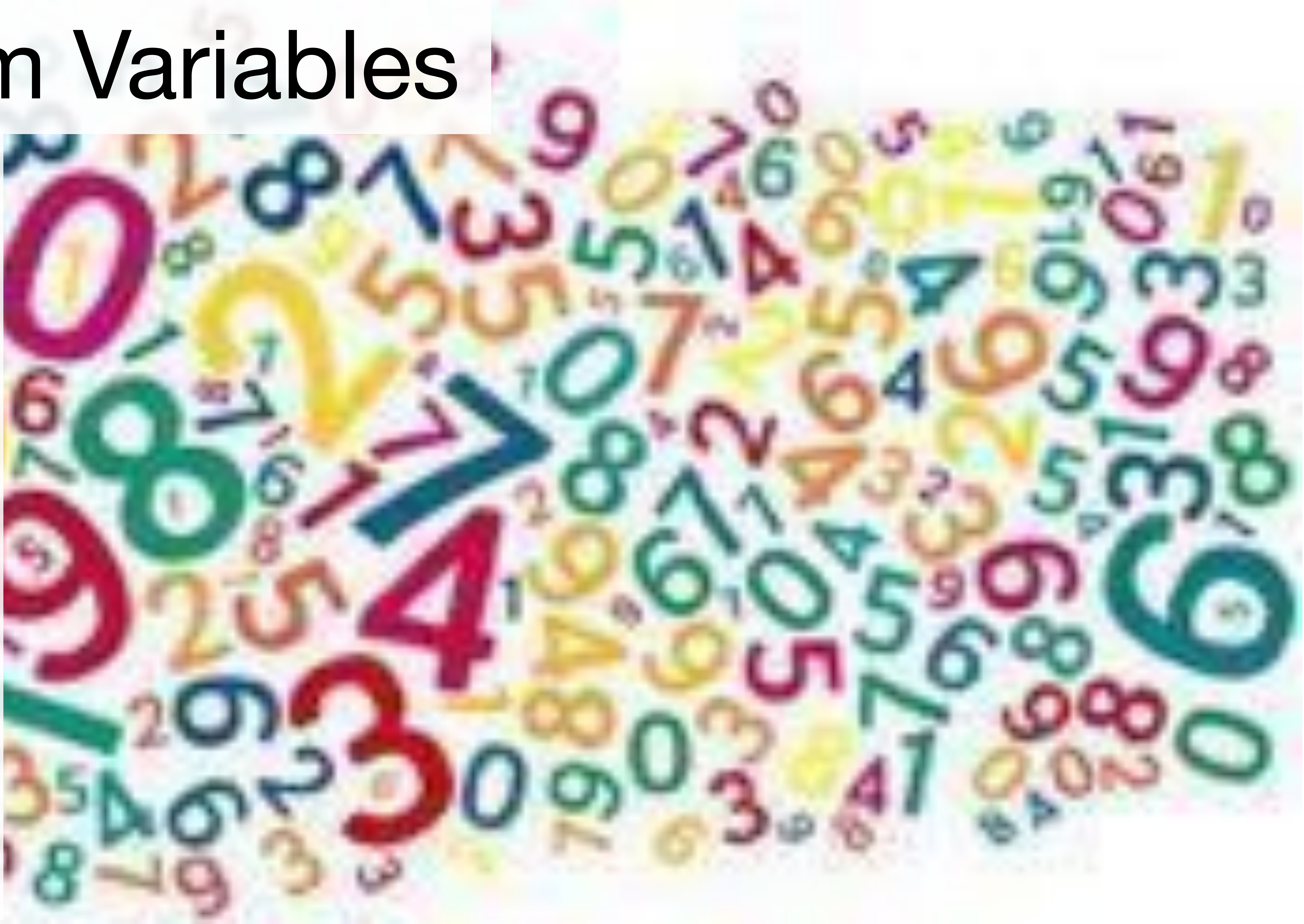
Random Variables

Motivation

Examples

Definitions

Visualization



What Matters

So far coins, dice, cards, dominoes, marbles, ….

Often subscribers clicks viewers yield weight sales

time congestion delay age temperature heart rate

GPA tuition assignment income cost

Numbers!

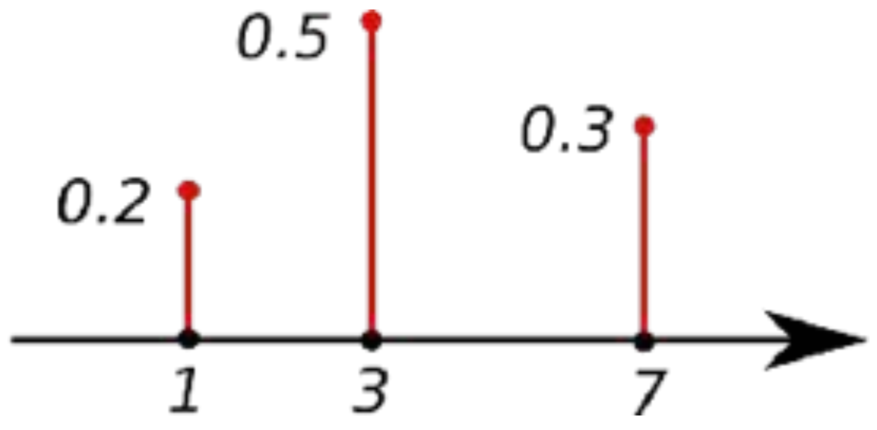
Random variable Number-valued random outcome

Xtra with Numbers

Distri-
bution

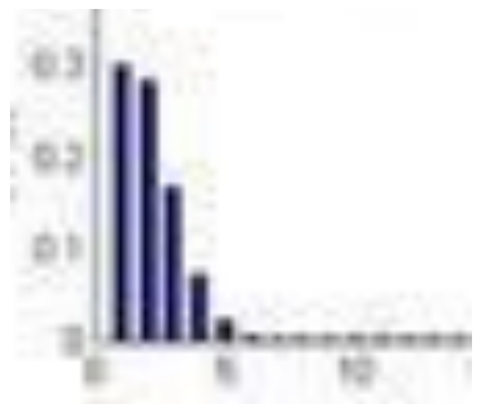
p(x)

View on a line



Express as function — $p(x)=1/x^2$

Consider properties — Decreasing



concentrated

Random
Variable

X

Perform operations — $X+1$ X^2

Combine variables — $X+Y$

Consider properties — average value of X

Two Types

Size of sample space Ω

Ω is finite $\{1, 2, 3\}$ $\{e, \pi\}$ or countably infinite $\mathbb{N}$ $\mathbb{Z}$ Discrete

Ω is uncountably infinite $[0,2]$ $(-1,3) \cup [4,5)$ $\mathbb{R}$ Continuous

Combination $[0,2] \cup \{e, \pi\}$ Mixed

Begin with discrete

Been There

Several past examples had number outcomes

Outcome of a die roll $\{1,\ldots,6\}$

Number of heads in 3 coin tosses $\{0,\ldots,3\}$

Values of a domino tile $\{0,\ldots,6\}$

Did not use numerical features → Use extensively

Familiar and new examples

General formulation

Heads

3 fair coins

Ω = { ttt, tth, tht, thh, htt, hth, hht, hhh }

$|\Omega| = 8$

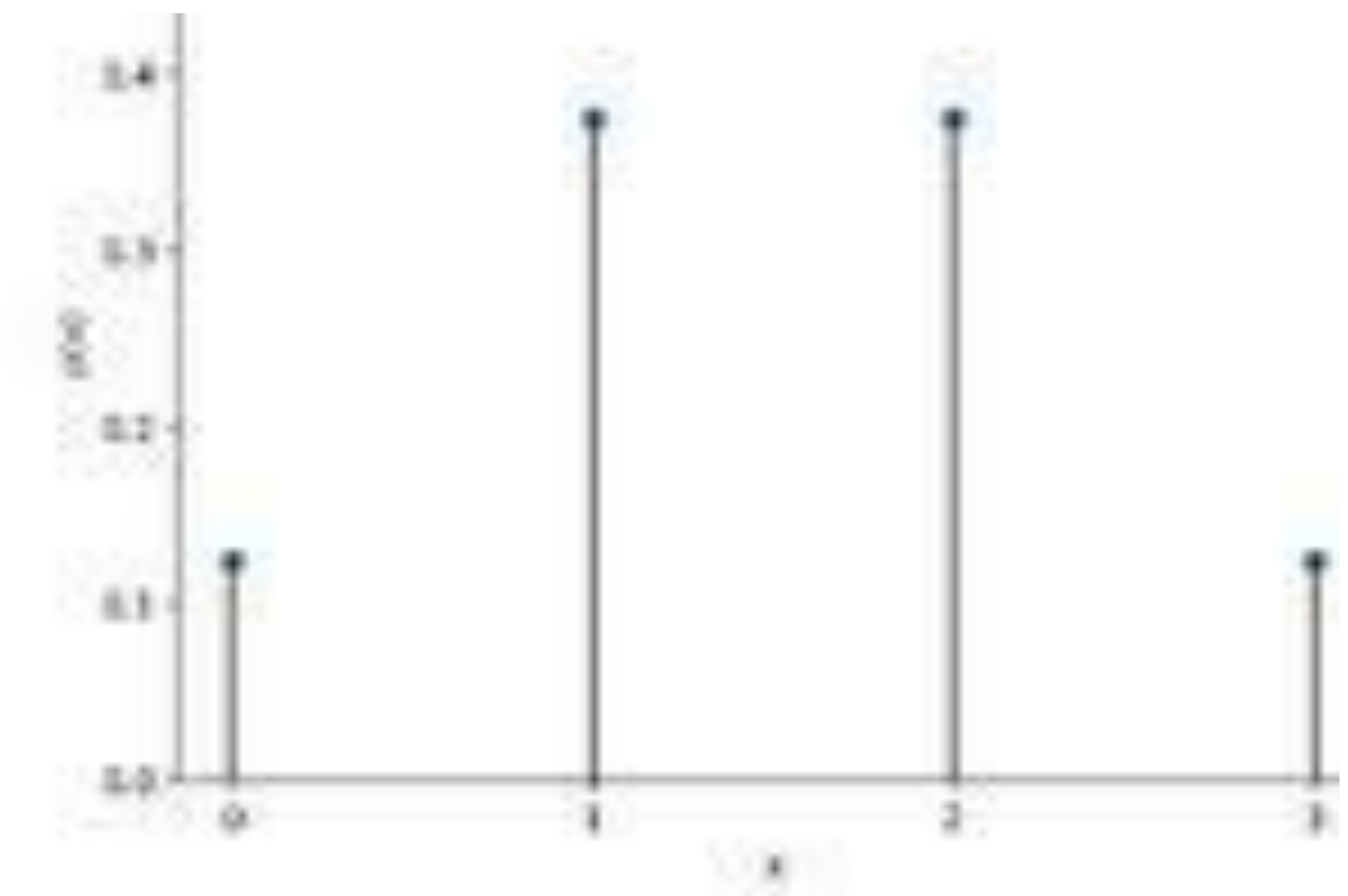
Equiprobable

$p = 1/8$

X

heads

x	Outcomes	p(x)
0	ttt	⅛
1	tth, tht, htt	⅜
2	thh, hth, hht	⅜
3	hhh	⅛



Specification

As before — **Explicit**: $p(1)=.1$ $p(2)=.2$ $p(3)=.3$ $p(4)=.4$

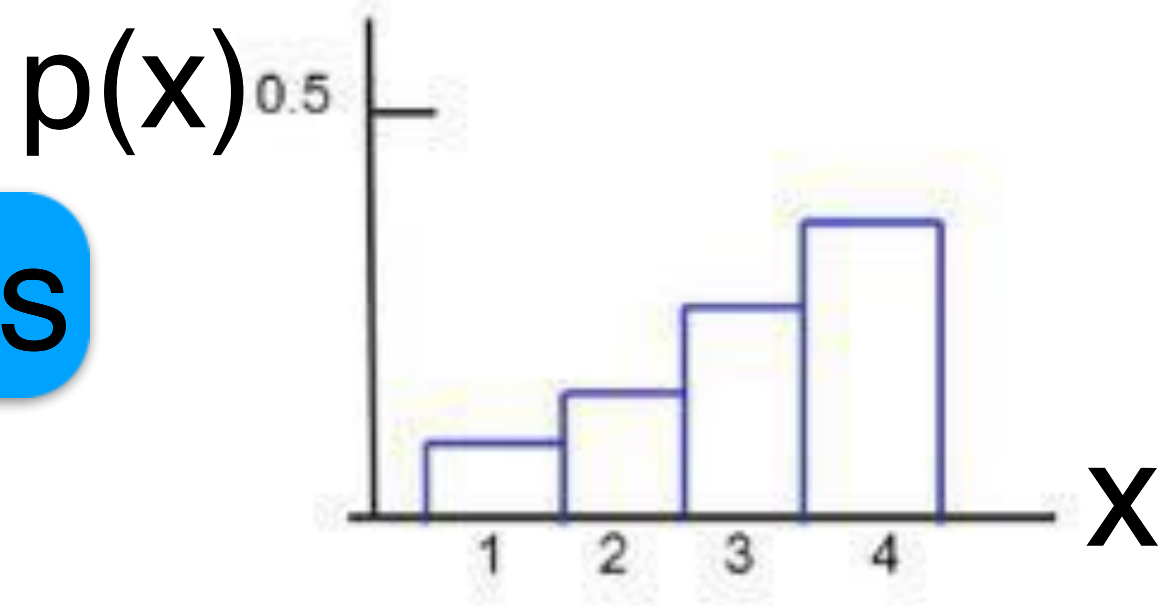
Table

x	1	2	3	4
p(x)	.1	.2	.3	.4

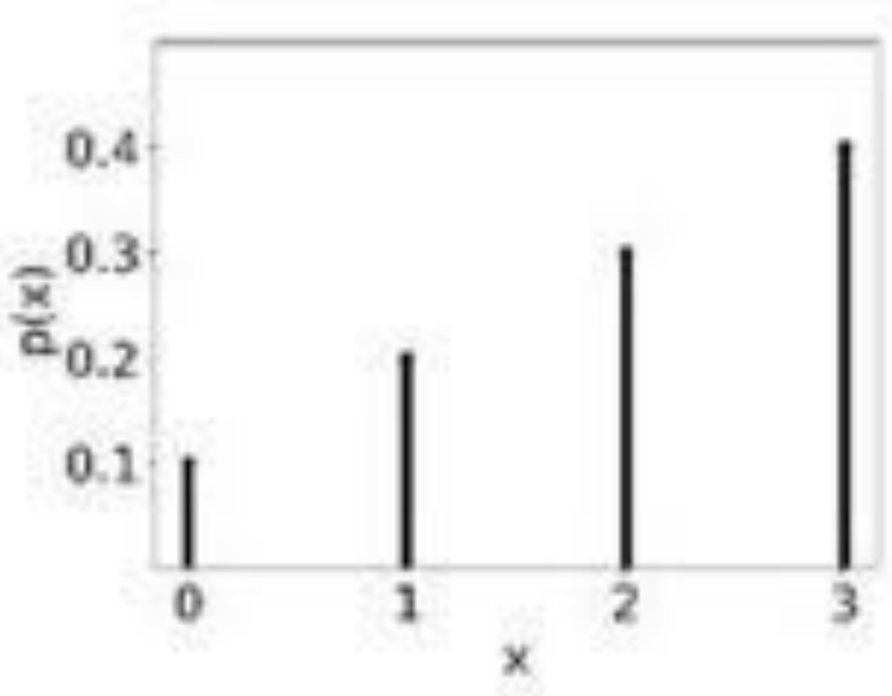
With numbers — **Function**: $p(x) = x / 10$ $x \in \{1,2,3,4\}$

Graphs

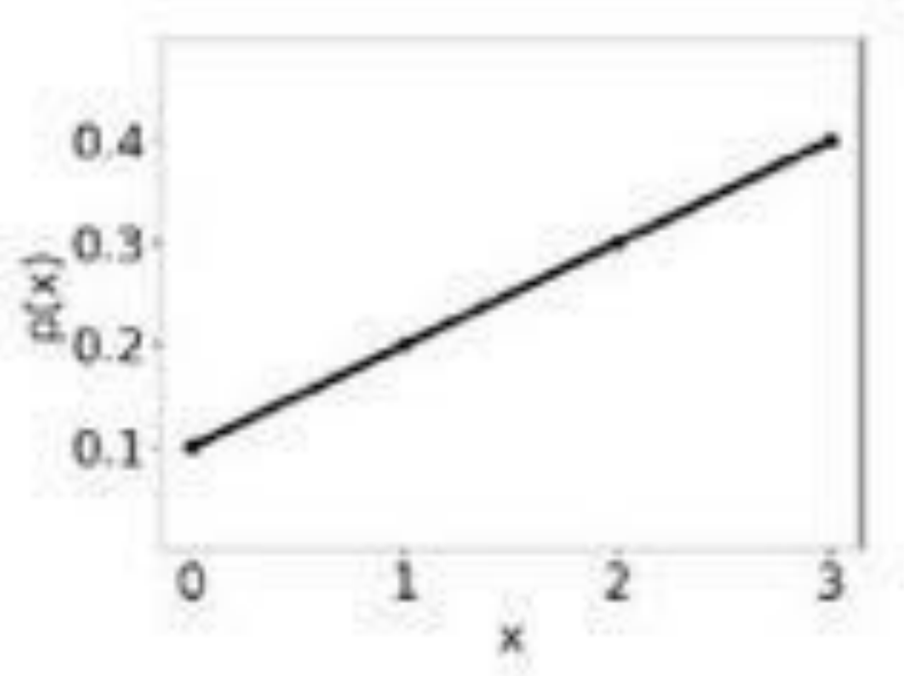
Histogram



Stem plot



Plot



Probability Mass Function

As before   pmf $p : \Omega \rightarrow \mathbb{R}$ Specify Ω and p

Ω Random variable $\rightarrow$ $\subseteq \mathbb{R}$

Discrete $\rightarrow$ finite or countably infinite

p $p(x) \geq 0 \quad \forall x \in \Omega$ $\sum_{x \in \Omega} p(x) = 1$

If X is distributed according to p, we write $X \sim p$

Alternative Notation

Discrete

$\Omega \subseteq \mathbb{R}$ Often $\mathbb{Z}$ $\mathbb{N}$ $\mathbb{P}$ $\{1, \ldots n\}$

$p(x) \rightarrow p_x$ p_i

$p_i \geq 0$ $\sum_i p_i = 1$

Types of Discrete Distributions

Finite $|\Omega| = n \in \mathbb{P}$

Infinite $|\Omega| = \infty = \aleph_0$

Finite Distributions

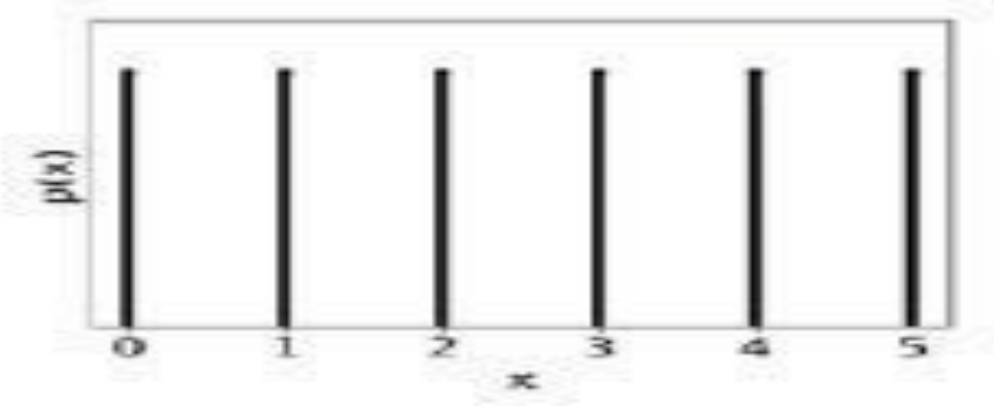
$|\Omega| = n$

Specify pmf

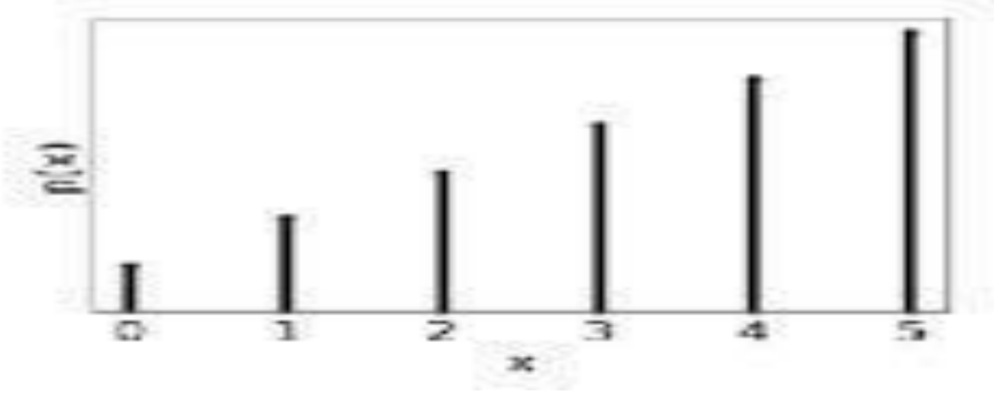
$p_1, p_2, \ldots, p_n$

$$\forall_{1 \le i \le n} \quad p_i \ge 0 \qquad \sum_{i=1}^{n} p_i = 1$$

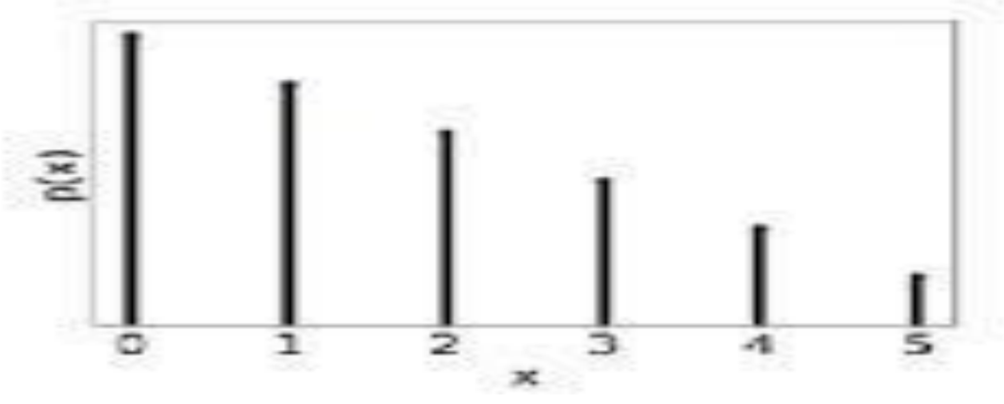
Uniform $p_1 = p_2 = \ldots = p_n$ $= 1/n$



Increasing $p_1 \le p_2 \le \ldots \le p_n$



Decreasing $p_1 \ge p_2 \ge \ldots \ge p_n$



Infinite Distributions

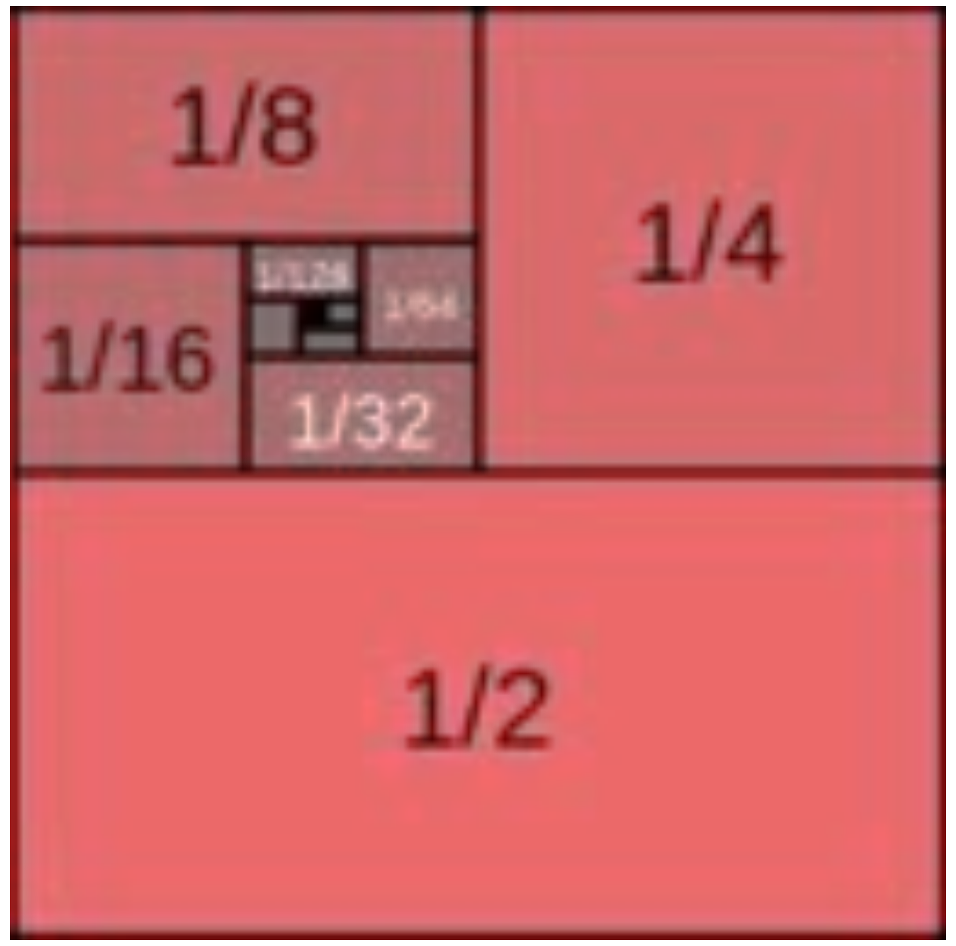
$|\Omega| = \infty$

One-sided infinite $p_1, p_2, p_3, \ldots$

Cannot be uniform $p = 0 \rightarrow \sum = 0$ $p > 0 \rightarrow \sum = \infty$

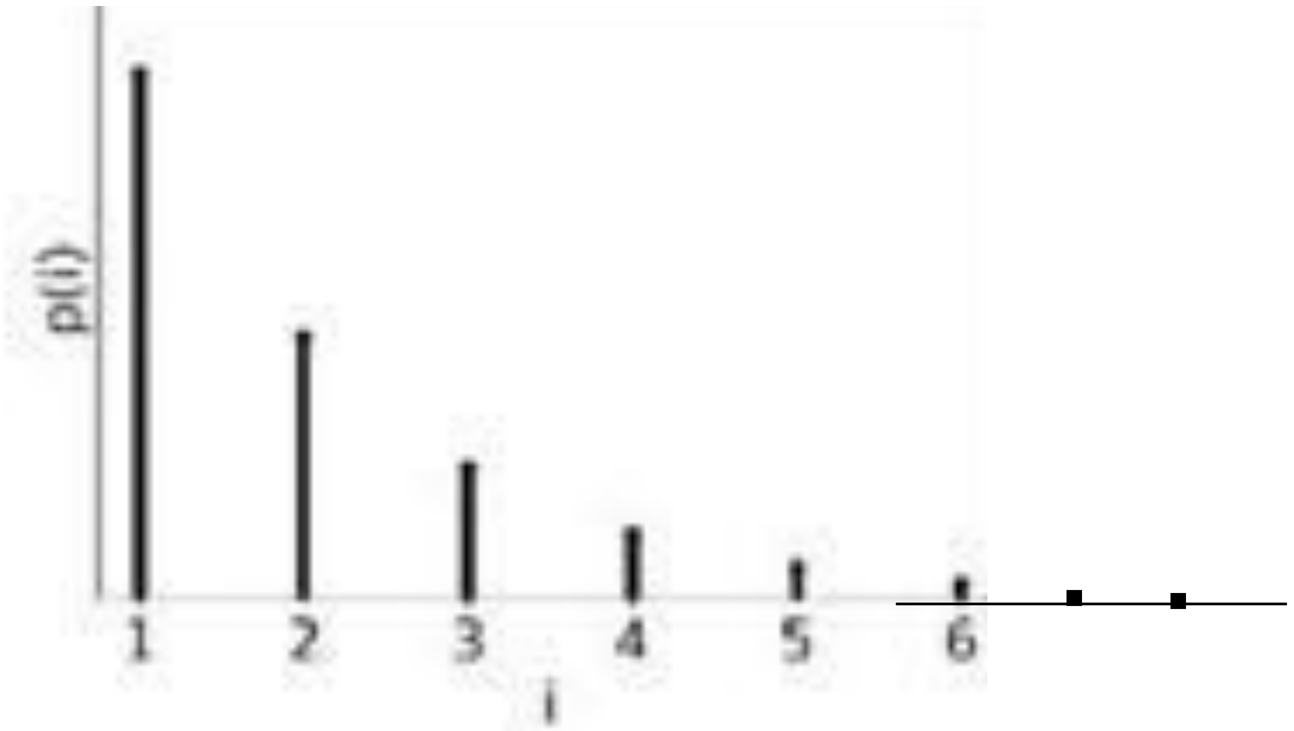
Cannot increase $p_i > 0 \rightarrow p_{i+1}, p_{i+2}, \ldots > 0 \rightarrow \sum = \infty$

Can decrease $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$



$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$$

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$



Doubly infinite $\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots$

$\ldots, \frac{1}{8}, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{8}, \ldots$

Formal Definition

Random variable is a mapping $f : \Omega \rightarrow \mathbb{R}$

Simplify terminology, focus on math

Number-valued random experiment

Random Variables

Motivation | Numbers | Operations

Examples

Definitions | pmf

Visualization | Histogram | Plot | Stem

Cumulative Distribution Functions

Areas of Interest

For random varialbe, often, interested in probability of intervals

Temperature between 20 and 80

Salary > 80K

GPA < 3.0

One function helps determine all interval probabilities

Cumulative Distribution Function

Probability mass function (pmf)

p: $\Omega \rightarrow \mathbb{R}$

Cumulative distribution function (cdf)

F: $\mathbb{R} \rightarrow \mathbb{R}$

$$F(x) \stackrel{\text{def}}{=} P(X \in (-\infty, x])$$

$$\stackrel{\text{def}}{=} P(X \leq x)$$

$$= \sum_{u \leq x} p(u)$$

X discrete, still F defined over $\mathbb{R}$

Example

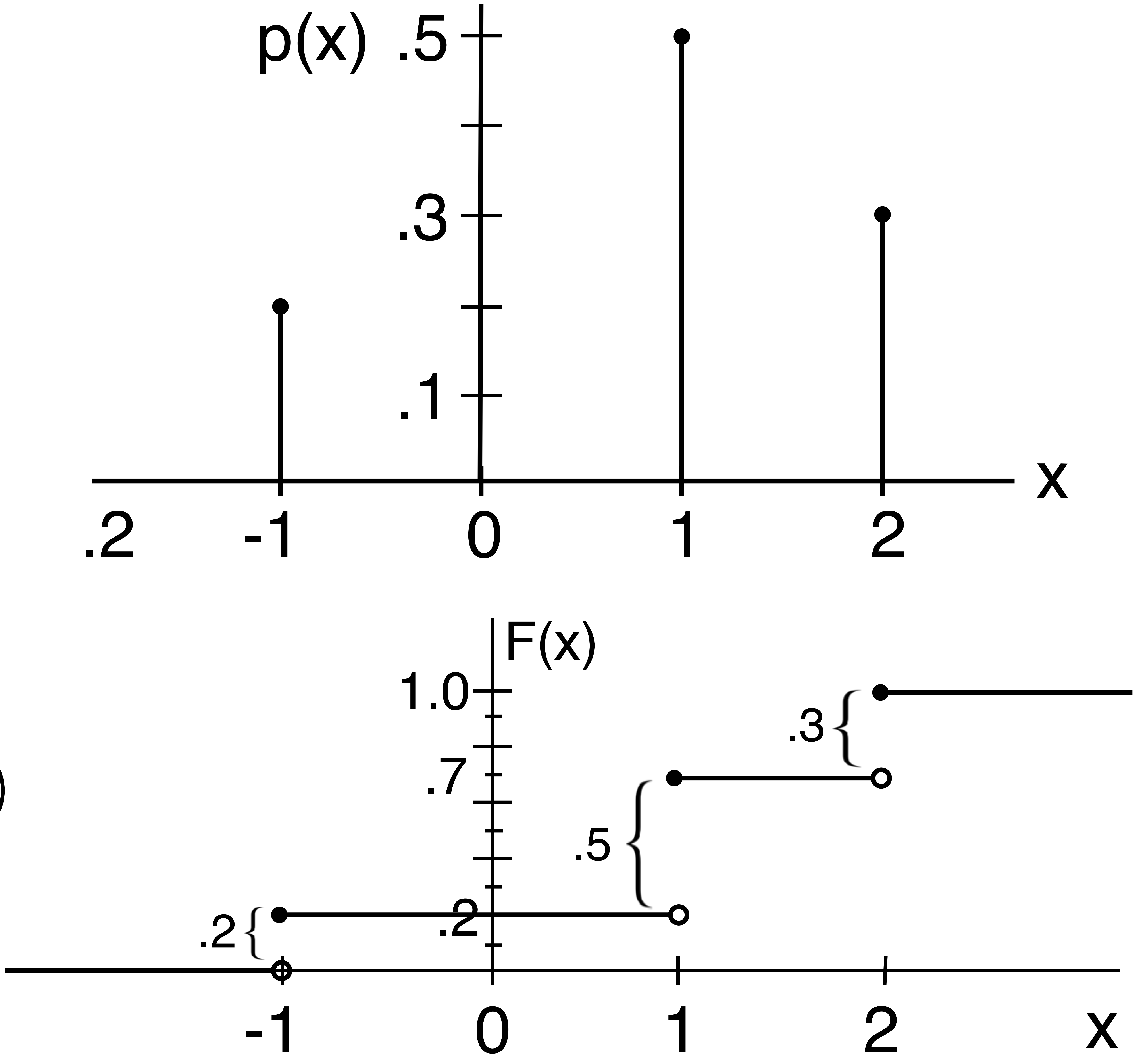
PMF

$$p(x) = \begin{cases} .2 & -1 \\ .5 & 1 \\ .3 & 2 \end{cases}$$

CDF

$$F(x) = P(X \leq x)$$

$$= \sum_{u \leq x} p(u)$$

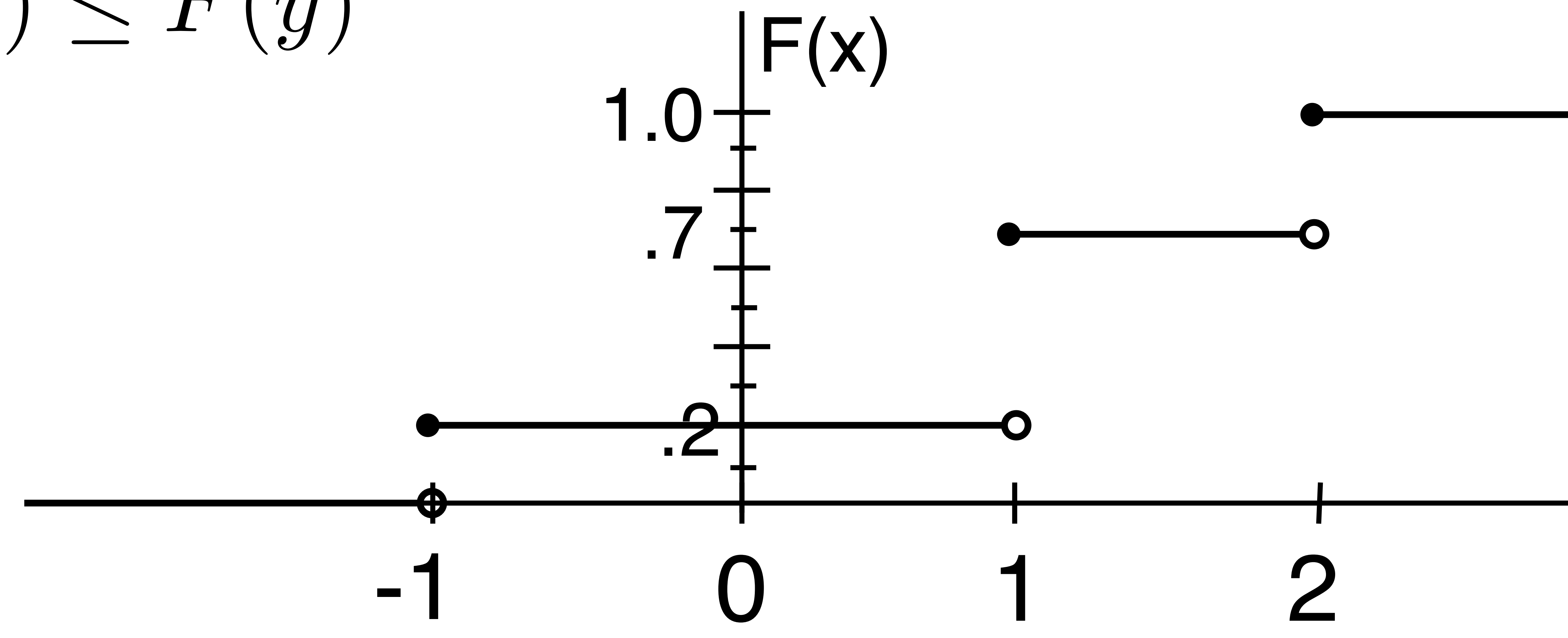


Properties

Nondecreasing $x \leq y \quad \rightarrow \quad F(x) \leq F(y)$

Limits $$\lim_{x \to -\infty} F(x) = 0$$

$$\lim_{x \to \infty} F(x) = 1$$



Right-continuous $$\lim_{x \searrow a} F(x) = F(a)$$

Interval Probabilities

$$P(X \le a) = F(a) \quad \text{– by definition}$$

$$P(X > a) = 1 - P(X \le a) = 1 - F(a)$$

$$P(a < X \le b) = P((X \le b) - (X \le a))$$

$$= P(X \le b) - P(X \le a)$$

$$= F(b) - F(a)$$

Expectations

What Matters

Important random-variable properties?

Range Min & max values of X Lowest & highest temperature / salary

$x_{min} = \min \{ x \in \Omega \mid p(x) > 0 \}$ $x_{max} = \max \{ x \in \Omega \mid p(x) > 0 \}$

Average Average temperature / salary

Range average $\dfrac{x_{\min} + x_{\max}}{2}$?

Element average $\dfrac{1}{|\Omega|} \sum_{x \in \Omega} x$? or over x s.t. $p(x)>0$

Sample Mean

$\Omega = \{0,\ldots,100\}$ $p(0)=.8$ $p(90)=.1$ $p(100)=.1$ all other $p(x)= 0$

Range average $(x_{min}+x_{max})/2$ $(0+100)/2 = 50$

Element average positive probabilities $(0+90+100)/3 = 63.3$

Ten samples Typical 0, 0, 0, 0, 90, 0, 0, 0, 100, 0

Sample mean $(8\cdot 0+1\cdot 90+1\cdot 100)/10 = 190/10 = 19$

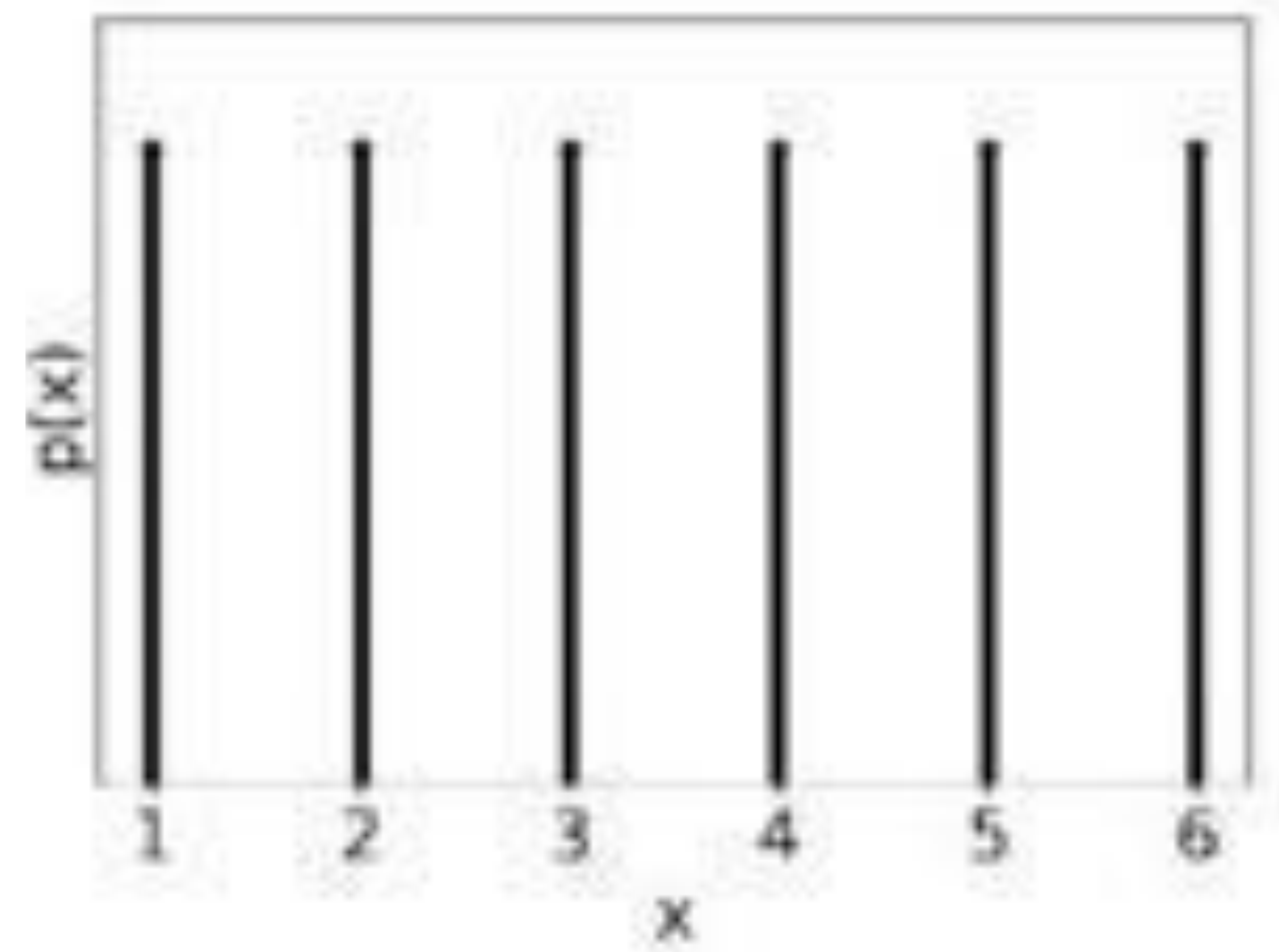
More representative of what we will observe

Fair Die

Roll a fair die n → ∞ times

Average of the observed values = ?

Each value ~n/6 times



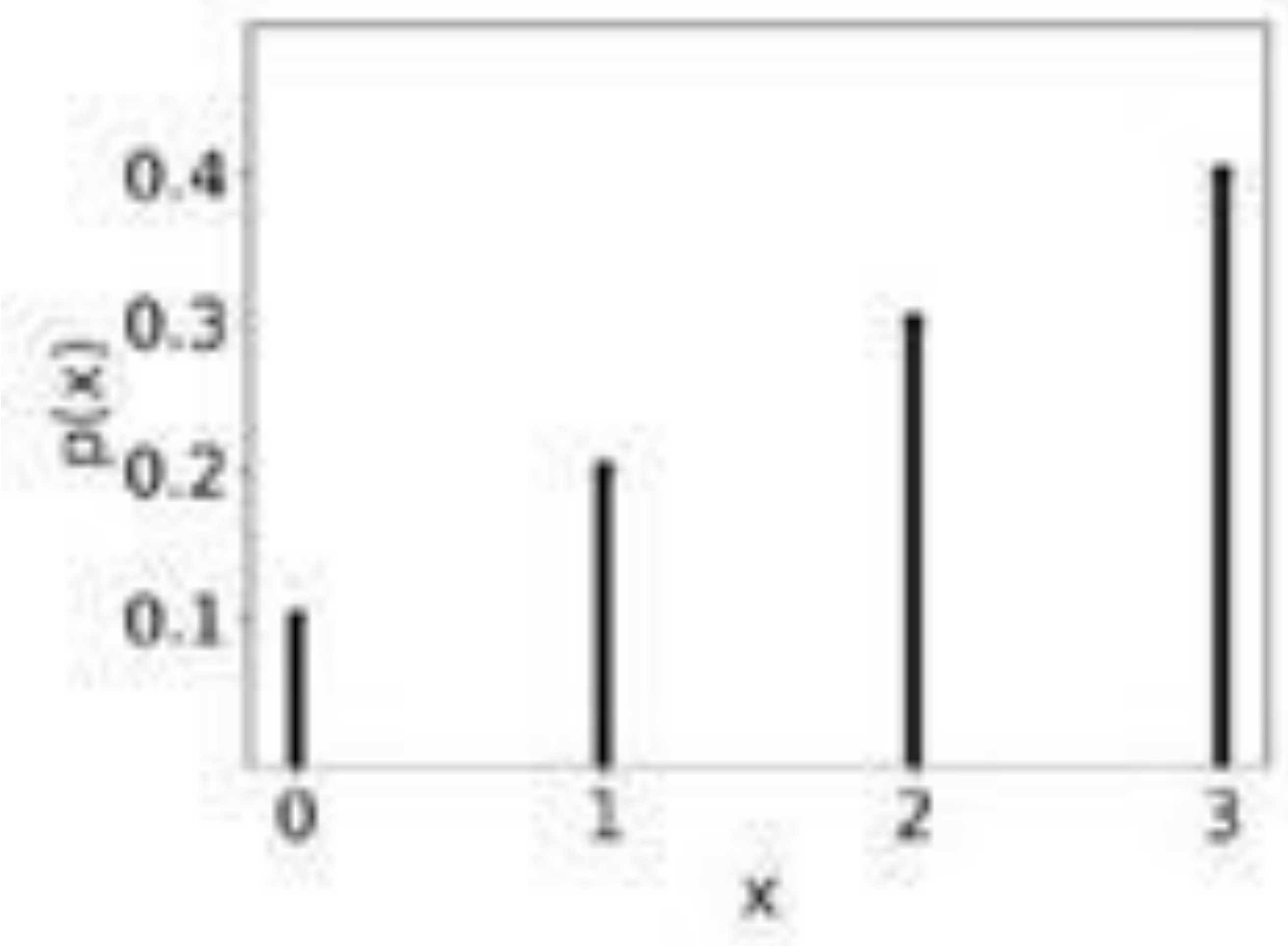
Average

$$\frac{\frac{n}{6}\cdot 1+\frac{n}{6}\cdot 2+\ldots+\frac{n}{6}\cdot 6}{n}=\frac{1+\ldots+6}{6}=\frac{1}{6}\frac{(1+6)\cdot 6}{2} = 3.5$$

1,…,6 → Average = 3.5

4-Sided Die

Side	Prob	Appear
1	.1	.1n
2	.2	.2n
3	.3	.3n
4	.4	.4n
	1	n



$$\text{Average} = \frac{.1n \cdot 1 + .2n \cdot 2 + .3n \cdot 3 + .4n \cdot 4}{n}$$

$$= 0.1 \cdot 1 + 0.2 \cdot 2 + 0.3 \cdot 3 + 0.4 \cdot 4 \quad = 3$$

Arithmetic average $(1+2+3+4)/4 = 2.5$

Probabilities skew to the right

Expectation

In n → ∞ samples x will appear p(x)·n times

$$\text{Average} = \frac{\sum_x [P(x) \cdot n] \cdot x}{n} = \boxed{\sum_x P(x) \cdot x \overset{\text{def}}{=} E(X)}$$

Expectation

Mean

E(X) also denoted EX μ_X μ

Not random constant property of the distribution

Fair Die

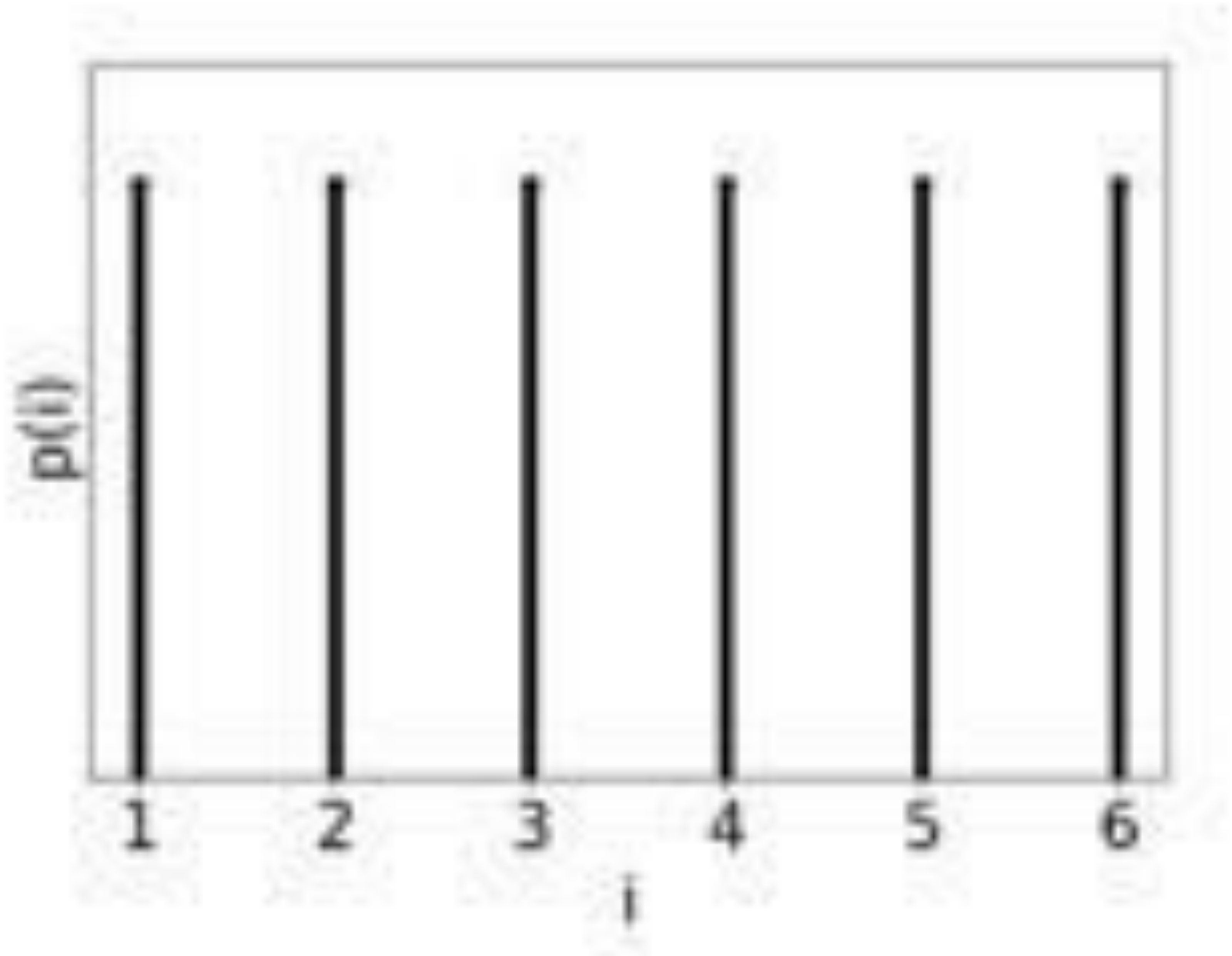
$$E(X) = \sum_{i=1}^{6} P(i) \cdot i$$

$$= \sum_{i=1}^{6} \frac{1}{6} \cdot i$$

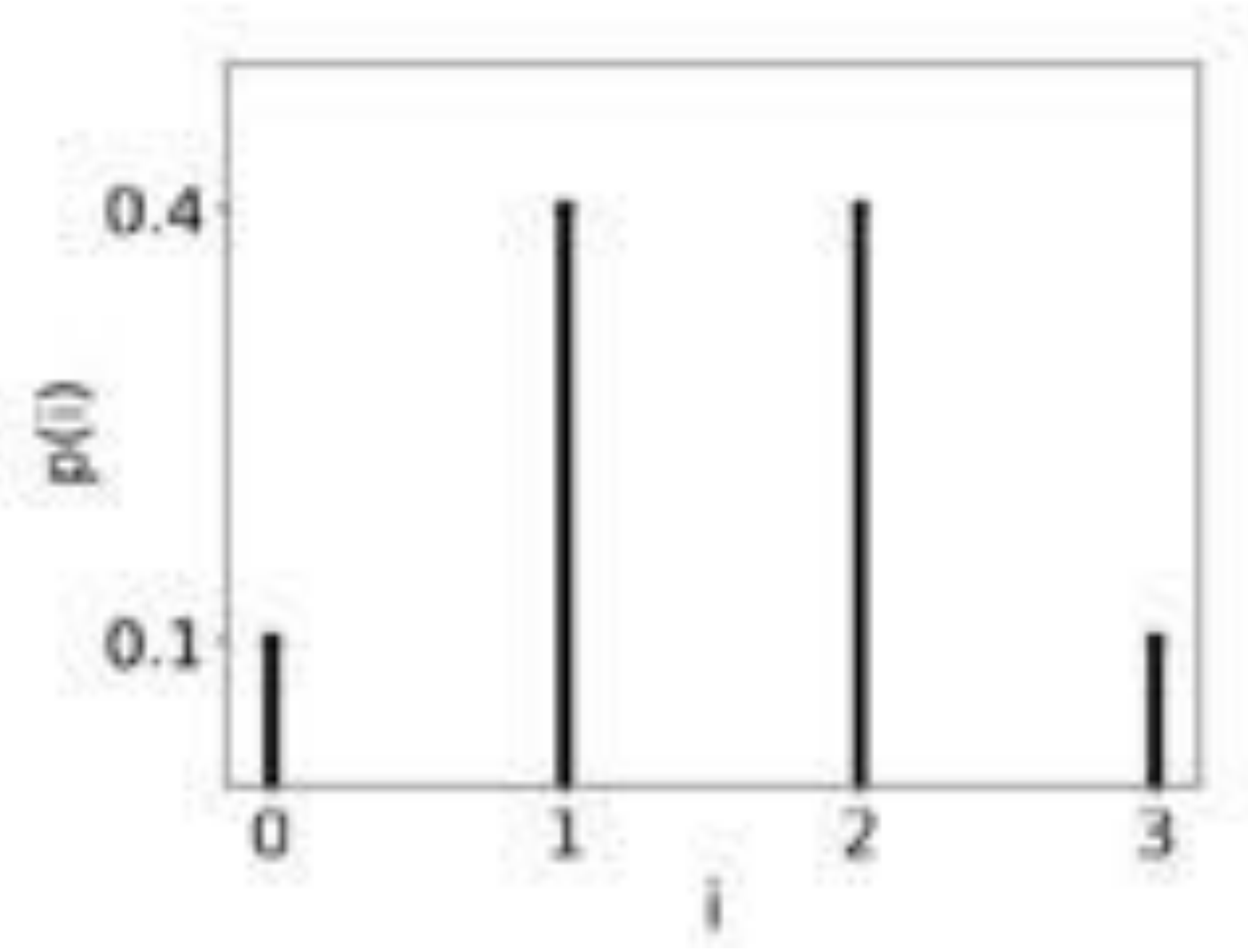
$$= \frac{1 + 2 + \ldots + 6}{6}$$

$$= \frac{1}{6} \frac{(1+6) \cdot 6}{2}$$

$$= \frac{7}{2} = 3.5$$ ✅



4 Sided- Die



$$E(X) = \sum_{i=1}^{4} p_i \cdot i$$

$$= 0.1 \cdot 1 + 0.2 \cdot 2 + 0.3 \cdot 3 + 0.4 \cdot 4$$

$$= 3$$ ✅

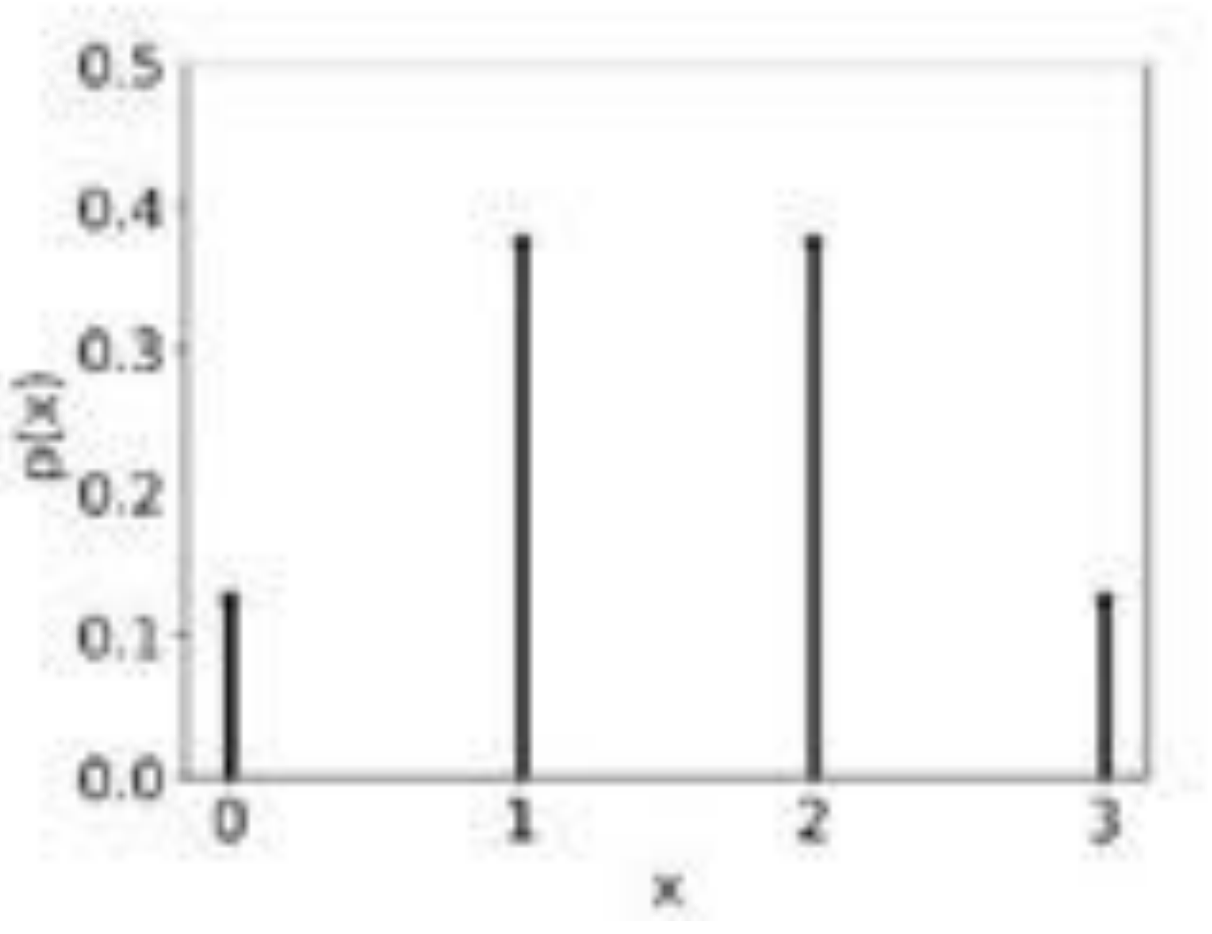
3 Coins

Toss a coin 3 times

X - # heads

E(X) = ?

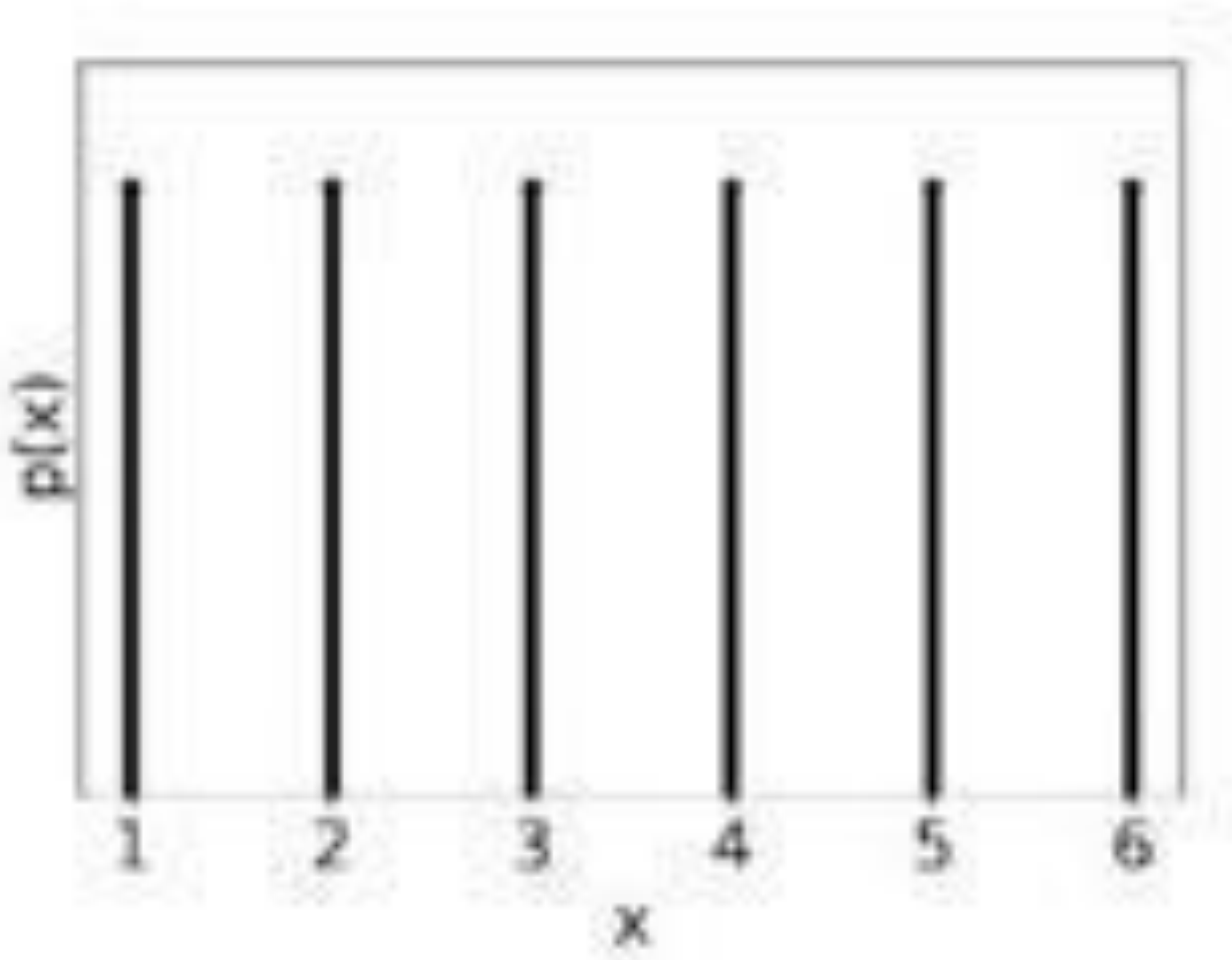
x	outcomes	p(x)
0	ttt	⅛
1	tth,tht,htt	⅜
2	thh, hth, hht	⅜
3	hhh	⅛



$\sum P(x) \cdot x$ $= \frac{1}{8} \cdot 0 + \frac{3}{8} \cdot 1 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 3$ $= 1.5$

heads ranges from 0 to 3, on average 1.5 ✅

Uniform Variables



X uniform over Ω

$$p(x) = \frac{1}{|\Omega|}$$

$$E(X) = \sum_{x\in\Omega} p(x)\cdot x = \sum_{x\in\Omega} \frac{1}{|\Omega|}\cdot x = \frac{1}{|\Omega|}\sum_{x\in\Omega} x$$

E(X) is the arithmetic average of elements in Ω

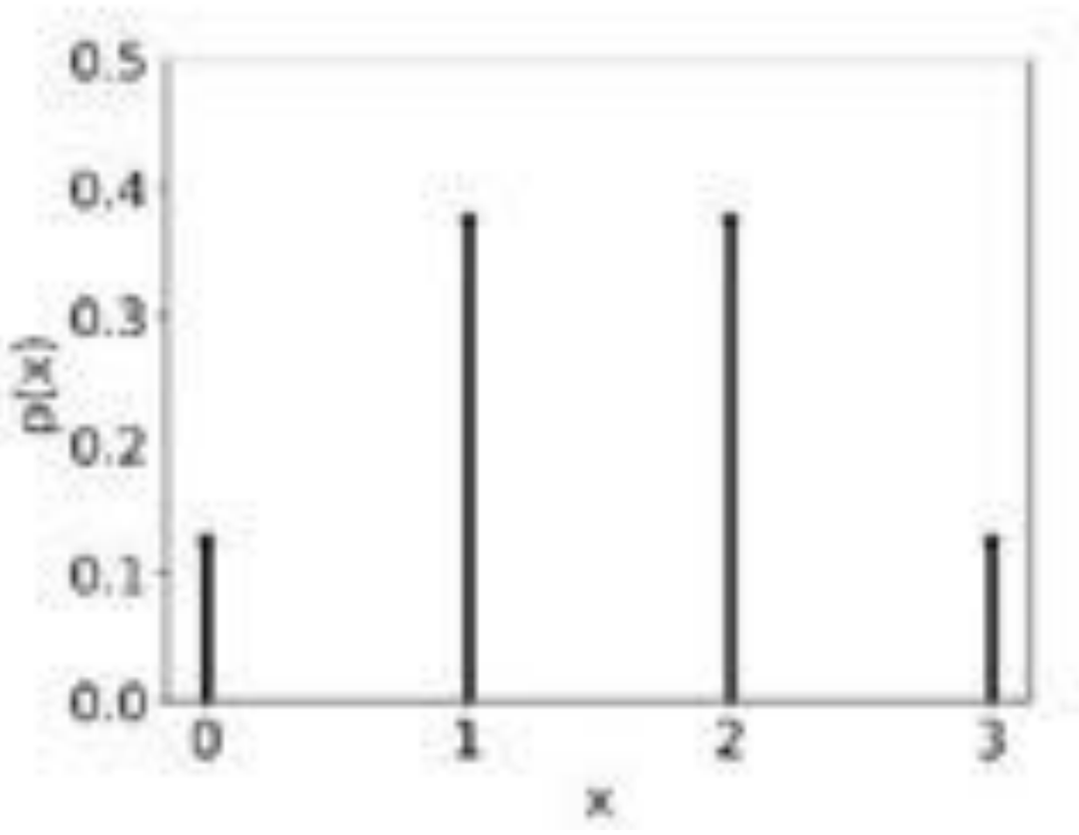
$$E(X) = \frac{1+2+\ldots+6}{6} = 3.5$$

Symmetry

A distribution p is symmetric around a if for all $x > 0$, $p(a+x) = p(a-x)$

If p is symmetric around a, then $E(X) = a$

x	outcomes	P(x)
0	ttt	⅛
1	tth,tht,htt	⅜
2	thh, hth, hht	⅜
3	hhh	⅛



Symmetric around 1.5

$E(X) = 1.5$

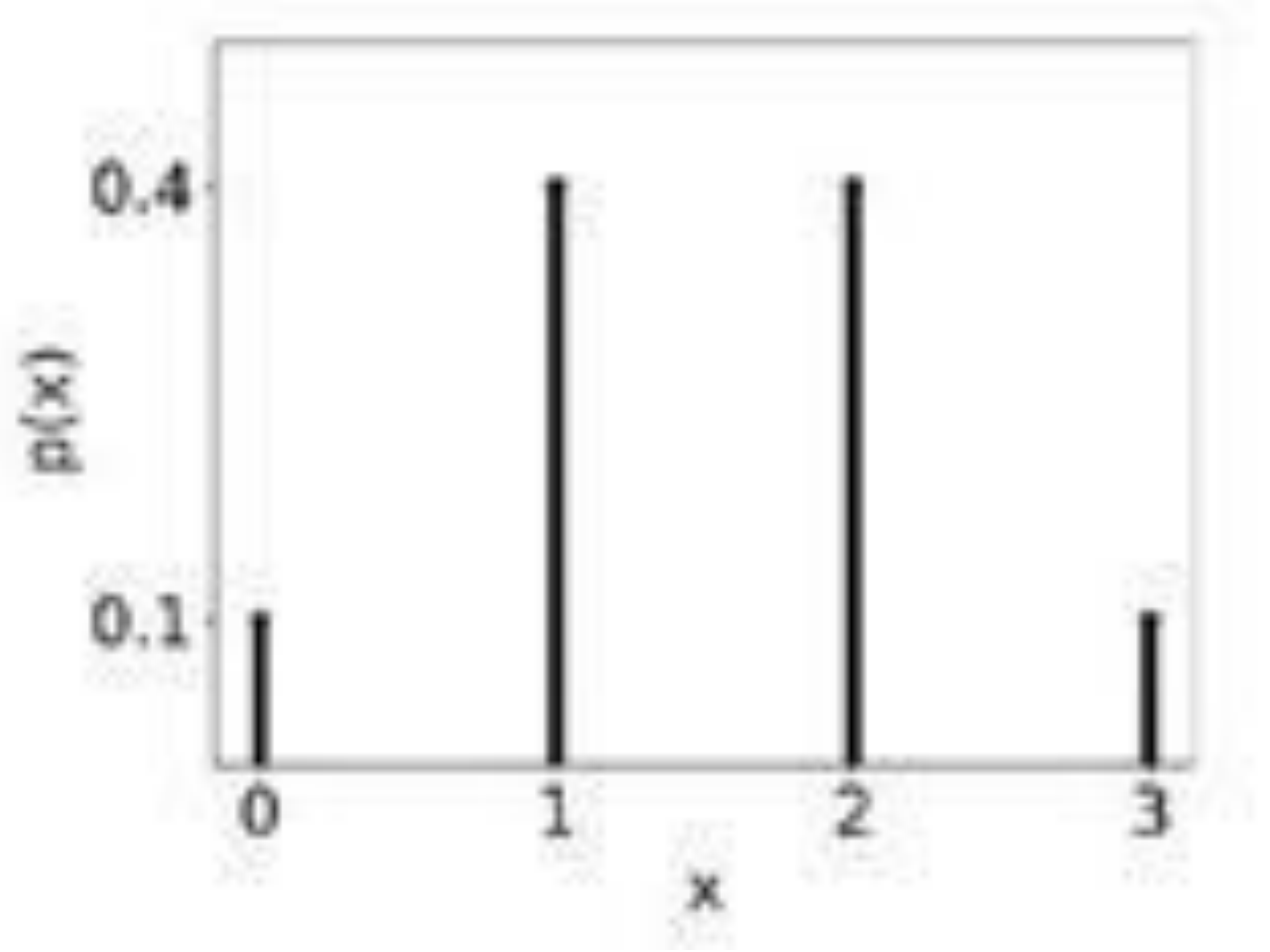
Properties

E(X) | Despite notation | Not random | Number | Property of distribution

$E(X) = 1.5$

$x_{min} \leq E(X) \leq x_{max}$ $= \text{ iff } X = c$

$0 \leq E(X) \leq 3$



X is a constant, namely $X=c$ → $E(X)=c$

$E(E(X)) = E(X)$

Is Expectation Expected?

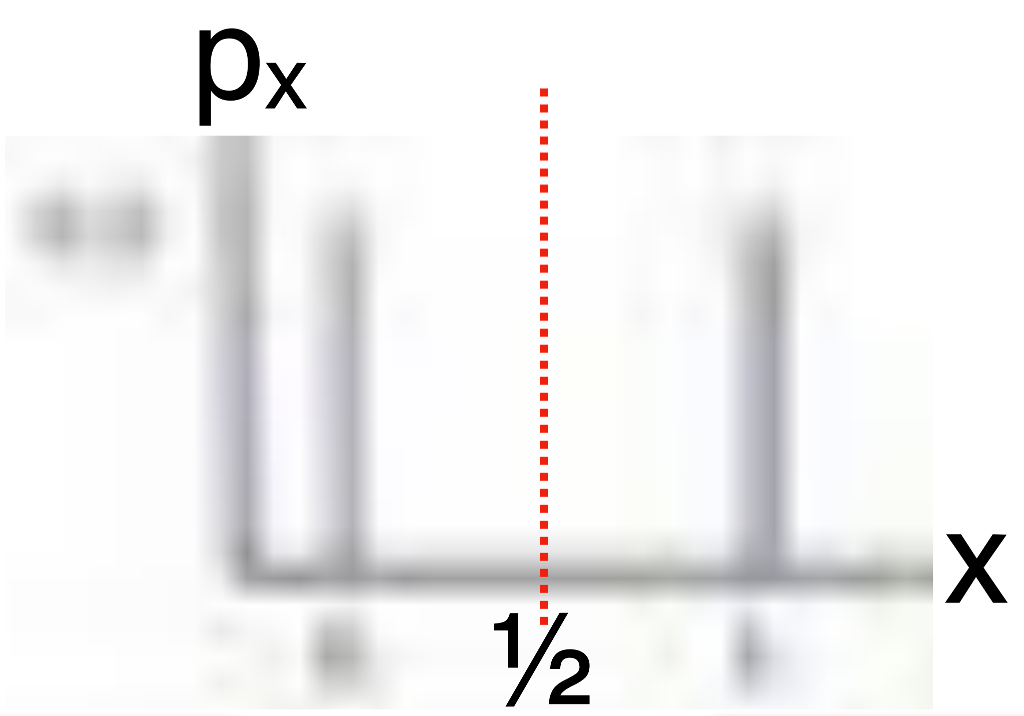
μ = EX - expectation of X

Do we expect to see it?

Is p_μ high?

Not necessarily

We may never see it!



$X \in \{0,1\}$

$p_0 = p_1 = 0.5$

$EX = 0 \cdot p_0 + 1 \cdot p_1 = 0 \cdot ½ + 1 \cdot ½ = ½$

Symmetric around ½

½ will never happen!

Many samples → average = ½

EX - average of large sample

Not necessarily likely

May not be observed at all

Infinite Expectation

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

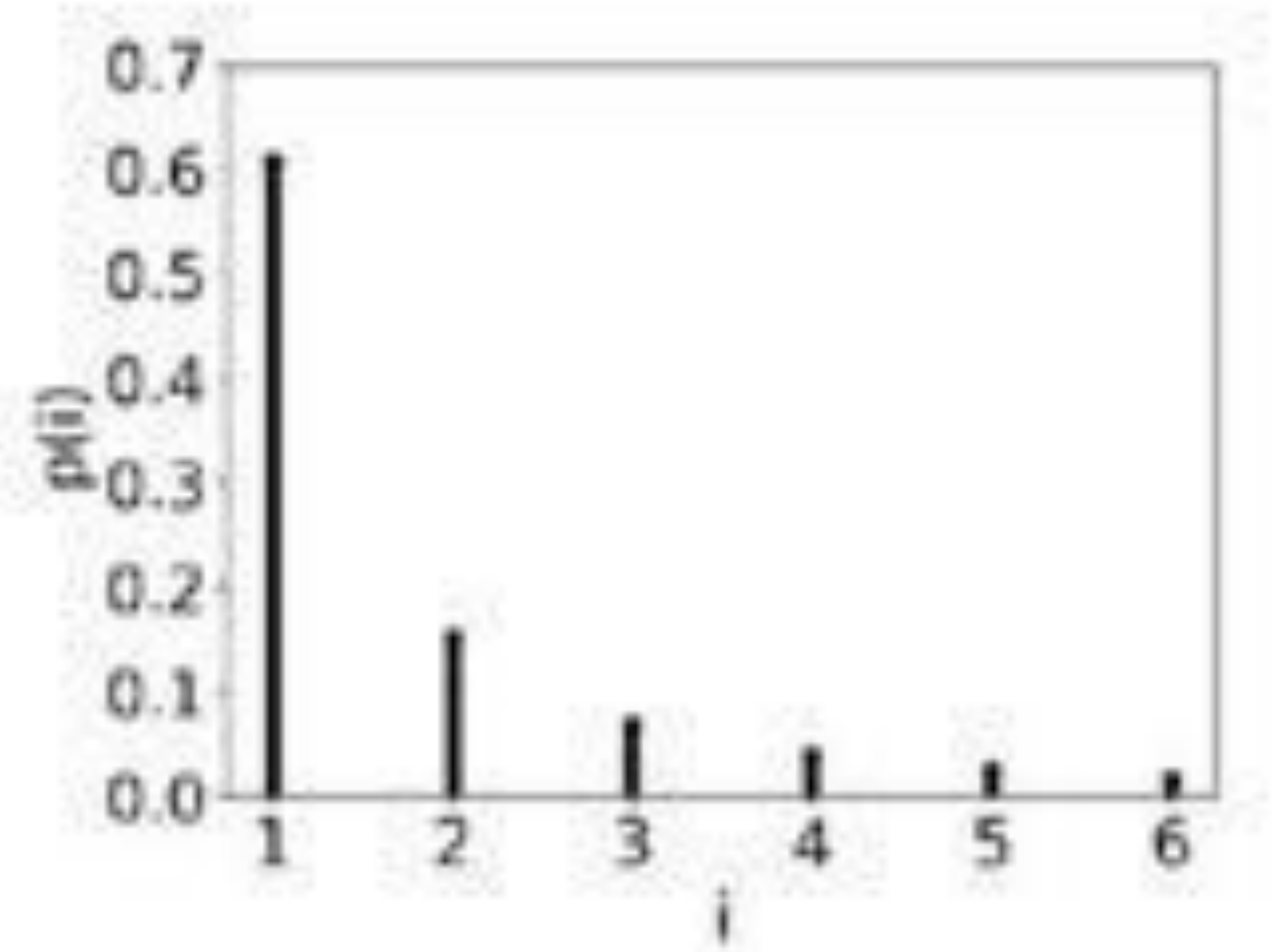
Basel problem

Euler → famous

$$\frac{6}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} = 1$$

$$p_i = \frac{6}{\pi^2} \cdot \frac{1}{i^2}$$

probability distribution over $\mathbb{P}$



$$E(X) = \sum_{i=1}^{\infty} i \cdot p_i \;\; = \frac{6}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i} \;\; = \infty$$

Many samples

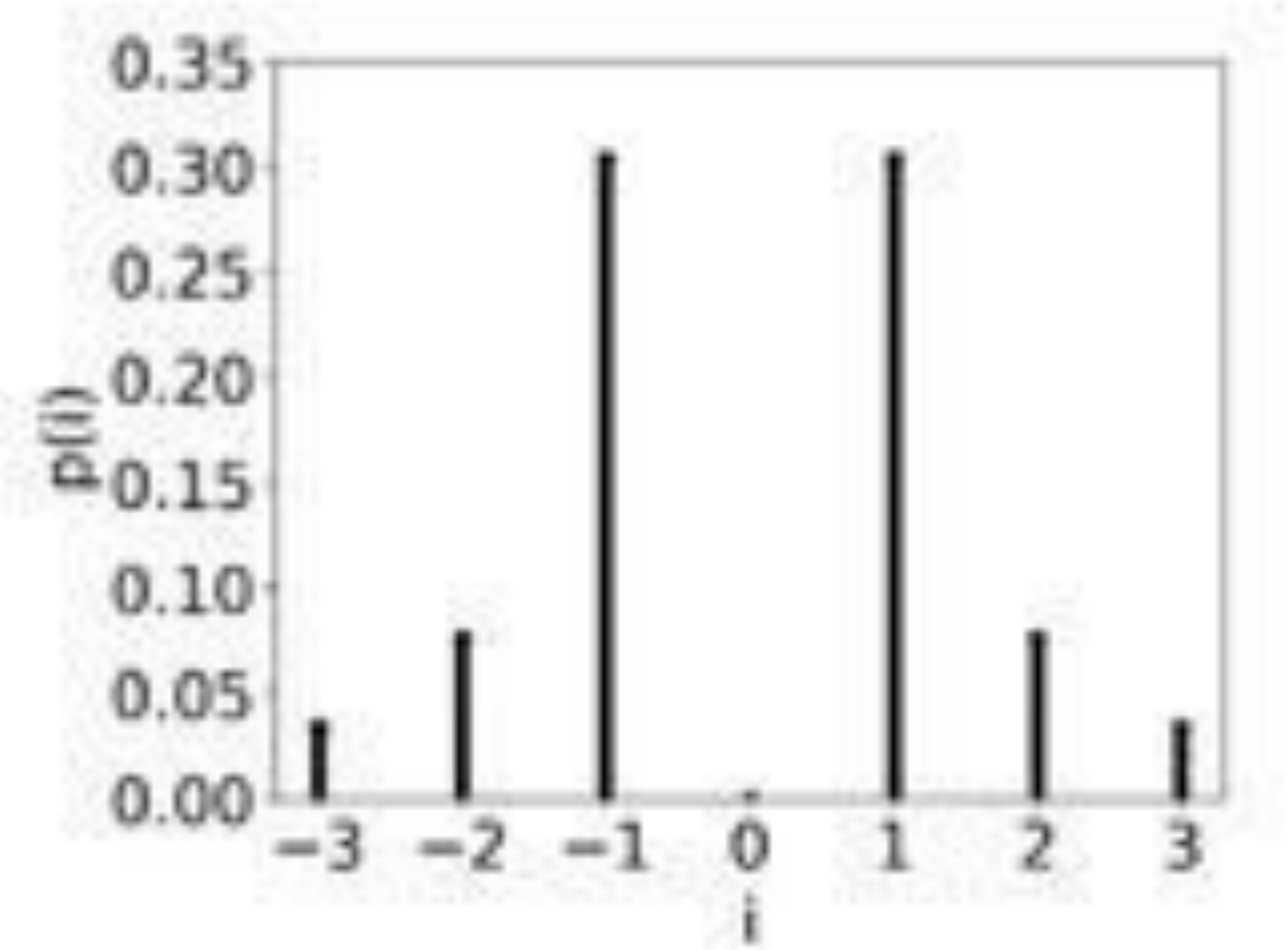
Average will go to ∞

Undefined Expectation

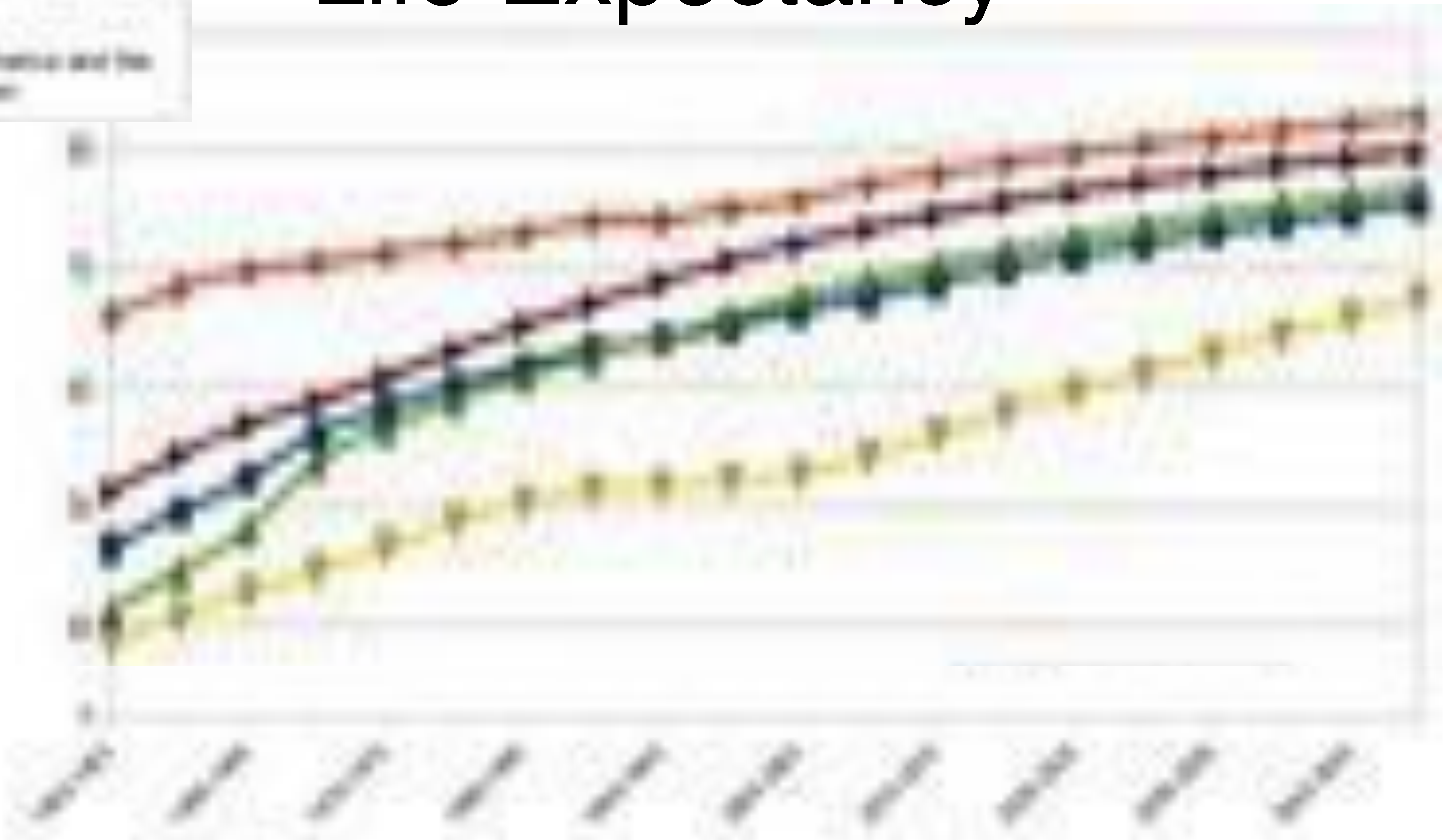
$$p_i = \frac{3}{\pi^2} \cdot \frac{1}{i^2} \quad \text{for} \quad i \neq 0$$

E(X) = ∞ - ∞

Undefined



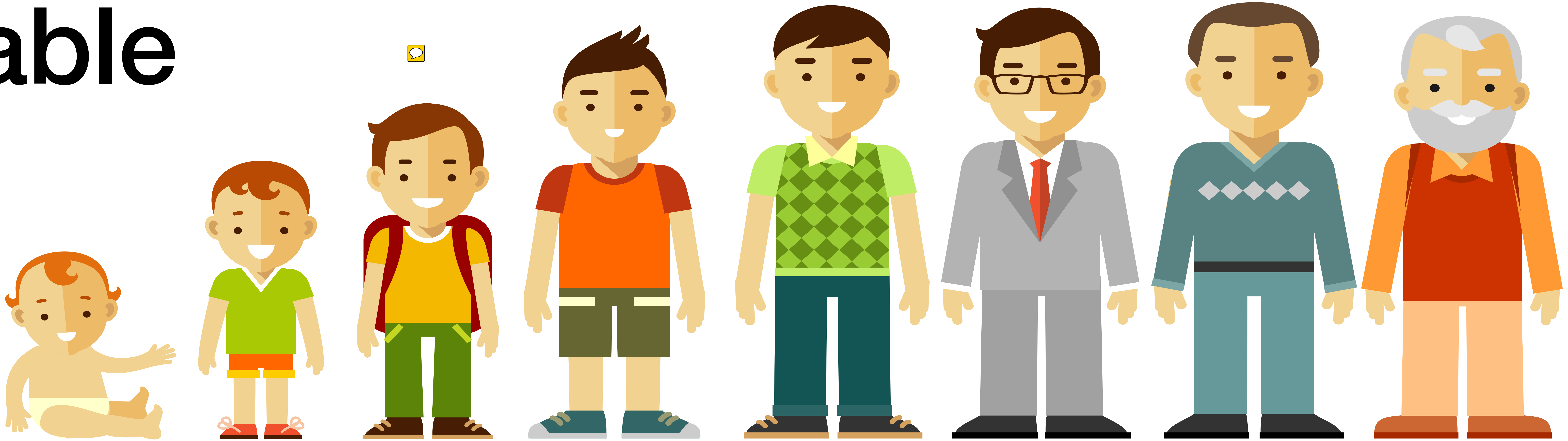
Life Expectancy



Expectation

Expectations of Functions of Random Variables

Variable



Modifications

(aka functions)

Functions of a Random Variable

Random variables X take values in $\mathbb{R}$

Often interested in related variable

$Y = g(X)$

$g: \mathbb{R} \rightarrow \mathbb{R}$ is a fixed function

X

Random salary in $

$10 raise

$Y = X + 10$

10% raise

$Y = 1.1X$

→ CEO

$Y = X^2$

Deterministic Functions

$Y = g(X)$

g is a deterministic function over $\mathbb{R}$ (or Ω)

$Y = X + 3$

All randomness in Y derives from X

Deterministically modified by g

$X = 5$

$Y = 8$

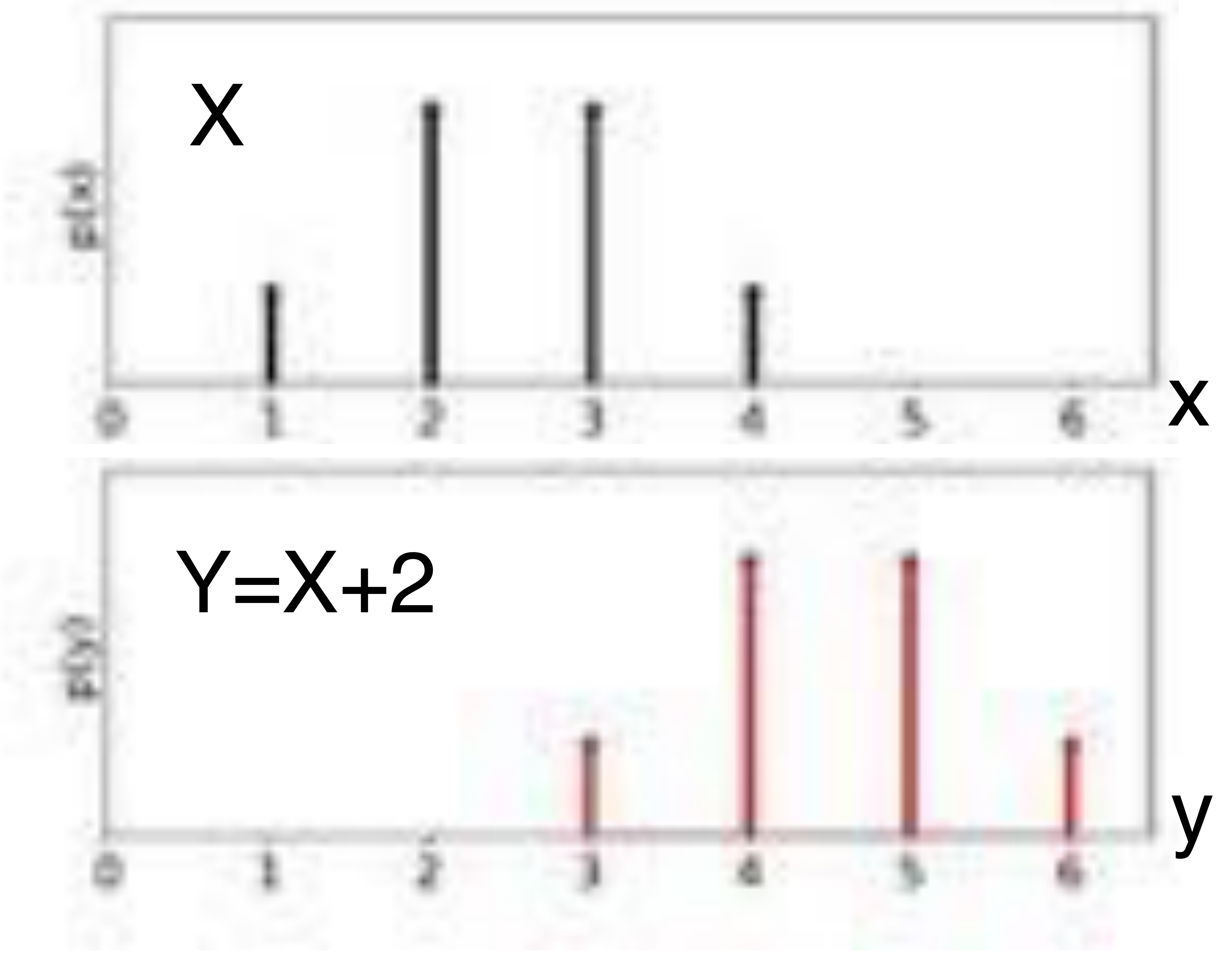
Translation

Add constant b to X

Translate X by b

Y = X + b

$P(Y=y) = P(X+b=y) = P(X=y-b)$



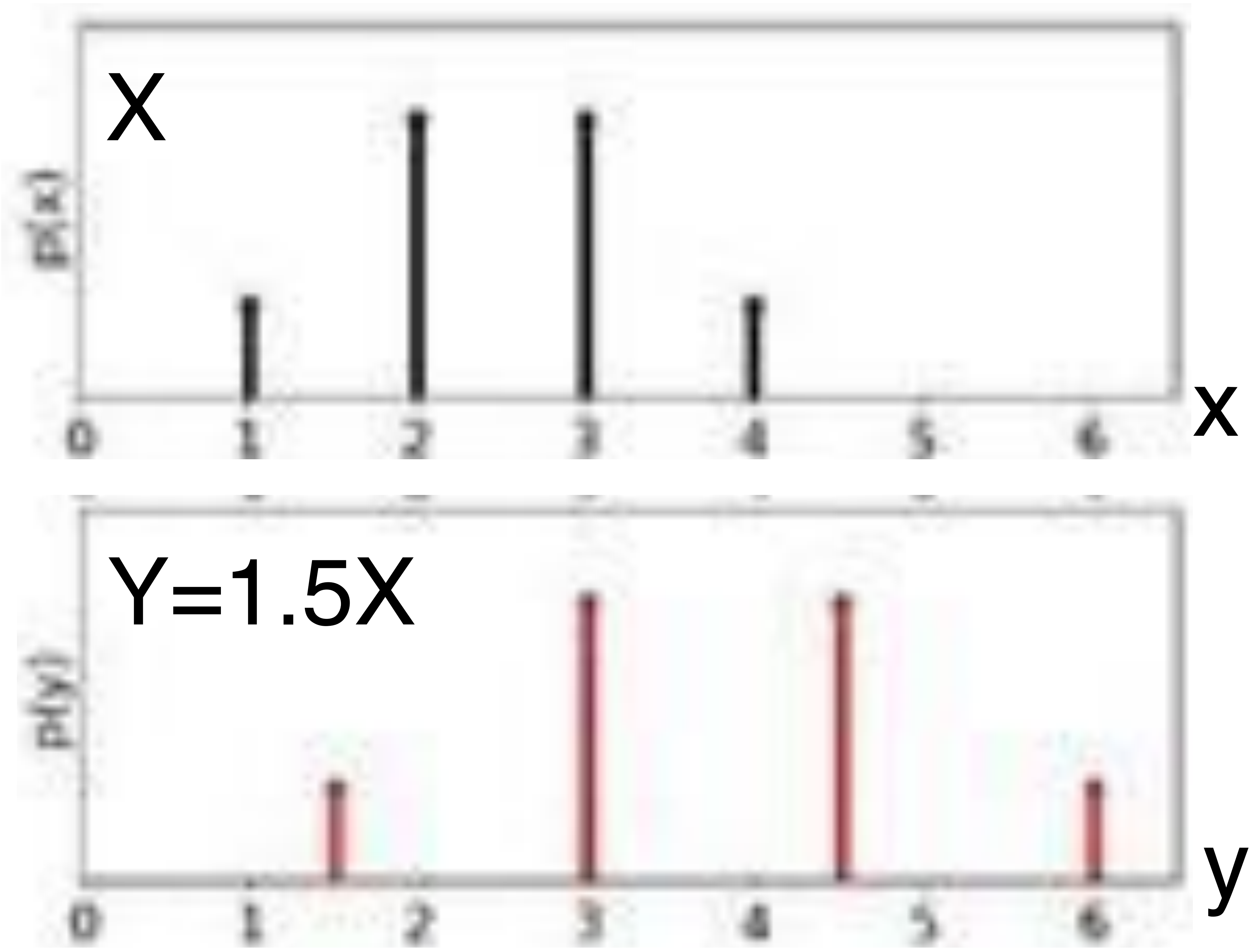
Scaling

Multiply X by a constant b

Scale X by a factor b

$Y = b \cdot X$

$P(Y=y) = P(bX=y) = P(X=y/b)$



Two Square Examples

Square is 1-1

X

x	0	1	2
p(X = x)	½	⅓	⅙

$Y = X^2$

y	0	1	4
p(Y = y)	½	⅓	⅙

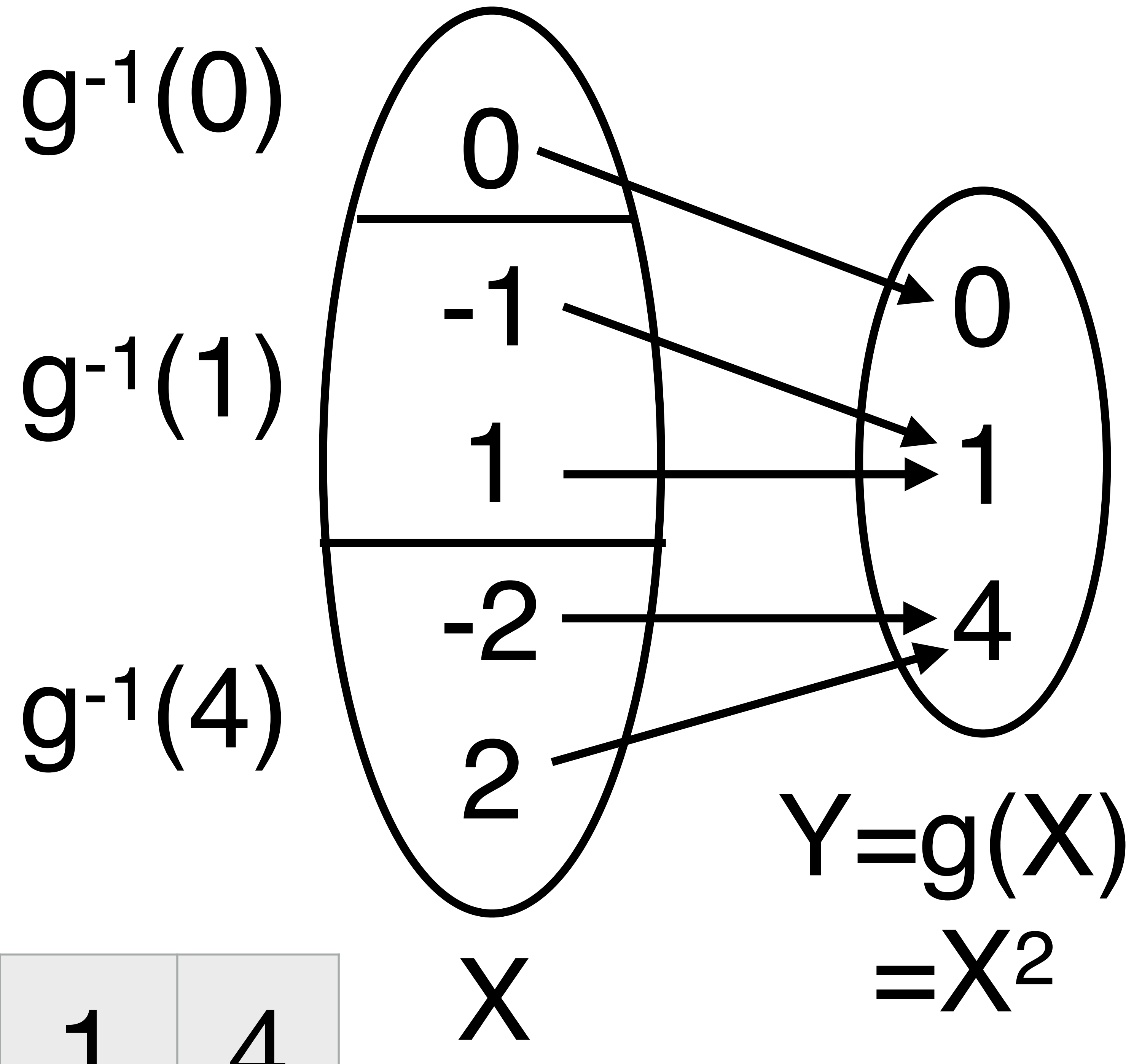
Square is many to 1

X

x	-2	-1	0	1	2
p(X = x)	⅕	⅕	⅕	⅕	⅕

$Y = X^2$

y	0	1	4
p(Y = y)	⅕	⅖	⅖



$$P(Y=y) = P(g(X)=y) = P(X \in g^{-1}(y)) = \sum_{x \in g^{-1}(y)} P(X=x)$$

Expectation of Functions of Random Variables

Expectation Reminder

X

x	-2	-1	0	1	2
p(x)	1/5	1/5	1/5	1/5	1/5

$$E(X) = \sum_x p(x) \cdot x$$

$$= \underbrace{-2 \cdot \tfrac{1}{5} + \underbrace{-1 \cdot \tfrac{1}{5} + 0 \cdot \tfrac{1}{5} + 1}_{0} \cdot \tfrac{1}{5} + 2}_{0} \cdot \tfrac{1}{5} = 0$$

"By Symmetry"

Expectation of a Square

X

x	-2	-1	0	1	2
p(x)	⅕	⅕	⅕	⅕	⅕

$Y = X^2$

y	0	1	4
p(y)	⅕	⅖	⅖

$P(Y = 0) = P(X^2 = 0) = P(X = 0) = 1/5$

$P(Y = 1) = P(X^2 = 1) = P(X \in \{-1, 1\}) = 2/5$

$P(Y = 4) = P(X^2 = 4) = P(X \in \{-2, 2\}) = 2/5$

$E(Y) = 1/5 \cdot 0 + 2/5 \cdot 1 + 2/5 \cdot 4 = 10/5 = 2$

Alternative Formulation

$$E(Y) = \sum_y y \cdot P(Y=y)$$

$$= \sum_y y \cdot P(X \in g^{-1}(y))$$

$$= \sum_y y \sum_{x \in g^{-1}(y)} p(x)$$

$$= \sum_y \sum_{x \in g^{-1}(y)} y \cdot p(x)$$

$$= \sum_y \sum_{x \in g^{-1}(y)} g(x) \cdot p(x)$$

$$= \sum_x g(x) \cdot p(x)$$

Example

Visualize

Square Again

X

x	-2	-1	0	1	2
p(x)	⅕	⅕	⅕	⅕	⅕

$Y = X^2$

y	0	1	4
p(y)	⅕	⅖	⅖

$$E(Y) = \sum_{y=0,1,4} y \cdot p(Y{=}y) = \tfrac{1}{5} \cdot 0 + \tfrac{2}{5} \cdot 1 + \tfrac{2}{5} \cdot 4 = 10/5 = 2$$

$$E(Y) = \sum_x x^2 \cdot p(x)$$

$$= (-2)^2 \cdot \tfrac{1}{5} + (-1)^2 \cdot \tfrac{1}{5} + 0^2 \cdot \tfrac{1}{5} + 1^2 \cdot \tfrac{1}{5} + 2^2 \cdot \tfrac{1}{5}$$

$$= \tfrac{4}{5} + \tfrac{1}{5} + \tfrac{1}{5} + \tfrac{4}{5} = 2$$

Visualization

$$E(Y) = \sum_{y} y \cdot P(Y{=}y)$$

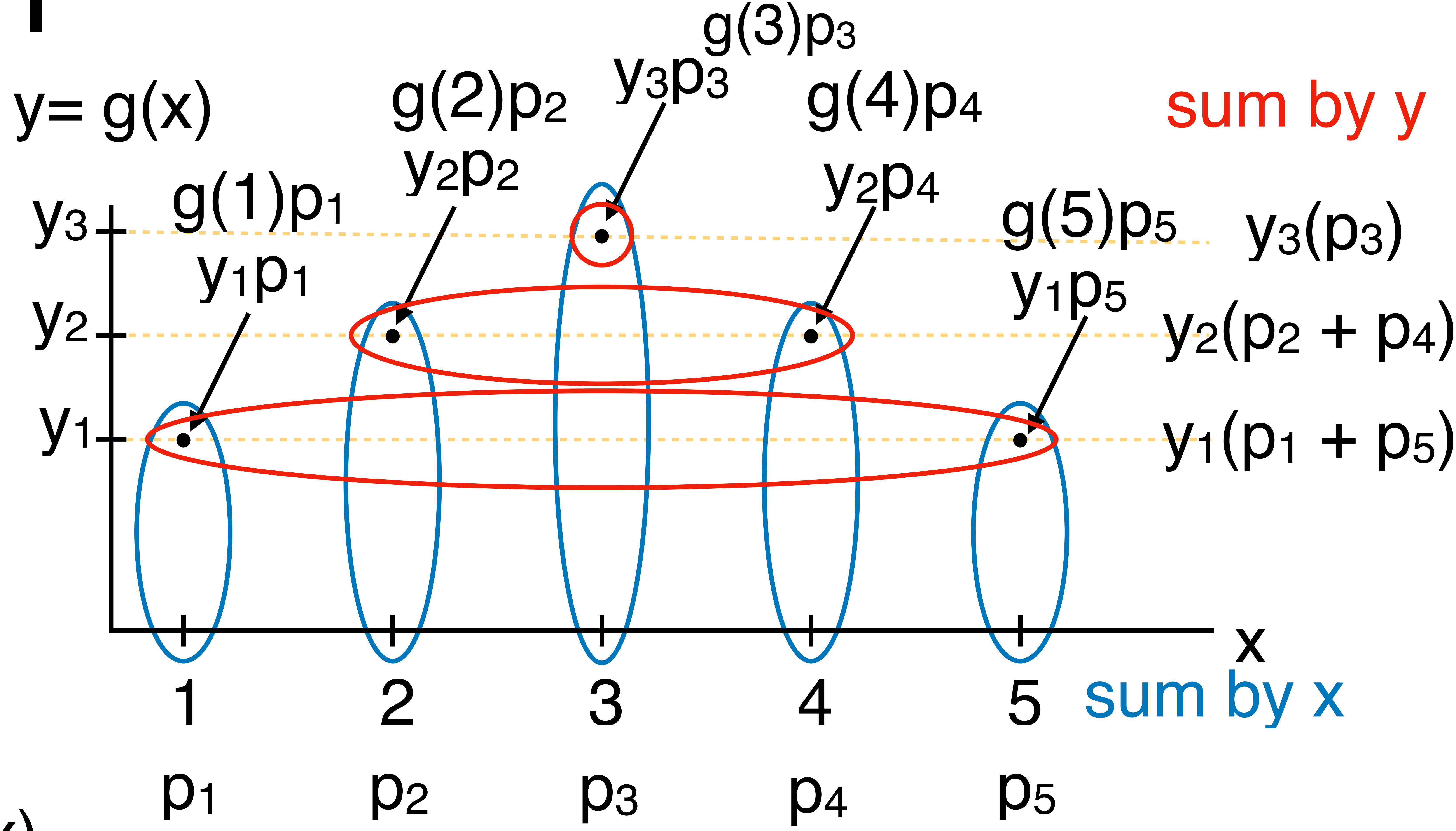
$$= \sum_{y} y \cdot P(X \in g^{-1}(y))$$

$$= \sum_{y} y \sum_{x \in g^{-1}(y)} p(x)$$

$$= \sum_{y} \sum_{x \in g^{-1}(y)} y \cdot p(x)$$

$$= \sum_{y} \sum_{x \in g^{-1}(y)} g(x) \cdot p(x)$$

$$= \sum_{x} g(x) \cdot p(x)$$



y: Fewer multiplication

x: Properties (next)

General Formulas

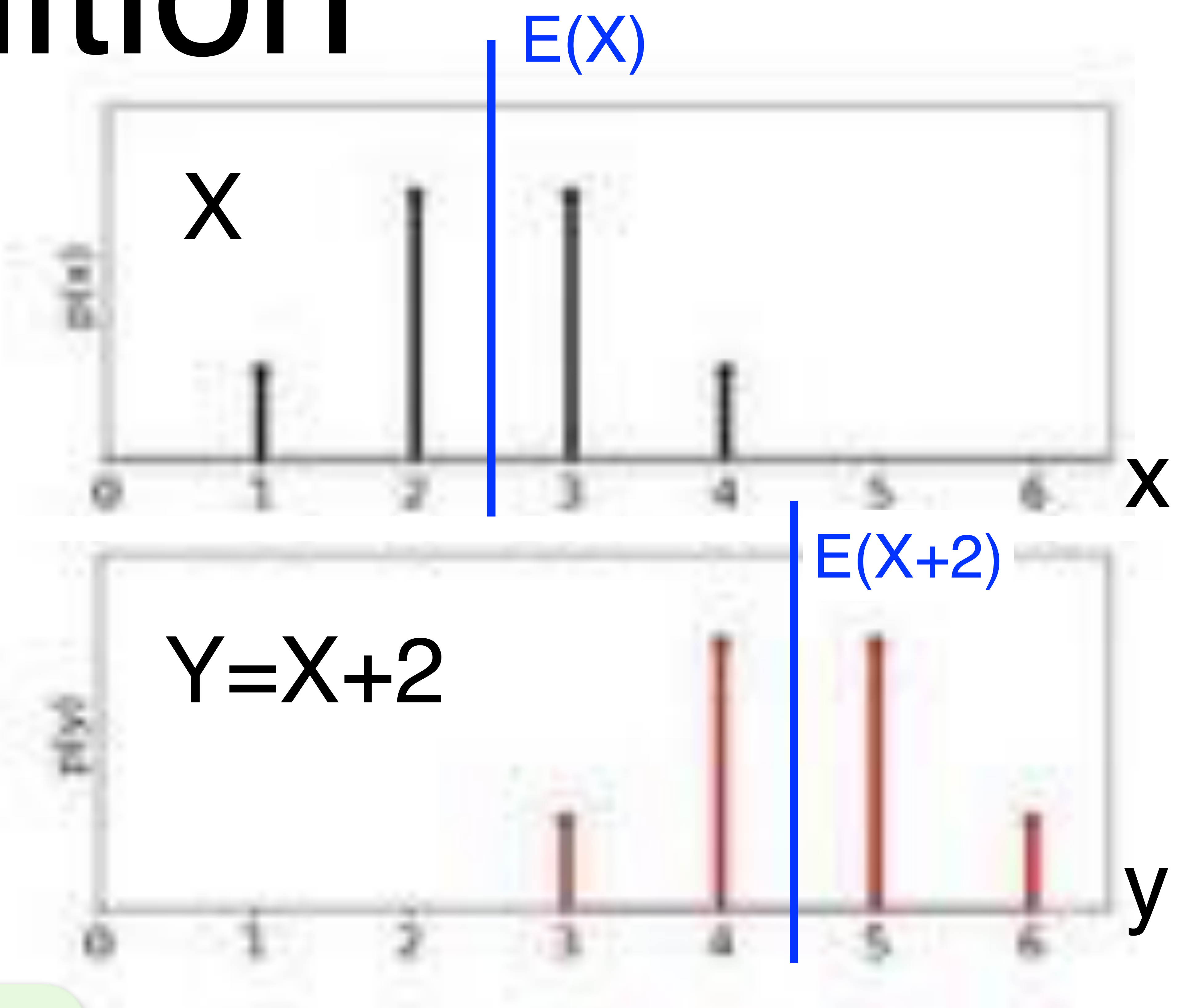
Constant Addition

$E(X + b)$ $= \sum p(x) \cdot (x + b)$

$= \sum p(x) \cdot x + \sum p(x) \cdot b$

$= E(X) + b \cdot \sum p(x)$

$= E(X) + b$



x	0	1
p(x)	1 - p	p

Bernoulli p

$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$

$E(X + 2) = (1 - p) \cdot (0 + 2) + p \cdot (1 + 2)$

$= 2 - 2p + 3p$

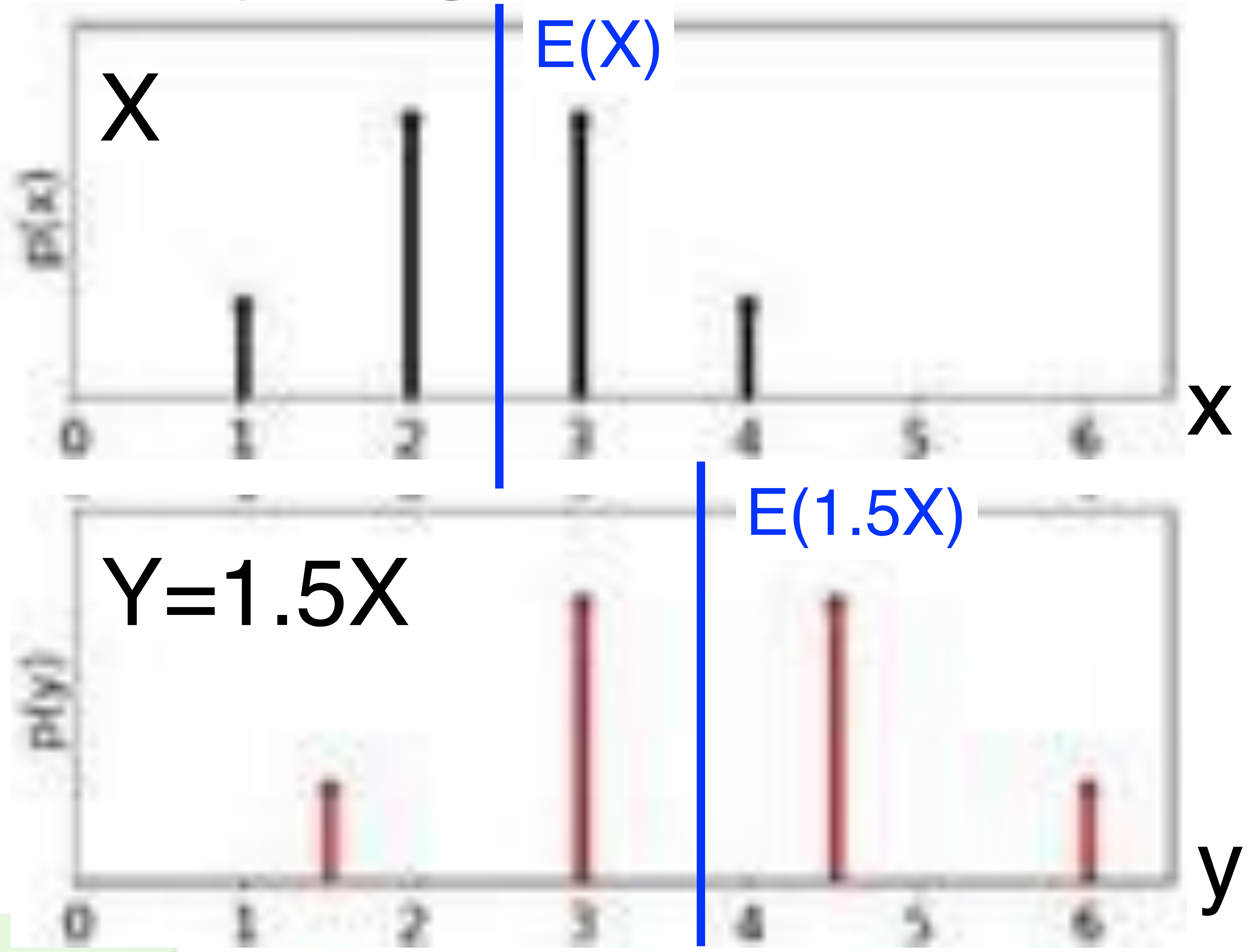
$= p + 2 = E(X) + 2$

Constant Multiplication

$$E(aX) = \sum p(x) \cdot (ax)$$

$$= a \sum p(x) \cdot x$$

$$= aE(X)$$



Bernoulli p

x	0	1
p(x)	1 - p	p

$$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$$

$$E(3X) = (1 - p) \cdot (3 \cdot 0) + p \cdot (3 \cdot 1)$$

$$= 3p = 3E(X)$$

Linearity of Expectation

$E(aX + b)$ $= E(aX) + b$

$= a\,E(X) + b$

Bernoulli p

x	0	1
p(x)	1 - p	p

$E(X) = (1 - p) \cdot 0 + p \cdot 1 = p$

$E(2X + 3) = (1 - p)(2 \cdot 0 + 3) + p(2 \cdot 1 + 3)$

$= 3 - 3p + 5p$

$= 2p + 3 = 2E(X) + 3$

Expectation of Functions of Variables

Next: Variance