SOME OF RAMANUJAN'S RESULTS

These are fifteen of the approximately 120 results that appear in Ramanujan's first letter to Hardy, as published by Hardy. The numbering is due to Hardy.

$$1-\frac{3!}{(1!2!)^3}x^2+\frac{6!}{(2!4!)^3}x^4-\cdots= \left(1+\frac{x}{(1!)^3}+\frac{x^2}{(2!)^3}+\cdots\right)\left(1-\frac{x}{(1!)^3}+\frac{x^2}{(2!)^3}-\cdots\right) \tag{1.1}$$

$$1-5\left(\frac{1}{2}\right)^3+9\left(\frac{1\cdot 3}{2\cdot 4}\right)^3-13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3+\cdots=\frac{2}{\pi} \tag{1.2}$$

$$1+9\left(\frac{1}{4}\right)^4+17\left(\frac{1\cdot 5}{4\cdot 8}\right)^4+25\left(\frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}\right)^4+\cdots=\frac{2\sqrt{2}}{\sqrt{\pi}\Gamma(\frac{3}{4})^2} \tag{1.3}$$

$$1-5\left(\frac{1}{2}\right)^5+9\left(\frac{1\cdot 3}{2\cdot 4}\right)^5-13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^5+\cdots=\frac{2}{\Gamma(\frac{3}{4})^4} \tag{1.4}$$

$$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\cdots dx=\frac{\sqrt{\pi}}{2}\,\frac{\Gamma(a+\frac{1}{2})\Gamma(b+1)\Gamma(b-a+\frac{1}{2})}{\Gamma(a)\Gamma(b+\frac{1}{2})\Gamma(b-a+1)} \tag{1.5}$$

$$\int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\cdots}=\frac{\pi}{2(1+r+r^3+r^6+r^{10}+\cdots)} \tag{1.6}$$

(1.7) If $\alpha\beta=\pi^2$, then

$$\alpha^{-\frac{1}{4}}\left(1+4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}dx\right)=\beta^{-\frac{1}{4}}\left(1+4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}dx\right).$$

$$\int_0^a e^{-x^2}dx=\frac{\sqrt{\pi}}{2}-\frac{e^{-a^2}}{2a+}\,\frac{1}{a+}\,\frac{2}{2a+}\,\frac{3}{a+}\,\frac{4}{2a+\cdots} \tag{1.8}$$

$$4\int_0^\infty \frac{xe^{-x\sqrt{5}}}{\cosh x}dx=\frac{1}{1+}\,\frac{1^2}{1+}\,\frac{1^2}{1+}\,\frac{2^2}{1+}\,\frac{2^2}{1+}\,\frac{3^2}{1+}\,\frac{3^2}{1+\cdots} \tag{1.9}$$

Typeset by $\mathcal{AMS}$-TEX

(1.10) If

$$u = \frac{x}{1+}\frac{x^5}{1+}\frac{x^{10}}{1+}\frac{x^{15}}{1+\cdots} \quad \text{and} \quad v = \frac{x^{\frac{1}{5}}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+\cdots},$$

then

$$v^5 = u\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$$

$$\frac{1}{1+}\frac{e^{-2\pi}}{1+}\frac{e^{-4\pi}}{1+\cdots} = \left[\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right]e^{\frac{2}{5}\pi} \tag{1.11}$$

$$\frac{1}{1+}\frac{e^{-2\pi\sqrt{5}}}{1+}\frac{e^{-4\pi\sqrt{5}}}{1+\cdots} = \left\{\frac{\sqrt{5}}{1+\left[5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1\right]^{\frac{1}{5}}} - \frac{\sqrt{5}+1}{2}\right\}e^{\frac{2\pi}{\sqrt{5}}} \tag{1.12}$$

(1.13) If $F(k) = 1 + (\frac{1}{2})^2 k + (\frac{1\cdot 3}{2\cdot 4})^2 k^2 + \cdots$ and $F(1-k) = \sqrt{210}F(k)$, then

$k = (\sqrt{2}-1)^4(2-\sqrt{3})^2(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2(\sqrt{10}-3)^4(4-\sqrt{15})^4(\sqrt{15}-\sqrt{14})^2(6-\sqrt{35})^2.$

(1.14) The coefficient of x^n in $(1-2x+2x^4-2x^9+\cdots)^{-1}$ is the integer nearest to

$$\frac{1}{4n}\left(\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right).$$

(1.15) The number of numbers between A and x which are either squares or sums of two squares is

$$K\int_A^x \frac{dt}{\sqrt{\log t}} + \theta(x),$$

where $K = 0.764\ldots$ and $\theta(x)$ is very small compared with the previous integral.