Basic Probabilities, Sampling Distribution

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Basic Probabilities

Review

The notes are based on [1] [2] [3] [4] [5]

Expectation

Expectation for Discrete Random Variable:

$$E(X) = \sum x_i f(x_i)$$

Expectation for Continuous Random Variable:

$$E(X) = \int x f(x) dx$$

Convergence of Sequence

Types of Convergence

In this section, we will develop the theoretical background to study the convergence of a sequence of random variables in more detail. In particular, we will define different types of convergence. When we say that the sequence X_n converges to X, it means that X_n 's are getting''closer and closer" to X. Different types of convergence refer to different ways of defining what ''closer" means. We also discuss how different types of convergence are related.

Types of Convergence

Convergence of Sequence

Convergence of Sequence: A sequence $a_1, a_2, a_3, \cdots, a_n$ *converges to a limit* L *if*

$$\lim_{n \to \infty} a_n = L$$

That is, for any $\epsilon > 0$*, there exists an* $N \in \mathbb{N}$ *such that*

$$|a_n - L| < \epsilon, \quad \text{for all } n > N$$

Basic Probability Theory

σ-algrbra, algebra and semi-ring

Definition: algebra, σ-algrbra, semi-ring

FAQ: Exampleof an algebra not a σ-algebra

Not required, just as an introduction. If you are very intersted in probability theory, it is a recommendation to find out why we need these definitions.

Convergence of Sequence

Sequence of Random Variables

In statistics, we draw a sample to make inference of the population, then, if we repeatly draw samples, we will have a sequence of samples from the same population, we usually refer them as i.i.d. (independent and identical distributed) or random samples. This can be denoted as

$$\{\Omega, \Sigma, P\}$$

where Ω is the sample space.

Convergence of Sequence

Sequence of Random Variables

$$\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}, \quad w_i \text{ are simple(single) events}$$

Σ is the σ-algebra (You may consider it is set of the sets of simple events in brief) and P is a probability measure.

Convergence of Sequence

However, if we consider the samples not necessarily from the same population, we may have a sequence of random variables $X_1, X_2, \cdots$, and an correspnded underlying sample space Ω. In particular, each X_n is a function from its Ω, to real numbers through the probability measure P.

In other words, a sequence of random variables is in fact a sequence of functions (Mapping, or P, or a probability measure) $X_n : \Omega \to \mathbb{R}$, such as

$$P(\omega_i) = x_i, \quad \omega_i \in \Omega \text{ and } \sum x_i = 1, \quad i = 1, \cdots, n$$

Example: Convergence of Sequence of R.V.

Consider the following random experiment: A fair coin is tossed once. Here, the sample space has only two elements $S = \{H, T\}$. We define a sequence of random variables $X_1, X_2, \cdots$ on this sample space as follows:

$$X_n(s) = \begin{cases} \frac{1}{n+1} & \text{if } s = H \\ 1 & \text{if } s = T \end{cases}$$

- Are the X_i independent?
  No, they are dependent as they are measuring the same coin.

Example: Convergence of Sequence of R.V.

- Find the PMF and CDF of X_n, $F_{X_n}(x)$ for $n = 1, 2, 3, \cdots$.

The PMF are

$$P_{X_n}(x) = P(X_n = x) = \begin{cases} \frac{1}{2} & \text{if } x = \frac{1}{n+1} \\ \frac{1}{2} & \text{if } x = 1 \end{cases}$$

Correspondingly, the CDF are

$$F_{X_n}(x) = P(X_n \leq x) = \begin{cases} 1 & \text{if } x \geq 1 \\ \frac{1}{2} & \text{if } \frac{1}{n+1} \leq x < 1 \\ 0 & \text{if } x < \frac{1}{n+1} \end{cases}$$

Example: Convergence of Sequence of R.V.

- As n goes to infinity, what does $F_{X_n}(x)$ look like?

Example: Suppose that T is R.V. as above, derive its p.d.f.

1. If T is given by $\frac{U}{\sqrt{V/k}}$, find the joint density of U and V.
2. Find the density function of T.

$$f_{U,V}(u,v) = \underbrace{\frac{1}{(2\pi)^{1/2}} e^{-u^2/2}}_{\text{pdf } N(0,1)} \quad \underbrace{\frac{1}{\Gamma\left(\frac{k}{2}\right) 2^{k/2}} v^{(k/2)-1} e^{-v/2}}_{\text{pdf } \chi^2_k}$$

Denote

$$t = \frac{u}{\sqrt{v/k}}, \quad w = v$$

where

$$u = t\left(\frac{w}{k}\right)^{1/2}, \quad v = w$$

Example: Calculation

The tensile strength for a type of wire is normally distributed with unknown mean μ and unknown variance σ^2. Six pieces of wire were randomly selected from a large roll; Y_i, the tensile strength for portion i, is measured for $i = 1, 2, ..., 6$. The population mean μ and variance σ^2 can be estimated by $\bar{Y}$ and s^2, respectively.

Find the approximate probability that $\bar{Y}$ will be within $2S/\sqrt{n}$ of the true population mean μ.

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[1] 0.8980605
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Example: Calculation

As

$$T = \frac{\bar{Y} - \mu}{S/\sqrt{n}}$$

Then

$$P(|\bar{Y} - \mu| \le 2S/\sqrt{n}) = P(-2 \le T \le 2) = P(T \le 2) - P(T \le -2) = ?$$

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pt(2,5)-pt(-2,5)
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[1] 0.8980605
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The F Distribution

Suppose that we want to compare the variances of two normal populations based on information contained in independent random samples from the two populations.

The F Distribution: Let W_1 and W_2 be independent χ^2 distributed random variables with v_1 and v_2 degree of freedom. Then,

$$F = \frac{W_1/v_1}{W_2/v_2} = \frac{(n-1)S_1^2/\sigma_1^2/(n_1-1)}{(n-1)S_2^2/\sigma_2^2/(n_2-1)} = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

is an F distribution, $F(v_1 = n_1 - 1, v_2 = n_2 - 1)$.

Example: Calculation

If there are two popluation with equal variance, we draw two sample with size $n_1 = 6$ and $n_2 = 10$, such that

$$P(\frac{S_1^2}{S_2^2} \leq b) = 0.95$$

What is the value of b?

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[1] 3.481659
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Reference

[1] Pishro-Nik, H. (2014). *Introduction to probability, statistics, and random processes*. Kappa Research, LLC Blue Bell, PA, USA.

[2] Larsen, R. J. and Marx, M. L. (2005). *An introduction to mathematical statistics*. Prentice Hall Hoboken, NJ.

[3] Casella, G. and Berger, R. (2024). *Statistical inference*. CRC Press.

[4] Tao, T. (2008). The strong law of large number.

[5] Anon. (2014). Derivation of t distribution density.