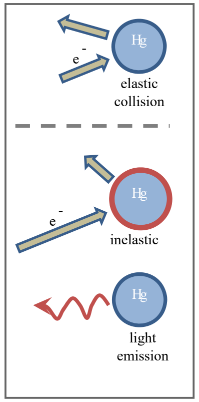
1.5. Linear momentum & collisions

Conservation of momentum powerful concept like conservation of energy → can be applied to many scenarios



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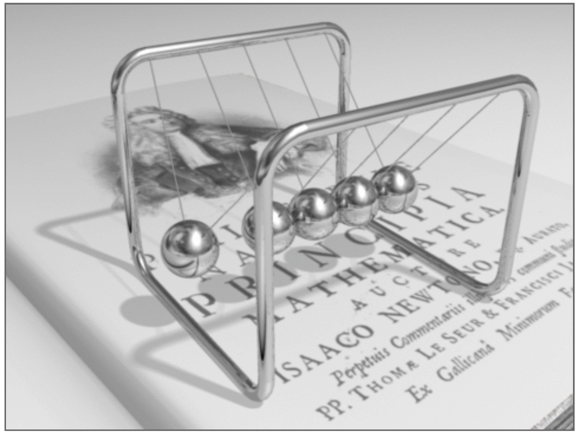
What is Momentum? - Empirical Approach

`mi07 - Newton-Wiege`

What remains constant before and after the collision?



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What is Momentum? - Empirical Approach (cont’)

`mi09 - Impulsübertragung Bälle`

What influences the height the ball reaches? How does this relate to collisions?

Revisiting Newton's Second Law

At any instant of time, the net force on a body equals its mass times its acceleration or, equivalently, the rate at which its momentum changes with time.

$$\vec{F}_{\text{net}} = m\vec{a} = m\frac{d\vec{v}}{dt}.$$

(Linear) Momentum

- **Momentum** unifies **inertia** (mass) and **motion** (velocity) into a single quantity:

$$\vec{p} = m \cdot \vec{v}$$

- Momentum is a **vector** pointing in the same direction as $\vec{v}$.
- Units: $\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}} = \frac{\mathrm{kg \cdot m}}{\mathrm{s}^2}\mathrm{s} = \mathrm{N} \cdot \mathrm{s}$

Revisiting Newton's Second Law (cont')

> *At any instant of time, the net force on a body equals its mass times its acceleration or, equivalently, the rate at which its momentum changes with time.*

$$\vec{F}_{\text{net}} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt}.$$

Thus, the **net force acting on an object is equal to the rate of change of its momentum**.

Conservation of Momentum

`mi07 - Newton-Wiege`

- Net force gives the rate of change of momentum:

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}$$

- If zero net external force → total momentum stays **constant**

$$0 = \frac{d\vec{p}}{dt} \;\leftrightarrow\; p = \text{const}$$

- This is the **law of conservation of momentum**
- Applies to any isolated system from colliding carts to galaxies and subatomic particles

Particle Systems & Total Momentum

`mi13 - Pendelwagen`

- A system with many particles has total momentum

$$\vec{P} = \sum_i m_i \cdot \vec{v}_i$$

- Internal forces cancel in equal and opposite pairs (Newton's third law)
- Therefore internal forces **cannot** change the total momentum
- Only **external forces** can change $\vec{P}$:

$$\vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt}$$

- Internal interactions may redistribute momentum among particles but leave the total unchanged

Momentum Conservation for an Isolated System

- Summary: If the net external force is zero:

$$\vec{F}_{\text{ext}} = 0 \;\rightarrow\; \frac{d\vec{P}}{dt} = 0 \;\rightarrow\; \vec{P} = \text{constant}$$

> ***Law of Conservation of Linear Momentum:***
> *Total momentum of an isolated system remains constant when no external forces act*

Momentum Conservation in Collisions

`mi06 - Ballistisches Pendel`

- Collisions involve very large **internal forces** acting for a **very short time**
- During this interval internal forces dominate and external forces are negligible $\rightarrow$ system can be treated as **isolated**
- Momentum remains **conserved** even if kinetic energy is changed into deformation, sound, or heat

$$\vec{P}_{\text{before}} = \vec{P}_{\text{after}}$$

$$\sum_i m_i \cdot \vec{v}_{i,\text{before}} = \sum_i m_i \cdot \vec{v}_{i,\text{after}}$$

Lost in translation:

Disclaimer

- *Momentum* [en] $\rightarrow$ *Impuls* [de]
- *Impulse* [en] $\rightarrow$ *Kraftstoß* [de]

Impulse: Idea & Dropping an Egg

- Stopping an object means changing its momentum.
- Floor vs gentle catch → **same momentum change**, but different stopping time length.
- Short time interval → large force → egg breaks.
- Longer time interval → smaller force.
- Impulse is the same; only the **force–time profile** differs.

Defining Impulse

- Force acting over time produces **impulse**

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}_{\text{net}} \, dt$$

- Impulse is the **area under the force–time curve**
- For a **constant force**, this reduces to

$$J = F_{\text{net}} \, \Delta t$$

- Larger Δt → smaller average force for the same momentum change.

Impulse–Momentum Theorem

- From Newton's second law:

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}_{\mathrm{net}}\, dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\, dt = \Delta\vec{p}$$

- **Impulse equals change in momentum**.
- Holds even when forces are large, brief, or time-dependent.
- During short impacts, other forces (e.g., gravity) are negligible.

Momentum Exchange & Newton's Third Law

`mi09 - Impulsübertragung Bälle`

- Interaction forces satisfy

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

- Integrating over the interaction time gives **equal and opposite impulses**:

$$\vec{J}_{AB} = -\vec{J}_{BA}$$

- Impulse gives momentum change:

$$\Delta\vec{p}_A = \vec{J}_{BA}, \qquad \Delta\vec{p}_B = \vec{J}_{AB}$$

- Therefore

$$\Delta\vec{p}_A = -\Delta\vec{p}_B$$

- One object’s momentum gain equals the other’s loss → **total momentum is conserved**.

Key Points So Far

- Momentum describes motion with inertia: $\vec{p} = m\vec{v}$.
- Only **external forces** change total momentum; internal forces cancel.
- Zero net external force → **momentum conservation**.
- Impulse links force and time: $\vec{J} = \Delta\vec{p}$.
- Interaction forces are equal and opposite → momentum is exchanged but total momentum stays constant.
- Short, intense interactions (collisions) are well described by impulse; longer stopping time reduces the average force.

Collisions: Momentum & Energy

`mi06 - Ballistisches Pendel`

- Short interaction → external forces negligible → isolated system.
- **Momentum is always conserved**:

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2'$$

- Collision type depends on **kinetic energy**:
    - Stays the same → elastic
    - Partly converted (heat, sound, deformation) → inelastic

Elastic Collision: Setup & Conservation

`mi01 - Elastischer Stoß`

- Two masses on a line: initial velocities v_1, v_2 → final velocities v_1', v_2'.
- **Momentum conserved**:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

- **Kinetic energy conserved**:

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 {v_1'}^2 + \frac{1}{2} m_2 {v_2'}^2$$

- These two conditions fully determine v_1' and v_2'.

Elastic Collision: Key Result

- Combining momentum & energy conservation yields a simple relation:

$$v_1 + v_1' = v_2 + v_2'$$

- Equivalent form:

$$v_1 - v_2 = -(v_1' - v_2')$$

- Interpretation:
  Relative speed of approach = relative speed of separation (direction reversed).

Inelastic Collisions: Basics

`mi02 - Unelastischer Stoß`

- **Momentum conserved**, but **kinetic energy decreases**:

$$K' < K$$

- Energy is transformed into heat, sound, deformation, internal energy.
- Momentum conservation:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

- Collision outcomes depend on how much kinetic energy is lost.

Perfectly Inelastic Collision

- Objects **stick together** → move with common final velocity v'.
- From momentum conservation:

$$v' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

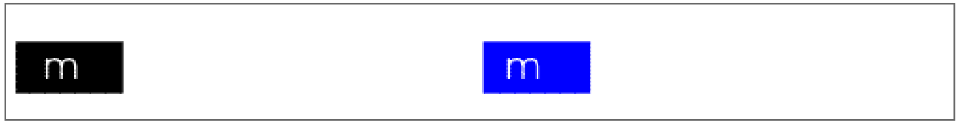
- Final kinetic energy:

$$K' = \tfrac{1}{2}(m_1 + m_2){v'}^2$$

- Energy loss:

$$\Delta K = -\tfrac{1}{2}\,\frac{m_1 m_2}{m_1 + m_2}\,(v_1 - v_2)^2$$

Seeing is believing (1):



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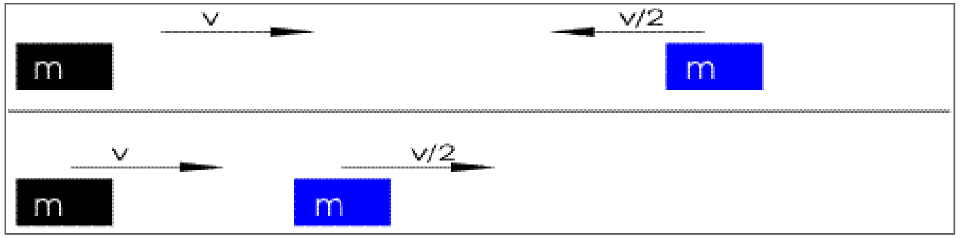
Seeing is believing (2):



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Seeing is believing (3):

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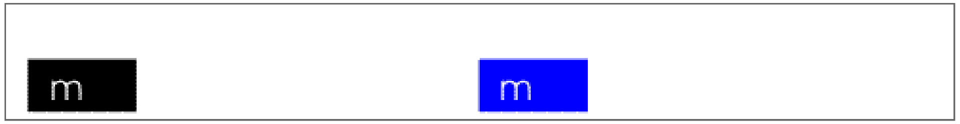
Seeing is believing (4):



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Elastic vs. Inelastic Collisions

Property	**Elastic Collision**	**Inelastic Collision**
Momentum conserved?	Yes	Yes
Kinetic energy conserved?	Yes	No
Energy loss to heat/sound/deformation?	No	Yes
Special case	–	Perfectly inelastic: objects stick together
Typical examples	Billiard balls	Clay balls, car crashes

Collisions in Higher Dimensions

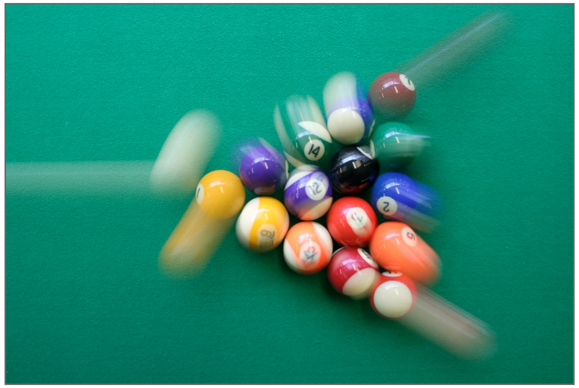
`sim 2D collision`

- Momentum is a **vector** $\vec{p}_{\text{before}} = \vec{p}_{\text{after}} \Rightarrow$ **conservation applied component-wise**, e.g. for 2D:

$$\begin{cases} m_1 v_{1,x} + m_2 v_{2,x} = m_1 v'_{1,x} + m_2 v'_{2,x} \\ m_1 v_{1,y} + m_2 v_{2,y} = m_1 v'_{1,y} + m_2 v'_{2,y} \end{cases}$$

- For **elastic** collisions, add energy condition (one equation only because scalar):

$$K_{\text{before}} = K_{\text{after}}$$



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Center of Mass (CM): Definition

- CM = mass-weighted average position of all particles:

$$\vec{r}_{\mathrm{CM}} = \frac{1}{M}\sum_i m_i \vec{r}_i, \qquad M = \sum_i m_i$$

- Component form (2D/3D):

$$x_{\mathrm{CM}} = \tfrac{1}{M}\sum_i m_i x_i, \quad y_{\mathrm{CM}} = \tfrac{1}{M}\sum_i m_i y_i, \quad z_{\mathrm{CM}} = \tfrac{1}{M}\sum_i m_i z_i$$

- For continuous bodies:

$$\vec{r}_{\mathrm{CM}} = \tfrac{1}{M}\int \vec{r}\, dm$$

- CM can lie **outside** the physical object.

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Center of Mass (CM): Implications

`mk27 - Wurfschnitzel`

- Total momentum:

$$\vec{P} = \sum_i p_i = \sum_i m_i \vec{v}_i = M \vec{v}_{\mathrm{CM}}$$

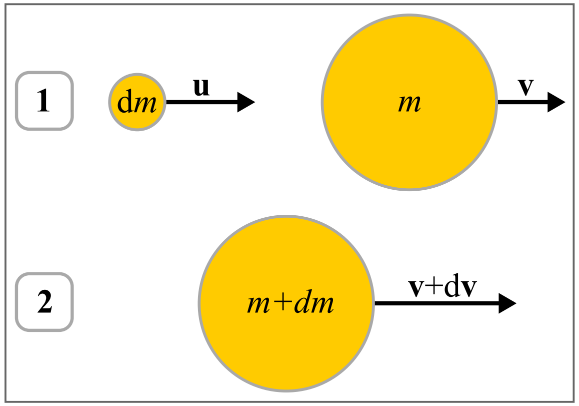
- CM acceleration obeys:

$$\frac{d\vec{P}}{dt} = M \vec{a}_{\mathrm{CM}} = \vec{F}_{\mathrm{ext}}$$

- Internal forces cancel → **only external forces** affect CM motion.
- The system **translates like a single particle of mass** M located at the CM.

Variable Mass Systems: Core Idea

- In some systems, mass **changes over time** (rockets, leaking carts, accreting snowballs).
- During a small interval dt, the system may **lose or gain a small mass** dm.
- To apply momentum conservation, we must include the momentum carried by this exchanged mass.
- Treat **system + exchanged mass** as an isolated system → total momentum is conserved at each step.



Momentum Conservation for Rocket

- In a short time dt, rocket mass changes by $dm < 0$, velocity by dv.
- Momentum conservation for rocket + exhaust (u is exhaust speed):

$$m\,dv = -u\,dm$$

- Divide by dt:

$$m\,\frac{dv}{dt} = u\left(-\frac{dm}{dt}\right)$$

- Mass ejection rate $R = -\frac{dm}{dt} > 0$ → **thrust**

$$F_{\text{thrust}} = uR, \quad m\,\frac{dv}{dt} = F_{\text{thrust}}$$

Tsiolkovsky Rocket Equation

`mi10 - Raketenschuss`

- Integrate the differential relation:

$$m\,dv = -u\,dm$$

- For constant exhaust speed u:

$$v = v_0 + u\,\ln\left(\frac{m_0}{m}\right)$$

- Key insight: **velocity gain grows with exhaust speed and mass ratio**.

Bye

`mi18 - Raketenauto`

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