Assignment 4

AI1110: Probability and Random Variables

Indian Institute of Technology Hyderabad

Bonthu Mani Hemanth Reddy
CS22BTECH11013

12.13.6.3 Question: In a game, a man wins a rupee for a six and loses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins / loses.

Solution: Let X represents number of times six appears in die rolls. Then,

$$\Pr(X = i|X = i) = \frac{5}{6} \tag{1}$$

$$\Pr(X = i+1|X = i) = \frac{1}{6} \tag{2}$$

Therefore, This is a markov process in which ith state describes number of sixes in i die rolls with transition probabilities as given below.

$$P_{i,i+1} = \frac{1}{6} \tag{3}$$

$$P_{i,i} = \frac{5}{6} \tag{4}$$

For 3 throws,

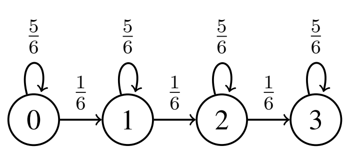

Fig. 1. Transition Graph

The transition matrix of this chain which represent the probabilities of transitions after one die roll is

$$A = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & 0 & 0 \\ 0 & \frac{5}{6} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{5}{6} \end{bmatrix} \tag{5}$$

To get probabilities after 2 and 3 die rolls, We need to raise this matrix to the power 2 and 3 respectively.

$$A^2 = \begin{bmatrix} 0.6944 & 0.2778 & 0.0278 & 0 \\ 0 & 0.6944 & 0.2778 & 0.0278 \\ 0 & 0 & 0.6944 & 0.2778 \\ 0 & 0 & 0 & 0.6944 \end{bmatrix} \tag{6}$$

$$A^3 = \begin{bmatrix} 0.5787 & 0.3472 & 0.0694 & 0.0046 \\ 0 & 0.5787 & 0.3472 & 0.0694 \\ 0 & 0 & 0.5787 & 0.3472 \\ 0 & 0 & 0 & 0.5787 \end{bmatrix} \tag{7}$$

Let $P_k(i, j)$ represents the probability of ending in jth state when started in ith state in k die rolls.

Here, We start with $X = 0$ and end with atmost 1 six ($X \leq 1$). So, $i = 0$.

The required probabilities are,

$$P_1(0,1) = \frac{1}{6} = 0.1667 \tag{8}$$

$$P_3(0,0) = 0.5787 \tag{9}$$

For $P_3(0,1)$ and $P_2(0,1)$, We need to exclude the transitions from 1 to 1. Hence,

$$P_2(0,1) = 0.2778/2 = 0.1389 \tag{10}$$

$$P_3(0,1) = 0.3472/3 = 0.1157 \tag{11}$$

In k die rolls ending at jth state, the winnings are,

$$W_(j,k) = 2 \times j - k \tag{12}$$

Let W denotes the winnings in the game

From the definition of expectation,

$$\begin{aligned} E(W) &= \sum_{k=1}^{k=3} P_k(0,1) \times (2 \times 1 - k) + P_3(0,0) \times (2 \times 0 - 3) \\ &= 0.1667 \times 1 + 0.1389 \times 0 \\ &\quad + 0.1157 \times -1 + 0.5787 \times -3 \end{aligned}$$

$$\therefore\ E(W) = -1.6851$$