Z and T Tests

Normal and similar distributions

σ known

Z test

σ unknown

T test

Examples

Motivation

Tests for mean

← Binomial and related distributions — Knew P_{H_0} exactly

→ Normal distributions — Know P_{H_0} approximately

Sample mean — Few samples, distributed ≈ Normal

Many (≥ 30) samples, any distribution

Central Limit Theorem - Review

$X_1, \ldots, X_n$ $\perp\!\!\!\perp$, $\sim$ any fixed distribution mean μ stdv σ

Sample mean $\overline{X} = \frac{1}{n}(X_1 + \ldots + X_n)$

$$\mu_{\overline{X}} = \mu \qquad V(\overline{X}) = \frac{\sigma^2}{n} \qquad \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$$

CLT For large n ≥ 30 $\overline{X} \dot{\sim} \mathcal{N}(\mu_{\overline{X}}, \sigma^2_{\overline{X}}) = \mathcal{N}(\mu, \frac{\sigma^2}{n})$

Tiny Transformation $\overline{X} - \mu \dot{\sim} \mathcal{N}(0, \frac{\sigma^2}{n})$

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \dot{\sim} \mathcal{N}(0, 1)$$

Typical Application

Unknown distribution or population

σ known

Relax later

H_0 : mean = μ

Given

H_A : mean > μ

Generalize later

Test

n independent samples

$X_1, X_2, \ldots, X_n$

Sample mean

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Moderate n

≥ 30

H_0

$$Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \mathrel{\dot\sim} \mathcal{N}(0,1)$$

mean 0

Z test

H_A

Normal

mean > 0

Z Test

H_0 $\mu_Z = 0$ $Z \dot{\sim} \mathcal{N}(0, 1)$

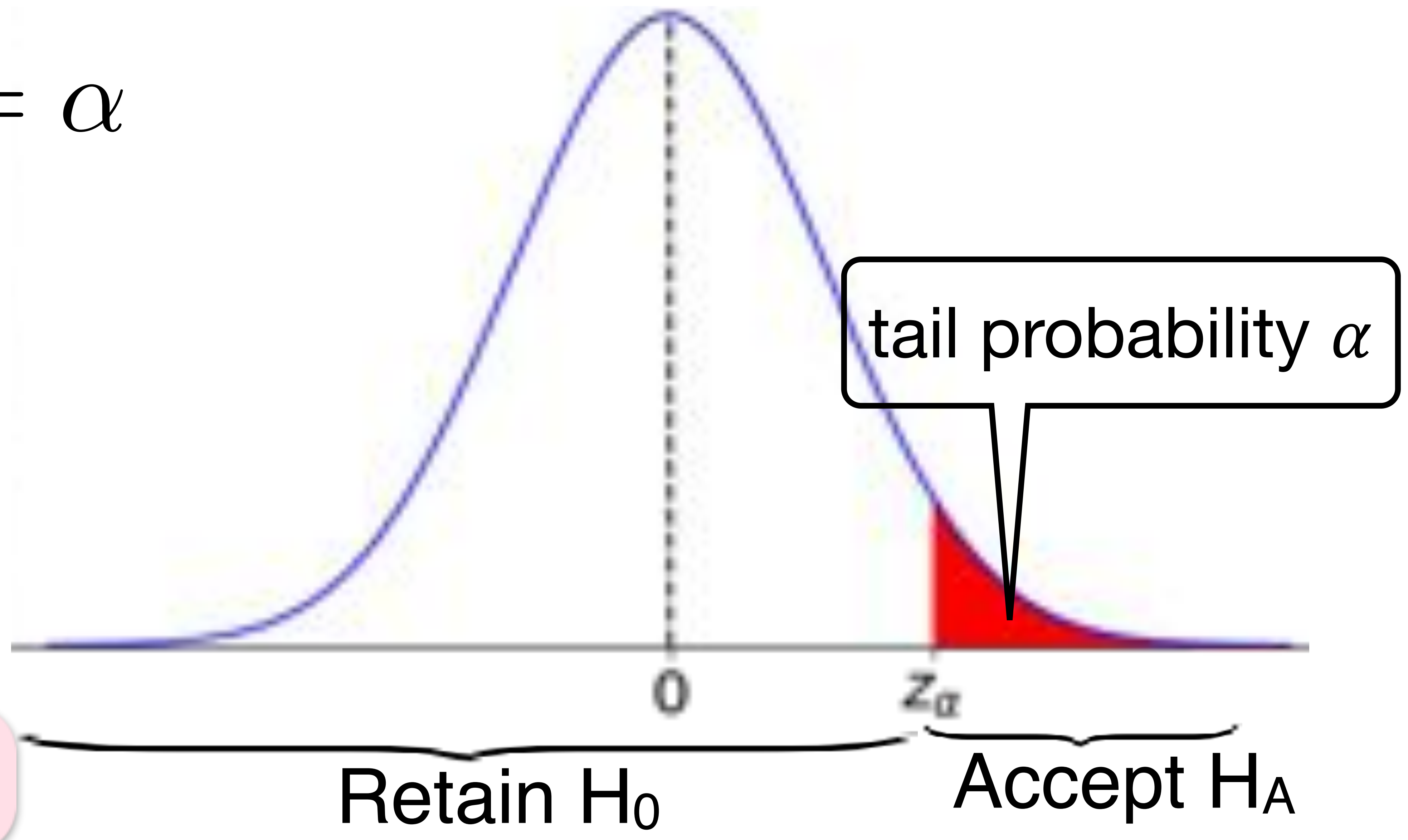
H_A $\mu_Z > 0$

Type-I error

Significance level α 5% 1%

Ensure $P_{H_0}(\text{Accept } H_A) \leq \alpha$

Critical value z_α $P(Z \geq z_\alpha) = \alpha$



Test Z $\geq z_\alpha$ Accept H_A

$< z_\alpha$ Retain H_0

$P_{H_0}(\text{Accept } H_A) = P(Z \geq z_\alpha) = \alpha$

p value of z $P(Z \geq z)$

$\leq \alpha$ $z \geq z_\alpha$ Accept H_A

$> \alpha$ $z < z_\alpha$ Retain H_0

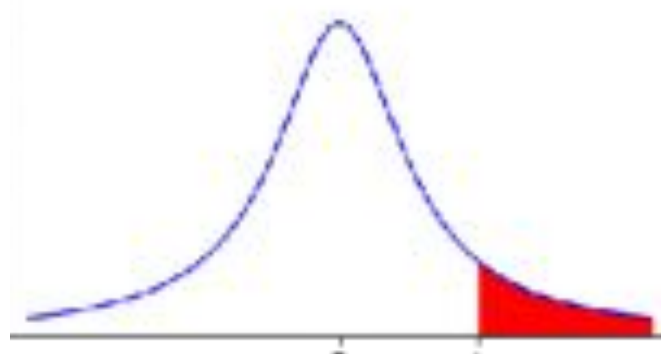
Computationally

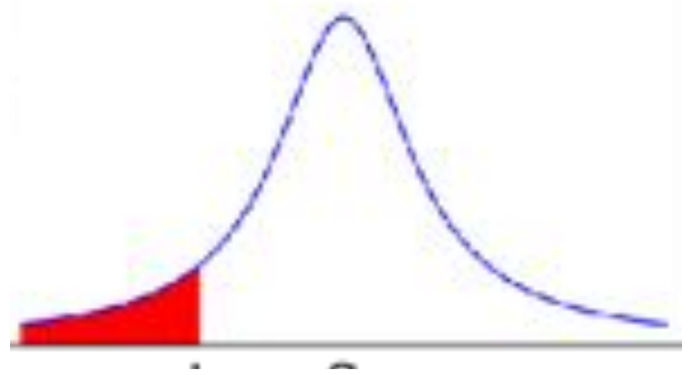
$P(Z \geq z) = 1 - \Phi(z)$

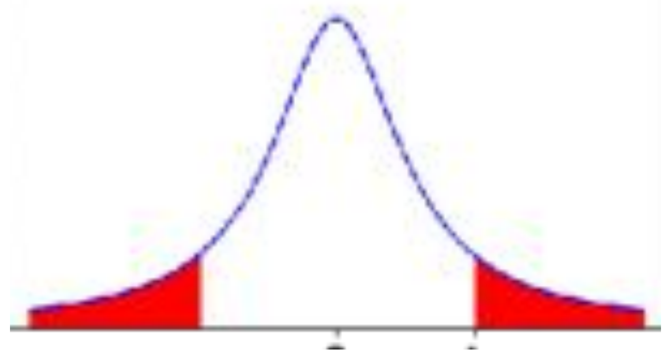
1-Tailed & 2-Tailed Tests

H_0 also
$\leq \mu$
$\geq \mu$

H_0	H_A	H_A is	Test
μ	$> \mu$	1-sided	1-Tailed
μ	$< \mu$	1-sided	1-Tailed
μ	$\neq \mu$	2-sided	2-tailed







p value
$P(Z \geq z)$
$P(Z \leq z)$
$P(\lvert Z \rvert \geq z)$

Examples Disclaimer

Data… for Demonstration Purposes Only!

Cocoa in Chocolate

100g dark chocolate bar contains 85 g ~~cocoa~~ cacao

Still a lot

Suspect less

Not quite

H_0 $\mu = 85$ g

H_A $\mu < 85$ g

Test with 5% significance level

Test

Sample size n=30

Buy 30 bars Measure cacao in each

Fact $\sigma = 0.5$ g

Maybe:
Know machine.
Want simple test.

n ≥ 30 $\overline{X} \dot{\sim} \mathcal{N}(\mu, \frac{\sigma}{\sqrt{n}})$

σ known

Z-Test

Z-Test

n=30

$X_1,\ldots,X_{30}$ = 84.1, 83.2, 85.7, …

$\overline{X} = 84.83$

Same calculation if H_0: $\mu \geq 85$ g

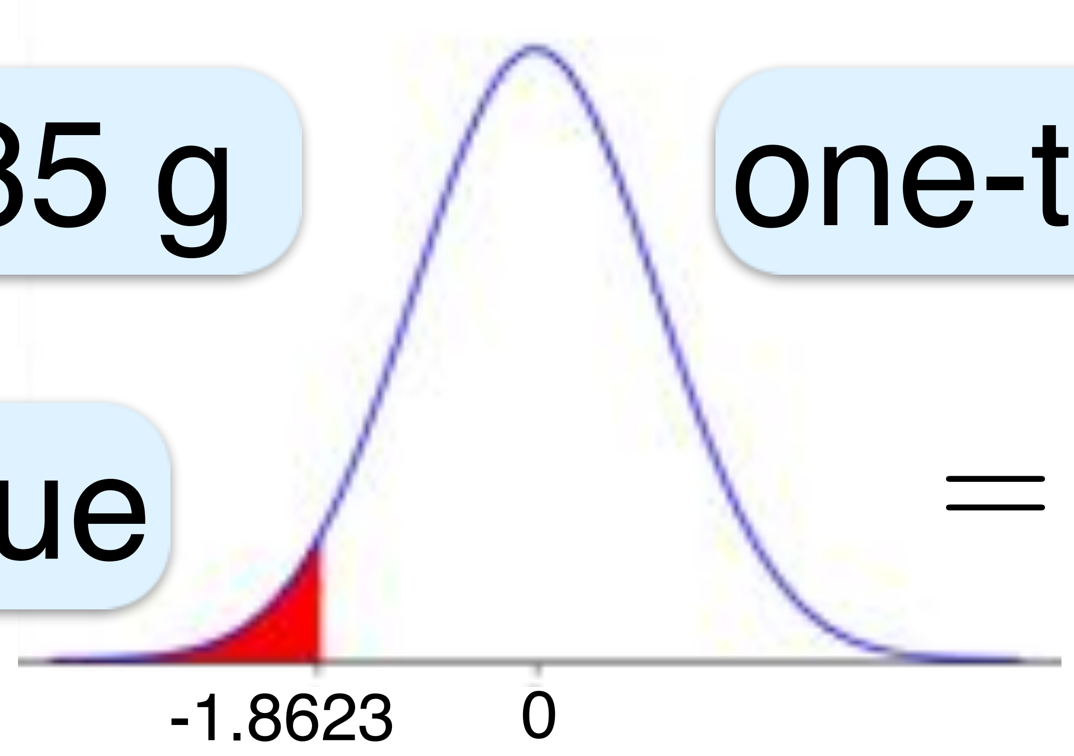
H_0 μ = 85 g σ = 0.5 g

Z score $Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \mathrel{\dot\sim} \mathcal{N}(0,1)$ $Z = \frac{84.83-85}{0.5/\sqrt{30}} \approx -1.8623$

Normal distribution

```
from scipy.stats import norm
print norm.cdf(-1.8623)
0.031280407744557l
```

H_A μ < 85 g one-tailed test



p value $= P(Z \leq -1.8623) = \Phi(-1.8623) \approx 0.0313 < 5\%$

Under H_0 μ = 85g, σ = 0.5 g, probability that 30 bars have average ≤ 84.83 g is ≈ 3.13%.

This low prob. is < 5% significance level.

We reject H_0 and accept H_A.

Accept H_A

Average bar contains < 85 g cacao

Two-Tailed Test

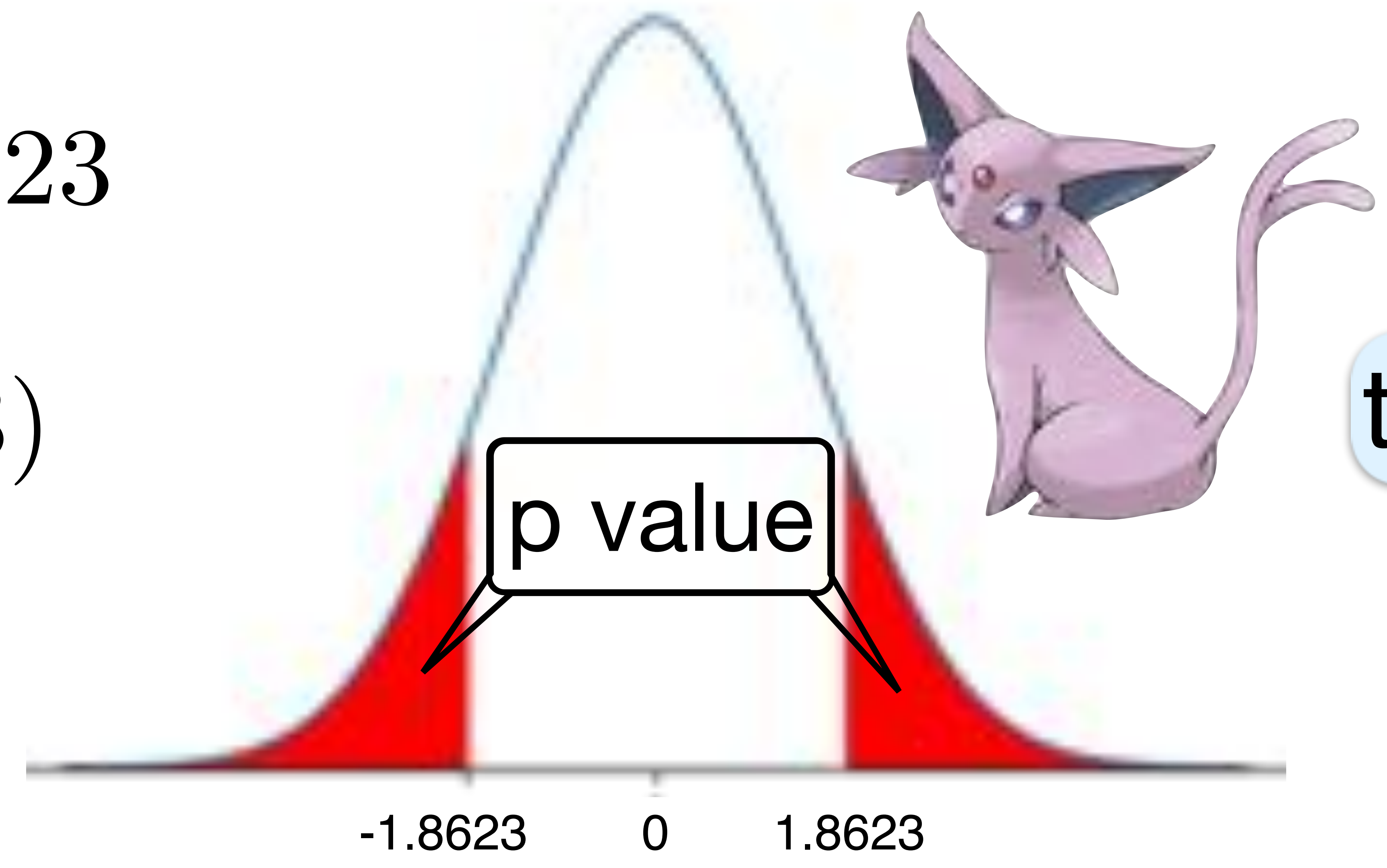
Two-Sided

H0 μ = 85 mg HA $\mu \neq$ 85 mg

Test with 5% significance level

Z score $Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \approx -1.8623$

p value
$$= P(|Z| \geq 1.8623)$$
$$= 2 \cdot \Phi(-1.8623)$$
$$\approx 0.0626 > 5\%$$



two-tailed test

Under H0 μ = 85g, σ = 0.5 g, the probability that 30 bars have average ≤ 84.83 or ≥ 85.17g is ≈ 6.26%.
This "high" prob. is > 5% Significance level.
"Likely" to happen under H0. Retain H0.

Retain H0

Retain possibility that on average a chocolate bar contains 85 g cocoa.

1- vs. 2-Sided Alternatives

1- 2- sided Alternative hypotheses

Same Significance 5%

Z score -1.8623

Yet 1-sided Accept H_A

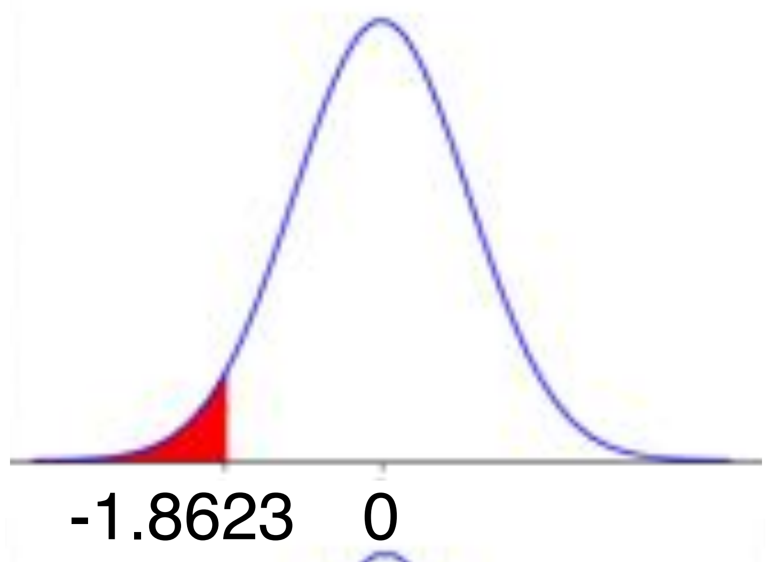
2-sided Retain H_0

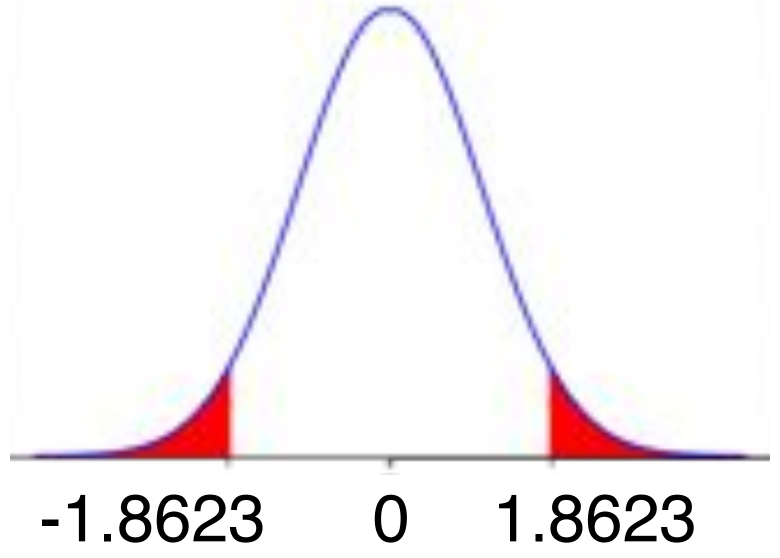
why?

p value P_{H_0}(observed value or one more towards H_A)

≤ 5% Accept H_A

> 5% Retain H_0





H_A	p Value	%	v. 5%	Prob.	Hyp
One-sided	P(Z < -1.86)	3.13%	<	Low	H_A
Two-sided	P(|Z| > 1.86)	6.26%	>	High	H_0

Relation to Confidence Intervals

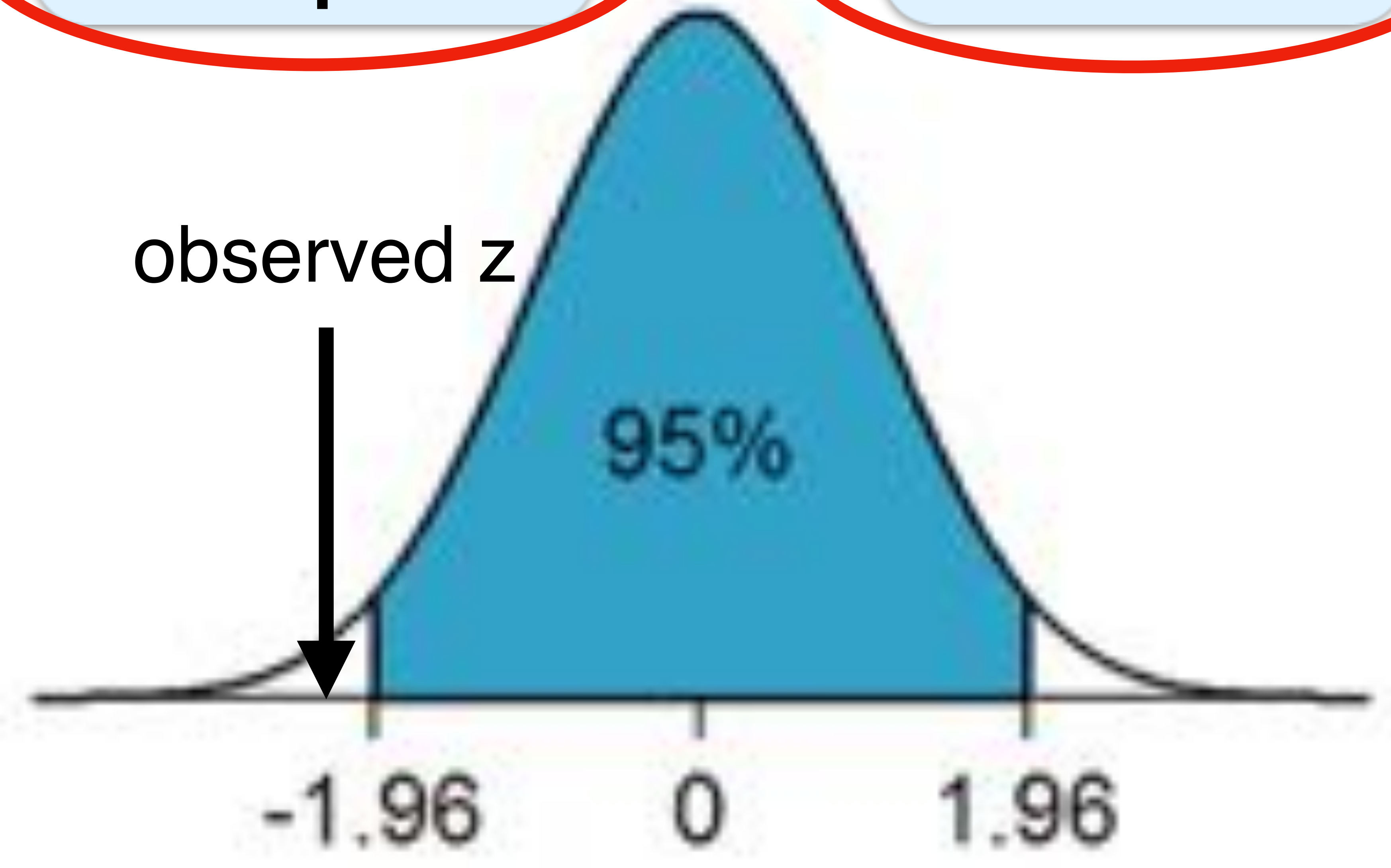
H_0 : mean = μ

H_A : 2-sided, mean $\neq \mu$

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

Significance level 5%

Accept H_A $\quad$ $|Z| \geq 1.96$



$\Longleftrightarrow$

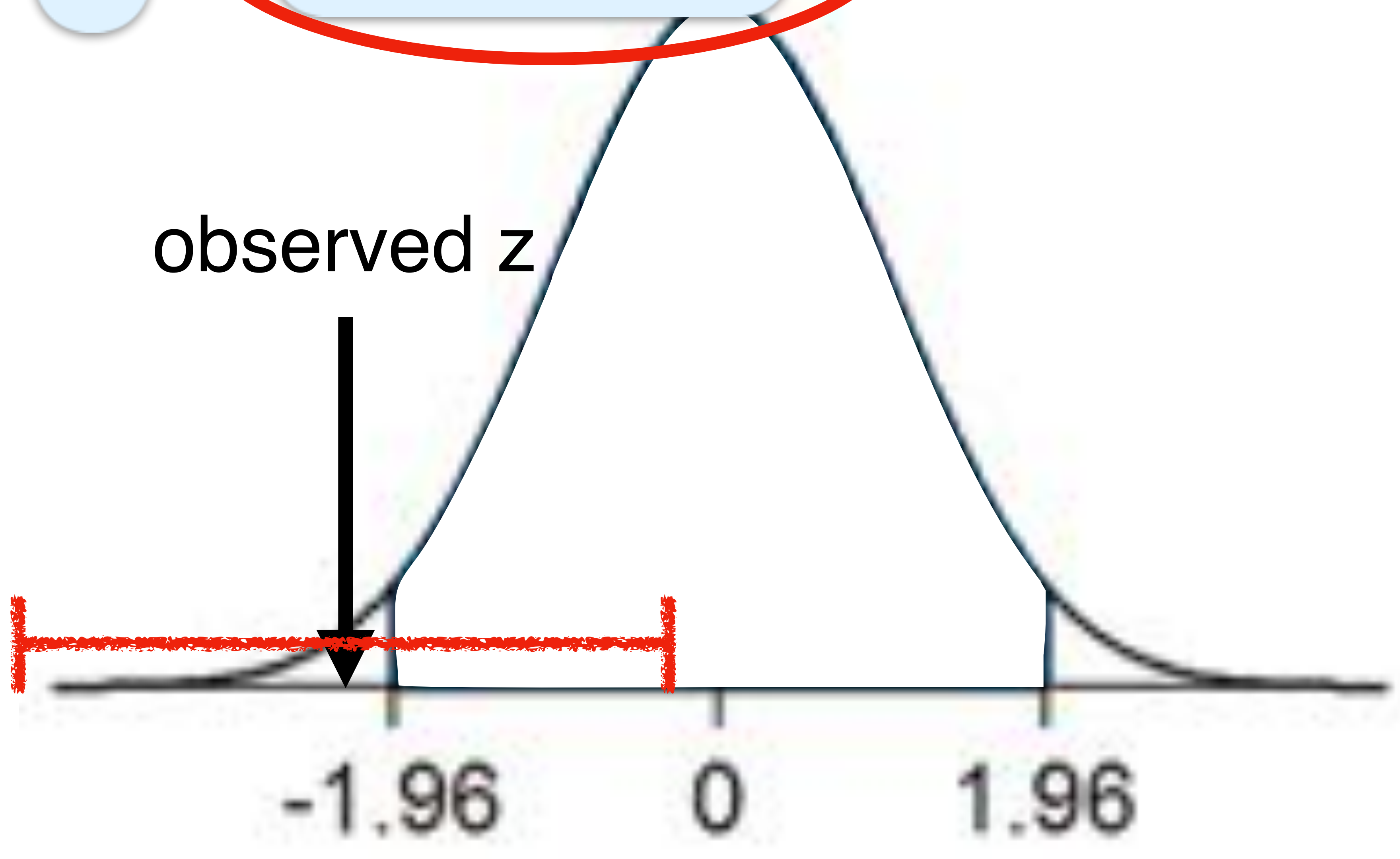
95% confidence interval

$$\left(\overline{X} - z_{95\%} \frac{\sigma}{\sqrt{n}}, \overline{X} + z_{95\%} \frac{\sigma}{\sqrt{n}}\right)$$

$\mu \notin$ confidence interval

iff $\quad 0 \notin (Z - z_{95\%}, Z + z_{95\%})$

iff $\quad |Z| \geq 1.96$



Unknown σ

What if we want to test μ

but don't know σ?

use t-distribution

William Sealy Gosset
(1876 - 1937)

Recall $X_1, X_2, \ldots, X_n \perp\!\!\!\perp \mathcal{N}_{\mu,\sigma}$

Sample mean $\overline{X} = \frac{1}{n}(X_1 + \ldots + X_n)$

Sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ Bessel corrected

$T_{n-1} = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ Student's t-distribution n-1 degrees of freedom

T-Test

Test statistic follows t distribution

H_0: mean is μ

Sample $X_1, ..., X_n$

n can be small

Roughly normal

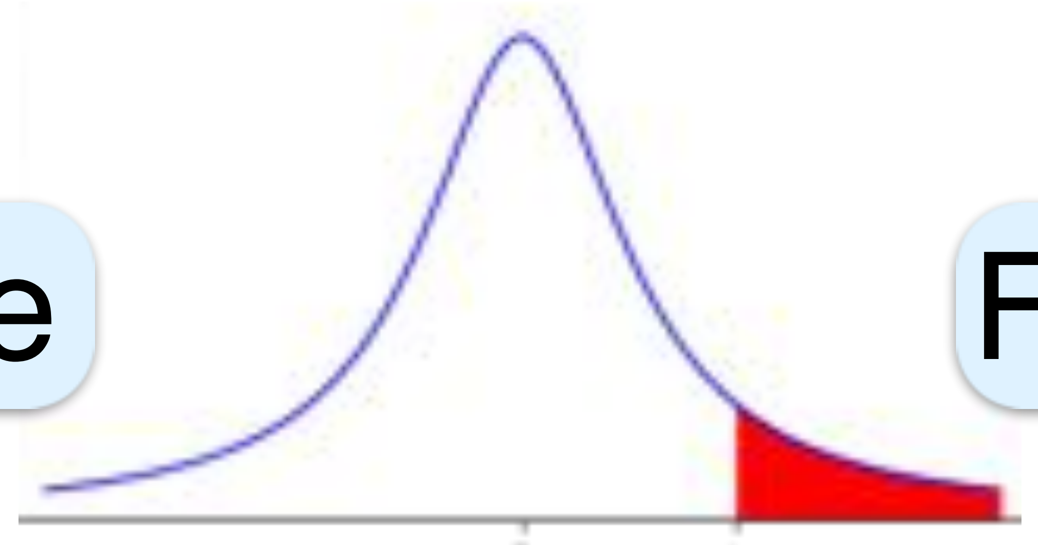
Normality test

Test statistic $T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$

Under H_0 $T \sim f_{n-1}(t)$ Student t-distribution with n-1 DoF

As before

Calculate p value



For t-distribution

Ludicrous Mode

How quickly does a Tesla Model X accelerate from 0 to 60 mph?

$\mu \leq 4$ sec

H_0

H_A

$\mu > 4$ sec

5% significance level

8 measurements

Assume acceleration time roughly normal

Don't know σ

T-Test

T-Test

Sample size

n = 8

3.74, 4.73, 3.85, 3.96, 4.11, 4.30, 4.28, 4.02 sec

```
import numpy as np
sample = [3.74, 4.73, 3.85, 3.96, 4.11, 4.30, 4.28, 4.02]
print np.mean(sample)
print np.var(sample, ddof=1)
```

4.12375 $= \overline{X}$

0.0975696428571$= S^2$

Under H_0

Distribution mean

$\mu = 4$

T-Test statistic

$$T = \frac{\overline{X}-\mu}{S/\sqrt{n}} \approx \frac{4.12375-4}{\sqrt{0.097570/8}} \approx 1.1206$$

p Value

Student's t-distribution with n-1=7 degrees of freedom

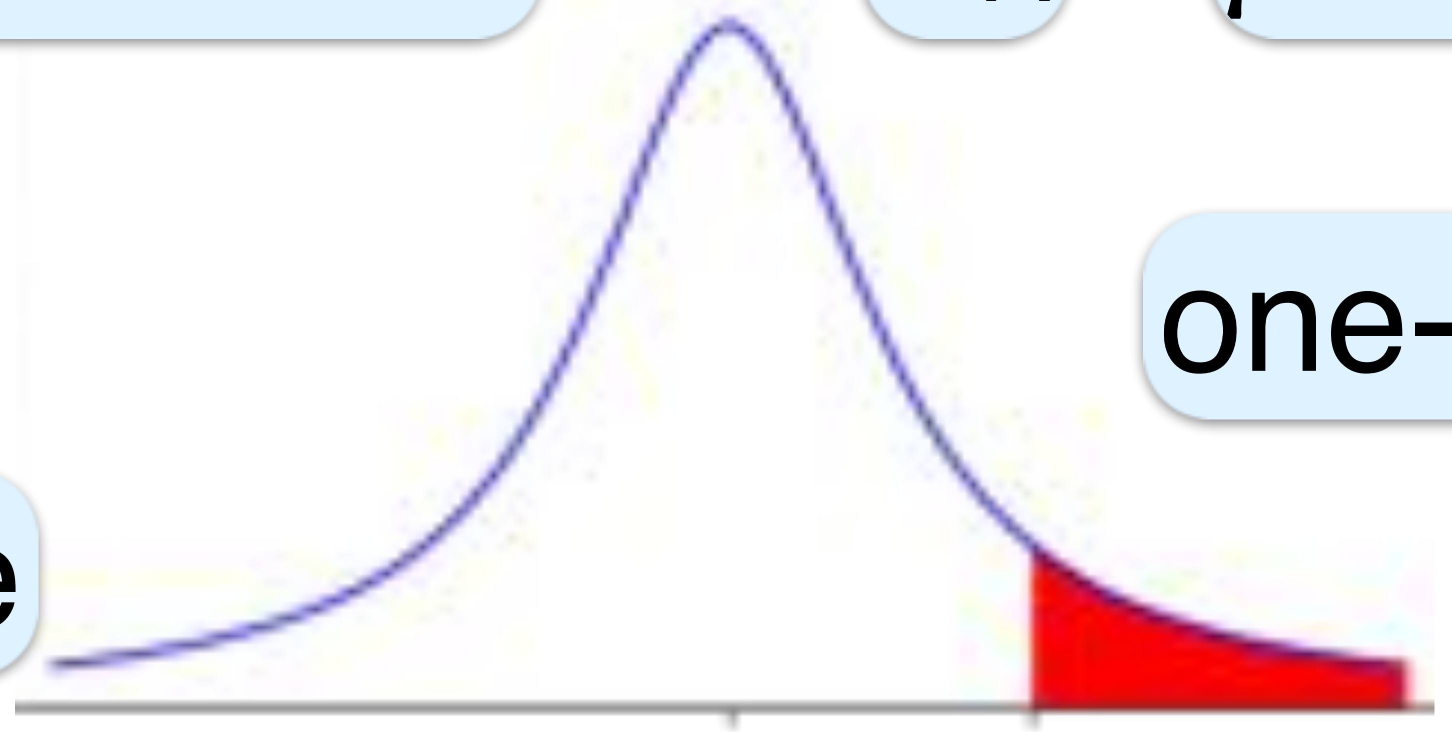
Test statistic $T = \frac{\overline{X}-\mu}{S/\sqrt{n}} \approx 1.1206$

Under H₀ $T \sim f_7(t)$

H_0 $\mu \leq 4$ sec

H_A $\mu > 4$ sec

one-tailed



Under the $\mu \leq 4$ sec H_0, probability of observing at least the 4.124 sec average and S^2 we saw, is ≈ 15%.

Exceeds our 5% significance level.

Retain H_0.

p value $= 1 - F_7(t) \approx 0.1497 > 5\%$

```
from scipy.stats import t
print 1-t.cdf(1.1206,7)
0.149713065157
```

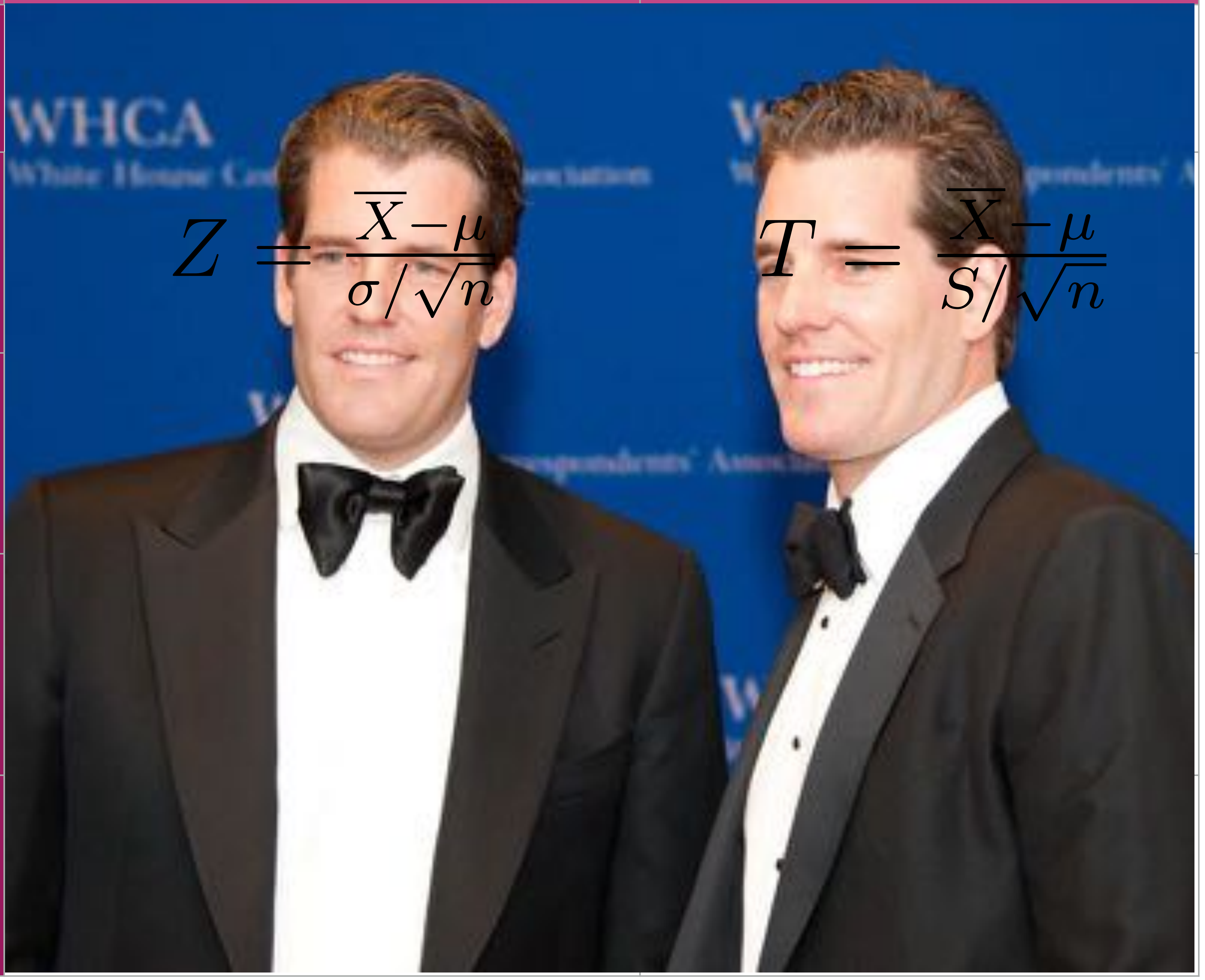
p value > 5%

Retain H_0

Retain possibility of average 0-60 acceleration in ≤ 4 sec.

Z- and T- Tests

	Z	T
Stdv σ		
Test Statistic	$Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$	$T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$
Typical Sample Size n		
Population Distribution		
Distribution used for p value		



similar

Cameron & Tyler Winklevoss

Winklevi Twins

Facebook & Bitcoin Early Investors

As sample size n increases

T → Z

Z and T Tests

Normal and similar distributions

σ known Z test

σ unknown T test

Examples