

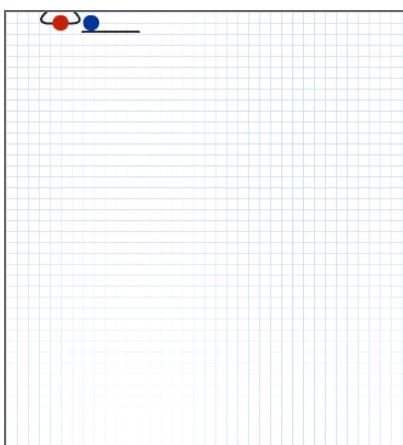
1.2. Kinematics beyond one dimension & projectile motion



mb09

Which object hits the ground first?

- both at the same time as their vertical acceleration is the same, i.e. g
- **superposition** of motions
- **projectile motion & circular motion**



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Describing motion in 2D and 3D

- Position vector: $\vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k}$
- **Cartesian basis** $\{\hat{i}, \hat{j}, \hat{k}\}$ is **orthonormal**:
 $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0, \quad |\hat{i}| = |\hat{j}| = |\hat{k}| = 1$
- Each component behaves like an independent 1D motion
 - \rightarrow All 1D kinematic equations apply separately to $x(t)$, $y(t)$ and $z(t)$
 - \rightarrow Vector operations (addition, subtraction, differentiation) act **component-wise**
- Equation for 2D motion:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j}, \quad \vec{a}(t) = \frac{d\vec{v}}{dt} =$$

$$\vec{r} = x \hat{i} + y \hat{j}, \quad \vec{v} = v_x \hat{i} + v_y \hat{j}, \quad \vec{a} = a$$

Superposition of perpendicular motions

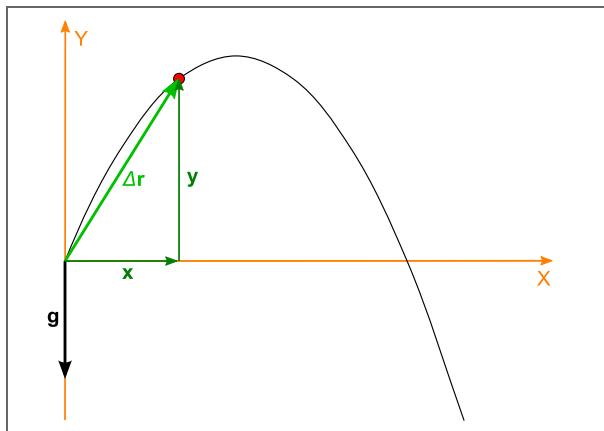
Garteneisenbahn

- Solve x and y components independently, then combine
- Position: $\mathbf{r}(t) = x(t) \hat{\mathbf{i}} + y(t) \hat{\mathbf{j}}$
- Velocity: $\mathbf{v}(t) = v_x(t) \hat{\mathbf{i}} + v_y(t) \hat{\mathbf{j}}$
- Independence holds when no coupling forces link x and y

Projectile motion: concept and assumptions

mb08 - Wasserstrahl

- Launch with v_0 at angle θ from (x_0, y_0)
 - → initial velocity:
$$\vec{v}(0) = \vec{v}_0 = (v_0 \cos \theta, v_0 \sin \theta)$$
- Neglect air resistance → only gravity acts:
 - → $\vec{a} = (0, -g)$ with $g \approx 9.81 \text{ m/s}^2$
- Superposition of **uniform motion** in x ,
uniformly accelerated motion in y
- Choose "up" as positive y ; signs must be used consistently



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Projectile motion: Kinematics along x

- No horizontal force $\Rightarrow a_x = 0$
- Constant horizontal velocity:

$$v_x(t) = v_{0x} = v_0 \cos \theta$$

- Horizontal position:

$$x(t) = x_0 + v_{0x} t = x_0 + v_0 \cos \theta t$$

- Uniform translation set by the initial horizontal component

Projectile motion: Kinematics along y

- Constant vertical acceleration: $a_y = -g$
- Vertical velocity:

$$v_y(t) = v_{y0} - g t = v_0 \sin \theta - g t$$

- Vertical position:

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2 = y_0 + v_0 \sin \theta t - \frac{1}{2}gt^2$$

- On the way up v_y decreases to zero at the apex; then $v_y < 0$ on the way down

Projectile motion: Combining x and y

- Eliminate t using

$$x = x_0 + v_{0x} t \leftrightarrow t = \frac{x - x_0}{v_{0x}}$$

- Substitute this expression for t into $y(t)$:

$$y = y_0 + v_{0y} t - \frac{1}{2} g t^2 = y_0 + v_{0y} \left(\frac{x - x_0}{v_{0x}} \right) - \frac{1}{2} g \left(\frac{x - x_0}{v_{0x}} \right)^2$$

- Rearrange:

$$y(x) = y_0 + \left(\frac{v_{0y}}{v_{0x}} \right) (x - x_0) - \frac{g}{2v_{0x}^2} (x - x_0)^2$$

- $\rightarrow y(x)$ is a parabola
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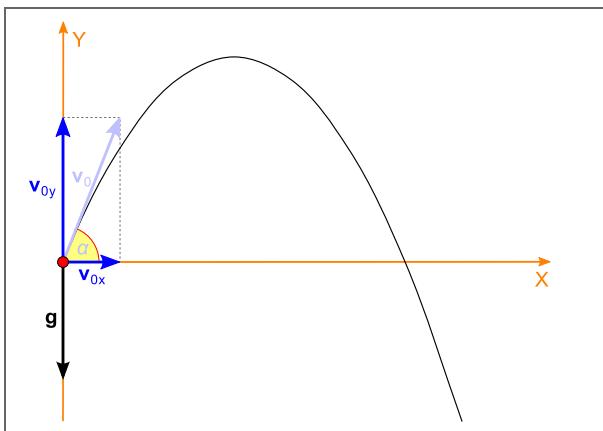
Projectile motion: Velocity vector

- Velocity vector: $\mathbf{v}(t) = (v_0 \cos \theta, v_0 \sin \theta - gt)$
- Magnitude:

$$|v(t)| = \sqrt{(v_0 \cos \theta)^2 + (v_0 \sin \theta - gt)^2}$$

- Angle: $\tan \theta_v(t) = \frac{v_y(t)}{v_x(t)} = \frac{v_0 \sin \theta - gt}{v_0 \cos \theta}$

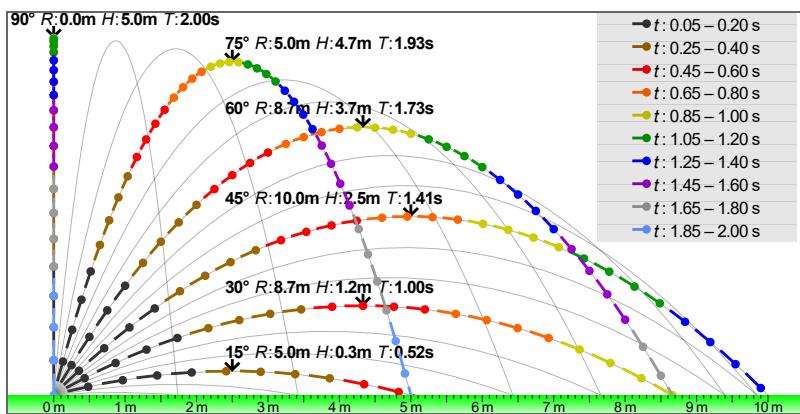
- The velocity direction \rightarrow tangent to the trajectory at each instant



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Projectile motion: Time of flight, maximum height, and range

- For $y_0 = 0$ and landing at $y = 0$:
 - Time of flight: $T = \frac{2v_{0y}}{g}$
 - Maximum height: $H = \frac{v_{0y}^2}{2g}$
 - Range: $x(T) = R = \frac{2 v_{0x} v_{0y}}{g}$
- Range is maximal at $\theta = 45^\circ$
- Angles θ and $(90^\circ - \theta)$ give same R but different T and H



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Projectile motion: concluding remarks

sim - projectile motion

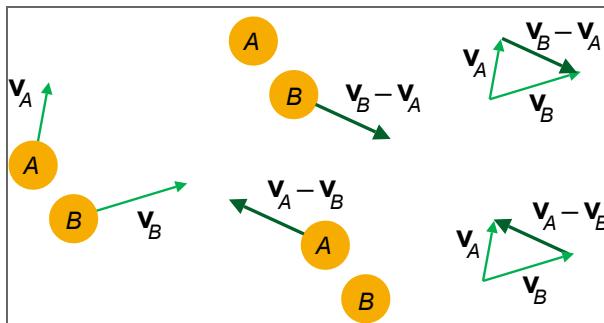
- Core idea: independence of x and y components (superposition)
- Changing y_0 shifts the optimal launch angle away from 45°
- Real trajectories deviate due to drag and lift; ideal results are first-order checks
- Use the full $y(t)$ and $x(t)$ with your specific initial/landing heights

Relative motion and Galilean kinematics

- Motion is always described **relative to a reference frame**
- In **Galilean kinematics** ($v \ll c$): space and time are absolute
- Velocities add vectorially between inertial frames:

$$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$$

- If $\vec{v}_{S'/S}$ is constant $\rightarrow \vec{a}_{P/S} = \vec{a}_{P/S'}$
- Examples: boat in a river, aircraft with tailwind, person on a moving train
- At very high speeds \rightarrow shift from Galilean kinematics to Einstein's relativity



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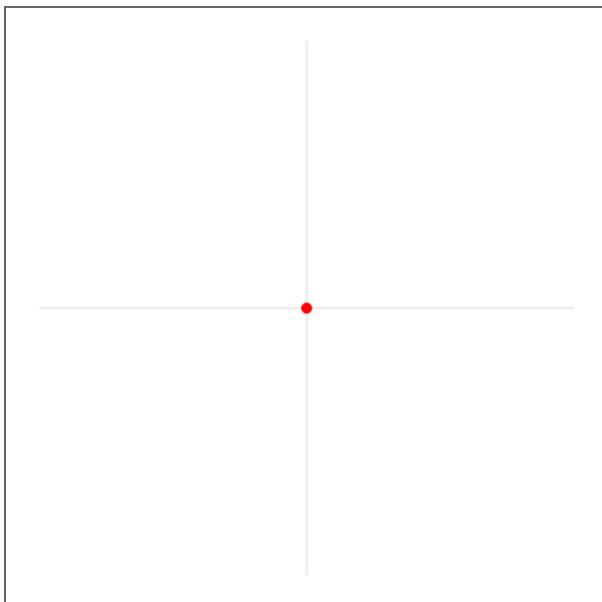
Uniform circular motion: angular quantities and relations

mb15 - Bewegung auf Kreisbahn

- The position on the circle is described by the **angular displacement** $\phi(t)$ (in radians) measured from the $+x$ -axis
- **Angular velocity** $\vec{\omega}$ is measured in **rad/s** and **angular acceleration** $\vec{\alpha}$ in **rad/s²**:

$$\omega = \frac{d\phi}{dt}, \quad \alpha = \frac{d\omega}{dt}$$

[nice video on angular velocity vs. angular frequency](#) and
[another one](#)



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Uniform circular motion: angular quantities and relations (cont')

mb17 - Schleifscheibe

- For **uniform circular motion** $\rightarrow \omega = \text{const}$,

$$\alpha = 0$$

$$\phi(t) = \phi_0 + \omega t, \quad T = \frac{2\pi}{\omega}, \quad f = \frac{1}{T}, \quad \omega =$$

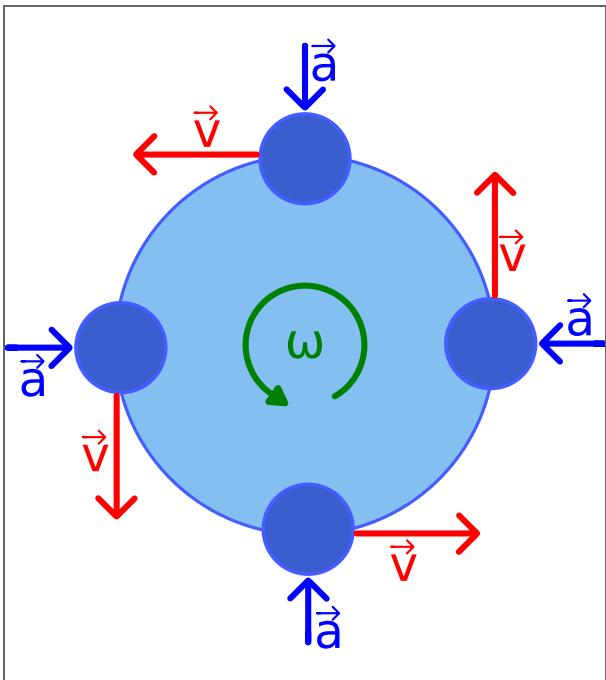
- Arc length and tangential speed:

$$s = r\phi, \quad |v| = \frac{ds}{dt} = \frac{d(r\phi)}{dt} = \frac{d(r\phi_0 + r\omega t)}{dt} =$$

- The velocity is tangent to the circle with constant magnitude
-

Centripetal acceleration: concept and direction

- Angular acceleration $\alpha = d\omega/dt$ may be zero, yet $\vec{a} \neq 0$
- $|\vec{v}|$ is constant, but the **direction** of \vec{v} changes with time \rightarrow acceleration exists
- This change in direction produces the **centripetal (radial)** acceleration \vec{a}_r
- As the object moves along the circle, \vec{v}_1 and \vec{v}_2 differ by $\Delta\phi$
- \rightarrow The change $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ points toward the center
- \rightarrow **Centripetal acceleration always points towards the center**



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Centripetal acceleration: magnitude

- From similar triangles of \vec{r} and \vec{v} :

$$\frac{|\Delta\vec{v}|}{|v|} \approx \frac{|\Delta\vec{r}|}{r} \leftrightarrow |\Delta\vec{v}| \approx \frac{|\Delta\vec{r}|}{r}\vec{v}$$

- Acceleration is the rate of change of velocity:

$$a_r = \lim_{\Delta t \rightarrow 0} \frac{|\Delta\vec{v}|}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\frac{|\Delta\vec{r}|}{r}\vec{v}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{|\Delta\vec{r}|\vec{v}}{r\Delta t} = \lim_{\Delta t \rightarrow 0}$$

- Since $\lim_{\Delta t \rightarrow 0} \frac{|\Delta\vec{r}|}{\Delta t} = \frac{dr}{dt} = v$, we obtain:

$$a_r = \frac{v^2}{r} = r\omega^2$$

- Magnitude increases with $|v|^2$ and decreases with $r \rightarrow$ tighter turns or higher speeds require larger inward acceleration
-

Conceptual link: superposition and Cartesian representation

ms35 - Projektion Kreisbewegung

- Circular motion can be viewed as **two perpendicular oscillations** with a 90° phase shift

$$x(t) = r \cos(\omega t), \quad y(t) = r \sin(\omega t) = r \cos\left(\omega t - \frac{\pi}{2}\right)$$

- Each coordinate oscillates harmonically → their combination produces the circular path $x^2 + y^2 = r^2$
- Equations:

$$\vec{r}(t) = r \cos(\omega t) \hat{i} + r \sin(\omega t) \hat{j},$$

$$\vec{v}(t) = -\omega r \sin(\omega t) \hat{i} + \omega r \cos(\omega t) \hat{j},$$

$$\vec{a}(t) = -\omega^2 r \cos(\omega t) \hat{i} - \omega^2 r \sin(\omega t) \hat{j} = -\omega^2 \vec{r}(t).$$

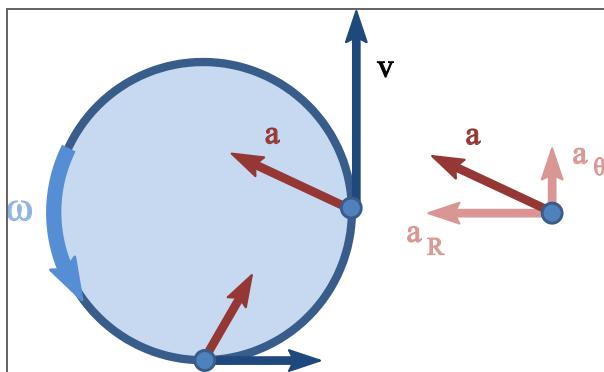
Uniformly accelerated circular motion

- When speed changes, \vec{a} splits into two perpendicular components:
 - **Radial (centripetal):** $a_r = \frac{v^2}{r} \rightarrow$ directs \vec{v} toward the center
 - **Tangential:** $a_{\tan} = \frac{dv}{dt} = r \alpha \rightarrow$ changes the speed along the path

- Total acceleration:

$$\vec{a} = \vec{a}_{\tan} + \vec{a}_r, \quad |\vec{a}| = \sqrt{a_{\tan}^2 + a_r^2}$$

- Applies to any curved path by using the local **radius of curvature r**



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Summary circular motion

Quantity	Expression	Description
Angular velocity	$\omega = \frac{d\phi}{dt}$	rate of change of angular position
Linear speed	$v = r\omega$	constant for uniform motion
Centripetal acceleration	$a_r = \frac{v^2}{r} = \omega^2 r$	directed toward the center
Angular acceleration	$\alpha = \frac{d\omega}{dt}$	zero for uniform motion

Universality of superposition

mb10 - Dart

- Powerful concept
- Will revisit for e.g. (standing) waves and charged particles moving in electromagnetic fields



Rotation



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