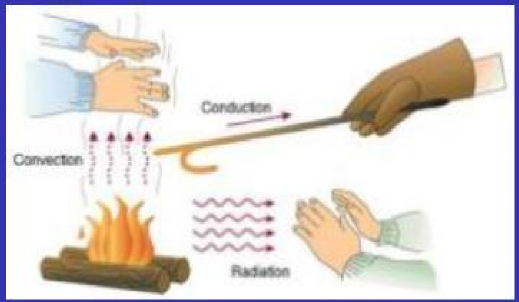
Heat Transfer



Introduction

Course Content: Heat Transfer

Introduction:

- What, How, and Where?
- Thermodynamics and Heat transfer
- Application
- Physical mechanism of heat transfer

Conduction:

- Introduction
- 1D, steady-state
- 2D, steady-state
- Transient

Convection:

- Introduction
- External and internal flows
- Free convection
- Boiling and condensation
- Heat exchangers

Radiation:

- Introduction
- View factors

Heat Transfer - What?

The science that deals with the determination of the rates of energy transfer due to temperature difference.

Driving force

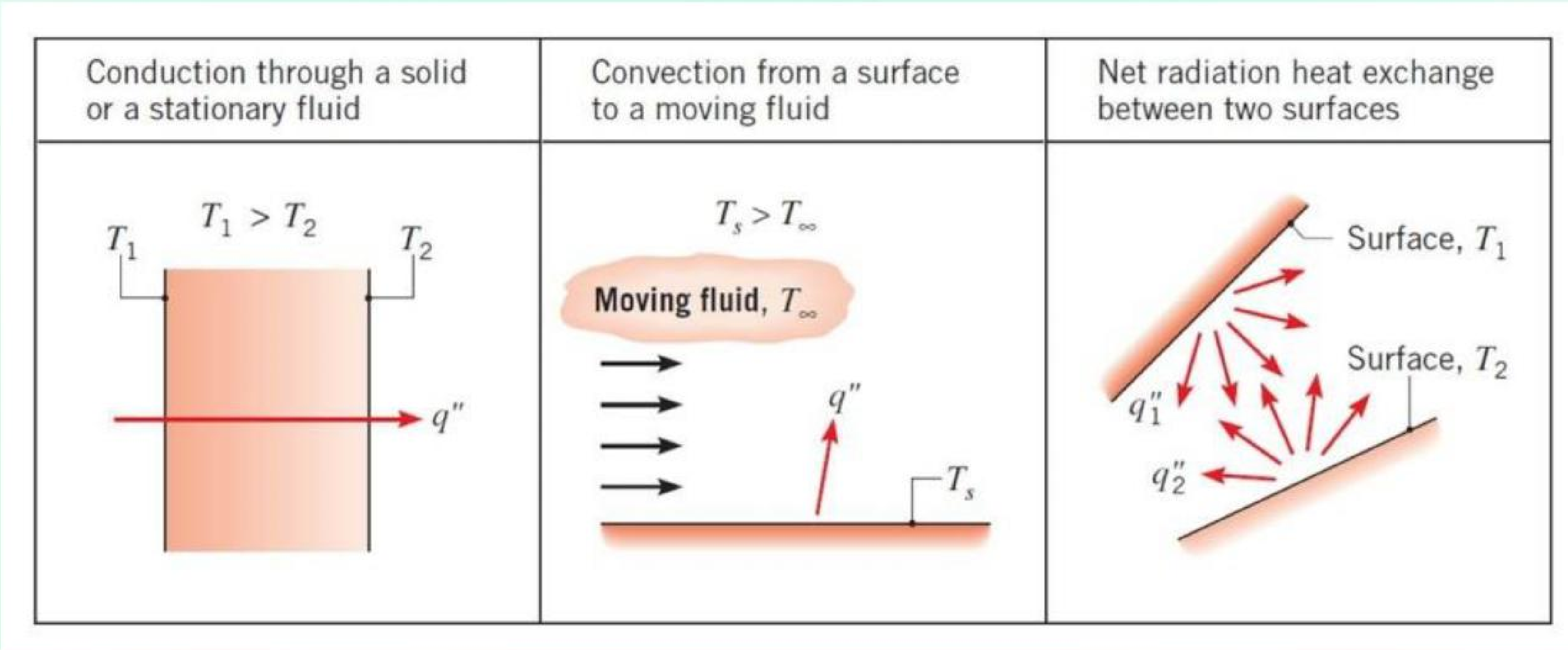
Temperature difference

as the voltage difference in electric current

as the pressure difference in fluid flow

Rate depends on magnitude of dT

Heat Transfer - How?



Heat Transfer - Where else?

Thermodynamics and Heat Transfer

Thermodynamics

Deals with the amount of energy (heat or work) during a process

Only considers the end states in equilibrium

Why?

Heat Transfer

Deals with the rate of energy transfer

Transient and non-equilibrium

How long?

Thermodynamics and Heat Transfer

Laws of Thermodynamics

Zeroth law - Temperature

First law Energy conserved

Second law Entropy

Third law $S \rightarrow$ constant as $T \rightarrow 0$

Laws of Heat Transfer

Fouriers law - Conduction

Newtons law of cooling - Convection

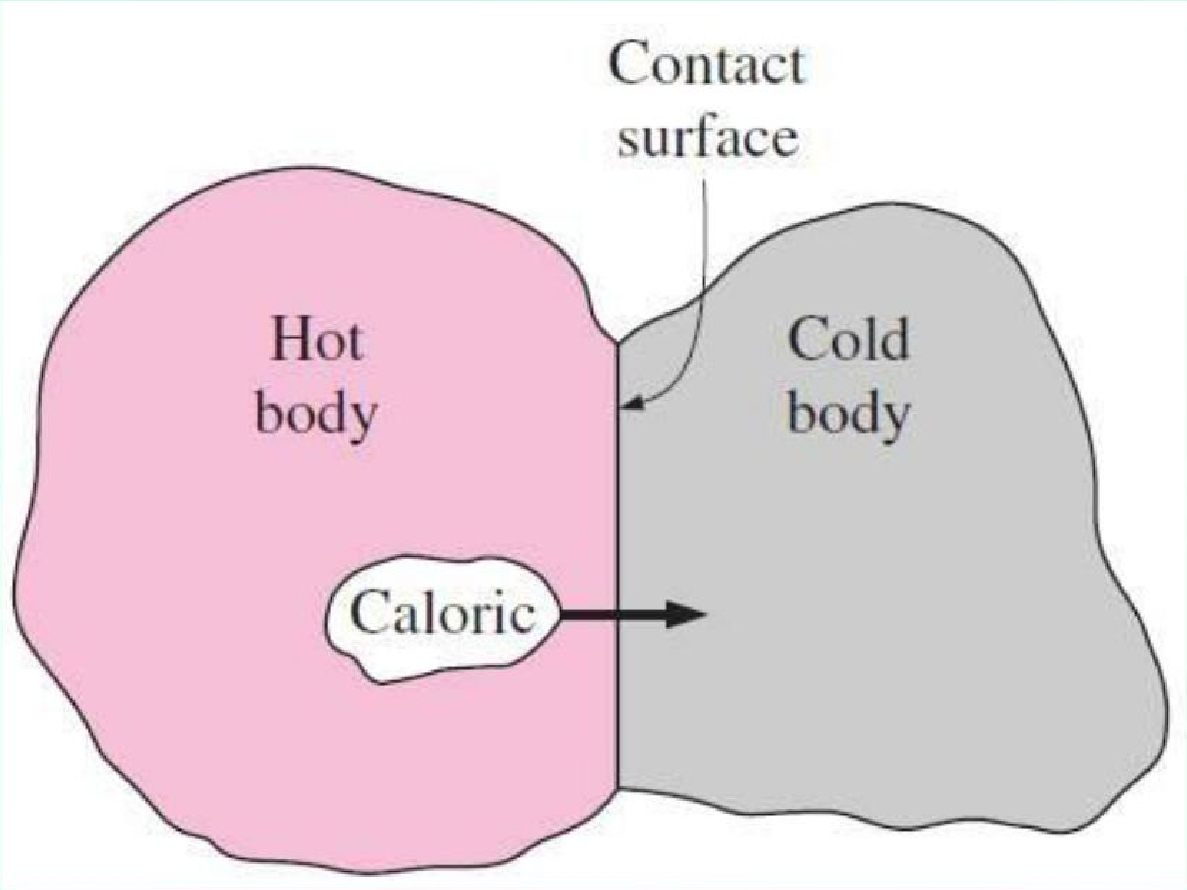
Stephan-Boltzmann law - Radiation

Heat Transfer - History

Caloric theory (18th Century)

Heat is a fluid like substance, '*caloric*' poured from one body into another.

Caloric: Massless, colorless, odorless, tasteless



Heat, Rate, Flux

Heat

The amount of heat transferred during a process, Q

Heat transfer rate

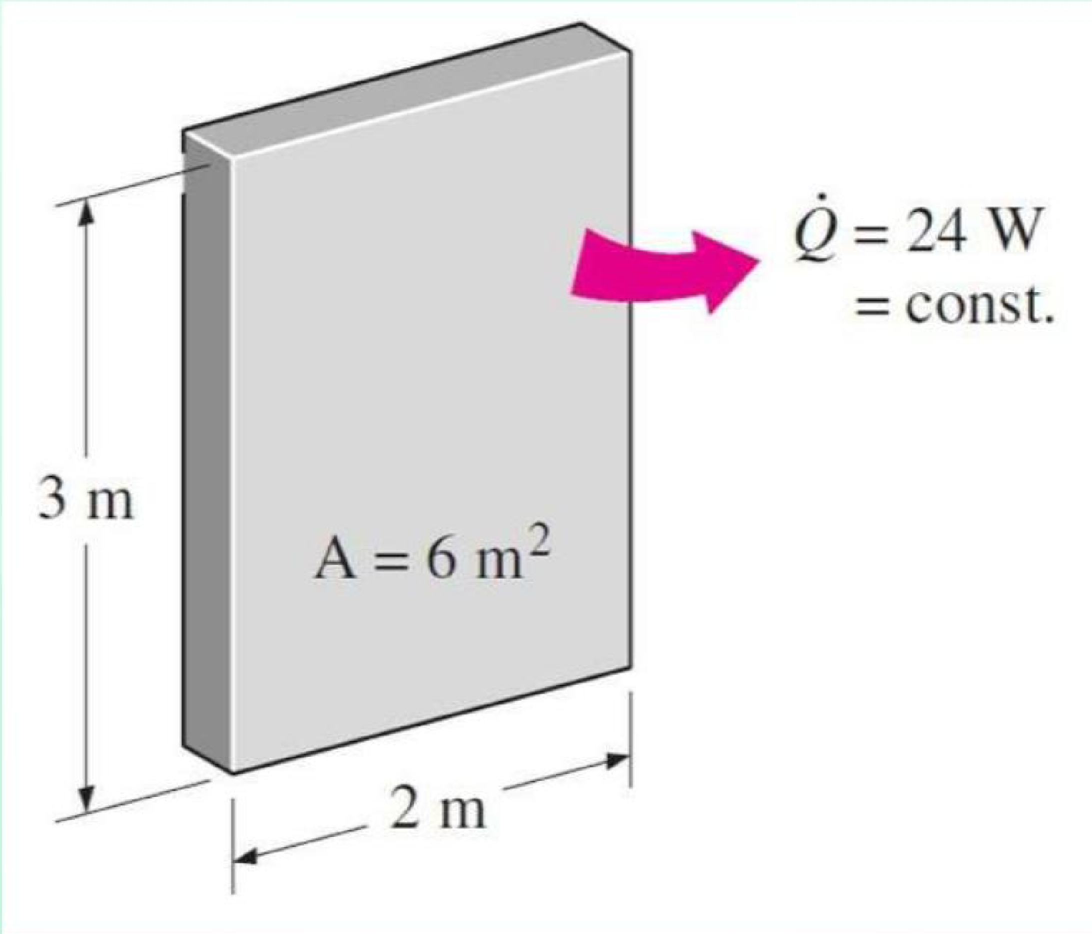
The amount of heat transferred per unit time, $\dot{Q}$

Heat flux

The rate of heat transfer per unit area normal to the direction of heat transfer:

$$q = \frac{\dot{Q}}{A}$$

Heat, Rate, Flux



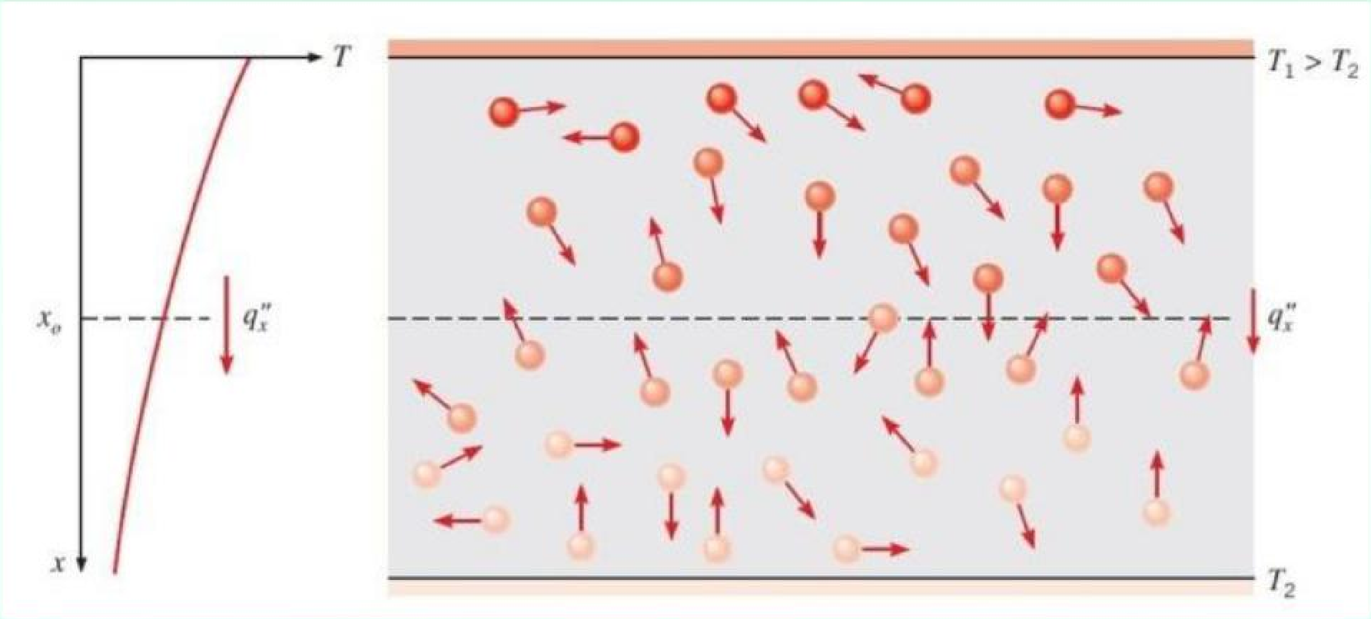
$$q = \frac{24\ \text{W}}{6\ \text{m}^2} = 4\ \text{W/m}^2$$

Conduction - Macroscopic View

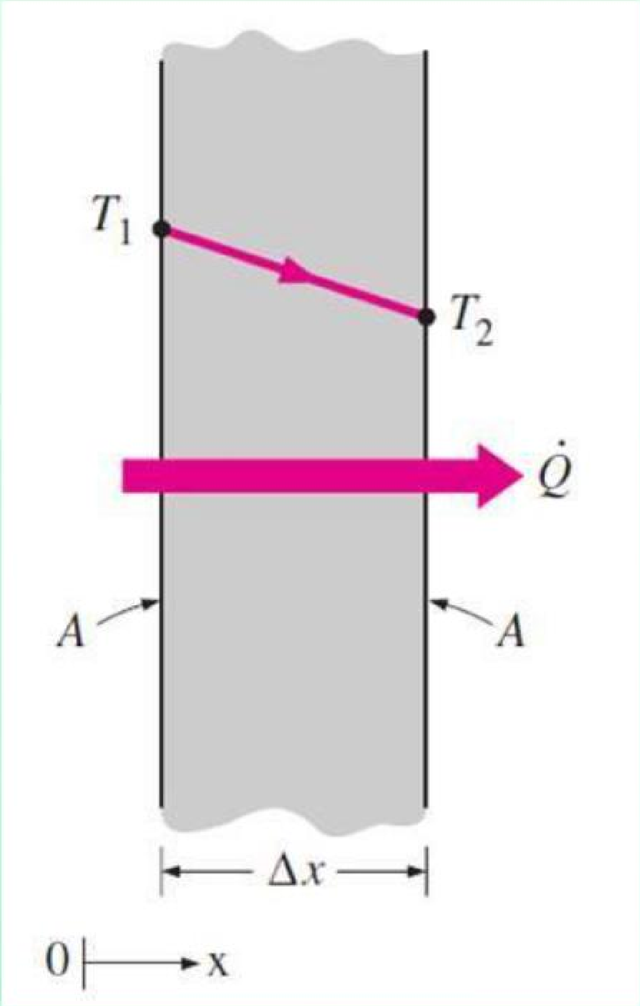
Viewed as

The transfer of energy from the more energetic to the less energetic particles of a substance due to interactions between the particles.

Net transfer by random molecules motion - *diffusion of energy*



Conduction: Fourier's Law of Heat Conduction



$$q_{cond} = -kA\frac{T_2 - T_1}{\Delta x} = -kA\frac{dT}{dx}$$

Problem: Conduction

The wall of an industrial furnace is constructed from 0.15 m thick fireclay brick having a thermal conductivity of 1.7 W/m K. Measurements made during steady-state operation reveal temperatures of 1400 and 1150 K at the inner and outer surfaces, respectively. What is the rate of heat loss through a wall that is $0.5 \times 1.2\ m^2$ on a side?

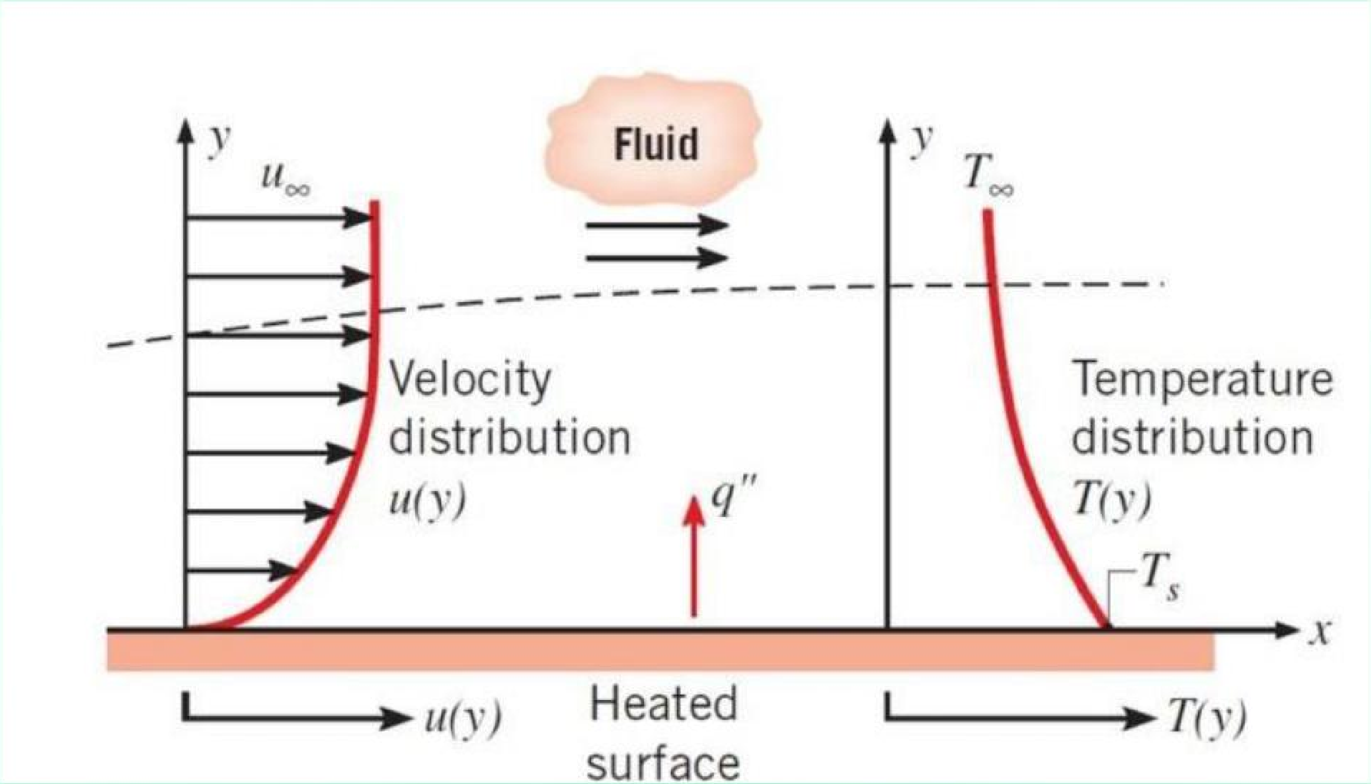
Ans: 1.7 kW

Convection

Comprised of two mechanisms

Energy transfer due to random molecular motion - *diffusion*

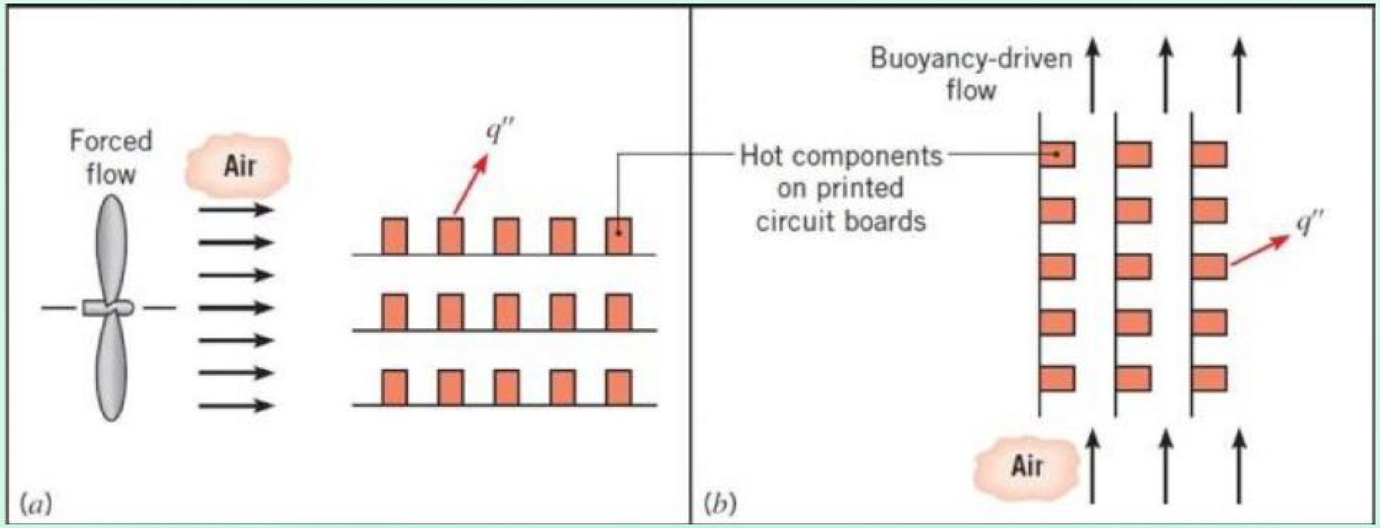
Energy transfer by the bulk motion of the fluid - *advection*



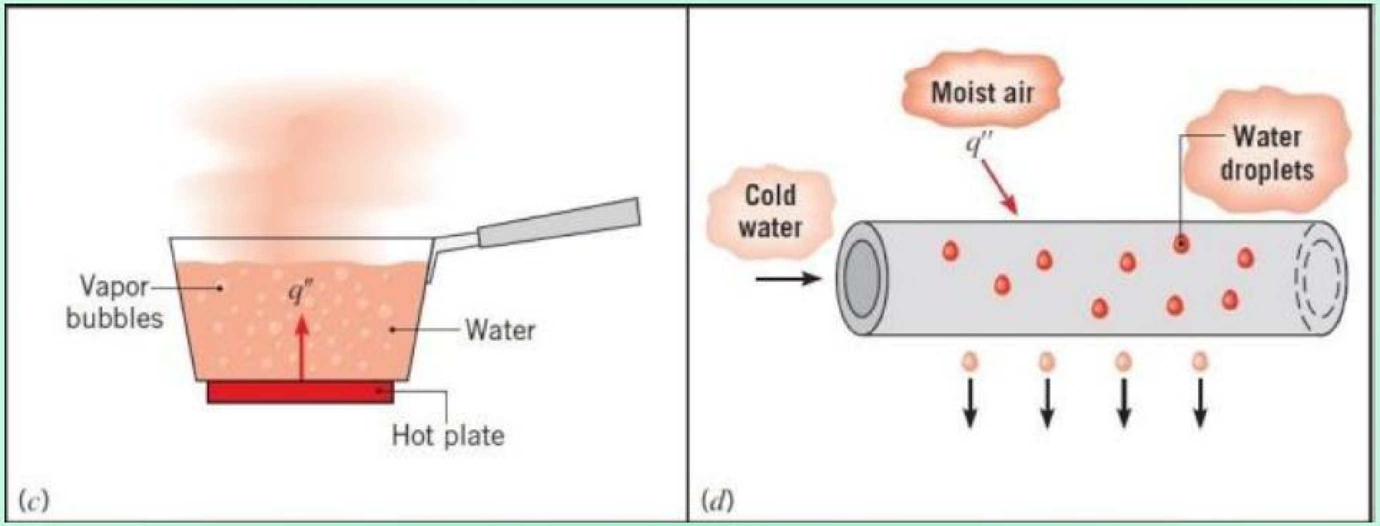
Boundary layer development in convection heat transfer

Convection - Classification

Forced and Free/Natural Convection



Boiling and Condensation



Convection: Newton's Law of Cooling

$$q_{conv} = hA_s(T_s - T_\infty)$$

Process	h (W/m^2 K)
Free convection Gases Liquids	 2-25 50-1000
Convection with phase change Boiling and Condensation	 2500-100,000

Thermal Radiation

Radiation

Energy emitted by matter that is at a nonzero temperature

Transported by electromagnetic waves (or photons)

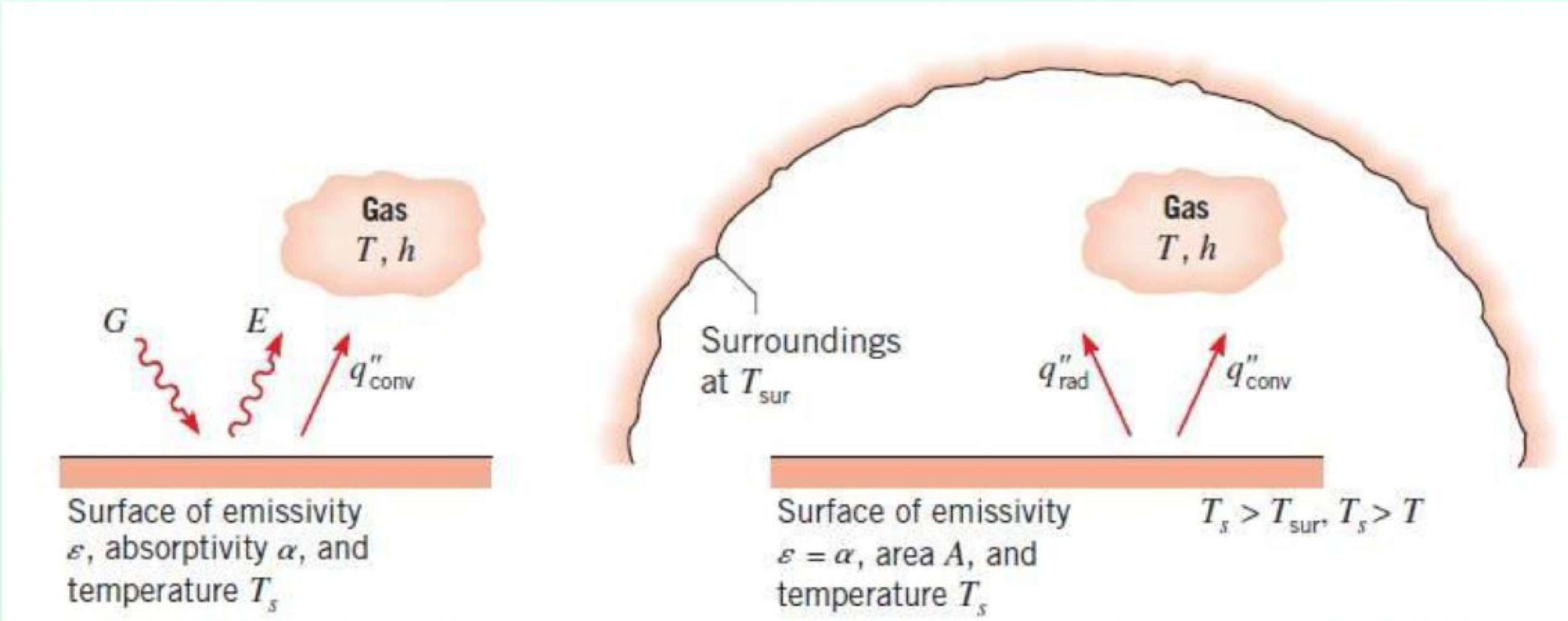
Medium?

Surface Emissive Power

The rate at which energy is released per unit area (W/m^2)

$$E_b = \sigma T_s^4$$

Radiation: Stefan-Boltzmann Law



For a real surface:

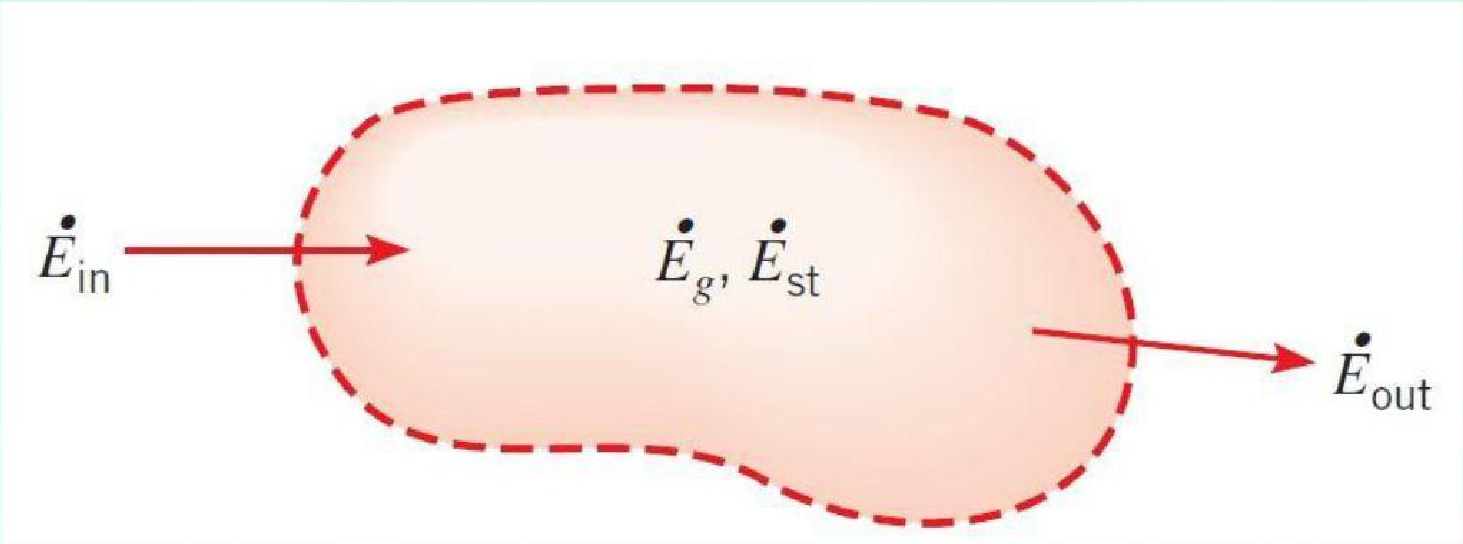
$$E = \varepsilon \sigma T_s^4$$

$$q''_{rad} = \varepsilon \sigma (T_s^4 - T_{sur}^4)$$

First Law of Thermodynamics

$$E_{in} - E_{out} = \Delta E_{st}$$

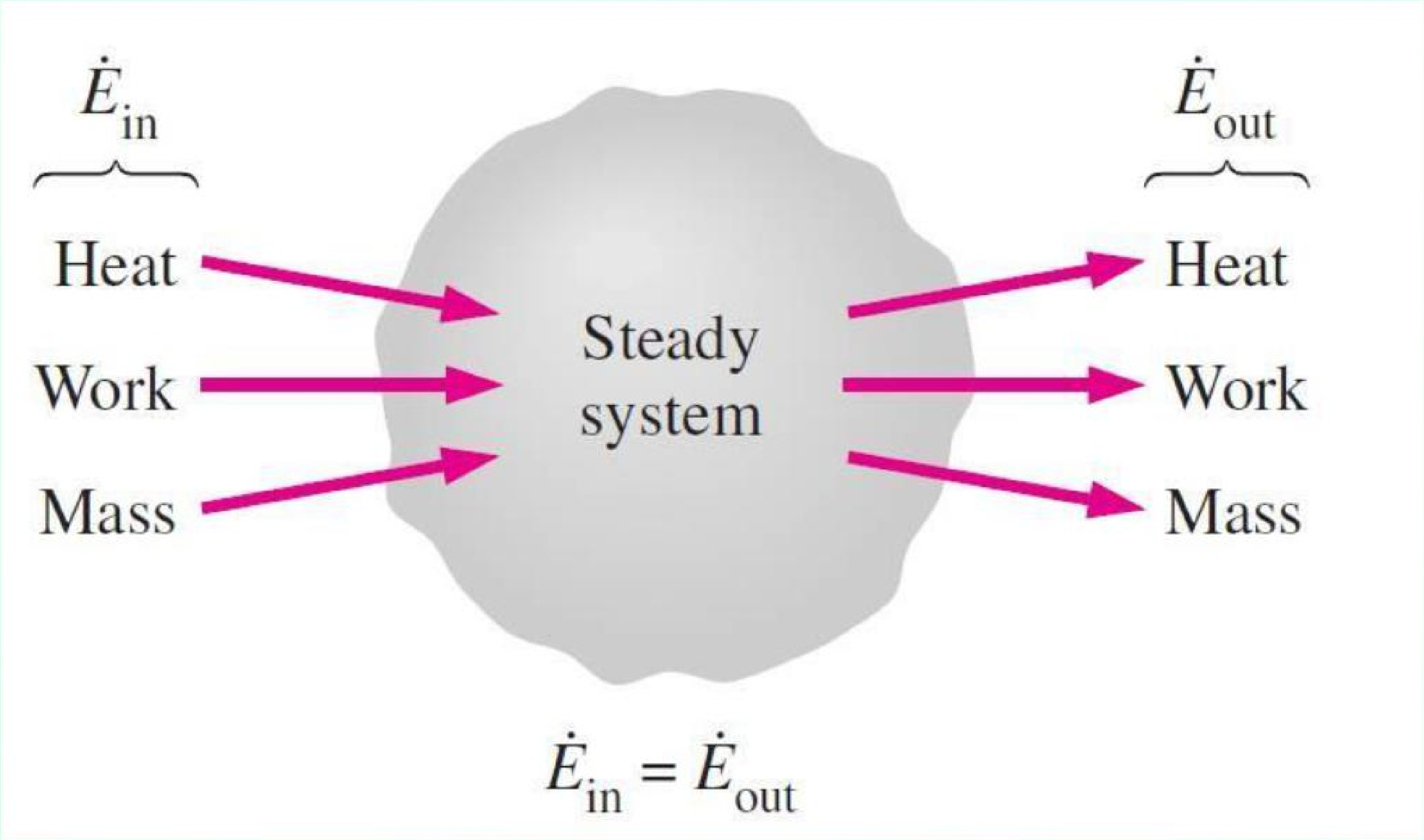
In rate form:



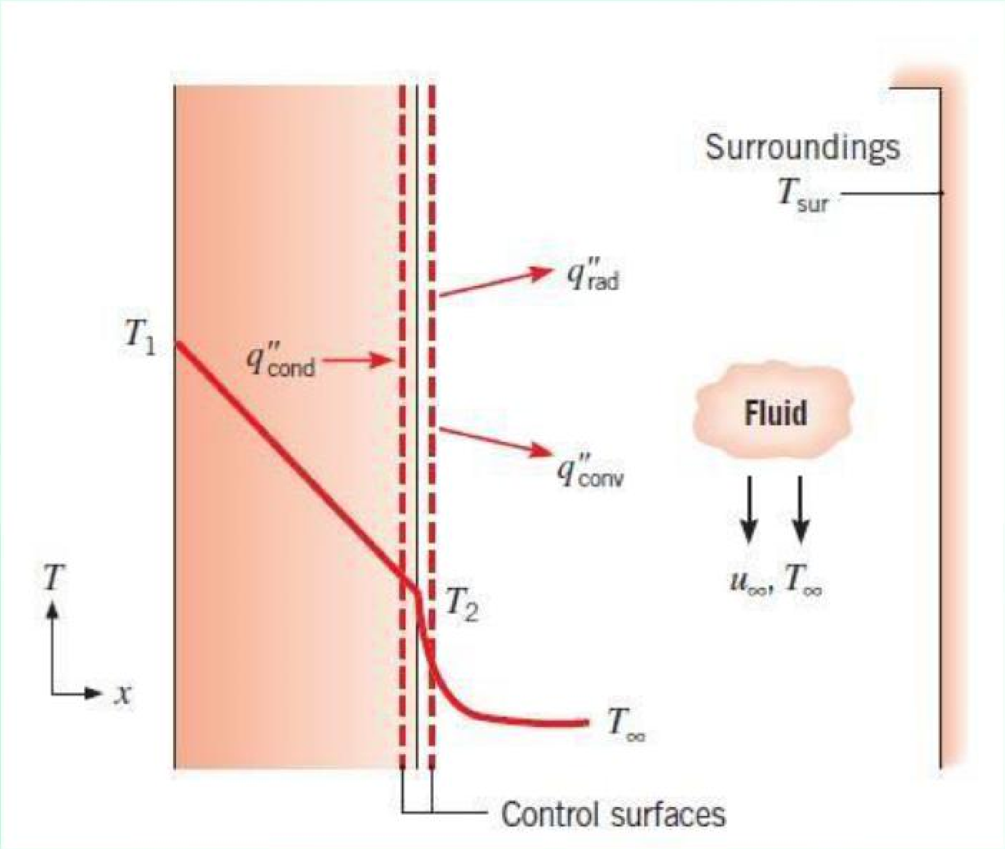
$$\dot{E}_{in} - \dot{E}_{out} = \frac{dE_{st}}{dt} = \dot{E}_{st}$$

First Law of Thermodynamics

Steady state with no heat generation



Surface Energy Balance



$$E_{in} - E_{out} = 0$$

$$q_{cond} - q_{conv} - q_{rad} = 0$$

Problem Solving: Methodology

Analysis of different problems will give a deeper appreciation for the fundamentals of the subject, and you will gain confidence in your ability to apply these fundamentals to the solution of engineering problems.

Be consistent in following these steps:

1. known
2. Find
3. Schematic
4. Assumptions
5. Properties
6. Analysis
7. Comments

Problem: Conduction

The hot combustion gases of a furnace are separated from the ambient air and its surrounding, which are at 25°C, by a brick wall 0.15 m thick. The brick has a thermal conductivity of 1.2 W/m K and a surface emissivity of 0.8. Under steady-state conditions an outer surface temperature of 100°C is measured. Free convection heat transfer to the air adjoining the surface is characterized by a convection coefficient of 20 W/m^2 K. What is the brick inner surface temperature.

Ans: 625 K

Problem: Convection

An experiment to determine the convection coefficient associated with airflow over the surface of a thick stainless steel casting involves the insertion of thermocouple into the casting at distances of 10 and 20 mm from the surface along a hypothetical line normal to the surface. The steel has a thermal conductivity of $15\,\mathrm{W/m\ K}$. If the thermocouples measure temperatures of 50 and $40^\circ\mathrm{C}$ in the steel when the air temperature is $100^\circ\mathrm{C}$, what is the convection coefficient?

Ans: $375\,\mathrm{W/m^2\,K}$

Problem: Radiation

The roof of a car in a parking lot absorbs a solar radiant flux of 800 W/m^2, and the underside is perfectly insulated. The convection coefficient between the roof and the ambient air is 12 W/m^2 K.

a) Neglecting radiation exchange with the surroundings, calculate the temperature of the roof under steady-state conditions if the ambient air temperature is 20°C.

b) For the same ambient air temperature, calculate the temperature of the roof if its surface emissivity is 0.8.

c) The convection coefficient depends on air flow conditions over the roof, increasing with increasing air speed. Compute and plot the roof temperature as a function of h for $2 \leq h \leq 200$ W/m^2 K.

Ans: 86.7°C

Steady-state vs Transient

Fourier's law of heat conduction

$$Q_{cond} = -kA\frac{dT}{dx}$$

transient

multidimensional - complex geometries

Steady-state heat transfer

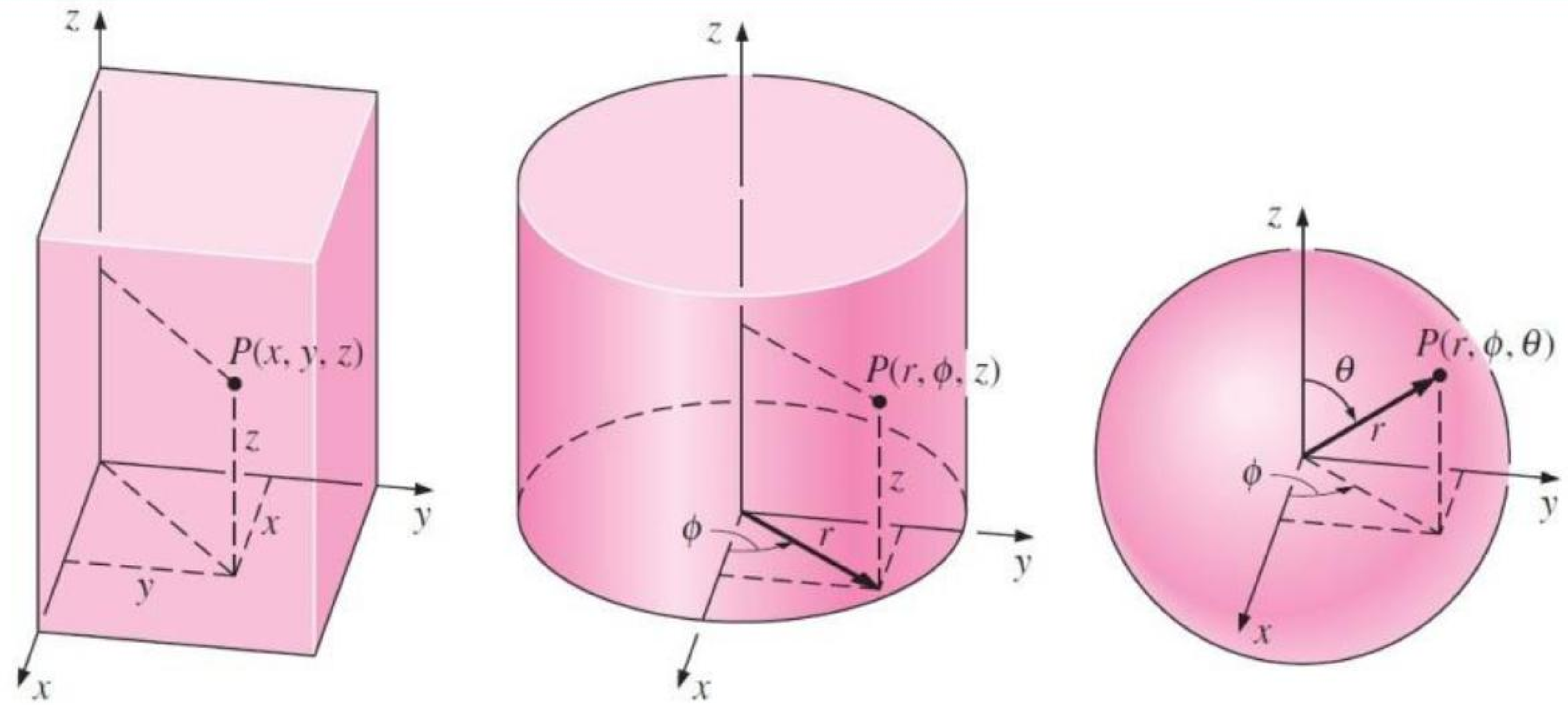
- No change with time at any point within the medium
- T and q'' remains unchanged with time
- $T = T(x, y, z)$
- Usually no but assumed

Transient heat transfer

- Time dependence
- $T = T(x, y, z, t)$
- Special case - *lumped* - T changes with time but not with location:

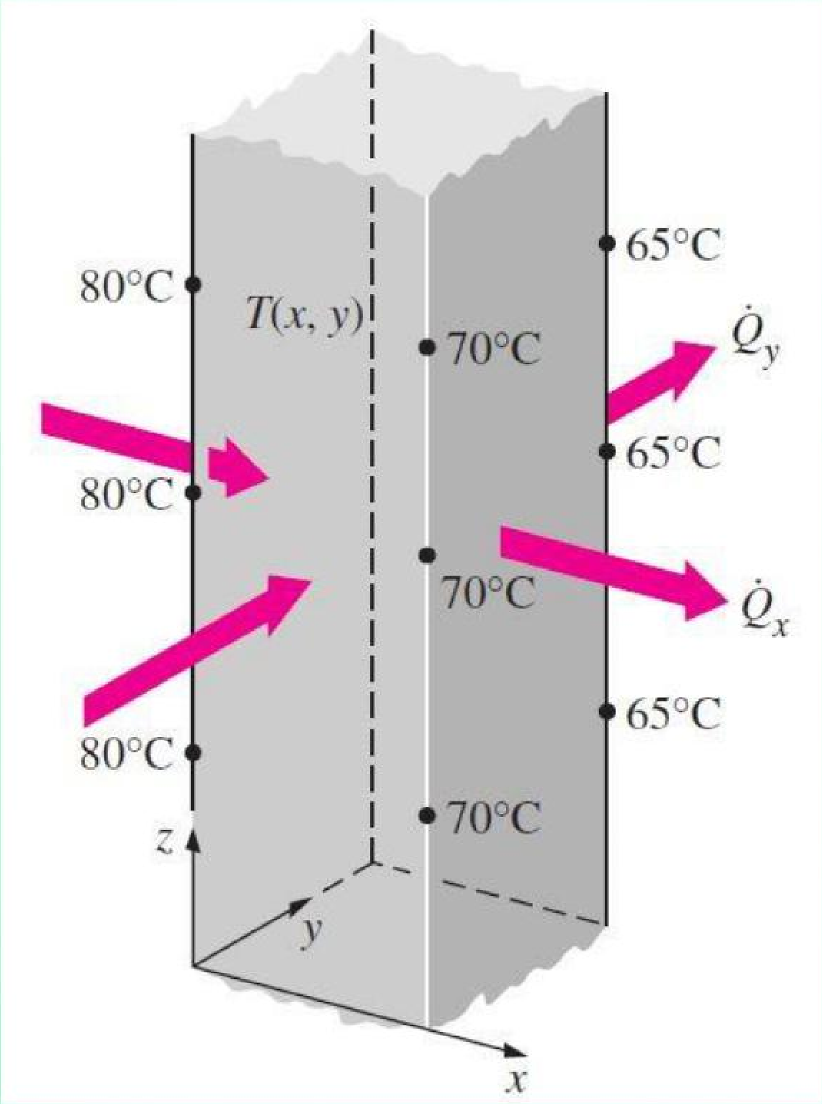
  $T = T(t)$

Coordinate System



(a) Rectangular coordinates (b) Cylindrical coordinates (c) Spherical coordinates

Multidimensional Heat Transfer



Thermal Conductivity

Thermal conductivity

$$k = \frac{q}{(\partial T/\partial x)}$$

The rate of heat transfer through a unit thickness of the material per unit area per unit temperature difference.

Specific heat, C_p

Ability to store thermal energy.
At room temperature,

$$\begin{aligned} C_p &= 4.18 \text{ kJ/kg K, water} \\ &= 0.45 \text{ kJ/kg K, iron} \end{aligned}$$

Thermal conductivity, k

Material's ability to conduct heat
At room temperature,

$$\begin{aligned} k &= 0.607 \text{ W/m K, water} \\ &= 80.2 \text{ W/m K, iron} \end{aligned}$$

Thermal Conductivity

- Transport property
- Indication of the rate at which energy is transferred by the diffusion process
- Depends on the physical structure of matter, atomic and molecular, related to the state of the matter
- *Isotropic* material - k is independent of the direction of transfer, $k_x = k_y = k_z$

  Laminated composite materials and wood

  k across grain is different than that parallel to grain

k for Different Materials at T_∞ and P_∞

Gases

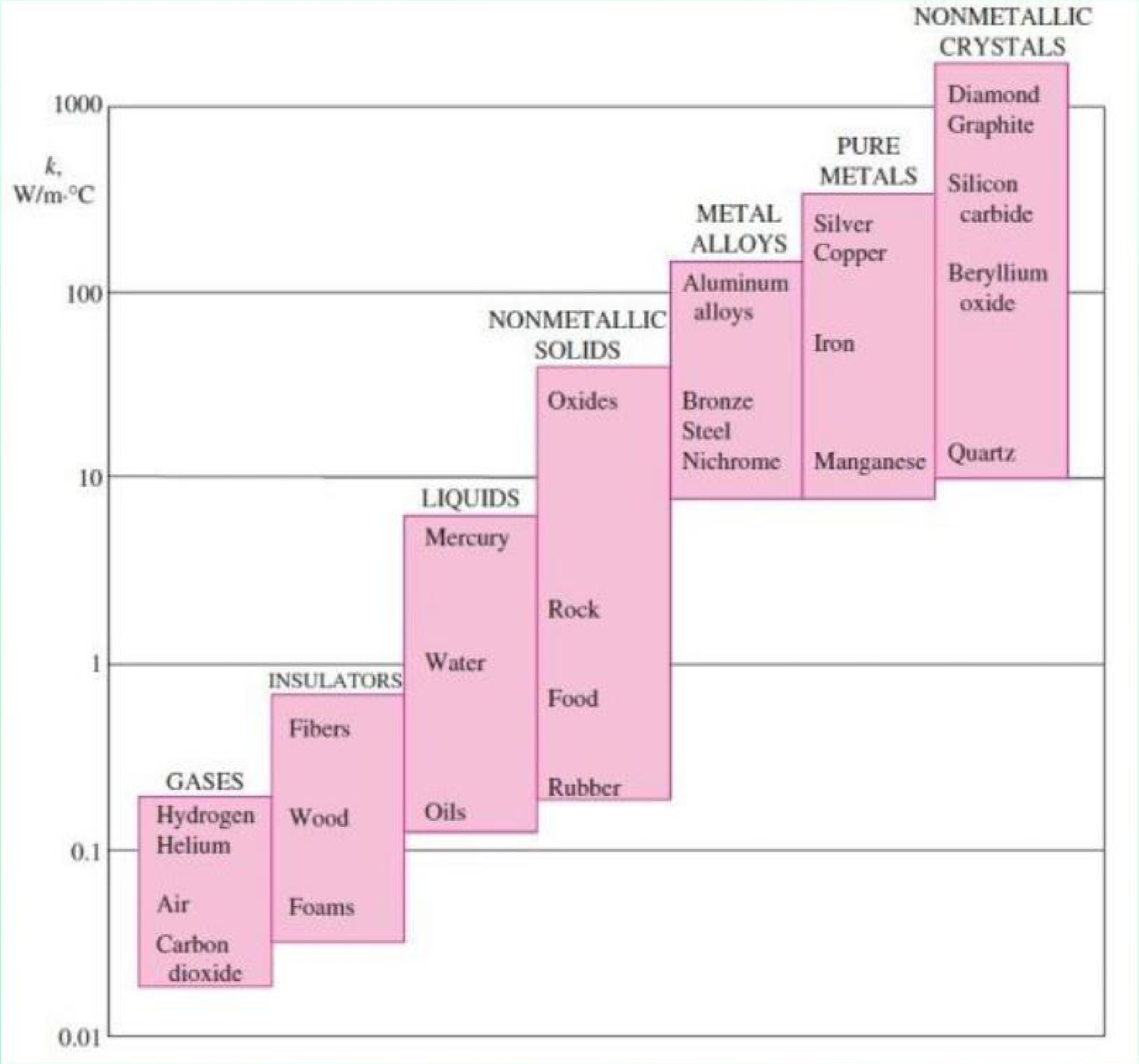
$T \uparrow \quad k \uparrow$

He(4), Air(29)

Liquids: Strong intermolecular forces

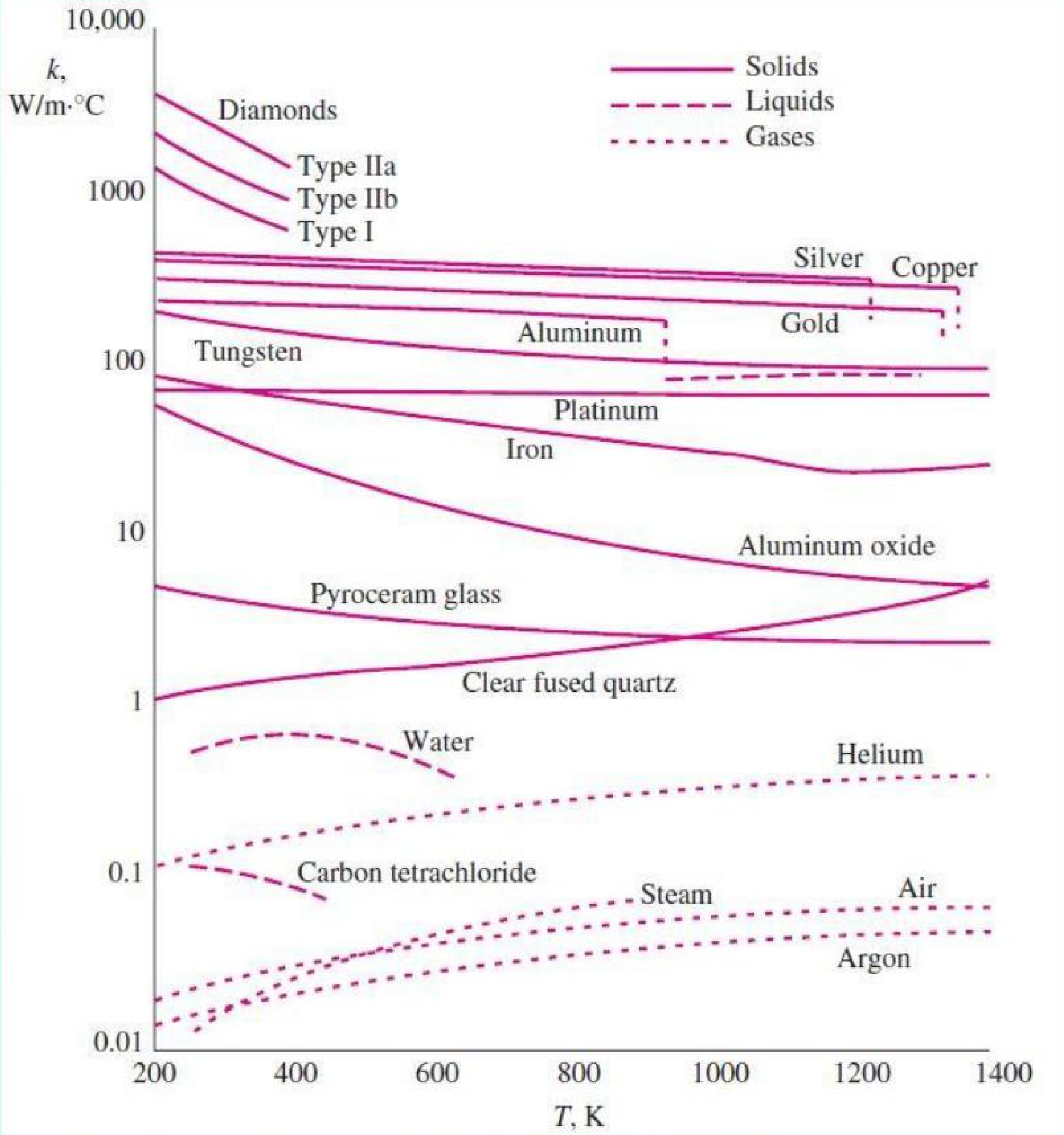
Most liquids: $T \uparrow \quad k \downarrow$

Except water: Not a linear trend



k - Temp. Dependency

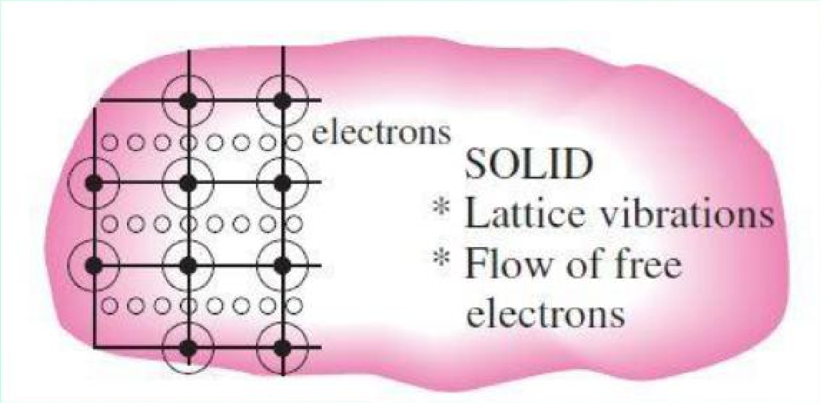
- Temp. dependency causes considerable complexity in conduction analysis
- $k_{average}$



k for Solids

$$k = k_l + k_e$$

High k for pure metals is due to k_e k_l depends on the way the molecules are arranged



Thermal Diffusivity α

- Thermophysical properties
  - k Transport property
  - $\rho,\ C_p$ Thermodynamic properties
- ρC_p is volumetric heat capacity (J/m^3 K)

- High $\boldsymbol{\alpha}$: faster propagation of heat into the medium
- Small $\boldsymbol{\alpha}$: heat is mostly absorbed by the material and a small amount of heat is conducted further

Thermal Diffusivity

c_p **Specific heat, J/kg · °C:** Heat capacity per unit mass

ρc_p **Heat capacity, J/m³·°C:** Heat capacity per unit volume

α **Thermal diffusivity, m²/s:** Represents how fast heat diffuses through a material

$$\alpha = \frac{\text{Heat conduction}}{\text{Heat storage}} = \frac{k}{\rho c_p} \qquad (\text{m}^2/\text{s})$$

A material that has a high thermal conductivity or a low heat capacity will obviously have a large thermal diffusivity.

The larger the thermal diffusivity, the faster the propagation of heat into the medium.

A small value of thermal diffusivity means that heat is mostly absorbed by the material and a small amount of heat is conducted further.

The thermal diffusivities of some materials at room temperature

Material	α, m²/s*
Silver	149×10^{-6}
Gold	127×10^{-6}
Copper	113×10^{-6}
Aluminum	97.5×10^{-6}
Iron	22.8×10^{-6}
Mercury (l)	4.7×10^{-6}
Marble	1.2×10^{-6}
Ice	1.2×10^{-6}
Concrete	0.75×10^{-6}
Brick	0.52×10^{-6}
Heavy soil (dry)	0.52×10^{-6}
Glass	0.34×10^{-6}
Glass wool	0.23×10^{-6}
Water (l)	0.14×10^{-6}
Beef	0.14×10^{-6}
Wood (oak)	0.13×10^{-6}

Heat Diffusion Equation: Application

Problem and application

- Determine *temperature distribution* in a medium resulting from conditions imposed on its boundaries
- The conduction heat flux at any point in the medium or on the surface may be computed from Fourier's law
- This information could be used to determine thermal stresses, expansions, deflections
- Temperature distribution may also be used to optimize the thickness of an insulating material or to determine the compatibility of special coatings or adhesives used with the material

Control Volume

Assumptions

Homogeneous medium

No bulk motion (advection)

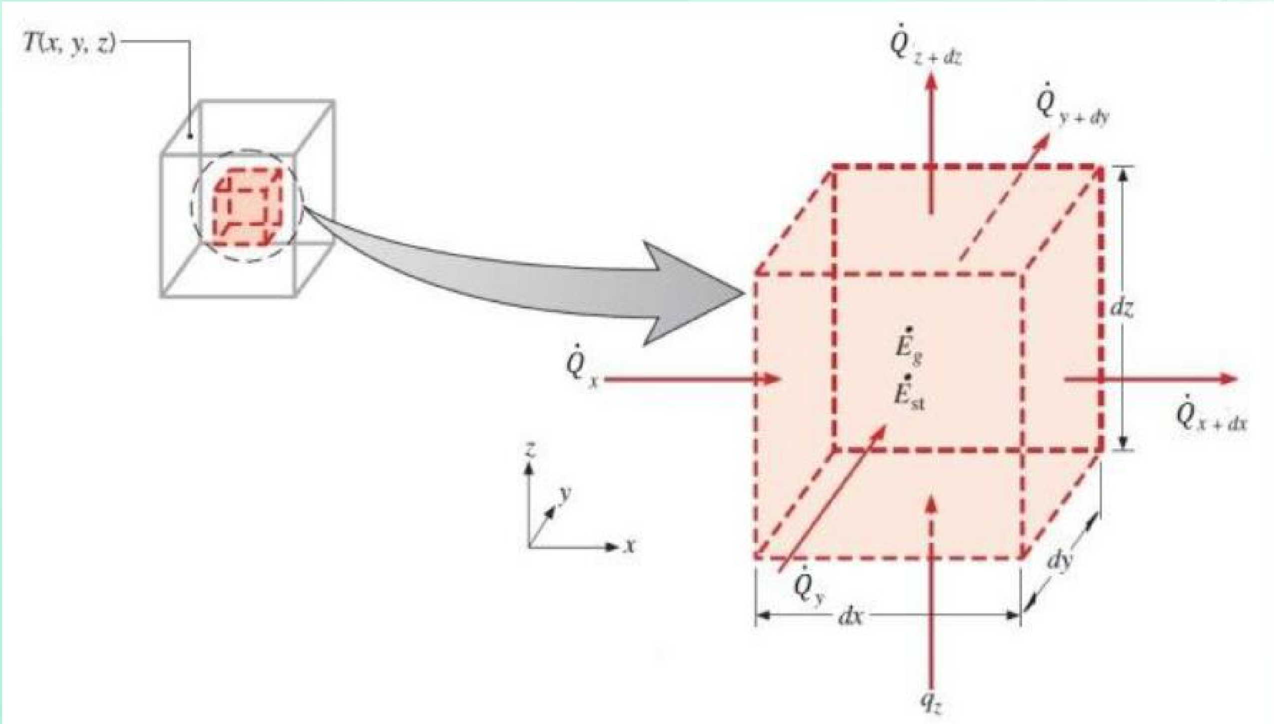
Schematic

Consider an infinitesimally small (differential) CV, $dx.dy.dz$

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$



Volumetric Properties

Generation

$$\dot{Q}_g = \dot{q}_g\, dxdydz$$

$\dot{q}_g$ is in W/m^3

Storage

$$Q_{st} = \underbrace{\rho C_p \frac{\partial T}{\partial t}}\, dxdydz$$

Rate of change of the sensible/thermal energy of the medium/volume

Heat Diffusion Equation

Governing Equation

$$Q_{in} - Q_{out} + Q_g = Q_{st}$$

$$q_x + q_y + q_z - q_{x+dx} - q_{y+dy} - d_{z+dz} + \dot{q}_g dxdydz = \rho C_p \frac{\partial T}{\partial t} dxdydz$$

$$-\frac{\partial q_x}{\partial x} dx - \frac{\partial q_y}{\partial y} dy - \frac{\partial q_z}{\partial z} dz + \dot{q}_g dxdydz = \rho C_p \frac{\partial T}{\partial t} dxdydz$$

However,

$$q_x = -kdydz\frac{\partial T}{\partial x}; \quad q_y = -kdxdz\frac{\partial T}{\partial y}; \quad q_z = -kdxdy\frac{\partial T}{\partial z}$$

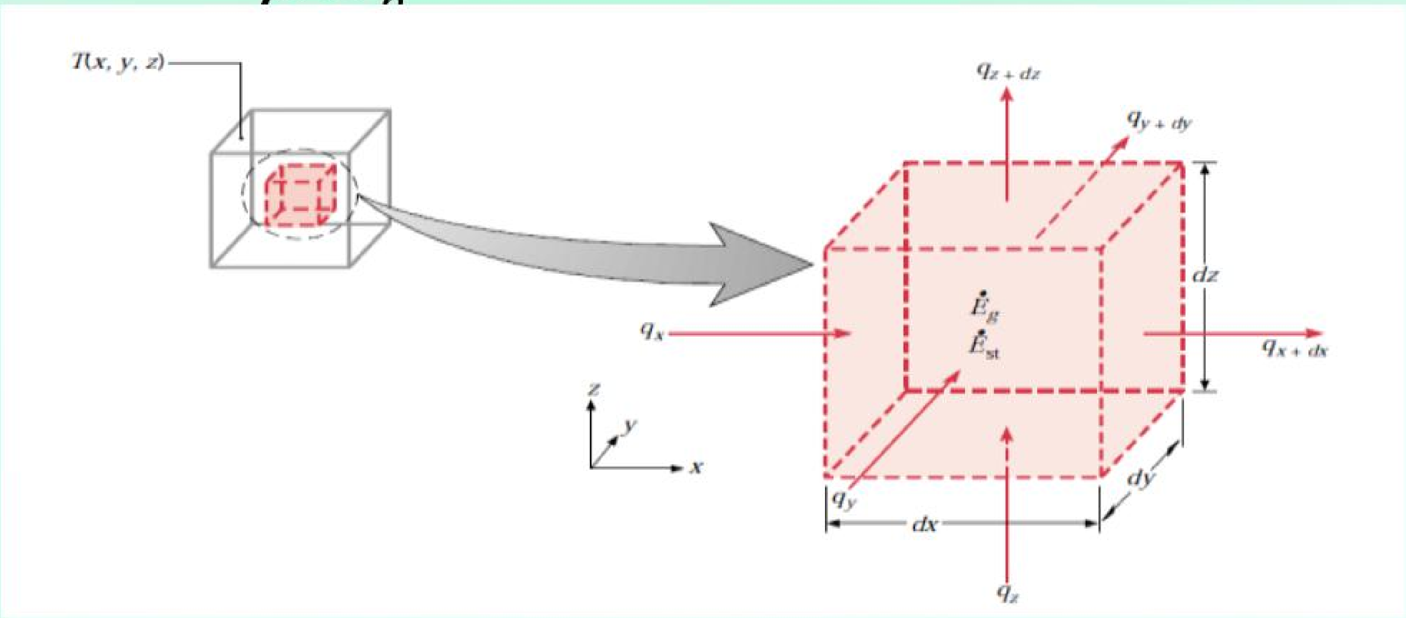
$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}_g = \rho C_p \frac{\partial T}{\partial t}$$

GENERAL HEAT CONDUCTION EQUATION

Cartesian Coordinate System

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q}{k} = \frac{1}{\alpha}\frac{dT}{dt}$$

$$where\ \alpha = \frac{k}{\rho C_p}\ is\ called\ as\ thermal\ diffusivity$$



Special Cases

Fourier-Biot equation - Isotropic

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}_g}{k} = \frac{1}{a}\frac{\partial T}{\partial t}$$

Diffusion equation - Transient, no heat generation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}$$

Poisson equation - Steady-state

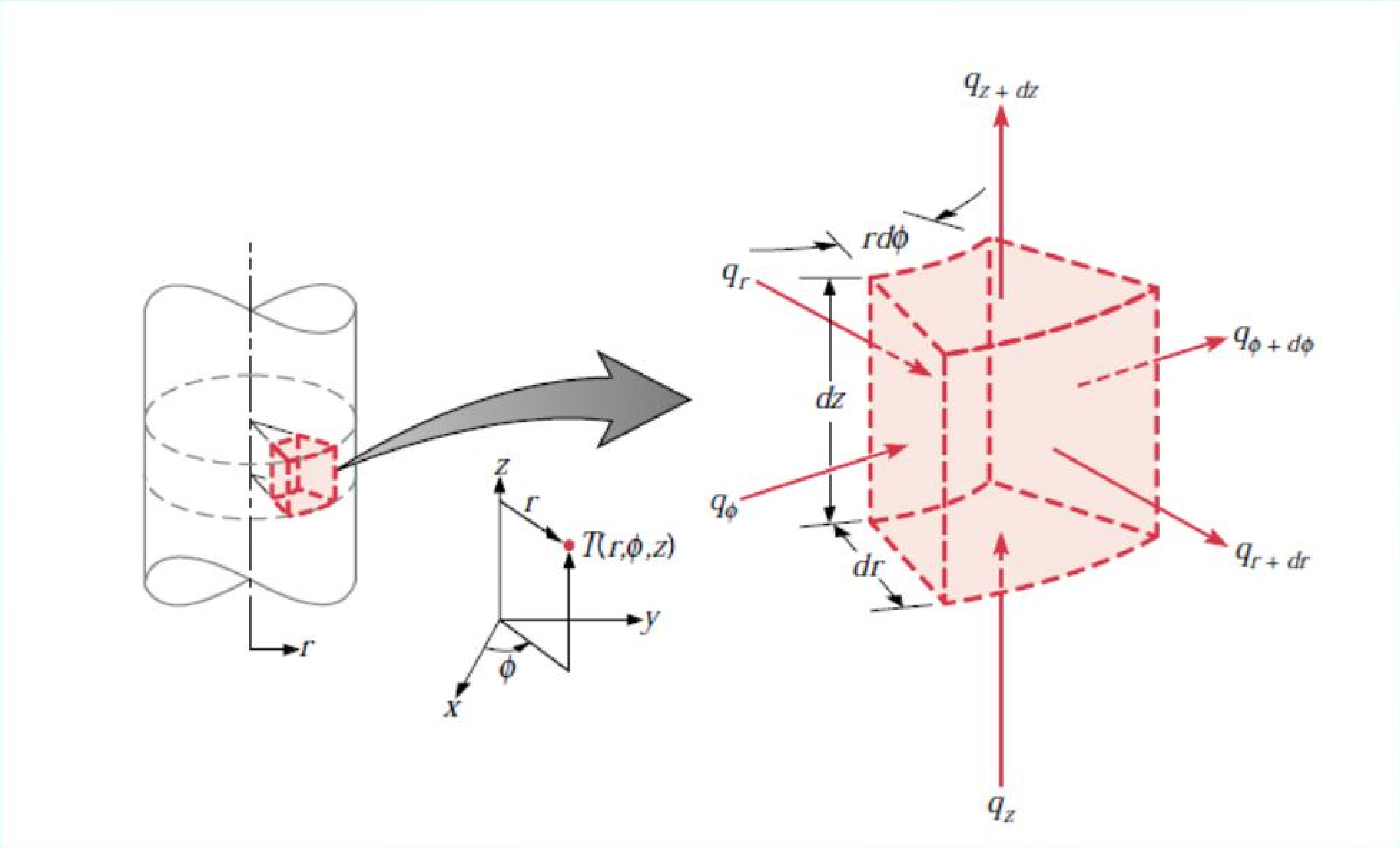
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}_g}{k} = 0$$

Laplace equation - Steady-state, no heat generation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} = 0$$

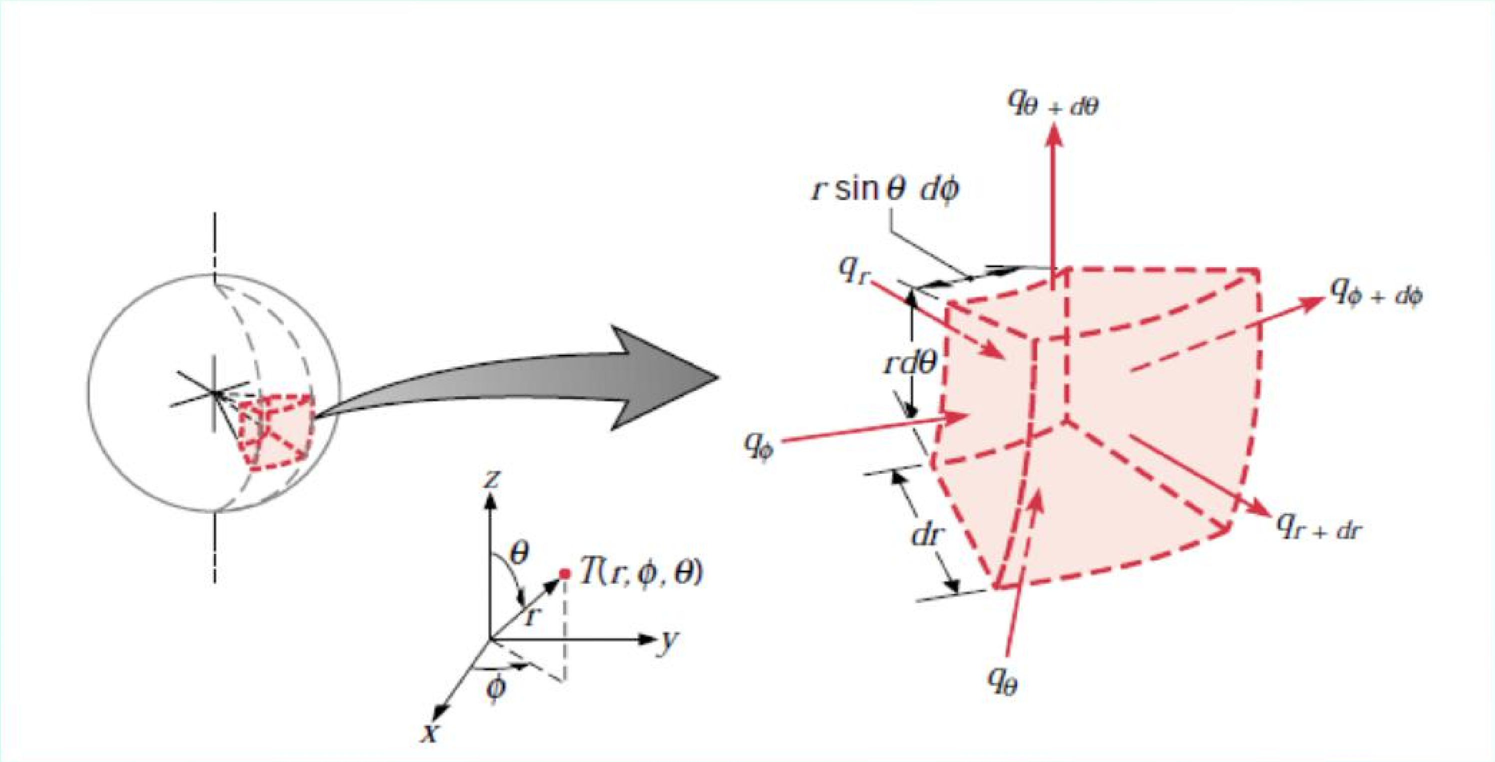
Cylindrical Coordinates

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q}{k}=\frac{1}{\alpha}\frac{dT}{dt}$$



Spherical Coordinates

$$\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial T}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}+\frac{q}{k}=\frac{1}{\alpha}\frac{dT}{dt}$$

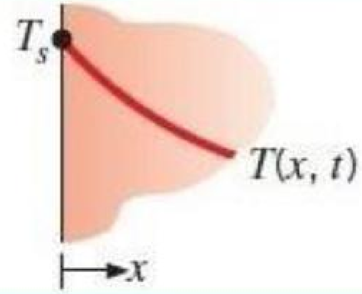


Boundary Conditions at $x = 0$

1. Constant surface temperature

$$T(0, t) = T_s$$

Dirichlet Condition
***BC* of first kind**

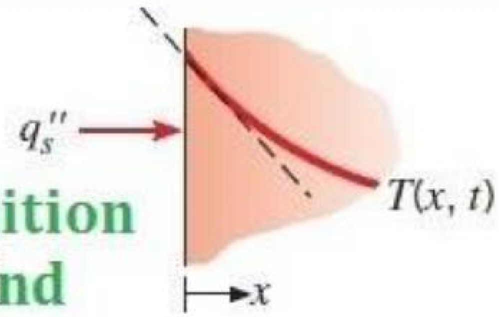


2. Constant surface heat flux

(a) Finite heat flux

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q_s''$$

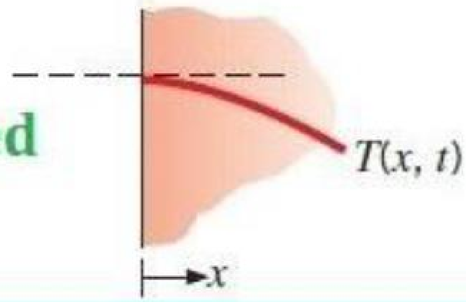
Neumann Condition
***BC* of second kind**



(b) Adiabatic or insulated surface

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0$$

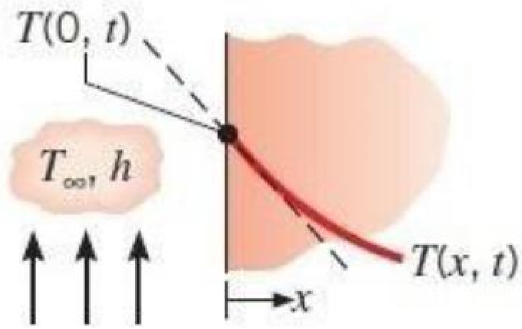
Perfectly insulated
Adiabatic



3. Convection surface condition

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0, t)]$$

BC of third kind



1 Specified Temperature Boundary Condition

The *temperature* of an exposed surface can usually be measured directly and easily.

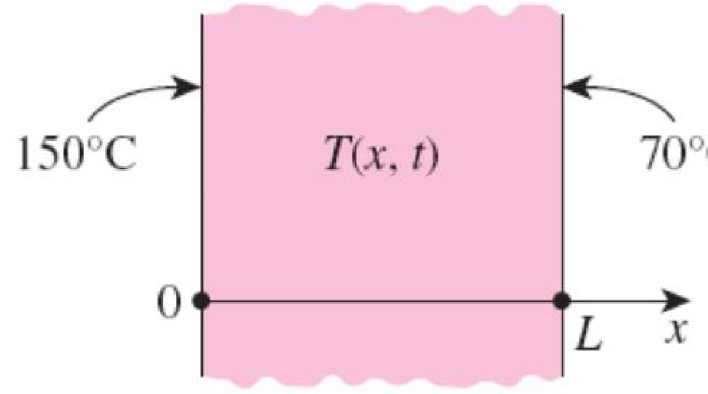
Therefore, one of the easiest ways to specify the thermal conditions on a surface is to specify the temperature.

For one-dimensional heat transfer through a plane wall of thickness *L*, for example, the specified temperature boundary conditions can be expressed as

$$T(0, t) = T_1$$
$$T(L, t) = T_2$$

where T_1 and T_2 are the specified temperatures at surfaces at $x = 0$ and $x = L$, respectively.

The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.



$T(0, t) = 150°C$

$T(L, t) = 70°C$

FIGURE 2–27

Specified temperature boundary conditions on both surfaces of a plane wall.

2 Specified Heat Flux Boundary Condition

The heat flux in the positive *x*-direction anywhere in the medium, including the boundaries, can be expressed by

$$\dot{q} = -k\frac{\partial T}{\partial x} = \begin{pmatrix} \text{Heat flux in the} \\ \text{positive } x - \text{direction} \end{pmatrix} \qquad (\text{W/m}^2)$$

For a plate of thickness *L* subjected to heat flux of 50 W/m² into the medium from both sides, for example, the specified heat flux boundary conditions

$$-k\frac{\partial T(0, t)}{\partial x} = 50 \qquad \text{and} \qquad -k\frac{\partial T(L, t)}{\partial x} = -50$$

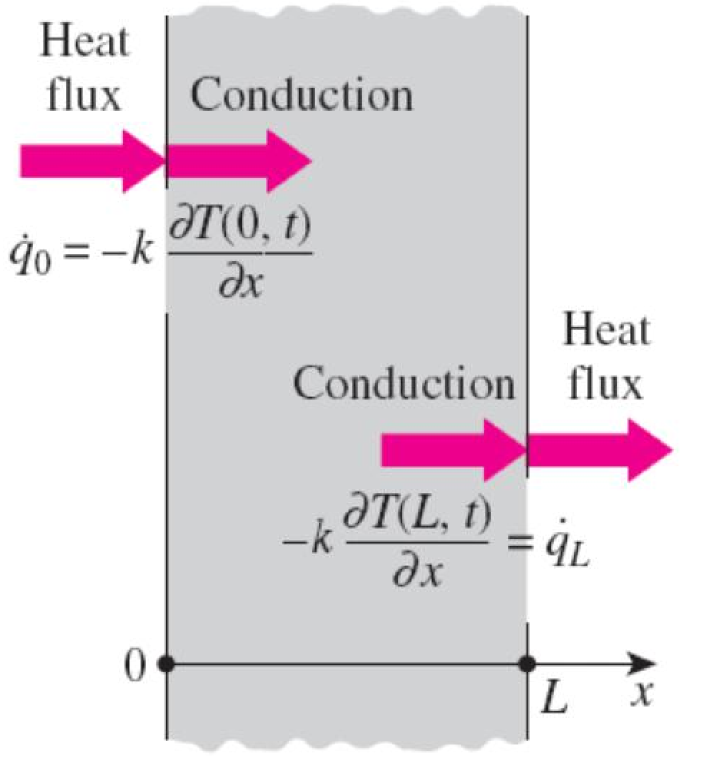


FIGURE 2–28

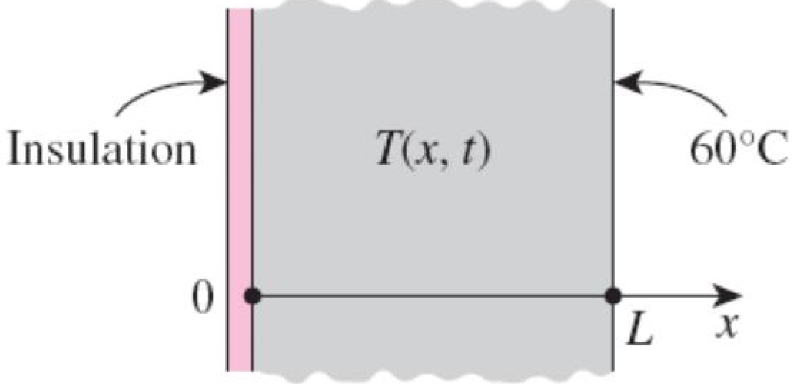
Specified heat flux boundary conditions on both surfaces of a plane wall.

Special Case: Insulated Boundary

A well-insulated surface can be modeled as a surface with a specified heat flux of zero. Then the boundary condition on a perfectly insulated surface (at x = 0, for example) can be expressed as

$$k\frac{\partial T(0,t)}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial T(0,t)}{\partial x} = 0$$

On an insulated surface, the first derivative of temperature with respect to the space variable (the temperature gradient) in the direction normal to the insulated surface is zero.



$$\frac{\partial T(0,t)}{\partial x} = 0$$

$$T(L,t) = 60°C$$

FIGURE 2–29

A plane wall with insulation and specified temperature boundary conditions.

Another Special Case: Thermal Symmetry

Some heat transfer problems possess *thermal symmetry* as a result of the symmetry in imposed thermal conditions.

For example, the two surfaces of a large hot plate of thickness *L* suspended vertically in air is subjected to the same thermal conditions, and thus the temperature distribution in one half of the plate is the same as that in the other half.

That is, the heat transfer problem in this plate possesses thermal symmetry about the center plane at $x = L/2$.

Therefore, the center plane can be viewed as an insulated surface, and the thermal condition at this plane of symmetry can be expressed as

$$\frac{\partial T(L/2, t)}{\partial x} = 0$$

which resembles the *insulation* or *zero heat flux* boundary condition.

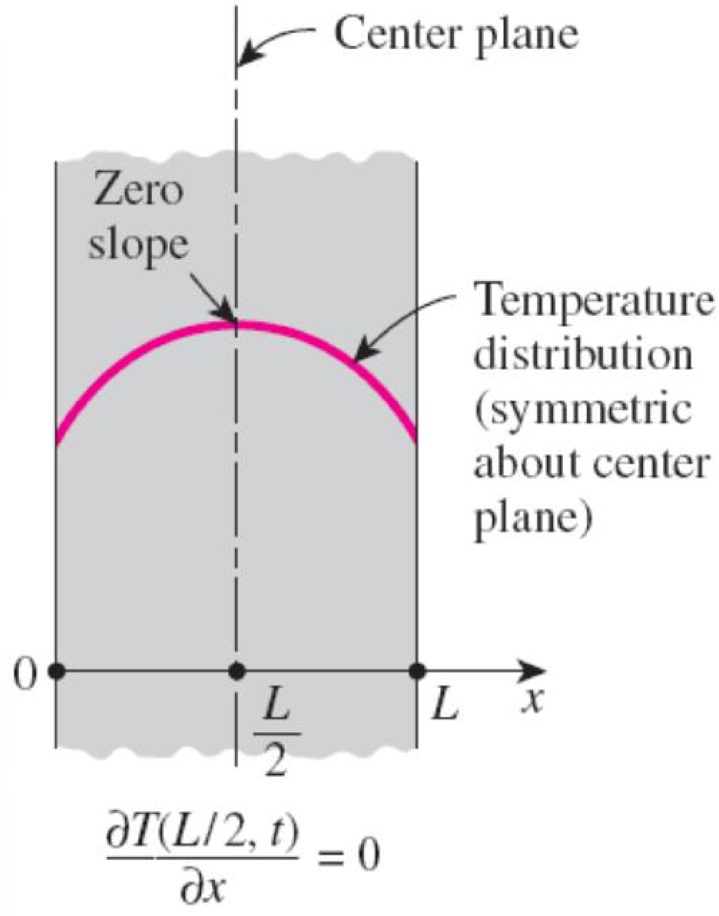


FIGURE 2–30

Thermal symmetry boundary condition at the center plane of a plane wall.

3 Convection Boundary Condition

For one-dimensional heat transfer in the *x*-direction in a plate of thickness *L*, the convection boundary conditions on both surfaces:

$$\begin{pmatrix}\text{Heat conduction}\\ \text{at the surface in a}\\ \text{selected direction}\end{pmatrix} = \begin{pmatrix}\text{Heat convection}\\ \text{at the surface in}\\ \text{the same direction}\end{pmatrix}$$

$$-k\frac{\partial T(0,t)}{\partial x} = h_1[T_{\infty 1} - T(0,t)]$$

$$-k\frac{\partial T(L,t)}{\partial x} = h_2[T(L,t) - T_{\infty 2}]$$

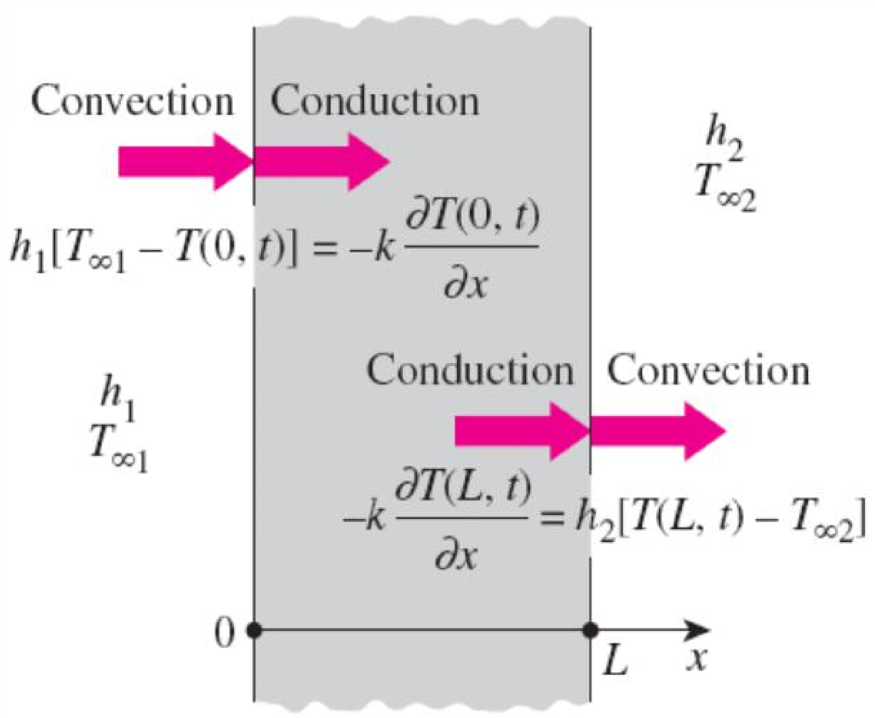


FIGURE 2–32

Convection boundary conditions on the two surfaces of a plane wall.

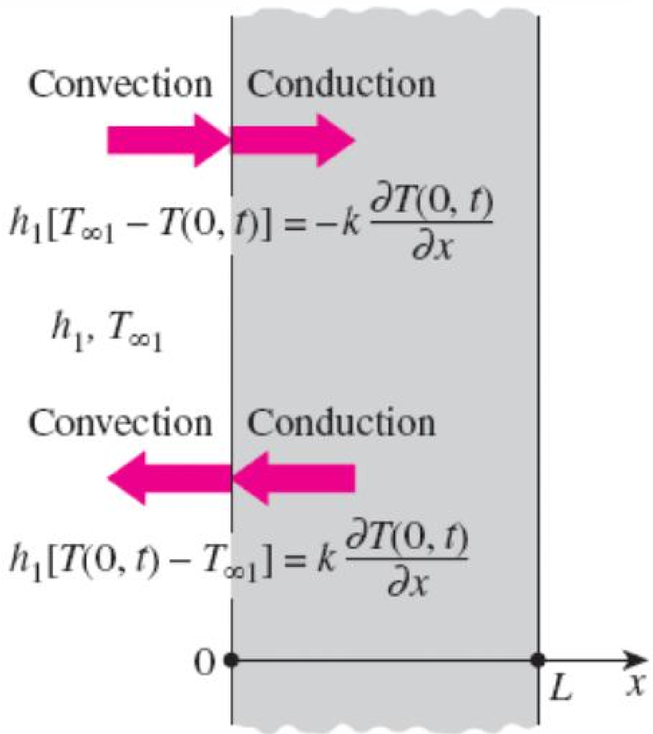


FIGURE 2–33

The assumed direction of heat transfer at a boundary has no effect on the boundary condition expression.

4 Interface Boundary Conditions

The boundary conditions at an interface are based on the requirements that

(1) two bodies in contact must have the *same temperature* at the area of contact and

(2) an interface (which is a surface) cannot store any energy, and thus the *heat flux* on the two sides of an interface *must be the same.*

The boundary conditions at the interface of two bodies *A* and *B* in perfect contact at $x = x_0$ can be expressed as

$$T_A(x_0, t) = T_B(x_0, t)$$

$$-k_A \frac{\partial T_A(x_0, t)}{\partial x} = -k_B \frac{\partial T_B(x_0, t)}{\partial x}$$

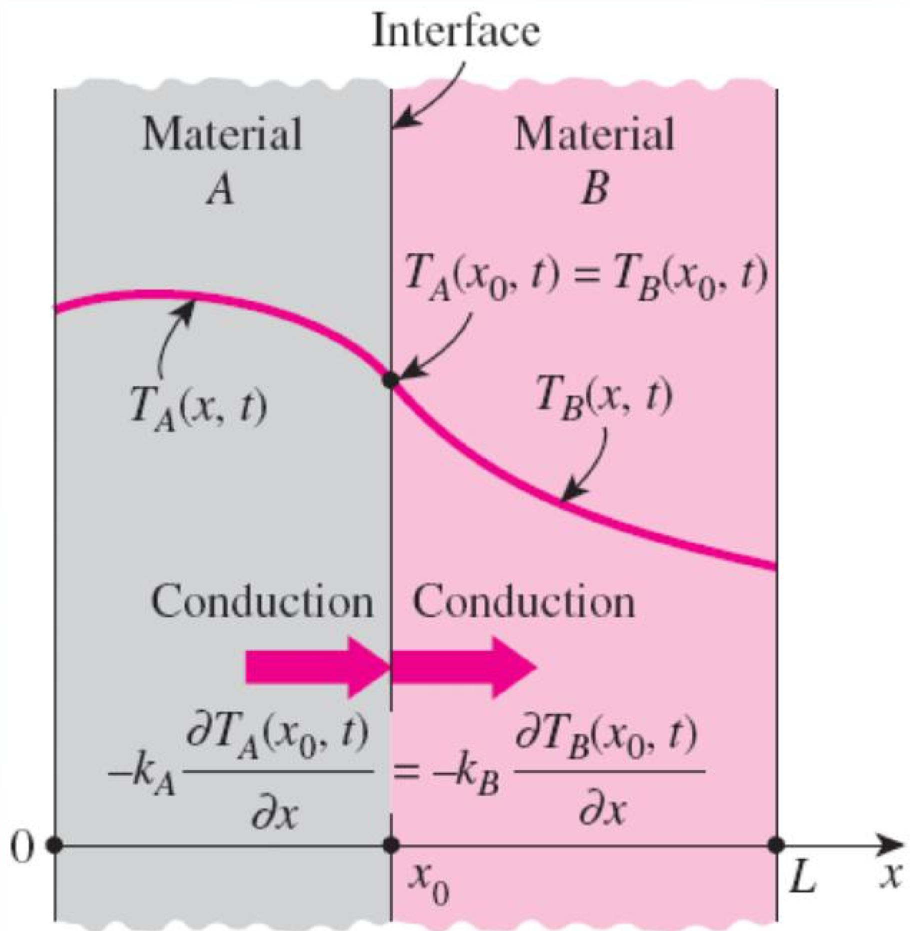
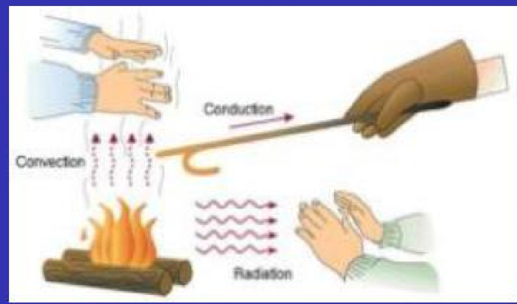


FIGURE 2–36

Boundary conditions at the interface of two bodies in perfect contact.

Heat Transfer



One-Dimensional, Steady-State Conduction

One-Dimensional, Steady-State

- Temp. gradients exist along only a single coordinate direction
- Heat transfer occurs exclusively in that direction
- Temp. at each point is independent of time

We will see:

- Temp. distribution & heat transfer rate in common (planar, cylindrical and spherical) geometries
- Thermal resistance
  - Thermal circuits to model heat flow
  - Electrical circuits to current flow

One Dimensional conduction equation in Cartesian Coordinate system

Case:1 Heat transfer without internal heat generation and steady state process

$$\frac{\partial^2 T}{\partial x^2} = 0 \qquad \frac{d^2 T}{dx^2} = 0$$

Case:2 Heat transfer with internal heat generation and steady state process

$$\frac{\partial^2 T}{\partial x^2} + \frac{q}{k} = 0$$

Case:3 Heat transfer with internal heat generation and unsteady state process

$$\frac{\partial^2 T}{\partial x^2} + \frac{q}{k} = \frac{1}{\alpha}\frac{dT}{dt}$$

Case:4 Heat transfer without internal heat generation and unsteady state process

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{dT}{dt}$$

One Dimensional conduction equation in Cylindrical Coordinate system

Case:1 Heat transfer without internal heat generation and steady state process

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}=0 \qquad r\frac{\partial^2 T}{\partial r^2}+\frac{\partial T}{\partial r}=0 \qquad \frac{d}{dr}\left(r\frac{\partial T}{\partial r}\right)=0$$

Case:2 Heat transfer with internal heat generation and steady state process

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{q}{k}=0 \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{q}{k}=0$$

Case:3 Heat transfer with internal heat generation and unsteady state process

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{q}{k}=\frac{1}{\alpha}\frac{dT}{dt}$$

Case:4 Heat transfer without internal heat generation and unsteady state process

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{dT}{dt}$$

One Dimensional conduction equation in Spherical Coordinate system

Case:1 Heat transfer without internal heat generation and steady state process

$$\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}=0 \qquad r^2\frac{\partial^2 T}{\partial r^2}+2r\frac{\partial T}{\partial r}=0 \qquad \frac{d}{dr}\left(r^2\frac{\partial T}{\partial r}\right)=0$$

Case:2 Heat transfer with internal heat generation and steady state process

$$r^2\frac{\partial^2 T}{\partial r^2}+2r\frac{\partial T}{\partial r}+\frac{q}{k}=0 \qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{q}{k}=0$$

Case:3 Heat transfer with internal heat generation and unsteady state process

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{q}{k}=\frac{1}{\alpha}\frac{dT}{dt}$$

Case:4 Heat transfer without internal heat generation and unsteady state process

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{dT}{dt}$$

Thermal Resistance Concept

$$\dot{Q}_{\text{cond, wall}} = kA\frac{T_1 - T_2}{L}$$

$$\dot{Q}_{\text{cond, wall}} = \frac{T_1 - T_2}{R_{\text{wall}}} \qquad \text{(W)}$$

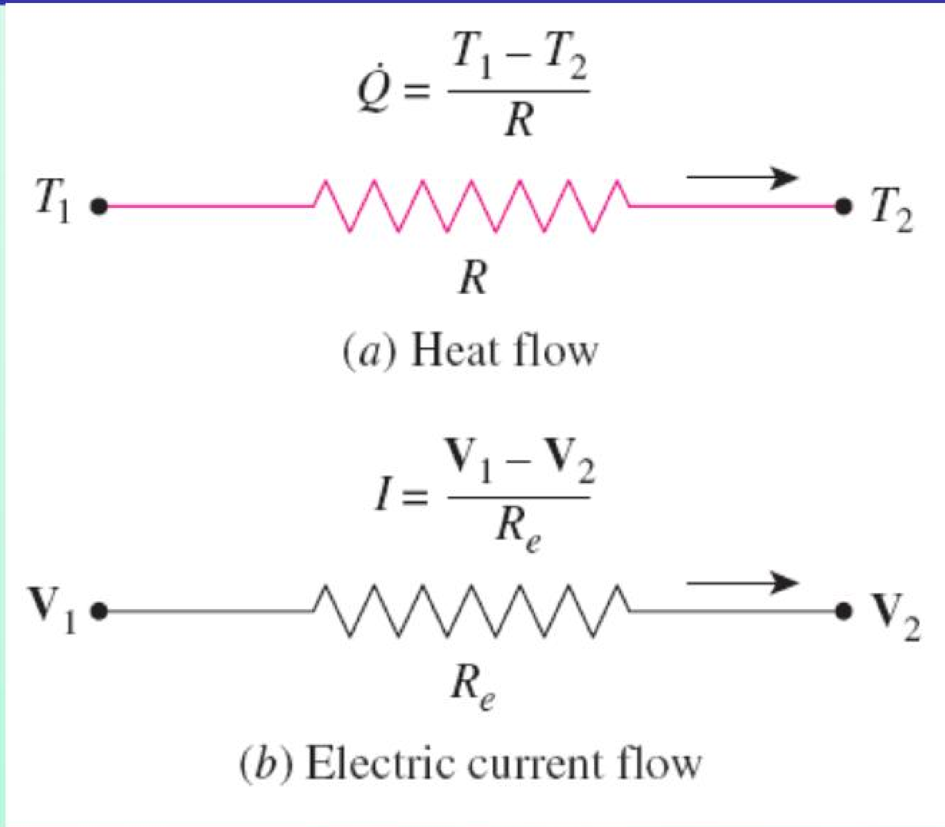
$$R_{\text{wall}} = \frac{L}{kA} \qquad (°\text{C/W})$$

Conduction resistance of the wall: *Thermal resistance* of the wall against heat conduction.

Thermal resistance of a medium depends on the *geometry* and the *thermal properties* of the medium.

$$I = \frac{\mathbf{V}_1 - \mathbf{V}_2}{R_e}$$

Electrical resistance



Analogy between thermal and electrical resistance concepts.

rate of heat transfer $\rightarrow$ electric current
thermal resistance $\rightarrow$ electrical resistance
temperature difference $\rightarrow$ voltage difference

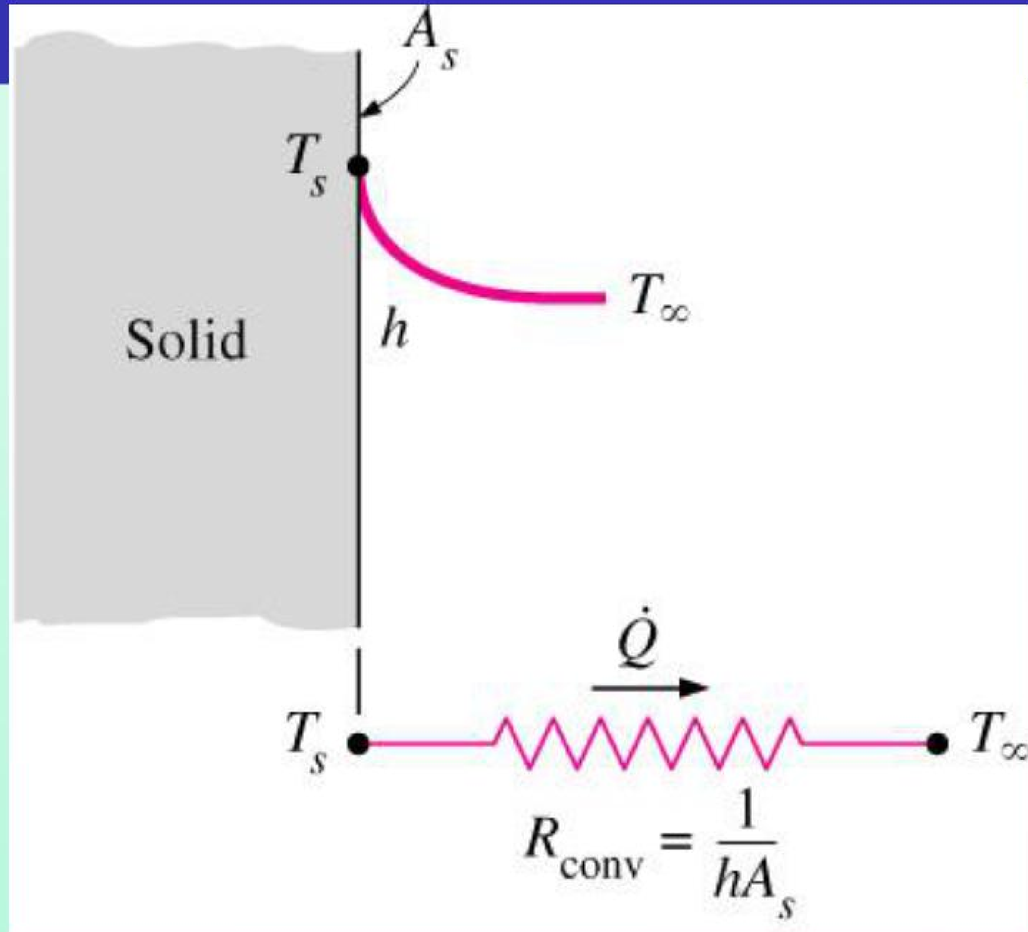
Newton's law of cooling

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty)$$

$$\dot{Q}_{conv} = \frac{T_s - T_\infty}{R_{conv}} \qquad \text{(W)}$$

$$R_{conv} = \frac{1}{hA_s} \qquad (^\circ\text{C/W})$$

Convection resistance of the surface: *Thermal resistance* of the surface against heat convection.



Schematic for convection resistance at a

When the convection heat transfer coefficient is very large ($h \to \infty$), the convection resistance becomes *zero* and $T_s \approx T$.

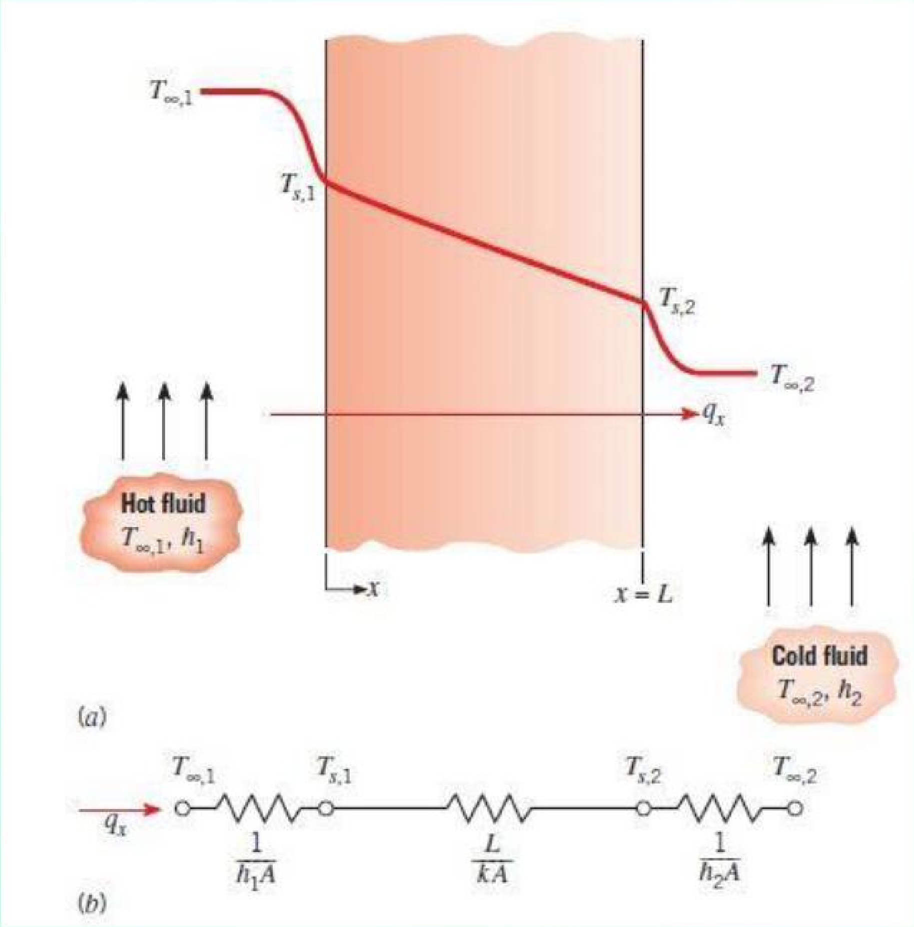
That is, the surface offers *no resistance to convection*, and thus it does not slow down the heat transfer process.

This situation is approached in practice at surfaces where boiling and condensation occur.

Cartesian Coordinates: $T\,(x)$

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

For 1-D, steady-state conduction in a plane wall with no heat generation, heat flux is a constant, independent of x.



Plane Wall

If k is constant then, $T(x) = C_1 x + C_2$

$$T(0) = T_{s,1} \quad \text{and} \quad T(L) = T_{s,2}$$

$$T(x) = (T_{s,2} - T_{s,1})\frac{x}{L} + T_{s,1}$$

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}(T_{s,2} - T_{s,1})$$

$$q''_x = \frac{k}{L}(T_{s,2} - T_{s,1})$$

Thermal Resistance

Ratio of driving potential to the corresponding transferrate

$$R_{t,cond} = \frac{(T_{s,1} - T_{s,2})}{q_x} = \frac{L}{kA}$$

$$R_e = \frac{E_{s,1} - E_{s,2}}{I}$$

$$R_{t,conv} = \frac{(T_s - T_\infty)}{q} = \frac{1}{hA}$$

Under steady state conditions:

Convection rate into the wall	=	Conduction rate through the wall	=	Convection rate from the wall

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2A}$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}} \qquad R_{tot} = \frac{1}{h_1A} + \frac{L}{kA} + \frac{1}{h_2A}$$

Thermal Resistance: Radiation

The **thermal resistance for radiation** - radiation exchange between the surface and its surroundings:

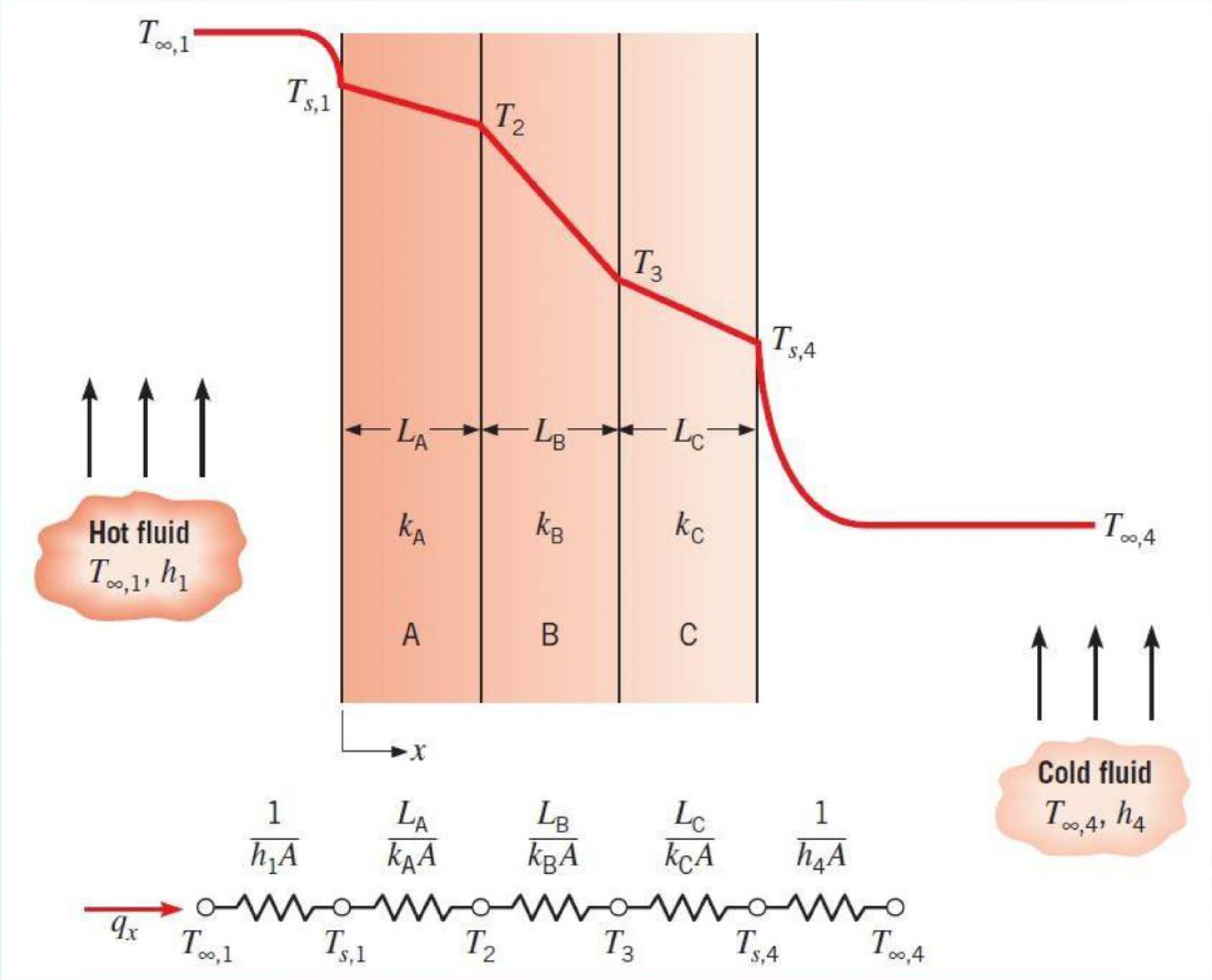
$$R_{t,rad} = \frac{T_s - T_{sur}}{q_{rad}} = \frac{1}{h_r A}$$

$$q_{rad} = h_r A \left(T_s - T_{sur}\right)$$

The radiation heat transfer coefficient, h_r:

$$h_r = \varepsilon\sigma \left(T_s + T_{sur}\right)\left(T_s^2 + T_{sur}^2\right)$$

The Composite Wall



The Composite Wall

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t}$$

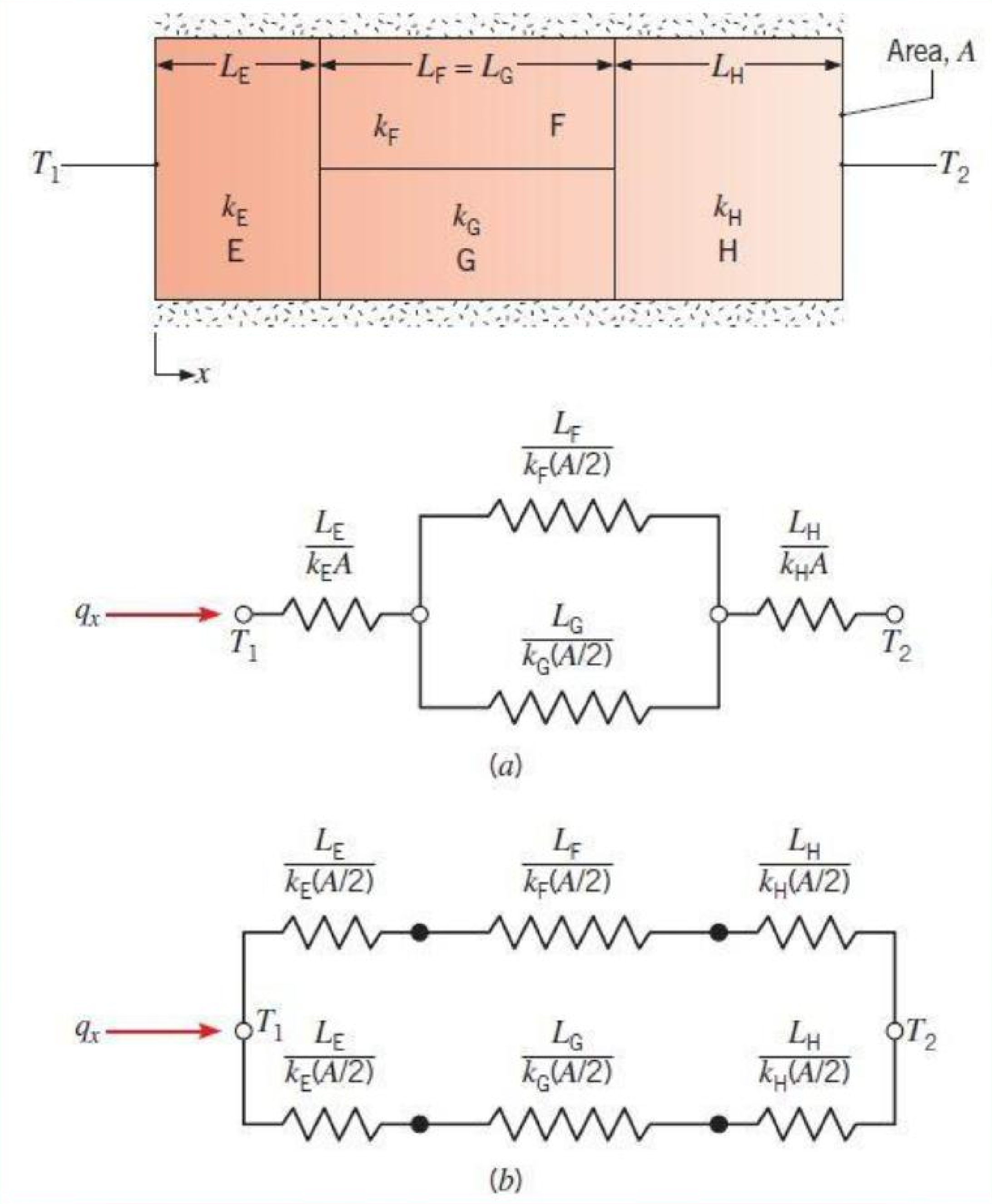
$$R_{tot} = \frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_4 A}$$

$$U = \frac{1}{R_{tot} A}$$

If U is the **overall heat transfer coefficient**

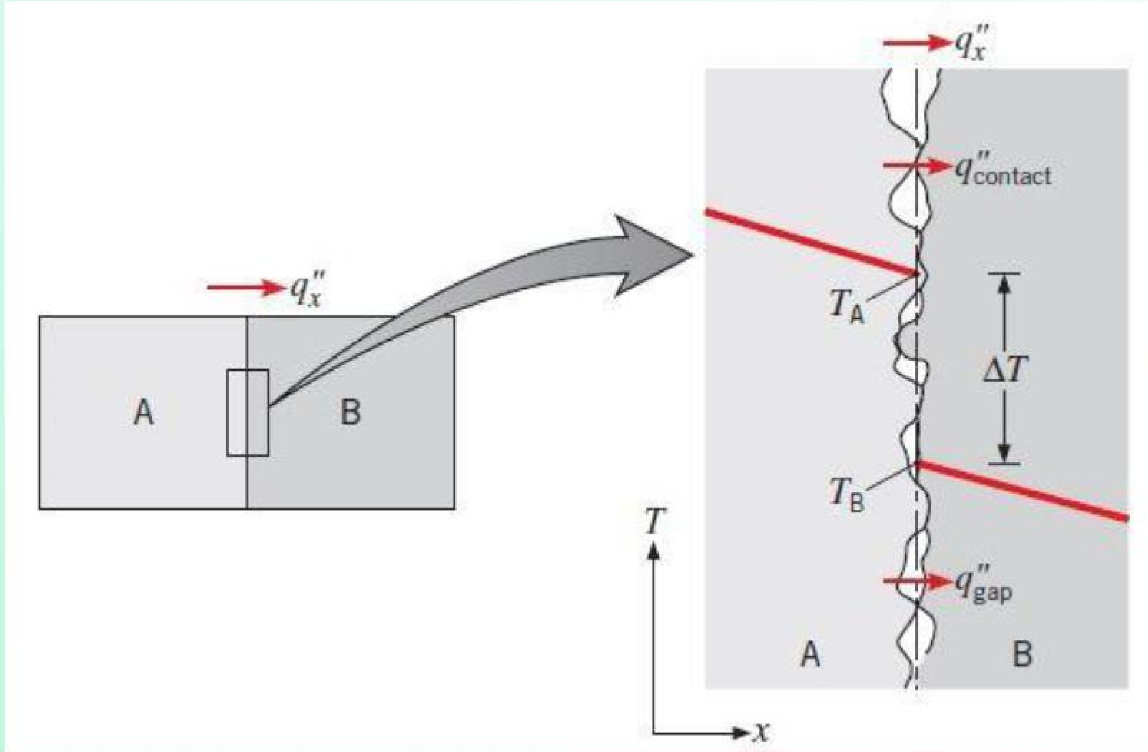
$$q_x = U \Delta T$$

Series-Parallel Composite Wall



Contact Resistance

$$R''_{t,c} = \frac{T_A - T_B}{q''_x}$$



Thermal Resistance, $R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)

(*a*) **Vacuum Interface**			(*b*) **Interfacial Fluid**	
Contact pressure	100 kN/m^2	10,000 kN/m^2	Air	2.75
Stainless steel	6–25	0.7–4.0	Helium	1.05
Copper	1–10	0.1–0.5	Hydrogen	0.720
Magnesium	1.5–3.5	0.2–0.4	Silicone oil	0.525
Aluminum	1.5–5.0	0.2–0.4	Glycerine	0.265

Thermal Resistance of Solid/Solid Interfaces

Interface	$R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)
Silicon chip/lapped aluminum in air (27–500 kN/m^2)	0.3–0.6
Aluminum/aluminum with indium foil filler (~100 kN/m^2)	~0.07
Stainless/stainless with indium foil filler (~3500 kN/m^2)	~0.04
Aluminum/aluminum with metallic (Pb) coating	0.01–0.1
Aluminum/aluminum with Dow Corning 340 grease (~100 kN/m^2)	~0.07
Stainless/stainless with Dow Corning 340 grease (~3500 kN/m^2)	~0.04
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9
Brass/brass with 15-μm tin solder	0.025–0.14

The Cylinder

The governing equation for 1D, steady state conduction in cylindrical coordinates:

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

The heat flux by Fourier's law of conduction,

$$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr}$$

- Here, $A = 2\pi rL$ is the area normal to the direction of heat transfer.
- The quantity $\frac{d}{dr}\left(kr\frac{dT}{dr}\right)$ is independent of r
- The conduction heat transfer rate q_r (not the heat flux, q_r'') is a constant in the radial direction

The Cylinder

Temperature distribution and heat transfer rate

$$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}$$

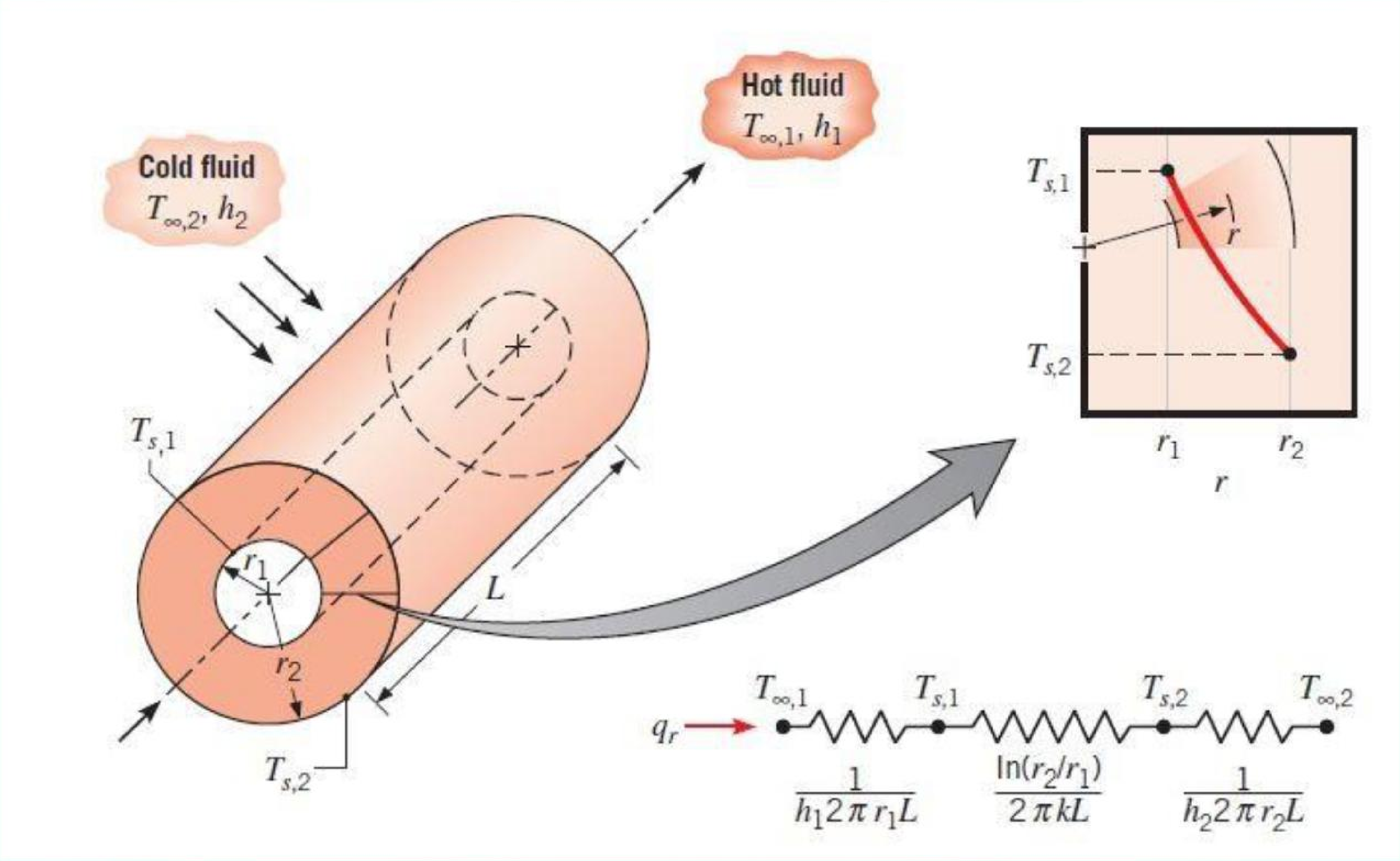
Note that the temperature distribution associated with radial conduction through a cylindrical wall is logarithmic, not linear, as it is for the plane wall.

$$q_r = \frac{2\pi Lk\,(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)}$$

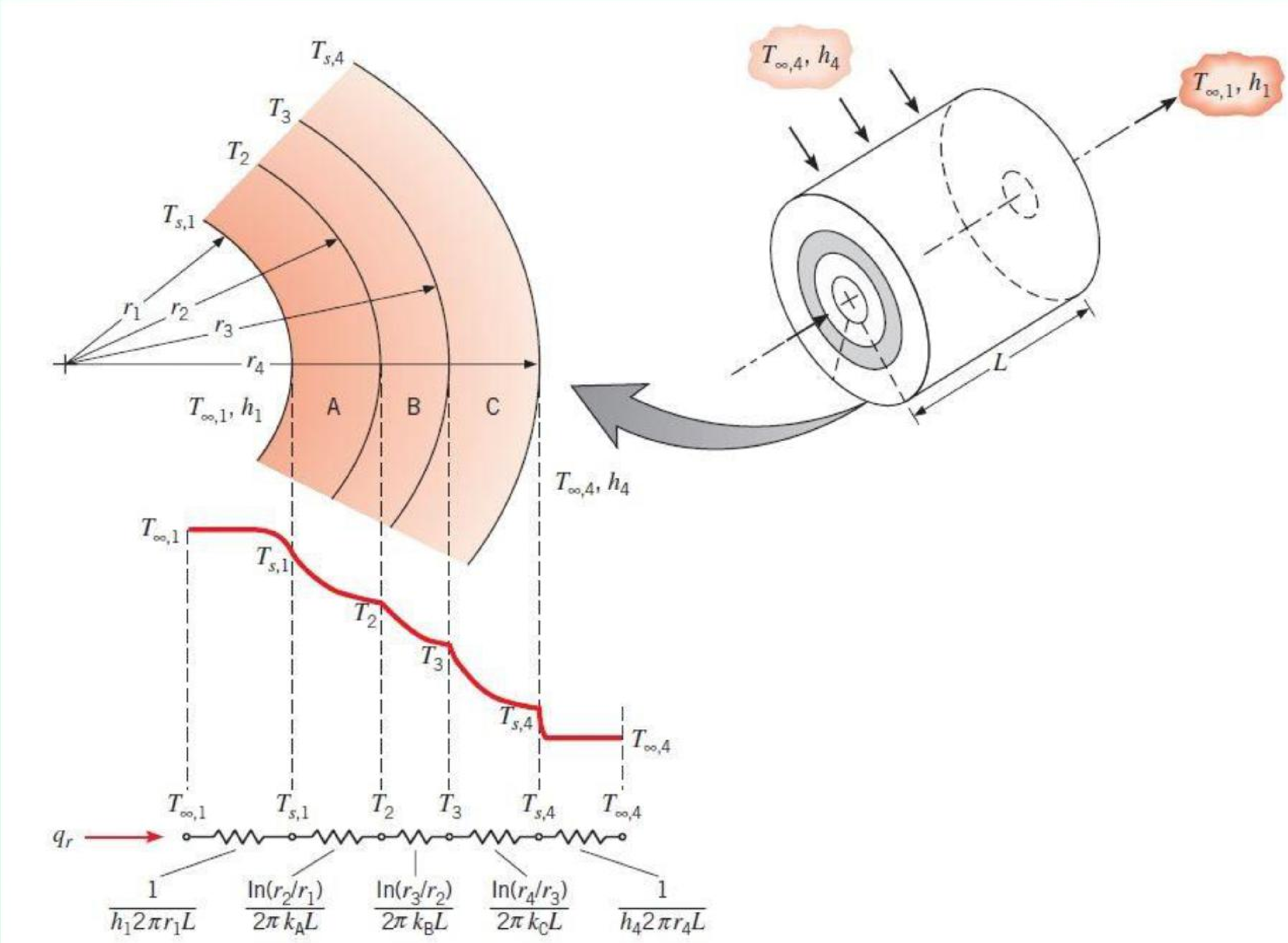
Note that q_r is independent of r.

$$R_{t,cond} = \frac{\ln(r_2/r_1)}{2\pi Lk}$$

The Cylinder



The Cylinder



The Cylinder

$$q_r = \frac{(T_{\infty,1} - T_{\infty,2})}{\frac{1}{2\pi r_1 L h_1} + \frac{\ln(r_2/r_1)}{2\pi k_A L} + \frac{\ln(r_3/r_2)}{2\pi k_B L} + \frac{\ln(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}}$$

$$q_r = \frac{(T_{\infty,1} - T_{\infty,2})}{R_{tot}} = UA\,(T_{\infty,1} - T_{\infty,2})$$

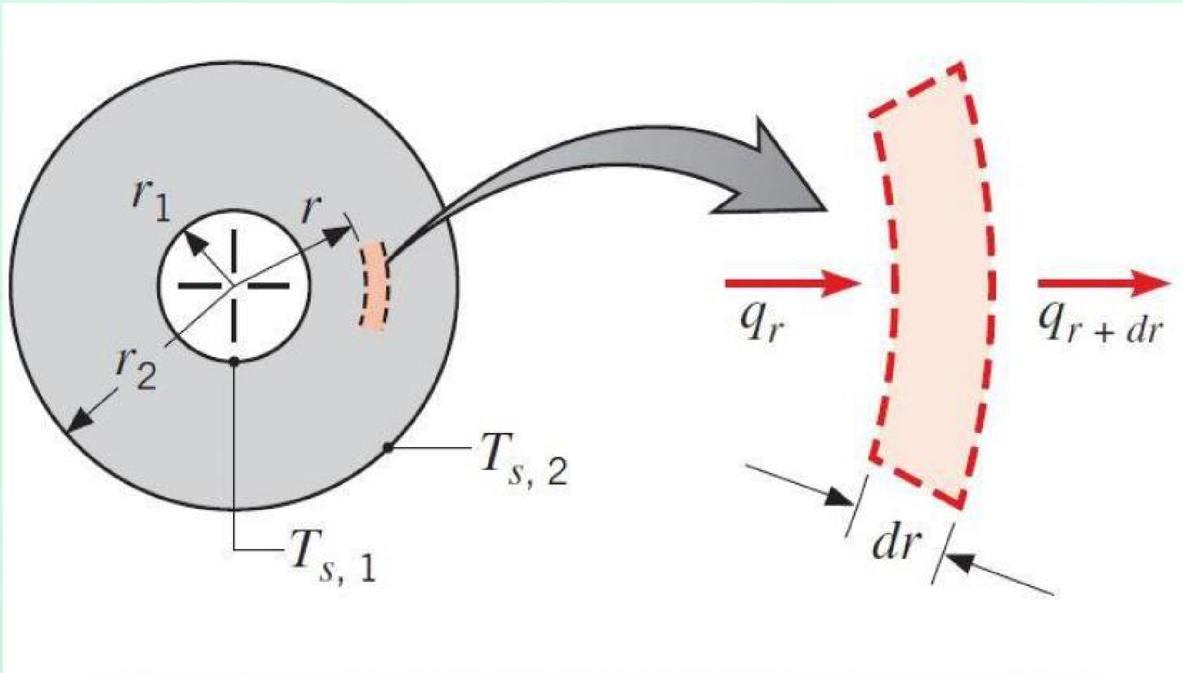
If U is defined in terms of the inside area, $A_1 = 2\pi r_1 L$, then:

$$U = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A}\ln\left(\frac{r_2}{r_1}\right) + \frac{r_1}{k_B}\ln\left(\frac{r_3}{r_2}\right) + \frac{r_1}{k_C}\ln\left(\frac{r_4}{r_3}\right) + \frac{r_1}{r_4}\frac{1}{h_4}}$$

- UA is constant, while U is not
- In radial systems, q is constant, while q'' is not

The Sphere

Consider a hollow sphere, whose inner and outer surfaces are exposed to fluids at different temperatures.



$$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right) = 0$$

$$q_r = -k(4\pi r)^2 \frac{dT}{dr}$$

The Sphere

Temperature distribution and heat transfer rate

$$T(r) = T_{s,1} + \frac{T_{s,1} - T_{s,2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \left(\frac{1}{r_1} - \frac{1}{r}\right)$$

$$q_r = \frac{4\pi k \left(T_{s,1} - T_{s,2}\right)}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

$$R_{t,cond} = \frac{1}{4\pi k} \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Problem: Sphere

A spherical thin walled metallic container is used to store liquid nitrogen at 80 K. The container has a diameter of 0.5 m and is covered with an evacuated, reflective insulation composed of silica powder. The insulation is 25 mm thick, and its outer surface is exposed to ambient air at 310 K. The convection coefficient is known to be 20 W/m^2 K. The latent heat of vaporization and the density of the liquid nitrogen are 2×10^5 J/kg and 804 kg/m^3, respectively. Thermal conductivity of evacuated silica powder (300 K) is 0.0017 W/m K.

1. What is the rate of heat transfer to the liquid nitrogen?
2. What is the rate of liquid boil-off?

Solution: Sphere

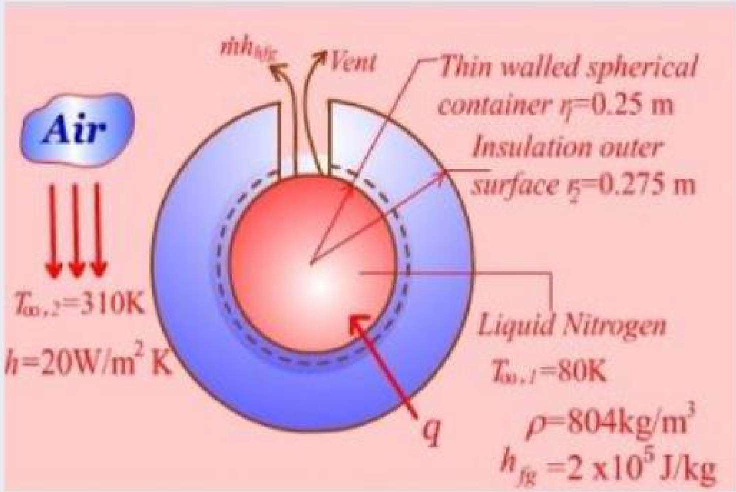
known

Liquid nitrogen is stored in a spherical container that is insulated and exposed to ambient air.

Find

- The rate of heat transfer to the nitrogen.
- The mass rate of nitrogen boil-off.

Schematic

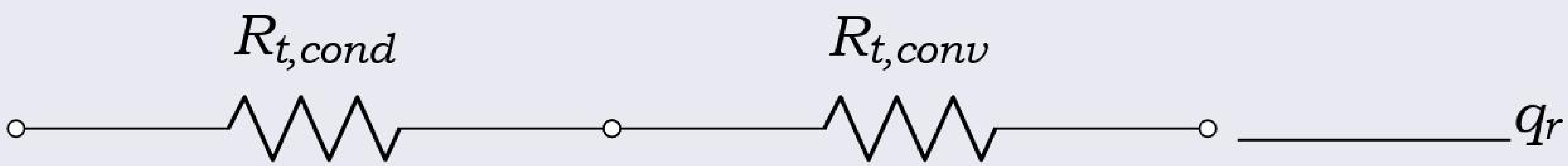


Solution: Sphere

Assumptions

1. Steady state conditions
2. One-dimensional transfer in the radial direction
3. Negligible resistance to heat transfer through the container wall and from the container to the nitrogen
4. Constant properties
5. Negligible radiation exchange between outer surface of insulation and surroundings

Analysis



Solution: Analysis

$$R_{t,cond} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$R_{t,conv} = \frac{1}{h\,4\pi r_2^2} \qquad q = \frac{(T_{\infty,2} - T_{\infty,1})}{R_{t,cond} + R_{t,conv}} = 12.88\,W$$

Energy balance for a control surface about the nitrogen:

$$\dot{E}_{in} - \dot{E}_{out} = 0$$

$$q - \dot{m} h_{fg} = 0$$

$$\Longrightarrow \dot{m} = 6.44 \times 10^{-5} \text{ kg/sec}$$

$$= 5.56 \text{ kg/day}$$

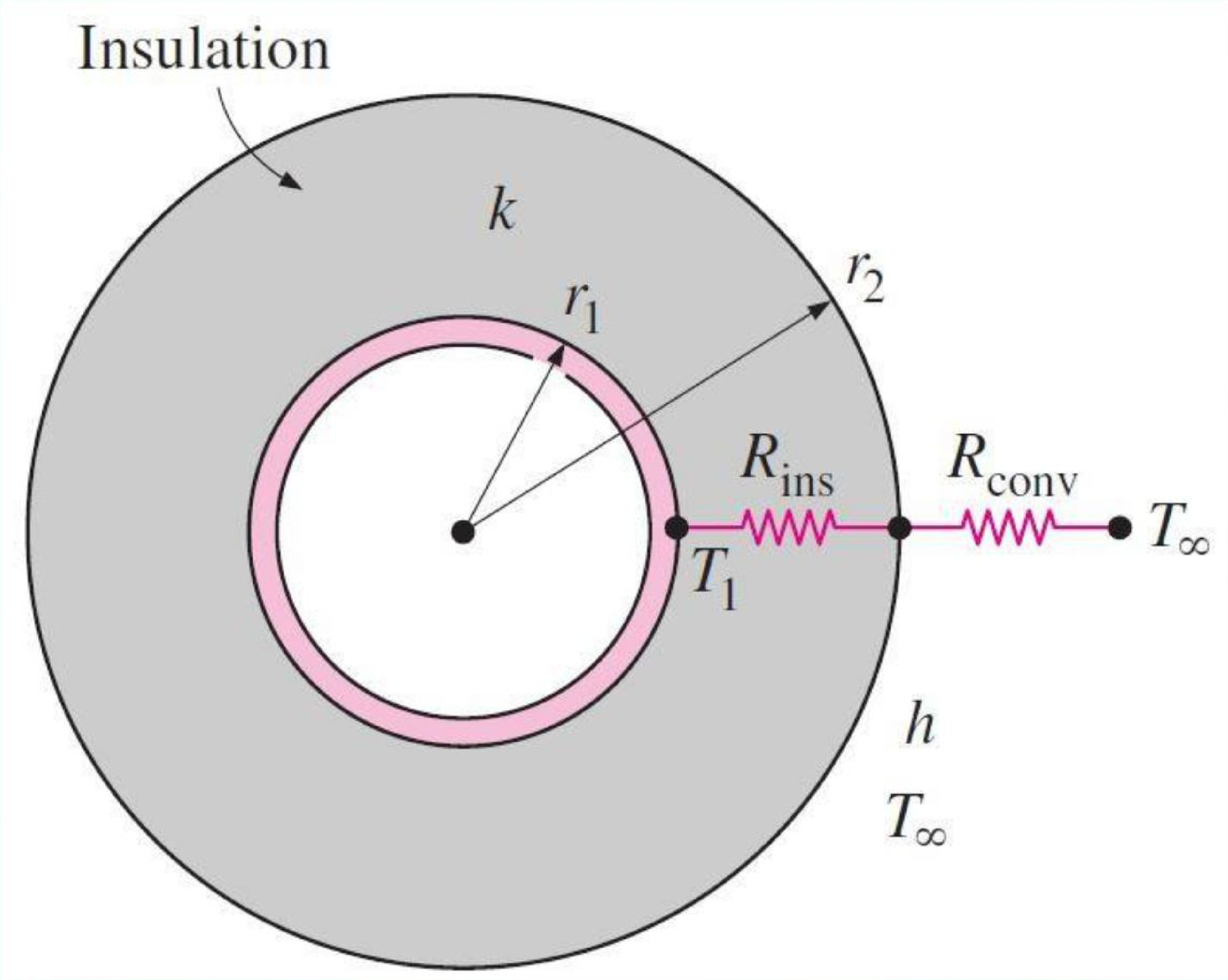
$$= \frac{\dot{m}}{\rho} = 0.00692 \text{ m}^3\text{/day}$$

$$\boxed{= 6.92 \text{ liters/day}}$$

The Critical Radius of Insulation

- We know that by adding more insulation to a wall always decreases heat transfer.
- This is expected, since the heat transfer area A is constant, and adding insulation will always increase the thermal resistance of the wall without affecting the convection resistance.
- However, adding insulation to a cylindrical piece or a spherical shell, is a different matter.
- The additional insulation increases the conduction resistance of the insulation layer but it also decreases the convection resistance of the surface because of the increase in the outer surface area for convection.
- Therefore, the heat transfer from a pipe may increase or decrease, depending on which effect dominates.

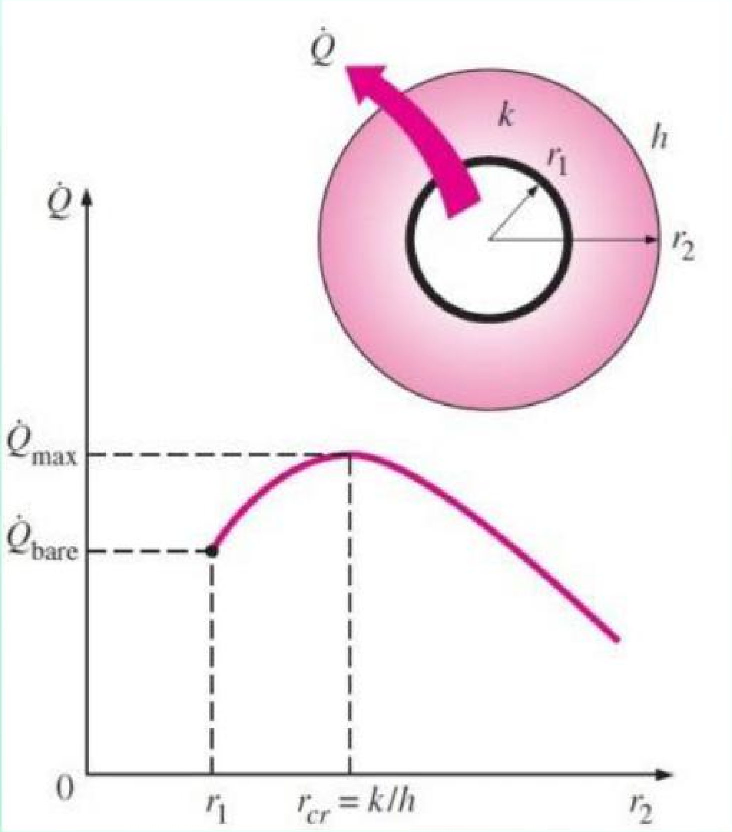
The Critical Radius of Insulation



The Critical Radius of Insulation

The rate of heat transfer from the insulated pipe to the surrounding air can be expressed as:

$$q_r = \frac{(T_1 - T_\infty)}{\frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi L k} + \frac{1}{h(2\pi r_2 L)}}$$



The value of r_2 at which heat transfer rate reaches max. is determined from the requirement that $\frac{dq_r}{dr}$ (zero slope):

$$r_{cr,cylinder} = \frac{k}{h}$$

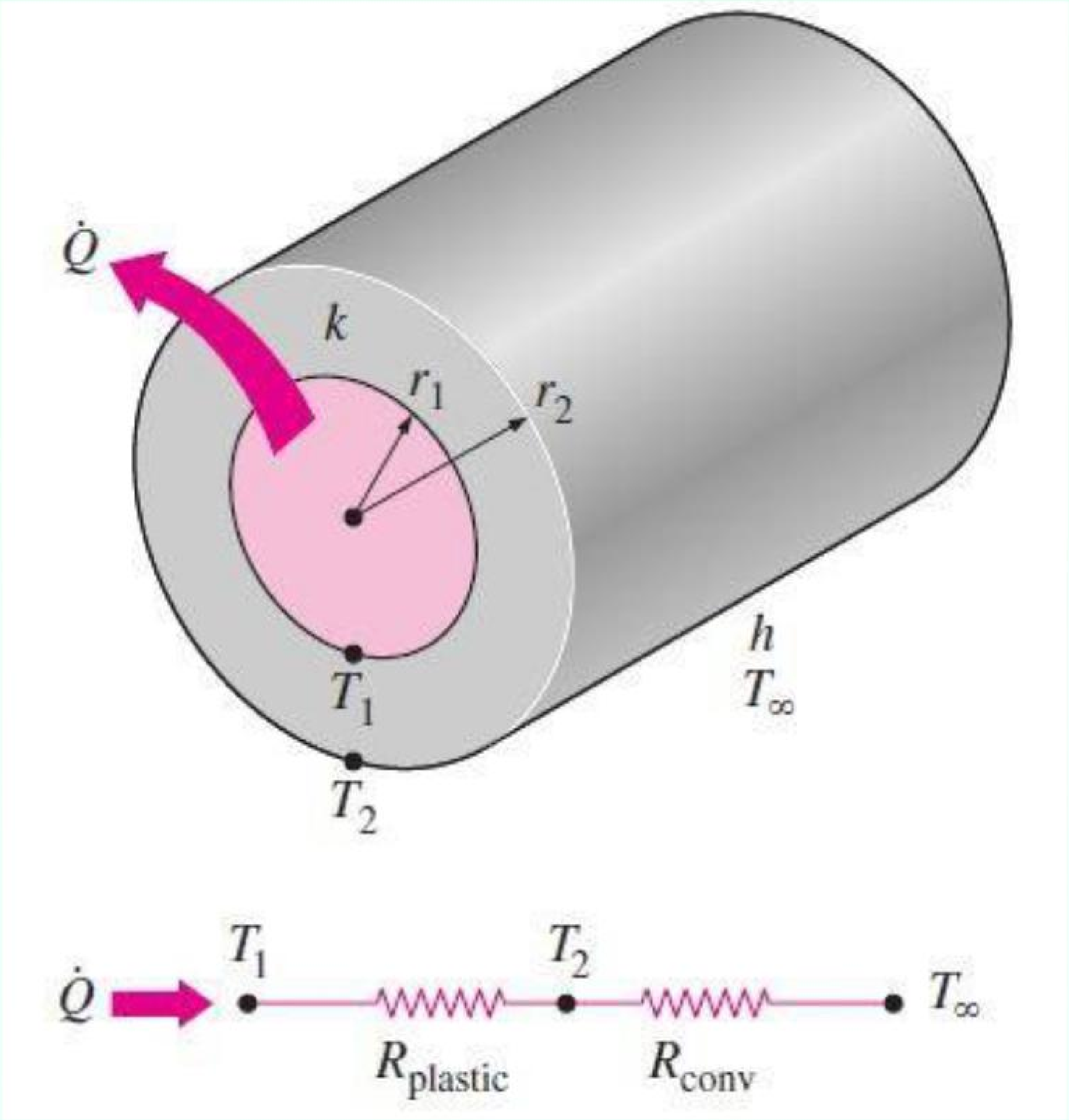
Problem: Critical Radius

A 3 mm diameter and 6 m long electric wire is tightly wrapped with a 2 mm thick plastic cover whose thermal conductivity is $k = 0.15$ W/m K. Electrical measurements indicate that a current of 10 A passes through the wire and there is a voltage drop of 8 V along the wire. If the insulated wire is exposed to a medium at 27°C with a heat transfer coefficient of $h = 12$ W/m^2 K, determine the temperature at the interface of the wire and the plastic cover in steady operation. Also determine whether doubling the thickness of the plastic cover will increase or decrease this interface temperature.

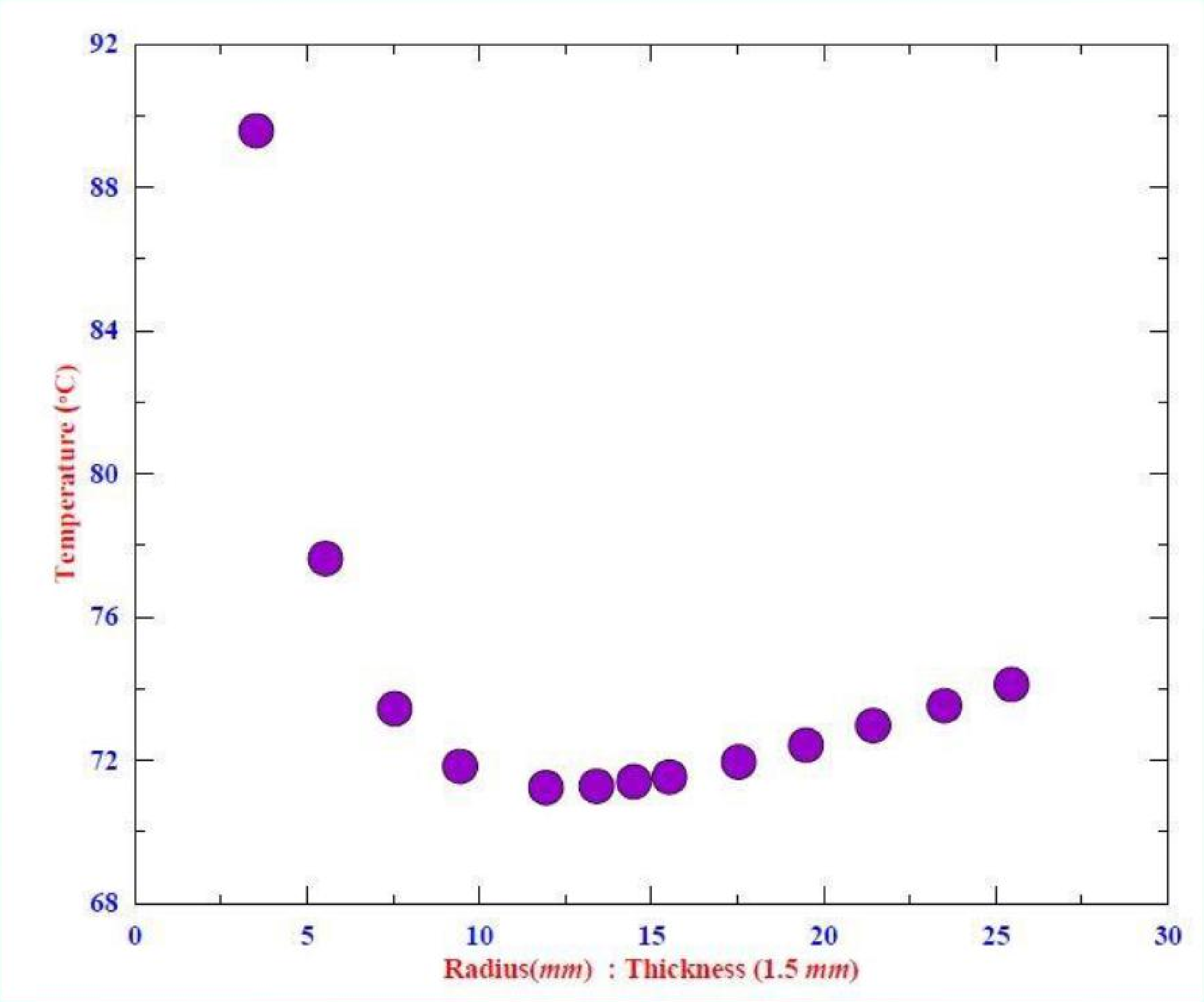
Ans: 89.5°C, 77.5°C

Hint: $q_r = VI$

Problem: Critical Radius



Problem: Critical Radius



Overall

	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\mathrm{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

[a]The critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.

Conduction with Thermal Energy Generation

A very common thermal energy generation process involves the conversion from electrical to thermal energy in a current carrying medium (resistance heating). The rate at which energy is generated by passing a current through a medium of electrical resistance R_e is:

$$\dot{E}_g = I^2 R_e$$

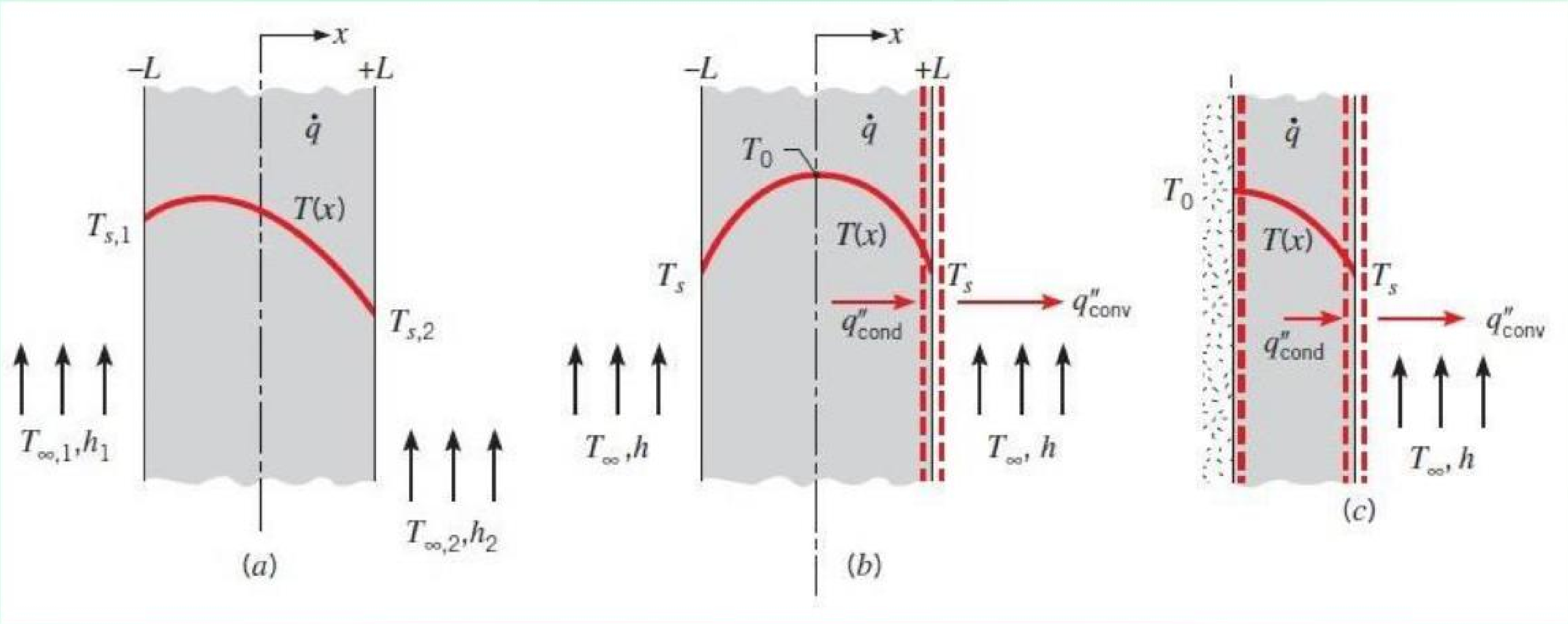
If this power generation occurs uniformly throughout the medium of volume V, the volumetric generation rate ($\mathrm{W/m^3}$) is:

$$\dot{q} = \frac{\dot{Q}_g}{V} = \frac{I^2 R_e}{V}$$

Conduction with $\dot{Q}_g$ in a Plane Wall

Consider a plane wall, in which there is uniform energy generation per unit volume ($\dot{q}$ is constant) and the surfaces are maintained at $T_{s,1}$ and $T_{s,2}$. The appropriate form of the heat equation:

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \qquad T(-L) = T_{s,1} \qquad T(L) = T_{s,2}$$



Conduction with Q_g in a Plane Wall

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$$

The heat flux at any point in the wall may be determined by Fourier's Law. The heat flux is not independent of x.

Special case: $T_{s,1} = T_{s,2} = T_s$

The temperature distribution is then symmetrical about the central plane:

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

The maximum temperature exists at the central plane:

$$T(0) = T_0 = \frac{\dot{q}L^2}{2k} + T_s$$

Conduction with $\dot{Q}_g$ in a Plane Wall

$$\frac{T(x) - T_0}{T_s - T_0} = \left(\frac{x}{L}\right)^2$$

- It is important to note that the temperature gradient $\frac{dT}{dx}\big|_{x=0} = 0$ at the plane of symmetry.
- No heat transfer across this plane - adiabatic surface.
- The above equation can also be applied to plane walls that are perfectly insulated on one side ($x = 0$) and a fixed T_s on the other side ($x = L$).

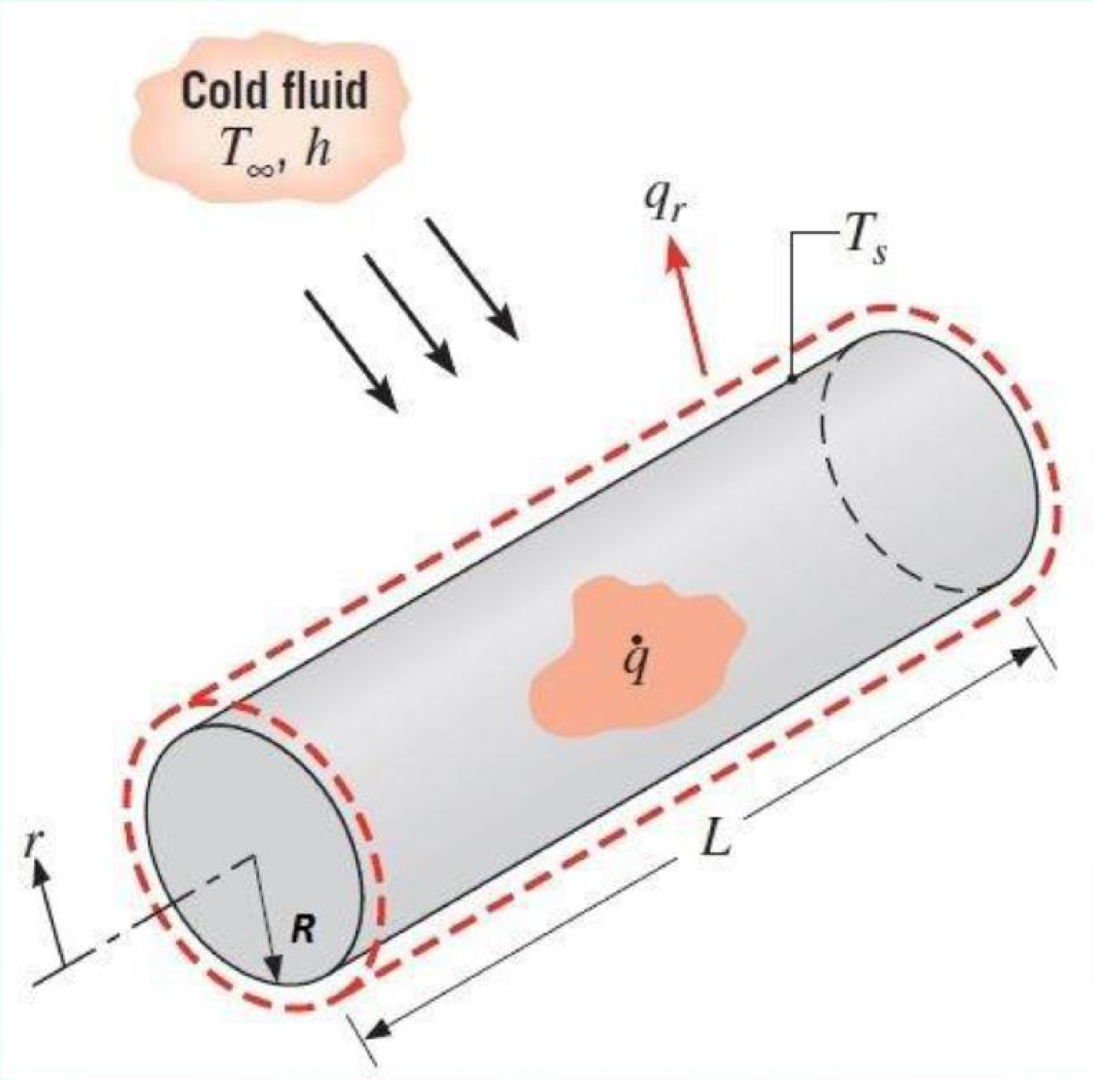
However, in most of the cases, T_s is an unknown. It is computed from the energy balance at the surface to the adjoining fluid:

$$-k \left.\frac{dT}{dx}\right|_{x=L} = h(T_s - T_\infty)$$

$$\Longrightarrow T_s = T_\infty + \frac{\dot{q}L}{h}$$

Conduction with $\dot{Q}_g$ in Radial Systems

Consider a long, solid cylinder (may be a current carrying wire).
For steady state conditions: $\dot{Q}_g = Q_{conv}$.



Conduction with $\dot{Q}_g$ in Radial Systems

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}}{k} = 0$$

Boundary conditions:

$$\left.\frac{dT}{dr}\right|_{r=0} = 0, \quad T(R) = T_s \text{ and } T(r=0) = T_0$$

$$\Longrightarrow T(r) = T_s + \frac{\dot{q}R^2}{4k}\left(1 - \frac{r^2}{R^2}\right)$$

or $$\frac{T(r) - T_s}{T_0 - T_s} = 1 - \left(\frac{r}{R}\right)^2$$

T_s can be obtained from the energy balance at the surface:

$$\dot{q}\pi R^2 L = h\,2\pi RL\,(T_s - T_\infty)$$

$$T_s = T_\infty + \frac{\dot{q}R}{2h}$$

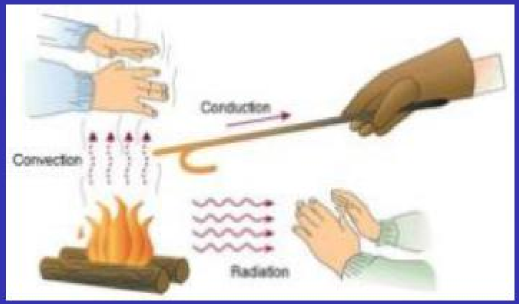
Problem

A plane wall of material A has uniform heat generation $\dot{q} = 1.5 \times 10^6$ W/m^3,
$k_A = 75$ W/m K, and thickness $L_A = 50$ mm. The inner surface of material is well insulated, while the outer surface is cooled by a water stream with $T_\infty = 30°$C and $= 1000$ W/m^2 K.

Sketch the temperature distribution that exists in the wall under steady state conditions.

- Determine the temperature T_0 of the insulated surface and the temperature T_1 of the cooled surface.
-

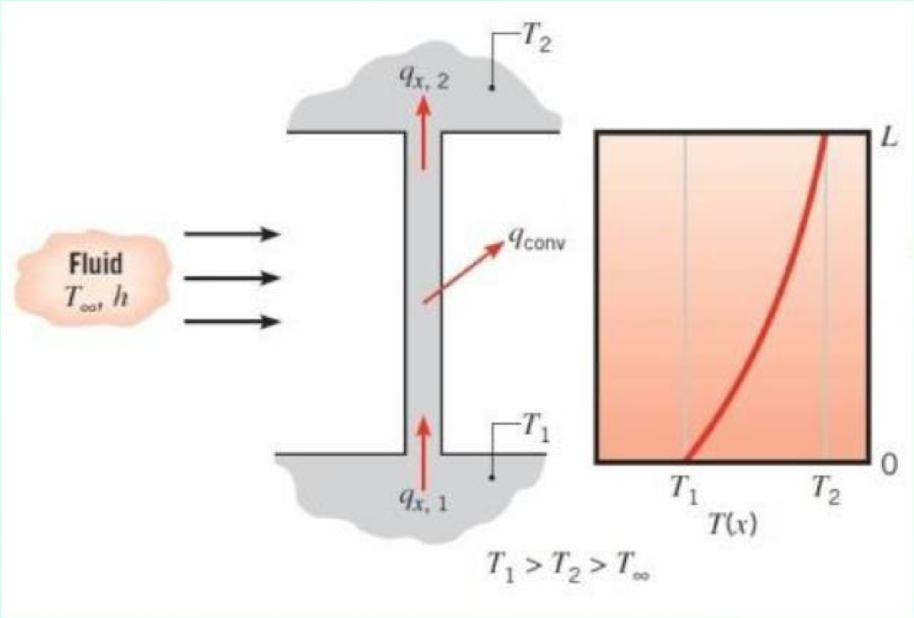
Heat Transfer



1D Heat Transfer from Extended Surfaces - Fins

Heat Transfer from Extended Surfaces

Extended surface: solid that experiences energy transfer by conduction within its boundaries, as well as energy transfer by convection and/or radiation between its boundaries and the surroundings



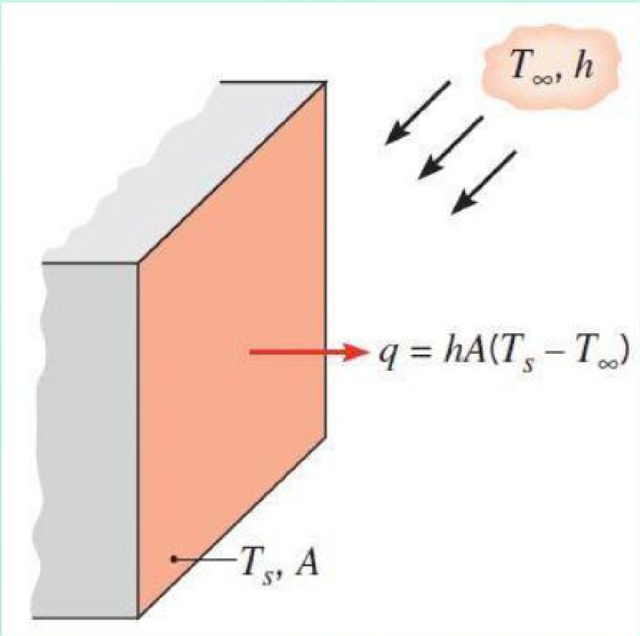
A strut is used to provide mechanical support to two walls at different T. A temperature gradient in the x-direction sustains heat transfer by conduction internally, at the same time there is energy transfer by convection from the surface.

Heat Transfer from Extended Surfaces

The most frequent application is one in which an extended surface is used specifically to enhance the heat transfer rate between a solid and an adjoining fluid - called as **fin**

Consider a plane wall:

$$q_{conv} = hA(T_s - T_\infty)$$



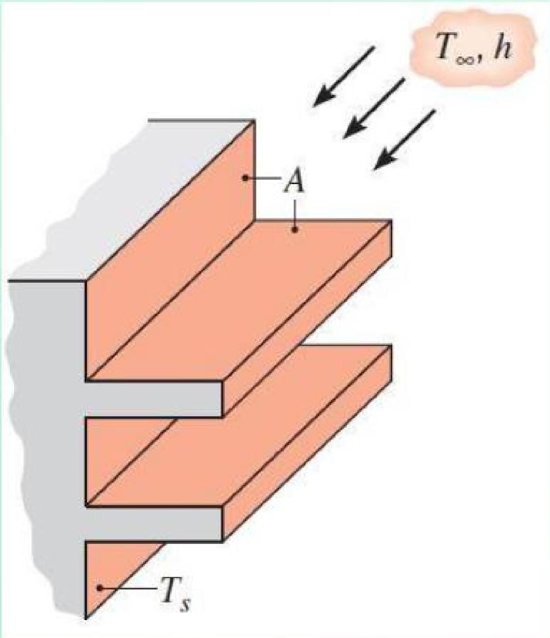
Heat Transfer from Extended Surfaces

For fixed T_s, 2 ways to enhance the rate of heat transfer:

- Increase the fluid velocity: cost of blower or pump power
- T_∞ could be reduced: impractical

Limitations: Many situations would be encountered in which increasing h to the max. possible value is either insufficient to obtain the desired heat transfer rate or the associated costs are prohibitively high.

How about increasing surface area for convection?
By providing fins that extend from the wall into the surrounding fluid.

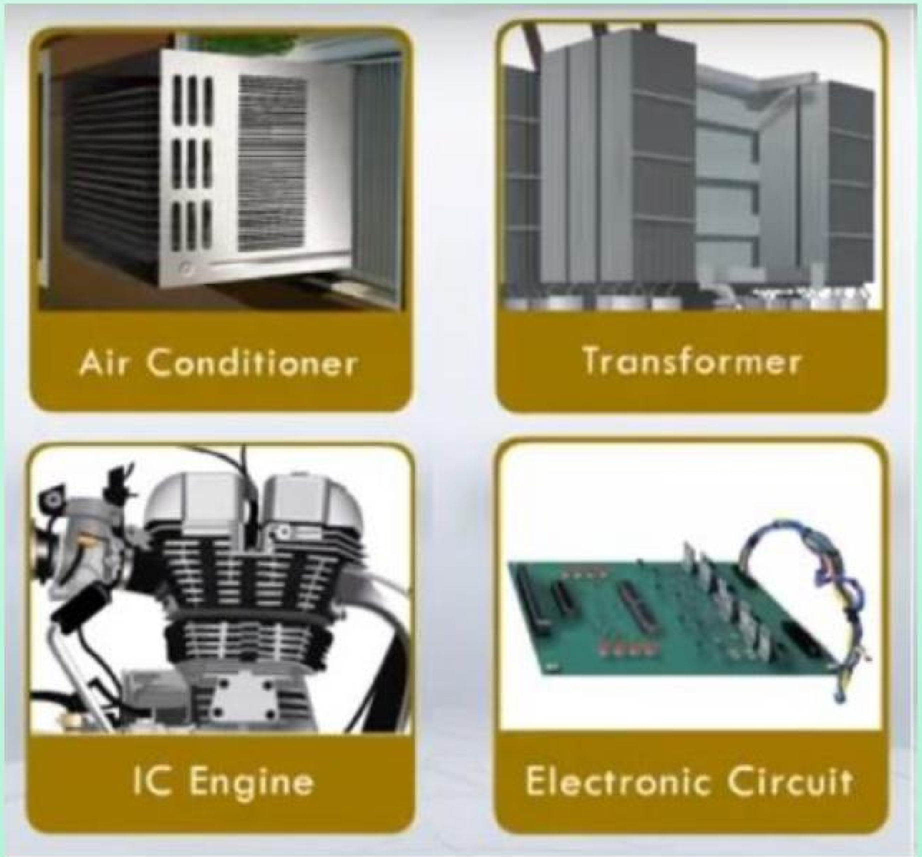


Fin Material

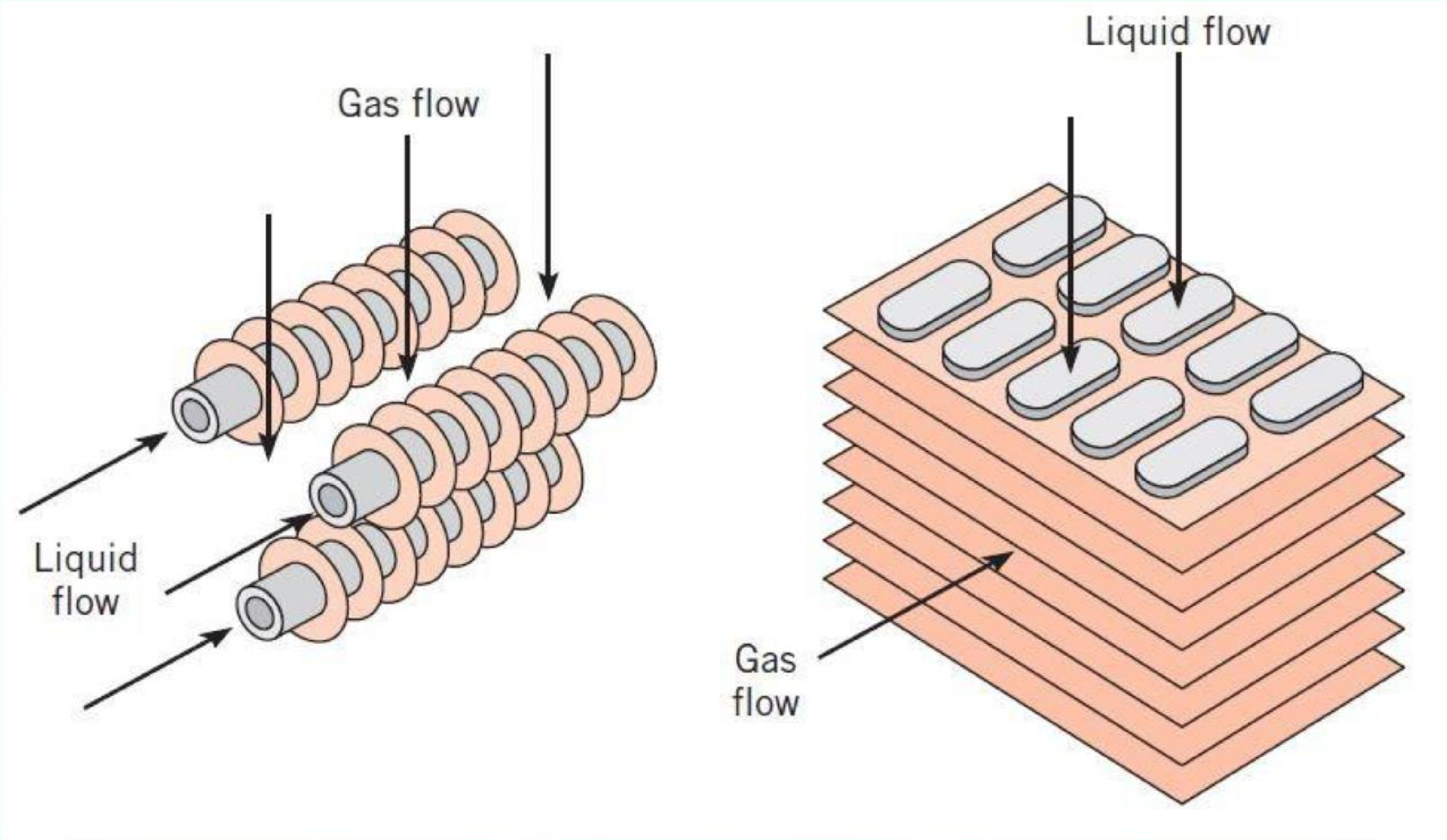
- k of the fin material has a strong effect on the temperature distribution along the fin and therefore influences the degree to which the heat transfer rate is enhanced.
- Ideally, the fin material should have a large k to minimize temperature variations from its base to its tip.
- In the limit of infinite thermal conductivity, the entire fin would be at the temperature of the base surface, thereby providing the maximum possible heat transfer enhancement.

Application of Fins

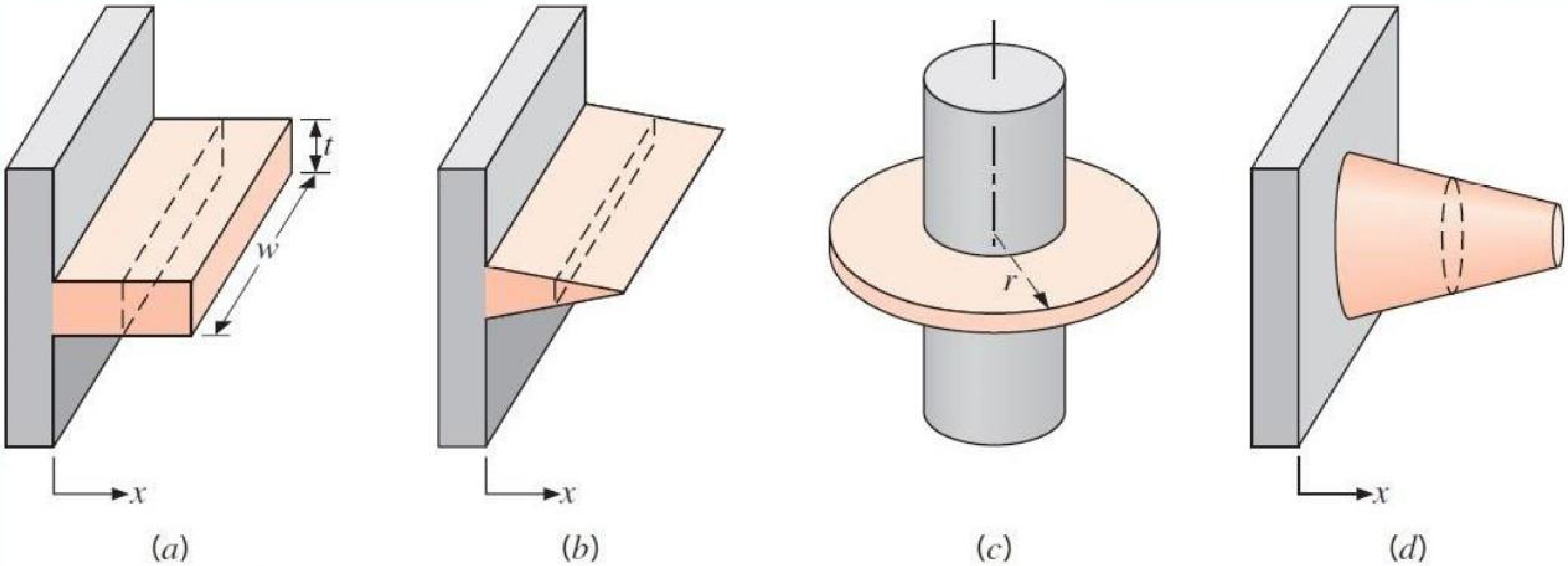
- The arrangement for cooling engine heads on motorcycles and lawn-mowers
- For cooling electric power transformers
- The tubes with attached fins used to promote heat exchange between air and the working fluid of an air conditioner



Typical Finned Tube Heat Exchangers



Fin Configurations

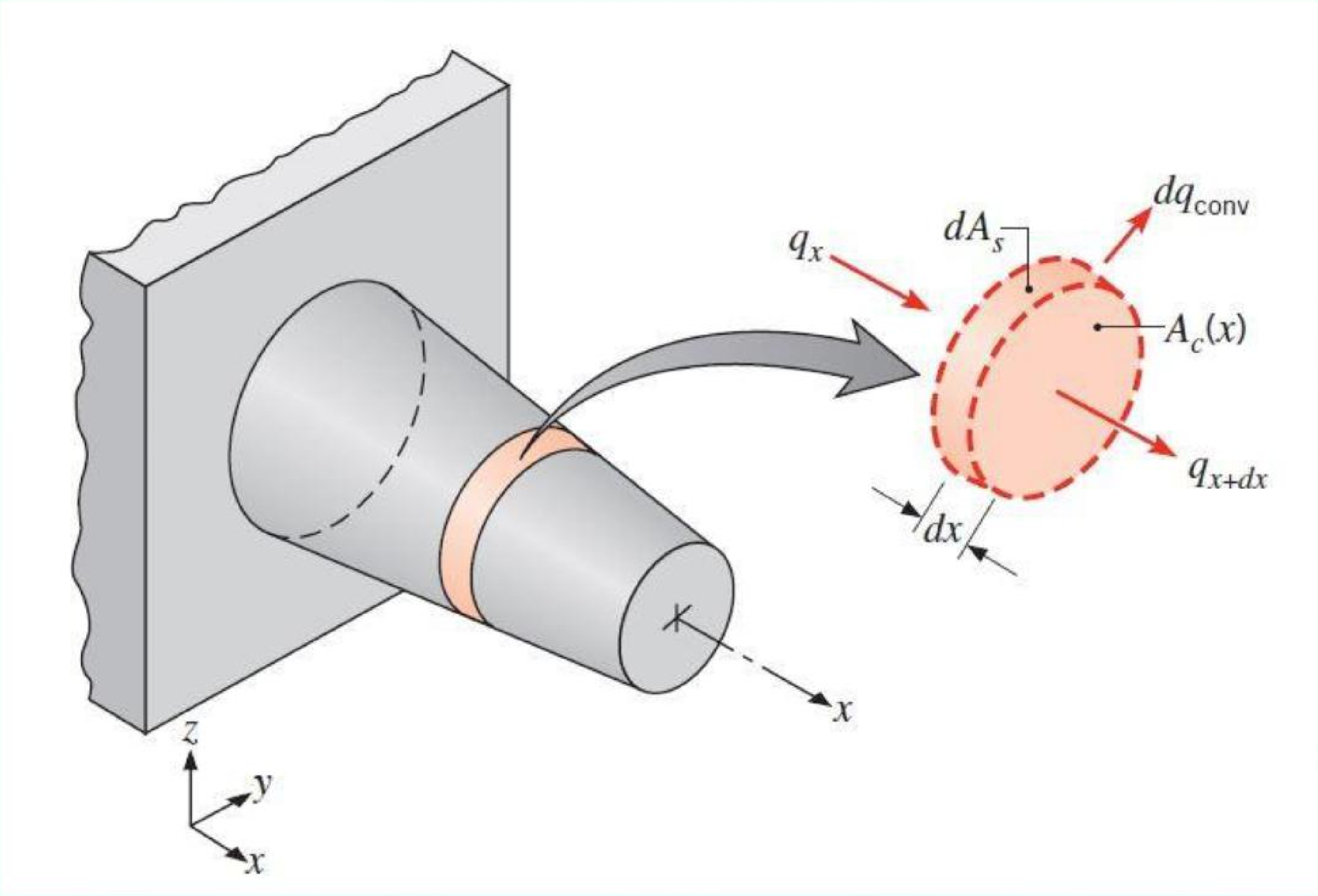


(a) Straight fin of uniform cross section. (b) Straight fin of nonuniform cross section. (c) Annular fin. (d) Pin fin.

For an extended surface, the direction of heat transfer from the boundaries is perpendicular to the principal direction of heat transfer in the solid.

General Conduction Analysis

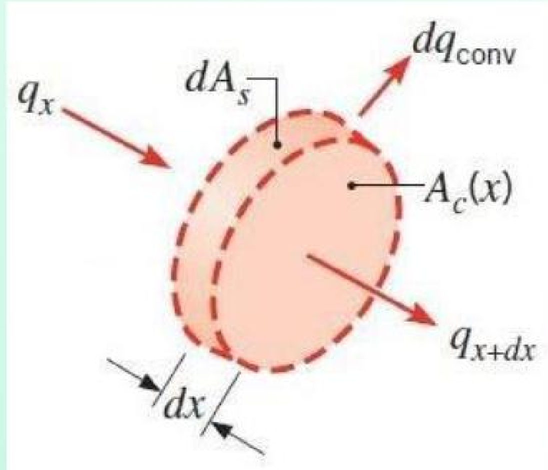
To determine the heat transfer rate associated with a fin, we must first obtain the temperature distribution along the fin.



Assumptions

- 1-D heat transfer (longitudinal x direction). In practice the fine is thin and the temperature changes in the longitudinal direction are much larger than those in the transverse direction.
- Steady state
- k is constant
- No heat generation
- Negligible radiation from the surface
- The rate at which the energy is convected to the fluid from any point on the fin surface must be balanced by the rate at which the energy reaches that point due to conduction in the transverse (y, z) direction.
- h is uniform over the surface

Derivation for fin



$$q_x = q_{x+dx} + dq_{conv}$$

However,

$$q_x = -kA_c\frac{dT}{dx}$$

A_c may vary with x.

$$q_{x+dx} = q_x + \frac{dq_x}{dx}dx$$

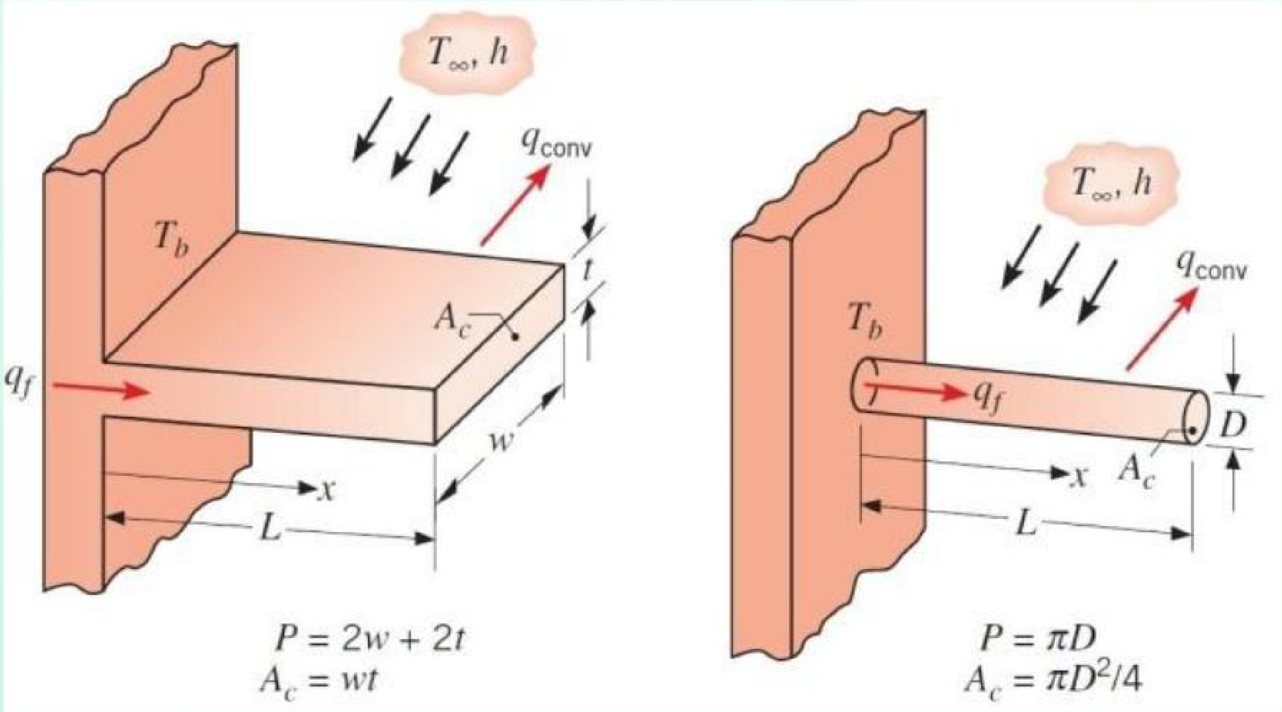
$$q_{conv} = h\,dA_s(T - T_\infty)$$

dA_s is the surface area of dx

$$\Longrightarrow k\frac{d}{dx}\left(A_c\frac{dT}{dx}\right)dx - h\,dA_s(T - T_\infty)$$

$$\boxed{\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T - T_\infty)}$$

Fins of Uniform Cross-Sectional Area



- $T(0) = T_b$
- A_c is constant, $dA_c/dx = 0$
- $A_s = Px$ where x is measured from base, P is fin perimeter
- $dA_s/dx = P$

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

Fins of Uniform Cross-Sectional Area

Excess temperature, θ

$$\theta(x) = T(x) - T_\infty$$

$$d\theta/dx = dT/dx$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

where $m^2 = \frac{hP}{kA_c}$

The above equation is a linear, homogeneous, second-order differential equation with constant coefficients. The general solution is of the form:

$$\boxed{\theta(x) = C_1 e^{mx} + C_2 e^{-mx}}$$

It is necessary to specify appropriate BCs for C_1 and C_2.

Fins of Uniform Cross-Sectional Area

One such condition may be specified in terms of the temperature at the base of the fin $(x = 0)$:

$$\theta(0) = T_b - T_\infty = \theta_b$$

The second condition, specified at the fin tip $(x = L)$, may correspond to any one of the four different physical conditions:

A. h at the fin tip
B. Adiabatic condition at the fin tip
C. Prescribed temperature maintained at the fin tip
D. Infinite fin (very long fin)

Fins of Uniform Cross-Sectional Area

A. Infinite fin (very long fin): As $L \to \infty$, $\theta_L \to 0$

B. Adiabatic condition at the fin tip

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

C. h at the fin tip

$$hA_c[T(L) - T_\infty] = -kAc \left.\frac{dT}{dx}\right|_{x=L} \Longrightarrow h\theta(L) = -k \left.\frac{d\theta}{dx}\right|_{x=L}$$

D. Prescribed temperature maintained at the fin tip: $\theta(L) = \theta_L$

Case A: Infinite fin (very long fin)

As $L \to \infty,\ \theta_L \to 0$ and $e^{-mL} \to 0$

$$\theta|_{x=0} = \theta_b; \qquad \theta|_{x=L} = 0$$

$$\theta_b = C_1 e^{mx} + C_2 e^{-mx}; \qquad C_1 e^{mL} + C_2 e^{-mL} = 0$$

Equation for infinite fin

$$\frac{\theta}{\theta_b} = e^{-mx}$$

$$q_f = -kA_c \left.\frac{d\theta}{dx}\right|_{x=0} = \sqrt{hPkA_c}\,\theta_b$$

Case B: Adiabatic Condition at the Fin Tip

$$\theta\big|_{x=0} = \theta_b; \qquad \left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

$$\theta_b = C_1 e^{mx} + C_2 e^{-mx}; \qquad C_1 e^{mx} - C_2 e^{-mx} = 0$$

$$\frac{\theta}{\theta_b} = \frac{e^{mx}}{1 + e^{2mL}} + \frac{e^{-mx}}{1 + e^{-2mL}}$$

Equation for adiabatic condition

$$\frac{\theta}{\theta_b} = \frac{\cosh[m(L - x)]}{\cosh mL}$$

Note: $\cosh A = \dfrac{e^{A} + e^{-A}}{2}$

Case B: Adiabatic Condition at the Fin Tip

$$q_f = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = -kA_c \left.\frac{d\theta}{dx}\right|_{x=0}$$

$$= -kA_c \, m\theta_b \left[\frac{1}{1+e^{2mL}} - \frac{1}{1+e^{-2mL}}\right]$$

$$= \sqrt{hPkA_c}\,\theta_b \left[\frac{e^{mL}-e^{-mL}}{e^{mL}+e^{-mL}}\right]$$

Rate of heat transfer: Adiabatic Condition

$$q_f = \sqrt{hPkA_c}\,\theta_b \tanh mL$$

Case C: h from the Fin Tip

A practical way is to account for the heat loss from the fin tip is to replace the *fin length* L in the relation for the adiabatic tip case by a corrected length.

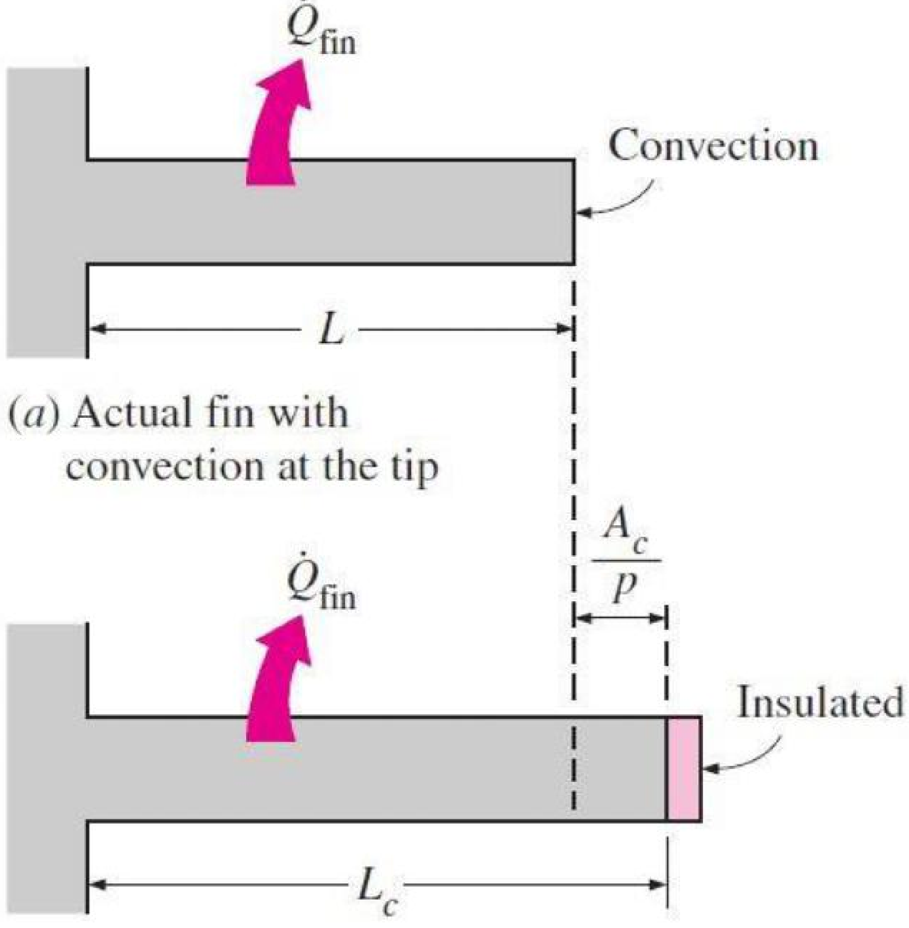
$$L_c = L + \frac{A_c}{P}$$

$$\boxed{\frac{\theta}{\theta_b} = \frac{\cosh[m(L_c - x)]}{\cosh mL_c}}$$

$$\boxed{q_f = \sqrt{hPkA_c}\,\theta_b \tanh mL_c}$$

$$L_{c,rectanular\ fin} = L + \frac{t}{2}$$

$$L_{c,cylindrical\ fin} = L + \frac{D}{4}$$



(*a*) Actual fin with convection at the tip

(*b*) Equivalent fin with insulated tip

Uniform Cross-Sectional Fin: Summary

Temperature distribution & heat loss for fins of uniform cross-section

Tip Cond.	**at** $x = L$	$\frac{\theta}{\theta_b}$	q_f
Infinite fin	$\theta(L) = 0$	e^{-mx}	M
Adiabatic	$\left.\frac{d\theta}{dx}\right\|_{x=L} = 0$	$\frac{\cosh[m(L-x)]}{\cosh mL}$	$M \tanh mL$
Convection	$h\theta_L = -k\left.\frac{d\theta}{dx}\right\|_{x=L_c}$	$\frac{\cosh[m(L_c-x)]}{\cosh mL_c}$	$M \tanh mL$

$$m = \sqrt{\frac{hP}{kA_c}}; \qquad M = \sqrt{hPkA_c}\,\theta_b; \qquad L_c = L + \frac{A_c}{P}$$

Problem

A very long rod 5 mm in diameter has one end maintained at 100°C. The surface of the rod is exposed to ambient air at 25°C with a convection heat transfer coefficient of 100 $W/m^2 K$.

- Determine the temperature distributions along rods constructed from pure copper, 2024 aluminium alloy and type AISI 316 stainless steel. What are the corresponding heat losses from the rods? Ans: 8.3 W, 5.6 W and 1.6 W
- Estimate how long the rods must be for the assumption of infinite length to yield an accurate estimate of the heat loss.

At $T = (T_b + T_\infty)/2 = 62.5°C = 335\ K$:

$$k_{copper} = 398\ W/m\,K$$

$$k_{aluminium} = 180\ W/mK$$

$$k_{stainless\ steel} = 14\ W/mK$$

Hint: For an infinitely long fin: $\theta/\theta_b = e^{-mx}$

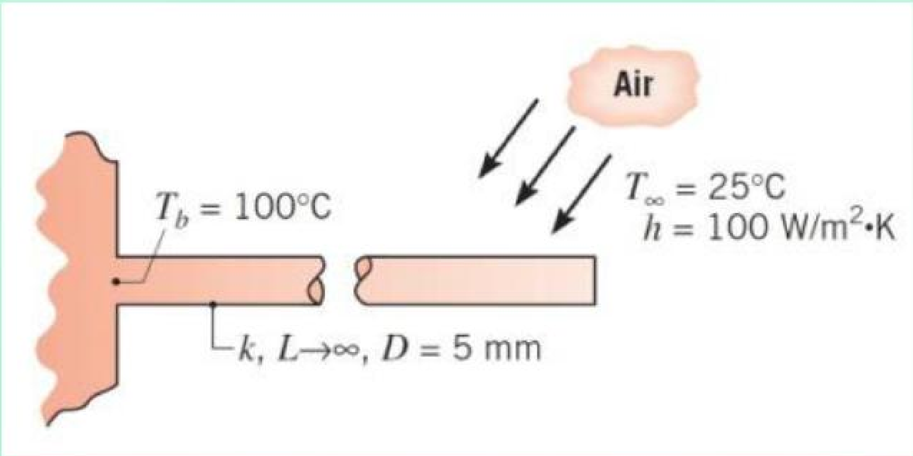
Solution

Find

- $T(x)$ and heat loss when rod is Cu, Al, SS.
- How long rods must be to assume infinite length.

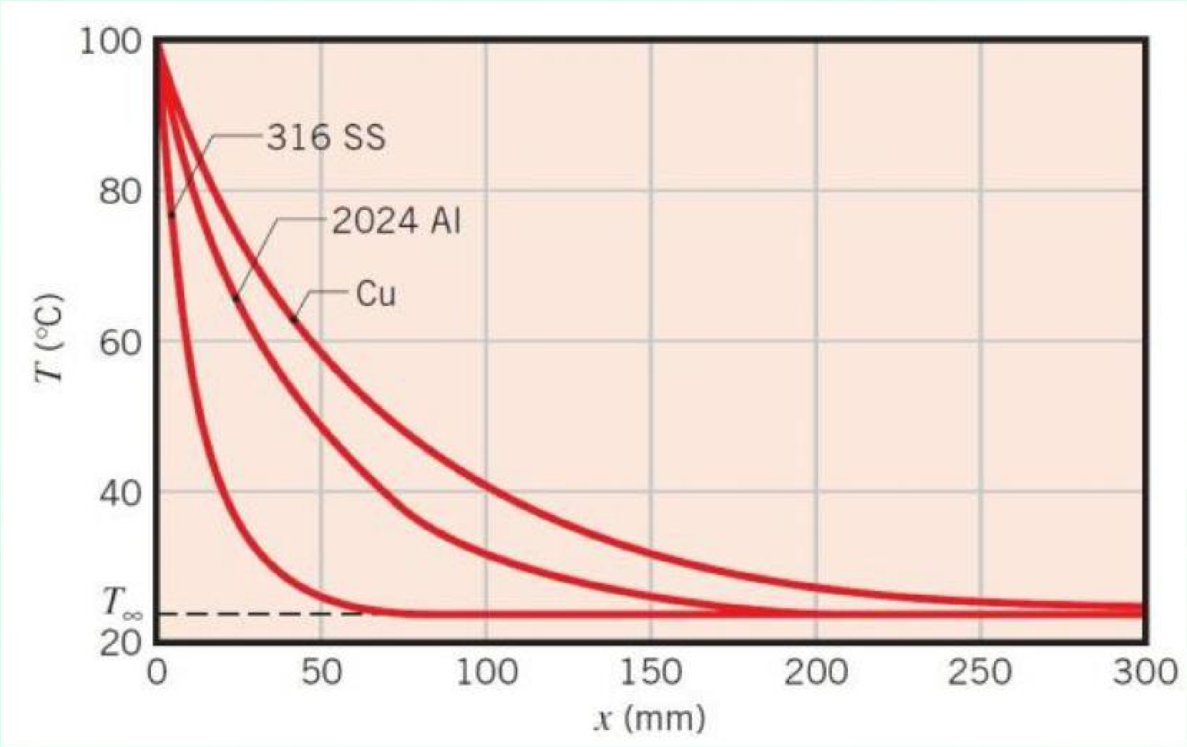
Assumptions

- Steady state conditions, 1-D along the rod
- Constant properties and uniform h
- Negligible radiation exchange with surroundings



Solution: Analysis - Part 1

$$T = T_\infty + (T_b - T_\infty)e^{-mx}$$



There is little additional heat transfer associated with lengths more than 50 mm (SS), 200 mm (Al), and 300 mm (Cu).

$$q_f = \sqrt{hPkA_c}\,\theta_b$$

Solution: Analysis - Part 2

Since there is no heat loss from the tip of an infinitely long rod, an estimate of the validity of the approximation may be made by comparing q_f for infinitely long fin and adiabatic fin tip.

$$\sqrt{hPkA_c}\ \theta_b = \sqrt{hPkA_c}\ \theta_b \tanh mL$$

$$\tanh 4 = 0.999 \qquad \text{and} \qquad \tanh 2.5 = 0.987$$

$$\Longrightarrow \quad mL \geq 2.5$$

$$L \geq \frac{2.65}{m} = 2.5 \sqrt{\frac{kA_c}{hP}}$$

$$L_{Cu} = 0.18\ m; \quad L_{Al} = 0.12\ m; \quad \text{and} \quad L_{SS} = 0.033\ m$$

Solution: Comments

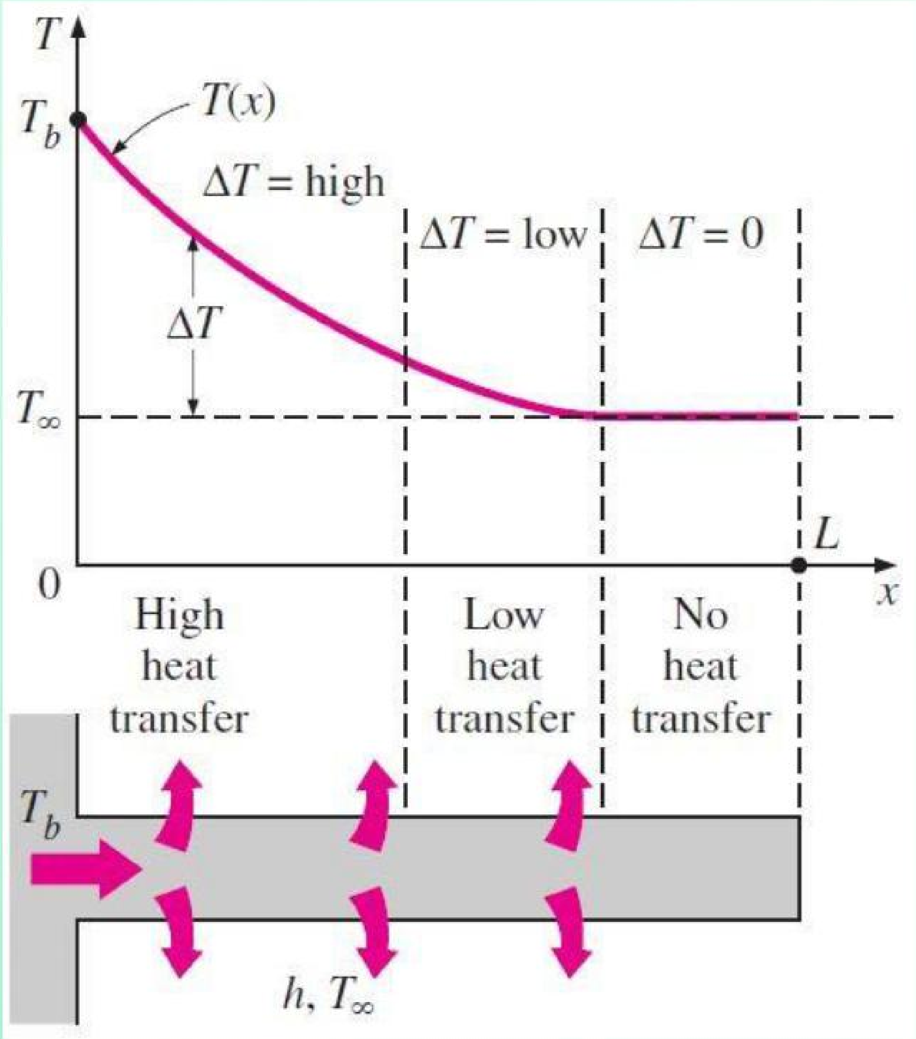
Comments

- The above results suggest that the fin heat transfer rate may accurately be predicted from the infinite fin approximation if $mL \geq 2.5$
- For more accuracy, if $mL \geq 4.6$:
  $L_\infty =$ 0.33 m (Cu), 0.23 m (Al) and 0.07 m (SS)

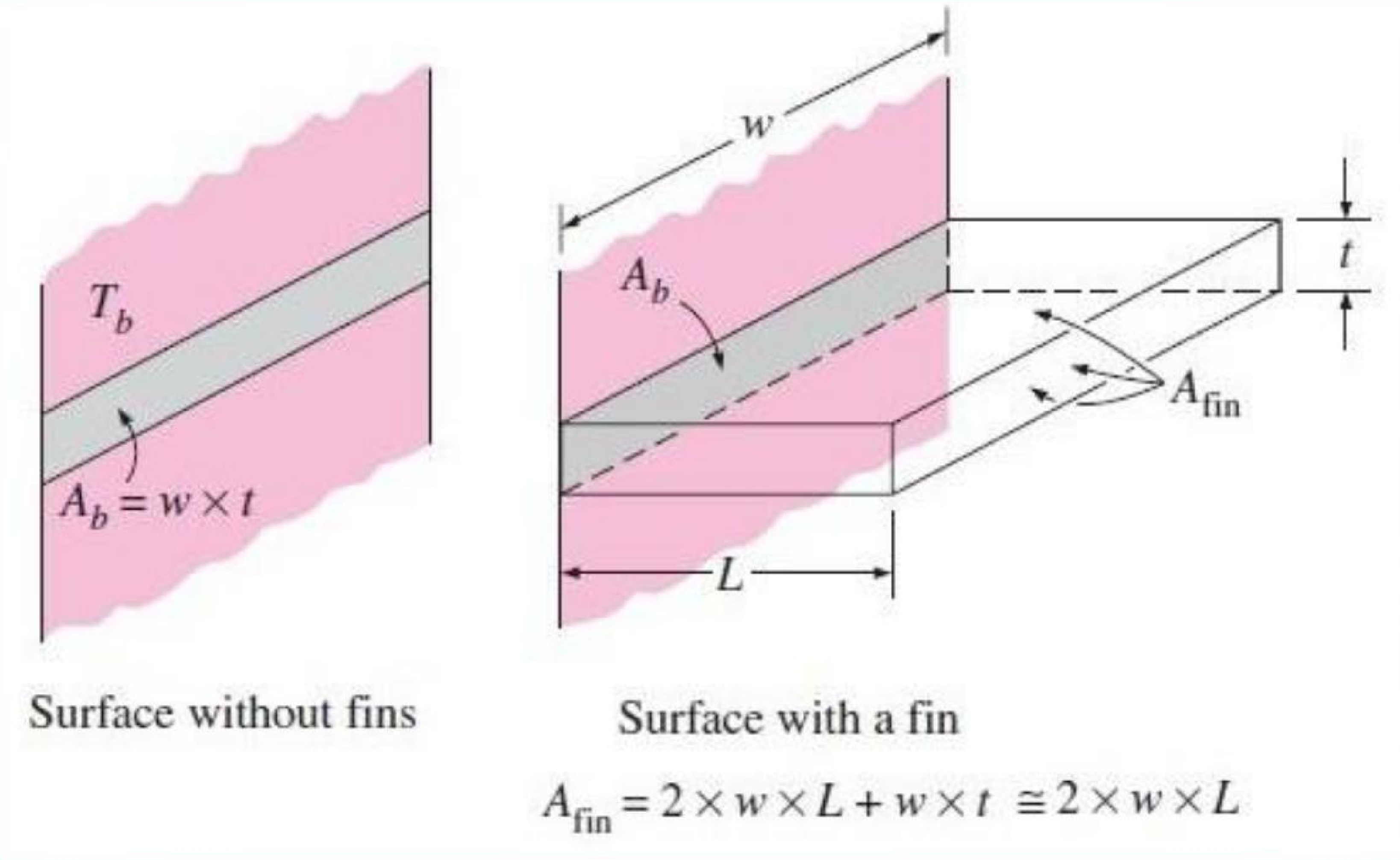
Proper Length of a Fin

$$\frac{q_{fin}}{q_{long\ fin}} = \tanh\ mL$$

mL	tanh *mL*
0.1	0.100
0.2	0.197
0.5	0.462
1.0	0.762
1.5	0.905
2.0	0.964
2.5	**0.987**
3.0	0.995
4.0	0.999
5.0	**1.000**



Fin Efficiency



No fin: $q_{conv} = hA_b(T_b - T_\infty)$

Fin Efficiency

- The temperature of the fin will be T_b at the fin base and gradually decrease towards the fin tip.
- Convection from the fin surface causes the temperature at any cross-section to drop somewhat from the midsection toward the outer surfaces.
- However, the cross-sectional area of the fins is usually very small, and thus the temperature at any cross-section can be considered to be uniform.
- Also, the fin tip can be assumed for convenience and simplicity to be adiabatic by using the corrected length for the fin instead of the actual length.

In the limiting case of zero thermal resistance or infinite k, the temperature of fin will be uniform at the value of T_b. The heat transfer from the fin will be maximum in this case ($k \to \infty$):

$$q_{fin,max} = hA_{fin}(T_b - T_\infty)$$

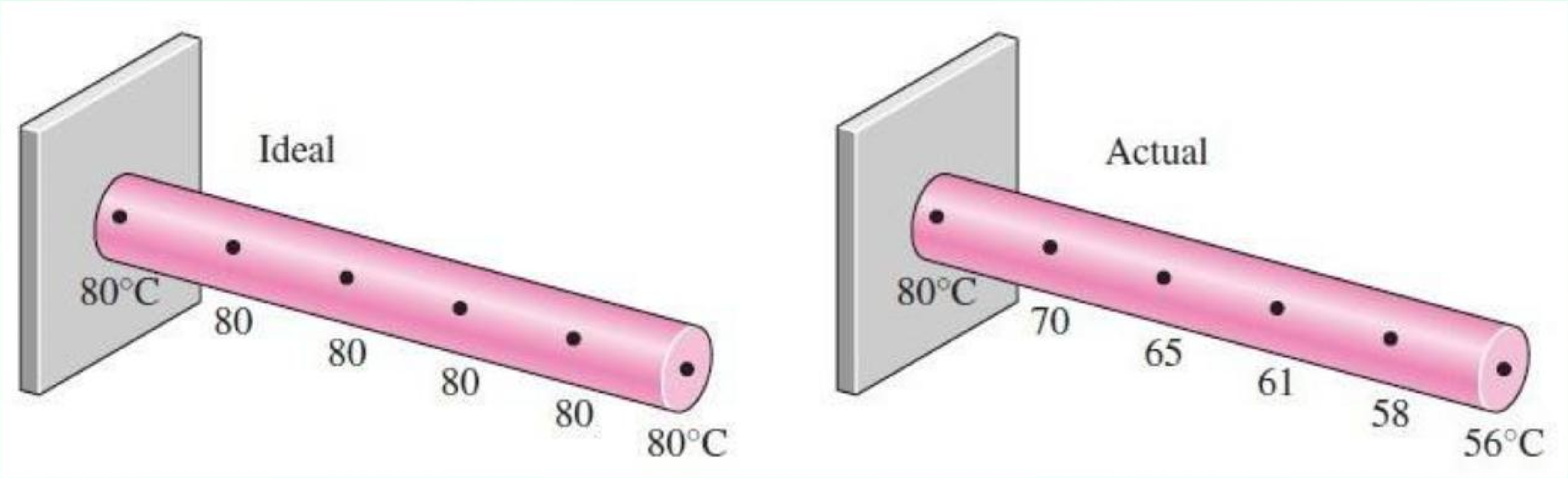
Fin Efficiency

In reality, however, the temperature of the fin will drop along the fin and thus the heat transfer from the fin will be less because of the decreasing $[T(x) - T_\infty]$ toward the fin tip.

To account for the effect of this decrease in temperature on heat transfer, we define fin efficiency as:

$$\eta_{fin} = \frac{q_{fin}}{q_{fin,max}} = \frac{\text{Actual heat transfer rate from the fin}}{\text{Ideal heat transfer rate from the fin if the entire fin were at base temperature}}$$

Fin Efficiency



Fin Efficiency: Uniform Cross-Sectional Area

Case A: Infinitely long fins

$$\eta_{\text{long fin}} = \frac{q_{fin}}{q_{fin,max}} = \frac{1}{mL}$$

$$\because A_{fin} = pL$$

Case B: Adiabatic tip

$$\eta_{\text{adiabatic}} = \frac{q_{fin}}{q_{fin,max}} = \frac{\tanh mL}{mL}$$

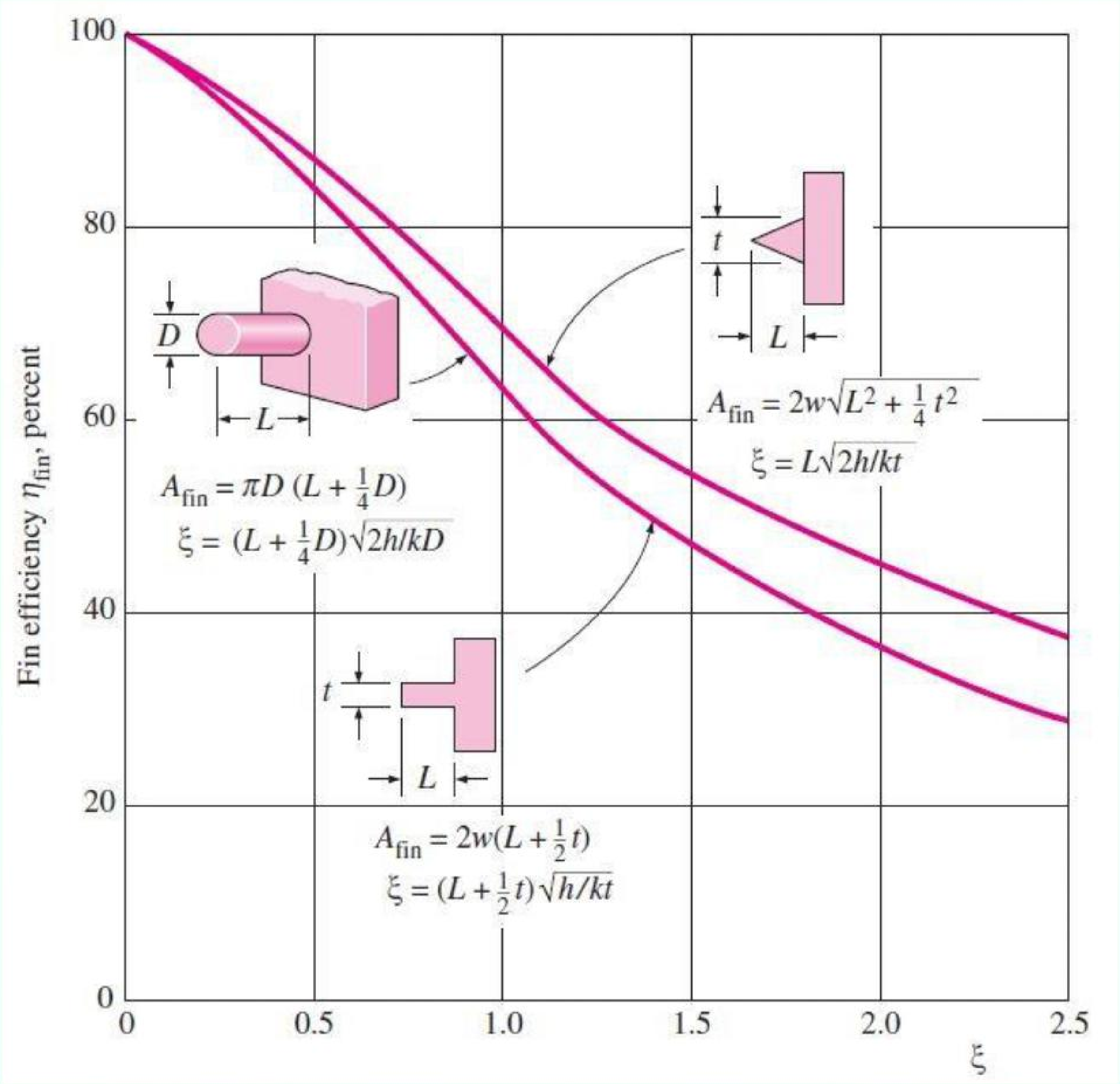
Case C: Convection at tip

$$\eta_{h\text{ at tip}} = \frac{q_{fin}}{q_{fin,max}} = \frac{\tanh mL_c}{mL_c}$$

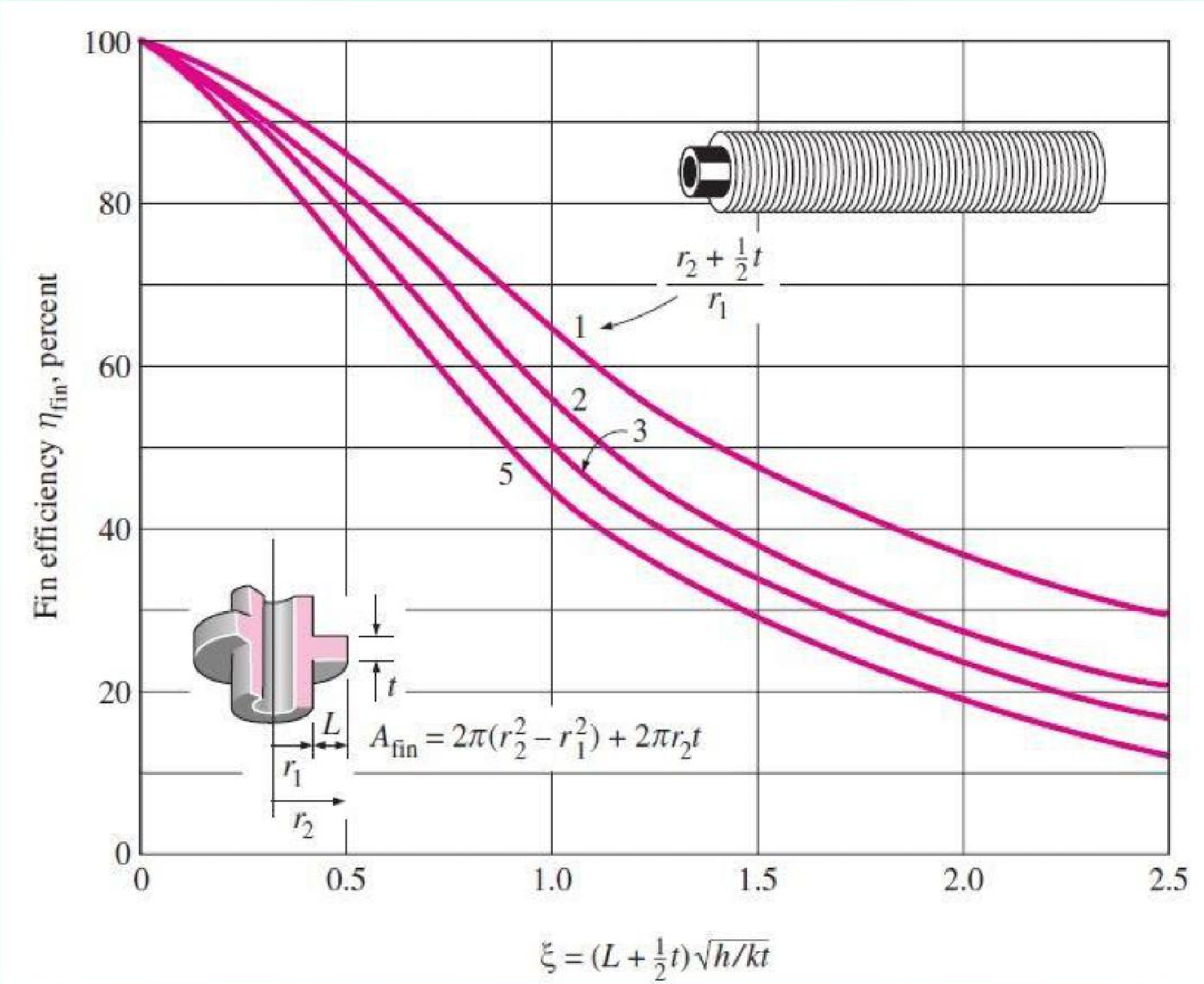
Fin Efficiency: Proper Length of a Fin

- An important consideration in the design of finned surace is the selection of the proper fin length, L.
- Normally the longer the fin, the larger the heat transfer and thus the higher the rate of heat transfer from the fin.
- Therefore, increasing the length of the fin beyond a certain value cannot be justified unless the added benefits outweigh the added cost.
- Also, η_{fin} decreases with increasing fin length because of the decrease in fin temperature with length.
- Fin lengths that cause the fin efficiency to drop below 60% usually cannot be justified economically and should be avoided.
- η of most fins used in practice is $> 90\%$.

η of Rectangular, Triangular, Parabolic Profiles



η of Annular fins of constant thickness, t

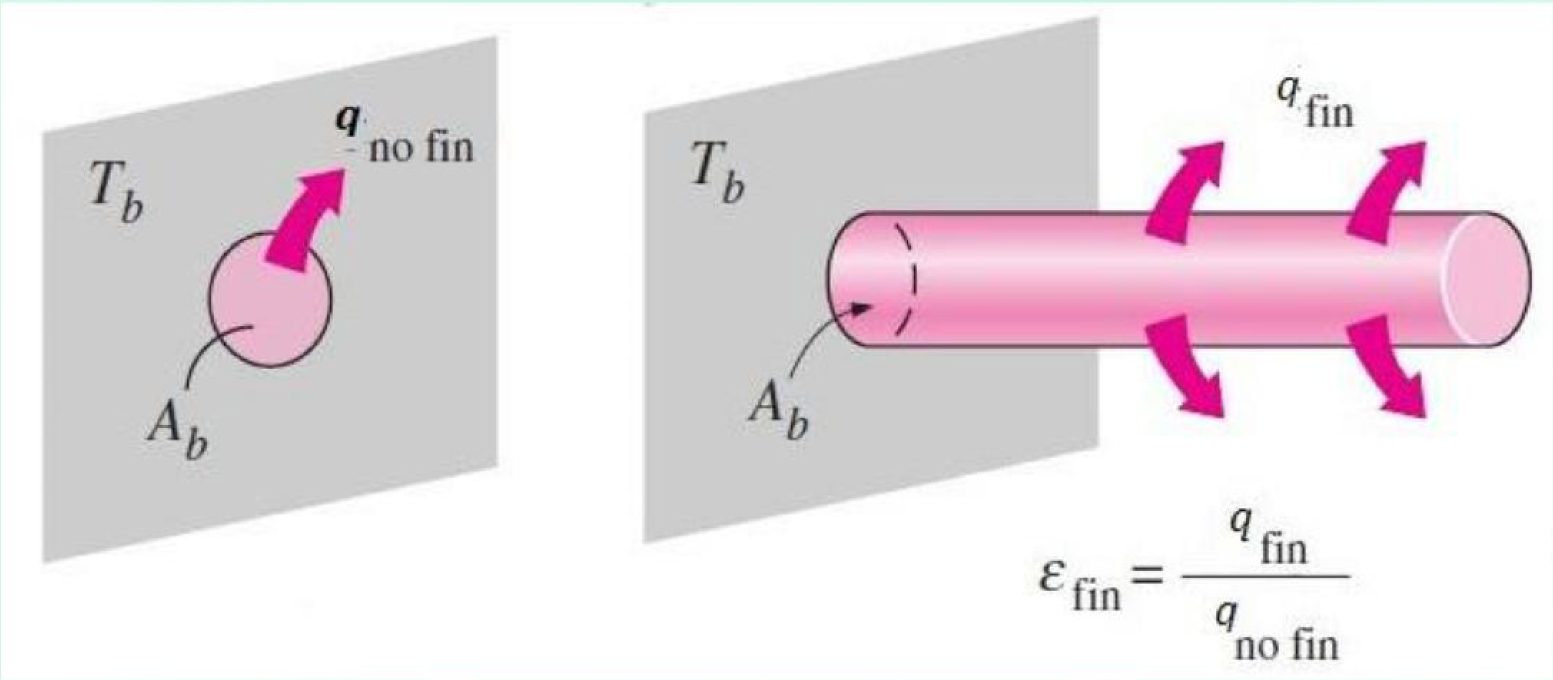


Fin Effectiveness

The performance of the fins is judged on the basis of enhancement of heat transfer relative to the no fin case.

$$\varepsilon_{fin} = \frac{q_{fin}}{q_{no\,fin}} = \frac{q_f}{hA_b(T_b - T_\infty)}$$

where A_b is the fin cross-sectional area at the base.



Fin Effectiveness: Physical Significance

- $\varepsilon_{fin} = 1$ indicates that the addition of fins to the surface does not affect heat transfer at all. That is, heat conducted to the fin through the base area A_b is equal to the heat transferred from the same area A_b to the surrounding medium.
- $\varepsilon_{fin} < 1$ indicates that the fin actually acts as insulation, slowing down the heat transfer from the surface. This situation can occur when fins made of low k are used.
- $\varepsilon_{fin} > 1$ indicates that the fins are enhancing heat transfer from the surface. However, the use of fins cannot be justified unless ε_{fin} is sufficiently larger than 1 (≥ 2). Finned surfaces are designed on the basis of maximizing effectiveness of a specified cost or minimizing cost for a desired effectiveness.

Efficiency and Effectiveness

η_{fin} and ε_{fin} are related to performance of the fin, but they are different quantities.

$$\varepsilon_{fin} = \frac{q_{fin}}{q_{no\ fin}}$$

$$= \frac{q_{fin}}{hA_b(T_b - T_\infty)}$$

$$= \frac{\eta_{fin} h A_{fin}(T_b - T_\infty)}{hA_b(T_b - T_\infty)}$$

$$\Longrightarrow \boxed{\varepsilon_{fin} = \frac{\eta_{fin} A_{fin}}{A_b}}$$

Therefore, η_{fin} can be determined easily when ε_{fin} is known, or vice versa.

ε for a Long Uniform Cross-Section Fin

$$\varepsilon_{fin} = \frac{q_{fin}}{q_{no\,fin}} = \frac{\sqrt{hPkA_c}\,\theta_b}{hA_b(T_b - T_\infty)} = \sqrt{\frac{kP}{hA_c}}$$

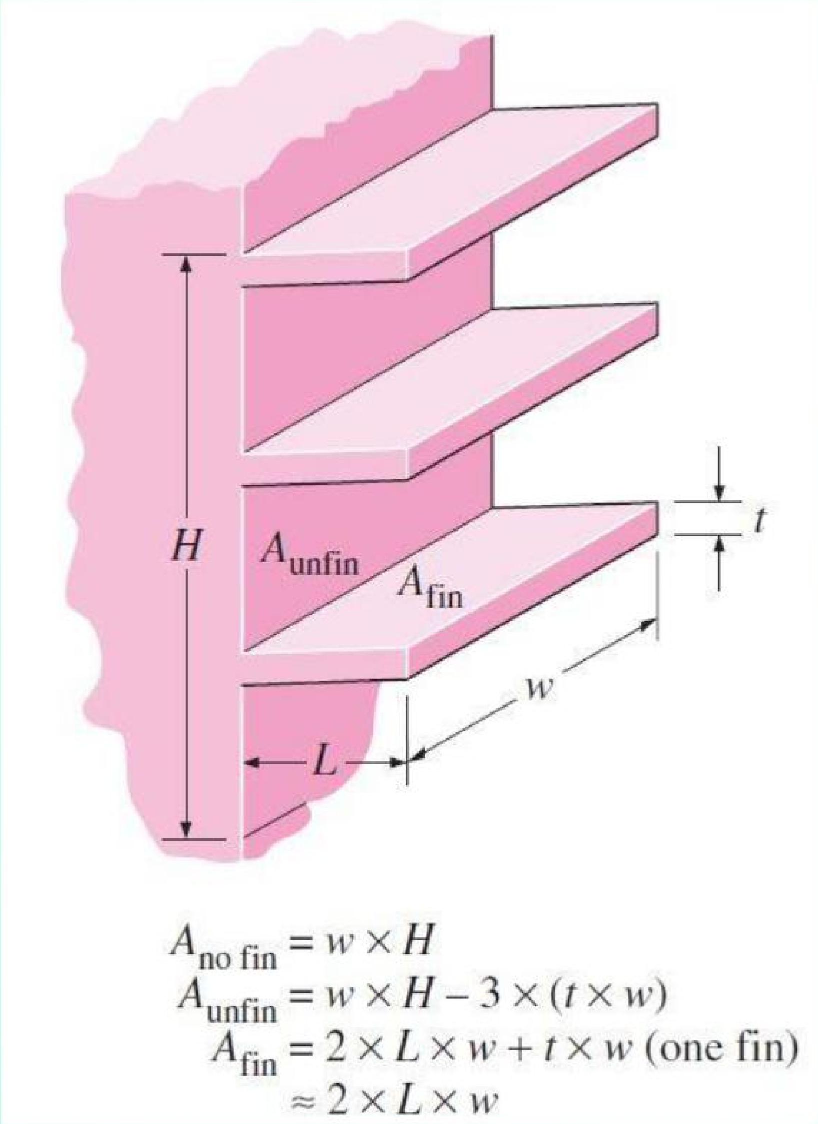
$\because A_c = A_b$ and $\theta_b = T_b - T_\infty$

- k of fin should be high. Ex: Cu, Al, Fe. Aluminium is low cost, weight, and resistant to corrosion.
- P/A_c should be high. Thin plates or slender pin fins
- h should be low. Gas instead of liquid; Natural convection instead of forced convection. Therefore, in liquid-to-gas heat exchangers (car radiators), fins are placed on the gas side.

Multiple Fins

Heat transfer rate for a surface containing n fins:

$$\begin{aligned} q_{tot,fin} &= q_{unfin} + q_{fin} \\ &= hA_{unfin}(T_b - T_\infty) \\ &\quad + \eta_{fin}A_{fin}(T_b - T_\infty) \end{aligned}$$



$$\boxed{q_{tot,fin} = h(A_{unfin} + \eta_{fin}A_{fin})(T_b - T_\infty)}$$

Overall Effectiveness

We can also define an overall effectiveness for a finned surface as the ratio of the total q from the finned surface to the q from the same surface if there were no fins:

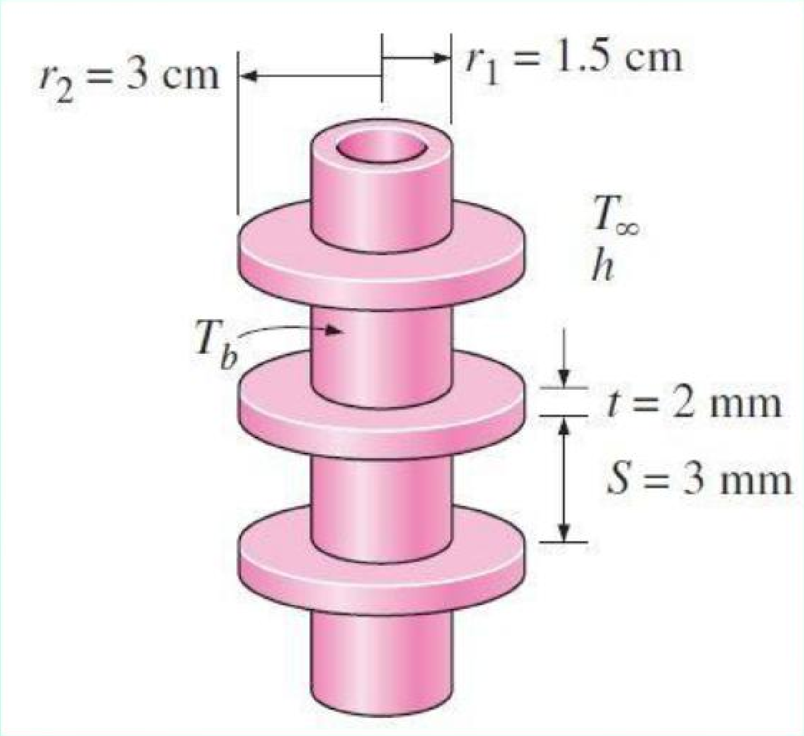
$$\varepsilon_{fin,overall} = \frac{q_{fin}}{q_{nofin}} = \frac{h(A_{unfin} + \eta_{fin}A_{fin})(T_b - T_\infty)}{hA_{nofin}(T_b - T_\infty)}$$

A_{nofin} is the area of the surface when there are no fins
A_{fin} is the total surface area of all the fins on the surface
A_{unfin} is the area of the unfinned portion of the surface.

$\varepsilon_{fin,overall}$ depends on number of fins per unit length as well as ε_{fin} of individual fins.
$\varepsilon_{fin,overall}$ is a better measure of the performance than ε_{fin} of individual fins.

Problem

Steam in a heating system flows through tubes: outer diameter is $D_1 = 3$ cm and whose walls are maintained at 125°C. Circular aluminium fins ($k = 180$ W/m K) of $D_2 = 6$ cm, $t = 2$ mm are attached. The space between the fins is 3 mm, and thus there are 200 fins per meter length of the tube. Surrounded air: $T_\infty = 27$°C, $h = 60$ W/m^2 K. Determine the increase in heat transfer from the tube per meter of its length as a result of adding fins.



Solution

Known

Properties of the fin, ambient conditions, heat transfer coefficient, dimensions of the fin.

Find

Increase in heat transfer from the tube per meter of its length as a result of adding fins.

Assumptions

- Steady state conditions, 1-D along the rod
- Constant properties and uniform h
- Negligible radiation exchange with surroundings

Solution: Analysis

In case of no fins (per unit length, $l = 1$ m:

$$A_{nofin} = \pi D_1 l = 0.0942\,\text{m}^2$$

$$q_{nofin} = hA_{nofin}(T_b - T_\infty) = 554\,\text{W}$$

$$r_1 = D_1/2 = 0.015\,\text{m}$$

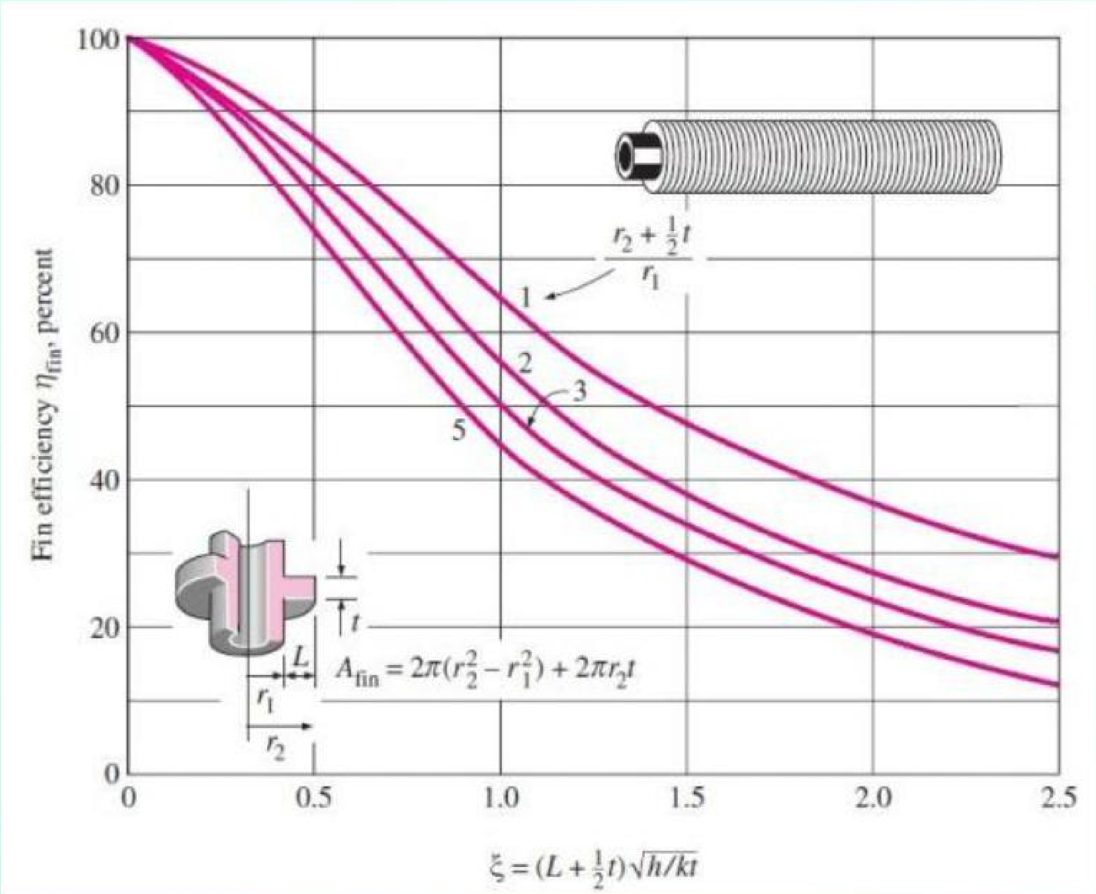
$$r_2 = D_2/2 = 0.03\,\text{m}$$

$$\frac{r_2 + \frac{t}{2}}{r_1} = \mathbf{2.07}$$

$$L = r_2 - r_1 = 0.015\,\text{m}$$

$$\xi = \left(L + \frac{t}{2}\right)\sqrt{\frac{h}{kt}} = \mathbf{0.207}$$

$$\Longrightarrow \boxed{\eta_{fin} = 0.95}$$



Solution: Analysis

q from finned portion

$$A_{fin} = 2\pi\left(r_2^2 - r_1^2\right) + 2\pi r_2 t$$
$$= 0.00462 \text{ m}^2$$
$$q_{fin} = \eta_{fin} q_{fin,max}$$
$$= \eta_{fin} h A_{fin}(T_b - T_\infty)$$
$$= 27.81 \text{ W}$$

q from unfinned portion of tube

$$A_{unfin} = 2\pi r_1 S$$
$$= 0.000283 \text{ m}^3$$
$$q_{unfin} = h A_{unfin}(T_b - T_\infty)$$
$$= 1.67 \text{ W}$$

There are 200 fins per meter length of the tube. The total heat transfer from the finned tube:

$$q_{tot,fin} = n(q_{fin} + q_{unfin}) = 5896 \text{ W}$$

$\therefore$ the increase in heat transfer from the tube per meter of its length as a result of the addition of fins is:

$$q_{increase} = q_{tot,fin} - q_{nofin} = \mathbf{5342\ W}$$ per meter tube length

Solution: Comments

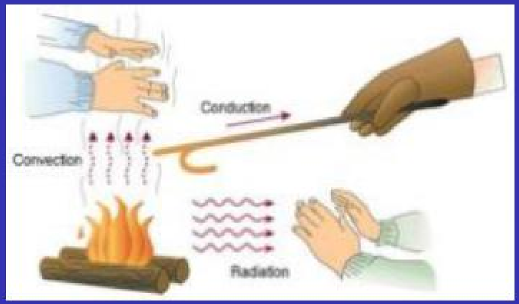
Effectiveness

The overall effectiveness of the finned tube is:

$$\varepsilon_{fin,overall} = \frac{q_{tot,fin}}{q_{tot,nofin}} = 10.6$$

That is, the rate of heat transfer from the steam tube increases by a factor of 10 as a result of adding fins.

Heat Transfer



Transient Heat Conduction

Transient Heat Conduction

Time dependent conduction - Temperature history inside a conducting body that is immersed suddenly in a bath of fluid at a different temperature.

Ex: Quenching of special alloys, heat treatment of bearings

The temperature of such a body varies with time as well as position.

$$T(x, y, z, t)$$

Transient Heat Conduction

A body is exposed to ambient

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{1}{a}\frac{\partial T}{\partial t}$$

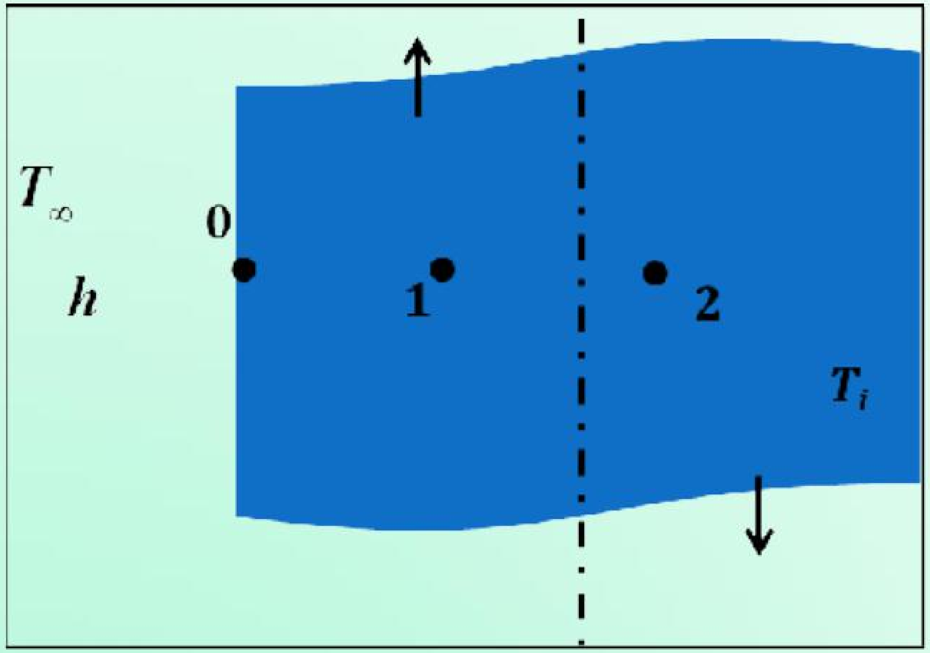
No heat generation

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}$$

$a \rightarrow$ Thermal diffusivity (m^2/s)

It appears only in the transient conduction

$$T = f(x, t)$$



$$T_{t=0} = T_i$$

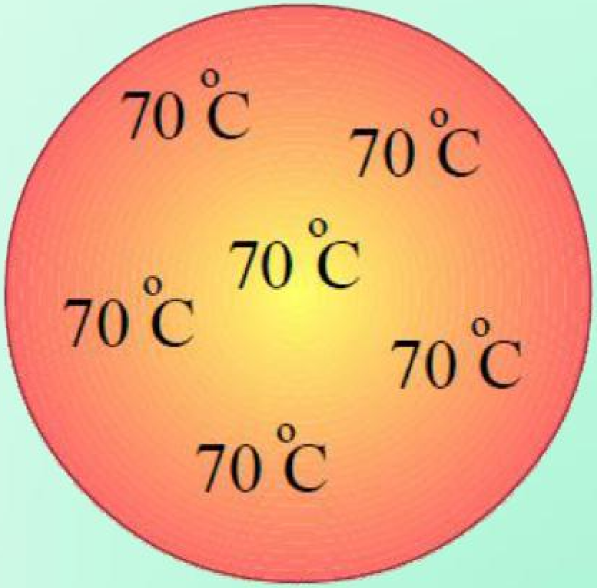
$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

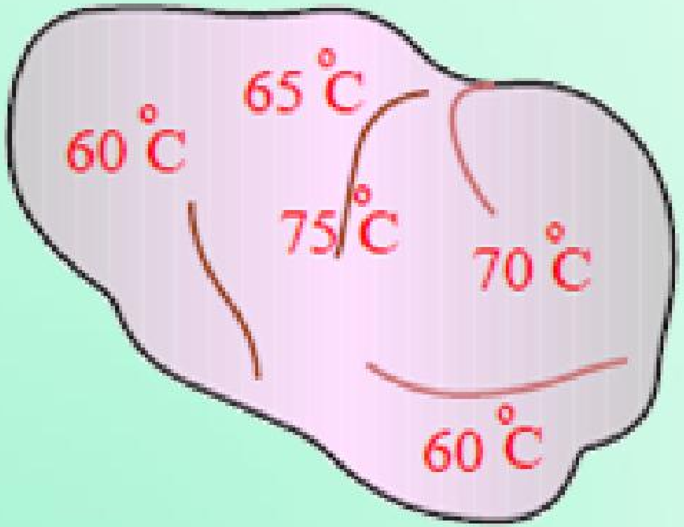
$$q_{x=\pm L} = h(T_\infty - T)$$

Lumped Capacitance Model

Lumped: Temperature is essentially uniform throughout the body.

$$T(x, y, z, t) = T(t)$$





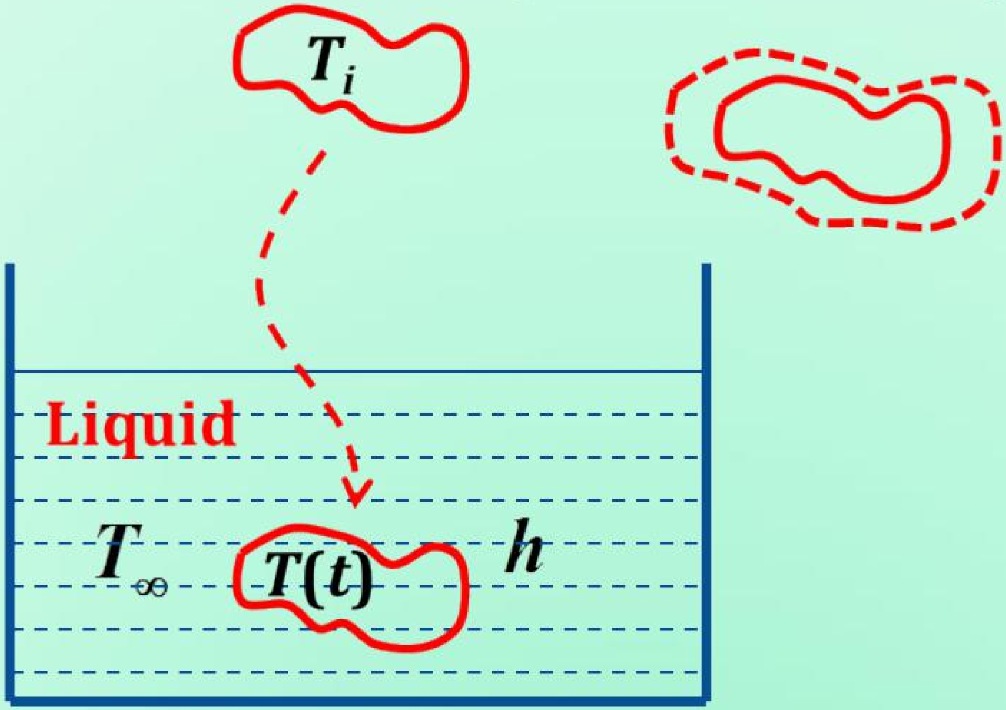
Copper ball with uniform temperature

Pot

Lumped Capacitance Model

Hot forging that is initially at uniform temperature, T_i and is quenched by immersing it in a liquid of lower temperature $T_\infty < T_i$

$$t = 0,\ T = T_i \qquad\qquad \delta \dot{q} = \frac{dE}{dt}$$



$$\dot{Q}_{in} - \dot{Q}_{out} + \dot{Q}_{gen} = \dot{Q}_{st}$$

Lumped Capacitance Model

$$-hA(T - T_\infty) = \rho V C_p \frac{dT}{dt}$$

$$\int_{T=T_i}^{T} \frac{dT}{T - T_\infty} = -\frac{hA}{\rho V C_p} \int_{t=0}^{t} dt$$

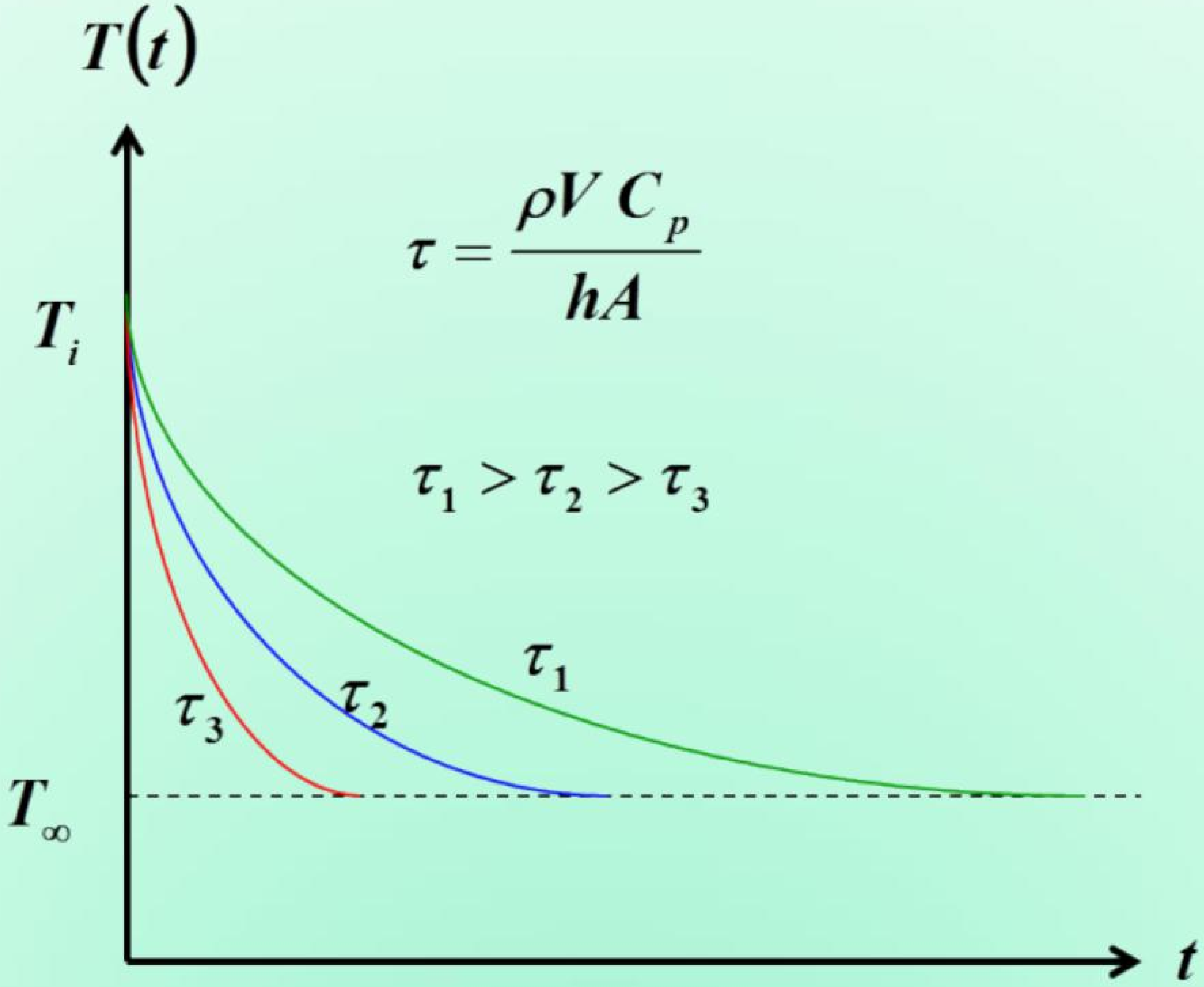
$$\boxed{\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{t}{\tau}} \qquad \tau = \frac{\rho V C_p}{hA}}$$

$$\tau = \frac{1}{hA} (\rho V C_p) = R_t C_t$$

R_t - Resistance to convection heat transfer

C_t - Lumped thermal capacitance of the solid

Time Constant



$$\left.\frac{\theta}{\theta_i}\right|_{t=\tau} = 0.368$$

Total Energy Transfer

The rate of convection heat transfer between the body and its environment at any time: $q = hA[T(t) - T_\infty]$

Total energy transfer occurring up to sometime, t:

$$Q = \int_{t=0}^{t} q\,dt$$

$$= \int_{t=0}^{t} hA[T(t) - T_\infty]\,dt$$

$$= hA(T_i - T_\infty) \int_{t=0}^{t} e^{-\frac{t}{\tau}}$$

$$\boxed{Q = \rho V C_p (T_i - T_\infty)\left(1 - e^{-\frac{t}{\tau}}\right)}$$

Criteria of the Lumped System Analysis

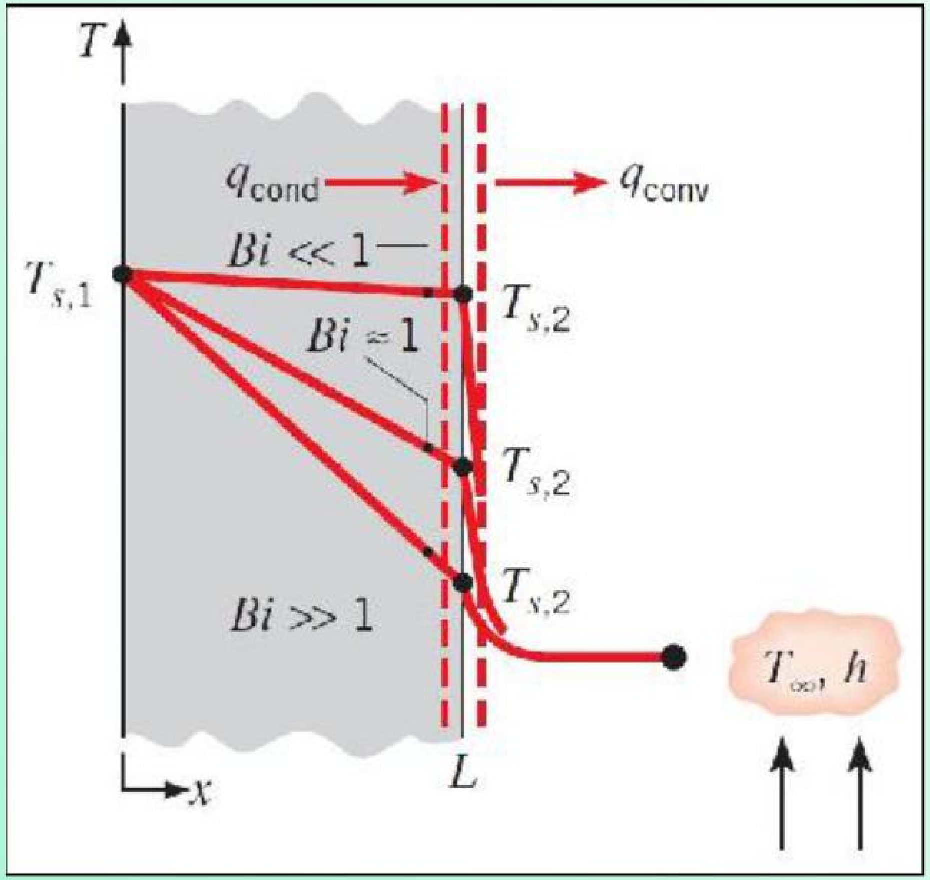
Consider a body exposed to ambient

$$q_{conv} = q_{cond}$$

$$h(T_{s,2} - T_{\infty}) = k\frac{T_{s,1} - T_{s,2}}{L_c}$$

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_{\infty}} = \frac{hL_c}{k} = \text{Bi}$$

$$L_c = \frac{V}{A}$$



Biot Number

$$\text{Bi} = \frac{hL_c}{k} = \frac{h\Delta T}{k\Delta T/L_c} = \frac{\text{Conv. at the surface of the body}}{\text{Conduction within the body}}$$

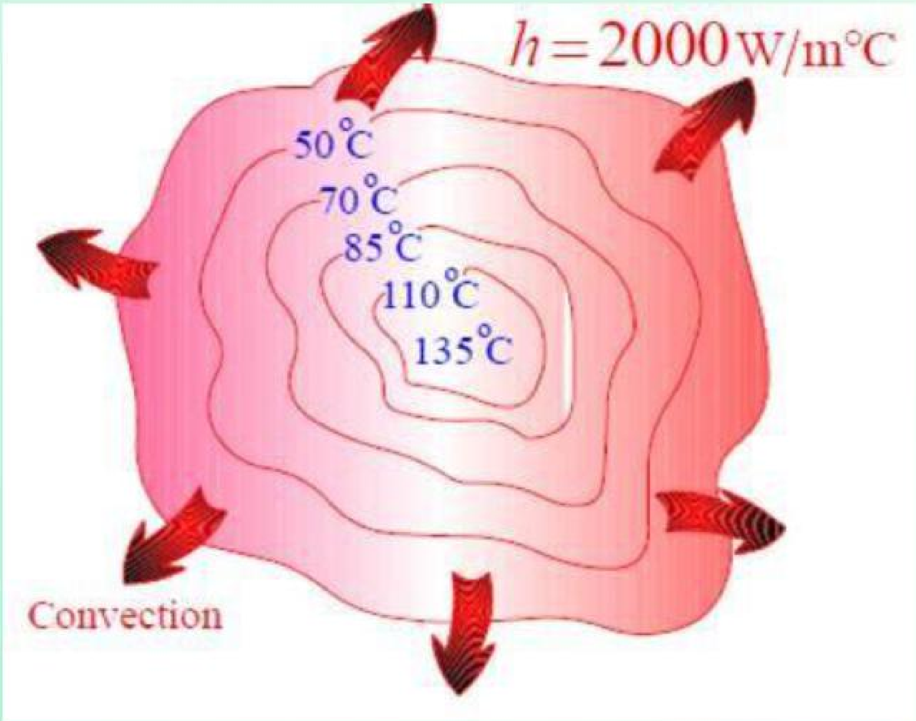
$$\text{Bi} = \frac{L_c/k}{1/h} = \frac{\text{Conduction resistance within the body}}{\text{Conv. resistance at the surface}}$$

Generally accepted, $\boxed{\text{Bi} \leq 0.1}$ for assuming lumped.

Biot Number

Small bodies with higher k and low h are most likely satisfy Bi ≤ 0.1.

When k is low and h is high, large temperature differences occur between the inner and outer regions of the body.



Professor Jean-Baptiste Biot

Jean-Baptiste Biot
(1774-1862)

- French physicist, astronomer, and mathematician born in Paris, France.
- Professor of mathematical physics at Coll`ege de France .
- At the age of 29, he worked on the analysis of heat conduction even earlier than Fourier did (unsuccessful). After 7 years, Fourier read Biot's work.
- Awarded the Rumford Medal of the Royal Society in 1840 for his contribution in the field of Polarization of light.

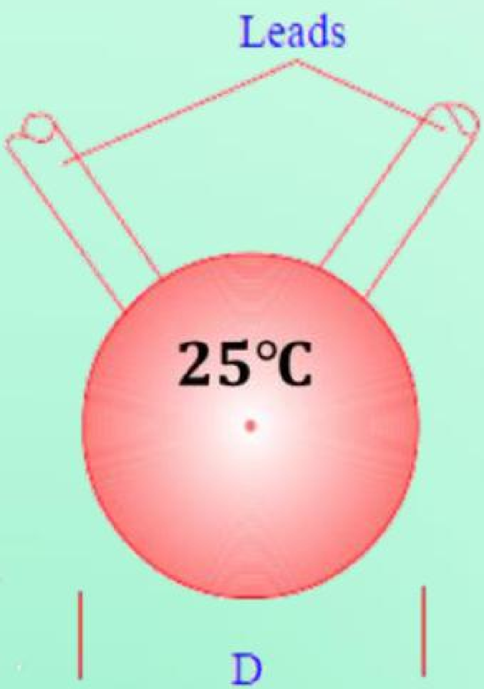
Problem: Thermocouple Diameter

Determine the thermocouple junction diameter needed to have a time constant of one second.

Ambient: $T_\infty = 200°$C, $h = 400$ W/m^2 K

Material

properties: $k = 20$ W/m K, $C_p = 400$ J/kg K, $\rho = 8500$ kg/m^{3}Ans:



0.706 mm

Problem: Solution

Known

Thermo-physical properties of the thermocouple junction used to measure the temperature of a gas stream.
Thermal environmental conditions.

Find

Junction diameter needed for a time constant of 1 second.

Assumptions

- Temperature of the junction is uniform at any instant.
- Radiation exchange with the surroundings is negligible.
- Losses by conduction through the leads is negligible.
- Constant properties.

Problem: Analysis

$$L_c = \frac{V}{A} = \frac{\pi D^3/6}{\pi D^2} = \frac{D}{6}$$

$$\tau = \frac{\rho C_p V}{hA} = \frac{\rho C_p D}{6h}$$

$$\boxed{D = 0.706\,\text{mm}}$$

$$\text{Bi} = \frac{hL_c}{k} = 2.35 \times 10^{-3} < 0.1$$

Criterion for using the lumped capacitance model is satisfied and the lumped capacitance method may be used to an excellent approximation.

Comments

Heat transfer due to radiation exchange between the junction and the surroundings and conduction through the leads would affect the time response of the junction and would, in fact, yield an equilibrium temperature that differs from T_∞.

Problem: Predict the Time of Death

A person is found dead at 5 PM in a room. The temperature of the body is measured to be 25°C when found. Estimate the time of death of that person.

$T_\infty = 20°\text{C}$

$h = 8\ \text{W/m}^2.\text{K}$

Known

T of the person at 5 PM.
Thermal environmental conditions.

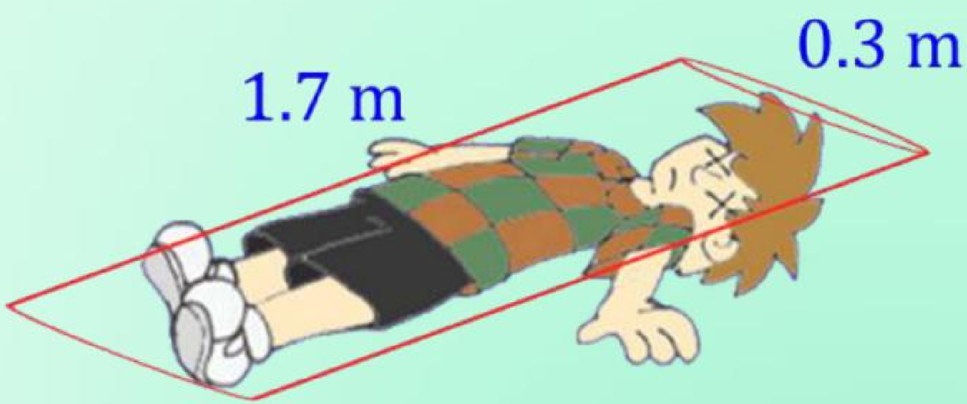
Find

The time of death of the person is to be estimated.

Problem: Solution

Assumptions

- The body can be modeled as a cylinder.
- Radiation exchange with the surroundings is negligible.
- The initial temperature of the person is 37°C. Assuming
- properties of water.



Water at (37 + 25)/2 = 31°C

$k = 0.617$ W/m.K

$Cp = 4.178$ kJ/kg.K

$\rho = 996$ kg/m^3

$$L_c = \frac{V}{A} = \frac{(\pi D^2/4)L}{\pi DL + 2(\pi D^2/4)} = 0.069 \text{ m}$$

Problem: Solution

$$\text{Bi} = \frac{hL_c}{k} = 0.9 > 0.1$$

Comment: Criterion for using the lumped capacitance model is not satisfied. However, let us get a rough estimate.

$$\tau = \frac{\rho C_p V}{hA} = \frac{\rho C_p V}{hA} = 35891\ \text{s}$$

$$\frac{T - T_\infty}{T_i - T_\infty} = \frac{25 - 20}{37 - 20} = e^{-\frac{t}{\tau}}$$

$$\boxed{t = 43923\ \text{s} = 12.2\ \text{hours}}$$

Therefore, the person would have died around 5 AM.