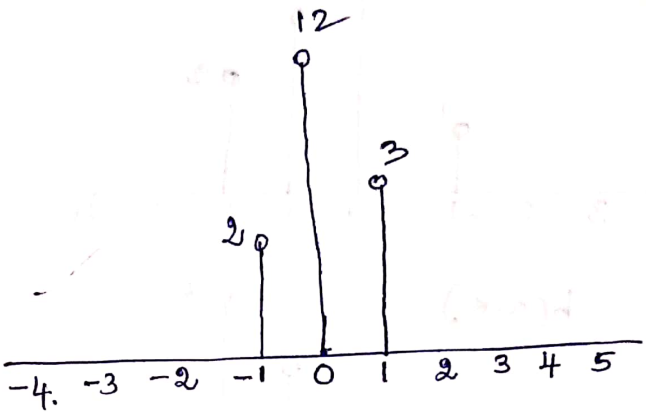
$$\boxed{y(0) = 7}$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k)$$

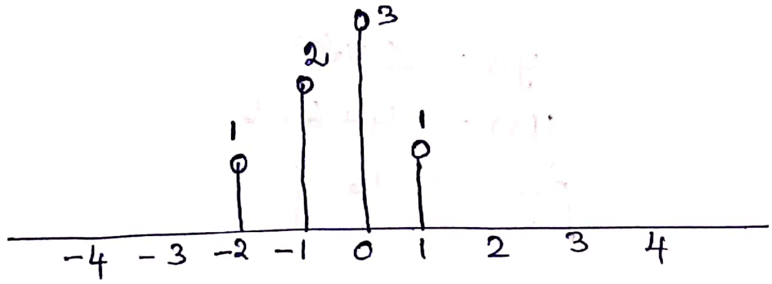
$x(k)$



$V_1(k)$



$h(1-k)$

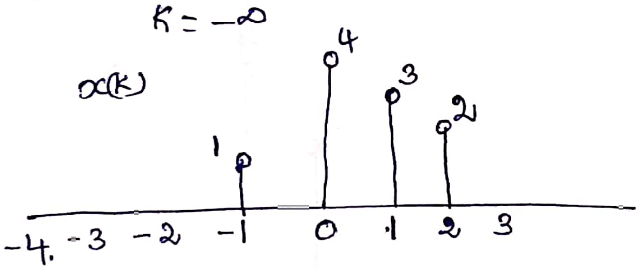


$$y(1) = 2 + 12 + 3$$

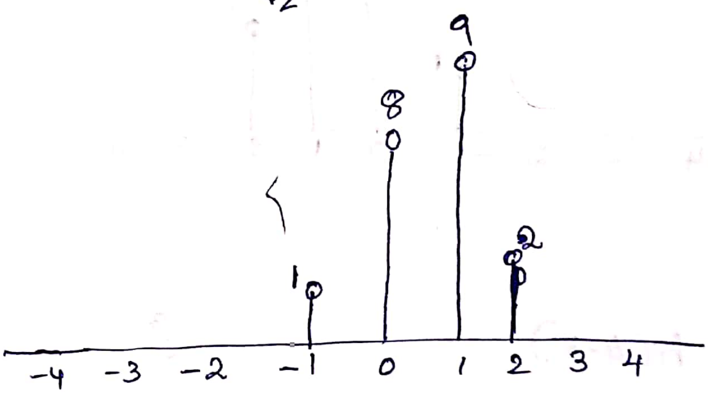
$$\boxed{y(1) = 17}$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)\, h(2-k)$$

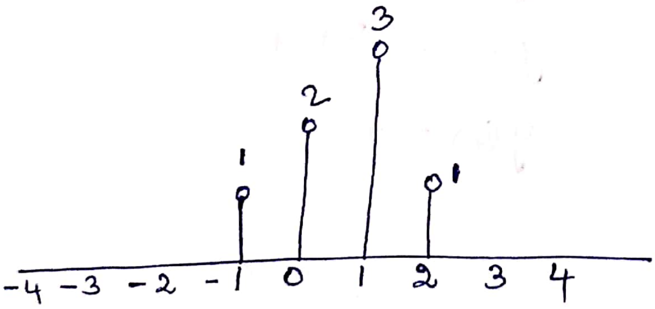
$x(k)$



$V_2(k)$



$h(2-k)$

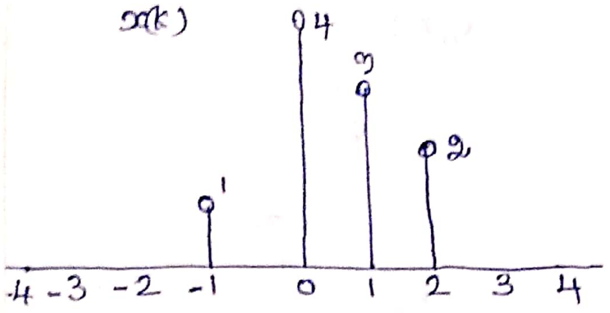


$$y(2) = 1 + 8 + 9 + 2$$

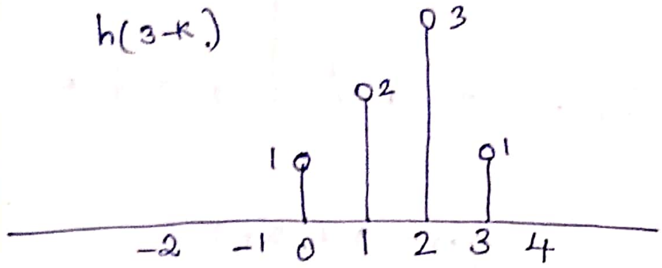
$$\boxed{y(2) = 20}$$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)\, h(3-k)$$

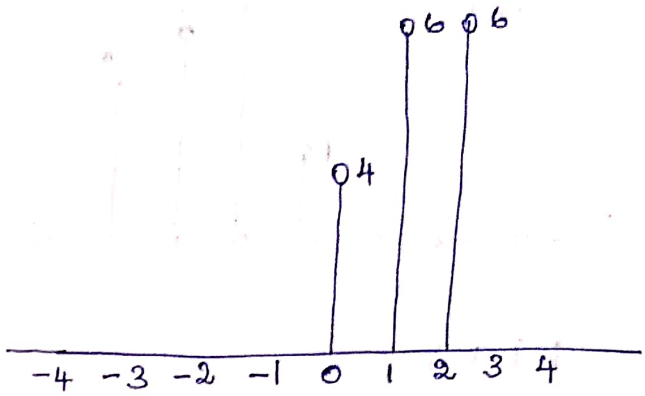
x(k)



h(3-k)



$V_3(k)$



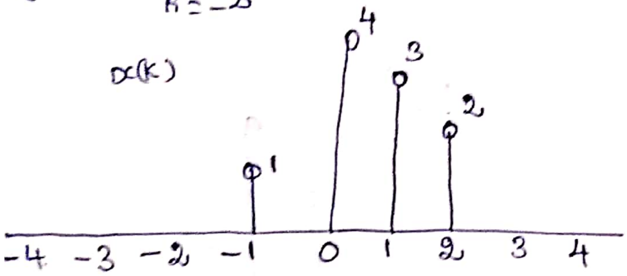
$y(3) = \sum V_3(k)$

$y(3) = 4+6+6$

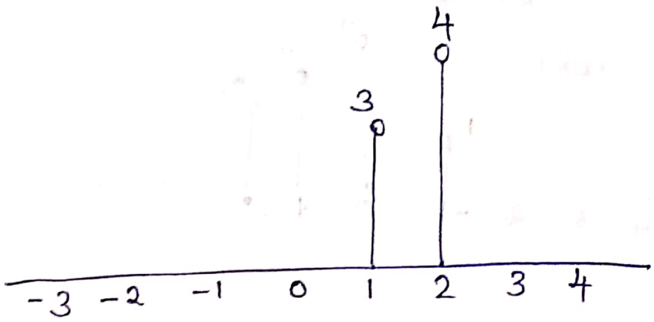
$\boxed{y(3) = 16}$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)\, h(4-k)$$

x(k)



$V_4(k)$

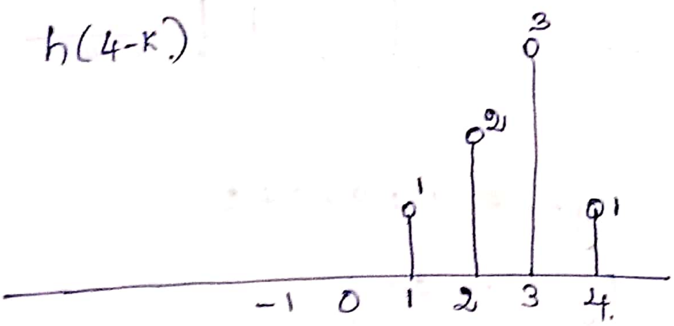


$y(4) = \sum V_4(k)$

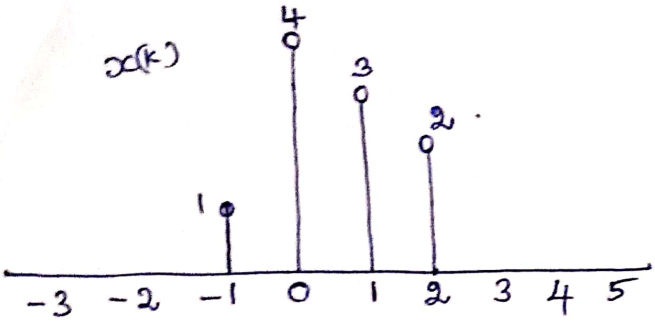
$y(4) = 3+4$

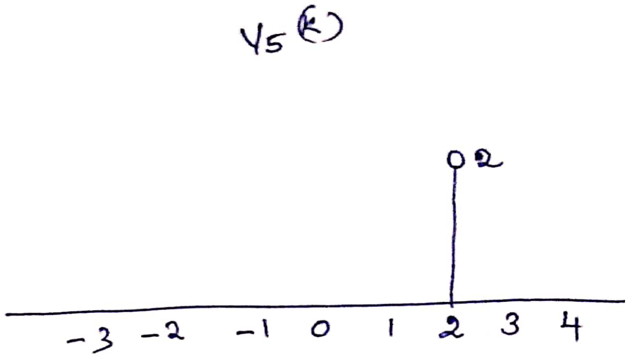
$y(4) = 7$

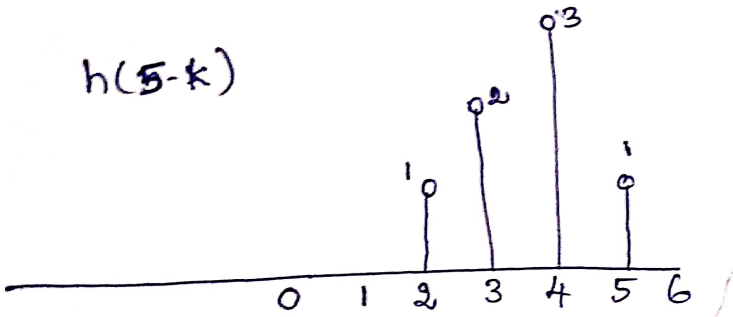
h(4-k)



$$y(5) = \sum_{k=-\infty}^{\infty} x(k)\, h(5-k)$$







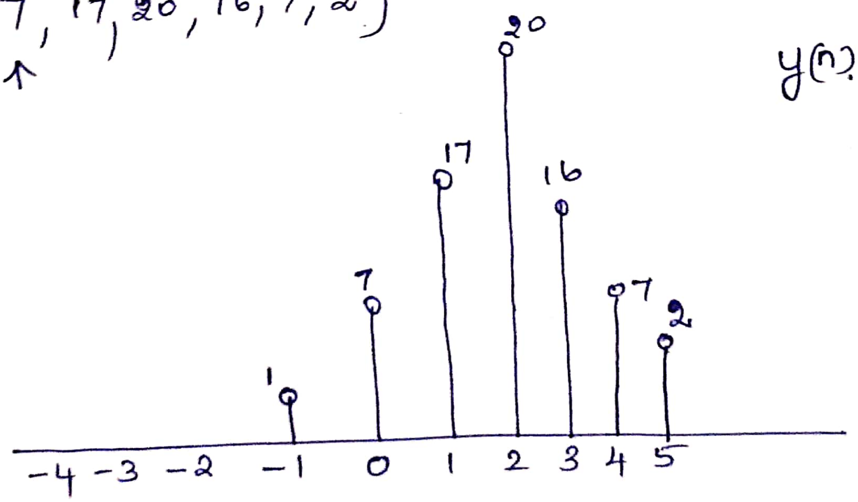
$y(5) = \sum v_5 k$

$$\boxed{y(5) = 2}$$

For $n \geq 6$ the $x(k)$ and $h(n-k)$ do not operlap and $y(n) = 0$

$y(n) = \{ y(-1), y(0), y(1), y(2), y(3), y(4)\ y(5) \}$

$y(n) = \{ 1, \underset{\uparrow}{7}, 17, 20, 16, 7, 2 \}$



Determine the convolution of the following Sequences

$x(n) = 2^n u(n)$ & $h(n) = \left(\frac{1}{2}\right)^n u(n)$

Solution:-

$x(n) = 2^n u(n)$

$h(n) = \left(\frac{1}{2}\right)^n u(n)$

if $x(n) = 0$ for $n < 0$
The Sequence is Causal sequence.

For a causal system the impulse response $h(n) = 0$ for $n < 0$

If the input to the causal System is a Causal sequence
the limit in the convolution
Sum is modified as.

$$y(n) = \sum_{K=0}^{n} x(K)\, h(n-K)$$

$$y(n) = \sum_{K=0}^{n} 2^K \cdot \left(\frac{1}{2}\right)^{n-K}.$$

$$y(n) = \sum_{K=0}^{n} 2^K \cdot \left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{2}\right)^{-K}.$$

$$y(n) = \sum_{K=0}^{n} 2^K \cdot \left(\frac{1}{2}\right)^n \cdot \frac{1}{\left(\frac{1}{2}\right)^K}.$$

$$y(n) = \sum_{K=0}^{n} 2^K \cdot \left(\frac{1}{2}\right)^n \cdot 2^K.$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} 2^k \cdot 2^k$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} 2^{2k}.$$

$$\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} 4^k.$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{4^{n+1}-1}{4-1}\right]$$

$$\boxed{y(n) = \left(\frac{1}{2}\right)^n \left[\frac{4^{n+1}-1}{3}\right]}$$

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* A Signal may be analysed with the help of Fourier series and Fourier transform.

* Fourier Series is used for periodic signals and Fourier transform is used for aperiodic Signals.

* Fourier transform may be used for both aperiodic and periodic signals.

The Discrete time Fourier transform (DTFT) $X(e^{j\omega})$ of a discrete-time Signal $x(n)$ is expressed as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}. \quad \text{--- (1)}$$

$$\text{DTFT } x(n) = X(e^{j\omega})$$

Inverse Discrete-Time Fourier Transform IDTFT is expressed as

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \, e^{j\omega n} \, d\omega. \quad \text{--- (2)}$$

* IDTFT equation is called the Synthesis equation.

* This Synthesis equation indicates that an aperiodic Signal $x(n)$ may be represented as a linear combination of complex exponentials infinitesimally close in frequency.

* The frequency spectrum of DTFT is continuous.
* The frequency spectrum of DTFT is periodic.

Frequency-Domain Representation of Discrete-Time Systems and Signals.

or

Sinusoidal Steady State Response.

Consider a LTI (Linear Time Invariant system)

$$x(n) = e^{j\omega n} \longrightarrow \overset{h(n)}{\boxed{\text{LTI System}}} \longrightarrow y(n)$$

where $x(n) = e^{j\omega n}$ $\quad -\infty < n < \infty$ is the input sequence.

$y(n)$ is the output

$h(n)$ is the impulse response

The output $y(n)$ is computed by the convolution of $x(n)$ and $h(n)$

$$y(n) = h(n) * x(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega n} \cdot e^{-j\omega k}.$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}.$$

Take $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}$.

$\boxed{y(n) = e^{j\omega n} H(e^{j\omega})}$ or $y(n) = x(n) H(e^{j\omega})$

* $H(e^{j\omega})$ describes the changes in complex amplitude of a complex exponential as a function of the frequency ω.

* The quantity $H(e^{j\omega})$ is called the frequency response of the system whose unit-Sample response is $h(n)$.

* $H(e^{j\omega})$ is complex and can be expressed in terms of its real and imaginary part as.

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j\, H_I(e^{j\omega})$$

* In terms of magnitude and phase $H(e^{j\omega})$ can be expressed as.

$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\arg[H(e^{j\omega})]}$$

* The phase is Some time refered as group delay.

* The group delay is defined as the negative of the first derivative with respect to ω, of the phase.

Response to a Sinusoidal Signal:-

Let the system under consideration be an LTI system.

The excitation of the system be a Sinusoid $x(n) = A\cos(\omega_0 n + \varphi)$

$$x(n) = A\left[\frac{1}{2}\left[e^{j(\omega_0 n+\varphi)} + e^{-j(\omega_0 n+\varphi)}\right]\right]$$

$$= \frac{A}{2}\left[e^{j\omega_0 n} \cdot e^{j\varphi} + e^{-j\omega_0 n} \cdot e^{-j\varphi}\right]$$

$$x(n) = \frac{A}{2} e^{j\omega_0 n} \cdot e^{j\varphi} + \frac{A}{2} e^{-j\omega_0 n} \cdot e^{-j\varphi}$$

$$y(n) = H(e^{j\omega_0}) \cdot x(n)$$

$$y(n) = \left[\frac{A}{2} \cdot e^{j\omega_0 n} \cdot e^{j\varphi} + \frac{A}{2} e^{-j\omega_0 n} \cdot e^{-j\varphi}\right] H(e^{j\omega_0})$$

$$y(n) = \frac{A}{2} \cdot e^{j\omega_0 n} \cdot e^{j\varphi} H(e^{j\omega_0}) + \frac{A}{2} \cdot e^{-j\omega_0 n} \cdot e^{-j\varphi} H(e^{j\omega_0})$$

$$y(n) = y_1(n) + y_2(n)$$

$y_2(n)$ is the conjugate of $y_1(n)$

So we can write

$$y(n) = 2\,\mathrm{Re}\left[\frac{A}{2} e^{j\omega_0 n} \cdot e^{j\varphi} \cdot H(e^{j\omega_0})\right]$$

$$= A\,\mathrm{Re}\left[e^{j(\omega_0 n+\varphi)} \left|H(e^{j\omega_0})\right| e^{j\arg[H(e^{j\omega_0})]}\right]$$

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$$= A \ \mathrm{Re}\left[e^{j(w_0 n + \varphi)} \ |H(e^{jw_0})| \ e^{j\theta} \right]$$

$$= A \ |H(e^{jw_0})| \ \mathrm{Re} \ e^{j(w_0 n + \varphi + \theta)}$$

Real part of $e^{j(w_0 n + \varphi + \theta)} = \cos(w_0 n + \varphi + \theta)$

$$\boxed{y(n) = A \ |H(e^{jw_0})| \ \cos(w_0 n + \varphi + \theta)}$$

A discrete-time system has a unit impulse response $h(n)$ given by

$$h(n) = \frac{1}{2}\delta(n) + \delta(n-1) + \frac{1}{2}\delta(n-2)$$

Find the frequency response $H(e^{j\omega})$

Solution:-

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}.$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2}\delta(n) + \delta(n-1) + \frac{1}{2}\delta(n-2)\right] e^{-j\omega n}.$$

$$= \frac{1}{2} e^{-j\omega n}\Big|_{n=0} + e^{-j\omega n}\Big|_{n=1} + \frac{1}{2} e^{-j\omega n}\Big|_{n=2}.$$

$$= \frac{1}{2} + e^{-j\omega} + \frac{1}{2} e^{-j\omega 2}$$

$$= e^{-j\omega}\left[\frac{1}{2e^{-j\omega}} + \frac{e^{-j\omega}}{e^{-j\omega}} + \frac{1 e^{-j2\omega}}{2\, e^{-j\omega}}\right]$$

$$= e^{-j\omega}\left[\frac{1}{2e^{-j\omega}} + 1 + \frac{1\, e^{+j(2\omega-(-\omega))}}{2}\right]$$

$$= e^{-j\omega}\left[\frac{1}{2} e^{j\omega} + 1 + \frac{1}{2} e^{+j(-2\omega+\omega)}\right]$$

$$= e^{-j\omega}\left[\frac{1}{2} e^{j\omega} + 1 + \frac{1}{2} e^{+j(-\omega)}\right]$$

$$= e^{-j\omega}\left[\frac{1}{2} e^{j\omega} + 1 + \frac{1}{2} e^{-j\omega}\right]$$

$$\cos\omega = \frac{1}{2}\left[e^{j\omega} + e^{-j\omega}\right]$$

$$H(e^{j\omega}) = e^{-j\omega}\left[1 + \cos\omega\right]$$

$$|H(e^{j\omega})| = |e^{-j\omega}|\,|1+\cos\omega|$$

$$= |1+\cos\omega|$$

$$e^{-j\omega} = \cos\omega - j\sin\omega$$

$$\theta(\omega) = \angle e^{-j\omega} + \angle 1+\cos\omega$$

$$\theta = \tan^{-1}\frac{b}{a}$$

$$= -\omega + 0$$

$$= \tan^{-1}\left(\frac{-\sin\omega}{\cos\omega}\right)$$

$$\theta(\omega) = -\omega$$

$$= \tan^{-1}(-\tan\omega)$$

$$\theta(\omega) = -\omega$$

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6. a) Prove that convolution in time domain leads to multiplication in frequency domain for Discrete time signals.

Statement: The time convolution property ~~of~~ States that the DTFT of convolution of two Sequences in time domain is equalent to multiplication of their DTFT. i.e

$$x_1(n) \xleftrightarrow{FT} X_1(\omega) \quad \text{and} \quad x_2(n) \xleftrightarrow{FT} X_2(\omega)$$

then

$$F\left[x_1(n) * x_2(n)\right] = X_1(\omega) \cdot X_2(\omega)$$

Proof: From the definition of DTFT, we have

$$F\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$\therefore F\left[x_1(n) * x_2(n)\right] = \sum_{n=-\infty}^{\infty} \left[x_1(n) * x_2(n)\right] e^{-j\omega n}.$$

But the Convolution of two Sequences is defined as,

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k)$$

$$\therefore F\left[x_1(n) * x_2(n)\right] = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k)\right] e^{-j\omega n}.$$

By Interchanging the order of Summations, we get

$$F\left[x_1(n) * x_2(n)\right] = \sum_{k=-\infty}^{\infty} x_1(k) \cdot \sum_{n=-\infty}^{\infty} x_2(n-k) \cdot e^{-j\omega n}$$

Substituting $(n-k) = m$ and $n = m+k$ in the Second Summation, We have,

$$F\left[x_1(n) * x_2(n)\right] = \sum_{k=-\infty}^{\infty} x_1(k) \cdot \sum_{m=-\infty}^{\infty} x_2(m) \cdot e^{-j\omega(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \cdot e^{-j\omega k} \cdot \sum_{m=-\infty}^{\infty} x_2(m) \cdot e^{-j\omega m}$$

$$\therefore \boxed{F\left[x_1(n) * x_2(n)\right] = X_1(\omega) \cdot X_2(\omega)}$$

$\therefore$ The Convolution in Time domain is equivalent to multiplication in frequency domain.

Determine the frequency response of an LTI System described by

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n)$$

Solution:

$y(n) = Y(e^{j\omega})$ $\quad\quad$ $x(n) = X(e^{j\omega})$

$y(n-1) = e^{-j\omega} Y(e^{j\omega})$

$y(n-2) = e^{-j2\omega} Y(e^{j\omega})$

$$Y(e^{j\omega}) - \frac{3}{4} e^{-j\omega} Y(e^{j\omega}) + \frac{1}{8} e^{-j2\omega} Y(e^{j\omega}) = X(e^{j\omega})$$

$$Y(e^{j\omega}) \left[1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-j2\omega}\right] = X(e^{j\omega})$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-j2\omega}}$$

$$H(e^{j\omega}) = \frac{1}{1 - \frac{3}{4}\left[\cos\omega - j\sin\omega\right] + \frac{1}{8}\left[\cos 2\omega - j\sin 2\omega\right]}$$

$$|H(e^{j\omega})| = \frac{1}{1 - \frac{3}{4}\cos\omega + j\frac{3}{4}\sin\omega + \frac{1}{8}\cos 2\omega - \frac{1}{8} j\sin 2\omega}$$

$$= \frac{1}{\left(1 - \frac{3}{4}\cos w + \frac{1}{8}\cos 2w\right) + j\frac{3}{4}\sin w - j\frac{1}{8}\sin 2w}$$

$$= \frac{1}{\left(1 - \frac{2}{4}\cos w + \frac{1}{8}\cos 2w\right) + j\left(\frac{3}{4}\sin w - \frac{1}{8}\sin 2w\right)}$$

$$|H(e^{jw})| = \frac{1}{\sqrt{\left(1 - \frac{3}{4}\cos w + \frac{1}{8}\cos 2w\right)^2 + \left(\frac{3}{4}\sin w - \frac{1}{8}\sin 2w\right)^2}}$$

$$\angle H(e^{jw}) = -\tan^{-1}\left(\frac{\frac{3}{4}\sin w - \frac{1}{8}\sin 2w}{1 - \frac{3}{4}\cos w + \frac{1}{8}\cos 2w}\right)$$

Determine the frequency response $H(e^{j\omega})$ for the System and plot magnitude response and phase response

$$y(n) + \frac{1}{4} y(n-1) = x(n) - x(n-1)$$

Solution:-

$$y(n) + \frac{1}{4} y(n-1) = x(n) - x(n-1)$$

Taking DTFT on bothsides.

$$Y(e^{j\omega}) + \frac{1}{4} e^{-j\omega} \cdot Y(e^{j\omega}) = X(e^{j\omega}) - e^{-j\omega} X(e^{j\omega})$$

$$Y(e^{j\omega}) \left[1 + \frac{1}{4} e^{-j\omega}\right] = X(e^{j\omega}) \left[1 - e^{-j\omega}\right]$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - e^{-j\omega}}{1 + \frac{1}{4} e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1 - [\cos\omega - j\sin\omega]}{1 + \frac{1}{4}[\cos\omega - j\sin\omega]}$$

$$H(e^{j\omega}) = \frac{1 - \cos\omega + j\sin\omega}{1 + \frac{1}{4}\cos\omega - j\frac{1}{4}\sin\omega}$$

$$|H(e^{j\omega})| = \frac{\sqrt{(1-\cos\omega)^2 + \sin^2\omega}}{\sqrt{\left(1+\frac{1}{4}\cos\omega\right)^2 + \left(\frac{1}{4}\sin\omega\right)^2}}$$

$$|H(e^{j\omega})| = \frac{\sqrt{(1-\cos\omega)^2 + \sin^2\omega}}{\sqrt{\left(1+\frac{1}{4}\cos\omega\right)^2 + \frac{1}{16}\sin^2\omega}}$$

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{\sin\omega}{1-\cos\omega}\right) - \tan^{-1}\left[\frac{\sin\omega}{\left(1+\frac{1}{4}\cos\omega\right)}\right]$$

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$$|H(e^{j\omega})| = \frac{\sqrt{(1-\cos\omega)^2 + \sin^2\omega}}{\sqrt{\left(1+\frac{1}{4}\cos\omega\right)^2 + \frac{1}{16}\sin^2\omega}}$$

$$= \frac{\sqrt{1 + \cos^2\omega - 2\cos\omega + \sin^2\omega}}{\sqrt{1 + 2\times\frac{1}{4}\cos\omega + \frac{1}{16}\cos^2\omega + \frac{1}{16}\sin^2\omega}}$$

$$\cos^2\omega + \sin^2\omega = 1$$

$$= \frac{\sqrt{1 - 2\cos\omega + 1}}{\sqrt{1 + \frac{1}{2}\cos\omega + \frac{1}{16}(\cos^2\omega + \sin^2\omega)}}$$

$$= \frac{\sqrt{2 - 2\cos\omega}}{\sqrt{1 + \frac{1}{2}\cos\omega + \frac{1}{16}}}$$

$$= \frac{\sqrt{2 - 2\cos\omega}}{\sqrt{1 + \frac{1}{16} + \frac{1}{2}\cos\omega}}$$

$$= \frac{\sqrt{2(1 - \cos\omega)}}{\sqrt{\frac{17}{16} + \frac{1}{2}\cos\omega}}$$

$$= \frac{\sqrt{2 \times 2 \sin^2 \frac{\omega}{2}}}{\sqrt{\frac{17}{16} + \frac{1}{2} \cos\omega}}$$

$$\sin^2\omega = \frac{1-\cos 2\omega}{2}$$

$$\sin^2\frac{\omega}{2} = \frac{1-\cos \cancel{2}\frac{\omega}{\cancel{2}}}{2}$$

$$\boxed{2\sin^2\frac{\omega}{2} = 1-\cos\omega}$$

$$= \frac{\sqrt{4\sin^2\frac{\omega}{2}}}{\sqrt{1.0625 + 0.5\cos\omega}}$$

$$= \frac{2\sin\frac{\omega}{2}}{\sqrt{1.0625 + 0.5\cos\omega}}.$$

$$\boxed{|H(e^{j\omega})| = \frac{2\sin\frac{\omega}{2}}{\sqrt{1.0625 + 0.5\cos\omega}}}$$

Phase is

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{\sin\omega}{1-\cos\omega}\right) - \tan^{-1}\left[\frac{(-0.25\sin\omega)}{(1+0.25\cos\omega)}\right]$$

$$= \tan^{-1}\left(\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\sin^2\frac{\omega}{2}}\right) - \tan^{-1}\left[\frac{(-0.25\sin\omega)}{(1+0.25\cos\omega)}\right]$$

$$= \tan^{-1}\left(\frac{\cos\frac{w}{2}}{\sin\frac{w}{2}}\right) - \tan^{-1}\left(\frac{-0.25\sin w}{1+0.25\cos w}\right)$$

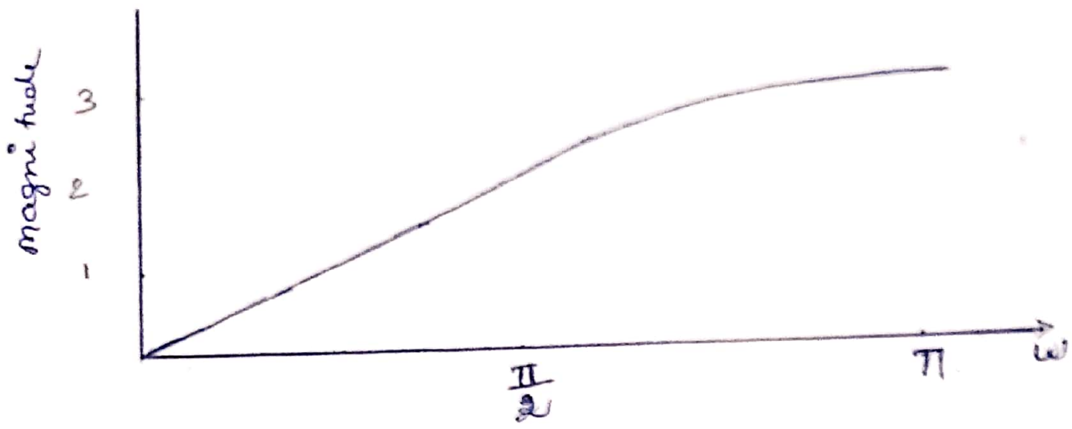
$$= \tan^{-1}\left(\cot\frac{w}{2}\right) - \tan^{-1}\left(\frac{-0.25\sin w}{1+0.25\cos w}\right)$$

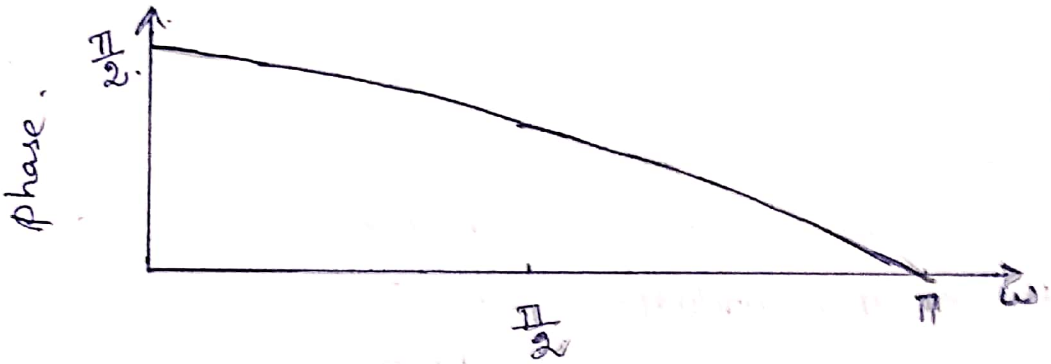
$$= \tan^{-1}\tan\left(90-\frac{w}{2}\right) - \tan^{-1}\left(\frac{-0.25\sin w}{1+0.25\cos w}\right)$$

$$\angle H(e^{jw}) = 90 - \frac{w}{2} - \tan^{-1}\left(\frac{-0.25\sin w}{1+0.25\cos w}\right)$$

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π.
w	0	30	45	60	75	90	105	120	135	150	180°
$\lvert H(e^{jw})\rvert$	0	0.423	0.636	0.873	1.115	1.372	1.642	1.92	2.195	2.435	2.666
$\angle H(e^{jw})$ degrees	90	80.86	76.04	70.89	65.27	59.03	51.97	43.89	34.62	24.06	0
$\angle H(e^{jw})$ radian.	0.5π	0.449π	0.422π	0.394π	0.363π	0.328π	0.288π	0.244π	0.192π	0.1336π	0

By putting the calculator in degree mode calculate the value of $\angle H(e^{jw})$ and multiply the result by $\frac{\pi}{180}$ to get the answer in radian.


magnitude
3
2
1
$\frac{\pi}{2}$
π
ω


Phase
$\frac{\pi}{2}$
$\frac{\pi}{2}$
π
ω

Problems in LTI Systems

Find the system function and impulse response of the system described by the difference equation.

$$y(n) = \frac{1}{3} y(n-1) + x(n)$$

Solution

$$y(n) = \frac{1}{3} y(n-1) + x(n)$$

Taking z transform on both sides.

$$Y(z) = \frac{1}{3} z^{-1} Y(z) + X(z)$$

$$Y(z) - \frac{1}{3} z^{-1} Y(z) = X(z)$$

$$Y(z)\left[1 - \frac{1}{3} z^{-1}\right] = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{3} z^{-1}}$$

But $\frac{Y(z)}{X(z)} = H(z)$

$$Z\{a^n u(n)\} = \frac{z}{z-a}$$

$$H(z) = \frac{1}{1 - \frac{1}{3} z^{-1}}$$

Taking Inverse z transform

$$h(n) = \left(\frac{1}{3}\right)^n u(n)$$

Find the system function and the impulse response of the system described by the difference equation.

$$y(n) = x(n) + 3x(n-1) - 4x(n-2) + 2x(n-3).$$

Solution.

$$y(n) = x(n) + 3x(n-1) - 4x(n-2) + 2x(n-3)$$

Taking z transform on both sides.

$$Y(z) = X(z) + 3z^{-1}X(z) - 4z^{-2}X(z) + 2z^{-3}X(z)$$

$$Y(z) = X(z)\left[1 + 3z^{-1} - 4z^{-2} + 2z^{-3}\right]$$

$$\frac{Y(z)}{X(z)} = 1 + 3z^{-1} - 4z^{-2} + 2z^{-3}$$

$$\frac{Y(z)}{X(z)} = H(z)$$

$$H(z) = 1 + 3z^{-1} - 4z^{-2} + 2z^{-3}.$$

By the definition of z transform.

$$H(z) = h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}$$

$$h(0) = 1, \quad h(1) = 3, \quad h(2) = -4, \quad h(3) = 2$$

$$h(n) = \{\underset{\uparrow}{1}, 3, -4, 2\}$$

Problem

Determine the pole-zero plot for the system described by the difference equation $y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) - x(n-1)$

Also comment on the stability of the system.

Solution:-

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) - x(n-1)$$

Taking z transform on both sides.

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)\left[1 - z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}.$$

$$H(z) = \frac{1 - z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}.$$

$$H(z) = \frac{1 - z^{-1}}{z^{-2}\left(\frac{1}{z^{-2}} - \frac{3}{4z^{-1}} + \frac{1}{8}\right)}$$

$$H(z) = \frac{1 - z^{-1}}{z^{-2}\left(z^{2} - \frac{3}{4}z + \frac{1}{8}\right)}$$

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$$H(z) = \frac{1-z^{-1}}{z^{-2}\left[\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)\right]}$$

$$H(z) = \frac{z^2\left(1-z^{-1}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}$$

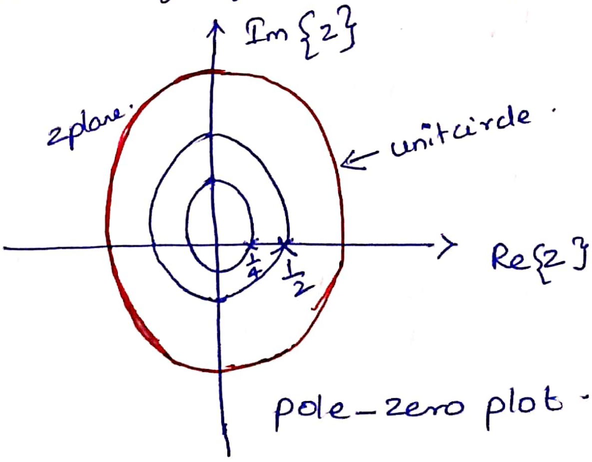
$$= \frac{z^2 - \frac{z^2}{z}}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}$$

$$= \frac{\left(z^2 - z\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}$$

$$H(z) = \frac{z(z-1)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}$$

The zeros of the transfer function $z=0$, $z=1$

The poles of the transfer function $z=\frac{1}{4}$, $z=\frac{1}{2}$.



pole-zero plot.

The pole of the given system function is at $z=\frac{1}{4}$ and $z=\frac{1}{2}$. Both the poles lies inside the unit circle of $|z|=1$. Hence the given System is stable.

Problem:-

Determine system function and unit sample response of the system described by diff eqn. $y(n) = \frac{1}{2}\, y(n-1) + 2\, x(n)$

Taking z- transform on both sides.

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2\, X(z)$$

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = 2X(z)$$

$$Y(z)\left[1-\frac{1}{2}z^{-1}\right] = 2\, X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$H(z) = \frac{2}{1-\frac{1}{2}z^{-1}}$$

$$H(z) = 2 \cdot \frac{1}{1-\frac{1}{2}z^{-1}}$$

Taking inverse z transform.

$$\boxed{h(n) = 2 \cdot \left(\frac{1}{2}\right)^{n} u(n)}$$

Problem.

Find the impulse response of the system described by the difference equation

$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$ using z transform.

Solution:-

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

Taking z- transform on both sides.

$$Y(z) - 3z^{-1}Y(z) - 4z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

$$Y(z)\left[1 - 3z^{-1} - 4z^{-2}\right] = X(z)\left[1 + 2z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}}.$$

$$H(z) = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}}.$$

To eliminate negative powers of z, multiply both Numerator and denominator by z^2.

$$H(z) = \frac{z^2(1 + 2z^{-1})}{z^2(1 - 3z^{-1} - 4z^{-2})}$$

$$= \frac{z^2 + 2z}{z^2 - 3z - 4}$$

$$= \frac{z(z+2)}{z^2 - 3z - 4}.$$

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$$H(z) = \frac{z(z+2)}{(z-4)(z+1)}$$

$$\frac{H(z)}{z} = \frac{z+2}{(z-4)(z+1)}$$

writing in partial fraction.

$$\frac{H(z)}{z} = \frac{A}{z-4} + \frac{B}{z+1} \quad \text{——} \quad 1$$

$$A = \left. \frac{H(z)}{z} \times z-4 \right|_{z=4}$$

$$A = \left. \frac{(z+2)}{\cancel{(z-4)}(z+1)} \times \cancel{(z-4)} \right|_{z=4}$$

$$A = \frac{4+2}{4+1}$$

$$\boxed{A = \frac{6}{5}}$$

$$B = \left. \frac{H(z)}{z} \times (z+1) \right|_{z=-1}$$

$$B = \left. \frac{(z+2)}{(z-4)\cancel{(z+1)}} \times \cancel{(z+1)} \right|_{z=-1}$$

$$B = \frac{-1+2}{-1-4}$$

$$\boxed{B = -\frac{1}{5}}$$

Substituting in ①

$$\frac{H(z)}{z} = \frac{6}{5}\frac{1}{(z-4)} - \frac{1}{5}\frac{1}{(z+1)}$$

$$z[a^n u(n)] = \frac{z}{z-}$$

$$H(z) = \frac{6}{5}\frac{z}{z-4} - \frac{1}{5}\frac{z}{(z+1)}$$

Taking Inverse ~~trans~~ z transform

$$\boxed{h(n) = \frac{6}{5}(4)^n u(n) - \frac{1}{5}(-1)^n u(n)}$$

Problem:-

Compute the response of the system $y(n) = 0.7\,y(n-1) - 0.12\,y($
$+ x(n-1) + x(n-2)$ to input $x(n) = u(n)$. Is the system Stable.

Solution:-

$$y(n) = 0.7y(n-1) - 0.12\,y(n-2) + x(n-1) + x(n-2)$$

Taking z- transform on both sides.

$$Y(z) = 0.7z^{-1}Y(z) - 0.12z^{-2}Y(z) + z^{-1}X(z) + z^{-2}X(z)$$

$$Y(z) - 0.7z^{-1}Y(z) + 0.12z^{-2}Y(z) = z^{-1}X(z) + z^{-2}X(z)$$

$$Y(z)\left[1 - 0.7z^{-1} + 0.12z^{-2}\right] = X(z)\left[z^{-1} + z^{-2}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{z^{-1}+z^{-2}}{1-0.7z^{-1}+0.12z^{-2}}$$

$$H(z) = \frac{z^{-1}+z^{-2}}{1-0.7z^{-1}+0.12z^{-2}}$$

multiplying NR and DR by z^2

$$H(z) = \frac{z^2(z^{-1}+z^{-2})}{z^2(1-0.7z^{-1}+0.12z^{-2})}$$

$$H(z) = \frac{(z+1)}{z^2-0.7z+0.12}$$

$$H(z) = \frac{(z+1)}{(z-0.4)(z-0.3)}$$

The poles of the system function are $z_1 = 0.4$ and $z_2 = 0.3$ and the ROC is $|z| > 0.4$. The poles are lying inside the unit circle.

$\therefore$ The system is stable.

The given input is

$x(n) = u(n)$, unit step input

$$X(z) = \frac{z}{z-1}$$

$$\frac{Y(z)}{X(z)} = \frac{(z+1)}{(z-0.4)(z-0.3)}$$

$$Y(z) = \frac{X(z) \cdot (z+1)}{(z-0.4)(z-0.3)}$$

$$= \frac{z \cdot (z+1)}{(z-1)\quad (z-0.4)(z-0.3)}$$

$$\frac{Y(z)}{z} = \frac{(z+1)}{(z-1)(z-0.4)(z-0.3)}$$

writing in partial fraction.

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-0.4} + \frac{C}{z-0.3} \quad \text{——} (1)$$

$$A = \frac{Y(z)}{z} \times (z-1) \Big|_{z=1}$$

$$= \frac{(z+1)\cancel{(z-1)}}{\cancel{(z-1)}(z-0.4)(z-0.3)} \Big|_{z=1}$$

$$A = \frac{2}{0.6 \times 0.7}$$

$$A = \frac{2}{0.42} \qquad \boxed{A = 4.76}$$

$$B = \frac{Y(z)}{z} \times (z-0.4) \Big|_{z=0.4}$$

$$B = \frac{(z+1)\cancel{(z-0.4)}}{(z-1)\cancel{(z-0.4)}(z-0.3)} \Big|_{z=0.4}$$

$$B = \frac{1.4}{-0.6 \times 0.1}$$

$$B = \frac{1.4}{-0.06}$$

$$\boxed{B = -23.33}$$

$$C = \frac{Y(z)}{z} \times (z-0.3) \Big|_{z=0.3}$$

$$C = \frac{(z+1)\cancel{(z-0.3)}}{(z-1)(z-0.4)\cancel{(z-0.3)}} \Big|_{z=0.3}$$

$$C = \frac{1.3}{-0.7 \times -0.1}$$

$$C = \frac{1.3}{0.07}$$

$$\boxed{C = 18.57}$$

Substituting the value of A, B, & C in ①

$$\frac{Y(z)}{z} = \frac{4.76}{z-1} - \frac{23.33}{z-0.4} + \frac{18.57}{z-0.3}$$

$$Y(z) = \frac{4.76z}{z-1} - 23.33\frac{z}{z-0.4} + 18.57\frac{z}{z-0.3}$$

Taking Inverse z transform.

$$\boxed{y(n) = 4.76(1)^n u(n) - 23.33(0.4)^n u(n) + 18.57(0.3)^n u(n)}$$

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Determine the pole zero plot for the System described by the difference equation.

$$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n) - x(n-1)$$

Solution:-

$$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n) - x(n-1)$$

Taking z transform on both sides, we get.

$$Y(z) = \frac{5}{6} z^{-1} Y(z) - \frac{1}{6} z^{-2} Y(z) + X(z) - z^{-1} X(z)$$

$$Y(z) - \frac{5}{6} z^{-1} Y(z) + \frac{1}{6} z^{-2} Y(z) = X(z) - z^{-1} X(z)$$

$$Y(z) \left[1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2} \right] = X(z) \left[1 - z^{-1} \right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}.$$

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{z^{-2} \left(\frac{1}{z^{-2}} - \frac{5}{6 z^{-1}} + \frac{1}{6} \right)}$$

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{z^{-2} \left(z^{2} - \frac{5}{6} z + \frac{1}{6} \right)}$$

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$$\therefore \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{z^{2}\left(\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)\right)}$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{z^{2}(1-z^{-1})}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{z^{2}-z}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

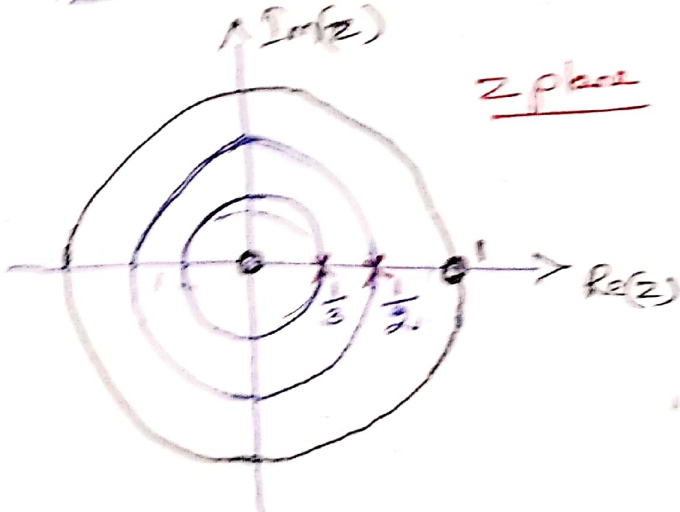
$\therefore$

$$\therefore \frac{Y(z)}{X(z)} = \frac{z(z-1)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$ and ROC: $|z| > \frac{1}{2}$

The zeros are $z=0$, $z=1$

The poles are $z=\frac{1}{3}$, $z=\frac{1}{2}$

Pole-zero plot



Z Transform

* Transform techniques are an important tool in the analysis of signals and linear time invariant (LTI) systems.

* Z transform is a mathematical tool which is used to convert the difference equations in time domain into the algebraic equations in frequency domain.

* Z transform is used for the digital signal

* The stability of the linear time - invariant (LTI) system can be determined using the Z transform.

* There are three types of Z transform

  (i) Right Sided.
  (ii) Left Sided
  (iii) Two Sided.

* Z transform gives the idea of the region where the signal may be stable, causal, bounded, zero etc.

* The Z transform is a operation that turns a sequence of numbers into a polynomial.

* Z transform provides a means of characterizing an LTI systems and its response to various signals, by its pole - zero locations.

The z transform of a discrete time signal x(n) is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{—— (1)}$$

$z \rightarrow$ complex variable.

This is called direct z-transform because it transforms the time-domain signal x(n) into its complex plane representation X(z).

In polar form z can be expressed as.

$$z = re^{j\omega} \quad \text{—— (2)}$$

where r is the radius of the unit circle.

Sub (2) in (1)

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$$

If $r = 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

The above equation is called Fourier transform for discrete time signal x(n) evaluated on unit circle.

Types of z transform.

1. Right sided or Right hand Sequence.

* A right hand Sequence is one for which $x(n)=0$ for all $n<n_0$, where n_0 can be positive or negative but finite.

* If n_0 is greater than or equal to zero, the resulting sequence is causal or positive time sequence.

* The ROC for right handed Sequence is entire z-plane except at $z=0$

2. Left sided or Left hand Sequence.

* A left-hand sequence $x(n)$ is one for which $x(n)=0$ for all $n \geq n_0$ where n_0 is positive or negative but finite.

* If $n_0 \leq 0$ the resulting sequence is anticausal sequence.

* The ROC for left handed Sequence. is entire z-plane except at $z=\infty$.

Region of Convergence (ROC) for a two-sided z transform.

The z transform of a discrete-time signal $x(n)$ is expressed as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{--- (1)}$$

Here z is a complex variable.

The above expression is called a two-sided z-transform.

The expression in (1) may be written as.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{-1} x(n) z^{-n} + \sum_{n=0}^{\infty} x(n) z^{-n}.$$

$$X(z) = \sum_{n=1}^{\infty} x(-n) z^{n} + \sum_{n=0}^{\infty} x(n) z^{-n}.$$

From above expression, it may be observed that the first series is a non-causal sequence which converges for $|z| < r_2$.

The second series is a causal sequence which converge for $|z| > r_1$.

Hence, the series represented by the expression

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

converges for $r_1 < |z| < r_2$ provided $r_1 < r_2$.

... ergence (ROC)

* Region of Convergence is the region where z-transform converges.

* z transform is an infinite power series.
* Infinite power series is not convergent for all values of z. Hence ROC is useful in mentioning z transform.

Significance of ROC :-

(i) ROC gives an idea about values of z for which z-transform can be calculated.
(ii) ROC can be used to determine causality of the system.
(iii) ROC can be used to determine stability of the system.

Properties of Region of Convergence (ROC)

1. The ROC consists of ring or disc in the z-plane centered at origin.

2. The ROC does not contain any poles.

3. If $x(n)$ is a causal sequence then the ROC is the entire z-plane except at $z=0$.

4. If $x(n)$ is a non-causal sequence then the Roc is the entire Z-plane except at $z=\infty$

5. If $x(n)$ is a finite duration, two sided sequence the Roc is entire z plane at $z=0$ and $z=\infty$.

6. The Roc of a LTI stable system contains the unit circle.

7. Roc must be a connected region.

A discrete-time signal is expressed as $x(n) = \cos \omega_0 n$ for $n \geq 0$. Find its z-transform:

Solution:-

$$x(n) = \cos \omega_0 n \quad \text{for } n \geq 0$$

$$\cos \omega_0 n = \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n} \right]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

for $n \geq 0$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n} \right] z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n} + \frac{1}{2} \sum_{n=0}^{\infty} e^{-j\omega_0 n} z^{-n}.$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(e^{j\omega_0} z^{-1} \right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(e^{-j\omega_0} z^{-1} \right)^n$$

$$\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.}$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0} z^{-1}} \right] + \frac{1}{2}\left[\frac{1}{1-e^{-j\omega_0} z^{-1}} \right]$$

Taking $\frac{1}{2}$ common

$$= \frac{1}{2}\left[\frac{1}{(1-e^{j\omega_0}z^{-1})} + \frac{1}{(1-e^{-j\omega_0}z^{-1})}\right]$$

$$X(z) = \frac{1}{2}\left[\frac{(1-e^{-j\omega_0}z^{-1}) + (1-e^{j\omega_0}z^{-1})}{(1-e^{j\omega_0}z^{-1})\,(1-e^{-j\omega_0}z^{-1})}\right]$$

$$= \frac{1}{2}\left[\frac{2-e^{-j\omega_0}z^{-1}-e^{j\omega_0}z^{-1}}{1-e^{j\omega_0}z^{-1}-e^{-j\omega_0}z^{-1}+e^{j\omega_0}z^{-1}\,e^{-j\omega_0}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{2-\left[e^{-j\omega_0}z^{-1}+e^{j\omega_0}z^{-1}\right]}{1-z^{-1}(e^{j\omega_0}+e^{-j\omega_0})+z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2-\left[e^{-j\omega_0}+e^{j\omega_0}\right]z^{-1}}{1-z^{-1}(e^{j\omega_0}+e^{-j\omega_0})+z^{-2}}\right]$$

$$= \frac{1}{2}\left[\frac{2-2\cos\omega_0\, z^{-1}}{1-z^{-1}\,2\cos\omega_0+z^{-2}}\right]$$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{2}\,(1-\cos\omega_0\, z^{-1})}{1-z^{-1}\,2\cos\omega_0+z^{-2}}$$

$$X(z) = \frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$$

$$X(z) = \frac{1 - \frac{1}{z}\cos\omega_0}{1 - \frac{2}{z}\cos\omega_0 + \frac{1}{z^2}}$$

$$X(z) = \frac{\dfrac{z - \cos\omega_0}{z}}{\dfrac{z^2 - 2z\cos\omega_0 + 1}{z^2}}$$

$$X(z) = \frac{(z - \cos\omega_0) \times z^2}{z\,(z^2 - 2z\cos\omega_0 + 1)}$$

$$\boxed{X(z) = \frac{(z - \cos\omega_0)\,z}{z^2 - 2z\cos\omega_0 + 1}}$$

Problem:

Determine Z - transform of the following Sequence.

$x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$

Solution:-

$x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$

The given sequence . is a causal signal.

From the given data $x_1(0) = 1,\ x_1(1) = 2,\ x_1(2) = 3,\ x_1(3) = 4$

$x_1(4) = 5,\ x_1(5) = 0,\ x_1(6) = 7$

By definition $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

But here in this question we are not having any negative values so

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$X_1(z) = \sum_{0}^{6} x_1(n) z^{-n}$$

$$= x_1(0) z^{0} + x_1(1) z^{-1} + x_1(2) z^{-2} + x_1(3) z^{-3} + x_1(4) z^{-4} + x_1(5) z^{-5} + x_1(6) z^{-6}$$

Substituting the values.

$$= 1.1 + 2.z^{-1} + 3.z^{-2} + 4z^{-3} + 5.z^{-4} + 0.z^{-5} + 7.z^{-6}$$

$$\boxed{X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 7z^{-6}}$$

$X_1(z) = \infty$ for $z=0$

ie $X_1(z)$ is convergent for all values of z, except $z=0$

Hence Region of convergence is entire z plane except $z=0$.

Problem

Determine z transform of the following sequence.

$x_2(n) = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$.

Solution: -

$x_2(n) = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$

The given sequence is two sided or bidirectional.

From the given question

$x_2(-3) = 1$, $x_2(-2) = 2$, $x_2(-1) = 3$, $x_2(0) = 4$, $x_2(1) = 5$, $x_2(2) = 0$

$x_2(3) = 7$

By definition $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$X_2(z) = \sum_{n=-3}^{3} x_2(n) z^{-n}$$

$$X_2(z) = x_2(-3) z^3 + x_2(-2) z^2 + x_2(-1) z^1 + x_2(0) z^0 + x_2(1) z^{-1} + x_2(2) z^{-2} + x_2(3) z^{-3}$$

Substituting the values.

$$= 1 \cdot z^3 + 2 \cdot z^2 + 3 \cdot z + 4 + 5 z^{-1} + 0 \cdot z^{-2} + 7 \cdot z^{-3}$$

$$\boxed{X_2(z) = z^3 + 2z^2 + 3z + 4 + 5z^{-1} + 7z^{-3}}$$

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$X_3(z) = \infty$ for $z=0$ and $z=\infty$

Hence Region of Convergence is entire z plane except $z=0$ and $z=\infty$

Problem

Determine z transform of the following sequence.

$$x(n) = \{-3, -2, -1, 0, \underset{\uparrow}{1}\}$$

Solution:-

$$x(n) = \{-3, -2, -1, 0, \underset{\uparrow}{1}\}$$

The given sequence is anticausal.

from the given question

$x(-4) = -3$, $x(-3) = -2$, $x(-2) = -1$, $x(-1) = 0$, $x(0) = 1$

By definition $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

Since the sequence is anticausal.

$$X(z) = \sum_{n=-\infty}^{0} x(n) z^{-n}$$

$$X(z) = \sum_{n=-4}^{0} x(n) z^{-n}$$

$$= x(-4) z^4 + x(-3) z^3 + x(-2) z^2 + x(-1) z^1 + x(0) z^0$$

$$= -3z^4 - 2z^3 - 1z^2 + 0.z + 1.$$

$$= -3z^4 - 2z^3 - 1z^2 + 1$$

$$\boxed{X(z) = 1 - z^2 - 2z^3 - 3z^4}$$

$X_1(z) = \infty$ for $z = \infty$

$X(z)$ is convergent for all values of z except at $z = \infty$.

Hence Region of convergence is entire z plane except $z = \infty$

Problem:

Determine the z transform of the signal.

$$x(n) = \left(\frac{1}{2}\right)^n u(n), \quad n \geq 0$$

Solution:-

The given signal is a causal signal and of infinite duration.

By definition

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Since the signal is causal

$$= \sum_{n=0}^{\infty} x(n)\, z^{-n}.$$

Sub the value of $x(n)$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n u(n) \cdot z^{-n}$$

But the value of $u(n) = 1$ for $n \geq 0$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

But $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$.

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}.$$

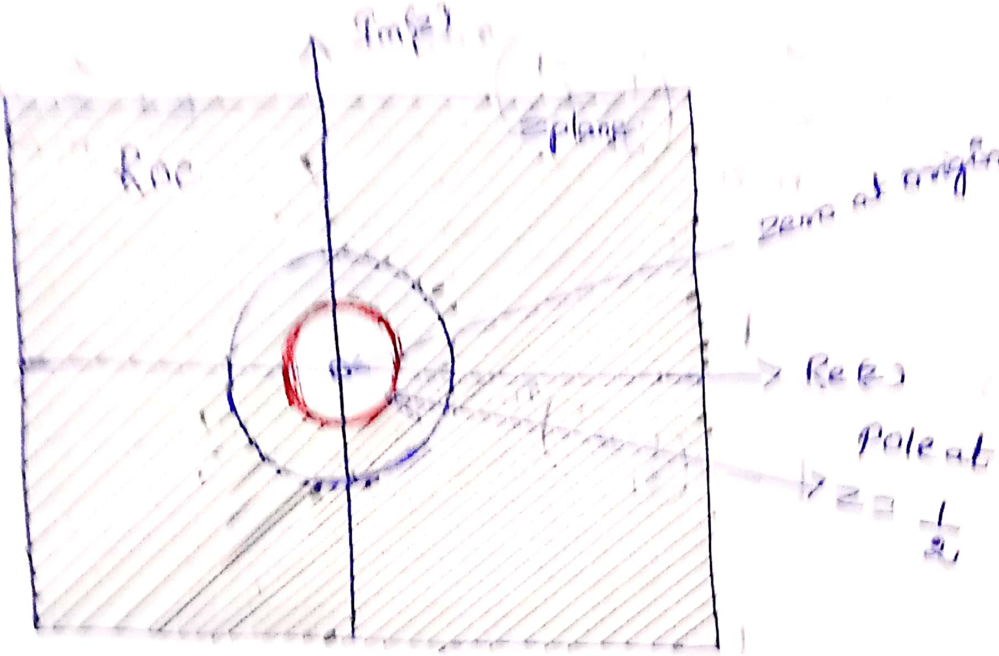
$$X(z) = \frac{1}{1 - \frac{1}{2z}}$$

multiply a z in the numerator and denominator.

$$X(z) = \frac{z}{z\left(1 - \frac{1}{2z}\right)}$$

$$X(z) = \frac{z}{z - \frac{1}{2}}$$

The region of convergence $|z| > \frac{1}{2}$.



Problem

Ex: Determine the z transform and Region of Convergence (ROC) of the signal $x(n) = a^n u(n)$, $n \geq 0$

Solution:-

The given signal is of infinite duration and is causal.

By definition

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n) z^{-n}$$

Sub the value of $x(n)$

$$= \sum_{n=0}^{\infty} a^n u(n) z^{-n}$$

But $u(n) = 1$, for $n \geq 0$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n \qquad \text{But } \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$= \frac{1}{1-az^{-1}}$$

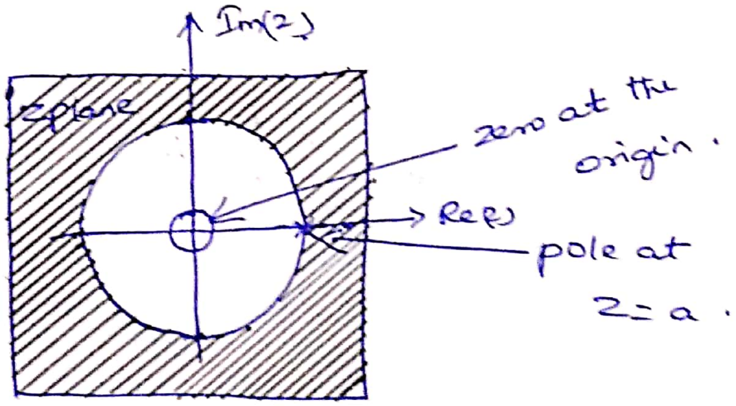
$$= \frac{1}{1-\frac{a}{z}}$$

multiply a z in the NR and DR.

$$= \frac{z}{z(1-\frac{a}{z})}$$

$$X(z) = \frac{z}{z-a}$$

ROC is $|z| > a$.



Find the z-transform and ROC of the Signal $x(n) = -a^n u(-n-1)$

Solution:-

The given signal is anticausal and is of infinite duration.

By definition of z transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$

$u(-n-1) = 0, n \geq 0$

$u(-n-1) = 1, n \leq -1$

$$X(z) = \sum_{-\infty}^{-1} -a^n z^{-n}.$$

$$= -\sum_{-\infty}^{-1} a^n z^{-n}.$$

$$= -\sum_{n=1}^{\infty} (a^{-1} z^{1})^n$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$

$$= -\left[\sum_{n=0}^{\infty} (a^{-1} z)^n - 1\right]$$

$$X(z) = -\left[\frac{1}{1-a^{-1}z} - 1\right]$$

$$= 1 - \frac{1}{1-a^{-1}z}.$$

$$= \frac{1-a^{-1}z-1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{-\frac{z}{a}}{1-\frac{z}{a}}$$

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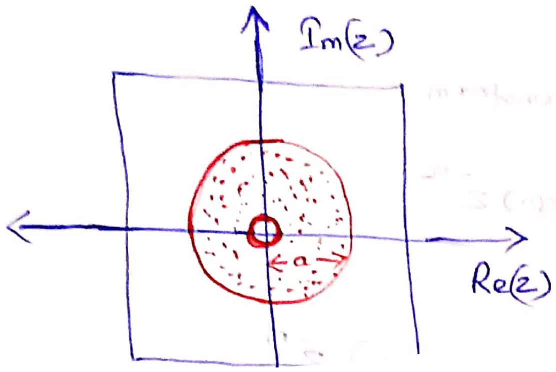
$$= \frac{-\frac{z}{a}}{\frac{a-z}{a}} = \frac{-z}{a-z} = \frac{z}{z-a}$$

Region of Convergence is $\frac{z}{z-a}$ $|z| < a$.



∴ From the above figure it is clear that ROC of a anticausal Signal is interior of a circle of given radius.

Find the z transform and ROC of the signal

$$x(n) = a^n u(n) + b^n u(-n-1)$$

By the definition of z transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}.$$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{n=-\infty}^{-1} (bz^{-1})^n.$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{n=1}^{\infty} (b^{-1}z)^n$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{n=0}^{\infty} (b^{-1}z)^n - 1$$

$$= \frac{1}{1-az^{-1}} + \frac{1}{1-b^{-1}z} - 1$$

$$= \frac{1}{1-az^{-1}} + \frac{1-1(1-b^{-1}z)}{1-b^{-1}z}$$

$$= \frac{1}{1-az^{-1}} + \frac{1-1+b^{-1}z}{1-b^{-1}z}$$

$$X(z) = \frac{1}{1-az^{-1}} + \frac{b^{-1}z}{1-b^{-1}z} = \frac{1}{1-\frac{a}{z}} + \frac{\frac{z}{b}}{1-\frac{z}{b}} = \frac{z}{z-a} + \frac{z}{b-z} = \frac{z}{z-a} - \frac{z}{z-b}$$

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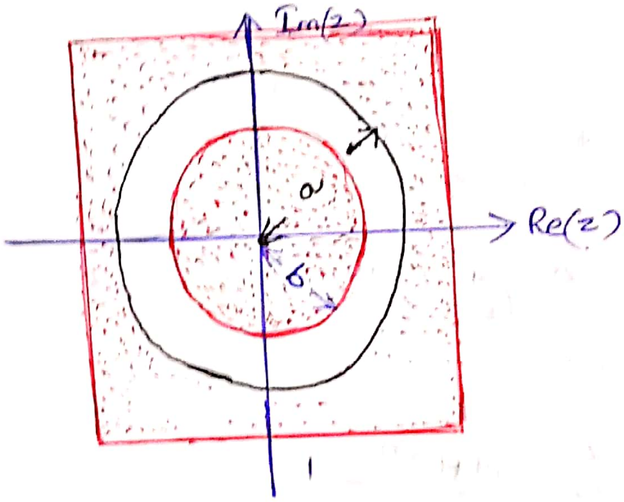
* The first power series converges if $|az^{-1}| < 1$ or $|z| > |a|$
and the second power series converges if $|b^{-1}z| < 1$ or $|z| < |b|$

* If $|b| < |a|$, the two ROCs do not overlap and
$X(z)$ does not exist.



If $|b| > |a|$ the two ROCs overlap and $X(z)$ exists.
Therefore the ROC of $X(z)$ is $|a| < |z| < |b|$.
That is for an infinite duration two-sided signal the ROC is a ring in the z plane.

$$X(z) = \frac{z}{z-a} - \frac{z}{z-b}$$

ROC : $|a| < |z| < |b|$

Discrete Fourier Series:

* Any periodic function can be expressed in a Fourier Series representation.

* Consider a discrete-time Signal $x(n)$ that is periodic with period N defined by $x(n) = x(n+kN)$ for any integer value of k.

* The periodic function $x(n)$ can be synthesised as the sum of sine and cosine sequences or equivalently a linear combination of complex exponentials whose frequencies are multiples of the fundamental frequency $\frac{2\pi}{N}$

* This is done by constructing a periodic sequence for which each period is identical to the finite-length sequence.

Exponential Form of Discrete Fourier Series.

* A real periodic discrete-time signal $x(n)$ of period N can be expressed as a weighted sum of complex exponential sequences.

Discrete Fourier Series

* DFS is a frequency analysis tool for periodic infinite duration discrete-time signals which is practical because it's discrete in frequency

Let $x(n)$ be a periodic sequence with fundamental period N where N is a positive integer.

$$x(n) = x(n+rN)$$

for any integer value of r

Let $x(t)$ be the continuous time counter part of $x(n)$.

According to Fourier Series expansion

$x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\Omega t} = \sum_{-\infty}^{\infty} a_k e^{j\frac{2\pi k}{T_p}t}$$

which has frequency components $\Omega = 0, \pm\Omega_0, \pm 2\Omega_0, \pm 3\Omega_0$

Sub $x(t) = x(n)$, $T_p = N$ and $t = n$

$$x(n) = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kn}{N}} \quad \text{——— (1)}$$

(1) is valid for discrete time signals as only the sample points of $x(t)$ are considered.

$x(n)$ has frequency components at $\omega = 0, \pm\frac{2\pi}{N}, \pm\left(\frac{2\pi}{N}\right)2, \ldots$

and the respective complex exponentials are $e^{j\left(\frac{2\pi}{N}(0)\right)}, e^{j\frac{2\pi}{N}(1)}, \ldots, e^{j\frac{2\pi}{N}(N-1)}$

and the infinite summation is reduced to

$$x(n) = \sum_{k=0}^{N-1} a_k e^{\frac{j2\pi kn}{N}}$$

Defining $X(k) = Na_k$, $k = 0, 1, \dots N-1$, as the DFS coefficients,

the inverse DFS formula is given as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} \quad , \; n = 0, 1, 2, \dots, N-1.$$

The formula for converting $x(n)$ to $X(k)$ is derived as follows

Multiplying both sides of above equation by $e^{-j\left(\frac{2\pi}{N}\right) rn}$ and

Summing from $n=0$ to $n=N-1$

$$\sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right) rn} = \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} \right) e^{\frac{-j2\pi rn}{N}}$$

$$= \sum_{n=0}^{N-1} \frac{1}{N} \left(\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi (k-r) n}{N}} \right)$$

$$= \sum_{n=0}^{N-1} \frac{1}{N} \left(\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi (k-r) n}{N}} \right)$$

$$= \sum_{k=0}^{N-1} X(k) \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{\frac{j2\pi (k-r) n}{N}} \right]$$

Using the orthogonality identity of complex exponentials

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{\frac{j2\pi (k-r)n}{N}} = \begin{cases} 1, & k-r = mN, \; m \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi rn}{N}} = X(r)$$

which is also periodic with period N

$$W_N = e^{\frac{-j2\pi}{N}}$$

7.b). Determine the frequency response of an LTI system described by

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n).$$

Soln :

By taking FT,

$$Y(e^{jw}) \left[1 - \frac{3}{4} e^{-jw} + \frac{1}{8} e^{-j2w}\right] = X(e^{jw})$$

$$\frac{Y(e^{jw})}{X(e^{jw})} = \frac{1}{1 - \frac{3}{4} e^{-jw} + \frac{1}{8} e^{-j2w}}.$$

The frequency response,

$$H(e^{jw}) = \frac{1}{1 - \frac{3}{4} e^{-jw} + \frac{1}{8} e^{-j2w}}.$$

The Amplitude response is,

$$|H(e^{jw})| = \frac{1}{1 - \left(\frac{3}{4} \cos w - j\frac{3}{4} \sin w\right) + \frac{1}{8}\left(\cos 2w - j\sin 2w\right)}$$

$$|H(e^{jw})| = \frac{1}{\sqrt{\left(1 - \frac{3}{4} \cos w + \frac{1}{8} \cos 2w\right)^2 + \left(\frac{3}{4} \sin w - \frac{1}{8} \sin 2w\right)^2}}$$

The phase response,

$$\angle H(e^{jw}) = -\tan^{-1}\left[\left(\frac{3}{4} \sin w - \frac{1}{8} \sin 2w\right) \Big/ \left(1 - \frac{3}{4} \cos w + \frac{1}{8} \cos 2w\right)\right]$$

7. a). What are the necessary and sufficient conditions for stability of a system? prove it.

Solution: Let $x(n)$ be a bounded i/p sequence satisfying $|x(n)| \le M_x < \infty$,

$h(n)$ be the Impulse response of the system, then the output of the system $y(n)$ can be found using Convolution Sum

$$y(n) = \sum_{k=-\infty}^{\infty} x(k).h(n-k). = \sum_{k=-\infty}^{\infty} h(k).x(n-k).$$

The magnitude of the output is given by

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k).x(n-k) \right|$$

WKT, the magnitude of the Sum of terms is less than or equal to the Sum of the magnitudes, hence

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k).x(n-k) \right| \le \sum_{k=-\infty}^{\infty} |h(k)|.|x(n-k)|.$$

Let the bounded value of the i/p as 'M', then

$$|y(n)| \le M \sum_{k=-\infty}^{\infty} |h(k)|.$$ This condition is

Satisfied when $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$.

$\therefore$ The necessary and sufficient condition for stability is $\boxed{\sum_{n=-\infty}^{\infty} |h(n)| < \infty}$.

6. b). Determine the convolution of the following signal sequences.

$$x(n) = 2^n u(n).$$

$$h(n) = \left(\frac{1}{2}\right)^n u(n).$$

Solution: The given signal is a causal signal.

$\therefore$ The convolution output can be found by using

$$y(n) = \sum_{k=0}^{n} x(k) \cdot h(n-k).$$

$$= \sum_{k=0}^{n} 2^k \cdot \left(\frac{1}{2}\right)^{n-k}.$$

$$= \left(\frac{1}{2}\right)^n \cdot \sum_{k=0}^{n} 2^k \cdot 2^k$$

$$= \left(\frac{1}{2}\right)^n \cdot \sum_{k=0}^{n} 4^k$$

$$= \left(\frac{1}{2}\right)^n \cdot \left[\frac{(4)^{n+1} - 1}{4-1}\right]$$

$$y(n) = \left(\frac{1}{2}\right)^n \left[\frac{(4)^{n+1} - 1}{3}\right].$$

Inverse z - Transform.

The inverse z-transform is defined as.

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz.$$

where the integral is a contour integral over a closed path c. that encloses the origin and lies within the region of convergence of X(z), the closed contour C is a circle in the ROC of X(z) in the z-plane.

The methods to determine inverse z-transform are.

(i) Power series or long division method

(ii) Partial fraction expansion method.

(iii) Cauchy Residue method

(iv) Convolution method.

Power series or long division method.

By the definition of one sided z-transform.

It can be expanded into an infinite powers of z^{-1} as follows.

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots +$$

when $x(z)$ is rational, the expansion can be performed by the value of $x(n)$ at any instant of time are the coefficients of z^{-k}.

If $X(z)$ can be expressed as a ratio of two polynomials the coefficients $x(0), x(1), \dots x(n)$ can be obtained by Synthetic division of the numerator by the denominator as follows.

$$X(z) = \frac{N(z)}{D(z)}$$

$$= \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}.$$

$$X(z) = x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots$$

The Values of the sequence represent the inverse z-transform of $x(n)$ for $n \geq 0$ is a causal sequence.

The z transform for non-causal or anticausal signal $x(n)$ represented as.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}} \dots + x(-2) z^{2} +$$

$$x(-1) z + x(0)$$

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Determine the inverse z transform of $X(z) = \dfrac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$.

when (a) ROC is $|z| > 1$ (b) ROC : $|z| < 0.5$

Solution:-

(i) ROC is $|z| > 1$

From ROC $x(n)$ is a causal sequence.

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}.$$

$$
\begin{array}{r|l}
 & 1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} \\
\hline
1 - 1.5z^{-1} + 0.5z^{-2} & 1 \\
 & (-)1 \overset{(+)}{-} 1.5z^{-1} \overset{(-)}{+} 0.5z^{-2}. \\
\hline
 & 0 + 1.5z^{-1} - 0.5z^{-2}. \\
 & \overset{(-)}{+} 1.5z^{-1} \overset{(+)}{-} 2.25z^{-2} \overset{(-)}{+} 0.75z^{-3} \\
\hline
 & 1.75z^{-2} - 0.75z^{-3} \\
 & (-)1.75z^{-2} \overset{(+)}{-} 2.625z^{-3} \overset{(-)}{+} 0.875z^{-4} \\
\hline
 & 1.875z^{-3} - 0.875z^{-4}. \\
 & (-)1.875z^{-3} \overset{(+)}{-} 2.8125z^{-4}. \overset{(-)}{+} 0.9375z^{-5} \\
\hline
 & 1.9375z^{-4} - 0.9375z^{-5}
\end{array}
$$

$$X(z) = 1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} + \ldots$$

We know that

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \ldots$$

Comparing the two equations.

$$x(n) = \{1, 1.5, 1.75, 1.875, \dots\}$$

(b) ROC : $|z| < 0.5$

The ROC is interior of the circle. Therefore the given Signal is anti-causal.

For anticausal signal the long-division method is obtained by the following.

$$2z^2 + 6z^3 + 14z^4 + 30z^5$$

$$0.5z^{-2} - 1.5z^{-1} + 1 \;\big)\; 1$$

$$(-)1 \;(+)3z \;(\mp)2z^2$$

$$3z - 2z^2$$

$$(-)3z \;(+)9z^2 \;(\mp)6z^3$$

$$7z^2 - 6z^3$$

$$(-)7z^2 \;(+)21z^3 \;(\mp)14z^4$$

$$15z^3 - 14z^4$$

$$(-)15z^3 \;(+)45z^4 \;(\mp)30z^5$$

$$31z^4 - 30z^5$$

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$$X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5$$

But

$$X(z) = \dots + z^4 x(-4) + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2}$$

$$X(z) = x(-5)z^5 + x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2}.$$

Comparing the equation we will get.

$$x(n) = \{30, 14, 6, 2, 0, \underset{\uparrow}{0}\}$$

Problem:-

Find the inverse z-transform of $X(z) = \dfrac{z+0.2}{(z+0.5)(z-1)}$ $|z|>1$

Solution:-

$$X(z) = \frac{z+0.2}{z^2 - 0.5z - 0.5}$$

$$\begin{array}{r|l}
 & z^{-1} + 0.7z^{-2} + 0.85z^{-3} + 0.775z^{-4} \\
\hline
z^2 - 0.5z - 0.5 & z + 0.2 \\
 & (-)z \overset{(+)}{-} 0.5 \overset{(+)}{-} 0.5z^{-1} \\
\hline
 & 0.7 + 0.5z^{-1} \\
 & (-)0.7 \overset{(+)}{-} 0.35z^{-1} \overset{(+)}{-} 0.35z^{-2} \\
\hline
 & 0.85z^{-1} + 0.35z^{-2} \\
 & (-)0.85z^{-1} \overset{(+)}{-} 0.425z^{-2} \overset{(+)}{-} 0.425z^{-3} \\
\hline
 & 0.775z^{-2} + 0.425z^{-3} \\
 & (-)0.775z^{-2} \overset{(+)}{-} 0.3875z^{-3} \overset{(+)}{-} 0.3875z^{-4} \\
\hline
 & 0.8125z^{-3} + 0.3875z^{-4}
\end{array}$$

$$X(z) = \frac{z+0.2}{z^2 - 0.5z - 0.5}$$

$$X(z) = z^{-1} + 0.7z^{-2} + 0.85z^{-3} + 0.775z^{-4}$$

But

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots +$$

Comparing we get

$$x(n) = \{\underset{\uparrow}{0}, 1, 0.7, 0.85, 0.775\}$$

Note!-

1) In each step of the long division process, the lowest power term of z^{-1} is eliminated.

2) Long division method is not suitable for $x(n)$ when n is large. The method becomes tedious. It is suitable to determine the values of the first few samples of the signal.

Partial fraction expansion method

Consider a rational function $X(z)$ given by

$$X(z) = \frac{N(z)}{D(z)}$$

$$= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \quad \text{—} (1)$$

A rational function of the form as given in (1) is called proper rational function if $a_N \neq 0$ and $M < N$.

An improper rational function $(M \geq N)$ can be written as the sum of a polynomial and a proper rational function.

Improper rational function can be expressed as.

$$X(z) = \frac{N(z)}{D(z)}$$

$$= c_0 + c_1 z^{-1} + \dots + C_{M-N} z^{-(M-N)} + \frac{N_1(z)}{D_1(z)}$$

Consider a rational function as shown in ①.

$$X(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}.$$

The negative powers of z is eliminated by multiplying both numerator and denominator by z^N.

$$X(z) = \frac{z^N \left(b_0 + b_1 z^{-1} + \dots + b_M z^{-M}\right)}{z^N \left(1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}\right)}$$

$$X(z) = \frac{z^N b_0 + b_1 z^{N-1} + \dots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N z^{N-N}}$$

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \dots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N} \quad -②$$

$\boxed{z^0 = 1}$

Dividing the equation ② by z.

$$\frac{X(z)}{z} = \frac{b_0 z^N + b_1 z^{N-1} + \dots + b_M z^{N-M}}{z\left(z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N\right)}$$

$$\frac{X(z)}{z} = \frac{z^{-1}\left(b_0 z^N + b_1 z^{N-1} + \dots + b_M z^{N-M}\right)}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N.}$$

$$= \frac{b_0 z^{N-1} + b_1 z^{N-2} + \dots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N}$$

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \dots + b_M z^{N-M-1}}{(z-P_1)(z-P_2)\dots(z-P_N)} \quad \text{--- (3)}$$

For distinct poles equation (3) can be expanded in the form as expressed by

$$\frac{X(z)}{(z)} = \frac{C_1}{z-P_1} + \frac{C_2}{z-P_2} + \dots + \frac{C_N}{z-P_N} \quad \text{--- (4)}$$

The coefficients $C_1, C_2, \dots C_N$ can be found by using the formula.

$$C_K = \left. \frac{(z-P_K)\,X(z)}{z} \right|_{z=P_K} \quad K = 1, 2, \dots, N.$$

If $X(z)$ has a pole of multiplicity l, (ie) the denominator contains the factor $(z-P_K)^l$ then equation (3) can be written as

$$\frac{X(z)}{z} = \frac{1}{(z-P_1)(z-P_2)^2} \quad \text{--- (5)}$$

The equation (5) can be expanded in partial fraction as.

$$\frac{X(z)}{z} = \frac{C_1}{z-P_1} + \frac{C_2}{z-P_2} + \frac{C_3}{(z-P_2)^2}.$$

where $C_1 = \frac{X(z)}{z}(z-P_1)\Big|_{z=P_1}$

$$C_2 = \frac{d}{dz}(z-P_2)^2 \frac{X(z)}{z}\Big|_{z=P_2}.$$

$$C_3 = (z-P_2)^2 \frac{X(z)}{z}\Big|_{z=P_2}.$$

The limits in the convolution sum can be modified according to the type of sequence and system.

For a causal system the impulse response $h(n) = 0$ for $n < 0$

The limits in the convolution sum of a causal system is modified as

$$y(n) = \sum_{k=-\infty}^{n} x(k)\, h(n-k)$$

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

If the input to the causal system is a causal sequence i.e $x(n) = 0$ for $n < 0$ the limits in the convolution sum is modified as

$$y(n) = \sum_{k=0}^{n} x(k)\, h(n-k)$$

$$= \sum_{k=0}^{n} h(k)\, x(n-k)$$

1.b). Determine the pole zero plot for the system described by the difference equation

$$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n) - x(n-1)$$

Solution:

Given the difference equation

$$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n) - x(n-1)$$

Take z-TFM on both sides, we get

$$Y(z) = \frac{5}{6} z^{-1} Y(z) - \frac{1}{6} z^{-2} Y(z) + X(z) - z^{-1} X(z).$$

$$Y(z)\left[1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}\right] = X(z)\left[1 - z^{-1}\right].$$

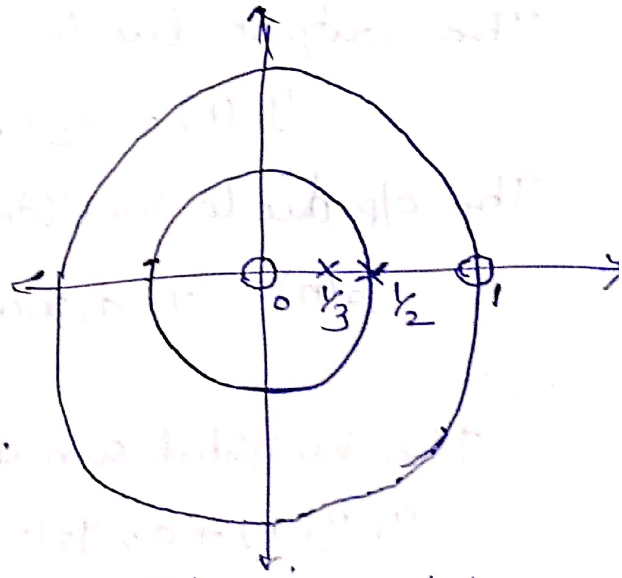
$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{\left[1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}\right]} = \frac{1 - z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - \frac{1}{3} z^{-1}\right)}.$$

and ROC : $|z| > \frac{1}{2}$.

$$\frac{Y(z)}{X(z)} = \frac{z(z-1)}{(z-\frac{1}{3})(z-\frac{1}{2})}.$$

∴ The Zeros are $z = 0, 1$.

and the poles are at $z = \frac{1}{3}, \frac{1}{2}$.



Pole-zero plot.

3. b). Determine whether the following signals are Energy or power signals.

(i) $x(n) = \left(\frac{1}{4}\right)^n u(n)$. (ii) $x(n) = \sin\left(\frac{n\pi}{3}\right)$.

(i) $x(n) = \left(\frac{1}{4}\right)^n u(n)$.

The Energy of the Signal $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$.

$$E = \sum_{n=\infty}^{\infty} \left[\left(\frac{1}{4}\right)^n u(n)\right]^2 = \sum_{n=0}^{\infty} \left[\left(\frac{1}{4}\right)^n\right]^2 = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n.$$

$$E = \frac{1}{1-\left(\frac{1}{16}\right)} = 1.06\cancel{67}$$

$\because \boxed{1 + a + a^2 + \dots + \infty = \frac{1}{1-a}}$

The Power $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{16}\right)^n$.

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{1-\left(\frac{1}{16}\right)^{N+1}}{1-\left(\frac{1}{9}\right)}\right].$$

$P = 0$.

The Energy is finite and power is Zero.

$\therefore$ The given signal is an <u>Energy Signal</u>.

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(ii) $x(n) = \sin\left(\frac{n\pi}{3}\right)$.

The Energy of the Signal $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$$E = \sum_{n=-\infty}^{\infty} \left|\sin^2\left(\frac{\pi n}{3}\right)\right|$$

$$E = \sum_{n=-\infty}^{\infty} \left[\frac{1-\cos\left(\frac{2\pi}{3}n\right)}{2}\right] = \infty.$$

The power of the Signal

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\sin^2\left(\frac{\pi}{3}n\right)\right|.$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1-\cos\frac{2\pi}{3}n}{2}$$

$$= \frac{1}{2} \lim_{N\to\infty} \frac{1}{2N+1} \boxed{\cancel{\sum_{n=-N}^{N} \left|\sin^2\right(}} \sum_{n=-N}^{N} 1$$

$$P = \frac{1}{2}$$

$\therefore$ The Energy is infinite and Power is finite.

Hence this signal is a power Signal.

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5. Determine the convolution sum of two sequences using graphical method.

$x(n) = \{3, 2, 1, 2\}$; $h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$

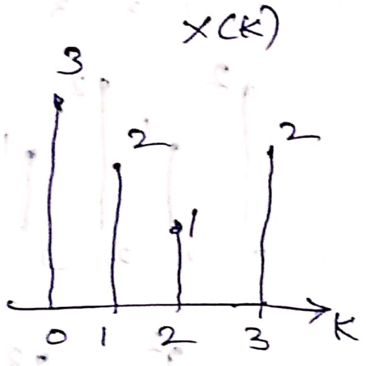
Solution:

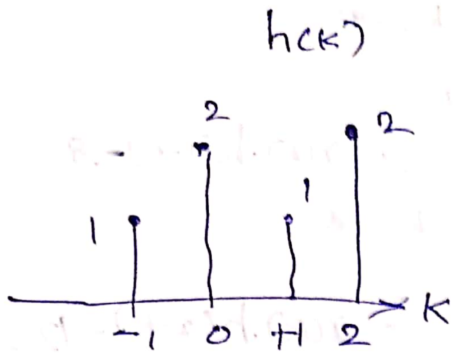
Step 1:

The sequence $x(n)$ starts at $n_1 = 0$ and $h(n)$ starts at $n_2 = -1$. Therefore the starting time for o/p sequence $y(n)$ is $n = n_1 + n_2 = 0 + (-1)$

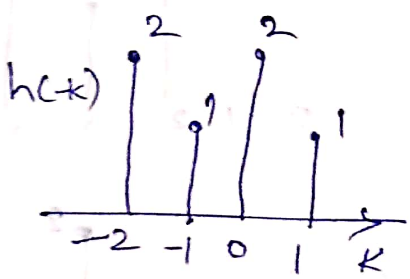
$n = -1$

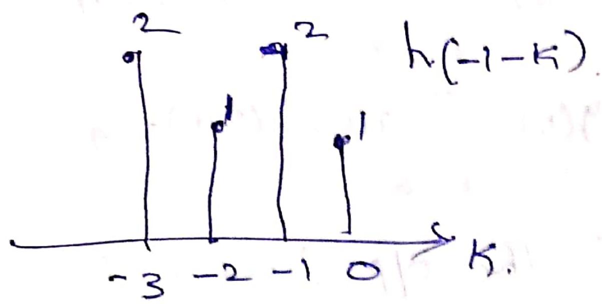
Step 2: Express both sequence in terms of index k.





Step 3 Fold (Time reverse) $h(k)$ about $k=0$ to obtain $h(-k)$.





(Shift $h(-k)$ left by one sample).

The convolution sum is

$$y(n) = \sum_{k=-\infty}^{\infty} x(k).h(n-k)$$

For $n=-1$, $y(-1) = \sum_{k=-\infty}^{\infty} x(k).h(-1-k)$

Multiply the two sequences $x(k)$ and $h(-1-k)$ element by element and sum the products.

$\therefore y(-1) = 0(2)+0(1)+0(2)+3(1)$
$+2(0)+1(0)+2(0).$

$$\boxed{y(-1) = 3}$$

For $n=0$,

$$y(0) = \sum_{k=-\infty}^{\infty} x(k).h(-k) = 8.$$

Similarly,

$$y(1) = \sum_{k=-\infty}^{\infty} x(k).h(1-k) = 8$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k).h(2-k) = 12.$$

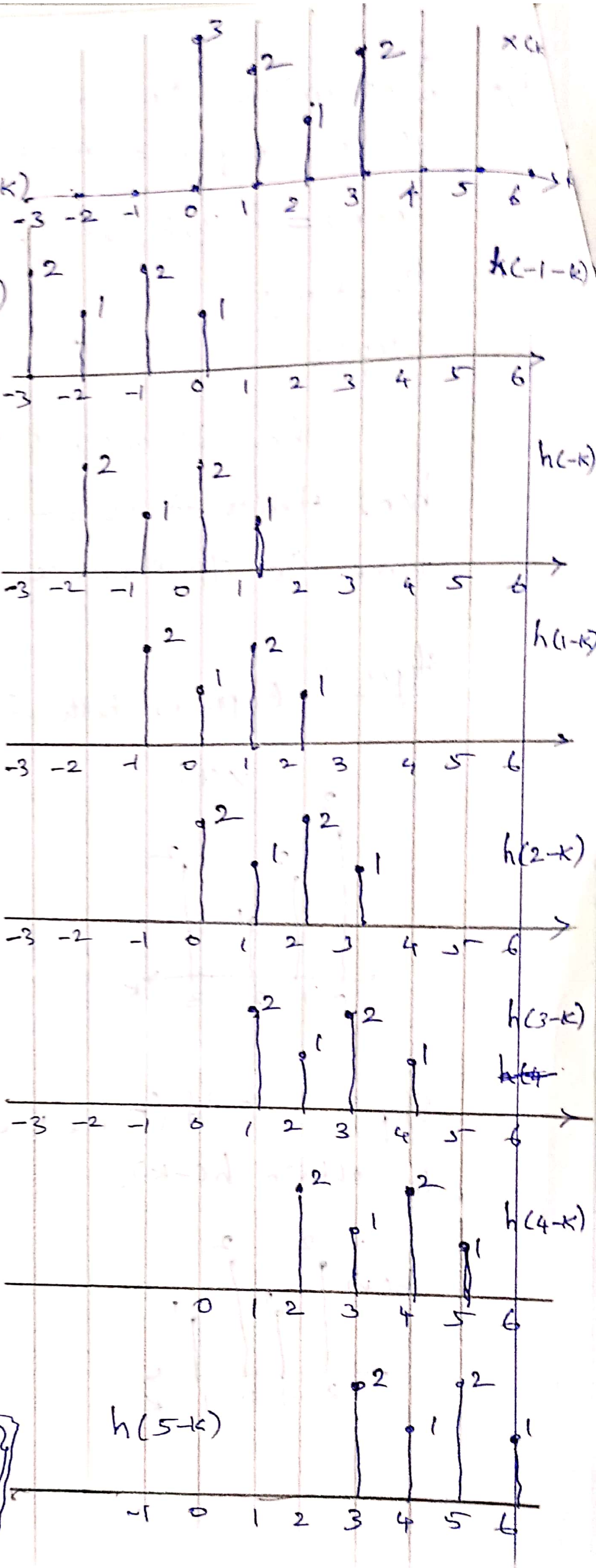
$$y(3) = \sum_{k=-\infty}^{\infty} x(k).h(3-k) = 9$$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k).h(4-k) = 4$$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k).h(5-k) = 4$$

$\therefore$ The o/p sequence is

$$\boxed{y(n) = \{3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4\}}$$



6. b). Determine the convolution of the following sequences $x(n) = 2^n u(n)$. & $h(n) = \left(\frac{1}{2}\right)^n u(n)$.

Solution:

Given that $\left.\begin{aligned} x(n) &= 2^n u(n) \\ h(n) &= \left(\frac{1}{2}\right)^n u(n) \end{aligned}\right\} \rightarrow$ causal sequence.

The convolution o/p sequence is given by

$$y(n) = \sum_{k=0}^{n} x(k) \cdot h(n-k).$$

$$= \sum_{k=0}^{n} 2^k \cdot \left(\frac{1}{2}\right)^{n-k}$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} 2^k \cdot \left(\frac{1}{2}\right)^{-k} = \left(\frac{1}{2}\right)^n \cdot \sum_{k=0}^{n} 2^{2k}.$$

$$= \left(\frac{1}{2}\right)^n \cdot \left[1 + 2^2 + 2^4 + 2^6 + \dots (n+1) \text{ terms}\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{(2^2)^{n+1} - 1}{(2)^2 - 1}\right]$$

$$y(n) = \left(\frac{1}{2}\right)^n \left[\frac{4 \cdot 4^n - 1}{3}\right] = \left(\frac{1}{2}\right)^n \left(\frac{4^{n+1} - 1}{3}\right)$$