MATH6646 Project Report

A Few Numerical Methods for Stochastic Differential Equations

Hemanth Sarabu, Venkata Ramana Makkapati, Vinodhini Comandur

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1 Introduction to SDEs

Stochastic differential equations generalize ordinary differential equations and their applications originated initially in the finance sector, later covering various other areas including insurance, marketing, robotics, biology, etc. A quick history to SDE has to start with Robert Brown observing Brownian motion in 1827 which was later used to model stock process in the Paris Bourse by Bachelier. It was Einstein (1906) and Wiener (1923) who gave formalism to what we now call Wiener process (or Brownian motion). However, we needed a tool like calculus invented by Leibniz and Newton to analyze stochastic processes and it was Ito who filled the gap in 1944 by proposing Ito calculus for SDEs. Subsequently, the topic gained prominence when in 1973, Nobel prize winners Merton, Black and Scholes applied it to study initiated entire derivatives and risk management.

1.1 Preliminaries

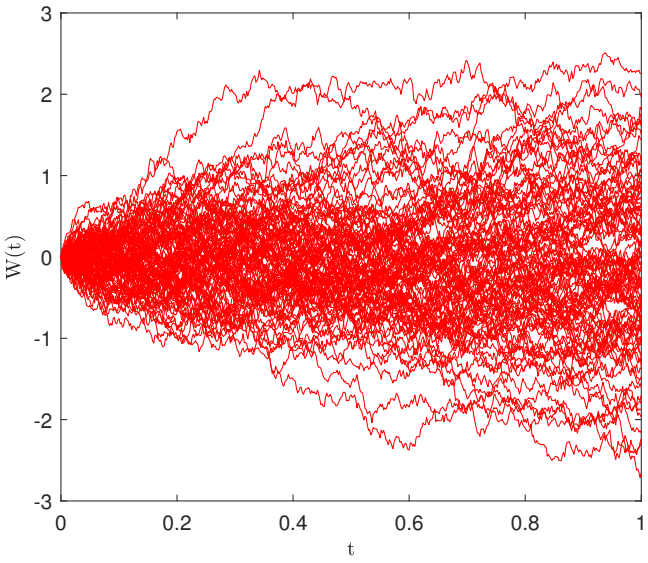


Figure 1: Wiener process: Observe the mean staying close to 0 and variance increasing with time.

The Wiener process over the time interval $[t_0, t_f]$ has the following properties

1. $W(t)$ is a random variable that depends continuously on $t \in [t_0, t_f]$;
2. $W(t_0) = 0$ (with probability 1);
3. $t_0 \leq s < t \leq t_f$, $W(t) - W(s)$ is normally distributed with zero mean and variance $t - s$;
4. $t_0 \leq s < t < u < v \leq t_f$, $W(t) - W(s)$ and $W(u) - W(v)$ are independent.

Furthermore, almost all trajectories of the Wiener process have unbounded variation, which implies a trajectory of Wiener process will eventually take any value. That is reason why even though the trajectories of Wiener process are continuous in time, they are not differentiable almost everywhere. Also, Wiener process is Markovian, which means the future depends on the past through the present.

Given a continuous function $f(t)$ (function of time), the integral $\int_{t_0}^{t_f} f(t)dt$ can be defined using the Riemann sum

$$\int_{t_0}^{t_f} f(t)\, dt = \lim_{h \to 0} \left(h \sum_{i=1}^{n} f(t_i^*) \right), \tag{1}$$

where h is the length of each interval, assumed to be a constant for a given discretization $t_0 < t_1 < t_2 < \cdots < t_n = t_f$, and t_i^* can be the left-end point (t_{i-1}) or the right end-point (t_i) or the mid-point $\left(\frac{t_{i-1} + t_i}{2} \right)$ of each interval i. The integral converges to the same value in the all the three cases. Now, an integral of a function $g(t)$ with respect to Brownian motion can be defined using a Riemann sum approximation as

$$\int_{t_0}^{t_f} g(t)\, dW(t) = \lim_{h \to 0} \left(\sum_{i=1}^{n} g(t_{i-1})(W(t_{i-1}) - W(t_i)) \right), \tag{2}$$

which is known as the Ito integral, and an alternative is to define

$$\int_{t_0}^{t_f} g(t) \circ dW(t) = \lim_{h \to 0} \left(\sum_{i=1}^{n} g\left(\frac{t_{i-1} + t_i}{2} \right) (W(t_{i-1}) - W(t_i)) \right), \tag{3}$$

which is known as the Stratonovich integral. Note that both the integrals do not necessarily converge to the same value. In this report, we will stick to the Ito's way of defining a stochastic differential equation based on the Ito integral defined above, as opposed to Statonovich's.

Consider the integral

$$X(t) = X_0 + \int_{t_0}^{t_f} f(X(t))\, dt + \int_{t_0}^{t_f} g(X(t))\, dW(t), \tag{4}$$

and assume that the functions f and g satisfy linear growth bound and are sufficiently smooth. Then it becomes a solution to the stochastic differential equation (or you can call stochastic process)

$$dX(t) = f(X(t))\, dt + g(X(t))\, dW(t), \quad X(t_0) = X_0 \tag{5}$$

Here, $f(X(t))$ is called the drift coefficient and $g(X(t))$ is called the diffusion coefficient. For the sake of simplicity, $X(t)$ is assumed to be a scalar for now however, the analysis can be easily extended to vectors. Some important results in stochastic calculus are discussed next.

1.2 Stochastic Taylor Series Expansion and Chain Rule

The well know chain rule on a function $F(x)$

$$dF(X(t)) = \frac{\partial F(X)}{\partial X}\, dX(t) = f(X(t)) \frac{\partial F(X)}{\partial X}\, dt, \tag{6}$$

is one of the most important operations in calculus, assuming there is no diffusion term in the dynamical equation for the state X. The equivalent in stochastic calculus is the Ito's lemma (based on the SDE given in equation (5))

$$dF(X(t)) = \left[f(X(t)) \frac{\partial F(X(t))}{\partial X} + \frac{1}{2} g^2(X(t)) \frac{\partial^2 F(X(t))}{\partial X^2} \right] dt + g(X(t)) \frac{\partial F(X(t))}{\partial X} \, dW(t). \tag{7}$$

Another important result in the general calculus is the Taylor series expansion which forms the basis for all numerical analysis in ordinary differential equations. Since we are interested in numerical methods for stochastic differential equations, an analogous is the stochastic Taylor series expansion

$$X(t) = X_0 + f(X_0) \int_{t_0}^{t} d\tau + g(X_0) \int_{t_0}^{t} dW(\tau) + g(X_0) g'(X_0) \left[\frac{1}{2} (W(t) - W(t_0))^2 - \frac{1}{2} (t - t_0) \right] + H.O.T \tag{8}$$

can be used to come up with numerical methods to propagate SDEs. The proof for the above Taylor series expansion can be found in Ref. [1].

2 Numerical Methods: Euler-Maruyama and Milstein methods

In 1955, Japanese mathematician, Gisiro Maruyama, came up with a numerical method that is a natural extension to the forward Euler method. The method is given by

$$X_j = X_{j-1} + f(X_{j-1}) \Delta t + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1})), \tag{9}$$

where $j = 1, 2, \ldots, N$, $\tau_j = j\Delta t$. The term $W(\tau_j) - W(\tau_{j-1})$ is called Brownian increment. When $g(X) = 0$, the emergence of forward Euler method can be observed, and hence the name Euler-Maruyama (EM) method.

Note that the EM method misses a term that is first order in time, observe equation (8). A more accurate method is the Misltein method given by

$$X_j = X_{j-1} + f(X_{j-1}) \Delta t + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1})) \tag{10}$$

$$+ \frac{1}{2} g(X_{j-1}) g'(X_{j-1}) ((W(\tau_j) - W(\tau_{j-1}))^2 - \Delta t). \tag{11}$$

The method was put forth by a Russian mathematician, Grigori N. Milstein in 1974. Note that the method involves finding a first order derivative for the diffusion coefficient, which can be computationally intensive.

The test equation to simulate and analyze the above methods is of the form

$$dX(t) = \lambda X(t) dt + \mu X(t) dW(t), \quad X(t_0 = 0) = X_0, \tag{12}$$

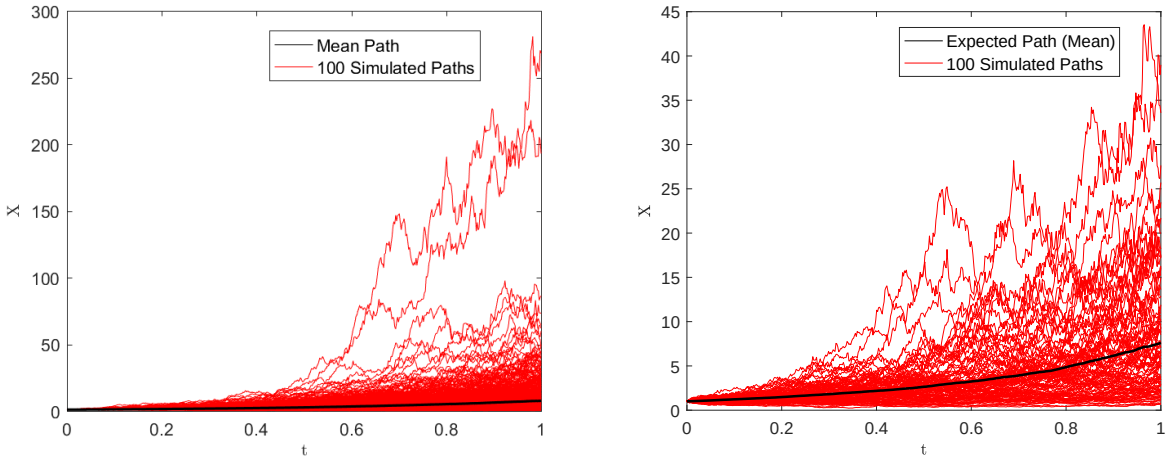
whose actual solution is

$$X(t) = X(t) exp \left((\lambda - \frac{1}{2} \mu^2) t + \mu W(t) \right). \tag{13}$$

For the numerical simulations, carried out in MATLAB, $\lambda = 2$, $\mu = 1$, $X_0 = 1$, $T = 1$.

2.1 Monte Carlo Simulations and Expected Value

The Monte Carlo method is a particularly useful tool when governing equations take the form of SDEs. The solutions to the deterministic analogue are often simpler to solve and tractable. In the case of SDEs, however, a given solution is conditioned on the specific Brownian motion path $W(t)$ which is used to compute the solution. By its nature, the process is random and hence a single generated path does not shed light on the characteristics of other possible solutions that represent the same SDE. This is where Monte Carlo



(a) Euler-Maruyama method

(b) Milstein method

Figure 2: Monte Carlo simulations of the test equation

simulations are employed; one can integrate to obtain solutions for multiple runs (with differing $W(t)$ paths) to study the evolution of the family of solutions. Information from these simulations can be used to compute likelihoods of certain events such as that of reaching a given endpoint $X(T)$. Often, the mean of all the generated paths is computed to derive the expected value of $W(t)$. The expected value is used to evaluate the weak order of convergence of a numerical method for SDEs which is discussed later. The Monte-Carlo simulations for EM and Milstein methods are plotted in Figure 2. The take-aways from these plots are: 1) Each trajectory is one possible solution for the given SDE; 2) The average trajectory, which is also called mean/expected path, is one such possible trajectory that can be analyzed to examine the set of solutions.

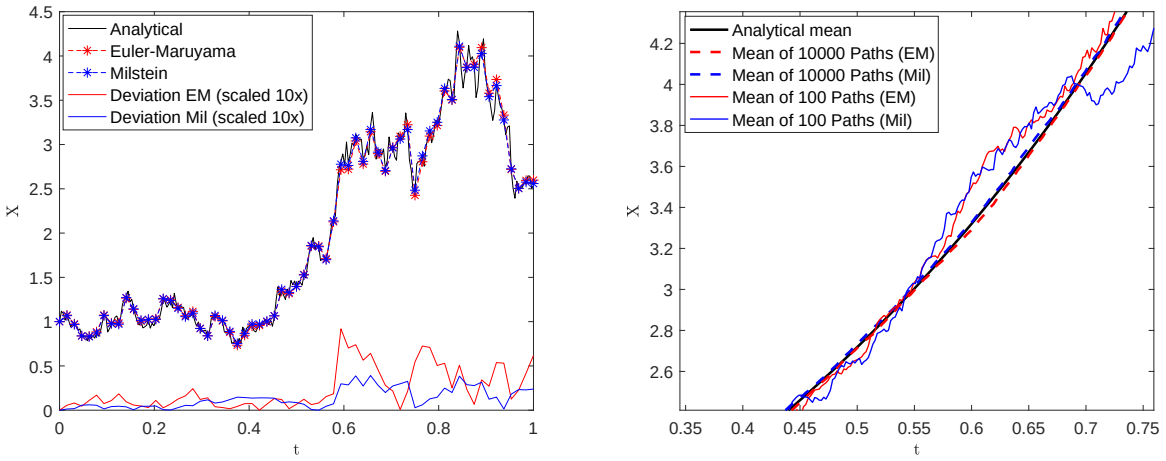
2.2 Comparison

The plots in Figure 3 can be analyzed to compare both methods. In Figure 3(a), an individual path is analyzed and it can be observed that both methods follow the path given by the actual solution obtained by from a fixed Brownian path reasonably well. However, the Milstein method is more accurate, which can also be observed from the errors shown in the same plot. It can be observed that the errors for the Milstein method are lower on average compared to the EM method.

In Figure 3(b), the mean path for the actual solution is compared with the average trajectory obtained from the trajectories propagated using both methods. Again, Milstein's superior performance can be clearly observed. Another obvious observation is that as the number of trajectories which are considered to take the average path increases, the accuracy of the mean path for a method also increases.

3 Strong and Weak Convergence

Convergence in SDEs is not as direct as in the case of deterministic ODEs, owing to the presence of a random variable satisfying Brownian motion or Wiener process properties. For SDEs, the order of convergence is discussed depending on the nature of the SDE solution being considered. In some applications, the solution is required for a given path which is known as *strong solution* or *path-wise solution*. Here, the process (or the solution) satisfies the equation and is defined on a given probability space. The ability of a method to compute strong solutions on average is quantified by the *strong order of convergence*. On the other hand, a *weak solution* consists of a probability space and a process that satisfies the integral equation. In these cases, where only expected value information about the solution is of interest, and not the complete solution itself,



(a) Analyzing a single path

(b) Analyzing the mean path

Figure 3: Analysis plots

leads to the concept of *weak order of convergence*. Thus, how the difference between the random variables $X(t = t_n)$ (true solution) and X_n (numerical solution) is measured leads to strong or weak convergence. This overall approach to convergence gives more freedom to select a specific numerical method depending on the focus of the problem.

Using $\mathbb{E}|X_n - X(\tau_n)|$, where $\mathbb{E}$ denotes the expected value, leads to the concept of strong convergence. A method is said to have *strong order of convergence* equal to γ if there exists a constant C such that

$$\mathbb{E}|X_n - X(\tau_n)| \leq C\Delta t^{\gamma} \tag{14}$$

where $\tau_n = n\Delta t \in [0, T]$ and Δt sufficiently small. Thus, the strong order of convergence measures the rate at which the 'mean of the error' decays as $\Delta t \to 0$. A less demanding alternative is to measure the rate of decay of the 'error of the means'. This leads to the concept of weak convergence. A method is said to have *weak order of convergence* equal to γ if there exists a constant C such that for all functions p in some class

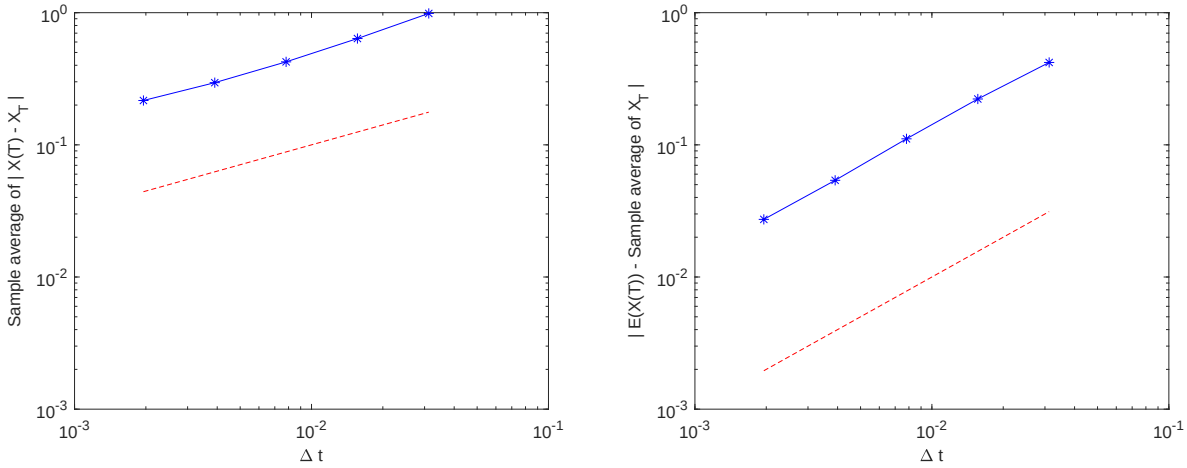
$$|\mathbb{E}p(X_n) - \mathbb{E}p(X(\tau_n))| \leq C\Delta t^{\gamma} \tag{15}$$

at any fixed $\tau_n = n\Delta t \in [0, T]$ and Δt sufficiently small. Typically, the functions p allowed in (15) must satisfy smoothness and polynomial growth conditions.

The numerical tests carried out in this project focus on the error at the endpoint $t = T$ for the linear SDE given in (12). Equation (12) is a special case of (5), with $f(X) = \lambda X$ and $g(X) = \mu X$. This SDE arises as an asset price model in financial mathematics. For consistency, the following values have been assumed and maintained across all simulations

$$\lambda = 2, \quad \mu = 1, \quad X_0 = 1, \quad \text{and} \quad T = 1.$$

The proofs for the order of convergence for the numerical methods have not been mentioned as they are beyond the scope of this study. The aforementioned values were obtained from [2].



(a) Strong (b) Weak

Figure 4: Convergence properties of EM method

3.1 Convergence of the Euler-Maruyama Method (EM)

3.1.1 Strong order of convergence

If f and g in (5) satisfy appropriate conditions, it can be shown that EM method has strong order of convergence $\gamma = \frac{1}{2}$. Thus,

$$e_{\Delta t}^{strong} := \mathbb{E}|X_T - X(T)| \leq C\Delta t^{\frac{1}{2}}, \quad where \quad T = L\Delta t \tag{16}$$

The numerical implementation for showing the strong convergence of EM method computes 1000 different discretized Brownian paths over the [0,1] with $\delta t = 2^{-9}$. For each path, EM method is applied with 5 different stepsizes $\Delta t = 2^{s-1}\delta t$ for $1 \leq s \leq 5$. If the inequality (16) holds with approximate equality, then

$$\log(e_{\Delta t}^{strong}) \approx \log(C) + \frac{1}{2}\log(\Delta t) \tag{17}$$

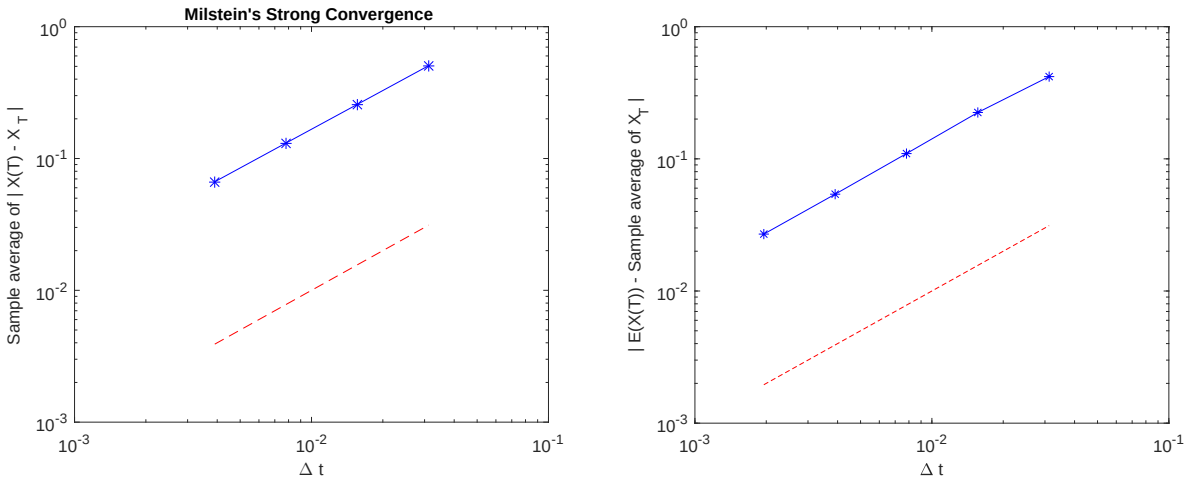
The results are plotted on a log-log scale which can be seen in Figure 4(a). For reference, a dashed red line of slope one-half is added. It is seen that the slopes of the two curves appear to match well, suggesting that (17) is valid. This is tested further by assuming that a power law relation $e_{\Delta t}^{strong} = C\Delta t^q$ exists for some constants C and q, so that $\log(e_{\Delta t}^{strong}) = \log(C) + q\log(\Delta t)$. A least squares fit for $\log(C)$ and q is computed at the end of the program for verification of the order of convergence. The numerical simulation resulted in $q = 0.5486$, which is close to the theoretical value of 0.5, with a least squares residual of 0.0796.

3.1.2 Weak order of convergence

For the numerical solution, the function p (in equation (15)) is taken as the identity function. For appropriate f and g (in equation (5)), it can be shown that EM method has weak order of convergence $\gamma = 1$.
Similar to the strong convergence tests,

$$e_{\Delta t}^{weak} := |\mathbb{E}p(X_n) - \mathbb{E}p(X(\tau_n))| \leq C\Delta t, \quad where \quad T = L\Delta t \tag{18}$$

denotes the weak endpoint error in EM method, with $p(X) \equiv X$. In the weak convergence case, the number of discretized Brownian paths had to be increased to 10^6 to obtain satisfactory results. The precise reason for this unknown but one possible explanation is that the strong convergence computes expected value of the terminal error while the weak convergence computes error in the expected values at the terminal. Error



(a) Strong (b) Weak

Figure 5: Convergence properties of Milstein's method

values are bound to be much lower than the actual values, and by obtaining expectation of these two different quantities, where the former could lead to a more concise value as opposed to the latter which could be more spread out, more paths would be required to lower the spread and obtain an acceptable value for the weak convergence method. In this case, 5 stepsizes $\Delta t = 2^{s-10}$ for $1 \leq s \leq 5$ were used and the path increment was chosen as $\delta t = \Delta t$ for extra efficiency. The results for weak convergence can be seen in Figure 4(b), where the dashed red reference line of slope one is added. For the weak convergence, the least squares power law fit gives $q = 0.9932$ and a residue of 0.0447, confirming the convergence order of one.

3.2 Convergence of the Milstein's Higher Order Method

3.2.1 Strong order of convergence

The Milstein's higher order method yields a strong order of 1. The numerical implementation considers $\delta t = 2^{-9}$ and 5 stepsizes $\Delta t = 2^{s-1}$, where $1 \leq s \leq 5$ over 1000 sample Brownian paths. The resulting log-log error plot is shown in Figure 5(a) along with a reference line of slope 1. The least-squares power law fit gives $q = 0.9756$, which matches well with theory, and a residue of 5.8645e-04.

3.2.2 Weak order of convergence

The Milstein's higher order method also yields a weak order of 1. The numerical implementation is very similar to that of the weak convergence of EM method by considering $\delta t = \Delta t$ and 5 stepsizes $\Delta t = 2^{s-10}$, where $1 \leq s \leq 5$ over 10^6 sample Brownian paths. The resulting log-log error plot is shown in Figure 5(b) along with a reference line of slope 1. The least-squares power law fit gives $q = 0.9971$ and a residue of 0.0493.

3.3 Weak Euler-Maruyama Method

Weak convergence concerns only the mean of the solution, so one is free to use any $\sqrt{\Delta t}N(0,1)$ sample for the Brownian path increment given by $W(\tau_j) - W(\tau_{j-1})$ on any step. Hence, the order of weak convergence is maintained if the increment is replaced by an independent two-point random variable $\sqrt{\Delta t}V_j$, where V_j takes the values +1 and -1 with equal probability (same mean and variance as the original normal distribution case). Therefore, the weak Euler-Maruyama (WEM) method is an extension of the EM method with weak

order of convergence by replacing the Brownian increment by $\sqrt{\Delta t}V_j$, which has a weak order of convergence $\gamma = 1$, but does not offer strong convergence since it does not use any pathwise information. The motivation behind WEM is that random number generators that sample from V_j can be made more efficient than those that sample from $N(0, 1)$. The resulting graph for WEM is displayed in Figure 6.

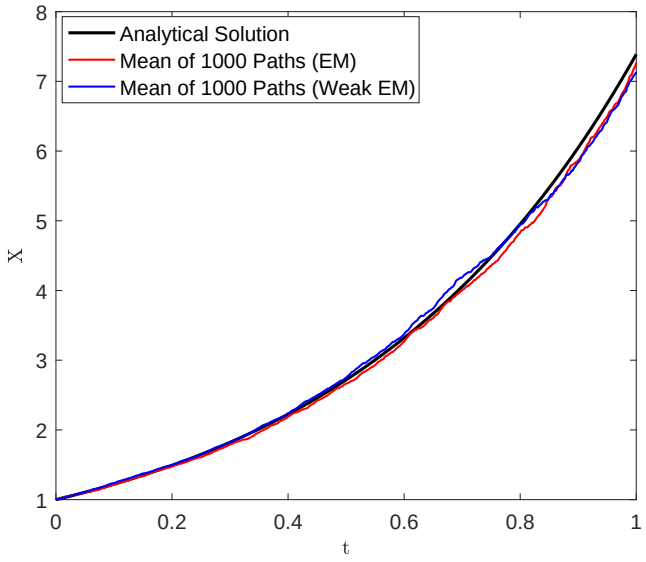


Figure 6: Results of the weak EM method

3.4 Key Observations and Conclusions

The numerical implementation for convergence led to the following key points -

- Weak EM method yields the same order of convergence as the EM method with weak convergence, although WEM method utilizes Brownian increment with an independent two-point random variable. This allows WEM to be more efficient than the EM method with weak convergence which samples from $N(0, 1)$.
- The numerical implementation for monitoring error in the strong convergence case assumes that a number of other sources of error are negligible, such as
  - *Sampling error*: the error arising from approximating an expected value by a sampled mean;
  - *Random number*: inherent errors in the random number generator;
  - *Rounding error*: floating point roundoff errors.
- For the numerical implementation of the methods, the number of samples has to be sufficiently large and the timestep must be sufficiently small for the predicted orders of convergence to be observable [2]. According to Ref. [2], as Δt decreases, lack of independence in the samples from a random umber generator typically degrades the computation before rounding errors become significant.
- As Δt decreases, the error between numerical and true solutions decreases.
- Milstein's method results in lower values of error between numerical and true solutions compared to EM method due to higher strong order of convergence.

4 Linear Stability

4.1 SDE Stability

The concepts of strong and weak convergence study the accuracy of a numerical method for a finite interval $[0, T]$ for small stepsizes Δt. However, the long-term ($t \to \infty$) behavior of an SDE is important to study as many applications are concerned with the evolution of an SDE solution over a long duration. Hence, convergence bounds of the form (14) or (15) are not relevant in this context, since the constant C generally grows unboundedly with T. For deterministic ODE methods, a large body of *stability theory* has been developed that gives insight into the behavior of numerical methods in the Δt fixed, $t_j \to \infty$ limit. The most popularly analyzed equation is the linear test equation $dX/dt = \lambda X$, where $\lambda \in \mathbb{C}$ is a constant parameter. For SDEs, it is possible to develop an analogous linear stability theory.
For a deterministic linear equation, which has solutions of the form $X_0 \exp(\lambda t)$, it is considered *stable* if

$$\lim_{t\to\infty} X(t) = 0 \quad \text{for any } X_0$$

For this condition to be satisfied, the stability of the equation is characterized by $Re(\lambda) < 0$. In order to generalize this idea to the SDE case, it is important to realize that random variables are infinite-dimensional objects and hence, norms are not equivalent in general. Two common measures of stability are used for SDEs: *mean-square* and *asymptotic*. Assuming that $X_0 \neq 0$ with probability 1, solutions of (12) satisfy

$$\lim_{t\to\infty} \mathbb{E}X(t)^2 = 0 \Leftrightarrow Re(\lambda) + \frac{1}{2}|\mu|^2 < 0 \tag{19}$$

$$\lim_{t\to\infty} |X(t)| = 0, \text{ with probability 1} \Leftrightarrow Re(\lambda - \frac{1}{2}\mu^2) < 0 \tag{20}$$

The left-hand side of (19) defines *mean-square stability* while the right=hand side characterizes this property in terms of the parameters λ and μ. Similarly, (20) defines and characterizes *asymptotic stability*. It follows immediately from (19) and (20) that if (12) is mean-square stable, then it is automatically asymptotic stable, but not vice versa. Hence, for this test SDE, mean-square stability is a more stringent requirement than asymptotic stability.

4.2 EM and Milstein's methods Stability

Considering the case where the parameters λ and μ are chosen so that the SDE (equation (5)) is stable in the mean-square or asymptotic sense. This subsequently leads to determining the stability of the numerical method, i.e. for what range of Δt is the EM or Milstein's solution stable in an analogous sense. The mean-square version for the numerical method is easy to analyze. Simple properties of the expected value show that

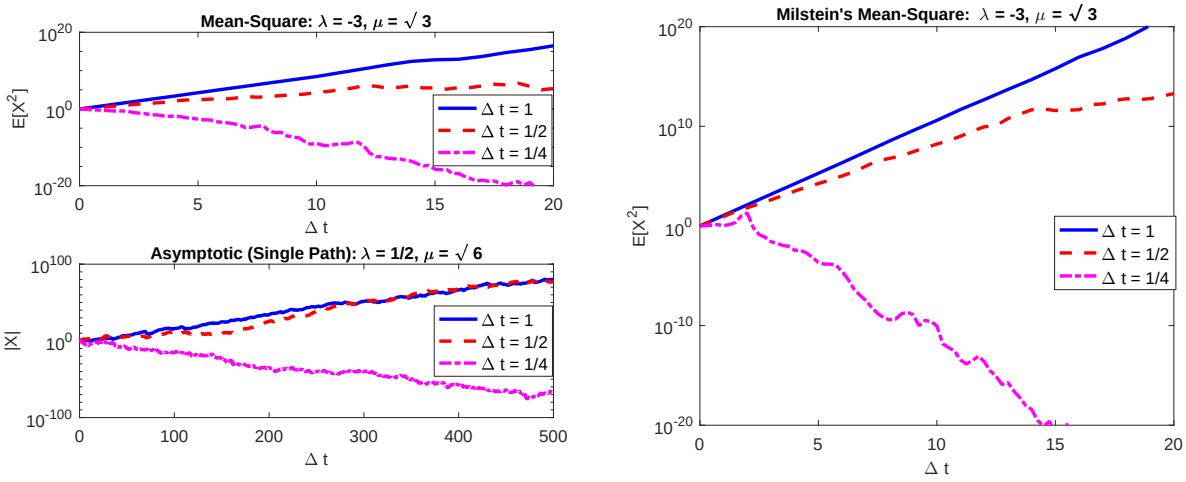
$$\lim_{j\to\infty} \mathbb{E}X_j^2 = 0 \Leftrightarrow |1 + \Delta t\lambda|^2 + \Delta t|\mu|^2 < 1 \tag{21}$$

$$\lim_{j\to\infty} \mathbb{E}X_j^2 = 0 \Leftrightarrow |1 + \Delta t\lambda|^2 + \Delta t|\mu|^2 + \frac{(\Delta t|\mu|^2)^2}{2} < 1 \tag{22}$$

where (21) represents EM applied to the test equation (5) and (22) is for Milstein's applied to (5). The asymptotic version of method stability can be studied via the strong law of large numbers and the law of the iterated logarithm, leading to

$$\lim_{j\to\infty} |X_j| = 0, \text{ with probability 1} \Leftrightarrow \mathbb{E}\log\left|1 + \Delta t\lambda + \sqrt{\Delta t}N(0,1)\right| < 0 \tag{23}$$

for EM on the test SDE. The asymptotic version of Milstein's method on the test SDE is not included as it is beyond the scope of this study.



(a) EM (b) Milstein

Figure 7: Stability analysis on numerical methods

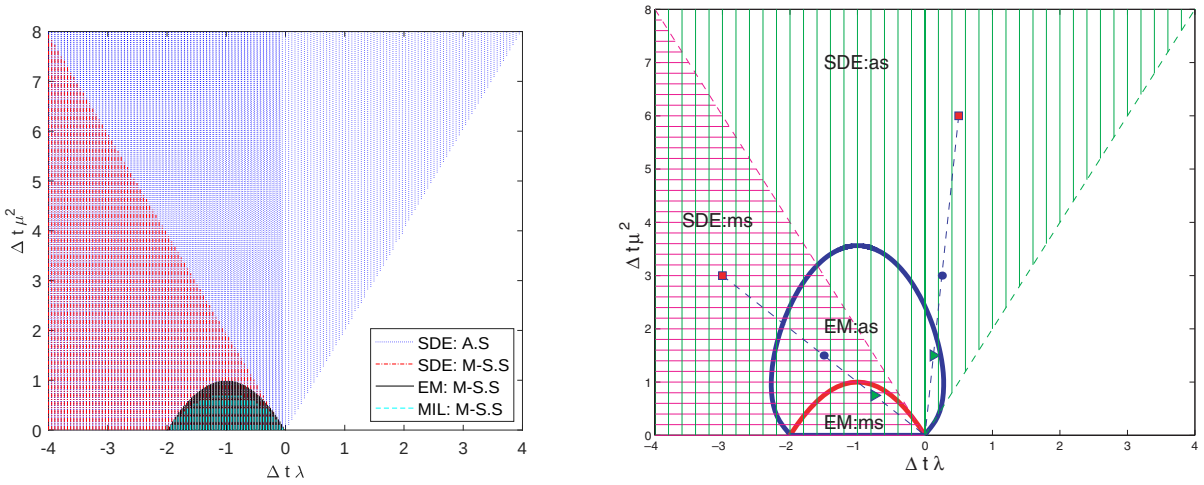
4.3 Results

The same test SDE is used to analyze stability. However, it can be noted that the values $\lambda = 2$ and $\mu = 1$ used in the previous sections of Monte-Carlo simulations and convergence violate both the mean-square (19) and asymptotic stability (20) conditions of the SDE. With the SDE stability requirements violated, it is ineffective to investigate the stability of the numerical methods as they would also yield unstable solutions.

To test mean-square stability, the test SDE (5) is solved with $X_0 = 1$ (constant) over the interval [0,20] for the parameter set $\lambda = -3$ and $\mu = \sqrt{3}$. These values clearly satisfy (19) and hence, the problem is mean-square stable. Both EM and Milstein's methods are applied over 50000 discrete Brownian paths for three different stepsizes: $\Delta t = 1,\ 1/2,\ 1/4$ as shown in Figure 7 respectively, which depict the sample average of X_j^2 against t_j. Note that the vertical axis is logarithmically scaled. For both methods, the $\Delta t = 1$ and $\Delta t = 1/2$ curves increase with t, while the $\Delta t = 1/4$ curve decays towards zero. Hence, this test correctly implies that for $\Delta t = 1,\ 1/2$, both EM and Milstein's methods are unstable while for $\Delta t = 1/4$, they are stable, in the mean-square sense. However, the number of Brownian sample paths used (50000) is not sufficient to resolve the behavior fully as the three curves should be straight lines. This highlights the fact that simplistic sampling without further checks may lead to misleading conclusions.

To examine asymptotic stability of EM, the parameter set of $\lambda = 1/2$ and $\mu = \sqrt{6}$ is chosen. It follows from (19) and (20) that the SDE is asymptotically stable but is not mean-square stable. Since asymptotic stability concerns a probability 1 event, EM is applied over a single discrete Brownian path for $\Delta t = 1,\ 1/2,\ 1/4$ as shown in Figure 7(a). Since the computation with a single path is cheap, the a larger interval [0,500] for integration is considered by plotting $|X_j|$ against t_j along the path. It can be shown that only the smallest of these timesteps, $\Delta t = 1/4$, satisfies the condition (23), which has been elaborated further below using Figure 8 for illustration. Also, for the chosen Δt, solution appears to decay to zero, in agreement with the theory.

The study of linear stability for numerical methods leads to the question of how to determine the choice of stepsize Δt for numerical evaluation of a given SDE. As explained earlier, stability is an attribute of both the SDE and the numerical method. The results presented earlier indicate a sample set of Δt values and how they affect the stability of the numerical methods for a chosen set of λ and μ of a stable test SDE. However, a more general assessment is necessary to determine suitable stepsize for the numerical method for any values of the SDE parameters. Figure 8(a) offers further insight into these computations and addresses the fundamental decision of stepsize value. Figure 8(a) plots the regions of stability for $\lambda, \mu \in \mathbb{R}$, with x-axis



(a) By us

(b) From the paper by Higham [2]

Figure 8: Stability regions

and y-axis representing $\Delta t\lambda$ and $\Delta t\mu^2$, respectively for the test SDE, as well as the EM and Milstein's numerical methods. The stability regions were obtained by considering the inequalities given by (19), (20), (21), and (22). Hence, Figure 8(a) only plots the mean-square as well as asymptotic stability regions of the test SDE, the mean-square stability region of the EM method and the mean-square stability region of the Milstein's method. Since asymptotic stability regions could not be plotted easily, Figure 8(b) obtained from [2] has also been included for verification.

4.4 Key Observations and Conclusions

On observing both figures of stability regions, the following points are made -

- Considering either the stability of the SDE or the stability of the numerical method, it is seen that the stability region for asymptotic stability is more compared to mean-square stability region for all cases. This is because mean-square stability is a stronger condition compared to asymptotic stability, thereby restricting the choices of λ, μ, and Δt more.
- On observing mean-square stability regions of both EM and Milstein's methods, it is observed that the stability region area is more for the EM method in comparison to Milstein's method. This is expected since the EM method has a lower strong order of convergence ($\gamma = \frac{1}{2}$), which would lead to more flexibility, as opposed to the Milstein's method ($\gamma = 1$).
- Based on the aforementioned points and the asymptotic stability region of the EM method shown in Figure 8, it can be concluded that the asymptotic stability region of Milstein's method is bound to be smaller than the EM method but definitely more than the mean-square stability region of Milstein's method.
- An A-stable numerical method would have a stability region which includes the stability region of the SDE at least. From this basic understanding, it is clear that both EM and Milstein's methods are not A-stable.

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