PROBABILITY

Definitions :-

Random Experiment :- If an experiment is conducted any number of times under identical conditions, there is a set of all possible outcomes associated with it. If the result is not certain and in any one of the several possible outcomes the experiment is called a Random experiment (or) a trial.

Events :- The outcomes of a random experiment are called events (or) cases.

Equally Likely Events :- Events are said to be Equally Likely if all of them have equal chance to happen.

Exhaustive Events :- The total number of possible outcomes in any trial is known as Exhaustive Events.

Mutually Exclusive Events :- Two or more events are said to be Mutually Exclusive if the occurrence of one of the events prevents the occurrence of any of the remaining events.

Dependent Events :- Events are said to be dependent if the happening of one event depends on the happening of other event.

Independent Event :- Events are said to be independent if the happening of one event does not depend on the happening of other event.

Sample space :- The set of all possible out comes of any experiment is called sample space.

Sample point :- Each element of a sample space is called a sample point.

* Mathematical Definition :-
The probability of the occurence of the given event is earual to the ratio between the number of cases favourable

to the given event and the total number of possible cases, provided all the cases are symmetric

Probability $= \frac{m}{n}$ $\quad 0 \leq$ probability ≤ 1

* Classical probability :- If there are n-exhaustive, mutually exclusive and equally likely cases, out of which m cases are favourable to an event A then the probability of the happening of the event A is given by the ratio m/n.

$$P(A) = m/n \qquad m \leq n, \quad 0 \leq P(A) \leq 1$$

<OR>

Classical (or) Mathematical Definition of probability :-

The probability of the happening of the event is the ratio of numbers of favourable cases [m] to the total number of possible cases [n].

$$\text{Probability} = \frac{\text{no. of favourable cases}}{\text{total no. of possible cases}}$$

ie $P(E) = \frac{m}{n}$

E = Success , $\bar{E}$ = Failure

$$P(\bar{E}) = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

$$P(\bar{E}) = 1 - P(E)$$

$$P(E) + P(\bar{E}) = 1$$

$$P + q = 1 \Rightarrow P = 1 - q \Rightarrow q = 1 - P$$

$$0 \leq P(E) \leq 1$$

$$0 \leq P(\bar{E}) \leq 1$$

* Statistical probability :- suppose an experiment is repeated n- times under essential identical conditions. Let an event A happens m- times then m/n is defined as the relative frequency of A. The limit of this relative frequency as $n \to \infty$ is defined as the probability of A.

$$P(A) = \lim_{n \to \infty} m/n \qquad 0 \leq \frac{m}{n} \leq 1.$$

<OR>

Statistical Definition of probability :- Let an experiment is repeated 'n' times under favourable results for the event 'A' are 'm', then the statistical probability defined as follows

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

Favourable Events :- The number of outcomes which entail the happening of an event ~~are~~ called ~~the~~ ~~number~~ ~~of~~ favourable events.

Axioms of probability :-

1) Non - Negative : If 'A' is any event then

$$P(A) \geq 0$$

2) Certainty : If 'S' is a certain even then

$$P(S) = 1$$

3) Additivity : If A and B are two mutually exclusive events then.

$$P(A \cup B) = P(A) + P(B)$$

Axiomatic Definition of probability :-

Let 'A' be any event associated with the sample space 'S'. The probability of event 'A' is denoted by P(A) is real number satisfy following properties. These properties are known as axioms.

1) $P(A) \geq 0$

2) $P(S) = 1$

3) If $A_1, A_2, \dots, A_n$ are mutually exclusive events then

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$$

$$P\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} P(A_i)$$

Theorem : 1

Statement :- show that $P(\phi) = 0$

ie show that probability of an impossible event is zero.

Proof :- Let S and ϕ are mutually exclusive events.

$$S \cap \phi = \phi$$

$$S \cup \phi = S$$

Taking probability on both sides

$$P(S \cup \phi) = P(S)$$

According to 3rd Axiom

$$P(S) + P(\phi) = P(S)$$

$$P(\phi) = P(S) - P(S)$$

$$P(\phi) = 0$$

Theorem : 2

Statement :~ show that $P(A) = 1 - P(\bar{A})$ <OR>

$P(\bar{A}) = 1 - P(A)$

Proof :~ Let A and $\bar{A}$ are mutually exclusive events

$A \cap \bar{A} = \phi$

$A \cup \bar{A} = S$

Taking probability on both sides

$P(A \cup \bar{A}) = P(S)$

According to 3^{rd} Axiom

$P(A) + P(\bar{A}) = P(S)$

According to 2^{nd} Axiom

$P(S) = 1$

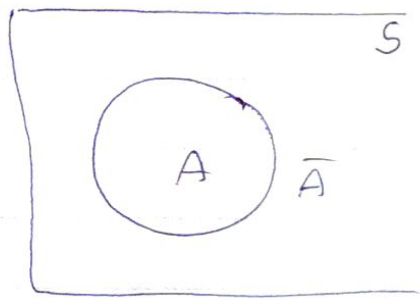
$P(A) + P(\bar{A}) = 1$

$P(A) = 1 - P(\bar{A})$

OR

$P(A) + P(\bar{A}) = 1$

$P(\bar{A}) = 1 - P(A)$.



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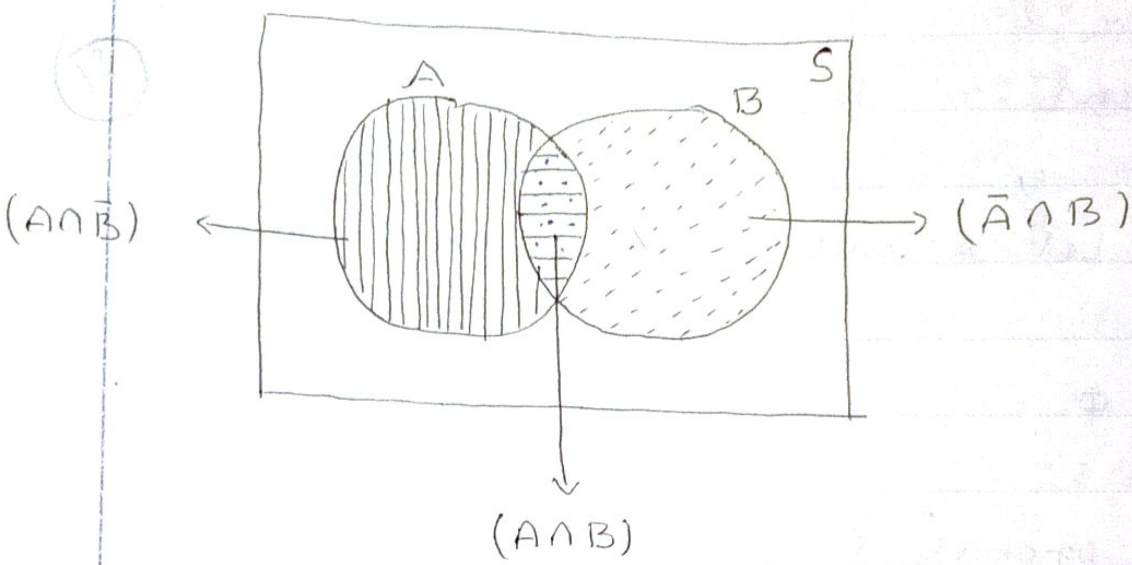


$A = (A \cap \bar{B}) \cup (A \cap B)$

$B = (\bar{A} \cap B) \cup (A \cap B)$

Theorem : 3

Statement :- If 'A' and 'B' are any two events then $P(A \cap \bar{B}) = P(A) - P(A \cap B)$.

Proof :- Let $(A \cap \bar{B})$ and $(A \cap B)$ are mutually exclusive events.

$(A \cap \bar{B}) \cap (A \cap B) = \phi$

$(A \cap \bar{B}) \cup (A \cap B) = A$

Taking probability on both sides

$P[(A \cap \bar{B}) \cup (A \cap B)] = P(A)$

According to 3^{rd} Axiom

$P(A \cap \bar{B}) + P(A \cap B) = P(A)$

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$

Theorem : 4 (15)

statement :- If 'A' and 'B' are any two events then $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

Proof :- Let $(\bar{A} \cap B)$ and $(A \cap B)$ are mutually exclusive events.

$(\bar{A} \cap B) \cap (A \cap B) = \phi$

$(\bar{A} \cap B) \cup (A \cap B) = B$.

Taking probability on both sides

$P[(\bar{A} \cap B) \cup (A \cap B)] = P(B)$

According to 3rd Axioms

$P(\bar{A} \cap B) + P(A \cap B) = P(B)$

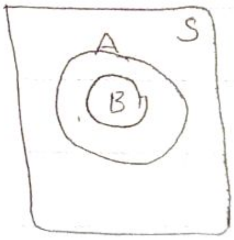
$P(\bar{A} \cap B) = P(B) - P(A \cap B)$

Theorem : 5 (16)

statement : If $B \subset A$ then show that

(i) $P(A \cap \bar{B}) = P(A) - P(B)$

(ii) $P(A) \geq P(B)$



Proof :- (i) If $B \subset A$ then

$A \cap B = B$

Taking probability on both sides

$P(A \cap B) = P(B)$ —— (1)

But we know that

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$ —— (2)

substituting (1) in (2)

$P(A \cap \bar{B}) = P(A) - P(B)$

(ii) probability is always positive

$P(A \cap \bar{B}) \geq 0$

$P(A) - P(B) \geq 0$

$P(A) \geq P(B)$

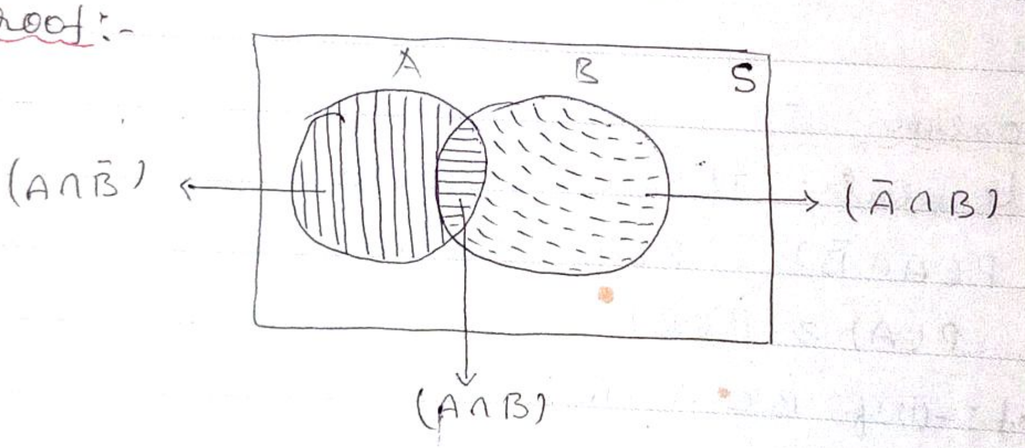
Addition Theorem of probability for Two Events

Theorem : 6

statement: If 'A' and 'B' are any two events then show that

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:-



~~Let $(A \cup B) = (A \cap \bar{B}) \cup (A \cap B) \cup (\bar{A} \cap B)$~~

~~Taking probability on both sides~~

~~$P(A \cup B)$~~ = Let $(A \cap \bar{B}), (A \cap B), (\bar{A} \cap B)$ are mutually exclusive events

$(A \cap \bar{B}) \cap (A \cap B) \cap (\bar{A} \cap B) = \phi$

$(A\cup B) = (A\cap\bar{B})\cup(A\cap B)\cup(\bar{A}\cap B)$

Taking probability on both sides

$P(A\cup B) = P[(A\cap\bar{B})\cup(A\cap B)\cup(\bar{A}\cap B)]$

According to 3rd Axiom

$P(A\cup B) = P(A\cap\bar{B}) + P(A\cap B) + P(\bar{A}\cap B)$ — (1)

but we know that

$P(A\cap\bar{B}) = P(A) - P(A\cap B)$ — (2)

$P(\bar{A}\cap B) = P(B) - P(A\cap B)$ — (3)

substitute (2), (3) in (1)

$P(A\cup B) = P(A\cap\bar{B}) + P(A\cap B) + P(\bar{A}\cap B)$

$P(A\cup B) = P(A) - P(A\cap B) + \cancel{P(A\cap B)} + P(B) - \cancel{P(A\cap B)}$

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$.

Addition Theorem of probability for 'n' Events

Theorem : 7

Statement :- If $A_1, A_2, \ldots A_n$ are n events then

$$P(A_1\cup A_2\cup\ldots\cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j=1}^{n} P(A_i\cap A_j) + \sum_{i<j<k=1}^{n} P(A_i\cap A_j\cap A_k) + \cdots + (-1)^{n-1} P(A_1\cap A_2\cap\ldots\cap A_n)$$

proof :- we know that

$P(A_1\cup A_2) = P(A_1) + P(A_2) - P(A_1\cap A_2)$

$P(A_1\cup A_2) = \sum_{i=1}^{2} P(A_i) + (-1)^{2-1} P(A_1\cap A_2)$

Hence the theorem is true for $n=2$ events

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

Substitute $A_2 = A_2 \cup A_3$ in the above equation.

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2 \cup A_3) - P[A_1 \cap (A_2 \cup A_3)]$$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2 \cup A_3) - P\{(A_1 \cap A_2) \cup (A_1 \cap A_3)\}$$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_2 \cap A_3) - \{P(A_1 \cap A_2) + P(A_1 \cap A_3) - P(A_1 \cap A_2 \cap A_3)\}$$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

$$P(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} P(A_i) - \sum_{i<j=1}^{2} P(A_i \cap A_j) + (-1)^{3-1} P(A_1 \cap A_2 \cap A_3).$$

Hence the theorem is true for $n = 3$ events

Let us suppose that for $n = r$ true say and also true for $n = r+1$

$$P(A_1 \cup A_2 \cup \ldots \cup A_r) = \sum_{i=1}^{r} P(A_i) - \sum_{i<j=1}^{r} P(A_i \cap A_j) + \sum_{i<j<k=1}^{r} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \cdots \cap A_r)$$

Let $(A_1 \cup A_2 \cup \ldots \cup A_r \cup A_{r+1}) = (A_1 \cup A_2 \cup \ldots \cup A_r) \cup (A_{r+1})$

Taking probability on both sides

$$P(A_1 \cup A_2 \cup \ldots \cup A_r \cup A_{r+1}) = P[(A_1 \cup A_2 \cup \ldots \cup A_r) \cup (A_{r+1})]$$

$$P(A_1 \cup A_2 \cup \ldots \cup A_r \cup A_{r+1}) = P(A_1 \cup A_2 \cup \ldots \cup A_r) +$$

$$P(A_{r+1}) - P[(A_1 \cup A_2 \cup \dots \cup A_r) \cap A_{r+1}]$$

$$P(A_1 \cup A_2 \cup \dots \cup A_r \cup A_{r+1}) = \sum_{i=1}^{r} P(A_i) - \sum_{i<j=1}^{r} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k=1}^{r} P(A_i \cap A_j \cap A_k) + \dots + (-1)^{r-1} P(A_1 \cap A_2 \cap \dots \cap A_r) +$$

$$P(A_{r+1}) - P\{(A_1 \cap A_{r+1}) \cup (A_2 \cap A_{r+1}) \cup \dots \cup (A_r \cap A_{r+1})\}$$

$$P(A_1 \cup A_2 \cup \dots \cup A_r \cup A_{r+1}) = \sum_{i=1}^{r+1} P(A_i) - \sum_{i<j=1}^{r+1} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k=1}^{r+1} P(A_i \cap A_j \cap A_k) + \dots + (-1)^{r} P(A_1 \cap A_2 \cap \dots \cap A_r \cap A_{r+1})$$

Hence the theorem is true for $n = r+1$ if it is true for $n = r$. But we have proved that the theorem is true for $n = 2$.

$\therefore$ By the principle of mathematical induction the theorem is true for all values of 'n'.

Boole's inequalities

(15) Theorem 8:-

Statement:- If $A_1, A_2, \dots A_n$ are n events then $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$

Proof:- we know that

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

probability always positive

$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$ — (1)

$P(\bigcup_{i=1}^{2} A_i) \leq \sum_{i=1}^{2} P(A_i)$ — (2)

Hence the theorem is true for $n = 2$

Let us suppose that the theorem is true for $n = r$

$P(\bigcup_{i=1}^{r} A_i) \leq \sum_{i=1}^{r} P(A_i)$

Let $(\bigcup_{i=1}^{r+1} A_i) = (\bigcup_{i=1}^{r} A_i) \cup (A_{r+1})$

Taking probability on both sides

$P(\bigcup_{i=1}^{r+1} A_i) = P\left[(\bigcup_{i=1}^{r} A_i) \cup (A_{r+1})\right]$

from (1)

$P(\bigcup_{i=1}^{r+1} A_i) \leq P(\bigcup_{i=1}^{r} A_i) + P(A_{r+1})$

$P(\bigcup_{i=1}^{r+1} A_i) \leq \sum_{i=1}^{r} P(A_i) + P(A_{r+1})$

$$P\left[\bigcup_{i=1}^{r+1} A_i\right] \leq \sum_{i=1}^{r+1} P(A_i)$$

Hence the theorem is true for $n = r+1$ if it is true for $n = r$. But we have proved that the theorem is true for $n = 2$.

$\therefore$ By the principle of mathematical induction the theorem is true for all values of 'n'.

Theorem: 9

Statement :- For 'n' events $A_1, A_2, A_3, \dots A_n$ then $P\left[\bigcap_{i=1}^{n} A_i\right] \geq \sum_{i=1}^{n} P(A_i) - (n-1)$.

Proof:- We know that

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

probability lies between 0 and 1.

ie $0 \leq P(A_1 \cup A_2) \leq 1$

so $P(A_1 \cup A_2) \leq 1$

$P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq 1$

$P(A_1) + P(A_2) - 1 \leq P(A_1 \cap A_2)$

$P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$ —— (1)

$P\left(\bigcap_{i=1}^{2} A_i\right) \geq \sum_{i=1}^{2} P(A_i) - (2-1)$

Hence the theorem is true for $n = 2$

Let us suppose that the theorem is true for $n = r$

$$P\left[\bigcap_{i=1}^{r} A_i\right] \geq \sum_{i=1}^{r} P(A_i) - (r-1)$$

Let $\left[\bigcap_{i=1}^{r+1} A_i\right] = \left(\bigcap_{i=1}^{r} A_i\right) \cap (A_{r+1})$

Taking probability on both sides

$$P\left[\bigcap_{i=1}^{r+1} A_i\right] = P\left\{\left[\bigcap_{i=1}^{r} A_i\right] \cap A_{r+1}\right\}$$

from (1)

$$P\left[\bigcap_{i=1}^{r+1} A_i\right] \geq P\left[\bigcap_{i=1}^{r} A_i\right] + P(A_{r+1}) - 1$$

$$P\left[\bigcap_{i=1}^{r+1} A_i\right] \geq \sum_{i=1}^{r} P(A_i) - (r-1) + P(A_{r+1}) - 1$$

$$P\left[\bigcap_{i=1}^{r+1} A_i\right] \geq \sum_{i=1}^{r} P(A_i) + P(A_{r+1}) - r + \cancel{1} - \cancel{1}$$

$$P\left[\bigcap_{i=1}^{r+1} A_i\right] \geq \sum_{i=1}^{r} P(A_i) + P(A_{r+1}) - r$$

Hence the theorem is true for $n = r+1$ if it is true for $n = r$. But we have proved that the theorem is true for $n = 2$.

$\therefore$ By the principle of mathematical induction the theorem is true for all values of n.

90 Conditional probability :- The conditional probability of the event 'A' given that event 'B' has already happened, it is denoted by $P(A/B)$ is defined as follows

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

The conditional probability of the event B given that event A has already happened, it is denoted by $P(B|A)$ is defined as follows.

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Multiplication theorem of probability for two events

Theorem 10:-

Statement :- If A and B are any two events then

$$P(A \cap B) = P(A) \cdot P(B|A)$$

OR

$$P(A \cap B) = P(B) \, P(A/B)$$

Proof :- Let us suppose that the sample space consists of 'N' outcomes out of which n_A, n_B, n_{AB} are favourable to the events A, B and $(A \cap B)$ respectively.

According to unconditional probability of events are

$$P(A) = \frac{n_A}{N}, \quad P(B) = \frac{n_B}{N} \text{ and } P(A \cap B) = \frac{n_{AB}}{N}$$

The sample space corresponding to the event (A/B) consists of n_B outcomes out of which n_{AB} are favourable to the happening of the event 'A'

According to the conditional probabilities are defined as follows.

$$P(A/B) = n_{AB}/n_B$$

$$P(B/A) = n_{AB}/n_A$$

$$P(A \cap B) = \frac{n_{AB}}{N}$$

dividing and multiplying by n_A

$$P(A \cap B) = \frac{n_{AB}}{N} \times \frac{n_A}{n_A}$$

$$P(A \cap B) = \frac{n_{AB}}{n_A} \times \frac{n_A}{N}$$

$$P(A \cap B) = P(B/A)\ P(A)$$

$$P(A \cap B) = P(A)\ P(B/A)$$

$$P(A \cap B) = \frac{n_{AB}}{N}$$

dividing and Multiplying by n_B

$$P(A \cap B) = \frac{n_{AB}}{N} \cdot \frac{n_B}{n_B}$$

$$P(A \cap B) = \frac{n_{AB}}{n_B} \cdot \frac{n_B}{N}$$

$$P(A \cap B) = P(A/B)\ P(B)$$

$$P(A \cap B) = P(B) \cdot P(A/B)$$

Multiplication Theorem of probability for 'n' Events

Theorem: II

Statement :- For 'n' events $A_1, A_2, \dots A_n$ then $P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)\ P(A_2/A_1)$ $P(A_3/A_1 \cap A_2) \cdots P(A_n/A_1 \cap A_2 \cap \dots \cap A_{n-1})$

Proof :- we know that

~~$P(A_1 \cap A_2) = P($~~ $\quad P(A_2/A_1) = \dfrac{P(A_1 \cap A_2)}{P(A_1)}$

By cross multiplication

$P(A_1)\, P(A_2/A_1) = P(A_1 \cap A_2)$

ie $P(A_1 \cap A_2) = P(A_1)\, P(A_2/A_1)$

Hence the theorem is true for $n=2$.

$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2)$

$\cancel{P(A_4/A_1 \cap A_2 \cap A_3)}$

Hence the theorem is true for $n=3$.

Let us assume that the theorem is true for $n=r$, we get

$P(A_1 \cap A_2 \cap \ldots \cap A_r) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2)$

$P(A_4/A_1 \cap A_2 \cap A_3) \ldots\ P(A_r/A_1 \cap A_2 \cap \ldots \cap A_{r-1})$

Let $A = A_1 \cap A_2 \cap \ldots \cap A_r$

$A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_r \cap A_{r+1} = A \cap A_{r+1}$

Taking probability on both sides

$P(A_1 \cap A_2 \cap \ldots \cap A_r \cap A_{r+1}) = P(A \cap A_{r+1})$

-do- $= P(A)\, P(A_{r+1}/A)$

-do- $= P(A_1 \cap A_2 \cap \ldots \cap A_r)\, P(A_{r+1}/A_1 \cap A_2 \cap \ldots \cap A_r)$

-do- $= P(A_1) \cdot P(A_2/A_1)\, P(A_3/A_1 \cap A_2) \cdot$

$P(A_r/A_1 \cap A_2 \cap \ldots \cap A_{r-1})\, P(A_{r+1}/A_1 \cap A_2 \cap \ldots \cap A_r)$

Hence the theorem is true for $n=r+1$, it is true for $n = r$. But the theorem is proved to be true for $n = 2$.

By the principle of mathematical induction the theorem is true for all values of 'n'.

Condition for independency of two events

An event 'B' is said to be independent of event 'A', if the conditional probability of $(B|A)$ is equal to unconditional probability of 'B'.

ie $P(B/A) = P(B)$

From the multiplication of probability of two events.

we know that

$P(A \cap B) = P(A)\ P(B|A)$

$P(A \cap B) = P(A)\ P(B)$ is the condition for independency of two events A and B.

$P(A \cap B) = P(B) . P(A/B)$

$P(A \cap B) = P(B) . P(A)$.

Pairwise independency of 'n' events:

The events $A_1, A_2, A_3, \ldots A_n$ are said to be pairwise independent

If $P(A_i \cap A_j) = P(A_i)\,P(A_j)\ \forall\, i, j$

Mutually independency of n events:

The events $A_1, A_2, \dots A_n$ are said to be mutually independent, if the following conditions are satisfied.

$P(A_i \cap A_j) = P(A_i)\,P(A_j) = {}^nC_2$

$P(A_i \cap A_j \cap A_k) = P(A_i)\,P(A_j)\,P(A_k) = {}^nC_3$

$\vdots \qquad\qquad \vdots \qquad\quad \vdots \qquad \vdots$

$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)\,P(A_2) \cdots P(A_n) = {}^nC_n$

condition $= {}^nC_2 + {}^nC_3 + \dots + {}^nC_n$

$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n) - {}^nC_0 - {}^nC_1$

$= 2^n - {}^nC_0 - {}^nC_1$

condition $= 2^n - 1 - n.$

Theorem : 12

statement: If A and B are two independent events then A and $\bar{B}$ are also independent events.

proof:- If A and B are independent events then $P(A \cap B) = P(A) \cdot P(B)$

$(A \cap \bar{B}) \cup (A \cap B) = A$

Taking probability on both sides

$P[(A \cap \bar{B}) \cup (A \cap B)] = P(A)$

According to 3rd Axiom

$P(A \cap \bar{B}) + P(A \cap B) = P(A)$

$P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$P(A \cap \bar{B}) = P(A) - P(A) \cdot P(B)$

$P(A \cap \bar{B}) = P(A)[1 - P(B)]$

Here $1 - P(B) = P(\bar{B})$

$P(A \cap \bar{B}) = P(A)\, P(\bar{B})$.

$\therefore$ A and $\bar{B}$ are independent events.

Theorem : 13

statement :- If A and B are two independent events then $\bar{A}$ and B are also independent.

Proof :- If A and B are independent events then $P(A \cap B) = P(A)\, P(B)$.

$(\bar{A} \cap B) \cup (A \cap B) = B$

Taking probability on both sides

$P[(\bar{A} \cap B) \cup (A \cap B)] = P(B)$

According to 3rd Axiom

$P(\bar{A} \cap B) + P(A \cap B) = P(B)$

$P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$P(\bar{A} \cap B) = P(B) - P(A) \cdot P(B)$

$P(\bar{A} \cap B) = P(B)[1 - P(A)]$

Here $1 - P(A) = P(\bar{A})$

$P(\bar{A} \cap B) = P(B) P(\bar{A})$

$P(\bar{A} \cap B) = P(\bar{A}) P(B)$

$\therefore$ $\bar{A}$ and B are independent events.

Theorem: 14

Statement :- If A and B are two independent events then $\bar{A}$ and $\bar{B}$ are also independent events.

Proof :- If A and B are independent events then $P(A \cap B) = P(A) \cdot P(B)$.

According to De-Margan's Law.

$(\bar{A} \cap \bar{B}) = (\overline{A \cup B})$

Taking probability on both sides

$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

$P(\bar{A} \cap \bar{B}) = 1 - [P(A) + P(B) - P(A \cap B)]$

$P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B) + P(A \cap B)$

$P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B) + P(A) \cdot P(B)$

$P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B)[1 - P(A)]$

$P(\bar{A} \cap \bar{B}) = [1 - P(A)][1 - P(B)]$

$P(\bar{A} \cap \bar{B}) = P(\bar{A}) P(\bar{B})$.

$\therefore$ $\bar{A}$ and $\bar{B}$ are also independent events.

Theorem: 15

Statement :- If A, B, C are mutually independent events then $(A \cup B)$, C are also independent events.

Proof :- If A, B, C are mutually independent events

$P(A \cap B) = P(A) \cdot P(B)$

$P(A \cap C) = P(A)\, P(C)$

$P(B \cap C) = P(B)\, P(C)$

Let $[(A \cup B) \cap C] = (A \cap C) \cup (B \cap C)$

Taking probability on both sides

$P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]$

$P[(A \cup B) \cap C] = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

$P[(A \cup B) \cap C] = P(A) \cdot P(C) + P(B)\, P(C) - P(A)\, P(B)\, P(C)$

$P[(A \cup B) \cap C] = P(C)\, [P(A) + P(B) - P(A)\, P(B)]$

$P[(A \cup B) \cap C] = P(C)\, [P(A) + P(B) - P(A \cap B)]$

$P[(A \cup B) \cap C] = P(C)\, P(A \cup B)$

$\therefore A[(A \cup B) \cap C] = P(A \cup B)\, P(C)$

$\therefore$ $A \cup B$, C are mutually independent events.

Theorem : 16

Statement :- For any two events A and B

Then P(A
Proof :-
Let (A∩
Taking
P(A∩B
Let A ⊂
Taking
P(A) ≤
we kno
P(A∪B)
probabi
P(A∪B)
From ①,
P(A∩B)

P(A∩B) :

Then $P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

Proof:- we know that

Let $(A \cap B) \subset A$

Taking probability on both sides

$P(A \cap B) \leq P(A)$ — (1)

Let $A \subset (A \cup B)$

Taking probability on both sides

$P(A) \leq P(A \cup B)$ — (2)

we know that

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

probability is always positive

$P(A \cup B) \leq P(A) + P(B)$ — (3)

From (1), (2) and (3) we get

$P(A \cap B) \leq P(A)$

$P(A) \leq P(A \cup B)$

$P(A \cup B) \leq P(A) + P(B)$

$P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

Theorem : 17

statement:- For any three events A, B and C then

$$P(A \cup B / C) = P(A/C) + P(B/C) - P(A \cap B / C)$$

Proof:- we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Let $[(A \cup B) \cap C] = (A \cap C) \cup (B \cap C)$

Taking probability on both sides

$$P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]$$

$$P[(A \cup B) \cap C] = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

Dividing through out by $P(C)$

$$\frac{P[(A \cup B) \cap C]}{P(C)} = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)}$$

$$\frac{P[(A \cup B) \cap C]}{P(C)} = \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

$$P(A \cup B / C) = P(A/C) + P(B/C) - P(A \cap B / C)$$

Baye's Theorem

Theorem: 18

Statement :- If $E_1, E_2, E_3, \cdots E_n$ are mutually (disjoint) exclusive events with $P(E_i) \neq 0$ $(i = 1, 2, \ldots n)$ Then for any event 'A' which is a subset of $(\bigcup_{i=1}^{n} E_i)$, such that $P(A) > 0$, we have

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)}$$

Proof:- According to conditional probability

$$P(A/E_i) = \frac{P(A \cap E_i)}{P(E_i)}$$

by cross multiplication

$$P(A \cap E_i) = P(E_i)\,P(A/E_i) \quad \text{--- (1)}$$

$$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)}$$

by cross Multiplication

$$P(A \cap E_i) = P(A)\,P(E_i/A) \quad \text{--- (2)}$$

equating (1) and (2), we get

$$P(A)\,P(E_i/A) = P(E_i)\,P(A/E_i)$$

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\cancel{\sum_{i=1}^{n}}\; P(A)} \quad — (3)$$

given that $A \subset \left(\bigcup_{i=1}^{n} E_i\right)$

$$A = A \cap \left(\bigcup_{i=1}^{n} E_i\right)$$

$$A = A \cap [E_1 \cup E_2 \cup \cdots \cup E_n]$$

$$A = (A \cap E_1) \cup (A \cap E_2) \cup \cdots \cup (A \cap E_n)$$

Taking probability on both sides

$$P(A) = P[(A \cap E_1) \cup (A \cap E_2) \cup \cdots \cup (A \cap E_n)]$$

Here $E_1, E_2, \ldots E_n$ are mutually exclusive events then

$[(A \cap E_1), (A \cap E_2) \ldots (A \cap E_n)]$ are also mutually exclusive events

~~P(A)=~~ According to 3rd Axiom.

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \cdots + P(A \cap E_n)$$

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i) \quad — (4)$$

$\cancel{P(A)} = \cancel{\sum_{i=1}^{n}}$ substituting (1) in (4)

$$P(A) = \sum_{i=1}^{n} P(E_i)\,P(A/E_i) \quad — (5)$$

substituting (5) in (3)

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)}$$

(5) Mutually Exclusive Events

Events are said to be mutually exclusive if the happening of anyone of them prevents the happening of all the
(precludes)

others ie if no two or more of them can happen simultaneously in the same trial.

22 **Statement** :- For any two events A and B

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof :- Let A and $(\bar{A} \cap B)$ are mutually exclusive events.

$\therefore$ ~~kat~~ $P[A \cup (\bar{A} \cap B)] = P(A) + P(\bar{A} \cap B)$ — e

let ~~but~~ $A \cup B = A \cup (\bar{A} \cap B)$ —— a

Taking probability on both sides — b

$P(A \cup B) = P[A \cup (\bar{A} \cap B)]$ — (1) — c

but we know that

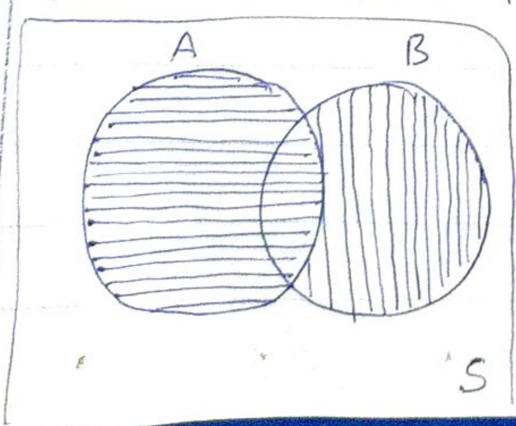
$P(\bar{A} \cap B) = P(B) - P(A \cap B)$ — (2)

substituting (2) in (1)

~~[illegible]~~ According to (3) 2d p.

$P(A \cup B) = P(A) + P(\bar{A} \cap B)$ — (1) — d

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



(14) State and prove addition theorem on probability for any three events.

Statement :- If A, B, C are any three events then

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$

Proof :- $P(A \cup B \cup C) = P[(A \cup B) \cup C]$

$= P(A \cup B) + P(C) - P[(A \cup B) \cap C]$

$= P(A) + P(B) - P(A \cap B) + P(C) - P[(A \cap C) \cup (B \cap C)]$
using distribution law.

$= P(A) + P(B) - P(A \cap B) + P(C) - \{P(A \cap C) + P(B \cap C)] - P[(A \cap C) \cap (B \cap C)]\}$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$

(25) An event is known to be independent of the events B, B∪C, and B∩C. show that is also independent of 'C'.

Proof:- given

$$P(A\cap B) = P(A)\cdot P(B)$$

$$P[A\cap(B\cap C)] = P(A)\cdot P(B\cap C) \quad —(1)$$

$$P[A\cap(B\cup C)] = P(A)\cdot P(B\cup C)$$

$$= P(A)\left[P(B)+P(C)-P(B\cap C)\right]$$

$$= P(A)P(B)+P(A)P(C) - P(A)\cdot P(B\cap C)$$

$$= P(A\cap B) + P(A)P(C) - P[A\cap(B\cap C)] \quad —(2)$$

But

$$P[A\cap(B\cup C)] = P[(A\cap B)\cup(A\cap C)]$$

By distributive law

$$= P(A\cap B)+P(A\cap C) - P[(A\cap B)\cap(A\cap C)]$$

$$= P(A\cap B)+P(A\cap C) - P[A\cap(B\cap C)] \quad —(3)$$

from (2) and (3)

$P(A \cap C) = P(A) \cdot P(C)$

$\therefore$ A and C are independent

UNIT - I

RANDOM VARIABLES

Problems :

1. A random variable 'x' has the probability function

x	0	1	2	3	4	5	6	7
$P(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

i) Determine 'k' value

ii) Evaluate $P[x<6]$, $P[x\geq 6]$, $P[0<x<5]$

iii) If $P(x\leq K)>\frac{1}{2}$, find the minimum value of 'k'.

iv) Determine the distribution function of 'x'

v) mean

vi) variance.

Sol: Given

x	0	1	2	3	4	5	6	7
$P(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

i) k value :

Total probability = 1

$$\Sigma P(x) = 1$$

$$0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1$$

$$10K^2+9K = 1$$

$$10K^2+9K-1=0$$

$$10K^2+10K-K-1=0$$

$$10K(K+1)-1(K+1)=0$$

$$K+1=0 \quad 10K-1=0$$

$$K=-1 \quad K=\frac{1}{10}$$

As probability is positive,

$$\boxed{K=\frac{1}{10}}$$

ii) $P[x<6] = P(x=0)+P(x=1)+P(x=2)+P(x=3)+P(x=4)+P(x=5)$

$= 0+k+2k+2k+3k+k^2$

$= k^2+8k$

$= \left(\frac{1}{10}\right)^2+8\left(\frac{1}{10}\right)$

$= \frac{1}{100}+\frac{8}{10}$

$= \frac{81}{100}$

$P[x<6] = 0.81$

$P[x\geq 6] = P(x=6)+P(x=7)$

$= 2k^2+7k^2+k$

$= 9k^2+k$

$= 9\left(\frac{1}{100}\right)+\frac{1}{10}$

$P[x\geq 6] = 0.19$

$P[0<x<5] = P(x=1)+P(x=2)+P(x=3)+P(x=4)$

$= k+2k+2k+3k$

$= 8k$

$= \frac{8}{10}$

$P[0<x<5] = 0.8$

iii) $P(x\leq k) > \frac{1}{2}$

The required minimum value of k is obtained as follows :

$P[x\leq 1] = P(x=0)+P(x=1)$

$= 0+k$

$= \frac{1}{10}$

$P[x\leq 1] = 0.1 < 0.5$

$P[x\leq 2] = P(x=0)+P(x=1)+P(x=2)$

$= 0+k+2k$

$= 3k$

$= \frac{3}{10}$

$P[x\leq 2] = 0.3 < 0.5$

$P[x \le 3] = p(x=0) + p(x=1) + p(x=2) + p(x=3)$

$= 0 + k + 2k + 2k$

$= 5k$

$P[x \le 3] = 0.5 = 0.5$

$P[x \le 4] = p(x=0) + p(x=1) + p(x=2) + p(x=3) + p(x=4)$

$= 0 + k + 2k + 2k + 3k$

$= 8k$

$P[x \le 4] = 0.8 > 0.5$

$\therefore$ The minimum value of k for which $P[x \le k] > \frac{1}{2}$ is $\boxed{k=4}$

iv) The distribution function of 'x' is given by the following table :

x	$F(x) = P(X \le x)$	
0	$P(x \le 0) = p(x=0)$	0
1	$P(x \le 1) = P(x=0) + P(x=1) = k = \frac{1}{10}$	0.1
2	$P(x \le 2) = P(x=0) + P(x=1) + P(x=2) + P(x=3) = k + 2k = \frac{3}{10}$	0.3
3	$P(x \le 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3) = k + 2k + 2k = 5/10$	0.5
4	$P(x \le 4) = P(x=0) + p(x=1) + p(x=2) + p(x=3) + p(x=4) = k + 2k + 2k + 3k = \frac{8}{10}$	0.8
5	$P(x \le 5) = p(x=0) + p(x=1) + p(x=2) + p(x=3) + p(x=4) + p(x=5) = k + 2k + 2k + 3k + k^2 = 8k + k^2$	0.81
6	$P(x \le 6) = P(x=0) + p(x=1) + p(x=2) + p(x=3) + p(x=4) + p(x=5) + p(x=6) = 8k + 3k^2$	0.83
7	$P(x \le 1) = P(x=0) + p(x=1) + p(x=2) + p(x=3) + p(x=4) + p(x=5) + p(x=6) + p(x=1) = 8k + k + 10k^2$	1

v) Mean $= \mu = \Sigma x p(x) = \Sigma x \cdot p(x)$

$= 0\times0 + 1\times K + 2\times2K + 3\times2K + 4\times3K + 5\times K^2 + 6\times2K^2 + 7\times(7K^2+K)$

$= K + 4K + 6K + 12K + 5K^2 + 12K^2 + 49K^2 + 7K$

$= 30K + 66K^2$

4

$= \frac{30}{10} + \frac{66}{100}$

$= 3 + 0.66$

Mean $= 3.66$

$\boxed{\mu = 3.66}$

vi) variance $= \sigma^2 = \Sigma x^2 p(x) - \mu^2$

$= 0 + K + 8K + 18K + 48K + 25K^2 + 72K^2 + 343K^2 + 49K$

$= 124K + 440K^2$

$= \frac{124}{10} + \frac{440}{100}$

$= 16.8$

$\sigma^2 = \Sigma x^2 p(x) - \mu^2 = 16.8 - 13.3956$

$= 3.4044$

2. From the probability distribution of a discrete random variable given below.

x	-2	-1	0	1	2	3
$P(x)$	0.1	K	0.2	2K	0.3	K

i) mean and variance.

Sol:

x	-2	-1	0	1	2	3
$P(x)$	0.1	K	0.2	2K	0.3	K

=> K value:

$0.1 + K + 0.2 + 2K + 0.3 + K = 1$

$4K + 0.6 = 1$

$4K = 1 - 0.6$

3.

Sol

$4k = 0.4$

$k = 0.1$

5

i)

Mean : $-2(0.1) + (-1)(0.1) + 0 + 2(0.1) + 0.3(2) + 3(0.1)$

$= -0.2 - 0.1 + 0.2 + 0.6 + 0.3$

$\boxed{\mu = 0.8}$

Variance $= \sigma^2 = 4(0.1) + 1(0.1) + 2(0.1) + 4(0.3) + 9(0.1) - 0.64$

$\boxed{\sigma^2 = 2.16}$

3. For the following probability distribution,

x	-3	6	9
$P(x)$	$1/6$	$1/2$	$1/3$

find i) $E(x)$ = mean

ii) $E(x^2)$

iii) $E((2x+1)^2)$

Sol:

x	-3	6	9
$P(x)$	$1/6$	$1/2$	$1/3$

i) mean : $E(x)$

$= -3\left(\frac{1}{6}\right) + 6\left(\frac{1}{2}\right) + 9\left(\frac{1}{3}\right)$

$= -\frac{1}{2} + 3 + 3$

$= 6 - \frac{1}{2}$

$= \frac{11}{2}$

$E(x) = 5.5$

ii) $E(x^2) = \mu = \Sigma x^2 p(x)$

$= 9\left(\frac{1}{6}\right) + 36\left(\frac{1}{2}\right) + 81\left(\frac{1}{3}\right)$

$= \frac{3}{2} + 18 + 27 = \frac{93}{2}$

$E(x^2) = 46.5$

iii) $E[(2x+1)^2] = E[(2x)^2 + 2(2x)\,1 + 1^2]$ (6)

$= E[4x^2+4x+1]$

$= 4E[x^2] + 4E[x] + 1$

$= 4[46.5] + 4[5.5] + 1$

$E[(2x+1)^2] = 209$

4. Two dice are thrown. Let 'x' assign to each point with in {a, b} in 'S'. The maximum of its numbers i.e., $x(a,b) = \max(a,b)$. Find the probability distribution. 'x' is a random variable with $x(S) = \{1,2,3,4,5,6\}$. Also find the mean and variance of distribution.

Sol: Total possible outcomes $n = 6^n$

$= 6^2$

$n = 36$

The maximum number could be 1, 2, 3, 4, 5, 6.

i.e., $x(S) = x(a,b) = \max(a,b)$

The number 1 will appearing only in 1 case (1,1).

So $P(1) = P(x) = 1 = \frac{1}{36}$

For maximum 2,

favourable cases are {1, 2} {2, 1} {2, 2} = 3

So $P(2) = P(x) = 2 = \frac{3}{36} = \frac{1}{12}$

For maximum 3,

favourable cases are {(1,3) (3,1) (2,3) (3,2), (3,3)}

So $P(3) = P(x) = 3 = \frac{5}{36}$

For max 4,

favourable cases are {(1,4) (4,1) (2,4) (4,2) (3,4) (4,3), (4,4)}

So $P(4) = P(x) = 4 = \frac{7}{36}$

for max 5, (7)

Favourable cases are {(1,5) (5,1) (2,5) (5,2) (3,5) (5,3) (4,5) (5,4) (5,5)}

$P(x) = P(5) = 5 = \frac{9}{36} = \frac{1}{4}$

for max 6,

favourable cases are {(1,6) (6,1) (2,6) (6,2) (3,6) (6,3) (4,6) (6,4) (5,6) (6,5), (6,6)}

$P(6) = p(x) = 6 = \frac{11}{36}$

⇒ The required discrete probability distribution,

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

i. Mean $= \mu = \Sigma x\, p(x)$

$$= 1\times\frac{1}{36} + 2\times\frac{3}{36} + 3\times\frac{5}{36} + 4\times\frac{7}{36} + 5\times\frac{9}{36} + 6\times\frac{11}{36}$$

$$\boxed{\mu = 4.47222}$$

ii. Variance $= \sigma^2 = \Sigma x^2 p(x) - \mu^2$

$$= 1\times\frac{1}{36} + 4\times\frac{3}{36} + 9\times\frac{5}{36} + 16\times\frac{7}{36} + 25\times\frac{9}{36} + 36\times\frac{11}{36} - (4.47222)^2$$

$$= 21.97222 - 20.00075$$

$$\boxed{\sigma^2 = 1.97147}$$

5. Let 'x' denote the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine the ~~m~~

1. Discrete probability distribution
2. Expectation
3. Variance.
4. Standard deviation.

Sol. Total possible outcomes $= n = 6^n$

$= 6^2$

$n = 36$

The minimum number could be $\{1, 2, 3, 4, 5, 6\}$

For min 1,

favourable cases are $\{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$
$(2,1)\ (3,1)\ (4,1)\ (5,1)\ (6,1)\}$

$P(x) = p(1) = \frac{11}{36}$

For min 2,

favourable cases are $\{(2,2)\ (2,3)\ (3,2)\ (2,4)\ (4,2), (2,5)$
$(5,2)\ (2,6)\ (6,2)\}$

$P(x) = p(2) = \frac{9}{36}$

for min 3,

favourable cases are $\{(3,3)\ (3,4)\ (4,3)\ (3,5)\ (5,3)$
$(3,6)\ (6,3)\}$

$P(x) = p(3) = \frac{7}{36}$

for min 4,

favourable cases are $\{(4,4)\ (4,5)\ (5,4)\ (4,6)\ (6,4)\}$

$P(x) = p(4) = \frac{5}{36}$

for min 5,

favourable cases are $\{(5,5)\ (5,6)\ (6,5)\}$

$P(x) = p(5) = \frac{3}{36}$

For min 6,

favourable cases are $\{(6,6)\}$

$P(x) = p(6) = \frac{1}{36}$

The required possibility discrete probability distribution

(9)

x	1	2	3	4	5	6
$p(x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

ii) Expectation: mean = $E(x)$

$$\mu = 1\times\frac{11}{36} + 2\times\frac{9}{36} + 3\times\frac{7}{36} + 4\times\frac{5}{36} + 5\times\frac{3}{36} + 6\times\frac{1}{36}$$

$$\mu = 2.52777$$

iii) variance σ^2

$$\sigma^2 = 1\times\frac{11}{36} + 4\times\frac{9}{36} + 9\times\frac{7}{36} + 16\times\frac{5}{36} + 25\times\frac{3}{36} + 36\times\frac{1}{36} - (2.52777)^2$$

$$= 8.3611 - 6.38962$$

$$\sigma^2 = 1.97148$$

iv) standard deviation,

$$\sigma = \sqrt{\text{variance}}$$

$$\sigma = 1.40409$$

6. Find the mean and variance of the uniform probability distribution given by $p(x) = \frac{1}{n}$ for $x = 1, 2, 3, \ldots n$

Sol: Given that

$$p(x) = \frac{1}{n}, \quad x = 1, 2, 3, \ldots n$$

The probability distribution is

x	1	2	3		n
$p(x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$		$\frac{1}{n}$

Mean = $\mu = \sum x\, p(x)$

$$= \frac{1}{n} + 2\times\frac{1}{n} + 3\times\frac{1}{n} + \cdots + n\times\frac{1}{n}$$

$$= \frac{1}{n}\left[1+2+\cdots n\right]$$

$$\mu = \frac{1}{n}\left[\frac{n(n+1)}{2}\right]$$

$1+2+3+\cdots+n = \frac{n(n+1)}{2}$

$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$$\boxed{\mu = \frac{n+1}{2}}$$

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ii) variance $= \sigma^2 = \Sigma x^2 p(x) - (\mu)^2$

$$= 1\times\frac{1}{n} + 4\times\frac{1}{n} + 9\times\frac{1}{n} + \cdots + n^2\times\frac{1}{n} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{n^2+1+2n}{4}\right)$$

$$= \frac{1}{n}\left(\frac{n^2+n(2n+1)}{6}\right) - \left(\frac{n^2+1+2n}{4}\right)$$

$$= \frac{1}{n}\left(\frac{2n^3+n^2+2n^2+n}{6}\right) - \frac{(n^2+1+2n)}{4}$$

$$= \frac{1}{n}\left(\frac{2n^3+3n^2+n}{6}\right) - \frac{(n^2+1+2n)}{4}$$

$$= \frac{4n^2+6n+2n - (3n^2+3+6n)}{12}$$

$$= \frac{4n^2+3n^2-4n-3}{12}$$

$$= \frac{4n^2+6n+2-3n^2-3-6n}{12}$$

$$= \frac{n^2-1}{12}$$

7 A sample of 4 items is selected at random from a box containing 12 items of which 5 are defective find the expected number of defective items.

Sol: Let x = the number of defective items among 4 items drawn from 12.

$x = 0, 1, 2, 3, 4$

NO. of good items = 7

NO. of defective items = 5

(11)

$p(x) = 0 = p(\text{no defective item})$

i.e., $\frac{\text{Favourable items}}{\text{Total items}} = \frac{5C_0 \times 7C_4}{^{12}C_4} = \frac{7}{99}$

$= 0.07070$

$p(x) = 1 = p(1 \text{ defective and } 3 \text{ are good})$

i.e., $= \frac{^5C_1 \times {^7C_3}}{^{12}C_4}$

$= \frac{35}{99} = 0.35353$

$p(x) = 2 = p(2 \text{ defective and } 2 \text{ are good})$

i.e., $= \frac{^5C_2 \times {^7C_2}}{^{12}C_4}$

$= \frac{14}{33}$

$= 0.42424$

$p(x) = 3 = p(3 \text{ defective and } 1 \text{ is good})$

i.e., $= \frac{^5C_3 \times {^7C_1}}{^{12}C_4}$

$= \frac{14}{99}$

$p(x) = 4 = p(4 \text{ are defective } \& \text{ no one is good})$

i.e., $= \frac{^5C_4 \times {^7C_0}}{^{12}C_4}$

$= \frac{1}{99}$

The required discrete probability distribution,

x	0	1	2	3	4
$p(x)$	$\frac{7}{99}$	$\frac{35}{99}$	$\frac{14}{33}$	$\frac{14}{99}$	$\frac{1}{99}$

Expected number of defective items,

Expected $= \text{mean} = E(x) = \Sigma x\, p(x)$

$$= 0 \times \frac{7}{99} + 1 \times \frac{35}{99} + 2 \times \frac{14}{33} + 3 \times \frac{14}{99} + 4 \times \frac{1}{99}$$

$$= \frac{5}{3}$$

$$= 1.6666$$

8. Find the distribution function which corresponds to the probability distribution defined by $f(x) = \frac{x}{15}$ for $x = 1, 2, 3, 4, 5$.

Sol: Given that,

$$f(x) = \frac{x}{15}$$

put $x = 1$

$$f(1) = \frac{1}{15}$$

put $x = 2$

$$f(2) = \frac{2}{15}$$

put $x = 3$

$$f(3) = \frac{3}{15} = \frac{1}{5}$$

put $x = 4$

$$f(4) = \frac{4}{15} = \frac{4}{15}$$

put $x = 5$

$$f(5) = \frac{5}{15} = \frac{1}{3}$$

on,

(12)

s,

$+4\times\frac{1}{99}$

corresponds

by

f = probability density func

(13)

Now $F(1) = f(1) = \frac{1}{15}$

$F(2) = F(1) + f(2) = \frac{1}{15} + \frac{2}{15}$

$= \frac{1}{5}$

$F(3) = \cancel{F}(1) + F(2) + f(3)$

$= \frac{2}{15} + \frac{1}{5}$

$= \frac{2}{5}$

$F(4) = F(3) + f(4)$

$= \frac{2}{5} + \frac{4}{15}$

$= \frac{2}{3}$

$F(5) = F(4) + f(5)$

$= \frac{2}{3} + \frac{1}{3}$

$= 1$

9. Show that the variance of a random variable x is given by $\sigma^2 = E(x^2) - [E(x)]^2$.

Sol: Proof:

We know that,

$$\sigma^2 = E[x - E(x)]^2$$

$$= E[x^2 - 2xE(x) + [E(x)]^2]$$

$$= E(x^2) - E((2xE(x)) + E[E(x)]^2$$

$$= E(x^2) - 2E(x)E(x) + [E(x)]^2$$

$$= E(x^2) - 2[E(x)]^2 + [E(x)]^2$$

$$\sigma^2 = E(x^2) - [E(x)]^2$$

10. If the probability density of a random variable is given by $f(x) = \begin{cases} K(1-x^2) & \text{for } 0<x<1 \\ 0, & \text{otherwise} \end{cases}$ 14

find the value of 'K' and the probabilities that a random variable having this probability density will take on a value

i. b/w 0.1 and 0.2

ii. greater than 0.5

Sol: Given that,

$$f(x) = \begin{cases} K(1-x^2) & \text{for } 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

we know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i. $\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$

$$\int_0^1 f(x)\,dx = 1$$

$$f(x) = K(1-x^2)$$

$$\int_0^1 K(1-x^2)\,dx = 1$$

$$K\left\{\int_0^1 1\,dx - \int_0^1 x^2\,dx\right\} = 1$$

$$K\left\{(x)_0^1 - \left(\frac{x^3}{3}\right)_0^1\right\} = 1$$

$$K\left[(1-0) - \left(\frac{1}{3}-0\right)\right] = 1$$

$$K\left[1-\frac{1}{3}\right] = 1$$

$$K\left[\frac{2}{3}\right] = 1$$

$$K = \frac{3}{2}$$

ii. The probability that variat (variable) will take on a value b/w 0.1 and 0.2.

(15)

$$P[0.1 < x < 0.2] = \int_{0.1}^{0.2} f(x)\,dx = 1$$

$$= \int_{0.1}^{0.2} \frac{3}{2}[1 - x^2]\,dx = 1$$

$$\Rightarrow \frac{3}{2}\left[(x)_{0.1}^{0.2} - \left(\frac{x^3}{3}\right)_{0.1}^{0.2}\right] = 1$$

$$= \frac{3}{2}\left[(0.2 - 0.1) - \left(\frac{(0.2)^3}{3} - \frac{(0.1)^3}{3}\right)\right]$$

$$= 0.1465$$

iii) $P(x > 0.5) = \int_{0.5}^{1} \frac{3}{2}[1 - x^2]\,dx$

$= P(0.5 < x < 1)$

$$= \frac{3}{2}\left[(x)_{0.5}^{1} - \left(\frac{x^3}{3}\right)_{0.5}^{1}\right]$$

$$= \frac{3}{2}\left[(1 - 0.5) - \left[\frac{1}{3} - \frac{(0.5)^3}{3}\right]\right]$$

$$= 0.3125$$

11. The probability density function of a continuous is given by $f(x) = ke^{-|x|}$, $-\infty < x < \infty$, find B, A

1. k value
2. mean
3. variance
4. The probability that the variant that lies between 0 and 4.

Sol: Given that,

$$f(x) = ke^{-|x|}, \quad -\infty < x < \infty$$

we know that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

i.e., The total probability is unity

$$\int_{-\infty}^{\infty} k e^{-|x|} dx = 1$$ (16)

$$\int_{-\infty}^{0} k e^{-|x|} dx + \int_{0}^{\infty} k e^{-|x|} dx = 1$$

from $-\infty$ to 0 $|x| = -x$

0 to ∞ $|x| = x$

$$\int_{-\infty}^{0} k e^{-(-x)} dx + \int_{0}^{\infty} k e^{-x} dx = 1$$

$$\int_{-\infty}^{0} k e^{x} dx + \int_{0}^{\infty} k \cdot e^{-x} dx = 1$$

$$\left\{ k \left[e^{x} \right]_{-\infty}^{0} + k \left[\frac{e^{-x}}{-1} \right]_{0}^{\infty} \right\} = 1$$

$$\left\{ k \left[e^{0} - e^{-\infty} \right] - k \left[e^{-\infty} - e^{-0} \right] \right\} = 1$$

$$k \left[1 - 0 \right] - k \left[0 - 1 \right] = 1$$

$$k - 0 - 0 + k = 1$$

$$2k = 1$$

$$\boxed{k = \frac{1}{2}}$$

$$\therefore f(x) = \frac{1}{2} e^{-|x|}$$

2. mean :

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{\infty} x \cdot k e^{-|x|} dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{2} e^{-|x|} dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x\cdot e^{-|x|}\, dx$$ (17)

$x(e^x-1)$
$e^x(x-1)$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} x\cdot e^{x} dx + \int_{0}^{\infty} x\cdot e^{-x} dx\right]$$

$$= \frac{1}{2}\left(\left[x\cdot e^{x} - 1\cdot e^{x} + 0\cdot e^{x}\right]_{-\infty}^{0} + \left(x\left(\frac{e^{-x}}{-1}\right) - 1\left(\frac{e^{-x}}{-1\cdot -1}\right) + 0\right)_{0}^{\infty}\right]$$

$$= \frac{1}{2}\left\{\left[x\cdot e^{x} - e^{x}\right]_{-\infty}^{0} + \left[-xe^{-x} - e^{-x}\right]_{0}^{\infty}\right\}$$

$$= \frac{1}{2}\left\{\left[(0\cdot e^{0} - e^{0}) - (-\infty e^{-\infty} - e^{-\infty})\right] + \left((-\infty e^{-\infty} - e^{-\infty}) - (0\cdot e^{-0} - e^{-0})\right]\right\}$$

$$= \frac{1}{2}(-1+1)$$

$E(x) = 0$

iii) variance $= \sigma^2 = E(x^2) - [E(x)]^2$

Here

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|} dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} x^2 e^{x} dx + \int_{0}^{\infty} x^2 e^{-x} dx\right]$$

$$= \frac{1}{2}\left\{\left[x^2\cdot e^{x} - 2x\cdot e^{x} + 2\cdot e^{x}\right]_{-\infty}^{0} + \left[x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(\frac{e^{-x}}{1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right]_{0}^{\infty}\right\}$$

$$= \frac{1}{2}\left\{\left((0 - 0 + (e^{0})2) - (-\infty^2 e^{-\infty} + \infty e^{-\infty} + e^{-\infty}\right) + \left[\infty^2 e^{-\infty} - \infty e^{-\infty} - e^{-\infty}\right] - (-2e^{-0})\right\}$$

$$= \frac{1}{2}\left\{(2+2)\right\}$$

$$= \frac{4}{2}$$

$E(x^2) = 2$

$\boxed{\sigma^2 = 2}$ (18)

$\sigma^2 = E(x^2) - (E(x))^2$

$\sigma^2 = 2 - 0$

$\boxed{\sigma^2 = 2}$

iv. $P(0 < x < 4) = \int_0^4 f(x)\,dx$

$= \int_0^4 \frac{1}{2} e^{-|x|}\,dx$

where,

0 to ∞ → 0 to 4, $|x| = x$

$\cancel{\frac{1}{2}\left[\int_{-\infty}^{0} e^{x}\,dx + \int_0^{\infty} e^{-x}\,dx\right]}$

$= \frac{1}{2}\int_0^4 e^{-x}\,dx$

$= 0.4908$

12. If a random variable has the probability function as $f(x) = \begin{cases} 2e^{-2x} & \text{for } x>0 \\ 0, & x \le 0 \end{cases}$ find the i) probabilities that it will take on a value between 1 and 3 ii) greater than 0.5.

Sol: Given that

$f(x) = \begin{cases} 2e^{-2x}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases}$

i) $P(1 < x < 3) = \int_1^3 f(x)\,dx$

$= \int_1^3 2\cdot e^{-2x}\,dx.$

$= 2\cdot\int_1^3 e^{-2x}\,dx$

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$$= 2 \cdot \left[\frac{e^{-2x}}{-2} \right]_1^3$$

$$= 2 \left[\frac{e^{-6}}{-2} - \frac{e^{-2}}{-2} \right]$$

$$= 0.13285$$

ii) $p(0.5 < x < \infty) = 2 \int_{0.5}^{\infty} e^{-2x}$

$$= 2 \left(\frac{e^{-2(0.5)}}{2} \right]$$

$$= 0.36787$$

13. A continuous random variable has the probability density function $f(x) = \begin{cases} kx e^{-\lambda x}, & x \geq 0, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$

Determine i. k value

ii. Mean

iii. variance

Sol: Given that,

$$f(x) = \begin{cases} kx e^{-\lambda x}, & x \geq 0, \lambda > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} kx e^{-\lambda x}\,dx = 1$$

$$k \left[x \cdot \frac{e^{-\lambda x}}{-\lambda} - 1 \cdot \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty} = 1$$

$$k \left[0 - \frac{e^{-\lambda \infty}}{\lambda^2} - \left[0 - \frac{e^{-\lambda(0)}}{\lambda^2} \right] \right] = 1$$

$$k \left[0 - \left[-\frac{1}{\lambda^2} \right] \right] = 1$$

$$k \left(\frac{1}{\lambda^2} \right] = 1$$

$$\boxed{k = \lambda^2}$$

ii) Mean : $\int_0^\infty x \cdot f(x)\,dx = 1$

$\int_0^\infty x \cdot k \cdot x e^{-\lambda x}\,dx = 1$

$= k \cdot \int_0^\infty x^2 e^{-\lambda x}\,dx$

$= k\left[x^2 \frac{e^{-\lambda x}}{-\lambda} - 2x\frac{e^{-\lambda x}}{\lambda^2} + 2\frac{e^{-\lambda x}}{-\lambda^3}\right]_0^\infty$

$= k\left[0 - \left(0 - 0 + 2\frac{e^{-\lambda(0)}}{-\lambda^3}\right)\right]$

$= k\left[0 + \frac{2}{\lambda^3}\right]$

$= \cancel{\lambda^2} \times \frac{2}{\cancel{\lambda^3}}$

$\boxed{\text{mean} = \frac{2}{\lambda} = E(x)}$

iii) Variance $= \sigma^2 = E(x^2) - (E(x))^2$

$\sigma^2 = k\int_0^\infty x^3 e^{-\lambda x}\,dx - \frac{4}{\lambda^2}$

$= \lambda^2\left[x^3\frac{e^{-\lambda x}}{-\lambda} - 3x^2\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + 6x\left(\frac{e^{-\lambda x}}{-\lambda^3}\right) - 6\left(\frac{e^{-\lambda x}}{\lambda 4}\right)\right]_0^\infty - \frac{4}{\lambda^2}$

$= \lambda^2\left[0 - \left(-6 \cdot \frac{e^{-\lambda(0)}}{\lambda 4}\right)\right] - \frac{4}{\lambda^2}$

$= \lambda^2\left[6\frac{1}{\lambda 4}\right] - \frac{4}{\lambda^2}$

$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$

$\sigma^2 = \frac{2}{\lambda^2}$

$\Rightarrow E(x^2) = \frac{6}{\lambda 2}$

$\sigma^2 = \frac{6}{\lambda^2} - \frac{4}{\lambda 2} \Rightarrow \boxed{\sigma^2 = \frac{2}{\lambda 2}}$

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14. Probability density function of a random variable of x is $f(x) = \begin{cases} \frac{1}{2}\sin x, & 0 \le x \le \pi \\ 0, & \text{elsewhere} \end{cases}$ find

i. Mean

ii. Mode

iii. Median of the distribution.

iv. Find probability b/w 0 & $\frac{5}{2}$

Sol: Given

$$f(x) = \begin{cases} \frac{1}{2}\sin x, & 0 \le x \le \pi \\ 0, & \text{elsewhere} \end{cases}$$

i. Mean = μ = $E(x)$

Mean is denoted by $E(x)$.

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = 1$$

$$f(x) = \frac{1}{2}\sin x$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{2}\sin x\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x \sin x\,dx$$

$$= \frac{1}{2}\int_{0}^{\pi} x \sin x\,dx$$

$$= \frac{1}{2}\left\{\left[x(-\cos x) - 1(-\sin x)\right]_0^{\pi}\right\}$$

$$= \frac{1}{2}\left(-x\cos x + \sin x\right]_0^{\pi}$$

$$= \frac{1}{2}\left[-\pi\cos\pi + \sin\pi - (-0 + \sin 0)\right]$$

$$= \frac{1}{2}\left[-\pi(-1) + 0 - (0)\right]$$

$$= \frac{1}{2}(\pi)$$

$$E(x) = \frac{\pi}{2}$$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\int \sin x = -\cos x$

$\int \cos x = \sin x$

$\cos\pi = -1$

$\sin\pi = 0$

$\frac{1}{\lambda^2}$

ii. mode :

mode is the value of x for which $f(x)$ is maximum.

Given that,

$$f(x) = \frac{1}{2}\sin x$$

$$f'(x) = \frac{1}{2}\cos x.$$

For $f(x)$ to be maximum,

i.e., $f'(x) = 0$

$$\frac{1}{2}\cos x = 0$$

$$\cos x = 0$$

$$x = \cos^{-1}(0)$$

$$\boxed{x = \frac{\pi}{2}}$$

$$f'(x) = \frac{1}{2}\cos x$$

$$f''(x) = \frac{1}{2}(-\sin x)$$

At $x = \frac{\pi}{2}$

$$f''\left(\frac{\pi}{2}\right) = \frac{-\sin\frac{\pi}{2}}{2}$$

$$f''\left(\frac{\pi}{2}\right) = \frac{-1}{2} < 0$$

Hence $f(x)$ is maximum at $x = \frac{\pi}{2}$

$\therefore$ mode of the distribution $= x = \frac{\pi}{2}$

iii. Median :

If M is the median of the distibution,

Note:

Median is divided the total area into equal parts. If 'x' is defined from a to b and 'M'

is the median.

(Q3)

$$\int_a^M f(x)dx = \int_M^b f(x)dx = \frac{1}{2}$$

According to problem,

$$\int_0^m f(x)dx = \int_m^\pi f(x)dx = \frac{1}{2}$$

By normal distribution table,

$$\int_0^m f(x)dx = \frac{1}{2}$$

$$\int_0^m \frac{1}{2}\sin x\, dx = \frac{1}{2}$$

$$\cancel{2}\int_0^m \frac{1}{\cancel{2}}\sin x\, dx = 1$$

$$(-\cos x)_0^m = 1$$

$$-\cos m + \cancel{\frac{\pi}{2}}1 = 1$$

$$-\cos m + 1\cancel{\frac{\pi}{2}} - 1 = 0$$

$$-\cos m = 0$$

$$\cos m = 0$$

$$\boxed{m = \frac{\pi}{2}}$$

$\therefore$ median of the distribution $= \frac{\pi}{2} = m$

mean = median = mode = $\frac{\pi}{2}$

iv) Probability between 0 & $\frac{\cancel{\pi}5}{2}$

$$P(0 < x < \frac{5}{2}) = \int_0^{\cancel{\pi}/2} f(x)\, dx$$

$$= \int_0^{\cancel{\pi}/2} \frac{1}{2}\sin x\, dx$$

$$= \frac{1}{2}(-\cos x)_0^{\pi/2} = \frac{1}{2}(+0 + \cos 0) = \frac{1}{2}$$

Note: Mode is gives by $f'(x)=0$ and $f''(x)<0$ for $a<x<b$

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15. A continuous random variable has the distribution function,

$$f(x) = \begin{cases} 0, & \text{if } x \le 1 \\ K(x-1)^4, & \text{if } 1<x<3 \\ 0, & \text{if } x>3 \end{cases}$$ determine

i. $f(x)$ ii. k iii. mean

Sol: Given that

$$F(x) = \begin{cases} 0, & \text{if } x \le 1 \\ K(x-1)^4, & \text{if } 1<x<3 \\ 0, & \text{if } x>3 \end{cases}$$

We know that

$$\boxed{f(x) = \frac{d}{dx} F(x)}$$

i. $$f(x) = \begin{cases} 0, & \text{if } x \le 1 \\ 4\cdot k(x-1)^3, & \text{if } 1<x<3 \\ 0, & \text{if } x>3 \end{cases}$$

ii. We know that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Acc to problem,

$$\Rightarrow \int_1^3 4k(x-1)^3\,dx = 1 \qquad (a-b)^3 = a^3-b^3-3a^2b+3ab^2$$

$$\Rightarrow 4k\int_1^3 (x-1)^3\,dx = 1 \qquad (x-1)^3 = x^3-1-3x^2+3x$$

$$= x^3-3x^2+3x-1$$

$$\Rightarrow \cancel{4k}\,\frac{\cancel{(x-1)^4}}{4} = 1$$

$$\Rightarrow 4k\int_1^3 x^3-3x^2+3x-1 = 1$$

$$\Rightarrow 4k\left(\frac{x^4}{4} - 3\left(\frac{x^3}{3}\right) + 3\left(\frac{x^2}{2}\right) - x\right]_1^3 = 1$$

iii.

for

distribution

ve

$3a^2b$

$3ab^2$

$+3x$

-1

$$\Rightarrow 4K\left[\left(\frac{81}{4} - 27 + 3\left(\frac{9}{2}\right) - 3 - \left(\frac{1}{4} - 1 + \frac{3}{2} - 1\right)\right)\right] = 1$$

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$$\Rightarrow 4K\left[\frac{15}{4} - \left(-\frac{1}{4}\right)\right] = 1$$

$$= 4K\left[\frac{16}{4}\right] = 1$$

$$= 16K = 1$$

$$\boxed{K = \frac{1}{16}}$$

(or)

$$\Rightarrow 4K\int_1^3 (x-1)^3 dx = 1$$

$$4K\left(\frac{(x-1)^4}{4}\right)_1^3 = 1$$

$$K\left[(3-1)^4 - 0\right] = 1$$

$$K\left[16\right] = 1$$

$$\boxed{K = \frac{1}{16}}$$

iii. Mean :

$$E(x) = \int_1^3 x f(x) dx$$

$$= \int_1^3 x \cdot 4K(x-1)^3 dx$$

$$= \frac{4}{16}\int_1^3 x(x-1)^3 dx$$

$$= \frac{1}{4}\int_1^3 x\left[x^3 - 3x^2 + 3x - 1\right] dx$$

$$= \frac{1}{4}\int_1^3 x^4 - 3x^3 + 3x^2 - x\, dx$$

$$= \frac{1}{4}\left(\frac{x^5}{5} - 3\cdot\frac{x^4}{4} + \cancel{3}\cdot\frac{x^3}{\cancel{3}} - \frac{x^2}{2}\right]_1^3$$

$$= \frac{1}{4}\left[\frac{207}{20} + \frac{1}{20}\right] = \left[\frac{208}{20}\right]\frac{1}{4} = \frac{52}{20} = \frac{26}{10} = \frac{13}{5}$$

16. The cummulative distribution function for a continuous random variable 'x' is $F(x) = \begin{cases} 1-e^{-2x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$

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find

i. density function $f(x)$

ii. Mean

iii. variance

A, B

Sol: Given that

$$F(x) = \begin{cases} 1-e^{-2x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

We know that

$$\frac{d}{dx} e^{ax} = a \cdot e^{ax}$$

i.

$$f(x) = \frac{d}{dx} F(x)$$

$$f(x) = \frac{d}{dx}(1-e^{-2x})$$

$$= 0 + e^{-2x}(2)$$

$$f(x) = 2 \cdot e^{-2x}$$

$$f(x) = \begin{cases} 2e^{-2x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

ii. Mean :

$$E(x) = \int_0^\infty x \cdot f(x)\, dx$$

$$= \int_0^\infty x \cdot 2e^{-2x}\, dx$$

$$= 2 \cdot \int_0^\infty x \cdot e^{-2x}\, dx$$

$$= 2 \cdot \left[x \cdot \frac{e^{-2x}}{-2} - 1 \cdot \frac{e^{-2x}}{4} \right]_0^\infty$$

$$= 2\left[0 - \left(0 - \frac{e^{-2(0)}}{4}\right)\right]$$

$$= \cancel{2}\left[\frac{1}{\cancel{4}_2}\right]$$

Mean $= \mu$

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$$E(x) = \frac{1}{2}$$

iii. Variance :

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$\sigma^2 = \int_0^\infty x^2 \cdot f(x)\,dx$$

$$E(x^2) = 2\int_0^\infty x^2 \cdot e^{-2x}\,dx$$

$$= 2\left[x^2 \cdot \frac{e^{-2x}}{-2} - 2x\,\frac{e^{-2x}}{4} + 2\,\frac{e^{-2x}}{-8}\right]_0^\infty$$

$$= 2\left[0 - \left(0 - 0 + 2\,\frac{e^{-2(0)}}{-8}\right)\right]$$

$$= 2\left[0 - (0 - \frac{1}{4}\right]$$

$$E(x^2)\ \sigma^2 = \frac{1}{2}$$

$$\sigma^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$$\sigma^2 = \frac{1}{2} - \frac{1}{4}$$

$$\sigma^2 = \frac{1}{4}$$

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17. A continuous random variable x has a probability density function $f(x) = 3x^2, 0 \le x \le 1$. Find a & b such that

i. $P[x \le a] = P[x > a]$

ii. $P[x \ge b] = 0.05$

Sol: Given that

$$f(x) = 3x^2, \quad 0 \le x \le 1$$

i. Since the total probability is equal to 1

$$P[x \le a] = P(x > a] = \frac{1}{2}$$

Now $P[x \le a] = \frac{1}{2}$

So $P[0 \le x \le a] = \frac{1}{2}$

$$\int_0^a f(x)\,dx = \frac{1}{2}$$

$$\int_0^a 3x^2\,dx = \frac{1}{2}$$

$$\cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_0^a = \frac{1}{2}$$

$$a^3 = \frac{1}{2}$$

$$a = \sqrt[3]{\frac{1}{2}}$$

$$a = \frac{1}{2^{1/3}}$$

$$a = 0.79370$$

ii. $P(x \geq b) = 0.05$

$$P(b \leq x \leq 1) = 0.05$$

$$\int_b^1 3x^2\,dx = 0.05$$

$$\cancel{3}\cdot\left(\frac{x^3}{\cancel{3}}\right)_b^1 = 0.05$$

$$1 - b^3 = 0.05$$

$$1 - 0.05 = b^3$$

$$b^3 = 0.95$$

$$b = \sqrt[3]{\frac{19}{20}}$$

$$b = \left(\frac{19}{20}\right)^{1/3}$$

$$b = 0.98304$$

18. If x is a continuous random variable and $y = ax+b$. Prove that $E(y) = aE(x)+b$ where a, b are constants.
$V(y) = a^2 V(x)$

Sol: Let $y = ax+b$ —①

Taking expectation on both sides

$$E(y) = E[ax+b]$$

$$E(y) = \int_{-\infty}^{\infty} (ax+b)\, f(x)\,dx$$

$$E(y) = \int_{-\infty}^{\infty} ax f(x)\,dx + \int_{-\infty}^{\infty} b\cdot f(x)\,dx$$

$$E(y) = a\int_{-\infty}^{\infty} x\cdot f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

Here

$$\int_{-\infty}^{\infty} x\cdot f(x)\,dx = E(x)$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$E(y) = aE(x) + b(1)$$

$$\boxed{E(y) = a\,E(x) + b} \quad \text{— ②}$$

① - ②

$$y - E(y) = ax - aE(x) + \cancel{b} - \cancel{b}$$

$$y - E(y) = a[x - E(x)]$$

Squaring on both sides

$$[y - E(y)]^2 = a^2[x - E(x)]^2$$

$$[y - E(y)]^2 = a^2[x - E(x)]^2$$

Taking E on both sides,

$$E[y - E(y)]^2 = E[a^2(x - E(x))^2]$$

Here $E(x - E(x))^2 = V(x)$

$$E(y - E(y))^2 = V(y)$$

$$V(y) = a^2 V(x)$$

19. If x is a continuous random variable and k is a constant, then prove that i. $V(x+k) = V(x)$ A, B

ii. $V(kx) = k^2 V(x)$

Sol: By definition,

$$E(x) = \int_{-\infty}^{\infty} x\cdot f(x)\,dx$$

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$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$V(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2$$

i. $$V(x+K) = \int_{-\infty}^{\infty} (x+K)^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} (x+K) f(x)\,dx\right]^2$$

$$= \int_{-\infty}^{\infty} (x^2+2Kx+K^2) f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx + \int_{-\infty}^{\infty} K\cdot f(x)\,dx\right]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx + \int_{-\infty}^{\infty} 2x\cdot K f(x)\,dx + \int_{-\infty}^{\infty} K^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx + K\int_{-\infty}^{\infty} f(x)\,dx\right]^2$$

Here $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx + 2K\int_{-\infty}^{\infty} x\cdot f(x)\,dx + K^2\int_{-\infty}^{\infty} f(x)\,dx - \left\{\int_{-\infty}^{\infty} x f(x)\,dx + K\int_{-\infty}^{\infty} f(x)\,dx\right\}^2$$

$$= E(x^2) + 2KE(x) + K^2 - \{E(x)+K\}^2$$

$$= E(x^2) + 2KE(x) + K^2 - [E(x)]^2 - 2KE(x) - K^2$$

$$= E(x^2) - (E(x))^2$$

$$= V(x)$$

$$\boxed{V(x+K) = V(x)}$$

ii. $$V(Kx) = E(Kx)^2 - [E(Kx)]^2$$

$$= \int_{-\infty}^{\infty} (Kx)^2\,dx - \left(\int_{-\infty}^{\infty} Kx f(x)\,dx\right)^2$$

$$= K^2\int_{-\infty}^{\infty} x^2 f(x)\,dx - K^2\left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2$$

$$= K^2\left[E(x^2) - (E(x))^2\right]$$

$$V(Kx) = K^2 V(x)$$