

1.6. Rotational dynamics & angular momentum

mk15

Which egg is cooked? How to check?



Mass vs. mass distribution?

mk14 - Abrollen versch. Körper

Which body is the fastest & why?

- Study rotational motion
- Introduce **torque, moment of inertia, & angular momentum**
- New concepts of **rotational kinetic energy & angular momentum conservation**

Recap & Primer Rotational Motion

- The position on the circle is described by the **angular displacement** $\phi(t)$ (in radians) measured from the $+x$ -axis
- **Angular velocity** ω is measured in **rad/s** and **angular acceleration** α in **rad/s²**:

$$\omega = \frac{d\phi}{dt}, \quad \alpha = \frac{d\omega}{dt} = \frac{d^2\phi}{dt^2}$$

- Linear quantities from angular ones:

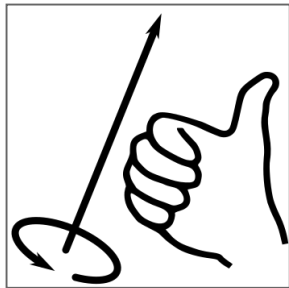
$$s = r\phi, \quad v = r\omega, \quad a_t = r\alpha, \quad a_r = \frac{v}{r} = r\omega^2$$

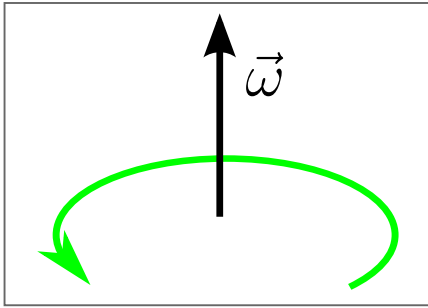
- **Key idea:** Angular motion is identical for all points, linear motion scales with radius r .

Angular motion with vectors

Question: Linear quantities are **vectors**? What about angular ones?

- $\vec{s}, \vec{v}, \vec{a}_t$ all tangential
- position vector \vec{r} and \vec{a}_r radially
- $\Rightarrow \vec{\phi}, \vec{\omega}, \vec{\alpha}$ oriented along rotation axis \Leftrightarrow they **define the axis of rotation**
- **direction** (positive vs. negative) determined by **Right-Hand Rule** (a.k.a. convention)





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Angular motion with vectors (cont')

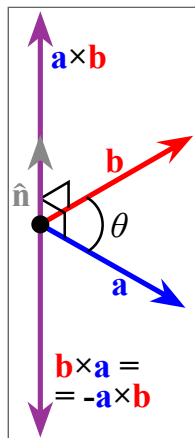
Cross product relates angular and linear quantities, e.g.

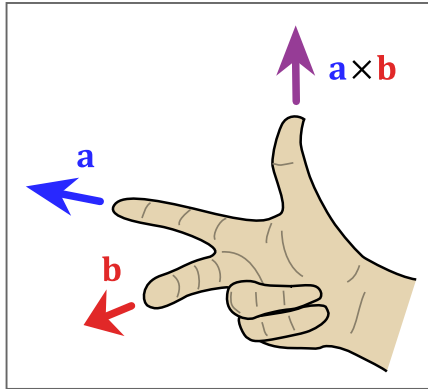
$$\vec{v} = \vec{\omega} \times \vec{r}$$

Key properties:

$$\vec{v} \perp (\vec{\omega} \times \vec{r})$$

$$\vec{\omega} \times \vec{r} = -\vec{r} \times \vec{\omega}$$



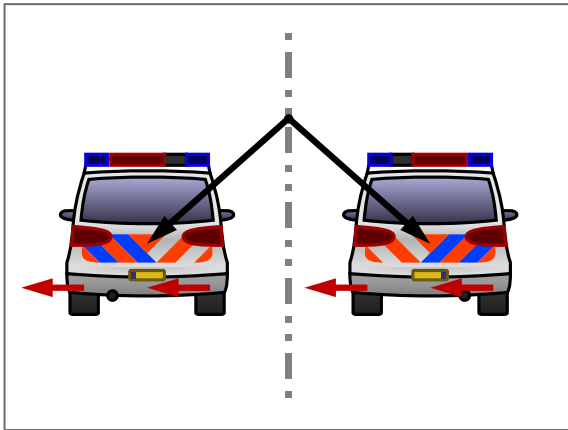


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Side note: Pseudovectors

- $\vec{\phi}, \vec{\omega}, \vec{\alpha}$ are pseudovectors, also called axial vectors
- Under reflection, pseudovectors **do not flip sign**
- Imagine doing the right-hand-rule in front of a mirror
- Pseudovectors are relevant e.g. for symmetry-based solutions in physics



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Revisit acceleration \vec{a} for circular motion

- To compute linear acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\vec{\omega} \times \vec{r})}{dt}$$

- Use **vector product rule**:

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

- Apply to $\vec{\omega}$ and \vec{r} :

$$\vec{a} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

Revisit acceleration \vec{a} for circular motion (cont')

Substitute $\frac{d\vec{\omega}}{dt} = \vec{\alpha}$, & $\frac{d\vec{r}}{dt} = \vec{v} = \vec{\omega} \times \vec{r}$

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{a}_t + \vec{a}_R$$

Tangential:

$$\vec{a}_t = \vec{\alpha} \times \vec{r}$$

Radial (centripetal):

$$\vec{a}_R = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

(note relation to magnitude of $a_R = \frac{v^2}{r} = \omega^2 r$)

Accelerated circular motion

mk23 - accelerated circular motion

What causes the angular acceleration?

Accelerated circular motion (cont')

mk02 - Drehmoment

What (else) causes the angular acceleration?

Torque

- Linear motion: force causes acceleration ($F = ma$)
- Rotational motion: effectiveness of a force depends on **magnitude**, **angle**, and **where** it is applied
- Torque:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

- Rotational analogue of force and measured in $\text{N} \cdot \text{m}$
- Direction:
 - Perpendicular to the plane of \vec{r} and \vec{F}
 - Set by **right-hand rule**
 - CCW \rightarrow out of page \odot , CW \rightarrow into page \otimes

Torque Magnitude

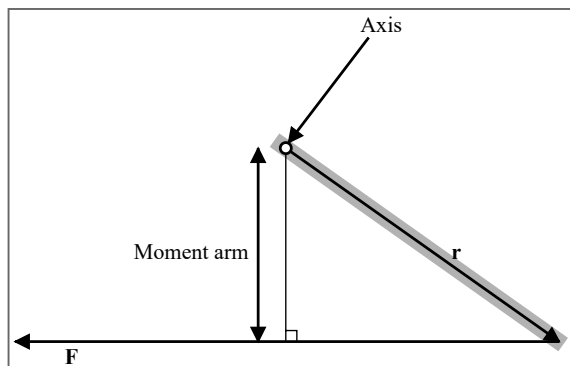
sim - Torque

- Torque magnitude:

$$\tau = rF \sin \phi = r_{\perp} F = r F_{\perp}$$

→ Maximal when $\vec{F} \perp \vec{r}$; zero when $\vec{F} \parallel \vec{r}$

- Lever arm (moment arm) $r_{\perp} = r \sin \phi \rightarrow$ only the **perpendicular** component of the force causes rotation.



from **wikipedia**, **CC0 1.0 Universal**

Connection to Newton's second law:

$$\vec{F} = m\vec{a}, \quad \vec{\tau} = \vec{r} \times \vec{F}$$

What component of \vec{a} is relevant for torque?

→ Tangential acceleration $a_t = r\alpha$

Because $\vec{a}_t \perp \vec{r}$, we obtain torque magnitude as:

$$\tau = rF_t = ra_tm = mr^2\alpha$$

Revisiting mass vs. mass distribution

mk14 - Abrollen versch. Körper

$$\tau = mr^2\alpha$$

\Rightarrow **mass distribution captured by $mr^2 \Rightarrow$ moment of inertia I**

Net torque & moment of inertia

- Moment of inertia I is the **rotational analogue of mass** \rightarrow measures **resistance to changes in rotational motion**.
- Larger $I \rightarrow$ smaller angular acceleration for the same torque
- For many particles in a rigid body:

$$\tau_{\text{net}} = \sum_i \tau_i = \sum_i m_i r_i^2 \alpha.$$

Since all particles share the same α , define

$$I = \sum_i m_i r_i^2, \quad \tau_{\text{net}} = I\alpha.$$

- Continuous mass distribution:

$$I = \int r^2 dm.$$

r = perpendicular distance from axis to element dm .

Net torque & moment of inertia (cont')

mk23 - torque vs. moment of inertia

$$I = \int r^2 dm, \quad \tau_{\text{net}} = I\alpha$$

- Key idea:
Mass farther from the axis contributes **much more** (because of r^2).
- Units of I : **kg·m²**.
- Rotational equilibrium:
If $\vec{\tau}_{\text{net}} = 0$, then $\alpha = 0 \rightarrow$ body at rest or rotating at constant ω .

Moment of inertia: Example

Uniform rod of length L , mass M , with linear mass density $\lambda = \frac{M}{L}$, rotating about its center:

- Mass element:

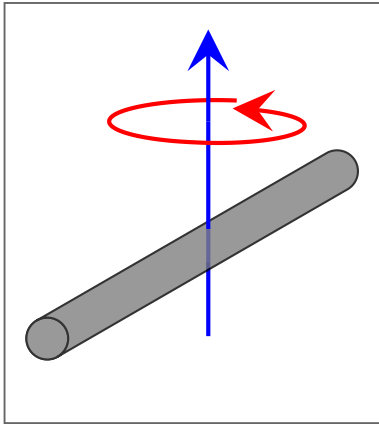
$$dm = \lambda dx = \frac{M}{L} dx.$$

- Compute I :

$$I = \frac{M}{L} \int_{-L/2}^{L/2} x^2 dx = \frac{1}{12} ML^2.$$

- Result for rod about its center:

$$I_{\text{center}} = \frac{1}{12} ML^2.$$



from [wikipedia](#), CC0 1.0 Universal

Moment of inertia: Example II

Uniform rod with L , M , rotating about one end:

$$I = \frac{M}{L} \int_0^L x^2 dx = \frac{1}{3}ML^2.$$

Interpretation: Moving the axis away from the center of mass increases I

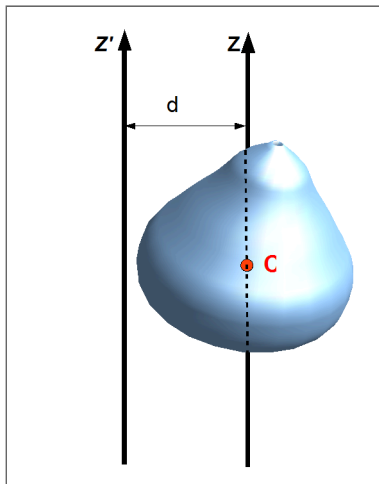
\Rightarrow **Moment of inertia depends on rotation axis**

Parallel-axis Theorem (Steiner's theorem)

- Used when the rotation axis is **shifted** from the center-of-mass (CM) axis.
- Theorem:

$$I = I_{\text{CM}} + Md^2,$$

with I_{CM} = moment of inertia about CM axis, d = distance between axes



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Moment of inertia: Revisit Example II

Uniform rod, length L , mass M , rotating about one end:

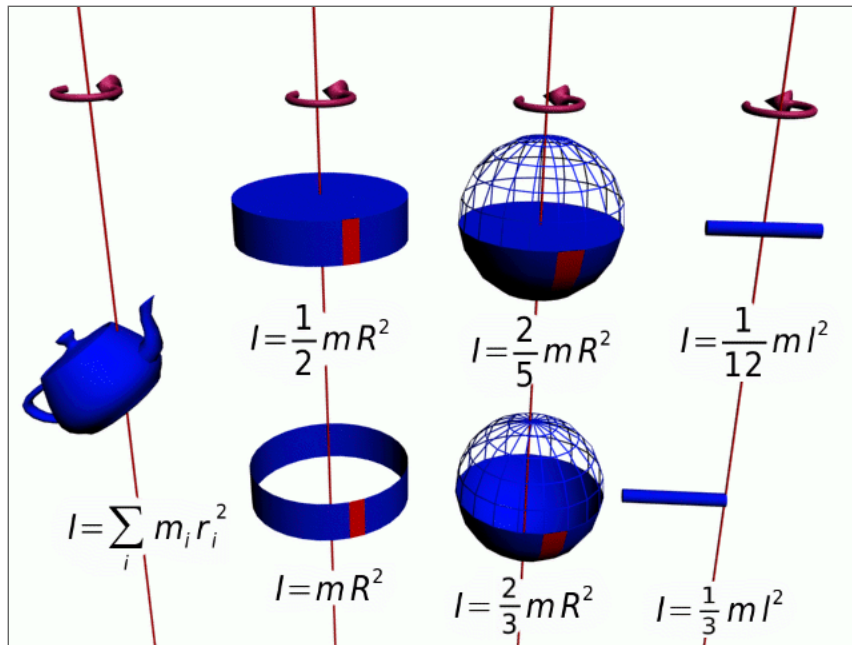
$$I_{\text{CM}} = \frac{1}{12}ML^2, \quad d = \frac{L}{2}.$$

Moment of inertia about end:

$$I_{\text{end}} = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2.$$

Practical note: Usually easier to start from a known **CM moment of inertia** and apply the **parallel-axis theorem** instead of re-integrating.

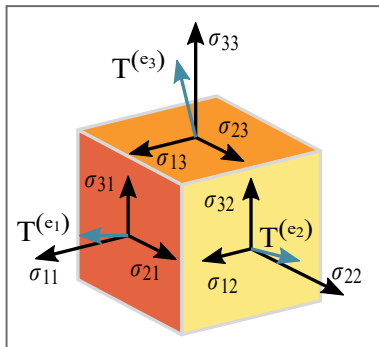
Common Moments of Inertia & the Tensor Perspective









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Moments of Inertia: Tensor Perspective

- Moment of inertia is an **inertia tensor** in 3D
- Different axes \rightarrow different values of I
- In this course: rotation about **one symmetry axis**, so I behaves like a scalar



Order-0 tensor: <i>Scalar</i>	Order-1 tensor: <i>Vector</i>	Order-2 tensor: <i>Matrix</i>	Order-3 tensor	Order-4 tensor	...
					
a	\mathbf{a}	\mathbf{A}	\mathbf{a}, \mathbf{A}	\mathbf{a}, \mathbf{A}	
1	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$	$\begin{bmatrix} 1, 9 & 3, 11 \\ 5, 13 & 7, 15 \end{bmatrix}$	
ALGEBRA					
VECTOR ALGEBRA					
MATRIX ALGEBRA					
TENSOR ALGEBRA					

*[left] **Stress tensor example** from wikipedia, Attribution-Share Alike 3.0 Unported; [right]*
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Rotational Kinetic Energy

- Rotating rigid body:
 - Treat body as many small masses m_i at distances r_i with velocity $v_i = r_i\omega$
 - Kinetic energy of each element:

$$K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2 \omega^2.$$

- Total rotational kinetic energy:

$$K_{\text{rot}} = \sum_i K_i = \frac{1}{2}\omega^2 \sum_i m_i r_i^2.$$

- Recognize the moment of inertia: $I = \sum_i m_i r_i^2$
- Final result:

$$K_{\text{rot}} = \frac{1}{2}I\omega^2$$

- Direct analogue of $K_{\text{trans}} = \frac{1}{2}mv^2$ with $m \rightarrow I$ and $v \rightarrow \omega$.

Work Done by a Torque: Rotational Work–Energy Theorem

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta.$$

- Using $\tau = I\alpha = I \frac{d\omega}{dt} = I \frac{d\omega}{d\phi} \frac{d\phi}{dt} = I \frac{d\omega}{d\phi} \omega$, the net work becomes:

$$W = \int_{\theta_i}^{\theta_f} I \frac{d\omega}{d\phi} \omega d\phi = I \int_{\theta_i}^{\theta_f} \omega d\omega$$

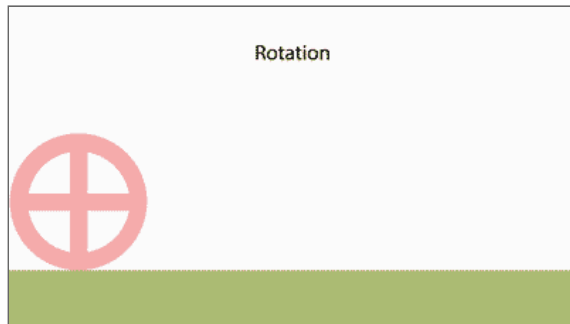
$$W_{\text{net}} = \Delta K_{\text{rot}} = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2.$$

Rolling Without Slipping

- Rolling = **translation of the center of mass** + **rotation about an axis**.
- No-slip condition: **contact point is instantaneously at rest** relative to the surface

$$v_{\text{CM}} - R\omega = 0 \leftrightarrow v_{\text{CM}} = R\omega \leftrightarrow a_{\text{CM}} = R\alpha$$

- Interpretation: one full revolution \rightarrow object advances by $2\pi R$.
- If $v_{\text{CM}} \neq R\omega \rightarrow$ **slipping**



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Flywheel: Storing rotational energy

stores rotational energy via conservation of angular momentum



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Angular momentum for a Particle

- **angular momentum** \vec{L} (linear analogue $\vec{p} = m\vec{v}$):

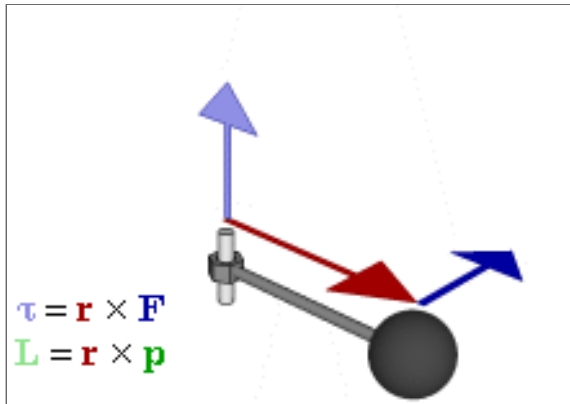
$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

- Angular momentum is a **vector** (axial/pseudovector) with direction from the **right-hand rule**
- Units: **kg·m²/s**
- Magnitude:

$$L = rp \sin \phi = rmv \sin \phi$$

- For particle in circular motion with $v = r\omega$ and $I = mr^2$:

$$L = mrv = mr(r\omega) = mr^2\omega = I\omega$$



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Angular momentum for a Particle (cont')

$$\vec{L} = \vec{r} \times \vec{p}$$

- Time derivative (product rule):

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

- Substitute $\frac{d\vec{r}}{dt} = \vec{v}$, $\vec{p} = m\vec{v}$, and $\vec{F} = \frac{d\vec{p}}{dt}$:
 - First term: $\vec{v} \times m\vec{v} = \vec{0}$ (parallel vectors).
 - Remaining term:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

- Analogue relation to translational dynamics:

$$\vec{\tau} = \frac{d\vec{L}}{dt} \leftrightarrow \vec{F} = \frac{d\vec{p}}{dt}$$

Angular momentum for a Rigid Object

$$\vec{L} = I\vec{\omega}$$

- Differentiate (rigid body $\rightarrow I$ constant):

$$\frac{d\vec{L}}{dt} = \frac{Id\vec{\omega}}{dt} = I\vec{\alpha}.$$

Therefore:

$$\vec{\tau}_{\text{net}} = I\vec{\alpha}.$$

Rotational analogue of $F = ma$.

Angular Momentum Summary:

- Particles: $\vec{L} = \vec{r} \times \vec{p}$
- Rigid bodies: $\vec{L} = I\vec{\omega}$ (for symmetry axes)
- Always:

$$\vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$$

Conservation of Angular Momentum - Theory

- If net external torque is zero:

$$\frac{d\vec{L}}{dt} = 0 \Leftrightarrow \vec{L} = \text{const.}$$

- **If the net external torque is zero, angular momentum stays constant**
- Internal mass redistribution can change angular velocities, **but not total angular momentum**
- Applies to all scales: atoms, planets, stars, skaters, rotating machinery, etc

Conservation of Angular Momentum - Seeing is believing

mk13 - Drehimpulserhaltung: $L_{\text{sys}} = 0$

mk13 - Drehimpulserhaltung: $L_{\text{sys}} = L$

- For rotation about a fixed axis (direction of \vec{L} unchanged):

$$L_1 = L_2, \quad I_1\omega_1 = I_2\omega_2$$

Conservation of Angular Momentum - Key points

- Angular momentum is conserved when the **net external torque** is zero.
- Internal forces cannot change total \vec{L} , only its distribution.
- Reducing moment of inertia increases angular velocity if L is fixed.
- Conservation applies at all scales: atoms, machinery, stars, galaxies.
- When torque-free, both the **magnitude** and **direction** of \vec{L} remain constant.
- Collisions or mass redistribution obey angular momentum conservation when isolated from external torques.

Solve initial question

mk15

Tasche mit Wasser und Papierschnipsel

Which egg is cooked? How to check?



Bonus Puzzle

mk11 - Maxwell'sches Rad

How can the scale show less weight?

$$\tau = Tr = I\alpha$$

$$ma = mg - T \leftrightarrow T = m(g - a)$$

$$\Rightarrow a < g$$