Lecture Tutorial 1B: SI System, Dimensions, & Measurement Errors

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Today focus on measuring physical quantities including units and uncertainties

Is this a realistic time or a measurement error?

from ***wikipedia****, public domain*

Leap second

- 1-s adjustment to **UTC** to keep it aligned with **solar time (UT1)**
- Compensates for **Earth's irregular, slowing rotation** vs **atomic time (TAI)**
- Introduced **1972** to keep UTC – UT1 < 0.9 s $\to$ **27 leap seconds** added (last: **Dec 31 2016**)
- Causes **issues in precise systems** (computers, GPS, etc.)
- **CGPM 2022**: leap seconds will be **abolished by** $\leq$ **2035**, let discrepancy increase to a full minute ($\approx$ 50-100 years) $\to$ last minute of the day taking

two minutes

General Conference on Weights and Measures (CGPM)

- Established **1875** by the **Metre Convention**, ensuring international coordination of measurement standards
- Defines, maintains, and updates base units and constants
  - $\rightarrow$ **universal, precise, and stable units** for science, engineering, and technology
  - $\rightarrow$ **very active & important part of science**, e.g. national institutes such as Physikalisch-Technische Bundesanstalt (PTB)
- Governs **SI system (Système International d'Unités)**:
  - **1960** $\rightarrow$ 11th CGPM: adoption of the **SI system**
  - **2018** $\rightarrow$ 26th CGPM: complete **redefinition of SI units** using **fundamental constants**

The SI System: Why?

- Physics requires **numbers & units**.
- Without units, a value is meaningless (18.6 m $\neq$ 18.6 in).
- Units must be **standardized and reproducible** to ensure **universality**, **consistency**, **comparability**, and **precision**

The SI System: A non-trivial task or the history of the meter

How to define one meter? `ow01`

Year	Definition / Event	Rationale
1790s	1 m = 1/10 000 000 of the distance from equator $\rightarrow$ pole	Geographic reference
1889	Meter defined as distance between two marks on a platinum–iridium bar (stored in Paris)	Physical artifact
1960	1 m = 1 650 763.73 wavelengths of orange–red Kr-86 radiation	Optical standard, atomic reference
1983	Meter redefined as the distance light travels in $1/299792458$ s in vacuum	Definition via exact c

Year	Definition / Event	Rationale
2019	Metre is defined by fixing $c = 299792458$ m s^{-1}, with the second defined from the Cs-133 frequency $\Delta\nu_{Cs}$	Definition via fixed fundamental constants

The SI System: Why to fix constants instead of measuring them

Meter definition:

$$1 \text{ m} = \frac{9\,192\,631\,770}{299\,792\,458} \frac{c}{\Delta\nu_{Cs}}$$

Since 2019, all 7 base units linked to a **fundamental constant**:

- Does not rely on artifacts such as kilogram prototype
- Constants have **exact values**.
- **Measured quantities have uncertainties, not the units themselves.**

The SI System: Units (since 2019)

Quantity	Unit	Symbol	Dimension	Defined by fixing the numerical value of
Length	meter	m	[L]	speed of light in vacuum $c = 299\,792\,458$ **m·s⁻¹**
Time	second	s	[T]	Cs-133 hyperfine transition frequency $\Delta\nu_{Cs} = 9\,192\,631\,770$ **Hz**
Mass	kilogram	kg	[M]	Planck constant $h = 6.626\,070\,15 \times 10^{-34}$ **J·s**
Electric current	ampere	A	[I]	elementary charge $e = 1.602\,176\,634 \times 10^{-19}$ **C**
Thermodynamic temperature	kelvin	K	[Θ]	Boltzmann constant $k_B = 1.380\,649 \times 10^{-23}$

Quantity	Unit	Symbol	Dimension	Defined by fixing the numerical value of
				J·K⁻¹
Amount of substance	mole	mol	[N]	Avogadro constant $N_A = 6.022\,140\,76 \times 10^{23}$ **mol⁻¹**
Luminous intensity	candela	cd	[J]	luminous efficacy $K_{cd} = 683$ **lm·W⁻¹ at 540×10¹² Hz**

- The **2019 revision** fixed these constants' values exactly, making all SI units **stable, artifact-free, and universally reproducible**.

Orders of Magnitude: Metric prefixes

Prefix	Symbol	Factor	Prefix	Symbol	Factor
quetta	Q	10^{30}	quecto	q	10^{-30}
ronna	R	10^{27}	ronto	r	10^{-27}
yotta	Y	10^{24}	yocto	y	10^{-24}
zetta	Z	10^{21}	zepto	z	10^{-21}
exa	E	10^{18}	atto	a	10^{-18}
peta	P	10^{15}	femto	f	10^{-15}
tera	T	10^{12}	pico	p	10^{-12}
giga	G	10^{9}	nano	n	10^{-9}
mega	M	10^{6}	micro	µ	10^{-6}
kilo	k	10^{3}	milli	m	10^{-3}
hecto	h	10^{2}	centi	c	10^{-2}

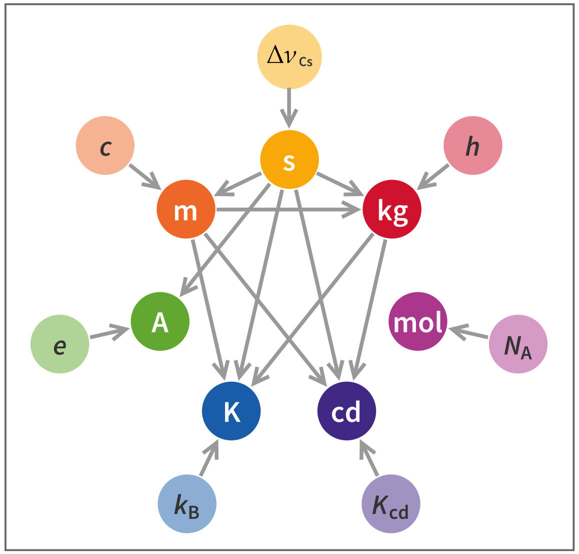
Prefix	Symbol	Factor	Prefix	Symbol	Factor
deka	da	10^{1}	deci	d	10^{-1}

Orders of Magnitude: Examples

Quantity	Example	Typical Scale
Length	Proton diameter	10^{-15} m
	Human height	1 m
	Earth–Sun distance	10^{11} m
Mass	Electron	10^{-30} kg
	Human	10^{2} kg
	Earth	6×10^{24} kg

The SI system: Summary

- SI base: 7 units defined by fixed constants.
- Derived units follow directly from base units and dimensions.
- Prefixes allow scaling by powers of 10.



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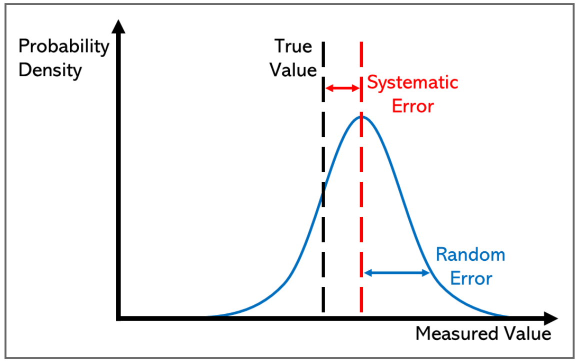
Measurement errors: Why it matters

`mb03 - Fehlerbetrachtungen zur Zeitmessung`

- Every measurement deviates from the (unknown) true value.
- Goal: state a **best estimate** and a **justified interval**.
- Distinguish **systematic** (bias) vs **random** (scatter).



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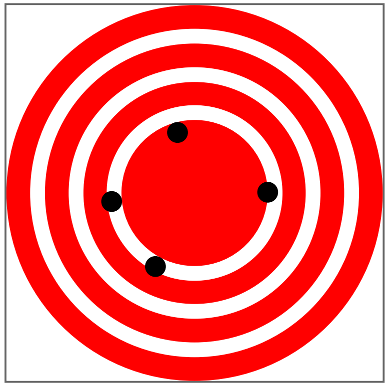
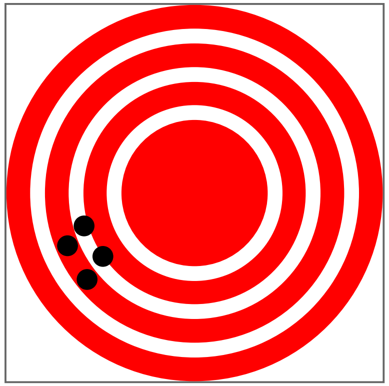
Measurement Errors: Reporting & assessing a result

- Report **central value + uncertainty interval**
- **Absolute:** $X = \bar{x} \pm \Delta x$ **Relative:** $X = \bar{x}(1 \pm \frac{\Delta x}{\bar{x}})$
- **Accuracy:** closeness of a measured value to the **true or accepted value**
- **Precision:** degree of **reproducibility** among repeated measurements (how tightly results cluster)





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Measurement errors: Sources of deviations

- **Objective**: environment (T, p, supply voltage), instrument drift.
- **Subjective**: parallax, reaction time, estimation.
- **Methodological**: wrong model, neglected effects (e.g., buoyancy).

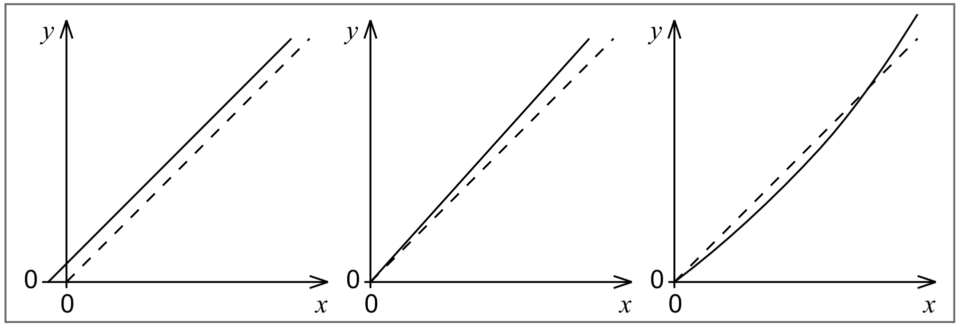
Measurement errors: Systematic vs random

Property	Systematic Error	Random Error
Behavior	Same sign and magnitude for each measurement	Fluctuates between measurements
Effect on mean	Persists in the mean	Statistically distributed around the mean
Correction / Reduction	Corrected by calibration	Reduces with repeated measurements

$\rightarrow$ Strategy: **correct** systematics; **estimate** random uncertainty.

Measurement errors: Manufacturer error limits

- Analog: Typically expressed as percentage of full scale or absolute value
- Digital: Typically expressed as percentage of reading + digits [+ possibly % of range].
- **Use as systematic limits for direct measurements** (if calibration not possible/imperfect)



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Measurement errors: Random scatter & Gaussian model

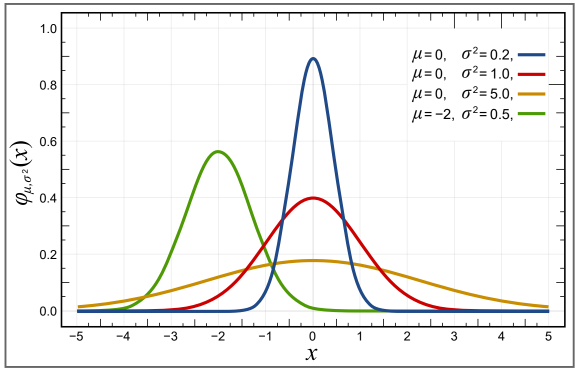
`mb02 - Galton – Brett`

- Repeat measurements ($n \geq 5$) $\rightarrow$ histogram approaches **Gaussian distribution**:

$$w(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

- Use sample statistics to estimate mean μ & standard deviation σ:

$$\mu = \tfrac{1}{n}\sum_{i=1}^{n} x_i\,, \qquad \sigma = \sqrt{\frac{1}{n-1}\sum(x_i-\bar{x})^2}$$



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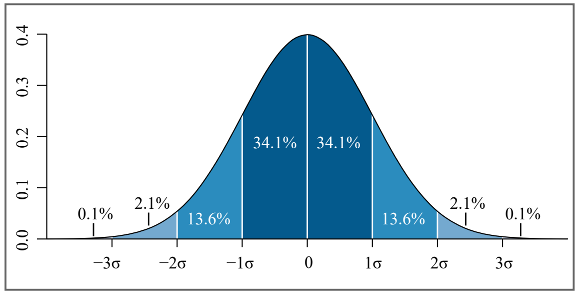
Measurement errors: Uncertainty of the mean (Student-t)

- In reality, we do not have infinite measurements $\rightarrow$ only **estimate** mean $\bar{x}$ & standard deviation s instead of the true (unkown) μ & σ
- Account for the finite number of samples:
  - Standard error: $s_{\bar{x}} = \frac{s}{\sqrt{n}}$.
  - For finite n at confidence level set by t: $\Delta x = t\,\frac{s}{\sqrt{n}}$

n	68.3 %	95.4 %	99.73 %
5	1.11	2.65	5.51
10	1.05	2.28	3.96
∞	1.00	2.00	3.00



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Measurement errors: Linear error propagation

- How to report the error for a quantity we do not measure directly? $\rightarrow$ **error propagation**
- For independent inputs and a single indirect measurement, estimated via:

$$\Delta F = \left|\frac{\partial F}{\partial x}\right|\Delta x + \left|\frac{\partial F}{\partial y}\right|\Delta y + \cdots$$

Measurement errors: Example error propagation

`mb13 - Geschwindigkeitsbestimmung mit rotierenden Scheiben`

The projectile velocity is $v = \frac{s}{t}$, but we do not measure t directly.

The angular velocity of the rotating disks (with n rotations per unit time) is

$$\omega = 2\pi n = \frac{\varphi}{t}$$

Therefore, the flight time t of the projectile between the two disks is

$$t = \frac{\varphi}{2\pi n}$$

Thus, the projectile velocity can be estimated indirectly via:

$$v = \frac{s}{t} = \frac{2\pi n s}{\varphi}$$

Example numerical values:

$$v = \frac{2\pi \cdot \frac{1600\frac{1}{\min}}{60} \cdot 0.50\mathrm{m}}{15^\circ \mathrm{rad} \cdot \frac{\pi}{180^\circ}} \approx 320 \text{ m/s}.$$

Measurement errors: Example error propagation (cont')

`mb13 - Geschwindigkeitsbestimmung mit rotierenden Scheiben`

For $v(n, s, \varphi) = \dfrac{2\pi n s}{\varphi}$ with measurement errors in s, φ, and n, the linear absolute error propagation

$$\Delta v \approx \left|\frac{\partial v}{\partial s}\right| \Delta s + \left|\frac{\partial v}{\partial \varphi}\right| \Delta \varphi + \left|\frac{\partial v}{\partial n}\right| \Delta n$$

uses the partial derivatives

$$\frac{\partial v}{\partial s} = \frac{2\pi n}{\varphi}, \qquad \frac{\partial v}{\partial \varphi} = -\frac{2\pi n s}{\varphi^2}, \qquad \frac{\partial v}{\partial n} = \frac{2\pi s}{\varphi}.$$

Thus

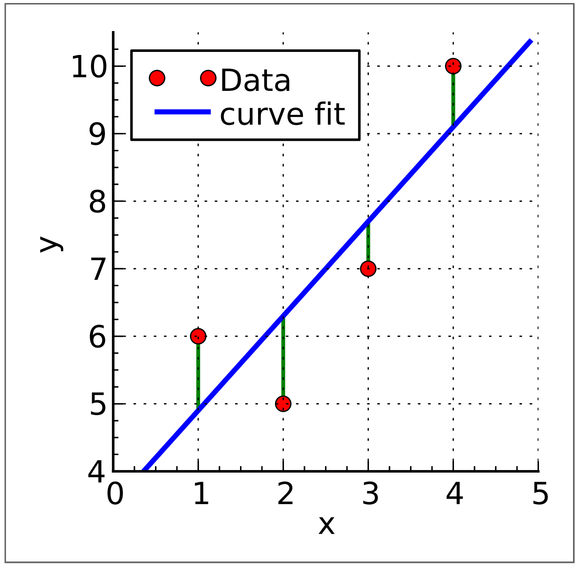
$$\boxed{\Delta v \approx \frac{2\pi n}{\varphi}\,\Delta s \;+\; \frac{2\pi n s}{\varphi^2}\,\Delta\varphi \;+\; \frac{2\pi s}{\varphi}\,\Delta n}$$

Measurement errors: Linear regression

- Sometimes easier to vary one measurement conditions precisely, instead of repeating the measurements over and over $\rightarrow$ linear regression
- Linear model: $y = a + bx$ for points (x_i, y_i).
- Estimates:

$$b = \frac{\overline{xy} - \bar{x}\,\bar{y}}{\overline{x^2} - (\bar{x})^2}, \qquad a = \bar{y} - b\,\bar{x}$$

- Residuals: $\Delta y_i = a + bx_i - y_i$.



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Measurement errors: Example linear regression

`mb05 - Fallbeschleunigung`

When an object is dropped from rest and air resistance is neglected, its motion follows a simple kinematic relation between **height** and **fall time**.
The distance fallen after time t is

$$h = \frac{1}{2} g t^2,$$

We could rearrange the equation into $g = \frac{2h}{t^2}$, but instead we will **vary the height**, **measure** t, and **estimate** g **via linear regression of** $h(t^2)$.
The **slope** of the regression line $h = \mathrm{fct}(t^2)$ equals $\frac{g}{2}$, hence

$$g = 2 \times \text{slope}.$$

Summary: SI System & Measurement Errors

- **SI System:** Universal framework ensuring consistent, reproducible quantities based on **7 base units**; all others (N, J, Pa, ...) derived systematically.
- **Measurement Errors:**
  - *Systematic:* same sign & magnitude
  - *Random:* fluctuate & statistically distributed $\rightarrow$ **Correct** systematics, **estimate** random uncertainty.
- **Statistics:** Repeated measurements follow a **Gaussian distribution**; mean $\bar{x}$, standard deviation s, and mean uncertainty $\Delta x = t\, s/\sqrt{n}$ (Student-t).
- **Error Propagation:** Combined uncertainty for indirect estimation

$$\Delta F = \sum_i \left| \frac{\partial F}{\partial x_i} \right| \Delta x_i$$

links measured errors to derived quantities.

- **Linear Regression:** Fit $y = ax + b$ to data; slope a and intercept b from least squares give **best estimates** and their uncertainties.