1.9. Oscillations

In this chapter we introduce **oscillatory motion**. We describe **simple harmonic motion** using the mass–spring system and the pendulum for small oscillations. We then include **damping** and **external driving forces**, and conclude with **resonance** as a key feature of forced oscillations.

1.9.1. Simple harmonic motion

Many oscillatory systems encountered in physics can be reduced to the same mathematical structure. A mass–spring system is particularly useful because it allows us to **derive** this structure explicitly and then recognize it as a general model for oscillatory motion, known as **simple harmonic motion (SHM)**.

In all that follows, the displacement from equilibrium is denoted by $x(t)$. Its first and second time derivatives are written as

$$\dot{x} = \frac{dx}{dt}, \qquad \ddot{x} = \frac{d^2x}{dt^2},$$

where $\dot{x}$ and $\ddot{x}$ represent the velocity and acceleration, respectively.

Horizontal mass–spring system

Consider a mass m attached to an ideal, massless spring on a frictionless horizontal surface. If the mass is displaced by a distance x from equilibrium, the spring exerts a restoring force given by Hooke's law,

$$F = -kx.$$

Applying Newton's second law gives

$$m\ddot{x} = -kx,$$

or, equivalently,

$$\ddot{x} + \frac{k}{m}x = 0.$$

We now need to find a function $x(t)$ that satisfies this equation. Any valid solution must make the **time dependence cancel identically** when inserted into the equation, so that it holds for all times t.

The motion of the mass repeatedly swings between two extreme positions, which strongly suggests that $x(t)$ should be **periodic**. Polynomial functions do not repeat, and exponential functions grow or decay, so neither can describe sustained oscillations. Trigonometric functions, however, are periodic and naturally describe motion that alternates between maximum positive and negative displacements.

Suppose the mass is initially displaced from equilibrium by an amount x_0 and released from rest. At $t = 0$, the displacement is maximal and the velocity is zero. A cosine function has exactly this behavior, whereas a sine function would start at zero displacement. We therefore **guess** a solution of the form

$$x(t) = x_0 \cos(\omega t),$$

where ω is a constant that determines how fast the system oscillates in time. This constant is called the **angular frequency**. Since the cosine function repeats when its argument increases by 2π, the period of the motion is

$$T = \frac{2\pi}{\omega}.$$

To test whether this guess is valid, we insert it into the equation of motion. The first and second time derivatives are

$$\dot{x}(t) = -x_0 \omega \sin(\omega t),$$

$$\ddot{x}(t) = -x_0 \omega^2 \cos(\omega t).$$

Substituting into the differential equation yields

$$\ddot{x} + \frac{k}{m} x = -x_0 \omega^2 \cos(\omega t) + \frac{k}{m} x_0 \cos(\omega t) = 0.$$

$$-\omega^2 + \frac{k}{m} = 0$$

The cosine factor is common to both terms. For the equation to hold for **all times** t, the remaining coefficients must cancel, which requires

$$\omega^2 = \frac{k}{m}.$$

With this choice, all explicit time dependence cancels, and the equation is satisfied identically. Thus,

$$x(t) = x_0 \cos(\omega t), \qquad \omega = \sqrt{\frac{k}{m}},$$

is indeed a solution of the equation of motion.

The period of the oscillation is therefore

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}},$$

and depends only on the mass and the spring constant, not on the amplitude.

More general initial conditions lead to a phase-shifted cosine function, which will be introduced later when discussing simple harmonic motion in full generality.

Vertical mass–spring system

Now consider the same mass–spring system oriented vertically. In addition to the spring force, the mass experiences its weight mg. Let x denote the downward displacement measured from the spring's **natural length**. The forces acting on the mass are the weight mg downward and the spring force $-kx$ upward, so Newton's second law gives

$$m\ddot{x} = mg - kx.$$

Before describing the oscillatory motion, we first determine the **equilibrium position**. At equilibrium the acceleration vanishes, so we set $\ddot{x} = 0$ and obtain

$$0 = mg - kx_0 \quad \Rightarrow \quad x_0 = \frac{mg}{k}.$$

This x_0 is the static extension of the spring due to the mass's weight.

To describe the oscillation, it is more convenient to measure displacement **from equilibrium** rather than from the natural length. We therefore redefine $x(t)$ to be the displacement from the equilibrium position. The total extension of the spring is then $x_0 + x$.

Inserting this into the equation of motion and using $x_0 = \frac{mg}{k}$ gives

$$m\ddot{x} = mg - k(x_0 + x) = mg - k(\frac{mg}{k} + x) = mg - mg - kx.$$

Thus, we obtain

$$m\ddot{x} = -kx,$$

or

$$\ddot{x} + \frac{k}{m}x = 0.$$

This equation is identical to the one obtained for the horizontal mass–spring system. The motion is therefore oscillatory, and by the same reasoning as before we try a cosine solution,

$$x(t) = x_0' \cos(\omega t) \quad \& \quad \ddot{x}(t) = -x_0'\omega^2 \cos(\omega t),$$

where x_0' is the maximum displacement from equilibrium.

Inserting this form into the equation again requires the time dependence to cancel for all t, which leads to

$$\omega^2 = \frac{k}{m}.$$

The period of the oscillation is the same as for the horizontal system:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

Thus, gravity shifts the equilibrium position but does not affect the oscillation frequency or period, as long as the spring obeys Hooke's law.

Simple harmonic motion in general

The two spring–mass systems illustrate a more general principle. A system executes **simple harmonic motion** whenever its equation of motion can be written in the form

$$\ddot{x} + \omega^2 x = 0,$$

with ω a positive constant. Equivalently, SHM is characterized by

$$\ddot{x} = -\omega^2 x,$$

which states that the acceleration is always proportional to the displacement from equilibrium and directed toward it. In this sense, **harmonic** motion refers to oscillatory motion that is sinusoidal, single-frequency, and generated by a linear restoring mechanism.

So far, we have focused on situations where the motion starts at a maximum displacement with zero initial velocity, for which a cosine function is a natural choice. In practice, however, an oscillating system can be released from **any** position and with **any** initial velocity. For example, the mass might pass through equilibrium with a finite speed, or be released from a position that is not an extreme point. To describe all such possibilities, we must allow for a shift of the time origin of the motion.

This leads to the introduction of a **phase constant** ϕ, which encodes the initial conditions. The general solution of the SHM equation can therefore be written as

$$x(t) = A\cos(\omega t + \phi).$$

An equivalent representation is

$$x(t) = A\sin(\omega t + \phi),$$

since sine and cosine differ only by a constant phase shift of $\pi/2$. Both expressions describe the same class of motions; the choice between sine and cosine is purely a matter of convenience.

In all cases, the motion is periodic with period

$$T = \frac{2\pi}{\omega},$$

where ω is called the **angular eigenfrequency** or **natural angular frequency** of the system.

Simple harmonic motion is therefore not tied to springs in particular. Any physical system whose restoring force leads to an equation of motion of the form $\ddot{x} + \omega^2 x = 0$ will exhibit sinusoidal oscillations, with the specific physical parameters of the system determining the value of ω.

1.9.2. Energy in simple harmonic motion

In simple harmonic motion, energy is continuously exchanged between **kinetic** and **potential** forms, while the total mechanical energy remains constant in the absence of damping.

For a spring–mass system obeying Hooke's law,

$$F = -kx,$$

the potential energy stored in the spring follows from the work done against the restoring force,

$$U(x) = \frac{1}{2}kx^2.$$

The kinetic energy of the oscillating mass is

$$K = \frac{1}{2}mv^2.$$

The total mechanical energy is the sum of kinetic and potential energy,

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2.$$

For simple harmonic motion, the displacement is bounded by the maximum value x_0, the **amplitude** of the motion. Using the relation $\omega^2 = k/m$ and evaluating the energy at a turning point, where $x = \pm x_0$ and $v = 0$, the total energy becomes

$$E = \frac{1}{2}kx_0^2 = \frac{1}{2}m\omega^2 x_0^2.$$

The total mechanical energy therefore depends only on the amplitude and not on time.

At the equilibrium position $x = 0$, the potential energy vanishes and the kinetic energy is maximal. At the turning points $x = \pm x_0$, the velocity is zero and all energy is stored as potential energy. At intermediate positions, energy is shared between kinetic and potential forms in such a way that their sum remains constant.

Energy conservation also provides a direct relation between speed and displacement,

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_0^2,$$

which yields

$$v = \pm\omega\sqrt{x_0^2 - x^2}.$$

The oscillator therefore moves fastest near equilibrium and slows to rest as it approaches the extreme positions.

Thus, simple harmonic motion can be understood as a periodic and reversible transformation between kinetic and potential energy, with a fixed total energy set by the amplitude of the motion.

1.9.3. Simple harmonic motion as a projection of circular motion

Simple harmonic motion can be understood geometrically as the **projection of uniform circular motion onto a straight line**. This analogy provides an intuitive interpretation of amplitude, phase, angular frequency, and the fixed phase relations between displacement, velocity, and acceleration.

Consider a point moving uniformly around a circle of radius r_0 with constant angular speed ω. The motion can be described not only in polar coordinates but also in **Cartesian coordinates**. If the angular position of the point is

$$\theta(t) = \omega t + \phi,$$

then its Cartesian coordinates are

$$x(t) = r_0 \cos\theta(t), \qquad y(t) = r_0 \sin\theta(t).$$

If we choose the x-axis as the axis of projection, the position of the projected point along this axis is

$$x(t) = r_0 \cos(\omega t + \phi),$$

which is exactly the displacement of a particle undergoing simple harmonic motion. Projecting onto the y-axis would lead to a sine function instead, showing that sine and cosine correspond simply to different choices of reference direction or phase.

In this picture, the radius r_0 corresponds to the **amplitude** of the oscillation, while the angular speed of rotation corresponds to the **angular eigenfrequency** ω. One full revolution of the circular motion corresponds to one complete oscillation in one dimension.

Differentiating the projected position gives the velocity,

$$v(t) = \dot{x} = -r_0\omega \sin(\omega t + \phi),$$

and differentiating once more gives the acceleration,

$$a(t) = \ddot{x} = -r_0\omega^2 \cos(\omega t + \phi) = -\omega^2 x(t).$$

The defining relation of simple harmonic motion therefore follows directly from the geometry of uniform circular motion.

The rotating radius vector, often called a **phasor**, provides a convenient visual representation of the oscillation. Its uniform rotation makes the phase $\omega t + \phi$ explicit, while its projection onto a chosen axis yields the instantaneous displacement. The constant angular speed of the phasor explains the fixed period of the motion,

$$T = \frac{2\pi}{\omega}.$$

This circular-motion model shows that simple harmonic motion is the one-dimensional shadow of a uniform rotational motion, offering a unified and physically transparent picture of oscillatory motion.

1.9.4. The mathematical pendulum

A **simple (mathematical) pendulum** consists of a point mass suspended from a fixed point by a massless, inextensible string of length L. When the pendulum is displaced slightly from its vertical equilibrium position and released, it swings back and forth under the action of gravity (no friction assumed).

Let $\theta(t)$ denote the angular displacement from the vertical. The weight mg acts downward, and only its tangential component contributes to the motion along the circular arc. The restoring force along the arc is

$$F_{\mathrm{r}} = -mg\sin\theta,$$

where the minus sign indicates that the force always points toward the equilibrium position.

For **small angular displacements** (measured in radians), the approximation $\sin\theta \approx \theta$ is valid. In this limit, the restoring force becomes

$$F_{\mathrm{r}} \approx -mg\,\theta,$$

showing that the restoring force is proportional to the angular displacement.

Newton's second law applied to the tangential motion can then be written as

$$ma_{\mathrm{t}} = F_{\mathrm{r}} = -mg\,\theta.$$

The tangential acceleration is related to the angular acceleration by $a_{\mathrm{t}} = L\ddot{\theta}$, so the equation of motion becomes

$$mL\ddot{\theta} = -mg\,\theta.$$

Dividing by mL yields

$$\ddot{\theta} + \frac{g}{L}\theta = 0.$$

This is exactly the differential equation of **simple harmonic motion**.

As in previous cases, we now look for a periodic solution. Since the pendulum swings back and forth between two extreme angles, we **guess** a cosine form,

$$\theta(t) = \theta_0\cos(\omega t + \phi),$$

where θ_0 is the maximum angular displacement. To verify this guess, we insert it explicitly into the equation of motion. The first and second time derivatives are

$$\dot{\theta}(t) = -\theta_0\omega\sin(\omega t + \phi),$$

$$\ddot{\theta}(t) = -\theta_0 \omega^2 \cos(\omega t + \phi).$$

Substituting $\theta(t)$ and $\ddot{\theta}(t)$ into

$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

gives

$$-\theta_0 \omega^2 \cos(\omega t + \phi) + \frac{g}{L}\,\theta_0 \cos(\omega t + \phi) = 0.$$

The cosine factor is common to both terms. For the equation to hold for **all times** t, the remaining coefficients must cancel, which requires

$$\omega^2 = \frac{g}{L}.$$

With this condition, all explicit time dependence cancels identically, confirming that the guessed solution satisfies the equation of motion. The quantity ω is therefore the **angular eigenfrequency** of the pendulum.

The period of the oscillation follows directly as

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}},$$

and the (ordinary) frequency is defined as

$$f = \frac{1}{T}.$$

For small oscillation angles, the period depends only on the pendulum length and the local gravitational acceleration. It is independent of both the mass of the pendulum bob and the oscillation amplitude. Gravity provides the restoring force, while the geometry of the circular motion ensures that the pendulum behaves as a simple harmonic oscillator near equilibrium.

1.9.5. The physical pendulum

In many oscillating systems the mass is distributed over an extended body rather than concentrated at a point. Such systems rotate about a fixed axis and are described as **physical pendulums**. As before, we restrict ourselves to small angular displacements and neglect friction.

Consider a rigid body of total mass m oscillating about a horizontal pivot. Let d be the distance between the pivot and the center of mass, and let I be the moment of inertia of the body about the pivot. When the body is displaced by a small angle θ from equilibrium, gravity exerts a restoring torque

$$\tau = -mgd\sin\theta.$$

For small angles, the approximation $\sin\theta \approx \theta$ applies, so that

$$\tau \approx -mgd\,\theta.$$

Using the rotational form of Newton's second law,

$$I\ddot{\theta} = \tau,$$

the equation of motion becomes

$$I\ddot{\theta} + mgd\,\theta = 0,$$

$$\ddot{\theta} + \frac{mgd}{I}\,\theta = 0.$$

This is again the equation of **simple harmonic motion**. By comparison with the standard SHM form, the angular eigenfrequency is

$$\omega = \sqrt{\frac{mgd}{I}},$$

and the corresponding period is

$$T = 2\pi\sqrt{\frac{I}{mgd}}.$$

A physical pendulum can therefore be associated with a simple pendulum of an **equivalent length**

$$L_{\text{eq}} = \frac{I}{md},$$

such that the period can be written in the familiar form

$$T = 2\pi\sqrt{\frac{L_{\text{eq}}}{g}}.$$

This equivalent length depends only on the mass distribution of the body and the location of the pivot.

1.9.6. The torsion pendulum

A closely related oscillatory system is the **torsion pendulum**, in which a rigid body oscillates by twisting a wire or fiber rather than swinging under gravity. In this case, the restoring torque is proportional to the angular displacement,

$$\tau = -\kappa\theta,$$

where κ is the **torsion constant** of the wire.

Applying Newton's second law for rotation gives

$$I\ddot{\theta} + \kappa\theta = 0,$$

$$\ddot{\theta} + \frac{\kappa}{I}\theta = 0,$$

which has exactly the form of the equation for simple harmonic motion. The angular eigenfrequency and the period of a torsion pendulum are therefore

$$\omega = \sqrt{\frac{\kappa}{I}}, \qquad T = 2\pi\sqrt{\frac{I}{\kappa}}.$$

Both the physical pendulum and the torsion pendulum are rotational realizations of simple harmonic motion. Despite their different physical origins, they are governed by the same mathematical structure as linear oscillators.

1.9.7. Damped harmonic motion

In real oscillating systems, mechanical energy is gradually lost due to friction, air resistance, or internal dissipation. As a result, the amplitude decreases with time. This behavior is known as **damped harmonic motion**.

A common model assumes a damping force proportional to the velocity and opposite to the direction of motion,

$$F_{\text{damp}} = -b\,\dot{x}.$$

For a mass–spring oscillator, the total force acting on the mass is therefore

$$F = -kx - b\,\dot{x},$$

and Newton’s second law gives the equation of motion

$$m\ddot{x} + b\dot{x} + kx = 0.$$

We already know that the undamped oscillator ($b = 0$) is described by a cosine, because the system repeatedly swings between two extreme positions. Damping should not destroy this oscillatory character (at least for weak damping), but it should cause the amplitude to decrease slowly in time. The simplest way to model such behavior is to multiply the oscillation by an exponential decay factor. We therefore **guess** a solution of the form

$$x(t) = x_0\,e^{-\gamma t}\cos(\omega t),$$

where $\gamma > 0$ controls the decay rate of the amplitude and ω is the (modified) angular frequency of the oscillation.

To determine which values of γ and ω are compatible with the equation of motion, we insert this form and require that all explicit time dependence cancels. We first compute the derivatives step by step.

For the first derivative, we apply the product rule,

$$x(t) = x_0\, e^{-\gamma t} \cos(\omega t),$$

which gives

$$\dot{x}(t) = x_0 \left(\frac{d}{dt} e^{-\gamma t} \right) \cos(\omega t) + x_0 e^{-\gamma t} \left(\frac{d}{dt} \cos(\omega t) \right).$$

Evaluating the derivatives yields

$$\dot{x}(t) = x_0 \left(-\gamma e^{-\gamma t} \cos(\omega t) - \omega e^{-\gamma t} \sin(\omega t) \right),$$

or

$$\dot{x}(t) = x_0 e^{-\gamma t} \left(-\gamma \cos(\omega t) - \omega \sin(\omega t) \right).$$

For the second derivative, we again use the product rule. Define

$$z(t) = -\gamma \cos(\omega t) - \omega \sin(\omega t), \qquad \dot{x}(t) = x_0 e^{-\gamma t} z(t).$$

Then

$$\ddot{x}(t) = x_0 \left(-\gamma e^{-\gamma t} z(t) + e^{-\gamma t} \dot{z}(t) \right) = x_0 e^{-\gamma t} \left(\dot{z}(t) - \gamma z(t) \right).$$

Differentiating $z(t)$ term by term gives

$$\dot{z}(t) = -\gamma \frac{d}{dt} \cos(\omega t) - \omega \frac{d}{dt} \sin(\omega t) = \gamma \omega \sin(\omega t) - \omega^2 \cos(\omega t).$$

Furthermore,

$$-\gamma z(t) = -\gamma \left(-\gamma \cos(\omega t) - \omega \sin(\omega t) \right) = \gamma^2 \cos(\omega t) + \gamma \omega \sin(\omega t).$$

Combining both contributions yields

$$\ddot{x}(t) = x_0 e^{-\gamma t} \left((\gamma^2 - \omega^2) \cos(\omega t) + 2\gamma\omega \sin(\omega t) \right).$$

Substituting $\dot{x}(t)$ and $\ddot{x}(t)$ into the equation of motion

$$m\ddot{x} + b\dot{x} + kx = 0$$

leads to

$$x_0 e^{-\gamma t} \Big[m(\gamma^2 - \omega^2) \cos(\omega t) + 2m\gamma\omega \sin(\omega t) + b\big(-\gamma \cos(\omega t) - \omega \sin(\omega t) \big) + k \cos(\omega t) \Big] = 0.$$

Collecting terms proportional to $\cos(\omega t)$ and $\sin(\omega t)$ gives

$$x_0 e^{-\gamma t} \Big[\big(m(\gamma^2 - \omega^2) - b\gamma + k \big) \cos(\omega t) + \big(2m\gamma\omega - b\omega \big) \sin(\omega t) \Big] = 0.$$

Since $\cos(\omega t)$ and $\sin(\omega t)$ are linearly independent functions, their coefficients must vanish separately. From the sine term we obtain

$$2m\gamma\omega - b\omega = 0 \quad \Rightarrow \quad \gamma = \frac{b}{2m}.$$

From the cosine term we obtain

$$m(\gamma^2 - \omega^2) - b\gamma + k = 0.$$

We first divide by m to isolate the frequencies,

$$\gamma^2 - \omega^2 - \frac{b}{m}\gamma + \frac{k}{m} = 0.$$

From the sine term we already found

$$\gamma = \frac{b}{2m} \quad \Rightarrow \quad \frac{b}{m} = 2\gamma,$$

so the cosine equation becomes

$$\gamma^2 - \omega^2 - 2\gamma^2 + \frac{k}{m} = 0.$$

Combining the γ terms gives

$$-\omega^2 - \gamma^2 + \frac{k}{m} = 0,$$

or equivalently

$$\omega^2 = \frac{k}{m} - \gamma^2.$$

Introducing the undamped angular eigenfrequency

$$\omega_0^2 = \frac{k}{m},$$

we finally obtain

$$\omega^2 = \omega_0^2 - \gamma^2.$$

Underdamped motion

For **weak damping**, the motion remains oscillatory. This requires the damped angular frequency to be real,

$$\omega^2 = \omega_0^2 - \gamma^2 > 0 \quad \Rightarrow \quad \gamma < \omega_0.$$

In this **underdamped** case, the displacement is

$$x(t) = x_0\, e^{-\gamma t} \cos(\omega t), \qquad \gamma = \frac{b}{2m}, \qquad \omega = \sqrt{\omega_0^2 - \gamma^2}.$$

The oscillation frequency is slightly reduced compared to the undamped case, while the exponential factor causes the amplitude to decay gradually with time. Further the angular frequency is altered from the eigenfrequency of in the undamped case.

Since both kinetic and potential energy scale with the square of the amplitude, the total mechanical energy decays as

$$E(t) = E_0 e^{-2\gamma t}.$$

Critical damping

The limiting case between oscillatory and non-oscillatory motion occurs when

$$\gamma = \omega_0.$$

This situation is known as **critical damping**. The general solution of the equation of motion then takes the form

$$x(t) = (C_1 + C_2 t)\, e^{-\gamma t},$$

where C_1 and C_2 are constants determined by the initial conditions.

In critically damped motion, the system returns to equilibrium **as fast as possible without oscillating**. This behavior is desirable in many technical applications, such as measuring instruments and shock absorbers, where overshoot must be avoided.

Overdamped motion

For **strong damping**, when

$$\gamma > \omega_0,$$

the motion is **overdamped**. The solution is a sum of two purely exponential terms,

$$x(t) = C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t},$$

with decay rates

$$\lambda_{1,2} = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}.$$

The two exponential terms correspond to **two distinct time scales** in the motion. One decay rate is large and leads to a fast initial relaxation, while the other is smaller and governs the slow approach to equilibrium at long times. Depending on the initial conditions, one of the terms may dominate early in the motion, but at sufficiently long times the term with the smaller decay rate always determines the behavior.

No oscillations occur in this case. Instead, the system approaches equilibrium monotonically and more slowly than in the critically damped case.

Summary damped harmonic motion

Damped harmonic motion therefore provides a realistic extension of simple harmonic motion. Depending on the strength of the damping, the system may oscillate with a decaying amplitude (underdamped), return to equilibrium as fast as possible without oscillation (critical damping), or relax slowly and non-oscillatorily (overdamped).

1.9.8. Forced oscillations and resonance

So far, we have studied oscillators that evolve freely once released. In many real situations, however, an oscillating system is acted upon by a **periodic external force** that continuously supplies energy. Such motion is called a **forced oscillation**. The competition between energy input from the driving force and energy loss due to damping leads to the phenomenon of **resonance**.

We consider a damped mass–spring system driven by a sinusoidal force

$$F(t) = F_0 \cos(\omega_d t),$$

where ω_d is the driving angular frequency. Newton's second law then gives

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega_d t).$$

At first, the motion depends on the initial conditions and resembles a freely damped oscillation, whose amplitude decreases with time. Because the system is continuously driven, this decaying motion is gradually replaced by a persistent oscillation maintained by the external force. After some time, only this driven motion remains, with a fixed amplitude and a fixed phase relation to the force.

Because the driving force is sinusoidal, we **guess** that the steady-state response has the same time dependence, but with a possibly different amplitude and a phase shift,

$$x(t) = A\cos(\omega_d t - \delta),$$

where A is the oscillation amplitude and δ is the phase difference between the displacement and the driving force. Substituting this ansatz into the equation of motion shows that the system oscillates at the **driving frequency** ω_d, not at its natural frequency $\omega_0 = \sqrt{k/m}$.

Carrying out the substitution and solving for the amplitude yields

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2}},$$

where $\gamma = b/(2m)$ is the damping constant. The amplitude therefore depends strongly on the driving frequency.

This dependence can be read directly from the expression for the amplitude. For **very low driving frequencies**, $\omega_d \ll \omega_0$, both ω_d^2 and $2\gamma\omega_d$ are small compared to ω_0^2. The denominator then simplifies to

$$\sqrt{(\omega_0^2-\omega_d^2)^2+(2\gamma\omega_d)^2} \;\approx\; \sqrt{\omega_0^4} \;=\; \omega_0^2.$$

Using $\omega_0^2 = k/m$, the amplitude becomes

$$A \approx \frac{F_0/m}{\omega_0^2} = \frac{F_0}{k}.$$

In this limit, the oscillator simply follows the applied force almost quasi-statically ($\delta \approx 0$, $x(t) \approx (F_0/k)\cos(\omega_d t)$), as if the spring were slowly stretched and compressed.

For **very high driving frequencies**, $\omega_d \gg \omega_0$, the term ω_d^2 dominates the expression. The denominator is then approximately

$$\sqrt{(\omega_0^2-\omega_d^2)^2+(2\gamma\omega_d)^2} \;\approx\; \sqrt{\omega_d^4} \;=\; \omega_d^2.$$

The amplitude therefore reduces to

$$A \approx \frac{F_0/m}{\omega_d^2} = \frac{F_0}{m\omega_d^2}.$$

In this regime, the response is small because the inertia of the mass prevents it from following the rapidly oscillating force.

Between these two limits, the amplitude reaches a maximum at the **resonance frequency**

$$\omega_r = \sqrt{\omega_0^2 - 2\gamma^2},$$

which lies slightly below ω_0 for weak damping. Inserting $\omega_d = \omega_r$ into the expression for A shows that the denominator reduces to approximately $2\gamma\omega_0$ for weak damping, so that

$$A(\omega_r) \approx \frac{F_0/m}{2\gamma\omega_0}.$$

The resonance amplitude is therefore inversely proportional to the damping strength.

At resonance, the driving force supplies energy to the oscillator at exactly the same average rate at which energy is dissipated by damping. When damping is small, this balance occurs at a large amplitude, which is why resonance can produce very strong oscillations.

The phase difference δ also depends on the driving frequency. At low frequencies, the motion is nearly in phase with the driving force. At resonance, the displacement lags the force by $\pi/2$. At high frequencies, the motion becomes nearly out of phase. Damping limits both the maximum amplitude and the sharpness of the resonance peak.

A convenient measure of the sharpness of the resonance is the **quality factor**

$$Q = \frac{\omega_0}{2\gamma}.$$

A large quality factor corresponds to weak damping and a sharp resonance, while strong damping leads to a broad and less pronounced resonance.

Forced oscillations and resonance explain how oscillatory systems can be selectively amplified or controlled by external driving. These ideas are fundamental in mechanical engineering, acoustics, electrical circuits, and many other areas of physics.