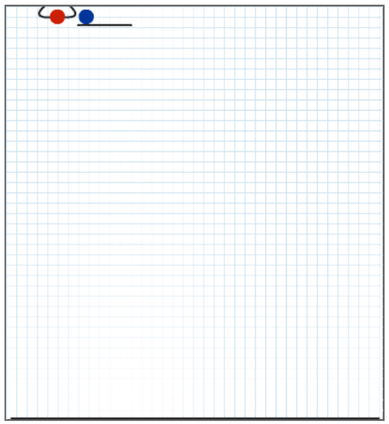
1.2. Kinematics beyond one dimension & projectile motion

`mb09`

Which object hits the ground first?

- both at the same time as their vertical acceleration is the same, i.e. g
- **superposition** of motions
- **projectile motion & circular motion**



Describing motion in 2D and 3D

- Position vector: $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$
- **Cartesian basis** $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is **orthonormal**:
  $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0, \qquad |\hat{\imath}| = |\hat{\jmath}| = |\hat{k}| = 1$
- Each component behaves like an independent 1D motion
    - $\rightarrow$ All 1D kinematic equations apply separately to $x(t)$, $y(t)$ and $z(t)$
    - $\rightarrow$ Vector operations (addition, subtraction, differentiation) act **component-wise**

- Equation for 2D motion:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{\imath} + \frac{dy}{dt}\,\hat{\jmath}, \qquad \vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}, \qquad \vec{v} = v_x\,\hat{\imath} + v_y\,\hat{\jmath}, \qquad \vec{a} = a_x\,\hat{\imath} + a_y\,\hat{\jmath}$$

Superposition of perpendicular motions

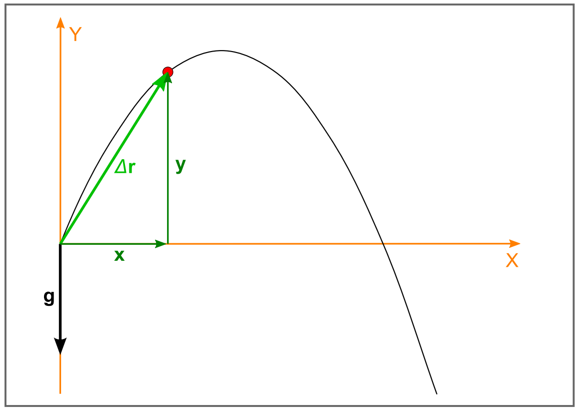
`Garteneisenbahn`

- Solve x and y components independently, then combine
- Position: $\mathbf{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}$
- Velocity: $\mathbf{v}(t) = v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}$
- Independence holds when no coupling forces link x and y

Projectile motion: concept and assumptions

`mb08 - Wasserstrahl`

- Launch with v_0 at angle θ from (x_0, y_0)
    - $\rightarrow$ initial velocity: $\vec{v}(0) = \vec{v}_0 = (v_0 \cos\theta,\ v_0 \sin\theta)$
- Neglect air resistance $\rightarrow$ only gravity acts:
    - $\rightarrow \vec{a} = (0, -g)$ with $g \approx 9.81\,\mathrm{m/s}^2$
- Superposition of **uniform motion** in x, **uniformly accelerated motion** in y
- Choose "up" as positive y; signs must be used consistently



Projectile motion: Kinematics along x

- No horizontal force $\Rightarrow a_x = 0$
- Constant horizontal velocity: $v_x(t) = v_{0x} = v_0 \cos\theta$
- Horizontal position: $x(t) = x_0 + v_{0x}\, t = x_0 + v_0 \cos\theta\, t$
- Uniform translation set by the initial horizontal component

Projectile motion: Kinematics along y

- Constant vertical acceleration: $a_y = -g$
- Vertical velocity: $v_y(t) = v_{y0} - g\,t = v_0 \sin\theta - g\,t$
- Vertical position: $y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2 = y_0 + v_0 \sin\theta\, t - \frac{1}{2}gt^2$
- On the way up v_y decreases to zero at the apex; then $v_y < 0$ on the way down

Projectile motion: Combining x and y

- Eliminate t using

$$x = x_0 + v_{0x}\,t \leftrightarrow t = \frac{x - x_0}{v_{0x}}$$

- Substitute this expression for t into $y(t)$:

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 = y_0 + v_{0y}\left(\frac{x - x_0}{v_{0x}}\right) - \tfrac{1}{2}g\left(\frac{x - x_0}{v_{0x}}\right)^2$$

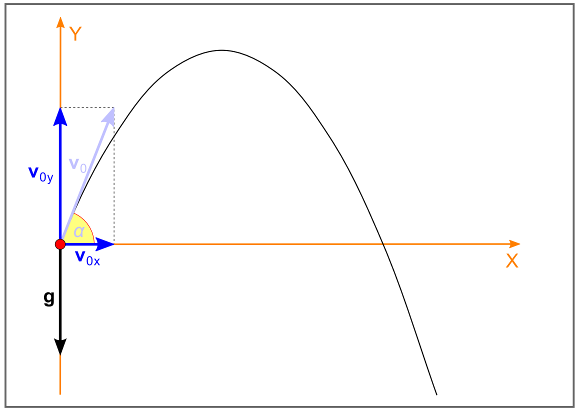
- Rearrange:

$$y(x) = y_0 + \left(\frac{v_{0y}}{v_{0x}}\right)(x - x_0) - \frac{g}{2v_{0x}^2}(x - x_0)^2$$

- $\rightarrow$ y(x) is a parabola

Projectile motion: Velocity vector

- Velocity vector: $\mathbf{v}(t) = (v_0 \cos\theta,\ v_0 \sin\theta - gt)$
- Magnitude: $|v(t)| = \sqrt{(v_0 \cos\theta)^2 + (v_0 \sin\theta - gt)^2}$
- Angle: $\tan\theta_v(t) = \frac{v_y(t)}{v_x(t)} = \frac{v_0 \sin\theta - gt}{v_0 \cos\theta}$
- The velocity direction $\rightarrow$ tangent to the trajectory at each instant



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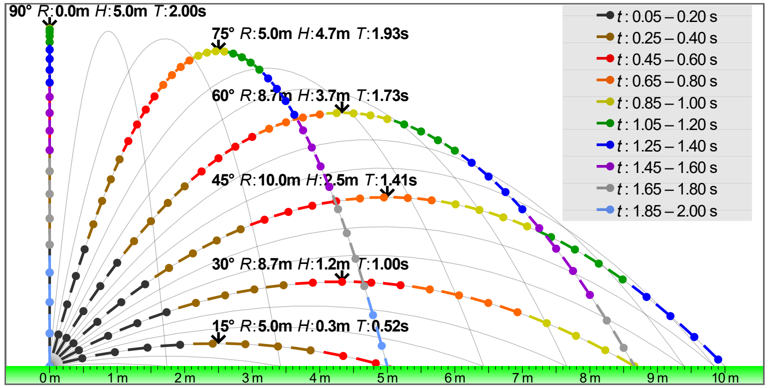
Projectile motion: Time of flight, maximum height, and range

- For $y_0 = 0$ and landing at $y = 0$:
  - Time of flight: $T = \dfrac{2v_{0y}}{g}$
  - Maximum height: $H = \dfrac{v_{0y}^2}{2g}$
  - Range: $x(T) = R = \dfrac{2\, v_{0x}\, v_{0y}}{g}$
- Range is maximal at $\theta = 45^\circ$
- Angles θ and $(90^\circ - \theta)$ give same R but different T and H



Projectile motion: concluding remarks

`sim - projectile motion`

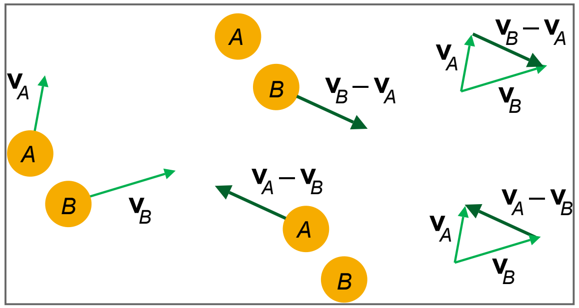
- Core idea: independence of x and y components (superposition)
- Changing y_0 shifts the optimal launch angle away from 45°
- Real trajectories deviate due to drag and lift; ideal results are first-order checks
- Use the full $y(t)$ and $x(t)$ with your specific initial/landing heights

Relative motion and Galilean kinematics

- Motion is always described **relative to a reference frame**
- In **Galilean kinematics** ($v \ll c$): space and time are absolute
- Velocities add vectorially between inertial frames:

$$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$$

- If $\vec{v}_{S'/S}$ is constant $\rightarrow \vec{a}_{P/S} = \vec{a}_{P/S'}$
- Examples: boat in a river, aircraft with tailwind, person on a moving train
- At very high speeds $\rightarrow$ shift from Galilean kinematics to Einstein's relativity



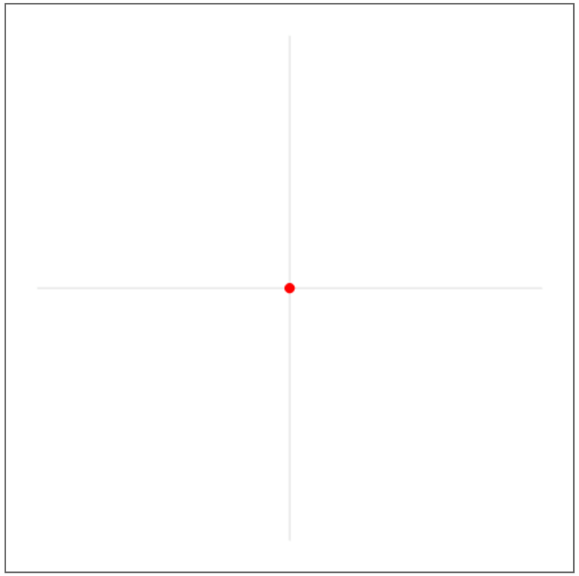
Uniform circular motion: angular quantities and relations

`mb15 - Bewegung auf Kreisbahn`

- The position on the circle is described by the **angular displacement** $\phi(t)$ (in radians) measured from the $+x$-axis
- **Angular velocity** $\vec{\omega}$ is measured in **rad/s** and **angular acceleration** $\vec{\alpha}$ in **rad/s²**:

$$\omega = \frac{d\phi}{dt}, \qquad \alpha = \frac{d\omega}{dt}$$

nice video on angular velocity vs. angular frequency *and* ***another one***



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Uniform circular motion: angular quantities and relations (cont’)

`mb17 - Schleifscheibe`

- For **uniform circular motion** $\rightarrow \omega = \mathrm{const}$, $\alpha = 0$

$$\phi(t) = \phi_0 + \omega t, \qquad T = \frac{2\pi}{\omega}, \qquad f = \frac{1}{T}, \qquad \omega = 2\pi f$$

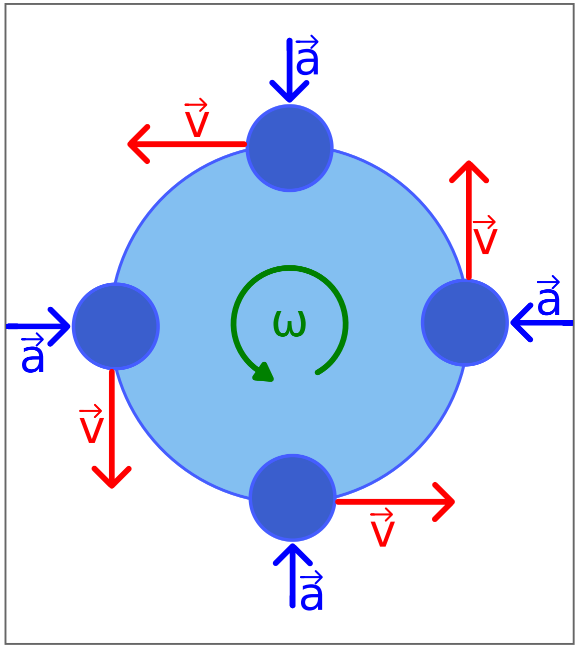
- Arc length and tangential speed:

$$s = r\phi, \qquad |v| = \frac{ds}{dt} = \frac{d(r\phi)}{dt} = \frac{d(r\phi_0 + r\omega t)}{dt} = r\omega$$

- The velocity is tangent to the circle with constant magnitude

Centripetal acceleration: concept and direction

- Angular acceleration $\alpha = d\omega/dt$ may be zero, yet $\vec{a} \neq 0$
- $|\vec{v}|$ is constant, but the **direction** of $\vec{v}$ changes with time $\rightarrow$ acceleration exists
- This change in direction produces the **centripetal (radial)** acceleration $\vec{a}_r$
- As the object moves along the circle, $\vec{v}_1$ and $\vec{v}_2$ differ by $\Delta\phi$
- $\rightarrow$ The change $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ points toward the center
- $\rightarrow$ **Centripetal acceleration always points towards the center**



$\vec{a}$
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Centripetal acceleration: magnitude

- From similar triangles of $\vec{r}$ and $\vec{v}$:

$$\frac{|\Delta\vec{v}|}{|v|} \approx \frac{|\Delta\vec{r}|}{r} \leftrightarrow |\Delta\vec{v}| \approx \frac{|\Delta\vec{r}|}{r}\vec{v}$$

- Acceleration is the rate of change of velocity:

$$a_r = \lim_{\Delta t\to 0}\frac{|\Delta\vec{v}|}{\Delta t} = \lim_{\Delta t\to 0}\frac{\frac{|\Delta\vec{r}|}{r}\vec{v}}{\Delta t} = \lim_{\Delta t\to 0}\frac{|\Delta\vec{r}|\vec{v}}{r\Delta t} = \lim_{\Delta t\to 0}\frac{|\Delta\vec{r}|}{\Delta t}\frac{\vec{v}}{r}$$

- Since $\lim_{\Delta t\to 0}\frac{|\Delta\vec{r}|}{\Delta t} = \frac{dr}{dt} = v$, we obtain:

$$a_r = \frac{v^2}{r} = r\omega^2$$

- Magnitude increases with $|v|^2$ and decreases with $r \rightarrow$ tighter turns or higher speeds require larger inward acceleration

Conceptual link: superposition and Cartesian representation

`ms35 - Projektion Kreisbewegung`

- Circular motion can be viewed as **two perpendicular oscillations** with a 90° phase shift

$$x(t) = r\cos(\omega t), \qquad y(t) = r\sin(\omega t) = r\cos\left(\omega t - \frac{\pi}{2}\right).$$

- Each coordinate oscillates harmonically $\rightarrow$ their combination produces the circular path $x^2 + y^2 = r^2$
- Equations:

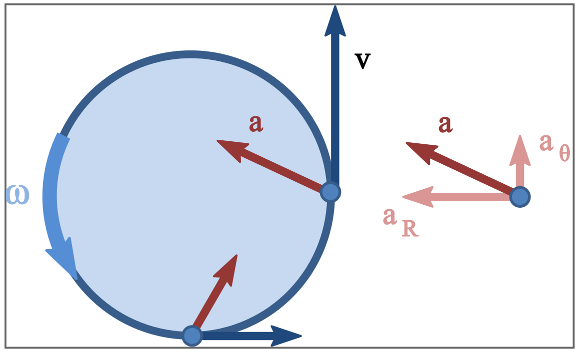
$$\vec{r}(t) = r\cos(\omega t)\,\hat{\imath} + r\sin(\omega t)\,\hat{\jmath},$$

$$\vec{v}(t) = -\omega r\sin(\omega t)\,\hat{\imath} + \omega r\cos(\omega t)\,\hat{\jmath},$$

$$\vec{a}(t) = -\omega^2 r\cos(\omega t)\,\hat{\imath} - \omega^2 r\sin(\omega t)\,\hat{\jmath} = -\omega^2\vec{r}(t).$$

Uniformly accelerated circular motion

- When speed changes, $\vec{a}$ splits into two perpendicular components:
  - **Radial (centripetal):** $a_r = \dfrac{v^2}{r}$ → directs $\vec{v}$ toward the center
  - **Tangential:** $a_{\tan} = \dfrac{dv}{dt} = r\,\alpha$ → changes the speed along the path
- Total acceleration: $\vec{a} = \vec{a}_{\tan} + \vec{a}_r, \qquad |\vec{a}| = \sqrt{a_{\tan}^2 + a_r^2}$
- Applies to any curved path by using the local **radius of curvature** r



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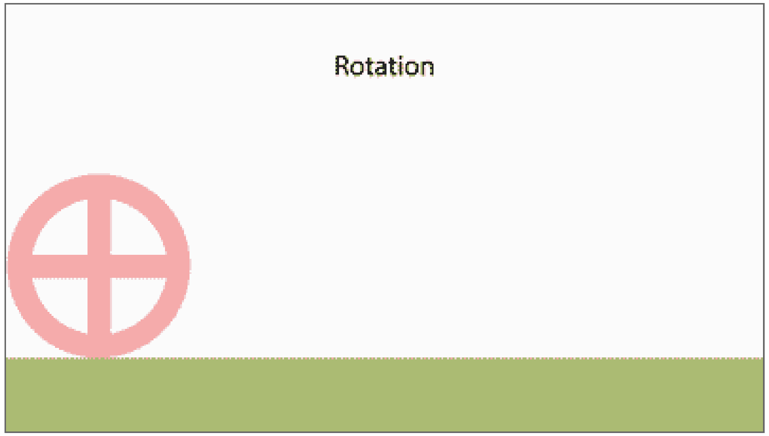
Summary circular motion

Quantity	Expression	Description
Angular velocity	$\omega = \frac{d\phi}{dt}$	rate of change of angular position
Linear speed	$v = r\omega$	constant for uniform motion
Centripetal acceleration	$a_r = \frac{v^2}{r} = \omega^2 r$	directed toward the center
Angular acceleration	$\alpha = \frac{d\omega}{dt}$	zero for uniform motion

Universality of superposition

`mb10 - Dart`

- Powerful concept
- Will revisit for e.g. (standing) waves and charged particles moving in electromagnetic fields



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