2.7. AC circuits & Electromagnetic oscillations

In this chapter, we will combine what we have learned so far to study electromagnetic oscillations. Further, we combine the three basic building blocks, i.e. resistance, capacity, and inductance, into AC circuits and study them.

2.7.1. The LR-circuit

Let begin our journey through circuits and oscillations, with a simple DC circuit. Imagine a simple circuit consisting of an ideal voltage source V_0, an ideal resistor R, and an ideal inductor L in series. This LR-circuit is fundamental in electronics, particularly in applications like power supplies, radio circuits, and more. Unlike the RC-circuit, which involves a capacitor, the LR-circuit focuses on the behavior of an inductor.

Let's investigate the LR-circuit is analyzed under two conditions: when the circuit is switched on and when it is switched off. Switching on the circuit allows the inductor to oppose the change in current, while switching off demonstrates the inductor's attempt to maintain current.

Switching on the DC supply

When the circuit is closed, the inductor initially opposes the change in current, causing the current to rise gradually over time instead of instantaneously from zero to its maximum value. To analyze this, we apply Kirchhoff's loop rule, which states that the sum of the voltage changes around a closed loop is zero. For the LR-circuit, the DC supply provides V_0, the voltage drop across the resistor is $V_R = IR$, and the voltage across the inductor is $V_L = L\frac{dI}{dt}$.

Thus, the loop equation becomes:

$$V_0 - L\frac{dI}{dt} - IR = 0$$

$$V_0 = L\frac{dI}{dt} + IR$$

Hence, the sum of the voltage drops over the resistor and inductor is constant over time, i.e. V_0, but their relative contribution changes. Immediately after switching on, the voltage drop appears completely over the inductor due to its induced voltages (opposed direction to in source, Lenz's law). Over time, the voltage over the inductor reduces as the change in magnetic field diminishes because the change in

current becomes smaller and smaller. Eventually, i.e. steady state, no voltage drop over the inductor occurs and the entire voltage drop is at the resistor with $V_0 = RI_0$.

Let's analyze this mathematically. To solve the first-order linear ordinary differential equation from above, we start by separating the variables:

$$L\frac{dI}{dt} + RI = V_0$$

$$\frac{dI}{dt} = \frac{V_0 - RI}{L}$$

$$\frac{dI}{V_0 - RI} = \frac{dt}{L}$$

Then, we integrate both sides from $t = 0$ with the current being zero to time t with the current being I:

$$\int_0^I \frac{dI}{V_0 - RI} = \int_0^t \frac{dt}{L}$$

Let us substitute the following: $u = V_0 - RI$. Therefore we also get $du = -RdI \leftrightarrow dI = -\frac{du}{R}$, and integrate:

$$-\frac{1}{R}\int_{V_0}^{V_0 - RI} \frac{du}{u} = \int_0^t \frac{dt}{L}$$

$$-\frac{1}{R}\ln\left(\frac{V_0 - RI}{V_0}\right) = \frac{t}{L}$$

Solving for I:

$$\ln\left(\frac{V_0 - RI}{V_0}\right) = -\frac{Rt}{L}$$

$$\frac{V_0 - RI}{V_0} = e^{-\frac{Rt}{L}}$$

$$V_0 - RI = V_0 e^{-\frac{Rt}{L}}$$

$$RI = V_0(1 - e^{-\frac{Rt}{L}})$$

$$I(t) = \frac{V_0}{R}(1 - e^{-\frac{t}{\tau}}) \quad \text{with} \quad \tau = \frac{L}{R}$$

The current asymptotically approaches $\frac{V_0}{R}$ with a time constant $\tau = \frac{L}{R}$. In line with the RC-circuit, τ is in seconds (Ω/H = (V/A)/(Vs/A) = s) and after τ the current $I(\tau)$ reached 63% of its maximum value. While the equations suggest, that $I(t)$ never reach their final value, it is fair to assume that after 5τ the process is completed.

The voltage over the inductor is therefore:

$$V_L = L\frac{dI}{dt} = \frac{d}{dt}\left(\frac{V_0}{R}\left(1 - e^{-tR/L}\right)\right)$$

$$V_L = L\left(\frac{V_0}{R} \cdot \frac{-R}{L} \cdot e^{-tR/L}\right)$$

$$V_L = -V_0 e^{-t/\tau}$$

Switching off the DC supply

When the circuit is switched off, the same logic applied, i.e. the inductor impedes the change in current. Thus, the inductor attempts to maintain the current, leading to a gradual decay instead of a suddenly vanishing current. With the voltage source removed ($V_0 = 0$), Kirchhoff's loop rule gives:

$$L\frac{dI}{dt} + RI = 0$$

Separating variables:

$$\frac{dI}{I} = -\frac{R}{L}dt$$

Integrating from $t = 0$ (where $I = I_0 = \frac{V_0}{R}$) to time t:

$$\int_{I_0}^{I} \frac{dI}{I} = -\frac{R}{L}\int_0^t dt$$

$$\ln\left(\frac{I}{I_0}\right) = -\frac{Rt}{L}$$

$$I(t) = I_0 e^{-\frac{t}{\tau}} \quad \text{with} \quad \tau = \frac{L}{R}$$

Again, τ is in units of seconds and after τ the current has decayed to 37% of its initial value. *To summarize, a resistor impedes current, i.e. "limits" the current flow, while a inductor impedes the *change in current*, i.e. "smooths out" changes in current flow.*

The voltage over the inductor can be found:

$$V_L = L\frac{dI}{dt} = \frac{d}{dt}\left(I_0 e^{-tR/L}\right) = \frac{-LR}{L}I_0 e^{-tR/L}$$

$$V_L = -V_0 e^{-t/\tau}$$

2.7.2. Electromagnetic oscillations

Before investigating circuits with AC power sources, we will analyze circuits which oscillate, i.e. have an inductance and capacitance in them. We start with the most simple case, the LC-circuit before finishing with the LRC-circuit.

LC-circuit

Let us explore the LC-circuit, a fundamental system in electronics that introduces the concept of electromagnetic oscillations. Unlike the LR-circuit, which includes a resistor, the LC-circuit consists of only three elements connected in series: an ideal capacitor, an ideal inductor, and an open switch. Imagine the capacitor is fully charged at time $t = 0$ with the charges $+Q$ and $-Q$ on each plate, respectively. There is no additional voltage source and at $t = 0$ the voltage across the plates of the capacitor is $V = \frac{Q}{C}$. When the circuit is closed, the capacitor begins to discharge through the inductor. The inductor opposes this change in current due to (self-)induction.

To analyze the LC-circuit, we apply Kirchhoff's loop rule. The total voltage around the loop must sum to zero:

$$-L\frac{dI}{dt} + \frac{Q}{C} = 0$$

Since the current of the capacitor is decreasing (positive charge plate discharges) and current is derivative of charge w.r.t. time, we can write $I = -\frac{dQ}{dt}$. Using this relation the minus sign vanishes. We rearrange the equation to to obtain a second order linear differential equation that describes simple harmonic motion:

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$

The general solution for this differential equation is:

$$Q = Q_0 \cos(\omega t + \phi)$$

with Q_0 as the initial charge on the capacitor and ϕ as the initial phase (determined by initial conditions). We should further examine the implication of these equation. Lets apply the general solution of our initial differential equation. With $\frac{d^2Q}{dt^2} = -\omega^2 Q_0 \cos(\omega t + \phi)$, we obtain:

$$-\omega^2 Q_0 \cos(\omega t + \phi) + \frac{1}{LC} Q_0 \cos(\omega t + \phi) = 0$$

$$(-\omega^2 + \frac{1}{LC})(Q_0 \cos(\omega t + \phi)) = 0$$

This equation is true for all case were $(-\omega^2 + \frac{1}{LC}) = 0$. Thus, the angular frequency of oscillation $\omega = 2\pi f$ is:

$$\omega = \sqrt{\frac{1}{LC}}$$

The current $I(t)$ is the time derivative of the charge:

$$I(t) = -\frac{dQ}{dt} = \omega Q_0 \sin(\omega t + \phi) = I_0 \sin(\omega t + \phi)$$

with the maximum current $I_0 = \omega Q_0 = \frac{Q}{\sqrt{LC}}$.

The energy in the LC-circuit is conversed and oscillates between the capacitor and the inductor. Initially, the capacitor is fully charged, storing electrical energy. As it discharges, electrical energy is converted into magnetic energy stored in the inductor. The capacitor is not an indefinite source of charge. Hence, after reaching the maximum current, i.e. the inductor has all energy stored in it, the current decreases. This change in current causes yet another induction which induces a voltage in opposite direction to before, effectively charging the capacitor (in opposite direction to the initial condition) to start the cycle all over again. This energy exchange between the capacitor and inductor continues, resulting in oscillations of charge, current, and voltage.

The energy in the inductor U_B and capacitor U_E are (see previous chapters):

$$U_B = \frac{1}{2}LI^2 = \frac{L\omega^2 Q_0^2}{2}\sin^2(\omega t + \phi) = \frac{Q_0^2}{2C}\sin^2(\omega t + \phi)$$

$$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{Q_0^2}{2C}\cos^2(\omega t + \phi)$$

Therefore the total energy U is:

$$U = U_B + U_E = \frac{Q_0^2}{2C}\sin^2(\omega t + \phi) + \frac{Q_0^2}{2C}\cos^2(\omega t + \phi) = \frac{Q_0^2}{2C}$$

The LC-circuit demonstrates the fundamental concept of electromagnetic oscillations, where energy is exchanged between an electric field (capacitor) and a magnetic field (inductor). This oscillatory behavior is the basis for many electronic circuits, including radio transmitters, receivers, and filters. The angular frequency ω determines the oscillation frequency, which is crucial for tuning and resonance in such applications.

By analyzing the LC-circuit, we gain insight into the principles of energy conservation and electromagnetic oscillations.

LRC-circuit

Let us explore the LRC-circuit, a fundamental system in electronics that (re-)introduces the concept of *damped oscillations*. The LRC-circuit consists of an ideal inductor L, an ideal resistor R, and an ideal capacitor C connected in series. Unlike the energy conversion in the LC-circuit, the resistor adds losses in form of heat and, therefore, introduces damping, causing the oscillations to decay over time.

Assume the capacitor is fully charged at $t = 0$, with an initial charge Q_0, and no battery or other energy source is present. The capacitor begins to discharge through the inductor and resistor. Applying Kirchhoff's loop rule to the LRC-circuit, give use the same equation as in the LC-circuit with the addition of the resistors voltage drop IR:

$$-L\frac{dI}{dt} - IR + \frac{Q}{C} = 0$$

Since the current of the capacitor is decreasing and current is derivative of charge w.r.t. time, i.e. $I = -\frac{dQ}{dt}$, we obtain:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

This is a second-order linear differential equation that describes damped oscillations. The behavior of the solution depends on the damping factor, which is determined by the relationship between R, L, and C. As we seen before, the general form a damped harmonic oscillator is $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$ and the conditions for the dampening case are specified as $b^2 \gtrless 4mk$. Hence, we need to evaluate $R^2 \gtrless \frac{4L}{C}$, which gives use three case:

1. $R^2 < \frac{4L}{C}$: Underdamped oscillation with exponential decay
2. $R^2 > \frac{4L}{C}$: Overdamped, i.e. the damping is too strong to allow any oscillations
3. $R^2 = \frac{4L}{C}$: Critically damped oscillation with the angular frequency $\omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ and the charge over time $Q(t) = Q_0 e^{-\frac{R}{2L}t}\cos(\omega_d t + \phi)$ with the initial charge Q_0 and phase ϕ.

The total energy in the LRC-circuit is not conserved due to the presence of the resistor, which dissipates energy as heat. Initially, the capacitor stores electrical energy, which is converted into magnetic energy in the inductor as the capacitor discharges. However, the resistor continuously dissipates energy, leading to a decay in the oscillations over time. Given the critical role of oscillators in electronic systems, thermal energy dissipation due to resistance in actual circuits necessitates the inclusion of an external power supply.

```
interactive(children=(FloatLogSlider(value=1.0, description='R (Ω)', max=3.0), FloatLogSlider
(value=1.0, descr…

<function __main__.simulate_lrc_wave(R, L, C, T_MAX)>
```

2.7.3. AC circuits

It is time to analyze circuits with an AC power supply. Before combining the basic building elements of circuits, i.e. resistor, inductor, and capacitor, we will investigate them briefly one by one to understand the complex (pun intended) relation of current, voltage, resistance and reactance.

Resistance vs. Reactance

In DC circuits, opposition to current flow is determined by resistance R, which remains constant regardless of frequency. However, in AC circuits, inductors and capacitors introduce an additional type of opposition known as *reactance* X. Unlike resistance, reactance depends on frequency and causes a phase shift between voltage and current. Reactance, measured in ohms (Ω), is different from resistance because it does not dissipate energy as heat but instead temporarily stores it in magnetic (inductors) or electric (capacitors) fields.

Resistor in an AC Circuit

A resistor obeys **Ohm's law**, which states that the voltage across it is proportional to the current flowing through it:

$$V = IR$$

If the current is sinusoidal, i.e. $I = I_0 \cos(\omega t)$, the voltage across the resistor is:

$$V = (I_0 \cos(\omega t))R = (I_0 R)\cos(\omega t) = V_0 \cos(\omega t)$$

where $V_0 = I_0 R$ is the peak voltage.

Since both voltage and current follow the same cosine function, they are *in phase*. That is, the peaks and zero crossings of the voltage and current waveforms occur at the same time.

Unlike inductors and capacitors, resistors dissipate energy as heat. The average rate at which energy is transformed into heat, i.e. the average power is:

$$\overline{P} = \overline{IV} = I_{rms} V_{rms} = I_{rms}^2 R = \frac{V_{rms}^2}{R}.$$

Inductor in an AC Circuit

A voltage applied to an inductor with inductance L will induce a voltage opposite to its origin in the inductor itself. According to Kirchhoff's loop rule, the sum of the external and induced voltage must be zero:

$$V - L\frac{dI}{dt} = 0 \leftrightarrow V = L\frac{dI}{dt}$$

For sinusoidal current $I(t) = I_0 \cos(\omega t)$, we obtain:

$$V = L\frac{d(I_0 \cos(\omega t))}{dt} = -L\omega I_0 \sin(\omega t)$$

Rewriting using trigonometric identities, i.e. $\sin\theta = -\cos(\theta + \frac{\pi}{2})$:

$$V(t) = \omega L I_0 \cos(\omega t + \frac{\pi}{2}) = V_0 \cos(\omega t + \frac{\pi}{2})$$

with $V_0 = \omega L I_0$.

The *voltage leads the current by* $90°$ *(or $\pi/2$ radians)*, meaning when the current is maximum, the voltage is zero, and vice versa. Alternatively, we could say the current lags the voltage by $90°$.

The inductor's opposition to AC is given by its *inductive reactance*:

$$X_L = \omega L = 2\pi f L$$

For the peak voltage V_0 and peak current I_0 as well as the their RMS equivalents, we can write:

$$V_0 = X_L I_0 \quad \& \quad V_{RMS} = X_L I_{RMS}$$

Note that, since X_L increases with frequency, inductors resist high-frequency currents more than low-frequency ones. Since voltage and current are out of phase by $90°$, the *average power dissipation is zero in an ideal inductor*. The energy is stored in the magnetic field and returned to the circuit.

Capacitor in an AC Circuit

A capacitor with capacitance C in an AC circuit continuously charges and discharges as the voltage oscillates. According to Kirchhoff's loop rule, the voltage applied to the capacitor is the same as the voltage across the capacitor:

$$V = \frac{Q}{C}$$

Since current is the rate of change of charge $I = \frac{dQ}{dt}$, and assuming sinusoidal current $I(t) = I_0 \cos(\omega t)$, we can relate both equations with each other:

$$I = \frac{dQ}{dt} = I_0 \cos(\omega t)$$

We integrate the equation to find the charge:

$$Q = \int_0^t dQ = \int_0^t I_0 \cos(\omega t) dt = \frac{I_0}{\omega} \sin(\omega t)$$

Substituting into the voltage equation:

$$V = \frac{Q}{C} = \frac{1}{\omega C} I_0 \sin(\omega t)$$

Using the identity $\sin\theta = \cos(\theta - \frac{\pi}{2})$:

$$V = \frac{1}{\omega C} I_0 \cos(\omega t - \frac{\pi}{2}) = V_0 \cos(\omega t - \frac{\pi}{2})$$

with $V_0 = \frac{I_0}{\omega C}$.

The *voltage lags the current by $90°$ (or $\pi/2$ radians)*, meaning when the current is maximum, the voltage is zero, and vice versa. Alternatively, we could say the current leads the voltage by $90°$. This is opposite to the phase relation at an inductor.

The capacitor's opposition to AC is given by its *capacitive reactance*:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

For the peak voltage V_0 and peak current I_0 as well as their RMS equivalents, we can write:

$$V_0 = X_C I_0 \quad \& \quad V_{RMS} = X_C I_{RMS}$$

Note that, since X_C decreases with frequency, capacitors allow high-frequency currents to pass more easily than low-frequency ones. Since voltage and current are out of phase by $90°$, the *average power dissipation is zero* in an ideal capacitor. The energy is stored in the electric field and returned to the circuit.

Summary of AC Circuit Components

Component	Voltage-Current Relationship	Phase Difference	Reactance
Resistor (R)	$V = IR$	In phase ($0°$)	R
Inductor (L)	$V = L\frac{dI}{dt}$	Voltage leads current by $90°$	$X_L = \omega L$
Capacitor (C)	$I = C\frac{dV}{dt}$	Current leads voltage by $90°$	$X_C = \frac{1}{\omega C}$

While resistance is associated with thermal losses, reactance does not dissipate energy as heat but instead temporarily stores it in magnetic (inductors) or electric (capacitors) fields. Resistance does not cause phase shifts between voltage and current, while reactance causes frequency-dependent phase shift between voltage and current. Inductive reactance (X_L) increases with frequency and causes voltage to lead current. Capacitive reactance (X_C) decreases with frequency and causes voltage to lag current.

High-Pass and Low-Pass Filters

Resistors, capacitors, and inductors can be combined to create frequency-dependent circuits, such as high-pass and low-pass filters. A high-pass filter allows high-frequency signals to pass while attenuating low-frequency signals while a low-pass filter allows low-frequency signals to pass while attenuating high-frequency signals. These filters are widely used in signal processing, audio electronics, and communication systems to isolate desired frequency components. In a typical two-way loudspeaker, a high-pass filter, often constructed with a capacitor in series with the tweeter, is used to ensure that only high-frequency signals are sent to the tweeter, which is designed to reproduce treble sounds. Conversely, a low-pass filter, often constructed with an inductor in series with the woofer, directs only low-frequency signals to the woofer, optimized for bass sounds. This division ensures that each speaker operates within its optimal frequency range, resulting in improved sound fidelity and clarity.

Impedance: Unified concept for resistance and reactance

While reactance describes the frequency-dependent opposition of inductors and capacitors, a more general term for opposition to AC is *impedance Z*. Impedance combines both *resistance R* and *reactance X* into a single complex quantity:

$$Z = R + jX$$

with R represents the real part (pure resistance), X represents the imaginary part (pure reactance), j is the imaginary unit ($j^2 = -1$). Impedance is measured in Ohms [Ω]. The magnitude of the impedance can be determined from the resistance R, inductive reactance X_L, and capacitive reactance X_C with Pythagorean theorem:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

The phase angle ϕ of the impedance is given by:

$$\tan \phi = \frac{X_L - X_C}{R}$$

Impedance determines how voltage and current interact in AC circuits, including magnitude and phase shifts:

- If $X_L > X_C$, the circuit behaves more like an inductor (current lags voltage).
- If $X_L < X_C$, the circuit behaves more like a capacitor (current leads voltage).
- If $X_L = X_C$, the circuit is in resonance, and impedance is purely resistive.

AC LRC-circuit & phasor diagrams

In this section, we analyze a circuit containing all three fundamental AC components, i.e. a resistor R, inductor L, and capacitor C, connected in series. We explore the voltage and current relationships using phasor diagrams and derive the circuit's impedance. With these tools we can also analyze simpler circuit by simply setting the resistance/reactance of one component to zero.

Let's start by applying Kirchhoff's loop rule: The sum of the voltage drops across each circuit element must equal the source voltage V at any given time:

$$V = V_R + V_L + V_C$$

We already know that the voltages across the inductor and capacitor are out of phase with the voltage across the resistor. Hence, the peak source voltage V_0 is not simply the sum of the individual peak voltages:

$$V_0 \neq V_{R0} + V_{L0} + V_{C0}$$

However, the current I in a series LRC circuit remains the same at all points in the circuit and, therefore, the current at all elements in the circuit is in phase:

$$I = I_0 \cos(\omega t)$$

Let's recap what we know about the individual elements:

- The resistor voltage is in phase with the current: $V_{R0} = I_0 R$
- The inductor voltage leads the current by 90°: $V_{L0} = I_0 X_L = I_0(\omega L)$
- The capacitor voltage lags the current by 90°: $V_{C0} = I_0 X_C = I_0 \left(\frac{1}{\omega C}\right)$

To analyze LRC circuits visually, we use *phasor diagrams*. Each voltage is represented as a rotating vector, so-called *phasor*, in the complex plane. At $t = 0$ we draw the phasors as vectors in the x-y-plane:

- V_{R0} is drawn along the positive x-axis (in phase with the current).
- V_{L0} is drawn 90° ahead of V_R (positive y-axis).
- V_{C0} is drawn 90° behind V_R (negative y-axis).

Each phasor as a fixed length/magnitude, i.e. V_{R0}, V_{L0} & V_{C0}, but its orientation changes over time as the current alternates as a sinusoidal function. The voltage we would measure for each element at time instance t, i.e. V_R, V_L & V_C, is the *projection of the vector onto the x-axis*.

The *resultant voltage* V_0 is found by vector summation. Its magnitude, i.e. vector length is found by applying the Pythagorean theorem to the voltage triangle:

$$V_0 = \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2} = I_0\sqrt{R^2 + (X_L - X_C)^2} = I_0 Z$$

Due to their phase relation, V_R is always perpendicular to V_L as well as V_C and V_L and V_C are in opposing direction. Hence, the legs/cathetus of the triangle is V_R and the difference of V_L and V_C. Beyond $V_0 = I_0 Z$, the same is true for the RMS values: $V_{RMS} = I_{RMS} Z$.

The same logic applies for the *impedance*:

$$Z = R + jX_L - jX_C$$

The magnitude of the total impedance Z is given by:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

with $X_L = \omega L$ and $X_C = \frac{1}{\omega C}$ as the inductive and capacitive reactance, respectively.

The *phase angle* ϕ between the source voltage and current is found using trigonometry:

$$\tan\phi = \frac{V_{L0} - V_{C0}}{V_{R0}} = \frac{I_0(X_L - X_C)}{I_0 R} = \frac{X_L - X_C}{R}$$

Alternatively, it can be determined as:

$$\cos\phi = \frac{V_{R0}}{V_0} = \frac{I_0 R}{I_0 Z} = \frac{R}{Z}$$

Therefore, the phase relation of current and voltage can be summarized as:

- If $X_L > X_C$, current lags voltage
- If $X_C > X_L$, current leads voltage

- If $X_L = X_C$, current and voltage are in phase (the circuit is in resonance, meaning the impedance is purely resistive)

Since only the resistor dissipates energy as heat, the *average power* dissipated is:

$$P = I_{rms}^2 R = I_{rms}^2 Z \cos \phi = V_{rms} I_{rms} \cos \phi$$

The *power factor* $\cos \phi$ indicates the efficiency of power transfer. In purely resistive circuits, $\cos \phi = 1$, all supplied power is dissipated, while for purely reactant circuits $\cos \phi = 0$, no power is dissipated.

These concepts are crucial for understanding AC circuit behavior, resonance, and energy transfer in electrical engineering applications.

```
interactive(children=(FloatSlider(value=65.0, description='Resistance (Ω)', max=1000.0, min=1
0.0, step=10.0), …
```

```
<function __main__.plot_phasor(R, L, C_mF, time, combine_VLC, show_Vtotal, show_Vreal, show_c
urrent)>
```

Resonance in AC circuits

Resonance in an LRC series circuit occurs when the inductive reactance X_L and capacitive reactance X_C are equal, resulting in a purely resistive impedance. This phenomenon has significant practical applications, such as in radio tuning and power transfer optimization.

To understand the phenomenon, lets start by investigating the *RMS current* in the circuit:

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V_{rms}}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

Since X_L and X_C are frequency dependent, the circuit's current varies with the source frequency. The current is maximized when *resonance occurs*, at which point the inductive reactance and capacitive reactance cancel out:

$$X_L = X_C$$

$$\omega L = \frac{1}{\omega C}.$$

Solving this equation gives us the *resonant frequency* ω_0:

$$\omega_0 = \sqrt{\frac{1}{LC}},$$

Therefore, the resonance frequency f_0 is:

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

So summarize, at resonance:

- The impedance is purely resistive ($Z = R$), minimizing resistance to current
- The phase angle between voltage and current is zero $\phi = 0$, meaning voltage and current are in phase
- The current is maximized, given by $I_{max} = \frac{V_{rms}}{R}$

The *average power dissipated* in the circuit at resonance is maximal, because $\cos\phi = \frac{R}{Z} = 1$:

$$P = V_{rms}I_{rms}\cos\phi = V_{rms}I_{rms}$$

The shape of the current vs. frequency response can be change by adapting the resistance R. When R is small, the resonance curve is narrow and the peak is high, indicating strong resonance. Consequently, when R is large, the resonance effect is weaker, and the peak is broader. This behavior is simply due to the fact that resistance impede currents and the concept of resonance in AC circuits is analogous to mechanical resonance, such as a swing being pushed at just the right frequency.

```
interactive(children=(FloatLogSlider(value=0.01, description='L (H)', max=-1.0, min=-3.0), Fl
oatLogSlider(valu…
<function __main__.plot_lc_impedance(L, C)>
```

Impedance Matching

When transferring power between circuits, *impedance matching* ensures maximum efficiency. This occurs when the impedance of the circuit providing power Z_1 matches the impedance of the *load* Z_2:

$$Z_1 = Z_2$$

To understand this, let's assume a simple circuit 1 with the voltage source V and a single resistor R_1 in series. Connected to this is circuit 2 which will be represented by a single load R_2 (could be the equivalent net resistance of the entire circuit). As the same current flows I to both circuits, the power delivered to circuit 2 would be $P = I^2 R_2$. The relation of the current I and source voltage V is simply $V = I(R_1 + R_2) \leftrightarrow I = \frac{V}{R_1+R_2}$. Combining this gives us:

$$P = I^2 R_2 = \left(\frac{V}{R_1 + R_2}\right)^2 R_2 = \frac{V^2 R_2}{(R_1 + R_2)^2}$$

To find the maximum delivered power, we take the derivate w.r.t. R_2 (usually we can design our load circuit but the power delivery circuit is fixed, thus R_1 is fixed):

$$\frac{dP}{dR_2} = \frac{d}{dR_2}\left(\frac{V^2 R_2}{(R_1 + R_2)^2}\right) = \frac{V^2(R_1 - R_2)}{(R_1 + R_2)^3} = 0$$

Thus, $R_1 = R_2$ satisfies the equation. As a pure Ohmic impedance, is still an impedance, we can generalize our finding to the above mentioned statement that the maximum power is delivered if the impedances match, i.e. $Z_1 = Z_2$. If impedances are mismatched the power transfer is inefficient and transferred signals can be **distorted**, leading to erroneous results in experimental or communication systems.