1.7. Hydrostatics

`mf19 - Connected soap bubbles`

If connected, what bubble grows?

Introduce **new concepts**:

- Phase of matter & density
- Pressure & buoyancy
- Surface tension & capillarity

Learn about **Pascal's principle & Archimedes' Principle**

A Matter of Phase

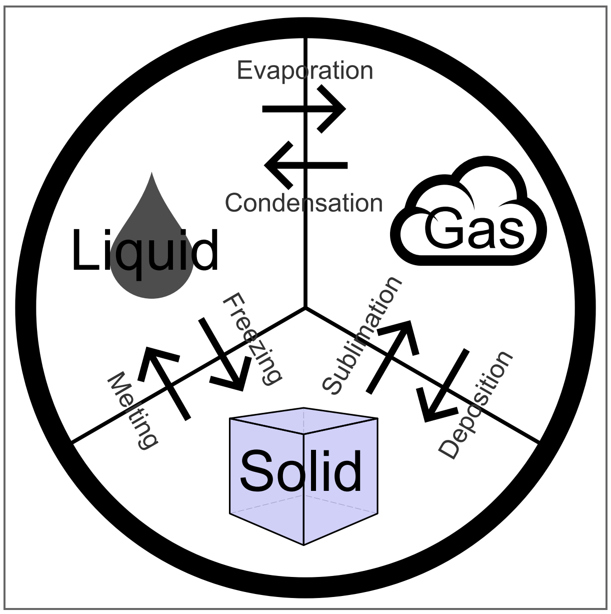
`mf01 - Flaschenexplosion`

`mf04 - Flaschenteufel`

Can we explain the outcome?

Phases of Matter

- **Fluids = liquids & gases** (since both flow under applied force)



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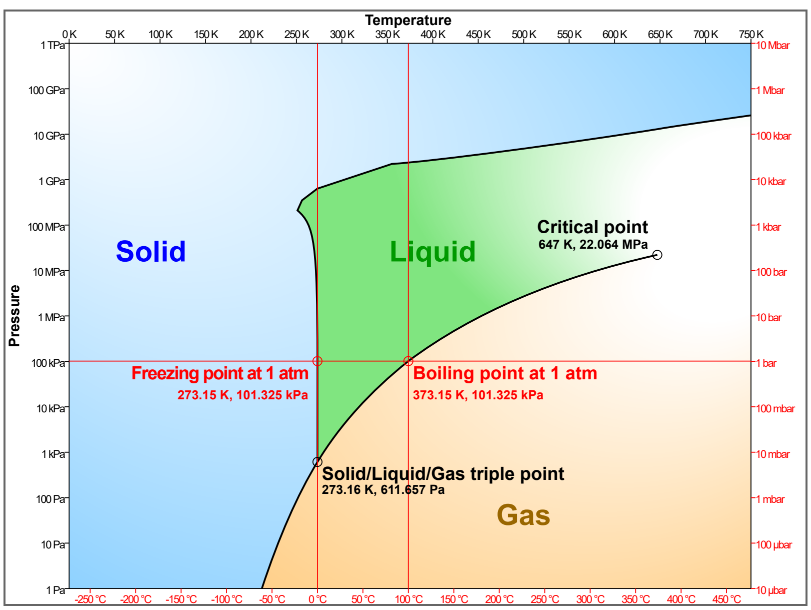
Phases of Matter (cont')

Property	**Solid**	**Liquid**	**Gas**
Molecular spacing	Very close	Close	Far apart
Forces between molecules	Strong	Moderate	Negligible
Shape	Fixed	Variable	Variable
Volume	Fixed	Fixed	Variable
Compressibility	Very small	Small	Large
Flow	No	Yes	Yes

Phase diagrams - All about that Phase

- Show how **phase depends on temperature and pressure**
- Lines mark **phase boundaries** where transitions occur (melting, boiling, sublimation)
- **Triple point**: solid, liquid, and gas coexist
- **Critical point**: end of liquid–gas boundary; beyond it → supercritical fluid



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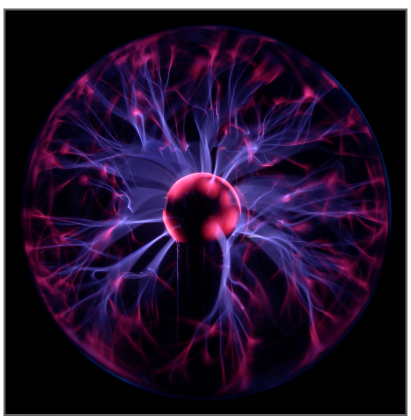
Plasma: The Fourth State

- Very high temperatures ionize atoms (electrons separate from nuclei)
- Mixture of free electrons and ions → **plasma**
- Conductive and strongly affected by electric and magnetic fields
- Occurs in stars, lightning, fluorescent lamps, etc.



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Density

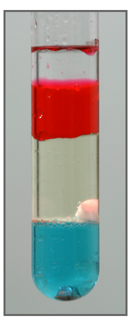
- Density is mass per volume measure in $\mathrm{kg/m^3}$

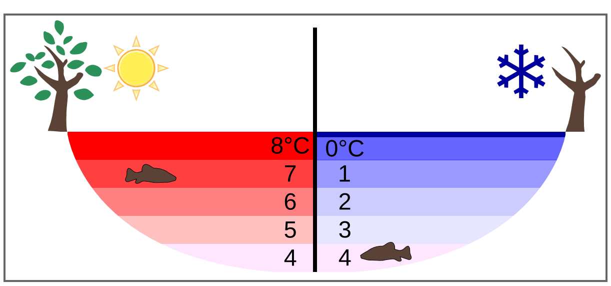
$$\rho = \frac{m}{V}$$

- Temperature dependence:

$$\rho(T) = \frac{\rho_0}{1 + \beta(T - T_0)}$$

- **Gases**: strong dependence on T and p; **solids/liquids**: weak dependence
- **Water anomaly**: maximum density at $4^\circ\mathrm{C}$





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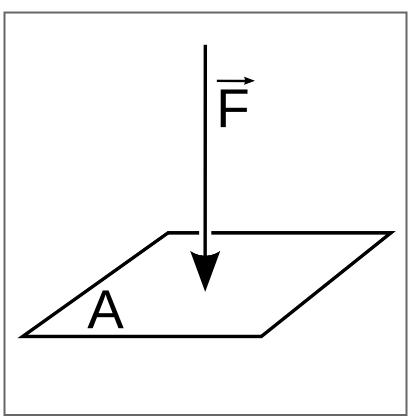
Pressure - Scalar Definition

- Pressure = normal force $F_\perp$ over contact area A:

$$P = \frac{F_\perp}{A}$$

- Pressure is a **scalar**
- Units:
  - SI: $1\ \mathrm{Pa} = 1\ \mathrm{N/m^2}$
  - $1\ \mathrm{atm} \approx 1.013 \times 10^5\ \mathrm{Pa}$
  - $1\ \mathrm{bar} = 10^5\ \mathrm{Pa}$
  - $1\ \mathrm{mmHg} = 133\ \mathrm{Pa}$
  - $1\ \mathrm{psi} = 6895\ \mathrm{Pa}$



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Pressure - Differential Form

- Introduce the **area vector**:

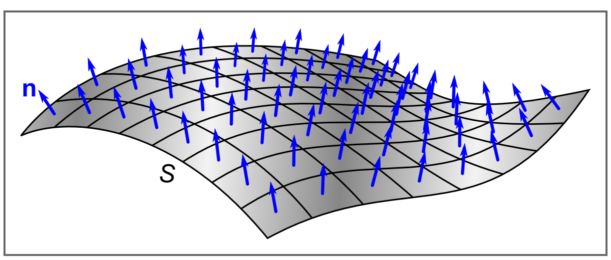
$$\mathrm{d}\vec{A} = \hat{n}\,\mathrm{d}A$$

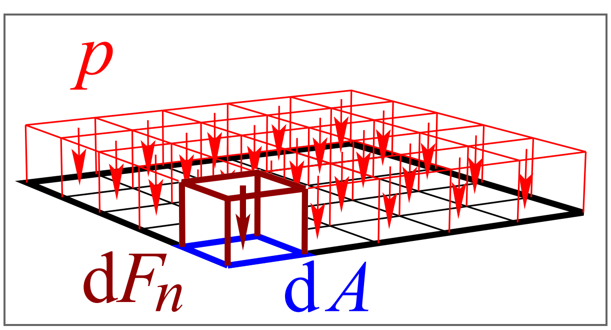
  - magnitude = area $\mathrm{d}A$
  - direction = outward surface normal $\hat{n}$

- **Force** on the surface element:

$$\mathrm{d}\vec{F} = -p\,\mathrm{d}\vec{A}$$

- **Minus sign**: Convention that force is considered towards the surface element, while the normal vector points outward





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Pressure at a Point in a Fluid

`mf05 - Schweredruckmessung`

- In a fluid at rest, pressure is **the same in all directions**
- Pressure in static fluids does **not** depend on orientation or container shape
- If pressure were not isotropic, there would be a **pressure difference** $\rightarrow$ a net force would act $\rightarrow$ the fluid would start to flow
- Fluids exert forces **perpendicular** to surfaces.

Pressure & Depth - Derivation

- thin fluid layer of thickness $\mathrm{d}y$ and area A
- mass: $\rho A\,\mathrm{d}y$, weight: $\rho g A\,\mathrm{d}y$
- Pressure forces:
  - upward: $(P + \mathrm{d}P)A$
  - downward: PA
- Force balance (no acceleration):

$$(P + \mathrm{d}P)A - PA = \rho g\,A\,\mathrm{d}y$$

- Simplifies to the **hydrostatic relation** for a fluid at rest:

$$\boxed{\mathrm{d}P = \rho g\,\mathrm{d}y} \qquad \leftrightarrow \qquad \boxed{\frac{\mathrm{d}P}{\mathrm{d}y} = \rho g}$$

Pressure & Depth - Absolute Pressure

- At the fluid surface ($y = 0$), pressure is P_0 (e.g. atmospheric pressure)
- Integrate from 0 to depth h:

$$\int_{P_0}^{P} \mathrm{d}P = \int_0^h \rho g \, \mathrm{d}y$$

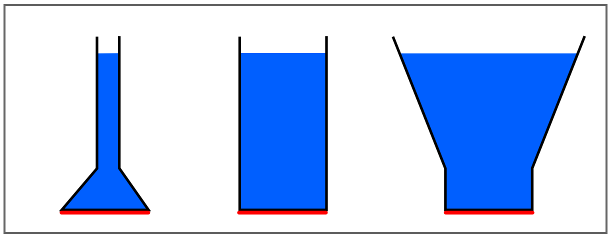
$$P - P_0 = \rho g h$$

$$P = P_0 + \rho g h$$

- Result: pressure in a uniform-density fluid **increases linearly with depth**

Pressure Differences & Hydrostatic Paradox

`mf13 - Hydrostat. Paradoxon`



*from **wikipedia**, gemeinfrei*

Pressure Differences & Hydrostatic Paradox (cont')

- Pressure difference in a uniform fluid:

$$\Delta P = \rho g(h_2 - h_1)$$

- Pressure depends only on **depth**, not container shape or volume
- **Hydrostatic paradox:** containers of different shape have the **same pressure** at the same depth
- **Reason**: Pressure is set by the **fluid column above the point**, not by total fluid amount

Atmospheric Pressure

- Air has mass $\rightarrow$ its weight creates **atmospheric pressure** on all surfaces
- Pressure at a point is the weight of the air column above:

$$P = \int_h^\infty \rho g \, \mathrm{d}h$$

- At sea level:

$$P_{\mathrm{atm}} \approx 1.013 \times 10^5 \ \mathrm{Pa} \approx 1 \ \mathrm{atm}$$

- Equivalent units (single line): **101.3 kPa = 760 mmHg = 14.7 psi = 1.013 bar**
- Atmospheric pressure corresponds to a **10.3 m water column**

Atmospheric Pressure vs. Altitude

- Higher altitude $\to$ **less air above** $\to$ lower pressure
- For small height changes, density $\approx$ constant, thus, linear approximation gives $P = P_0 - \rho g h$
- For larger height differences, density varies (ideal gas assumed): $\rho = \frac{MP}{RT}$
- Substituting into hydrostatic relation:

$$\frac{\mathrm{d}P}{\mathrm{d}h} = -\rho g \Rightarrow \ \frac{\mathrm{d}P}{\mathrm{d}h} = -\frac{Mg}{RT}P$$

- Solve by separation:

$$P = P_0\, e^{-h/h_0}$$

- **Scale/characteristic height**:

$$h_0 = \frac{RT}{Mg}$$

  The height over which pressure drops by a factor of e (Earth: $h_0 \approx 8.4\ \mathrm{km}$ at 20°C)

An Ode to Ordinary Differential Equations (ODE)

- Hydrostatic relation is a **differential equation**:

$$\frac{\mathrm{d}P}{\mathrm{d}h} = -\rho(h)\, g$$

- Its solution depends on the model for density:
    - **Constant** ρ → linear law

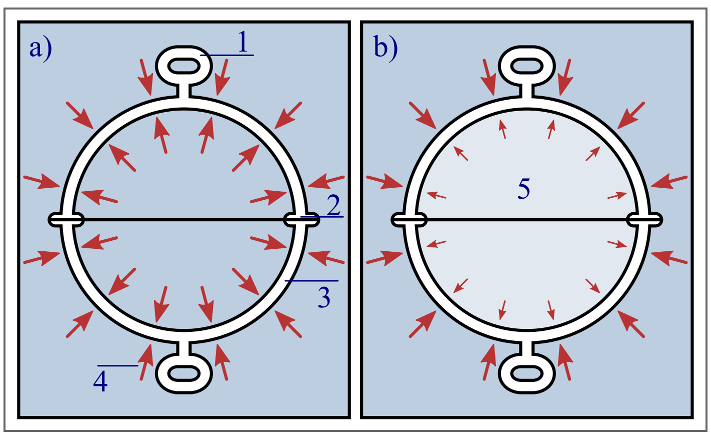
      $P = P_0 + \rho g h$
    - **Ideal-gas atmosphere** → exponential law

      $P = P_0\, e^{-h/h_0}$
- This ODE framework allows us to model any atmosphere
  (e.g., variable temperature, variable composition)

Magdeburger Halbkugeln

`mf40 - Halbkugeln`

How does it work?



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Absolute, Gauge, Differential Pressure & Vacuum

- **Absolute pressure:** measured relative to **vacuum** ($P = 0$)
- **Atmospheric pressure:** ambient air pressure P_{atm}
- **Gauge pressure:** relative to atmosphere

$$P_g = P_{\text{abs}} - P_{\text{atm}}$$

  - $P_g > 0$: above atmospheric
  - $P_g < 0$: **partial vacuum**

- **Differential pressure:**

$$\Delta P = P_2 - P_1$$

Measuring Pressure with Fluid Columns (Manometers)

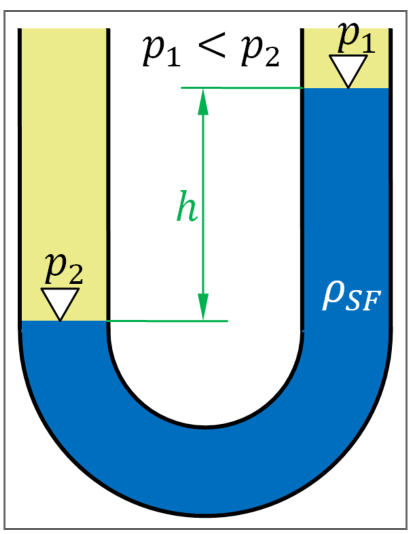
A manometer measures the pressure difference between **system 1** and **system 2**:

$$P_1 - P_2 = \rho g(h_2 - h_1)$$

- If **system 2 = atmosphere**, this reduces to:

$$P_{\text{system}} = P_{\text{atm}} - \rho g h$$

- **positive & negative pressure differences can be detected**

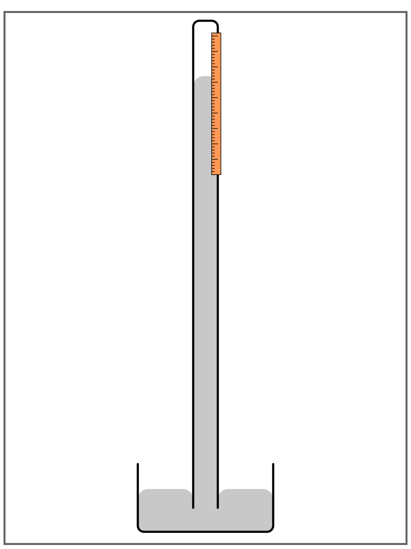


Mercury Barometer

- Invented by Evangelista Torricelli (1608–1647) to measure **absolute** atmospheric pressure
- A tube filled with mercury is inverted into a reservoir $\rightarrow$ a small **vacuum** forms at the top (no trapped air)
- Atmospheric pressure supports the mercury column:

$$P_{\text{atm}} = \rho_{\text{Hg}} g h$$

- At sea level: $h \approx 0.760 \text{ m} \quad (= 760 \text{ mm Hg})$



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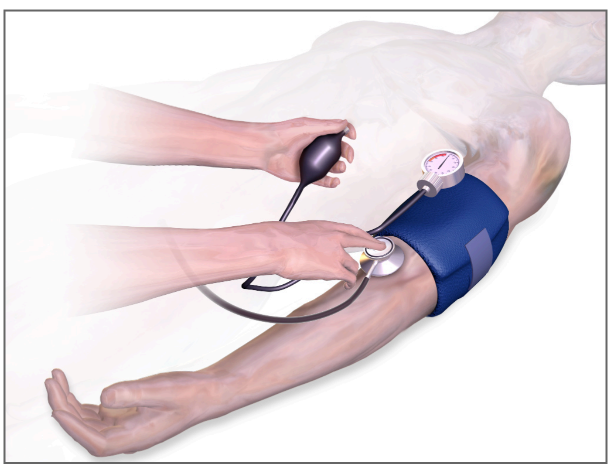
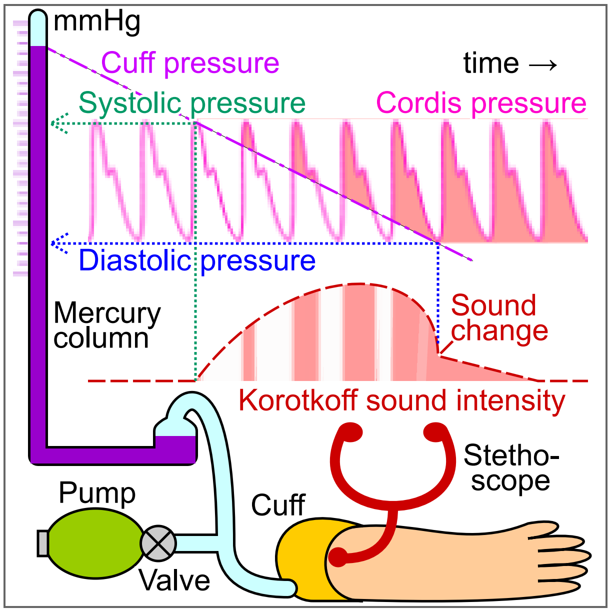
Measuring Blood Pressure (Auscultatory Method)

- Inflate cuff → artery temporarily **closed**
- Slowly release pressure:
    - **First Korotkoff sound** → blood just starts to flow → **systolic pressure**
    - **Sounds disappear** → flow becomes smooth → **diastolic pressure**
- Cuff pressure at these moments gives systolic and diastolic values





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Pascal’s Principle

`mf02 - Allseitge Druckausbreitung`

> ***Pascal’s Principle:***
> *A change in pressure applied to an enclosed fluid is transmitted undiminished to every point of the fluid and to the walls of the container.*

This means that an applied pressure change has the same value everywhere in the fluid:

$$\Delta P = \frac{\Delta F}{A}$$

Hydraulic Press - Demo

`mf45- Hydraulik`

How does it work?
Is there energy created "from nothing"?

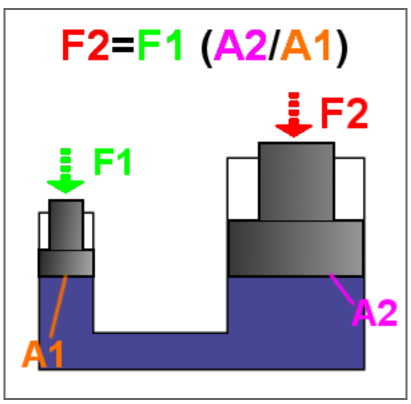
Hydraulic Press - Theory

- **Pascal's principle**:

$$F_2 = PA_2 = \frac{A_2}{A_1} F_1$$

- **Mechanical advantage**: $\dfrac{F_2}{F_1} = \dfrac{A_2}{A_1}$
- **Energy conserved**: large force ↔ small displacement

$$W_1 = PA_1 d_1 = PA_2 d_2 = W_2, \Rightarrow \; F_1 d_1 = F_2 d_2$$



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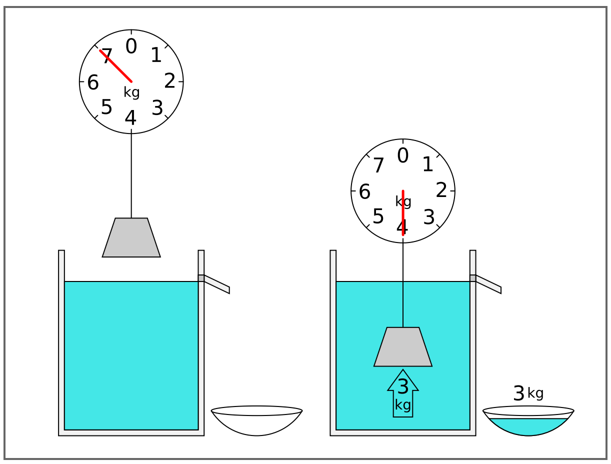
Buoyancy - Demo

`mf43 - Abh. Auftrieb`

Which body has the smallest apparent weight in water?



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Buoyant Force - Origin

- Pressure increases with depth: $P = \rho g h$
- Forces on top and bottom of a submerged object (area A):

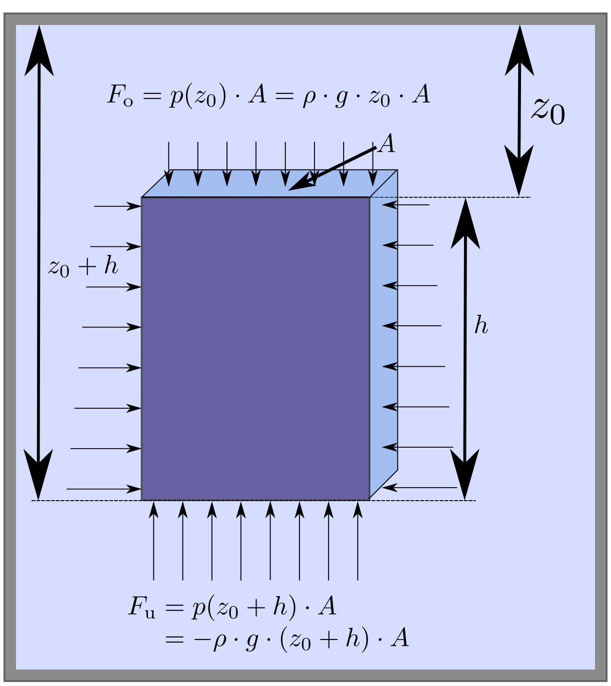
$$F_{\text{top}} = \rho g h_t A, \qquad F_{\text{bottom}} = \rho g h_b A$$

- Net upward (buoyant) force:

$$F_B = F_{\text{bottom}} - F_{\text{top}} = \rho g (h_b - h_t) A = \rho g V_{\text{disp}}$$

- Apparent weight:

$$W_{\text{app}} = W - F_B$$

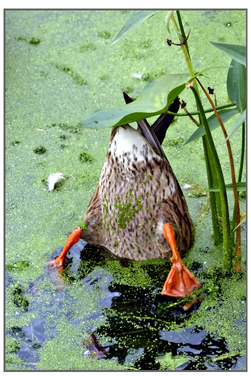


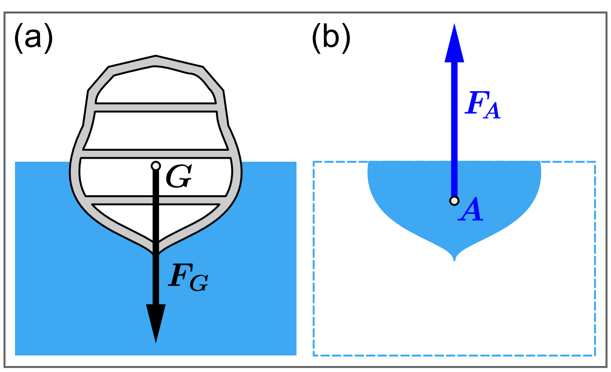
Buoyancy - Floating condition

`mf04 - Flaschenteufel`

- Net upward (buoyant) force: $F_B = \rho_{\text{fluid}} g V_{\text{disp}}$
- Apparent weight:
  $W_{\text{app}} = W - F_B = \rho_{\text{obj}} g V_{\text{obj}} - \rho_{\text{fluid}} g V_{\text{disp}}$
- Floating condition ($W_{\text{app}} = 0 \;\leftrightarrow\; W = F_B$):

$$\frac{V_{\text{sub}}}{V_{\text{obj}}} = \frac{\rho_{\text{obj}}}{\rho_{\text{fluid}}}$$





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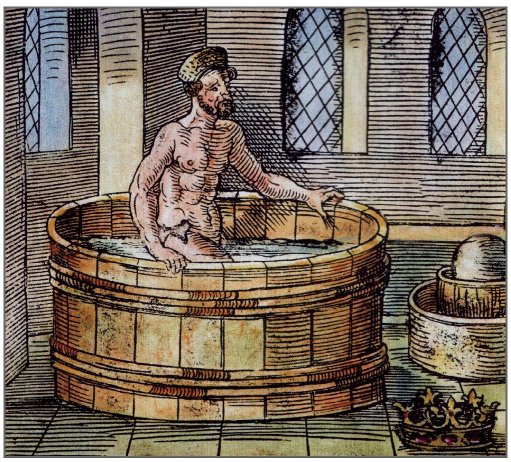
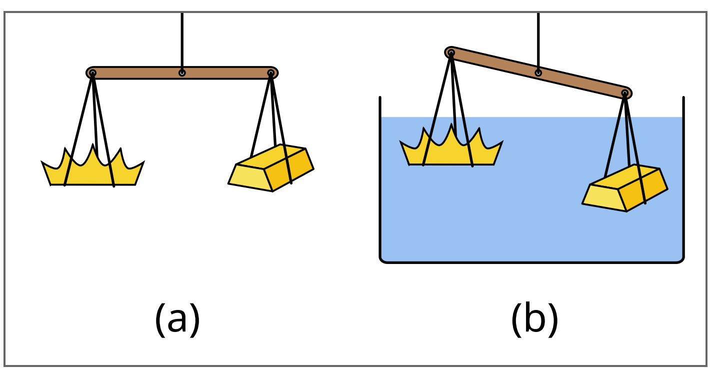
Archimedes' Principle & Buoyancy

> ***Archimedes' Principle:***
> *Any object wholly or partially immersed in a fluid experiences an upward buoyant force equal to the **weight of the fluid displaced**.*





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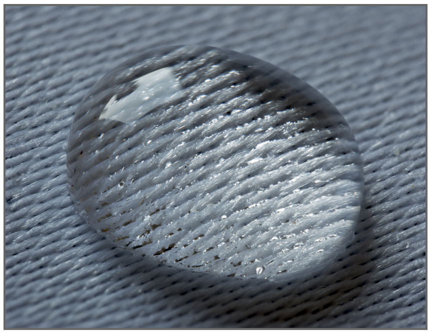
Surface Tension - Demo

`mf41 - Oberflächenspannung`



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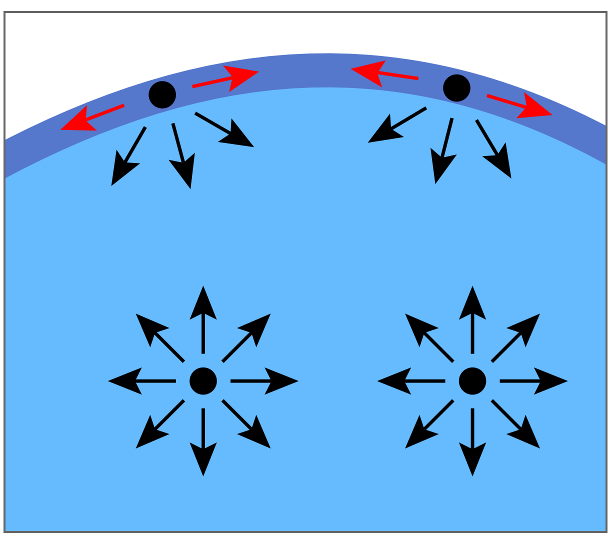
Surface Tension - Molecular Perspective

- **Cohesion**: attraction between molecules of the same substance
- **Adhesion**: attraction between molecules of different substances
- **Bulk molecules**: many neighbors → many cohesive bonds → low energy
- **Surface molecules**: fewer neighbors → fewer bonds → higher energy (costs work to create surface)



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Definition of Specific Surface Energy & Surface Tension

- Enlarging a liquid surface by ΔA requires work ΔW.
  Specific surface energy:

$$\varepsilon = \frac{\Delta W}{\Delta A}$$

- A liquid surface pulls tangentially along any line in the interface.
  Surface tension:

$$\gamma = \frac{F}{L}$$

- In static equilibrium:

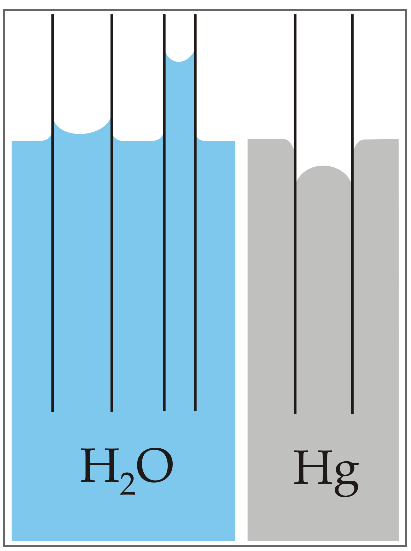
$$\gamma = \varepsilon$$

Units: N/m = J/m² (since J/m² = N·m/m² = N/m)

Capillarity: Cohesion vs Adhesion

`mf17 - Kapillarität`

- **Adhesion > Cohesion**: $\rightarrow$ liquid wets the tube $\rightarrow$ concave meniscus $\rightarrow$ liquid **rises**
- **Cohesion > Adhesion**: $\rightarrow$ liquid does not wet the tube $\rightarrow$ convex meniscus $\rightarrow$ liquid **falls** (depression)
- Capillary height is **inversely proportional** to tube radius



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Revisit initial experiment

`mf19 - Connected soap bubbles`

If connected, what bubble grows?

$$\Delta P = \frac{4\varepsilon}{r} = \frac{4\gamma}{r}.$$

$\rightarrow$ pressure difference large the smaller the droplet

Excess Pressure in a Spherical Droplet (Surface–Energy Derivation)

Small radial increase: $r \to r + \Delta r$

Work by pressure

$$\Delta V = 4\pi r^2 \Delta r, \qquad \Delta W = p\,\Delta V = p(4\pi r^2 \Delta r)$$

Increase in surface energy

$$E = \varepsilon(4\pi r^2)$$

$$\Delta E = 4\pi\varepsilon\left[(r + \Delta r)^2 - r^2\right] \approx 8\pi r\varepsilon\,\Delta r$$

Energy balance

$$\Delta W = \Delta E$$

$$p(4\pi r^2 \Delta r) = 8\pi r\varepsilon\,\Delta r$$

Result

$$p = \frac{2\varepsilon}{r} \qquad (\gamma = \varepsilon) \qquad \Rightarrow \qquad \Delta P = \frac{2\gamma}{r}$$

Soap Bubble (two surfaces)

Twice the surface energy:

$$\Delta E_{\text{bubble}} = 2 \times (8\pi r \varepsilon \, \Delta r)$$

Result:

$$\Delta P = \frac{4\gamma}{r}$$

A soap bubble has **twice** the excess pressure of a droplet of the same radius.