Functions of Random Variables

Functions of Random Variables

X - income, Y = 0.3X - income tax

X - driving speed, Y= 2^x - speeding ticket

$X \sim f_X$ $\quad$ f_X known distribution

$Y = g(X)$ $\quad$ g known deterministic function

f_Y ? $\quad$ What is the distribution of Y?

Power-law pdf’s

$a > -1$

$\int_0^1 \underbrace{(a+1)x^a}_{\geq 0} dx = x^{a+1}\Big|_0^1 = 1$

$a < -1$ later

$$f(x) = \begin{cases} (a+1)x^a & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

pdf!

$F(x) = \int_0^x (a+1)u^a du = u^{a+1}\Big|_0^x = x^{a+1}$

$a \leq x \leq 1$

Use $f(x) = 3x^2$

$F(x) = x^3$

$0 \leq x \leq 1$

$$\boxed{\begin{array}{c} X \sim f_X(x) = 3x^2 \\ F_X(x) = x^3 \\ 0 \le x \le 1 \end{array}}$$

$g \nearrow$

$$Y = g(X) \triangleq X^{\frac{3}{2}}$$

$$F_Y(y) = P(Y \le y) \qquad 0 \le y \le 1$$

$$= P(X^{\frac{3}{2}} \le y)$$

$$= P(X \le y^{\frac{2}{3}})$$

$$= F_X(y^{\frac{2}{3}})$$

$$= y^2 \qquad F_Y(0) = 0$$

$$F_Y(1) = 1 \text{ ✅}$$

$$f_Y(y) = F_Y'(y) = 2y$$

$g \searrow$

$$Y = g(X) \triangleq X^{-3}$$

$$F_Y(y) = P(Y \le y) \qquad y \ge 1$$

$$= P(X^{-3} \le y)$$

$$= P(X \ge y^{-\frac{1}{3}})$$

$$= 1 - P(X \le y^{-\frac{1}{3}})$$

$$F_Y(1) = 0 \qquad = 1 - F_X(y^{-\frac{1}{3}})$$

$$\text{✅ } F_Y(\infty) = 1 \qquad = 1 - y^{-1}$$

$$f_Y(y) = F_Y'(y) = y^{-2}$$

$$g \nearrow$$

$$\boxed{f_X(x) = 3x^2 \quad F_X(x) = x^3 \quad g(x) = x^{\frac{3}{2}} \quad h(y) = y^{\frac{2}{3}} \quad f_Y(y) = 2y}$$

$F_Y(y) \triangleq P(Y \leq y)$

$f_Y(y) = F'_Y(y)$

$= P(g(X) \leq y)$

$P(X^{\frac{3}{2}} \leq y)$

$= [F_X(h(y))]'$

$[(y^{\frac{2}{3}})^3]'$

$= P(X \leq g^{-1}(y))$

$P(X \leq y^{\frac{2}{3}})$

$= F'_X(h(y)) \cdot h'(y)$

$3(y^{\frac{2}{3}})^2 \cdot \frac{2}{3} y^{-\frac{1}{3}}$

$= F_X(g^{-1}(y))$

$F_X(y^{\frac{2}{3}})$

$= f_X(h(y)) \cdot h'(y)$

$2y$

$= F_X(h(y))$

$h(y) \overset{\text{def}}{=} g^{-1}(y)$

g $\searrow$

$$F_Y(y) \triangleq P(Y \leq y)$$

$$= P(g(X) \leq y)$$

$$= P(X \geq g^{-1}(y))$$

$$= 1 - P(X \leq g^{-1}(y))$$

$$= 1 - F_X(g^{-1}(y))$$

$$= 1 - F_X(h(y))$$

$$f_Y(y) = F_Y'(y)$$

$$= [1 - [F_X(h(y))]'$$

$$= -F_X'(h(y)) \cdot h'(y)$$

$$= -f_X(h(y)) \cdot h'(y)$$

Combining

$g \nearrow$ $$f_Y(y) = f_X(h(y)) \cdot h'(y)$$

$g \searrow$ $$f_Y(y) = - f_X(h(y)) \cdot h'(y)$$

For both $$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$

Alternative formulation $$f_Y(y) = \frac{1}{|g'(x)|} f_X(x) \Big|_{y=g(x)}$$

Functions of Random Variables

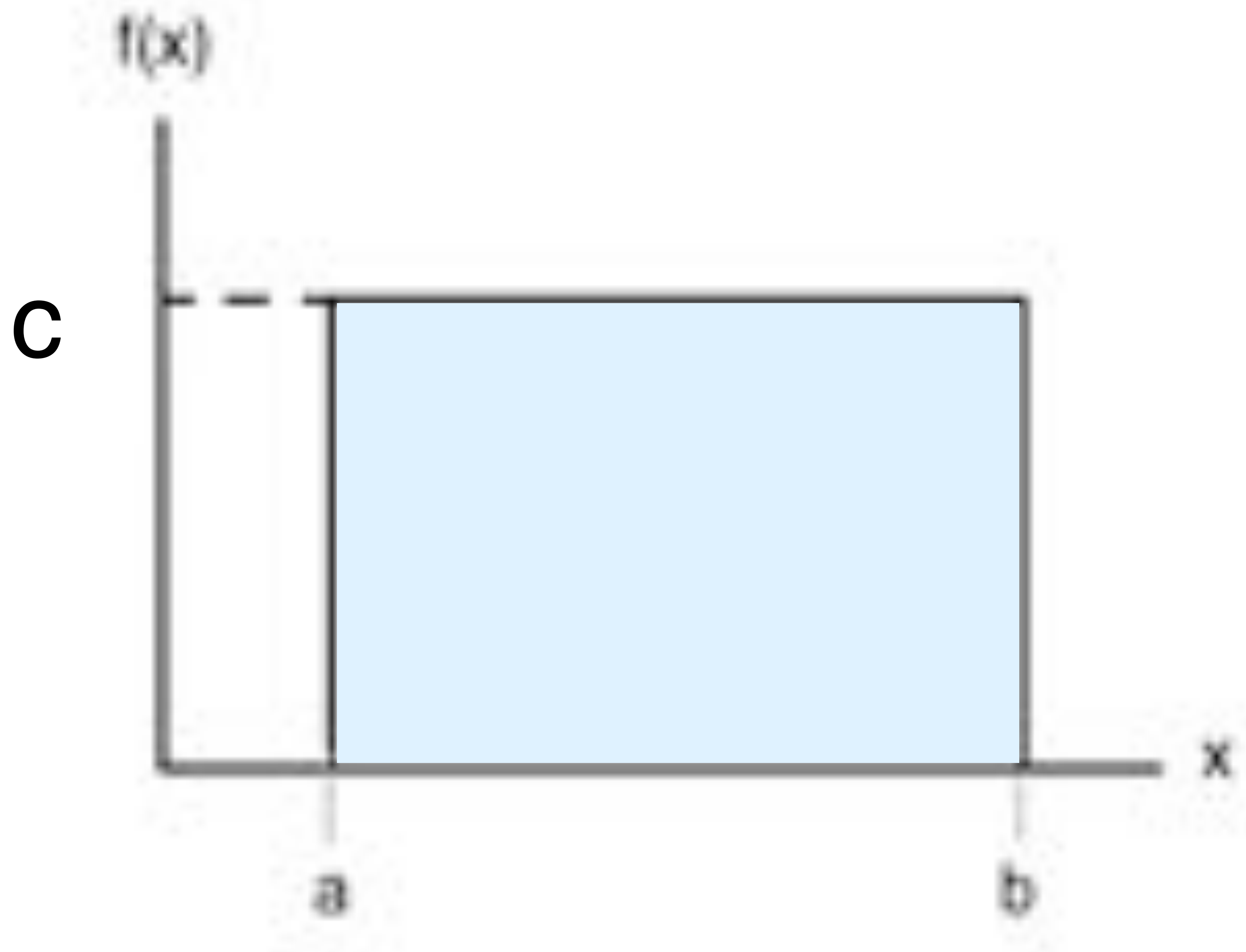
Uniform Distributions

Uniform distribution

Definintuition

For a<b, the ***uniform*** distribution $U_{[a,b]}$ is constant inside [a,b] and 0 outside

$$f(x) = \begin{cases} c & x \in [a,b] \quad \text{equally likely} \\ 0 & x \notin [a,b] \quad \text{never happen} \end{cases}$$



For $(\alpha, \beta) \subseteq [a,b]$

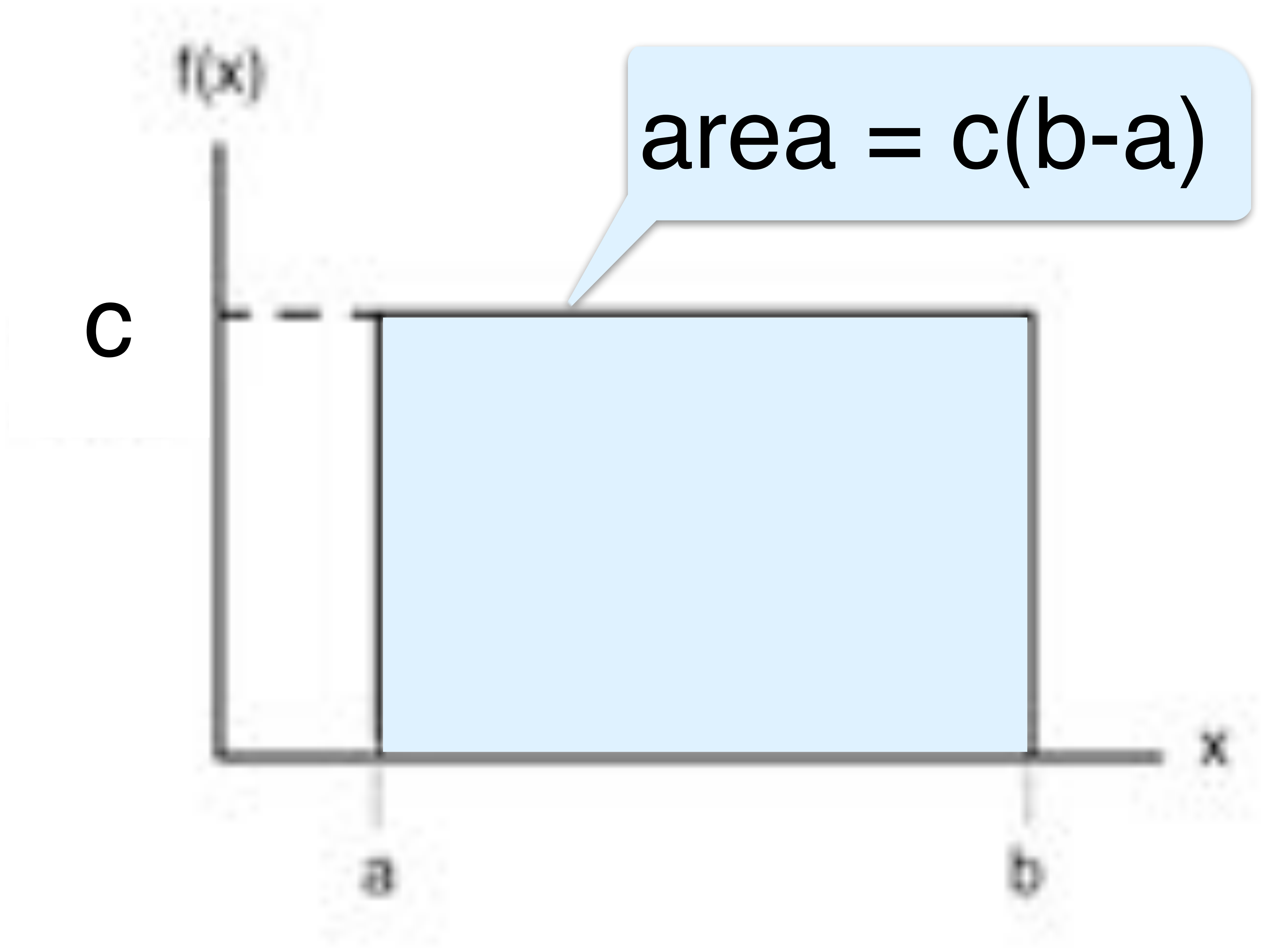
Probability determined by, and $\propto$, length $\beta-\alpha$

Area under curve is always a rectangle

Integrals are just height x width

c?

$$f(x) = \begin{cases} c = \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$$



$$1 = \int_{-\infty}^{\infty} f(x)dx = c(b-a)$$

$$c = \frac{1}{b-a}$$

ΣWILL IT ADD?

YES IT ADDS!

Who’s Uniform

Departure times

Wait time for a bus

Location of chip defect

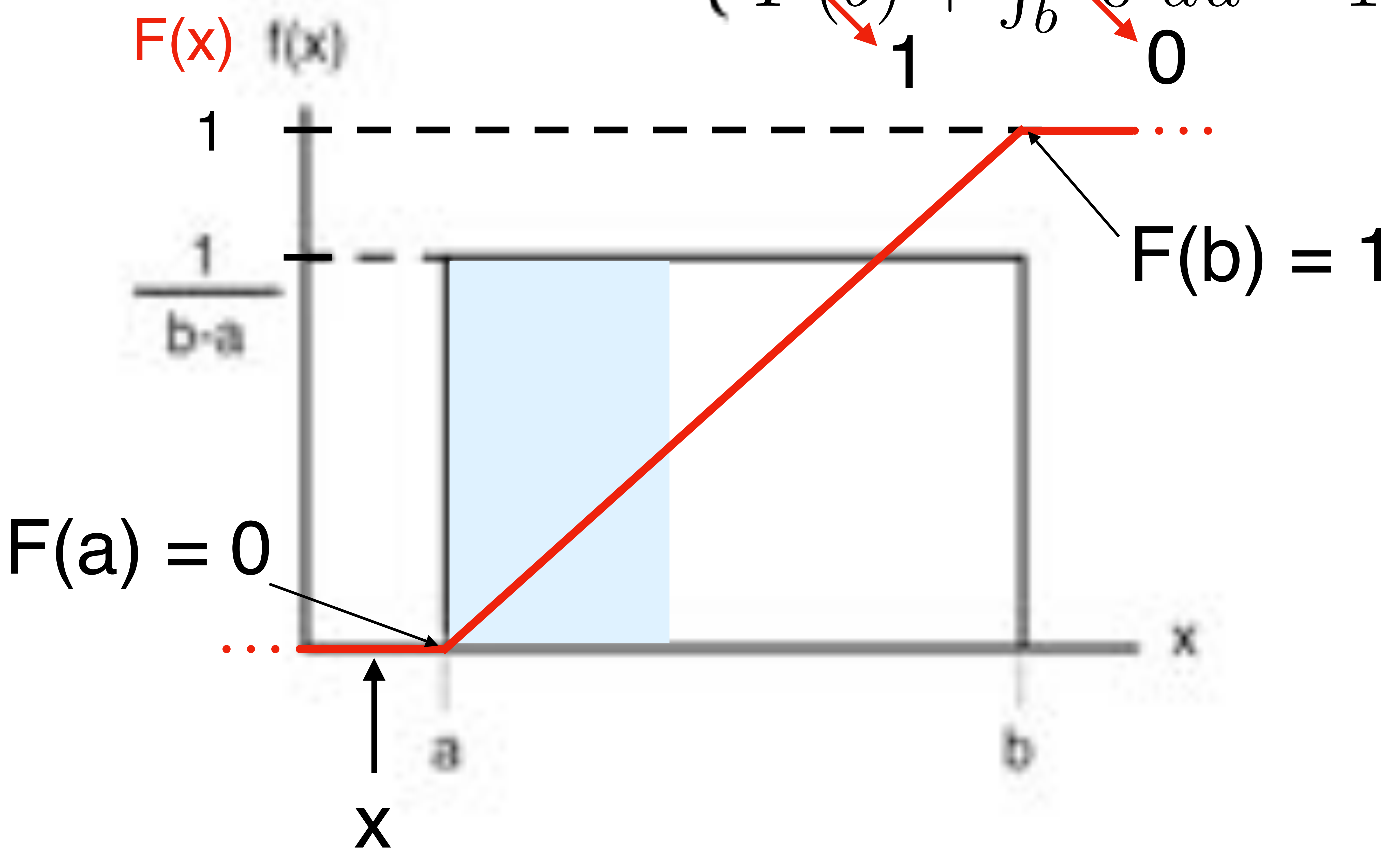
Location of a molecule in space

Considering a small area in time or space

Not so many…

CDF

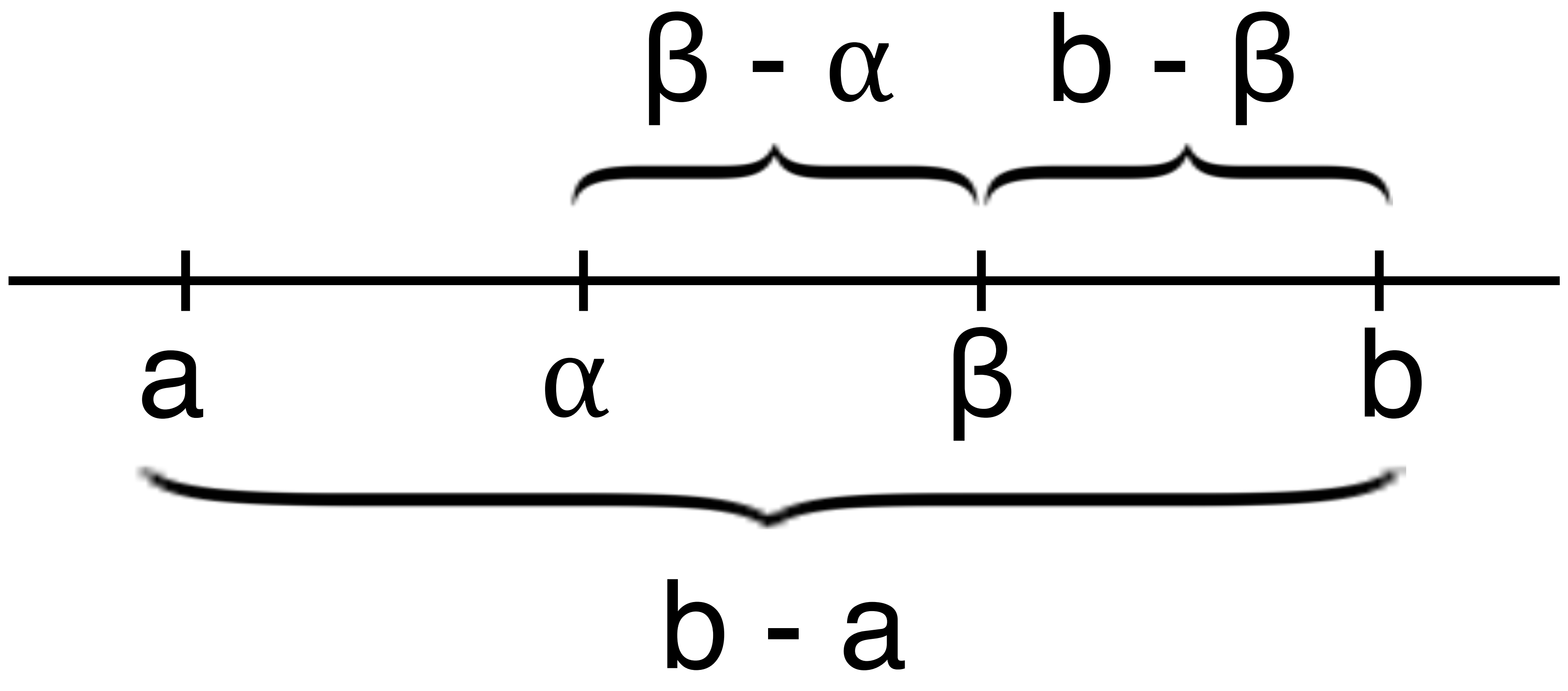
$$F(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} \int_{-\infty}^{x} 0 \ du = 0 & x \leq a \\ \cancelto{0}{F(a)} + \int_{a}^{x} \frac{1}{b-a} du = \frac{x-a}{b-a} & a \leq x \leq b \\ \cancelto{1}{F(b)} + \cancelto{0}{\int_{b}^{x} 0 \ du} = 1 & x \geq b \end{cases}$$



Interval Probabilities

For $a \leq \alpha \leq \beta \leq b$

Interval	Probability
$(\alpha, \beta]$	$F(\beta) - F(\alpha) = \frac{\beta - a}{b - a} - \frac{\alpha - a}{b - a} = \frac{\beta - \alpha}{b - a}$
$[\beta, \infty)$	$F(\infty) - F(\beta) = 1 - \frac{\beta - a}{b - a} = \frac{b - \beta}{b - a}$
$\{\alpha\}$	$F(\alpha) - F(\alpha) = 0$



μ & σ

X ~ $U_{[0,\,1]}$ first

$f(x) = 1, \quad 0 \le x \le 1$

$$EX = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$EX^2 = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$V(X) = EX^2 - (EX)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\sigma = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}} \approx 0.29$$

Translation and Scaling

Uniformity preserved under translation and scaling

$X \sim U_{[0,\,1]}$ wolog $(a \neq 0)$

For any constants $a > 0$ and b, $Y \stackrel{\text{def}}{=} aX + b \sim U_{[b,\,a+b]}$

Range: $0 \to b$ $1 \to a+b$

PDF $Y = aX + b \stackrel{\text{def}}{=} g(X)$

$$f_Y(y) = \left.\frac{f_X(x)}{|g'(x)|}\right|_{x=g^{-1}(y)} = \frac{1}{a}$$

Or: equal-length interval map to =-length intervals

General μ & σ

$Y \sim U_{[a,\, b]}$

$Y = (b - a)X + a$

$$EX = \frac{1}{2}$$

$$EY = (b-a)EX + a = \frac{b-a}{2} + a = \frac{a+b}{2}$$

$$V(X) = \frac{1}{12}$$

$$V(Y) = V[(b-a)X + a] = (b-a)^2 V(X) = \frac{(b-a)^2}{12}$$

$$\sigma = \frac{b-a}{2\sqrt{3}} \approx 0.29(b-a)$$

Uniform Distributions

$U_{[a,b]}$ $a<b$

PDF $f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$

CDF $F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x \geq b \end{cases}$

$\mu = \frac{a+b}{2}$ $V = \frac{(b-a)^2}{12}$ $\sigma = \frac{b-a}{2\sqrt{3}}$

Exponential Distribution

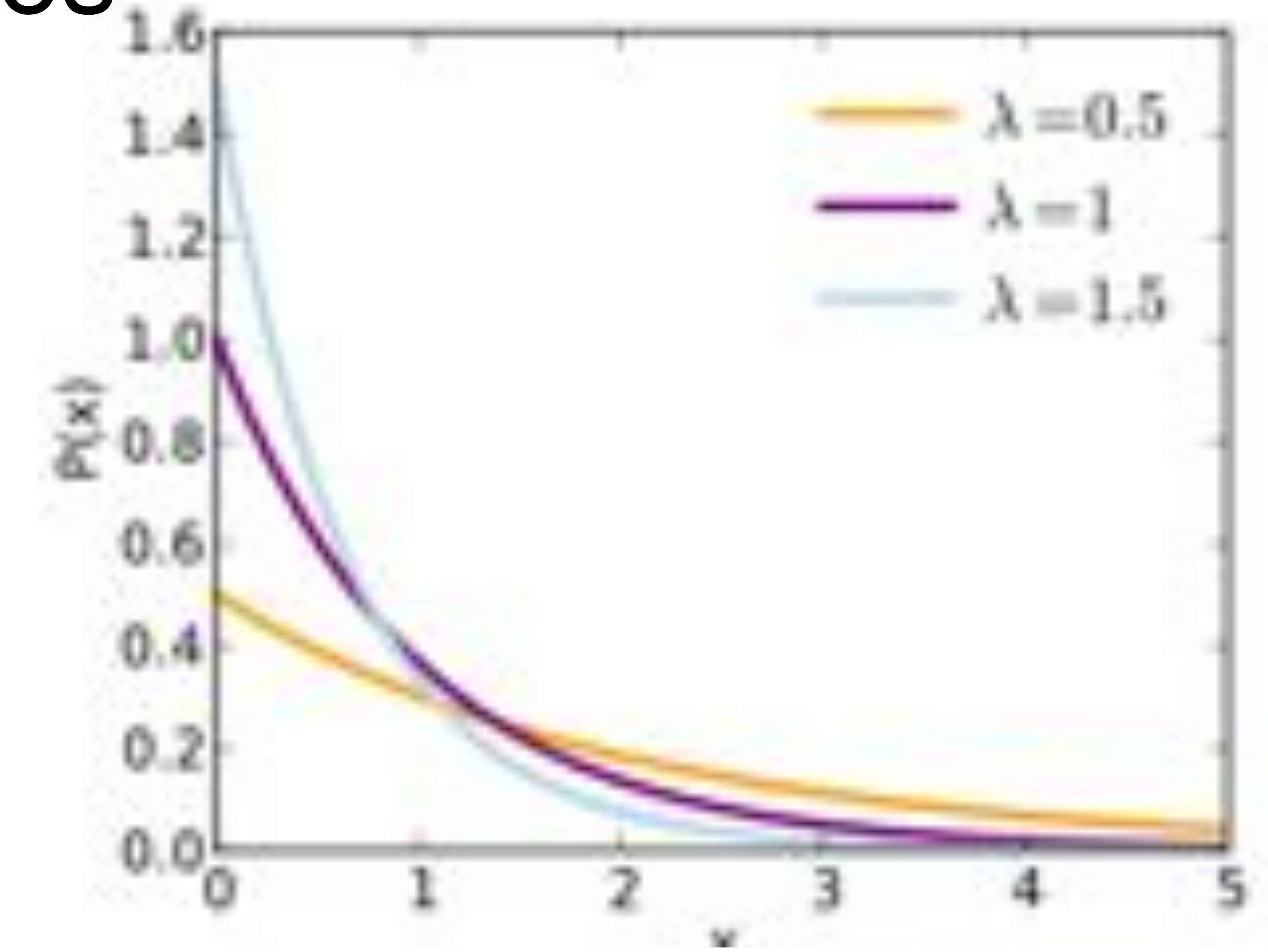
Definition

Extends geometric distribution to continuous values

$\lambda > 0$

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

≥ 0 ✅



ΣWILL IT ADD?

$$\int f(x)dx = \int_0^\infty \lambda e^{-\lambda x}\, dx$$
$$= -\, e^{-\lambda x}\Big|_0^\infty$$
$$= 0 - (-1)$$
$$= 1$$

YES IT ADDS!

Who’s Exponential

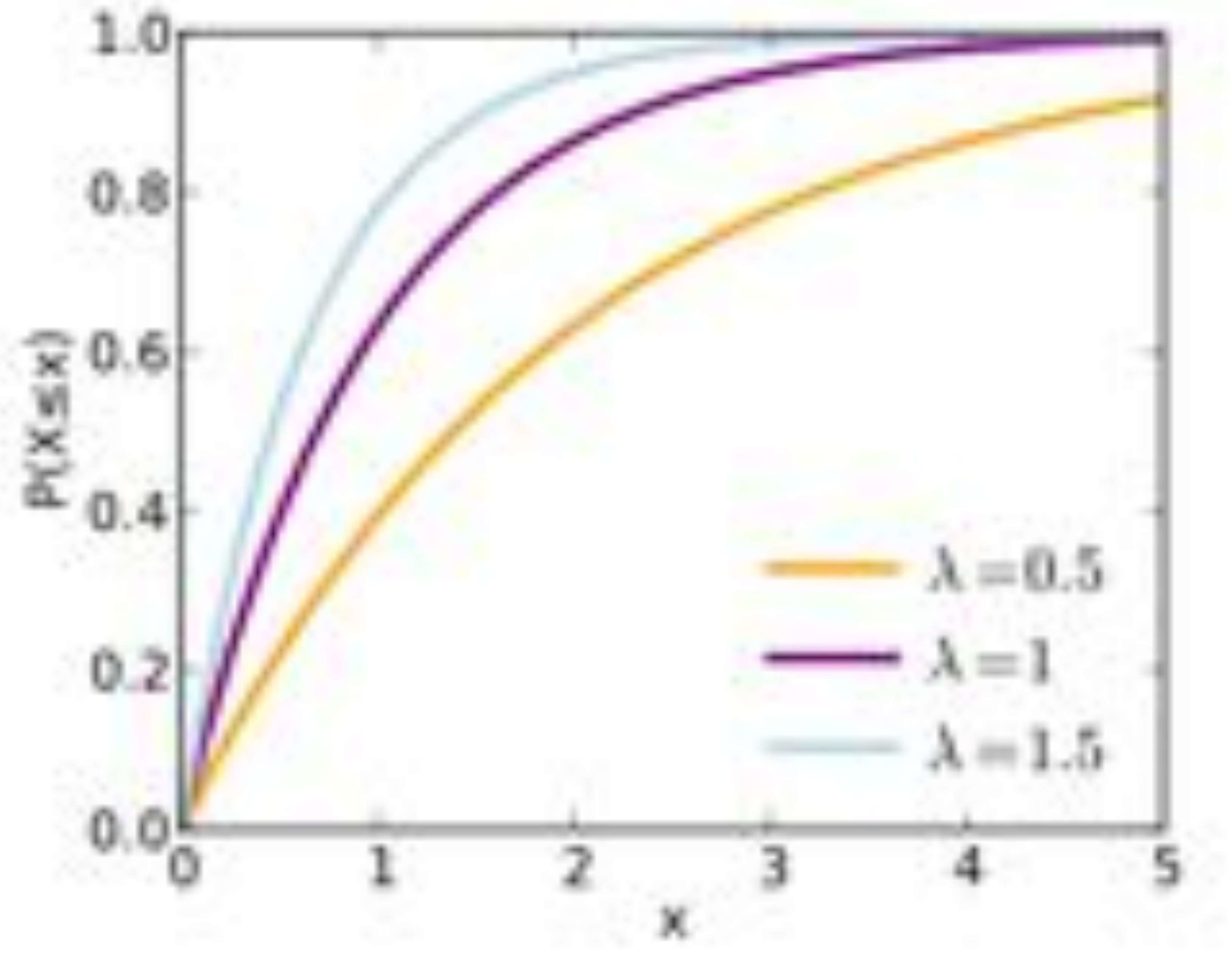
Duration of a phone call

Wait time when you call an airline

Lifetime of a car

Time between accidents

CDF



$x \geq 0$

$$P(X > x) = \int_x^\infty \lambda e^{-\lambda u} du$$

$$= -e^{-\lambda u}\Big|_x^\infty$$

$$= e^{-\lambda x}$$

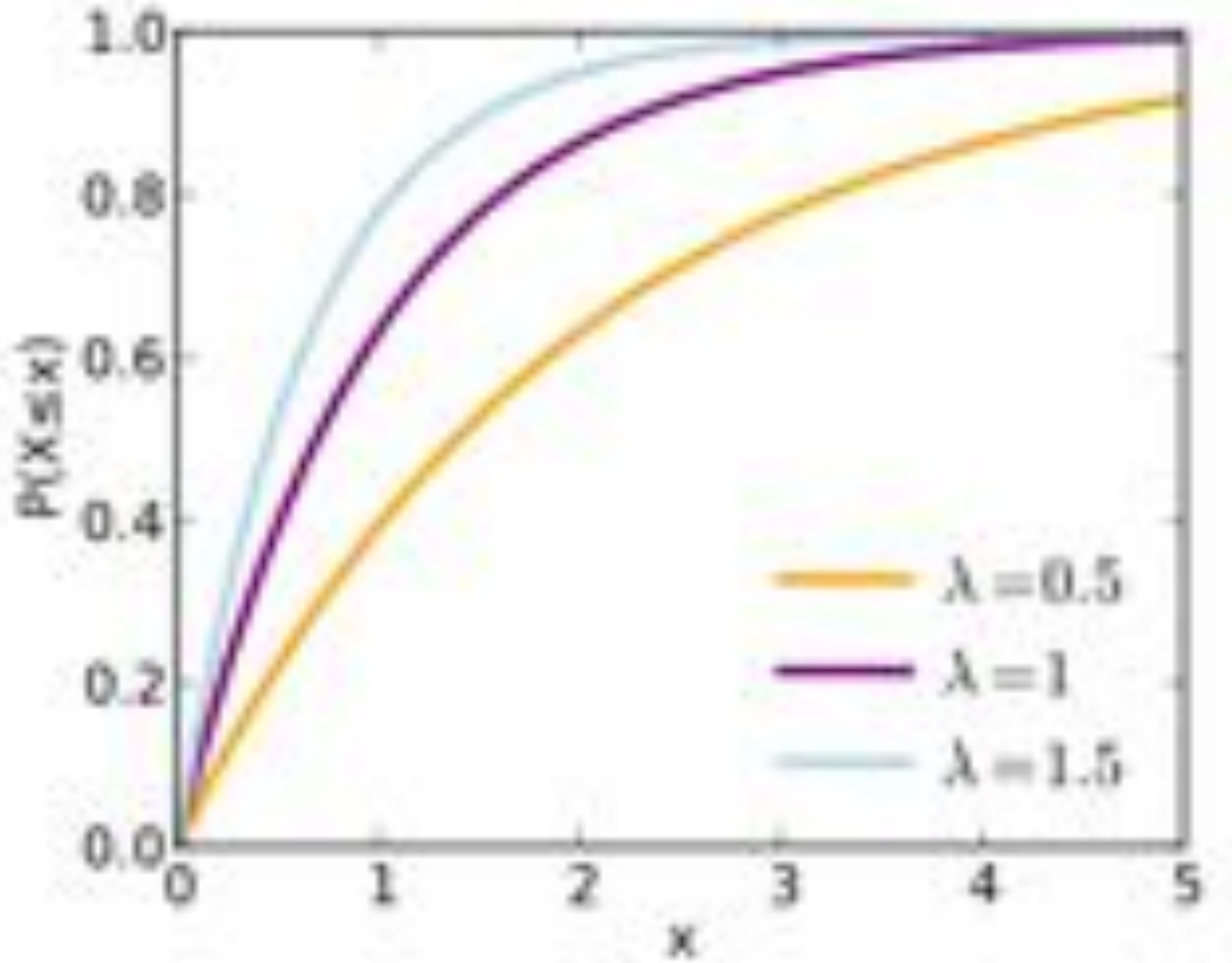
$$F(x) = P(X \leq x) \;\; = 1 - P(X > x) = 1 - e^{-\lambda x} \qquad x \geq 0$$

$$F(x) = 0 \qquad x \leq 0$$

CDF

$$P(X > x) = \begin{cases} \int_x^\infty \lambda e^{-\lambda u} du \ = -e^{-\lambda u}\Big|_x^\infty \ = e^{-\lambda x} & x \geq 0 \\ 1 & x \leq 0 \end{cases}$$

$$F(x) = P(X \leq x) = \begin{cases} 1 - P(X > x) \ = 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$



Example

$0 \leq a \leq b$

$$P(a \leq X \leq b) = P(a < X < b)$$

$$= F(b) - F(a)$$

$$= (1 - e^{-\lambda b}) - (1 - e^{-\lambda a})$$

$$= e^{-\lambda a} - e^{-\lambda b}$$

Expectation

$$EX = \int_0^\infty x\lambda e^{-\lambda x} dx$$

$$u = x \qquad dv = \lambda e^{-\lambda x} dx$$

$$\int u \, dv = uv - \int v \, du$$

$$du = 1 \qquad v = -e^{-\lambda x}$$

$$= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx$$

$$= 0 - \frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty$$

$$= \frac{1}{\lambda}$$

Variance

$$EX^2 = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$u = x^2 \qquad dv = \lambda e^{-\lambda x} dx$$

$$du = 2x \ dx \qquad v = -e^{-\lambda x}$$

$$\int u \ dv = uv - \int v \ du$$

$$= -x^2 e^{-\lambda x}\Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x} dx$$

$$EX = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$= 0 + \frac{2}{\lambda} EX = \frac{2}{\lambda^2}$$

$$V(X) = EX^2 - (EX)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\sigma = \frac{1}{\lambda}$$

Memoryless

$X \sim f_\lambda$ $a, b \geq 0$

$$P(X \geq a + b | X \geq a) = \frac{P(X \geq a+b, X \geq a)}{P(X \geq a)}$$

$$= \frac{P(X \geq a+b)}{P(X \geq a)}$$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$$

$$= e^{-\lambda b}$$

$$= P(X \geq b)$$

$$P(X < a + b | x \geq a) = 1 - P(X \geq a + b | X \geq a) = 1 - P(X \geq b) = P(X < b)$$

$$f(X = a + b | X \geq a) = f(X = b)$$

While Waiting in Line

DMV has 2 clerks, each with exponential service time

When you arrive, one person is in line 🙁

While you wait, someone cuts in front of you 😡

At some point a clerk becomes available and starts serving the first person

Before first person finishes, other clerk starts serving second person

If all three of you served randomly, P(you finish last) = ⅓

P(you finish last now)?

Evaluation

A - time first person finishes

B - time second person finishes

C - time you finish

Service	P(A < B < C)
Fixed	1
Exponential	?

Orders	Probability
A < B < C	¼
A < C < B	¼
B < A < C	¼
B < C < A	¼
C < A < B	0
C < B < A	0

?

$$P(A<B<C) = \underbrace{P(A < B)}_{\frac{1}{2}} \cdot \underbrace{P(B < C \mid A < B)}_{\frac{1}{2}} = \tfrac{1}{4}$$

$$P(B<C<A) = \overbrace{P(B < A)}^{\frac{1}{2}} \cdot \overbrace{P(C < A \mid B < A)}^{\frac{1}{2}} = \tfrac{1}{4}$$

Conclusion

All three of you served randomly, P(you finish last) = ⅓

Fixed service time, P(you finish last) = 1

Exponential (memoryless) service time

- You won't finish first
- All 4 other orders equally likely
- P(you finish last) = ¼ + ¼ = ½
- Only slightly larger than ⅓

Orders	P
A < B < C	¼
A < C < B	¼
B < A < C	¼
B < C < A	¼
C < A < B	0
C < B < A	0

Summary

Exponential

PDF $f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

CDF $F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

$EX = \frac{1}{\lambda}$ $V(X) = \frac{1}{\lambda^2}$ $\sigma = \frac{1}{\lambda}$

Memoryless

Normal Distribution