1.2. Kinematics beyond one dimension & projectile motion

In this chapter, motion is described as a vector phenomenon extending from one to multiple dimensions, introducing component analysis for 2D and 3D motion, projectile trajectories, relative velocity, and circular motion.

1.2.1. Moving to higher dimensions: Basic Concept

To describe motion in a plane, we need **two perpendicular coordinate axes**, usually x (horizontal) and y (vertical). The corresponding **unit vectors** $\hat{\imath}$ and $\hat{\jmath}$ point in the positive directions of these axes and have unit magnitude:

$$|\hat{\imath}| = |\hat{\jmath}| = 1, \qquad \hat{\imath} \cdot \hat{\jmath} = 0.$$

Any vector quantity—position, velocity, or acceleration—can then be written in **component form**:

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}, \qquad \vec{v} = v_x\,\hat{\imath} + v_y\,\hat{\jmath}, \qquad \vec{a} = a_x\,\hat{\imath} + a_y\,\hat{\jmath}.$$

Each component behaves like an independent one-dimensional motion, so all 1D kinematic equations still apply separately to $x(t)$ and $y(t)$. Vector operations such as addition, subtraction, or differentiation act **component-wise**:

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{\imath} + \frac{dy}{dt}\,\hat{\jmath}.$$

The chosen **reference frame** defines the origin and the orientation of the coordinate axes. Once set, all components must be expressed relative to that frame. Changing frames (for example, from a moving train to the ground) alters the numerical components of $\vec{r}$ or $\vec{v}$ but never the physical event itself.

The concept can be extended to beyond 2D. In three dimensions, a third perpendicular axis z is added together with the unit vector $\hat{k}$. The **Cartesian basis** $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is **orthonormal**:

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0, \qquad |\hat{\imath}| = |\hat{\jmath}| = |\hat{k}| = 1.$$

To conclude, beyond 1D, position $\vec{r}(t)$, velocity $\vec{v}(t)$, and acceleration $\vec{a}(t)$ must be treated as vectors. The **position vector** $\vec{r}(t)$ points from the chosen origin to the object (represented as an infinitesimal small point). The **displacement** over an interval is $\Delta\vec{r} = \vec{r} - \vec{r}_0$. The **velocity** is the time derivative of position,

$$\vec{v}(t) = \frac{d\vec{r}}{dt},$$

and the **acceleration** is the time derivative of velocity,

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}.$$

1.2.2. Projectile motion & the superposition of motions

A simple yet striking observation reveals a key idea of two-dimensional motion: if one ball is dropped from the edge of a table and another is rolled horizontally off the same table at the same instant, **both hit the floor simultaneously**. The horizontal motion of the second ball has no influence on how fast it falls vertically. This experiment demonstrates the **independence of motion along perpendicular directions**, or in mathematical terms, the **superposition principle** for kinematics.

Superposition means that the overall motion of an object can be seen as the **vector sum of independent component motions**. In the case of a projectile, one motion occurs horizontally (constant velocity, no acceleration), and the other vertically (uniform acceleration due to gravity). These act simultaneously but independently, linked only by the shared time variable t.

This idea allows a **component-wise analysis** of motion. Each coordinate axis is treated separately using the one-dimensional equations of motion, while the full trajectory and velocity are obtained by recombining the components into vector form:

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}, \qquad \vec{v}(t) = v_x(t)\,\hat{\imath} + v_y(t)\,\hat{\jmath}, \qquad \vec{a}(t) = a_x(t)\,\hat{\imath} + a_y(t)\,\hat{\jmath}.$$

Because $\hat{\imath}$ and $\hat{\jmath}$ are perpendicular, their contributions do not interfere. One can analyze e.g. $x(t)$ and $y(t)$ independently and then reconstruct $\vec{r}(t)$.

The superposition principle is **not limited to projectiles**. It is a powerful concept that reappears throughout physics. For instance, in **electromagnetic motion** (a charged particle moving simultaneously under electric and magnetic fields), in **wave interference**, and in **circular motion** (as the sum of two perpendicular harmonic motions).

Projectile motion thus serves as the first and simplest illustration of this universal idea: complex motion can often be understood as the sum of simpler, independent parts.

1.2.3. Kinematics of projectile motion

Having established that motion along perpendicular directions can be treated independently, we now apply this principle to a projectile launched near Earth's surface, where gravity is the only acting force and air resistance is neglected. The acceleration is constant and directed downward:

$$\vec{a} = -g\,\hat{\jmath}, \qquad g \approx 9.81\,\mathrm{m/s^2}.$$

Horizontal motion

Along the x-axis, no acceleration acts once the object is in flight:

$$a_x = 0.$$

Hence, the horizontal velocity remains constant:

$$v_x(t) = v_{0x} = v_0 \cos \theta_0,$$

and the horizontal position changes linearly with time:

$$x(t) = x_0 + v_{0x}\, t.$$

This is a case of **uniform motion**, i.e. the projectile covers equal horizontal distances in equal time intervals.

Vertical motion

Along the y-axis, gravity produces a constant downward acceleration:

$$a_y = -g.$$

The velocity and position along y evolve as

$$v_y(t) = v_{0y} - g\,t, \qquad y(t) = y_0 + v_{0y}\, t - \tfrac{1}{2} g\, t^2,$$

where $v_{0y} = v_0 \sin \theta_0$.
This is **uniformly accelerated motion**, identical in form to one-dimensional free fall.

Combining the components

The total position and velocity vectors follow from their components:

$$\vec{r}(t) = x(t)\, \hat{\imath} + y(t)\, \hat{\jmath}, \qquad \vec{v}(t) = v_x(t)\, \hat{\imath} + v_y(t)\, \hat{\jmath}.$$

Eliminating t between $x(t)$ and $y(t)$ yields the trajectory:

- rearrange $x(t)$:

$$x = x_0 + v_{0x}\, t \leftrightarrow t = \frac{x - x_0}{v_{0x}}$$

- substitute this expression for t into $y(t)$:

$$y = y_0 + v_{0y} \left(\frac{x - x_0}{v_{0x}} \right) - \tfrac{1}{2} g \left(\frac{x - x_0}{v_{0x}} \right)^2$$

- rearrange:

$$y(x) = y_0 + \left(\frac{v_{0y}}{v_{0x}} \right) (x - x_0) - \frac{g}{2 v_{0x}^2} (x - x_0)^2$$

This a **parabola** opening downward.
At the apex, the vertical velocity vanishes ($v_y = 0$), while the horizontal component remains unchanged.

Characteristic quantities

If the projectile lands at the same height from which it was launched ($y = y_0$), several useful quantities follow directly:

- **Time of flight:**

$$y = 0 = v_{0y}T - \tfrac{1}{2}gT^2$$

$$T = \frac{2v_{0y}}{g} = \frac{2v_0 \sin \theta_0}{g}$$

- **Range (horizontal distance):**

$$x(t) = R = v_{0x}T = \frac{2\, v_{0x}\, v_{0y}}{g} = \frac{v_0^2 \sin 2\theta_0}{g}$$

- **Maximum height:**

$$v_y(t) = 0 = v_{0y} - gt_H \leftrightarrow t_H = \frac{v_{0y}}{g}$$

$$y(t_H) = H = 0 + v_{0y}t_H - \frac{1}{2}gt_H^2$$

$$H = \frac{v_{0y}^2}{2g} = \frac{v_0^2 \sin^2 \theta_0}{2g}$$

The range R is maximized when $\sin 2\theta_0 = 1$, i.e. at $\theta_0 = 45°$. Launch angles θ_0 and $(90° - \theta_0)$ produce equal ranges, though with different flight times and heights.

Relation between velocity and direction

At any instant,

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2}, \qquad \tan \phi = \frac{v_y}{v_x},$$

where ϕ is the angle of motion relative to the horizontal. During flight, ϕ continuously decreases as v_y changes from positive to negative while v_x remains constant.

Limiting and special cases

For a purely **vertical launch** ($\theta_0 = 90°$), $v_{0x} = 0$ and the motion reduces to **one-dimensional free fall**. For a **horizontal launch** ($\theta_0 = 0°$), $v_{0y} = 0$ and the path simplifies to

$$y = y_0 - \tfrac{1}{2} g\, t^2,$$

with the same vertical fall time as a dropped object from height y_0.

Concluding remarks

Projectile motion elegantly demonstrates the **superposition principle**: horizontal and vertical components evolve independently yet describe one coherent trajectory when combined. This **decomposition by components is a universal method**. We can apply the same reasoning to more complex cases such as charged particles in electric or magnetic fields, where forces act in perpendicular directions but can be analyzed using the same vector framework.

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1.2.4. Relative velocity

Velocity is always defined **relative to a chosen reference frame**.
In **Galilean kinematics**—the framework of Newtonian mechanics valid at speeds much smaller than the speed of light—**space and time are absolute**, and **velocities add vectorially** between inertial observers.

If object A moves with velocity $\vec{v}_{AB}$ relative to observer B, and observer B moves with velocity $\vec{v}_{BC}$ relative to observer C, then the velocity of A relative to C is

$$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}.$$

Each term represents how motion appears **from the perspective of a different observer**. In two or three dimensions, components add independently along each axis. Classic examples include a **boat crossing a river**. Its velocity relative to the shore equals its velocity relative to the water plus the river's current.
While different observers may measure different **velocities**, all **inertial frames** (those moving at constant velocity relative to each other) agree on **accelerations** caused by the same physical forces, such as gravity g.

At very high speeds ($v \sim c$), this Galilean picture no longer holds, and the addition law is replaced by **Einstein's relativistic velocity formula**. However, for all everyday and and most engineering-scale motions, Galilean kinematics provides an excellent approximation.

1.2.5 Uniform circular motion

Circular motion occurs whenever an object moves around a fixed point at constant distance r, continuously changing the **direction** of its velocity.
Even if the **speed** remains constant, the change in direction means the object is accelerating (acceleration vector always directed toward the center of the circle).

Angular quantities and their relations

The position of the object on the circle can be described by the **angular position** $\phi(t)$ measured in radians and with respect to the $+x$-axis.
The corresponding angular quantities are defined as

$$\omega = \frac{d\phi}{dt}, \qquad \alpha = \frac{d\omega}{dt}.$$

Here, ω is the **angular velocity**, and α is the **angular acceleration**.
For uniform circular motion, ω is constant and $\alpha = 0$, giving

$$\phi(t) = \phi_0 + \omega t, \qquad T = \frac{2\pi}{\omega}, \qquad f = \frac{1}{T}.$$

The arc length s along the circle relates to the angle via $s = r\phi$. Differentiating links linear and angular quantities:

$$v = \frac{ds}{dt} = r\frac{d\phi}{dt} = r\omega.$$

Thus, the familiar linear relation "$v = s/t$" generalizes to "$\omega = \phi/t$" for uniform angular motion.

Centripetal acceleration

The angular acceleration α measures how fast the angular velocity ω changes in time. For a uniform circular motion $\alpha = \frac{d\omega}{dt} = 0$, but there is still an acceleration at play due to the circular motion which causes the direction of the **velocity vector** $\vec{v}$ to change over time. This change in velocity (direction) is captured by the **centripetal acceleration** a_r. Let's derive its direction and magnitude.

At any instant, the velocity vector $\vec{v}$ is **tangent** to the circular path.
As the object moves from point A to point B, the direction of $\vec{v}$ changes by an angle $\Delta\phi$, though its magnitude remains constant.
The **change in velocity vector** $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ points toward the center of the circle for small $\Delta\vec{v}$.

This can be seen geometrically: for small $\Delta\phi$, the velocity vectors $\vec{v}_1$ and $\vec{v}_2$ form a triangle similar to the one formed by the radius vectors $\vec{r}_1$ and $\vec{r}_2$.
As $\Delta t \to 0$, $\Delta\vec{v}$ becomes perpendicular to $\vec{v}$, pointing directly inward.

Since acceleration is the rate of change of velocity, $\vec{a}$ must point in the same direction as $\Delta\vec{v}$, i.e. toward the center.
This inward-directed acceleration is the **centripetal** (center-seeking) or **radial acceleration**, denoted by $\vec{a}_r$.

From the geometric similarity of the small triangles formed by $\vec{r}$ and $\vec{v}$:

$$\frac{|\Delta\vec{v}|}{v} \approx \frac{|\Delta\vec{r}|}{r},$$

which we can rearrange into:

$$|\Delta\vec{v}| \approx \frac{|\Delta\vec{r}|}{r}\vec{v}.$$

Acceleration is defined as the rate of change of the velocity. Combining this with our previous equation gives us :

$$a_r = \lim_{\Delta t \to 0}\frac{|\Delta\vec{v}|}{\Delta t} = \lim_{\Delta t \to 0}\frac{\frac{|\Delta\vec{r}|}{r}\vec{v}}{\Delta t} = \lim_{\Delta t \to 0}\frac{|\Delta\vec{r}|\vec{v}}{r\Delta t} = \lim_{\Delta t \to 0}\frac{|\Delta\vec{r}|}{\Delta t}\frac{\vec{v}}{r}.$$

Since $\lim_{\Delta t \to 0}\frac{|\Delta\vec{r}|}{\Delta t} = \frac{dr}{dt} = v$, we obtain our final equation for a_r:

$$a_r = \frac{v^2}{r} = \omega^2 r.$$

Thus, the magnitude of $\vec{a}_r$ increases quadratically with speed and decreases inversely with radius. The faster the motion or the tighter the curve, the larger the inward acceleration required.

Conceptual link: superposition viewpoint & Cartesian representation

Uniform circular motion can also be seen as the **superposition of two perpendicular simple harmonic motions (SHMs)** of equal amplitude r, equal frequency ω, and a $90°$ phase difference:

$$x(t) = r\cos(\omega t), \qquad y(t) = r\sin(\omega t) = r\cos\left(\omega t - \frac{\pi}{2}\right).$$

Each coordinate oscillates harmonically; their combination produces the circular path

$$x^2 + y^2 = r^2.$$

This interpretation highlights the general **superposition principle** introduced earlier: Therefore, complex motion (like circular motion) can be decomposed into the sum of simpler, independent motions along perpendicular axes. We can apply $\vec{v} = \frac{d\vec{r}}{dt}$ and $\vec{a} = \frac{d\vec{v}}{dt}$ to describe kinematics for uniform circular motion in Cartesian coordinates as:

$$\vec{r}(t) = r\cos(\omega t)\,\hat{\imath} + r\sin(\omega t)\,\hat{\jmath},$$

$$\vec{v}(t) = -\omega r\sin(\omega t)\,\hat{\imath} + \omega r\cos(\omega t)\,\hat{\jmath},$$

$$\vec{a}(t) = -\omega^2 r\cos(\omega t)\,\hat{\imath} - \omega^2 r\sin(\omega t)\,\hat{\jmath} = -\omega^2\vec{r}(t).$$

1.2.7. Non-uniform circular motion

When the speed changes, the acceleration decomposes into perpendicular components. The **radial** component,

$$a_r = \frac{v^2}{r},$$

continually redirects $\vec{v}$ toward the center. The **tangential** component,

$$a_{\text{tan}} = \frac{dv}{dt},$$

changes the speed along the path (parallel to $\vec{v}$ when speeding up, antiparallel when slowing down). The total acceleration is

$$\vec{a} = \vec{a}_{\text{tan}} + \vec{a}_r, \qquad |\vec{a}| = \sqrt{a_{\text{tan}}^2 + a_r^2}.$$

This framework extends to any curved trajectory by introducing the instantaneous **radius of curvature** r; locally the path is an arc of a circle, so the same decomposition applies with the local r and instantaneous v.

1.2.8. Summary circular motion

Quantity	Expression	Description
Angular velocity	$\omega = \dfrac{d\phi}{dt}$	rate of change of angular position
Linear speed	$v = r\omega$	constant for uniform motion
Centripetal acceleration	$a_r = \dfrac{v^2}{r}$ $= \omega^2 r$	directed toward the center
Angular acceleration	$\alpha = \dfrac{d\omega}{dt}$	zero for uniform motion