1.6. Rotational dynamics & angular momentum

In this chapter we introduce the physics of **rotational motion**, where angular quantities play roles analogous to those in linear dynamics. We define **torque, moment of inertia,** and **angular momentum**, and explore how they determine the rotational response of an object. We also develop the concept of **rotational kinetic energy** and show how **angular momentum conservation** governs systems ranging from everyday rotating bodies to astronomical motion.

1.6.1. Primer & Recap of Rotational Kinematics

In linear motion, we describe a particle's position, velocity, and acceleration using the quantities x, v, and a. In rotational motion we introduce completely analogous quantities: the angular position θ, the angular velocity ω, and the angular acceleration α, which describe how an object rotates about a fixed axis. These quantities form a direct bridge between translational and rotational kinematics.

We consider a rigid body rotating in the xy-plane about a fixed axis perpendicular to that plane, namely the z-axis. The angular position is given by the angle θ between a chosen reference line in the body and the positive x-axis. A point P in the body moves along a circular path of radius r, and its arc length s measured from the x-axis is

$$s = r\,\theta,$$

with θ is measured in radians. One full revolution corresponds to an angular displacement of 2π radians. The radian (rad) is the SI unit of angular measurement and is defined such that $1\ \mathrm{rad}$ corresponds to an arc length equal to the radius, $s = r$. The usual conversion relations are

$$360^\circ = 2\pi\ \mathrm{rad}, \qquad 1^\circ = \frac{\pi}{180}\ \mathrm{rad}.$$

Thus, to convert an angle from radians to degrees, use

$$\theta_{\mathrm{deg}} = \theta_{\mathrm{rad}} \cdot \frac{180}{\pi}.$$

The change in angular position is the angular displacement

$$\Delta\theta = \theta_2 - \theta_1.$$

Angular displacement has both magnitude and direction. Its direction is defined by the right-hand rule: curl the fingers of your right hand in the direction of rotation, and the thumb points along the axis of rotation in the direction of the angular displacement vector.

The average angular velocity is defined by

$$\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t},$$

and the instantaneous angular velocity is the derivative of angular position,

$$\omega = \frac{d\theta}{dt}.$$

We adopt the standard convention that **counterclockwise (CCW) rotation is positive** and **clockwise (CW) rotation is negative**, as seen from the positive direction of the rotation axis. This follows the right-hand rule. For a **rigid body**, all points rotate together, meaning **every point shares the same angular velocity ω regardless of its radius**; only the linear velocity depends on r. The SI unit of angular velocity is radians per second ($\mathrm{rad/s}$).

If the angular velocity changes with time, the body has an angular acceleration. The average angular acceleration is

$$\alpha_{\text{avg}} = \frac{\Delta\omega}{\Delta t},$$

and the instantaneous angular acceleration is

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Positive angular acceleration corresponds to an increase in ω in the CCW direction. Again, for a **rigid body, every point has the same angular acceleration** α, even though their linear accelerations differ. The SI unit of angular acceleration is radians per second squared ($\mathrm{rad/s^2}$).

For a point on a rotating rigid body at a distance r from the axis, the linear quantities follow directly from the angular ones. Differentiating $s = r\theta$ gives the tangential velocity

$$v = \frac{ds}{dt} = \frac{rd\theta}{dt} = r\frac{d\theta}{dt} = r\omega,$$

and differentiating once more gives the tangential acceleration

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha.$$

There is also a radial (centripetal) acceleration associated with the changing direction of the velocity,

$$a_r = \frac{v^2}{r} = \frac{r^2\omega^2}{r} = r\omega^2,$$

directed toward the center. **Thus the linear velocity and acceleration of a point depend on its distance from the axis, but the angular motion does not.**

Recap of Uniform Circular Motion

Uniform circular motion appears as a special case of rotational motion in which ω is constant (i.e. $\alpha = 0$). Then $a_t = 0$, while the radial acceleration $a_r = r\omega^2$ remains nonzero.

The analogies between linear and angular quantities as well as linear and rotational kinematics are:

Linear quantity / relation	Angular quantity / relation
x	θ
Δx	$\Delta\theta$
$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$

Pseudovectors $\vec{\omega}$ & $\vec{\alpha}$

Angular quantities also have an important vector character. The angular position is represented by an angular vector $\vec{\theta}$ pointing **along the rotation axis**, according to the right-hand rule. Thus, for rotation in the xy-plane, $\vec{\theta}$ is perpendicular to that plane and points along $\pm\hat{z}$ depending on the rotation sense.

The angular velocity vector is defined by

$$\vec{\omega} = \frac{d\vec{\theta}}{dt},$$

and the angular acceleration vector by

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}.$$

Both $\vec{\omega}$ and $\vec{\alpha}$ are **pseudovectors (axial vectors)**: their directions do not correspond to physical displacements in space but arise from the right-hand rule applied to the sense/direction of rotation. Their magnitudes are the scalar values ω and α, respectively.

The linear velocity of a particle at position $\vec{r}$ relative to the axis is given by the vector relation

$$\vec{v} = \vec{\omega} \times \vec{r},$$

which ensures that $\vec{v}$ is perpendicular to both $\vec{\omega}$ and $\vec{r}$ and therefore points **tangentially** to the circular path.

To derive $\vec{a}$, we start from the velocity and differentiate with respect to time:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\vec{\omega} \times \vec{r})}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}).$$

This is the vector analogue of the scalar function $a = \frac{dv}{dt} = \frac{d\omega r}{dt}$. We have two time-dependent entities, i.e. ω and r. Thus, we need the product rule to find the derivative. Further, these are vectors. For the two time-dependent vectors $\vec{\omega}$ and $\vec{r}$, the product rule (vector notation) $\frac{d}{dt}\left(\vec{A} \times \vec{B}\right) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$ gives:

$$\frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}.$$

Thus, we get:

$$\vec{a} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

Using the identities $\frac{d\vec{\omega}}{dt} = \vec{\alpha}$ and $\frac{d\vec{r}}{dt} = \vec{v} = \vec{\omega} \times \vec{r}$, we get the final equation:

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{a_R} + \vec{a_t}$$

with the **tangential component** $\vec{a_t} = \vec{\alpha} \times \vec{r}$ and **radial (centripetal) component** $\vec{a_R} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$.

This vector notation directly connects with the summary of circular motion in scalar notation introduced earlier:

$$\omega = \frac{d\phi}{dt}, \qquad v = r\omega, \qquad a_r = \omega^2 r, \qquad \alpha = \frac{d\omega}{dt}.$$

1.6.2. Torque

In linear motion a force causes an object to accelerate in a straight line according to Newton's second law, $F = ma$. In rotational motion we are interested in how effectively a force can make an object rotate about some axis, and this depends not only on the magnitude and direction of the force, but also on **where** it is applied and at **what angle**. Intuitively we know that pushing a door near its hinges feels much harder than pushing at the outer edge, even if we use the same force, and that pushing perpendicular to the door is more effective than pushing at a shallow angle; these observations motivate the concept of **torque**, the rotational analogue of force. Consider a force $\vec{F}$ applied to a particle at position $\vec{r}$ measured from the axis (typically the origin). The torque is defined as the **vector cross product**

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

The direction of the torque vector $\vec{\tau}$ is perpendicular to the plane formed by $\vec{r}$ and $\vec{F}$ and is determined by the **right-hand rule**: point your index fingers along $\vec{r}$ and your middle finger along $\vec{F}$, and your thumb points in the direction of $\vec{\tau}$. If the force tends to produce **counterclockwise (CCW)** rotation, the torque vector points **out of the page** and is positive (indicated in 2D plot with ⊙); if it tends to produce **clockwise (CW)** rotation, the torque vector points **into** the page and is negative (indicated in 2D plot with ⊗). As with angular velocity and acceleration, torque is a **pseudovector (axial vector)** whose direction encodes the sense of rotation.

Torque is a specific instance of the **vector cross product**. For two vectors $\vec{A}$ and $\vec{B}$, their cross product is defined by

$$|\vec{A} \times \vec{B}| = AB \sin \phi,$$

where ϕ is the angle between them, and its direction is perpendicular to both $\vec{A}$ and $\vec{B}$ following the right-hand rule. The cross product is anticommutative, $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$, and vanishes when the vectors are parallel or antiparallel.

In Cartesian coordinates, writing $\vec{r} = (x, y, z)$ and $\vec{F} = (F_x, F_y, F_z)$, the torque can be computed from the determinant

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = (yF_z - zF_y)\hat{i} + (zF_x - xF_z)\hat{j} + (xF_y - yF_x)\hat{k}.$$

Therefore, the torque's magnitude is

$$\tau = rF \sin \phi,$$

where r is the distance from the rotation axis, F is the force magnitude, and ϕ is the angle between $\vec{r}$ and $\vec{F}$. This formula shows explicitly that the torque is largest when the force is applied **perpendicular** to $\vec{r}$ (where $\sin \phi = 1$) and vanishes when the force is directed along $\vec{r}$ (where $\sin \phi = 0$).

It is often useful to introduce the **lever arm** (or **moment arm**) $r_\perp = r \sin \phi$, which is the perpendicular distance from the axis of rotation to the **line of action** of the force. In terms of this lever arm the torque magnitude becomes

$$\tau = Fr_\perp,$$

which emphasizes that only the component of the force perpendicular to the radius contributes to the tendency to rotate.

Torque is directly connected to rotational acceleration through the rotational analogue of Newton's second law. For a particle of mass m moving in a circle of radius r,

$$\vec{F} = m\vec{a}.$$

The torque about the axis is

$$\vec{\tau} = \vec{r} \times \vec{F} = m(\vec{r} \times \vec{a}).$$

For circular motion, the tangential component of the acceleration is $a_t = r\alpha$, so the torque magnitude becomes

$$\tau = mr^2\alpha = I\alpha,$$

where $I = mr^2$ is the **moment of inertia** of the particle (see next section).

For a rigid body composed of **many particles**, each contributes a torque $\vec{\tau}_i = \vec{r}_i \times \vec{F}_i$, and the **net torque** is

$$\vec{\tau}_{\text{net}} = \sum_i \vec{\tau}_i.$$

The rotational equation of motion is then

$$\vec{\tau}_{\text{net}} = I\vec{\alpha}.$$

If $\vec{\tau}_{\text{net}} = 0$, the angular acceleration is zero and the body either remains at rest or rotates with constant angular velocity; this is **rotational equilibrium**.

Thus, torque measures how forces create rotation, both in magnitude and direction, and forms the basis for all rotational dynamics.

```
interactive(children=(FloatSlider(value=0.0, description='F [N]', step=5.0), FloatSlider(valu
e=0.2, descriptio…
<function __main__.plot_wrench_clock(F, r_hand, phi_deg, theta_deg)>
```

1.6.3. Moment of Inertia

In translational motion, the mass m of an object measures its resistance to changes in linear motion and appears in Newton's second law $\vec{F}_{\text{net}} = m\vec{a}$. In rotational motion, the analogous quantity is the **moment of inertia** I, which measures how strongly an object resists changes in its rotational motion about a given axis. A larger moment of inertia means that, for the same torque, the resulting angular acceleration will be smaller, and a larger torque is needed to achieve a given angular acceleration.

We first recall the connection between torque and angular acceleration for a single particle of mass m moving in a circle of radius r under a tangential force F_t. The torque about the axis is $\tau = rF_t$, and the tangential component of Newton's second law gives $F_t = ma_t$. Using the relation $a_t = r\alpha$ between tangential acceleration and angular acceleration, we obtain

$$\tau = rF_t = r(ma_t) = mr(r\alpha) = mr^2\alpha.$$

Thus, for a single particle, the torque is proportional to the angular acceleration with proportionality constant mr^2.

For an extended rigid body composed of many particles, each particle i with mass m_i at perpendicular distance r_i from the axis contributes a torque $\tau_i = m_i r_i^2 \alpha$ (for a given angular acceleration α). The net torque about the axis is

$$\tau_{\text{net}} = \sum_i \tau_i = \sum_i m_i r_i^2 \alpha = \alpha \sum_i m_i r_i^2.$$

Because the body is rigid, all particles share the same angular acceleration α, so we can factor it out and define the moment of inertia I about the rotation axis as

$$I = \sum_i m_i r_i^2.$$

The rotational analog of Newton's second law then becomes

$$\tau_{\text{net}} = I\alpha.$$

This equation plays the same role for rotational motion as $\vec{F}_{\text{net}} = m\vec{a}$ does for translational motion.

For a continuous mass distribution, the sum becomes an integral. If dm denotes a small mass element at perpendicular distance r from the axis, the moment of inertia is

$$I = \int r^2\, dm.$$

Here r is always the perpendicular distance from the chosen axis of rotation to the mass element dm. **The value of I therefore depends both on the total mass and on how that mass is distributed relative to the axis**. Mass distributed farther from the axis contributes more strongly because of the r^2 factor. The SI unit of moment of inertia is $\mathrm{kg\ m}^2$.

To illustrate how to compute I by integration, consider a thin **uniform** rod of length L and mass M, rotating about an axis through its center and perpendicular to its length.
For a uniform rod the **linear mass density** is constant:

$$\lambda = \frac{M}{L}.$$

A small segment of length dx at position x then has mass

$$dm = \lambda\, dx = \frac{M}{L}\, dx.$$

The moment of inertia becomes

$$I = \int_0^M x^2\, dm = \frac{M}{L}\int_{-L/2}^{L/2} x^2\, dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \frac{1}{12}ML^2.$$

Thus, for a uniform rod about its center,

$$I_{\text{center}} = \frac{1}{12}ML^2.$$

Parallel-axis Theorem

If we want the moment of inertia of the same rod about an axis through one end (still perpendicular to the rod), we can use the **parallel-axis theorem**.
It states that for any axis parallel to a center-of-mass axis:

$$I = I_{\text{CM}} + Md^2,$$

where I_{CM} is the moment of inertia about the center-of-mass axis, and d is the distance between the two axes.

For the rod:

- $I_{\text{CM}} = \frac{1}{12}ML^2$,
- distance from center to end: $d = L/2$.

Thus,

$$I_{\text{end}} = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2.$$

This demonstrates that moving the rotation axis **away from the center of mass increases the moment of inertia**.

Alternatively, we can compute it directly by shifting the integration region to the interval 0 to L:

$$I = \frac{M}{L}\int_0^L x^2\,dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{1}{3}ML^2.$$

In practice, however, it is usually easier to start from the known **center-of-mass** moment of inertia and apply the **parallel-axis theorem**, rather than re-deriving I from scratch for every new axis.

Common Moments of Inertia & the Tensor Perspective

Moments of inertia for common shapes about symmetry axes are well defined. A few examples are:

Shape / Object	Axis of Rotation	Moment of Inertia
Thin hoop / cylindrical shell	Through center, perpendicular	$I = MR^2$
Solid cylinder or disk	Through center, perpendicular	$I = \frac{1}{2}MR^2$
Solid sphere	Through center	$I = \frac{2}{5}MR^2$
Thin spherical shell	Through center	$I = \frac{2}{3}MR^2$
Rod of length L	Through center, perpendicular	$I = \frac{1}{12}ML^2$
Rod of length L	Through one end, perpendicular	$I = \frac{1}{3}ML^2$

For systems composed of multiple parts, the total moment of inertia about a given axis is the sum of the moments of inertia of the individual parts about that axis,

$$I_{\text{total}} = \sum_i I_i,$$

possibly after using the parallel-axis theorem for each part.

The connection between moment of inertia and torque is given by the rotational equation of motion

$$\tau_{\text{net}} = I\alpha.$$

Moment of inertia is, in full generality, a **tensor quantity**. For arbitrary 3D bodies, the resistance to rotation depends on the **chosen axis**, and the principal moments can differ along different directions. In this course we restrict ourselves to rotation about a **single symmetry axis**, where I behaves as a scalar.

In experiments, moments of inertia can be determined either by measuring the response to a known torque, using $I = \tau/\alpha$, or by analyzing the period of a torsional pendulum. In the torsional pendulum method, a disk is suspended by a wire that exerts a restoring torque $\tau = -\kappa\theta$, where κ is the torsion constant. The system oscillates with period

$$T = 2\pi\sqrt{\frac{I}{\kappa}},$$

so measuring T and knowing κ allows us to determine I.

Overall, the moment of inertia is the key quantity that links torque and angular acceleration through $\tau_{\text{net}} = I\alpha$. It depends strongly on mass distribution and the choice of axis, and it controls how “easy” or “hard” it is to spin an object, in complete analogy with how mass controls resistance to linear acceleration in translational dynamics.

1.6.4. Rotational Kinetic Energy

In linear motion, a moving object of mass m and speed v has kinetic energy $K = \frac{1}{2}mv^2$. A rigid body rotating about a fixed axis also possesses kinetic energy, but this energy depends on both how fast it rotates and how its mass is distributed relative to the axis. The corresponding quantity is the **rotational kinetic energy**, which will turn out to depend on the moment of inertia I and the angular velocity ω.

To derive the expression for rotational kinetic energy, we regard a rotating rigid body as a collection of many small mass elements m_i, each at a perpendicular distance r_i from the rotation axis. If the body rotates with angular velocity ω, each mass element moves in a circle with tangential speed $v_i = r_i\omega$. Its kinetic energy is therefore

$$K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i(r_i\omega)^2 = \frac{1}{2}m_i r_i^2\omega^2.$$

The total kinetic energy of the rigid body is the sum over all elements,

$$K = \sum_i K_i = \frac{1}{2}\omega^2 \sum_i m_i r_i^2.$$

Recognizing that $\sum_i m_i r_i^2$ is the moment of inertia I about the rotation axis, we obtain

$$K = \frac{1}{2} I\omega^2,$$

which is the rotational kinetic energy of a rigid body rotating with angular velocity ω about a fixed axis. This expression is directly analogous to $K = \frac{1}{2}mv^2$, with the substitutions $m \to I$ and $v \to \omega$; mass measures resistance to linear acceleration, while moment of inertia measures resistance to angular acceleration.

Many physical systems involve objects that both translate and rotate. A rolling wheel, for instance, has kinetic energy associated with the motion of its center of mass and additional kinetic energy associated with its rotation about the center. For such an object of total mass M and center-of-mass speed v_{CM}, the total kinetic energy is

$$K_{\mathrm{total}} = K_{\mathrm{trans}} + K_{\mathrm{rot}} = \frac{1}{2} M v_{\mathrm{CM}}^2 + \frac{1}{2} I_{\mathrm{CM}} \omega^2,$$

where I_{CM} is the moment of inertia about an axis through the center of mass. For pure rolling without slipping, the kinematic condition $v_{\mathrm{CM}} = R\omega$ holds, and the translational and rotational contributions are directly related.

Work done by a torque

The work–energy theorem has a direct rotational counterpart. In linear motion, the net work done by the net force equals the change in kinetic energy, $W_{\mathrm{net}} = \Delta K$. For rotational motion about a fixed axis, the **work done by a torque** τ over an angular displacement $\Delta\theta$ is

$$W = \tau \Delta\theta,$$

if the torque is constant. For a varying torque, the work is given by the integral

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta.$$

Using $\tau = I\alpha$ and the kinematic relation $d\theta = \omega \, dt$, one can show that the net work done by the torque equals the change in rotational kinetic energy:

$$W_{\mathrm{net}} = \Delta K_{\mathrm{rot}} = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2.$$

This is the rotational work–energy theorem and mirrors the translational version with torque and angular displacement replacing force and linear displacement.

Under ideal conditions, if no external torque acts on a rotating rigid body, its rotational kinetic energy remains constant, just as the kinetic energy of a freely moving particle remains constant in the absence of net work.

1.6.5. Rolling Motion

Many familiar objects—wheels, balls, cylinders—move by **rolling**, a motion that combines rotation about an axis with translation of the center of mass. Rolling motion is a central application of rotational dynamics because it links angular and linear quantities through the no-slip condition, and it shows how energy and forces are shared between translational and rotational forms.

Rolling Without Slipping

The condition for **rolling without slipping** is that the point of the body touching the surface is instantaneously **at rest relative to the surface**.

For a rolling wheel, the velocity of the contact point is the vector sum of:

- the **translational** velocity of the center of mass, v_{CM}, and
- the **rotational** velocity of the point relative to the center, $R\omega$ (directed backward at the bottom).

Without slipping, at the point of contact these must cancel:

$$v_{\mathrm{CM}} - R\omega = 0.$$

Therefore,

$$v_{\mathrm{CM}} = R\omega.$$

where v_{CM} is the center-of-mass linear velocity, ω the angular velocity, and R the radius of the rolling object. This means the object advances by one circumference $2\pi R$ per full revolution. If the rolling condition does not hold (i.e., if $v_{\mathrm{CM}} \neq R\omega$), the object is slipping.

Differentiating the no-slip condition leads to the acceleration relation:

$$a_{\mathrm{CM}} = R\alpha,$$

relating linear and angular accelerations.

Rolling Down an Incline

When a rigid body rolls down an incline without slipping, gravitational potential energy is converted into both translational and rotational kinetic energy. Energy conservation yields:

$$Mgh = \frac{1}{2}Mv_{\mathrm{CM}}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv_{\mathrm{CM}}^2\left(1 + \frac{I}{MR^2}\right).$$

Solving for the speed at the bottom gives:

$$v_{\mathrm{CM}} = \sqrt{\frac{2gh}{1 + I/(MR^2)}}.$$

Objects with smaller $I/(MR^2)$ accelerate faster.

1.6.6. Angular Momentum

In linear motion, momentum is defined as $\vec{p} = m\vec{v}$ and characterizes the state of translational motion. In rotational motion, the analogous quantity is the **angular momentum** $\vec{L}$, which characterizes how the motion of a particle or an extended body contributes to rotation about a chosen point or axis. Angular momentum is a vector (more precisely, an axial or pseudovector) whose direction is given by the right-hand rule and whose magnitude depends on both the distribution of mass and the velocity. For rigid rotation about a fixed axis, the angular momentum is proportional to the angular velocity, $L = I\omega$, in close analogy to $p = mv$. The connection between torque and angular momentum is given by $\vec{\tau} = \dfrac{d\vec{L}}{dt}$, directly paralleling the linear relation $\vec{F} = \dfrac{d\vec{p}}{dt}$ and providing the basis for the conservation of angular momentum, which we will discuss in the next section.

Angular Momentum of a Particle

For a single particle of mass m at position $\vec{r}$ (measured from some origin O) with linear momentum $\vec{p} = m\vec{v}$, the angular momentum about O is defined by the vector cross product

$$\vec{L} = \vec{r} \times \vec{p}.$$

The magnitude of $\vec{L}$ is

$$L = rp\sin\phi = rmv\sin\phi,$$

where ϕ is the angle between $\vec{r}$ and $\vec{p}$. The direction of $\vec{L}$ is perpendicular to the plane defined by $\vec{r}$ and $\vec{p}$. Angular momentum has SI units $\mathrm{kg\ m^2/s}$, the same as torque multiplied by time.

For a particle moving in a circle of radius r with speed v, the magnitude of the angular momentum about the center is

$$L = mrv.$$

Using $v = r\omega$ and $I = mr^2$ for a single particle, this can be written as

$$L = mrv = mr(r\omega) = mr^2\omega = I\omega,$$

showing that the relation $L = I\omega$ holds for a particle in circular motion. Notice the similarity to it's linear counterpart, i.e. $p = mv$.

The time rate of change of the angular momentum follows from differentiating $\vec{L} = \vec{r} \times \vec{p}$ with the product rule:

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}.$$

Let's rewrite this using $\frac{d\vec{r}}{dt} = \vec{v}$ and $\vec{p} = m\vec{v}$ as well as $\vec{F} = \frac{d\vec{p}}{dt}$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}dt.$$

Since $\frac{d\vec{r}}{dt} = \vec{v}$ is parallel to $\vec{p} = m\vec{v}$, the first term vanishes, and we obtain

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}.$$

Thus torque is the time derivative of angular momentum:

$$\vec{\tau} = \frac{d\vec{L}}{dt}.$$

If the net external torque about the chosen origin is zero, then $\frac{d\vec{L}}{dt} = 0$ and the angular momentum of the particle is conserved.

Angular Momentum and Torque for a System of Particles; General Motion

Most physical systems consist of many particles. For a system with particles of mass m_i, positions $\vec{r}_i$, and momenta $\vec{p}_i = m_i\vec{v}_i$, the total angular momentum about a reference point O is

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i.$$

Differentiating with respect to time gives

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) = \sum_i \left(\frac{d\vec{r}_i}{dt} \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt} \right).$$

As before, $\frac{d\vec{r}_i}{dt} = \vec{v}_i$ is parallel to $\vec{p}_i$, so $\vec{v}_i \times \vec{p}_i = \vec{0}$ and the first term vanishes. We are left with

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i,$$

where $\vec{F}_i$ is the total force on particle i. Each $\vec{F}_i$ can be decomposed into external and internal contributions,

$$\vec{F}_i = \vec{F}_{i,\text{ext}} + \sum_j \vec{F}_{ij},$$

where $\vec{F}_{ij}$ is the force exerted on particle i by particle j. Substituting into the expression for $d\vec{L}/dt$ gives

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_{i,\text{ext}} + \sum_i \sum_j \vec{r}_i \times \vec{F}_{ij}.$$

If the internal forces satisfy Newton's third law and act along the line joining particle pairs (central forces), the internal torque contributions cancel in pairs:

$$\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji} = \vec{0},$$

so the double sum over internal forces vanishes. Only external forces remain, and we obtain

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_{i,\text{ext}} \equiv \vec{\tau}_{\text{ext}},$$

where $\vec{\tau}_{\text{ext}}$ is the net external torque about the chosen origin. This result is completely general and for any system of particles, we can state:

$$\vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}.$$

It is often useful to decompose the total angular momentum into the motion of the center of mass and the motion relative to the center of mass. Let the center of mass be at position

$$\vec{R} = \frac{1}{M} \sum_i m_i \vec{r}_i,$$

where $M = \sum_i m_i$ is the total mass. We write each position as

$$\vec{r}_i = \vec{R} + \vec{r}_i',$$

with $\vec{r}_i'$ measured from the center of mass, and each velocity as

$$\vec{v}_i = \vec{V} + \vec{v}_i',$$

where $\vec{V} = \dfrac{d\vec{R}}{dt}$ is the center-of-mass velocity. The total angular momentum becomes

$$\vec{L} = \sum_i \vec{r}_i \times m_i \vec{v}_i = \sum_i m_i (\vec{R} + \vec{r}_i') \times (\vec{V} + \vec{v}_i').$$

Expanding the cross products gives four terms:

$$\vec{L} = \sum_i m_i (\vec{R} \times \vec{V}) + \sum_i m_i (\vec{R} \times \vec{v}_i') + \sum_i m_i (\vec{r}_i' \times \vec{V}) + \sum_i m_i (\vec{r}_i' \times \vec{v}_i').$$

Using $\sum_i m_i \vec{r}_i' = \vec{0}$ (the center of mass is the origin of the primed coordinates), the middle two sums vanish, leaving

$$\vec{L} = M(\vec{R} \times \vec{V}) + \sum_i m_i (\vec{r}_i' \times \vec{v}_i').$$

We can identify

$$\vec{L}_{\text{CM}} = M(\vec{R} \times \vec{V})$$

as the angular momentum associated with the translational motion of the center of mass, and

$$\vec{L}' = \sum_i m_i(\vec{r}_i' \times \vec{v}_i')$$

as the angular momentum about the center of mass (rotation). Thus, the total angular momentum decomposes into

$$\vec{L} = \vec{L}_{\text{CM}} + \vec{L}'.$$

This separation mirrors the decomposition of kinetic energy into translational and rotational parts and is especially useful for analyzing complex systems where both the center of mass moves and the body rotates about its center of mass.

Angular Momentum and Torque for a Rigid Object

For a rigid body rotating about a fixed axis (for example, the z-axis) with angular velocity ω, each mass element m_i at perpendicular distance r_i from the axis has tangential speed $v_i = r_i\omega$ and linear momentum $p_i = m_i v_i$. Its angular momentum magnitude about the axis is $L_i = r_i p_i = m_i r_i^2 \omega$. Summing over all mass elements gives

$$L = \sum_i L_i = \sum_i m_i r_i^2 \omega = I\omega,$$

where $I = \sum_i m_i r_i^2$ is the moment of inertia about the rotation axis. In vector form, for rotation about a symmetry axis of the body, we can write

$$\vec{L} = I\vec{\omega},$$

with $\vec{L}$ and $\vec{\omega}$ both pointing along the rotation axis according to the right-hand rule. Angular momentum is again an axial or pseudovector, changing sign when we reverse the sense of rotation.

The connection to torque is obtained by differentiating $\vec{L} = I\vec{\omega}$ with respect to time. If I is constant (fixed axis and rigid body),

$$\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha},$$

so the net external torque about the axis is

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} = I\vec{\alpha}.$$

This is the rotational analog of Newton’s second law, directly relating the angular acceleration to the applied torque.

For symmetric objects rotating about a symmetry axis (disks, cylinders, spheres), $\vec{L}$ and $\vec{\omega}$ are parallel and the simple relation $\vec{L} = I\vec{\omega}$ holds. For more general three-dimensional motion of irregular bodies, the distribution of mass can be described by different principal moments of inertia I_x, I_y, I_z about mutually perpendicular axes, and the components of angular momentum are then

$$\vec{L} = I_x \omega_x \hat{i} + I_y \omega_y \hat{j} + I_z \omega_z \hat{k}.$$

In such cases, $\vec{L}$ and $\vec{\omega}$ need not be parallel, leading to more complex rotational behavior such as wobbling or nutation.

Finally, the angular momentum of a rigid body about an arbitrary point P can be written as

$$\vec{L}_P = \vec{R}_P \times (M\vec{V}_{\mathrm{CM}}) + \vec{L}_{\mathrm{CM}},$$

where $\vec{R}_P$ is the position of the center of mass relative to P, $\vec{V}_{\mathrm{CM}}$ is the center-of-mass velocity, and $\vec{L}_{\mathrm{CM}}$ is the angular momentum about the center of mass. This expression again separates the contribution from the translation of the center of mass and the rotation about the center of mass and applies to any rigid body, regardless of its motion.

In summary, angular momentum generalizes the concept of momentum to rotational motion: for particles, $\vec{L} = \vec{r} \times \vec{p}$; for systems and rigid bodies, $\vec{L}$ is the sum over all mass elements and can be decomposed into center-of-mass and rotational parts; and in all cases, the net external torque equals the time rate of change of total angular momentum, $\vec{\tau}_{\mathrm{ext}} = \dfrac{d\vec{L}}{dt}$. In the next section, we will focus on the powerful consequences of this relation in the form of the conservation of angular momentum.

1.6.7. Conservation of Angular Momentum

The **conservation of angular momentum** is a central and remarkably powerful principle in mechanics. It states that if the **net external torque** on a system is zero, then the system’s total angular momentum remains **constant in time**—both in magnitude and direction. This principle applies to systems ranging from rotating atoms and molecules to the collapse of stars, from figure skaters pulling in their arms to collisions involving rotating machinery.

When a system is isolated from external torques, the redistribution of mass inside the system may alter the individual angular velocities of its components, but the **total angular momentum** remains unchanged. The law follows directly from the rotational analog of Newton’s second law,

$$\vec{\tau}_{\mathrm{ext}} = \frac{d\vec{L}}{dt},$$

so that if $\vec{\tau}_{\mathrm{ext}} = \vec{0}$, then

$$\vec{L} = \mathrm{constant}.$$

Conservation Law

For rotation about a fixed axis where the direction of $\vec{L}$ is constant, the law reduces to scalars:

$$L_i = L_f, \qquad I_i\omega_i = I_f\omega_f.$$

Thus, if a system changes its **moment of inertia** I while isolated from external torques, its angular velocity ω must adjust to keep $I\omega$ constant.

As an example, imagine a figure skater pulling their arms in. This changes the mass distribution, therefore the moment of inertia. Due to conservation of angular momentum, if I decreases ω has to increase.

External vs. Internal Torques

Conservation of angular momentum requires that the **net external torque** be zero. Internal forces within the system (e.g., explosions, ejections of mass, tightening of muscles) come in equal-and-opposite pairs and do not change the total $\vec{L}$. They can redistribute angular momentum among components but cannot alter the total.

Conceptual Summary

- Angular momentum is conserved when the **net external torque** is zero.
- Internal forces cannot change total $\vec{L}$, only its distribution.
- Reducing moment of inertia increases angular velocity if L is fixed.
- Conservation applies at all scales: atoms, machinery, stars, galaxies.
- When torque-free, both the **magnitude** and **direction** of $\vec{L}$ remain constant.
- Collisions or mass redistribution obey angular momentum conservation when isolated from external torques.

In the next section, we explore how a torque **perpendicular** to angular momentum alters only the **direction** of $\vec{L}$, producing **precession**, the hallmark behavior of spinning tops and gyroscopes.

1.6.8. Gyroscope

A spinning top or gyroscope exhibits one of the most striking behaviors in rotational dynamics: when a torque acts **perpendicular** to the angular momentum vector, the object does not simply tip over but instead undergoes **precession**, a steady rotation of its spin axis around the vertical. This motion follows directly from the vector relationship between torque and angular momentum, and it illustrates vividly how rotational quantities differ fundamentally from their linear counterparts.

In rotational dynamics, the governing equation

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

shows that torque determines how the angular momentum vector changes in time. If the torque is aligned **parallel** to $\vec{L}$, it changes the **magnitude** of angular momentum, increasing or decreasing the spin rate. If the torque is **perpendicular** to $\vec{L}$—as gravity is for a spinning top—then it changes the

direction of $\vec{L}$ while keeping its magnitude constant. This is the essence of precession: the angular momentum vector moves sideways, causing the top's axis to sweep a cone around the vertical.

Spinning Top Under Gravity

Consider a symmetric spinning top supported at a fixed pivot. Its center of mass is a horizontal distance r from the pivot, and gravity produces a downward force $M\vec{g}$ at that point. The torque about the pivot is

$$\vec{\tau} = \vec{r} \times M\vec{g},$$

with magnitude

$$\tau = Mgr\sin\theta,$$

where θ is the angle between the top's axis and the vertical. This torque is horizontal—perpendicular to both the displacement $\vec{r}$ and gravity—and therefore perpendicular to the top's angular momentum $\vec{L}$, which points along the symmetry axis of rotation.

Because $\vec{\tau} \perp \vec{L}$, the torque does not slow the spin: instead it changes the **direction** of $\vec{L}$, causing it to rotate about the vertical axis. The visible consequence is precession: the tip of the top's axis traces out a cone.

Precessional Angular Velocity

Consider a spinning top whose angular momentum $\vec{L}$ points along its symmetry axis, tilted by an angle θ from the vertical. Gravity exerts a downward force $M\vec{g}$ at the center of mass. The **perpendicular lever arm** relative to the pivot is

$$r_\perp = r\sin\theta,$$

so the gravitational torque has magnitude

$$\tau = Mg\,r_\perp = Mg\,r\sin\theta.$$

The torque vector is given by the full cross product

$$\vec{\tau} = \vec{r} \times M\vec{g},$$

where $\vec{r}$ points from the pivot to the center of mass and $M\vec{g}$ is vertical. These two vectors lie in the **same vertical plane** that also contains the tilted spin axis (and thus $\vec{L}$). By the right-hand rule, $\vec{\tau}$ is perpendicular to this plane and therefore **perpendicular to** $\vec{L}$. Consequently, $\tau = d\vec{L}/dt$ changes only the **direction** of $\vec{L}$, not its magnitude.

Vector directions:

- $\vec{L}$: along the tilted spin axis.
- $\vec{\tau}$: perpendicular to the vertical plane defined by $\vec{r}$ and $\vec{g}$ as well as perpendicular to $\vec{L}$.
- $\vec{\Omega}$: horizontal, describing the slow rotation of $\vec{L}$ around the vertical.

During precession, the tip of $\vec{L}$ moves on a horizontal circle. The horizontal component of $\vec{L}$ determines its radius:

$$L_\perp = L\sin\theta$$

If the precession angular velocity is Ω, then in a short time Δt the tip of $\vec{L}$ sweeps an arc (imagine a circle with radius with $L_\perp$ and a circle segment with long side $L_\perp$ and short side $\Omega\Delta t$):

$$|\Delta\vec{L}| = L_\perp(\Omega\Delta t) = L\sin\theta\,\Omega\,\Delta t.$$

From the torque relation,

$$|\Delta\vec{L}| = \tau\,\Delta t.$$

Equating the two expressions gives

$$\tau\,\Delta t = L\sin\theta\,\Omega\,\Delta t$$

$$\tau = L\sin\theta\,\Omega, \qquad \Omega = \frac{\tau}{L\sin\theta}.$$

Insert $\tau = Mg\,r_\perp = Mg\,r\sin\theta$ and $L = I\omega$:

$$\Omega = \frac{Mg\,r\sin\theta}{I\omega\sin\theta} = \frac{Mg\,r}{I\omega}.$$

Thus,

$$\boxed{\Omega = \frac{Mg\,r}{I\omega}}.$$

A fast spin (large ω) or a large moment of inertia I produces **slow precession**. If the top spins counterclockwise (seen from above), the precession direction is clockwise by the right-hand rule.

Nutation: Wobbling of the Axis

If the top is not released smoothly, the initial torque does not act uniformly and the axis undergoes a small oscillation superimposed on precession, known as **nutation**. Friction and air resistance damp nutation over time, leaving steady precession.

Energy Considerations

Because the gravitational torque is perpendicular to the spin angular velocity, it does **no work**, and therefore the rotational kinetic energy remains

$$K = \frac{1}{2}I\omega^2$$

during steady precession. Only processes that change ω (such as friction) change the rotational kinetic energy.

Conceptual Summary

- A torque perpendicular to $\vec{L}$ changes the **direction**, not the magnitude, of angular momentum.
- The resulting slow horizontal rotation of the spin axis is **precession**.
- Precession slows as the top spins faster or has larger moment of inertia.
- When spin decreases, precession accelerates until the top can no longer remain upright.
- Gyroscopes exploit these effects for stability in aircraft, ships, and smartphones.

In the next section, we turn to **rotating reference frames** and examine how fictitious forces such as the **Coriolis** and **centrifugal** forces arise from viewing motion in a rotating coordinate system.

1.6.9. Rotating Reference Frames and Inertial Forces

Newton's laws hold in their standard form only in inertial frames.
When describing motion from a rotating frame (such as standing on the rotating Earth), additional apparent forces appear. These are **inertial** or **fictitious forces**: the **centrifugal**, **Coriolis**, and **Euler** forces. They do not correspond to physical interactions but arise because the reference frame itself accelerates.

Velocity transformation: why $\vec{\omega} \times \vec{r}$ appears

Let S be an inertial frame and S' a frame rotating with angular velocity $\vec{\omega}$. A particle at position $\vec{r}$ has velocities related by

$$\vec{v}_S = \vec{v}_{S'} + \vec{\omega} \times \vec{r}.$$

The term $\vec{\omega} \times \vec{r}$ is the linear velocity due to the rotation of the coordinate axes themselves.

Acceleration transformation

Differentiate the velocity relation:

$$\vec{a}_S = \frac{d}{dt}\big(\vec{v}_{S'} + \vec{\omega} \times \vec{r}\big) = \frac{d\vec{v}_{S'}}{dt} + \frac{d}{dt}\big(\vec{\omega} \times \vec{r}\big).$$

The first term is simply $\frac{d\vec{v}_{S'}}{dt} = \vec{a}_{S'}$.

For the second term, use the product rule for the cross product:

$$\frac{d}{dt}\big(\vec{\omega} \times \vec{r}\big) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}.$$

Since $\frac{d\vec{r}}{dt} = \vec{v}_S = \vec{v}_{S'} + \vec{\omega} \times \vec{r}$, we obtain

$$\vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\omega} \times (\vec{v}_{S'} + \vec{\omega} \times \vec{r}) = \vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}).$$

Putting it together:

$$\vec{a}_S = \vec{a}_{S'} + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\omega} \times \vec{v}_{S'}.$$

Combine the two identical $\vec{\omega} \times \vec{v}_{S'}$ terms:

$$\vec{a}_S = \vec{a}_{S'} + 2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}.$$

Newton’s second law and the origin of fictitious forces

In the inertial frame:

$$m\vec{a}_S = \sum \vec{F}_{\text{real}}.$$

Insert the acceleration transformation and solve for $m\vec{a}_{S'}$:

$$m\left[\vec{a}_{S'} + 2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}\right] = \sum \vec{F}_{\text{real}}$$

$$m\vec{a}_{S'} + m\left[2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}\right] = \sum \vec{F}_{\text{real}}$$

$$m\vec{a}_{S'} = \sum \vec{F}_{\text{real}} - m\left[2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}\right].$$

A rotating observer wants Newton’s form:

$$m\vec{a}_{S'} = \sum \vec{F}_{\text{real}} + \sum \vec{F}_{\text{fictitious}}.$$

Thus the total fictitious force is:

$$\vec{F}_{\text{fict}} = -m\left[2\vec{\omega} \times \vec{v}_{S'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}\right].$$

Individual fictitious forces

- **Coriolis force**

$$\vec{F}_C = -2m(\vec{\omega} \times \vec{v}_{S'})$$

- **Centrifugal force**

$$\vec{F}_{\text{cf}} = -m\left[\vec{\omega} \times (\vec{\omega} \times \vec{r})\right]$$

- **Euler force** (only if the rotation rate changes)

$$\vec{F}_E = -m\frac{d\vec{\omega}}{dt} \times \vec{r}$$

Centrifugal Force

The centrifugal force is a **fictitious force** that arises from the rotating (non-inertial) reference frame. It arises from the component of the acceleration vector $\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$ and points **radially outward**.

For motion at distance r from the rotation axis, the **magnitude** of the centrifugal force is:

$$F_{\text{cf}} = m\omega^2 r = \frac{mv^2}{r}$$

Note: The magnitude of the centrifugal force is **equal** to the magnitude of the centripetal force, but their directions are **opposite**. Centrifugal force is outward, while centripetal force is inward.

Coriolis Force

The Coriolis force acts on objects that move relative to the rotating frame:

$$\vec{F}_C = -2m(\vec{\omega} \times \vec{v}).$$

Magnitude:

$$F_C = 2m\omega v \sin\phi,$$

where ϕ is the angle between $\vec{\omega}$ and $\vec{v}$.

Note that the Coriolis force does not depend on the position $\vec{r}$ but only on the linear and angular velocity.

On Earth, the Coriolis force shapes **wind patterns**, **ocean currents**, and **projectile trajectories**:

- **Northern Hemisphere:** deflection to the **right**
- **Southern Hemisphere:** deflection to the **left**

Foucault Pendulum

A **Foucault pendulum** is the classical demonstration of Earth's rotation. Its swing plane precesses (rotates) relative to the ground at the rate:

$$\Omega_p = \omega \sin\phi$$

Where:

- Ω_p: Rate of precession (angular frequency).
- ω: Earth's angular rotation speed ($\approx 15^\circ$ per hour).
- ϕ: **Latitude** (angle from the equator) of the pendulum.

Key Precession Rates (Time T for one full rotation of the swing plane):

- At the North/South Pole ($\phi = 90^\circ$): $T = 24$ hours (one full rotation per sidereal day).
- At the Equator ($\phi = 0^\circ$): **No precession** ($\Omega_p = 0$).
- At mid-latitudes (e.g., $\phi = 45^\circ$): $T \approx 34$ hours.

Euler Force (Azimuthal Force)

The Euler force is a fictitious force that appears when the **angular velocity of the rotating coordinate system ($\vec{\omega}$) is changing** ($d\vec{\omega}/dt \neq 0$).

$$\vec{F}_E = -m\frac{d\vec{\omega}}{dt} \times \vec{r}$$

Direction: The force is always **tangential** (perpendicular) to the position vector $\vec{r}$. It acts opposite to the direction of the tangential acceleration.

Magnitude (Example: Turntable Speeding Up): The tangential acceleration ($\vec{a}_E$) experienced by an object at distance r due to the changing rotation is:

$$a_E = r\frac{d\omega}{dt}$$

The magnitude of the Euler force is therefore $F_E = ma_E$.

Conceptual Summary

- Rotating frames require fictitious forces to maintain Newton's laws.
- **Centrifugal force:** outward radial force reducing apparent weight.
- **Coriolis force:** sideways deflection of moving objects; crucial on global scales.
- **Euler force:** appears only when ω changes with time.
- These forces disappear in an inertial frame; they reflect the acceleration of the rotating frame itself.