Controller Design for Robust Output Regulation of Regular Linear Systems

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***Abstract*—We present three dynamic error feedback controllers for robust output regulation of regular linear systems. These controllers are i) a minimal order robust controller for exponentially stable systems, ii) an observer-based robust controller, and iii) a new internal model based robust controller structure. In addition, we present two controllers that are by construction robust with respect to predefined classes of perturbations. The results are illustrated with an example where we study robust output tracking of a sinusoidal reference signal for a 2-D heat equation with boundary control and observation.**

***Index Terms*—Controller design, feedback, regular linear systems, robust output regulation.**

I. Introduction

The topic of this paper is the construction of controllers for robust output regulation of linear infinite-dimensional systems. The goal in this control problem is to design a control law for a linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t), \quad x(0) = x_0 \in X \tag{1a}$$

$$y(t) = Cx(t) + Du(t) \tag{1b}$$

in such a way that the output $y(t)$ converges asymptotically to a given reference signal $y_{ref}(t)$ despite the external disturbance signal $w(t)$. In addition, the controller must tolerate small perturbations and uncertainties in the parameters (A, B, C, D) of the plant (1). The robust output regulation problem was first studied for finite-dimensional systems in the 1970's most notably by Francis and Wonham [6], [7], and Davison [4], and since then the theory of output regulation has been been actively developed for infinite-dimensional systems [2], [9], [11], [21], [22], [24].

The most recent developments in the field are related to the study of output regulation and robust output regulation for infinite-dimensional systems with unbounded input and output operators, and especially for *regular linear systems* [26], [29], [30] which are often encountered in the study of partial differential equations with boundary control and observation [3]. In particular, the characterization of the solvability of the output regulation problem using the so-called *regulator equations* was extended for systems with unbounded operators B and C in [14], [17], and the *internal model principle* of robust output regulation was generalized for regular linear systems in [19].

In this paper we continue the work begun in [19]. The main emphasis in the reference [19] was on studying the properties of robust controllers and on characterizing the solvability of the robust output regulation problem. In this paper we concentrate on designing actual controllers that achieve robust output regulation for the regular linear system (1). As our main results we present three different robust controllers. Two of these controllers employ structures that are familiar from the control of systems with bounded operators B and C, and the third employs a completely new complementary controller structure.

The reference signal $y_{ref}(\cdot)$ and the disturbance signal $w(\cdot)$ are assumed to be generated by an *exosystem*

$$\dot{v}(t) = Sv(t), \quad v(0) = v_0 \in W \tag{2a}$$

$$w(t) = Ev(t) \tag{2b}$$

$$y_{ref}(t) = -Fv(t) \tag{2c}$$

on a finite-dimensional space $W = \mathbb{C}^r$. Here S is a matrix with eigenvalues $\sigma(S) = \{i\omega_1, \dots, i\omega_q\} \subset i\mathbb{R}$. The main objective in this paper is to achieve robust output regulation for the system (1) by choosing appropriate parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ for the dynamic error feedback controller

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \quad z(0) = z_0 \in Z \tag{3a}$$

$$u(t) = Kz(t) \tag{3b}$$

where $e(t) = y(t) - y_{ref}(t)$ is the *regulation error*.

The main tool in constructing robust controllers is the internal model principle, which provides a complete characterization of the controllers that achieve robust output regulation for the system (1) and for the reference and disturbance signals generated by the exosystem (2). In particular, this fundamental result tells us that the control problem can be solved by including a suitable *internal model* of the dynamics of the exosystem into the controller (3), and by choosing the rest of the parameters of the controller in such a way that the closed-loop system consisting of the plant and the controller is stable. The classical definition of the internal model (also referred to as the *p-copy internal model*) requires that if p is the dimension of the output space Y and if S has a Jordan block of dimension n_k associated to an eigenvalue $i\omega_k$, then the operator $\mathcal{G}_1$ must have at least p independent Jordan chains of length greater or equal to n_k associated to the same eigenvalue $i\omega_k$ [6], [16]. In this paper we also use an alternative definition for an internal

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model, called the $\mathcal{G}$-*conditions*, which is applicable even if Y is infinite-dimensional [10], [19].

The first controller in this paper presented in Section IV is constructed by choosing $\mathcal{G}_1$ as the internal model of the exosystem (2) and by stabilizing the closed-loop system with suitable choices of $\mathcal{G}_2$ and K. It is well-known that if the plant (1) is exponentially stable and S is diagonal, then this very simple structure is extremely effective. Indeed, this controller structure has been successfully used on several occasions for infinite-dimensional systems with bounded and unbounded input and output operators [8], [9], [13], [22], [27]. The most important advantages of this controller structure is that due to the internal model principle, this controller is of minimal possible order, and if $\dim Y < \infty$ then the resulting controller is finite-dimensional. In [23] this type of structure was used for regular linear systems on Hilbert spaces and with $U = Y$. In this paper we present a minimal order controller that solves the robust output regulation for a regular linear system (1) on a Banach space X, without restrictions on the input and output spaces, and with the most general choices for the stabilizing operators $\mathcal{G}_2$ and K.

In Section IV we in addition present a separate version of the minimal order controller for a situation where robustness is only required with respect to a predefined class $\mathcal{O}_0$ of perturbations. The design is motivated by the recent observation [15], [18] that in such a situation the robust output regulation problem may be solvable with a controller incorporating a *reduced order internal model* that is strictly smaller than the full p-copy internal model. In particular, in [15] such a controller was successfully designed for a given class $\mathcal{O}_0$ of perturbations. In this paper we present a new controller that solves the robust output regulation problem for a stable regular linear system and for a given class $\mathcal{O}_0$. This new controller has the advantage over the one presented in [15] in that the controller is of minimal order, and it is finite-dimensional whenever $\dim Y < \infty$. This controller is new even for finite-dimensional linear systems and for infinite-dimensional systems with bounded operators B and C.

The second robust controller of this paper presented in Section V employs a novel structure that was introduced in [15] for construction of controllers with reduced order internal models. In particular, the system operator $\mathcal{G}_1$ of the controller has a triangular structure that is naturally complementary to the structure of observer-based robust controllers [4], [10]. The main advantages of this new controller are that it has the natural structure for the inclusion of the p-copy internal model into the dynamics of the controller, and that it can robustly regulate plants that have a larger number of inputs than outputs. The construction of this second controller is a new result even for finite-dimensional linear systems and for infinite-dimensional systems with bounded operators B and C. In Section V we also use the same structure to generalize the original reduced order internal model based controller in [15] for regular linear systems.

Finally, the third robust controller presented in Section VI is an observer-based controller that employs the triangular structure that was successfully used for robust output regulation of systems with bounded B and C together with infinite-dimensional diagonal exosystems in [10]. In this paper we generalize the observer-based construction in [10] to regular linear systems with nondiagonal exosystems.

As the first main result in this paper we present the internal model principle. This result was first generalized for regular linear systems in [19] in the more general setting of infinite-dimensional exosystems and strong stability of the closed-loop system. In this paper we introduce it for regular linear systems with finite-dimensional exosystems and exponential closed-loop stability. We demonstrate that the exponential closed-loop stability allows simplifying general assumptions of the theorem, and show that the regulation error has an exponential rate of decay.

We illustrate the construction of controllers by considering the robust output regulation problem for a two-dimensional heat equation with boundary control and observation. We begin by stabilizing the system with negative output feedback, and we subsequently construct a minimal order controller that achieves robust tracking of a sinusoidal reference signal.

The paper is organized as follows. The standing assumptions on the plant, the exosystem and the controller are stated in Section II. In Section III we formulate the robust output regulation problem and present the internal model principle. The minimal order controller for stable systems is presented in Section IV. In Section V we introduce the new controller structure for robust output regulation. Finally, the observer-based robust controller is constructed in Section VI. The robust output tracking of the two-dimensional heat equation is considered in Section VII.

II. The Plant, The Exosystem and The Controller

If X and Z are Banach spaces and $A : X \to Z$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Z is denoted by $\mathcal{L}(X, Z)$. If $A : X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the spectrum, the point spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda, A) = (\lambda - A)^{-1}$. The inner product on a Hilbert space is denoted by $\langle \cdot, \cdot \rangle$. For an operator $A : \mathcal{D}(A) \subset X \to Z_1 \times \cdots \times Z_n$ we use the notation $A = (A_k)_{k=1}^n$, where $A_k : \mathcal{D}(A) \subset X \to Z_k$ for all $k \in \{1, \ldots, n\}$, to signify that $Ax = (A_k x)_{k=1}^n$ for $x \in \mathcal{D}(A)$. On the other hand, for an operator $A \in \mathcal{L}(X_1 \times \cdots \times X_n, Z)$ we use the notation $A = (A_1, \ldots, A_n)$, meaning that $Ax = \sum_{k=1}^n A_k x_k$ for all $x = (x_k)_{k=1}^n \in X_1 \times \cdots \times X_n$.

We consider a linear system (1) on a Banach space X with state $x(t) \in X$, output $y(t) \in Y$, and input $u(t) \in U$. The spaces U and Y are Hilbert spaces. The operator $A : \mathcal{D}(A) \subset X \to X$ generates a strongly continuous semigroup $T(t)$ on X. For a fixed $\lambda_0 \in \rho(A)$ we define the scale spaces $X_1 = (\mathcal{D}(A), \|(\lambda_0 - A) \cdot \|)$ and $X_{-1} = \overline{(X, \|R(\lambda_0, A) \cdot \|)}$ (the completion of X with respect to the norm $\|R(\lambda_0, A) \cdot \|$) [5, Sec. 2.5]. The extension of A to X_{-1} is denoted by $A_{-1} : X \subset X_{-1} \to X_{-1}$.

Throughout the paper we assume that (1) is a *regular linear system* [25], [26], [29], [30]. In particular, $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$ are admissible with respect to A and

$D \in \mathcal{L}(U,Y)$. The operator C in (1b) is replaced with its Λ-extension

$$C_\Lambda x = \lim_{\lambda\to\infty} \lambda C R(\lambda,A)x$$

with $\mathcal{D}(C_\Lambda)$ consisting of those $x \in X$ for which the limit exists. If $C \in \mathcal{L}(X,Y)$, then $C_\Lambda = C$. For a regular linear system we have $\mathcal{R}(R(\lambda,A)B) \subset \mathcal{D}(C_\Lambda)$ for all $\lambda \in \rho(A)$, and the transfer function of (1) is [26, Sec. 4]

$$P(\lambda) = C_\Lambda R(\lambda,A)B + D \quad \forall\, \lambda \in \rho(A).$$

Finally, we define $X_B = \mathcal{D}(A) + \mathcal{R}(R(\lambda_0,A_{-1})B) \subset \mathcal{D}(C_\Lambda)$, which is independent of the choice of $\lambda_0 \in \rho(A)$.

Assumption 1: The pair (A,B) is exponentially stabilizable and there exists $L \in \mathcal{L}(Y,X)$ such that $A + LC_\Lambda$ generates an exponentially stable semigroup.

The stabilizability of (A,B) means that there exists $K \in \mathcal{L}(X_1,U)$ such that (A,B,K_Λ) is a regular linear system for which I is an admissible feedback operator, and $(A+BK_\Lambda)|_X$ generates an exponentially stable semigroup [28].

The exosystem (2) is a linear system on the finite-dimensional space $W = \mathbb{C}^r$ for some $r \in \mathbb{N}$, and $S \in \mathcal{L}(W) = \mathbb{C}^{r\times r}$, $E \in \mathcal{L}(W,X)$, and $F \in \mathcal{L}(W,Y)$. We assume the geometric multiplicity of each of the eigenvalues $\sigma(S) = \{i\omega_k\}_{k=1}^q \subset i\mathbb{R}$ is equal to one. We denote by $n_k \in \mathbb{N}$ the size of the Jordan block associated to $i\omega_k \in \sigma(S)$. The following standing assumption is crucial for the solvability of the robust output regulation problem. An immediate consequence of this assumption is that in order to achieve robust output regulation it is necessary that $\dim U \geq \dim Y$.

Assumption 2: For every $k \in \{1,\dots,q\}$ we have $i\omega_k \in \rho(A)$ and $P(i\omega_k) \in \mathcal{L}(U,Y)$ is surjective.

The dynamic error feedback controller (3) is an abstract linear system on a Banach space Z. The operator $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a semigroup on Z, and $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$ and $K \in \mathcal{L}(Z_1,U)$ is admissible with respect to $\mathcal{G}_1$. The operator K in (3) is replaced with its Λ-extension K_Λ.

The closed-loop system consisting of the plant (1) and the controller (3) on the Banach space $X_e = X \times Z$ with state $x_e(t) = (x(t), z(t))^T$ is of the form

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \quad x_e(0) = x_{e0} \in X_e$$
$$e(t) = C_{e\Lambda} x_e(t) + D_e v(t)$$

where $e(t) = y(t) - y_{ref}(t)$ is the regulation error, $x_{e0} = (x_0, z_0)^T$, $C_e = (C_\Lambda, DK_\Lambda)$, $D_e = F$

$$A_e = \begin{pmatrix} A_{-1} & BK_\Lambda \\ \mathcal{G}_2 C_\Lambda & \mathcal{G}_1 + \mathcal{G}_2 DK_\Lambda \end{pmatrix}, \quad B_e = \begin{pmatrix} E \\ \mathcal{G}_2 F \end{pmatrix}.$$

The operator $A_e : \mathcal{D}(A_e) \subset X_e \to X_e$ has the domain

$$\mathcal{D}(A_e) = \{(x,z) \in X_B \times \mathcal{D}(\mathcal{G}_1) \mid A_{-1}x + BK_\Lambda z \in X\}$$

where $X_B = \mathcal{D}(A) + \mathcal{R}(R(\lambda_0,A_{-1})B)$, and $\mathcal{D}(C_e) = \mathcal{D}(C_\Lambda) \times \mathcal{D}(K_\Lambda) \supset \mathcal{D}(A_e)$, $B_e \in \mathcal{L}(W, X\times Z)$ and $D_e \in \mathcal{L}(W,Y)$. Here $C_{e\Lambda}$ is the Λ-extension of C_e.

Theorem 3: The closed-loop system (A_e,B_e,C_e,D_e) is a regular linear system.

Proof: See [19, Sec. 8]. □

III. The Robust Output Regulation Problem and the Internal Model Principle

We can now formulate the robust output regulation problem. We consider perturbations $(\tilde{A},\tilde{B},\tilde{C},\tilde{D},\tilde{E},\tilde{F}) \in \mathcal{O}$ of (A,B,C,D,E,F) where the operators in the class $\mathcal{O}$ of admissible perturbations are such that (i) the perturbed plant $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ is a regular linear system and (ii) $i\omega_k \in \rho(\tilde{A})$ for all $k \in \{1,\dots,q\}$. These two conditions are in particular satisfied for all bounded and sufficiently small perturbations to (A,B,C,D), and for arbitrary bounded perturbations to the operators E and F.

The Robust Output Regular Problem. Choose the controller $(\mathcal{G}_1,\mathcal{G}_2,K)$ in such a way that the following are satisfied:

(a) The closed-loop semigroup $T_e(t)$ is exponentially stable.
(b) For all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\alpha\cdot}e(\cdot) \in L^2(0,\infty;Y)$ for some $\alpha > 0$.
(c) If the operators (A,B,C,D,E,F) are perturbed to $(\tilde{A},\tilde{B},\tilde{C},\tilde{D},\tilde{E},\tilde{F}) \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\tilde{\alpha}\cdot}e(\cdot) \in L^2(0,\infty;Y)$ for some $\tilde{\alpha} > 0$.

We have from [19, Sec. 4] that for initial states $x_{e0} \in \mathcal{D}(A_e)$ the regulation error $e(\cdot)$ is a continuous function and $\lim_{t\to\infty} e(t) = 0$ whenever the property (b) holds. Thus for such initial states the condition $e^{\alpha\cdot}e(\cdot) \in L^2(0,\infty;Y)$ for an $\alpha > 0$ implies that the regulation error decays to zero at an exponential rate.

In the following we present two definitions for an internal model [16], [19]. In Definition 4 "independent Jordan chains" refer to chains originating from linearly independent eigenvectors of $\mathcal{G}_1$.

Definition 4: Assume $\dim Y < \infty$. A controller $(\mathcal{G}_1,\mathcal{G}_2, K)$ is said to *incorporate a p-copy internal model* of the exosystem S if for all $k \in \{1,\dots,q\}$ we have

$$\dim \mathcal{N}(i\omega_k - \mathcal{G}_1) \geq \dim Y$$

and $\mathcal{G}_1$ has at least $\dim Y$ independent Jordan chains of length greater than or equal to n_k associated to the eigenvalue $i\omega_k$.

Definition 5: A controller $(\mathcal{G}_1,\mathcal{G}_2,K)$ is said to satisfy the *$\mathcal{G}$-conditions* if

$$\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\} \quad \forall\, k \in \{1,\dots,q\} \tag{4a}$$
$$\mathcal{N}(\mathcal{G}_2) = \{0\} \tag{4b}$$
$$\mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k-1} \subset \mathcal{R}(i\omega_k - \mathcal{G}_1) \quad \forall\, k \in \{1,\dots,q\}. \tag{4c}$$

The following lemma gives a sufficient condition for invariance of the $\mathcal{G}$-conditions in the situation where the matrix S of the exosystem is diagonal.

Lemma 6: Let S be a diagonal matrix. If the operators $(\mathcal{G}_1,\mathcal{G}_2)$ satisfy the $\mathcal{G}$-conditions, and if $K : \mathcal{D}(\mathcal{G}_1) \subset Z \to Y$

is such that $\mathcal{N}(i\omega_k - \mathcal{G}_1) \subset \mathcal{N}(K)$ for all $k \in \{1,\dots,q\}$, then also $(\mathcal{G}_1 + \mathcal{G}_2 K, \mathcal{G}_2)$ satisfy the $\mathcal{G}$-conditions.

Proof: Since S is a diagonal matrix, we have $n_k = 1$ for all $k \in \{1,\dots,q\}$ and the condition (4c) is trivially satisfied. Because the condition $\mathcal{N}(\mathcal{G}_2) = \{0\}$ is identical for both $(\mathcal{G}_1 + \mathcal{G}_2 K, \mathcal{G}_2)$ and $(\mathcal{G}_1, \mathcal{G}_2)$, it is sufficient to show that $\mathcal{R}(i\omega_k - \mathcal{G}_1 + \mathcal{G}_2 K) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ for all k. To this end, let $w = (i\omega_k - \mathcal{G}_1 - \mathcal{G}_2 K)z = \mathcal{G}_2 y$ for some $k \in \{1,\dots,q\}$, $z \in \mathcal{D}(\mathcal{G}_1)$ and $y \in Y$. This implies $(i\omega_k - \mathcal{G}_1)z = \mathcal{G}_2(y + Kz) \in \mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$, and we thus have $z \in \mathcal{N}(i\omega_k - \mathcal{G}_1)$. Due to our assumptions we then also have $Kz = 0$ and $w = (i\omega_k - \mathcal{G}_1)z = \mathcal{G}_2 y$, which finally imply $w = 0$ due to (4a). □

The following theorem presents the internal model principle for regular linear systems with finite-dimensional exosystems and exponential closed-loop stability.

Theorem 7: Assume that the controller stabilizes the closed-loop system exponentially. Then the controller solves the robust output regulation problem if and only if it satisfies the $\mathcal{G}$-conditions.

Moreover, if $\dim Y < \infty$, then the controller solves the robust output regulation problem if and only if it incorporates a p-copy internal model of the exosystem.

Proof: Since A_e generates an exponentially stable semigroup and S is a matrix with spectrum on $i\mathbb{R}$, the Sylvester equation $\Sigma S = A_e \Sigma + B_e$ has a unique solution $\Sigma \in \mathcal{L}(W, X_e)$ satisfying $\mathcal{R}(\Sigma) \subset \mathcal{D}(A_e)$ [20]. Because an exponentially stable semigroup is also strongly stable, and since $i\mathbb{R} \subset \rho(A_e)$, we have from [19, Thm. 7.2] that the controller satisfies the $\mathcal{G}$-conditions if and only if it solves the robust output regulation problem as defined in the reference [19]. The definition of the robust output regulation problem in [19] can be obtained from our problem statement with the following modifications:

(1) The exponential closed-loop stability is replaced by strong stability.
(2) It is assumed that for all admissible perturbations the Sylvester equation $\Sigma S = \tilde{A}_e \Sigma + \tilde{B}_e$ has a solution.
(3) The condition $e^{\alpha \cdot} e(\cdot) \in L^2(0,\infty;Y)$ for $x_{e0} \in X_e$ is replaced by $\lim_{t\to\infty} e(\cdot) = 0$ for $x_{e0} \in \mathcal{D}(A_e)$.

We begin by showing that under the assumption of exponential closed-loop stability the two conditions in (iii) are equivalent. We prove this only for the nominal closed-loop system (A_e, B_e, C_e, D_e). For perturbed parameters the situation can be handled analogously. We have from [19, Lem. 4.3] that

$$x_e(t) = T_e(t)x_{e0} - T_e(t)\Sigma v_0 + \Sigma T_S(t) v_0$$

for all $x_{e0} \in X_e$ and $v_0 \in W$, and since (A_e, B_e, C_e, D_e) is a regular linear system

$$e(t) = C_{e\Lambda} T_e(t)(x_{e0} - \Sigma v_0) + (C_e \Sigma + D_e) T_S(t) v_0$$

is defined for almost all $t \geq 0$. In addition, if $x_{e0} \in \mathcal{D}(A_e)$, then $e(t)$ is continuous and is given by the above formula for all $t \geq 0$. The error contains the two terms $e(t) = e_1(t) + e_2(t)$. The second term $e_2(\cdot) = (C_e \Sigma + D_e) T_S(\cdot) v_0$ is continuous and it is either nonvanishing or identically zero [19, Lem. A.1]. Since $T_e(t)$ is exponentially stable, for some $\alpha > 0$ the first term satisfies $e^{\alpha \cdot} e_1(\cdot) \in L^2(0,\infty;Y)$ for all $x_{e0} \in X_e$ and $v_0 \in W$. These properties imply, under the assumption of exponential stability of the closed-loop system, that the regulation error satisfies $e^{\alpha \cdot} e(\cdot) \in L^2(0,\infty;Y)$ for all $x_{e0} \in X_e$ and $v_0 \in W$ if and only if $\lim_{t\to\infty} e(t) = 0$ for all $x_{e0} \in \mathcal{D}(A_e)$ and $v_0 \in W$. Thus the conditions in (iii) are equivalent.

Assume now that the controller satisfies the $\mathcal{G}$-conditions. The class of admissible perturbations in this paper is strictly smaller than the class of perturbations in [19] because exponential stability is stronger than strong stability, and because $\Sigma S = \tilde{A}_e \Sigma + \tilde{B}_e$ has a solution for any perturbations for which the closed-loop system is exponentially stable [20]. Because of this, and because we assumed the exponential closed-loop stability, we have from [19, Thm. 7.2] that the controller satisfying the $\mathcal{G}$-conditions solves the robust output regulation problem as defined in this paper.

Conversely, we can now assume that the controller solves the robust output regulation problem. It then follows from the proof of [19, Thm. 7.2] that the controller must satisfy the $\mathcal{G}$-conditions provided that the class of admissible perturbations contains $\tilde{E} = 0$ (corresponding to the zero disturbance signal) and arbitrary bounded perturbations to the operator F of the exosystem. Because these perturbations do not affect the stability of the closed-loop system, they also belong to the class $\mathcal{O}$ of perturbations in this paper. This concludes that the controller indeed satisfies the $\mathcal{G}$-conditions.

Finally, if $\dim Y < \infty$, we similarly have from [19, Thm. 6.2] that the controller solves the robust output regulation problem if and only if it incorporates a p-copy internal model of the exosystem. □

IV. The Minimal Robust Controller for Stable Systems

In this section we construct a minimal order robust controller under the assumption that the system operator A of the regular linear system (1) generates an exponentially stable semigroup and the matrix S of the exosystem is diagonal, i.e.

$$S = \operatorname{diag}(i\omega_1, i\omega_2, \dots, i\omega_q) \in \mathbb{C}^{q\times q}.$$

We begin by choosing the parameters of the controller. In this controller structure the system operator $\mathcal{G}_1$ contains precisely the internal model of the exosystem (2). This is achieved by defining $Z = Y^q$, and

$$\mathcal{G}_1 = \operatorname{diag}(i\omega_1 I_Y, \dots, i\omega_q I_Y), \quad K = \varepsilon K_0 = \varepsilon \left(K_0^1, \dots, K_0^q\right)$$

where $\varepsilon > 0$ and $K_0 \in \mathcal{L}(Z, U)$. We choose the components $K_0^k \in \mathcal{L}(Y, U)$ of K_0 in such a way that the operators $P(i\omega_k) K_0^k$ are invertible. This is possible due to the assumption of surjectivity of $P(i\omega_k)$, and can be achieved, for example, by choosing $K_0^k = P(i\omega_k)^\dagger$ (the Moore-Penrose pseudoinverse of $P(i\omega_k)$) for all $k \in \{1,\dots,q\}$. Finally, we choose

$$\mathcal{G}_2 = \left(\mathcal{G}_2^k\right)_{k=1}^q = \left(-\left(P(i\omega_k) K_0^k\right)^*\right)_{k=1}^q \in \mathcal{L}(Y, Z).$$

If we make the choice $K_0^k = P(i\omega_k)^\dagger$, then $\mathcal{G}_2^k = -I_Y$ for all $k \in \{1,\dots,q\}$.

Theorem 8: Assume that the semigroup $T(t)$ generated by A is exponentially stable and S is a diagonal matrix. Then there exists $\varepsilon^* > 0$ such that for any $0 < \varepsilon \le \varepsilon^*$ the controller with the above choices of parameters solves the robust output regulation problem.

In particular, the operators $(\mathcal{G}_1, \mathcal{G}_2)$ satisfy the $\mathcal{G}$-conditions and the closed-loop system is exponentially stable for all $0 < \varepsilon \le \varepsilon^*$.

Proof: We begin by showing that the controller satisfies the $\mathcal{G}$-conditions. Since K_0^k were chosen in such a way that $P(i\omega_k)K_0^k$ are invertible for all $k \in \{1, \ldots, q\}$, we have that $\mathcal{N}(\mathcal{G}_2) = \{0\}$. Let $k \in \{1, \ldots, q\}$, $z, z_1 \in Z$, and $y \in Y$ be such that $z = (i\omega_k - \mathcal{G}_1)z_1 = \mathcal{G}_2 y$. The diagonal structure of $\mathcal{G}_1$ implies that we then necessarily have $0 = \mathcal{G}_2^k y = -(P(i\omega_k)K_0^k)^* y$, which is only possible if $y = 0$ since $P(i\omega_k)K_0^k$ is invertible. This further implies $z = \mathcal{G}_2 y = 0$. Since $k \in \{1, \ldots, q\}$ and $z \in \mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{N}(\mathcal{G}_2)$ were arbitrary, this concludes $\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$. Finally, since $n_k = 1$ for all $k \in \{1, \ldots, q\}$, the condition (4c) is trivially satisfied.

We define $H = (H_1, H_2, \ldots, H_q) \in \mathcal{L}(Z, X)$ by choosing

$$H_k = R(i\omega_k, A_{-1})BK_0^k$$

for all $k \in \{1, \ldots, q\}$. Due to the diagonal structure of $\mathcal{G}_1$, it is easy to see that this operator is the unique solution of the Sylvester equation $H\mathcal{G}_1 = A_{-1}H + BK_0$. Clearly $\mathcal{R}(H) \subset X_B$ and we can define $C_0 = C_\Lambda H + DK_0 \in \mathcal{L}(Z, Y)$. The operator C_0 is of the form $C_0 = (C_0^1, \ldots, C_0^q)$. A direct computation shows that

$$\begin{aligned} C_0^k &= C_\Lambda H_k + DK_0^k = C_\Lambda R(i\omega_k, A_{-1})BK_0^k + DK_0^k \\ &= P(i\omega_k)K_0^k \end{aligned}$$

and thus $C_0 = -\mathcal{G}_2^*$.

It remains to show that there exists $\varepsilon^* > 0$ such that the closed-loop system is exponentially stable for all $0 < \varepsilon \le \varepsilon^*$. The closed-loop system operator is given by

$$A_e = \begin{pmatrix} A_{-1} & \varepsilon BK_0 \\ \mathcal{G}_2 C_\Lambda & \mathcal{G}_1 + \varepsilon \mathcal{G}_2 DK_0 \end{pmatrix}$$

$$\mathcal{D}(A_e) = \{(x, z) \in X_B \times Z \mid A_{-1}x + \varepsilon BK_0 z \in X\}.$$

If we choose a similarity transformation

$$Q_e = \begin{pmatrix} -I & \varepsilon H \\ 0 & I \end{pmatrix} = Q_e^{-1} \in \mathcal{L}(X \times Z)$$

we can define $\hat{A}_e = Q_e A_e Q_e^{-1}$ with domain $\mathcal{D}(\hat{A}_e) = \{x_e \in X_e \mid Q_e^{-1}x_e \in \mathcal{D}(A_e)\}$. Using $\mathcal{R}(H) \subset X_B$ and $\mathcal{R}(A_{-1}H + BK_0) = \mathcal{R}(H\mathcal{G}_1) \subset X$ the condition $Q_e^{-1}x_e \in \mathcal{D}(A_e)$ for $x_e = (x, z) \in X \times Z$ becomes

$$Q_e^{-1}x_e \in \mathcal{D}(A_e) \Leftrightarrow \begin{cases} -x + \varepsilon Hz \in X_B \\ -A_{-1}x + \varepsilon A_{-1}Hz + \varepsilon BK_0 z \in X \end{cases}$$
$$\Leftrightarrow x_e \in \mathcal{D}(A) \times Z$$

and thus $\mathcal{D}(\hat{A}_e) = \mathcal{D}(A) \times Z$. Now for any $x_e = (x, z) \in \mathcal{D}(\hat{A}_e)$ a direct computation using $H\mathcal{G}_1 = A_{-1}H + BK_0$ and $C_\Lambda H + DK_0 = -\mathcal{G}_2^*$ shows that

$$\begin{aligned} \hat{A}_e x_e &= Q_e A_e \begin{pmatrix} -x + \varepsilon Hz \\ z \end{pmatrix} \\ &= \begin{pmatrix} (A - \varepsilon H\mathcal{G}_2 C_\Lambda)x - \varepsilon^2 H\mathcal{G}_2\mathcal{G}_2^* z \\ -\mathcal{G}_2 C_\Lambda x + (\mathcal{G}_1 - \varepsilon\mathcal{G}_2\mathcal{G}_2^*)z \end{pmatrix} \\ &= \left[\begin{pmatrix} A - \varepsilon H\mathcal{G}_2 C_\Lambda & 0 \\ -\mathcal{G}_2 C_\Lambda & \mathcal{G}_1 - \varepsilon\mathcal{G}_2\mathcal{G}_2^* \end{pmatrix} \right. \\ &\qquad \left. + \varepsilon^2 \begin{pmatrix} 0 & -H\mathcal{G}_2\mathcal{G}_2^* \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ z \end{pmatrix}. \end{aligned}$$

Since C is admissible with respect to A, we have from the results in [5, Sec. 3.3.c] that there exists $\varepsilon_1 > 0$ such that $A + \varepsilon H\mathcal{G}_2 C_\Lambda$ generates an exponentially stable semigroup provided that $0 < \varepsilon \le \varepsilon_1$. Moreover, Lemma 17 shows that the semigroup generated by $\mathcal{G}_1 - \varepsilon\mathcal{G}_2\mathcal{G}_2^*$ is exponentially stable for all $\varepsilon > 0$, since $\sqrt{\varepsilon}\mathcal{G}_2^k = -\sqrt{\varepsilon}(P(i\omega_k)K_0^k)^*$ are invertible for all $k \in \{1, \ldots, q\}$. Because C_Λ is an admissible input operator for $A - \varepsilon H\mathcal{G}_2 C_\Lambda$, $\mathcal{G}_2 \in \mathcal{L}(Y, Z)$, and the diagonal operators generate exponentially stable semigroups, the semigroup generated by the triangular operator is exponentially stable for all $0 < \varepsilon \le \varepsilon_1$. Furthermore, because the second term is a bounded operator, it follows from standard perturbation theory of semigroups and similarity that there there exists $\varepsilon^* > 0$ such that A_e is exponentially stable for all $0 < \varepsilon \le \varepsilon^*$.

Since the controller satisfies the $\mathcal{G}$-conditions and the closed-loop system is exponentially stable for all $0 < \varepsilon \le \varepsilon^*$, we have from Theorem 7 that for any $0 < \varepsilon \le \varepsilon^*$ the controller solves the robust output regulation problem. □

Remark 9: The controller presented in this section can also be used if the plant is initially unstable but can be stabilized with output feedback, i.e., there exists an admissible feedback element $K_1 \in \mathcal{L}(Y, U)$ such that the semigroup generated by $(A + BK_1(I - DK_1)^{-1}C_\Lambda)|_X$ is exponentially stable. Indeed, in such a case the controller can be designed for the stabilized system $((A + BK_1(I - DK_1)^{-1}C_\Lambda)|_X, B(I - K_1 D)^{-1}, (I - DK_1)^{-1}C_\Lambda, (I - DK_1)^{-1}D)$. This procedure is demonstrated in Section VII.

Remark 10: If the plant is *real* in the sense that $P(-i\omega) = \overline{P(i\omega)}$ for all $\omega \in \mathbb{R}$, if $Y = \mathbb{C}^p$, $U = \mathbb{C}^m$, and if the exosystem is of the form

$$S = \operatorname{diag}(i\omega_1, -i\omega_1, \ldots, i\omega_q, -i\omega_q, 0) \in \mathbb{C}^{(2q+1)\times(2q+1)}$$

then $(\mathcal{G}_1, \mathcal{G}_2, K)$ can be chosen to be real matrices. Indeed, in this case we can choose

$$\mathcal{G}_1 = \operatorname{diag}\left(G_1^1, \ldots, G_1^q, 0_{p\times p}\right)$$

where $G_1^k = \begin{pmatrix} 0 & \omega_k I_Y \\ -\omega_k I_Y & 0 \end{pmatrix}$, $K = \varepsilon(K_0^1, \ldots, K_0^q, K_0^{q+1})$ where $K_0^k = (\operatorname{Re}P(i\omega_k)^\dagger, \operatorname{Im}P(i\omega_k)^\dagger) \in \mathbb{R}^{m\times 2p}$ for $k \in \{1, \ldots, q\}$ and $K_0^{q+1} = P(0)^\dagger \in \mathbb{R}^{m\times p}$, and finally $\mathcal{G}_2 = (G_2^k)_{k=1}^{p+1}$ where $G_2^k = \begin{pmatrix} -I_Y \\ 0 \end{pmatrix} \in \mathbb{R}^{2p\times p}$ for $k \in \{1, \ldots, q\}$ and $G_2^{q+1} = -I_Y \in \mathbb{R}^{p\times p}$. The controller incorporates a p-copy internal model of the exosystem, and if we apply a unitary similarity transformation

$$Q = \operatorname{diag}(Q_0, \dots, Q_0, I_Y), \quad Q_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} I_Y & I_Y \\ iI_Y & -iI_Y \end{pmatrix}$$

then $(Q^*\mathcal{G}_1 Q, Q^*\mathcal{G}_2, KQ)$ coincides with the controller constructed in the beginning of this section. From this it follows that there exists $\varepsilon^* > 0$ such that the closed-loop system is exponentially stable and the real controller solves the robust output regulation problem for all $0 < \varepsilon \leq \varepsilon^*$.

A. Controller With a Reduced Order Internal Model

In this section we construct a minimal order controller for a version of the robust output regulation problem where the controller is only required to tolerate uncertainties from a given class $\mathcal{O}_0$ of admissible perturbations [18], [19]. More precisely, in part (c) of the robust output regulation problem we only consider perturbations such that $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}) \in \mathcal{O}_0$ and for which the perturbed closed-loop system is exponentially stable. We again assume that the plant is exponentially stable, the matrix S is diagonal, and we in addition assume that $P(i\omega_k)$ are boundedly invertible for all $k \in \{1, \dots, q\}$.

The class $\mathcal{O}_0$ in the control problem can be chosen freely, but it is assumed that all perturbations $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F})$ in $\mathcal{O}_0$ are such that (i) the perturbed plant $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is a regular linear system and (ii) $i\omega_k \in \rho(\tilde{A})$ and the transfer function $\tilde{P}(i\omega_k) = \tilde{C}_\Lambda R(i\omega_k, \tilde{A})\tilde{B} + \tilde{D}$ is boundedly invertible for all $k \in \{1, \dots, q\}$. Both of these requirements are in particular satisfied for sufficiently small bounded perturbations of A, B, C, and D. Being given such a class $\mathcal{O}_0$, we begin the construction of the controller by defining

$$\mathcal{S}_k = \operatorname{span}\left\{ \tilde{P}(i\omega_k)^{-1}\left(\tilde{C}R(i\omega_k, \tilde{A})\tilde{E}e_k + \tilde{F}e_k\right) \Big| \; (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}) \in \mathcal{O}_0 \right\} \subset U$$

for $k \in \{1, \dots, q\}$, where $(e_k)_{k=1}^q$ is the Euclidean basis of $W = \mathbb{C}^q$. We further define $p_k = \dim \mathcal{S}_k$. The controller that we construct contains a reduced order internal model where the number of copies of each of the frequencies $i\omega_k$ of the exosystem is exactly p_k. It should be noted that this controller differs from the minimial order controller with a full internal model only in the situation where $p_k < \dim Y$ for at least one $k \in \{1, \dots, q\}$.

Define $Z = Y_1 \times \cdots \times Y_q$ where $Y_k = \mathbb{C}^{p_k}$ if $p_k < \dim Y$ and $Y_k = Y$ if $p_k = \dim Y$ or $p_k = \infty$. We choose

$$\mathcal{G}_1 = \operatorname{diag}\left(i\omega_k I_{Y_k}\right)_{k=1}^q, \quad K = \varepsilon K_0 = \varepsilon\left(K_0^1, \dots, K_0^q\right)$$

where $\varepsilon > 0$ and $K_0^k \in \mathcal{L}(Y_k, U)$ are such that

$$K_0^k = \begin{cases} \left(u_k^1, \dots, u_k^{p_k}\right) & \text{if } p_k < \dim Y \\ P(i\omega_k)^{-1} & \text{if } p_k = \dim Y \text{ or } p_k = \infty \end{cases}$$

where $\{u_k^l\}_{l=1}^{p_k} \subset U$ is a basis of the subspace $\mathcal{S}_k$. Finally, we choose

$$\mathcal{G}_2 = \left(-\left(P(i\omega_k)K_0^k\right)^*\right)_{k=1}^q \in \mathcal{L}(Y, Z).$$

For those $k \in \{1, \dots, q\}$ for which $p_k = \dim Y$ or $p_k = \infty$ we then have $\mathcal{G}_2^k = -I_Y$.

Theorem 11: Assume that the semigroup $T(t)$ generated by A is exponentially stable, $S = \operatorname{diag}(i\omega_1, \dots, i\omega_q)$, and $P(i\omega_k)$ are boundedly invertible for all $k \in \{1, \dots, q\}$. Then there exists $\varepsilon^* > 0$ such that for any $0 < \varepsilon \leq \varepsilon^*$ the controller with the above choices of parameters solves the robust output regulation problem for the class $\mathcal{O}_0$ of perturbations.

Proof: Let $\varepsilon > 0$, $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}) \in \mathcal{O}_0$ and $k \in \{1, \dots, q\}$ and denote $y_k = -\tilde{C}R(i\omega_k, \tilde{A})\tilde{E}e_k - \tilde{F}e_k$. We begin by showing that there exists $z \in \mathcal{N}(i\omega_k - \mathcal{G}_1)$ such that $\tilde{P}(i\omega_k)Kz = y_k$. If k is such that $Y_k = Y$, we can choose $z = (z_1, \dots, z_q) \in Z$ such that $z_l = 0$ for $l \neq k$ and $z_k = (1/\varepsilon)P(i\omega_k)\tilde{P}(i\omega_k)^{-1}y_k$. Then clearly $z \in \mathcal{N}(i\omega_k - \mathcal{G}_1)$ and $\tilde{P}(i\omega_k)Kz = \varepsilon\tilde{P}(i\omega_k)K_0^k z_k = \tilde{P}(i\omega_k)\tilde{P}(i\omega_k)^{-1}y_k = y_k$. It remains to consider the situation $p_k < \dim Y$. Since $\tilde{P}(i\omega_k)^{-1}y_k \in \mathcal{S}_k$ by definition, and since $\{u_k^1, \dots, u_k^{p_k}\}$ is a basis of $\mathcal{S}_k$, there exist $\{\alpha_l\}_{l=1}^{p_k} \subset \mathbb{C}$ such that

$$\tilde{P}(i\omega_k)^{-1}y_k = \sum_{l=1}^{p_k} \alpha_l u_k^l.$$

Choose $z = (z_1, \dots, z_q)$ such that $z_l = 0$ for $l \neq k$ and $z_k = (1/\varepsilon)(\alpha_l)_{l=1}^{p_k} \in Y_k = \mathbb{C}^{p_k}$. Then clearly $z \in \mathcal{N}(i\omega_k - \mathcal{G}_1)$ and

$$\begin{aligned} \tilde{P}(i\omega_k)Kz &= \varepsilon\tilde{P}(i\omega_k)K_0^k z_k = \tilde{P}(i\omega_k)\sum_{l=1}^{p_k}\alpha_l u_k^l \\ &= \tilde{P}(i\omega_k)\tilde{P}(i\omega_k)^{-1}y_k = y_k. \end{aligned}$$

Since $k \in \{1, \dots, q\}$ and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}) \in \mathcal{O}_0$ were arbitrary, we have from [19, Thm. 5.1] that the controller solves the robust output regulation problem for the class $\mathcal{O}_0$ of perturbations if the closed-loop system is exponentially stable (see the proof of Theorem 7).

It remains to show that there exists $\varepsilon^* > 0$ such that for every $0 < \varepsilon \leq \varepsilon^*$ the closed-loop system is exponentially stable. However, if we define $H = (H_1, \dots, H_q) \in \mathcal{L}(Z, X)$ by choosing $H_k = R(i\omega_k, A_{-1})BK_0^k$, then H is the solution of the Sylvester equation $H\mathcal{G}_1 = A_{-1}H + BK_0$, and the stability closed-loop system can be established exactly as in the proof of Theorem 8, since we again have $C_\Lambda H + DK_0 = -\mathcal{G}_2^*$. □

V. The New Robust Controller Structure

In this section we introduce the new controller structure for robust output regulation of linear systems. This controller has the natural structure for the inclusion of a p-copy internal model into the dynamics of the controller. The construction of the controller is completed in steps. Some of the choices of the parameters require certain properties from the associated operators, and these properties are verified in Theorem 12. We begin by assuming that $\dim Y < \infty$. The case of an infinite-dimensional output space is considered separately for a diagonal exosystem in Section V-A.

Step 1°: We begin by choosing the state space of the controller as $Z = Z_0 \times X$, and choosing the general structure of the operators $(\mathcal{G}_1, \mathcal{G}_2, K)$ as

$$\mathcal{G}_1 = \begin{pmatrix} G_1 & G_2\left(C_\Lambda + DK_2^\Lambda\right) \\ 0 & A_{-1} + BK_2^\Lambda + L\left(C_\Lambda + DK_2^\Lambda\right) \end{pmatrix}, \quad \mathcal{G}_2 = \begin{pmatrix} G_2 \\ L \end{pmatrix}$$

and $K = (K_1, -K_2^\Lambda)$. The operator G_1 is the internal model of the exosystem (2), and it is defined by choosing $Z_0 = Y^{n_1} \times \cdots \times Y^{n_q}$, and

$$G_1 = \operatorname{diag}\left(J_1^Y, \ldots, J_q^Y\right) \in \mathcal{L}(Z_0), \quad K_1 = \left(K_1^1, \ldots, K_1^q\right).$$

Here for each $k \in \{1, \ldots, q\}$ we have

$$J_k^Y = \begin{pmatrix} i\omega_k I_Y & I_Y & & \\ & i\omega_k I_Y & \ddots & \\ & & \ddots & I_Y \\ & & & i\omega_k I_Y \end{pmatrix} \in \mathcal{L}(Y^{n_k}) \quad (5)$$

and $K_1^k = (K_1^{k1}, \ldots, K_1^{kn_k}) \in \mathcal{L}(Y^{n_k}, U)$, where $n_k \in \mathbb{N}$ is the dimension of the Jordan block in S associated to the eigenvalue $i\omega_k \in \sigma(S)$. We choose the components $K_1^{k1} \in \mathcal{L}(Y, U)$ of each K_1^k in such a way that $P(i\omega_k)K_1^{k1} \in \mathcal{L}(Y)$ are boundedly invertible. This is possible since $P(i\omega_k)$ are surjective by assumption, and can be achieved, for example, by choosing $K_1^{k1} = P(i\omega_k)^\dagger$ for all $k \in \{1, \ldots, q\}$. For $l \geq 2$ we can choose K_1^{kl} freely, e.g., $K_1^{kl} = 0$.

Step 2°: By Assumption 1 we can choose $K_2 \in \mathcal{L}(X_1, U)$ and $L_1 \in \mathcal{L}(Y, X)$ in such a way that $(A_{-1} + BK_2^\Lambda)|_X$ (here K_2^Λ is the Λ-extension of K_2) and $A + L_1 C_\Lambda$ generate exponentially stable semigroups. For $\lambda \in \rho(A + L_1 C_\Lambda)$ we define

$$P_L(\lambda) = C_\Lambda R(\lambda, A_{-1} + L_1 C_\Lambda)(B + L_1 D) + D.$$

The identity $P_L(i\omega_k) = (I - C_\Lambda R(i\omega_k, A)L_1)^{-1} P(i\omega_k)$ and the choice of K_1 imply that $P_L(i\omega_k)K_1^{1k} \in \mathcal{L}(Y)$ are boundedly invertible for all $k \in \{1, \ldots, q\}$. The domain of the operator $\mathcal{G}_1$ is chosen as

$$\mathcal{D}(\mathcal{G}_1) = \left\{(z_1, x_1) \in Z_0 \times X_B \mid A_{-1}x_1 + BK_2^\Lambda x_1 \in X\right\}.$$

Step 3°: We define $H = (H_1, H_2, \ldots, H_q) \in \mathcal{L}(Z_0, X)$ where $H_k = (H_k^1, H_k^2, \ldots, H_k^{n_k}) \in \mathcal{L}(Y^{n_k}, X)$ and

$$H_k^l = \sum_{j=1}^{l} (-1)^{l-j} R(i\omega_k, A_{-1} + L_1 C_\Lambda)^{l+1-j}(B + L_1 D)K_1^{kj}.$$

We have from [30, Sec. 7] that $(A + L_1 C_\Lambda, B + L_1 D, C_\Lambda, D)$ is a regular linear system, and the resolvent identity in [30, Prop. 6.6] implies $\mathcal{R}(H) \subset X_B$. We can therefore define $C_1 = C_\Lambda H + DK_1 \in \mathcal{L}(Z_0, Y)$.

Step 4°: We choose $G_2 \in \mathcal{L}(Y, Z_0)$ in such a way that the semigroup generated by $G_1 + G_2 C_1 \in \mathcal{L}(Z_0)$ is exponentially stable (i.e., the matrix is Hurwitz). The detectability of the pair (C_1, G_1) is proved in Theorem 12 below. Finally, we define $L = L_1 + HG_2 \in \mathcal{L}(Y, Z)$.

Theorem 12: Assume $\dim Y < \infty$. The controller with the above choices of parameters solves the robust output regulation problem.

In particular, the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has the following properties:

(1) The operator $\mathcal{G}_1$ generates a semigroup on Z and the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ incorporates a p-copy internal model of the exosystem.

(2) The operator H is the unique solution of the Sylvester equation

$$HG_1 = (A_{-1} + L_1 C_\Lambda)H + (B + L_1 D)K_1 \quad (6)$$

and the pair (C_1, G_1) where $C_1 = C_\Lambda H + DK_1 \in \mathcal{L}(Z_0, Y)$ is exponentially detectable.

(3) The semigroup generated by A_e is exponentially stable.

Proof: We begin by proving part (i). We have that

$$\mathcal{G}_1 = \begin{pmatrix} G_1 & 0 \\ 0 & A_{-1} \end{pmatrix} + \begin{pmatrix} 0 & G_2 \\ B & L \end{pmatrix} \begin{pmatrix} I & 0 \\ D & I \end{pmatrix} \begin{pmatrix} 0 & K_2^\Lambda \\ 0 & C_\Lambda \end{pmatrix}$$

where $\left(\begin{smallmatrix} 0 & G_2 \\ B & L \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & K_2^\Lambda \\ 0 & C_\Lambda \end{smallmatrix}\right)$ are admissible with respect to $\left(\begin{smallmatrix} G_1 & 0 \\ 0 & A \end{smallmatrix}\right)$. It is now straightforward to use the results in [30, Sec. 7] to verify that $\mathcal{G}_1$ with the proposed domain generates a strongly continuous semigroup on $Z = Z_0 \times X$. Moreover, it is easy to show that $K_\Lambda = K = (K_1, -K_2^\Lambda)$. For every $k \in \{1, \ldots, q\}$ the matrix G_1 clearly satisfies $\dim \mathcal{N}(i\omega_k - G_1) = \dim Y = p$ and it has exactly p Jordan blocks of size $n_k \times n_k$ associated to $i\omega_k$. Due to the triangular structure of $\mathcal{G}_1$, the controller therefore incorporates a p-copy internal model of the exosystem.

We will now show that H is the solution of the Sylvester (6). Denote $A_L = A_{-1} + L_1 C_\Lambda$ and $B_L = B + L_1 D$ for brevity. Due to the structure of the operator G_1 it is straigtforward to see that an operator $H \in \mathcal{L}(Z_0, X)$ such that $\mathcal{R}(H) \subset \mathcal{D}(C_\Lambda)$ is the solution of $HG_1 = A_L H + B_L K_1$ if and only if for all $k \in \{1, \ldots, q\}$ we have

$$\begin{aligned} (i\omega_k - A_L)H_k^1 &= B_L K_1^{k1} \\ (i\omega_k - A_L)H_k^2 + H_k^1 &= B_L K_1^{k2} \\ &\vdots \\ (i\omega_k - A_L)H_k^{n_k} + H_k^{n_k - 1} &= B_L K_1^{kn_k} \end{aligned}$$

where $H = (H_1, \ldots, H_q)$, and $H_k = (H_k^1, \ldots, H_k^{n_k}) \in \mathcal{L}(Y^{n_k}, X)$. For each $k \in \{1, \ldots, q\}$ the above system of equations has a unique solution

$$H_k^l = \sum_{j=1}^{l} (-1)^{l-j} R(i\omega_k, A_L)^{l+1-j} B_L K_1^{kj}.$$

Thus H defined in Step 3° is the unique solution of (6).

We will now show that (C_1, G_1) is exponentially detectable. We can do this by showing that for all $k \in \{1, \ldots, q\}$ and $z \in \mathcal{N}(i\omega_k - G_1)$ with $z \neq 0$ we have $C_1 z \neq 0$ [12, Thm. 6.2-5]. To this end, let $k \in \{1, \ldots, q\}$ and $z \in \mathcal{N}(i\omega_k - G_1)$ such that $z \neq 0$ be arbitrary. From the structure of G_1 we have that $z = (z_1, \ldots, z_q)$ where $z_l = 0$ for $l \neq k$, and further

$z_k = (z_k^1, 0, \dots, 0) \in Y^{n_k}$. Using $H_k^1 = R(i\omega_k, A_L)B_L K_1^{k1}$ we see that

$$\begin{aligned} C_1 z &= C_\Lambda H z + D K_1 z = C_\Lambda H_k z_k + D K_1^k z_k \\ &= C_\Lambda H_k^1 z_k^1 + D K_1^{k1} z_k^1 = P_L(i\omega_k) K_1^{k1} z_k^1 \neq 0 \end{aligned}$$

since $z_k^1 \neq 0$, and since we chose K_1^{k1} in such a way that $P(i\omega_k)K_1^{k1}$ and $P_L(i\omega_k)K_1^{k1}$ are boundedly invertible.

It remains to show that the closed-loop system is exponentially stable. With the chosen controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ the operator A_e becomes

$$A_e = \begin{pmatrix} A_{-1} & BK_1 & -BK_2^\Lambda \\ G_2 C_\Lambda & G_1 + G_2 D K_1 & G_2 C_\Lambda \\ L C_\Lambda & L D K_1 & A_{-1} + BK_2^\Lambda + L C_\Lambda \end{pmatrix}$$

with domain $\mathcal{D}(A_e)$ equal to

$$\mathcal{D}(A_e) = \left\{ (x, z_1, x_1) \in X_B \times Z_0 \times X_B \;\middle|\; \begin{cases} A_{-1}x + BK_1 z_1 - BK_2^\Lambda x_1 \in X \\ A_{-1}x_1 + BK_2^\Lambda x_1 \in X \end{cases} \right\}.$$

If we choose a similarity transform $Q_e \in \mathcal{L}(X \times Z_0 \times X)$

$$Q_e = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -I & H & -I \end{pmatrix} = Q_e^{-1}$$

we can define $\hat{A}_e = Q_e A_e Q_e^{-1}$ on $X \times Z_0 \times X$. If we denote $x_e = (x, z_1, x_1) \in X \times Z_0 \times X$ and use $\mathcal{R}(H) \subset X_B$, the domain of the operator $\hat{A}_e$ satisfies

$$\begin{aligned} \mathcal{D}(\hat{A}_e) &= \left\{ x_e \in X \times Z_0 \times X \mid Q_e^{-1} x_e \in \mathcal{D}(A_e) \right\} \\ &= \left\{ x_e \in X_B \times Z_0 \times X_B \mid Q_e^{-1} x_e \in \mathcal{D}(A_e) \right\}. \end{aligned}$$

For $x_e = (x, z_1, x_1) \in X_B \times Z_0 \times X_B$ we thus have

$$\begin{aligned} & Q_e^{-1} x_e \in \mathcal{D}(A_e) \\ \Leftrightarrow & \begin{cases} A_{-1}x + BK_1 z_1 - BK_2^\Lambda(-x + Hz_1 - x_1) \in X \\ \left(A_{-1} + BK_2^\Lambda\right)(-x + Hz_1 - x_1) \in X \end{cases} \\ \Leftrightarrow & \begin{cases} \left(A_{-1} + BK_2^\Lambda\right)x + B\left(K_1 - K_2^\Lambda H\right)z_1 + BK_2^\Lambda x_1 \in X \\ BK_1 z_1 + A_{-1}Hz_1 - A_{-1}x_1 \in X \end{cases} \\ \Leftrightarrow & \begin{cases} \left(A_{-1} + BK_2^\Lambda\right)x + B\left(K_1 - K_2^\Lambda H\right)z_1 + BK_2^\Lambda x_1 \in X \\ x_1 \in \mathcal{D}(A) \end{cases} \end{aligned}$$

since (6) implies $A_{-1}Hz_1 + BK_1 z = HG_1 z_1 - L_1(C_\Lambda H + DK_1)z_1 \in X$. The above conditions also imply $x \in X_B$, and thus

$$\begin{aligned} \mathcal{D}(\hat{A}_e) = \big\{ x_e \in X \times Z_0 \times \mathcal{D}(A) \mid & \left(A_{-1} + BK_2^\Lambda\right)x \\ & + B\left(K_1 - K_2^\Lambda H\right)z_1 + BK_2^\Lambda x_1 \in X \big\}. \end{aligned}$$

For $x_e = (x, z_1, x_1) \in \mathcal{D}(\hat{A}_e)$ a direct computation using $L = L_1 + G_2 H$, $C_1 = C_\Lambda H + DK_1$, and $HG_1 z_1 = (A_{-1} + L_1 C_\Lambda)Hz_1 + (B + L_1 D)K_1 z_1$ yields

$$\begin{aligned} \hat{A}_e x_e &= Q_e A_e \begin{pmatrix} x \\ z_1 \\ -x + Hz_1 - x_1 \end{pmatrix} \\ &= \begin{pmatrix} \left(A_{-1} + BK_2^\Lambda\right)x + B\left(K_1 - K_2^\Lambda H\right)z_1 + BK_2^\Lambda x_1 \\ \left(G_1 + G_2(C_\Lambda H + DK_1)\right)z_1 - G_2 C_\Lambda x_1 \\ (A_{-1} + L_1 C_\Lambda)x_1 \end{pmatrix} \\ &= \begin{pmatrix} A_{-1} + BK_2^\Lambda & B(K_1 - K_2^\Lambda H) & BK_2^\Lambda \\ 0 & G_1 + G_2 C_1 & -G_2 C_\Lambda \\ 0 & 0 & A + L_1 C_\Lambda \end{pmatrix} \begin{pmatrix} x \\ z_1 \\ x_1 \end{pmatrix}. \end{aligned}$$

The operator $G_2 \in \mathcal{L}(Y, Z_0)$ was chosen in such a way that $G_1 + G_2 C_1 \in \mathcal{L}(Z_0)$ is Hurwitz. Since $(A_{-1} + BK_2^\Lambda)|_X$ and $A + L_1 C_\Lambda$ generate exponentially stable semigroups, since B is an admissible input operator for $(A + BK_2^\Lambda)|_X$, C_Λ and K_2^Λ are admissible input operators for $A + L_1 C_\Lambda$, and $K_1 - K_2^\Lambda H$ and G_2 are bounded, we have that the semigroup generated by $\hat{A}_e$ is exponentially stable, and due to similarity, the same is true for A_e. We thus conclude that the closed-loop system is exponentially stable.

Because the controller incorporates a p-copy internal model of the exosystem and the closed-loop system is exponentially stable, we have from Theorem 7 that the controller solves the robust output regulation problem. □

A. *Controller for a Diagonal Exosystem*

In this section we consider the situation where the output space Y is allowed to be infinite-dimensional and the matrix S in the exosystem is diagonal. We will show that in this situation the robust output regulation problem can be solved with particularly simple choice for the parameter G_2 of the controller. For a diagonal matrix $S = \mathrm{diag}(i\omega_1, \dots, i\omega_q)$ we choose $Z_0 = Y^q$ and the internal model (G_1, K_1) of the exosystem is defined as

$$G_1 = \mathrm{diag}(i\omega_1 I_Y, \dots, i\omega_q I_Y), \quad K_1 = \left(K_1^1, \dots, K_1^q\right)$$

where K_1^k are chosen in such a way that $P(i\omega_k)K_1^k$ are boundedly invertible for all $k \in \{1, \dots, q\}$. The following is the main result of this section.

Theorem 13: Assume $S = \mathrm{diag}(i\omega_1, \dots, i\omega_q)$. If the other parameters of the controller are chosen as in the beginning of Section V and if we choose

$$G_2 = \left(G_2^k\right)_{k=1}^q = \left(-\left(P_L(i\omega_k)K_1^k\right)^*\right)_{k=1}^q \in \mathcal{L}(Y, Z_0)$$

then the controller solves the robust output regulation problem.

If we choose $K_1^k = P_L(i\omega_k)^\dagger = P(i\omega_k)^\dagger(I - C_\Lambda R(i\omega_k, A)L_1)$ for all k, then $G_2^k = -I_Y$ for all k.

Proof: Since $n_k = 1$ for all $k \in \{1, \dots, q\}$, we have $H = (H_1, \dots, H_q) \in \mathcal{L}(Z_0, X)$, where $H_k = R(i\omega_k, A_{-1} + L_1 C_\Lambda)(B + L_1 D)K_1^k$. Because of this, the operator $C_1 = (C_1^1, \dots, C_1^q)$ satisfies

$$C_1^k = C_\Lambda H_k + DK_1^k = P_L(i\omega_k)K_1^k$$

for all $k \in \{1, \dots, q\}$, which shows that $G_2 = -C_1^*$. The last claim of the theorem follows immediately from

$P_L(i\omega_k)=(I-C_\Lambda R(i\omega_k,A)L_1)^{-1}P(i\omega_k)$. The same identity and the fact that K_1^k were chosen so that $P(i\omega_k)K_1^k$ are boundedly invertible imply that the components G_2^k of G_2 are boundedly invertible for all $k\in\{1,\dots,q\}$. We thus have from Lemma 17 that the semigroup generated by $G_1+G_2C_1=G_1-G_2G_2^*$ is exponentially stable. The exponential stability of the closed-loop system can now be shown exactly as in the proof of Theorem 15.

Due to the fact that Y may be infinite-dimensional, we cannot use the concept of p-copy internal model. Instead, we will verify that the controller satisfies the $\mathcal{G}$-conditions. For this we will in particular use Lemma 6.

Since S is diagonal, the condition (4c) is trivially satisfied. The components $G_2^k=-(P_L(i\omega_k)K_1^k)^*$ of $G_2=(G_2^k)_{k=1}^q$ are boundedly invertible for all $k\in\{1,\dots,q\}$. This implies $\mathcal{N}(G_2)=\{0\}$, and also further shows that $\mathcal{N}(\mathcal{G}_2)=\{0\}$. Moreover, if for some $k\in\{1,\dots,q\}$ the elements $(z,x)\in Z$, $(w,v)\in\mathcal{D}(\mathcal{G}_1)$ with $w=(w_k)_{k=1}^q\in Z_0$, and $y\in Y$ are such that

$$\begin{pmatrix}z\\x\end{pmatrix}=\begin{pmatrix}i\omega_k-G_1 & 0\\ 0 & i\omega_k-\left(A_{-1}+BK_2^\Lambda\right)\end{pmatrix}\begin{pmatrix}w\\v\end{pmatrix}=\begin{pmatrix}G_2\\L\end{pmatrix}y$$

then we in particular have $z=(i\omega_k-G_1)w=G_2y$ and $G_2^ky=(i\omega_k-i\omega_k)w_k=0$. The invertibility of G_2^k implies $y=0$ and $(z,x)=\mathcal{G}_2y=0$. Since $k\in\{1,\dots,q\}$ was arbitrary, this shows that the operators $\left(\left(\begin{smallmatrix}G_1 & 0\\ 0 & A_{-1}+BK_2^\Lambda\end{smallmatrix}\right),\mathcal{G}_2\right)$ satisfy the $\mathcal{G}$-conditions. Since

$$\begin{aligned}\mathcal{G}_1&=\begin{pmatrix}G_1 & G_2\left(C_\Lambda+DK_2^\Lambda\right)\\ 0 & A_{-1}+BK_2^\Lambda+L\left(C_\Lambda+DK_2^\Lambda\right)\end{pmatrix}\\ &=\begin{pmatrix}G_1 & 0\\ 0 & A_{-1}+BK_2^\Lambda\end{pmatrix}+\begin{pmatrix}G_2\\L\end{pmatrix}\begin{pmatrix}0 & C_\Lambda+DK_2^\Lambda\end{pmatrix}\end{aligned}$$

where for any $k\in\{1,\dots,q\}$ we have $\mathcal{N}\left(i\omega_k-\left(\begin{smallmatrix}G_1 & 0\\ 0 & A_{-1}+BK_2^\Lambda\end{smallmatrix}\right)\right)\subset Z_0\times\{0\}\subset\mathcal{N}((0,C_\Lambda c+DK_2^\Lambda))$, Lemma 6 shows that the operators $(\mathcal{G}_1,\mathcal{G}_2)$ satisfy the $\mathcal{G}$-conditions as well.

Since the controller satisfies the $\mathcal{G}$-conditions and the closed-loop system is exponentially stable, we have from Theorem 7 that the controller solves the robust output regulation problem. □

B. Controller With a Reduced Order Internal Model

It was shown in [15] that the triangular structure used in this section is ideal for controllers with reduced order internal models. Indeed, if the internal model (G_1,K_1) is replaced with an appropriate reduced order internal model, the controller will solve the robust output regulation problem for a given class $\mathcal{O}_0$ of perturbations. As the final result in this section we present a generalization of the controller introduced in [15] for regular linear systems with diagonal exosystems. For this purpose we again assume that $P(i\omega_k)$ are invertible for all $k\in\{1,\dots,q\}$.

Let $\mathcal{O}_0$ be a class of admissible perturbations. Similarly as in Section IV-A we define $Z_0=Y_1\times\cdots\times Y_q$, and

$$G_1=\operatorname{diag}\left(i\omega_1I_{Y_1},\dots,i\omega_qI_{Y_q}\right),\quad K_1=\left(K_1^1,\dots,K_1^q\right)$$

where $K_1^k\in\mathcal{L}(Y_k,U)$ are such that

$$K_1^k=\begin{cases}\left(u_k^1,\dots,u_k^{p_k}\right) & \text{if } p_k<\dim Y\\ P(i\omega_k)^{-1} & \text{if } p_k=\dim Y \text{ or } p_k=\infty\end{cases}$$

in the notation of Section IV-A. Moreover, we define $G_2=(-(P_L(i\omega_k)K_1^k)^*)_{k=1}^q\in\mathcal{L}(Y,Z)$. The rest of the parameters of the controller $(\mathcal{G}_1,\mathcal{G}_2,K)$ are chosen as in the beginning of Section VI.

Theorem 14: Assume $S=\operatorname{diag}(i\omega_1,\dots,i\omega_q)$ and $P(i\omega_k)$ are invertible for all $k\in\{1,\dots,q\}$. Then the controller with the above choices of parameters solves the robust output regulation problem for the class $\mathcal{O}_0$ of perturbations.

Proof: If $(\tilde{A},\tilde{B},\tilde{C},\tilde{D},\tilde{E},\tilde{F})\in\mathcal{O}_0$ and $k\in\{1,\dots,q\}$, and if we choose z as in the proof of Theorem 11 (for $\varepsilon=1$), then it is easy to see that $\tilde{P}(i\omega_k)K\binom{z}{0}=y_k$ and $\binom{z}{0}\in\mathcal{N}(i\omega_k-\mathcal{G}_1)$. By [19, Thm. 5.1] the controller solves the robust output regulation problem for the class $\mathcal{O}_0$ of perturbations provided that the closed-loop system is exponentially stable. The stability of the closed-loop system can be shown exactly as in the proof of Theorem 13. □

VI. The Observer-Based Robust Controller

The observer-based robust controller structure presented in this section is based on the controller Hämäläinen and Pohjolainen [10] for systems with bounded input and output operators. The construction of the controller is again completed in steps and its properties are given in Theorem 15. For this controller structure it is necessary to assume that the spaces U and Y are isomorphic. We begin by assuming that the plant has the same finite number of inputs and outputs, that is, $U=Y=\mathbb{C}^p$. The case of an infinite-dimensional output space is again considered separately for a diagonal exosystem in Theorem 16.

Step 1°: We begin by choosing the state space of the controller as $Z=Z_0\times X$, and choosing

$$\mathcal{G}_1=\begin{pmatrix}G_1 & 0\\ (B+LD)K_1 & A_{-1}+BK_2+L(C_\Lambda+DK_2)\end{pmatrix}$$

$\mathcal{G}_2=\binom{G_2}{-L}$, and $K=(K_1,K_2^\Lambda)$. The operators (G_1,G_2) make up the internal model of the exosystem (2), and they are defined by choosing $Z_0=Y^{n_1}\times\cdots\times Y^{n_q}$, and

$$G_1=\operatorname{diag}\left(J_1^Y,\dots,J_q^Y\right)\in\mathcal{L}(Z_0),\quad G_2=\left(G_2^k\right)_{k=1}^q.$$

Here J_k^Y are as in (5) and $G_2^k=(G_2^{kl})_{l=1}^{n_k}\in\mathcal{L}(Y,Y^{n_k})$ for all $k\in\{1,\dots,q\}$, where $n_k\in\mathbb{N}$ is the dimension of the Jordan block in S associated to the eigenvalue $i\omega_k\in\sigma(S)$. We choose the components $G_2^{kn_k}\in\mathcal{L}(Y)$ of each G_2^k to be boundedly invertible (e.g., it is possible to choose $G_2^{kn_k}=I_Y$ for every $k\in\{1,\dots,q\}$).

Step 2°: By Assumption 1 we can choose $K_{21}\in\mathcal{L}(X_1,U)$ and $L\in\mathcal{L}(Y,X)$ in such a way that $(A_{-1}+BK_{21}^\Lambda)|_X$ (here K_{21}^Λ is the Λ-extension of K_{21}) and $A+LC_\Lambda$ generate exponentially stable semigroups. For $\lambda\in\rho(A_{-1}+BK_{21}^\Lambda)$ we define

$$P_K(\lambda)=\left(C_\Lambda+DK_{21}^\Lambda\right)R\left(\lambda,A_{-1}+BK_{21}^\Lambda\right)B+D.$$

Since $P(i\omega_k)$ were assumed to be surjective for all $k \in \{1,\dots,q\}$ and since $U = Y = \mathbb{C}^q$, the identity $P_K(i\omega_k) = P(i\omega_k)(I - K_{21}^\Lambda R(i\omega_k, A_{-1})B)^{-1}$ implies that $P_K(i\omega_k)$ are boundedly invertible for all $k \in \{1,\dots,q\}$.

Step 3°: We define an operator $H : \mathcal{D}(H) \subset X_{-1} \to Z_0$ in such a way that $H = (H_k)_{k=1}^q$ and $H_k = (H_k^l)_{l=1}^{n_k}$, where

$$H_k^l = \sum_{j=l}^{n_k} (-1)^{j-l} G_2^{kj}\left(C_\Lambda + DK_{21}^\Lambda\right) R\left(i\omega_k, A_{-1} + BK_{21}^\Lambda\right)^{j+1-l}.$$

Since we have from [30, Sec. 7] that $(A + BK_{21}^\Lambda, B, C_\Lambda + DK_{21}^\Lambda, D)$ is a regular linear system and $X_B \subset \mathcal{D}(C_\Lambda) \cap \mathcal{D}(K_{21}^\Lambda)$, it is immediate that $H \in \mathcal{L}(X, Z_0)$ and $\mathcal{R}(B) \subset \mathcal{D}(H)$, and we can thus define $B_1 = HB + G_2 D \in \mathcal{L}(U, Z_0)$.

Step 4°: We choose the operator $K_1 \in \mathcal{L}(Z_0, U)$ in such a way that the semigroup generated by $G_1 + B_1 K_1 \in \mathcal{L}(Z_0)$ is exponentially stable (i.e., the matrix is Hurwitz). The stabilizability of the pair (G_1, B_1) is shown in Theorem 15 below. Finally, we define $K_2^\Lambda = K_{21}^\Lambda + K_1 H \in \mathcal{L}(X, U)$ and choose the domain of the operator $\mathcal{G}_1$ as

$$\mathcal{D}(\mathcal{G}_1) = \left\{(z_1, x_1) \in Z_0 \times X_B \,|\, A_{-1}x_1 + B\left(K_1 z_1 + K_2^\Lambda x_1\right) \in X\right\}.$$

Theorem 15: Assume $U = Y = \mathbb{C}^p$. The controller with the above choices of parameters solves the robust output regulation problem.

In particular, the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ has the following properties:

(1) The operator $\mathcal{G}_1$ generates a semigroup on Z and the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ satisfies the $\mathcal{G}$-conditions in Definition 5.

(2) The operator H is the unique solution of the Sylvester equation

$$G_1 H = H\left(A_{-1} + BK_{21}^\Lambda\right) + G_2\left(C_\Lambda + DK_{21}^\Lambda\right) \quad (7)$$

on $\mathcal{D}(C_\Lambda) \cap \mathcal{D}(K_{21}^\Lambda)$. Moreover, (G_1, B_1) where $B_1 = HB + G_2 D \in \mathcal{L}(U, Z_0)$ is exponentially stabilizable.

(3) The semigroup generated by A_e is exponentially stable.

Proof: The property that $\mathcal{G}_1$ with the given domain generates a strongly continuous semigroup can be seen analogously as in the proof of Theorem 12.

We will now show that H defined in Step 3° is the solution of (7). Denote $A_K = A_{-1} + BK_{21}^\Lambda$ and $C_K = C_\Lambda + DK_{21}^\Lambda$ for brevity. The structure of G_1 implies that an operator H is the solution of $G_1 H = HA_K + G_2 C_K$ if and only if $H = (H_k)_{k=1}^q$ with $H_k = (H_k^l)_{l=1}^{n_k}$ for all k, and for all $k \in \{1,\dots,q\}$ we have

$$\begin{aligned} H_k^1(i\omega_k - A_K) + H_k^2 &= G_2^{k1} C_K \\ &\vdots \\ H_k^{n_k-1}(i\omega_k - A_K) + H_k^{n_k} &= G_2^{k,n_k-1} C_K \\ H_k^{n_k}(i\omega_k - A_K) &= G_2^{kn_k} C_K \end{aligned}$$

on $\mathcal{D}(C_\Lambda) \cap \mathcal{D}(K_{21}^\Lambda)$. For every $k \in \{1,\dots,q\}$ the above system of equations has a unique solution which is exactly H_k in step 3°.

We will now show that the pair (G_1, B_1) with $B_1 = HB + G_2 D$ is exponentially stabilizable. This is equivalent to the pair (B_1^*, G_1^*) being exponentially detectable. Let $k \in \{1,\dots,q\}$ and $z = (z_1,\dots,z_q) \in \mathcal{N}(-i\omega_k - G_1^*)$. Then $z_l = 0$ for $l \neq k$, and $z_k = (0,\dots,0,z_k^{n_k})$ with $z_k^{n_k} \in Y$. For any $u \in U$ we have

$$\begin{aligned} \langle u, B_1^* z\rangle = \langle B_1 u, z\rangle &= \left\langle \left(H_k^{n_k} B + G_2^{kn_k} D\right) u, z_k^{n_k}\right\rangle \\ &= \left\langle G_2^{kn_k}\left(C_K R(i\omega_k, A_K)B + D\right) u, z_k^{n_k}\right\rangle \\ &= \left\langle u, \left(G_2^{kn_k} P_K(i\omega_k)\right)^* z_k^{n_k}\right\rangle \end{aligned}$$

which immediately implies that we can have $B_1^* z = 0$ only if $z_k^{n_k} = 0$ due to the fact that $G_2^{kn_k}$ and $P_K(i\omega_k)$ are invertible. Since this also implies $z = 0$ and since $k \in \{1,\dots,q\}$ was arbitrary, we have that the pair (G_1, B_1) is exponentially stabilizable [12, Thm. 6.2-5]. Because of this it is possible to choose K_1 in such a way that $G_1 + B_1 K_1$ is Hurwitz.

We will now show that the closed-loop system is exponentially stable. When the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is chosen as suggested, we have that

$$A_e = \begin{pmatrix} A_{-1} & BK_1 & BK_2^\Lambda \\ G_2 C_\Lambda & G_1 + G_2 DK_1 & G_2 DK_2^\Lambda \\ -LC_\Lambda & BK_1 & A_{-1} + BK_2^\Lambda + LC_\Lambda \end{pmatrix}$$

with domain

$$\mathcal{D}(A_e) = \left\{(x, z_1, x_1) \in X_B \times Z_0 \times X_B \;\middle|\; \begin{cases} A_{-1}x + BK_1 z_1 + BK_2^\Lambda x_1 \in X \\ BK_1 z_1 + \left(A_{-1} + BK_2^\Lambda\right) x_1 \in X \end{cases}\right\}.$$

If we choose a similarity transform $Q_e \in \mathcal{L}(X \times Z_0 \times X)$

$$Q_e = \begin{pmatrix} -I & 0 & 0 \\ H & I & 0 \\ -I & 0 & I \end{pmatrix} = Q_e^{-1}$$

we can define $\hat{A}_e = Q_e A_e Q_e^{-1}$ on $X \times Z_0 \times X$. If we denote $x_e = (x, z_1, x_1) \in X \times Z_0 \times X$, we have

$$\begin{aligned} \mathcal{D}(\hat{A}_e) &= \left\{x_e \in X \times Z_0 \times X \,|\, Q_e^{-1} x_e \in \mathcal{D}(A_e)\right\} \\ &= \left\{x_e \in X_B \times Z_0 \times X_B \,|\, Q_e^{-1} x_e \in \mathcal{D}(A_e)\right\} \end{aligned}$$

and for $x_e = (x, z_1, x_1) \in X_B \times Z_0 \times X_B$ we have

$$Q_e^{-1} x_e \in \mathcal{D}(A_e)$$

$$\Leftrightarrow \begin{cases} -A_{-1}x + BK_1(Hx + z_1) + BK_2^\Lambda(-x + x_1) \in X \\ BK_1(Hx + z_1) + (A_{-1} + BK_2^\Lambda)(-x + x_1) \in X \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(A_{-1} + BK_{21}^\Lambda\right) x - BK_1 z_1 - BK_2^\Lambda x_1 \in X \\ x_1 \in \mathcal{D}(A) \end{cases}$$

where we have used $K_{21}^{\Lambda} = K_2^{\Lambda} - K_1 H$. Since the above condition also implies $x \in X_B$, the domain of $\hat{A}_e$ becomes

$$\mathcal{D}(\hat{A}_e) = \big\{x_e \in X \times Z_0 \times \mathcal{D}(A) | \\ \big(A_{-1} + BK_{21}^{\Lambda}\big)\, x - BK_1 z_1 - BK_2^{\Lambda} x_1 \in X\big\}.$$

For any $x_e = (x, z_1, x_1) \in \mathcal{D}(\hat{A}_e)$ a direct computation using $K_2^{\Lambda} = K_{21}^{\Lambda} + K_1 H$, $B_1 = HB + G_2 D$, and $G_1 H x = H(A_{-1} + BK_{21}^{\Lambda})x + G_2(C_{\Lambda} + DK_{21}^{\Lambda})x$ yields

$$\hat{A}_e x_e = Q_e A_e \begin{pmatrix} -x \\ Hx + z_1 \\ -x + x_1 \end{pmatrix} = \begin{pmatrix} \big(A_{-1}+BK_{21}^{\Lambda}\big)\, x - BK_1 z_1 - BK_2^{\Lambda} x_1 \\ (G_1 + (HB + G_2 DK_1))\, z_1 + (HB + G_2 D)K_2^{\Lambda} x_1 \\ (A_{-1} + LC_{\Lambda}) x_1 \end{pmatrix} = \begin{pmatrix} A_{-1}+BK_{21}^{\Lambda} & -BK_1 & -BK_2^{\Lambda} \\ 0 & G_1 + B_1 K_1 & B_1 K_2^{\Lambda} \\ 0 & 0 & A_{-1}+LC_{\Lambda} \end{pmatrix} \begin{pmatrix} x \\ z_1 \\ x_1 \end{pmatrix}.$$

The operator $K_1 \in \mathcal{L}(Z_0, U)$ was chosen in such a way that $G_1 + B_1 K_1 \in \mathcal{L}(Z_0)$ is Hurwitz. Since $(A_{-1} + BK_{21}^{\Lambda})|_X$ and $A + LC_{\Lambda}$ generate exponentially stable semigroups, since B and K_2^{Λ} are admissible with respect to $(A + BK_2^{\Lambda})|_X$ and $(A_{-1} + LC_{\Lambda})|_X$, respectively, and since K_1 and B_1 are bounded, the semigroup generated by $\hat{A}_e$ is exponentially stable, and because of similarity, the same is also true for A_e. This concludes that the closed-loop system is exponentially stable.

It remains to show that the controller satisfies the $\mathcal{G}$-conditions. We begin by showing that (G_1, G_2) satisfy the $\mathcal{G}$-conditions. We have $\mathcal{N}(G_2) = \{0\}$ since $G_2^{kn_k}$ are boundedly invertible for all $k \in \{1, \ldots, q\}$. If $z \in \mathcal{R}(i\omega_k - G_1) \cap \mathcal{R}(G_2)$ for some $k \in \{1, \ldots, q\}$, there exist z_1 and y such that $z = (i\omega_k - G_1)z_1 = G_2 y$. Here $z = (z_1, \ldots, z_q)$ with $z_k = (z_k^1, \ldots, z_k^{n_k}) \in Y^{n_k}$, and structure of G_1 implies that necessarily $z_k^{n_k} = 0$. On the other hand, we have $0 = z_k^{n_k} = G_2^{kn_k} y$, which implies $y = 0$ since $G_2^{kn_k}$ is invertible, and thus $z = G_2 y = 0$. This concludes that $\mathcal{R}(i\omega_k - G_1) \cap \mathcal{R}(G_2) = \{0\}$. Finally, a direct computation can be used to verify that $\mathcal{N}(i\omega_k - G_1)^{n_k - 1} = \{z = (z_1, \ldots, z_q) | z_k^{n_k} = 0,\ z_l = 0 \text{ for } l \neq k\} \subset \mathcal{R}(i\omega_k - G_1)$. This concludes that (G_1, G_2) satisfy the $\mathcal{G}$-conditions. Moreover, the surjectivity of the operators $G_2^{kn_k}$ implies $Z_0 = \mathcal{R}(i\omega_k - G_1) + \mathcal{R}(G_2)$, and we thus have $Z_0 = \mathcal{R}(i\omega_k - G_1) \oplus \mathcal{R}(G_2)$.

We will now show that $(\mathcal{G}_1, \mathcal{G}_2)$ satisfy the $\mathcal{G}$-conditions. The condition $\mathcal{N}(\mathcal{G}_2) = \{0\}$ follows immediately from $\mathcal{N}(G_2) = \{0\}$. If $(z, x) \in \mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$, there exist $(z_1, x_1) \in \mathcal{D}(\mathcal{G}_1)$ and $y \in Y$ such that

$$\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} i\omega_k - G_1 & 0 \\ B_L K_1 & A_{-1} + B_L K_2^{\Lambda} + LC_{\Lambda} \end{pmatrix} \begin{pmatrix} z_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} G_2 \\ -L \end{pmatrix} y$$

where we have denoted $B_L = B + LD$. The first line implies $z \in \mathcal{N}(i\omega_k - G_1) \cap \mathcal{R}(G_2) = \{0\}$, and since $\mathcal{N}(G_2) = \{0\}$, we have $y = 0$. Thus $(z, x) = \mathcal{G}_2 y = 0$ and we conclude that $\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$.

Finally, let $(z, x) \in \mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k - 1}$. Since the closed-loop is exponentially stable, we have $Z = \mathcal{R}(i\omega_k - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$ for all $k \in \{1, \ldots, q\}$ [16, Lem. 5.7]. Thus there exist $(z_1, x_1) \in \mathcal{D}(\mathcal{G}_1)$ and $y \in Y$ such that

$$\begin{pmatrix} z \\ x \end{pmatrix} = \begin{pmatrix} i\omega_k - G_1 & 0 \\ B_L K_1 & A_{-1} + B_L K_2^{\Lambda} + LC_{\Lambda} \end{pmatrix} \begin{pmatrix} z_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} G_2 \\ -L \end{pmatrix} y.$$

We will show that $y = 0$, which will conclude that $(z, x) \in \mathcal{R}(i\omega_k - \mathcal{G}_1)$. From the above equation we see that $z = (i\omega_k - G_1)z_1 + G_2 y$. The property $(z, x) \in \mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k - 1}$ and the triangular structure of $\mathcal{G}_1$ imply $z \in \mathcal{N}(i\omega_k - G_1)^{n_k - 1} \subset \mathcal{R}(i\omega_k - G_1)$. However, since $Z_0 = \mathcal{R}(i\omega_k - G_1) \oplus \mathcal{R}(G_2)$, in the decomposition $z = (i\omega_k - G_1)z_1 + G_2 y$ we must then necessarily have $G_2 y = 0$, which further implies $y = 0$ due to $\mathcal{N}(G_2) = \{0\}$. Since $(z, x) \in \mathcal{N}(i\omega_k - \mathcal{G}_1)^{n_k - 1}$ was arbitrary, we have that (4c) is satisfied.

Since the controller satisfies the $\mathcal{G}$-conditions and the closed-loop system is exponentially stable, we have from Theorem 7 that the controller solves the robust output regulation problem. □

Finally, we consider the situation where Y is infinite-dimensional and the matrix S in the exosystem is diagonal. We choose $Z_0 = Y^q$ and the internal model in the controller is of the form

$$G_1 = \operatorname{diag}(i\omega_1 I_Y, \ldots, i\omega_q I_Y), \quad G_2 = \big(G_2^k\big)_{k=1}^q \in \mathcal{L}(Y, Z_0)$$

where the components G_2^k are chosen to be boundedly invertible for all $k \in \{1, \ldots, q\}$.

Theorem 16: Assume $S = \operatorname{diag}(i\omega_1, \ldots, i\omega_q)$ and $P(i\omega_k) \in \mathcal{L}(U, Y)$ are boundedly invertible for all $k \in \{1, \ldots, q\}$. If the other parameters of the controller are chosen as in the beginning of Section VI and if we choose

$$K_1 = \Big(-\big(G_2^1 P_K(i\omega_1)\big)^*, \ldots, -\big(G_2^q P_K(i\omega_q)\big)^*\Big) \in \mathcal{L}(Z_0, U)$$

then the controller solves the robust output regulation problem.

If $G_2 = ((I - K_{21}^{\Lambda} R(i\omega_k, A_{-1})B)P(i\omega_k)^{-1})_{k=1}^q$, then $K_1 = (-I_Y, \ldots, -I_Y)$.

Proof: To show that the controller solves the robust output regulation problem, it is sufficient to show that the closed-loop system is exponentially stable, because the property that the controller satisfies the $\mathcal{G}$-conditions and all the other properties considered in the proof of Theorem 15 remain valid for a general Hilbert space Y.

Since $G_1 = \operatorname{diag}(i\omega_k I_Y)_{k=1}^q$, the operator B_1 is of the form $B_1 = (B_1^k)_{k=1}^q \in \mathcal{L}(U, Y^q)$, where for all $k \in \{1, \ldots, q\}$ we have

$$B_1^k = H_k B + G_2^k D = G_2^k P_K(i\omega_k)$$

since $H_k = G_2^k (C_{\Lambda} + DK_{21}^{\Lambda}) R(i\omega_k, A_{-1} + BK_{21}^{\Lambda})$. This shows that $K_1 = -B_1^*$. The last claim of the theorem follows from $P_K(i\omega_k) = P(i\omega_k)(I - K_{21}^{\Lambda} R(i\omega_k, A_{-1})B)^{-1}$, and the invertibility of G_2^k and $P(i\omega_k)$ imply that B_1^k are boundedly invertible for all $k \in \{1, \ldots, q\}$. We thus have from Lemma 17 that the operator $G_1 + B_1 K_1 = G_1 - B_1 B_1^*$ generates an exponentially stable semigroup. The exponential stability of the closed-loop system can now be shown as in the proof of Theorem 15. □

VII. Robust Control of a 2D Heat Equation

In this section we consider robust output regulation for a two-dimensional heat equation with boundary control and observation. Set-point regulation without the robustness requirement was considered for the same system in [14, Ex. 6.2].

We study the heat equation

$$x_t(\xi,t) = \Delta x(\xi,t), \quad x(\xi,0) = x_0(\xi)$$

on the unit square $\xi = (\xi_1,\xi_2) \in \Omega = [0,1]\times[0,1]$. The control and observation are located on the parts Γ_1 and Γ_2 of the boundary $\partial\Omega$, where $\Gamma_1 = \{\xi = (\xi_1,0) | 0 \le \xi_1 \le 1/2\}$ and $\Gamma_2 = \{\xi = (\xi_1,1) | 1/2 \le \xi_1 \le 1\}$. We denote $\Gamma_0 = \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$. The boundary control and the additional boundary conditions are defined as

$$\frac{\partial x}{\partial n}(\xi,t)|_{\Gamma_1} = u_1(t), \quad \frac{\partial x}{\partial n}(\xi,t)|_{\Gamma_2} = u_2(t), \quad \frac{\partial x}{\partial n}(\xi,t)|_{\Gamma_0} = 0$$

for $u(t) = (u_1(t),u_2(t)) \in U = \mathbb{C}^2$. The outputs $y(t) = (y_1(t), y_2(t)) \in Y = \mathbb{C}^2$ of the system are defined as averages of the value of $x(\xi,t)$ over the parts Γ_1 and Γ_2 of the boundary, i.e.,

$$y_1(t) = 2\int_0^{\frac{1}{2}} x(\xi_1,0;t)d\xi_1, \quad y_2(t) = 2\int_{\frac{1}{2}}^1 x(\xi_1,1;t)d\xi_1.$$

We define $A_0 = \Delta$ with domain $\mathcal{D}(A_0) = \{x \in H^2(\Omega) | (\partial x/\partial n) = 0 \text{ on } \partial\Omega\}$. We have from [3, Cor. 1] that with the above control and observation, the heat equation is a regular linear system (A_0,B,C,D) with $D=0$. The system becomes exponentially stable with negative output feedback, $u = -\kappa Cx + \tilde{u}$ where $\kappa > 0$ (cf. [14, Ex. 6.2]). We choose $\kappa = 1$, and define $A = ((A_0)_{-1} - BC)|_X$.

Our aim is to design a minimal order controller for the stabilized system $(A,B,C,0)$ to achieve robust output tracking of the reference signal $y_{ref}(t) = (-1,\cos(\pi t))$. To this end, we choose the exosystem as $W = \mathbb{C}^3$, $S = \operatorname{diag}(-i\pi,0,i\pi)$, $E=0$, and $F = -\begin{pmatrix} 0 & -1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$. The reference signal $y_{ref}(t)$ is then generated with the choice $v_0 = (1,1,1)$ of the initial state of the exosystem.

Since $p = \dim Y = 2$, the internal model and the parameters of the minimal order controller are given by

$$\begin{aligned}\mathcal{G}_1 &= \operatorname{diag}(-i\pi,-i\pi,0,0,i\pi,i\pi) \in \mathbb{C}^{6\times 6}\\ K &= \varepsilon K_0 = \varepsilon\left(K_0^1,K_0^2,K_0^3\right) \in \mathbb{C}^{2\times 6}\end{aligned}$$

where $\varepsilon > 0$ and K_0^k are to be chosen in such a way that the matrices $P(-i\pi)K_0^1$, $P(0)K_0^2$ and $P(i\pi)K_0^3$ are nonsingular. We choose $K_0^1 = P(-i\pi)^{-1}$, $K_0^2 = P(0)^{-1}$, and $K_0^3 = P(i\pi)^{-1}$. Finally, $\mathcal{G}_2 = (\mathcal{G}_2^k)_{k=1}^3$ where $\mathcal{G}_2^k = -I_{2\times 2}$ for $k \in \{1,\ldots,3\}$. We have from Theorem 8 that for small values of $\varepsilon > 0$ the controller achieves asymptotic tracking of the reference signal $y_{ref}(\cdot)$, and the control structure is robust with respect to perturbations in $(A,B,C,0)$ that preserve the property $\{0,\pm i\pi\} \subset \rho(\tilde{A})$) and the exponential stability of the closed-loop system. In particular, this includes small bounded perturbations to the operators A, B, C, and $D=0$.

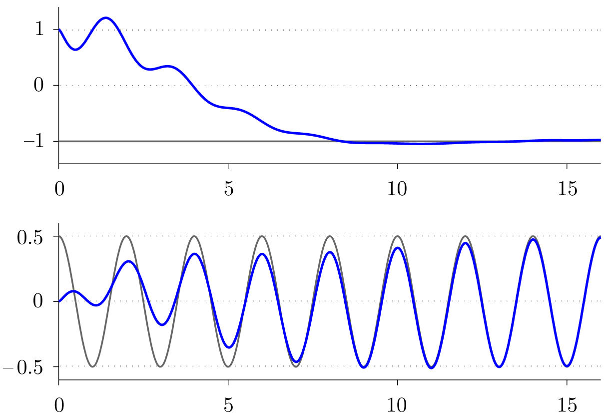

Fig. 1. Outputs $y_1(\cdot)$ and $y_2(\cdot)$ of the controlled system.

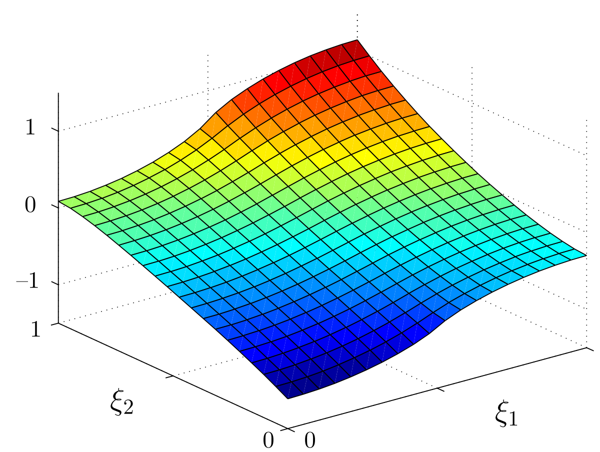

Fig. 2. State of the controlled system at time $t = 16$.

The robust controller also tolerates small perturbations and inaccuracies in the parameters K and G_2 of the controller (although robustness with respect to these operators is not required in the statement of the robust output regulation problem). Because of this property, we can use approximations for the values $P(\pm i\pi)^{-1}$ and $P(0)^{-1}$ in K_0. In this example we use a truncated eigenfunction expansion of A_0 in approximating the matrices $P(0)$ and $P(\pm i\pi)$. Finally, the parameter $\varepsilon > 0$ needs to be chosen in such a way that the closed-loop is stable.

The solution of the controlled heat equation can be approximated numerically using the truncated eigenfunction expansion of the operator A_0. For the simulation, the parameter ε is chosen to be $\varepsilon = 1/4$. Fig. 1 depicts the simulated behaviour of the two outputs of the plant. The solution of the controlled partial differential equation at time $t = 16$ is plotted in Fig. 2.

Appendix

Lemma 17: Let $G_1 = \operatorname{diag}(i\omega_1 I_Y,\ldots,i\omega_q I_Y) \in \mathcal{L}(Y^q)$ and $G_2 = (G_2^k)_{k=1}^q \in \mathcal{L}(U,Y^q)$ where U and Y are Hilbert spaces. If the components G_2^k of G_2 are boundedly invertible for all $k \in \{1,\ldots,q\}$, then the semigroup generated by $G_1 - G_2G_2^*$ is exponentially stable.

Proof: Since $G_1 - G_2G_2^*$ is a bounded operator, it is sufficient to show that $\sigma(G_1 - G_2G_2^*) \subset \mathbb{C}^-$. Since G_1 generates a contraction semigroup, the same is true for $G_1 - G_2G_2^*$, and thus $\sigma(G_1 - G_2G_2^*) \subset \overline{\mathbb{C}^-}$. It therefore remains to show that $i\mathbb{R} \subset \rho(G_1 - G_2G_2^*)$.

Let $i\omega \in i\mathbb{R}$ be such that $\omega \neq \omega_k$ for all $k \in \{1, \ldots, q\}$. We then have $i\omega \in \rho(G_1)$. If $I + G_2^*R(i\omega, G_1)G_2$ is boundedly invertible, then the Woodbury formula implies that $i\omega - G_1 + G_2G_2^*$ has a bounded inverse. However, since $G_2^*R(i\omega, G_1)G_2$ is bounded and skew-adjoint, we have $1 \in \rho(-G_2^*R(i\omega, G_1)G_2)$. This finally implies $i\omega \in \rho(G_1 - G_2G_2^*)$.

It remains to consider the case where $i\omega = i\omega_n$ for some $n \in \{1, \ldots, q\}$. We will show $\|(i\omega_n - G_1 + G_2G_2^*)z\| \geq c\|z\|$ for some constant $c > 0$ and for all $z \in Z$. If this is not true, there exists a sequence $(z_k)_{k\in\mathbb{N}} \subset Z$ such that $\|z_k\| = 1$ for all $k \in \mathbb{N}$ and $\|(i\omega_n - G_1 + G_2G_2^*)z_k\| \to 0$ as $k \to \infty$. Since $i\omega_n - G_1$ is skew-adjoint, we have

$$\begin{aligned} \|(i\omega_n - G_1 + G_2G_2^*)\, z_k\| &\geq |\langle (i\omega_n - G_1 + G_2G_2^*)\, z_k, z_k\rangle| \\ &\geq |\mathrm{Re}\,\langle (i\omega_n - G_1 + G_2G_2^*)\, z_k, z_k\rangle)| = \|G_2^* z_k\|^2 \end{aligned}$$

and thus $\|G_2^*z_k\| \to 0$ as $k \to \infty$. For every $k \in \mathbb{N}$ denote $z_k = z_k^1 + z_k^2$ where $z_k^1 \in \mathcal{R}(i\omega_n - G_1)$, $z_k^2 \in \mathcal{N}(i\omega_n - G_1)$, and $1 = \|z_k\|^2 = \|z_k^1\|^2 + \|z_k^2\|^2$. There exists $c_1 > 0$ such that $\|(i\omega_n - G_1)z_k^1\| \geq c_1\|z_k^1\|$ for all $k \in \mathbb{N}$. Thus

$$\begin{aligned} c_1 \left\|z_k^1\right\| &\leq \left\|(i\omega_n - G_1)z_k^1\right\| = \|(i\omega_n - G_1)z_k\| \\ &\leq \|(i\omega_n - G_1 + G_2G_2^*)\, z_k\| + \|G_2\|\, \|G_2^* z_k\| \to 0 \end{aligned}$$

as $k \to \infty$. Moreover, $\|G_2^*z_k^2\| \geq \|(G_2^n)^{-1}\|^{-1}\|z_k^2\|$, and

$$\left\|(G_2^n)^{-1}\right\|^{-1} \left\|z_k^2\right\| \leq \left\|G_2^* z_k^2\right\| \leq \|G_2^* z_k\| + \left\|G_2^* z_k^1\right\| \to 0$$

as $k \to \infty$. We have now shown that $z_k^1 \to 0$ and $z_k^2 \to 0$, but this contradicts the assumption that $\|z_k\|^2 = 1$ for all $k \in \mathbb{N}$, and thus the original claim holds. In particular $i\omega_n \notin \sigma_p(G_1 + G_2G_2^*)$ and the range of $i\omega_n - G_1 + G_2G_2^*$ is closed. Finally, the Mean Ergodic Theorem [1, Sec. 4.3] implies that the range of $i\omega_n - G_1 + G_2G_2^*$ is dense, and $i\omega_n \in \rho(G_1 + G_2G_2^*)$. □

References

[1] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*. Basel, Switzerland: Birkhäuser, 2001.

[2] C. I. Byrnes, I. G. Laukó, D. S. Gilliam, and V. I. Shubov, "Output regulation problem for linear distributed parameter systems," *IEEE Trans. Autom. Control*, vol. 45, no. 12, pp. 2236–2252, 2000.

[3] C. I. Byrnes, D. S. Gilliam, V. I. Shubov, and G. Weiss, "Regular linear systems governed by a boundary controlled heat equation," *J. Dyn. Control Syst.*, vol. 8, no. 3, pp. 341–370, 2002.

[4] E. J. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE Trans. Autom. Control*, vol. 21, no. 1, pp. 25–34, 1976.

[5] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*. New York: Springer-Verlag, 2000.

[6] B. A. Francis and W. M. Wonham, "The internal model principle for linear multivariable regulators," *Appl. Math. Optim.*, vol. 2, no. 2, pp. 170–194, 1975.

[7] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica J. IFAC*, vol. 12, pp. 457–465, 1976.

[8] T. Hämäläinen and S. Pohjolainen, "Robust control and tuning problem for distributed parameter systems," *Int. J. Robust Nonlin. Control*, vol. 6, no. 5, pp. 479–500, 1996.

[9] T. Hämäläinen and S. Pohjolainen, "A finite-dimensional robust controller for systems in the CD-algebra," *IEEE Trans. Autom. Control*, vol. 45, no. 3, pp. 421–431, 2000.

[10] T. Hämäläinen and S. Pohjolainen, "Robust regulation of distributed parameter systems with infinite-dimensional exosystems," *SIAM J. Control Optim.*, vol. 48, no. 8, pp. 4846–4873, 2010.

[11] E. Immonen, "On the internal model structure for infinite-dimensional systems: Two common controller types and repetitive control," *SIAM J. Control Optim.*, vol. 45, no. 6, pp. 2065–2093, 2007.

[12] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[13] H. Logemann and S. Townley, "Low-gain control of uncertain regular linear systems," *SIAM J. Control Optim.*, vol. 35, no. 1, pp. 78–116, 1997.

[14] V. Natarajan, D. S. Gilliam, and G. Weiss, "The state feedback regulator problem for regular linear systems," *IEEE Trans. Autom. Control*, vol. 59, no. 10, pp. 2708–2723, 2014.

[15] L. Paunonen, "Designing controllers with reduced order internal models," *IEEE Trans. Autom. Control*, vol. 60, no. 3, pp. 775–780, 2015.

[16] L. Paunonen and S. Pohjolainen, "Internal model theory for distributed parameter systems," *SIAM J. Control Optim.*, vol. 48, no. 7, pp. 4753–4775, 2010.

[17] L. Paunonen and S. Pohjolainen, "Output regulation theory for distributed parameter systems with unbounded control and observation," in *Proc. 52nd IEEE Conf. Decision Control*, Florence, Italy, pp. 1083–1088.

[18] L. Paunonen and S. Pohjolainen, "Reduced order internal models in robust output regulation," *IEEE Trans. Autom. Control*, vol. 58, no. 9, pp. 2307–2318, 2013.

[19] L. Paunonen and S. Pohjolainen, "The internal model principle for systems with unbounded control and observation," *SIAM J. Control Optim.*, vol. 52, no. 6, pp. 3967–4000, 2014.

[20] V. Phông, "The operator equation $AX - XB = C$ with unbounded operators A and B and related abstract Cauchy problems," *Math. Z.*, vol. 208, pp. 567–588, 1991.

[21] S. A. Pohjolainen, "Robust multivariable PI-controller for infinite-dimensional systems," *IEEE Trans. Autom. Control*, vol. 27, no. 1, pp. 17–31, 1982.

[22] R. Rebarber and G. Weiss, "Internal model based tracking and disturbance rejection for stable well-posed systems," *Automatica J. IFAC*, vol. 39, no. 9, pp. 1555–1569, 2003.

[23] R. Saij, "Robust Regulation of Control Systems," Ph.D. dissertation, Cadi Ayyad University, Marrakesh, Morocco, 2015.

[24] J. M. Schumacher, "Finite-dimensional regulators for a class of infinite-dimensional systems," *Syst. Control Lett.*, vol. 3, pp. 7–12, 1983.

[25] O. Staffans, *Well-Posed Linear Systems*. Cambridge, U.K.: Cambridge Univ. Press, 2005.

[26] O. Staffans and G. Weiss, "Transfer functions of regular linear systems. II. The system operator and the Lax-Phillips semigroup," *Trans. Amer. Math. Soc.*, vol. 354, no. 8, pp. 3229–3262, 2002.

[27] H. Ukai and T. Iwazumi, "Design of servo systems for distributed parameter systems by finite dimensional dynamic compensator," *Int. J. Syst. Sci.*, vol. 21, no. 6, pp. 1025–1046, 1990.

[28] G. Weiss and R. F. Curtain, "Dynamic stabilization of regular linear systems," *IEEE Trans. Autom. Control*, vol. 42, no. 1, pp. 4–21, 1997.

[29] G. Weiss, "The representation of regular linear systems on Hilbert spaces," in *Control and Estimation of Distributed Parameter Systems (Vorau, 1988)*. Basel, Switzerland: Birkhäuser, 1989, vol. 91, pp. 401–416.

[30] G. Weiss, "Regular linear systems with feedback," *Math. Control Signals Syst.*, vol. 7, no. 1, pp. 23–57, 1994.

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