1.10. Waves

In this chapter we introduce **wave motion** as the propagation of **oscillations through space and time**. We begin by defining the **types and basic characteristics of waves** and develop their **mathematical description** using traveling wave functions and the **wave equation**. We then study how waves **transport energy**, how they behave at **boundaries** through reflection and transmission, and how **superposition** leads to interference and standing waves. The chapter concludes with **refraction, dispersion, and diffraction**, highlighting the universal principles that govern wave phenomena across mechanics, acoustics, and optics.

1.10.1 Characteristics of Waves

A **wave** is a disturbance that travels through space and time, transferring **energy** and **momentum** without transporting matter. Typical examples include water waves, sound waves, seismic waves, and electromagnetic waves. All waves share common properties such as amplitude, wavelength, frequency, period, and speed, which together describe how the disturbance evolves and propagates.

The Nature of Waves

When energy is imparted to a medium, its particles are displaced from their equilibrium positions. These particles exert restoring forces on neighboring particles, thereby transmitting the disturbance through the medium. Wave motion is therefore the propagation of energy through oscillations of the medium's particles. Each particle oscillates about its equilibrium position, but there is **no net transport of matter**, and only the disturbance itself propagates through the medium.

Mechanical, Electromagnetic, and Matter Waves

Mechanical waves require a material medium for their propagation, such as water, air, or solids, and examples include sound waves, water waves, and waves on strings. Their behavior depends on the **elastic** and **inertial** properties of the medium. Electromagnetic waves do not require a material medium and can propagate through vacuum, with examples including light, radio waves, and X-rays. They arise from oscillating electric and magnetic fields that sustain each other during propagation.

Key Terms in Wave Motion

The **medium** is the substance through which the wave travels, while the **source** is the origin of the disturbance, such as a vibrating object. **Propagation** refers to the transfer of energy through oscillations of particles or variations of fields. The **direction of propagation** is the direction in which energy is transported. A **wavefront** is a surface connecting points that oscillate in the same phase and is useful for visualizing how waves spread through space.

Important Characteristics of a Wave

The **amplitude** A is the maximum displacement of particles of the medium from their equilibrium position and is a measure of the wave's energy content, with larger amplitudes corresponding to greater energy. The **wavelength** λ is the distance between two consecutive points that are in the same phase of motion, such as two neighboring crests or compressions. The **frequency** f is the number of complete oscillations per second and is measured in hertz, while the **period** T is the time required for one complete oscillation and is related to the frequency by

$$T = \frac{1}{f}.$$

The **wave speed** v describes how fast the disturbance propagates through the medium and is given by

$$v = f\lambda = \frac{\lambda}{T}.$$

Conceptual Notes

Waves transmit **energy rather than matter**, and their form and motion depend on both the **properties of the medium** and the **frequency of the source**. The frequency of a wave is determined by the source and does not change when the wave enters a different medium. The wave speed depends on the medium's **elasticity** and **inertia**, and the wavelength adjusts as necessary so that the relation

$$v = f\lambda$$

remains valid.

1.10.2 Types of Waves

Waves can be classified according to the **direction of particle motion** relative to the direction of wave propagation. This distinction defines the two principal types of mechanical waves, namely **transverse** and **longitudinal** waves.

Independently of particle motion, waves can also be classified by the **geometry of their wavefronts**, leading to **plane waves** and **spherical waves**, which describe how energy is distributed and propagates in space.

Transverse Waves

In a **transverse wave**, the particles of the medium oscillate **perpendicular** to the direction in which the wave propagates. The disturbance travels along the medium, while individual particles move about their equilibrium positions in a direction orthogonal to propagation. Typical examples include waves on a stretched string or rope, ripples on the surface of water, and electromagnetic waves, although electromagnetic waves are not mechanical waves. Points of maximum positive displacement are called **crests**, while points of maximum negative displacement are called **troughs**.

Longitudinal Waves

In a **longitudinal wave**, the particles of the medium oscillate **parallel** to the direction of wave propagation. Instead of crests and troughs, the wave consists of alternating regions of **compression**, where density and pressure are higher than average, and **rarefaction**, where they are lower. These regions move through the medium as the wave propagates. Common examples include sound waves in air, compression waves in springs or slinkies, and seismic primary waves, also known as P-waves.

Plane Waves and Spherical Waves

Waves can also be classified by the **geometry of their wavefronts**. A **plane wave** has wavefronts that are parallel planes, meaning that the disturbance propagates in a single direction with uniform amplitude and phase across each wavefront. Plane waves are a good approximation for waves far from their source or in confined systems such as waves on a long string. A **spherical wave** originates from a **point source** and has wavefronts that are concentric spheres expanding outward. In spherical waves, the energy spreads over an increasing area, so the amplitude decreases with distance from the source, even though the frequency remains constant.

Summary: Types of Waves

Wave type	Particle motion relative to propagation	Medium requirement	Typical examples
Transverse	Perpendicular to propagation	Requires shear rigidity	String waves, water surface waves, electromagnetic waves
Longitudinal	Parallel to propagation	Requires compressibility	Sound in air, compression waves, seismic P-waves

Waves are classified independently by how particles move and by how wavefronts propagate, with medium properties controlling allowed motion and geometry controlling how energy spreads in space.

Wave type	Wavefront shape	Energy distribution	Typical situation
Plane wave	Parallel planes	Uniform across wavefront	Far from source, guided waves
Spherical wave	Expanding spheres	Spreads as distance increases	Point sources, explosions, speakers

1.10.3 Kinematics of Wave Motion

To describe wave motion quantitatively, we consider a one-dimensional wave propagating along the x-axis, which may be transverse, as on a stretched string, or longitudinal, as in a solid rod or fluid-filled tube. Further, we assume that the wave propagates without changing shape so that its displacement can be expressed as a function of position and time.

Sinusoidal Traveling Waves

At a fixed instant in time, a sinusoidal wave of wavelength λ is described by the spatial displacement

$$D(x) = A\sin\left(\frac{2\pi}{\lambda}x\right),$$

where $D(x)$ is the displacement of the medium from equilibrium at position x and A is the amplitude, and this function is periodic in space with period λ.

If the wave travels in the positive x-direction with speed v, the entire wave pattern shifts rigidly along the x-axis without changing shape, so that after a time t the displacement becomes

$$D(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right).$$

Using the relations $v = \lambda f$ and $T = 1/f$, this expression can be written equivalently as

$$D(x,t) = A\sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T}\right).$$

Introducing the wave number

$$k = \frac{2\pi}{\lambda}$$

and the angular frequency

$$\omega = 2\pi f,$$

the traveling wave takes the compact and widely used form

$$D(x,t) = A\sin(kx - \omega t).$$

The quantity $(kx - \omega t)$ is called the **phase** of the wave, and the wave speed can be written as

$$v = \frac{\omega}{k},$$

which is why v is also referred to as the **phase velocity**. A wave traveling in the negative x-direction is described by

$$D(x,t) = A\sin(kx + \omega t).$$

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Phase and Physical Interpretation

The **phase** specifies the state of oscillation of a particle and determines how its motion compares to that of other particles in the wave. At a fixed time, the function $D(x,t)$ describes the **spatial shape** of the wave, which is sinusoidal and translates along the x-axis without changing form. At a fixed position, the displacement varies **sinusoidally in time**, so that each particle undergoes simple harmonic motion with angular frequency ω. Particles with the same phase oscillate in sync, while a phase difference $\Delta\phi$ indicates how far their oscillations are shifted relative to one another.

To allow for an arbitrary initial configuration of the wave, a phase constant ϕ may be included, giving the general sinusoidal traveling wave

$$D(x,t) = A\sin(kx - \omega t + \phi).$$

General Form of a Traveling Wave

A traveling wave does not need to be sinusoidal, and if dissipative effects are negligible, a wave of arbitrary shape maintains its form as it propagates. If the shape of the wave at time $t = 0$ is described by a function $D(x)$, then a wave traveling in the positive x-direction with speed v is given by

$$D(x,t) = D(x - vt),$$

while a wave traveling in the negative x-direction is described by

$$D(x,t) = D(x + vt).$$

These expressions provide the most general mathematical description of one-dimensional traveling waves and show explicitly that wave motion corresponds to a translation of the waveform at constant speed.

Particle Velocity and Wave Velocity

The **particle velocity** is the velocity of individual particles of the medium as they oscillate about their equilibrium positions and generally varies with time and position, whereas the **wave velocity** is the speed at which the waveform or disturbance propagates through the medium and is fixed by the properties of the medium. In simple harmonic motion, particles have zero velocity at their maximum displacement even though the wave itself continues to propagate through the medium.

Transverse and Longitudinal Waves in Mathematical Form

In a transverse wave, particle motion is perpendicular to the direction of propagation and can be described by an equation such as $y(x,t) = A\sin(kx - \omega t)$, while in a longitudinal wave, particle motion is parallel to the direction of propagation and is often written as $s(x,t) = s_m\cos(kx - \omega t)$. In longitudinal waves, regions of compression and rarefaction propagate through the medium, and associated pressure variations are ninety degrees out of phase with the particle displacement.

Summary of Wave Parameters

Symbol	Quantity	Expression	Unit
λ	Wavelength	—	m
f	Frequency	$f = 1/T$	Hz
T	Period	$T = 1/f$	s
v	Wave speed	$v = f\lambda$ $= \omega/k$	m/s
k	Wave number	$k = 2\pi/\lambda$	rad/m
ω	Angular frequency	$\omega = 2\pi f$	rad/s

Conceptual Notes

A traveling wave represents the propagation of a disturbance through space rather than the transport of matter, with each particle executing local oscillatory motion. The frequency and angular frequency are determined by the source and remain unchanged when the wave enters a different medium, while the wave speed is determined by the medium and the wavelength adjusts accordingly. The mathematical form $kx - \omega t = \mathrm{const}$ expresses the fact that points of constant phase move with speed $v = \omega/k$, linking the abstract wave function directly to the physical motion of the wave.

1.10.4 The Wave Equation

In the previous section, we introduced the wave function as a mathematical description of how a disturbance varies in space and time. We now ask a deeper question: **which equation governs the time evolution of such wave functions and determines how waves propagate dynamically through a medium?**

The answer is the **wave equation**, a fundamental relation that links the spatial curvature of a wave to its temporal acceleration. It applies to a wide range of physical systems, including vibrating strings, sound waves, water waves, and electromagnetic waves, and expresses a universal structure underlying wave motion.

To see how the wave equation arises, consider a sinusoidal traveling wave propagating along the x-axis,

$$y(x,t) = A\sin(kx - \omega t),$$

where A is the amplitude, $k = 2\pi/\lambda$ is the wave number, $\omega = 2\pi f$ is the angular frequency, and the wave speed is $v = \omega/k$. This function represents a wave of fixed shape traveling with constant speed.

We now examine how this function varies in space and time. Taking the second derivative with respect to position gives

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A \sin(kx - \omega t) = -k^2 y,$$

while taking the second derivative with respect to time yields

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(kx - \omega t) = -\omega^2 y.$$

Comparing these results shows that the second time derivative is proportional to the second spatial derivative,

$$\frac{\partial^2 y}{\partial t^2} = \frac{\omega^2}{k^2} \frac{\partial^2 y}{\partial x^2}.$$

Using the relation $v = \omega/k$, this becomes

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}.$$

This equation is called the **one-dimensional wave equation**. It governs all waves that propagate without changing shape at a constant speed v.

Although derived here from a sinusoidal wave, the wave equation is far more general. Any function of the form

$$y(x,t) = f(x - vt) \quad \text{or} \quad y(x,t) = f(x + vt),$$

representing a waveform traveling to the right or to the left with speed v, satisfies the same equation. Sinusoidal waves are therefore not special solutions but simply the most convenient ones for analysis.

The wave equation has a clear physical interpretation. The second spatial derivative measures the **curvature** of the wave profile, while the second time derivative represents the **acceleration** of the medium's particles. The equation therefore states that regions of greater curvature correspond to larger accelerations, reflecting stronger restoring forces within the medium. In this way, the wave equation links the geometry of the wave to the dynamics of the system.

In three spatial dimensions, the wave equation generalizes to

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2},$$

where the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

accounts for spatial curvature in all directions. This equation governs sound waves in air, electromagnetic waves in space, and many other physical wave phenomena.

1.10.5 Energy Transported by Waves

A wave transports **energy** through a medium without transporting matter. Individual particles of the medium oscillate about their equilibrium positions, repeatedly gaining and losing **kinetic** and **potential** energy, while the disturbance itself propagates through space. The coordinated motion of many oscillating particles results in a net flow of energy along the direction of wave propagation.

Energy of a Sinusoidal Wave on a String

To make this idea quantitative, consider a transverse sinusoidal wave traveling along a stretched string,

$$y(x,t) = A\sin(kx - \omega t),$$

where A is the amplitude, $k = 2\pi/\lambda$ is the wave number, and $\omega = 2\pi f$ is the angular frequency. The string has tension T and linear mass density μ.

For infinitesimal small string elements, each element of length dx and mass $dm = \mu\, dx$ oscillates vertically while the wave propagates horizontally along the string.

The transverse velocity of a string element is obtained by differentiating the displacement with respect to time,

$$v_y = \frac{\partial y}{\partial t} = -A\omega\cos(kx - \omega t).$$

The instantaneous **kinetic energy** of the element is therefore

$$dK = \frac{1}{2} dm\, v_y^2 = \frac{1}{2}\mu A^2\omega^2\cos^2(kx - \omega t)\, dx.$$

When the string is displaced, it is also slightly stretched, storing **elastic potential energy**. For small slopes, the potential energy of a string element is

$$dU = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx.$$

We can simply compute $\frac{\partial y}{\partial x}$

$$\frac{\partial y}{\partial x} = Ak\cos(kx - \omega t),$$

thus, we obtain

$$dU = \frac{1}{2}TA^2k^2\cos^2(kx - \omega t)\, dx.$$

For a stretched string, applying Newton's second law to a small string element shows that the transverse restoring force is set by the tension T, while the inertia is set by the linear mass density μ. Comparing the resulting equation of motion with the wave equation yields the wave speed

$$v = \sqrt{\frac{T}{\mu}}.$$

and from $v = \frac{\omega}{k} \iff \omega = vk$, the angular frequency is

$$\omega = vk = k\sqrt{\frac{T}{\mu}}.$$

Using this relation, the potential energy can be written as

$$dU = \frac{1}{2}\mu A^2\omega^2 \cos^2(kx - \omega t)\, dx.$$

The two expressions for the potential energy are fully equivalent. Using the wave speed on a stretched string,

$$v = \sqrt{\frac{T}{\mu}},$$

together with the dispersion relation $\omega = vk$, one finds

$$Tk^2 = \mu\omega^2.$$

Substituting this relation shows that

$$dK = dU$$

$$\frac{1}{2}TA^2k^2 \cos^2(kx - \omega t)\, dx = \frac{1}{2}\mu A^2\omega^2 \cos^2(kx - \omega t)\, dx.$$

This equivalence highlights that the elastic restoring forces and the inertial response of the string contribute symmetrically, which is why the instantaneous kinetic and potential energies of a sinusoidal wave have the same spatial and temporal dependence.

Average Energy and Power Transport

The total instantaneous energy of a string element is the sum of its kinetic and potential contributions,

$$dE = dK + dU = \mu A^2\omega^2 \cos^2(kx - \omega t)\, dx.$$

For a sinusoidal wave, the time average of $\cos^2(kx - \omega t)$ over one full oscillation is

$$\langle \cos^2 \rangle = \frac{1}{2}.$$

Averaging the energy over one period therefore yields the mean energy per unit length of the string,

$$\frac{dE_{\text{avg}}}{dx} = \frac{1}{2}\mu A^2\omega^2.$$

This energy is transported along the string together with the wave at speed v. In a time interval dt, a segment of string of length $v\,dt$ passes a given point, carrying an energy

$$\left(\frac{dE_{\text{avg}}}{dx}\right) v\,dt.$$

The **average power**, defined as the energy transported per unit time, is therefore

$$P = \frac{dE}{dt} = \frac{dE_{\text{avg}}}{dx}\,v = \frac{1}{2}\mu A^2\omega^2 v.$$

The power carried by a wave increases strongly with amplitude and is proportional to the **square of the amplitude**, reflecting the quadratic dependence of both kinetic and potential energy on the displacement.

Intensity, Energy Flow, and Geometric Spreading

The **intensity** I of a wave is the **average power transmitted per unit area** perpendicular to the direction of propagation,

$$I = \frac{P}{A_{\text{cross}}}.$$

It quantifies how strongly energy is delivered to a region of space and is therefore the relevant measure for waves that spread over an area, such as sound waves, water waves, and electromagnetic waves.

Because intensity is power per area, it depends not only on the total energy transport but also on how that energy is distributed in space. In particular, for a wave emitted uniformly from a localized source, the wavefronts are **spherical**, so the same total power is spread over an increasing wavefront area as the wave propagates outward.

For a source that emits total power P equally in all directions, the energy at distance r is distributed over a spherical surface of area $4\pi r^2$, so the intensity is

$$I = \frac{P}{4\pi r^2}.$$

Thus, the intensity decreases with the square of the distance from the source, a behavior known as the **inverse-square law**. This geometric spreading explains why sound, light, and other radiating waves become weaker as they travel farther from their source.

Conceptual Notes

Waves transport **energy rather than matter**, with energy alternating between kinetic and potential forms in the medium. For a sinusoidal wave on a string, the average power transported is $P = \frac{1}{2}\mu A^2\omega^2 v$, while in sound waves a corresponding expression involves the density of the medium and the amplitude of particle motion. In all cases, the intensity of a wave is proportional to the **square of its amplitude**.

When waves spread out from a point source, geometric effects lead to an intensity that decreases as $1/r^2$, even though the total emitted power remains constant.

1.10.6 Wave Reflection and Transmission at Boundaries

When a traveling wave encounters a **boundary** between two different media, or reaches a fixed or free end of a string, part of the wave's energy is **reflected** back into the original medium and part is **transmitted** into the new medium. The relative amounts of reflected and transmitted energy depend on the physical properties of the media, often summarized qualitatively in terms of their **wave impedance**, which determines how the wave interacts with the boundary.

Reflection at a Fixed End

Consider a transverse wave traveling along a stretched string that is rigidly **fixed** at one end. Because the end of the string cannot move, the displacement at the boundary must always be zero. If the incident wave is described by

$$y_i(x,t) = A\sin(kx - \omega t),$$

the reflected wave must cancel the incident displacement at the fixed end. This requirement leads to a reflected wave of the form

$$y_r(x,t) = -A\sin(kx + \omega t).$$

The negative sign indicates that the reflected wave is **inverted** relative to the incident wave. This inversion corresponds to a phase shift of 180°, meaning that crests reflect as troughs and vice versa.

The phase reversal can also be understood from **Newton's third law**. When the incident wave exerts a transverse force on the rigid support, the support responds with an equal and opposite force on the string. This reaction generates a reflected wave that enforces zero displacement at the boundary, resulting in the observed 180° phase shift.

Reflection at a Free End

If the end of the string is **free to move vertically**, there is no constraint forcing the displacement to vanish at the boundary. In this case, the reflected wave has the form

$$y_r(x,t) = A\sin(kx + \omega t),$$

and is therefore **not inverted**. The reflected wave returns with the same orientation as the incident wave, and no phase reversal occurs upon reflection at a free end.

Reflection and Transmission at a Boundary Between Two Strings

More generally, consider a wave traveling along a string that encounters a junction where it is connected to a second string with different physical properties. Let the two strings have linear mass densities μ_1 and

μ_2 and be under the same tension T. A wave incident from the first string toward the junction gives rise to a reflected wave in the first string and a transmitted wave in the second string.

The wave speeds in the two strings are

$$v_1 = \sqrt{\frac{T}{\mu_1}}, \qquad v_2 = \sqrt{\frac{T}{\mu_2}},$$

and the corresponding wave numbers satisfy $k_i = \omega / v_i$. At the junction, physical continuity requires that both the **displacement** of the string and the **transverse force** remain continuous. These boundary conditions lead to simple expressions for the ratios of reflected and transmitted amplitudes to the incident amplitude,

$$\frac{A_r}{A_i} = \frac{v_2 - v_1}{v_2 + v_1}, \qquad \frac{A_t}{A_i} = \frac{2v_2}{v_2 + v_1}.$$

Interpretation of Reflection Behavior

The sign and magnitude of the reflection coefficient determine the phase behavior of the reflected wave. If the wave encounters a medium in which the wave speed is smaller than in the original medium, corresponding to a heavier or more rigid string ahead, the reflection coefficient is negative and the reflected wave is inverted. If the wave encounters a medium with a higher wave speed, corresponding to a lighter string, the reflection coefficient is positive and the reflected wave is upright. In the limiting case of an infinitely heavy second medium, the boundary behaves like a fixed end and the reflected wave is fully inverted. In the opposite limit of a negligibly light second medium, the boundary behaves like a free end and the reflected wave is not inverted.

Energy Reflection and Transmission

The transport of energy by a wave depends on both its amplitude and its speed. Since the average power carried by a wave is proportional to vA^2, the **fractions of energy** reflected and transmitted at a boundary are given by

$$R = \left(\frac{A_r}{A_i}\right)^2, \qquad T = \frac{v_2}{v_1}\left(\frac{A_t}{A_i}\right)^2.$$

These quantities represent the reflected and transmitted energy fractions, respectively. Conservation of energy requires that

$$R + T = 1,$$

so that all incident energy is accounted for by reflection and transmission.

Because reflection necessarily produces two waves traveling in the same medium at the same time, boundaries provide the simplest physical situation in which wave superposition becomes unavoidable.

Reflection of Sound and Light Waves

The same principles apply to other types of waves. For **sound waves**, reflection occurs at boundaries where the acoustic impedance changes, such as at interfaces between air and solid walls or between air and water. Reflected sound waves give rise to phenomena such as echoes and resonances in enclosed spaces. For **light waves**, reflection and transmission occur at boundaries between media with different refractive indices, and the corresponding amplitude ratios are determined by the electromagnetic properties of the media.

Conceptual Notes

A wave reflected from a **fixed end** undergoes a phase reversal, while reflection from a **free end** does not. At a boundary between two media, waves are generally both reflected and transmitted, with the relative amplitudes determined by the wave speeds or impedances of the media. Energy is conserved during this process, with reflected and transmitted energy fractions summing to unity. Phase reversal occurs when reflection takes place from a more rigid or denser medium, and the same general principles govern reflection and transmission for mechanical, acoustic, and electromagnetic waves.

1.10.7 Superposition, Interference, and Standing Waves

Whenever more than one wave is present in the same region of space at the same time, the resulting motion of the medium is governed by the **principle of superposition**. This principle follows directly from the linearity of the wave equation and provides the conceptual foundation for interference patterns, standing waves, and resonance phenomena that arise naturally once waves overlap or reflect from boundaries.

The Principle of Superposition and Linearity

The **principle of superposition** states that when two or more waves occupy the same region of space simultaneously, the resulting displacement of the medium is the **algebraic sum** of the individual displacements produced by each wave. This principle applies to **linear waves**, for which restoring forces are proportional to displacement and the governing equation of motion remains linear.

The one-dimensional wave equation,

$$\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2},$$

is linear in the displacement variable D. As a consequence, if two functions $D_1(x,t)$ and $D_2(x,t)$ are solutions of the wave equation, then any linear combination,

$$D(x,t) = aD_1(x,t) + bD_2(x,t),$$

with constants a and b, is also a valid solution. This mathematical property provides the formal basis for superposition.

Physically, superposition implies that overlapping waves do not collide or permanently alter one another. During overlap, the medium responds to the combined displacement, but once the waves pass through each other, each continues to propagate independently with its original shape and speed. Energy and momentum are not destroyed or absorbed during overlap; they are redistributed temporarily through interference.

Conditions and Limitations of Superposition

The principle of superposition is valid only when wave amplitudes are sufficiently small that the restoring forces remain linear, as described by Hooke's law. If amplitudes become large, nonlinear effects can arise and superposition no longer holds. Such behavior occurs in strong sound waves, shock waves, or specially engineered nonlinear media. Electromagnetic waves in vacuum always obey superposition because Maxwell's equations are inherently linear.

Interference of Traveling Waves

Interference is a direct consequence of superposition and occurs when two or more coherent waves overlap. The resulting wave pattern depends on the **relative phase** of the waves and can lead to regions of enhanced or reduced amplitude.

Consider two sinusoidal waves of equal frequency, wavelength, and amplitude traveling in the same direction,

$$y_1(x,t) = A\sin(kx - \omega t), \qquad y_2(x,t) = A\sin(kx - \omega t + \phi),$$

where ϕ is the phase difference. Using the trigonometric identity $\sin\alpha + \sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$, their superposition is

$$y(x,t) = y_1 + y_2 = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right).$$

The combined wave remains sinusoidal with the same frequency and wavelength, but its amplitude depends on the phase difference,

$$A_R = 2A\cos\left(\frac{\phi}{2}\right).$$

Since wave intensity is proportional to the square of the amplitude, the resulting intensity is

$$I = I_0\cos^2\left(\frac{\phi}{2}\right),$$

where I_0 is the maximum intensity obtained when the waves are exactly in phase.

Interference is **constructive** when $\phi = 0, 2\pi, 4\pi, \ldots$, resulting in maximum amplitude, and **destructive** when $\phi = \pi, 3\pi, 5\pi, \ldots$, resulting in cancellation. In many physical situations, the phase difference arises from a path difference Δx traveled by the waves,

$$\phi = \frac{2\pi}{\lambda}\Delta x.$$

Constructive interference occurs for $\Delta x = n\lambda$, while destructive interference occurs for $\Delta x = (n + \frac{1}{2})\lambda$, where n is an integer.

For two coherent waves that overlap continuously, a stationary interference pattern of alternating constructive and destructive regions can form. Such patterns require the waves to have the same frequency and a fixed phase relationship. If coherence is lost and the phase varies randomly, the interference pattern disappears.

Interference does not violate energy conservation. Even in regions of destructive interference where the displacement is small or zero, energy is not destroyed but redistributed to regions of constructive interference. The total energy carried by the waves remains constant.

Standing Waves from Superposition of Opposite Waves

A **standing wave** is a special interference pattern formed when two waves of the same frequency and amplitude travel in **opposite directions** through the same medium. This situation commonly arises due to reflection at boundaries.

For two such waves,

$$y_1(x,t) = A\sin(kx - \omega t), \qquad y_2(x,t) = A\sin(kx + \omega t).$$

Using the trigonometric identity $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$, their superposition is (with $\alpha = kx - \omega t$ and $\beta = kx + \omega t$ and $\cos(-\omega t) = \cos(\omega t)$)

$$y(x,t) = 2A\sin(kx)\cos(\omega t).$$

This expression represents a standing wave. The factor $\sin(kx)$ determines the fixed spatial pattern of the wave, while $\cos(\omega t)$ describes the temporal oscillation, which has the same frequency at every point along the medium.

Points where $\sin(kx) = 0$ are **nodes**, where the displacement is always zero and occur at

$$x_n = n\frac{\lambda}{2}.$$

Points where $|\sin(kx)| = 1$ are **antinodes**, where the displacement amplitude is maximal. Adjacent nodes or antinodes are separated by half a wavelength.

The amplitude of oscillation depends on position according to

$$A(x) = 2A\sin(kx).$$

Thus, every point of the medium executes simple harmonic motion with the same angular frequency ω, but with an amplitude that varies from zero at nodes to a maximum at antinodes.

Energy in Standing Waves

Unlike traveling waves, standing waves do **not** transport energy along the medium. Instead, energy oscillates locally between kinetic and potential forms. At nodes, the displacement and kinetic energy are always zero, while potential energy is maximal; energy oscillates locally but does not propagate. At antinodes the energy alternates between maximum kinetic and maximum potential values as the medium oscillates in time.

Standing Waves on a String and Resonance

For a string of length L fixed at both ends, the displacement must vanish at $x = 0$ and $x = L$. Applying this boundary condition to the standing-wave form requires

$$\sin(kL) = 0,$$

which implies

$$kL = n\pi, \qquad n = 1, 2, 3, \ldots.$$

Using $k = \frac{2\pi}{\lambda}$ and $v = \lambda f$, the allowed wavelengths and frequencies are therefore

$$\lambda_n = \frac{2L}{n}, \qquad f_n = \frac{nv}{2L}.$$

These discrete frequencies are the **natural frequencies** or **harmonics** of the string. Each allowed standing-wave pattern is called a **mode of vibration**. The lowest-frequency mode is the **fundamental**, while higher modes are overtones.

Resonance occurs when a system is driven by an external periodic force whose frequency matches one of its natural frequencies. Under this condition, energy transfer is highly efficient and the amplitude of oscillation increases significantly. In real systems, the amplitude is ultimately limited by damping and energy losses.

In a standing wave, kinetic energy is largest near nodes where the displacement changes most rapidly, while potential energy is largest near antinodes where the displacement is greatest. While resonance is essential for sound production and signal amplification, excessive resonance can lead to structural damage if oscillations grow too large.

Conceptual Notes

Superposition is a direct consequence of the linearity of the wave equation and governs interference, standing waves, and resonance. Interference redistributes energy spatially without loss, while standing waves result from the superposition of oppositely traveling waves and involve no net energy transport. Resonance occurs when a system is driven at one of its natural frequencies, leading to large oscillation amplitudes.

1.10.8 Wave Speed Variations: Refraction and Dispersion

Wave propagation is fundamentally controlled by the **wave speed** v, which is determined by the physical properties of the medium. When the wave speed varies, either because the wave enters a different medium or because different frequency components travel at different speeds, the observable behavior of the wave changes. Two central phenomena arise from this dependence: **refraction**, which results from spatial changes in wave speed, and **dispersion**, which results from frequency-dependent wave speed.

Dispersion

In a **dispersive medium**, the wave speed depends on frequency. As a result, different sinusoidal components of a composite wave propagate with different speeds. Since any realistic waveform can be decomposed into sinusoidal components by Fourier analysis, dispersion causes the overall shape of a wave packet or pulse to change as it travels through the medium. In contrast, in a **non-dispersive medium**, the wave speed is independent of frequency, so all components propagate together and the waveform maintains its shape during propagation.

Dispersion is therefore a property of the medium rather than of a specific waveform. It plays a central role in pulse broadening, signal distortion, and the spreading of wave packets, and it becomes especially important when waves contain a wide range of frequencies.

Refraction

Refraction is the change in direction and speed of a wave as it passes obliquely from one medium into another in which the wave velocity is different. During refraction, the **frequency of the wave remains constant**, because it is determined by the source. However, the **wave speed** and therefore the **wavelength** change, leading to a change in the direction of propagation.

Refraction occurs for all types of waves, including water waves, sound waves, seismic waves, and electromagnetic waves such as light. The phenomenon is a direct consequence of spatial variations in wave speed.

Wavefront Interpretation of Refraction

Refraction can be understood most clearly using wavefronts. When a wavefront crosses a boundary at an angle, different parts of the wavefront enter the new medium at different times. If the wave speed changes across the boundary, the portion that enters first changes speed while the remainder continues at the original speed. This difference causes the wavefront to rotate, producing a change in the direction of propagation.

In regions where the wave speed is lower, wavefronts are closer together, corresponding to a shorter wavelength. In regions where the wave speed is higher, wavefronts are more widely spaced. The bending of the wave is therefore a geometric consequence of unequal wavefront spacing across the boundary.

Law of Refraction for Waves

Consider a wave incident on a boundary between two media at an angle θ_1 relative to the normal and refracted into the second medium at an angle θ_2. If the wave speeds in the two media are v_1 and v_2, the angles are related by

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

If the wave slows down upon entering the second medium, so that $v_2 < v_1$, the refracted wave bends **toward the normal**. If the wave speeds up, it bends **away from the normal**.

Change of Wavelength at a Boundary

Because wave speed, frequency, and wavelength are related by $v = f\lambda$, and because the frequency remains unchanged during refraction, the wavelengths in the two media satisfy

$$\frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2}.$$

A decrease in wave speed therefore leads to a shorter wavelength, while an increase in wave speed leads to a longer wavelength. The change in wavelength is directly responsible for the change in propagation direction.

Unified View: Refraction and Dispersion

Refraction and dispersion arise from the same physical cause: **variations in wave speed**. In refraction, the wave speed varies **in space**, leading to changes in wavelength and propagation direction. In dispersion, the wave speed varies **with frequency**, causing different spectral components to separate and the waveform to change shape. Together, these phenomena illustrate how the properties of a medium govern both the geometry and the evolution of wave motion.

1.10.9 Diffraction

Diffraction is the spreading or bending of waves when they encounter an obstacle, slit, or aperture whose size is comparable to the wavelength. It is a defining property of wave motion and allows waves to propagate into regions that would otherwise lie in geometric shadow, explaining many everyday and scientific observations.

Nature of Diffraction

When a wavefront passes through an opening or around an obstacle, each point on the wavefront acts as a source of new waves, leading to a redistribution of energy beyond the obstacle or aperture. The extent of diffraction depends on the ratio between the wavelength and the characteristic size of the opening. If the opening is much larger than the wavelength, diffraction is weak and the wave travels almost in straight lines. If the opening is comparable to the wavelength, diffraction becomes pronounced and the wave spreads significantly. If the opening is smaller than the wavelength, the emerging wave resembles a nearly spherical wave.

Huygens' Principle

Huygens' principle states that every point on a wavefront can be regarded as a source of secondary spherical wavelets that propagate outward with the same speed as the wave itself. The new wavefront at a later time is formed by the envelope of these secondary wavelets. This principle provides a unified qualitative explanation of reflection, refraction, interference, and diffraction.

Diffraction Through a Single Slit

When a plane wave of wavelength λ is incident on a slit of width a, each point within the slit acts as a source of secondary wavelets that interfere with one another in the region beyond the slit. The resulting interference pattern consists of a central bright maximum flanked by alternating dark and bright regions. The condition for destructive interference, corresponding to dark minima in the diffraction pattern, is

$$a \sin \theta = m\lambda, \quad m = \pm 1, \pm 2, \pm 3, \ldots$$

where θ is the angle measured from the central axis. The central maximum is the brightest feature and is wider than the secondary maxima, whose intensity decreases with increasing angle.

Diffraction by an Obstacle or Edge

Diffraction also occurs when waves encounter an obstacle or a sharp edge. The portions of the wavefront that are not blocked act as secondary sources, producing wave propagation into the region behind the obstacle. This explains why water waves spread behind barriers, why sound can be heard around corners, and why light can bend slightly around small apertures under suitable conditions.

Diffraction Through Multiple Slits

When waves pass through multiple equally spaced slits, forming a diffraction grating, the superposition of waves from all slits produces sharp and well-defined maxima. The condition for constructive interference in a grating is

$$d \sin \theta = m\lambda, \quad m = 0, \pm 1, \pm 2, \ldots$$

where d is the spacing between adjacent slits. Diffraction gratings are widely used because they produce narrow peaks that allow precise wavelength measurements.

Applications and Everyday Examples

Diffraction plays a central role in optical spectroscopy, acoustic engineering, radio-wave transmission, microscopy, astronomy, and X-ray crystallography. In everyday life, diffraction explains why sound can be heard through doorways, why shadows have fuzzy edges, why compact discs display colorful patterns, and why light spreads after passing through narrow slits.

Conceptual Notes

Diffraction occurs whenever waves encounter edges, apertures, or obstacles comparable in size to their wavelength. The amount of spreading increases for larger wavelengths and smaller openings. Single-slit diffraction produces a broad central maximum accompanied by weaker side maxima. Diffraction and interference are complementary manifestations of wave superposition, and diffraction ultimately limits the resolving power of optical instruments. More details follow in the next semester/Physics II.