UNIT-IV : Vector differentiation.

Vector function : Let 'S' be a set of real numbers. Corresponding to each scalar $t \in S$, let there will be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a vector function. We write $\bar{f} = \bar{f}(t)$.

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in three dimensional space.

Then we write $\bar{f} = \bar{f}(t) = f_1(t)\,\bar{i} + f_2(t)\,\bar{j} + f_3(t)\,\bar{k}$ where $f_1(t)$, $f_2(t)$ and $f_3(t)$ are real valued functions.

Derivative : We define derivative of a vector function $\bar{f}(t)$ as $\lim_{t \to a} \frac{\bar{f}(t) - \bar{f}(a)}{t-a}$ where 'a' is any point in the interval I (or) domain of $\bar{f}$. It is denoted by $\bar{f}'(a)$ (or) $\left(\frac{d\bar{f}}{dt}\right)_{at\ t=a}$.

If $\bar{f}'(a)$ exists, then we say that $\bar{f}$ is differentiable at $t=a$.

General rules of differentiation:

If $\bar{a}$ and $\bar{b}$ are differentiable vector functions, then

(1) $\frac{d}{dt}(\bar{a} \pm \bar{b}) = \frac{d\bar{a}}{dt} \pm \frac{d\bar{b}}{dt}$

(2) $\frac{d}{dt}(\bar{a} \cdot \bar{b}) = \frac{d\bar{a}}{dt} \cdot \bar{b} + \bar{a} \cdot \frac{d\bar{b}}{dt}$

(3) $\frac{d}{dt}(\bar{a} \times \bar{b}) = \frac{d\bar{a}}{dt} \times \bar{b} + \bar{a} \times \frac{d\bar{b}}{dt}$

4) $\frac{d}{dt}(\phi \bar{F}) = \phi \frac{d\bar{F}}{dt} + \frac{d\phi}{dt} \cdot \bar{F}$, where ϕ is a scalar function and $\bar{F}$ is a vector function.

5) If $\bar{F} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$, then $\frac{d\bar{F}}{dt} = \frac{df_1}{dt}\bar{i} + \frac{df_2}{dt}\bar{j} + \frac{df_3}{dt}\bar{k}$

6) If $\bar{F}$ is a constant vector, then $\frac{d\bar{F}}{dt} = \bar{0}$.

Note: similarly we can define the Partial derivatives $\frac{\partial \bar{F}}{\partial t}$, $\frac{\partial \bar{F}}{\partial x}$ etc.

Scalar and vector point functions:

Consider a region in three dimensional space. To each point $P(x,y,z)$, if we associate a unique real number (i.e., scalar) say ϕ, then $\phi(x,y,z)$ is called a scalar point function.

Similarly to each point $P(x,y,z)$, if we associate a unique vector $\bar{F}(x,y,z)$, then $\bar{F}$ is called a vector point function.

Ex: Temperature and speed are scalar point functions.

velocity and acceleration are vector point functions.

Vector Differential operator: The vector differential operator is defined as $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$.

Gradient of a scalar point function:

Defn: Let $\varphi(x,y,z)$ be a scalar point function defined in some region of space. Then the gradient of φ denoted by grad φ (or) $\nabla\varphi$ is defined as

$$\text{grad}\,\varphi = \nabla\varphi = \bar{i}\frac{\partial\varphi}{\partial x} + \bar{j}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z}$$

Geometrical Interpretation of $\nabla\varphi$:

The gradient of a scalar function $\varphi(x,y,z)$ at a point $P(x,y,z)$ is a vector along the normal to the level surface $\varphi(x,y,z)=c$ at P and is in increasing direction.

Directional derivatives: Let $\varphi(x,y,z)$ be a scalar function defined throughout some region of space. Let P be any point in the space, then $\overline{OP}=\bar{r}$. Let Q be a neighbouring point and $PQ=\delta r$, then $\lim\limits_{\delta r\to 0}\frac{\delta\varphi}{\delta r} = \frac{\partial\varphi}{\partial r}$ is called the directional derivative of φ at P in the direction PQ.

Theorem: The directional derivative of a scalar point function φ at a point $P(x,y,z)$ in the direction of a unit vector $\bar{e}$ is equal to $\bar{e}\cdot\text{grad}\,\varphi$.

ie, Directional derivative of φ along the unit vector e at $P(x,y,z) = \bar{e}\cdot\nabla\varphi$.

Note: $\nabla\varphi$ gives the maximum rate of change of φ and the magnitude of this maximum is $|\nabla\varphi|$.

$\therefore$ Greatest value of directional derivative of φ at $P = |\nabla\varphi|_{at\,P}$.

Problems:

1) If $\bar{A} = 5t^2\bar{i} + t\bar{j} - t^3\bar{k}$, $\bar{B} = \sin t\,\bar{i} - \cos t\,\bar{j}$, then find (i) $\frac{d}{dt}(\bar{A}\cdot\bar{B})$ (ii) $\frac{d}{dt}(\bar{A}\times\bar{B})$.

Sol: (i) $\frac{d}{dt}(\bar{A}\cdot\bar{B}) = \bar{A}\cdot\frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt}\cdot\bar{B}$

$= (5t^2\bar{i} + t\bar{j} - t^3\bar{k})(\cos t\,\bar{i} + \sin t\,\bar{j}) +$

$(10t\bar{i} + \bar{j} - 3t^2\bar{k})(\sin t\,\bar{i} - \cos t\,\bar{j})$

$= (5t^2\cos t + t\sin t) + (10t\sin t - \cos t)$ $\left(\because \bar{i}\cdot\bar{i} = 1,\ \bar{i}\cdot\bar{j} = 0,\ \bar{i}\cdot\bar{k} = 0\right)$ etc

$= 5t^2\cos t + 11t\sin t - \cos t$

(ii) $\frac{d}{dt}(\bar{A}\times\bar{B}) = \bar{A}\times\frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt}\times\bar{B}$

$= (5t^2\bar{i} + t\bar{j} - t^3\bar{k})\times(\cos t\,\bar{i} + \sin t\,\bar{j}) + (10t\bar{i} + \bar{j} - 3t^2\bar{k})\times$
$(\sin t\,\bar{i} - \cos t\,\bar{j})$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$= \left[\bar{i}(0 + t^3\sin t) - \bar{j}(0 + t^3\cos t) + \bar{k}(5t^2\sin t - t\cos t)\right]$
$+ \left[\bar{i}(0 - 3t^2\cos t) - \bar{j}(0 + 3t^2\sin t) + \bar{k}(-10t\cos t - \sin t)\right]$

$= (t^3\sin t - 3t^2\cos t)\bar{i} - t^2(t\cos t + 3\sin t)\bar{j} +$
$((5t^2 - 1)\sin t - 11t\cos t)\bar{k}$.

2) Find grad ϕ where $\phi = 3x^2y - y^3z^2$ at the point $(1, -2, -1)$

Sol: Given $\phi = 3x^2y - y^3z^2$.

we have grad $\phi = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$$\nabla\phi = \bar{i}\,\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \bar{j}\,\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \bar{k}\,\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \bar{i}(6xy - 0) + \bar{j}(3x^2 - 3y^2z^2) + \bar{k}(0 - 2y^3z)$$

$$= 6xy\,\bar{i} + (3x^2 - 3y^2z^2)\,\bar{j} - 2y^3z\,\bar{k}$$

At the Point $(1,-2,-1)$, $\nabla\phi = 6(1)(-2)\bar{i} + (3(1) - 3(-2)^2(-1)^2)\bar{j}$
$- 2(-2)^3(-1)\,\bar{k}$

$$(\nabla\phi)_{(1,-2,-1)} = -12\,\bar{i} + (3-12)\,\bar{j} - 16\,\bar{k}$$

3) Find $\text{grad}\,\phi$ where $\phi(x,y,z) = \log(x^2+y^2+z^2)$ at $(1,1,1)$

Sol: Given $\phi(x,y,z) = \log(x^2+y^2+z^2)$

Then $\dfrac{\partial\phi}{\partial x} = \dfrac{1}{x^2+y^2+z^2}\,\dfrac{\partial}{\partial x}(x^2+y^2+z^2) = \dfrac{2x}{x^2+y^2+z^2}$

$\dfrac{\partial\phi}{\partial y} = \dfrac{1}{x^2+y^2+z^2}\,\dfrac{\partial}{\partial y}(x^2+y^2+z^2) = \dfrac{2y}{x^2+y^2+z^2}$

$\dfrac{\partial\phi}{\partial z} = \dfrac{1}{x^2+y^2+z^2}\,\dfrac{\partial}{\partial z}(x^2+y^2+z^2) = \dfrac{2z}{x^2+y^2+z^2}$

$\therefore \text{grad}\,\phi = \nabla\phi = \bar{i}\,\dfrac{\partial\phi}{\partial x} + \bar{j}\,\dfrac{\partial\phi}{\partial y} + \bar{k}\,\dfrac{\partial\phi}{\partial z}$

$\Rightarrow \nabla\phi = \bar{i}\left(\dfrac{2x}{x^2+y^2+z^2}\right) + \bar{j}\left(\dfrac{2y}{x^2+y^2+z^2}\right) + \bar{k}\left(\dfrac{2z}{x^2+y^2+z^2}\right)$

At the Point $(1,1,1)$, $\nabla\phi = \frac{2}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k}$

4) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, Show that

(i) $\text{grad}\,r = \dfrac{\bar{r}}{r}$ (ii) $\text{grad}\left(\dfrac{1}{r}\right) = \dfrac{-\bar{r}}{r^3}$

(iii) $\text{grad}\,r^n = n r^{n-2}\,\bar{r}$ (or) $\nabla r^n = n r^{n-2}\,\bar{r}$

(iv) $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$, where $\bar{a}$ is a constant vector.

(v) $\nabla(f(r)) = \dfrac{f'(r)}{r}\,\bar{r}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol: (i) To Prove $\text{grad}\, r = \frac{\bar{r}}{r}$:

we have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

and $r^2 = x^2 + y^2 + z^2$.

Diffng. Partially w.r.t x, y, z, we get

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

III^{ly} $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$.

Now $\text{grad}\, r = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$

(or) $\text{grad}\, r = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) r$

$$= \bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}$$

$$= \bar{i}\left(\frac{x}{r}\right) + \bar{j}\left(\frac{y}{r}\right) + \bar{k}\left(\frac{z}{r}\right)$$

$$= \frac{x\bar{i} + y\bar{j} + z\bar{k}}{r} = \frac{\bar{r}}{r}$$

(ii) To Prove $\text{grad}\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$

we have $r^2 = x^2 + y^2 + z^2$.

Diffng Partially w.r.t x, y, z, we get

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$

$$= \sum \bar{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \sum \bar{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}$$

$$= \sum -\bar{i}\frac{1}{r^2}\left(\frac{x}{r}\right) = -\sum \bar{i}\frac{x}{r^3}$$

$$= -\frac{1}{r^3}\sum x\bar{i} = -\frac{1}{r^3}\left(x\bar{i} + y\bar{j} + z\bar{k}\right)$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$$

(iii) we have $r^2 = x^2 + y^2 + z^2$

Diffng Partially w.r.t x, y & z we get

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $\nabla r^n = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) r^n = \sum \bar{i} \frac{\partial}{\partial r}(r^n)$

$$= \sum \bar{i}\, n\, r^{n-1} \frac{\partial r}{\partial x} = \sum \bar{i}\, n\, r^{n-1} \frac{x}{r}$$

$$= n r^{n-1} \sum x\bar{i} = n r^{n-1}(x\bar{i} + y\bar{j} + z\bar{k}) = n r^{n-2}\bar{r}$$

$$\therefore \nabla r^n = n r^{n-1}\, \bar{r}$$

(iv) Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, a_1, a_2, a_3 are constants.

Then $\bar{a}\cdot\bar{r} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k})(x\bar{i} + y\bar{j} + z\bar{k})$

$$= a_1 x + a_2 y + a_3 z$$

Now $\nabla(\bar{a}\cdot\bar{r}) = \sum \bar{i} \frac{\partial}{\partial x}(a_1 x + a_2 y + a_3 z)$

$$= \sum \bar{i}(a_1) = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$$

(v) we have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

Diffng Partially w.r.t x, y & z, we get

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $\nabla(f(r)) = \sum \bar{i} \frac{\partial}{\partial x}(f(r))$

$$= \sum \bar{i} f'(r) \frac{\partial r}{\partial x}$$

$$= \sum \bar{i} f'(r) \cdot \frac{x}{r}$$

$$= \frac{f'(r)}{r} \sum x\bar{i} = \frac{f'(r)}{r}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\nabla(f(r)) = \frac{f'(r)}{r} \cdot \bar{r}$$

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5) Find a unit normal vector to the surface $x^3+y^3+3xyz=3$ at the Point $(1,2,-1)$.

Sol: Let $\phi = x^3+y^3+3xyz-3=0$

Then $\frac{\partial\phi}{\partial x} = 3x^2+3yz$, $\frac{\partial\phi}{\partial y} = 3y^2+3xz$, $\frac{\partial\phi}{\partial z} = 3xy$

The normal vector to the given surface ϕ at the Point $(1,2,-1)$ is $[\nabla\phi]_{at\ (1,2,-1)}$.

$\therefore \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$= \bar{i}(3x^2+3yz) + \bar{j}(3y^2+3xz) + \bar{k}(3xy)$

At $(1,2,-1)$; $\nabla\phi = \bar{i}(3+3(2)(-1)) + \bar{j}(3(4)+3(1)(-1))$
$+\bar{k}(3(1)(-2))$

$\Rightarrow \nabla\phi = -3\bar{i}+9\bar{j}+6\bar{k}$.

Hence a unit vector normal to the given surface at $(1,2,-1)$

is $\frac{\nabla\phi}{|\nabla\phi|} = \frac{-3\bar{i}+9\bar{j}+6\bar{k}}{\sqrt{9+81+36}} = \frac{-3\bar{i}+9\bar{j}+6\bar{k}}{3\sqrt{14}} = \frac{-\bar{i}+3\bar{j}+2\bar{k}}{\sqrt{14}}$

6) Find a unit normal vector to the surface $x^2-y^2+z=2$ at the Point $(1,-1,2)$.

Sol: Let $\phi = x^2-y^2+z-2$.

Then $\frac{\partial\phi}{\partial x} = 2x$, $\frac{\partial\phi}{\partial y} = -2y$, $\frac{\partial\phi}{\partial z} = 1$

$\therefore \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2x\bar{i} - 2y\bar{j} + \bar{k}$

$(\nabla\phi)_{(1,-1,2)} = 2(1)\bar{i} - 2(-1)\bar{j} + \bar{k} = 2\bar{i} + 2\bar{j} + \bar{k}$

This is the normal vector to the given surface at $(1,-1,2)$

Hence a unit normal vector normal to the given surface at $(1,-1,2)$ is $\frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i}+2\bar{j}+\bar{k}}{\sqrt{4+4+1}} = \frac{2\bar{i}+2\bar{j}+\bar{k}}{\sqrt{9}} = \frac{2\bar{i}+2\bar{j}+\bar{k}}{3}$

7) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\bar{i}-\bar{j}-2\bar{k}$.

Sol: Given $\phi = x^2yz + 4xz^2$

$\therefore \frac{\partial \phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial \phi}{\partial y} = x^2z, \quad \frac{\partial \phi}{\partial z} = x^2y + 8xz$

we have $\nabla \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$

$\Rightarrow \nabla \phi = (2xyz + 4z^2)\bar{i} + x^2z\,\bar{j} + (x^2y + 8xz)\bar{k}$

$\therefore \nabla \phi$ at $(1,-2,-1) = (2(1)(-2)(-1) + 4(-1)^2)\,\bar{i} + (1)(-1)\,\bar{j}$
$+ (1(-2) + 8(1)(-1))\,\bar{k}$

$(\nabla \phi)_{at\,(1,-2,-1)} = (4+4)\bar{i} - \bar{j} + (-2-8)\bar{k} = 8\bar{i} - \bar{j} - 10\bar{k}$

Given vector is $\bar{a} = 2\bar{i} - \bar{j} - 2\bar{k}$

$\Rightarrow |\bar{a}| = \sqrt{4+1+4} = 3$

The unit vector in the direction of $2\bar{i}-\bar{j}-2\bar{k}$ is

$\bar{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\bar{i}-\bar{j}-2\bar{k}}{3}$

$\therefore$ Required directional derivatives along the unit vector $\bar{e}$

at $(1,-2,-1) =$ grad $\phi \cdot \bar{e}$ (or) $\nabla \phi \cdot \bar{e}$

$= (8\bar{i} - \bar{j} - 10\bar{k})\frac{(2\bar{i}-\bar{j}-2\bar{k})}{3}$

$= \frac{(16+1+20)}{3} = \frac{37}{3}$

8) Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the Point $P = (1,2,3)$ in the direction of the line PQ where $Q = (5,0,4)$

Sol: Given $f = x^2 - y^2 + 2z^2$

Then $\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -2y, \quad \frac{\partial f}{\partial z} = 4z$

$\therefore \nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2x\,\bar{i} - 2y\,\bar{j} + 4z\,\bar{k}$

$(\nabla f)_{\text{at } P(1,2,3)} = 2(1)\bar{i} - 2(2)\bar{j} + 4(3)\bar{k} = 2\bar{i} - 4\bar{j} + 12\bar{k}$

Given $Q = (5,0,4)$.

$\therefore \overline{PQ} = \overline{OQ} - \overline{OP} = (5\bar{i} + 4\bar{k}) - (\bar{i} + 2\bar{j} + 3\bar{k})$

$= 4\bar{i} - 2\bar{j} + \bar{k}$

If $\bar{e}$ is the unit vector in the direction PQ, then

$$\bar{e} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{|4\bar{i} - 2\bar{j} + \bar{k}|} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{16+4+1}} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$$

$\therefore$ Directional derivative of f in the direction PQ =

$$(\nabla f)\cdot\bar{e}$$

$$= (2\bar{i} - 4\bar{j} + 12\bar{k})\frac{(4\bar{i} - 2\bar{j} + \bar{k})}{\sqrt{21}} = \frac{8+8+12}{\sqrt{21}}$$

$$= \frac{28}{\sqrt{21}}$$

9) Find the directional derivative of the function $\phi = xy^2 + yz^3$ at the Point $(2,-1,1)$ in the direction of the normal to the surface $x\log z - y^2 + 4 = 0$ at $(-1,2,1)$.

Sol: Given $\phi = xy^2 + yz^3$ — (1)

Then $\frac{\partial \phi}{\partial x} = y^2$, $\frac{\partial \phi}{\partial y} = 2xy + z^3$, $\frac{\partial \phi}{\partial z} = 3yz^2$

$\therefore \nabla\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = y^2\bar{i} + (2xy + z^3)\bar{j} + 3yz^2\bar{k}$

$(\nabla\phi)_{\text{at}(2,-1,1)} = (-1)^2\bar{i} + (2(2)(-1)+1)\bar{j} + 3(-1)(1)\bar{k} = \bar{i} - 3\bar{j} - 3\bar{k}$

Let $f = x\log z - y^2 + 4$.

Then $\frac{\partial f}{\partial x} = \log z$, $\frac{\partial f}{\partial y} = -2y$, $\frac{\partial f}{\partial z} = \frac{x}{z}$

$\therefore \nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \log z\,\bar{i} - 2y\bar{j} + \frac{x}{z}\bar{k}$

Normal to the surface f at $(-1,2,1) = \bar{n} = (\nabla f)_{(-1,2,1)}$

$\therefore \bar{n} = -4\bar{j} - \bar{k}$

unit normal vector $= \bar{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{-4\bar{j}-\bar{k}}{\sqrt{16+1}} = \frac{-4\bar{j}-\bar{k}}{\sqrt{17}}$

Hence the directional derivative of ϕ at $(2,-1,1)$ in the direction of $\bar{n}$ $= (\nabla\phi)\cdot\bar{e}$

$= (\bar{i}-3\bar{j}-3\bar{k})\cdot\left(\frac{-4\bar{j}-\bar{k}}{\sqrt{17}}\right) = \frac{0+12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$

10) In what direction from $(3,1,-2)$ is the directional derivative of $f(x,y,z) = x^2y^2z^4$ maximum and what is its magnitude?

Sol: Given $f(x,y,z) = x^2y^2z^4$.

$\therefore \frac{\partial f}{\partial x} = 2xy^2z^4, \quad \frac{\partial f}{\partial y} = 2x^2yz^4, \quad \frac{\partial f}{\partial z} = 4x^2y^2z^3$

we have $\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$\Rightarrow \nabla f = (2xy^2z^4)\bar{i} + (2x^2yz^4)\bar{j} + (4x^2y^2z^3)\bar{k}$

The normal vector to the surface f at $(3,1,-2) = (\nabla f)_{at(3,1,-2)}$

ie, $\bar{n} = (2(3)(1)(-2)^4)\bar{i} + (2(3)^2(1)(-2)^4)\bar{j} + [4(3)^2(1)(-2)^3]\bar{k}$

$\Rightarrow \bar{n} = 96\bar{i} + 288\bar{j} - 288\bar{k} = 96(\bar{i}+3\bar{j}-3\bar{k})$

The directional derivative is maximum in the direction of $\bar{n}$ and the magnitude of this maximum is $|\nabla f|$

ie, $|\nabla f| = \sqrt{(96)^2(1+9+9)} = 96\sqrt{19}$

Angle between the two surfaces: Let $\phi_1(x,y,z)=0$ and $\phi_2(x,y,z)=0$ be the two surfaces and $\overline{n_1}$ and $\overline{n_2}$ be the normal vectors to the two surfaces at a point $P(x,y,z)$ resply. If 'θ' is the angle between the normals of the two surfaces, then

$$\cos\theta = \frac{\overline{n_1}\cdot\overline{n_2}}{|\overline{n_1}|\,|\overline{n_2}|} \quad (\text{or}) \quad \theta = \cos^{-1}\left(\frac{\overline{n_1}\cdot\overline{n_2}}{|\overline{n_1}|\cdot|\overline{n_2}|}\right)$$

10) Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Sol: The angle between the two surfaces is the angle between the two normals at a point P.

Let $\phi_1 = x^2+y^2+z^2-9=0$ and $\phi_2 = x^2+y^2-z-3=0$

Then $\nabla\phi_1 = \bar{i}\frac{\partial\phi_1}{\partial x}+\bar{j}\frac{\partial\phi_1}{\partial y}+\bar{k}\frac{\partial\phi_1}{\partial z} = 2x\bar{i}+2y\bar{j}+2z\bar{k}$

$\nabla\phi_2 = \bar{i}\frac{\partial\phi_2}{\partial x}+\bar{j}\frac{\partial\phi_2}{\partial y}+\bar{k}\frac{\partial\phi_2}{\partial z} = 2x\bar{i}+2y\bar{j}-\bar{k}$

Let $\overline{n_1}$ and $\overline{n_2}$ be the normals to the given surfaces at the point $P=(2,-1,2)$.

$\therefore \overline{n_1} = (\nabla\phi_1)_{(2,-1,2)} = 4\bar{i}-2\bar{j}+4\bar{k} \Rightarrow |\overline{n_1}| = \sqrt{16+4+16} = 6$

$\overline{n_2} = (\nabla\phi_2)_{(2,-1,2)} = 4\bar{i}-2\bar{j}-\bar{k} \Rightarrow |\overline{n_2}| = \sqrt{16+4+1} = \sqrt{21}$

Let 'θ' be the angle between the two surfaces.

Then $\cos\theta = \frac{\overline{n_1}\cdot\overline{n_2}}{|\overline{n_1}|\,|\overline{n_2}|} = \frac{(4\bar{i}-2\bar{j}+4\bar{k})\cdot(4\bar{i}-2\bar{j}-\bar{k})}{6\sqrt{21}}$

$\Rightarrow \cos\theta = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$

$\therefore \theta = \cos^{-1}\frac{8}{3\sqrt{21}}$.

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Divergence of a vector Point function:

Defn: Let $\bar{F}$ be any continuously differentiable vector Point function. Then the divergence of $\bar{F}$, denoted by $\operatorname{div}\bar{F}$, is defined as

$$\operatorname{div}\bar{F} = \bar{i}\cdot\frac{\partial \bar{F}}{\partial x} + \bar{j}\cdot\frac{\partial \bar{F}}{\partial y} + \bar{k}\cdot\frac{\partial \bar{F}}{\partial z}$$

$$(\text{or})\ \operatorname{div}\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\bar{F} = \nabla\cdot\bar{F}$$

Note: Divergence of a vector Point function is a scalar.

Theorem: If $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then $\operatorname{div}\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Solenoidal vector: A vector Point function $\bar{F}$ is said to be solenoidal if $\operatorname{div}\bar{F} = 0$.

Problems:

1) If $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find $\operatorname{div}\bar{F}$ at $(1,-1,1)$

Sol: Given $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$

$$\Rightarrow \bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$$

$$\because f_1 = xy^2,\quad f_2 = 2x^2yz,\quad f_3 = -3yz^2$$

we have $\operatorname{div}\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$$\Rightarrow \operatorname{div}\bar{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$\Rightarrow \operatorname{div}\bar{F} = y^2 + 2x^2z - 6yz$$

$$\therefore (\operatorname{div}\bar{F})_{at\,(1,-1,1)} = 1 + 2 + 6 = 9$$

2) Find div $\bar{F}$ where $\bar{F} = \text{grad}(x^3+y^3+z^3-3xyz)$.

Sol: Let $\phi = x^3+y^3+z^3-3xyz$.

Then $\frac{\partial\phi}{\partial x} = 3x^2-3yz$, $\frac{\partial\phi}{\partial y} = 3y^2-3xz$, $\frac{\partial\phi}{\partial z} = 3z^2-3xy$

$\therefore \bar{F} = \text{grad}\,\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$\Rightarrow \bar{F} = (3x^2-3yz)\bar{i} + (3y^2-3xz)\bar{j} + (3z^2-3xy)\bar{k}$

$= f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

Hence $\text{div}\,\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$

$= 6x+6y+6z = 6(x+y+z)$

3) If $\bar{F} = x(y+z)\bar{i} + y(z+x)\bar{j} + z(x+y)\bar{k}$ then find div$\bar{F}$.

Sol: Given $\bar{F} = x(y+z)\bar{i} + y(z+x)\bar{j} + z(x+y)\bar{k}$

$\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$\therefore f_1 = xy+xz$, $f_2 = yz+yx$, $f_3 = zx+zy$

$\Rightarrow \frac{\partial f_1}{\partial x} = y+z$, $\frac{\partial f_2}{\partial y} = z+x$, $\frac{\partial f_3}{\partial z} = x+y$

Hence $\text{div}\,\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = y+z+z+x+x+y$

$= 2(x+y+z)$

4) Show that $\nabla\cdot(r^n\bar{r}) = (n+3)r^n$. Find n if it is solenoidal.

Sol: we have $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and

$r^2 = x^2+y^2+z^2$

Diffng. Partially w.r.t x, y, z we get

$\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$.

Now $r^n \bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$$\nabla(r^n\bar{r}) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= \sum \frac{\partial}{\partial x}(r^n x) = \sum\left[n r^{n-1}\frac{\partial r}{\partial x}\cdot x + r^n(1)\right]$$

$$= \sum\left[n r^{n-1}\frac{x}{r}\cdot x + r^n\right]$$

$$= n r^{n-2}\sum x^2 + \sum r^n$$

$$= n r^{n-2}(x^2+y^2+z^2) + 3r^n$$

$$= n r^{n-2}\cancel{r^2} + 3r^n \quad (\because r^2 = x^2+y^2+z^2)$$

$$\nabla(r^n\bar{r}) = n r^n + 3r^n = (n+3)r^n$$

If $\bar{F} = r^n\bar{r}$ is solenoidal, then $\nabla(r^n\bar{r}) = 0$

$$\Rightarrow (n+3)r^n = 0 \Rightarrow \boxed{n=-3} \quad (\because r \text{ is non-zero})$$

5) Show that $\frac{\bar{r}}{r^3}$ is solenoidal. (or) $\nabla\left(\frac{\bar{r}}{r^3}\right) = 0$

Sol: we have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r^2 = x^2+y^2+z^2$.

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\frac{\bar{r}}{r^3} = \frac{1}{r^3}(x\bar{i} + y\bar{j} + z\bar{k}) = r^{-3}(x\bar{i} + y\bar{j} + z\bar{k})$

$$\therefore \nabla\left(\frac{\bar{r}}{r^3}\right) = \frac{\partial}{\partial x}(r^{-3}x) + \frac{\partial}{\partial y}(r^{-3}y) + \frac{\partial}{\partial z}(r^{-3}z)$$

$$= \sum \frac{\partial}{\partial x}(r^{-3}x) = \sum\left[(-3)r^{-4}\frac{\partial r}{\partial x}x + r^{-3}(1)\right]$$

$$= \sum -3r^{-4}\frac{x}{r}x + \sum r^{-3}$$

$$= -3r^{-5}\sum x^2 + 3r^{-3}$$

$$= -3r^{-5}(r^2) + 3r^{-3} \quad (\because \sum x^2 = x^2+y^2+z^2 = r^2)$$

$$= -3r^{-3} + 3r^{-3} = 0$$

Hence $\frac{\bar{r}}{r^3}$ is solenoidal.

6) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, then find div r

Sol: we have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$\therefore \operatorname{div} \bar{r} = \nabla \cdot \bar{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

H.W:

1) S.T $(x+3y)\bar{i} + (y-2z)\bar{j} + (x-2z)\bar{k}$ is solenoidal

2) Find the divergence of the vector $\bar{V} = xyz\,\bar{i} + 3x^2y\,\bar{j} + (xz^2 - y^2z)\bar{k}$ at the point $(2,-1,1)$

3) P.T $\operatorname{div}\left(\frac{\bar{r}}{r}\right) = \frac{2}{r}$

Curl of a vector:

Defn: Let $\bar{F}$ be any continuously differentiable vector point function. Then curl $\bar{F}$ denoted by $(\nabla \times \bar{F})$ is defined as $\bar{i} \times \frac{\partial \bar{F}}{\partial x} + \bar{j} \times \frac{\partial \bar{F}}{\partial y} + \bar{k} \times \frac{\partial \bar{F}}{\partial z}$.

$$\therefore \operatorname{curl} \bar{F} = \nabla \times \bar{F} = \bar{i} \times \frac{\partial \bar{F}}{\partial x} + \bar{j} \times \frac{\partial \bar{F}}{\partial y} + \bar{k}\,\frac{\partial \bar{F}}{\partial z} = \sum \bar{i} \times \frac{\partial \bar{F}}{\partial x}$$

Note: 1) Curl $\bar{F}$ is a vector function.

2) If $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then $\operatorname{curl} \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix} = \nabla \times \bar{F}$

3) If $\bar{F}$ is a constant vector, then curl $\bar{F} = \bar{0}$.

Defn: A vector $\bar{F}$ is said to be irrotational if curl $\bar{F} = \bar{0}$.

<u>Note:</u> If $\bar{F}$ is irrotational, then there exists a scalar function $\phi(x,y,z)$ such that $\bar{F}=\nabla\phi$.

This function ϕ is called <u>scalar potential</u> of $\bar{F}$ and $\bar{F}$ is called <u>conservative</u>.

$\therefore \nabla\times\bar{F}=\bar{0} \Leftrightarrow$ there exists a scalar function ϕ s.t $\bar{F}=\nabla\phi$.

1) Find curl $\bar{F}$ where $\bar{F} = grad(x^3+y^3+z^3-3xyz)$.

<u>Sol:</u> Let $\phi = x^3+y^3+z^3-3xyz$

Then $\frac{\partial\phi}{\partial x} = 3x^2-3yz$, $\frac{\partial\phi}{\partial y} = 3y^2-3xz$, $\frac{\partial\phi}{\partial z} = 3z^2-3xy$

$\therefore \bar{F} = grad\,\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$\Rightarrow \bar{F} = (3x^2-3yz)\bar{i} + (3y^2-3xz)\bar{j} + (3z^2-3xy)\bar{k}$

$= F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

Now Curl $\bar{F}$ = $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$

$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix} = \bar{i}(-3x+3x) - \bar{j}(-3y+3y) + \bar{k}(-3z+3z) = \bar{0}$

$\therefore$ Curl $\bar{F} = \bar{0} \Rightarrow \bar{F}$ is irrotational.

2) Given $\bar{A} = x^2yz\,\bar{i} + y^2zx\,\bar{j} + z^2xy\,\bar{k}$ find div $\bar{A}$ and curl $\bar{A}$.

<u>Sol:</u> Given $\bar{A} = x^2yz\,\bar{i} + y^2zx\,\bar{j} + z^2xy\,\bar{k}$

$= A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$

$\therefore$ Curl $\bar{A}$ = $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & y^2zx & z^2xy \end{vmatrix}$$

$$= \bar{i}(z^2x - y^2x) - \bar{j}(z^2y - x^2y) + \bar{k}(y^2z - x^2z)$$

Find $\operatorname{div} \bar{A}$.

3) Show that $\operatorname{curl}(r^n \bar{r}) = \bar{0}$

Sol: We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r^2 = x^2 + y^2 + z^2$

Diffg. Partially w.r.t x, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

|||ly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $r^n \bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$$\therefore \operatorname{curl}(r^n \bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \sum \bar{i}\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right]$$

$$= \sum \bar{i}\left(z n r^{n-1}\frac{\partial r}{\partial y} - y n r^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= \sum \bar{i}\left(zn r^{n-1}\frac{y}{r} - yn r^{n-1}\frac{z}{r}\right)$$

$$= \frac{n r^{n-1}}{r}\sum \bar{i}(zy - yz)$$

$$= n r^{n-2}\sum 0\bar{i} = n r^{n-2}(0\bar{i} + 0\bar{j} + 0\bar{k})$$

$$= n r^{n-2}(\bar{0}) = \bar{0}$$

Hence $r^n \bar{r}$ is irrotational

4) Prove that $\operatorname{curl} \bar{r} = \bar{0}$.

Sol: $\operatorname{curl} \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}$

4) A vector field is given by $\bar{A} = (x^2+xy^2)\bar{i} + (y^2+x^2y)\bar{j}$. Show that the field is irrotational and find the scalar potential.

Sol: Given $\bar{A} = (x^2+xy^2)\bar{i} + (y^2+x^2y)\bar{j}$

Then $\text{Curl}\,\bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+xy^2 & y^2+x^2y & 0 \end{vmatrix}$

$= \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(2xy - 2xy) = \bar{0}$

$\therefore \bar{A}$ is irrotational.

$\therefore$ There exists ϕ such that $\text{grad}\,\phi = \bar{A}$, where ϕ is the scalar potential function.

$\nabla\phi = \bar{F} \Rightarrow \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k} = (x^2+xy^2)\bar{i} + (y^2+x^2y)\bar{j}$

comparing the like terms, we get

$\frac{\partial\phi}{\partial x} = x^2+xy^2 \Rightarrow \phi = \int(x^2+xy^2)\,dx + c_1(y,z)$

$\Rightarrow \phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + c_1(y,z)$

$\frac{\partial\phi}{\partial y} = y^2+x^2y \Rightarrow \phi = \int(y^2+x^2y)\,dy + c_2(x,z)$

$\Rightarrow \phi = \frac{y^3}{3} + \frac{x^2y^2}{2} + c_2(x,z)$

$\therefore \phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + \frac{y^3}{3} + c$ (or) $\phi = 2x^3 + 3x^2y^2 + 2y^3 + c$

5) Find constants a, b, c so that the vector $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$.

Sol: Given $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$

Then $\nabla\times\bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$

$\Rightarrow \bar{i}(c+1) + \bar{j}(a-4) + \bar{k}(b-2)$

If $\bar{A}$ is irrotation $\Rightarrow \nabla \times \bar{A} = \bar{0}$

$\therefore (c+1)\bar{i} + (a-4)\bar{j} + (b-2)\bar{k} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$\Rightarrow c+1=0,\ a-4=0,\ b-2=0 \Rightarrow a=4, b=2, c=-1$

$\therefore \bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y-2z)\bar{k}$

we have $\bar{A} = \nabla\phi$

$\Rightarrow (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k} = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

Comparing both sides, we get

$\frac{\partial\phi}{\partial x} = x+2y+4z \Rightarrow \phi = \int(x+2y+4z)\,dx = \frac{x^2}{2} + 2xy + 4xz + c_1(y,z)$

$\frac{\partial\phi}{\partial y} = 2x-3y-z \Rightarrow \phi = \int(2x-3y-z)\,dy = 2xy - \frac{3y^2}{2} - yz + c_2(x,z)$

$\frac{\partial\phi}{\partial z} = 4x-y+2z \Rightarrow \phi = \int(4x-y+2z)\,dy = 4xz - yz + z^2 + c_3(x,y)$

Hence $\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - yz + c$

6) If $\bar{A}$ is irrotational vector, evaluate $\text{div}(\bar{A}\times\bar{r})$.

Sol: we have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Since $\bar{A}$ is irrotational vector, we have $\boxed{\nabla\times\bar{A} = \bar{0}}$ — (1)

Now $\text{div}(\bar{A}\times\bar{r}) = \nabla\cdot(\bar{A}\times\bar{r})$

$= (\nabla\times\bar{A})\cdot\bar{r} - \bar{A}(\nabla\times\bar{r})$

$= 0\cdot\bar{r} - \bar{A}(\nabla\times\bar{r}) = -\bar{A}(\nabla\times\bar{r})$ — (2)

Now $\nabla\times\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \bar{i}(0) + \bar{j}(0) + \bar{k}(0) = \bar{0}$

$\therefore$ From (2), we get $\text{div}(\bar{A}\times\bar{r}) = 0$

Extra problems. Contd.

1) Find the values of a, b, c if the directional derivative of the function $\phi = axy^2 + byz + cz^2x^3$ at the point $(1, 2, -1)$ has maximum magnitude 64 in the direction parallel to z-axis.

$$\phi = axy^2 + byz + cz^2x^3$$

$$\nabla\phi = \bar{i}(ay^2 + 3cx^2z^2) + \bar{j}(2yax + bz) + \bar{k}(by + 2czx^3)$$

$$(\nabla\phi)_{(1,2,-1)} = \bar{i}(4a+3c) + \bar{j}(4a-b) + \bar{k}(2b-2c)$$

Unit vector parallel to z-axis $= \bar{k}$.

Given DD of $\nabla\phi$ has maximum magnitude 64 in direction of $\bar{k}$. $\Rightarrow \nabla\phi \cdot \bar{k} = 64$

$$\Rightarrow 2b - 2c = 64$$

$$\Rightarrow b - c = 32 \quad \text{— (1)}$$

$\nabla\phi$ is in the direction of z-axis $\Rightarrow$ it is $\perp$r to x and y axes.

$$\therefore \nabla\phi \cdot \bar{i} = 0 \quad \text{and} \quad \nabla\phi \cdot \bar{j} = 0$$

$$4a + 3c = 0 \quad \text{— (2)}, \quad 4a - b = 0 \quad \text{— (3)}$$

Solving (1), (2) and (3). (2) − (3) $\Rightarrow 3c + b = 0$

(1) $\Rightarrow -c + b = 32$

$$4c = -32$$

$$c = -8$$

$b = 32 + c = 32 - 8$

$b = 24$

$4a = b$

$a = \frac{b}{4} = 6$

$\therefore a = 6, b = 24, c = -8$

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Laplacian Operator

$\text{div}(\text{grad}\,\phi) = \nabla\cdot(\nabla\phi) = \nabla^2\phi$.

$\nabla^2\phi = 0$ is called Laplacian equation.

∇^2 is called Laplacian operator

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

ϕ is called harmonic function.

ie; Any function satisfying the Laplacian equation is called harmonic function.

If $\phi(x,y,z)$ is a constant function, then The surface represented by ϕ is called potential surface

Result: If $\nabla^2\phi = 0$, then $\nabla\phi$ is both solenoidal and irrotational.

$\nabla^2\phi = 0 \Rightarrow \text{div}(\text{grad}\,\phi) = \nabla\cdot\nabla\phi = 0 \Rightarrow \nabla\phi$ is solenoidal.

$$\text{curl}(\text{grad}\,\phi) = \nabla\times\nabla\phi = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right) - \bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right) = \bar{0}$$

$\therefore \nabla\phi$ is irrotational.

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1. Prove that div. (grad r^m) $= m(m+1)\, r^{m-2}$

(or) $\nabla^2 r^m = m(m+1)\, r^{m-2}$.

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

Diff wrt x partially, $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\text{grad}(r^m) = \nabla(r^m) = \bar{i}\frac{\partial}{\partial x}r^m + \bar{j}\frac{\partial}{\partial y}r^m + \bar{k}\frac{\partial}{\partial z}r^m$$

$$= \sum \bar{i}\,\frac{\partial}{\partial x}(r^m)$$

$$= \sum \bar{i}\, m r^{m-1}\frac{\partial r}{\partial x}$$

$$= \sum i\, m r^{m-1}\frac{x}{r}$$

$$= \sum i\, m r^{m-2}\, x$$

$$\text{div}(\text{grad } r^m) = m\sum\left[r^{m-2} + x(m-2)r^{m-3}\frac{\partial r}{\partial x}\right]$$

$$= m\sum\left[r^{m-2} + x(m-2)r^{m-3}\frac{x}{r}\right]$$

$$= m\sum\left[r^{m-2} + x^2(m-2)r^{m-4}\right]$$

$$= m\left[3r^{m-2} + (m-2)r^{m-4}(x^2+y^2+z^2)\right]$$

$$= m\left[3r^{m-2} + (m-2)r^{m-4}\cdot r^2\right]$$

$$= m\left[3r^{m-2} + (m-2)r^{m-2}\right]$$

$$= m r^{m-2}\left[3+m-2\right]$$

$$= m r^{m-2}(m+1) = m(m+1)r^{m-2}$$

(3)

2. Show that $\nabla^2[f(r)] = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}$

$$= f''(r) + \frac{2}{r} f'(r)$$

and hence find $\nabla^2\left(\frac{1}{r}\right)$ and $\nabla^2(\log r)$.

Sol:

$$\text{grad } f(r) = \nabla f(r) = \sum i \frac{\partial}{\partial x} f(r)$$

$$= \sum i f'(r)\frac{\partial r}{\partial x} = \sum i f'(r)\frac{x}{r}$$

$$\nabla^2(f(r)) = \text{div}(\text{grad } f(r)) = \sum \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right]$$

$$= \sum \left[\frac{r\frac{\partial}{\partial x}[x f'(r)] - x f'(r)\frac{\partial r}{\partial x}}{r^2}\right]$$

$$= \sum \left[\frac{r\left[x f''(r)\frac{x}{r} + f'(r)\right] - x f'(r)\frac{x}{r}}{r^2}\right]$$

$$= \sum \left[\frac{x^2 f''(r) + r f'(r) - x^2 \frac{f'(r)}{r}}{r^2}\right]$$

$$= \frac{f''(r)}{r^2}\sum x^2 + \sum \frac{\cancel{r} f'(r)}{r^{\cancel{2}}} - \frac{1}{r^3} f'(r)\sum x^2$$

$$= \frac{f''(r)}{r^2}(x^2+y^2+z^2) + \frac{3 f'(r)}{r} - \frac{1}{r^3} f'(r)(x^2+y^2+z^2)$$

$$= \frac{f''(r)}{\cancel{r^2}}\cdot \cancel{r^2} + \frac{3 f'(r)}{r} - \frac{1}{r^{\cancel{3}}} f'(r)\cdot \cancel{r^2}$$

$$= f''(r) + \frac{2}{r} f'(r).$$

1) $\nabla^2\left(\frac{1}{r}\right) =$

Put $f(r) = \frac{1}{r}$, $f'(r) = \frac{-1}{r^2}$, $f''(r) = \frac{2}{r^3}$

$$\therefore \nabla^2\left(\frac{1}{r}\right) = \frac{2}{r^3} + \frac{2}{r}\left(\frac{-1}{r^2}\right) = \frac{2}{r^3} - \frac{2}{r^3} = 0$$

2) $\nabla^2(\log r)$

$f(r) = \log r$, $f'(r) = \frac{1}{r}$, $f''(r) = \frac{-1}{r^2}$

$\nabla^2(\log r) = \frac{-1}{r^2} + \frac{2}{r}\frac{1}{r} = \frac{-1}{r^2} + \frac{2}{r^2} = \underline{\underline{\frac{1}{r^2}}}$.

Vector Identities

1) $\text{curl}(\phi\bar{a}) = (\text{grad}\,\phi)\times\bar{a} + \phi\,\text{curl}\,\bar{a}$.

Pf: $\text{curl}(\phi\bar{a}) = \nabla\times(\phi\bar{a})$

$= \sum \bar{i}\times\frac{\partial}{\partial n}(\phi\bar{a})$

$= \sum \bar{i}\times\left[\frac{\partial\phi}{\partial n}\bar{a} + \phi\frac{\partial\bar{a}}{\partial n}\right]$

$= \sum\left(\bar{i}\frac{\partial\phi}{\partial n}\right)\times\bar{a} + \sum\left(\bar{i}\times\frac{\partial\bar{a}}{\partial n}\right)\phi$

$= \nabla\phi\times\bar{a} + \cancel{\nabla\times(\phi\bar{a})}(\nabla\times\bar{a})\phi$

$= (\text{grad}\,\phi\times\bar{a}) + \phi\,\text{curl}\,\bar{a}$.

2) $\text{div}(\bar{a}\times\bar{b}) = \bar{b}\cdot\text{curl}\,\bar{a} - \bar{a}\cdot\text{curl}\,\bar{b}$

$\text{div}(\bar{a}\times\bar{b}) = \nabla\cdot(\bar{a}\times\bar{b}) = \sum\bar{i}\cdot\frac{\partial}{\partial n}(\bar{a}\times\bar{b})$

$= \sum\bar{i}\cdot\left(\frac{\partial\bar{a}}{\partial n}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial n}\right)$

$= \sum\bar{i}\cdot\left(\frac{\partial\bar{a}}{\partial n}\times b\right) + \sum\bar{i}\cdot\left(\bar{a}\times\frac{\partial\bar{b}}{\partial n}\right)$

$= \sum\left(\bar{i}\times\frac{\partial\bar{a}}{\partial n}\right)\cdot\bar{b} + \sum\left(\bar{i}\times\frac{\partial\bar{b}}{\partial n}\right)\cdot\bar{a}$

$= (\nabla\times\bar{a})\cdot\bar{b} - (\nabla\times\bar{b})\cdot\bar{a}$

$= \underline{\underline{\bar{b}\cdot\text{curl}\,\bar{a} - \bar{a}\cdot\text{curl}\,\bar{b}}}$.

3) curl $(\phi\bar{a}) = (\text{grad}\,\phi)\times\bar{a} + \phi\,\text{curl}\,\bar{a}$ (Similar to 1)

$$\text{curl}(\phi\bar{a}) = \nabla\times(\phi\bar{a}) = \sum \bar{i}\times\frac{\partial}{\partial x}(\phi\bar{a})$$

$$= \sum \bar{i}\times\left[\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right]$$

$$= \sum\left(\bar{i}\frac{\partial\phi}{\partial x}\right)\times\bar{a} + \sum\left(\bar{i}\times\frac{\partial\bar{a}}{\partial x}\right)\phi$$

$$= \nabla\phi\times\bar{a} + (\nabla\times\bar{a})\phi$$

$$= \underline{(\text{grad}\,\phi)\times\bar{a}} + \phi\,\text{curl}\,\bar{a}.$$

~~4) curl $(\bar{a}\times\bar{b}) = \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\,\bar{a}$~~

4) P.T div curl $\bar{f} = 0$.

Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\text{curl}\,\bar{f} = \nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\text{div}(\text{curl}\,\bar{f}) = \nabla\cdot(\text{curl}\,\bar{f})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial f_3}{\partial x} + \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 f_3}{\partial x\partial y}} - \cancel{\frac{\partial^2 f_2}{\partial x\partial z}} - \cancel{\frac{\partial^2 f_3}{\partial y\partial x}} + \cancel{\frac{\partial^2 f_1}{\partial y\partial z}} + \cancel{\frac{\partial^2 f_2}{\partial z\partial x}} - \cancel{\frac{\partial^2 f_1}{\partial z\partial y}}$$

$$= \underline{0}$$

$\therefore$ curl $\bar{f}$ is always solenoidal.

Problems :

1) For a solenoidal vector, prove that

curl curl curl curl $\bar{f} = \nabla^4 \bar{f}$

~~curl We know that ($\bar{a} \times b$~~

We know that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

curl curl $\bar{f} = \nabla \times (\nabla \times \bar{f}) = \nabla(\nabla \cdot \bar{f}) - (\nabla \cdot \nabla)\bar{f}$

$= \nabla(\nabla \cdot \bar{f}) - \nabla^2 \bar{f}$

Given $\bar{f}$ is solenoidal. $\therefore \nabla \cdot \bar{f} = 0$.

$\therefore$ curl curl $\bar{f} = -\nabla^2 \bar{f} = -\bar{a}$ (say).

$\therefore$ curl curl (curl curl $\bar{f}$) = curl curl $(-\bar{a})$

$= -$ curl curl $(\bar{a})$

$= -[\nabla \times (\nabla \times \bar{a})]$

$= -[\nabla(\nabla \cdot \bar{a}) - \nabla^2 \bar{a}]$

$= -$ grad (div $\bar{a}$) $+ \nabla^2 \bar{a}$

$= -$ grad div $(\nabla^2 \bar{f}) + \nabla^2(\nabla^2 \bar{f})$

$= -$ grad div $(\nabla^2 \bar{f}) + \nabla^4 \bar{f}$

$$\nabla^2 \bar{f} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k})$$

$$= \sum \bar{i}\left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)$$

$$= \sum i \nabla^2 f_1$$

$$\text{div}(\nabla^2 \bar{f}) = \nabla \cdot (\nabla^2 \bar{f})$$

$$= \frac{\partial}{\partial x}(\nabla^2 f_1) + \frac{\partial}{\partial y}(\nabla^2 f_2) + \frac{\partial}{\partial z}(\nabla^2 f_3)$$

$= \nabla^2 \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right)$

$= \nabla^2 (\nabla \cdot \bar{f})$

$= \nabla^2 (0)$ $\quad [\because \bar{f}$ is solenoidal$]$

$= 0.$

$\therefore$ curl curl curl curl $\bar{f} = \nabla^4 \bar{f}$.

2. Find $(\bar{A} \cdot \nabla)\phi$ at $(1,-1,1)$ if

$\bar{A} = 3xyz^2 \bar{i} + 2xy^3 \bar{j} - x^2yz \bar{k}$ and $\phi = 3x^2 - yz$.

$(\bar{A} \cdot \nabla)\phi = (\bar{A} \cdot \bar{i}) \frac{\partial \phi}{\partial x} + (\bar{A} \cdot \bar{j}) \frac{\partial \phi}{\partial y} + (\bar{A} \cdot \bar{k}) \frac{\partial \phi}{\partial z}$

$= (3xyz^2)(6x) + (2xy^3)(-z) + (-x^2yz)(-y)$

$= 18x^2yz^2 - 2xy^3z + x^2y^2z$

$(\bar{A} \cdot \nabla)\phi$ at $(1,-1,1)$

$= 18(1)(-1)(1) - 2(1)(-1)(1) + (1)(1)(1)$

$= -18 + 2 + 1 = \underline{\underline{-15}}.$

3. Find $(\bar{A} \times \nabla)\phi$ if $\bar{A} = yz^2 \bar{i} - 3xz^2 \bar{j} + 2xyz \bar{k}$

and $\phi = xyz$.

$[\bar{i} \times \bar{j} = \bar{k}$, $\bar{j} \times \bar{k} = \bar{i}$ and $\bar{k} \times \bar{i} = \bar{j}$

$\bar{j} \times \bar{i} = -\bar{k}$, $\bar{k} \times \bar{j} = -\bar{i}$ and $\bar{i} \times \bar{k} = -\bar{j}]$

$\bar{i} \quad \bar{j} \quad \bar{k}$

$(\bar{A} \times \nabla)\phi = (\bar{A} \times \bar{i}) \frac{\partial \phi}{\partial x} + (\bar{A} \times \bar{j}) \frac{\partial \phi}{\partial y} + (\bar{A} \times \bar{k}) \frac{\partial \phi}{\partial z}$

$= \left(-3xz^2(-\bar{k}) + 2xyz\bar{j}\right) yz + \left[yz^2 \bar{k} + 2xyz(-\bar{i})\right] xz$

$+ \left[\left(yz^2(-\bar{j})\right) + \left(-3xz^2 \bar{i}\right)\right] xy$

$= -5x^2yz^2 \bar{i} + \underline{\underline{xy^2z^2 \bar{j} + 4xyz^3 \bar{k}}}$

4. Evaluate $\nabla \cdot \left[r \, \nabla\left(\frac{1}{r^3}\right)\right]$ (8)

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\nabla\left(\frac{1}{r^3}\right) = \sum \bar{i} \frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) = \sum \bar{i} \left(-3r^{-4}\right)\frac{\partial r}{\partial x}$$

$$= \sum \bar{i}\left(-3r^{-4}\right)\frac{x}{r}$$

$$= \sum \bar{i}\left(-3r^{-5}x\right).$$

$$r \, \nabla\left(\frac{1}{r^3}\right) = \sum i\left(-3r^{-4}x\right)$$

$$\nabla \cdot \left(r \, \nabla\left(\frac{1}{r^3}\right)\right) = \frac{\partial}{\partial x}\left(-3r^{-4}x\right) + \frac{\partial}{\partial y}\left(-3r^{-4}y\right) + \frac{\partial}{\partial z}\left(-3r^{-4}z\right)$$

$$= -3r^{-4} + -3x\left(-4r^{-5}\right)\frac{x}{r}$$

$$-3r^{-4} + -3y\left(-4r^{-5}\right)\frac{y}{r}$$

$$-3r^{-4} + -3z\left(-4r^{-5}\right)\frac{z}{r}$$

$$= -9r^{-4} + 12r^{-6}\left(x^2 + y^2 + z^2\right)$$

$$= -9r^{-4} + 12r^{-6}r^2$$

$$= -9r^{-4} + 12r^{-4}$$

$$= 3r^{-4} = \frac{3}{r^4}.$$