Lecture Tutorial 1D: Precession & Rotating Reference Frames

`mk19 - Präzession (Fahrradfelge)`

Precession $==$ **Witchcraft?**

Reminder from Previous Lecture

`Basket + Ball`

$$\vec{p} = m \cdot \vec{v}, \quad \& \quad \vec{F}_{\text{ext}} = \frac{d\vec{p}}{dt}$$

$\rightarrow$ change in momentum follows external force

$$\vec{\tau} = \vec{r} \times \vec{F}, \quad \& \quad \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$$

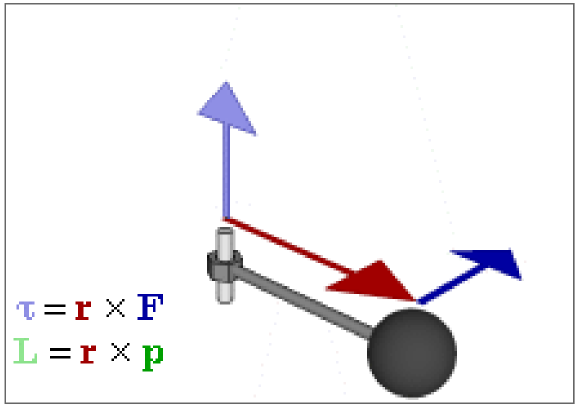
$\rightarrow$ change in angular momentum follows external torque

Direction of the Torque & its consequences:

`mk18 - Flugzeugkreisel`

$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt}$$

Aspect	**Parallel torque $\vec{\tau}_{\parallel}$**	**Perpendicular torque $\vec{\tau}_{\perp}$**
Relation to $\vec{L}$ & governing idea	Acts **along** $\vec{L}$. Changes $\frac{d\vec{L}}{dt}$ in the same direction.	Acts **orthogonal** to $\vec{L}$. Changes $\frac{d\vec{L}}{dt}$ sideways to $\vec{L}$.
Effect on $\vec{L}$	**Magnitude changes**, **direction stays the same**	**Magnitude stays constant**, **direction changes**
Intuitive summary	"Spin faster or slower"	"Axis changes direction"



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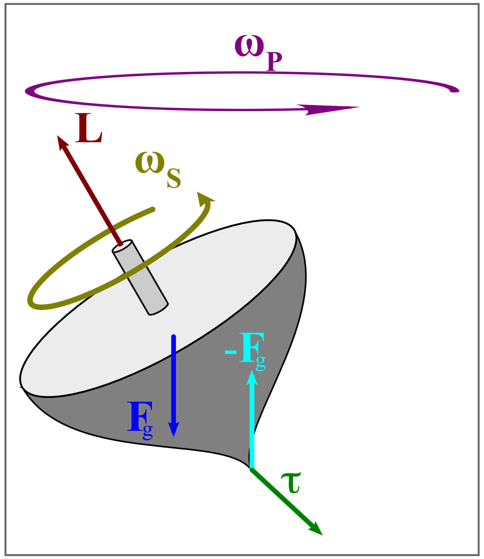
Spinning Tops under Gravity

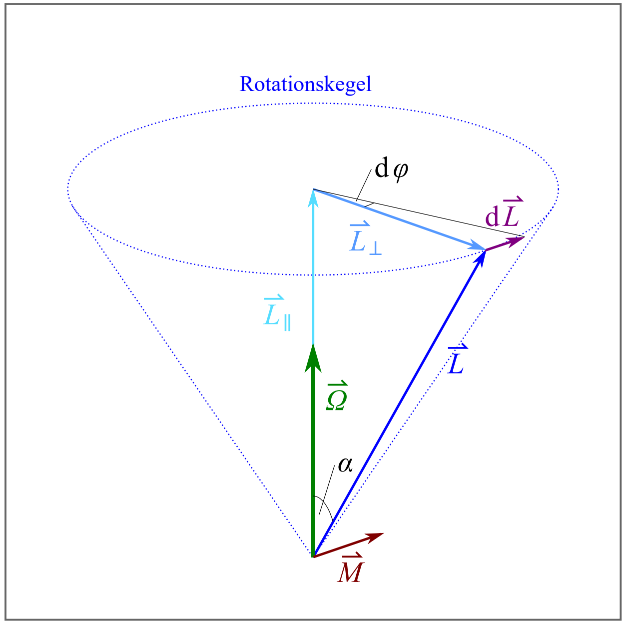
`mk19 - Präzession (Fahrradfelge)`

- **Symmetric top** supported at a pivot; **center of mass** located a horizontal distance r from pivot; **angular momentum** $\vec{L}$ points along the symmetry axis
- Resulting torque due to gravity:

$$\vec{\tau} = \vec{r} \times M\vec{g}$$

- This torque is **horizontal** → perpendicular to both $\vec{r}$ and $\vec{g}$.
- Since $\vec{\tau} = \frac{d\vec{L}}{dt} \perp \vec{L} \to$ torque changes **direction**, not magnitude, of $\vec{L} \to$ **precession** (spin axis rotates around the vertical, tracing a cone)





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Precession Rate

- Vectors:
  - $\vec{L}$: along the tilted spin axis
  - $\vec{\tau}$: perpendicular to the plane containing $\vec{r}$ & $\vec{g}$; **and** perpendicular to $\vec{L}$
  - $\vec{\Omega}$: horizontal; describes slow rotation of $\vec{L}$ around vertical
- Tip of $\vec{L}$ moves on a horizontal circle; horizontal component sets radius:

$$L_{\perp} = L\sin\theta.$$

- In time Δt, geometric change of $\vec{L}$ (arc length):

$$|\Delta\vec{L}| = L_{\perp}\,(\Omega\Delta t) = L\sin\theta\,\Omega\,\Delta t$$

- From $\vec{\tau} = \frac{d\vec{L}}{dt}$:

$$|\Delta \vec{L}| = \tau \, \Delta t$$

Precession Rate (cont')

- Equate and solve:

$$L \sin\theta \, \Omega \, \Delta t = \tau \, \Delta t$$

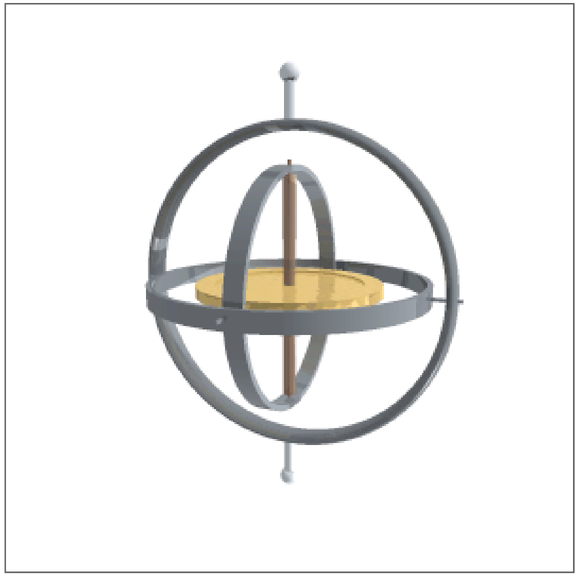
$$L \sin\theta \, \Omega = \tau$$

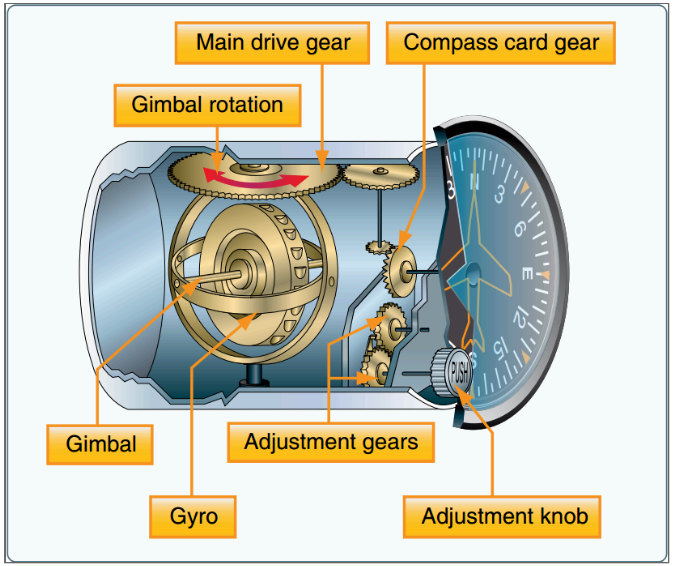
- Insert $\tau = M g \, r_\perp = M g \, r \sin\theta$ and $L = I\omega$:

$$\Omega = \frac{\tau}{L \sin\theta} = \frac{M g \, r \sin\theta}{I\omega \sin\theta} = \boxed{\frac{M g \, r}{I\omega}}$$

- A fast spin (large ω) or large moment of inertia I → **slow precession**. Counterclockwise spin (viewed from above) produces clockwise precession (right-hand rule).

Gyroscope - Heading Indicator





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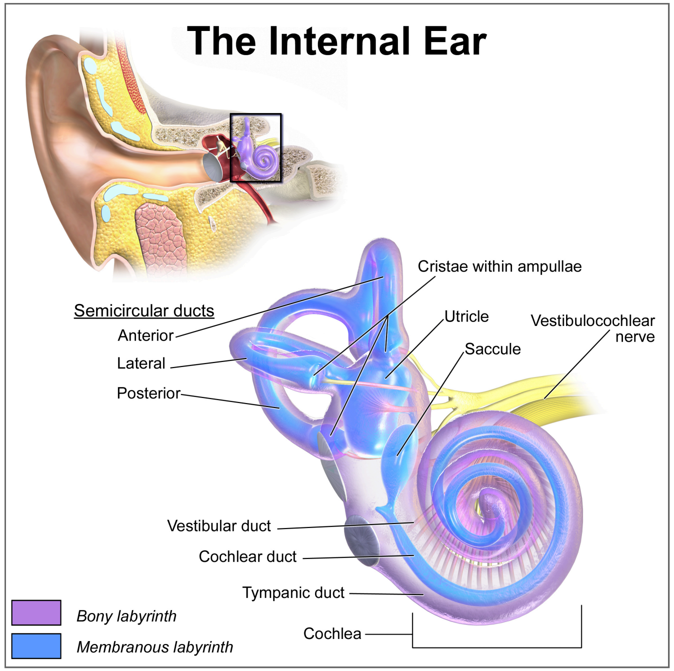
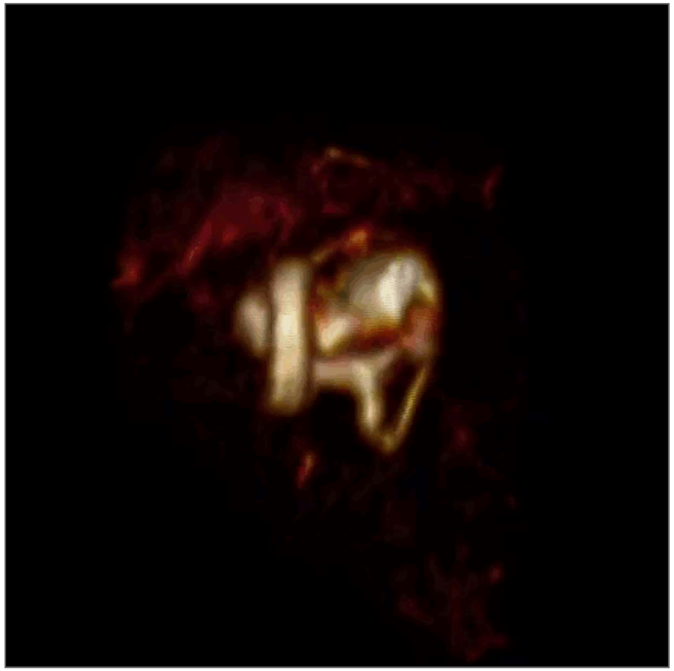
Gyro in the Ear - Vestibular System





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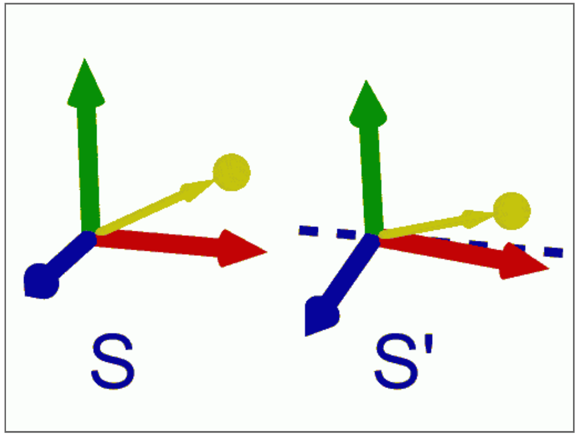
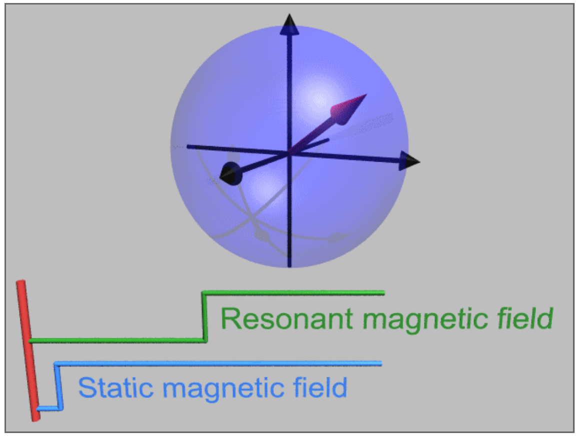
Rotating Reference Frames - Primer & Motivation

`md47 - Feder auf dem Drehschemel`

- Sometimes **easier** to study and describe physics in rotation coordinate system (e.g. **Bloch simulator; structure demo**)
- However, **rotating reference frame** (e.g., Earth-bound observer) introduces **apparent accelerations**
- **Newton's laws do not apply in non-inertial frames** such as rotating frames





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Velocity Transformation

- Consider two observers:
  - Frame S: inertial (non-rotating)
  - Frame S': rotates with angular velocity $\vec{\omega}$
- A particle at position $\vec{r}$ has different measured velocities in the two frames

$$\vec{v}_S = \vec{v}_{S'} + \vec{\omega} \times \vec{r}$$

- Meaning $\vec{\omega} \times \vec{r}$:
  - **Velocity of the rotating axes themselves**
  - The cross product guarantees direction perpendicular & magnitude $= \omega r_\perp$ (tangential speed)
- Core idea: **Velocity differences between rotating and inertial frames arise solely from the motion of the rotating coordinate system.**

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Acceleration Transformation

- Want to explain forces (dynamics) $\to$ study motion (kinematics) first
- Differentiate the velocity relation to obtain the acceleration transformation

$$\vec{v}_S = \vec{v}_{S'} + \vec{\omega} \times \vec{r}$$

- Take time derivative to get velocity in S (inertial, non-rotating frame):

$$\vec{a}_S = \frac{d}{dt}\big(\vec{v}_{S'} + \vec{\omega} \times \vec{r}\big) = \frac{d\vec{v}_{S'}}{dt} + \frac{d}{dt}\big(\vec{\omega} \times \vec{r}\big)$$

Acceleration Transformation - First Term

$$\vec{a}_S = \frac{d\vec{v}_{S'}}{dt} + \frac{d}{dt}\left(\vec{\omega} \times \vec{r}\right)$$

The first term is just the acceleration seen in the rotating frame $\frac{d\vec{v}_{S'}}{dt} = \vec{a}_{S'}$

$$\vec{a}_S = \vec{a}_{S'} + \frac{d}{dt}\left(\vec{\omega} \times \vec{r}\right)$$

Acceleration Transformation - Second Term

$$\vec{a}_S = \vec{a}_{S'} + \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

Apply the product rule for the second term (cross product):

$$\frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

For the second term of the second term, use $\frac{d\vec{r}}{dt} = \vec{v}_S = \vec{v}_{S'} + \vec{\omega} \times \vec{r}$:

$$\vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\omega} \times (\vec{v}_{S'} + \vec{\omega} \times \vec{r}) = \vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Thus,

$$\vec{a}_S = \vec{a}_{S'} + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\omega} \times \vec{v}_{S'}$$

Acceleration Transformation - Final Equation

$$\vec{a}_S = \vec{a}_{S'} + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\omega} \times \vec{v}_{S'}$$

Combine the identical $\vec{\omega} \times \vec{v}_{S'}$ terms:

$$\vec{a}_S = \vec{a}_{S'} + 2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

Newton's second law and the origin of fictitious forces

In the inertial frame:

$$m\vec{a}_S = \sum \vec{F}_{\text{real}}$$

Insert the acceleration transformation and solve for $m\vec{a}_{S'}$:

$$m\left[\vec{a}_{S'} + 2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}\right] = \sum \vec{F}_{\text{real}}$$

$$m\vec{a}_{S'} + m\left[2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}\right] = \sum \vec{F}_{\text{real}}$$

$$m\vec{a}_{S'} = \sum \vec{F}_{\text{real}} - m\left[2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}\right].$$

In the rotating reference frame we observe real and fictitious forces, solely due to the rotation of the frame:

$$m\vec{a}_{S'} = \sum \vec{F}_{\text{real}} + \sum \vec{F}_{\text{fictitious}}$$

The Fictitious Forces

- **Fictitious forces** appear only in **rotating (non-inertial) frames due to the acceleration of the frame itself**

$$\vec{F}_{\text{fict}} = -m\left[2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}\right]$$

- **Coriolis force**

$$\vec{F}_C = -2m(\vec{\omega} \times \vec{v}_{S'})$$

- **Centrifugal force**

$$\vec{F}_{\text{cf}} = -m\left[\vec{\omega} \times (\vec{\omega} \times \vec{r})\right]$$

- **Euler force** (only if the rotation rate changes)

$$\vec{F}_E = -m\frac{d\vec{\omega}}{dt} \times \vec{r}$$

Centrifugal Force

`md47 - Feder auf dem Drehschemel`

$$\vec{F}_{\text{cf}} = -m\big[\vec{\omega} \times (\vec{\omega} \times \vec{r})\big]$$

- Acceleration term:

$$\vec{a} = -\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

- Direction: always **radially outward** from the rotation axis
- Magnitude at distance R to rotation axis (not radius r):

$$F_{\text{cf}} = m\omega^2 R = \frac{mv^2}{R}$$

- Magnitude equals the **centripetal force**, but the **direction is opposite** (centrifugal outward, centripetal inward)

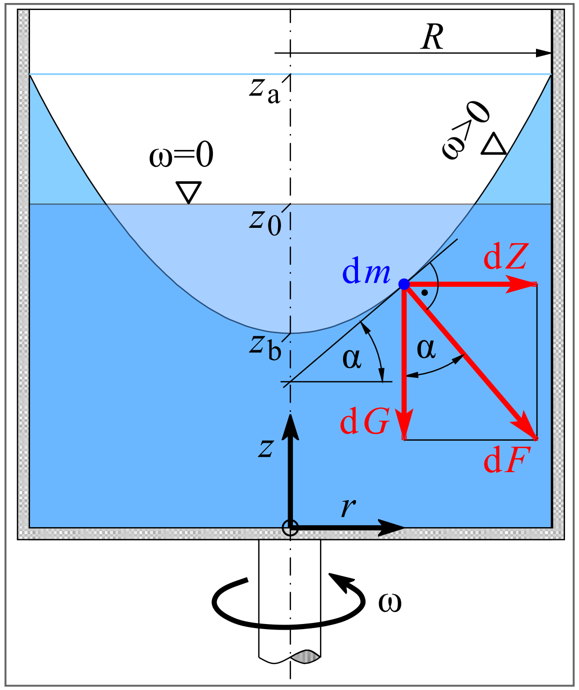
Centrifugal Force (cont')

`md20 - Demonstration der Zentrifugalkraft`

$$\vec{F}_{\text{cf}} = -m\big[\vec{\omega} \times (\vec{\omega} \times \vec{r})\big], \quad \& \quad F_{\text{cf}} = m\omega^2 R = \frac{mv^2}{R}$$

$$\vec{a} = -\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

- **Magnitude at distance R to rotation axis (not radius r) & always radially outward from the rotation axis**


R
z_a
ω>0
ω=0
z_0
dm
dZ
z_b
α
α
dG
dF
z
r
ω

Coriolis Force

`md14 - Coriolis pendulum`

$$\vec{F}_C = -2m(\vec{\omega} \times \vec{v}_{S'})$$

- Coriolis term in the acceleration:

$$\vec{a}_C = -2\,\vec{\omega} \times \vec{v}_{S'}$$

- Appears only for objects **moving in the rotating frame** $\rightarrow$ no Coriolis force when $\vec{v}_{S'} = 0$
- **Independent of position**
- Direction: always **perpendicular** to both the rotation axis and the velocity
- Magnitude for speed v at angle ϕ between $\vec{\omega}$ and $\vec{v}_{S'}$:

$$F_C = 2m\omega v \sin\phi$$

Foucault Pendulum

`md12 - Foucault Pendulum`

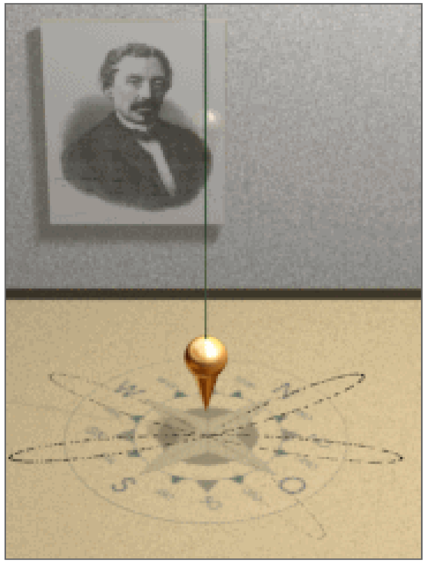
- A **Foucault pendulum** demonstrates Earth's rotation through the **apparent precession** Ω_p of its swing plane
- Earth is rotating underneath the pendulum's swing plane
- Precession rate:

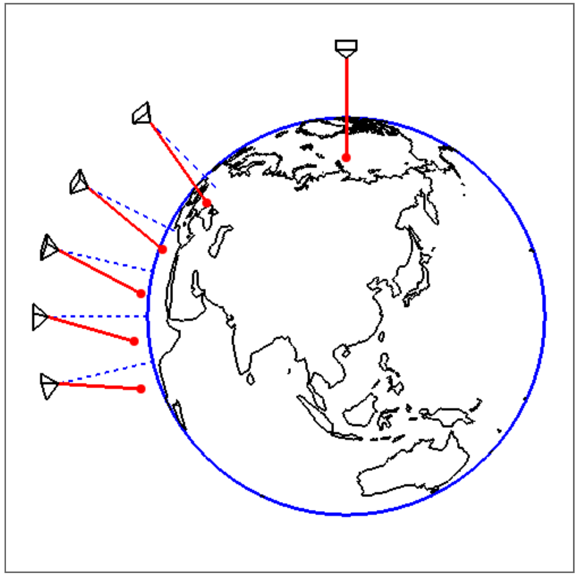
$$\Omega_p = \omega \sin \phi$$

  where

    - Ω_p: relative precession angular frequency of pendulum relative of Earth (rotating frame)
    - ω: Earth's rotation rate ($\approx 15^\circ$/h)
    - ϕ: geographic latitude
- **Precession periods** (time T for one full rotation of the swing plane):
    - Poles ($\phi = 90^\circ$): $T = 24$ h

- Equator ($\phi = 0^\circ$): $\Omega_p = 0$ (no precession)
- Mid-latitudes ($\phi = 45^\circ$): $T \approx 34$ h





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Coriolis Force – Effect on Global Scale

The Coriolis force arises in a rotating coordinate system:

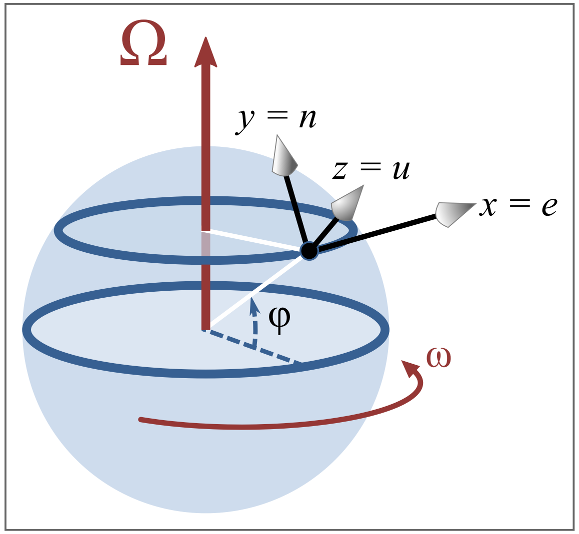
$$\vec{F}_C = -2m(\vec{\Omega} \times \vec{v}_{S'}), \qquad \vec{a}_C = -2\,\vec{\Omega} \times \vec{v}_{S'}.$$

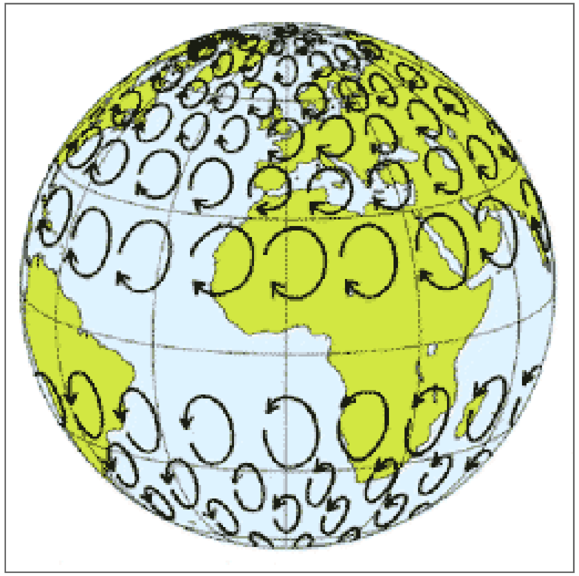
Only **moving** air or water experiences the Coriolis force. Its *horizontal* magnitude is

$$a_{C,h} = 2\omega v \sin\phi,$$

with $\vec{\Omega}$ = Earth’s rotation vector, ω = Earth’s rotation rate (scalar magnitude; same everywhere), $\vec{v}_{S'}$ = velocity **relative to Earth**, ϕ = latitude

- **Northern Hemisphere** ($\phi > 0$): always **90° to the right** of motion
- **Southern Hemisphere** ($\phi < 0$): always **90° to the left**
- **Equator** ($\phi = 0$): no horizontal Coriolis





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Coriolis Force – Intuition

`md15 – Globus`

1. Shot from the equator toward Magdeburg
    - At the equator the ground moves fastest.
    - The body keeps this larger tangential velocity as it travels north.
    - The ground beneath it moves slower.
    - Therefore the body appears to drift to the right of its path (eastward relative to the ground).
2. Shot from Magdeburg toward the equator
    - Magdeburg has a smaller tangential velocity than the equator.
    - The body keeps this smaller tangential speed while moving south.
    - The equatorial ground is faster.
    - Therefore the body again appears to drift to the right of its path (westward relative to the ground).

Summary:

- A body always keeps the **tangential velocity of the place where it is released**.
- Direction follows from the vector product $\vec{\Omega} \times \vec{v}_{S'}$ on a rotating sphere.

Euler Force (Azimuthal Force)

$$\vec{F}_E = -m\frac{d\vec{\omega}}{dt} \times \vec{r}$$

- Tangential acceleration for changing rotation rate:

$$\vec{a}_E = -r\frac{d\vec{\omega}}{dt}$$

- Euler (azimuthal) force appears when the **angular velocity changes** ($d\vec{\omega}/dt \neq 0$) & **scales with** r
- Direction: always **tangential** to $\vec{r}$, opposite the tangential acceleration
- Magnitude of Euler force:

$$F_E = ma_E.$$

- Example: starting carousel / merry-go-round
    - apparent force pushing the person to the back of their seat
    - larger apparent force the further away from the rotation axis

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Comparison of Fictitious Forces in a Rotating Frame

Force	Condition for Appearance	Acceleration Term	Direction	Scaling
Centrifugal	Object has position $\vec{r} \neq 0$ in rotating frame	$\vec{a}_{\mathrm{cf}} = -\vec{\omega} \times (\vec{\omega} \times \vec{r})$	Radially outward from rotation axis	$\propto \omega^2 r$
Coriolis	Object moves in rotating frame ($\vec{v}_{S'} \neq 0$)	$\vec{a}_C = -2\,\vec{\omega} \times \vec{v}_{S'}$	Perpendicular to $\vec{\omega}$ and $\vec{v}_{S'}$	$\propto \omega\, v$

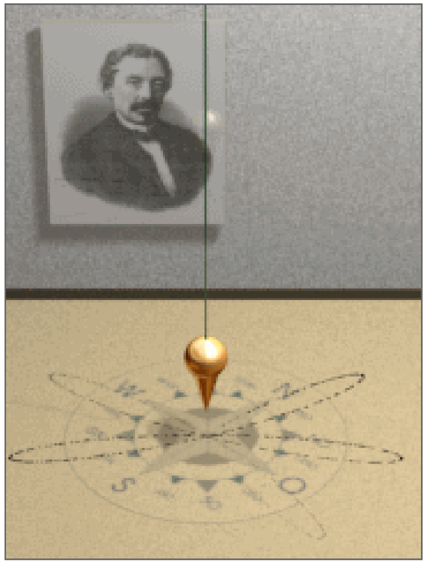
Force	Condition for Appearance	Acceleration Term	Direction	Scaling
Euler	Angular velocity changes ($d\vec{\omega}/dt \neq 0$)	$\vec{a}_E = -\frac{d\vec{\omega}}{dt} \times \vec{r}$	Tangential (azimuthal)	$\propto r\,\frac{d\omega}{dt}$

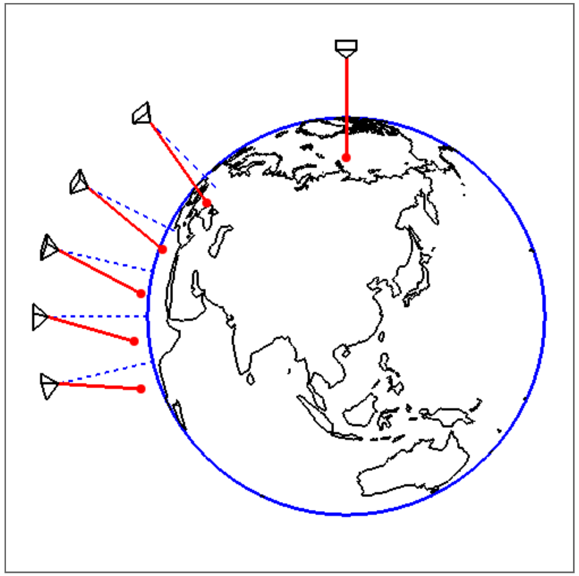
Summary of Fictitious Forces in a Rotating Frame

- Rotating frames require fictitious forces to maintain Newton's 2. law (does not fix 3. law)
- These forces do **not** arise from physical interactions; they appear **because the frame's axes accelerate**, **even when body at rest**
- These forces disappear in an inertial frame; they reflect the acceleration of the rotating frame itself

Foucault Pendulum - Check Experiment

`md12 - Foucault Pendulum`





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Fictitious forces in action:

`md16 - Beibehaltung der Schwingungsebene` **Pendulum swinging**

Force	Condition	Acceleration	Direction	Scale
Centrifugal	$\vec{r} \neq 0$	$-\vec{\omega} \times (\vec{\omega} \times \vec{r})$	Radially outward from rot. axis	$\omega^2 r$
Coriolis	$\vec{v}_{S'} \neq 0$	$-2\vec{\omega} \times \vec{v}_{S'}$	Perpendicular to $\vec{\omega}$ and $\vec{v}_{S'}$	ωv
Euler	$d\vec{\omega}/dt \neq 0$	$-(d\vec{\omega}/dt) \times \vec{r}$	Tangential	$r\, d\omega/dt$

Fictitious forces in action:

`md16 - Beibehaltung der Schwingungsebene` **Circle with equal, smaller, and faster** ω

Force	Condition	Acceleration	Direction	Scale
Centrifugal	$\vec{r} \neq 0$	$-\vec{\omega} \times (\vec{\omega} \times \vec{r})$	Radially outward from rot. axis	$\omega^2 r$
Coriolis	$\vec{v}_{S'} \neq 0$	$-2\vec{\omega} \times \vec{v}_{S'}$	Perpendicular to $\vec{\omega}$ and $\vec{v}_{S'}$	ωv
Euler	$d\vec{\omega}/dt \neq 0$	$-(d\vec{\omega}/dt) \times \vec{r}$	Tangential	$r\, d\omega/dt$

Fictitious forces in action:

`md16 - Beibehaltung der Schwingungsebene` **Ellipse**

Force	Condition	Acceleration	Direction	Scale
Centrifugal	$\vec{r} \neq 0$	$-\vec{\omega} \times (\vec{\omega} \times \vec{r})$	Radially outward from rot. axis	$\omega^2 r$
Coriolis	$\vec{v}_{S'} \neq 0$	$-2\vec{\omega} \times \vec{v}_{S'}$	Perpendicular to $\vec{\omega}$ and $\vec{v}_{S'}$	ωv
Euler	$d\vec{\omega}/dt \neq 0$	$-(d\vec{\omega}/dt) \times \vec{r}$	Tangential	$r\, d\omega/dt$

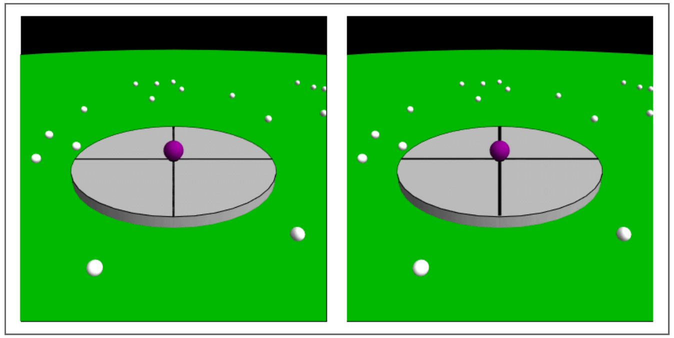
Fictitious forces in action:

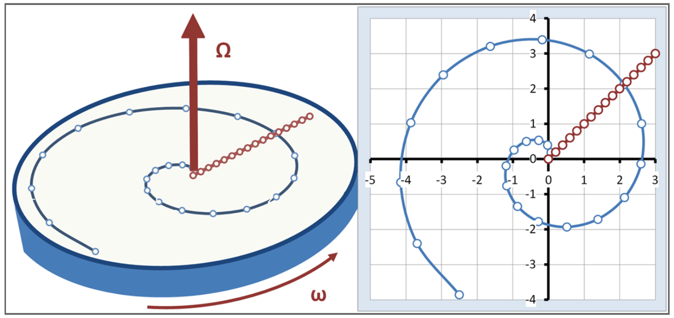
`md16 - Beibehaltung der Schwingungsebene` **Resting body**

Force	Condition	Acceleration	Direction	Scale
Centrifugal	$\vec{r} \neq 0$	$-\vec{\omega} \times (\vec{\omega} \times \vec{r})$	Radially outward from rot. axis	$\omega^2 r$
Coriolis	$\vec{v}_{S'} \neq 0$	$-2\vec{\omega} \times \vec{v}_{S'}$	Perpendicular to $\vec{\omega}$ and $\vec{v}_{S'}$	ωv
Euler	$d\vec{\omega}/dt \neq 0$	$-(d\vec{\omega}/dt) \times \vec{r}$	Tangential	$r\, d\omega/dt$

Fictitious forces in action:

`md16 - Beibehaltung der Schwingungsebene` **Linear uniform motion (fast & slow)**





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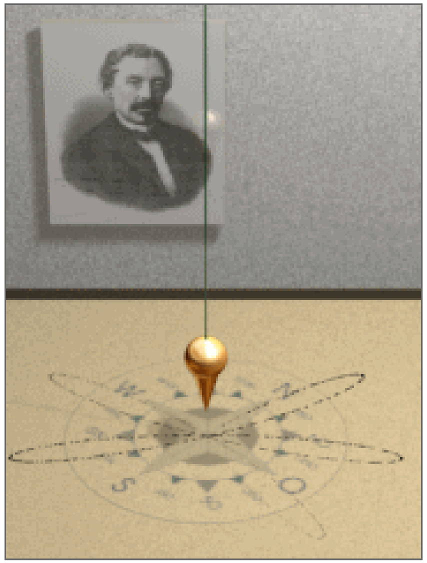
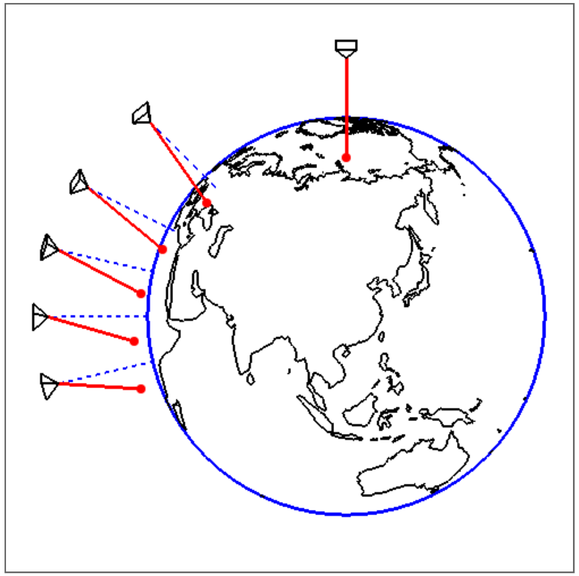
Foucault Pendulum - Check Experiment Last Time

`md12 - Foucault Pendulum`





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