Finite Automata

DFA Definition: A DFA is 5-tuple or quintuple M = (Q, Σ, δ, q_0, A) where

Q is non-empty, finite set of states.

Σ is non-empty, finite set of input alphabets.

δ is transition function, which is a mapping from Q x Σ to Q.

$q_0 \in Q$ is the start state.

$A \subseteq Q$ is set of accepting or final states.

Note: For each input symbol *a*, from a given state there is exactly one transition (there can be no transitions from a state also) and we are sure (or can determine) to which state the machine enters. So, the machine is called *deterministic* machine. Since it has finite number of states the machine is called Deterministic finite machine or Deterministic Finite Automaton or Finite State Machine (FSM).

The language accepted by DFA is

$$L(M) = \{ w \mid w \in \Sigma^* \text{ and } \delta^*(q_0, w) \in A \}$$

The non-acceptance of the string *w* by an FA or DFA can be defined in formal notation as:

$$\overline{L(M)} = \{ w \mid w \in \Sigma^* \text{ and } \delta^*(q_0, w) \notin A \}$$

DFA to accepting strings of a's and b's starting with the string ab

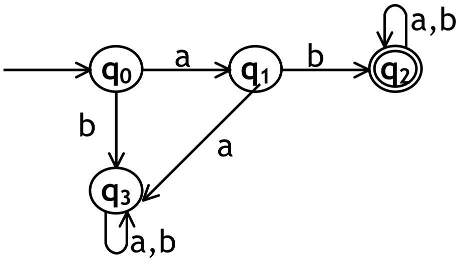


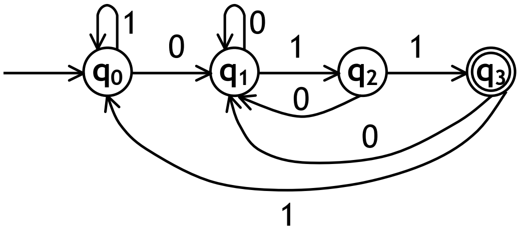
Fig.2.7 Transition diagram to accept string ab(a+b)*

So, the DFA which accepts strings of a's and b's starting with the string *ab* is given by M = (Q, Σ , δ, q_0, A) where

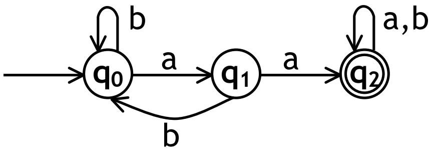
$Q = \{q_0, q_1, q_2, q_3\}$
$\Sigma = \{a, b\}$
q_0 is the start state
$A = \{q_2\}$.
δ is shown the transition table 2.4.

	δ	←Σ→ a	b
← States →	→q_0	q_1	q_3
	q_1	q_3	q_2
	q_2	q_2	q_2
	q_3	q_3	q_3

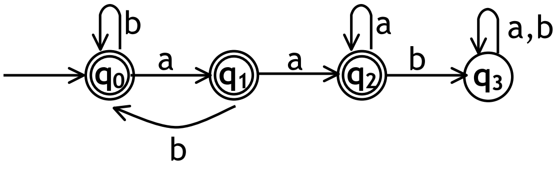
Draw a DFA to accept string of 0’s and 1’s ending with the string 011.



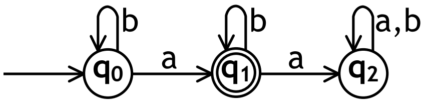
Obtain a DFA to accept strings of a’s and b’s having a sub string aa

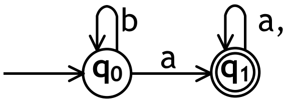


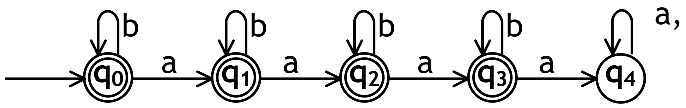
Obtain a DFA to accept strings of a’s and b’s except those containing the substring aab.



Obtain DFAs to accept strings of a’s and b’s having exactly one a,







Obtain a DFA to accept strings of a’s and b’s having even number of a’s and b’s

The machine to accept even number of a’s and b’s is shown in fig.2.22.

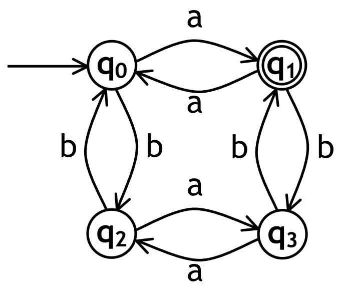


Fig.2.22 DFA to accept even no. of a’s and b’s

Regular language

Definition: Let M = (Q, Σ, δ, q_0, A) be a DFA. The language L is regular if there exists a machine M such that L = L(M).

Applications of Finite Automata

String matching/processing

Compiler Construction

The various compilers such as C/C++, Pascal, Fortran or any other compiler are designed using the finite automata. The DFAs are extensively used in the building the various phases of compiler such as

- Lexical analysis (To identify the tokens, identifiers, to strip of the comments etc.)
- Syntax analysis (To check the syntax of each statement or control statement used in the program)
- Code optimization (To remove the un wanted code)
- Code generation (To generate the machine code)

Other applications

The concept of finite automata is used in wide applications. It is not possible to list all the applications, as there are infinite numbers of applications. This section lists some applications:

1. Large natural vocabularies can be described using finite automaton which includes the applications such as spelling checkers and advisers, multi-language dictionaries, to indent the documents, in calculators to evaluate complex expressions based on the priority of an operator etc. to name a few. Any editor that we use uses finite automaton for implementation.
2. Finite automaton is very useful in recognizing difficult problems i.e., sometimes it is very essential to solve an un-decidable problem. Even though there is no general solution exists for the specified problem, using theory of computation, we can find the approximate solutions.

3. Finite automaton is very useful in hardware design such as circuit verification, in design of the hardware board (mother board or any other hardware unit), automatic traffic signals, radio controlled toys, elevators, automatic sensors, remote sensing or controller etc.
4. In game theory and games wherein we use some control characters to fight against a monster, economics, computer graphics, linguistics etc., finite automaton plays a very important role.

Non-deterministic finite automata (NFA)

Definition: An NFA is a 5-tuple or quintuple M = ($Q, \Sigma, \delta, q_0, A$) where

Q = finite non empty set of states.

Σ = Non-empty, finite set of input alphabets.

δ = Transition function which is a mapping from $Q \times \{\Sigma \cup \varepsilon\} \longrightarrow 2^Q$.

This function shows the change of state from one state to a set of states based on the input symbol.

$q_0 \in Q$ is the start state.

$A \subseteq Q$ is set of final states.

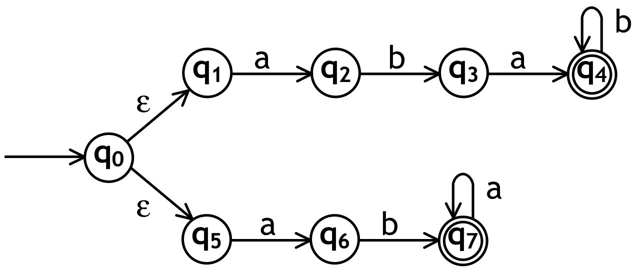
Acceptance of language

Definition: Let M = ($Q, \Sigma, \delta, q_0, A$) be a DFA where Q is set of finite states, Σ is set of input alphabets (from which a string can be formed), δ is transition function from $Q \times \{\Sigma U \varepsilon\}$ to 2^Q, q_0 is the start state and A is the final or accepting state. The string (also called language) *w* accepted by an NFA can be defined in formal notation as:

L(M) = { w | $w \in \Sigma^*$ and $\delta^*(q_0, w)$ = Q with atleast one
Component of Q in A}

Obtain an NFA to accept the following language L = {w | $w \in abab^n$ or aba^n where $n \geq 0$}

The machine to accept either $abab^n$ or aba^n where $n \geq 0$ is shown below:



2.1 Conversion from NFA to DFA

Let $M_N = (Q_N, \Sigma_N, \delta_N, q_0, A_N)$ be an NFA and accepts the language $L(M_N)$. There should be an equivalent DFA $M_D = (Q_D, \Sigma_D, \delta_D, q_0, A_D)$ such that $L(M_D) = L(M_N)$. The procedure to convert an NFA to its equivalent DFA is shown below:

Step1:

The start state of NFA M_N is the start state of DFA M_D. So, add q_0(which is the start state of NFA) to Q_D and find the transitions from this state. The way to obtain different transitions is shown in step2.

Step2:

For each state $[q_i, q_j, \ldots. q_k]$ in Q_D, the transitions for each input symbol in Σ can be obtained as shown below:

1. $\delta_D([q_i, q_j, \ldots. q_k], a) = \delta_N(q_i, a) \cup \delta_N(q_j, a) \cup \ldots\ldots \delta_N(q_k, a)$

   $= [q_l, q_m, \ldots. q_n]$ say.
2. Add the state $[q_l, q_m, \ldots. q_n]$ to Q_D, if it is not already in Q_D.
3. Add the transition from $[q_i, q_j, \ldots. q_k]$ to $[q_l, q_m, \ldots. q_n]$ on the input symbol *a* iff the state $[q_l, q_m, \ldots. q_n]$ is added to Q_D in the previous step.

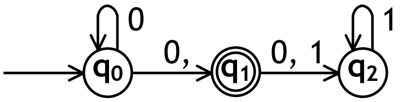
Step3:

The state $[q_a, q_b, \ldots. q_c] \in Q_D$ is the final state, if at least one of the state in $q_a, q_b, \ldots.. q_c \in A_N$ i.e., at least one of the component in $[q_a, q_b, \ldots. q_c]$ should be the final state of NFA.

Step4:

If epsilon ($\in$) is accepted by NFA, then start state q_0 of DFA is made the final state.

Convert the following NFA into an equivalent DFA.



Step1: q_0 is the start of DFA (see step1 in the conversion procedure).

So, $Q_D = \{[q_0]\}$ (2.7)

Step2: Find the new states from each state in Q_D and obtain the corresponding transitions.

Consider the state $[q_0]$:

When a = 0

$$\begin{aligned} \delta_D([q_0], 0) &= \delta_N([q_0], 0) \\ &= [q_0, q_1] \\ &\quad (2.8) \end{aligned}$$

When a = 1

$$\begin{aligned} \delta_D([q_0], 1) &= \delta_N([q_0], 1) \\ &= [q_1] \\ &\quad (2.9) \end{aligned}$$

Since the states obtained in (2.8) and (2.9) are not in Q_D(2.7), add these two states to Q_D so that

$$Q_D = \{[q_0], [q_0, q_1], [q_1] \} \qquad (2.10)$$

The corresponding transitions on a = 0 and a = 1 are shown below.

$\longleftarrow \Sigma \longrightarrow$

	δ	0	1
$\uparrow$	$[q_0]$	$[q_0, q_1]$	$[q_1]$
Q	$[q_0, q_1]$		
$\downarrow$	$[q_1]$		

Consider the state $[q_0, q_1]$:

When a = 0

$$\begin{aligned} \delta_D([q_0, q_1], 0) &= \delta_N([q_0, q_1], 0) \\ &= \delta_N(q_0, 0) \cup \delta_N(q_1, 0) \\ &= \{q_0, q_1\} \quad \cup \{q_2\} \\ &= [q_0, q_1, q_2] \\ &\quad (2.11) \end{aligned}$$

When a = 1

$$\begin{aligned} \delta_D([q_0, q_1], 1) &= \delta_N([q_0, q_1], 1) \\ &= \delta_N(q_0, 1) \cup \delta_N(q_1, 1) \\ &= \{q_1\} \cup \{q_2\} \\ &= [q_1, q_2] \\ &\qquad\qquad (2.12) \end{aligned}$$

Since the states obtained in (2.11) and (2.12) are the not defined in Q_D(see 2.10), add these two states to Q_D so that

$$Q_D = \{[q_0], [q_0, q_1], [q_1], [q_0, q_1, q_2], [q_1, q_2] \} \quad (2.13)$$

and add the transitions on *a* = 0 and *a* = 1 as shown below:

$\longleftarrow \Sigma \longrightarrow$

	δ	0	1
$\uparrow$	$[q_0]$	$[q_0, q_1]$	$[q_1]$
Q	$[q_0, q_1]$	$[q_0, q_1, q_2]$	$[q_1, q_2]$
$\downarrow$	$[q_1]$		
	$[q_0, q_1, q_2]$		
	$[q_1, q_2]$		

Consider the state $[q_1]$:

When *a* = 0

$$\begin{aligned} \delta_D([q_1], 0) &= \delta_N([q_1], 0) \\ &= [q_2] \\ &\quad (2.14) \end{aligned}$$

When *a* = 1

$$\begin{aligned} \delta_D([q_1], 1) &= \delta_N([q_1], 1) \\ &= [q_2] \\ &\quad (2.15) \end{aligned}$$

Since the states obtained in (2.14) and (2.15) are same and the state q_2 is not in Q_D(see 2.13), add the state q_2 to Q_D so that

$$Q_D = \{[q_0], [q_0, q_1], [q_1], [q_0, q_1, q_2], [q_1, q_2], [q_2]\} \qquad (2.16)$$

and add the transitions on *a* = 0 and *a* = 1 as shown below:

$\longleftarrow \Sigma \longrightarrow$

	δ	0	1
$\uparrow$	$[q_0]$	$[q_0, q_1]$	$[q_1]$
Q	$[q_0, q_1]$	$[q_0, q_1, q_2]$	$[q_1, q_2]$
$\downarrow$	$[q_1]$	$[q_2]$	$[q_2]$
	$[q_0, q_1, q_2]$		
	$[q_1, q_2]$		
	$[q_2]$		

Consider the state $[q_0,q_1,q_2]$:

When a = 0

$$\begin{aligned}\delta_D([q_0,q_1,q_2], 0) &= \delta_N([q_0,q_1,q_2], 0)\\ &= \delta_N(q_0, 0) \text{ U } \delta_N(q_1, 0) \text{ U } \delta_N(q_2, 0)\\ &= \{q_0,q_1\} \text{ U } \{q_2\} \text{ U } \{\phi\}\\ &= [q_0,q_1,q_2]\\ &\quad (2.17)\end{aligned}$$

When a = 1

$$\begin{aligned}\delta_D([q_0,q_1,q_2], 1) &= \delta_N([q_0,q_1,q_2], 1)\\ &= \delta_N(q_0, 1) \text{ U } \delta_N(q_1, 1) \text{ U } \delta_N(q_2, 1)\\ &= \{q_1\} \text{ U } \{q_2\} \text{ U } \{q_2\}\\ &= [q_1, q_2]\\ &\quad (2.18)\end{aligned}$$

Since the states obtained in (2.17) and (2.18) are not new states (are already in Q_D, see 2.16), do not add these two states to Q_D. But, the transitions on a = 0 and a = 1 should be added to the transitional table as shown below:

← Σ →

	δ	0	1
↑	$[q_0]$	$[q_0, q_1]$	$[q_1]$
	$[q_0, q_1]$	$[q_0, q_1, q_2]$	$[q_1, q_2]$
Q	$[q_1]$	$[q_2]$	$[q_2]$
	$[q_0, q_1, q_2]$	$[q_0,q_1,q_2]$	$[q_1, q_2]$
↓	$[q_1, q_2]$		
	$[q_2]$		

Consider the state $[q_1,q_2]$:

When a = 0

$$\begin{aligned}\delta_D([q_1,q_2], 0) &= \delta_N([q_1,q_2], 0)\\ &= \delta_N(q_1, 0) \text{ U } \delta_N(q_2, 0)\\ &= \{q_2\} \text{ U } \{\phi\}\\ &= [q_2]\\ &\quad (2.19)\end{aligned}$$

When a = 1

$$
\begin{aligned}
\delta_D([q_1,q_2],\ 1) &= \delta_N([q_1,q_2],\ 1) \\
&= \delta_N(q_1,\ 1)\ U\ \delta_N(q_2,\ 1) \\
&= \{q_2\}\ U\ \{q_2\} \\
&= [q_2] \\
&\ (2.20)
\end{aligned}
$$

Since the states obtained in (2.19) and (2.20) are not new states (are already in Q_D see 2.16), do not add these two states to Q_D. But, the transitions on *a* = 0 and *a* = 1 should be added to the transitional table as shown below:

$\longleftarrow \Sigma \longrightarrow$

Q	δ	0	1
$\uparrow$	$[q_0]$	$[q_0,\ q_1]$	$[q_1]$
	$[q_0,\ q_1]$	$[q_0,\ q_1,\ q_2]$	$[q_1,\ q_2]$
Q	$[q_1]$	$[q_2]$	$[q_2]$
	$[q_0,\ q_1,\ q_2]$	$[q_0,\ q_1,\ q_2]$	$[q_1,\ q_2]$
$\downarrow$	$[q_1,\ q_2]$	$[q_2]$	$[q_2]$
	$[q_2]$		

Consider the state $[q_2]$:

When *a* = 0

$$
\begin{aligned}
\delta_D([q_2],\ 0) &= \delta_N([q_2],\ 0) \\
&= \{\phi\} \\
&\ (2.21)
\end{aligned}
$$

When *a* = 1

$$
\begin{aligned}
\delta_D([q_2],\ 1) &= \delta_N([q_2],\ 1) \\
&= [q_2] \\
&\ (2.22)
\end{aligned}
$$

Since the states obtained in (2.21) and (2.22) are not new states (are already in Q_D, see 2.16), do not add these two states to Q_D. But, the transitions on *a* = 0 and *a* = 1 should be added to the transitional table. The final transitional table is shown in table 2.14. and final DFA is shown in figure 2.35.

δ	0	1
$[q_0]$	$[q_0,\ q_1]$	$[q_1]$
$[q_0,q_1]$	$[q_0,\ q_1,\ q_2]$	$[q_1,\ q_2]$
$[q_1]$	$[q_2]$	$[q_2]$
$[q_0,q_1,q_2$	$[q_0,q_1,q_2]$	$[q_1,\ q_2]$
$[q_1,q_2]$	$[q_2]$	$[q_2]$
$[q_2]$	ϕ	$[q_2]$

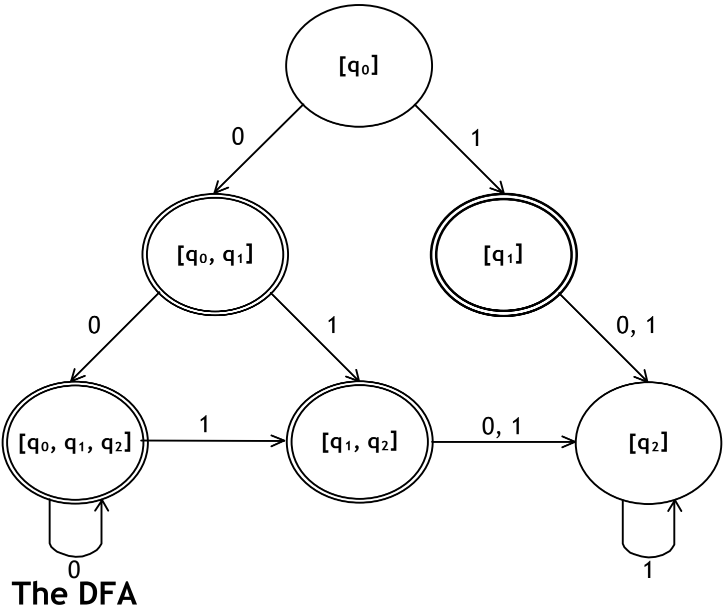
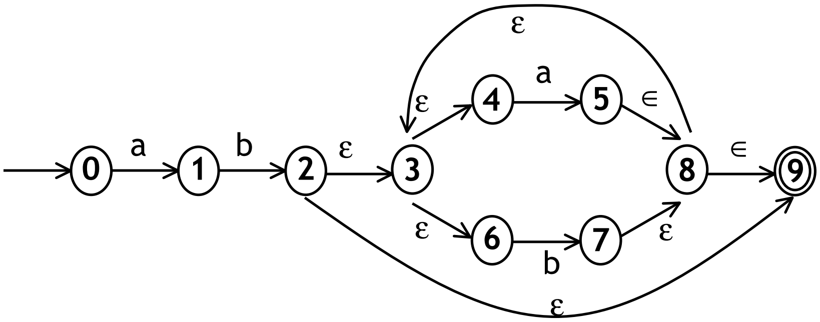


Fig.2.35 The DFA

Convert the following NFA to its equivalent DFA.



Let Q_D = {0} (A)

Consider the state [A]:

When input is *a*:

$\delta(A, a) = \delta_N(0, a)$
$= \{1\}$
(B)

When input is *b*:

$\delta(A, b) = \delta_N(0, b)$
$= \{\phi\}$

Consider the state [B]:

When input is *a*:

$\delta(B, a) \quad = \quad \delta_N(1, a)$

$\quad = \quad \{\phi\}$

When input is *b*:

$\delta(B, b) \quad = \quad \delta_N(1, b)$

$\quad = \quad \{2\}$

$\quad = \quad \{2,3,4,6,9\}$ (C)

This is because, in state 2, due to ε-transitions (or without giving any input) there can be transition to states 3,4,6,9 also. So, all these states are reachable from state 2. Therefore,

$\delta(B, b) = \{2,3,4,6,9\} = C$

Consider the state [C]:

When input is *a*:

$\delta(C, a) \quad = \quad \delta_N(\{2,3,4,6,9\}, a)$

$\quad = \quad \{5\}$

$\quad = \quad \{5, 8, 9, 3, 4, 6\}$

$\quad = \quad \{3, 4, 5, 6, 8, 9\}$ (ascending order) (D)

This is because, in state 5 due to ε-transitions, the states reachable are {8, 9, 3, 4, 6}. Therefore,

$\delta(C, a) = \{3, 4, 5, 6, 8, 9\} = D$

When input is *b*:

$\delta(C, b) \quad = \quad \delta_N(\{2, 3, 4, 6, 9\}, b)$

$\quad = \quad \{7\}$

$\quad = \quad \{7, 8, 9, 3, 4, 6\}$

$\quad = \quad \{3, 4, 6, 7, 8, 9\}$(ascending order) (E)

This is because, from state 7 the states that are reachable without any input (i.e., ε-transition) are {8, 9, 3, 4, 6}. Therefore,

$\delta(C, b) = \{3, 4, 6, 7, 8, 9\} = E$

Consider the state [D]:

When input is *a*:

$\delta(D, a) \quad = \quad \delta_N(\{3,4,5,6,8,9\}, a)$

$\quad = \quad \{5\}$

$\quad = \quad \{5, 8, 9, 3, 4, 6\}$

$\quad = \quad \{3, 4, 5, 6, 8, 9\}$ (ascending order) (D)

When input is *b*:

$$\begin{aligned}\delta(D, b) &= \delta_N(\{3,4,5,6,8,9\}, b)\\ &= \{7\}\\ &= \{7, 8, 9, 3, 4, 6\}\\ &= \{3, 4, 6, 7, 8, 9\} \quad \text{(ascending order)} \quad (E)\end{aligned}$$

Consider the state [E]:

When input is *a*:

$$\begin{aligned}\delta(E, a) &= \delta_N(\{3,4,6,7,8,9\}, a)\\ &= \{5\}\\ &= \{5, 8, 9, 3, 4, 6\}\\ &= \{3, 4, 5, 6, 8, 9\}\text{(ascending order)} \quad (D)\end{aligned}$$

When input is *b*:

$$\begin{aligned}\delta(E, b) &= \delta_N(\{3,4,6,7,8,9\}, b)\\ &= \{7\}\\ &= \{7, 8, 9, 3, 4, 6\}\\ &= \{3, 4, 6, 7, 8, 9\}\text{(ascending order)} \quad (E)\end{aligned}$$

Since there are no new states, we can stop at this point and the transition table for the DFA is shown in table 2.15.

Σ

Q

δ	a	b
A	B	-
B	-	C
C	D	E
D	D	E
E	D	E

Table 2.15 Transitional table

The states C,D and E are final states, since 9 (final state of NFA) is present in C, D and E. The final transition diagram of DFA is shown in figure 2.36

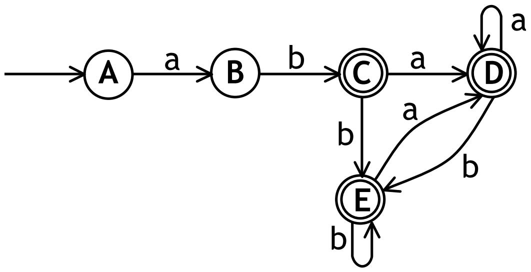


Fig. 2.36 The DFA

Regular Languages

Regular expression

Definition: A regular expression is recursively defined as follows.

1. ϕ is a regular expression denoting an empty language.
2. ε-(epsilon) is a regular expression indicates the language containing an empty string.
3. *a* is a regular expression which indicates the language containing only {a}
4. If R is a regular expression denoting the language L_R and S is a regular expression denoting the language L_S, then
   a. R+S is a regular expression corresponding to the language L_RUL_S.
   b. R.S is a regular expression corresponding to the language $L_R.L_{S.}$.
   c. R* is a regular expression corresponding to the language L_R^*.
5. The expressions obtained by applying any of the rules from 1-4 are regular expressions.

The table 3.1 shows some examples of regular expressions and the language corresponding to these regular expressions.

Regular expressions	Meaning
(a+b)*	Set of strings of a's and b's of any length including the NULL string.
(a+b)*abb	Set of strings of a's and b's ending with the string abb
ab(a+b)*	Set of strings of a's and b's starting with the string ab.
(a+b)*aa(a+b)*	Set of strings of a's and b's having a sub string aa.
a*b*c*	Set of string consisting of any number of a's(may be empty string also) followed by any number of b's(may include empty string) followed by any number of c's(may include empty string).
$a^+b^+c^+$	Set of string consisting of at least one 'a' followed by string consisting of at least one 'b' followed by string consisting of at least one 'c'.
aa*bb*cc*	Set of string consisting of at least one 'a' followed by string consisting of at least one

	‘b’ followed by string consisting of at least one ‘c’.
(a+b)* (a + bb)	Set of strings of a’s and b’s ending with either *a* or *bb*
(aa)*(bb)*b	Set of strings consisting of even number of a’s followed by odd number of b’s
(0+1)*000	Set of strings of 0’s and 1’s ending with three consecutive zeros(or ending with 000)
(11)*	Set consisting of even number of 1’s

Table 3.1 Meaning of regular expressions

Obtain a regular expression to accept a language consisting of strings of a’s and b’s of even length.

String of a’s and b’s of even length can be obtained by the combination of the strings aa, ab, ba and bb. The language may even consist of an empty string denoted by ε. So, the regular expression can be of the form

(aa + ab + ba + bb)*

The * closure includes the empty string.
Note: This regular expression can also be represented using set notation as

$$L(R) = \{(aa + ab + ba + bb)^n \mid n \geq 0\}$$

Obtain a regular expression to accept a language consisting of strings of a’s and b’s of odd length.

String of a’s and b’s of odd length can be obtained by the combination of the strings aa, ab, ba and bb followed by either *a* or *b*. So, the regular expression can be of the form

(aa + ab + ba + bb)* (a+b)

String of a’s and b’s of odd length can also be obtained by the combination of the strings aa, ab, ba and bb preceded by either *a* or *b*. So, the regular expression can also be represented as

(a+b) (aa + ab + ba + bb)*

Note: Even though these two expression are seems to be different, the language corresponding to those two expression is same. So, a variety of regular expressions can be obtained for a language and all are equivalent.

Obtain NFA from the regular expression

Theorem: Let R be a regular expression. Then there exists a finite automaton M = (Q, Σ, δ, q_0, A) which accepts L(R).

Proof: By definition, ϕ, ε and *a* are regular expressions. So, the corresponding machines to recognize these expressions are shown in figure 3.1.a, 3.1.b and 3.1.c respectively.

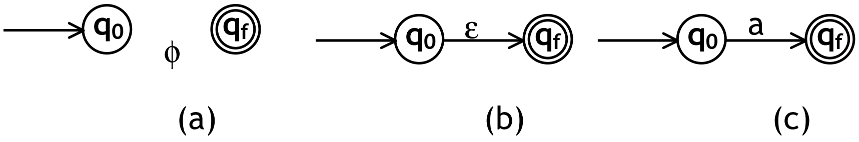


Fig 3.1 NFAs to accept ϕ, ε and a

The schematic representation of a regular expression R to accept the language L(R) is shown in figure 3.2. where q is the start state and f is the final state of machine M.

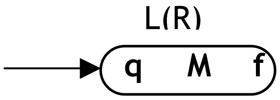


Fig 3.2 Schematic representation of FA accepting L(R)

In the definition of a regular expression it is clear that if R and S are regular expression, then R+S and R.S and R* are regular expressions which clearly uses three operators '+', '-' and '.'. Let us take each case separately and construct equivalent machine. Let M_1 = (Q_1, Σ_1, δ_1, q_1, f_1) be a machine which accepts the language L(R_1) corresponding to the regular expression R_1. Let M_2 = (Q_2, Σ_2, δ_2, q_2, f_2) be a machine which accepts the language L(R_2) corresponding to the regular expression R_2.

Case 1: R = R_1 + R_2. We can construct an NFA which accepts either L(R_1) or L(R_2) which can be represented as L(R_1 + R_2) as shown in figure 3.3.

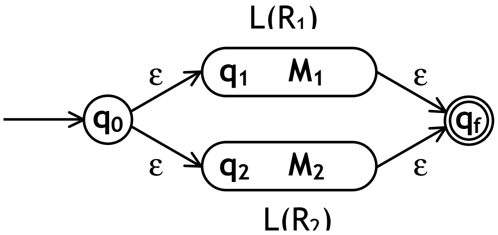


Fig. 3.3 To accept the language L(R1 + R2)

It is clear from figure 3.3 that the machine can either accept $L(R_1)$ or $L(R_2)$. Here, q_0 is the start state of the combined machine and q_f is the final state of combined machine M.

Case 2: $R = R_1 . R_2$. We can construct an NFA which accepts $L(R_1)$ followed by $L(R_2)$ which can be represented as $L(R_1 . R_2)$ as shown in figure 3.4.

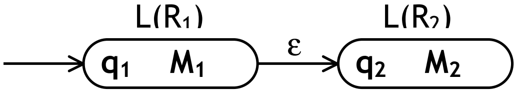


Fig. 3.4To accept the language L(R1 . R2)

It is clear from figure 3.4 that the machine after accepting $L(R_1)$ moves from state q_1 to f_1. Since there is a ε-transition, without any input there will be a transition from state f_1 to state q_2. In state q_2, upon accepting $L(R_2)$, the machine moves to f_2 which is the final state. Thus, q_1 which is the start state of machine M_1 becomes the start state of the combined machine M and f_2 which is the final state of machine M_2, becomes the final state of machine M and accepts the language $L(R_1.R_2)$.

Case 3: $R = (R_1)^*$. We can construct an NFA which accepts either $L(R_1)^*$) as shown in figure 3.5.a. It can also be represented as shown in figure 3.5.b.

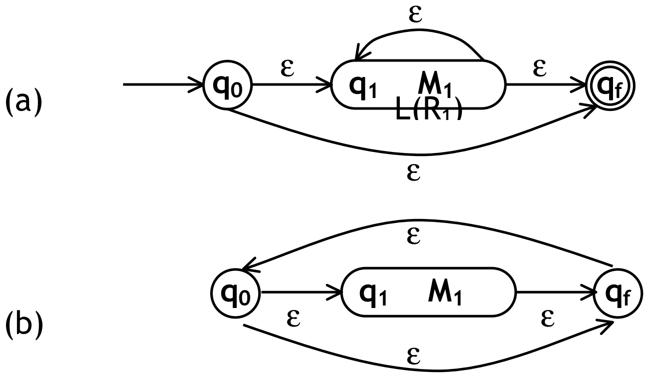
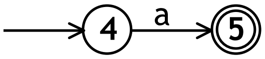


Fig. 3.5 To accept the language $L(R1)^*$

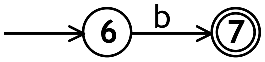
It is clear from figure 3.5 that the machine can either accept ε or any number of $L(R_1)$s thus accepting the language $L(R_1)^*$. Here, q_0 is the start state q_f is the final state.

Obtain an NFA which accepts strings of a's and b's starting with the string ab.
The regular expression corresponding to this language is ab(a+b)*.

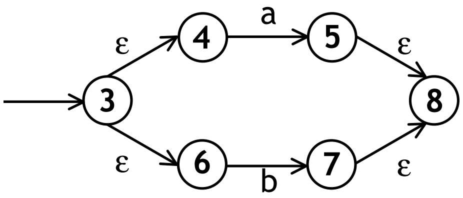
Step 1: The machine to accept 'a' is shown below.



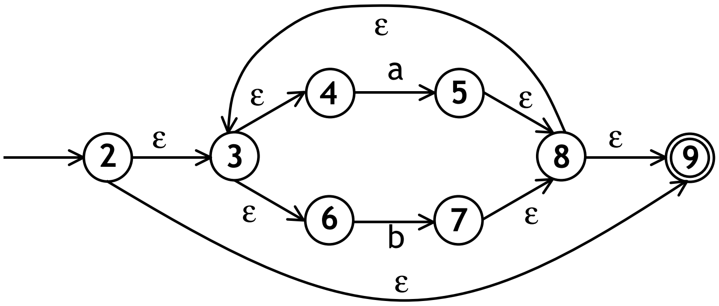
Step 2: The machine to accept ‘b’ is shown below.



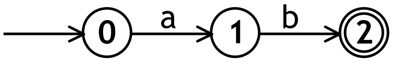
Step 3: The machine to accept (a + b) is shown below.



Step 4: The machine to accept (a+b)* is shown below.



Step 5: The machine to accept ab is shown below.



Step 6: The machine to accept ab(a+b)* is shown below.

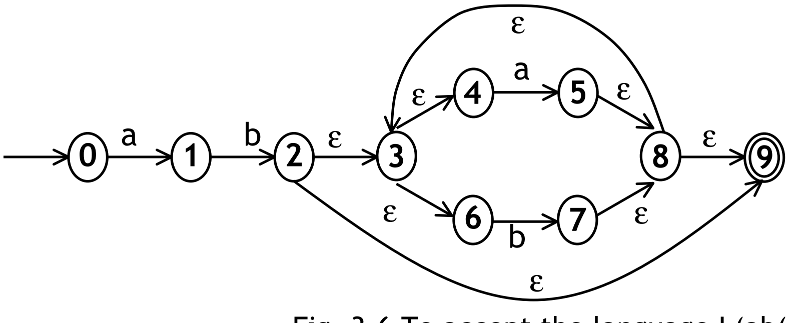


Fig. 3.6 To accept the language L(ab(a+b)*)

Obtain the regular expression from FA

Theorem: Let M = (Q, Σ, δ, q_0, A) be an FA recognizing the language L. Then there exists an equivalent regular expression R for the regular language L such that L = L(R).

The general procedure to obtain a regular expression from FA is shown below. Consider the generalized graph

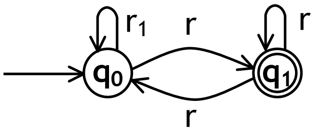


Fig. 3.9 Generalized transition graph

where r_1, r_2, r_3 and r_4 are the regular expressions and correspond to the labels for the edges. The regular expression for this can take the form:

$$r = r_1^* r_2 (r_4 + r_3 r_1^* r_2)^* \qquad (3.1)$$

Note:

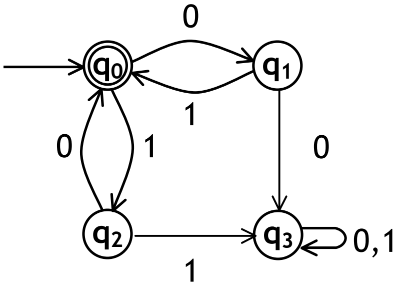
1. Any graph can be reduced to the graph shown in figure 3.9. Then substitute the regular expressions appropriately in the equation 3.1 and obtain the final regular expression.
2. If r_3 is not there in figure 3.9, the regular expression can be of the form

$$r = r_1^* r_2\, r_4^* \qquad (3.2)$$

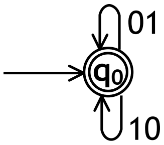
3. If q_0 and q_1 are the final states then the regular expression can be of the form

$$r = r_1{*} + r_1^* r_2\, r_4^* \qquad (3.3)$$

Obtain a regular expression for the FA shown below:



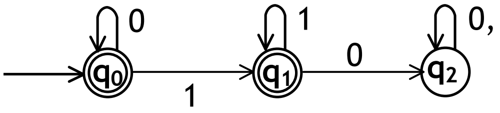
The figure can be reduced as shown below:



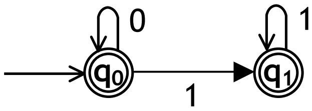
It is clear from this figure that the machine accepts strings of 01's and 10's of any length and the regular expression can be of the form

$$(01 + 10)^*$$

What is the language accepted by the following FA



Since, state q_2 is the dead state, it can be removed and the following FA is obtained.



The state q_0 is the final state and at this point it can accept any number of 0's which can be represented using notation as

$$0^*$$

q_1 is also the final state. So, to reach q_1 one can input any number of 0's followed by 1 and followed by any number of 1's and can be represented as

$$0^*11^*$$

So, the final regular expression is obtained by adding 0^* and 0^*11^*. So, the regular expression is

$$\begin{aligned} R.E &= 0^* + 0^*11^* \\ &= 0^*\ (\in\ + 11^*) \\ &= 0^*\ (\in\ + 1^+) \\ &= 0^*\ (1^*) = 0^*1^* \end{aligned}$$

It is clear from the regular expression that language consists of any number of 0's (possibly ε) followed by any number of 1's(possibly ε).

Applications of Regular Expressions

Pattern Matching refers to a set of objects with some common properties. We can match an identifier or a decimal number or we can search for a string in the text.

An application of regular expression in UNIX editor ed.

In UNIX operating system, we can use the editor *ed* to search for a specific pattern in the text. For example, if the command specified is

```
/acb*c/
```

then the editor searches for a string which starts with *ac* followed by zero or more b's and followed by the symbol *c*. Note that the editor ed accepts the regular expression and searches for that particular pattern in the text. As the input can vary dynamically, it is challenging to write programs for string patters of these kinds.

Properties of regular languages

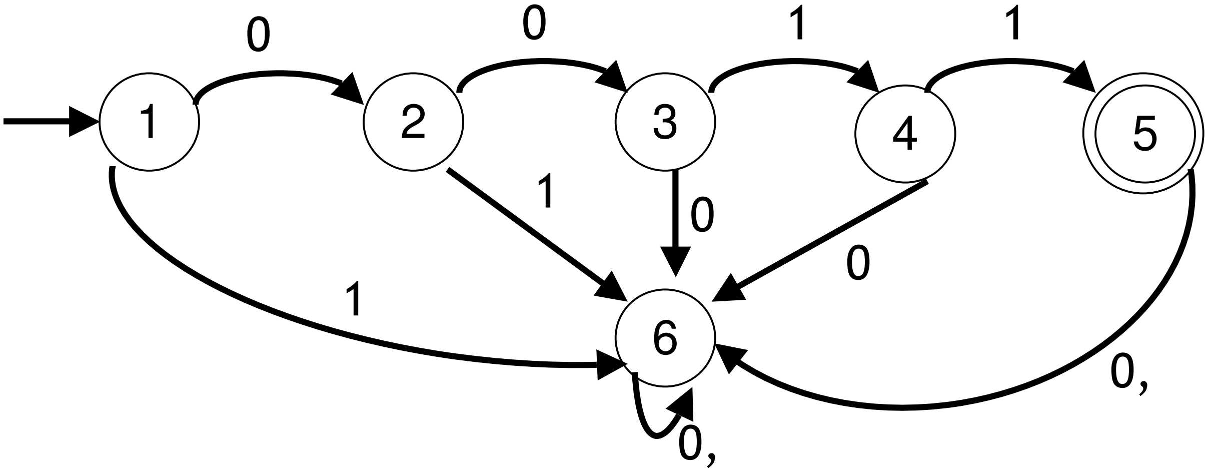
- Pumping Lemma Used to prove certain languages like $L = \{0^n1^n \mid n \geq 1\}$ are not regular.

- Closure properties of regular languages Used to build recognizers for languages that are constructed from other languages by certain operations. Ex. Automata for intersection of two regular languages
- Decision properties of regular languages
- Used to find whether two automata define the same language
- Used to minimize the states of DFA eg. Design of switching circuits.

Pumping Lemma for regular languages

Let $L = \{0^n1^n \mid n \geq 1\}$

There is no regular expression to define L. 00*11* is not the regular expression defining L. Let $L= \{0^21^2\}$



State 6 is a trap state, state 3 remembers that two 0's have come and from there state 5 remembers that two 1's are accepted.

This implies DFA has no memory to remember arbitrary 'n'. In other words if we have to remember n, which varies from 1 to ∞, we have to have infinite states, which is not possible with a finite state machine, which has finite number of states.

Pumping Lemma for Regular Languages

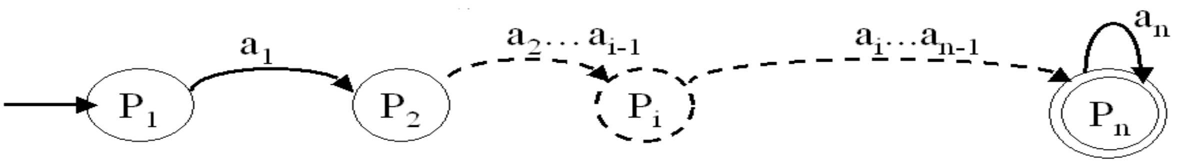
Theorem:

Let L be a regular language. Then there exists a constant ‘n’ (which depends on L) such that for every string w in L such that $|w| \geq n$, we can break w into three strings, w=xyz, such that:

1. $|y| > 0$
2. $|xy| \leq n$
3. For all $k \geq 0$, the string xy^kz is also in L.

PROOF:

Let L be regular defined by an FA having ‘n’ states. Let w= a_1,a_2,a_3----a_n and is in L. $|w| = n \geq n$. Let the start state be P_1. Let w = xyz where x= a_1,a_2,a_3 ----- a_{n-1} , y=a_n and z = ε.



$\delta(P_1,a_1)=P_2$

$\delta(P_2,a_2)=P_3$

$\delta(P_n,a_n)=P_{n+1}$

But there are only n states. => there must be a loop. Let there be a loop in P_n State.

Let $x=a_1,\ldots\ldots a_{n-1}$

$y=a_n$

$z= \varepsilon$

Therefore, $xy^kz = a_1$ ------ a_{n-1} $(a_n)^k$ ε

k=0 a_1 ------ a_{n-1} is accepted

k=1 a_1 ------ a_n is accepted

k=2 a_1 ------ a_{n+1} is accepted

k=10 a_1 ------ a_{n+9} is accepted and so on.

Uses of Pumping Lemma: - This is to be used to show that, certain languages are not regular. It should never be used to show that some language is regular. If you want to show that language is regular, write separate expression, DFA or NFA.

General Method of proof: -

(i) Select w such that $|w| \geq n$

(ii) Select y such that $|y| \geq 1$

(iii) Select x such that $|xy| \leq n$

(iv) Assign remaining string to z

(v) Select k suitably to show that, resulting string is not in L.

Example 1.

To prove that $L=\{w|w \in a^n b^n$, where $n \geq 1\}$ is not regular

Proof:

Let L be regular. Let n is the constant (PL Definition). Consider a word w in L. Let $w = a^n b^n$, such that $|w|=2n$. Since $2n > n$ and L is regular it must satisfy PL.

$$\text{Consider} \qquad w= \underbrace{\overbrace{aa\text{-----}a}^{n}}_{xy}\ \underbrace{\overbrace{bb\text{-----}b}^{n}}_{z}$$

xy contain only a's. (Because $|xy| \leq n$).

Let $|y|=l$, where $l > 0$ (Because $|y| > 0$).

Then, the break up of x. y and z can be as follows

$$w = \overbrace{a^{n-l}}^{x}\ \overbrace{a^{l}}^{y}\ \overbrace{b^{n}}^{z}$$

from the definition of PL , $w=xy^kz$, where k=0,1,2,------∞, should belong to L.

That is $a^{n-l}\ (a^l)^k\ b^n \in L$, for all k=0,1,2,------ ∞

Put k=0. we get $a^{n-l}\ b^n \notin L$.

Contradiction. Hence the Language is not regular.

Example 2.

To prove that L={w|w is a palindrome on {a,b}*} is not regular. i.e., L={aabaa, aba, abbbba,…}

Proof:

Let L be regular. Let n is the constant (PL Definition). Consider a word w in L. Let w = $a^nba^{n,}$ such that $|w|=2n+1$. Since $2n+1 > n$ and L is regular it must satisfy PL.

Consider $$w= \underbrace{\overbrace{aa\text{-----}a}^{n}}_{xy}\ \underbrace{b\ \overbrace{aa\text{-----}a}^{n}}_{z}$$

xy contain only a's. (Because **$|xy| \leq n$).**

Let $|y|=l$, where $l > 0$ (Because **$|y| > 0$).**

That is, the break up of x. y and z can be as follows

$$w = \overbrace{a^{n-1}}^{x}\ \ \overbrace{a^{l}}^{y}\ \ \overbrace{ba^{n}}^{z}$$

from **the definition** of PL $w=xy^kz$, where $k=0,1,2,\text{------}\infty$, should belong to L.

That is $a^{n-l}\ (a^l)^k\ ba^n \in L$, for all $k=0,1,2,\text{------}\ \infty$.

Put k=0. we get a^{n-l} b $a^n \notin$ L, because, it is not a palindrome. Contradiction, hence the language is not regular

.

Example 3.

To prove that L={ all strings of 1's whose length is prime} is not regular. i.e., L={1^2, 1^3 ,1^5 ,1^7 ,1^{11} ,----}

Proof: Let L be regular. Let w = 1^p where p is prime and | p| = n +2

Let y = m.

by PL $xy^kz \in L$

$| xy^kz |$ = $| xz | + | y^k |$ Let k = p-m

= (p-m) + m (p-m)

= (p-m) (1+m) -----this can not be prime if $p-m \geq 2$ or $1+m \geq 2$

1. $(1+m) \geq 2$ because $m \geq 1$

2. Limiting case p=n+2

   $(p-m) \geq 2$ since $m \leq n$

Example 4. To prove that L={ 0^{i^2} | i is integer and i >0} is not regular. i.e., L={0^2, 0^4 ,0^9 ,0^{16} ,0^{25} ,----}

Proof: Let L be regular. Let $w = 0^{n^2}$ where $|w| = n^2 \geq n$

by PL $xy^kz \in L$, for all k = 0,1,---

Select k = 2

$| xy^2z | = | xyz | + | y |$

$= n^2 +$ Min 1 and Max n

Therefore $n^2 < | xy^2z | \leq n^2 + n$

$n^2 < | xy^2z | < n^2 + n + 1+n$ adding 1 + n (Note that less than or equal to is $n^2 < | xy^2z | < (n + 1)^2$ replaced by less than sign)

Say n = 5 this implies that string can have length > 25 and < 36 which is not of the form 0^{i^2}.

Exercises for Readers: -

a) Show that following languages are not regular

- $L=\{a^nb^m \mid n, m \geq 0 \text{ and } n<m \}$
- $L=\{a^nb^m \mid n, m \geq 0 \text{ and } n>m \}$
- $L=\{a^nb^mc^md^n \mid n, m \geq 1 \}$
- $L=\{a^n \mid n \text{ is a perfect square} \}$
- $L=\{a^n \mid n \text{ is a perfect cube} \}$

b) Apply pumping lemma to following languages and understand why we cannot complete proof

- $L=\{a^naba \mid n \geq 0 \}$
- $L=\{a^nb^m \mid n, m \geq 0 \}$

Closure Properties of Regular Languages

1. The union of two regular languages is regular.
2. The intersection of two regular languages is regular.
3. The complement of a regular language is regular.
4. The difference of two regular languages is regular.
5. The reversal of a regular language is regular.
6. The closure (star) of a regular language is regular.
7. The concatenation of regular languages is regular.
8. A homomorphism (substitution of strings for symbols) of a regular language is regular.
9. The inverse homomorphism of a regular language is regular

Closure under Union

Theorem: If L and M are regular languages, then so is L $\cup$M.

Example 1.

L1={a,a^3,a^5,-----}

L2={a^2,a^4,a^6,-----}

L1$\cup$L2 = {a,a^2,a^3,a^4,----}

RE=a$(a)^*$

Example 2.

L1={ab, a^2 b^2, a^3b^3, a^4b^4,-----}

L2={ab,a^3 b^3,a^5b^5,-----}

L1$\cup$L2 = {ab,a^2b^2, a^3b^3, a^4b^4, a^5b^5----}

RE=ab$(ab)^*$

Closure Under Complementation

Theorem : If L is a regular language over alphabet S, then L = Σ^* - L is also a regular language.

Example 1.

$L1=\{a,a^3,a^5,-----\}$

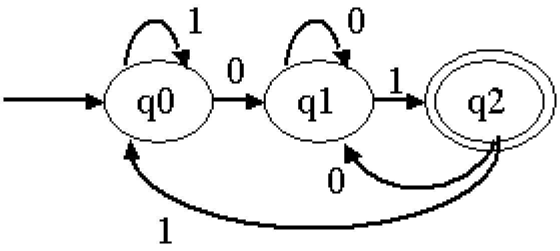
$\Sigma^* -L1 = \{e,a^2,a^4,a^6,-----\}$

RE=(aa)*

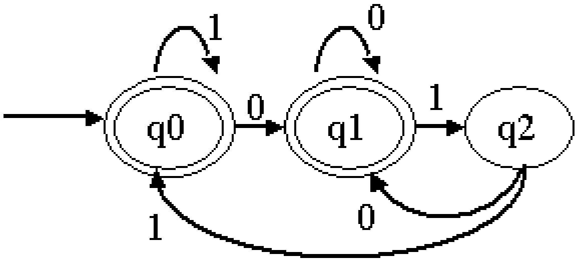
Example 2.

Consider a DFA, A that accepts all and only the strings of 0’s and 1’s that end in 01. That is $L(A) = (0+1)^*01$. The complement of L(A) is therefore all string of 0’s and 1’s that do not end in 01

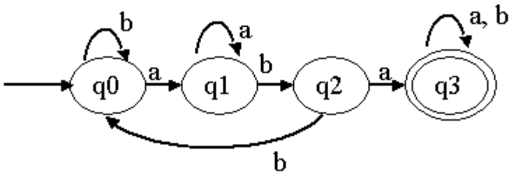
$L(A)=(0+1)^*01$



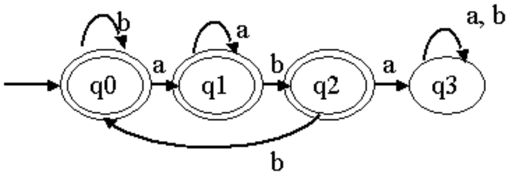
$\overline{L(A)}=\{0,1\}^* - L(A)$



$L(A)=(a+b)^*aba\ (a+b)^*$



$\overline{L(A)}=\{a,b,c\}^* - L(A)$



Theorem: - If L is a regular language over alphabet Σ, then, $\overline{L} = \Sigma^* - L$ is also a regular language.

Proof: - Let L =L(A) for some DFA. A=(Q, **Σ, δ,** q_0, F). Then L = L(B), where B is the DFA (Q, **Σ, δ,** q_0, Q-F). That is, B is exactly like A, but the accepting states of A have become non-accepting states of B, and vice versa, then w is in L(B) if and only if δ^ **(** $q_{0,}$ w) is in Q-F, which occurs if and only if w is not in L(A).

Closure Under Intersection

Theorem : If L and M are regular languages, then so is L ∩ M.

Example 1.

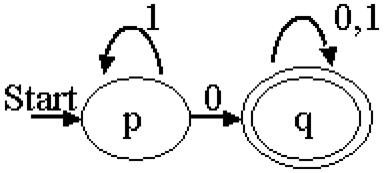
$L1=\{a,a^2,a^3,a^4,a^5,a^6,-----\}$
$L2=\{a^2,a^4,a^6,-----\}$
$L1L2 = \{a^2,a^4,a^6,----\}$
$RE=aa(aa)^*$

Example 2.

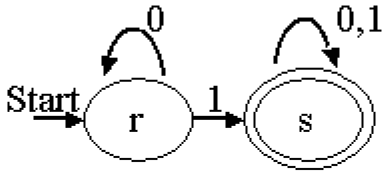
$L1=\{ab,a^3b^3,a^5b^5,a^7b^7-----\}$
$L2=\{a^2\ b^2,\ a^4b^4,\ a^6b^6,-----\}$
$L1\cap L2 = \phi$
$RE= \phi$

Example 3.

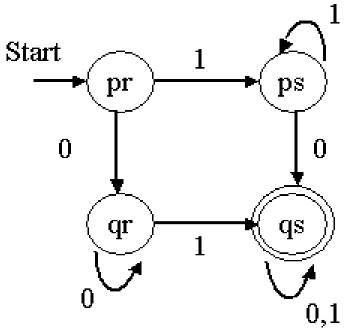
Consider a DFA that accepts all those strings that have a 0.



Consider a DFA that accepts all those strings that have a 1.



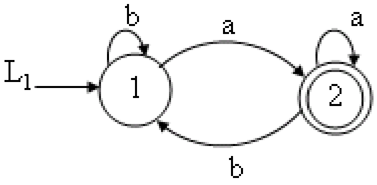
The product of above two automata is given below.

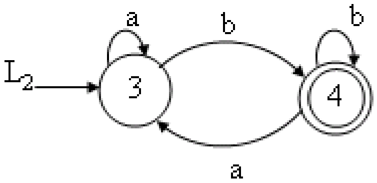


This automaton accepts the intersection of the first two languages: Those languages that have both a 0 and a 1. Then pr represents only the initial condition, in which we have seen neither 0 nor 1. Then state qr means that we have seen only once 0's, while state ps represents the condition that we have seen only 1's. The accepting state qs represents the condition where we have seen both 0's and 1's.

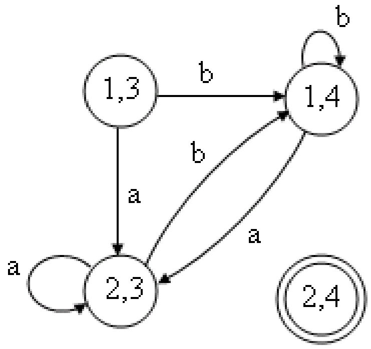
Example 4

Write a DFA to accept the intersection of L1=(a+b)*a and L2=(a+b)*b that is for L1 ∩ L2.



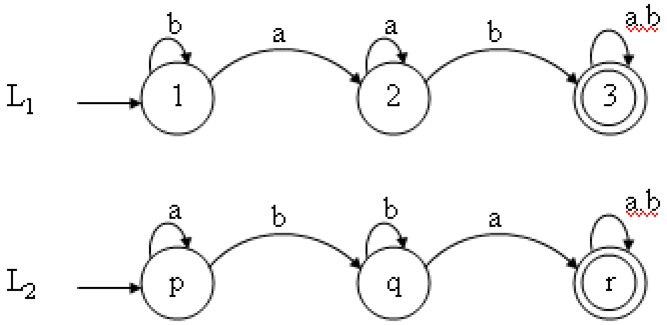


DFA for L1 ∩ L2 = ϕ (as no string has reached to final state (2,4))

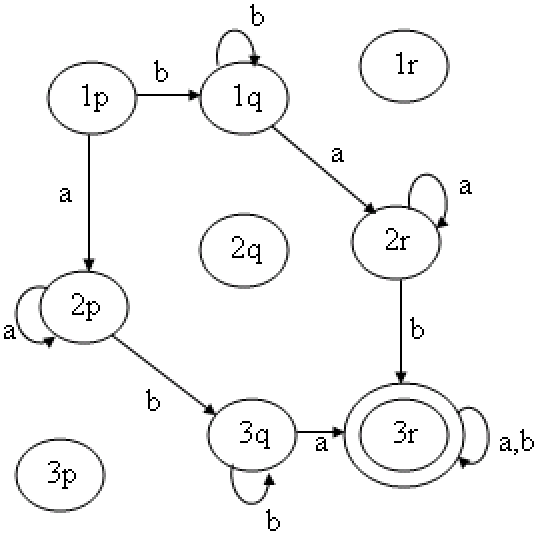


Example 5

Find the DFA to accept the intersection of L1=(a+b)*ab (a+b)* and L2=(a+b)*ba (a+b)* that is for L1 ∩ L2



DFA for L1 ∩ L2



Closure Under Difference

Theorem: If L and M are regular languages, then so is L – M.

Example.

$L1=\{a,a^3,a^5,a^7,-----\}$

$L2=\{a^2,a^4,a^6,-----\}$

$L1-L2 = \{a,a^3,a^5,a^7----\}$

$RE=a(a)^*$

Reversal

Theorem : If L is a regular language, so is L^R

Ex.

$L=\{001,10,111,01\}$

$L^R=\{100,01,111,10\}$

To prove that regular languages are closed under reversal.

Let L = {001, 10, 111}, be a language over $\Sigma=\{0,1\}$. L^R is a language consisting of the reversals of the strings of L. That is $L^R = \{100,01,111\}$.

If L is regular we can show that L^R is also regular.

Proof.

As L is regular it can be defined by an FA, $M = (Q, \Sigma , \delta, q_0, F)$, having only one final state. If there are more than one final states, we can use ∈- transitions from the final states going to a common final state.

Let FA, $M^R = (Q^R, \Sigma^R , \delta^R, q_0^R, F^R)$ defines the language L^R,

Where $Q^R = Q$, $\Sigma^R = \Sigma$, $q_0^R=F$, $F^R=q_0$, and $\delta^R (p,a)\text{->} q$, iff $\delta (q,a) \text{->} p$

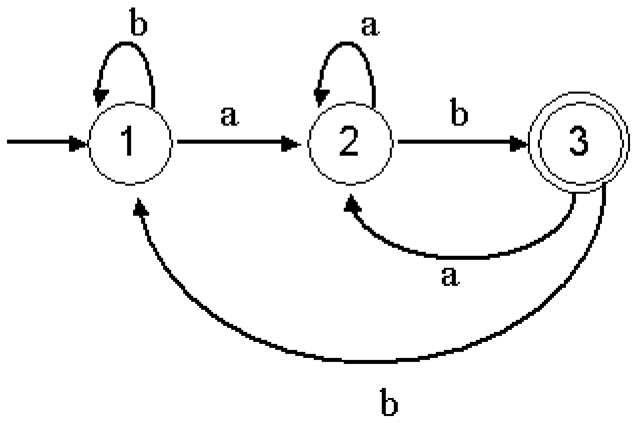
Since M^R is derivable from M, L^R is also regular.

The proof implies the following method

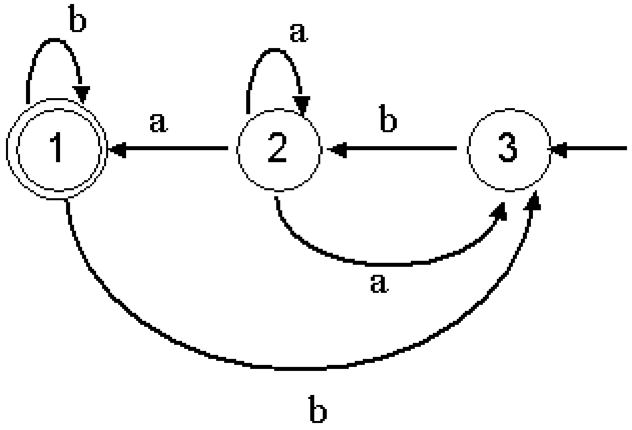
1. Reverse all the transitions.
2. Swap initial and final states.
3. Create a new start state p0 with transition on ∈ to all the accepting states of original DFA

Example

Let r=(a+b)* ab define a language L. That is L = {ab, aab, bab,aaab, -----}. The FA is as given below



The FA for L^R can be derived from FA for L by swapping initial and final states and changing the direction of each edge. It is shown in the following figure.



Homomorphism

A string homomorphism is a function on strings that works by substituting a particular string for each symbol.

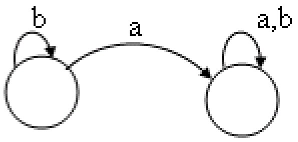
Theorem: If L is a regular language over alphabet Σ, and h is a homomorphism on Σ, then h (L) is also regular.

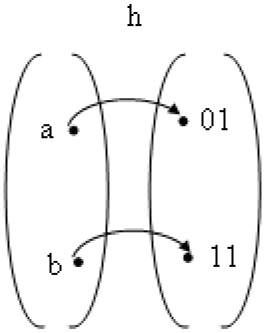
Example.

The function h defined by h(0)=ab h(1)=c is a homomorphism.

h applied to the string 00110 is ababccab

L1= (a+b)* a (a+b)*





$h : \{a, b\} \longrightarrow \{0, 1\}^*$

$h : \{a, b\} \rightarrow \{0, 1\}^*$

Ex.

$h_1(a) = 01$

$h_1(b) = 11$

Ex.

$h_2(a) = 101$

$h_2(b) = 010$

Ex.

$h_2(a) = 01$

$h_3(a) = 101$

Resulting :

h1(L) = (01 + 11)* 01 (01 + 11)*

h2(L) = (101 + 010)* 101 (101 + 010)*

h3(L) = (01 + 101)* 01 (01 + 101)*

Inverse Homomorphism

Theorem: If h is a homomorphism from alphabet S to alphabet T, and L is a regular language over T, then h$^{-1\ (}$L) is also a regular language.

Example. Let L be the language of regular expression (00+1)*. Let h be the homomorphism defined by h(a)=01 and h(b)=10. Then h^{-1}(L) is the language of regular expression (ba)*.

Decision Properties of Regular Languages

1. Is the language described empty?

2. Is a particular string w in the described language?

3. Do two descriptions of a language actually describe the same language?

This question is often called "equivalence" of languages.

Converting Among Representations

Converting NFA's to DFA's

Time taken for either an NFA or -NFA to DFA can be exponential in the number of states of the NFA. Computing ε-Closure of n states takes O(n^3) time. Computation of DFA takes O(n^3) time where number of states of DFA can be 2^n. The running time of NFA to DFA conversion including ε transition is O(n^3 2^n). Therefore the bound on the running time is O(n^3s) where s is the number of states the DFA actually has.

DFA to NFA Conversion

Conversion takes O(n) time for an n state DFA.

Automaton to Regular Expression Conversion

For DFA where n is the number of states, conversion takes O($n^3 4^n$) by substitution method and by state elimination method conversion takes O(n^3) time. If we convert an NFA to DFA and then convert the DFA to a regular expression it takes the time O($n^3 4^{n^3} 2^n$)

Regular Expression to Automaton Conversion

Regular expression to ε-NFA takes linear time – O(n) on a regular expression of length n. Conversion from ε-NFA to NFA takes O(n^3) time.

Testing Emptiness of Regular Languages

Suppose R is regular expression, then

1. R=R1 + R2. Then L(R) is empty if and only if both L(R1) and L(R2) are empty.

2. R= R1R2. Then L(R) is empty if and only if either L(R1) or L(R2) is empty.

3. R=R1* Then L(R) is not empty. It always includes at least ε

4. R=(R1) Then L(R) is empty if and only if L(R1) is empty since they are the same language.

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3. R=(R1)* Then L(R) is not empty. It always includes at least ε
4. R=(R1) Then L(R) is empty if and only if L(R1) is empty since they are the same language.

Testing Membership in a Regular Language

Given a string w and a Regular Language L, is w in L.

If L is represented by a DFA, simulate the DFA processing the string of input symbol w, beginning in start state. If DFA ends in accepting state the answer is 'Yes', else it is 'no'. This test takes O(n) time

If the representation is NFA, if w is of length n, NFA has s states, running time of this algorithm is $O(ns^2)$

If the representation is ε - NFA, ε - closure has to be computed, then processing of each input symbol, a, has 2 stages, each of which requires $O(s^2)$ time.

If the representation of L is a Regular Expression of size s, we can convert to an ε -NFA with almost 2s states, in O(s) time. Simulation of the above takes $O(ns^2)$ time on an input w of length n

Minimization of Automata (Method 1)

Let p and q are two states in DFA. Our goal is to understand when p and q ($p \neq q$) can be replaced by a single state.

Two states p and q are said to be distinguishable, if there is at least one string, w, such that one of $\hat{\delta}$ (p,w) and $\hat{\delta}$ (q,w) is accepting and the other is not accepting.

Algorithm 1:

List all unordered pair of states (p,q) for which $p \neq q$. Make a sequence of passes through these pairs. On first pass, mark each pair of which exactly one element is in F. On each subsequent pass, mark any pair (r,s) if there is an $a \in \Sigma$ for which δ (r,a) = p, δ (s,a) = q, and (p,q) is already marked. After a pass in which no new pairs are marked, stop. The marked pair (p,q) are distinguishable.

Examples:

1. Let L = $\{\in, a^2, a^4, a^6, \ldots\}$ be a regular language over $\Sigma = \{a,b\}$. The FA is shown in Fig 1.

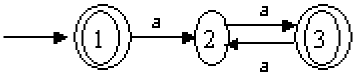


Fig 1.

Fig 2. gives the list of all unordered pairs of states (p,q) with p ≠ q.

2	1	
3		1
	1	2

Fig 2.

The boxes (1,2) and (2,3) are marked in the first pass according to the algorithm 1. In pass 2 no boxes are marked because, $\delta(1,a) \rightarrow \phi$ and $\delta\ (3,a) \rightarrow 2$. That is $(1,3) \xrightarrow{a} (\phi,2)$, where ϕ and 3 are non final states.

$\delta(1,b) \rightarrow \phi$ and $\delta\ (3,b) \rightarrow \phi$. That is $(1,3) \xrightarrow{b} (\phi,\phi)$, where ϕ is a non-final state. This implies that (1,3) are equivalent and can replaced by a single state A.

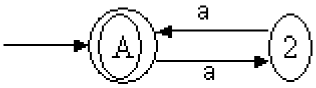
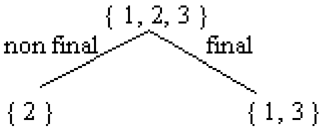


Fig 3. Minimal Automata corresponding to FA in Fig 1

Minimization of Automata (Method 2)



Consider set {1,3}. $(1,3) \xrightarrow{a} (2,2)$ and $(1,3) \xrightarrow{b} (\phi,\phi)$. This implies state 1 and 3 are equivalent and cannot be divided further. This gives us two states 2,A. The resultant FA is shown is Fig 3.

Example 2. (Method1):

Let r= (0+1)*10, then L(r) = {10,010,00010,110, ---}. The FA is given below

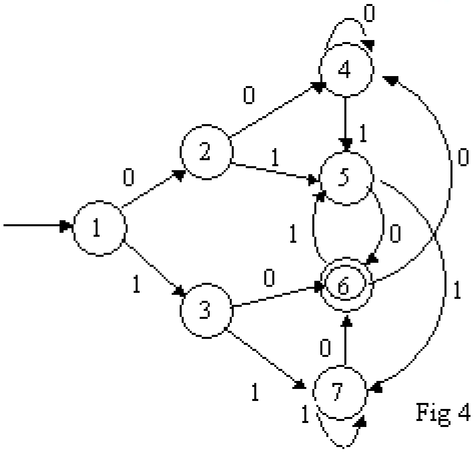


Fig 4

Following fig shows all unordered pairs (p,q) with p ≠ q

2						
3	2	2				
4			2			
5	2	2		2		
6	1	1	1	1	1	
7	2	2		2		1
	1	2	3	4	5	6

In pass 2,

$(3,1) \xrightarrow{0} (6,2)$

$(3,2) \xrightarrow{0} (6,4)$

$(5,1) \xrightarrow{0} (6,2)$

$(5,2) \xrightarrow{0} (6,4)$

$(7,4) \xrightarrow{0} (6,4)$

$(7,2) \xrightarrow{0} (6,4)$

and so on.

The pairs marked 1 are those of which exactly one element is in F; They are marked on pass 1. The pairs marked 2 are those marked on the second pass. For example (5,2) is one of these, since (5,2) → (6,4), and the pair (6,4) was marked on pass 1.

From this we can make out that 1, 2, and 4 can be replaced by a single state 124 and states 3, 5, and 7 can be replaced by the single state 357. The resultant minimal FA is shown in Fig. 6

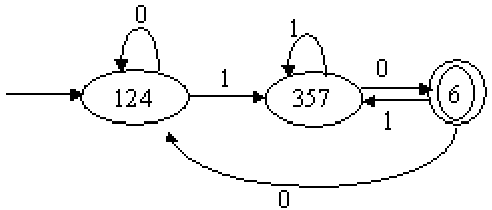


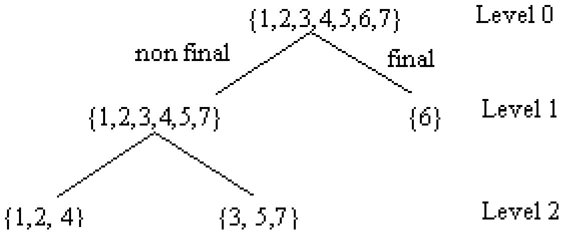
Fig 6

The transitions of fig 4 are mapped to fig 6 as shown below

Original DFA	Minimal DFA
$1 \xrightarrow{0} 2$	$124 \xrightarrow{0} 124$
$2 \xrightarrow{1} 5$	$124 \xrightarrow{1} 357$
$3 \xrightarrow{0} 6$	$357 \xrightarrow{0} 6$

and so on.

Example 2. (Method1):



(2,3) $\xrightarrow{0}$ (4,6) this implies that 2 and 3 belongs to different group hence they are split in level 2. Similarly it can be easily shown for the pairs (4,5) (1,7) and (2,5) and so on.

Context Free Grammar

Context Free grammar or CGF, G is represented by four components that is G= (V, T, P, S), where V is the set of variables, T the terminals, P the set of productions and S the start symbol.

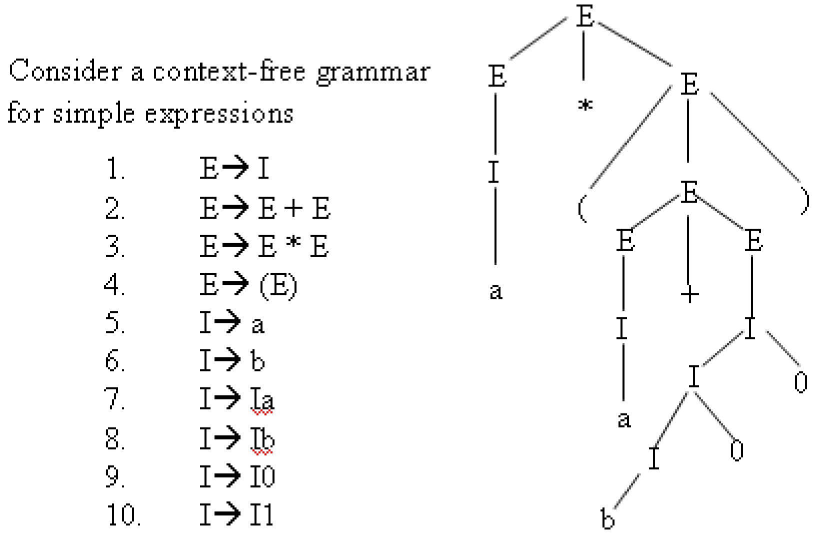
Example: The grammar G_{pal} for palindromes is represented by

G_{pal} = ({P}, {0, 1}, A, P)

Where A represents the set of five productions

1. P→∈
2. P→0
3. P→1
4. P→0P0
5. P→1P1

Derivation using Grammar



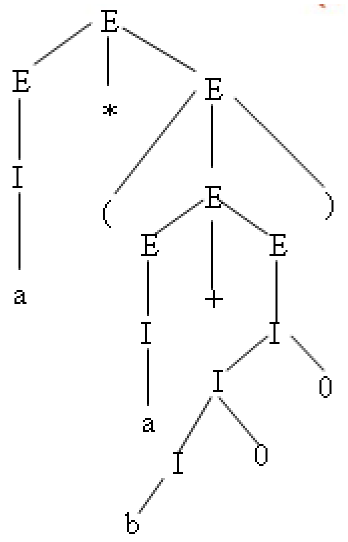
Consider a context-free grammar for simple expressions

1. E→ I
2. E→ E + E
3. E→ E * E
4. E→ (E)
5. I→ a
6. I→ b
7. I→ Ia
8. I→ Ib
9. I→ I0
10. I→ I1

Example 1: Leftmost Derivation
The inference that a * (a+b00) is in the language of variable E can be reflected in a derivation of that string, starting with the string E. Here is one such derivation:

E →E * E → I * E → a * E →
a * (E) → a * (E + E) → a * (I + E) → a * (a + E) →
a * (a + I) → a * (a + I0) → a * (a + I00) → a * (a + b00)

Leftmost Derivation - Tree



Example 2: Rightmost Derivations

The derivation of Example 1 was actually a leftmost derivation. Thus, we can describe the same derivation by:

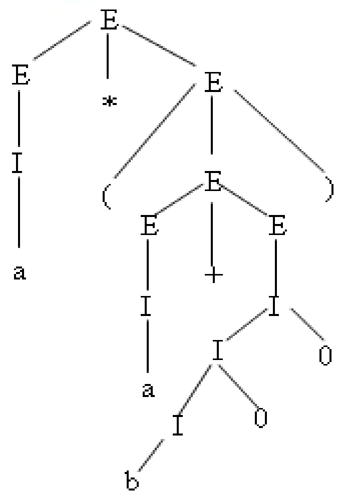
E➔ E * E ➔ E *(E) ➔ E * (E + E) ➔

E * (E + I) ➔ E * (E +I0) ➔ E * (E + I00) ➔ E * (E + b00) ➔

E * (I + b00) ➔ E * (a +b00) ➔ I * (a + b00) ➔ a * (a + b00)

We can also summarize the leftmost derivation by saying E ➔ a * (a + b00), or express several steps of the derivation by expressions such as E * E ➔ a * (E).

Rightmost Derivation - Tree



There is a rightmost derivation that uses the same replacements for each variable, although it makes the replacements in different order. This rightmost derivation is:

E ➔ E * E ➔ E * (E) ➔ E * (E + E) ➔

E * (E + I) ➔ E * (E + I0) ➔ E * (E + I00) ➔ E * (E + b00) ➔

E * (I + b00) ➔ E * (a + b00) ➔ I * (a + b00) ➔ a * (a + b00)

This derivation allows us to conclude E ➔ a * (a + b00)

Consider the Grammar for string(a+b)*c

E→E + T | T

T→ T * F | F

F→ (E) | a | b | c

Leftmost Derivation

E→T➔T*F➔F*F➔(E)*F➔(E+T)*F➔(T+T)*F➔(F+T)*F ➔(a+T)*F ➔(a+F)*F ➔(a+b)*F➔(a+b)*c

Rightmost derivation

E→T➔T*F➔T*c➔F*c➔(E)*c➔(E+T)*c➔(E+F)*c ➔(E+b)*c➔(T+b)*c➔(F+b)*c➔(a+b)*c

Example 2:

Consider the Grammar for string (a,a)

S->(L)|a

L->L,S|S

Leftmost derivation

S→(L)→(L,S)→(S,S)→(a,S)→(a,a)

Rightmost Derivation

S→(L)→(L,S)→(L,a)→(S,a)→(a,a)

The Language of a Grammar

If G = (V,T,P,S) is a CFG, the language of G, denoted by L(G), is the set of terminal strings that have derivations from the start symbol. L(G) = {w in T | S ➔ w}

Sentential Forms

Derivations from the start symbol produce strings that have a special role called "sentential forms". That is if G = (V, T, P, S) is a CFG, then any string in (V ∪ T)* such that S ➔α is a sentential form. If S ➔α, then is a left - sentential form, and if S ➔α, then is a right - sentential form. Note that the language L(G) is those sentential
forms that are in T*; that is they consist solely of terminals.

For example, E * (I + E) is a sentential form, since there is a derivation

E ➔ E * E ➔ E * (E) ➔ E * (E + E) ➔ E * (I + E)

However this derivation is neither leftmost nor rightmost, since at the last step, the middle E is replaced.

As an example of a left – sentential form, consider a * E, with the leftmost derivation.

E → E * E → I * E → a * E

Additionally, the derivation

E → E * E → E * (E) → E * (E + E)

Shows that

E * (E + E) is a right – sentential form.

Ambiguity

A context – free grammar G is said to be ambiguous if there exists some w ∈L(G) which has at least two distinct derivation trees. Alternatively, ambiguity implies the existence of two or more left most or rightmost derivations.

Example

Consider the grammar G=(V, T, E, P) with V={E, I}, T={a, b, c, +, *, (,)}, and productions.

E→I,

E→E+E,

E→E*E,

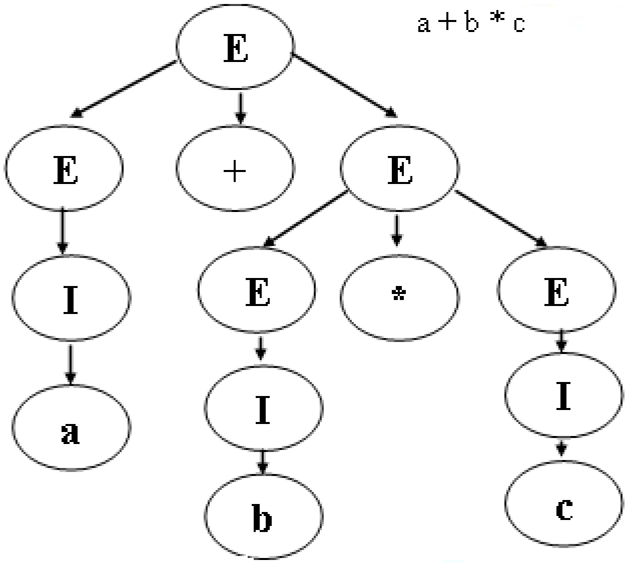
E→(E),

I→a|b|c

Consider two derivation trees for a + b * c.

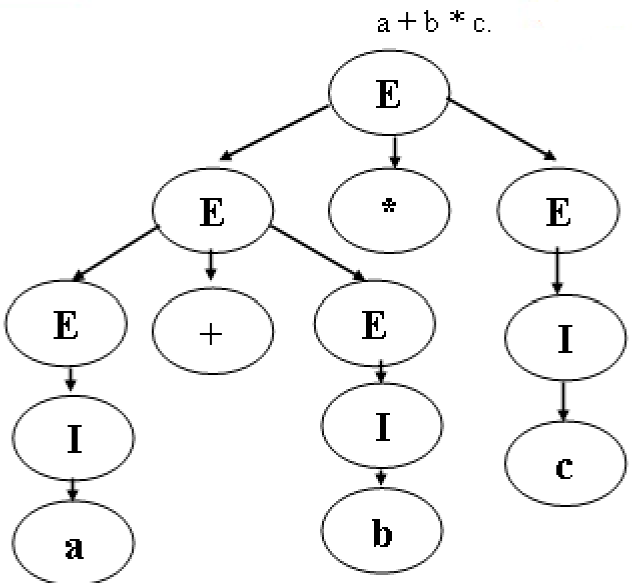
Tree I

Let a=5, b=6, c=7
The value for Tree I
will be 47



Tree II

Let a=5, b=6, c=7
The value for Tree II
will be 77



Now unambiguous grammar for the above

Example:

E→T, T→F, F→I, E→E+T, T→T*F,

F→(E), I→a|b|c

Inherent Ambiguity

A CFL L is said to be inherently ambiguous if all its grammars are ambiguous. **Example:** Condider the Grammar for string aabbccdd

S→AB | C

A→ aAb | ab

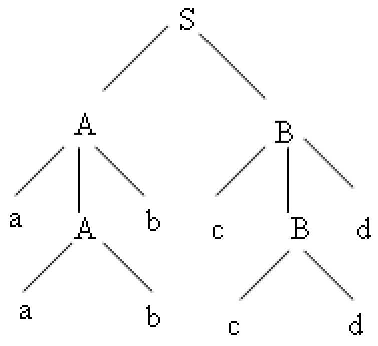
B→cBd | cd

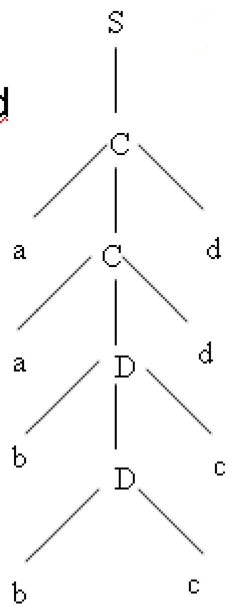
C→ aCd | aDd

D->bDc | bc

Parse tree for string aabbccdd

Parse tree for string aabbccdd





Applications of Context - Free Grammars

- Parsers
- The YACC Parser Generator
- Markup Languages
- XML and Document typr definitions

The YACC Parser Generator

```
E→ E+E | E*E | (E)|id
%{ #include <stdio.h>
%}
%token ID id
%%
Exp   :  id  { $$ = $1 ; printf ("result is %d\n", $1);}
                | Exp '+' Exp    {$$ = $1 + $3;}
                | Exp '*' Exp    {$$ = $1 * $3; }
                |  '(' Exp ')'      {$$ = $2; }
                ;
%%

int main (void) {
return yyparse ( );
}
void yyerror (char *s) {
fprintf (stderr, "%s\n", s);
}
%{
```

```
#include "y.tab.h"
%}
%%
[0-9]+          {yylval.ID = atoi(yytext); return id;}
[ \t \n]         ;
[+ * ( )]                 {return yytext[0];}
.                         {ECHO; yyerror ("unexpected character");}
%%
```

Example 2:

```
%{
 #include <stdio.h>
%}
%start line
%token <a_number> number
%type <a_number> exp term factor
%%
line : exp ';' {printf ("result is %d\n", $1);}
;
exp : term {$$ = $1;}
         | exp '+' term {$$ = $1 + $3;}
         | exp '-' term {$$ = $1 - $3;}
term : factor {$$ = $1;}
         | term '*' factor {$$ = $1 * $3;}
         | term '/' factor {$$ = $1 / $3;}
 ;
factor : number {$$ = $1;}
| '(' exp ')' {$$ = $2;}
;
 %%
 int main (void) {
return yyparse ( );
}
void yyerror (char *s) {
fprintf (stderr, "%s\n", s);
}
%{
#include "y.tab.h"
%}
%%
[0-9]+  {yylval.a_number = atoi(yytext); return number;}
[ \t\n] ;
[-+*/();]  {return yytext[0];}
.                   {ECHO; yyerror ("unexpected character");}
%%
```

Markup Languages:

Functions

- Creating links between documents
- Describing the format of the document

Example

The Things I *hate*

1. Moldy bread
2. People who drive too slow
   In the fast lane

HTML Source

```
<P> The things I <EM>hate</EM>:
<OL>
<LI> Moldy bread
<LI>People who drive too slow
In the fast lane
</OL>
```

HTML Grammar

✓	Char	a | A | …	
✓	Text	e | Char Text	
✓	Doc	e | Element Doc	
✓	Element	Text | <EM> Doc </EM>| <p> Doc |<OL> List </OL>| …	
✓	List-Item	<LI> Doc	
✓	List	e | List-Item List	Start symbol

XML and Document type definitions.

1. A→E1,E2.
   A→BC
   B→E1
   C→E2
2. A→E1 | E2.
   A→E1
   A→E2
3. A→(E1)*
   A→BA

A→ε
B→E1

4. A→(E1)+

A→BA
A→B
B→E1

5. A→(E1)?

A→ε
A→E1

EXERCISE QUESTIONS

1) Design context-free grammar for the following cases
   a) L={ 0n1n | n≥l }
   b) L={aibjck| i≠j or j≠k}

2) The following grammar generates the language of RE
   0*1(0+1)*
   S → A|B
   A → 0A|ε
   B → 0B|1B|ε
   Give leftmost and rightmost derivations of the following strings
   a) 00101 b) 1001 c) 00011

3) Consider the grammar
   S → aS|aSbS|ε
   Show that deviation for the string aab is ambiguous

3) Suppose h is the homomorphism from the alphabet {0,1,2} to the alphabet { a,b} defined by h(0) = a; h(1) = ab & h(2) = ba
   a) What is h(0120) ?
   b) What is h(21120) ?
   c) If L is the language L(01*2), what is h(L) ?
   d) If L is the language L(0+12), what is h(L) ?
   e) If L is the language L(a(ba)*) , what is h-1(L) ?

Simplification of CFG

The goal is to take an arbitrary Context Free Grammar G = (V, T, P, S) and perform transformations on the grammar that preserve the language generated by the grammar but reach a specific format for the productions. A CFG can be simplified by eliminating

1. Useless symbols

Those variables or terminals that do not appear in any derivation of a terminal string starting from Start variable, S.

2. ε - Productions

   $A \rightarrow \varepsilon$, where A is a variable

3. Unit production

   $A \rightarrow B$, A and B are variables

1. Eliminate useless symbols:

Definition: Symbol X is useful for a grammar G = (V, T, P, S) if there is $S \overset{*}{\Rightarrow} \alpha X\beta \overset{*}{\Rightarrow} w$, $w \in \Gamma^*$. If X is not useful, then it is useless.

Omitting useless symbols from a grammar does not change the language generated

Example:

$S \rightarrow aSb \mid \varepsilon \mid A$, $A \rightarrow aA$. Here A is a useless symbol
$S \rightarrow A$, $A \rightarrow aA \mid \varepsilon$, $B \rightarrow bB$. Here B is a useless symbol

Symbol X is useful if both the conditions given below are satisfied in that order itself.

1. X is generating variable. ie if $X \overset{*}{\Rightarrow} w$, where $w \in \Gamma^*$
2. X is reachable. ie if $S \overset{*}{\Rightarrow} \alpha X\beta$

Theorem:

Let G = (V, T, P, S) be a CFG and assume that $L(G) \neq \phi$, then $G_1=(V_1, T_1, P_1, S)$ be a grammar without useless symbols by

1. Eliminating non generating symbols
2. Eliminating symbols that are non reachable

Elimination has to be performed only in the order of 1 followed by 2. Otherwise the grammar produced will not be completely useless symbols eliminated.

Example: Consider a grammar with productions: $S \rightarrow AB|a$, $A \rightarrow a$. If the order of eliminations is 1 followed by 2 gives $S \rightarrow a$. If the order of eliminations is 2 followed by 1 gives S → a, A →a. Here A is still useless symbol, so completely useless symbols are not eliminated.

Eliminate non-generating symbols:

Generating symbols follow to one of the categories below:

1. Every symbol of T is generating
2. If A → α and α is already generating, then A is generating

Non-generating symbols = V- generating symbols.

Example:

G= ({S, A, B}, {a}, S → AB | a, A →a}, S)

Solution:

B is a non-generating symbol. After eliminating B, we get

G1= ({S, A}, {a}, {S → a, A →a}, S)

Example:

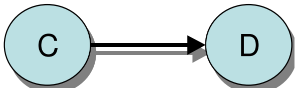
Eliminate the non-generating symbols from S → aS | A | C, A →a, B → aa, C → aCb

Solution:

C is a non-generating symbol. After eliminating C gets, S → aS | A, A →a, B → aa

Eliminate symbols that are non reachable:

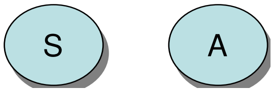
Dependency Graph: For the production C → xDy, the dependency graph is given below.



Draw dependency graph for all productions. If there is no edge reaching a variable X from Start symbol S, then X is non reachable.

Example: Eliminate non-reachable symbols from G= ({S, A}, {a}, {S → a, A →a}, S)

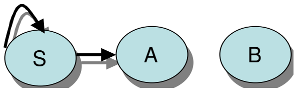
Solution:



Draw the dependency graph as given above. A is non-reachable from S. After eliminating A, G1= ({S}, {a}, {S → a}, S)

Example:

Eliminate non-reachable symbols from S → aS | A, A → a, B → aa



Draw the dependency graph as given above. B is non-reachable from S. After eliminating B, we get the grammar with productions S → aS | A, A → a

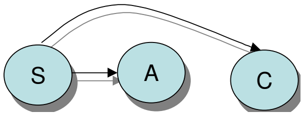
Example:

Eliminate useless symbols from the grammar with productions S → AB | CA, B →BC | AB, A →a, C → AB | b

Step 1: Eliminate non-generating symbols

V_1 = {A, C, S}
P_1 = {S → CA, A →a, C → b}

Step 2: Eliminate symbols that are non reachable



Draw the dependency graph as given above. All Variables are reachable. So the final variables and productions are same V_1 and P_1.

V_2 = {A, C, S}
P_2 = {S → CA, A →a, C → b}

Exercises:

Eliminate useless symbols from the grammar

1. P= {S → aAa, A →Sb | bCC, C →abb, E → aC}
2. P= {S → aBa | BC, A → aC | BCC, C →a, B → bcc, D → E, E →d}
3. P= {S → aAa, A → bBB, B → ab, C → aB}
4. P= {S → aS | AB, A → bA, B → AA}

III. Eliminate ε - Productions:

Most theorems and methods about grammars G assume L(G) does not contain ε. So if ε is not there in L(G), then we have to find out an equivalent G without ε-productions. Example for a grammar G with ε - productions is

S → ABA, A → aA | ε, B → bB | ε

The procedure to find out an equivalent G with out ε-productions

1. Find nullable variables
2. Add productions with nullable variables removed.
3. Remove ε-productions and duplicates

Step 1: Find set of nullable variables

Nullable variables: Variables that can be replaced by null (ε). If A $\overset{*}{\Rightarrow}$ ε then A is a nullable variable.

In the grammar with productions S → ABA, A → aA | ε, B → bB | ε, A is nullable because of the production A → ε. B is nullable because of the production B → ε. S is nullable because both A and B are nullable.

Algorithm to find nullable variables is given below:

```
V: set of variables
$N_0 \leftarrow \{A \mid A \text{ in } V, \text{ production } A \rightarrow \varepsilon\}$
Repeat
    $N_i \leftarrow N_{i-1} U \{A \mid A \text{ in } V, A \rightarrow \boldsymbol{\alpha}, \boldsymbol{\alpha} \text{ in } N_{i-1}\}$
until $N_i = N_{i-1}$
```

Step 2: Add productions with nullable variables removed

For each production of the form A → w, create all possible productions of the form A → w', where w' is obtained from w by removing one or more occurrences of nullable variables.

Example:

In the grammar with productions S → ABA, A → aA | ε, B → bB | ε, A, B and S are nullable variables. So after removing nullable variables we get the productions

S → ABA | BA | AA | AB | A | B | ε
A → aA | ε | a
B → bB | ε | b

Step 3: Remove ε-productions and duplicates

The desired grammar consists of the original productions together with the productions constructed in step 2, minus any productions of the form A → ε.

Example:

For the above example we get the final grammar with productions

S → ABA | BA | AA | AB | A | B
A → aA | a
B → bB | b

Example:

Find out the grammar without ε - Productions

$G = (\{S, A, B, D\}, \{a\}, \{S \rightarrow aS \mid AB, A \rightarrow \varepsilon, B \rightarrow \varepsilon, D \rightarrow b\}, S)$

Solution:

Nullable variables = {S, A, B}

New Set of productions:

$S \rightarrow aS \mid a$
$S \rightarrow AB \mid A \mid B$
$D \rightarrow b$

G_1= $(\{S, B, D\}, \{a\}, \{S \rightarrow aS \mid a \mid AB \mid A \mid B, D \rightarrow b\}, S)$

Exercise:

Eliminate ε - productions from the grammar

1. $S \rightarrow a \mid Xb \mid aYa, X \rightarrow Y \mid \varepsilon, Y \rightarrow b \mid X$
2. $S \rightarrow Xa, X \rightarrow aX \mid bX \mid \varepsilon$
3. $S \rightarrow XY, X \rightarrow Zb, Y \rightarrow bW, Z \rightarrow AB, W \rightarrow Z, A \rightarrow aA \mid bB \mid \varepsilon, B \rightarrow Ba \mid Bb \mid \varepsilon$
4. $S \rightarrow ASB \mid \varepsilon, A \rightarrow aAS \mid a, B \rightarrow SbS \mid A \mid bb$

But if you have to get a grammar without ε - productions and useless symbols, follow the sequence given below:

1. Eliminate ε - productions and obtain G_1
2. Eliminate useless symbols from G_1 and obtain G_2

Example:

Eliminate ε - productions and useless symbols from the grammar

$S \rightarrow a \mid aA \mid B \mid C, A \rightarrow aB \mid \varepsilon, B \rightarrow aA, C \rightarrow aCD, D \rightarrow ddd$

Step 1: Eliminate ε - productions

Nullable ={A}

$P1=\{S \rightarrow a \mid aA \mid B \mid C, A \rightarrow aB, B \rightarrow aA \mid a, C \rightarrow aCD, D \rightarrow ddd$

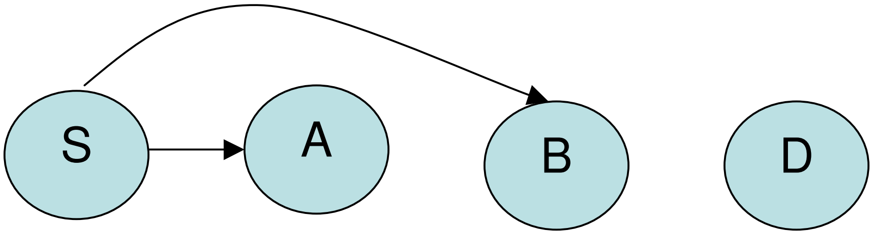
Step 2: Eliminating useless symbols

Step 2a: Eliminate non-generating symbols

Generating ={D, B, A, S}

P2={S →a | aA| B, A → aB, B → aA|a, D → ddd}

Step 2b: Eliminate non -reachable symbols



C is non-reachable, eliminating C gets

P3= S →a |aA|B, A → aB, B →aA|a}

I. Eliminate unit productions

Definition:

Unit production is of form A → B, where A and B are variables. Unit productions could complicate certain proofs and they also introduce extra steps into derivations that technically need not be there. The algorithm for eliminating unit productions from the set of production P is given below:

1. Add all non unit productions of P to P1
2. For each unit production A → B, add to P1 all productions A → α, where B → α is a non-unit production in P.
3. Delete all the unit productions

Example:

Eliminate unit productions from S → ABA | BA | AA | AB | A | B, A → aA | a, B → bB|b

Solution:

The unit productions are S → A, S →B. So A and B are derivable. Now add productions from derivable.

S→ ABA | BA | AA | AB | A | B | aA | a | bB | b
A → aA | a
B → bB | b

Remove unit productions from above productions to get

S→ ABA | BA | AA | AB | aA | a | bB | b
A → aA | a
B → bB | b

Example:

Eliminate unit productions from S → Aa | B, A → a | bc | B, B →A | bb

Solution:

Unit productions are S → B, A → B and B → A. So A and B are derivable. Add productions from derivable and eliminate unit productions to get,

S → Aa | bb | a | bc
A → a | bc | bb
B → bb | a | bc

Reduced Grammar

If you have to get a grammar without ε - productions, useless symbols and unit productions, follow the sequence given below:

1. Eliminate ε - productions from G and obtain G_1
2. Eliminate unit productions from G_1 and obtain G_2
3. Eliminate useless symbols from $G_2$and obtain G_3

Example:

Eliminate useless symbols, ε -productions and unit productions from the grammar with productions:

S → a | aA | B| C, A → aB | ε, B → aA, C → cCD, D → ddd

Step 1: Eliminate ε -productions

Nullable = {A}

P_1 = {S → a | aA | B | C, A → aB, B → aA | a, C → cCD, D → ddd}

Step 2: Eliminate unit productions

Unit productions are S → B and S→ C. So Derivable variables are B and C.

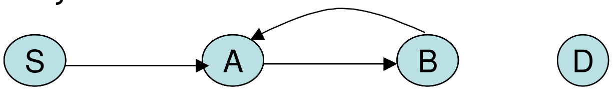
P_2 = {S → a | aA | cCD, A → aB, B → aA | a, C → cCD, D →ddd}

Step 3: Eliminate useless symbols

After eliminate non-generating symbol C we get

P_3 = {S → a | aA, A → aB, B → aA | a, D → ddd}

After eliminate symbols that are non reachable



P_4 = {S → a | aA, A → aB, B → aA | a}

So the equivalent grammar G_1 = ({S, A, B}, {a}, {S → a | aA, A → aB, B → aA | a}, S)

II. Chomsky Normal Form (CNF)

Every nonempty CFL without ε, has a grammar in CNF with productions of the form:

1. A → BC, where A, B, C ∈ V
2. A → a, where A ∈ V and a ∈ T

Algorithm to produce a grammar in CNF:

1. Eliminate useless symbols, ε -productions and unit productions from the grammar
2. Elimination of terminals on RHS of a production
   a) Add all productions of the form A → BC or A → a to P_1
   b) Consider a production A → $X_1X_2…X_n$ with some terminals of RHS. If X_i is a terminal say a_i, then add a new variable C_{ai} to V_1 and a new production C_{ai} → a_i to P_1. Replace X_i in A production of P by C_{ai}

c) Consider $A \rightarrow X_1X_2 \ldots X_n$, where $n \geq 3$ and all X_i's are variables. Introduce new productions $A \rightarrow X_1C_1$, $C_1 \rightarrow X_2C_2$, … , $C_{n-2} \rightarrow X_{n-1}X_n$ to P_1 and $C_1, C_2, \ldots, C_{n-2}$ to V_1

Example:

Convert to CNF: $S \rightarrow aAD$, $A \rightarrow aB \mid bAB$, $B \rightarrow b$, $D \rightarrow d$

Solution:

Step1: Simplify the grammar

Grammar is already simplified.

Step2a: Elimination of terminals on RHS

Change $S \rightarrow aAD$ to $S \rightarrow C_aAD$, $C_a \rightarrow a$
$A \rightarrow aB$ to $A \rightarrow C_aB$
$A \rightarrow bAB$ to $A \rightarrow C_bAB$, $C_b \rightarrow b$

Step2b: Reduce RHS with 2 variables

Change $S \rightarrow C_aAD$ to $S \rightarrow C_aC_1$, $C_1 \rightarrow AD$
$A \rightarrow C_bAB$ to $A \rightarrow C_bC_2$, $C_2 \rightarrow AB$

Grammar converted to CNF is given below:

$G_1=(\{S, A, B, D, C_a, C_b, C_1, C_2\}, \{a, b\}, S \rightarrow C_aC_1, A \rightarrow C_aB \mid C_bC_2, C_a \rightarrow a, C_b \rightarrow b, C_1 \rightarrow AD, C_2 \rightarrow AB\}, S)$

Example:

Convert the grammar with following productions to CNF:

$P=\{S \rightarrow ASB \mid \varepsilon, A \rightarrow aAS \mid a, B \rightarrow SbS \mid A \mid bb\}$

Solution:

Step1: Simplify the grammar

Step 1a: Eliminate ε -productions: Consists of $S \rightarrow \varepsilon$

Eliminating ε -productions from P to get:

P_1={S →ASB|AB, A → aAS | aA | a, B→ SbS | Sb | bS | b | A | bb}

Step 1b: Eliminate unit productions: B→ A

P_2= {S → ASB|AB, A →aAS|aA|a, B →SbS | Sb | bS | b | bb | aAS | aA | a}

Step 1c: Eliminate useless symbols: no useless symbols

Step2: Convert to CNF

P_3={S → AC_1 | AB, A → C_aC_2 | C_aA | a,
B → SC_3 | SC_b | C_bS | b | C_bC_b | C_aC_2 | C_aA | a,
C_a→ a, C_b → b, C_1 → SB, C_2 → AS, C_3 → C_bS}

Exercises:

Convert the following grammar with productions to CNF:

1. S → aSa | bSb | a | b | aa | bb
2. S → bA | aB, A → bAA | aS | a, B → aBB | bS | b
3. S→ Aba, A → aab, B → AC
4. S → 0A0 |1B1 | BB, A → C, B → S|A, C → S| ε
5. S → aAa | bBb| ε, A → C|a, B → C | b, C → CDE |ε, D → A | B | ab

III. Greibach Normal Form (GNF)

Every nonempty language without ε is L(G) for some grammar G with productions are of the form

A → aα,

where a ∈ T and α is a string of zero or more variables. Conversion to GNF is a complex process. Converting CFG in GNF to PDA gets a PDA without ε-rules.

Example:

Transform into Greibach normal form the grammar with productions

S → 0S1 | 01

Solution:

S → 0SA | 0A, A → 1

Uses of GNF

Construction of PDA from a GNF grammar can be made more meaningful with GNF. Assume that the PDA has two states: start state q_0 and accepting state q_1. An S rule of the form S → $aA_1A_2 \ldots A_n$ generates a transition that processes the terminal a, pushes the variables $A_1A_2 \ldots A_n$ on the stack and enters q_1. The remainder of the computation uses the input symbol and the stack top to determine the appropriate transition.

Pumping Lemma for CFL

The *pumping lemma for regular languages* states that every sufficiently long string in a regular language contains a short sub-string that can be pumped. That is, inserting as many copies of the sub-string as we like always yields a string in the regular language.

The *pumping lemma for CFL's* states that there are always two short sub-strings close together that can be repeated, both the same number of times, as often as we like.

For example, consider a CFL $L=\{a^nb^n \mid n \geq 1\}$. Equivalent CNF grammar is having productions S → AC | AB, A → a, B → b, C → SB. The parse tree for the string a^4b^4 is given in figure 1· Both leftmost derivation and rightmost derivation have same parse tree because the grammar is unambiguous.

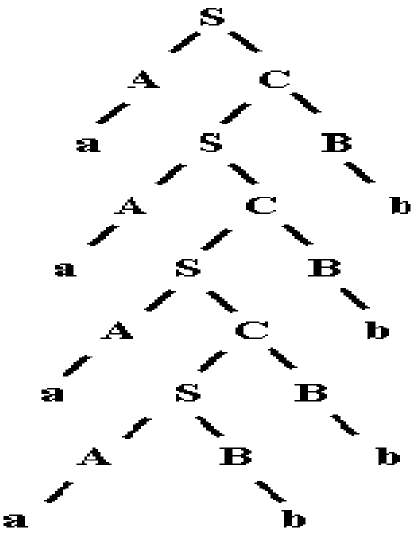


Figure 1: Parse tree for a^4b^4

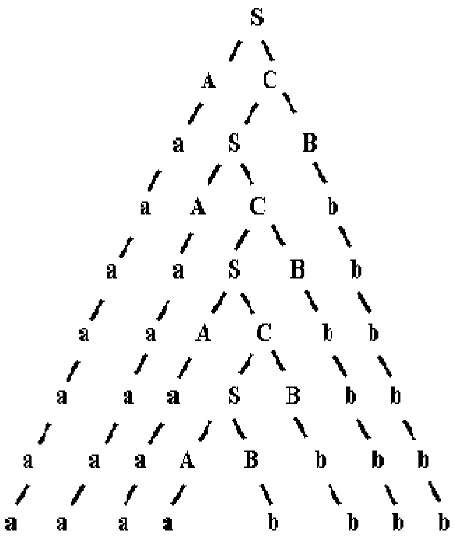


Figure 2: Extended Parse tree for a^4b^4

Extend the tree by duplicating the terminals generated at each level on all lower levels. The extended parse tree for the string a^4b^4 is given in figure 2. Number of symbols at each level is at most twice of previous level. 1 symbols at level 0, 2 symbols at 1, 4 symbols at 2 …2^i symbols at level i. To have 2^n symbols at bottom level, tree must be having at least depth of n and level of at least n+1.

Pumping Lemma Theorem:

Let L be a CFL. Then there exists a constant $k \geq 0$ such that if z is any string in L such that $|z| \geq k$, then we can write z = uvwxy such that

1. $|vwx| \leq k$ (that is, the middle portion is not too long).
2. $vx \neq \varepsilon$ (since v and x are the pieces to be "pumped", at least one of the strings we pump must not be empty).
3. For all $i \geq 0$, uv^iwx^iy is in L.

Proof:

The parse tree for a grammar G in CNF will be a binary tree. Let $k = 2^{n+1}$, where n is the number of variables of G. Suppose $z \in L(G)$ and $|z| \geq k$. Any parse tree for z must be of depth at least n+1. The longest path in the parse tree is at least n+1, so this path must contain at least n+1 occurrences of the variables. By pigeonhole principle, some variables occur more than once along the path. Reading from bottom to top, consider the first pair of same variable along the path. Say X has 2 occurrences. Break z into uvwxy such that w is the string of terminals generated at the lower occurrence of X and vwx is the string generated by upper occurrence of X.

Example parse tree:

For the above example S has repeated occurrences, and the parse tree is shown in figure 3. w = ab is the string generated by lower occurrence of S and vwx = aabb is the string generated by upper occurrence of S. So here u=aa, v=a, w=ab, x=b, y=bb.

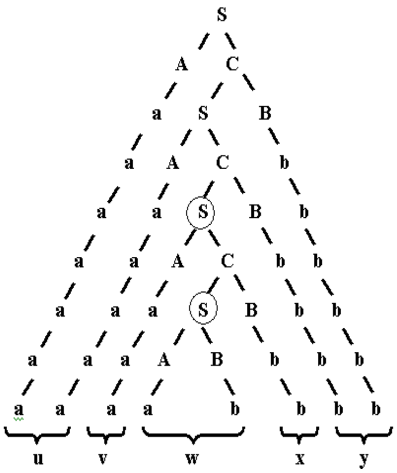


Figure 3: Parse tree for a^4b^4 with repeated occurrences of S circled.

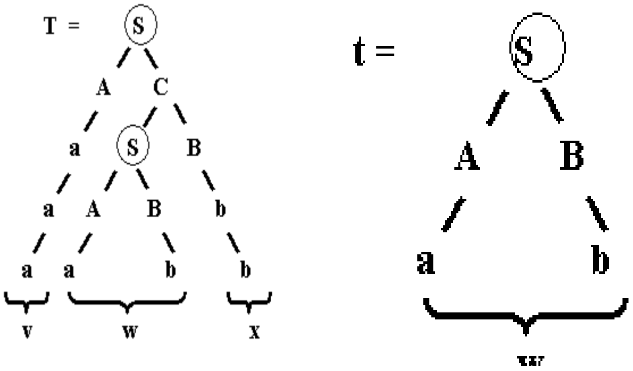


Figure 4: sub- trees

Let T be the subtree rooted at upper occurrence of S and t be subtree rooted at lower occurrence of S. These parse trees are shown in figure 4. To get $uv^2wx^2y \in L$, cut out t and replace it with copy of T. The parse tree for $uv^2wx^2y \in L$ is given in figure 5. Cutting out t and replacing it with copy of T as many times to get a valid parse tree for uv^iwx^iy for $i \geq 1$.

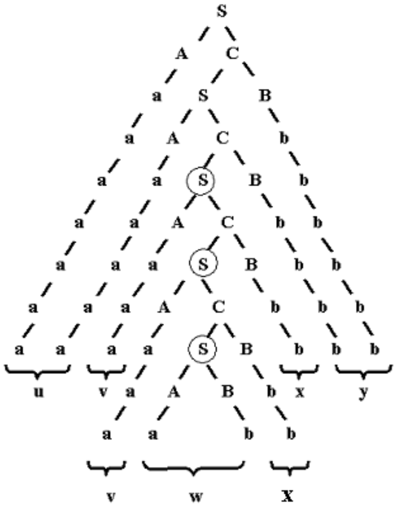


Figure 5: Parse tree for $uv^2wx^2y \in L$

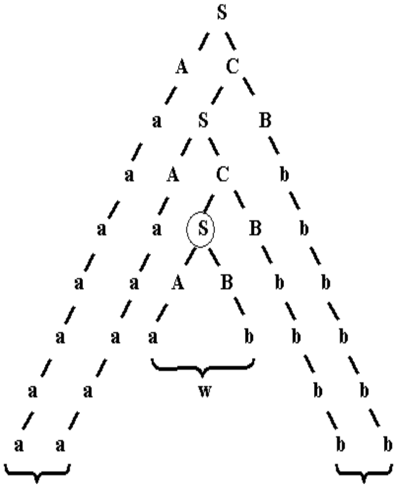


Figure 6: Parse tree for $uwy \in L$

To get $uwy \in L$, cut T out of the original tree and replace it with t to get a parse tree of $uv^0wx^0y = uwy$ as shown in figure 6.

Pumping Lemma game:

1. To show that a language L is not a CFL, assume L is context free.
2. Choose an "appropriate" string z in L
3. Express z = uvwxy following rules of pumping lemma
4. Show that uv^kwx^ky is not in L, for some k
5. The above contradicts the Pumping Lemma
6. Our assumption that L is context free is wrong

Example:

Show that $L = \{a^ib^ic^i \mid i \geq 1\}$ is not CFL

Solution:

Assume L is CFL. Choose an appropriate $z = a^nb^nc^n = uvwxy$. Since $|vwx| \leq n$ then vwx can either consists of

1. All a's or all b's or all c's
2. Some a's and some b's
3. Some b's and some c's

Case 1: vwx consists of all a's

If $z = a^2b^2c^2$ and $u = \varepsilon$, $v = a$, $w = \varepsilon$, $x = a$ and $y = b^2c^2$ then, uv^2wx^2y will be $a^4b^2c^2 \neq L$

Case 2: vwx consists of some a's and some b's

If $z = a^2b^2c^2$ and $u = a$, $v = a$, $w = \varepsilon$, $x = b$, $y = bc^2$, then uv^2wx^2y will be $a^3b^3c^2 \neq L$

Case 3: vwx consists of some b's and some c's

If $z = a^2b^2c^2$ and $u = a^2b$, $v = b$, $w = c$, $x = \varepsilon$, $y = c$, then uv^2wx^2y will be $a^2b^3c^2 \neq L$

If you consider any of the above 3 cases, uv^2wx^2y will not be having an equal number of a's, b's and c's. But Pumping Lemma says $uv^2wx^2y \in L$. Can't contradict the pumping lemma! Our original assumption must be wrong. So L is not context-free.

Example:

Show that $L = \{ww \mid w \in \{0, 1\}^*\}$ is not CFL

Solution:

Assume L is CFL. It is sufficient to show that $L1= \{0^m1^n0^m1^n \mid m,n \geq 0\}$, where n is pumping lemma constant, is a CFL. Pick any $z = 0^n1^n0^n1^n = uvwxy$, satisfying the conditions $|vwx| \leq n$ and $vx \neq \varepsilon$.

This language we prove by taking the case of $i = 0$, in the pumping lemma satisfying the condition uv^iwx^iy for $i \geq 0$.

z is having a length of 4n. So if $|vwx| \leq n$, then $|uwy| \geq 3n$. According to pumping lemma, $uwy \in L$. Then uwy will be some string in the form of tt, where t is repeating. If so, n $|t| \geq 3n/2$.

Suppose vwx is within first n 0's: let vx consists of k 0's. Then uwy begins with $0^{n-k}1^n$ $|uwy| = 4n-k$. If uwy is some repeating string tt, then $|t| =2n-k/2$. t does end in 0 but tt ends with 1. So second t is not a repetition of first t.

Example: z = $0^3 1^3 0^3 1^3$, vx = 0^2 then uwy = tt = $01^3 0^3 1^3$, so first t = $01^3 0$ and second t = $0^2 1^3$. Both t's are not same.

Suppose vwx consists of 1st block of 0's and first block of 1's: vx consists of only 0's if x= ε, then uwy is not in the form tt. If vx has at least one 1, then |t| is at least 3n/2 and first t ends with a 0, not a 1.

Very similar explanations could be given for the cases of vwx consists of first block of 1's and vwx consists of 1st block of 1's and 2nd block of 0's. In all cases uwy is expected to be in the form of tt. But first t and second t are not the same string. So uwy is not in L and L is not context free.

Example:

Show that L=$\{0^i 1^j 2^i 3^j \mid i \geq 1, j \geq 1\}$ is not CFL

Solution:

Assume L is CFL. Pick z = uvwxy = $0^n 1^n 2^n 3^n$ where |vwx| ≤ n and vx ≠ ε. vwx can consist of a substring of one of the symbols or straddles of two adjacent symbols.

Case 1: vwx consists of a substring of one of the symbols

Then uwy has n of 3 different symbols and fewer than n of 4th symbol. Then uwy is not in L.

Case 2: vwx consists of 2 adjacent symbols say 1 & 2

Then uwy is missing some 1's or 2's and uwy is not in L.

If we consider any combinations of above cases, we get uwy, which is not CFL. This contradicts the assumption. So L is not a CFL.

Exercises:
Using pumping lemma for CFL prove that below languages are not context free

1. $\{0^p \mid p \text{ is a prime}\}$
2. $\{a^n b^n c^i \mid i \leq n\}$

Closure Properties of CFL

Many operations on Context Free Languages (CFL) guarantee to produce CFL. A few do not produce CFL. *Closure properties* consider operations on CFL that are guaranteed to produce a CFL. The CFL's are closed under *substitution*, *union*, *concatenation*, *closure (star)*, *reversal*, *homomorphism* and *inverse homomorphism*. CFL's are not closed under *intersection* (but the intersection of a CFL and a regular language is always a CFL), *complementation*, and *set-difference*.

I. Substitution:

By substitution operation, each symbol in the strings of one language is replaced by an entire CFL language.

Example:

$S(0) = \{a^nb^n \mid n \geq 1\}$, $S(1)=\{aa,bb\}$ is a substitution on alphabet $\Sigma = \{0, 1\}$.

Theorem:

If a substitution s assigns a CFL to every symbol in the alphabet of a CFL L, then s(L) is a CFL.

Proof:

Let $G = (V, \Sigma, P, S)$ be grammar for the CFL L. Let $G_a = (V_a, T_a, P_a, S_a)$ be the grammar corresponding to each terminal $a \in \Sigma$ and $V \cap V_a = \phi$. Then $G' = (V', T', P', S)$ is a grammar for s(L) where

- $V' = V \cup V_a$
- T'= union of T_a's all for $a \in \Sigma$
- P' consists of
- All productions in any P_a for $a \in \Sigma$
- The productions of P, with each terminal a is replaced by S_a everywhere a occurs.

Example:

$L = \{0^n1^n \mid n \geq 1\}$, generated by the grammar $S \rightarrow 0S1 \mid 01$, $s(0) = \{a^nb^m \mid m \leq n\}$, generated by the grammar $S \rightarrow aSb \mid A$; $A \rightarrow aA \mid ab$, $s(1) = \{ab, abc\}$, generated by the grammar $S \rightarrow abA$, $A \rightarrow c \mid \varepsilon$. Rename second and third S's to S_0 and S_1, respectively. Rename second A to B. Resulting grammars are:

$S \rightarrow 0S1 \mid 01$
$S_0 \rightarrow aS_0b \mid A;\ A \rightarrow aA \mid ab$
$S_1 \rightarrow abB;\ B \rightarrow c \mid \varepsilon$

In the first grammar replace 0 by S_0 and 1 by S_1. The resulted grammar after substitution is:

$S \rightarrow S_0SS_1 \mid S_0S_1$ $\quad S_0 \rightarrow aS_0b \mid A;\ A \rightarrow aA \mid ab$ $\quad S_1 \rightarrow abB;\ B \rightarrow c \mid \varepsilon$

II. Application of substitution:

a. *Closure under union of CFL's L_1 and L_2:*

Use L={a, b}, s(a)=L_1 and s(b)=L_2. Then s(L)= $L_1 \cup L_2$.

How to get grammar for $L_1 \cup L_2$?

Add new start symbol S and rules $S \rightarrow S_1 \mid S_2$

The grammar for $L_1 \cup L_2$ is G = (V, T, P, S) where V = $\{V_1 \cup V_2 \cup S\}$, $S \notin (V_1 \cup V_2)$ and P = $\{P_1 \cup P_2 \cup \{S \rightarrow S_1 \mid S_2\}\}$

Example:

$L_1 = \{a^nb^n \mid n \geq 0\}$, $L_2 = \{b^na^n \mid n \geq 0\}$. Their corresponding grammars are

G_1: $S_1 \rightarrow aS_1b \mid \varepsilon$, G_2 : $S_2 \rightarrow bS_2a \mid \varepsilon$

The grammar for $L_1 \cup L_2$ is

$G = (\{S, S_1, S_2\}, \{a, b\}, \{S \rightarrow S_1 \mid S_2, S_1 \rightarrow aS_1b \mid \varepsilon, S_2 \rightarrow bS_2a\}, S)$.

b. *Closure under concatenation of CFL's L_1 and L_2:*

Let L={ab}, s(a)=L_1 and s(b)=L_2. Then s(L)=L_1L_2

How to get grammar for L_1L_2?

Add new start symbol and rule $S \rightarrow S_1S_2$

The grammar for L_1L_2 is G = (V, T, P, S) where V = $V_1 \cup V_2 \cup \{S\}$, $S \notin V_1 \cup V_2$ and P = $P_1 \cup P_2 \cup \{S \rightarrow S_1S_2\}$

Example:

$L_1 = \{a^n b^n \mid n \geq 0\}$, $L_2 = \{b^n a^n \mid n \geq 0\}$ then $L_1L_2 = \{a^n b^{\{n+m\}} a^m \mid n, m \geq 0\}$

Their corresponding grammars are

G_1: $S_1 \rightarrow aS_1b \mid \varepsilon$, $G_2 : S_2 \rightarrow bS_2a \mid \varepsilon$

The grammar for L_1L_2 is

$G = (\{S, S_1, S_2\}, \{a, b\}, \{S \rightarrow S_1S_2, S_1 \rightarrow aS_1b \mid \varepsilon, S_2 \rightarrow bS_2a\}, S)$.

c. *Closure under Kleene's star (closure * and positive closure $^+$) of CFL's L_1:*

Let $L = \{a\}^*$ (or $L = \{a\}^+$) and $s(a) = L_1$. Then $s(L) = L_1^*$ (or $s(L) = L_1^+$).

Example:

$L_1 = \{a^n b^n \mid n \geq 0\}$ $(L_1)^* = \{a^{\{n1\}} b^{\{n1\}} \ldots a^{\{nk\}} b^{\{nk\}} \mid k \geq 0$ and $ni \geq 0$ for all $i\}$

$L_2 = \{a^{\{n^2\}} \mid n \geq 1\}$, $(L_2)^* = a^*$

How to get grammar for $(L_1)^*$:

Add new start symbol S and rules $S \rightarrow SS_1 \mid \varepsilon$.

The grammar for $(L_1)^*$ is

$G = (V, T, P, S)$, where $V = V_1 \cup \{S\}$, $S \notin V_1$, $P = P_1 \cup \{S \rightarrow SS_1 \mid \varepsilon\}$

d. Closure under homomorphism of CFL L_i for every $a_i \in \Sigma$:

Suppose L is a CFL over alphabet Σ and h is a homomorphism on Σ. Let s be a substitution that replaces every $a \in \Sigma$, by h(a). ie $s(a) = \{h(a)\}$. Then $h(L) = s(L)$. ie $h(L) = \{h(a_1)\ldots h(a_k) \mid k \geq 0\}$ where $h(a_i)$ is a homomorphism for every $a_i \in \Sigma$.

III. Closure under Reversal:

L is a CFL, so L^R is a CFL. It is enough to reverse each production of a CFL for L, i.e., to substitute each production $A \rightarrow \alpha$ by $A \rightarrow \alpha^R$.

IV. Intersection:

The CFL's are not closed under intersection

Example:

The language $L = \{0^n1^n2^n \mid n \geq 1\}$ is not context-free. But $L_1 = \{0^n1^n2^i \mid n \geq 1, i \geq 1\}$ is a CFL and $L_2 = \{0^i1^n2^n \mid n \geq 1, i \geq 1\}$ is also a CFL. But $L = L1 \cap L_2$.

Corresponding grammars for L_1: $S \rightarrow AB$; $A \rightarrow 0A1 \mid 01$; $B \rightarrow 2B \mid 2$ and corresponding grammars for L_2: $S \rightarrow AB$; $A \rightarrow 0A \mid 0$; $B \rightarrow 1B2 \mid 12$.

However, $L = L_1 \cap L_2$, thus intersection of CFL's is not CFL

a. Intersection of CFL and Regular Language:

Theorem: If L is CFL and R is a regular language, then $L \cap R$ is a CFL.

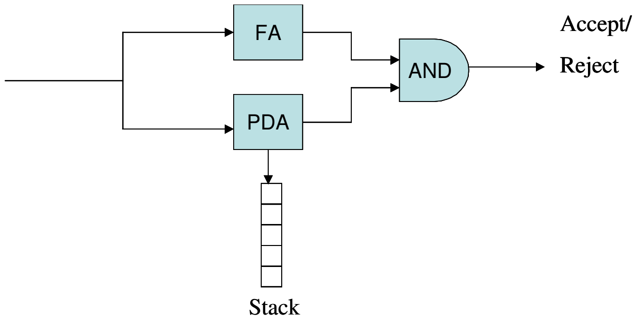


Figure 1: PDA for $L \cap R$

Proof:

$P = (Q_P, \Sigma, \Gamma, \delta_P, q_P, Z_0, F_P)$ be PDA to accept L by final state. Let $A = (Q_A, \Sigma, \delta_A, q_A, F_A)$ for DFA to accept the Regular Language R. To get $L \cap R$, we have to run a Finite Automata in parallel with a push down automata as shown in figure 1. Construct PDA $P' = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ where

- $Q = (Q_p \times Q_A)$
- $q_0 = (q_p, q_A)$
- $F = (F_P \times F_A)$
- δ is in the form $\delta((q, p), a, X) = ((r, s), g)$ such that
  1. $s = \delta_A(p, a)$
  2. (r, g) is in $\delta_P(q, a, X)$

That is for each move of PDA P, we make the same move in PDA P′ and also we carry along the state of DFA A in a second component of P′. P′ accepts a string w if and only if both P and A accept w. ie w is in $L \cap R$. The moves $((q_p, q_A), w, Z) \mid\!-^*_{P'} ((q, p), \varepsilon, \gamma)$ are possible if and only if $(q_p, w, Z) \mid\!-^*_{P} (q, \varepsilon, \gamma)$ moves and $p = \delta^*(q_A, w)$ transitions are possible.

b. CFL and RL properties:

Theorem: The following are true about CFL's L, L_1, and L_2, and a regular language R.

1. **Closure of CFL's under set-difference with a regular language. ie L - R is a CFL.**

    Proof:

    R is regular and regular language is closed under complement. So R^C is also regular. We know that $L - R = L \cap R^C$. We have already proved the closure of intersection of a CFL and a regular language. So CFL is closed under set difference with a Regular language.

2. **CFL is not closed under complementation**

    L^C is not necessarily a CFL

    Proof:

    Assume that CFLs were closed under complement. ie if L is a CFL then L^C is a CFL. Since CFLs are closed under union, $L_1^C \cup L_2^C$ is a CFL. By our assumption $(L_1^C \cup L_2^C)^C$ is a CFL. But $(L_1^C \cup L_2^C)^C = L_1 \cap L_2$, which we just showed isn't necessarily a CFL. Contradiction! . So our assumption is false. CFL is not closed under complementation.

3. **CFLs are not closed under set-difference. ie $L_1 - L_2$ is not necessarily a CFL.**

    Proof:

    Let $L1 = \Sigma^* - L$. Σ^* is regular and is also CFL. But $\Sigma^* - L = L^C$. If CFLs were closed under set difference, then $\Sigma^* - L = L^C$ would always be a CFL. But CFL's are not closed under complementation. So CFLs are not closed under set-difference.

V. Inverse Homomorphism:

Recall that if h is a homomorphism, and L is any language, then $h^{-1}(L)$, called an *inverse homomorphism,* is the set of all strings w such that $h(w) \in L$. The CFL's are closed under inverse homomorphism.

Theorem:

If L is a CFL and h is a homomorphism, then $h^{-1}(L)$ is a CFL

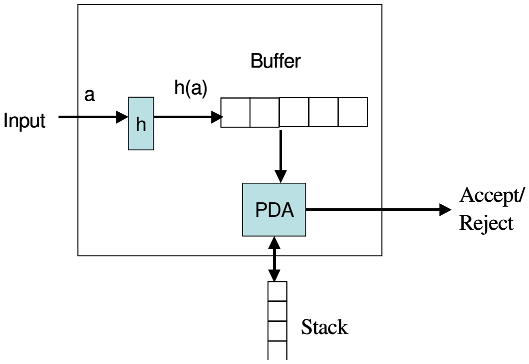


Figure 2: PDA to simulate inverse

We can prove closure of CFL under inverse homomorphism by designing a new PDA as shown in figure 2. After input a is read, h(a) is placed in a buffer. Symbols of h(a) are used one at a time and fed to PDA being simulated. Only when the buffer is empty does the PDA read another of its input symbol and apply homomorphism to it.

Suppose h applies to symbols of alphabet Σ and produces strings in T*. Let PDA P = (Q, T, Γ, δ, q_0, Z_0, F) that accept CFL L by final state. We construct a new PDA P′ = (Q′, Σ, Γ, δ′, (q_0, ε), Z_0, (F x ε)) to accept $h^{-1}(L)$, where

- Q′ is the set of pairs (q, x) such that
- q is a state in Q
- x is a suffix of some string h(a) for some input string a in Σ
- δ′ is defined by
- δ′ ((q, ε), a, X) = {((q, h(a)),X)}
- If δ(q, b, X) = {(p, γ)} where b∈ T or b = ε then δ′ ((q, bx), ε, X) = {((p, x), γ)}
- The start state of P′ is (q_0, ε)
- The accepting state of P′ is (q, ε), where q is an accepting state of P.

Once we accept the relationship between P and P′, P accepts h(w) if and only if P′ accepts w, because of the way the accepting states of P′ are defined.

Thus $L(P')=h^{-1}(L(P))$

Introduction to Push Down Automata

Just as finite-state automata correspond to regular languages, the context-free languages (CFL) have corresponding machines called pushdown automata (PDA).

Regular expressions are generators for regular languages and Finite Automata's are recognizers for them. Similarly for Context-free Languages, Context Free Grammars (CFG) are generators and Pushdown Automata (PDA) are recognizers.

PDA is more powerful than FA. An FA cannot recognize the language a^nb^n, $n \geq 0$, because FA does not have any memory to remember the number of a's it has already seen, for equating with number of b's found. PDA is NFA with an added memory. Stack functions as the required memory. So, a PDA is an NFA with a stack. Figure 1 shows a diagrammatic representation of PDA. The Finite State Control (FSC) reads inputs, one symbol at a time. Based on the input symbol, current state and the top symbol on the stack, FSC does some state transitions and does some operations to the stack content. Stack could be kept unchanged, or some thing could be pushed into the stack and could be popped out of the stack.

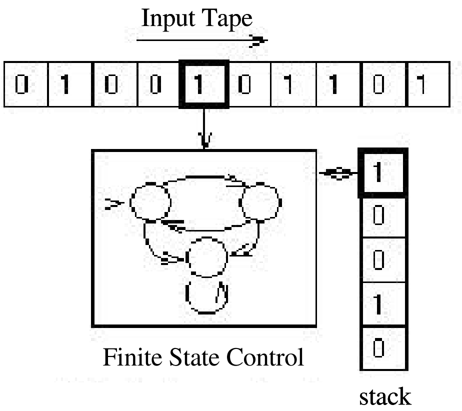


Figure 1: PushDown Automaton

Formal Definition:

A *nondeterministic pushdown automaton* or *npda* is a 7-tuple defined as $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$, where

Q = a finite set of *states*,
Σ = a the *input alphabet*,
Γ = the *stack alphabet*,
δ = *transition function*, has the form
δ: $Q \text{ X } (\Sigma \cup \{\in\}) \text{ X } \Gamma \rightarrow$ finite subsets of $Q \text{ X } \Gamma^*$
$q_0 \in Q$ is the *initial state*,
$z \in \Gamma$ is the *stack start symbol*, and
$F \subseteq Q$ is a set of *final states*.

Transition function: for any given state, input symbol and stack symbol, gives a new state and stack symbol; i.e. it has the form: (P, a, t) → (Q, u)

Basically, if, $a \in \Sigma$, $t \in \Gamma$ and P and Q are states. Means "read the symbol 'a' from the input, move from state P to state Q, and replace the symbol 't' on top of the stack with the symbol 'u' ".

If u = t, then stack is unchanged.

If u = $\in$, then stack is popped

If u= wx, then t is replaced with x and w is pushed into the stack.

Example 1:

Construct PDA for the language L= $\{a^n b^n \mid a, b \in \Sigma\ n \geq 0\}$.

Start at state q_0 and keep Z_0 in the stack. The following transitions are possible:

1. If current state is q_0, and symbol on input tape is at ε, and stack top is Z_0, then move to q_2 the final state.
2. If current state is q_0, and input tape symbol is a, and stack top Z_0, then stay in q_0 and push 'a' to the stack.
3. If current state is q_0, input tape symbol is 'a', and stack top is a, stay in q_0 and push 'a' to the stack.
4. If current state is q_0, input tape symbol is b, stack top is a, move to state q_1 and pop the top symbol of the stack.
5. If current state is q_1, input tape symbol is b, stack top is a, stay in q_1 and pop the top symbol of the stack
6. If current state is q_1, input tape symbol is ε and stack top is Z_0, move to q_2 the final state.

So we can define PDA as M = $(\{q_0, q_1, q_2\}, \{a, b\}, \{a, b, Z_0\}, \delta, q_0, Z_0, \{q_2\})$, where δ is defined by following rules:

$\delta(q_0, a, Z_0) = \{(q_0, aZ_0)\}$

$\delta(q_0, a, a) = \{(q_0, aa)\}$

$\delta(q_0, b, a) = \{(q_1, \varepsilon)\}$

$\delta(q_1, b, a) = \{(q_1, \varepsilon)\}$

$\delta(q_1, \varepsilon, Z_0) = \{(q_2, \varepsilon)\}$

$\delta(q_0, \varepsilon, Z0) = \{(q_2, \varepsilon)\}$

$\delta(q,x,Y) = \phi$ for all other possibilities

Graphical Notation of PDA:

To understand the behavior or PDA clearer, the transition diagram of PDA can be used. Transition diagram of PDA is generalization of transition diagram of FA.

1. Node corresponds to states of PDA
2. Arrow labeled Start indicates start state
3. Doubly circled states are final states
4. Arc corresponds to transitions of PDA. If δ(q, a, X) = {(p, α)...} is an arc labeled (a, X/α) from state q to state p means that an input tape head positioned at symbol a and stack top with X, moves automaton to state q and replaces the stack top with α.

The transition diagram for the above example PDA is given in Figure 2.

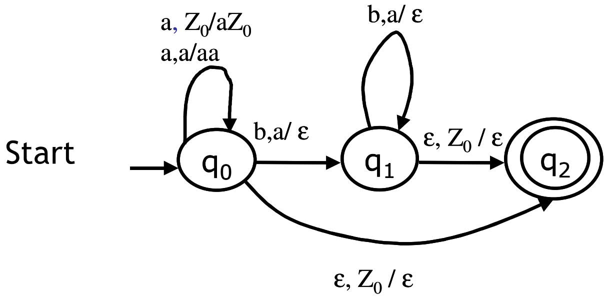


Figure 2: Transition diagram

Instantaneous Description:

Instantaneous Description or configuration of a PDA describes its execution status at any time. Instantaneous Description is a represented by a triplet (q, w, u), where

1. q is the current state of the automaton,
2. w is the unread part of the input string, w ∈ Σ*
3. u is the stack contents, written as a string, with the leftmost symbol at the top of the stack. So u ∈ Γ^*

Moves of A PDA:

Let the symbol "|-" indicates a move of the nPDA. There are two types of moves possible for a PDA.

1. **Move by consuming input symbol**

Suppose that $\delta(q_1, a, x) = \{(q_2, y), ...\}$. Then the following move by consuming an input symbol is possible:

(q_1, aW, xZ) |- (q_2, W, yZ),

where W indicates the rest of the input string following the a, and Z indicates the rest of the stack contents underneath the x. This notation says that in moving from state q_1 to state q_2, an input symbol 'a' is consumed from the input string aW, and the symbol 'x' at the top (left) of the stack xZ is replaced with symbol 'y', leaving yZ on the stack.

The above example PDA with a few example input strings, the moves are given below:

a) Moves for the input string aabb:

$(q_0, aabb, Z_0)$ |- (q_0, abb, aZ_0) as per transition rule $\delta(q_0, a, Z_0) = \{(q_0, aZ_0)\}$

|- (q_0, bb, aaZ_0) as per transition rule $\delta(q_0, a, a) = \{(q_0, aa)\}$

|- (q_1, b, aZ_0) as per transition rule $\delta(q_0, b, a) = \{(q_1, \varepsilon)\}$

|- (q_1, ε, Z_0) as per transition rule $\delta(q_0, b, a) = \{(q_1, \varepsilon)\}$

|- $(q_2, \varepsilon,\varepsilon)$ as per transition rule $\delta(q_1, \varepsilon, Z_0) = \{(q_2, \varepsilon)\}$

PDA reached a configuration of $(q_2, \varepsilon,\varepsilon)$. The input tape is empty, stack is empty and PDA has reached a final state. So the string is accepted.

b) Moves for the input string aaabb:

$(q_0, aaabb, Z_0)$ |- $(q_0, aabb, aZ_0)$ as per transition rule $\delta(q_0, a, Z_0) = \{(q_0, aZ_0)\}$

|- (q_0, abb, aaZ_0) as per transition rule $\delta(q_0, a, a) = \{(q_0, aa)\}$

|- $(q_0, bb, aaaZ_0)$ as per transition rule $\delta(q_0, a, a) = \{(q_0, aa)\}$

|- (q_1, b, aaZ_0) as per transition rule $\delta(q_0, b, a) = \{(q_1, \varepsilon)\}$

|- (q_1, ε, aZ_0) as per transition rule $\delta(q_0, b, a) = \{(q_1, \varepsilon)\}$

|- There is no defined move.

So the automaton stops and the string is not accepted.

c) Moves for the input string aabbb:

$(q_0, aabbb, Z_0)$ |- $(q_0, abbb, aZ_0)$ as per transition rule $\delta(q_0, a, Z_0) = \{(q_0, aZ_0)\}$

|- (q_0, bbb, aaZ_0) as per transition rule $\delta(q_0, a, a) = \{(q_0, aa)\}$

|- (q_1, bb, aZ_0) as per transition rule $\delta(q_0, b, a) = \{(q_1, \varepsilon)\}$

|- (q_1, b, Z_0) as per transition rule $\delta(q_0, b, a) = \{(q_1, \varepsilon)\}$

|- There is no defined move.

So the automaton stops and the string is not accepted.

2. ε- move

Suppose that $\delta(q_1, \varepsilon, x) = \{(q_2, y), ...\}$. Then the following move without consuming an input symbol is possible:

(q_1, aW, xZ) |- (q_2, aW, yZ),

This notation says that in moving from state q_1 to state q_2, an input symbol ‘a’ is not consumed from the input string aW, and the symbol ‘x’ at the top (left) of the stack xZ is replaced with symbol ‘y’, leaving yZ on the stack. In this move, the tape head position will not move forward. This move is usually used to represent non-determinism.

The relation $|-^*$ is the reflexive-transitive closure of |- used to represent zero or more moves of PDA. For the above example, $(q_0, aabb, Z_0)$ |-* $(q_2, \varepsilon,\varepsilon)$.

Example 2:

Design a PDA to accept the set of all strings of 0’s and 1’s such that no prefix has more 1’s than 0’s.

Solution: The transition diagram could be given as figure 3.

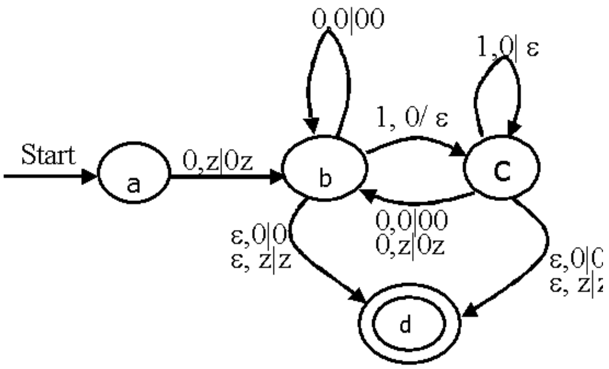


Figure 3: Transition diagram

M = ({a, b, c, d}, {0,1}, {0,1,Z}, δ, a, Z, {d}), where δ is given by:

δ (a, 0, Z) ={(b, 0Z)}

δ (b, 0, 0) ={(b, 00)}

δ (b, 1, 0) ={(c, ε)}

δ (c, 0, 0) ={(b, 00)}

δ (c, 1, 0) ={(c, ε)}

δ (b, ε, 0) ={(d, 0)}

δ (b, ε, Z) ={(d, Z)}
δ (c, 0, Z) ={(b, 0Z)}
δ (c, ε, 0) ={(d, 0)}
δ (c, ε, Z) ={(d, Z)}
For all other moves, PDA stops.

Moves for the input string 0010110 is given by:

(a, 0010110, Z) |- (b, 010110, 0Z) |- (b,10110, 00Z) |- (c, 0110, 0Z) |- (b, 110, 00Z) |- (c, 10, 0Z) |- (c, 0, Z) |- (b, ε, 0Z) |- (d, ε, 0Z).

So the input string is accepted by the PDA.

Moves for 011

(a,011,Z) |- (b,11,0Z) |- (c,1,Z) |-no move, so PDA stops

Exercises:

Construct PDA:

1. For the language L = {wcw^R | w $\in$ {a, b}*, c $\in$ Σ }
2. Accepting the set of all strings over {a, b} with equal number of a’s and b’s. Show the moves for abbaba
3. For the language L = {a^nb^{2n} | a, b $\in$ Σ, n $\geq$ 0}
4. Accepting the language of balanced parentheses, (consider any type of parentheses)

Languages of PDA

1. Languages of PDA

There are 2 ways of accepting an input string PDA

a. Accept by Final state

After consuming the input, PDA enters a final state. The content of the stack is irrelevant.

b. Accept by empty stack

After consuming the input, stack of the PDA will be empty. The current state could be final or non-final state.

Both methods are equivalent. It is possible to covert a PDA accept by final state to another PDA accept by empty stack and also the vice versa. Usually the languages that a PDA accept by final state and PDA by empty stack are different. For example the language $\{L = \mathbf{a}^n\mathbf{b}^m \mid n \geq m\}$, the appropriate PDA could be by final state. After consuming the input, the stack may not be empty.

Accept by Final state:

Let P = $(Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. Then L(P), the language accepted by P by the final state is

$\{w \mid (q_0, w, Z_0) \mid\!-^* (q, \varepsilon, \alpha)\}$, where $q \in F$ and $\alpha \in \Gamma^*$

Example: $L = \{ww^R \mid w$ is in $(0 + 1)^*\}$, the language for even length palindrome. Acceptable input strings are like 00, 1111, 0110, 101101, and 110011. In the string 0110, the difficulty is how to decide the middle of the input string? The 3rd 1 can be part of w or can be part of w^R. The PDA could be constructed as below.

$M = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, Z_0\}, \delta, q_0, Z_0, q_2)$, where δ is defined by:

$\delta(q_0, 0, Z_0) = \{(q_0, 0Z_0)\}$

$\delta(q_0, 1, Z_0) = \{(q_0, 1Z_0)\}$

$\delta(q_0, 0, 0) = \{(q_0, 00)\}$

$\delta(q_0, 1, 1) = \{(q_0, 11)\}$

$\delta(q_0, 0, 1) = \{(q_0, 01)\}$
$\delta(q_0, 1, 0) = \{(q_0, 10)\}$
$\delta(q_0, \varepsilon, Z_0) = \{(q_1, Z_0)\}$
$\delta(q_0, \varepsilon, 0) = \{(q_1, 0)\}$
$\delta(q_0, \varepsilon, 1) = \{(q_1, 1)\}$

$\delta(q_1, 0, 0) = \{(q_1, \varepsilon)\}$
$\delta(q_1, 1, 1) = \{(q_1, \varepsilon)\}$
$\delta(q_1, \varepsilon, Z_0) = \{(q_2, Z_0)\}$

$(q_0, ww^R, Z_0) \mid\text{-}^* (q_0, w^R, w^RZ_0) \mid\text{-} (q_1, w^R, w^RZ_0) \mid\text{-}^* (q_1, \varepsilon, Z_0) \mid\text{-} (q_2, \varepsilon, Z_0)$

The transition diagram for the PDA is given in figure 1.

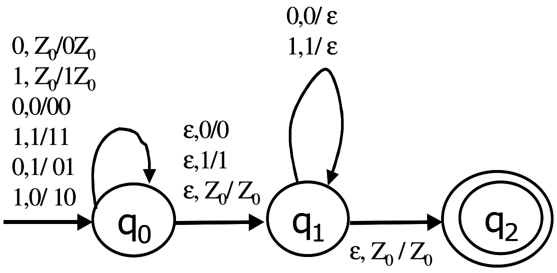


Figure 1: Transition Diagram for L = $\{ww^R\}$

The moves of the PDA for the input string 101101 are given figure 2.

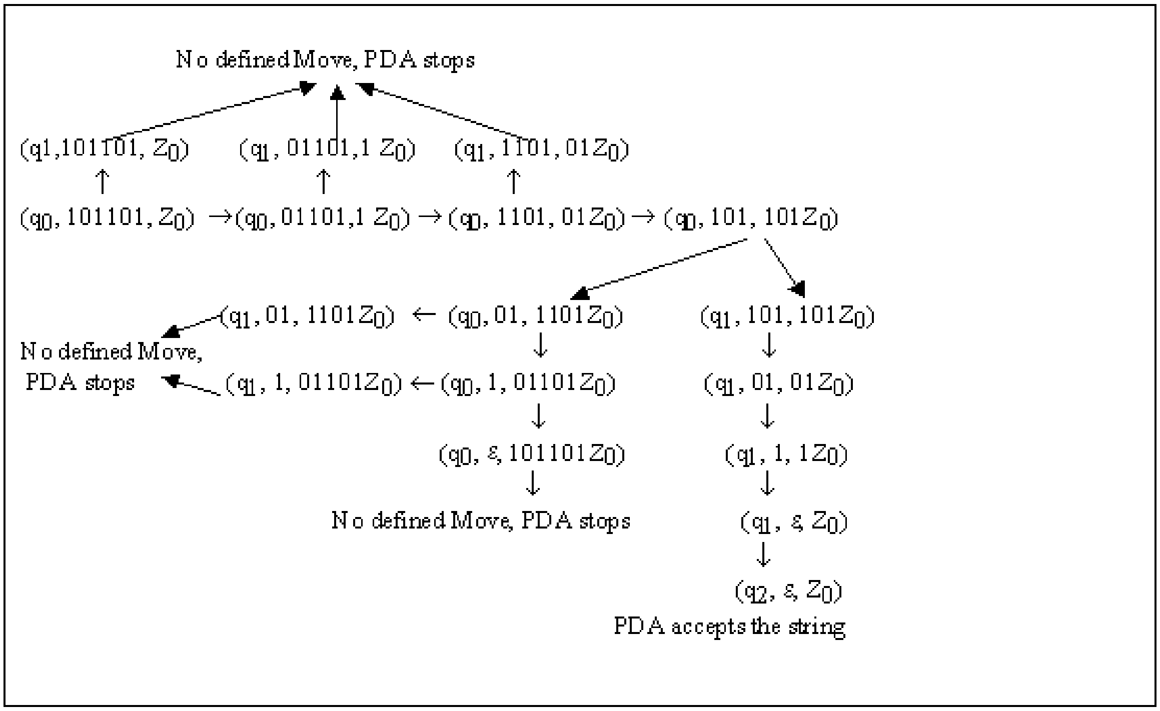


Figure 2: moves of PDA for string 101101

Accept by empty stack:

Let PDA P = $(Q, \Sigma, \Gamma, \delta, q_0, Z_0)$. We define the language accepted by empty stack by

$N(P) = \{w \mid (q_0, w, Z_0) \mid\!-^* (q, \varepsilon, \varepsilon)\}$, where $q \in Q$

Example:

Construct PDA to accept by empty stack for the language $L = \{ww^R \mid w$ is in $(0 + 1)^*\}$

Instead of the transition $\delta(q_1, \varepsilon, Z_0) = \{(q_2, Z_0)\}$ give $\delta(q_1, \varepsilon, Z_0) = \{(q_2, \varepsilon)\}$ to get accept by empty stack. The set of accepting states are irrelevant. This example also shows that L(P) = N(P)

Example:

Construct PDA to accept by empty stack for the language $L=\{0^i1^j \mid 0 \leq i \leq j\}$

The transition diagram for the PDA is given in Figure 3.

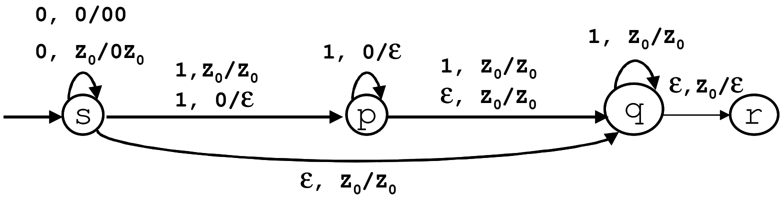


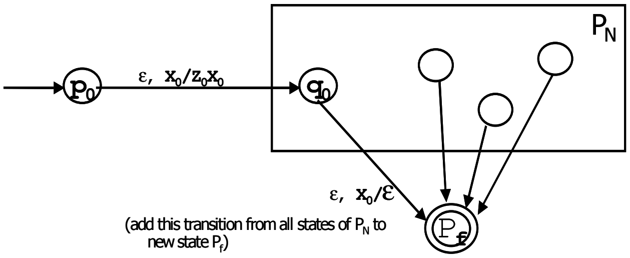
Figure 3: transition diagram of $0^i1^j \mid 0 \leq i \leq$

2. Conversion between the two forms:

a. From Empty Stack to Final State:

Theorem: If $L = N(P_N)$ for some PDA $P_N= (Q, \Sigma, \Gamma , \delta_N, q_0, Z_0)$, then there is a PDA P_F such that $L = L(P_F)$

Proof:



P_N
P₀
ε, x₀/z₀x₀
q₀
ε, x₀/ε
(add this transition from all states of P_N to new state P_f)
P_f

Figure 4: P_F simulates P_N

The method of conversion is given in figure 4.

We use a new symbol X_0, which must be not symbol of Γ to denote the stack start symbol for P_F. Also add a new start state p_0 and final state p_f for P_F. Let P_F = $(Q \cup \{p_0, p_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta_F, p_0, X_0, \{P_f\})$, where δ_F is defined by

$\delta_F(p_0, \varepsilon, X_0) = \{(q_0, Z_0 X_0)\}$ to push X_0 to the bottom of the stack

$\delta_F(q, a, y) = \delta_N(q, a, y)$ $a \in \Sigma$ or $a = \varepsilon$ and $y \in \Gamma$, same for both P_N and P_F.

$\delta_F(q, \varepsilon, X_0) = \{(P_f, \varepsilon)\}$ to accept the string by moving to final state.

The moves of P_F to accept a string w can be written like:

$(p_0, w, X_0) \mid\!-_{P_F} (p_0, w, Z_0X_0) \mid\!-^*_{P_F} (q, \varepsilon, X_0) \;\; \mid\!- (P_f , \varepsilon, \varepsilon)$

b. From Final State to Empty Stack:

Theorem: If $L = L(P_F)$ for some PDA $P_F= (Q, \Sigma, \Gamma, \delta_F, q_0, Z_0, F)$, then there is a PDA P_N such that $L = N(P_N)$

Proof:

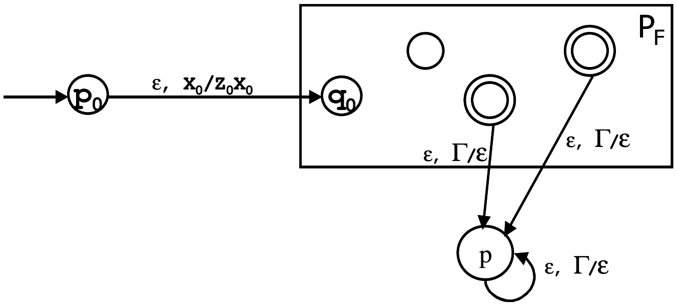


Figure 5: P_N simulates P_F

The method of conversion is given in figure 5.

To avoid P_F accidentally empting its stack, initially change the stack start content from Z_0 to Z_0X_0. Also add a new start state p_0 and final state p for P_N. Let P_N = $(Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{X_0\}, \delta_N, p_0, X_0)$, where δ_N is defined by:

$\delta_N(p_0, \varepsilon, X_0) = \{(q_0, Z_0X_0)\}$ to change the stack content initially

$\delta_N(q , a, y) = \delta_F(q , a, y)$, $a \in \Sigma$ or $a = \varepsilon$ and $y \in \Gamma$, same for both

$\delta_N(q , \varepsilon, y) = \{(p , \varepsilon)\}$, $q \in F$, $y \in \Gamma$ or $y = X_0$, same for both

$\delta_N(p , \varepsilon, y) = \{(p, \varepsilon)\}$, $y \in \Gamma$ or $y = X_0$, to pop the remaining stack contents.

The moves of P_N to accept a string w can be written like:

$(p_0, w, X_0) \mid\!-_{P_N} (q_0, w, Z_0X_0) \mid\!-^*_{P_N} (q, \varepsilon, X_0) \;\; \mid\!- (p, \varepsilon, \varepsilon)$

Example:

Construct PDA to accept by final state the language of all strings of 0's and 1's such that number of 1's is less than number of 0's. Also convert the PDA to accept by empty stack.

Solution:

PDA by final state is given by $M = (\{q_0, q_1\}, \{0, 1\}, \{0, 1, Z\}, \delta, q_0, Z, \{q_1\})$, where δ is given by:

$\delta(q_0, 0, Z) = \{(q_0, 0Z)\}$

$\delta(q_0, 1, Z) = \{(q_0, 1Z)\}$

$\delta(q_0, 0, 0) = \{(q_0, 00)\}$

$\delta(q_0, 0,1) = \{(q_0, \varepsilon)\}$

$\delta(q_0, 1, 1) = \{(q_0, 11)\}$

$\delta(q_0, 1,0) = \{(q_0, \varepsilon)\}$

$\delta(q_0, \varepsilon,Z) = \{(q_1, Z)\}$

$\delta(q_0, \varepsilon,0) = \{(q_1, 0)\}$

For all other moves, PDA stops.

PDA by empty stack is given by $M = (\{q_0, q_1, q_2\}, \{0, 1\}, \{0, 1, Z\}, \delta', q_0, Z)$, where δ' is the union of δ and the transitions given below:

$\delta(q_1, \varepsilon,Z) = \{(q_2, \varepsilon)\}$

$\delta(q_1, \varepsilon,0) = \{(q_2, \varepsilon)\}$

$\delta(q_2, \varepsilon,0) = \{(q_2, \varepsilon)\}$

$\delta(q_2, \varepsilon,Z) = \{(q_2, \varepsilon)\}$

Exercises:

Design nPDA to accept the language:

1. $\{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$
2. $\{a^i b^j c^{i+j} \mid i, j \geq 0\}$
3. $\{a^i b^{i+j} c^j \mid i \geq 0, j \geq 1\}$

Equivalence of PDA and CFG

I. Equivalence of PDA and CFG

The aim is to prove that the following three classes of languages are same:

1. Context Free Language defined by CFG
2. Language accepted by PDA by final state
3. Language accepted by PDA by empty stack

It is possible to convert between any 3 classes. The representation is shown in figure 1.

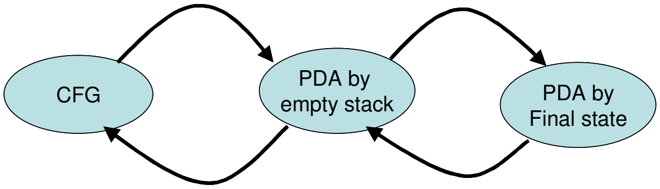


Figure 1: Equivalence of PDA and CFG

From CFG to PDA:

Given a CFG G, we construct a PDA P that simulates the leftmost derivations of G. The stack symbols of the new PDA contain all the terminal and non-terminals of the CFG. There is only one state in the new PDA; all the rest of the information is encoded in the stack. Most transitions are on ε, one for each production. New transitions are added, each one corresponding to terminals of G. For every intermediate sentential form uAα in the leftmost derivation of w (initially w = uv for some v), M will have Aα on its stack after reading u. At the end (case u = w) the stack will be empty.

Let G = (V, T, Q, S) be a CFG. The PDA which accepts L(G) by empty stack is given

by:

P = ({q}, T, V ∪ T, δ, q, S) where δ is defined by:

1. For each variable A include a transition,
   δ(q, ε, A) = {(q, b) | A → b is a production of Q}

2. For each terminal a, include a transition
   δ(q, a, a) = {(q, ε)}

CFG to PDA conversion is another way of constructing PDA. First construct CFG, and then convert CFG to PDA.

Example:

Convert the grammar with following production to PDA accepted by empty stack:

$S \rightarrow 0S1 \mid A$
$A \rightarrow 1A0 \mid S \mid \varepsilon$

Solution:

$P = (\{q\}, \{0, 1\}, \{0, 1, A, S\}, \delta, q, S)$, where δ is given by:

$\delta(q, \varepsilon, S) = \{(q, 0S1), (q, A)\}$
$\delta(q, \varepsilon, A) = \{(q, 1A0), (q, S), (q, \varepsilon)\}$
$\delta(q, 0, 0) = \{(q, \varepsilon)\}$
$\delta(q, 1, 1) = \{(q, \varepsilon)\}$

From PDA to CFG:

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$ be a PDA. An equivalent CFG is $G = (V, \Sigma, R, S)$, where $V = \{S, [pXq]\}$, where $p, q \in Q$ and $X \in \Gamma$, productions of R consists of

1. For all states p, G has productions $S \rightarrow [q_0Z_0\ p]$
2. Let $\delta(q,a,X) = \{(r, Y_1Y_2 \ldots Y_k)\}$ where $a \in \Sigma$ or $a = \varepsilon$, k can be 0 or any number and $r_1r_2 \ldots r_k$ are list of states. G has productions

   $[qXr_k] \rightarrow a[rY_1r_1]\ [r_1Y_2r_2] \ldots [r_{k-1}Y_kr_k]$

   If $k = 0$ then $[qXr] \rightarrow a$

Example:

Construct PDA to accept if-else of a C program and convert it to CFG. (This does not accept if –if –else-else statements).

Let the PDA $P = (\{q\}, \{i, e\}, \{X,Z\}, \delta, q, Z)$, where δ is given by:

$\delta(q, i, Z) = \{(q, XZ)\}$, $\delta(q, e, X) = \{(q, \varepsilon)\}$ and $\delta(q, \varepsilon, Z) = \{(q, \varepsilon)\}$

Solution:

Equivalent productions are:

S → [qZq]
[qZq] → i[qXq][qZq]
[qXq] → e
[qZq] → ε

If [qZq] is renamed to A and [qXq] is renamed to B, then the CFG can be defined by:

G = ({S, A, B}, {i, e}, {S→A, A→iBA | ε, B→ e}, S)

Example:
Convert PDA to CFG. PDA is given by P = ({p,q}, {0,1}, {X,Z}, δ, q, Z)), Transition

function δ is defined by:

δ(q, 1, Z) = {(q, XZ)}
δ(q, 1, X) = {(q, XX)}
δ(q, ε, X) = {(q, ε)}
δ(q, 0, X) = {(p, X)}
δ(p, 1, X) = {(p, ε)}
δ(p, 0, Z) = {(q, Z)}

Solution:

Add productions for start variable
S → [qZq] | [qZp]

For δ(q, 1, Z)= {(q, XZ)}
[qZq] → 1[qXq][qZq]
[qZq] → 1[qXp][pZq]
[qZp] → 1[qXq][qZp]
[qZp] → 1[qXp][pZp]

For δ(q, 1, X)= {(q, XX)}
[qXq] → 1[qXq][qXq]
[qXq] → 1[qXp][pXq]
[qXp] → 1[qXq][qXp]
[qXp] → 1[qXp][pXp]

For δ(q, ε, X) = {(q, ε)}
[qXq] → ε

For δ(q, 0, X) = {(p, X)}

$[qXq] \rightarrow 0[pXq]$
$[qXp] \rightarrow 0[pXp]$

For $\delta(p, 1, X) = \{(p, \varepsilon)\}$
$[pXp] \rightarrow 1$

For $\delta(p, 0, Z) = \{(q, Z)\}$
$[pZq] \rightarrow 0[qZq]$
$[pZp] \rightarrow 0[qZp]$

Renaming the variables [qZq] to A, [qZp] to B, [pZq] to C, [pZp] to D, [qXq] to E [qXp] to F, [pXp] to G and [pXq] to H, the equivalent CFG can be defined by:

G = ({S, A, B, C, D, E, F, G, H}, {0,1}, R, S). The productions of R also are to be renamed accordingly.

Exercises:

a. Convert to PDA, CFG with productions:

1. $A \rightarrow aAA$, $A \rightarrow aS \mid bS \mid a$
2. $S \rightarrow SS \mid (S) \mid \varepsilon$
3. $S \rightarrow aAS \mid bAB \mid aB$, $A \rightarrow bBB \mid aS \mid a$, $B \rightarrow bA \mid a$

b. Convert to CFG, PDA with transition function:

$\delta(q, 0, Z) = \{(q, XZ)\}$
$\delta(q, 0, X) = \{(q, XX)\}$
$\delta(q, 1, X) = \{(q, X)\}$
$\delta(q, \varepsilon, X) = \{(p, \varepsilon)\}$
$\delta(p, 1, X) = \{(p, XX)\}$
$\delta(p, \varepsilon, X) = \{(p, \varepsilon)\}$
$\delta(p, 1, Z) = \{(p, \varepsilon)\}$

II. Deterministic PDA:

NPDA provides non-determinism to PDA. Deterministic PDA’s (DPDA) are very useful for use in programming languages. For example Parsers used in YACC are DPDA’s.

Definition:

A PDA $P= (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is deterministic if and only if,

1.$\delta(q,a,X)$ has at most one move for $q \in Q$, $a \in \Sigma$ or $a= \varepsilon$ and $X \in \Gamma$

2.If $\delta(q,a,X)$ is not empty for some $a\in\Sigma$, then $\delta(q, \varepsilon,X)$ must be empty

DPDA is less powerful than nPDA. The Context Free Languages could be recognized by nPDA. The class of language DPDA accept is in between than of Regular language and CFL. NPDA can be constructed for accepting language of palindromes, but not by DPDA.

Example:

Construct DPDA which accepts the language $L = \{wcw^R \mid w \in \{a, b\}^*, c \in \Sigma\}$.

The transition diagram for the DPDA is given in figure 2.

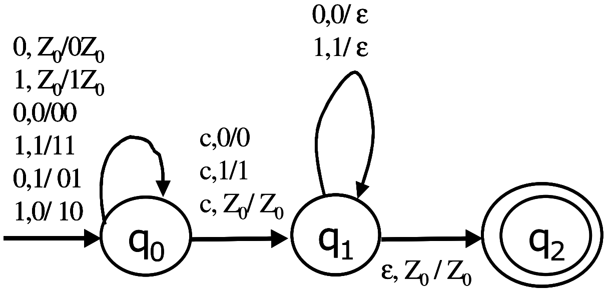


Figure 2: DPDA $L = \{wcw^R\}$

Exercises:

Construct DPDA for the following:

1. Accepting the language of balanced parentheses. (Consider any type of parentheses)
2. Accepting strings with number of a's is more than number of b's
3. Accepting $\{0^n1^m \mid n \geq m\}$

DPDA and Regular Languages:

The class of languages DPDA accepts is in between regular languages and CFLs. The DPDA languages include all regular languages. The two modes of acceptance are not same for DPDA.

Acceptance with final state:

If L is a regular language, L=L(P) for some DPDA P. PDA surely includes a stack, but the DPDA used to simulate a regular language does not use the stack. The stack is inactive always. If A is the FA for accepting the language L, then $\delta_P(q,a,Z)=\{(p,Z)\}$ for all p, $q \in Q$ such that $\delta_A(q,a)=p$.

Acceptance with empty stack:

Every regular language is not N(P) for some DPDA P. A language L = N(P) for some DPDA P if and only if L has prefix property. Definition of prefix property of L states that if x, $y \in$ L, then x should not be a prefix of y, or vice versa. Non-Regular language $L=WcW^R$ could be accepted by DPDA with empty stack, because if you take any x, $y\in$ $L(WcW^R)$, x and y satisfy the prefix property. But the language, L={0*} could be accepted by DPDA with final state, but not with empty stack, because strings of this language do not satisfy the prefix property. So N(P) are properly included in CFL L, ie. $N(P) \subseteq L$

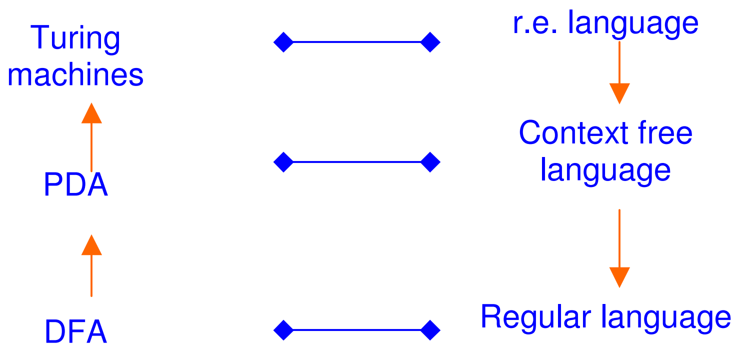
DPDA and Ambiguous grammar:

DPDA is very important to design of programming languages because languages DPDA accept are unambiguous grammars. But all unambiguous grammars are not accepted by DPDA. For example $S \rightarrow 0S0|1S1|\ \varepsilon$ is an unambiguous grammar corresponds to the language of palindromes. This is language is accepted by only nPDA. If L = N(P) for DPDA P, then surely L has unambiguous CFG.

If L = L(P) for DPDA P, then L has unambiguous CFG. To convert L(P) to N(P) to have prefix property by adding an end marker \$ to strings of L. Then convert N(P) to CFG G'. From G' we have to construct G to accept L by getting rid of \$ .So add a new production $\$\rightarrow\varepsilon$ as a variable of G.

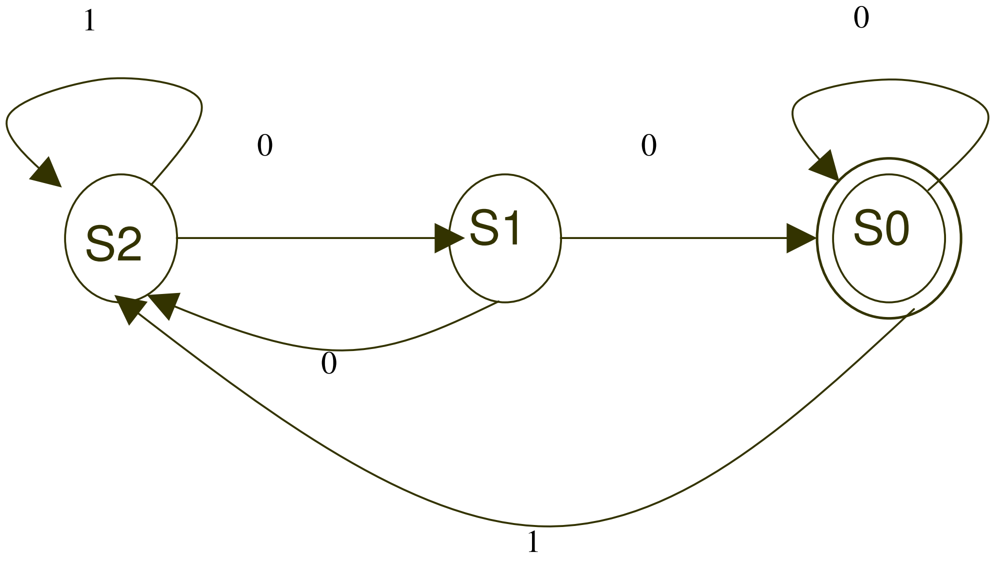
Language Hierarchy and Turing Machine (TM)

A Hierarchy of Formal Languages



Regular language

A language is called a regular language if some finite automaton recognizes it. For example to recognize string that is a multiple of 4



But can regular language recognize strings of the form 0^n1^n ? **No**

Context Free Language

A language is called Context free language if and only if some pushdown automaton recognizes it.

Ex: To recognize a string of the form 0^n1^n

$$S \rightarrow 0S1 / \lambda$$

Limitation again

Can context free language recognize strings of the form $0^n1^n2^n$? No

Recursively Enumerable Language (r.e. language)

A language is called r.e language if some Turing machine recognizes it.

Ex: To recognize strings of the form $0^n1^n2^n$

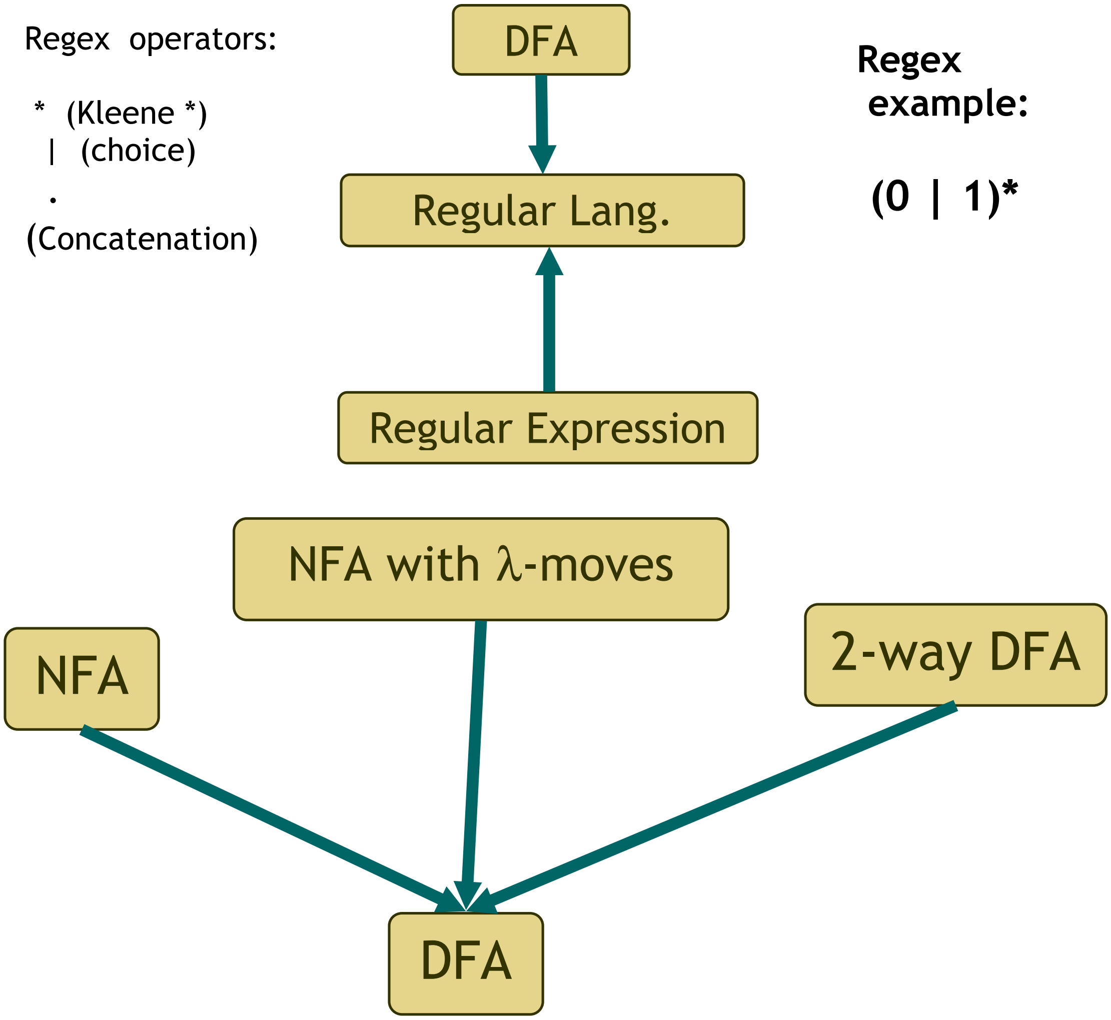
$$\boxed{\begin{array}{c}
(S_0,0) \rightarrow (S_1,X,R) \\
(S_1,0) \rightarrow (S_1,0,R) \\
(S_1,1) \rightarrow (S_2,Y,R) \\
(S_2,1) \rightarrow (S_2,1,R) \\
(S_2,2) \rightarrow (S_3,Z,L) \\
(S_3,1) \rightarrow (S_3,1,L) \\
(S_3,Y) \rightarrow (S_3,Y,L) \\
(S_3,0) \rightarrow (S_3,0,L) \\
(S_3,X) \rightarrow (S_0,X,R) \\
(S_0,Y) \rightarrow (S_A,Y,R)
\end{array}}$$

S_A : Accepting state

Formal Machines Overview

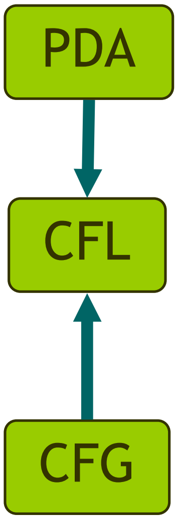
Regex operators:

* (Kleene *)
| (choice)
.
(Concatenation)



Regex example:

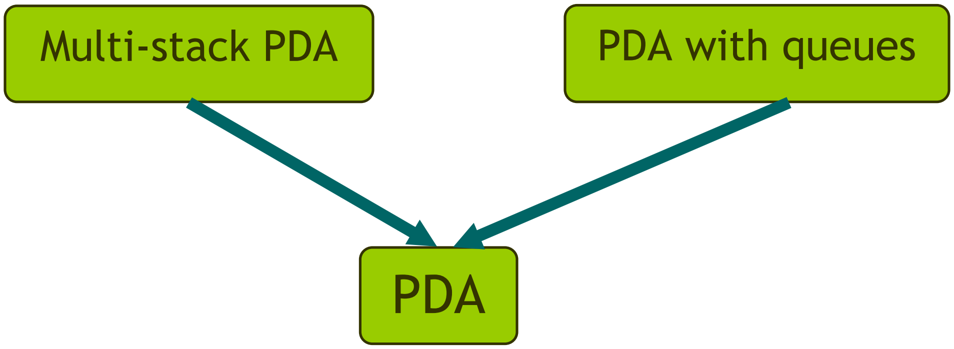
(0 | 1)*



CFL example:

0^n1^n

But there are strings that cannot be recognized by PDAs. For example: **$a^nb^nc^n$**



Some Historical Notes

At the turn of the 20th century the German mathematician David Hilbert proposed the
Entscheidungsproblem. Given a set X, and a universe of elements U, and a criteria for membership, is it possible to formulate a *general* procedureto decide whether a given element of U is a member of X?

Hilbert's grand ideas were watered down by the *incompleteness theorem* proposed by the Austrian Kurt Gödel. His theorem states that in any mathematical system, there exist certain obviously true assertions that cannot be *proved* to be true by the system.

In 1936 the British cryptologist Alan Turing, addressed Hilbert's *Entscheidungsproblem* using a different approach. He proposed two kinds of mathematical machines called the *a-machine* and the *c-machine* respectively, and showed the existence of some problems where membership cannot be determined.

But Turing's *a-machine* became famous for something else other than its original intentions. The *a-machine* was found to be *more expressive* than CFGs. It can recognize strings that cannot be modeled by CFLs like $a^nb^nc^n$. The *a-machines* came to be more popularly known as Turing Machines

The Princeton mathematician Alonzo Church recognized the power of *a-machines*. He invited Turing to Princeton to compare Turing Machines with his own l-calculus.

Church and Turing proved the equivalence of Turing Machines and l-calculus, and showed that they represent algorithmic computation. This is called the Church-Turing thesis.

Turing Machines

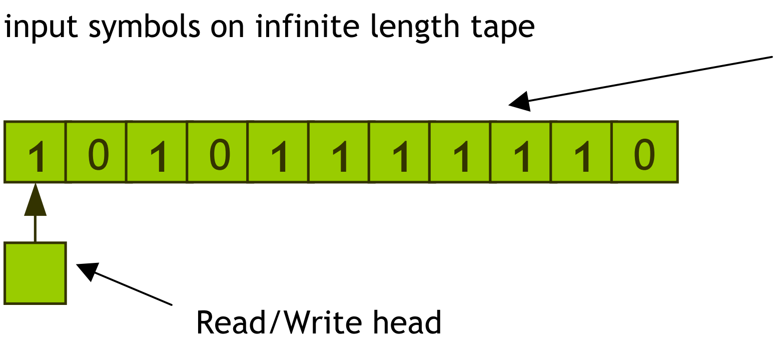
Definition:

A Turing Machine (TM) is an abstract, mathematical model that describes what can and cannot be computed. A Turing Machine consists of a tape of infinite length, on which input is provided as a finite sequence of symbols. A *head* reads the input tape. The Turing Machine starts at "start state" S_0. On reading an input symbol it optionally replaces it with another symbol, changes its internal state and moves one cell to the right or left.

Notation for the Turing Machine:

TM = $<Q, T, q_0, \delta, H>$ where,

Q = a set of TM states
T = a set of tape symbols
q_0 = the start state
$H \subset S$ is a set of halting states
$\delta : Q \times T \rightarrow Q \times T \times \{L,R\}$ is the transition function {L,R} is direction in which the head moves L : Left R: Right



The Turing machine model uses an infinite tape as its unlimited memory. (This is important because it helps to show that there are tasks that these machines cannot perform, even though unlimited memory and unlimited time is given.) The input symbols occupy some of the tape's cells, and other cells contain blank symbols.

Some of the characteristics of a Turing machine are:

1. The symbols can be both read from the tape and written on it.
2. The TM head can move in either direction – Left or Right.
3. The tape is of infinite length
4. The special states, Halting states and Accepting states, take immediate effect.

Solved examples:

TM Example 1:

Turing Machine U+1:

Given a string of 1s on a tape (followed by an infinite number of 0s), add one more 1 at the end of the string.

Input :#111100000000…….

➔

Output : #1111100000000……….

Initially the TM is in Start state S_0. Move right as long as the input symbol is 1. When a 0 is encountered, replace it with 1 and halt.
Transitions:

$(S_0, 1) \rightarrow (S_0, 1, R)$
$(S_0, 0) \rightarrow (h, 1, STOP)$

TM Example 2 :

TM: X-Y

Given two unary numbers x and y, compute |x-y| using a TM. For purposes of simplicity we shall be using multiple tape symbols.

Example: 5 (11111) – 3 (111) = 2 (11)
#11111b1110000….. ➔
#___11b___000…

a) Stamp out the first 1 of x and seek the first 1 of y.

(S_0, 1) → (S_1, _, R)
(S_0, b) → (h, b, STOP)
(S_1, 1) → (S_1, 1, R)
(S_1, b) → (S_2, b, R)

b) Once the first 1 of y is reached, stamp it out. If instead the input ends, then y has finished. But in x, we have stamped out one extra 1, which we should replace. So, go to some state s5 which can handle this.

(S_2, 1) → (S_3, _, L)
(S_2,_) → (S_2, _, R)
(S_2, 0) → (S_5, 0, L)

c) State s3 is when corresponding 1s from both x and y have been stamped out. Now go back to x to find the next 1 to stamp. While searching for the next 1 from x, if we reach the head of tape, then stop.

(S_3, _) → (S_3, _, L)
(S_3,b) → (S_4, b, L)
(S_4, 1) → (S_4, 1, L)
(S_4, _) → (S_0, _, R)
(S_4, #) → (h, #, STOP)

d) State s5 is when y ended while we were looking for a 1 to stamp. This means we have stamped out one extra 1 in x. So, go back to x, and replace the blank character with 1 and stop the process.

(S_5, _) → (S_5, _, L)
(S_5,b) → (S_6, b, L)
(S_6, 1) → (S_6, 1, L)
(S_6, _) → (h, 1, STOP)

Design of Turing Machines and Universal Turing Machine

Solved examples:

TM Example 1: Design a Turing Machine to recognize $0^n1^n2^n$
ex: #000111222_ _ _ _ _.......

Step 1: Stamp the first 0 with X, then seek the first 1 and stamp it with Y, and then seek the first 2 and stamp it with Z and then move left.

$$\boxed{\begin{array}{l} (S_0,0) \rightarrow (S_1,X,R) \\ (S_1,0) \rightarrow (S_1,0,R) \\ (S_1,1) \rightarrow (S_2,Y,R) \\ (S_2,1) \rightarrow (S_2,1,R) \\ (S_2,2) \rightarrow (S_3,Z,L) \end{array}}$$

S_0 = Start State, seeking 0, stamp it with X
S1 = Seeking 1, stamp it with Y
S2 = Seeking 2, stamp it with Z

Step 2: Move left until an X is reached, then move one step right.

$$\boxed{\begin{array}{l} (S_3,1) \rightarrow (S_3,1,L) \\ (S_3,Y) \rightarrow (S_3,Y,L) \\ (S_3,0) \rightarrow (S_3,0,L) \\ (S_3,X) \rightarrow (S_0,X,R) \end{array}}$$

S3 = Seeking X, to repeat the process.

Step 3: Move right until the end of the input denoted by blank(_) is reached passing through X Y Z s only, then the final state S_A is reached.

$$\boxed{\begin{array}{l} (S_0,Y) \rightarrow (S_4,Y,R) \\ (S_4,Y) \rightarrow (S_4,Y,R) \\ (S_4,Z) \rightarrow (S_4,Z,R) \\ (S_4,\_) \rightarrow (S_A,\_,STOP) \end{array}}$$

S4 = Seeking blank

These are the transitions that result in halting states.

$$\boxed{\begin{array}{l}
(S_4,1)\rightarrow(h,1,STOP)\\
(S_4,2)\rightarrow(h,2,STOP)\\
(S_4,\_)\rightarrow(S_A,\_,STOP)\\
(S_0,1)\rightarrow(h,1,STOP)\\
(S_0,2)\rightarrow(h,2,STOP)\\
(S_1,2)\rightarrow(h,2,STOP)\\
(S_2,\_)\rightarrow(h,\_,STOP)
\end{array}}$$

TM Example 2 : Design a Turing machine to accept a Palindrome

ex: #1011101_ _ _ _ _.......

Step 1: Stamp the first character (0/1) with _, then seek the last character by moving till a _ is reached. If the last character is not 0/1 (as required) then halt the process immediately.

$$\boxed{\begin{array}{l}
(S_0,0)\rightarrow(S_1,\_,R)\\
(S_0,1)\rightarrow(S_2,\_,R)\\
(S_1,\_)\rightarrow(S_3,\_,L)\\
(S_3,1)\rightarrow(h,1,STOP)\\
(S_2,\_)\rightarrow(S_5,\_,L)\\
(S_5,0)\rightarrow(h,0,STOP)
\end{array}}$$

Step 2: If the last character is 0/1 accordingly, then move left until a blank is reached to start the process again.

$$\boxed{\begin{array}{l}
(S_3,0)\rightarrow(S_4,\_,L)\\
(S_4,1)\rightarrow(S_4,1,L)\\
(S_4,0)\rightarrow(S_4,0,L)\\
(S_4,\_)\rightarrow(S_0,\_,R)\\
(S_5,1)\rightarrow(S_6,\_,L)\\
(S_6,1)\rightarrow(S_6,1,L)\\
(S_6,0)\rightarrow(S_6,0,L)\\
(S_6,\_)\rightarrow(S_0,\_,R)
\end{array}}$$

Step 3: If a blank (_) is reached when seeking next pair of characters to match or when seeking a matching character, then accepting state is reached.

$$\boxed{\begin{array}{l}
(S_3,\_)\rightarrow(S_A,\_,STOP)\\
(S_5,\_)\rightarrow(S_A,\_,STOP)\\
(S_0,\_)\rightarrow(S_A,\_,STOP)
\end{array}}$$

The sequence of events for the above given input are as follows:

#$s_0$10101_ _ _
#_$s_2$0101_ _ _
#_0$s_2$101_ _ _
. . . .
#_0101s_5_ _ _
#_010s_6_ _ _ _
#_$s_6$0101_ _ _
#_$s_0$0101_ _ _
. . . .
#_ _ _ _ s_5 _ _ _ _ _ _
#_ _ _ _ s_A _ _ _ _ _ _ _

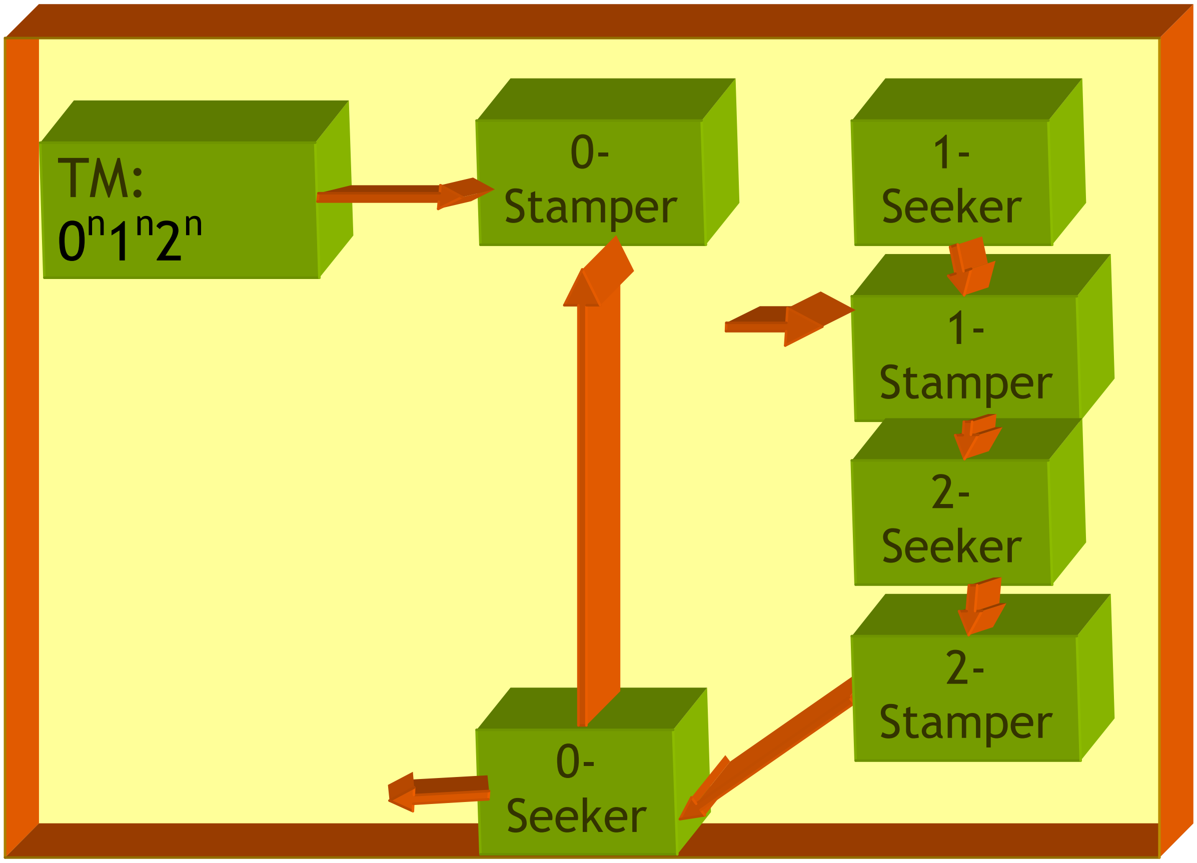
Exercises:

1. Design a TM to recognize a string of the form $a^n b^{2n}$.
2. Design a TM to recognize a string of 0s and 1s such that the number of 0s is not twice as that of 1s.

Modularization of TMs

Designing complex TM s can be done using modular approach. The main problem can be divided into sequence of modules. Inside each module, there could be several state transitions.

For example, the problem of designing Turing machine to recognize the language $0^n1^n2^n$ can be divided into modules such as 0-stamper, 1-stamper, 0-seeker, 1-seeker, 2-seeker and 2-stamper. The associations between the modules are shown in the following figure:



Load → Decode → Execute → Store

Universal Turing Machine (UTM)

A Universal Turing Machine UTM takes an encoding of a TM and the input data as its input in its tape and behaves as that TM on the input data.

A TM spec could be as follows:

TM = (S,S0,H,T,d)
Suppose, S={a,b,c,d}, S0=a, H={b,d} T={0,1}
d : (a,0) (b,1,R) , (a,1) (c,1,R) , (c,0) (d,0,R) and so on
then TM spec: $abcd$a$bd$01$a0b1Ra1c1Rc0d0R.......
where $ is delimiter

This spec along with the actual input data would be the input to the UTM.
This can be encoded in binary by assigning numbers to each of the characters appearing in the TM spec.

The encoding can be as follows:

$: 0000	0 : 0101
a : 0001	1 : 0110
b : 0010	L : 0111
c : 0011	R : 1000
d : 0100	

So the TM spec given in previous slide can be encoded as:

0000.0001.0010.0011.0100.0000.0001.0000.0010.0100 ……

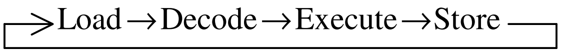
Hence TM spec can be regarded just as a number.

Sequence of actions in UTM:

Initially UTM is in the start state S0.

- Load the input that is TM spec.
- Go back and find which transition to apply.
- Make changes, where necessary.
- Then store the changes.
- Then repeat the steps with next input.

Hence, the sequence goes through the cycle:



Load → Decode → Execute → Store

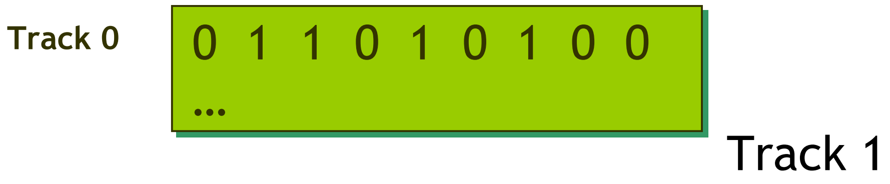
Extensions to Turing Machines

Proving Equivalence

For any two machines M_1 from class C_1 and M_2 from class C_2:

M_2 is said to be at least as expressive as M_1
if $L(M_2) = L(M_1)$ or if M_2 can *simulate* M_1.
M_1 is said to be at least as expressive as M_2
if $L(M_1) = L(M_2)$ or if M_1 can simulate M_2.

Composite Tape TMs



A composite tape consists of many *tracks* which can be read or written *simultaneously*.

A composite tape TM (CTM) contains more than one tracks in its tape.

Equivalence of CTMs and TMs

A CTM is simply a TM with a complex alphabet..

T = {a, b, c, d}

T' = {00, 01, 10, 11}

Turing Machines with Stay Option

Turing Machines with stay option has a third option for movement of the TM head:
left, right or *stay*.

STM = <S, T, δ, s_0, H>

δ: S x T à S x T x {L, R, S}

Equivalence of STMs and TMs

STM = TM:

Just don't use the S option…

TM = STM:

For L and R moves of a given STM build a TM that moves correspondingly L or R…

TM = STM:

For S moves of the STM, do the following:

1. Move right,
2. Move back left without changing the tape

3.STM: δ(s,a) |-- (s',b,S)
TM: δ(s,a) |-- (s'', b, R)
δ(s'',*) |-- (s',*,L)

2-way Infinite Turing Machine

In a 2-way infinite TM (2TM), the tape is infinite on both sides.
There is no # that delimits the left end of the tape.

Equivalence of 2TMs and TMs

2TM = TM:

Just don't use the left part of the tape…

TM = 2TM:

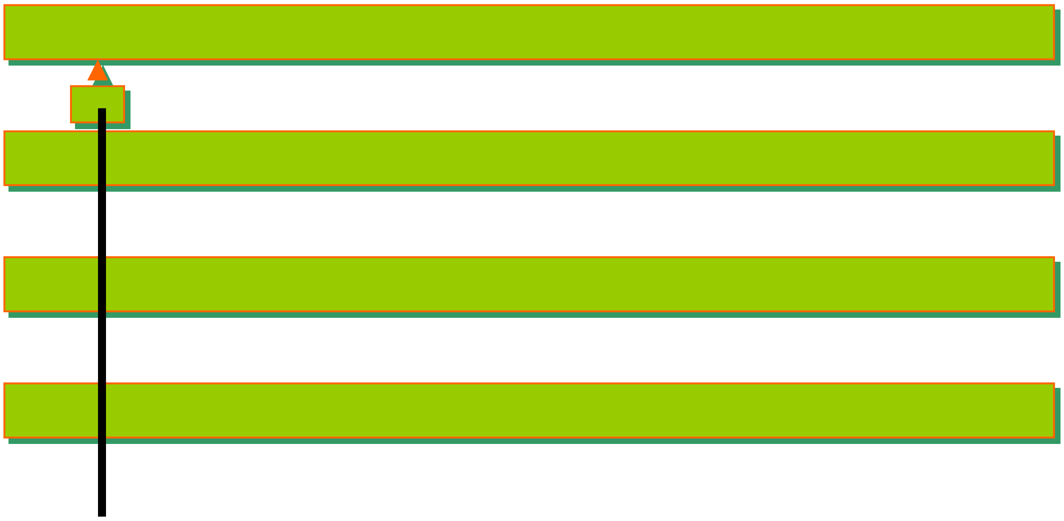
Simulate a 2-way infinite tape on a one-way infinite tape…

… -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6

0 -1 1 -2 2 -3 3 -4 4 -5 5

Multi-tape Turing Machines

A multi-tape TM (MTM) utilizes many tapes.



Equivalence of MTMs and TMs

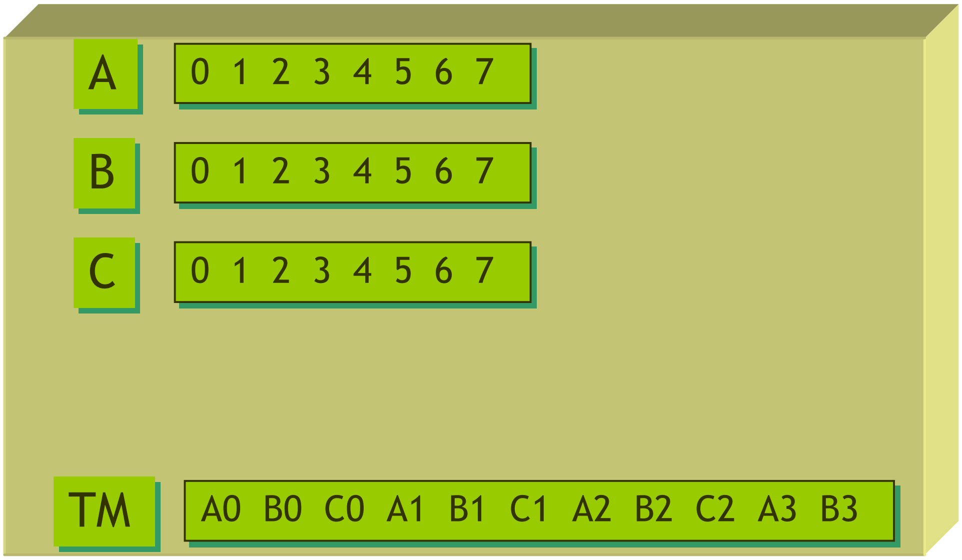
MTM = TM:

Use just the first tape…

TM = MTM:

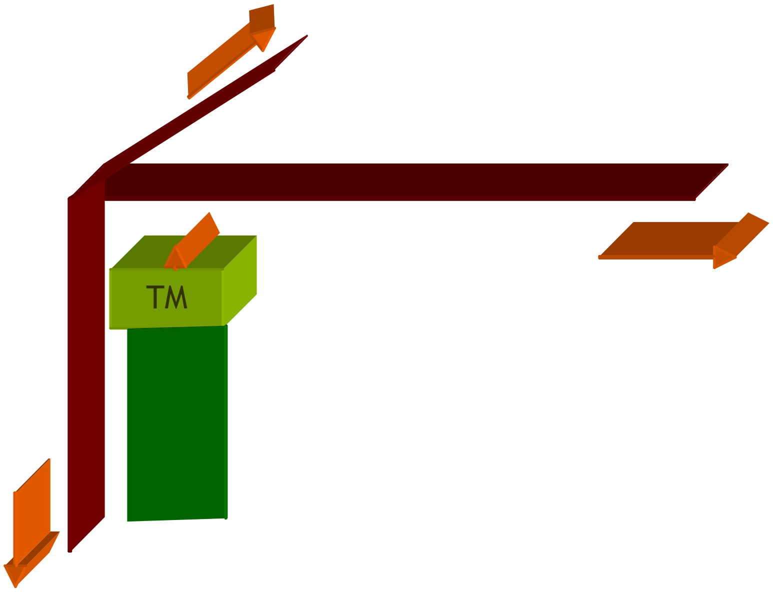
Reduction of multiple tapes to a single tape.

Consider an MTM having *m* tapes. A single tape TM that is equivalent can be constructed by reducing *m* tapes to a single tape.



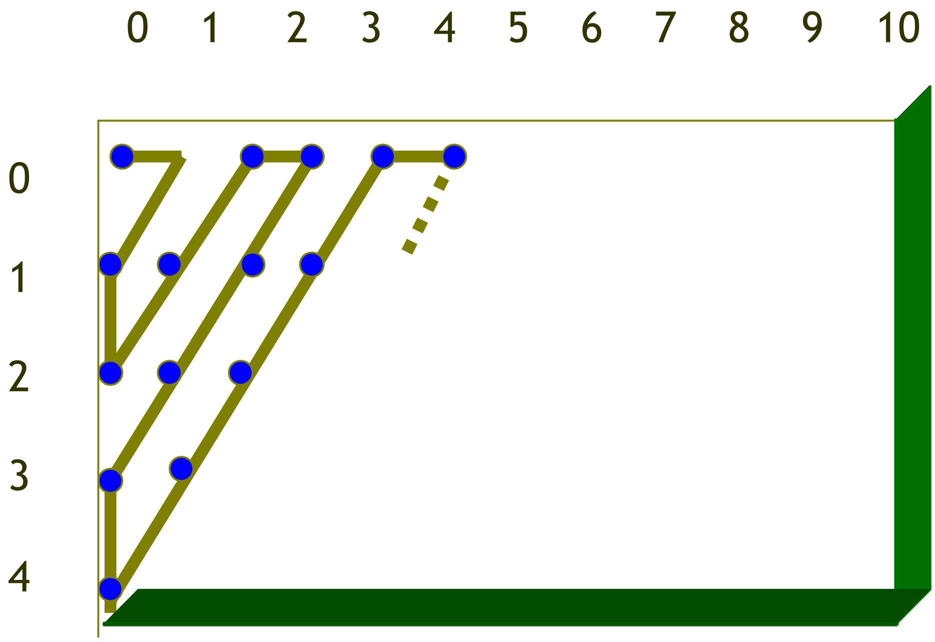
Multi-dimensional TMs

Multi-dimensional TMs (MDTMs) use a multi-dimensional space instead of a single dimensional tape.



Equivalence of MDTMs to TMs

Reducing a multi-dimensional space to a single dimensional tape.



Non-deterministic TM

A non-deterministic TM (NTM) is defined as:

NTM = <S, T, s_0, δ, H>

where δ: S x T x $2^{SxTx\{L,R\}}$

Ex: (s_2, a) |-- {(s_3, b, L), (s_4, a, R)}

Equivalence of NTMs and TMs

A “concurrent” view of an NTM:

(s_2,a) |-- {(s_3,b,L) (s_4,a,R)}

|-- at (s_2,a), two TMs are spawned:

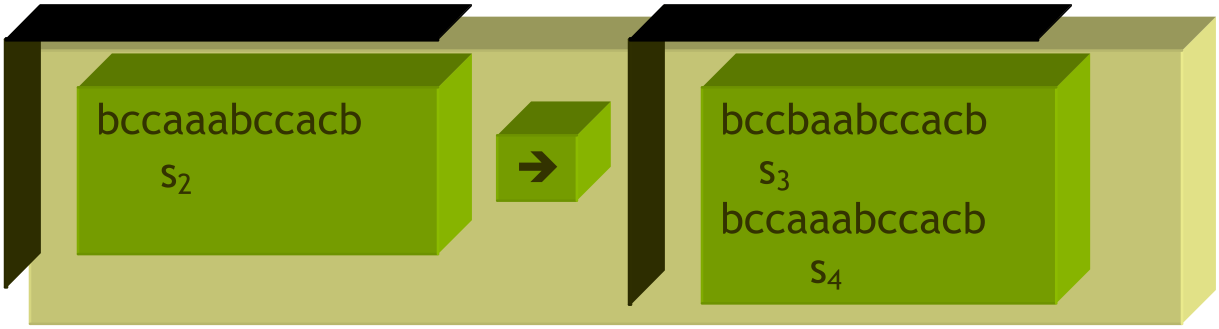
(s_2,a) |-- (s_3,b,L)

(s_2,a) |-- (s_4,a,R)

Simulating an NTM with an MDTM

Consider an MDTM to simulate an NTM:

(s_2,a) |-- {(s_3, b, L) (s_4, a, R)}



bccaaabccacb

s_2

➔

bccbaabccacb

s_3

bccaaabccacb

s_4