unit - 5

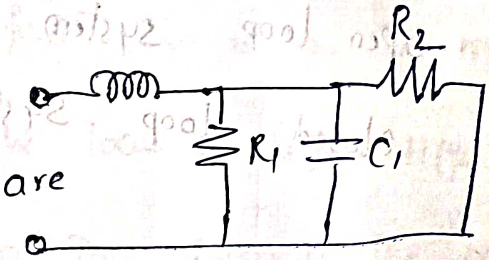
Introduction :-

The future behaviour of the system is based on present input and past history of the system. Past history of the system can be described by state variables.

No. of State Variables :-

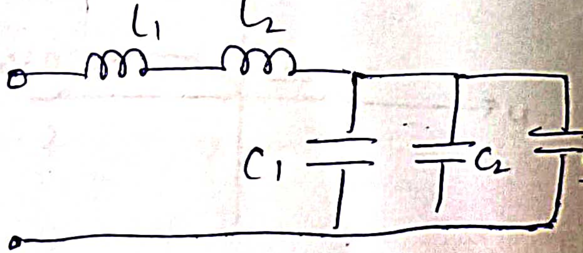
1) If any electrical n/w is given, the no. of state variables is equal to the sum of the inductors and capacitors.

2) If there are 3 storage elements then there are 3 states. If same kind of elements are connected in series or parallel then it should be treated as single component.



Circuit - I

For example, from the above circuit, there are 5 storage elements are present but there are only two states exists.



Circuit - II

3) If differential equation is given, the no. of state variables is equal to the order of differential equation.

Limitations of Transfer function analysis :-

1) Transfer function analysis is more suitable for single input and single output systems, whereas state analysis is used for multi-input and multi output systems.

2) Transfer function (T/F) analysis cannot give any idea about Controllability and Observability.

Initial conditions :- T/F analysis is valid for only LTI systems whereas State space Analysis is valid for dynamic systems i.e systems may be linear or non-linear, Time Variant or Time invariant.

Standard form of state Model :-

$$\dot{X}_{n\times1} = A_{n\times n} X_{n\times1} + B_{n\times m} U_{m\times1} \longrightarrow (1)$$

$$Y_{p\times1} = C_{p\times n} X_{n\times1} + D_{p\times m} U_{m\times1} \longrightarrow (2)$$

Equation (1) is called as state Equation

Equation (2) is called as output Equation

where A - State Matrix

$\dot{X} = \frac{dx}{dt}$ - Differential State vector

X - State vector

U - input vector

Y - output vector

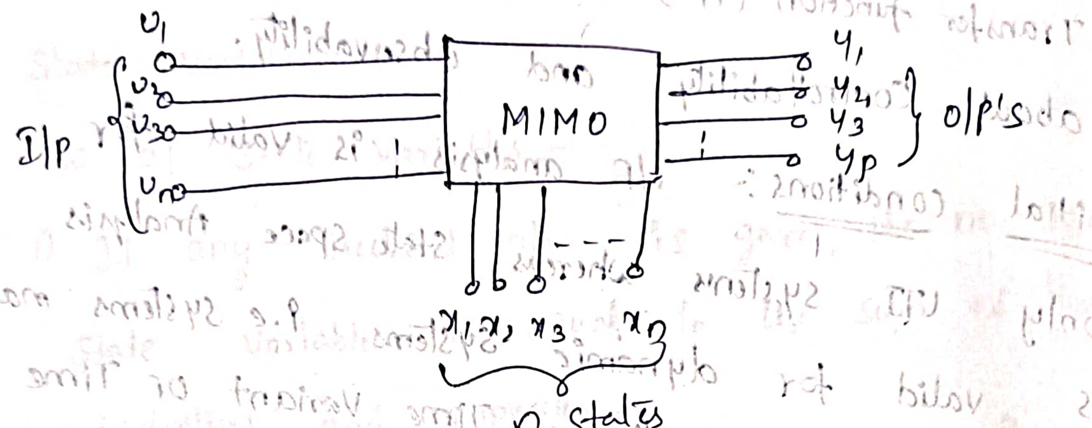
B - input matrix

C - output matrix

D - Transition matrix

Order of the matrix :-

Consider a MIMO system with n inputs, n states and p outputs as shown in figure.



Then

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}_{m\times 1} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

and $$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_p \end{bmatrix}_{p\times 1}$$

Note :- If the D matrix is always zero because if the network not consists of any active elements.

State Space Models :-

1) Differential Equations.
2) Transfer functions.
3) Signal flow graph
4) Electrical networks

Problems on State Model to Differential Equations:-

write the state model for the following differential equation

$$\dddot{y} + 2\ddot{y} + 3\dot{y} + 4y = 10\,u(t)$$

Given D.E is

$$\dddot{y} + 2\ddot{y} + 3\dot{y} + 4y = 10\,u(t) \longrightarrow (1)$$

The order of given D.E is 3

$$\boxed{n = 3}$$

let $y = x_1$

$$\dot{y} = \frac{dx_1}{dt} = x_2$$

$$\ddot{y} = \frac{dx_2}{dt} = x_3$$

$$\dddot{y} = \frac{dx_3}{dt} = \dot{x}_3$$

Sub the above values in eq (1)

we get

$$\dot{x}_3 + 2x_3 + 3x_2 + 4x_1 = 10u(t)$$

$$\dot{x}_3 = [-4x_1 - 3x_2 - 2x_3] + 10u(t)$$

The above equation is in the form of state equation given by

$$\dot{X}_{n\times1} = A_{n\times n} X_{n\times1} + B_{n\times m} U_{m\times1}$$

$$\underset{3\times1}{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}} = \underset{3\times3}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} + \underset{3\times1}{\begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}} U_{1\times1}$$

Now the output equation is given by

$$y = x_1$$

which is in the form of

$$y = CX + DU$$

where $D = [0]$

$$y = CX$$

$$[y]_{1\times1} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}_{1\times3}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}_{3\times1}$$

2. obtain the state equation for the differential equation given by

$$\ddddot{y} + 6\dddot{y} + 5\ddot{y} + 3\dot{y} + 2y = 5u(t)$$

Given D.E is

$$\ddddot{y} + 6\dddot{y} + 5\ddot{y} + 3\dot{y} + 2y = 5u(t)$$

order of D.E is 4

$$n = 4$$

let $y = x_1$

$$\dot{y} = \frac{dx_1}{dt} = x_2$$

$$\ddot{y} = \frac{dx_2}{dt} = x_3$$

$$\dddot{y} = \frac{dx_3}{dt} = x_4$$

$$\ddddot{y} = \frac{dx_4}{dt} = \dot{x}_4$$

Sub the above values in eq ①

$$\dot{x}_4 + 6x_4 - 5x_3 + 3x_2 + 2x_1 = 5u(t)$$

$$\dot{x}_4 = \left[-2x_1 - 3x_2 + 5x_3 - 6x_4\right] + 5u(t)$$

This is the required state Equation which is in the form of

$$\dot{X} = AX + BU$$

$$\underset{4\times1}{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}} = \underset{4\times4}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & 5 & -6 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} + \underset{4\times1}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}} \underset{1\times1}{[U]}$$

Now the output Equation is

$$y = x_1$$

which is in the form of

$$y = CX + DU$$

where $D = [0]$

$$\underset{1\times1}{[y]} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Problems on State Model to Transfer function :-

* Write the State Model to the given T/F

$$G(s)\,H(s) = \frac{2s+3}{s^3+5s^2+6s+7}$$

Given

$$T/F = \frac{Y(s)}{U(s)} = \frac{2s+3}{s^3+5s^2+6s+7}$$

$y(s) = 2s + 3s^0 \longrightarrow (1)$

and $u(s) = s^3 + 5s^2 + 6s + 7s^0 \longrightarrow (2)$

let $s^0 = x_1$

$s^1 = \frac{dx_1}{dt} = x_2$

$s^2 = \frac{dx_2}{dt} = x_3$

$s^3 = \frac{dx_3}{dt} = \dot{x}_3$

By substituting above values in eq (2) we get

$u(s) = \dot{x}_3 + 5x_3 + 6x_2 + 7x_1$

$\dot{x}_3 = [-7x_1 - 6x_2 - 5x_3] + u(s)$

The above Equation is the required State Equation which is in the form of

$\dot{X} = AX + BU$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [U]$$

By substituting s^0, s^1, s^2 & s^3 values in Equation (1). The output Equation is obtained as

$y(s) = [2x_2 + 3x_1]$

which is in the form of

$y = CX + DU$

where $[D] = 0$

$$[Y] = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}_{1\times3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1} + [0]_{1\times1} [U]_{1\times1}$$

③ Write the state model for the given T/F

$$T/F = \frac{s^3+6s^2+10}{s^5+10s^4-8s^2+7s+9}$$

Sol :- Given

$$T/F = \frac{Y(s)}{U(s)} = \frac{s^3+6s^2+10}{s^5+10s^4-8s^2+7s+9}$$

$$Y(s) = 6s^2+s^3+10s^0 \longrightarrow ①$$

and $U(s) = s^5+10s^4-8s^2+7s+9s^0 \longrightarrow ②$

The order of the given system is 5

$\boxed{n=5}$

let $s^0 = x_1$

$$s^1 = \frac{dx_1}{dt} = x_2$$

$$s^2 = \frac{dx_2}{dt} = x_3$$

$$s^3 = \frac{dx_3}{dt} = x_4$$

$$s^4 = \frac{dx_4}{dt} = x_5$$

and $s^5 = \frac{dx_5}{dt} = \dot{x}_5$

By substituting the above values, in eq ② we get

$$U(s) = \dot{x}_5 + 10x_5 + -8x_3 + 7x_2 + 9x_1$$

$$\dot{x}_5 = [-9x_1 - 7x_2 + 8x_3 - 10x_5] + U(s)$$

This is the required state equation which is in the form of

$$\dot{X} = AX + BU$$

$$\underset{5\times1}{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix}} = \underset{5\times5}{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -9 & -7 & 8 & 0 & -10 \end{bmatrix}} \underset{5\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}} + \underset{5\times1}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} U$$

By substituting the s^3, s^2 and s^0 values in Eq ①

the O/P equation is obtained as

$$y(s) = (x_4 + 6x_3 + 10x_1) + 0 \cdot U(s)$$

$$[y] = \underset{1\times5}{[10 \quad 0 \quad 6 \quad 1 \quad 0]} \underset{5\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}} + \underset{1\times1}{[0]} \underset{1\times1}{[U]}$$

Procedure for the Conversion of signal flow graph to State Model :-

① First node should be input node

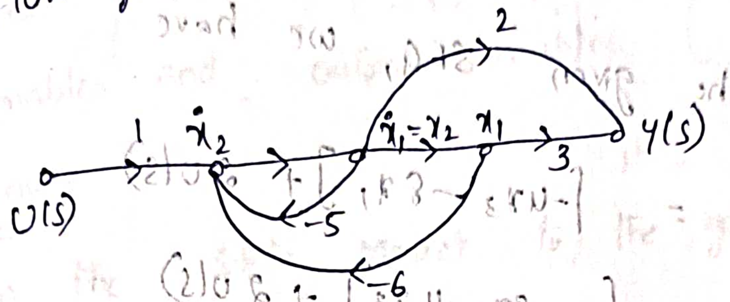
② Next node is the highest power of 's' equivalent to differential state variable.

③ Successive nodes are integrable nodes until to get x_1

(ii) last node should be the output node.

Problem :-

* obtain the State model for the following signal flow graph.



Sol: From the given signal flow graph (SFG)

$$\dot{x}_2 = -5x_2 - 6x_1 + U(s)$$

$$\dot{x}_2 = -6x_1 - 5x_2 + U(s)$$

This is the required state equation which is in the form of

$$\dot{X} = AX + BU$$

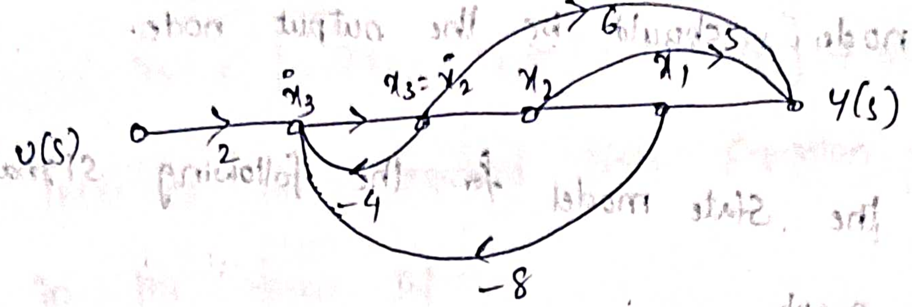
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}_{2\times1} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}_{2\times2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times1} [U]_{1\times1}$$

Now the output equation is

$$y(s) = 3x_1 + 2x_2$$

$$[y]_{1\times1} = [3 \;\; 2]_{1\times2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times1} + [0]_{1\times1} [U]_{1\times1}$$

* obtain the ~~signal flow graph~~ State Model for the following Signal flow graph. (SFG)



From the given SFG, we have

$$\dot{x}_3 = [-4x_3 - 8x_1] + 2U(s)$$

$$(or)\quad \dot{x}_3 = [-8x_1 - 4x_3] + 2U(s)$$

$$\underset{3\times1}{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}} = \underset{3\times3}{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -8 & 0 & -4\end{bmatrix}}\underset{3\times1}{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}} + \underset{3\times1}{\begin{bmatrix}0\\ 0\\ 2\end{bmatrix}}[U]_{1\times1}$$

Now, the output equation is

$$Y(s) = 3x_1 + 5x_2 + 6x_3$$

$$[Y]_{1\times1} = \underset{1\times3}{[3 \;\; 5 \;\; 6]}\underset{3\times1}{\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}} + \underset{1\times1}{[0]}[U]_{1\times1}$$

State Model for Electrical Networks :-

Procedure :-

1) Select the state variables as voltage across capacitor, current through Inductor.

2) No. of state variables equals to sum of Inductor and capacitors

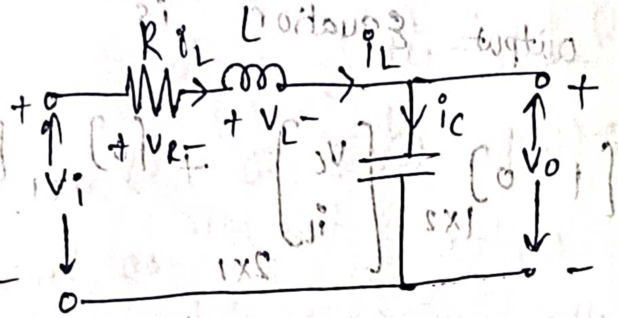
3) write Independent KCL and KVL at capacitor

junction. Apply KCL across inductor & KVL through inductors.

4) The resultant Equation should consists of only state variables, differential state variables, input variables and output variables.

Problems :-

Obtain the state Model to the given circuit.



Apply KVL to the input loop

$$-V_i + i_L R + L\frac{di_L}{dt} + V_c = 0$$

$$L\frac{di_L}{dt} = -V_c - i_L R + V_i$$

$$\dot{i}_L = \left[-\frac{1}{L}V_c - \frac{R}{L}i_L\right] + \frac{V_i}{L} \longrightarrow (1)$$

Here the state variables are V_c and i_L

Hence the corresponding state vector is $\begin{bmatrix} V_c \\ i_L \end{bmatrix}$

From the given circuit, we have

$$i_L = C\frac{dV_c}{dt}$$

$$\therefore \dot{V}_c = \frac{1}{C}i_L \longrightarrow (2)$$

From eq ① & ②, the state equation is obtained as

$$\begin{bmatrix} \dot{V_C} \\ \dot{i_L} \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_i$$

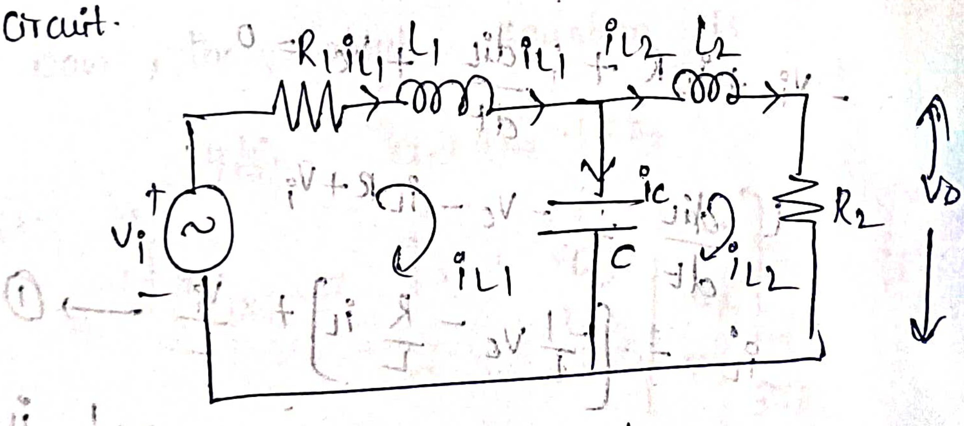
As the output is taken across C

$\therefore V_o = V_C$

Hence the output equation is

$$[V_o]_{1\times1} = [1 \quad 0]_{1\times2} \begin{bmatrix} V_C \\ i_L \end{bmatrix}_{2\times1} + [0]_{1\times1} [V_i]_{1\times1}$$

② Obtain the state model for the given electrical circuit.



Apply KVL to the loop

$$-V_i + i_{L_1} R_1 + L_1 \frac{di_L}{dt} + V_C = 0$$

$$L_1 \dot{i}_{L_1} = -V_C - i_{L_1} R_1 + V_i$$

$$\dot{i}_{L_1} = \left[-\frac{1}{L_1} V_C - \frac{R_1}{L_1} i_{L_1} \right] + \frac{1}{L_1} V_i \rightarrow ①$$

Apply KVL at loop ②

$$-V_C + L_2 \frac{di_{L_2}}{dt} + i_{L_2} R_2 = 0$$

(c) $\dot{i}_{L_2} L_2 = V_C - i_{L_2} R_2$

$$\dot{i}_{L_2} = \frac{1}{L_2} V_C - \frac{R_2}{L_2} i_{L_2} \longrightarrow (2)$$

Apply KCL at node (1)

$$i_{L_1} = i_C + i_{L_2}$$

$$i_{L_1} = C \frac{dV_C}{dt} + i_{L_2}$$

$$C \dot{V}_C = i_{L_1} - i_{L2}$$

$$\dot{V}_C = \frac{1}{C} i_{L_1} - \frac{1}{C} i_{L_2} \longrightarrow (3)$$

From eq. (1), (2) & (3) the State Equation is obtained as

$$\begin{bmatrix} \dot{i}_{L_1} \\ \dot{i}_{L_2} \\ \dot{V}_C \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & -1/L_1 \\ 0 & -R_2/L_2 & 1/L_2 \\ 1/C & -1/C & 0 \end{bmatrix} \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ V_C \end{bmatrix} + \begin{bmatrix} 1/L_1 \\ 0 \\ 0 \end{bmatrix} [V_i]$$

~~Transfer function for State Model~~

~~consider $\dot{X} = AX + BU \longrightarrow (1)$~~

~~$Y = CX + DU \longrightarrow (2)$~~

Since the output is taken across R_2

$$\therefore V_0 = i_{L_2} R_2$$

Hence the output Equation is

$$[V_0] = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ V_C \end{bmatrix} + [0][V_i]$$

Transfer function for State Model :-

Consider

$$\dot{x} = Ax + BU \rightarrow (1)$$

$$y = Cx + DU \rightarrow (2)$$

Apply Laplace transform for Equation (1)

$$sX(s) = AX(s) + BU(s)$$

$$X(s)[sI - A] = BU(s)$$

$$X(s) = B[sI - A]^{-1} U(s) \rightarrow (3)$$

Now Apply laplace transform for Equation (2)

$$Y(s) = CX(s) + DU(s) \rightarrow (4)$$

Now Substitute eq (3) in eq (4)

$$Y(s) = CB[sI - A]^{-1} U(s) + DU(s)$$

$$Y(s) = U(s)[C[sI - A]^{-1} B + D]$$

$$\frac{Y(s)}{U(s)} = C[sI - A]^{-1} B + D$$

But as the matrix D is equals to 0

$$\therefore \boxed{\frac{Y(s)}{U(s)} = C[sI - A]^{-1} B}$$

Note :-

(1) The Equation $|sI - A| = 0$ is known as Characteristic Equation

(2) The roots of above Equation (C.E) are called

as Eigen values.

problems :-

Find the T/F to the given state Model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} [u]$$

$$[y] = [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol) From the given state Model we have

$$A = \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \text{and} \quad C = [1 \quad 1]$$

we have

$$\frac{Y(s)}{U(s)} = C\,[sI - A]^{-1}\,B \longrightarrow (1)$$

Now $$[sI - A]^{-1} = \frac{\mathrm{Adj}\,[sI - A]}{|sI - A|}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix}$$

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix}$$

$$[sI - A] = \begin{bmatrix} s+2 & 3 \\ -4 & s+2 \end{bmatrix}$$

Now, $$\mathrm{Adj}\,[sI - A] = \begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}.$$

And $|SI - A| = \begin{vmatrix} S+2 & 3 \\ -4 & S-2 \end{vmatrix} = (S+2)(S-2) + 12$

$$|SI - A| = S^2 - 4 + 12 = S^2 + 8$$

$$[SI - A]^{-1} = \frac{Adj. [SI - A]}{|SI - A|}$$

$$[SI - A]^{-1} = \frac{1}{S^2+8} \begin{bmatrix} S-2 & -3 \\ 4 & S+2 \end{bmatrix}$$

$$[SI - A]^{-1} B = \frac{1}{S^2+8} \begin{bmatrix} S-2 & -3 \\ 4 & S+2 \end{bmatrix}_{2\times 2} \begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2\times 1}$$

$$= \frac{1}{S^2+8} \begin{bmatrix} 3S - 6 - 15 \\ 12 + 5S + 10 \end{bmatrix}_{2\times 1}$$

$$[SI - A]^{-1} B = \frac{1}{S^2+8} \begin{bmatrix} 3S - 21 \\ 5S + 22 \end{bmatrix}$$

Now from eq ①

$$\frac{Y(S)}{U(S)} = C\,(SI - A)^{-1} B$$

$$\frac{Y(S)}{U(S)} = \frac{1}{S^2+8} \begin{bmatrix} 1 & 1 \end{bmatrix}_{1\times 2} \begin{bmatrix} 3S - 21 \\ 5S + 22 \end{bmatrix}_{2\times 1}$$

$$= \frac{1}{S^2+8} \left[3S - 21 + 5S + 22 \right]_{1\times 1}$$

$$\boxed{\frac{Y(S)}{U(S)} = \frac{8S + 1}{S^2 + 8}}$$

* obtain the T/F to the given state Model :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} S & -3 \\ 2 & S+5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [U]$$

$$[Y] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and also find the step response.

Sol

From the given state Model, we have

$$[SI - A] = \begin{bmatrix} S & -3 \\ 2 & S+5 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

we have

$$\frac{Y(S)}{U(S)} = C[SI - A]^{-1} B \longrightarrow (1)$$

where $[SI - A]^{-1} = \dfrac{Adj[SI - A]}{|SI - A|}$

Now, $Adj[SI - A] = \begin{bmatrix} S+5 & 3 \\ -2 & S \end{bmatrix}$

and $|SI - A| = \begin{vmatrix} S & -3 \\ 2 & S+5 \end{vmatrix} = S(S+5)+6$

$$|SI - A| = S^2 + 5S + 6$$

$$[SI - A]^{-1} = \frac{1}{S^2+5S+6} \begin{bmatrix} S+5 & 3 \\ -2 & S \end{bmatrix}$$

$$C \cdot [SI - A]^{-1} = \frac{1}{S^2+5S+6} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} S+5 & 3 \\ -2 & S \end{bmatrix}$$

$$= \frac{1}{s^2+5s+6} \begin{bmatrix} 0-2 & 0+s \end{bmatrix}$$

$$C \cdot [sI - A]^{-1} = \frac{1}{s^2+5s+6} \begin{bmatrix} -2 & s \end{bmatrix}$$

Now

$$C \cdot [sI - A]^{-1} B = \frac{1}{s^2+5s+6} \begin{bmatrix} -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = C \cdot [sI - A]^{-1} B = \frac{1}{s^2+5s+6} \begin{bmatrix} -2 + s \end{bmatrix}$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{s-2}{s^2+5s+6}}$$

To find step response

Given that

Input is step signal

$$U(s) = 1/s$$

Now, we have

$$\frac{Y(s)}{U(s)} = \frac{s-2}{s^2+5s+6}$$

$$Y(s) = \frac{s-2}{s^2+5s+6} \cdot U(s)$$

$$Y(s) = \frac{s-2}{s^2+5s+6} \cdot \frac{1}{s}$$

$$Y(s) = \frac{s-2}{s(s^2+5s+6)}$$

Consider $\dfrac{s-2}{s(s^2+5s+6)} = \dfrac{s-2}{s(s+2)(s+3)}$

$$\frac{S-2}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$s-2 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2)$$

At $s=0$

$$-2 = A(2)(3)$$

$$\boxed{A = -1/3}$$

At $s=-2$

$$-4 = B(-2)(1)$$

$$\boxed{B = 2}$$

At $s=-3$

$$-5 = C(-3)(-1)$$

$$\boxed{C = -5/3}$$

$$Y(s) = \frac{s-2}{s(s+2)(s+3)} = \frac{-1}{3s} + \frac{2}{s+2} - \frac{5}{3(s+3)}$$

Apply inverse laplace transform on B.S

$$y(t) = L^{-1}\left\{\frac{-1}{3s} + \frac{2}{s+2} - \frac{5}{3(s+3)}\right\}$$

$$y(t) = \frac{-1}{3}u(t) + 2e^{-2t}u(t) - \frac{5}{3}e^{-3t}u(t)$$

$$\therefore \boxed{y(t) = u(t)\left[\frac{-1}{3} + 2e^{-2t} + \frac{-5}{3}e^{-3t}\right]}$$

Solution to state equations:-

1 Laplace transform method

Consider the state equation and o/p equation

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

Apply L.T for both equations

$$SX(S) - X(0) = AX(S) + BU(S) \longrightarrow ①$$

$$Y(S) = CX(S) + DU(S)$$

From Equation ①

$$X(S)[SI - A] = BU(S) + X(0)$$

$$X(S) = [SI-A]^{-1} BU(S) + [SI-A]^{-1} X(0)$$

Apply Inverse laplace transform

$$x(t) = L^{-1}\{(SI-A)^{-1} BU(S)\} + L^{-1}\{[SI-A]^{-1} X(0)\} \longrightarrow (2)$$

The Equation (2) contains two parts

1st part is called as "zero State Response"

2nd part is called as "zero Input Response"

2. Classical Method :-

We have

$$x(t) = \underbrace{\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau}_{ZSR} + \underbrace{e^{At} x(0)}_{ZIR} \longrightarrow ③$$

From eq ② & ③ we have

$$L^{-1}\{[SI-A]^{-1}x(0)\} = e^{At}x(0) \longrightarrow ④$$

And $$L^{-1}\{[SI-A]^{-1}BU(s)\} = \int_0^t e^{A(t-\tau)} BU(\tau)\, d\tau \longrightarrow ⑤$$

From equation ④,

$$L^{-1}\{[SI-A]^{-1}\}\cancel{x(0)} = e^{At}\cancel{x(0)}$$

$$L^{-1}\{[SI-A]^{-1}\} = e^{At} = \phi(t)$$

$$[SI-A]^{-1} = L\{e^{At}\} = L\{\phi(t)\}$$

$$[SI-A]^{-1} = \phi(s)$$

where $\phi(t)$ = State Transition matrix

$\phi(s)$ = Resolvent matrix

Now from Equation (5)

$$L^{-1}\{[SI-A]^{-1}BU(s)\} = \int_0^t e^{A(t-\tau)} BU(\tau)\, d\tau$$

we have $$L^{-1}\{[SI-A]^{-1}\} = \phi(t)$$

$$[SI-A]^{-1} = \phi(s)$$

$$L^{-1}\{\phi(s)\, BU(s)\} = \int_0^t e^{A(t-\tau)} BU(\tau)\, d\tau$$

$$\therefore \boxed{x(t) = L^{-1}\{\phi(s)BU(s)\} + L^{-1}\{\phi(s)x(0)\}}$$

Properties of STM:-

We have

$$\phi(t) = e^{At} = L^{-1}\{(SI-A)^{-1}\}$$

1) $\phi(0) = I$

2) $\phi^{-1}(t) = (e^{At})^{-1} = e^{-At} = e^{A(-t)} = \phi(-t)$

$\phi^{-1}(t) = \phi(-t)$

3) $\phi^{k}(t) = \phi(kt)$

4) $\phi(t_1+t_2) = e^{A(t_1+t_2)} = e^{At_1 + At_2} = e^{At_1} \cdot e^{At_2}$

$\phi(t_1+t_2) = \phi(t_1) \cdot \phi(t_2)$

5) $\phi(t_2-t_1) = \phi(t_2-t_0) * \phi(t_1-t_0)$

Problems on STM:-

Obtain the complete Time response of the s/m given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x$$

$$y = [1 \quad -1]\, x$$

and $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Given That

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x \; ; \; y = [1 \quad -1]\, x$$

and $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ and $C = [1 \quad -1]$

Now the required general Solution is

$$x(t) = L^{-1}\{\phi(s)\, B\, U(s)\} + L^{-1}\{\phi(s)\, x(0)\}$$

From the given State Equations i.e.,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x$$

Since $B = 0$

$$L^{-1}\{\phi(s)\, B\, U(s)\} = 0$$

Hence $x(t) = L^{-1}\{\phi(s)\, x(0)\} \longrightarrow 1$

But we know that

$$\phi(s) = [sI - A]^{-1} = \frac{Adj\,[sI - A]}{|sI - A|}$$

Now, $sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

$$sI - A = \begin{bmatrix} s & -1 \\ 2 & s \end{bmatrix}$$

$$Adj\,[sI - A] = \begin{bmatrix} s & -1 \\ 2 & s \end{bmatrix}$$

$$|sI - A| = \begin{vmatrix} s & -1 \\ 2 & s \end{vmatrix} = s^2 + 2$$

$$\therefore \phi(s) = \frac{Adj\,[sI - A]}{|sI - A|} = \frac{1}{s^2+2}\begin{bmatrix} s & 1 \\ -2 & s \end{bmatrix}$$

Now, $\phi(s)\,x(0) = \frac{1}{s^2+2}\begin{bmatrix} s & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\phi(s)\,x(0) = \frac{1}{s^2+2}\begin{bmatrix} s+1 \\ -2+s \end{bmatrix}$$

Apply ILT on b.s

$$L^{-1}\{\phi(s)x(0)\} = L^{-1}\left\{\frac{1}{s^2+2}\begin{bmatrix} s+1 \\ s-2 \end{bmatrix}\right\}$$

From Equation ①

$$x(t) = L^{-1}\{\phi(s)\,x(0)\} = L^{-1}\left\{\begin{bmatrix} \frac{s+1}{s^2+2} \\ \frac{s-2}{s^2+2} \end{bmatrix}\right\}$$

$$x(t) = \begin{bmatrix} L^{-1}\left\{\frac{s+1}{s^2+2}\right\} \\ L^{-1}\left\{\frac{s-2}{s^2+2}\right\} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} L^{-1}\left\{\frac{s}{s^2+2}\right\} + L^{-1}\left\{\frac{1}{s^2+2}\right\} \\ L^{-1}\left\{\frac{s}{s^2+2}\right\} - 2L^{-1}\left\{\frac{2}{s^2+2}\right\} \end{bmatrix}$$

$$\therefore x(t) = \begin{bmatrix} \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t \\ \cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{bmatrix}$$

To find the complete time response, we have i.e., $y(t) = [1 \ \ -1] \, x(t)$

$$y(t) = [1 \ \ -1] \begin{bmatrix} \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t \\ \cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{bmatrix}$$

$$y(t) = \cancel{\cos\sqrt{2}t} + \frac{1}{\sqrt{2}}\sin\sqrt{2}t - \cancel{\cos\sqrt{2}t} + \sqrt{2}\sin\sqrt{2}t$$

$$y(t) = \frac{3}{\sqrt{2}}\sin\sqrt{2}t.$$

2. Find out the time response for unit step input of the system given by.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $y = [-2 \ \ -3] \, x$

Sol) Given state Equation is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

The given system is non-Homogenous s/m.

Hence the general solution is

$$x(t) = L^{-1}\{\phi(s) B U(s)\} + L^{-1}\{\phi(s) x(0)\} \rightarrow (1)$$

But $\phi(s) = [sI - A]^{-1} = \dfrac{\text{Adj}[sI - A]}{|sI - A|} \rightarrow (2)$

NOW, $sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$|sI - A| = \begin{vmatrix} s+3 & 1 \\ -2 & s \end{vmatrix} = s(s+3) + 2$$

$$= s^2 + 3s + 2$$

From Equation ②

$$\phi(s) = \frac{\text{Adj}\,[sI - A]}{|sI - A|} = \frac{1}{s^2+3s+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\phi(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

NOW $\phi(s)\,x(0) = \dfrac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\phi(s)\,x(0) = \frac{1}{(s+1)\cdot(s+2)} \begin{bmatrix} s+3 \\ -2 \end{bmatrix}$$

Apply ILT on both sides

$$L^{-1}\{\phi(s)\,x(0)\} = L^{-1}\left\{\frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 \\ -2 \end{bmatrix}\right\}$$

$$L^{-1}\{\phi(s)\,x(0)\} = L^{-1}\begin{Bmatrix} \dfrac{s+3}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} \end{Bmatrix}$$

$$L^{-1}\{\phi(s)x(0)\} = \begin{bmatrix} L^{-1}\left\{\dfrac{s+3}{(s+1)(s+2)}\right\} \\ L^{-1}\left\{\dfrac{-2}{(s+1)(s+2)}\right\} \end{bmatrix}$$

$$L^{-1}\{\phi(s)x(0)\} = \begin{bmatrix} L^{-1}\left\{\dfrac{2}{s+1} - \dfrac{1}{s+2}\right\} \\ L^{-1}\left\{\dfrac{-2}{s+1} + \dfrac{2}{s+2}\right\} \end{bmatrix}$$

$$L^{-1}\{\phi(s)x(0)\} = \begin{bmatrix} 2e^{-t}u(t) - e^{-2t}u(t) \\ -2e^{-t}u(t) + 2e^{-2t}u(t) \end{bmatrix} \rightarrow 3$$

Now $\phi(s)\,B = \dfrac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix}$

$$\phi(s)\cdot B = \frac{1}{(s+1)(s+2)}\begin{bmatrix} (s+3)0+1\times 5 \\ -2\times 0+s\times 5 \end{bmatrix}$$

$$\phi(s)\cdot B\cdot U(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} 5 \\ 5s \end{bmatrix} U(s)$$

Given that,

The input of the system is a step input

$$U(s) = \frac{1}{s}$$

$$\phi(s)\,B\,U(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} 5 \\ 5s \end{bmatrix}\cdot\frac{1}{s}$$

$$\phi(s)\,B\,U(s) = \frac{1}{s(s+1)(s+2)}\begin{bmatrix} 5 \\ 5s \end{bmatrix}$$

Apply ILT on both sides

$$L^{-1}\left\{\phi(s)\,B\cdot U(s)\right\} = L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\begin{bmatrix}5\\5s\end{bmatrix}\right\}$$

$$L^{-1}\left\{\phi(s)\,B\,U(s)\right\} = L^{-1}\begin{bmatrix}\dfrac{5}{s(s+1)(s+2)}\\[2ex] \dfrac{5\cancel{s}}{\cancel{s}(s+1)(s+2)}\end{bmatrix}$$

$$L^{-1}\left\{\phi(s)\,B\,U(s)\right\} = \begin{bmatrix}L^{-1}\left\{\dfrac{5}{s(s+1)(s+2)}\right\}\\[2ex] L^{-1}\left\{\dfrac{5}{(s+1)(s+2)}\right\}\end{bmatrix}$$

$$L^{-1}\left\{\phi(s)\,B\,U(s)\right\} = \begin{bmatrix}L^{-1}\left\{\dfrac{5}{2s}-\dfrac{5}{s+1}+\dfrac{5}{2(s+2)}\right\}\\[2ex] L^{-1}\left\{\dfrac{5}{s+1}-\dfrac{5}{s+2}\right\}\end{bmatrix}$$

$$\therefore L^{-1}\left\{\phi(s)\,B\,U(s)\right\} = \begin{bmatrix}\frac{5}{2}u(t)-5e^{-t}u(t)+\frac{5}{2}e^{-2t}u(t)\\ 5e^{-t}u(t)-5e^{-2t}u(t)\end{bmatrix} \longrightarrow (4)$$

Substitute eq. (3) & (4) in equation (1)

$$x(t) = \begin{bmatrix}\frac{5}{2}u(t)-5e^{-t}u(t)+\frac{5}{2}e^{-2t}u(t)\\ 5e^{-t}u(t)-5e^{-2t}u(t)\end{bmatrix} + \begin{bmatrix}2e^{-t}u(t)-e^{-2t}u(t)\\ -2e^{-t}u(t)+2e^{-2t}u(t)\end{bmatrix}$$

$$\therefore x(t) = \begin{bmatrix} \frac{5}{2}u(t) - 3e^{-t}u(t) + \frac{3}{2}e^{-2t}u(t) \\ 3e^{-t}u(t) - 3e^{-2t}u(t) \end{bmatrix}$$

For Time response, we have

$$y(t) = \begin{bmatrix} -2 & -3 \end{bmatrix} \begin{bmatrix} \frac{5}{2}u(t) - 3e^{-t}u(t) + \frac{3}{2}e^{-2t}u(t) \\ 3e^{-t}u(t) - 3e^{-2t}u(t) \end{bmatrix}$$

$$= -5u(t) + 6e^{-t}u(t) - 3e^{-2t}u(t) - 9e^{-t}u(t) + 9e^{-2t}u(t)$$

$$= -5u(t) - 3e^{-t}u(t) + 6e^{-2t}u(t)$$

$$\therefore \boxed{y(t) = \left[6e^{-2t} - 3e^{-t} - 5\right]u(t)}$$

<u>Controllability</u> :- A system is said to be Controllable if it is possible to transfer the initial state to any other state in a finite time by controlled vector.

Controllability is verified by kalmn's Test.

$$Q_c = \left[B \;\; AB \;\; A^2B \; --- \; A^{n-1}B\right]$$

where n - order of the matrix 'A'

<u>Conditions</u> :-

① Rank of Q_c = Rank of A

② $|Q_c| \neq 0$

If the system is satisfying the above two conditions, then it is said to be Controllable

Problems of Controllability :-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ +2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Given system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\therefore$ The order of $A = 2$

$\Rightarrow \boxed{n=2}$

Now $Q_c = [B \quad AB] \longrightarrow$ ①

But, $AB = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Substitute B & AB values in Equation ①

$$Q_c = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Rank of Q_c :-

$$|Q_c| = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 0(-1) - 1(1) = -1 \neq 0$$

$\therefore$ Rank of $Q_c = 2$

Rank of A :-

$$|A| = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1-2 = -3 \neq 0$$

$\therefore$ Rank of $A = 2$

Conclusions :-

① Rank of Q_c = Rank of A

② $|Q_c| \neq 0$

Therefore, the given system is Controllable.

2. Check the controllability of

$$\dot{x}_1 = -2x_1 + u$$

$$\dot{x}_2 = -3x_2 + 5x_1$$

Given

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$A = \begin{bmatrix} -2 & 0 \\ 5 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore$ Order of matrix A, $n = 2$

$\therefore Q_c = [B \quad AB]$

Now, $AB = \begin{bmatrix} -2 & 0 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

$Q_c = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

Rank of Q_c :-

$|Q_c| = \begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} = 5 + 0 = 5 \neq 0$

$\therefore$ Rank of $Q_c = 2$

Rank of A :-

$|A| = \begin{vmatrix} -2 & 0 \\ 5 & -3 \end{vmatrix} = 6 - 0 = 6 \neq 0$

$\therefore$ Rank of $A = 2$

Conclusions :-

1) Rank of Q_c = Rank of A

$\Rightarrow |Q_c| \neq 0$

Hence the given system is Controllable.

Observability :-

A system is said to be observable if it is possible to determine initial states of the system by observing the output for a finite time interval.

$$Q_0 = [C^T \quad A^T C^T \quad (A^T)^2 C^T \quad --- \quad (A^T)^{n-1} C^T]$$

$$Q_0 = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

<u>conditions</u> :-

① Rank of Q_0 = Rank of A

② $|Q_0| \neq 0$

<u>problems on</u> <u>Controllability</u> and <u>observability</u> :-

check the controllability and observability for

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

and $y = [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Sol) Given

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and $C = [1 \quad 1]$

<u>Controllability</u> :-

$Q_C = [B \quad AB]$ $[\because n = 2]$

NOW, $AB = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\therefore \quad Q_c = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

NOW, $|Q_c| = \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = -2+2 = 0$

$\therefore$ Rank of $Q_c = 1$

NOW, $|A| = \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = 2+0 = 2 \neq 0$

$\therefore$ Rank of $A = 2$

Conclusions :-

1) Rank of $Q_c \neq$ Rank of A

2) $|Q_c| = 0$

Hence the given system is not controllable

Observability :-

$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix}$

NOW $CA = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$

$CA = \begin{bmatrix} 2-1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$\therefore \quad Q_o = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

NOW $|Q_o| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

$\therefore$ Rank of $Q_0 = 1$

we have Rank of $A = 2$

<u>conclusions</u> :-

1) Rank of $Q_0 \neq$ Rank of A

2) $|Q_0| = 0$

Hence the given system is not observable.

* Check the Controllability and observability for

$\dot{x}_1 = -2x_1 + x_2 + U$

$\dot{x}_2 = -x_2 + U$

$y = x_1 + x_2$

Given that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U,$$

and $[y] = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

<u>Controllability</u> :-

Since $n = 2$

$Q_C = [B \quad AB]$

Now $AB = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$$\therefore \quad Q_C = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

NOW $|Q_c| = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = -1+1=0$

Rank of $Q_c = 1$

NOW, $|A| = \begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix} = 2-0=2$

$\therefore$ Rank of $A = 2$

Conclusions :-

1) Rank of $Q_c \neq$ Rank of A

2) $|Q_c| = 0$

Hence, the given system is not controllable

Observability :-

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

NOW, $CA = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \end{bmatrix}$

$$\therefore \quad Q_0 = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$$

NOW, $|Q_0| = \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = 0+2 = 2 \neq 0$

Rank of $Q_0 = 2$

we have,

Rank of $A = 2$

conclusions :-

1) Rank of Q_0 = Rank of A

2) $|Q_0| \neq 0$

Hence the given system is observable

* Determine the Controllability and observability of

$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C = [1 \quad 2]$

sol) Given $A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

and $C = [1 \quad 2]$

order of the matrix A, $n = 2$

Controllability :-

$$Q_c = [B \quad AB]$$

Now, $AB = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$\therefore Q_c = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix}$

Now, $|Q_c| = \begin{vmatrix} 0 & 0 \\ 1 & -3 \end{vmatrix} = 0 - 0 = 0$

$\therefore$ Rank of Q_c

Now, $|A| = \begin{vmatrix} -1 & 0 \\ 0 & -3 \end{vmatrix} = 3 - 0 = 3 \neq 0$

$\therefore$ Rank of $A = 2$

Conclusions:-

1) Rank of $Q_c \neq$ Rank of A

2) $|Q_c| = 0$

Hence the given system is not controllable.

observability:-

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

Now, $CA = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -6 \end{bmatrix}$

$$\therefore Q_o = \begin{bmatrix} 1 & 2 \\ -1 & -6 \end{bmatrix}$$

Now, $|Q_o| = \begin{vmatrix} 1 & 2 \\ -1 & -6 \end{vmatrix} = -6+2 = -4 \neq 0$

$\therefore$ Rank of $Q_o = 2$

and also we have

$\therefore$ Rank of $A = 2$

Conclusions:-

1) Rank of Q_o = Rank of A

2) $|Q_o| \neq 0$

Hence, the given system is observable.