IIITB COMET FOUNDATION

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Tenth International Olympiad, 1968

1. Prove that there is one and only one triangle whose side lengths are consecutive integers, and one of whose angles is twice as large as another.

2. Find all natural numbers x such that the product of their digits (in decimal notation) is equal to $x^2 - 10x - 22$.

3. Consider the system of equations $ax_1^2 + bx_1 + c = x_2$
$ax_2^2 + bx_2 + c = x_3$
$\vdots$
$ax_n^2 + bx_n + c = x_1$, with unknowns $x_1, x_2, \ldots, x_n$, where a, b, c are real and $a \neq 0$. Let $\Delta = (b-1)^2 - 4ac$. Prove that for this system:
   (a) if $\Delta < 0$, there is no solution,
   (b) if $\Delta = 0$, there is exactly one solution,
   (c) if $\Delta > 0$, there is more than one solution.

4. Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which are the sides of a triangle.

5. Let f be a real-valued function defined for all real numbers x such that, for some positive constant a, the equation $f(x+a) = \frac{1}{2} + \sqrt{f(x) - [f(x)]^2}$ holds for all x.
   (a) Prove that the function f is periodic (i.e., there exists a positive number b such that $f(x+b) = f(x)$ for all x).
   (b) For $a = 1$, give an example of a non-constant function with the required properties.

6. For every natural number n, evaluate the sum $\sum_{k=0}^{\infty} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \cdots + \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor + \cdots$ (The symbol $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.)

Eleventh International Olympiad, 1969

1. Prove that there are infinitely many natural numbers a with the following property: the number $z = n^4 + a$ is not prime for any natural number n.

2. Let $a_1, a_2, \ldots, a_n$ be real constants, x a real variable, and $f(x) = \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{4}\cos(a_3 + x) + \cdots + \frac{1}{2^{n-1}}\cos(a_n + x)$. Given that $f(x_1) = f(x_2) = 0$, prove that $x_2 - x_1 = m\pi$ for some integer m.

3. For each value of $k = 1, 2, 3, 4, 5$, find necessary and sufficient conditions on the number $a > 0$ so that there exists a tetrahedron with k edges of length a, and the remaining $6 - k$ edges of length 1.

4. A semicircular arc γ is drawn on AB as diameter. C is a point on γ other than A and B, and D is the foot of the perpendicular from C to AB. We consider three circles, $\gamma_1, \gamma_2, \gamma_3$, all tangent to the line AB. Of these, γ_1 is inscribed in $\triangle ABC$, while γ_2 and γ_3 are both tangent to CD and to γ, one on each side of CD. Prove that γ_1, γ_2, and γ_3 have a second tangent in common.

5. Given $n > 4$ points in the plane such that no three are collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.

6. Prove that for all real numbers $x_1, x_2, y_1, y_2, z_1, z_2$, with $x_1 > 0$, $x_2 > 0$, $x_1y_1 - z_1^2 > 0$, $x_2y_2 - z_2^2 > 0$, the inequality $\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2} \leq \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2}$ is satisfied. Give necessary and sufficient conditions for equality.

Twelfth International Olympiad, 1970

1. Let M be a point on the side AB of triangle $\triangle ABC$. Let r_1, r_2, r be the radii of the inscribed circles of triangles $\triangle AMC, \triangle BMC$, and $\triangle ABC$, respectively. Let q_1, q_2, q be the radii of the escribed circles of the same triangles that lie in the angle $\angle ACB$. Prove that $\frac{r_1}{q_1} \cdot \frac{r_2}{q_2} = \frac{r}{q}$.

2. Let a, b, n be integers greater than 1, and let a and b be the bases of two number systems. Let A_n, A_{n-1} be numbers in the system with base a, and B_n, B_{n-1} be numbers in the system with base b, defined as follows: $A_n = x_n x_{n-1} \cdots x_0, \quad A_{n-1} = x_{n-1} x_{n-2} \cdots x_0,$
$B_n = x_n x_{n-1} \cdots x_0, \quad B_{n-1} = x_{n-1} x_{n-2} \cdots x_0$, with $x_n \neq 0,\ x_{n-1} \neq 0$. Prove that $\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n}$ if and only if $a > b$.

3. The real numbers $a_0, a_1, \ldots, a_n, \ldots$ satisfy the condition: $1 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$ Define the numbers $b_1, b_2, \ldots, b_n, \ldots$ by $b_n = \sum_{k=1}^{n} \left(\frac{1 - a_{k-1}/a_k}{\sqrt{a_k}} \right)$.

   (a) Prove that $0 \leq b_n < 2$ for all n.

   (b) Given c with $0 \leq c < 2$, prove that there exist numbers $a_0, a_1, \ldots$ with the above properties such that $b_n > c$ for large enough n.

4. Find the set of all positive integers n such that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two subsets whose products are equal.

5. In the tetrahedron $ABCD$, $\angle BDC = 90^\circ$. Suppose the foot H of the perpendicular from D to the plane ABC is the intersection of the altitudes of $\triangle ABC$. Prove that $(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2)$. For what tetrahedra does equality hold?

6. In a plane there are 100 points, no three of which are collinear. Consider all possible triangles having these points as vertices. Prove that no more than 70% of these triangles are acute-angled.