Chapter 9 Series & Sequences

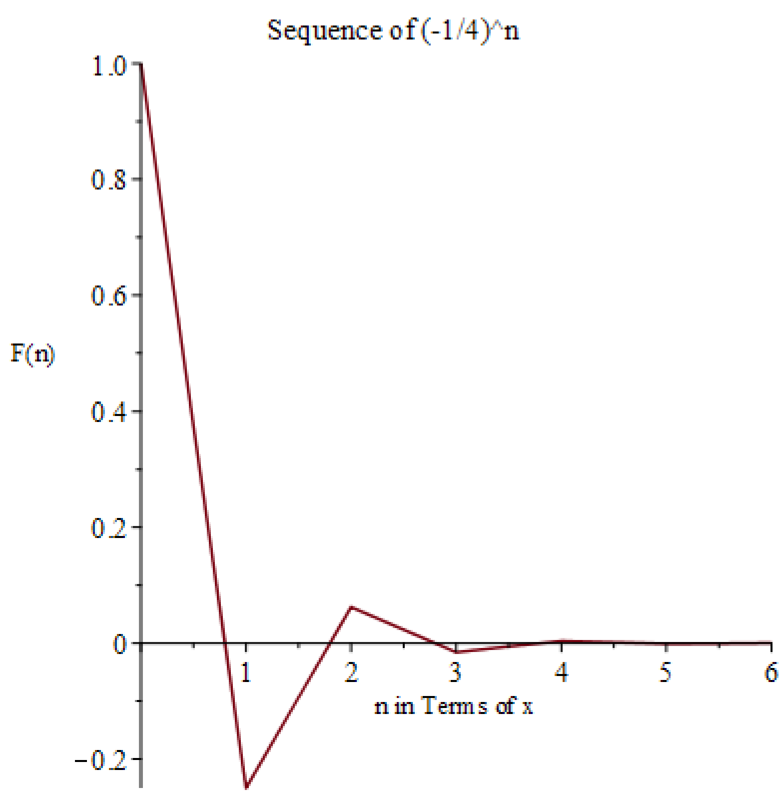
Jared A. Hembree

Surry Community College

1. Considering the following sequence $1, \frac{-1}{4}, \frac{1}{16}, \frac{-1}{64}, \ldots$ the next two apparent terms of the sequence are as follows: $\frac{1}{256}, \frac{-1}{1024}$. Knowing the sequence is as follows $1, \frac{-1}{4}, \frac{1}{16}, \frac{-1}{64}, \frac{1}{256}, \frac{-1}{1024}, \ldots$ we can begin to form an expression for the nth term in the sequence. $\left(\frac{-1}{4}\right)^0 = 1$, $\left(\frac{-1}{4}\right)^1 = \frac{-1}{4}$, $\left(\frac{-1}{4}\right)^2 = \frac{1}{16}$, …, therefore, the expression for the nth term is $\left(\frac{-1}{4}\right)^n$. When considering

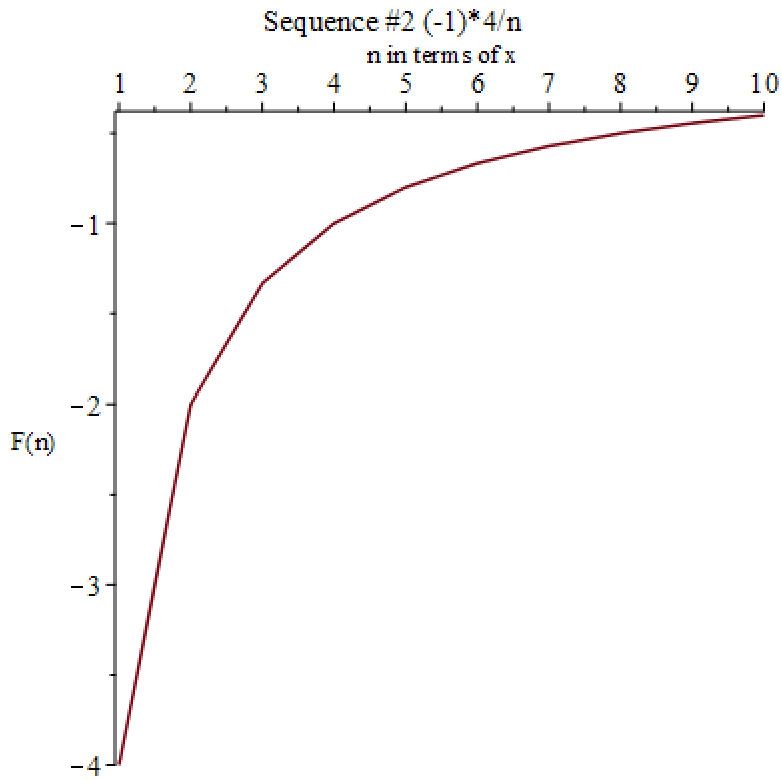
$$\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n$$

the sequence will converge using the alternating series test. $A_n = \left(\frac{-1}{4}\right)^n$, $B_n = \frac{1}{4^n}$. The limit of B_n as n approaches infinity is zero and since B_n is a decreasing sequence, A_n is convergent.



2. Considering the sequence

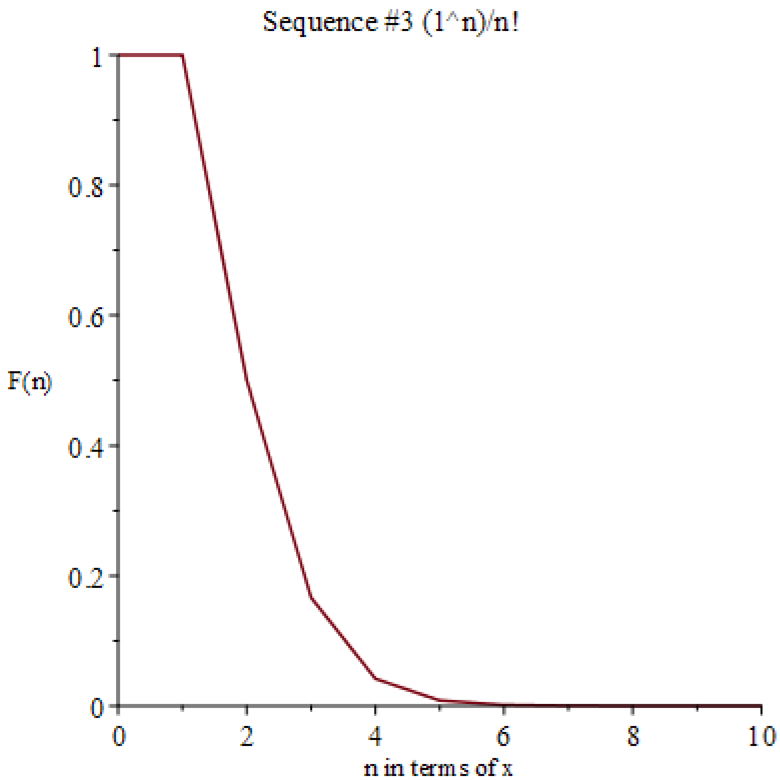
$$A_n = (-1)\frac{4}{n}$$



The limit of A_n as n approaches infinity is zero, therefore, the sequence converges absolutely to zero and we can see graphically that as n head towards infinity our values are approaching zero.

3. Consider the sequence

$$A_n = \frac{1^n}{n!}$$



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The following sequence converges absolutely and the limit of A_n as n approaches infinity is zero and we can see graphically that as n head towards infinity our values are approaching zero.

4. Knowing that a government program costs taxpayer 3.5 billion per year and has a cut back of 30 percent per year we can formulate the sequence $3.5\left(\frac{7}{10}\right)^{n}$.

$3.5\left(\frac{7}{10}\right)^{1}$	$\frac{49}{20} = 2.45$
$3.5\left(\frac{7}{10}\right)^{2}$	$\frac{343}{200} = 1.715$
$3.5\left(\frac{7}{10}\right)^{3}$	$\frac{2401}{2000} \approx 1.2005$
$3.5\left(\frac{7}{10}\right)^{4}$	$\frac{16807}{20000} \approx .84035$
$3.5\left(\frac{7}{10}\right)^{5}$	$\frac{117649}{200000} \approx .588245$

After a period of five years the taxpayers will be paying approximately 588 million dollars.

5. Knowing that the rate of inflation is 3.5 percent per year and the average price is currently 40,000 dollars, the total price after six years will be approximately 49,170.21 dollars according to the sequence $40{,}000(1.035)^{n}$. The average or mean after six years however, is approximately 45,196.05 dollars.

$40{,}000(1.035)^{1}$	41400
$40{,}000(1.035)^{2}$	42849
$40{,}000(1.035)^{3}$	886743/200 ≈ 44348.7
$40{,}000(1.035)^{4}$	1836036801/40000 ≈ 45900.9
$40{,}000(1.035)^{5}$	380059617807/8000000 ≈ 47507.5
$40{,}000(1.035)^{6}$	7872340886049/1600000000 ≈ 49170.2

6. Considering the sequence

$$\sum_{n=1}^{\infty} \frac{-5}{(-6)^{n-1}}$$

the first five terms of the are as follows:

$$\frac{-5}{(-6)^{1-1}} = -5$$

$$\frac{-5}{(-6)^{2-1}} = \frac{5}{6}$$

$$\frac{-5}{(-6)^{3-1}} = \frac{-5}{36}$$

$$\frac{-5}{(-6)^{4-1}} = \frac{5}{216}$$

$$\frac{-5}{(-6)^{5-1}} = \frac{-5}{1296}$$

7. When finding the sum of the series

$$\sum_{\mathrm{n}=1}^{\infty} \frac{9}{(\mathrm{n}+9)(\mathrm{n}+11)}$$

we can use the form $\frac{A}{(n+9)} + \frac{B}{(n+11)} = 9$. After solving for A and B, $A = \frac{9}{2}$ and $B = \frac{-9}{2}$.

setting our formula back up we can factor out 9/2 and get $\frac{9}{2}\left(\frac{1}{n+9} - \frac{1}{n+11}\right)$. S_n =

$$\frac{9}{2}\left(\frac{1}{1+9} - \frac{1}{1+11}\right) + \frac{9}{2}\left(\frac{1}{2+9} - \frac{1}{2+11}\right) + \frac{9}{2}\left(\frac{1}{3+9} - \frac{1}{3+11}\right) + \frac{9}{2}\left(\frac{1}{4+9} - \frac{1}{4+11}\right) + \cdots$$

$$\lim_{n\to\infty} \frac{9}{2}\left(\frac{1}{10} + \frac{1}{12} + \frac{1}{n+9} - \frac{1}{n+11}\right) = \frac{189}{220}$$

Therefore $S_n = \frac{189}{220}$.

8. Considering the repeating decimal .777… as a geometric series we must know first that geometric series have the form $\Sigma\ ar^{n}$ and that r must be 0 < r < 1. By breaking down the repeating decimal into parts

$$\frac{7}{10}=\frac{7}{10^1}\,, \qquad \frac{7}{100}=\frac{7}{10^2}\,, \qquad \frac{7}{1000}=\frac{7}{10^3}$$

we can now determine the following

$$\frac{7}{10}\left(\frac{1}{10}\right)^{0}+\frac{7}{10}\left(\frac{1}{10}\right)^{1}+\frac{7}{10}\left(\frac{1}{10}\right)^{2}+\frac{7}{10}\left(\frac{1}{10}\right)^{3}+\cdots$$

$$\sum_{a=0}^{\infty}\frac{7}{10}\left(\frac{1}{10}\right)^{n}$$

A = 7/10 and r = 1/10 and by using $\frac{a}{1-r}$ and plugging in our values we get 7/9, therefore the repeating decimal .777… is equivalent to 7/9.

9. Supposing that a ball drops from a height of 14 feet and after each bounce only rebounds .91h feet, to find the total distance traveled we begin by making a pattern.

$$14+14\left(\frac{91}{100}\right)^{1}+14\left(\frac{91}{100}\right)^{2}+14\left(\frac{91}{100}\right)^{3}+\cdots$$

Since we rebound we must consider that the lengths with the exception of the first bounce must accounted for twice.

$$14+2*14\left(\frac{91}{100}\right)^{1}+2*14\left(\frac{91}{100}\right)^{2}+2*14\left(\frac{91}{100}\right)^{3}+\cdots$$

After creating this new pattern we can now determine that our series will look like

$$D = 14 + 28 \sum_{n=0}^{\infty} \left(\frac{91}{100}\right)^{n+1}$$

$$D = 14 + 28 \sum_{n=0}^{\infty} 1 * \left(\frac{91}{100}\right)^{n+1}$$

$$D = 14 + 28 * \left(\frac{91}{100}\right) \sum_{n=0}^{\infty} \left(\frac{91}{100}\right)^{n}$$

$$D = 14 + \left(\frac{637}{9}\right) \sum_{n=0}^{\infty} \left(\frac{91}{100}\right)^{n}$$

$$14 + \left(\frac{637}{29}\right)\left(\frac{1}{1 - \frac{91}{100}}\right) = \frac{2674}{9} \approx 297.11\, ft$$

10. A resort city spends 100 million annually and approximately 90% of the revenue is spent again in the resort every year and 90% of that revenue is spent again so on and so forth. Using this information, we can imply that the expression for the nth term with 100 million spent initially is as follows,

$$\sum_{n=1}^{\infty} 100(.9)^{n}$$

Using the Geometric Series Test we can determine that the sum of the series diverges. Since $S_n = \frac{a}{1-r}$ and a = 100 and r = .9 the series will converge to 1000.

11. Considering the series

$$\sum(-1)^n\left(\frac{n^8+11}{n^8+10}\right)$$

Using the Alternating series test the conditions state that $A_{n+1} \leq A_n$ and that B_n must be a decreasing sequence and the $\lim_{n\to\infty} b_n = 0$. We know that $A_n = (-1)^n\left(\frac{n^8+11}{n^8+10}\right)$ and $B_n = \left(\frac{n^8+11}{n^8+10}\right)$ The series passes the first condition of $A_{n+1} \leq A_n$, however, it fails the second condition because the $\lim_{n\to\infty} b_n \neq 0$, therefore the series will diverge according to the alternating series test.

12. Considering the series

$$\mathrm{A_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$$

Using the alternating series, we know that the series will go as follows: $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5}$ ... and since the condition states that $A_{n+1} \leq A_n$ and this condition cannot be shown to be true for all n, the alternating series test cannot be applied.

13. Considering the series

$$\sum_{\mathrm{n}=1}^{\infty}\left(\frac{\mathrm{n}}{10\mathrm{n}+11}\right)$$

Using the nth term test we can determine that whether the series will converge or diverge. $A_n = \left(\frac{\mathrm{n}}{10\mathrm{n}+11}\right)$ and if we take the limit of A_n as n approaches infinity the limit is approaching 1/10. According to the nth term divergence test in order for a series to divergent $\lim_{n\to\infty} \mathrm{A_n} \neq 0$ and since A_n approaches 1/10 the series diverges.

14. Considering the series

$$\sum_{n=1}^{\infty}\left(\frac{-7}{8}\right)^{n}$$

Using the geometric series test we can determine whether the series is divergent or convergent. A_n = $-1^{n}\left(\frac{7}{8}\right)$ and r = $\left(\frac{7}{8}\right)$. Since the absolute value of r < 1 the following series will converge absolutely to $\frac{8}{15}$ because S_n = $\frac{A_n}{1-r}$.

15. Considering the series

$$\sum_{n=0}^{\infty}\left(\frac{1}{5^n}-\frac{1}{6^n}\right)$$

where A_n = $\left(\frac{1}{5^n}-\frac{1}{6^n}\right)$ using the telescoping series,

$$s_N\sum_{a=0}^{N}\left(\frac{1}{5^n}-\frac{1}{6^n}\right)$$

$\left(\frac{1}{5^1}-\frac{1}{6^1}\right)$	$\left(\frac{1}{5}-\frac{1}{6}\right)$	$\frac{1}{30}$
$\left(\frac{1}{5^2}-\frac{1}{6^2}\right)$	$\left(\frac{1}{25}-\frac{1}{36}\right)$	$\frac{1}{30}-\left(\frac{1}{25}-\frac{1}{36}\right)=\frac{11}{900}$
$\left(\frac{1}{5^3}-\frac{1}{6^3}\right)$	$\left(\frac{1}{125}-\frac{1}{216}\right)$	$\frac{11}{900}-\left(\frac{1}{125}-\frac{1}{216}\right)=\frac{91}{27000}$
$\left(\frac{1}{5^4}-\frac{1}{6^4}\right)$	$\left(\frac{1}{625}-\frac{1}{1296}\right)$	$\frac{91}{27000}-\left(\frac{1}{625}-\frac{1}{1296}\right)=\frac{671}{810000}$

Therefore, since $\lim_{n\to\infty}\left(\frac{1}{5^n}-\frac{1}{6^n}\right)=\frac{1}{20}$ the series converges absolutely to $\frac{1}{20}$.

16. Considering the series

$$\sum_{n=8}^{\infty} \frac{1}{6n^2 + 7}$$

Using Integral Test, we know that the conditions state that if the series is positive, continuous and decreasing from 1 to infinity, $A_n = f(n)$, and if and only if the integral of f(n) is convergent will the series A_n be convergent. Letting $A_n = \frac{1}{6n^2+7}$ then integral is $\frac{\tan^{-1}\left(\frac{\sqrt{42}x}{7}\right)}{\sqrt{42}} = \frac{1}{\sqrt{42}\tan\left(\frac{\sqrt{42}}{7}x\right)} = f(n)$. Since f(n) is converging, according to our condition, A_n must also converge.

17. Considering the series

$$\sum_{n=1}^{\infty} \frac{6}{n\sqrt[3]{n}}$$

Using the P-series Test the conditions are when a series is in the form $\frac{1}{n^P}$, when P > 1 the series converges and when P ≤ 1 then the series diverges. Since we can rewrite our series as $\frac{6}{n^{4/3}}$, $P = \frac{4}{3}$ which is greater than 1 indicating that the series will converge.

18. Considering the series

$$\sum_{n=1}^{\infty} \frac{10}{\sqrt[9]{n^9 + 10}}$$

Using the Root Test the conditions state that the $\lim_{n\to\infty} \sqrt[n]{|a_n|} = |a_n|^{1/n} = L$ and when L < 1 the series is absolutely convergent, when L > 1 the series diverges and when L = 1 the series is inconclusive. Letting $a_n = \frac{10}{\sqrt[9]{n^9+10}}$ we take the limit of a_n which equals 0 therefore the series converges.

19. Considering the series

$$\sum_{n=1}^{\infty} \frac{4^n + 1}{6^n + 1}$$

Using Direct Comparison, the conditions state that $A_n \leq B_n$ and A_n, $B_n \geq 0$ and if B_n converges so must A_n and if A_n diverges so must B_n. $A_n = \frac{4^n+1}{6^n+1}$ and $B_n = \frac{1}{4^n+1}$. B_n is larger than A_n therefore the first condition is met. A_n and B_n are also both larger than zero satisfying the second condition meaning we may now continue with the Direct Comparison Test. The limit of B_n as n approaches infinity will equal zero meaning that B_n converges and since B_n converges, following the direct comparison rules, A_n must also converge, therefore, A_n will converges.

20. Considering the series

$$\sum_{\mathrm{n}=1}^{\infty} (-1)^{\mathrm{n}} \frac{8\mathrm{n}^7 + 5}{5\mathrm{n}^9 + 2\mathrm{n} + 8}$$

Using the Alternating series test the conditions state that $A_{n+1} \leq A_n$ and that B_n must be a decreasing sequence and the $\lim_{n\to\infty} b_n = 0$. Letting $A_n = (-1)^{\mathrm{n}} \frac{8\mathrm{n}^7+5}{5\mathrm{n}^9+2\mathrm{n}+8}$ and $B_n = \frac{8\mathrm{n}^7+5}{5\mathrm{n}^9+2\mathrm{n}+8}$. A_n and B_n both pass the conditions indicating that if B_n converges or diverges, A_n will act in the same manner. Therefore, since the $\lim_{n\to\infty} \left|\frac{8\mathrm{n}^7+5}{5\mathrm{n}^9+2\mathrm{n}+8}\right| = 0$, A_n not only converges but converges absolutely according to the conditions of the Alternating Series Test.

21. Considering the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{4}{5}\right)^n}{n^2}$$

Using the Alternating series test the conditions state that $A_{n+1} \leq A_n$ and that B_n must be a decreasing sequence and the $\lim_{n\to\infty} b_n = 0$. Letting $A_n = \frac{(-1)^{n-1}\left(\frac{4}{5}\right)^n}{n^2}$ and $B_n = \frac{\left(\frac{4}{5}\right)^n}{n^2}$. A_n and B_n both pass the conditions indicating that if B_n converges or diverges, A_n will act in the same manner. Therefore, since the $\lim_{n\to\infty} \left|\frac{\left(\frac{4}{5}\right)^n}{n^2}\right| = 0$, A_n not only converges but converges absolutely according to the conditions of the Alternating Series Test.

22. Considering the series

$$\sum_{n=0}^{\infty} \frac{n!}{15^n}$$

The Ratio Test conditions state that the $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \text{p}$ and if p >1 a_n will diverge and if p < 1 a_n will converge and if a_n= 1 the series will be either convergent or divergent. Letting $a_n = \frac{n!}{15^n}$ we know that as the $\lim_{n\to\infty} \left|\frac{\frac{n+1!}{15^{n+1}}}{\frac{n!}{15^n}}\right| = \infty$, therefore the series diverges.

23. Considering the series

$$\sum_{n=1}^{\infty} \left(\frac{3n}{4n+1}\right)^n$$

The Ratio Test conditions state that the $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \text{p}$ and if p >1 a_n will diverge and if p < 1 a_n will converge and if a_n= 1 the series will be either convergent or divergent. Letting $a_n = \left(\frac{3n}{4n+1}\right)^n$ we know that the $\lim_{n\to\infty} \left|\frac{\left(\frac{3n+1}{4n+2}\right)^{n+1}}{\left(\frac{3n}{4n+1}\right)^n}\right| = \frac{3}{4}$, therefore the series will converge.

24. Considering the series

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + 3n + 2}$$

Using the Limit Comparison Test the conditions are A_n must be ≥ 0 and B_n must be > 0 and if $\lim_{n\to\infty} \frac{A_n}{B_n}$ is positive and finite then they will both either converge or diverge. $A_n = \frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)}$ and $B_n = \frac{1}{(n+2)}$. The $\lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{(n+2)}} = \lim_{n\to\infty} \frac{1}{(n+1)} = 0$

therefore the series converges.