Unit - III

Concepts of Stability and Algebraic criteria

Definitions of Stability :-

The term stability refers to the stable working condition of a control system.

The different definitions of the stability are the following

1. A system is stable, if its o/p is bounded for any bounded input

2. A system is asymptotically stable, if in the absence of input, the output tends towards zero irrespective of initial conditions.

3. A system is stable if for a bounded disturbing input signal the output vanishes ultimately as t approaches ∞.

4. A system is unstable if for a bounded disturbing input signal the output is of infinite amplitude or oscillatory.

5. For a bounded input signal, if the output has a constant amplitude oscillations then the systems may be stable or unstable under some limited constraints. Such a system is called limitedly stable.

6. If a system output is stable for all variations of its parameters then the system is called absolutely stable system

7 If the system output is stable for a limited range

of variations of its parameters, then the system is called conditionally stable system.

Impulse response of a system

Let $M(s)$ = closed loop transfer function of a system

$C(s)$ = output in s-domain

$R(s)$ = input in s-domain

Now $M(s) = \frac{C(s)}{R(s)}$

$\therefore$ output in s-domain $C(s) = M(s)\,R(s)$

Now, Response in time domain $c(t) = L^{-1}\{C(s)\}$

Input in time domain $r(t) = L^{-1}\{R(s)\}$

For an impulse input $r(t) = \delta(t)$; $R(s) = L\{\delta(t)\} = 1$

$\therefore$ Impulse response $= L^{-1}\{C(s)\} = L^{-1}\{M(s)\,R(s)\}$

$= L^{-1}\{M(s)\} = m(t)$

Hence, impulse response of a system is the inverse Laplace transform of system transfer function.

→ The importance of impulse response is that, the output of a system for any arbitrary input can be obtained by convolution of input and impulse response.

Response $c(t) = m(t) * r(t)$

where $*$ is the symbol for convolution.

Mathematically, the convolution operation is defined as

$$c(t) = \int_{-\infty}^{\infty} m(\tau)\, r(t-\tau)\, d\tau \longrightarrow (1)$$

where τ is the dummy variable used for Integration.

Bounded - input Bounded output stability :-

A linear relaxed system is said to have BIBO stability if every bounded (finite) input results in a bounded output.

A condition for BIBO stability can be obtained from Convolution operation defined by equation ①

$$C(t) = \int_0^\infty m(\tau)\, r(t-\tau)\, d\tau \longrightarrow (2)$$

If the input $r(t)$ is bounded then there exist a constant A_1 such that $|r(t)| \leq A_1 < \infty$. The condition for bounded output for this bounded input condition can be derived as follows.

On taking the absolute value on both sides of equation (2)

$$|c(t)| = \left| \int_0^\infty m(\tau)\, r(t-\tau)\, d\tau \right|$$

$$|c(t)| \leq \int_0^\infty |m(\tau)\, r(t-\tau)|\, d\tau$$

$$|c(t)| \leq \int_0^\infty |m(\tau)|\, |r(t-\tau)|\, d\tau$$

$$|c(t)| \leq \int_0^\infty |m(\tau)|\, A_1\, d\tau$$

$$\therefore |c(t)| \leq A_1 \int_0^\infty |m(\tau)|\, d\tau$$

If the output $C(t)$ is bounded then there exists a constant A_2 such that $|c(t)| \leq A_2 < \infty$

$$\therefore A_1 \int_0^\infty |m(\tau)|\, d\tau \leq A_2 < \infty$$

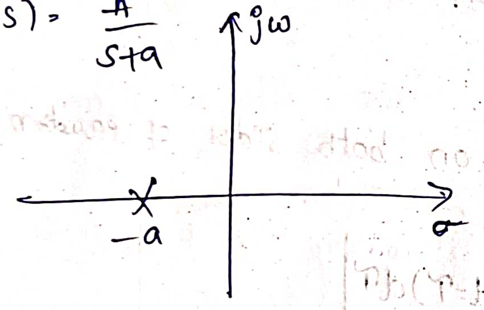
The above condition is satisfied if, $\int_0^{\infty} |m(\tau)| d\tau < \infty$

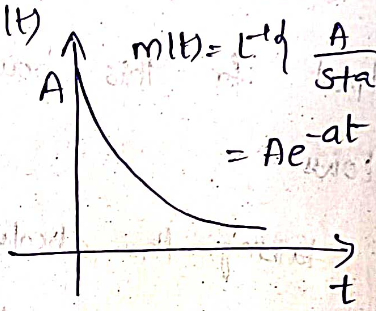
Hence for bounded output, $\int_0^{\infty} |m(t)| dt < \infty$.

Therefore we can conclude that a system with impulse response $m(t)$ is BIBO stable if and only if the impulse response is absolutely integrable. (i.e., $\int_0^{\infty} |m(t)| dt$ is finite. This means that area under the absolute value curve of the impulse response $m(t)$ evaluated from $t=0$ to $t=\infty$ must be finite)

Location of Poles on S-Plane for stability :-

$M(s) = \frac{A}{s+a}$



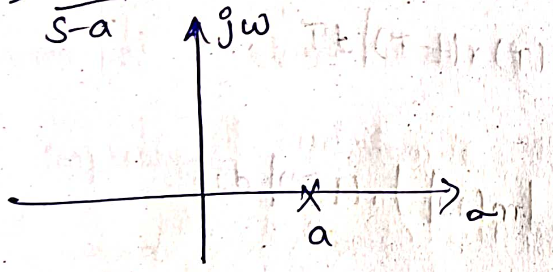


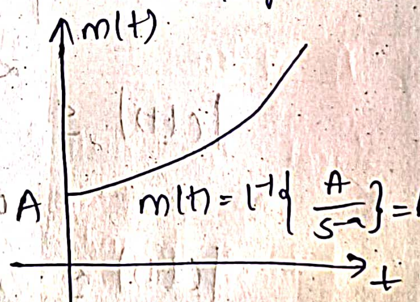
$m(t) = L^{-1}\left\{\frac{A}{s+a}\right\}$

$= Ae^{-at}$

Impulse response is exponentially decaying. Stable system

$M(s) = \frac{A}{s-a}$



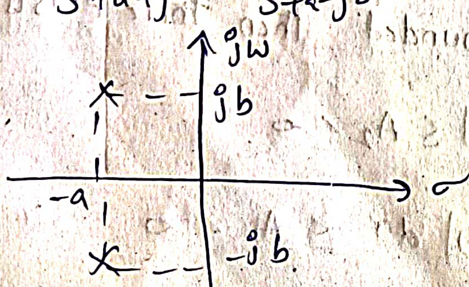


$m(t) = L^{-1}\left\{\frac{A}{s-a}\right\} = Ae^{at}$

Root on positive real axis

Impulse response is exponentially increasing. unstable system

$M(s) = \frac{A}{s+a+jb} + \frac{A}{s+a-jb}$

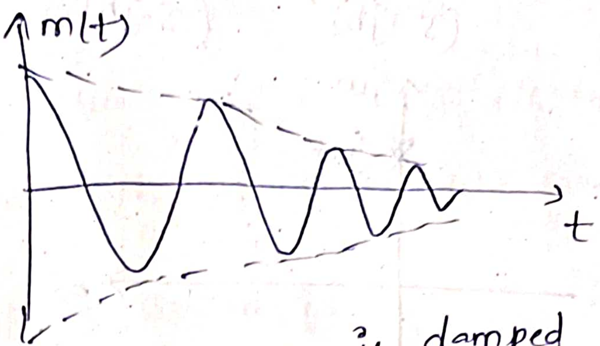


$m(t) = L^{-1}\left\{\frac{A}{s+a+jb} + \frac{A}{s+a-jb}\right\}$

$= Ae^{-(a+jb)t} + Ae^{-(a-jb)t}$

$= 2Ae^{-at}\cos bt$

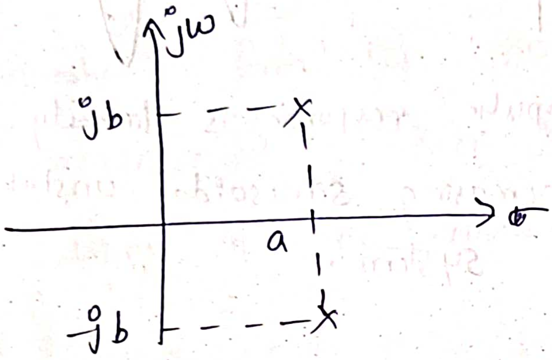
$= 2Ae^{-at} \sin(bt + 90^\circ)$



Complex conjugate roots on left half of s-plane

Impulse response is damped sinusoidal. Stable system

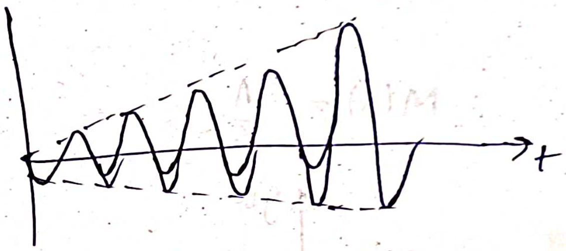
$$M(s) = \frac{A}{s-a+jb} + \frac{A}{s-a-jb}$$



$$m(t) = L^{-1}\left\{ \frac{A}{s-a+jb} + \frac{A^*}{s-a-jb} \right\}$$

$$= Ae^{-(a+jb)t} + A^* e^{-(a-jb)t}$$

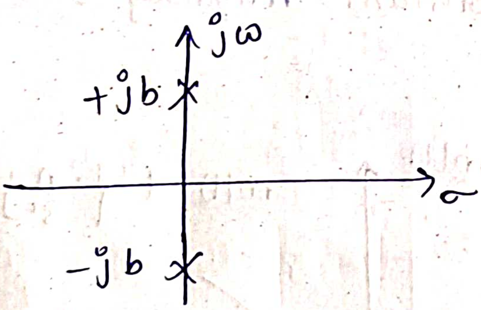
$$= 2Ae^{at} \cos bt = 2Ae^{at} \sin(bt+90^\circ)$$



Complex conjugate roots on right half of s-plane.

Impulse response is exponentially increasing sinusoidal. unstable System.

$$M(s) = \frac{A}{s+jb} + \frac{A^*}{s-jb}$$

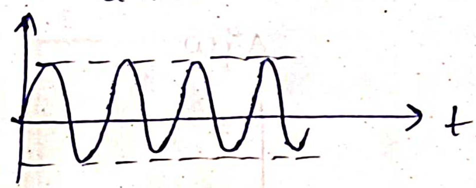


$$m(t) = L^{-1}\left\{ \frac{A}{s+jb} + \frac{A^*}{s-jb} \right\}$$

$$= Ae^{jbt} + A^* e^{jbt}$$

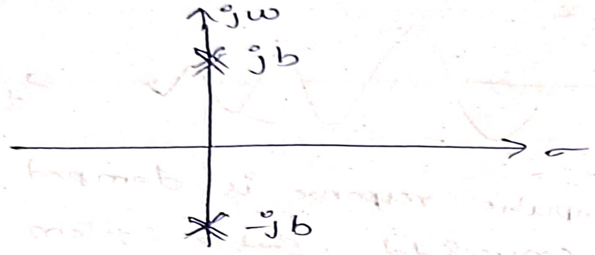
$$= 2A \cos bt = 2A \sin(bt+90^\circ)$$

m(t)



Impulse response is oscillatory Marginally Stable System

$$M(s) = \frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}$$



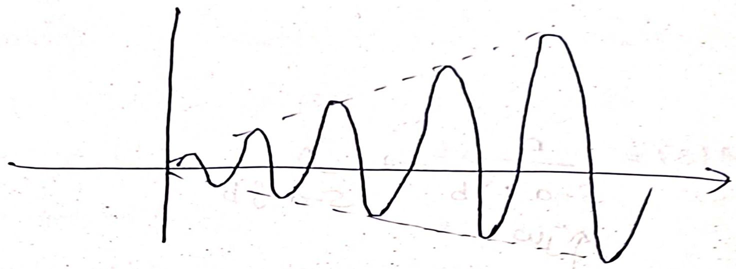
Double pair of roots on imaginary axis

$$m(t) = L^{-1}\left\{ \frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2} \right\}$$

$$= At\,e^{-jbt} + A^* t\, e^{jbt}$$

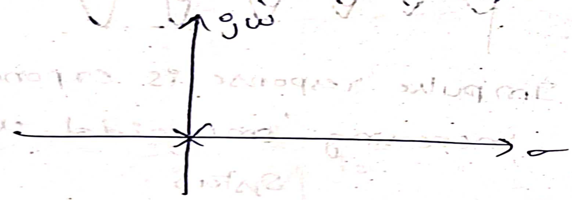
$$= 2At\cos bt$$

$$= 2At \sin(bt + 90^\circ)$$



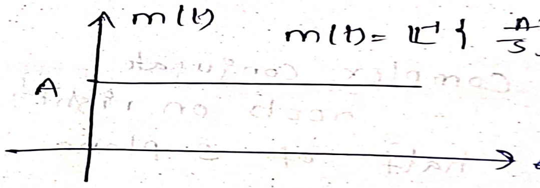
Impulse response is linearly increasing sinusoidal. unstable system.

$$M(s) = \frac{A}{s}$$



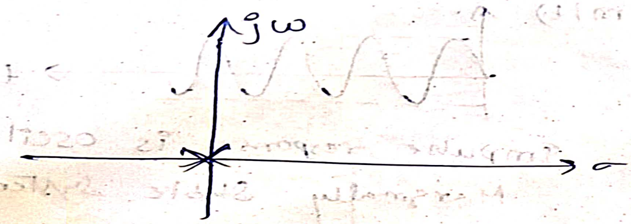
single root at origin.

$$m(t) = L^{-1}\left\{ \frac{A}{s} \right\} = A$$

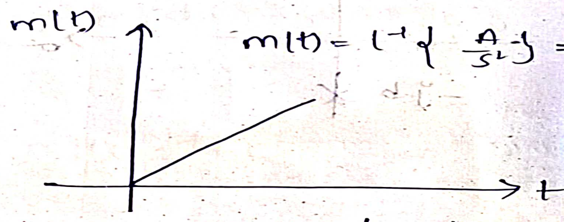


Impulse response is Constant. Marginally stable system

$$M(s) = A/s^2$$



$$m(t) = L^{-1}\left\{ \frac{A}{s^2} \right\} = At$$



Impulse response linearly increases with time. unstable system.

Routh Hurwitz Criteria

Using Routh criterion, Determine the stability of the system represented by the characteristic equation $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of characteristic Equation.

Sol) The characteristic equation of the system is

$$s^4+8s^3+18s^2+16s+5=0$$

The given characteristic equation is 4^{th} order equation and so it has 4 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd power of s.

$$\begin{array}{lccc} s^4 & : & 1 & 18 & 5 \\ s^3 & : & 8 & 16 & \end{array}$$

The elements of s^3 row can be divided by 8 to simplify the computations

$$\begin{array}{lccc} s^4 & : & 1 & 18 & 5 \\ s^3 & : & 1 & 2 & \\ s^2 & : & 16 & 5 & \\ s^1 & : & 1.7 & & \\ s^0 & : & 5 & & \end{array}$$

$s^2 : \frac{1\times18-2\times1}{1} \quad \frac{1\times5-0\times1}{1}$

$s^2 : 16 \quad 5$

$s^1 : \frac{16\times2-5\times1}{16}$

$s^1 : 1.6875 \approx 1.7$

$s^0 : \frac{1.7\times5-0\times16}{1.7}$

$s^0 : 5$

On examining the elements of first column of routh array it is observed that all the elements are positive and there is no sign change. Hence all the roots are lying on the left half of S-plane and the system is stable.

Result :-

1. Stable system

2. All the four roots are lying on the left half of S-pl

5.2

Construct Routh array and determine the stability of the system whose characteristic equation $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the no. of roots lying on the right half of s-plane, left half of s-plane and on imaginary axis.

Sol)

s^6 :	1	8	20	16
s^5 :	2	12	16	
s^6 :	1	8	20	16
s^5 :	1	6	8	
s^4 :	2	12	16	
s^6 :	1	8	20	16
s^5 :	1	6	8	
s^4 :	1	6	8	
s^3 :	0	0		
s^3 :	1	3		
s^2 :	3	8		
s^1 :	0.33			
s^0 :	8			

s^4 : $\frac{1\times8-6\times1}{1}$ $\frac{1\times20-8\times1}{1}$

s^4 : 2 12 16

s^3 : $\frac{1\times6-6\times1}{1}$ $\frac{1\times8-8\times1}{1}$

s^3 : 0 0

Auxiliary equation is

$A = s^4+6s^2+8$

$\frac{dA}{ds} = 4s^3+12s$

The coefficients of $\frac{dA}{ds}$ are used to form s^3 row

s^3 : 4 12

s^3 : 1 3

On examining the elements of 1st column of routh array it is observed that there is no sign change. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

$$s^2 : \frac{1\times 6-3\times 1}{1} \qquad \frac{1\times 8-0\times 1}{1}$$

$$s^2 : 3 \qquad 8$$

$$s^1 : \frac{3\times 3-8\times 1}{3}$$

$$s^1 : 0.33$$

$$s^0 : \frac{0.33\times 8-0\times 3}{0.33}$$

$$s^0 : 8$$

Auxiliary polynomial is

$$s^4+6s^2+8=0$$

Let $s^2=x$

$$\therefore x^2+6x+8=0$$

Roots of quadratic are $x=\dfrac{-6\pm\sqrt{6^2-4\times 8}}{2}$

$$=-3\pm 1 = -2 \text{ or } -4$$

The roots of auxiliary polynomial is,

$$s=\pm\sqrt{x}=\pm\sqrt{-2} \text{ and } \pm\sqrt{-4}$$

$$=+j\sqrt{2},\ -j\sqrt{2},\ +j2 \text{ and } -j2$$

→ The roots of auxiliary polynomial are also roots of characteristic equation. Hence 4 roots are lying on imaginary axis and the remaining 2 roots are lying on left half of s-plane.

* Construct Routh array and determine the stability of the system represented by the characteristic equation $s^5+s^4+2s^3+2s^2+3s+5=0$. Comment on the location of the roots of characteristic equation?

s^5	: 1	2	3
s^4	: 1	2	5
s^3	: ϵ	-2	
s^2	: $\frac{2\epsilon+2}{\epsilon}$	5	
s^1	: $\frac{-(5\epsilon^2+4\epsilon+4)}{2\epsilon+2}$	0	
s^0	: 5		

$s^3 : \frac{1\times 2 - 2\times 1}{1} \quad \frac{1\times 3 - 5\times 1}{1}$

$s^3 : 0 \quad -2$

Replace 0 by ϵ

$s^3 : \epsilon \quad -2$

$s^2 : \frac{\epsilon\times 2 + 2\times 1}{\epsilon} \quad \frac{\epsilon\times 5 - 0\times 1}{\epsilon}$

$s^2 : \frac{2\epsilon+2}{\epsilon} \quad 5$

$$s^1 : \frac{\frac{2\epsilon+2}{\epsilon}(-2) - 5\times\epsilon}{\frac{2\epsilon+2}{\epsilon}} \quad 0$$

$$s^1 : \frac{-4\epsilon-4-5\epsilon^2}{2\epsilon+2} \quad 0$$

$$s^0 : \frac{\frac{-(5\epsilon^2+4\epsilon+4)\times 5}{2\epsilon+2}}{\frac{-(5\epsilon^2+4\epsilon+4)}{2\epsilon+2}} \quad 0$$

On letting $\epsilon \to 0$ we get

s^5	: 1	2	3
s^4	: 1	2	5
s^3	: 0	-2	
s^2	: ∞	5	
s^1	: -2		
s^0	: 5		

On observing the elements of first column of routh array, it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and the system is unstable. The remaining three roots are lying on the left half of s-plane.

* By routh stability criterion determine the stability of the system represented by the characteristic equation

$9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$. Comment on the location of roots of characteristic equation?

Sol

$s^5 : 9 \quad 10 \quad -9$

$s^4 : -20 \quad -1 \quad -10$

$s^3 : 9.55 \quad -13.5$

$s^2 : -29.3 \quad -10$

$s^1 : -16.8$

$s^0 : -10$

$s^3 : \dfrac{-20\times10+1\times9}{-20} \quad \dfrac{-20(-9)+10\times}{-20}$

$s^3 : 9.55 \quad -13.5$

$s^2 : \dfrac{9.55(-1)+13.5(-20)}{9.55} \quad \dfrac{9.55(-10)}{9.55}$

$s^2 : -29.3 \quad -10$

$s^1 : \dfrac{-29.3(-13.5)+10(9.55)}{-29.3}$

$s^1 : -16.8$

$s^0 : \dfrac{-16.8(-10)-0}{-16.8}$

$s^0 = -10$

3 sign changes ⇒ 3 roots are lying on the right half of s-plane and the remaining two roots are lying on the left half of s-plane

* The characteristic polynomial of a system is $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$. Determine the location of roots on s-plane and hence the stability of the system

Sol

$s^7 : 1 \quad 24 \quad 24 \quad 23$

$s^6 : 9 \quad 24 \quad 24 \quad 15$

$s^7 : 1 \quad 24 \quad 24 \quad 23$

$s^6 : 3 \quad 8 \quad 8 \quad 5$

$s^5 : 1 \quad 1 \quad 1$

$s^4 : 5 \quad 5 \quad 5$

$s^4 : 1 \quad 1 \quad 1$

$s^3 : 0 \quad 0$

$s^5 : \dfrac{9\times24-24\times1}{9} \quad \dfrac{9\times24-24}{9} \quad \dfrac{9\times23-15}{9}$

$s^5 : \dfrac{3\times24-8}{3} \quad \dfrac{3\times24-8}{3} \quad \dfrac{3\times23-5}{3}$

$s^5 : 21.33 \quad 21.33 \quad 21.33$

Divide by 21.33

$s^5 : 1 \quad 1 \quad 1$

$s^4 : \dfrac{1\times8-3}{1} \quad \dfrac{1\times8-3}{1} \quad \dfrac{1\times5-0}{1}$

$s^4 : 5 \quad 5 \quad 5$

s^3 : 2 1

s^2 : 0.5 1

s^1 : -3

s^0 : 1

Auxilary polyn

$A = s^4 + s^2 + 1$

$\frac{dA}{ds} = 4s^3 + $

s^3 : 4

Divide by

s^3 : 2

$s^2 : \frac{2\times1 - 1\times1}{2}$

s^2 : 0.5 1

$s^1 : \frac{0.5\times1 - 2\times1}{0.5}$

: -3

$s^0 : \frac{-3\times1 - 0}{-3}$

s^0 : 1

The auxiliary equation is $s^4 + s^2 + 1 = 0$

Put $s^2 = x$ in the auxiliary equation,

$$x^2 + x + 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$= 1\angle 120^\circ \text{ or } 1\angle -120^\circ$$

But $s^2 = x$, $\therefore$ $s = \pm\sqrt{x} = \pm\sqrt{1\angle 120^\circ}$

$$= \pm\sqrt{1}\angle 120^\circ/2$$

$$= \pm 1\angle 60^\circ$$

$$= \pm(0.5 + j0.866)$$

or $\pm\sqrt{1\angle -120^\circ}$

$\pm\sqrt{1}\angle -120^\circ/2$, $\pm 1\angle -60^\circ$, $\pm(0.5 - j0.866)$

Two roots of auxiliary polynomial are lying on the right half of s-plane and the remaining two on left half of s-plane.

Result:

1. The system is unstable.
2. Two roots are lying on right half of s-plane and five roots are lying on left half of s-plane.

* The characteristic polynomial of a system is $s^7 + 5s^6 +$ $9s^5 + 9s^4 + 4s^3 + 20s^2 + 36s + 36 = 0$. Determine the location of roots on the s-plane and hence the stability of the system?

Sol)

s^7 : 1 9 4 36

s^6 : 5 9 20 36

Divide s^6 row by 5 to simplify the computations

s^7 :	1	9	4	36
s^6 :	1	1.8	4	7.2
s^5 :	1	0	4	
s^4 :	1	0	4	
s^3 :	0	0		

s^5 : $\frac{1\times9-1\times1.8}{1}$ $\frac{1\times4-4\times1}{1}$ $\frac{1\times36-7.2\times1}{1}$

s^5 : 7.2 0 28.8

s^5 : 1 0 4

s^4 : $\frac{1\times1.8}{1}$ $\frac{1\times4-4}{1}$ $\frac{1\times7.2}{1}$

s^4 : 1.8 0 7.2

s^4 : 1 0 4

The row of all zeros indicate the existence of auxiliary polynomial.

Divide the characteristic equation by auxiliary equation to get the quotient polynomial

$$\begin{array}{r|l} & s^3 + 5s^2 + 9s + 9 \\ s^4+4 & s^7 + 5s^6 + 9s^5 + 9s^4 + 4s^3 + 20s^2 + 36s + 36 \\ & \cancel{s^7} \qquad\qquad\qquad\qquad + \cancel{4s^3} \\ \hline & 5s^6 + 9s^5 + 9s^4 + 20s^2 + 36s + 36 \\ & \cancel{5s^6} + \qquad\qquad \cancel{20s^2} \\ \hline & 9s^5 + 9s^4 \qquad + 36s + 36 \\ & \cancel{9s^5} \qquad\qquad\quad \cancel{+36s} \\ \hline & 9s^4 \qquad\qquad + 36 \\ & 9s^4 \qquad\qquad + 36 \end{array}$$

The characteristic equation can be expressed as a product of quotient polynomial and auxiliary equation

$\therefore \quad s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$

$(s^4+4)\quad (s^3+5s^2+9s+9)=0$

The routh array is constructed for quotient polynomial

$$\begin{array}{lcll} s^3 & : & 1 & 9 \\ s^2 & : & 5 & 9 \\ s^1 & : & 7.2 & \\ s^0 & : & 9 & \end{array}$$

There is no sign change in the elements of first column of routh array of quotient polynomial. Hence all the roots of quotient polynomial are lying on left half of s-plane

To determine the stability, the roots of auxiliary polynomial should be evaluated.

$s^4+4=0$

Put $s^2=x \Rightarrow s^4+4=x^2+4=0$

$x^2=-4 \Rightarrow x=\pm\sqrt{-4}=\pm j2=2\angle 90^\circ$ or $2\angle -90^\circ$

But $s=\pm\sqrt{x}=\pm\sqrt{2\angle 90^\circ}$ or $\pm\sqrt{2\angle -90^\circ}$

$=\pm\sqrt{2}\angle 90^\circ/2$ or $\pm\sqrt{2}\angle -90^\circ/2$

$=\pm\sqrt{2}\angle +45^\circ$ or $\pm\sqrt{2}\angle -45^\circ$

$x=r\cos\theta=\sqrt{2}\cdot\cos 45^\circ=\sqrt{2}\cdot\frac{1}{\sqrt{2}}=1$

$= \pm(1+j1)$ or $\pm(1-j1)$

Two roots of auxiliary equation are lying on right half of s-plane and the other two on the left half of s-plane.

<u>Result</u>:- The system is unstable.

Two roots are lying on the right half of s-plane and five roots are lying on the left half of s-plane.

<u>5.7</u> Use the routh stability criterion to determine the location of roots on s-plane and hence the stability for the system represented by the characteristic equation

$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$

<u>Sol</u>

$s^5 : 1 \quad 8 \quad 7$

$s^4 : 4 \quad 8 \quad 4$

$s^3 : 6 \quad 6$

$s^3 : \frac{4\times8-8\times1}{4} \quad \frac{4\times7-4\times1}{4}$

$= \frac{24}{4} = 6 \quad \frac{24}{4} = 6$

$s^5 : 1 \quad 8 \quad 7$

$s^4 : 1 \quad 2 \quad 1$

$s^3 : 1 \quad 1$

$s^2 : 1 \quad 1$

$s^1 : \epsilon \quad 0$

$s^0 : 1$

$s^2 : \frac{1\times2-1\times1}{1} \quad \frac{1\times1-0}{1}$

$: 1 \quad 1$

When $\epsilon \to 0$ there is no sign change in first column of routh array. But we have a row of all zeros and so there is a possibility of roots on imaginary axis

This can be found from the roots of auxiliary polynomial.

The auxiliary polynomial is $s^2 + 1 = 0$

But $s^2 = -1$ or $s = \pm j1$

The roots of auxiliary polynomial are $+j1$ and $-j1$ lying on imaginary axis.

<u>Result</u>:-

1. The system is limitedly or marginally stable.
2. Two roots are lying on imaginary axis and three roots are lying on left half of s-plane.

5.8 Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$

Sol

$$\begin{array}{lllll}
s^6 & : & 1 & 3 & 3 & 1 \\
s^5 & : & 1 & 3 & 2 \\
s^4 & : & \epsilon & 1 & 1 \\
s^3 & : & \frac{3\epsilon-1}{\epsilon} & \frac{2\epsilon-1}{\epsilon} & 0 \\
s^2 & : & \frac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1} & 1 & 0 \\
s^1 & : & \frac{4\epsilon^2-\epsilon}{2\epsilon^2-4\epsilon+1} \\
s^0 & : & 1
\end{array}$$

$s^4 : \frac{1\times3-3\times1}{1} \quad \frac{1\times3-2\times1}{1} \quad \frac{1\times1}{1}$

$s^4 : 0 \quad 1 \quad 1$

$s^3 : \frac{\epsilon\times3-1}{\epsilon} \quad \frac{\epsilon\times2-1}{\epsilon}$

$s^2 : \dfrac{\frac{3\epsilon-1}{\epsilon} - (2\epsilon-1)}{\frac{3\epsilon-1}{\epsilon}} \quad \dfrac{\frac{3\epsilon-1}{\epsilon}\times1}{\frac{3\epsilon-1}{\epsilon}}$

$s^2 : \frac{3\epsilon-1-2\epsilon^2+2\epsilon}{3\epsilon-1} \quad 1$

$s^1 \quad \dfrac{\left(\frac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1}\right)\left(\frac{2\epsilon-1}{\epsilon}\right) - \left(\frac{3\epsilon-1}{\epsilon}\right)}{\frac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1}}$

on letting $\epsilon \to 0$ we get

s^6	1	3	3	1
s^5	1	3	2	
s^4	0	1	1	
s^3	$-\infty$	$-\infty$		
s^2	1	1		
s^1	0			
s^0	1			

$$s^1 \quad \frac{\dfrac{-4\epsilon^3+2\epsilon^2+8\epsilon^2-4\epsilon-2\epsilon+1-(3\epsilon-1)}{(3\epsilon-1)\epsilon}}{\dfrac{-2\epsilon^2+4\epsilon-1}{(3\epsilon-1)}}$$

$$s^1 \quad \frac{-4\epsilon^3+10\epsilon^2-\cancel{6\epsilon}+\cancel{1}-9\epsilon^2-\cancel{1}+\cancel{6\epsilon}}{\epsilon(-2\epsilon^2+4\epsilon-1)}$$

$$s^1 \quad \frac{-4\epsilon^3+\epsilon^2}{\epsilon(-2\epsilon^2+4\epsilon-1)}$$

$$s^1 \quad \frac{\cancel{\epsilon}(-4\epsilon^2+1)}{\cancel{\epsilon}(-2\epsilon^2+4\epsilon-1)}$$

Since there is a row of all zeros (s^1 row), there is possibility of roots on imaginary axis. The auxiliary polynomial is

$s^2+1=0$

The roots of auxiliary polynomial are $s=\pm\sqrt{-1}=\pm j1$

The roots of auxiliary polynomial are also roots of Characteristic Equation. Hence two roots are lying on imaginary axis. $\therefore$ divide the characteristic polynomial by auxiliary equation and construct the routh array for quotient polynomial to find the roots lying on right half of s-plane.

$$s^6+s^5+3s^4+3s^3+3s^2+2s+1=0 \Rightarrow (s^2+1)(s^4+s^3+2s^2+2s+1)=0$$

$$
\begin{array}{r}
s^4 + s^3 + 2s^2 + 2s + 1 \\
s^2+1 \overline{\big)\, s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1} \\
\underline{-\,s^6 \qquad\quad + s^4} \\
s^5 + 2s^4 + 3s^3 + 3s^2 + 2s + 1 \\
\underline{-\,s^5 \qquad\quad + s^3} \\
2s^4 + 2s^3 + 3s^2 + 2s + 1 \\
\underline{-\,2s^4 \qquad\quad + 2s^2} \\
2s^3 + s^2 + 2s + 1 \\
\underline{-\,2s^3 \qquad\quad + 2s} \\
s^2 + 1
\end{array}
$$

$$\frac{s^2+1}{0}$$

s^4 : 1   2   1

s^3 :

s^2 : $\frac{1\times 2 - 2\times 1}{1}$ $\frac{1\times 1}{1}$

s^1 : $\frac{+2\epsilon - 1}{\epsilon}$

s^0

On letting $\epsilon \to 0$ above negative

s^4 : 1   2   1

s^3 : 1   2

s^2 : 0   1

s^1 : $-\infty$

s^0 : 1

There are two sign changes. Hence two roots are lying on right half of s-plane and other two roots of quotient polynomial are lying on the left half of s-plane.

Result:-

→ The system is stable

→ Two roots are lying on imaginary axis, two roots are lying on right half of s-plane and two roots are lying on left half of s-plane

* Determine the range of K for stability of unity feed back system whose open loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Sol Closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+K}$$

The Characteristic Equation $s(s+1)(s+2)+K=0$

$$s(s^2+3s+2)+K=0 \Rightarrow s^3+3s^2+2s+K=0$$

Routh array is constructed as shown below

s^3 : 1 2

s^2 : 3 K

s^1 : $\frac{6-K}{3}$

s^0 : K

s^1 : $\frac{3\times 2-K\times 1}{3}$

s^0 : $\frac{\frac{6-K}{3}\times K}{\frac{6-K}{3}}$

s^0 : K

From s^0 row, for the system to be stable $K > 0$

From s^1 row, for the system to be stable $\frac{6-K}{3} > 0$

$$\frac{6-K}{3} > 0 \Rightarrow 6-K > 0 \Rightarrow K < 6$$

$\therefore$ Range of K for the system to be stable is

$$0 < K < 6$$

Ex:- 5.10 The open loop transfer function of a unity feedback control system is given by $G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}$. By applying routh criterion, discuss the stability of the closed

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{(s+2)(s+4)(s^2+6s+25)}}{1+\frac{K}{(s+2)(s+4)(s^2+6s+25)}}$$

$$\frac{C(s)}{R(s)} = \frac{K}{(s+2)(s+4)(s^2+6s+25)+K}$$

Characteristic Equation $(s+2)(s+4)(s^2+6s+25)+K=0$

$$(s^2+6s+8)(s^2+6s+25)+K = 0$$

$$s^4+12s^3+69s^2+198s+200+K=0$$

s^4	1	69	200+K
s^3	12	198	

s^4	1	69	200+K
s^3	1	16.5	
s^2	52.5	200+K	
s^1	$\frac{666.25-K}{52.5}$		

$s^2 : \frac{69-16.5}{1} \quad \frac{200+K}{1}$

$s^2 : 52.5 \quad 200+K$

$s^1 : \frac{52.5\times16-(200+K)}{52.5}$

$s^1 : \frac{666.25-K}{52.5}$

s^0 : $200+K$

From s^1 row, for the system to be stable

$$(666.25-K) > 0$$

$K < 666.25$

From s^0 row, for the system to be stable

$$(200+K) > 0$$

$K > -200$

But practical values of K starts from 0. Hence K should be greater than 0.

Range of K for the system to be stable

$$0 \leq K < 666.25$$

When $K = 666.25$ the s^1 row becomes zero, which indicates the possibility of roots on imaginary axis.

When $K = 666.25$, the coefficients of auxiliary equation are given by s^2 row

$$52.5s^2 + 200 + K = 0 \Rightarrow 52.5s^2 + 200 + 666.25 = 0$$

$$s^2 = \frac{-200 - 666.25}{52.5} = -16.5$$

$$s = \pm\sqrt{-16.5} = \pm j\sqrt{16.5} = \pm j\,4.06$$

When $K = 666.25$ the system has roots on imaginary axis and so it oscillates. The frequency of oscillation is given by the value of root on imaginary axis

$\therefore$ frequency of oscillation, $\omega = 4.06$ rad/sec

Result:

1. The range of k for stability $0 < k < 666.25$

2. The system oscillates when $k = 666.25$

3. frequency of oscillation $\omega = 4.06$ rad/sec.

5.11 The open loop transfer function of a unity feedback system is given by $G(s) = \dfrac{k(s+1)}{s^3+as^2+2s+1}$. Determine the value of k and a so that the system oscillates at a frequency of 2 rad/sec.

Sol closed loop transfer function $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)}$

$$\frac{C(s)}{R(s)} = \frac{\dfrac{k(s+1)}{s^3+as^2+2s+1}}{1+\dfrac{k(s+1)}{s^3+as^2+2s+1}} = \frac{k(s+1)}{s^3+as^2+2s+1+k(s+1)}$$

$$= \frac{k(s+1)}{s^3+as^2+s(k+2)+1+k}$$

The characteristic equation is $s^3+as^2+s(k+2)+1+k=0$

$$\begin{array}{l|ll}
s^3 & 1 & k+2 \\
s^2 & a & 1+k \\
s^1 & \dfrac{a(k+2)-(1+k)}{a} & \\
s^0 & 1+k &
\end{array}$$

s^1 $\dfrac{a(k+2)-(1+k)}{a}$

From s^2 row, the auxiliary polynomial is

$as^2 + (1+k) = 0$

$as^2 = -(1+k)$

$s = \pm j\sqrt{\frac{1+k}{a}}$

Given that $s = \pm j2$ $\quad \sqrt{\frac{1+k}{a}} = 2 \Rightarrow \frac{1+k}{a} = 4$

$k = 4a - 1$

from s^1 row $\quad \frac{a(2+k) - (1+k)}{a} = 0$

$a(2+k) = 1+k \Rightarrow 2a + ak = 1+k$

$2a = 1 + k(1-a) \Rightarrow k = \frac{2a-1}{1-a}$

$4a - 1 = \frac{2a-1}{1-a} \Rightarrow (4a-1)(1-a) = 2a-1$

$4a - 4a^2 - \cancel{1} + a = 2a - \cancel{1}$

$-4a^2 + 5a - 2a = 0 \Rightarrow a(-4a+3) = 0$

$a = 0$ or $a = 3/4$ since $a \neq 0$

when $a = 3/4$, $k = 4a - 1 \Rightarrow \cancel{4} \times \frac{3}{\cancel{4}} - 1 = 2$

when the system oscillates at a frequency of 2 rad/sec

$k = 2$ and $a = 3/4$

5.12 A feedback system has open loop transfer function of $G(s) = \frac{ke^{-s}}{s(s^2+5s+9)}$. Determine the maximum value of k for stability of closed loop system.

Sol Generally control systems have very low bandwidth which implies that it has very low frequency range of operation. Hence for low-frequency ranges the term e^{-s} can be replaced by $1-s$

$$\therefore G(s) = \frac{ke^{-s}}{s(s^2+5s+9)} = \frac{k(1-s)}{s(s^2+5s+9)}$$

Closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{k(1-s)}{s(s^2+5s+9)+k(1-s)}$

$$= \frac{k(1-s)}{s(s^2+5s+9)+k(1+s)}$$

The characteristic equation $s(s^2+5s+9)+k(1-s)=0$

$$s(s^2+5s+9)+k(1-s) = s^3+5s^2+9s+k-ks=0$$

$$s^3+5s^2+(9-k)s+k=0$$

s^3	1	$9-k$	
s^2	5	k	$\frac{5(9-k)-k}{5}$
s^1	$9-1.2k$		$\frac{45-6k}{5} = 9-1.2k$
s^0	k		

from s^1 row, for stability of the system $9-1.2k>0$

If $9-1.2k>0$ then $1.2k<9$ $\therefore k<\frac{9}{1.2} = 7.5$

from s^0 row, for stability of the system $k>0$

For stability of the system k should be in the range of $0<k<7.5$

Root locus:-

Root locus technique was introduced by W.R. Evans in 1948 for the analysis of control system. Consider the open loop transfer function of system $G(s) = \frac{k}{s(s+P_1)(s+P_2)}$

The closed loop transfer function of the system with unity feedback is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{k}{s(s+P_1)(s+P_2)}}{1+\frac{k}{s(s+P_1)(s+P_2)}} = \frac{k}{s(s+P_1)(s+P_2)+k}$$

The characteristic equation is given by

$$s(s+P_1)(s+P_2)+k=0$$

The roots of characteristic equation is a function of open loop gain k. when the gain k is varied from 0 to ∞, the roots of characteristic equation will take different values. when $k=0$, the roots are given by open loop poles. when $k \to \infty$, the roots will take the values of open loop zeros.

→ The path taken by the roots of characteristic equation when open loop gain k is varied from 0 to ∞ are called root loci.

* A unity feedback control system has an open loop transfer function $G(s) = \frac{k}{s(s^2+4s+13)}$. Sketch the root locus.

Step 1 :- To locate poles and zeros

The poles of open loop transfer function are the roots of the equation $s(s^2+4s+13)=0$

The roots of the quadratic are $s = \frac{-4 \pm \sqrt{4^2-4\times13}}{2}$

$= -2 \pm j3$

∴ The poles are lying at $s = 0, -2+j3$ and $-2-j3$

Let us denote the poles as P_1, P_2, P_3

$P_1 = 0 \quad P_2 = -2+j3 \quad P_3 = -2-j3$

Step 2 :- To find the root locus on real axis

There is only one pole on real axis at the origin.

Hence if we choose any test point on the negative real axis then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus.

Step 3 :- To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches are 3. There is no finite zero. Hence all the three root locus branches ends at zeros at ∞.

The number of asymptotes required are three.

Angles of asymptotes $= \frac{\pm 180^\circ (2q+1)}{n-m}$; $q = 0, 1, \ldots n-m$

Here $n = 3$ and $m = 0$ $\therefore q = 0, 1, 2, 3$

when $q = 0$, Angles $= \frac{\pm 180^\circ}{3} = \pm 60^\circ$

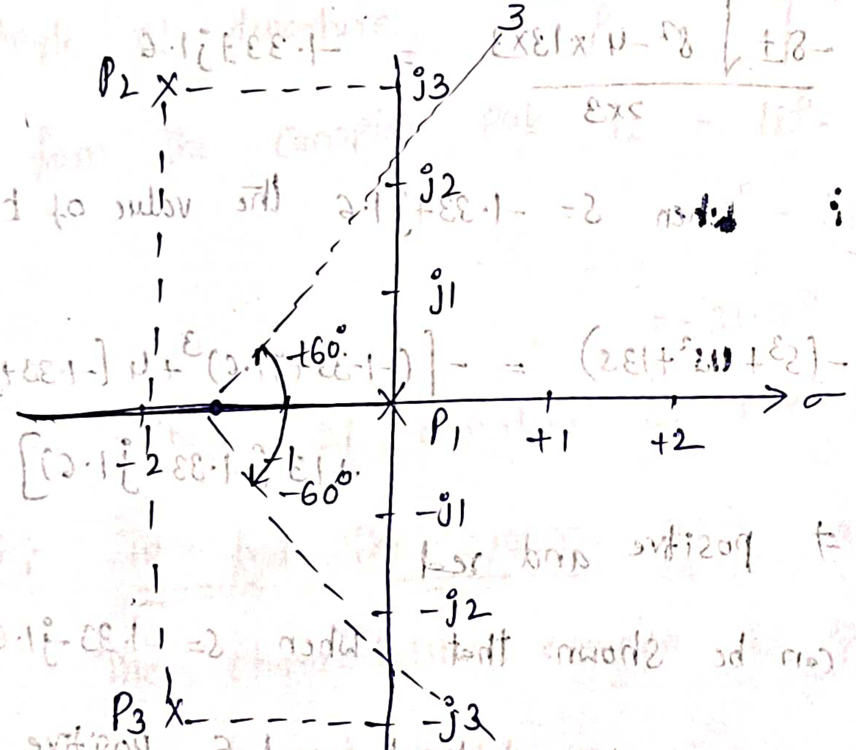
when $q = 1$, Angles $= \frac{\pm 180^\circ \times 3}{3} = \pm 180^\circ$

when $q = 2$, Angles $= \frac{\pm 180^\circ \times 5}{3} = \pm 60^\circ = \pm 300^\circ$

when $q = 3$, Angles $= \frac{\pm 180^\circ \times 7}{3} = \pm 420^\circ = \pm 60^\circ$

$$\text{centroid} = \frac{\text{Sum of poles} - \text{sum of zeros}}{n - m}$$

$$= \frac{0 - 2 + j3 - 2 - j3}{3} = \frac{-4}{3} = -1.33$$



The centroid is marked on real axis and from centroid the angles of asymptotes are marked using a protractor.

<u>Step 4: To find the break away and breakin points</u>

Closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\frac{K}{s(s^2+4s+13)}}{1 + \frac{K}{s(s^2+4s+13)}}$

$$= \frac{k}{s(s^2+4s+13)+k}$$

The characteristic Equation is $s(s^2+4s+13)+k=0$

$s^3+4s^2+13s+k=0 \Rightarrow k=-s^3-4s^2-13s$

Differentiating the equation of 'k' w.r.t 's' we get

$$\frac{dk}{ds} = -(3s^2+8s+13)$$

Put $\frac{dk}{ds}=0$

$-(3s^2+8s+13)=0 \Rightarrow 3s^2+8s+13=0$

$$s = \frac{-8 \pm \sqrt{8^2-4\times13\times3}}{2\times3} = -1.33 \pm j1.6$$

Check for k: when $s=-1.33+j1.6$ the value of k is

$$k = -(s^3+4s^2+13s) = -[(-1.33+j1.6)^3+4(-1.33+j1.6)^2+13(-1.33+j1.6)]$$

$k \neq$ positive and real

Also it can be shown that when $s=-1.33-j1.6$ the value of k is not equal to real & positive

Since the values of k for $s=-1.33 \pm j1.6$ are not real & positive, these points are not ~~equal to real~~ an actual breakaway or breakin points. The root locus has neither breakaway nor breakin point.

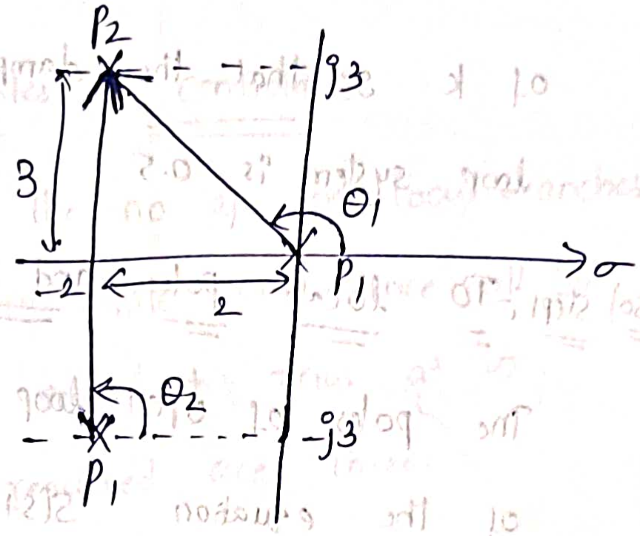
Step 5 :- To find the angle of departure

let us consider the complex pole P_2 as shown in fig.

Draw vectors from all other poles to the pole P_2

as shown in fig. let the angles of these vectors be θ_1

and θ_2



Here $\theta_1 = 180^\circ - \tan^{-1}\left(\frac{3}{2}\right)$

$= 123.7^\circ$

$\theta_2 = 90^\circ$

Angle of departure

from the complex pole $P_2 = 180^\circ - (\theta_1 + \theta_2)$

$= 180^\circ - (123.7 + 90^\circ)$

$= -33.7^\circ$

$\therefore$ Angle of departure at pole $P_3 = +33.7^\circ$

Step 6 :- To find the crossing point on imaginary axis

The characteristic equation is given by

$$s^3 + 4s^2 + 13s + k = 0$$

put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + k = 0$$

$$-j\omega^3 - 4\omega^2 + 13j\omega + k = 0$$

Equating Imaginary part $= 0$

$-\omega^3 + 13\omega = 0$

$\omega^2 = 13 \Rightarrow \omega = \pm\sqrt{13} = \pm 3.6$

Real part $= 0$

$-4\omega^2 + k = 0$

$k = 4\omega^2$

$= 4 \times 13 = 52$

The crossing point of root locus is $\pm j3.6$. The value of k at this crossing point is $k=52$.

* Sketch the root locus of the system whose open loop transfer function $G(s) = \frac{k_1}{s(s+2)(s+4)}$. Find the value of k so that the damping ratio of the closed loop system is 0.5

Sol) Step 1 :- To locate poles and zeros

The poles of open loop transfer function are the roots of the equation $s(s+2)(s+4)=0$

Poles are lying at $s=0, -2, -4$

Here $P_1=0$ $P_2=-2$ $P_3=-4$

Step 2 :- To find the root locus on real axis

There are three poles on real axis.

(i) Choose a test point on real axis between $S=0$ & $S=-2$.

To the right of this point the total no. of real poles & zeros is one, which is an odd no. Hence the real axis between $S=0$ & $S=-2$ will be a part of root locus.

(ii) Choose a test point on real axis between $S=-2$ & $S=-4$

To the right of this point the total no. of real poles & zeros is two which is an even no. Hence the real axis is not part of root locus

(iii) choose a test point on real axis to the left of $s=-4$. To the right of this point, the total no. of real poles & zeros is three, which is an odd no. Hence the entire negative real axis from $s=-4$ to $-\infty$ will be a part of root locus.

Step 3 :- To find asymptotes & centroid :-

Since there are 3 poles the no. of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at ∞. The no. of asymptotes required are three.

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m} \quad q = 0, 1, 2 \ldots n-m$$

Here $n=3$ & $m=0$ $\therefore q = 0, 1, 2, 3$

$$\text{when } q=0, \text{ Angles} = \pm \frac{180^\circ}{3} = \pm 60^\circ$$

$$\text{when } q=1, \text{ Angles} = \frac{\pm 180^\circ (\cancel{3})}{\cancel{3}} = \pm 180^\circ$$

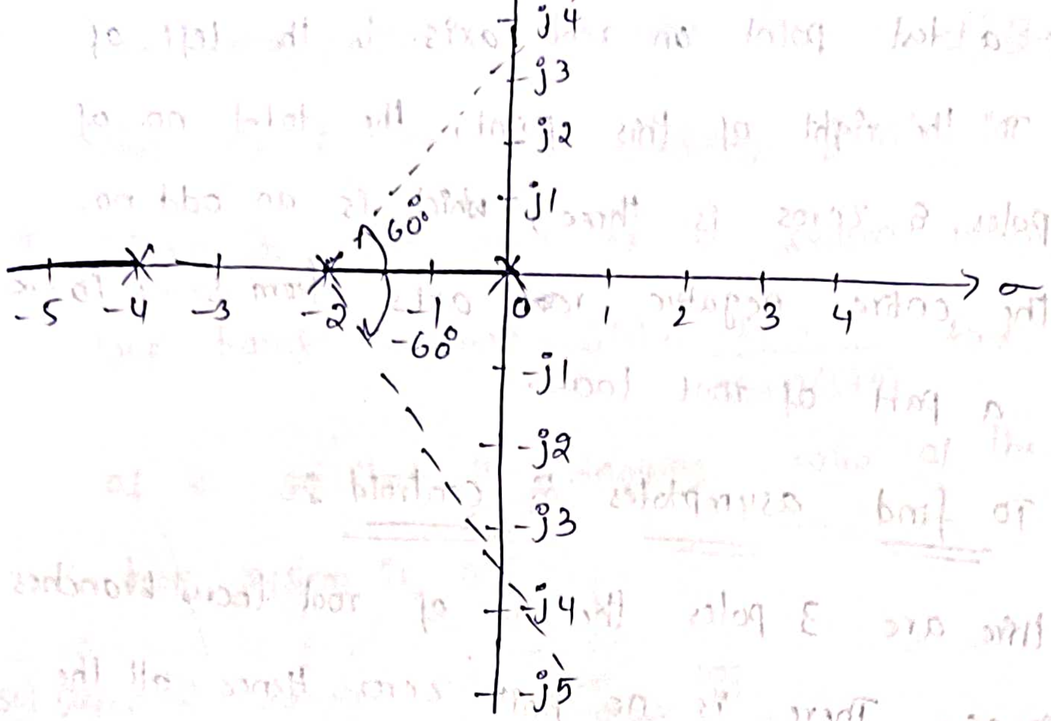
$$\text{Centroid} = \frac{\text{Sum of Poles} - \text{Sum of zeros}}{n-m}$$

$$= \frac{0-2-4-0}{3} = \frac{-6}{3} = -2$$

Step 4 :- To find the break away & breakin points

$$\text{Closed loop transfer function } \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+2)(s+4)}}{1+\frac{K}{s(s+2)(s+4)}} = \frac{K}{s(s+2)(s+4)+K}$$



characteristic equation is given by

$$s(s+2)(s+4) + k = 0$$

$$s(s^2+6s+8) + k = 0$$

$$s^3 + 6s^2 + 8s + k = 0$$

$$k = -s^3 - 6s^2 - 8s$$

$$\frac{dk}{ds} = -(3s^2 + 12s + 8)$$

$$-(3s^2+12s+8) = 0 \Rightarrow 3s^2+12s+8 = 0$$

$$s = \frac{-12 \pm \sqrt{12^2 - 4\times3\times8}}{2\times3} = -0.845 \text{ or } -3.154$$

<u>Check for k</u> :- when $s = -0.845$

$$k = -\left((-0.845)^3 - 6(-0.845)^2 - 8(-0.845)\right)$$

$$k = 3.08$$

Since k is +ve & real for, $s = -0.845$ this point is actual breakaway point.

When $S = -3.154$

$$K = -[(-3.154)^3 + 6(-3.154)^2 + 8(-3.154)]$$

$$K = -3.08$$

Since K is -ve for $S = -3.154$ this is not actual breakaway point.

Steps :- To find angle of departure

Since there are no complex pole or zero, we need not find angle of departure or arrival.

Step 6 :- To find the crossing point of imaginary axis

characteristic Equation is given by

$$S^3 + 6S^2 + 8S + K = 0$$

put $S = j\omega$

$$(j\omega)^3 + 6(j\omega)^2 + 8(j\omega) + K = 0$$

$$-j\omega^3 - 6\omega^2 + j8\omega + K = 0$$

Imaginary part to zero	real part to zero
$-j\omega^3 + j8\omega = 0$	$-6\omega^2 + K = 0$
$\cancel{+j}\omega^2 = \cancel{+j}8\cancel{\omega}$	$K = 6\omega^2$
$\omega^2 = 8$	$= 6 \times 8 = 48$
$\omega = \pm\sqrt{8} = \pm 2.8$	

The crossing point of root locus is $\pm j2.8$. The value of K corresponding to this point is $K = 48$.

The root locus has three branches. One branch starts at the pole $S=-4$ & travel through -ve real axis to meet the zero at ∞. The other two root locus branches starts at $S=0$ & $S=-2$ & travel through negative real axis, break away from real axis is at $S=-0.845$, then crosses imaginary axis at $S=\pm j2.8$ & travel parallel to asymptotes to meet the zeros at ∞.

To find the value of k corresponding to $\delta=0.5$:-

Given that $\delta=0.5$

Let $\alpha=\cos^{-1}\delta=\cos^{-1}(\frac{1}{2})=60^{\circ}$

Draw a line OP, such that the angle b/w line OP and negative real axis is 60°. The meeting point of the line OP and root locus gives the dominant pole Sd.

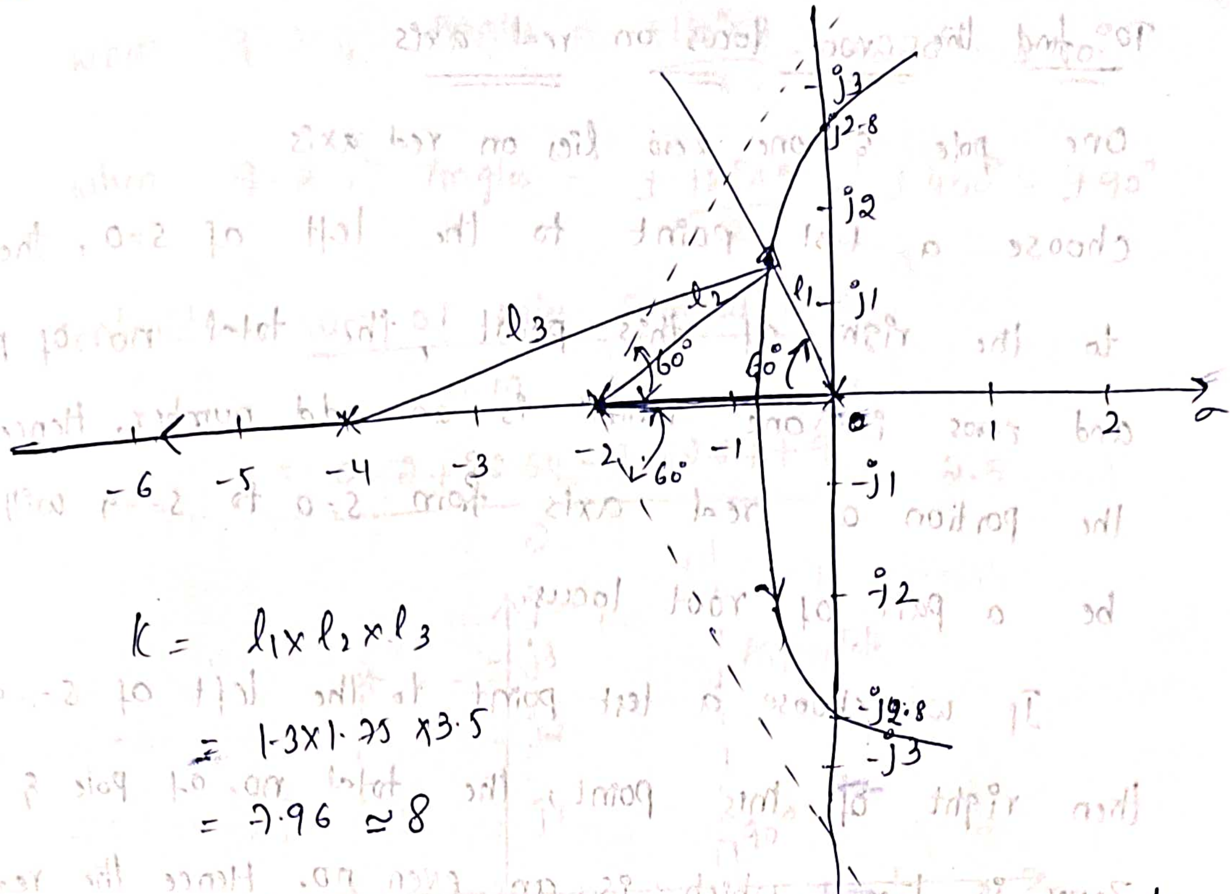
Let Ksd be value of k corresponding to the point S=Sd

$$K_{sd}=\frac{\text{product of length of vector from all poles to the point } S=S_d}{\text{Product of length of vector from all zeros to the point } S=S_d}$$

$l_1=2.6cm$ $=2.6\times\frac{1}{2}=1.3$ units

$l_2=3.5cm$ $=3.5\times\frac{1}{2}=1.75$ units

$l_3=7cm$ $=7\times\frac{1}{2}=3.5$ units



$K = l_1 \times l_2 \times l_3$

$= 1.3 \times 1.75 \times 3.5$

$= 7.96 \simeq 8$

5.24 The open loop transfer function of a unity feedback system is given by $G(s) = \dfrac{K(s+9)}{s(s^2+4s+11)}$

Sketch the root locus of the system.

Sol To locate poles & zeros

The poles of open loop transfer function are the roots of the equations $s^2+4s+11=0$

$$s = \frac{-4 \pm \sqrt{16-4(11)}}{2} = -2 \pm j2.64$$

The poles are lying at $s = 0, -2+j2.64, -2-j2.64$

The zeros are lying at $s=-9$ & ∞

Let us denote the poles as $P_1\ P_2\ P_3$, finite zero by Z_1

Here $P_1 = 0$ $P_2 = -2+j2.64$, $P_3 = -2-j2.64$ & $Z_1 = -9$

To find the root locus on real axis

One pole & one zero lies on real axis.

choose a test point to the left of $s=0$, then to the right of this point, the total no. of poles and zeros is one which is an odd number. Hence the portion of real axis from $s=0$ to $s=-9$ will be a part of root locus.

If we choose a test point to the left of $s=-9$ then right of this point, the total no. of pole & zeros is two, which is an even no. Hence the real axis from $s=-9$ to $-\infty$ will not be part of root locus.

Step 3:- To find angles of asymptotes & Centroid

Since there are 3 poles, the number of root locus branches are 3. one root locus branch start at pole at origin and travel along -ve real axis to meet the zero at $s=-9$. The other two root locus branches meet the zeros at ∞. The no. of asymptotes required are two.

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ(2q+1)}{n-m} \quad q=0,1,2 \ldots n-m$$

Here $n=3$ and $m=1$. $\therefore q = 0, 1, 2, 3$.

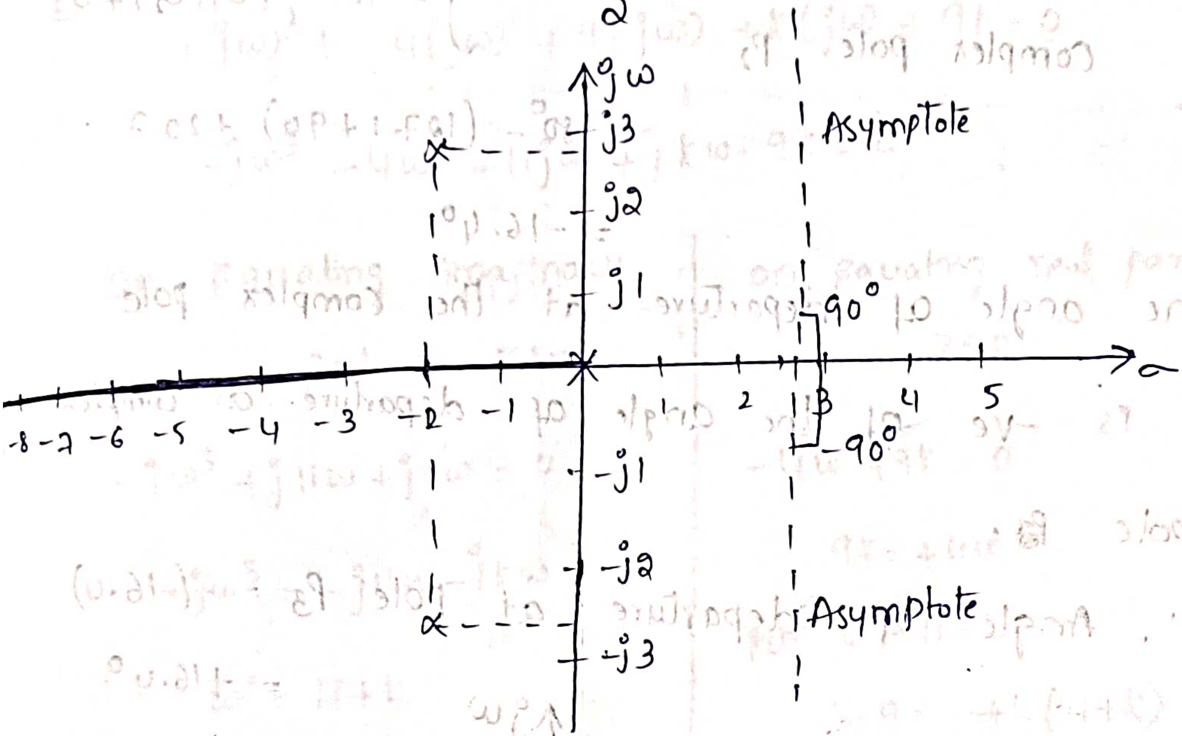
when $q=0$, $\text{Angles} = \frac{\pm 180^\circ}{2} = \pm 90^\circ$

when $q = 1$, Angles $= \frac{\pm 180^\circ \times 3}{2} = \pm 270^\circ = \mp 90^\circ$

when $q = 2$, Angles $= \frac{\pm 180^\circ \times 5}{2} = \pm 450^\circ = \pm 90^\circ$

$$\text{centroid} = \frac{\text{sum of poles} - \text{Sum of zeros}}{n-m}$$

$$= \frac{0 - 2 + j2.64 - 2 - j2.64 + 9}{2} = 2.5$$



step 4 :- To find the breakaway and break in points :

From the location of poles and zeros and from the knowledge of typical sketches of root locus, it can be concluded that there is no possibility of breakaway or breakin points.

step 5 :- To find the angle of departure :-

let us consider the complex pole P_2 as shown in fig.

Draw vectors from all other poles & zero to the Pole P_2 as shown in fig. Let the angles of these vectors be θ_1, θ_2 & θ_3.

Here $\theta_1 = 180^\circ - \tan^{-1}\frac{2.64}{2} = 127.1$

$\theta_2 = 90^\circ$

$\theta_3 = \tan^{-1}\frac{2.64}{7} = 20.7^\circ$

$$\left.\begin{array}{l}\text{Angle of departure from the}\\ \text{complex pole } P_2\end{array}\right\} = 180^\circ - (\theta_1+\theta_2)+\theta_3$$

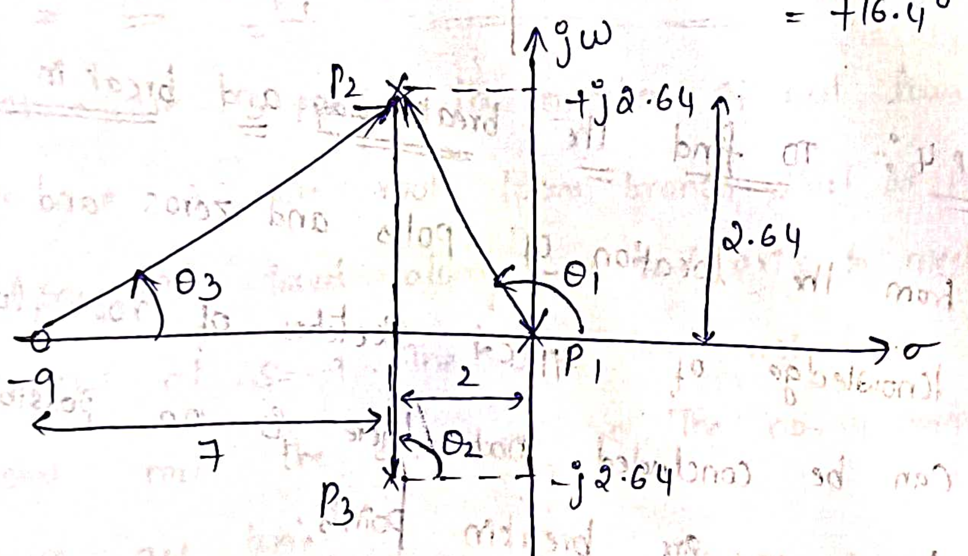
$= 180^\circ - (127.1+90) + 20.7$

$= -16.4^\circ$

The angle of departure at the complex pole P_3 is -ve of the angle of departure at complex pole P_2.

$\therefore$ Angle of departure at pole $P_3 = -(-16.4)$

$= +16.4^\circ$



Step 6 :- To find the crossing point of imaginary axis :-

Closed loop transfer function:
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K(s+9)}{s(s^2+4s+11)}}{1+\frac{K(s+9)}{s(s^2+4s+11)}}$$

$$= \frac{K(s+9)}{s(s^2+4s+11)+K(s+9)}$$

characteristic Equation is $s(s^2+4s+11)+K(s+9)=0$

$$s^3+4s^2+11s+Ks+9K=0 \Rightarrow (s^3+4s^2+11s)+Ks+9K=0$$

put $s=j\omega$,

$$(j\omega)^3+4(j\omega)^2+11(j\omega)+K(j\omega)+9K=0$$

$$-j\omega^3-4\omega^2+11j\omega+jK\omega+9K=0$$

On equating imaginary part to zero

$$-j\omega^3+j11\omega+jK\omega=0$$

$$-j\omega^3=-j11\omega-jK\omega$$

$$\omega^2=11+K$$

put $K=8.8$ $\therefore \omega^2=11+8.8$

$$\omega=\pm\sqrt{19.8}=\pm 4.4$$

On equating real part to zero

$$-4\omega^2+9K=0$$

$$9K=+4\omega^2$$

put $\omega^2=11+K$

$$\therefore 9K=+4(11+K)$$

$$9K-4K=44$$

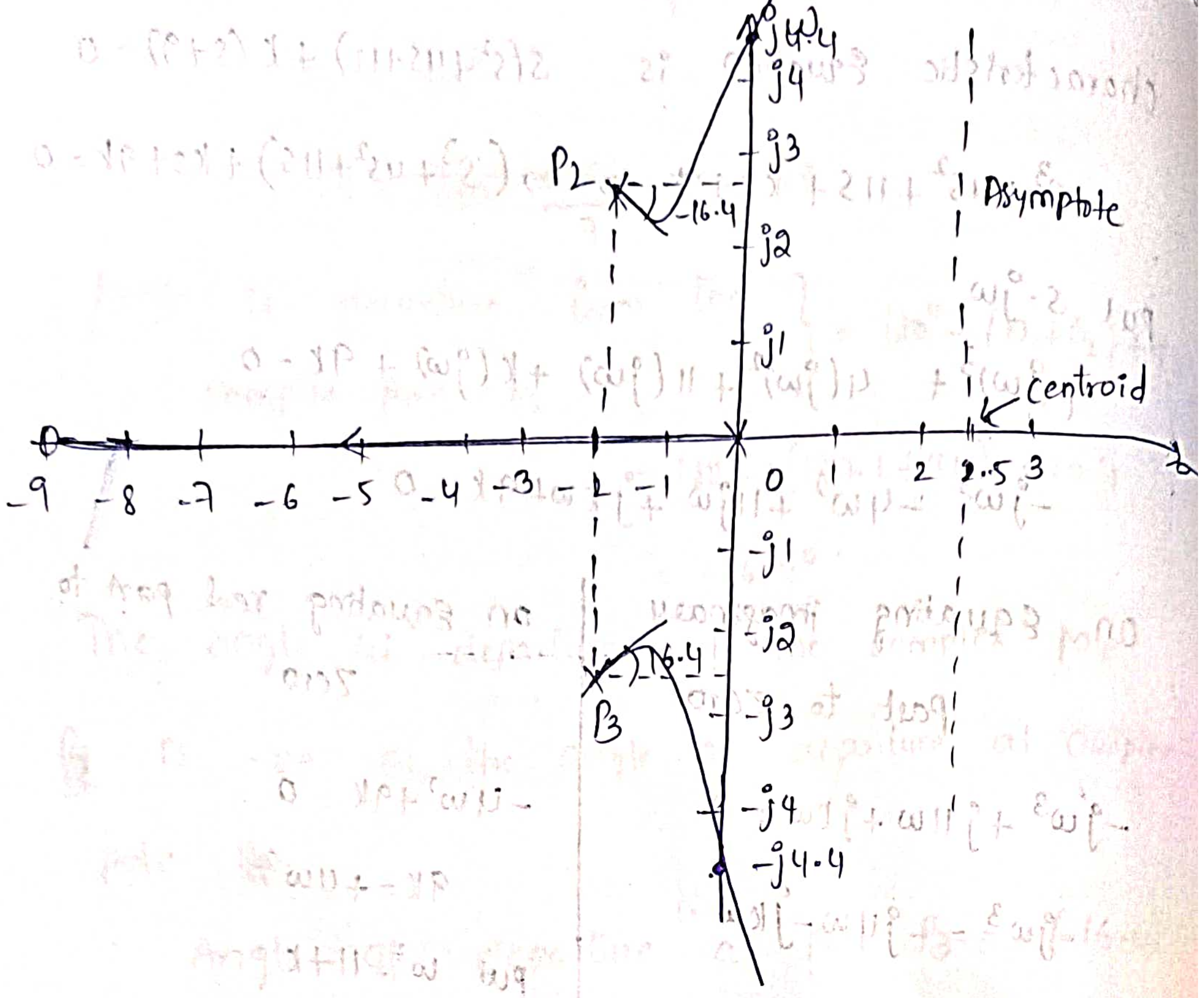
$$5K=44$$

$$K=\frac{44}{5}=8.8$$

The crossing point of root locus is $\pm j4.4$. The value of K at this crossing point is $K=8.8$

The root locus has 3 branches. One branch starts at pole at origin & travel through -ve real axis to meet the zero at $s=-9$.

The other two root locus branches starts at complex poles crosses the imaginary axis at $\pm j4.4$ and travel parallel to asymptotes to meet the zeros at



* Sketch the root locus for the unity feedback system whose open loop transfer function is,

$$G(s)\,H(s) = \frac{k}{s(s+4)\,(s^2+4s+20)}$$

Sol: Step 1:- To locate poles & zeros:-

The poles of open loop transfer function are the roots of the equation $s(s+4)\,(s^2+4s+20)=0$

Roots of the quadratic are $s = \frac{-4 \pm \sqrt{16-4(20)}}{2}$

$s = -2 \pm j4$

$\therefore$ The poles are lying at $s=0$, $s=-4$, $-2+j4$ & $-2-j4$

The zeros are lying at ∞.

Here $P_1 = 0$ $P_2 = -4$ $P_3 = -2+j4$ $P_4 = -2-j4$

Step 2 :- To find root locus on real axis

There are two poles on the real axis. choose a test point on real axis b/w $S=0$ and $S=-4$. To the right of this point, the total number of real poles is one which is an odd number. Hence the real axis b/w $S=0$ & $S=-4$, will be a part of root locus.

→ choose a test point to the left of $S=-4$, now to the right of this test point, the total number of poles & zeros is two which is an even number. Hence the real axis from $S=-4$ to $S=-\infty$ will not be part of root locus.

Step 3 :- To find angles of asymptotes and centroid

Since there are 4 poles. The number of root locus branches are four. There is no finite zero. Hence all the four root locus branches ends at zeros at ∞.

Hence the asymptotes required are 4.

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ(2q+1)}{n-m} \quad q = 0,1,2 \ldots n-m$$

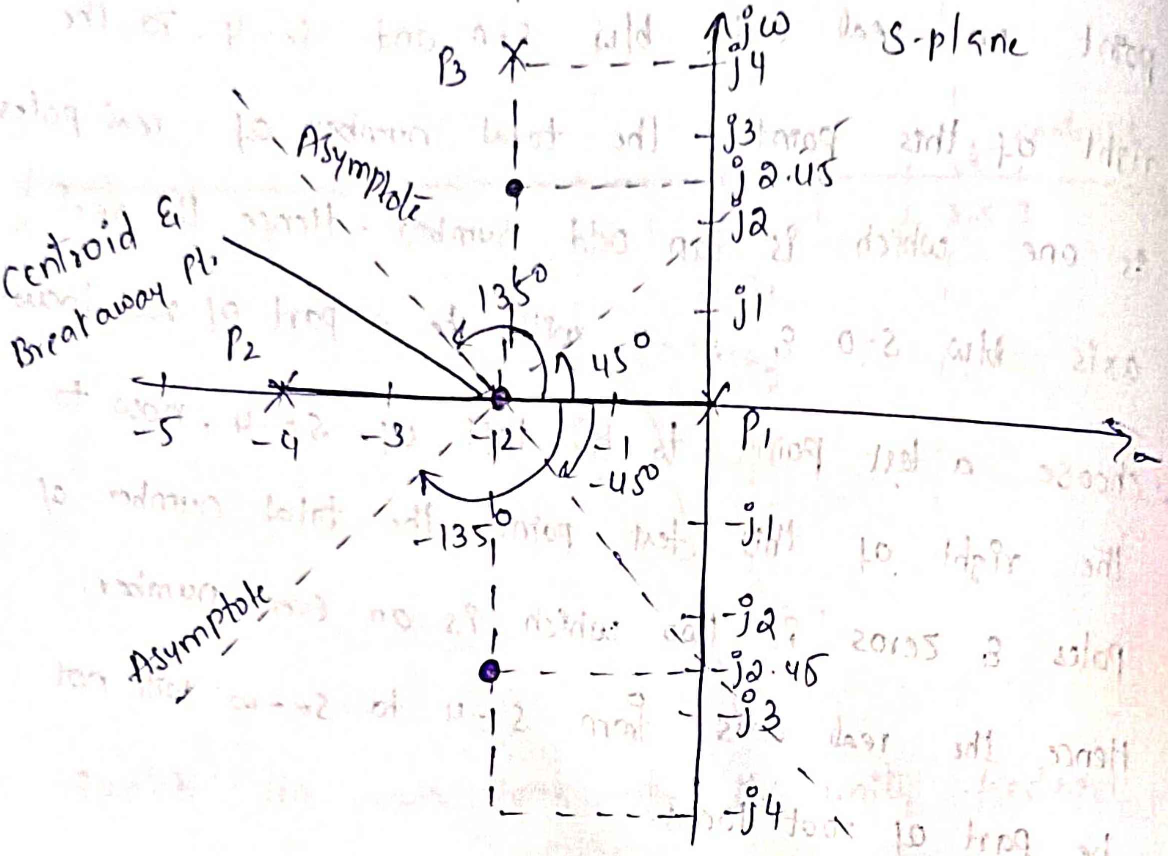
Here $n=3$ and $m=0$ $\therefore q = 0,1,2,3.$

When $q=0$, $\text{Angles} = \frac{\pm 180^\circ}{4} = \pm 45^\circ$

When $q = 1$, Angles $= \pm \frac{180^\circ \times 3}{4} = \pm 135^\circ$

$$\text{centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n - m}$$

$$= \frac{0 - 4 - 2 + j4 - 2 - j4}{4 - 0} = \frac{-8}{4} = -2$$



Step 4 :- To find the breakaway and breakin point :-

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+4)(s^2+4s+20)}}{1 + \frac{K}{s(s+4)(s^2+4s+20)}} = \frac{K}{s(s+4)(s^2+4s+20)+K}$$

Characteristic Equation $s(s+4)(s^2+4s+20)+K = 0$

$$K = -s(s+4)(s^2+4s+20)$$

$$= -(s^2+4s)(s^2+4s+20)$$

$$K = -(s^4 + 8s^3 + 36s^2 + 80s)$$

$$\frac{dK}{ds} = -(4s^3 + 24s^2 + 72s + 80)$$

put $\frac{dK}{ds} = 0$ $\therefore$ $-(4s^3 + 24s^2 + 72s + 80) = 0$

$$4s^3 + 24s^2 + 72s + 80 = 0$$

$$s^3 + 6s^2 + 18s + 20 = 0$$

The equation $s^3 + 6s^2 + 18s + 20 = 0$ will have atleast one real root. By trial & error, the real root is found to be $s = -2$.

$s^3 + 6s^2 + 18s + 20 = 0$ can be expressed as

$$(s+2)(s^2 + 4s + 10) = 0$$

The root of the quadratic $s^2 + 4s + 10 = 0$ are given by

$$s = \frac{-4 \pm \sqrt{16 - 4 \times 10}}{2} = -2 \pm j2.45$$

Check for K :- when $s = -2$

$$K = -(s^4 + 8s^3 + 36s^2 + 80s)$$

$$= -[(-2)^4 + 8(-2)^3 + 36(-2)^2 + 80(-2)]$$

$$\therefore K = 64$$

when $s = -2 \pm j2.45 = 3.16 \angle \pm 129^\circ$

$$K = -(s^4 + 8s^3 + 36s^2 + 80s)$$

$$= -((3.16 \angle \pm 129^\circ)^4 + 8(3.16 \angle \pm 129^\circ)^2 + 36(3.16 \angle \pm 129^\circ)^2$$

$$+ 80(3.16 \angle \pm 129^\circ))$$

$$= -\left[99.7\angle \pm 156^\circ + 252.4\angle \pm 27^\circ + 359.5\angle \pm 258^\circ + 252.8\angle \pm 129^\circ\right)$$

For +ve values of angles

$$K = -\left[91 + j40 + 225 + j115 - 75 - j351 - 159 + j196\right)$$

$$= -\left[-100\right) = 100$$

For -ve values of angles

$$K = -\left[-91 - j40 + 225 + j115 - 75 + j351 - 159 - j196\right]$$

$$= -\left[-100\right) = 100$$

Step 5 :- To find angle of departure

→ Let us consider the complex pole P_3 as shown in fig. Draw vectors from all other poles to the pole P_3 as shown in fig. Let angles of these vectors be θ_1, θ_2 & θ_3

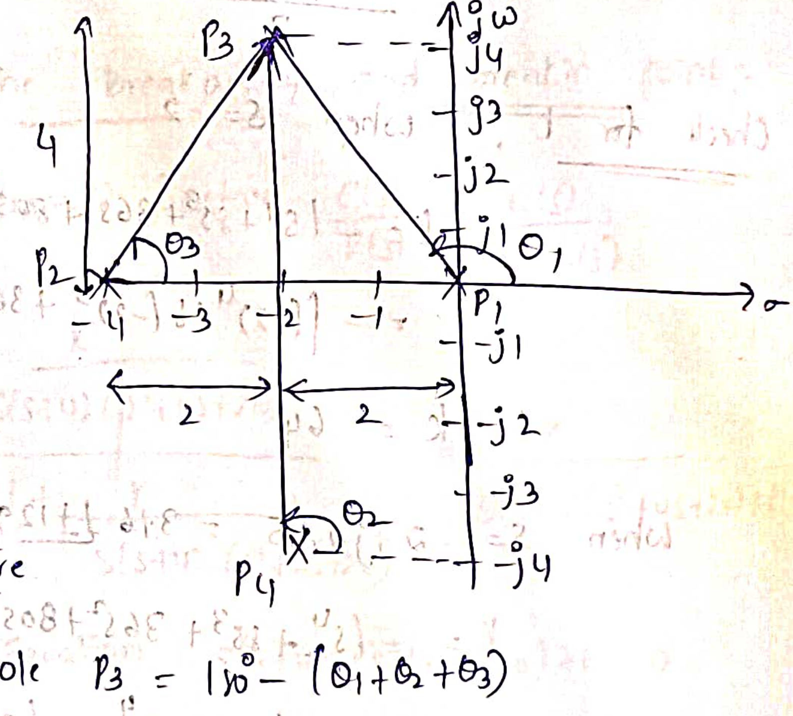
$\theta_1 = 180^\circ - \tan^{-1}\left(\frac{4}{2}\right)$

$= 117^\circ$

$\theta_2 = 90^\circ$

$\theta_3 = \tan^{-1}\left(\frac{4}{2}\right)$

$= 63^\circ$

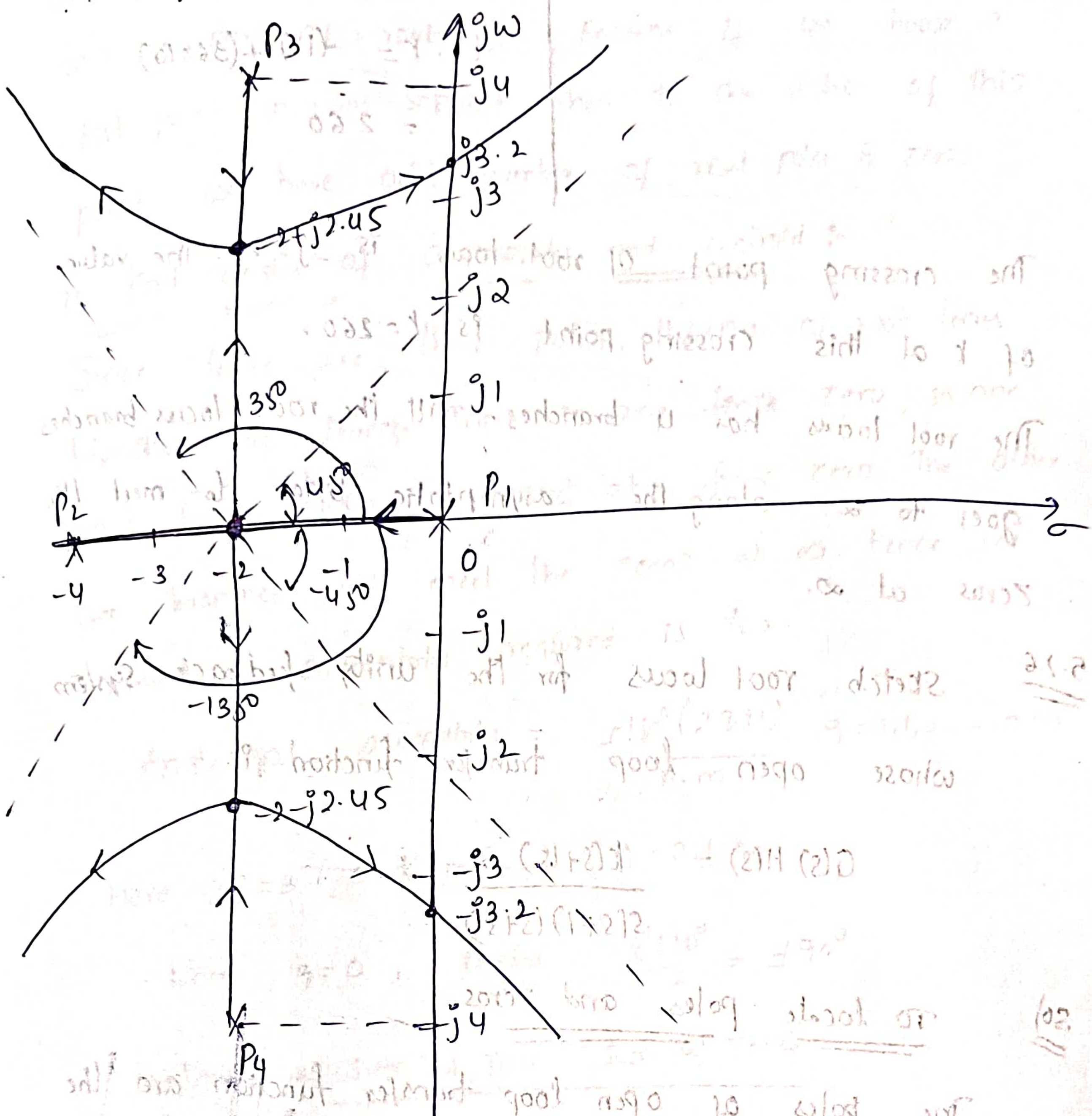


Angle of departure from complex pole $P_3 = 180^\circ - (\theta_1 + \theta_2 + \theta_3)$

$= 180^\circ - (117^\circ + 90^\circ + 63^\circ)$

$= -90^\circ$

∴ Angle of departure from complex pole $P_4 = +90^\circ$



step 6 :-

To find the crossing point on imaginary axis :-

The characteristic equation is given by

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

put $s = j\omega$

$$(j\omega)^4 + 8(j\omega)^3 + 36(j\omega)^2 + 80(j\omega) + K = 0$$

$$\omega^4 - j8\omega^3 - 36\omega^2 + j80\omega + K = 0$$

Imaginary part to zero

$$-j8\omega^3 + j80\omega = 0$$

$$\omega^2 = 10$$

Real part to zero

$$\omega^4 - 36\omega^2 + K = 0$$

$$K = -\omega^4 + 36\omega^2$$

$w = \pm\sqrt{10} = \pm 3.2$

put $w^2 = 10$

$\therefore K = -(10)^2 + (36 \times 10)$

$= 260$

The crossing point of root locus is $\pm j3.2$. The value of k at this crossing point is $k = 260$.

The root locus has 4 branches. All the root locus branches goes to ∞ along the asymptotic lines to meet the zeros at ∞.

5.26 Sketch root locus for the unity feed back system whose open loop transfer function is

$$G(s)\,H(s) = \frac{k(s+1.5)}{s(s+1)(s+5)}$$

Sol) TO locate poles and zeros.

The poles of open loop transfer function are the roots of the equation $s(s+1)(s+5) = 0$ and zeros are the roots of the equation $(s+1.5) = 0$

The poles are lying at $s = 0, -1, -5$

The zeros are lying at $s = -1.5$ and ∞

Here $P_1 = 0$, $P_2 = -1$, $P_2 = -5$ and $z_1 = -1.5$

Step 2 :- TO find root locus on real axis

The segment of real axis between $s=0$ and $s=-1$

and the segment of real axes between $s=-1.5$ and $s=-5$ will be part of root locus. Because if we choose a test point in this segment then to the right of this point we have odd number of real poles & zeros.

To find angles of asymptotes and centroid :-

Since there are three poles, the no. of root locus branches are three. There is one finite zero, so one root locus branch will end at finite zero. The other two branches will meet the zeros at ∞. Hence the number of asymptotes required is two.

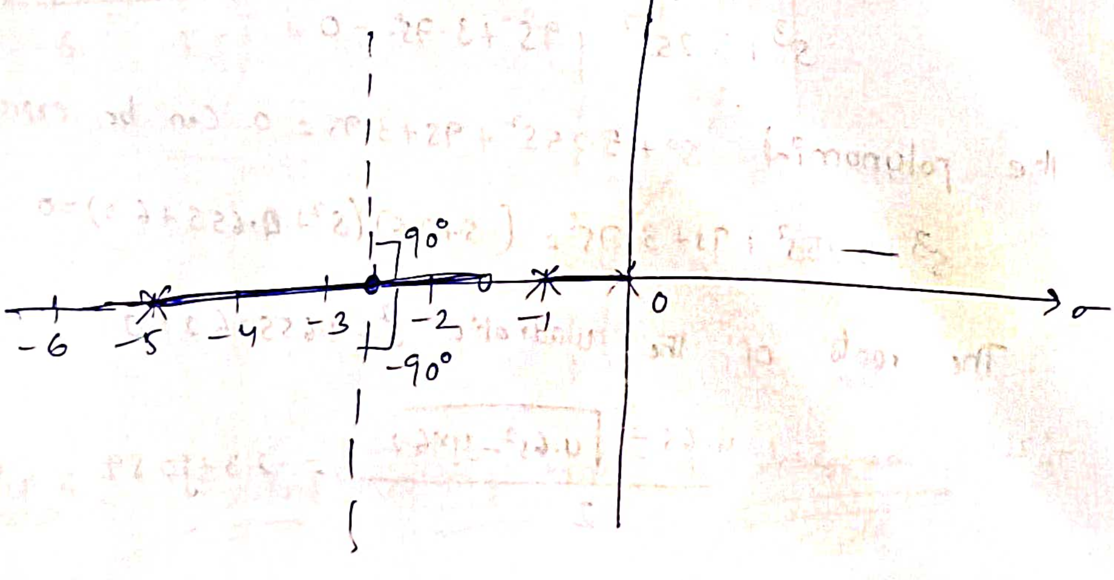
$$\text{Angle of asymptotes} = \frac{\pm 180^\circ(2q+1)}{n-m} \quad q=0,1,2 \ldots n-m$$

Here $n=3$ and $m=1$ $\therefore q=0,1,2$

when $q=0$, $\text{Angles} = \frac{\pm 180^\circ}{2} = \pm 90^\circ$

$$\text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{n-m}$$

$$= \frac{0-1-5-(-1.5)}{2} = -2.25$$



Step 4 :- To find the break away and breakin points

Closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\frac{C(s)}{R(s)} = \frac{\frac{K(s+1.5)}{s(s+1)(s+5)}}{1+\frac{K(s+1.5)}{s(s+1)(s+5)}} = \frac{K(s+1.5)}{s(s+1)(s+5)+K(s+1.5)}$$

Characteristic Equation $s(s+1)(s+5)+K(s+1.5)=0$

$$K = \frac{-s(s+1)(s+5)}{(s+1.5)} = \frac{-s(s^2+6s+5)}{s+1.5} = \frac{-(s^3+6s^2+5s)}{s+1.5}$$

$$\frac{dK}{ds} = \frac{-(s+1.5)(3s^2+12s+5)-[-(s^3+6s^2+5s)(1)]}{(s+1.5)^2}$$

$$= \frac{-3s^3-4.5s^2-12s^2-18s-5s-7.5+s^3+6s^2+5s}{(s+1.5)^2}$$

$$= \frac{-2s^3-10.5s^2-18s-7.5}{(s+1.5)^2}$$

$$= \frac{-2(s^3+5.25s^2+9s+3.75)}{(s+1.5)^2}$$

For $\frac{dK}{ds}=0$, the numerator should be zero

$$s^3+5.25s^2+9s+3.75=0$$

The polynomial $s^3+5.25s^2+9s+3.75=0$ can be expressed as

$$s^3+5.25s^2+9s+3.75 = (s+0.6)(s^2+4.65s+6.2)=0$$

The roots of the quadratic $s^2+4.65s+6.2=0$

$$s = \frac{-4.65 \pm \sqrt{4.65^2-4\times 6.2}}{2} = -2.3 \pm j0.89$$

check for k :- when $s = -0.6$

$$k = \frac{-(s^3+6s^2+5s)}{s+1.5} = \frac{-[(-0.6)^3+6(0.6)^2+5(-0.6)]}{-0.6+1.5}$$

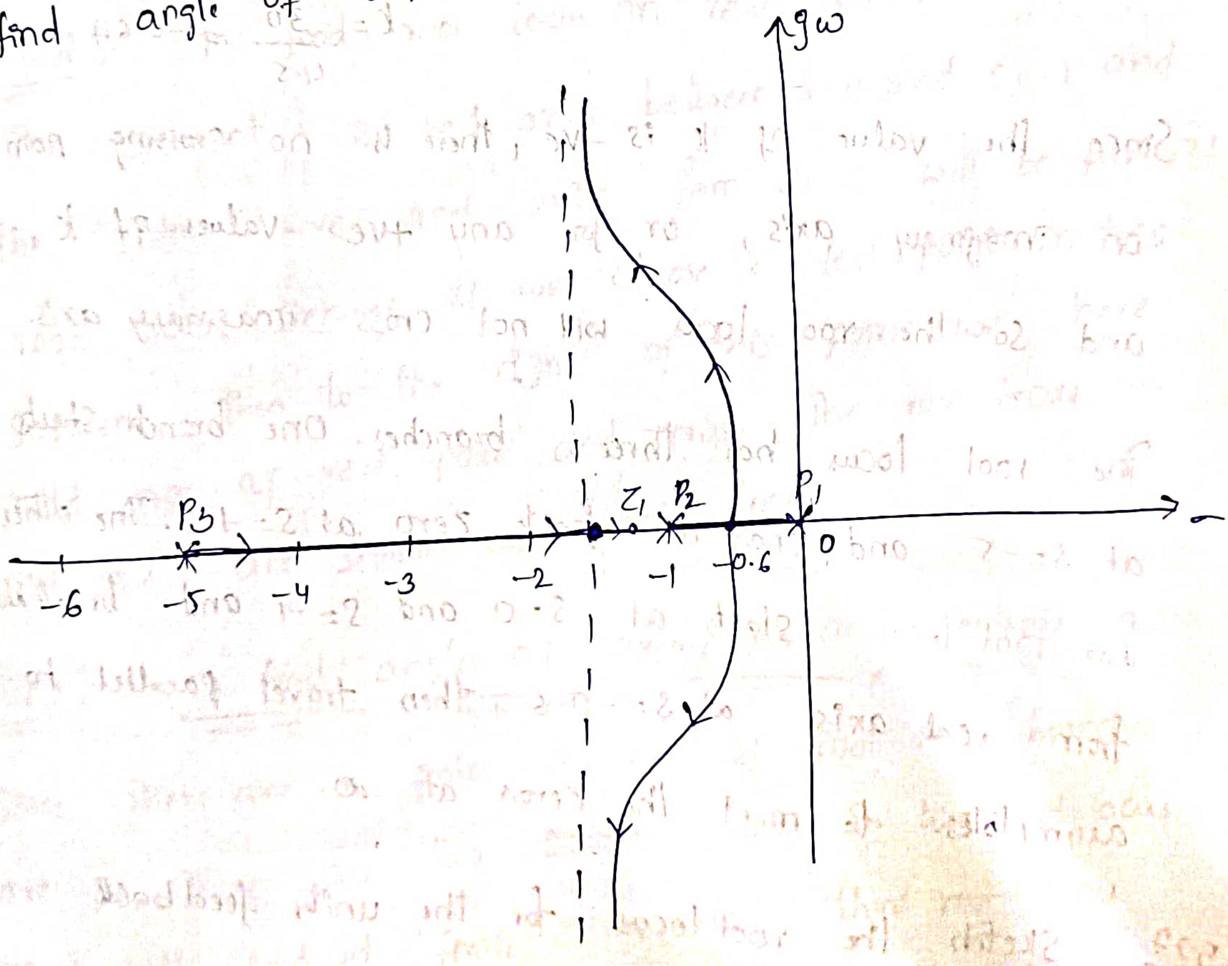
$= 1.17$

For $s = -0.6$, the value of k is +ve and real & so it is actual breakaway point.

It can be shown that for $s = -2.3 \pm j0.86$ the value of k is not +ve and real and so they cannot be breakaway points.

step 5 :- To find angle of departure :-

Since there are no complex pole or zero we need not find angle of departure or arrival.



step 6 :- To find crossing point at Imaginary axis :-

The characteristic Equation is

$$s(s+1)(s+5)+k(s+1.5)=0$$

$$s(s^2+6s+5)+ks+1.5k=0$$

$$s^3+6s^2+5s+ks+1.5k=0$$

put $s=j\omega$

$$(j\omega)^3+6(j\omega)^2+5(j\omega)+k(j\omega)+1.5k=0$$

$$-j\omega^3-6\omega^2+5j\omega+jk\omega+1.5k=0$$

Imaginary part to zero

$$j\omega^3+j5\omega+jk\omega=0$$

$$j\omega^3=-j5\omega-jk\omega$$

$$\omega^2=5+k$$

Real part to zero

$$-6\omega^2+1.5k=0$$

$$\omega^2=5+k$$

$$-6(5+k)+1.5k=0$$

$$-30-6k+1.5k=0$$

$$k=\frac{-30}{4.5}=-6.67$$

Since the value of k is $-$ve, there is no crossing point on imaginary axis, or for any +ve values of k, and so the root locus will not cross imaginary axis.

The root locus has three branches. One branch starts at $s=-5$ and ends at finite zero at $s=-1.5$. The other two root locus starts at $s=0$ and $s=-1$ and breakaway from real axis at $s=-0.6$, then travel parallel to asymptotes to meet the zeros at ∞.

5.27 Sketch the root locus for the unity feedback system whose open loop transfer function is

$$G(s) = \frac{k(s^2+6s+25)}{s(s+1)(s+2)}$$

Sol)

To locate poles and zeros

The poles of open loop transfer function are the roots of the equations $(s+1)(s+2)=0$ and the zeros are the roots of the equation $s^2+6s+25=0$

The roots of quadratic are $s = \frac{-6 \pm \sqrt{6^2-4\times 25}}{2}$

$s = -3 \pm j4$

The poles are lying at $s = 0, -1, -2$

The zeros are lying at $s = -3+j4$, $-3-j4$

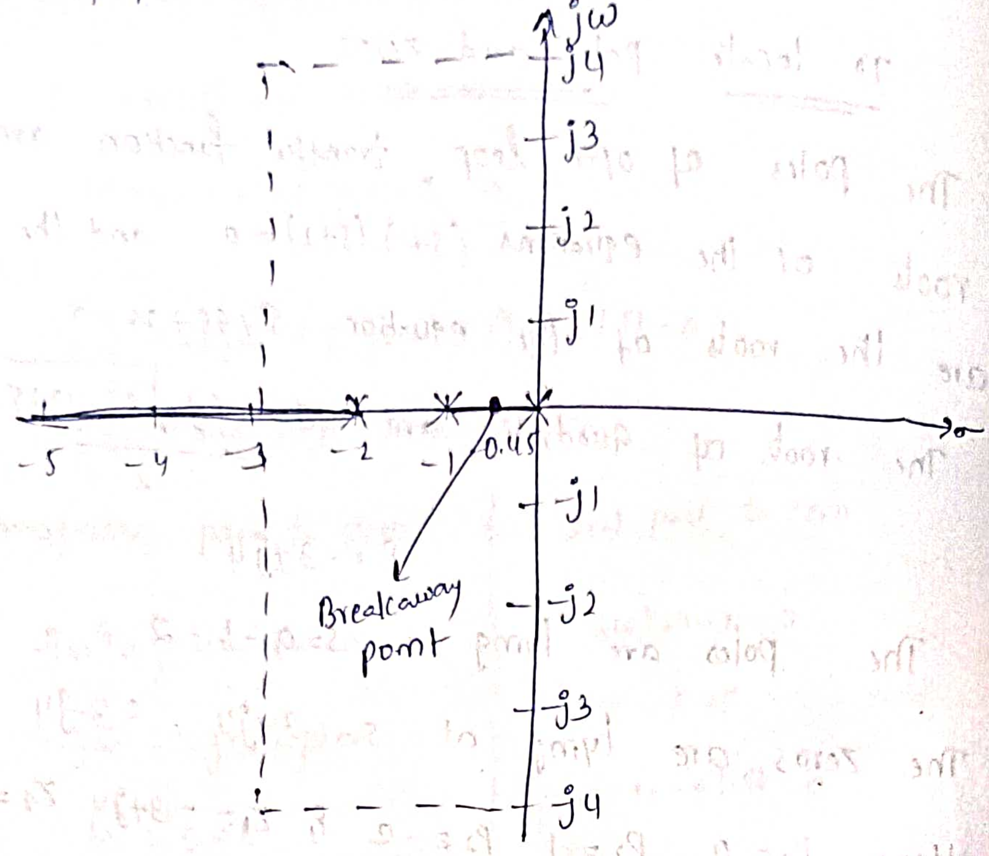
Here $P_1 = 0$ $P_2 = -1$ $P_3 = -2$ & $z_1 = -3+j4$ $z_2 = -3-j4$

step 2 :- To find root locus on real axis :-

The segment of real axis between $s=0$ and $s=-1$ and the entire -ve real axis from $s=-2$ will be part of root locus. Because if we choose a test point in this segment then to the right of this point we have odd no. of real poles and zeros. The root locus on real axis are shown as a bold line

Step 3 :- To find angles of asymptotes and centroid :-

Since there are 3 poles the no. of root locus branches are 3. There are two finite zeros. So two root locus branch will end at finite zeros. The third root locus will meet the zero at ∞ by travelling through -ve real axis. Here the no. of asymptote is one & angle of asymptote is $\pm 180°$



step 4: To find the break away and breakin points

closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\frac{C(s)}{R(s)} = \frac{\frac{k(s^2+6s+25)}{s(s+1)(s+2)}}{1+\frac{k(s^2+6s+25)}{s(s+1)(s+2)}} = \frac{k(s^2+6s+25)}{s(s+1)(s+2)+k(s^2+6s+25)}$$

Characteristic Equation $s(s+1)(s+2)+k(s^2+6s+25)=0$

$$k = \frac{-s(s+1)(s+2)}{s^2+6s+25} = \frac{-s(s^2+3s+2)}{s^2+6s+25}$$

$$= \frac{-s^3-3s^2-2s}{s^2+6s+25}$$

$$\frac{dk}{ds} = \frac{s^2+6s+25(-3s^2-6s-2)-(-s^3-3s^2-2s)(2s+6)}{(s^2+6s+25)^2}$$

$$= \frac{-3s^4 - 18s^3 - 75s^2 - 6s^3 - 36s^2 - 150s - 2s^2 - 12s - 50 + 2s^4 + 6s^3 + 6s^2 + 18s^2 + 4s^2 + 12s}{(s^2+6s+25)^2}$$

$$= \frac{-(s^4 + 12s^3 + 91s^2 + 150s + 50)}{(s^2+6s+25)^2}$$

For $\frac{dk}{ds} = 0$ the numerator should be zero

$$\therefore s^4 + 12s^3 + 91s^2 + 150s + 50 = 0$$

$$s^4 + 12s^3 + 91s^2 + 150s + 50 \approx (s^2 + 2s + 0.7)(s^2 + 10s + 70.3)$$

The roots of the quadratic $s^2 + 2s + 0.7 = 0$ are

$$s = \frac{-2 \pm \sqrt{2^2 - 4 \times 0.7}}{2} = -0.45, -1.55$$

The roots of the quadratic $s^2 + 10s + 70.3 = 0$ are

$$s = \frac{-10 \pm \sqrt{10^2 - 4 \times 70.3}}{2} = -5 \pm j6.73$$

Here $s = -1.55$ is not a point on root locus, hence it cannot be a break away point.

Check the other three values for actual breakaway point

When $s = -0.45$; $K = \frac{-s^3 - 3s^2 - 2s}{s^2 + 6s + 25}$

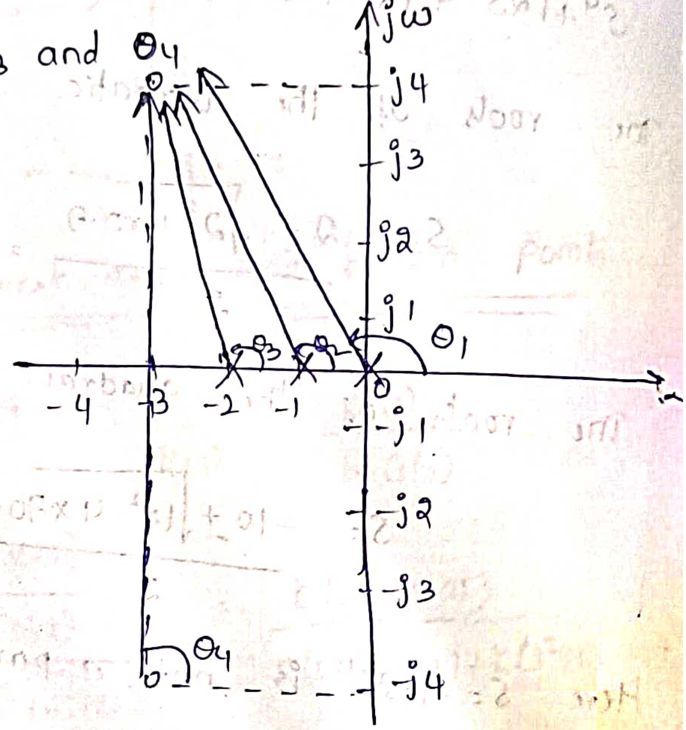
$$= \frac{-(-0.45)^3 - 3(-0.45)^2 - 2(-0.45)}{(-0.45)^2 + 6(-0.45) + 25}$$

$$= 0.017$$

For $s = -0.45$, the value of k is positive and real and so it is actual breakaway point. It can be shown that for $s = -5 \pm j6.73$ the value of k is not +ve and real and so they cannot be break away points.

step 5 :- To find angle of arrival

Let us consider the complex zero z_1 as shown in fig. Draw vectors from all other poles and zero to the zero z_1 as shown in fig. Let the angles of these vectors be θ_1, θ_2, θ_3 and θ_4



$\theta_1 = 180^\circ - \tan^{-1}\left(\frac{4}{3}\right)$

$= 126.9^\circ$

$\theta_2 = 180^\circ - \tan^{-1}\left(\frac{4}{2}\right)$

$= 116.6^\circ$

$\theta_3 = 180^\circ - \tan^{-1}\left(\frac{4}{1}\right)$

$= 104^\circ$

$\theta_4 = 90^\circ$

Angle of arrival at complex zero z_1 $\Big\}$ $180^\circ - \theta_4 + (\theta_1 + \theta_2 + \theta_3)$

$= 180^\circ - 90^\circ + 126.9^\circ + 116.6^\circ + 104^\circ$

$= 437.5^\circ \Rightarrow 77.5^\circ$

Angle of arrival at complex zero z_2 $= -77.5^\circ$

step 6 :- To find the crossing point on imaginary axis

The characteristic Equation is

$$s(s+1)(s+2) + k(s^2+6s+25) = 0$$

$$s(s^2+3s+2) + ks^2+6sk+25k = 0$$

$$s^3+3s^2+2s+ks^2+6sk+25k = 0$$

$$s^3+(3+k)s^2+(2+6k)s+25k = 0$$

put $s = j\omega$

$$(j\omega)^3 + (3+k)(j\omega)^2 + (2+6k)(j\omega) + 25k = 0$$

$$-j\omega^3 - (3+k)\omega^2 + j(2+6k)\omega + 25k = 0$$

on equating imaginary part to zero

$$-j\omega^3 + j(2+6k)\omega = 0$$

$$-j\omega^3 = -j(2+6k)\omega$$

$$\omega^2 = (2+6k)$$

Real part to zero

$$-(3+k)\omega^2 + 25k = 0.$$

put $\omega^2 = 2+6k$

$$-(3+k)(2+6k)+25k = 0$$

$$-6k^2+5k-6 = 0$$

$$k = \frac{-5 \pm \sqrt{5^2-4(-6)(-6)}}{2\times -6}$$

$$= 0.4 \pm j0.9$$

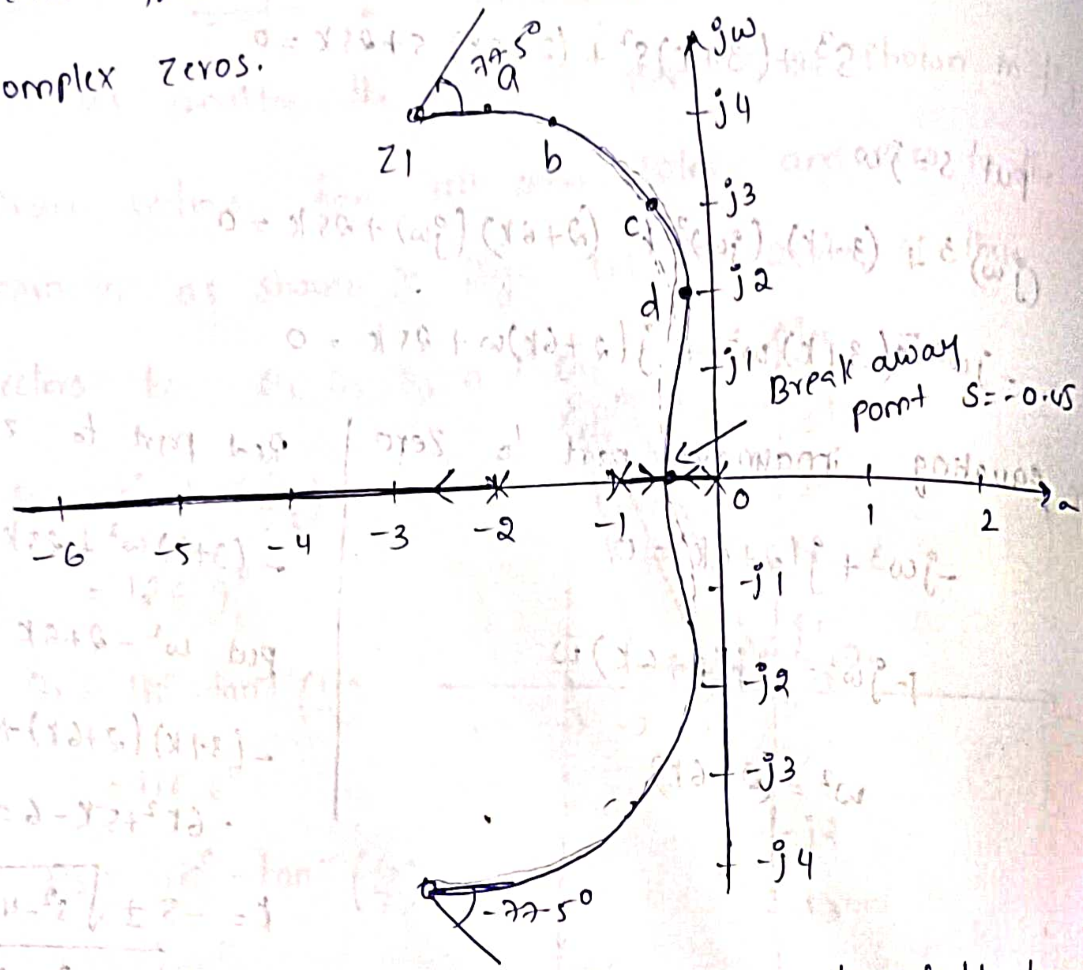
Since the value of k is not real and positive,

there is no crossing point on imaginary axis.

Step 7 :- To find points on root locus

choose test points a, b, c, d on the s-plane and adjust the test points to satisfy angle criterion. on the upper half of s-plane the root locus is sketched through the test points a, b, c and d.

The root locus has 3 branches. one branch starts at $s = -2$ and goes to ∞ along -ve real axis. The other two root locus branches starts at $s = 0$ and $s = -1$. & breaks from real axis at $s = -0.45$ then meets the complex zeros.



5.28 Sketch the root locus for the unity feedback system whose open loop transfer function is

$$G(s) = \frac{k}{s(s^2+6s+10)}$$

Step 1:- To locate poles and zeros

The poles of open loop transfer function are the roots of the equation $s(s^2+6s+10) = 0$

The roots of the quadratic are

$$S = \frac{-6 \pm \sqrt{6^2 - 4 \times 10}}{2} = -3 \pm j1$$

The poles are lying at $S = 0, -3+j1$ and $-3-j1$

Here $P_1 = 0$ $P_2 = -3+j1$ & $P_3 = -3-j1$

Step 2 :- To find the root locus on real axis :-

There is only one pole on real axis at the origin. Hence if we choose any test point on the -ve real axis then to the right of that point the total no. of real poles and zeros is one, which is an odd no. Hence the -ve real axis will be part of root locus.

Step 3 :- To find angles of asymptotes and centroid

Since there are 3 poles, the no. of root locus branches are 3. There is no finite zero. Hence all the three root locus branches ends at zero at ∞. The no. of asymptotes required are 3.

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m}$$

Here $n = 3$ and $m = 0$ $\therefore q = 0, 1, 2, 3$

When $q = 0$, Angles $= \frac{\pm 180^\circ}{3} = \pm 60^\circ$

When $q = 1$, Angles $= \frac{\pm 180^\circ \times 3}{3} = \pm 180^\circ$

$$\text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{n-m}$$

$$= \frac{-3+j1 - 3-j1}{3} = -2$$

Step 4: To find the break away and break in points

Closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$= \frac{\frac{K}{s(s^2+6s+10)}}{1+\frac{K}{s(s^2+6s+10)}} = \frac{K}{s(s^2+6s+10)+K}$$

Characteristic equation is $s(s^2+6s+10)+K=0$

$$\therefore K = -s(s^2+6s+10) = -s^3-6s^2-10s$$

$$\frac{dK}{ds} = -3s^2-12s-10$$

Put $\frac{dK}{ds} = 0$

$$-3s^2-12s-10=0 \Rightarrow 3s^2+12s+10=0$$

$$s = \frac{-12 \pm \sqrt{12^2-4\times3\times10}}{2\times3} = -1.18 \text{ or } -2.82$$

Check for K : when $s=-1.18$, $K = -s^3-6s^2-10s$

$$= -(-1.18)^3 - 6(-1.18)^2 - 10(-1.18)$$

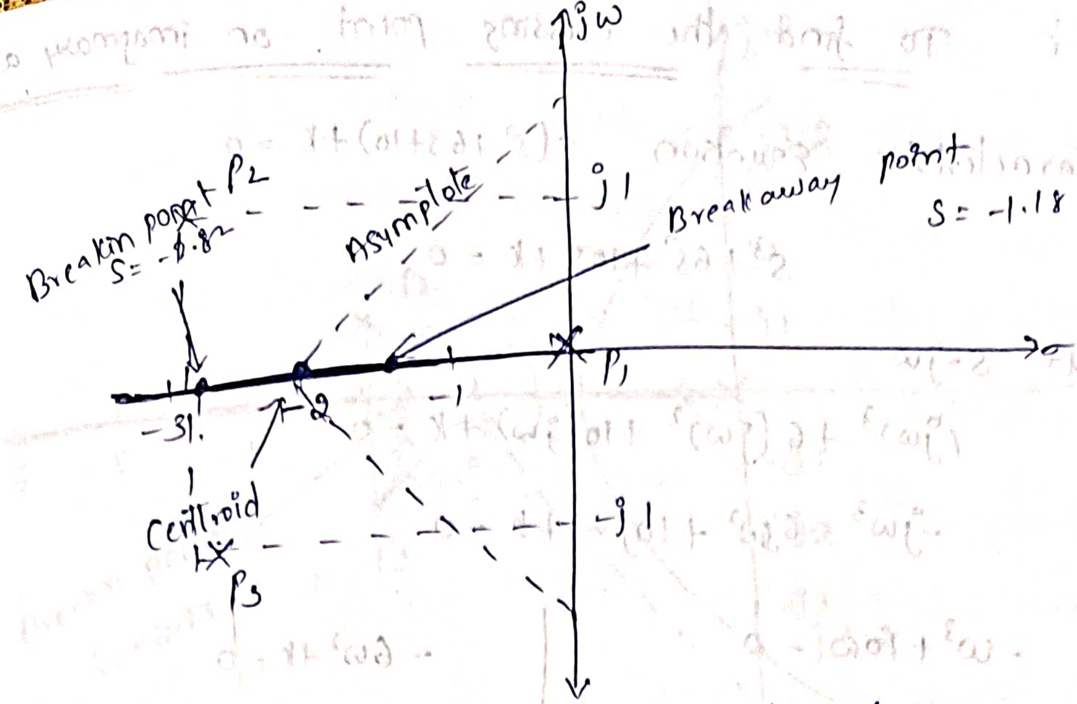
$$= +5.09$$

When $s=-2.82$, $K = -s^3-6s^2-10s$

$$= -(-2.82)^3 - 6(-2.82)^2 - 10(-2.82)$$

$$= 2.91$$

Since the values of K for $s=-1.18$ and -2.82 are positive & real, both the points are actual break away or breakin points. It can be proved that $s=-2.82$ is a break in point and $s=-1.18$ is a break away point.



Step 5 :- To find angle of departure :-

Consider the complex pole P_2 shown in fig. Draw vectors from all other poles to the pole P_2 as shown in fig. Let the angle of these vectors be θ_1 and θ_2.

Here $\theta_1 = 180° - \tan^{-1}(1/3)$

$= 161.6°$

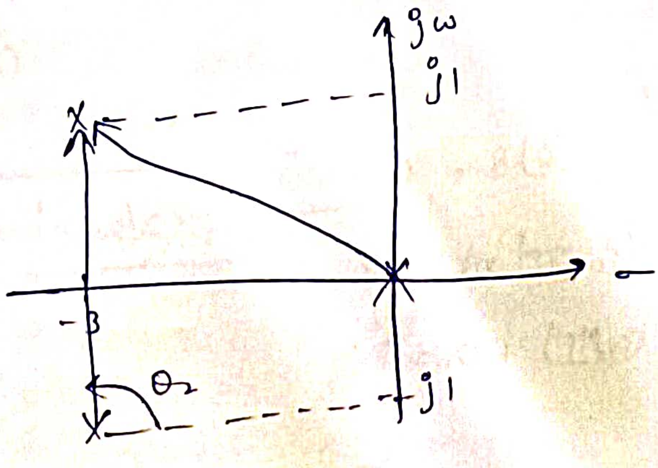
$\theta_2 = 90°$

Angle of departure from complex pole P_2 $\Big\} = 180 - (\theta_1 + \theta_2)$

$= 180° - (161.6 + 90°)$

$= -71.6 = -72°$

$\therefore$ Angle of departure at pole $P_3 = 72°$



Step 6 :- To find the crossing point on imaginary axis

Characteristic equation $s(s^2+6s+10)+K=0$

$$s^3+6s^2+10s+K=0$$

put $s=j\omega$

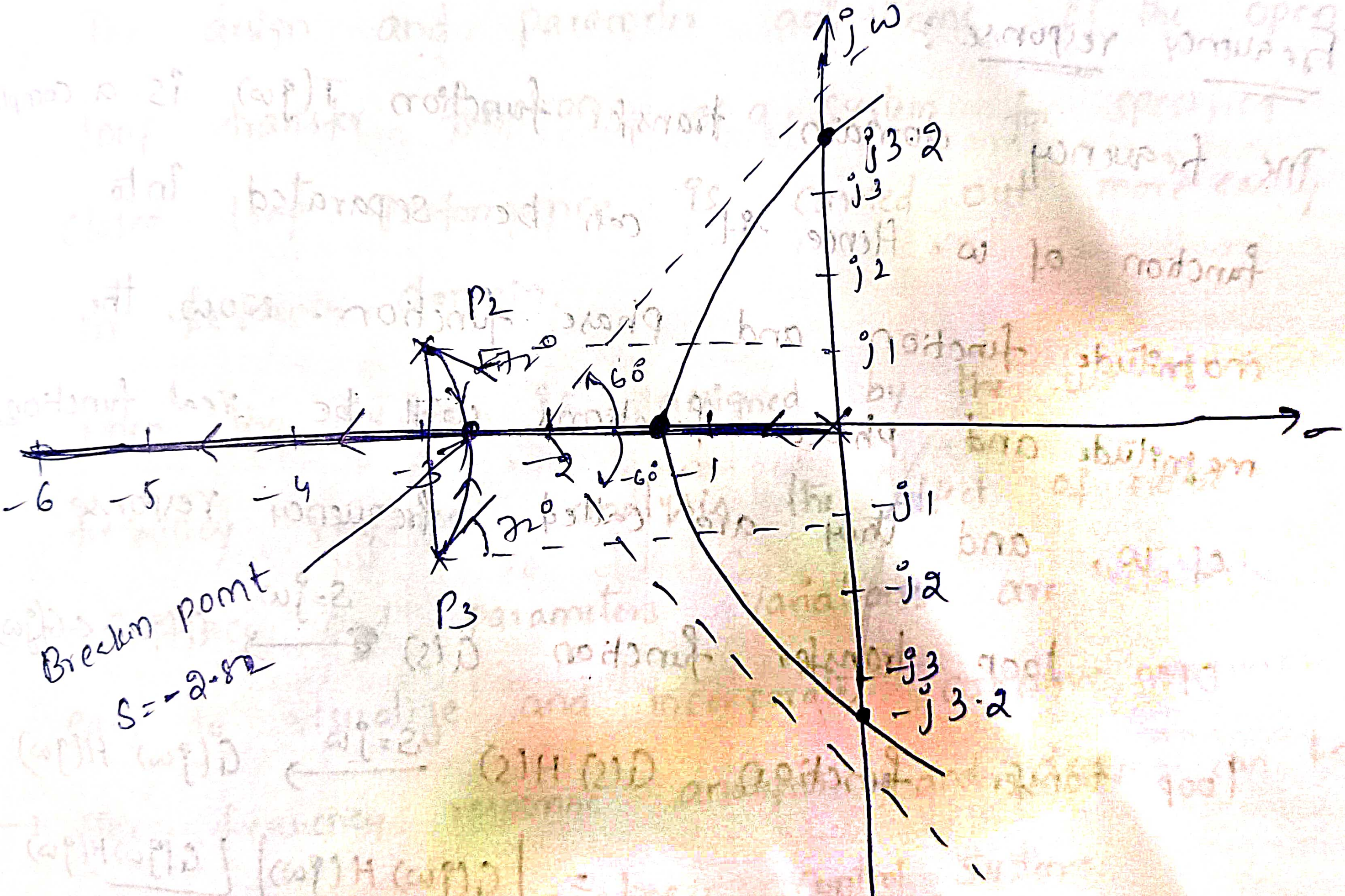
$$(j\omega)^3+6(j\omega)^2+10(j\omega)+K=0$$

$$-j\omega^3-6\omega^2+10j\omega+K=0$$

$-\omega^3+10\omega=0$	$-6\omega^2+K=0$
$\omega^3=10\omega$	$K=6\omega^2$
$\omega^2=10$	$K=6\times10=60$
$\omega=\pm\sqrt{10}=\pm3.2$	

The root locus crosses imaginary axis at $\pm j3.2$ and the gain K corresponding to this point is 60.

The root locus has 3 branches. One branch starts at $s=0$ and goes to ∞ along -ve real axis. The other two locus branches start at $s=-3\pm j1$ and enter the real axis at $s=-2.82$ and then breakaway from real axis at $s=-1.18$. Finally they travel parallel to asymptotes to meet the zeros at ∞.


jω
j3.2
j3
j2
j1
P2
120°
60°
-60°
-120°
-1
-2
-3
-4
-5
-6
σ
-j1
-j2
-j3
-j3.2
P3
Breakin point
S = -2.82