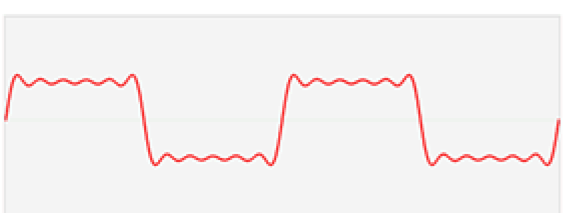
Lecture Tutorial 1F: Fourier Analysis

- **Complex numbers**
  & revisiting **oscillations**
- **Fourier series**: From **single frequencies** to **many frequencies**
- **Fourier transform**: analyzing non-periodic signals
- **Examples & applications**: Why the **frequency domain** is to powerful

⇒ Please download the **<u>phyphox app</u>** for some live demos



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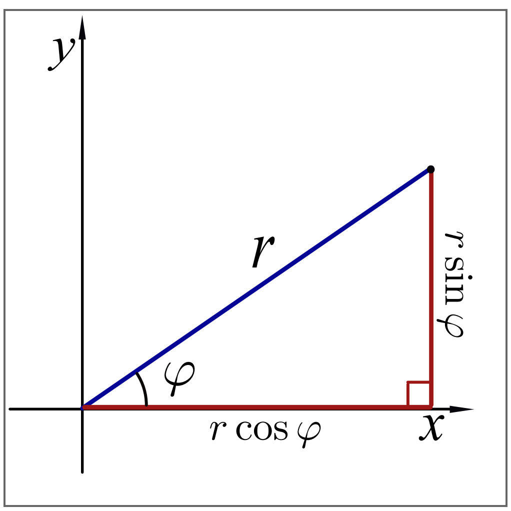
Reminder: Polar Coordinates

- Represent point by (r, θ) instead of (x, y).
- Conversion:
  - $x = r\cos\theta,$
    $y = r\sin\theta$
- Inverse:
  - $r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x)$



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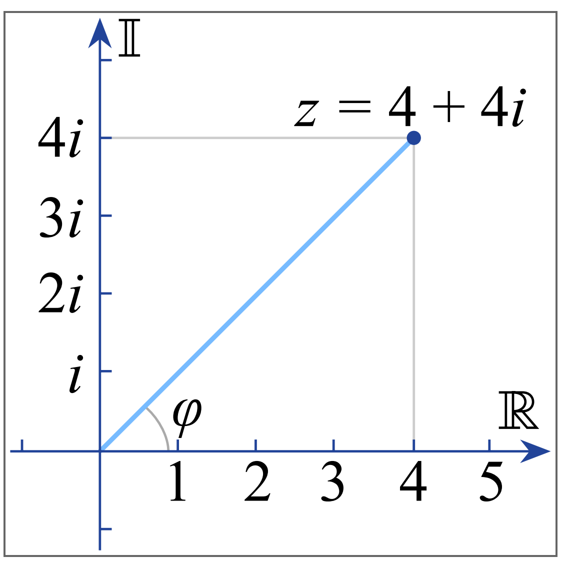
Complex Numbers — A Geometric View

- A complex number is **not mysterious**: it is a 2D quantity:

$$z = x + iy, \qquad i^2 = -1$$

- Can be viewed as a **vector in a plane** with **real part** x on horizontal axis & **imaginary part** y on vertical axis
- Length (magnitude) & direction (phase)

$$|z| = \sqrt{x^2 + y^2} \qquad \varphi = \tan^{-1}\left(\frac{y}{x}\right)$$



Complex numbers – Polar form

- Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$:

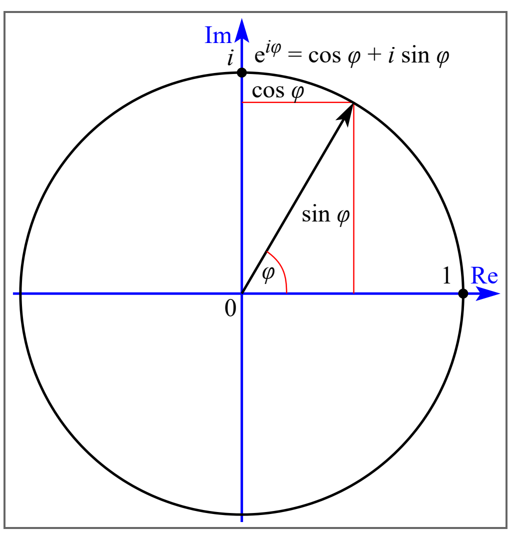
$$z = x + iy \quad \Leftrightarrow \quad z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

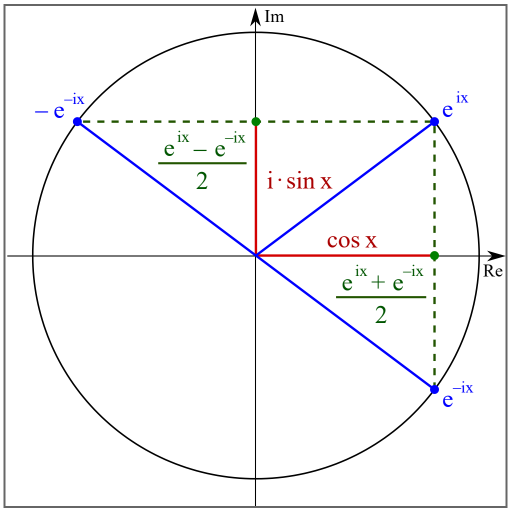
$\Rightarrow e^{i\theta}$ = **point at angle θ on the unit circle**

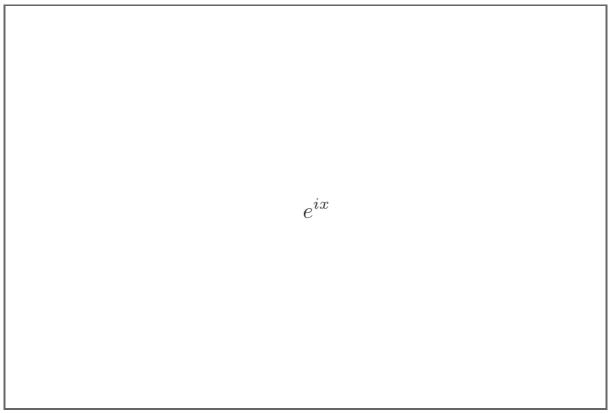
- Definitions of cosine and sine via complex exponentials:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

- Essential for SHM, damping, resonance, waves, AC circuits, ...







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Complex conjugate & multiplication

Complex conjugate reflects the vector (mirror at the real axis):

$$z = x + iy \quad \Rightarrow \quad z^* = x - iy$$

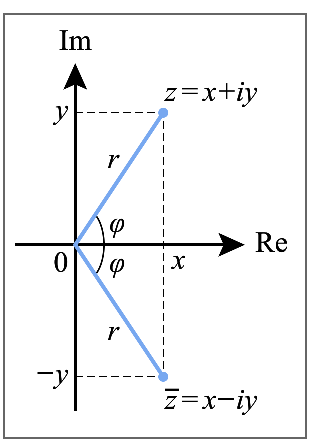
Multiplication of complex numbers (rotation + scaling):

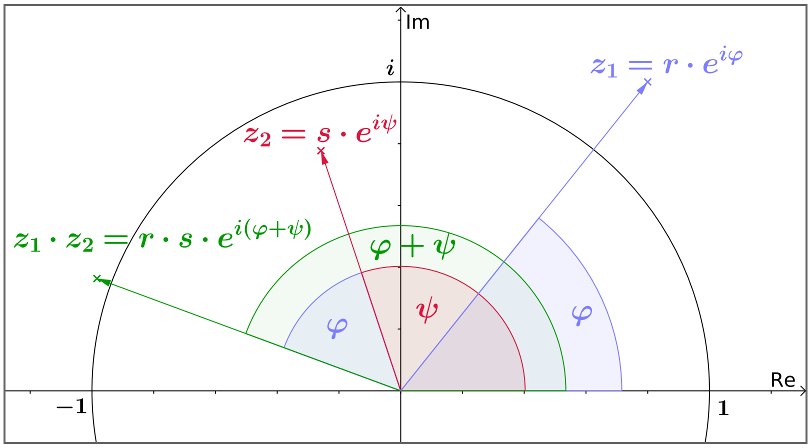
$$z_1 z_2 = r_1 r_2 \, e^{i(\varphi_1 + \varphi_2)}$$

Multiplication with the complex conjugate:

$$z = r\, e^{i\varphi}, \qquad z^* = r\, e^{-i\varphi}$$

$$z\, z^* = \left(r\, e^{i\varphi}\right) \left(r\, e^{-i\varphi}\right) = r^2\, e^{\,i\varphi - i\varphi} = r^2\, e^0 = r^2 = x^2 +$$





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Complex Exponentials for Oscillations: How They Appear Naturally

- Equation of motion (real coefficients):

$$\ddot{x} = -\frac{k}{m}x$$

- Inserting **exponential trial solution** $x(t) = X\,e^{\lambda t}$ gives:

$$\lambda^2 X\,e^{\lambda t} = -\frac{k}{m}\,X\,e^{\lambda t}$$

- Cancel the common factor $X\,e^{\lambda t}$:

$$\lambda^2 = -\frac{k}{m}$$

- **Allowing complex solutions gives two eigenvalues**:

$$\lambda = \pm i\omega, \qquad \omega = \sqrt{\frac{k}{m}}$$

From Complex Solutions to Real Oscillations

- **General solution as a linear combination of the two eigenmodes**:

$$x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$

- Real motion requires **complex conjugate coefficients**, $c_2 = c_1^*$:

$$x(t) = c_1 e^{i\omega t} + c_1^* e^{-i\omega t}$$

- Write the complex amplitude in polar form, $c_1 = |c_1| e^{i\phi}$:

$$x(t) = |c_1| e^{i\phi} e^{i\omega t} + |c_1| e^{-i\phi} e^{-i\omega t} = |c_1| \left(e^{i(\omega t + \phi)} + \epsilon \right.$$

- Using Euler's identity, $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, the motion becomes:

$$x(t) = 2|c_1| \cos(\omega t + \phi)$$

- Final physical solution with $A = 2|c_1|$:

$$x(t) = A\cos(\omega t + \phi)$$

The Superpower of Euler's Formula

- Exponentials reproduce themselves under differentiation:

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$$

- Allowing complex numbers unlocks a **larger solution space** (e.g. $\sqrt{-1}$ is permitted)
- Negative eigenvalues lead to imaginary exponents:

$$\lambda = \pm i\omega$$

- Imaginary exponents describe **periodic motion**:

$$e^{i\omega t}$$

- Euler's formula: $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$
- **Dynamics reduces to complex-number algebra**

Wave synthesis – From single to many frequencies

`sim – Wave Synthesis`

- Single oscillation:

$$x(t) = A\cos(\omega t + \phi_n)$$

- Superposition:

$$x(t) = \sum_n A_n \cos(\omega_n t + \phi_n)$$

- Adding frequencies $\rightarrow$ more complex signals

Fourier series – Building periodic signals

- Any **periodic signal** can be written as a sum of sine and cosine waves:

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right]$$

- $\omega_0 = 2\pi/T$ is the **fundamental frequency**
- Higher terms ($n = 2, 3, \ldots$) are **harmonics**
- Each harmonic contributes a **frequency** $n\omega_0$, a **phase**, an **amplitude**
- More terms $\rightarrow$ better approximation

*from **wikipedia**.gif), public domain*

Fourier transform – Analyzing non-periodic signals

- Real signals are often **not periodic**
- Idea:
    - Fourier **series** $\rightarrow$ discrete frequencies (periodic signals)
    - Fourier **transform** $\rightarrow$ continuous frequencies (non-periodic signals)
- Time signal $\rightarrow$ frequency spectrum

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt$$

- Inverse transform (back to time domain):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{i\omega t}\, d\omega$$

- Reminder: Euler's formula
  $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$

Fourier transform – Building intuition

`md16 - Beibehaltung der Schwingungsebene - Foucaultpendel`

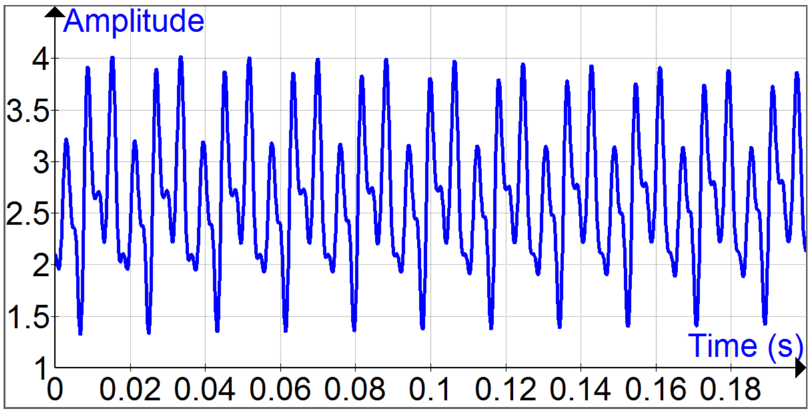
`sim - Winding Machine`

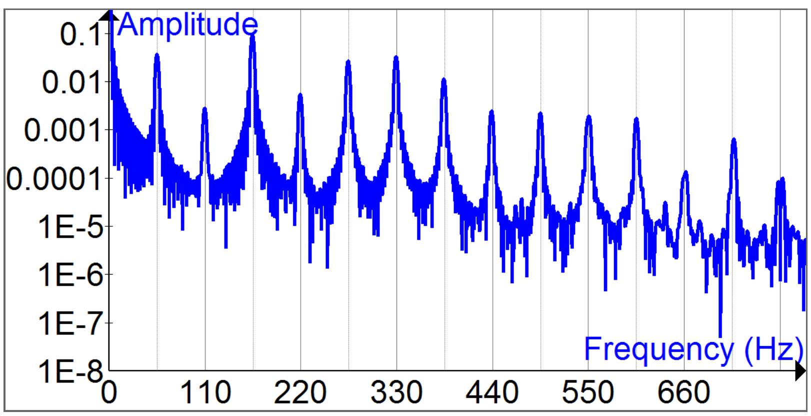
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt$$

- **Idea**: Compare the signal with a **test oscillation** $e^{-i\omega t}$ to measure how strongly this oscillation is present in $x(t)$
- **Output**: **Complex frequency spectrum**
  - $|X(\omega)| \to$ strength (amplitude) of frequency ω
  - $\arg X(\omega) \to$ phase of that frequency
- **Key message**:
  - **Any signal = sum of many oscillations**

- Fourier transform reveals which frequencies are present and how strongly

Fourier transform – Frequency spectrum of bass guitar (A at 55 Hz)





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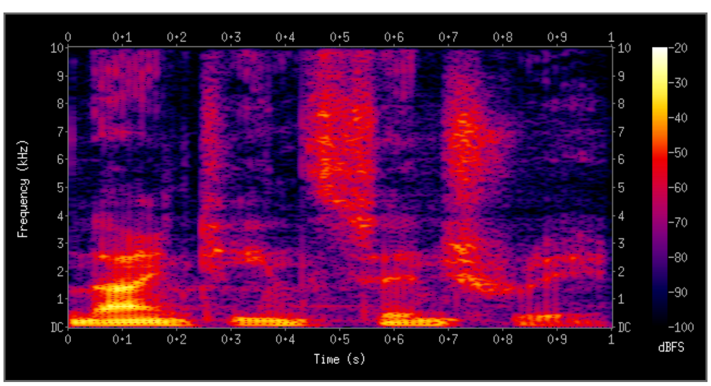
Time-windowed Fourier transform & spectrograms

`phyphox / ma19+ma05`

- Real signals often change in time
- Apply Fourier transform on **short time windows**
- Reveals **how frequencies evolve over time**
- Result visualized as a **spectrogram** with the frequencies' amplitude encoded as **intensity** of the plot
- Key trade-off:
    - short window → good time resolution
    - long window → good frequency resolution

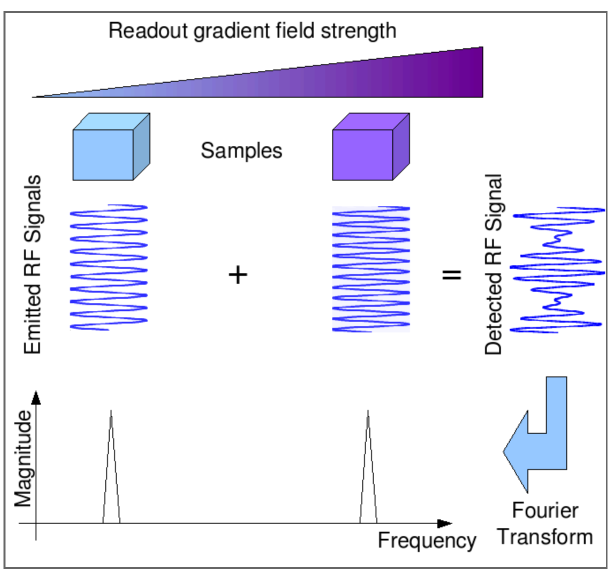


*from **wikipedia**, public domain*

Fourier in action

- Please open **phyphox** $\rightarrow$ **Acoustics** $\rightarrow$ **Tone gnerator**
- Set **frequency** to:
  $\text{Row number of your seat} \times 100\text{Hz}$
- **Press play** and maximize volume

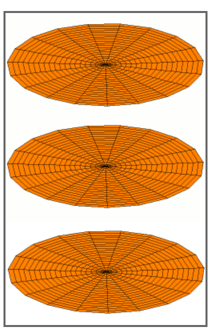
$\Rightarrow$ **MRI frequency encoding** shows how spatial position is converted into frequency information — the same Fourier principles behind FT-NMR (Ernst, Nobel Chemistry 1991) and modern MRI imaging (Lauterbur & Mansfield, Nobel Medicine 2003).



2D Fourier transform – spatial frequencies

`sim - 2D FT` **link** and **github**

- Fourier analysis applied to images (space instead of time)
- Performed as **two 1D Fourier transforms**:
  - once in x-direction
  - once in y-direction
- Low spatial frequencies → smooth, large-scale structure
- High spatial frequencies → edges and fine detail



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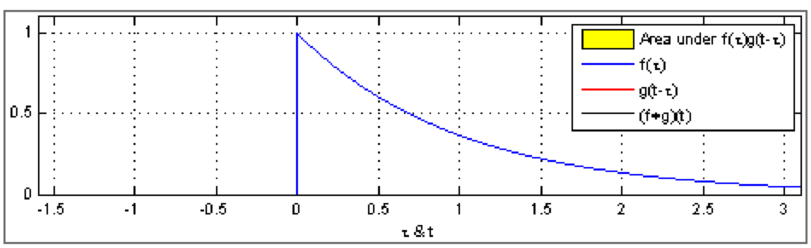
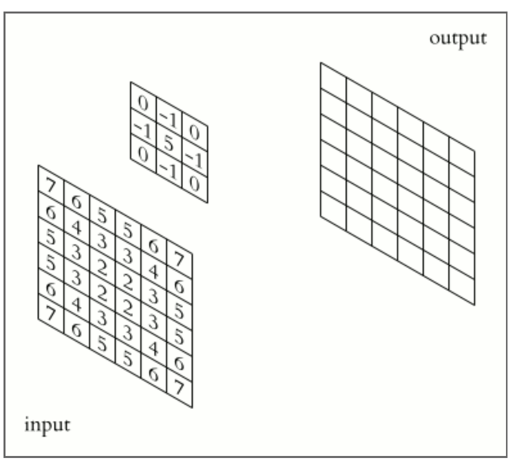
Why frequency space simplifies convolution

- Convolution in real space is computationally expensive
- In frequency space:
    - convolution → **multiplication**
- Filtering becomes simple and intuitive
    - low-pass → blur
    - high-pass → edge enhancement
- Core tool in signal and image processing





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Fourier Transform - Big picture

- Waves and oscillations all around us
- Complex signals become simple spectra $\rightarrow$ underlying patterns becomes visible
- Dynamics become algebra (differential $\rightarrow$ algebraic equations e.g. wave equation, heat equation, diffusion)
- Basis for/related to other transforms such as Laplace & Wavelet
- **Fourier analysis changes how we think about problems**