

1.5. Linear momentum & collisions



In the previous chapters we examined how forces change motion and energy.

We now turn to **linear momentum**, a fundamental quantity that ties force to motion in a simple and powerful way.

Momentum is central to understanding collisions and **conservation laws**, and it also leads naturally to the concepts of **center of mass motion** and **propulsion** in systems with changing mass such as rockets.

1.5.1. Momentum

The **linear momentum** \vec{p} of an object is defined as its mass m multiplied by its velocity \vec{v} :

$$\vec{p} = m \cdot \vec{v}$$

Momentum is a **vector quantity** that points in the same direction as the velocity and is measured in $\text{kg} \cdot \frac{\text{m}}{\text{s}}$ or $\text{N} \cdot \text{s}$ (reminder: $\text{kg} \frac{\text{m}}{\text{s}} = \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \text{s} = \text{N} \cdot \text{s}$).

Mass measures an object's **inertia**, and velocity specifies an object's motion. Momentum is powerful because it unifies these two ideas into a single quantity that captures an object's resistance to changes in its motion.

1.5.2. Revisiting Newton's Second Law

Newton's **second law of motion** states:

At any instant of time, the net force on a body equals its mass times its acceleration or, equivalently, the rate at which its momentum changes with time.

Thus, we can re-write Newton's second law (for constant mass) as

$$\vec{F}_{\text{net}} = m\vec{a} = m \frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt}.$$

Thus, the **net force acting on an object is equal to the rate of change of its momentum**. Further, the re-written second law is helpful to study systems with changing mass (such as rockets).

1.5.3. Conservation of Momentum

We defined momentum as $\vec{p} = m\vec{v}$ and showed that the net force equals the rate of change of momentum, $\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}$.

Thus, **if the net external force on a system is zero, its total momentum remains constant**. This is the **law of conservation of momentum**, a universal principle that applies to any isolated system, from colliding billiard balls to interacting galaxies and even subatomic particles.

Particle Systems & Total Momentum

Let's assume we have a system with many particles. Each particle has its own mass m_i and velocity \vec{v}_i . We can compute the **total momentum** as the vector sum of all individual momenta:

$$\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \cdot \vec{v}_i$$

Applying Newton's second law gives

$$\vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt}$$

which states that the **net external force** determines how the system's total momentum changes. It is sensible that no internal forces between the particles occur as, according to Newton's third law, the internal forces always occur in equal and opposite pairs along the same line, so their vector sum is zero.

Thus, the system's total momentum can change **only** through external influences. Internal interactions may redistribute momentum among the particles, but they **cannot** change the total \vec{P} of the system.

Momentum Conservation for an Isolated System

If the net external force is zero,

$$\vec{F}_{\text{ext}} = 0$$

then

$$\frac{d\vec{P}}{dt} = 0$$

and therefore the total momentum is constant:

$$\vec{P} = \text{constant}$$

Law of Conservation of Linear Momentum:

The total momentum of an isolated system remains constant in both magnitude and direction when no external forces act on it.

This holds regardless of the nature or strength of **internal** interactions.

Momentum Conservation in Collisions

A collision is an interaction where large **internal forces** act over a **very short time**.

During this short interval:

- internal forces dominate
- external forces (friction, gravity) have negligible effect on total momentum

Thus the system can be treated as isolated, and the total momentum before and after the collision is the same.

Even if kinetic energy is transformed into deformation, sound, or heat, **momentum remains conserved as long as external forces can be neglected**.

Symbolically, for any two interacting bodies in an isolated system:

$$\vec{P}_{\text{before}} = \vec{P}_{\text{after}}$$
$$\sum_i m_i \cdot \vec{v}_{i,\text{before}} = \sum_i m_i \cdot \vec{v}_{i,\text{after}}$$

This vector equation applies separately to each component (x, y, z) and forms the basis for analyzing collisions, explosions, and short-time interactions.

1.5.4. Impulse, Momentum, & Impulsive Forces

When a force acts over a finite time interval, Newton's second law integrated over time defines the **impulse**

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}_{\text{net}} dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt} dt = \Delta\vec{p}$$

which equals the resulting change in momentum. This is often called the **impulse–momentum theorem** $\vec{J} = \Delta\vec{p}$.

Impulse captures the **total effect** of a force acting over time. This becomes especially important when forces act briefly and vary rapidly, as in collisions or impacts.

A helpful way to build intuition is to imagine stopping a fragile object such as an egg. Whether the egg hits the floor or is caught gently, the **momentum change is the same**. What differs is the **time interval** over which that change happens. Stopping the egg abruptly in a very short time produces a large force, likely breaking it. Letting your hands move backward while catching the egg increases the stopping time and dramatically reduces the force. The impulse is the same in both cases, but the **force–time profile** is different.

Impulse can be visualized using a **force–time diagram**: the **area under the curve** represents the magnitude of the impulse J and therefore the change in momentum. A corresponding momentum–time

graph would have a **slope equal to the force**, showing directly how force accumulates impulse over time.

For a **constant force**, the impulse is

$$J = F_{\text{net}} \Delta t.$$

Thus, to maintain the same impulse, increasing Δt results in decreasing F_{avg} and therefore reduces the stress on the object.

In one dimension, we can further simplify in case of a constant force

$$F_{\text{net}} = \frac{dp}{dt} = \frac{\Delta p}{\Delta t} \leftrightarrow F_{\text{net}} \Delta t = \Delta p$$

Real collisions often involve **impulsive forces**: extremely large forces acting over extremely short times. Even though their precise time dependence is difficult to measure, the momentum change and therefore the (total) impulse remains well defined (see impulse–momentum theorem $\vec{J} = \Delta\vec{p}$). During such brief intervals, forces like gravity are negligible compared to the impulsive contact force. Impulse thus provides a powerful way to understand short, intense interactions, linking how forces act over time to how momentum changes.

When two bodies interact (such as a bat hitting a ball), the forces they exert on each other are **equal in magnitude and opposite in direction**, according to Newton's third law:

$$\vec{F}_{AB} = -\vec{F}_{BA}.$$

Integrating both sides over the interaction time shows that the **impulses** are also equal and opposite:

$$\vec{J}_{AB} = \int_{t_1}^{t_2} \vec{F}_{AB} dt = - \int_{t_1}^{t_2} \vec{F}_{BA} dt = -\vec{J}_{BA}.$$

Because impulse equals change in momentum, this directly gives

$$\Delta\vec{p}_A = \vec{J}_{BA}, \quad \Delta\vec{p}_B = \vec{J}_{AB},$$

and therefore

$$\Delta\vec{p}_A = -\Delta\vec{p}_B.$$

This shows that during any short, forceful interaction, one object's gain in momentum is exactly the other object's loss.

The **total momentum of the two-body system remains constant** throughout the interaction — a direct consequence of Newton's third law and another way to see the **conservation of momentum**.

Key points for momentum, its conservation, & impulse

- Momentum measures an object's "inertia in motion": $\vec{p} = m\vec{v}$.
- Newton's second law in momentum form links **net force** to **momentum change**.

- For a system of particles, the **total momentum** is $\vec{P} = \sum_i m_i \cdot \vec{v}_i$.
- Internal forces cancel in pairs (Newton's third law), so only **external forces** can change \vec{P} .
- If the net external force is zero, an isolated system has **constant total momentum**.
- Collisions can be treated as isolated because internal forces dominate and external forces act too weakly and too briefly to matter.
- **Impulse** gives the total effect of a force over time: $\vec{J} = \Delta\vec{p}$.
- Interaction forces satisfy $\vec{F}_{AB} = -\vec{F}_{BA} \rightarrow$ impulses are equal and opposite \rightarrow momentum exchanged but total momentum unchanged.
- Impulsive forces act briefly but strongly; momentum change is well defined even without knowing the force–time curve.
- Longer interaction time \rightarrow smaller average force for the same momentum change.

1.5.5. Elastic & Inelastic Collisions

We have developed the impulse–momentum theorem, i.e. $J = \Delta p$, and the law of momentum conservation, i.e. $\vec{P}_{\text{before}} = \vec{P}_{\text{after}}$, and we now extend this framework by including **energy considerations** to analyze how different kinds of collisions behave.

A collision is a short interaction between two bodies 1 and 2. If, during this brief interval, external forces (such as friction or gravity) are negligible compared to the forces the bodies exert on each other, the pair can be treated as an **isolated system**. In that case, the total momentum is conserved:

$$\vec{P}_{\text{before}} = \vec{P}_{\text{after}}$$

$$\vec{p}_1 + \vec{p}_2 = \vec{p}'_1 + \vec{p}'_2$$

(primes denote quantities after the collision).

While the **total momentum is always conserved in a collision**, there are **different types of collisions** which can be classified as follows:

- **Elastic Collision:**

- Total momentum is conserved.
- Total kinetic energy is also conserved:

$$K_{\text{before}} = K_{\text{after}}.$$

- Examples: billiard balls, gas molecule collisions, steel ball bearings.

- **Inelastic Collision:**

- Total momentum is conserved.
- Some kinetic energy is transformed into internal energy, sound, heat, or deformation:

$$K_{\text{after}} < K_{\text{before}}.$$

- If colliding objects **stick together** after impact, i.e. moving as one combined mass, we call it a **completely (perfectly) inelastic collision**
- Example: clay balls sticking together, car crashes.

Elastic Collision in One Dimension

Consider a head-on collision of two objects with masses m_1 and m_2 moving along the x-axis. Their initial velocities are v_1 and v_2 , and their final velocities after the collision are v'_1 and v'_2 .

For an **elastic collision**, both **momentum** and **kinetic energy** are conserved:

Momentum:

$$m_1 \cdot v_1 + m_2 \cdot v_2 = m_1 \cdot v'_1 + m_2 \cdot v'_2$$

Kinetic energy:

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v'^2_1 + \frac{1}{2}m_2v'^2_2$$

We can combine these two equations to obtain a simple and very useful relation.

1. Start from momentum conservation and rearrange:

$$m_1(v_1 - v'_1) = m_2(v'_2 - v_2)$$

2. Start from kinetic energy conservation, multiply both sides by 2, and rearrange:

$$m_1(v_1^2 - v'^2_1) = m_2(v'^2_2 - v_2^2)$$

3. Use the difference of squares, $a^2 - b^2 = (a - b)(a + b)$:

$$m_1(v_1 - v'_1)(v_1 + v'_1) = m_2(v'_2 - v_2)(v'_2 + v_2)$$

4. Now divide this equation by the rearranged momentum equation from step 1. The factors with m_1 , m_2 and $(v_1 - v'_1)$, $(v'_2 - v_2)$ cancel, leaving

$$v_1 + v'_1 = v_2 + v'_2$$

This result can be rewritten as

$$v_1 - v_2 = -(v'_1 - v'_2)$$

which says that in a one-dimensional elastic collision, the **relative speed of approach equals the relative speed of separation**, but with **reversed direction**. In other words, the **magnitude** of the relative velocity between the two objects stays the same, while its **sign changes** during the collision.

Inelastic Collision in One Dimension

In an **inelastic collision**, total momentum is conserved but kinetic energy is **not**. Some of the initial kinetic energy is transformed into internal energy, heat, sound, or permanent deformation.

Momentum conservation:

$$m_1 \cdot v_1 + m_2 \cdot v_2 = m_1 \cdot v'_1 + m_2 \cdot v'_2$$

Kinetic energy:

$$K' < K$$

A particularly important special case is the **completely (perfectly) inelastic collision**, where the two objects stick together and move with a **common final velocity** (v').

Using momentum conservation:

$$m_1 \cdot v_1 + m_2 \cdot v_2 = (m_1 + m_2) \cdot v'$$

so that

$$v' = \frac{m_1 \cdot v_1 + m_2 \cdot v_2}{m_1 + m_2}$$

To compute the kinetic energy transformed, we need to consider the following: The initial kinetic energy is:

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

The final kinetic energy is:

$$K' = \frac{1}{2} (m_1 + m_2) v'^2$$

Insert the expression for v' :

$$K' = \frac{1}{2} (m_1 + m_2) \left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \right)^2$$

Cancel the denominator:

$$K' = \frac{1}{2} \frac{(m_1 v_1 + m_2 v_2)^2}{m_1 + m_2}$$

The change in kinetic energy is

$$\Delta K = K' - K$$

Substitute the expressions for K' and K :

$$\Delta K = \frac{1}{2} \frac{(m_1 v_1 + m_2 v_2)^2}{m_1 + m_2} - \left[\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \right]$$

Expand the square in the numerator and collect terms carefully:

$$\Delta K = \frac{1}{2} \frac{m_1^2 v_1^2 + 2m_1 m_2 v_1 v_2 + m_2^2 v_2^2}{m_1 + m_2} - \frac{1}{2} (m_1 v_1^2 + m_2 v_2^2)$$

Use common denominator:

$$\Delta K = \frac{1}{2} \frac{m_1^2 v_1^2 + 2m_1 m_2 v_1 v_2 + m_2^2 v_2^2 - (m_1 + m_2)(m_1 v_1^2 + m_2 v_2^2)}{m_1 + m_2}$$

Simplify to obtain final result:

$$\Delta K = -\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2)^2$$

The negative sign indicates kinetic energy is lost; the “missing” energy has been converted into other forms.

A perfectly inelastic collision therefore produces the **maximum possible loss of kinetic energy** consistent with momentum conservation.

Collisions in Higher Dimensions

Most real collisions occur in **two or three dimensions**. Momentum is a **vector**, so momentum conservation must be applied **component by component**.

For two objects with masses m_1 and m_2 , initial velocities \vec{v}_1 and \vec{v}_2 , and final velocities \vec{v}'_1 and \vec{v}'_2 , momentum conservation reads

$$m_1 \cdot \vec{v}_1 + m_2 \cdot \vec{v}_2 = m_1 \cdot \vec{v}'_1 + m_2 \cdot \vec{v}'_2.$$

In two dimensions, this corresponds to two independent scalar equations:

$$\begin{cases} m_1 \cdot v_{1,x} + m_2 \cdot v_{2,x} = m_1 \cdot v'_{1,x} + m_2 \cdot v'_{2,x} \\ m_1 \cdot v_{1,y} + m_2 \cdot v_{2,y} = m_1 \cdot v'_{1,y} + m_2 \cdot v'_{2,y} \end{cases}$$

In three dimensions, a third equation (the z-component) is added.

If the collision is **elastic**, kinetic energy conservation adds one additional scalar constraint:

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2.$$

Because kinetic energy is a **scalar**, it contributes only **one equation**, independent of the number of spatial dimensions. Momentum conservation is what splits into multiple component equations.

To analyze such collisions, one typically:

- chooses axes so that one axis aligns with the initial direction of motion of one object
- applies momentum conservation separately to each component

- adds kinetic energy conservation only for elastic collisions
- uses the velocity components to determine angles (or vice versa)

```
VBox(children=(HBox(children=(FloatSlider(value=1.0, description='v1 (m/s)', max=5.0), FloatSlider(value=1.0, ...
Output()
```

1.5.6. Center of Mass

In our momentum-based description of interactions, we have so far treated each object as if it were a single point mass. Real bodies, however, consist of many particles distributed in space. To describe the translational motion of such systems in a simple and unified way, we introduce the **center of mass (CM)** — the effective point at which the total mass of a system can be considered to act.

Definition of the Center of Mass

For a system of particles with masses m_i and position vectors \vec{r}_i , the **center-of-mass position** \vec{r}_{CM} is the mass-weighted average of all positions:

$$\vec{r}_{\text{CM}} = \frac{1}{M} \sum_i m_i \vec{r}_i$$

with total mass:

$$M = \sum_i m_i.$$

For a continuous mass distribution, the sum becomes an integral:

$$\vec{r}_{\text{CM}} = \frac{1}{M} \int \vec{r} \, dm.$$

In Cartesian coordinates this yields for discrete masses:

$$x_{\text{CM}} = \frac{1}{M} \sum_i m_i x_i, \quad y_{\text{CM}} = \frac{1}{M} \sum_i m_i y_i, \quad z_{\text{CM}} = \frac{1}{M} \sum_i m_i z_i.$$

For continuous mass distributions, the Cartesian components are defined as:

$$x_{\text{CM}} = \frac{1}{M} \int x \, dm, \quad y_{\text{CM}} = \frac{1}{M} \int y \, dm, \quad z_{\text{CM}} = \frac{1}{M} \int z \, dm.$$

Note:

The center of mass does **not** need to lie inside the material of the object.

For example, an L-shaped body or a hollow cylinder can have their center of mass located *outside* the object itself.

Properties of the Center of Mass

To understand why the CM behaves like a single particle, we relate its motion to the system's total momentum (\vec{P}). The **total momentum** is defined as:

$$\vec{P} = \sum_i m_i \vec{v}_i$$

The **center-of-mass velocity** is defined as:

$$\vec{v}_{\text{CM}} = \frac{1}{M} \sum_i m_i \vec{v}_i$$

Combining these definitions yields the key identity: **the system's total momentum is determined by the motion of its CM.**

$$\vec{P} = M \vec{v}_{\text{CM}}$$

Differentiating \vec{P} with respect to time gives the rate of change of momentum, which is $\frac{d\vec{P}}{dt} = M \vec{a}_{\text{CM}}$.

Applying Newton's second law ($\frac{d\vec{P}}{dt} = \vec{F}_{\text{net}}$): The total force on the system is

$$\vec{F}_{\text{net}} = \sum_i \vec{F}_{i,\text{ext}} + \sum_i \vec{F}_{i,\text{int}}.$$

According to **Newton's third law**, the sum of all internal forces vanishes ($\sum_i \vec{F}_{i,\text{int}} = 0$).

Thus, the net force is only the **net external force**: $\vec{F}_{\text{net}} = \vec{F}_{\text{ext}}$.

Substituting this back into the second law yields the **equation of motion for the center of mass**:

$$M \vec{a}_{\text{CM}} = \vec{F}_{\text{ext}}$$

To conclude:

1. **Internal interactions cannot change the CM motion.** They may redistribute momentum among particles, but they do not affect the total \vec{P} or \vec{v}_{CM} .
2. The **CM moves exactly like a single particle of mass M** acted upon by the same net external force \vec{F}_{ext} .

Thus, any extended system **translates as if all its mass were concentrated at the CM** and acted on by the net external force alone.

1.5.7. Variable Mass & Propulsion

So far we have considered systems with constant total mass. Many important systems have **changing mass**, for example rockets (burning and expelling fuel), leaking carts, or snowballs that pick up mass. To describe their motion, we must apply **momentum conservation** carefully to systems where mass flows in or out.

We focus on the classic example of a rocket that expels mass as exhaust.

Momentum balance for a rocket in free space

Consider a rocket moving in one dimension along the x -axis. We describe everything in an inertial **lab frame** \mathcal{S} (for example, an observer far away from the rocket, in deep space). In this frame:

- The rocket has mass m and velocity v (both measured in the lab frame \mathcal{S}).
- Exhaust gases are ejected **backwards** with speed u **relative to the rocket**.

During a short time interval dt the rocket ejects a small mass dm (with $dm < 0$ since the rocket **loses** mass). Here dm is the change in the rocket's mass; the ejected gas has mass $-dm > 0$.

After this time the rocket's velocity has changed from v to v' (again, measured in the lab frame).

We now write down the total momentum of the rocket plus the ejected gas before and after the short time interval and then use **momentum conservation** (no external forces in the lab frame).

Before:

All mass is still in the rocket and moves with velocity v in the lab frame, so the total momentum is

$$p = m \cdot v.$$

After:

- The rocket has mass $m + dm$ and velocity v' in the lab frame.
- The ejected gas has mass $-dm$ and some velocity v_{gas} in the lab frame.

To find v_{gas} , we use **Galilean velocity addition**. In the rocket's instantaneous rest frame, the exhaust moves backward with speed u , i.e. its velocity relative to the rocket is

$$v_{\text{gas, rel to rocket}} = -u,$$

if we choose the positive x -direction along the rocket's forward motion.

In the lab frame \mathcal{S} , the rocket itself moves with velocity v' , so the gas velocity is

$$v_{\text{gas}} = v' + v_{\text{gas, rel to rocket}} = v' + (-u) = v' - u.$$

Thus the total momentum after the interval is

$$p' = (m + dm) \cdot v' + (-dm) \cdot (v' - u).$$

Expanding:

$$p' = (m + dm) \cdot v' - dm \cdot v' + dm \cdot u = m \cdot v' + dm \cdot u,$$

since $(m + dm) \cdot v' - dm \cdot v' = m \cdot v'$.

Momentum conservation ($p' = p$) gives

$$m \cdot v' + dm \cdot u = m \cdot v.$$

Rearrange:

$$m \cdot (v' - v) = -u \cdot dm.$$

Define the small change in velocity $dv = v' - v$, then

$$m \cdot dv = -u \cdot dm.$$

This is the **differential rocket equation** in free space.

It shows that the rocket's velocity increases by dv in such a way that the forward momentum gain $m \cdot dv$ equals the backward momentum carried away in the exhaust, $-u \cdot dm$.

Dividing by dt :

$$m \cdot \frac{dv}{dt} = -u \cdot \frac{dm}{dt}.$$

The term $-\frac{dm}{dt} > 0$ is the **mass ejection rate**. Let us define

$$R = -\frac{dm}{dt} \quad (> 0),$$

then the **thrust** is

$$F_{\text{thrust}} = u R = u \left(-\frac{dm}{dt} \right),$$

and the rocket equation can be written as

$$m \frac{dv}{dt} = F_{\text{thrust}}.$$

The rocket equation (Tsiolkovsky equation)

To find how the rocket's velocity changes as its mass decreases from an initial value m_0 to a later value m , we integrate the differential equation

$$m dv = -u dm.$$

Separate variables:

$$dv = -u \frac{dm}{m}.$$

Integrate from initial state (v_0, m_0) to final state (v, m) :

$$\int_{v_0}^v dv = -u \int_{m_0}^m \frac{1}{m} dm.$$

Carry out the integrals:

$$v - v_0 = -u [\ln m]_{m_0}^m = -u (\ln m - \ln m_0) = u \ln \left(\frac{m_0}{m} \right).$$

Thus, in free space (no external forces) and for constant exhaust speed u ,

$$v = v_0 + u \ln \left(\frac{m_0}{m} \right).$$

This is the **Tsiolkovsky rocket equation**. It relates the rocket's velocity change to the ratio of initial and final mass and to the exhaust speed.

Conceptual summary

- The rocket equation follows directly from **momentum conservation in an inertial lab frame** applied to the combined system "rocket + exhaust".
- A rocket accelerates by expelling mass: the rocket's forward momentum increase equals the backward momentum carried away by the exhaust.
- The **thrust** is the rate at which momentum is carried away in the exhaust stream. In one dimension, with exhaust speed u (relative to the rocket),

$$F_{\text{thrust}} = u \left(-\frac{dm}{dt} \right),$$

where $-\frac{dm}{dt} > 0$ is the positive mass ejection rate.

- Achieving a large change in speed requires a large **mass ratio** m_0/m ; this is the essence of the Tsiolkovsky rocket equation

$$\Delta v = u \ln \left(\frac{m_0}{m} \right).$$