

# Lecture Tutorial 1D: Precession & Rotating Reference Frames



**Precession == Witchcraft?**

mk19 - Präzession (Fahrradfelge)

## Reminder from Previous Lecture

Basket + Ball

$$\vec{p} = m \cdot \vec{v}, \quad \& \quad \vec{F}_{\text{ext}} = \frac{d\vec{p}}{dt}$$

→ change in momentum follows external force

$$\vec{\tau} = \vec{r} \times \vec{F}, \quad \& \quad \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$$

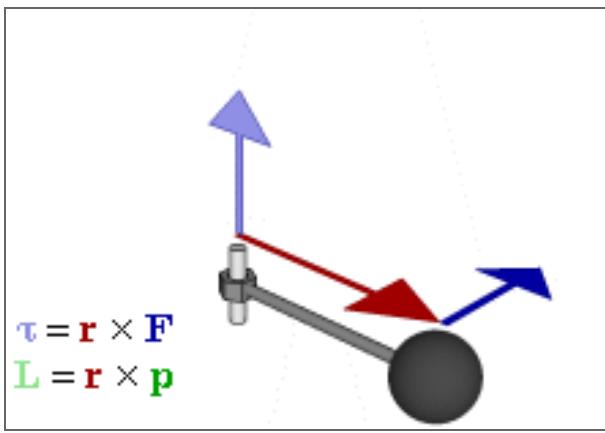
→ change in angular momentum follows external torque

# Direction of the Torque & its consequences:

mk18 - Flugzeugkreisel

$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt}$$

Aspect	Parallel torque $\vec{\tau}_{\parallel}$	Perpendicular torque $\vec{\tau}_{\perp}$
Relation to $\vec{L}$ & governing idea	Acts <b>along</b> $\vec{L}$ . Changes $\frac{d\vec{L}}{dt}$ in the same direction.	Acts <b>orthogonal</b> to $\vec{L}$ . Changes $\frac{d\vec{L}}{dt}$ sideways to $\vec{L}$ .
Effect on $\vec{L}$	<b>Magnitude changes, direction stays the same</b>	<b>Magnitude stays constant, direction changes</b>
Intuitive summary	"Spin faster or slower"	"Axis changes direction"



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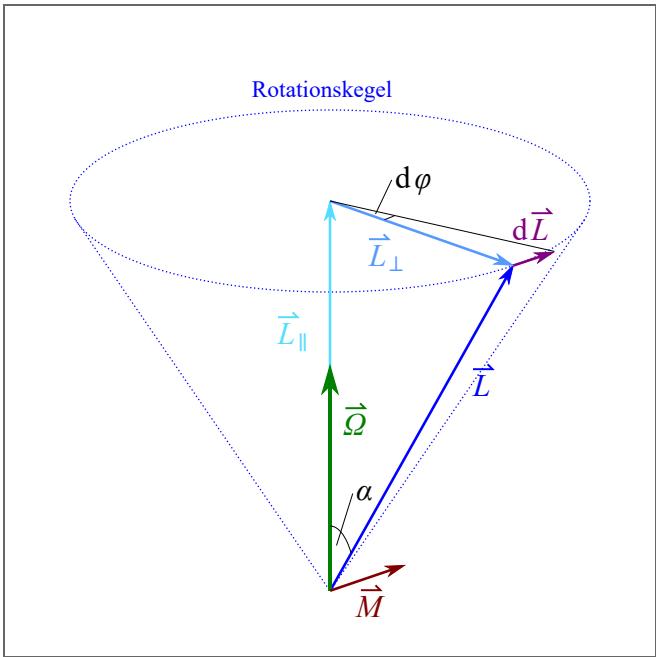
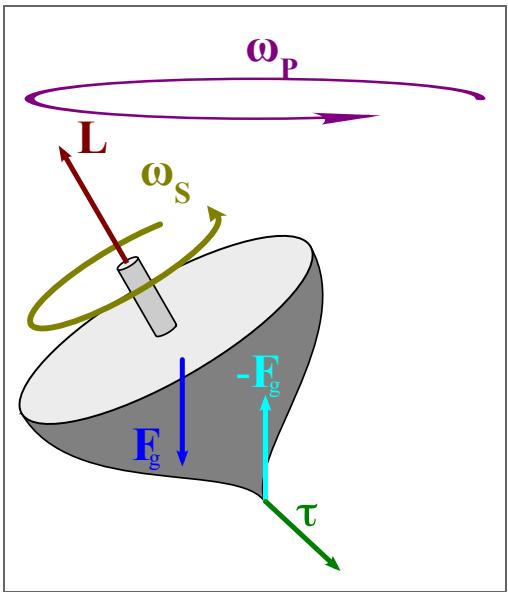
# Spinning Tops under Gravity

## mk19 - Präzession (Fahrradfelge)

- **Symmetric top** supported at a pivot; **center of mass** located a horizontal distance  $r$  from pivot; **angular momentum**  $\vec{L}$  points along the symmetry axis
- Resulting torque due to gravity:

$$\vec{\tau} = \vec{r} \times M\vec{g}$$

- This torque is **horizontal** → perpendicular to both  $\vec{r}$  and  $\vec{g}$ .
- Since  $\vec{\tau} = \frac{d\vec{L}}{dt} \perp \vec{L}$  → torque changes **direction**, not magnitude, of  $\vec{L}$  → **precession** (spin axis rotates around the vertical, tracing a cone)



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# Precession Rate

- Vectors:
  - $\vec{L}$ : along the tilted spin axis
  - $\vec{\tau}$ : perpendicular to the plane containing  $\vec{r}$  &  $\vec{g}$ ; **and** perpendicular to  $\vec{L}$
  - $\vec{\Omega}$ : horizontal; describes slow rotation of  $\vec{L}$  around vertical
- Tip of  $\vec{L}$  moves on a horizontal circle; horizontal component sets radius:

$$L_{\perp} = L \sin \theta.$$

- In time  $\Delta t$ , geometric change of  $\vec{L}$  (arc length):

$$|\Delta \vec{L}| = L_{\perp} (\Omega \Delta t) = L \sin \theta \Omega \Delta t$$

- From  $\vec{\tau} = \frac{d\vec{L}}{dt}$ :

$$|\Delta \vec{L}| = \tau \Delta t$$

## Precession Rate (cont')

- Equate and solve:

$$L \sin \theta \Omega \Delta t = \tau \Delta t$$

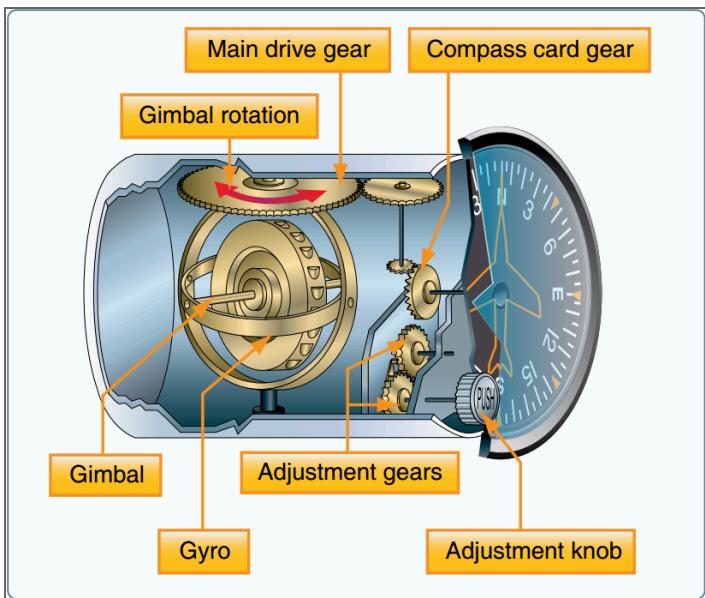
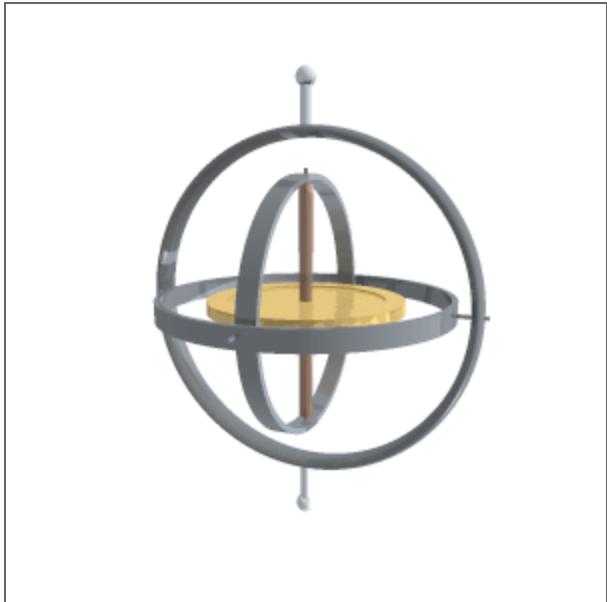
$$L \sin \theta \Omega = \tau$$

- Insert  $\tau = Mg r_{\perp} = Mg r \sin \theta$  and  $L = I\omega$ :

$$\Omega = \frac{\tau}{L \sin \theta} = \frac{Mg r \sin \theta}{I\omega \sin \theta} = \boxed{\frac{Mg r}{I\omega}}$$

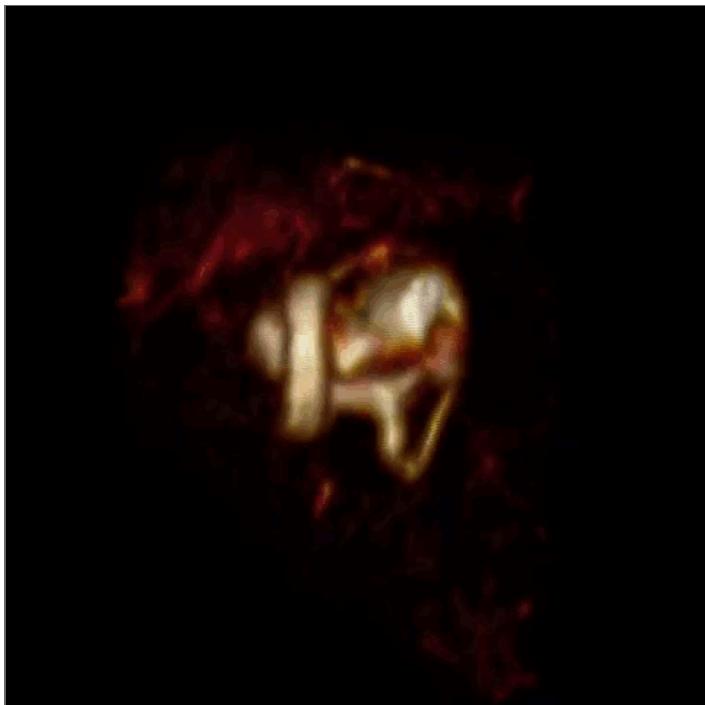
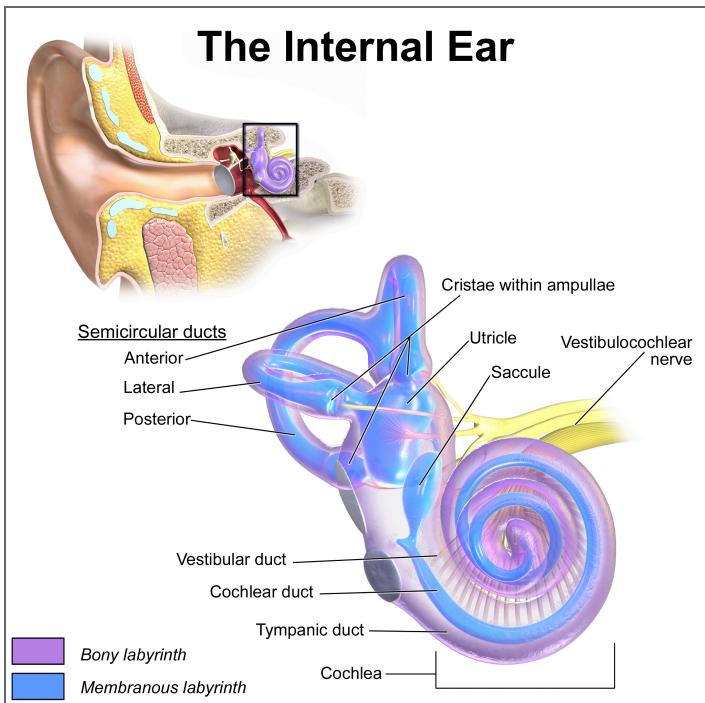
- A fast spin (large  $\omega$ ) or large moment of inertia  $I \rightarrow \textbf{slow precession}$ .  
Counterclockwise spin (viewed from above) produces clockwise precession (right-hand rule).

# Gyroscope - Heading Indicator



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# Gyro in the Ear - Vestibular System



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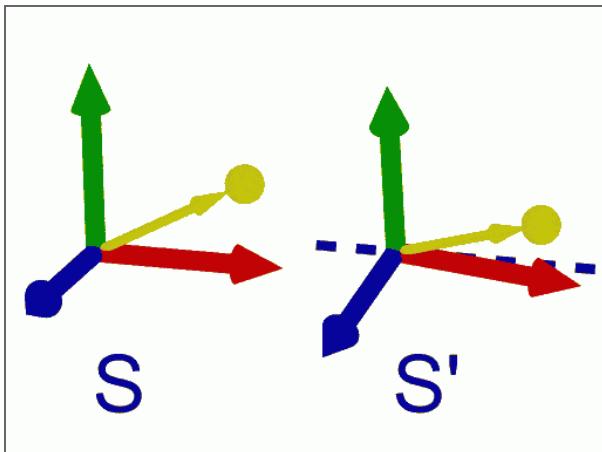
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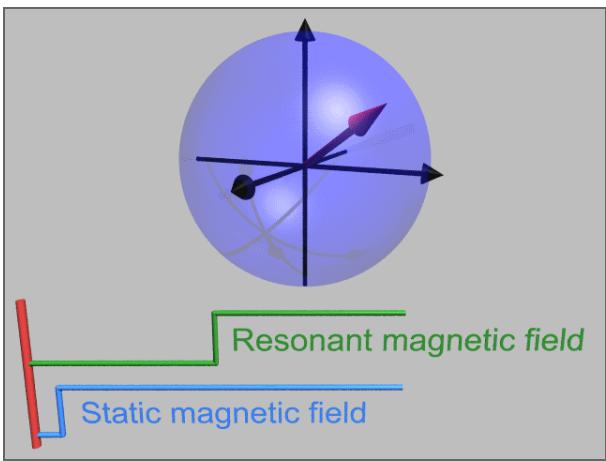
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# Rotating Reference Frames - Primer & Motivation

md47 - Feder auf dem Drehschemel

- Sometimes **easier** to study and describe physics in rotation coordinate system (e.g. **Bloch simulator; structure demo**)
- However, **rotating reference frame** (e.g., Earth-bound observer) introduces **apparent accelerations**
- **Newton's laws do not apply in non-inertial frames** such as rotating frames





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# Velocity Transformation

- Consider two observers:
  - Frame  $S$ : inertial (non-rotating)
  - Frame  $S'$ : rotates with angular velocity  $\vec{\omega}$
- A particle at position  $\vec{r}$  has different measured velocities in the two frames

$$\vec{v}_S = \vec{v}_{S'} + \vec{\omega} \times \vec{r}$$

- Meaning  $\vec{\omega} \times \vec{r}$ :
  - **Velocity of the rotating axes themselves**
  - The cross product guarantees direction perpendicular & magnitude  $= \omega r_{\perp}$  (tangential speed)
- Core idea: **Velocity differences between rotating and inertial frames arise solely from the motion of the rotating coordinate system.**



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# Acceleration Transformation

- Want to explain forces (dynamics)  $\rightarrow$  study motion (kinematics) first
- Differentiate the velocity relation to obtain the acceleration transformation

$$\vec{v}_S = \vec{v}_{S'} + \vec{\omega} \times \vec{r}$$

- Take time derivative to get velocity in  $S$  (inertial, non-rotating frame):

$$\vec{a}_S = \frac{d}{dt}(\vec{v}_{S'} + \vec{\omega} \times \vec{r}) = \frac{d\vec{v}_{S'}}{dt} + \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

## Acceleration Transformation - First Term

$$\vec{a}_S = \frac{d\vec{v}_{S'}}{dt} + \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

The first term is just the acceleration seen in the rotating frame  $\frac{d\vec{v}_{S'}}{dt} = \vec{a}_{S'}$

$$\vec{a}_S = \vec{a}_{S'} + \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

## Acceleration Transformation - Second Term

$$\vec{a}_S = \vec{a}_{S'} + \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

Apply the product rule for the second term (cross product):

$$\frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

For the second term of the second term, use

$$\frac{d\vec{r}}{dt} = \vec{v}_S = \vec{v}_{S'} + \vec{\omega} \times \vec{r}:$$

$$\vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\omega} \times (\vec{v}_{S'} + \vec{\omega} \times \vec{r}) = \vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Thus,

$$\vec{a}_S = \vec{a}_{S'} + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\omega} \times \vec{v}_S$$

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## Acceleration Transformation - Final Equation

$$\vec{a}_S = \vec{a}_{S'} + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\omega} \times \vec{v}_S$$

Combine the identical  $\vec{\omega} \times \vec{v}_{S'}$  terms:

$$\vec{a}_S = \vec{a}_{S'} + 2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

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Newton's second law and the origin of fictitious forces

In the inertial frame:

$$m\vec{a}_S = \sum \vec{F}_{\text{real}}$$

Insert the acceleration transformation and solve for  $m\vec{a}_{S'}$ :

$$m[\vec{a}_{S'} + 2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}] = \sum \vec{F}_{\text{real}}$$

$$m\vec{a}_{S'} + m[2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}] = \sum \vec{F}_{\text{re}}$$

$$m\vec{a}_{S'} = \sum \vec{F}_{\text{real}} - m[2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}]$$

**In the rotating reference frame we observe real and fictitious forces, solely due to the rotation of the frame:**

$$m\vec{a}_{S'} = \sum \vec{F}_{\text{real}} + \sum \vec{F}_{\text{fictitious}}$$

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# The Fictitious Forces

- **Fictitious forces** appear only in **rotating (non-inertial) frames due to the acceleration of the frame itself**

$$\vec{F}_{\text{fict}} = -m[2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}]$$

- **Coriolis force**

$$\vec{F}_C = -2m(\vec{\omega} \times \vec{v}_{S'})$$

- **Centrifugal force**

$$\vec{F}_{\text{cf}} = -m[\vec{\omega} \times (\vec{\omega} \times \vec{r})]$$

- **Euler force** (only if the rotation rate changes)

$$\vec{F}_E = -m \frac{d\vec{\omega}}{dt} \times \vec{r}$$

# Centrifugal Force

md47 - Feder auf dem Drehschemel  
md20 - Demonstration der  
Zentrifugalkraft

$$\vec{F}_{\text{cf}} = -m[\vec{\omega} \times (\vec{\omega} \times \vec{r})]$$

- Acceleration term:

$$\vec{a} = -\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

- Direction: always **radially outward** from the rotation axis
- Magnitude at distance  $R$  to rotation axis (not radius  $r$ ):

$$F_{\text{cf}} = m\omega^2 R = \frac{mv^2}{R}$$

- Magnitude equals the **centripetal force**, but the **direction is opposite** (centrifugal outward, centripetal inward)

# Coriolis Force

## md14 - Coriolis pendulum

$$\vec{F}_C = -2m(\vec{\omega} \times \vec{v}_{S'})$$

- Coriolis term in the acceleration:

$$\vec{a}_C = -2\vec{\omega} \times \vec{v}_{S'}$$

- Appears only for objects **moving in the rotating frame** → no Coriolis force when

$$\vec{v}_{S'} = 0$$

- **Independent of position**
- Direction: always **perpendicular** to both the rotation axis and the velocity
- Magnitude for speed  $v$  at angle  $\phi$  between  $\vec{\omega}$  and  $\vec{v}_{S'}$ :

$$F_C = 2m\omega v \sin \phi$$

# Foucault Pendulum

## md12 - Foucault Pendulum

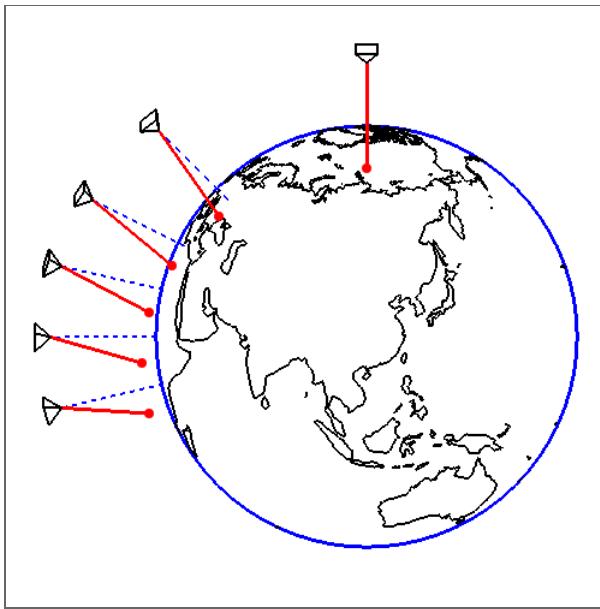
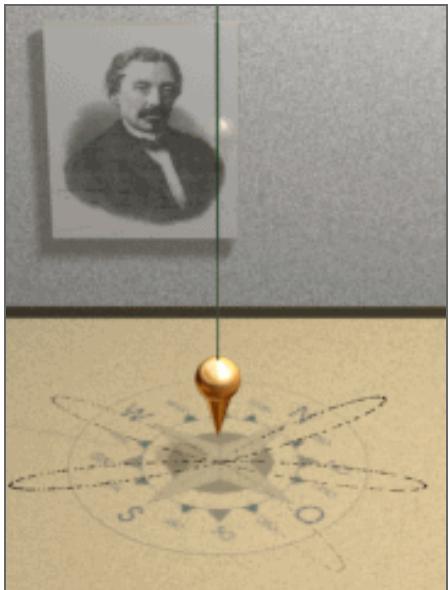
- A **Foucault pendulum** demonstrates Earth's rotation through the **apparent precession**  $\Omega_p$  of its swing plane
- Earth is rotating underneath the pendulum's swing plane
- Precession rate:

$$\Omega_p = \omega \sin \phi$$

where

- $\Omega_p$ : relative precession angular frequency of pendulum relative of Earth (rotating frame)
  - $\omega$ : Earth's rotation rate ( $\approx 15^\circ/\text{h}$ )
  - $\phi$ : geographic latitude
- **Precession periods** (time  $T$  for one full rotation of the swing plane):
    - Poles ( $\phi = 90^\circ$ ):  $T = 24 \text{ h}$

- Equator ( $\phi = 0^\circ$ ):  $\Omega_p = 0$  (no precession)
- Mid-latitudes ( $\phi = 45^\circ$ ):  $T \approx 34$  h



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## Coriolis Force – Effect on Global Scale

The Coriolis force arises in a rotating coordinate system:

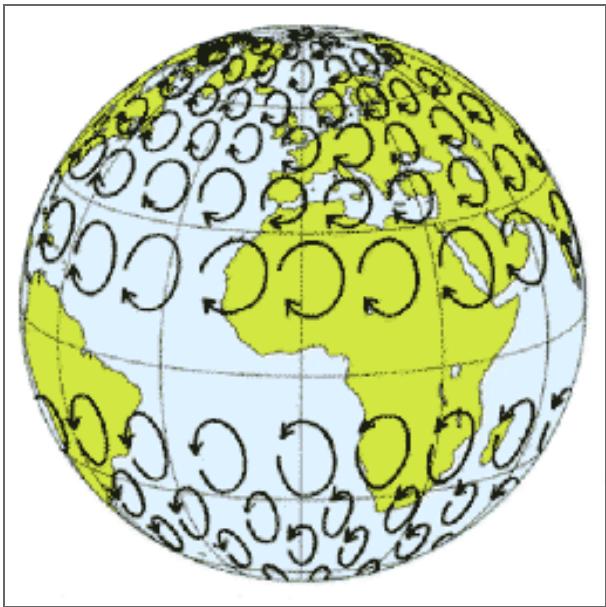
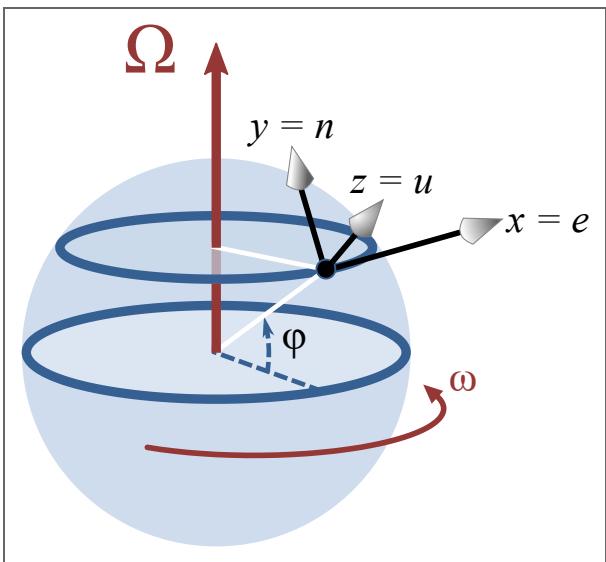
$$\vec{F}_C = -2m(\vec{\Omega} \times \vec{v}_{S'}), \quad \vec{a}_C = -2\vec{\Omega} \times \vec{v}_{S'}.$$

Only **moving** air or water experiences the Coriolis force. Its *horizontal* magnitude is

$$a_{C,h} = 2\omega v \sin \phi,$$

with  $\vec{\Omega}$  = Earth's rotation vector,  $\omega$  = Earth's rotation rate (scalar magnitude; same everywhere),  $\vec{v}_{S'}$  = velocity **relative to Earth**,  $\phi$  = latitude

- **Northern Hemisphere** ( $\phi > 0$ ): always **90° to the right** of motion
- **Southern Hemisphere** ( $\phi < 0$ ): always **90° to the left**
- **Equator** ( $\phi = 0$ ): no horizontal Coriolis



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# Coriolis Force – Intuition

## md15 – Globus

1. Shot from the equator toward Magdeburg

- At the equator the ground moves fastest.
- The body keeps this larger tangential velocity as it travels north.
- The ground beneath it moves slower.
- Therefore the body appears to drift to the right of its path (eastward relative to the ground).

2. Shot from Magdeburg toward the equator

- Magdeburg has a smaller tangential velocity than the equator.
- The body keeps this smaller tangential speed while moving south.
- The equatorial ground is faster.
- Therefore the body again appears to drift to the right of its path (westward relative to the ground).

## **Summary:**

- A body always keeps the **tangential velocity of the place where it is released.**
- Direction follows from the vector product  $\vec{\Omega} \times \vec{v}_{S'}$  on a rotating sphere.

## Euler Force (Azimuthal Force)

$$\vec{F}_E = -m \frac{d\vec{\omega}}{dt} \times \vec{r}$$

- Tangential acceleration for changing rotation rate:

$$\vec{a}_E = -r \frac{d\vec{\omega}}{dt}$$

- Euler (azimuthal) force appears when the **angular velocity changes** ( $d\vec{\omega}/dt \neq 0$ ) & **scales with  $r$**
- Direction: always **tangential** to  $\vec{r}$ , opposite the tangential acceleration
- Magnitude of Euler force:

$$F_E = ma_E.$$

- Example: starting carousel / merry-go-round
  - apparent force pushing the person to the back of their seat
  - larger apparent force the further away from the rotation axis



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# Comparison of Fictitious Forces in a Rotating Frame

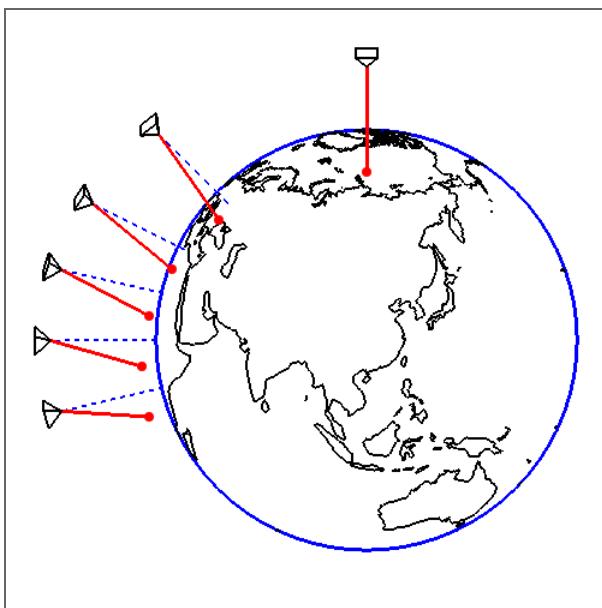
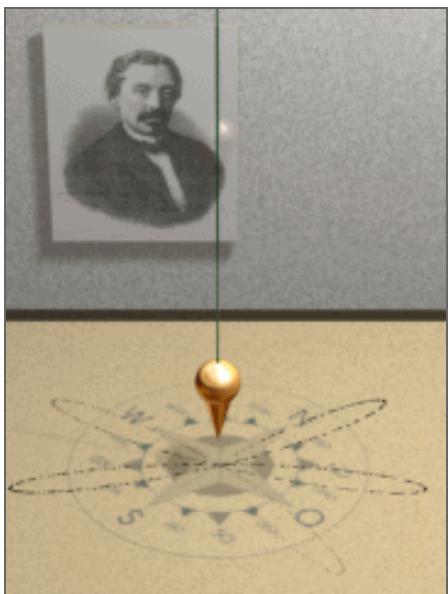
Force	Condition for Appearance	Acceleration Term	Direction
<b>Centrifugal</b>	Object has position $\vec{r} \neq 0$ in rotating frame	$\vec{a}_{cf} = -\vec{\omega} \times (\vec{\omega} \times \vec{r})$	Radial outward from axis
<b>Coriolis</b>	Object moves in rotating frame ( $\vec{v}_{S'} \neq 0$ )	$\vec{a}_C = -2\vec{\omega} \times \vec{v}_{S'}$	Pendulum to left
<b>Euler</b>	Angular velocity changes ( $d\vec{\omega}/dt \neq 0$ )	$\vec{a}_E = -\frac{d\vec{\omega}}{dt} \times \vec{r}$	Tangential (azimuthal)

## Summary of Fictitious Forces in a Rotating Frame

- Rotating frames require fictitious forces to maintain Newton's 2. law (does not fix 3. law)
- These forces do **not** arise from physical interactions; they appear **because the frame's axes accelerate, even when body at rest**
- These forces disappear in an inertial frame; they reflect the acceleration of the rotating frame itself

# Foucault Pendulum - Check Experiment

md12 - Foucault Pendulum



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Fictitious forces in action:

md16 - Beibehaltung der Schwingungsebene **Pendulum swinging**

Force	Condition	Acceleration	Direction
Centrifugal	$\vec{r} \neq 0$	$-\vec{\omega} \times (\vec{\omega} \times \vec{r})$	Radially outward from rot. axis
Coriolis	$\vec{v}_{S'} \neq 0$	$-2\vec{\omega} \times \vec{v}_{S'}$	Perpendicular to $\vec{\omega}$ and $\vec{v}_{S'}$
Euler	$d\vec{\omega}/dt \neq 0$	$-(d\vec{\omega}/dt) \times \vec{r}$	Tangential

Fictitious forces in action:

md16 - Beibehaltung der Schwingungsebene **Circle with equal, smaller, and faster  $\omega$**

Force	Condition	Acceleration	Direction
Centrifugal	$\vec{r} \neq 0$	$-\vec{\omega} \times (\vec{\omega} \times \vec{r})$	Radially outward from rot. axis
Coriolis	$\vec{v}_{S'} \neq 0$	$-2\vec{\omega} \times \vec{v}_{S'}$	Perpendicular to $\vec{\omega}$ and $\vec{v}$
Euler	$d\vec{\omega}/dt \neq 0$	$-(d\vec{\omega}/dt) \times \vec{r}$	Tangential

Fictitious forces in action:

md16 - Beibehaltung der Schwingungsebene **Ellipse**

Force	Condition	Acceleration	Direction
Centrifugal	$\vec{r} \neq 0$	$-\vec{\omega} \times (\vec{\omega} \times \vec{r})$	Radially outward from rot. axis
Coriolis	$\vec{v}_{S'} \neq 0$	$-2\vec{\omega} \times \vec{v}_{S'}$	Perpendicular to $\vec{\omega}$ and $\vec{v}_{S'}$
Euler	$d\vec{\omega}/dt \neq 0$	$-(d\vec{\omega}/dt) \times \vec{r}$	Tangential

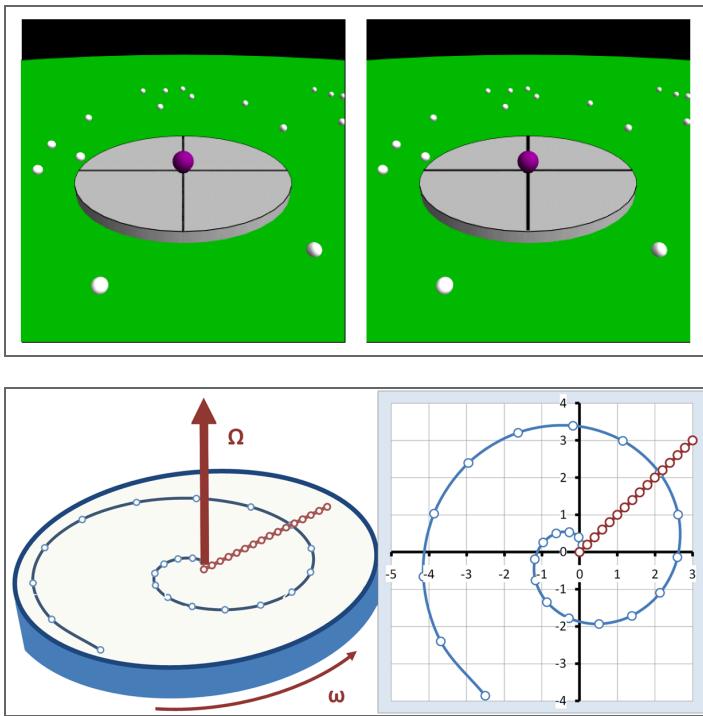
Fictitious forces in action:

md16 - Beibehaltung der Schwingungsebene **Resting body**

Force	Condition	Acceleration	Direction
Centrifugal	$\vec{r} \neq 0$	$-\vec{\omega} \times (\vec{\omega} \times \vec{r})$	Radially outward from rot. axis
Coriolis	$\vec{v}_{S'} \neq 0$	$-2\vec{\omega} \times \vec{v}_{S'}$	Perpendicular to $\vec{\omega}$ and $\vec{v}_{S'}$
Euler	$d\vec{\omega}/dt \neq 0$	$-(d\vec{\omega}/dt) \times \vec{r}$	Tangential

Fictitious forces in action:

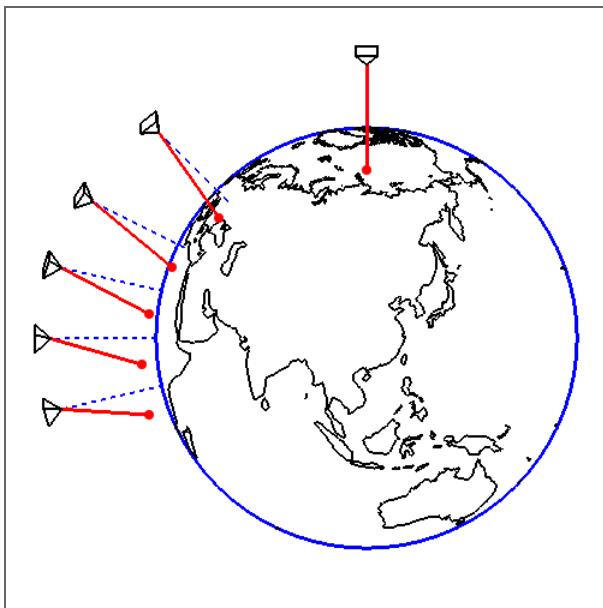
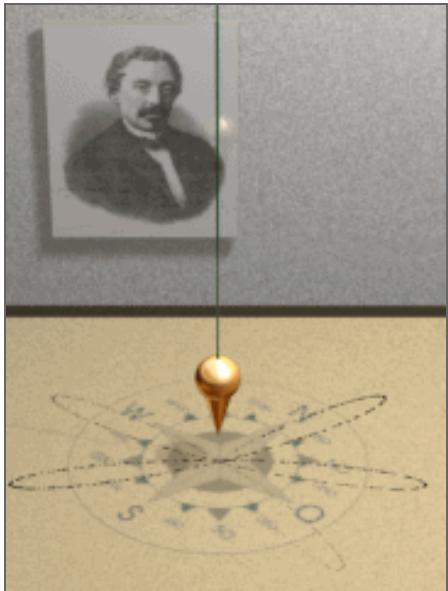
md16 - Beibehaltung der  
Schwingungsebene **Linear uniform motion**  
**(fast & slow)**



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# Foucault Pendulum - Check Experiment Last Time

md12 - Foucault Pendulum



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