Lecture Tutorial 1A: Mathematical tools for physics

Scalars & vectors

- **Scalars**: magnitude only (e.g., mass, time, energy)
- **Vectors**: magnitude + direction (e.g., displacement, velocity, force)
- Notation: $\vec{v}$ or bold **v**; magnitude written $|\vec{v}|$ or v
- Link to physics (kinematics): speed is a scalar, velocity is a vector

Primer on vectors

- Graphical representation: arrow (length $\propto$ magnitude, arrow shows direction)
- Has components for each direction
- For 2D:

$$\vec{v} = v_x \hat{i} + v_y \hat{j} \qquad \text{or} \qquad \vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

- Resolve vector into axes: $v_x = |\vec{v}| \cos\theta$, $v_y = |\vec{v}| \sin\theta$
- Magnitude: $|\vec{v}| = v = \sqrt{v_x^2 + v_y^2}$
- Direction: $\theta = \tan^{-1}(v_y / v_x)$
- Components & angles: $\cos\theta = v_x / |\vec{v}| \quad \& \quad \sin\theta = v_y / |\vec{v}|$

Unit vectors

- Unit vectors: magnitude 1.
- Cartesian basis: $\hat{i}, \hat{j}, \hat{k}$.
- **Orthonormal**: $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$, and mutually perpendicular.

Vector operations: addition & subtraction

- Same direction $\rightarrow$ simple arithmetic
- Opposite direction $\rightarrow$ subtraction
- Different directions
    - $\rightarrow$ graphically with tail-to-tip or parallelogram
    - $\rightarrow$ numerically **if perpendicular** Pythagorean theorem for magnitude or trigonometry
    - $\rightarrow$ **general** numerical approach: compute for each components individually

Example: 2D vector addition

$$\vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Their sum is obtained by adding corresponding components:

$$\vec{c} = \vec{a} + \vec{b} = \begin{bmatrix} 3+1 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

Equivalently, in unit-vector notation:

$$\vec{a} = 3\hat{\imath} + 2\hat{\jmath}, \qquad \vec{b} = 1\hat{\imath} + 4\hat{\jmath}.$$

$$\vec{c} = \vec{a} + \vec{b} = (3\hat{\imath} + 2\hat{\jmath}) + (1\hat{\imath} + 4\hat{\jmath}).$$

Grouping the $\hat{\imath}$ and $\hat{\jmath}$ terms gives:

$$\vec{c} = (3+1)\hat{i} + (2+4)\hat{j} = 4\hat{i} + 6\hat{j}.$$

Scalar multiplication

- Multiplying a scalar c with a vector means multiplying **each component**:

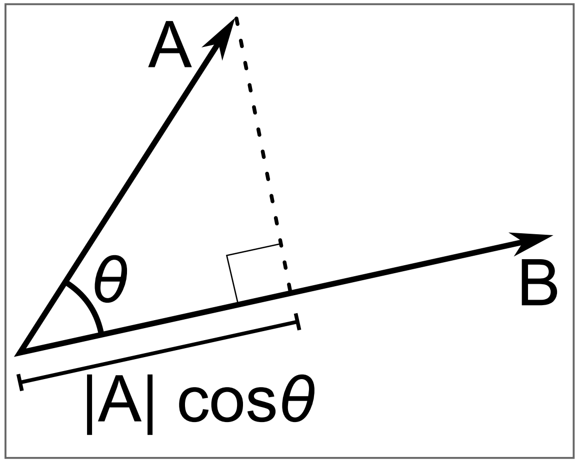
$$\text{If } \vec{v} = (v_x, v_y, v_z),\ \text{then } c\vec{v} = (cv_x, cv_y, cv_z).$$

- The **magnitude** of $\vec{v}$ changes by a factor of $|c|$ and the **direction** is reversed if $c < 0$.

Dot product

- Dot product $\rightarrow$ result is a **scalar**.

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = ab\cos\theta$$



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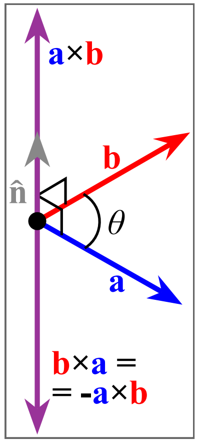
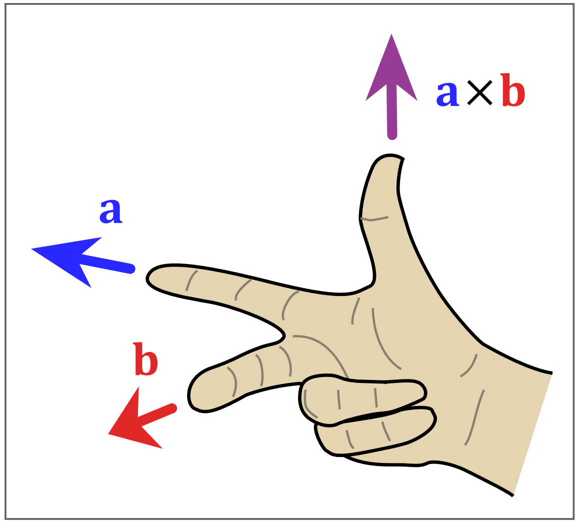
Cross product

- Cross product $\rightarrow$ result is a **vector** perpendicular to both $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\hat{i} - (a_x b_z - a_z b_x)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$





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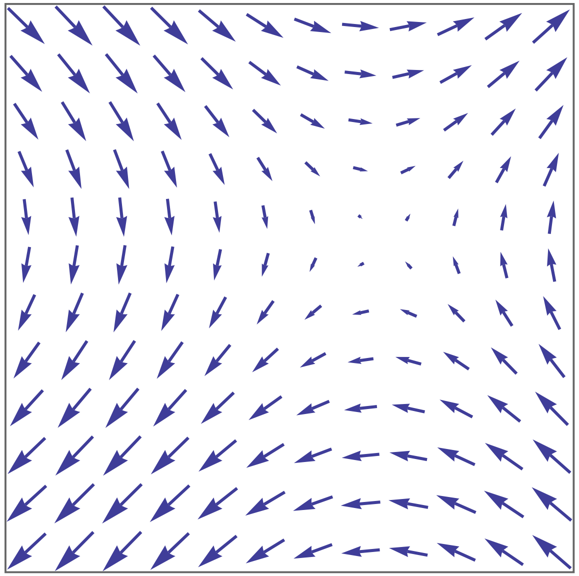
Vector fields

- A **vector field** assigns a vector to **each point in space**.
- Applications: Gravitational field, electric & magnetic field, fluid dynamics, etc.
- Example: Time-dependent velocity field:

$$\vec{v}(x, y, z, t) = v_x(x, y, z, t)\,\hat{i} + v_y(x, y, z, t)\,\hat{j} + v_z(x, y, z, t)\,\hat{k}.$$

- $\rightarrow$ Interpretation: Each point in the fluid has a local velocity vector that can **change in space and time**.



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Derivatives: Concept

- **derivative** measures the **instantaneous rate of change** or **slope** of a function.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- $\rightarrow$ It tells how much $f(x)$ changes when x changes a little bit.

Derivatives: Basic rules

Type	Formula	Example
Power	$(x^n)' = nx^{n-1}$	$(x^3)' = 3x^2$
Constant	$(c)' = 0$	$(5)' = 0$
Scalar multiple	$(cf)' = cf'$	$(4x^2)' = 4(2x) = 8x$
Sum / difference	$(f \pm g)' = f' \pm g'$	$(x^2 + \sin x)' = 2x + \cos x$

Derivatives: Chain Rule

Used for **nested functions** $f(g(x))$:

$$(f(g(x)))' = f'(g(x))\, g'(x)$$

Example:
If $f(u) = \sin u$ and $u = 3x^2$, then

$$\frac{d}{dx}[\sin(3x^2)] = \cos(3x^2) \cdot (6x) = 6x \cos(3x^2)$$

Common derivatives

- $(e^x)' = e^x$
- $(a^x)' = a^x \ln a$
- $(\ln x)' = 1/x$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x$

Integrals – Concept

- An **integral** is the **inverse operation** of differentiation.
- The integral measures the **signed area** between $f(x)$ and the x-axis (above axis = positive, below = negative).
- $\rightarrow$ It represents the **net accumulated change**.

Indefinite integral:

$$\int f(x)\,dx = F(x) + C$$

- where C is the **constant of integration**.

Definite integral:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- gives the **net accumulated change** of $f(x)$ between a and b.

Common integrals

Function $f(x)$	**Integral** $\int f(x)\,dx$	**Notes**
x^n	$\frac{x^{n+1}}{n+1} + C$	$n \neq -1$
e^x	$e^x + C$	Exponential function
a^x	$\frac{a^x}{\ln a} + C$	Base $a > 0$, $a \neq 1$
$\frac{1}{x}$	$\ln x + C$	Natural logarithm
$\sin x$	$-\cos x + C$	
$\cos x$	$\sin x + C$	
$\frac{1}{1+x^2}$	$\arctan x + C$	Inverse tangent

Integral calculus for kinematics

Differentiation and integration connect **acceleration**, **velocity**, and **position**. By definition:

$$a(t) = \frac{dv}{dt}, \qquad v(t) = \frac{dx}{dt}.$$

Thus, integrating step by step:

$$v(t) = \int a(t)\,dt, \qquad x(t) = \int v(t)\,dt.$$

Integral calculus for kinematics: Constant acceleration

For $a(t) = a_0$:

$$v = v_0 + a_0 t, \qquad x = x_0 + v_0 t + \tfrac{1}{2} a_0 t^2.$$

Integral calculus for kinematics: Time-dependent acceleration

If $a(t) = \alpha t + \beta$:

$$v(t) = \tfrac{1}{2}\alpha t^2 + \beta t + v_0, \qquad x(t) = \tfrac{1}{6}\alpha t^3 + \tfrac{1}{2}\beta t^2 + v_0 t + x_0.$$

Integral Calculus for Kinematics: When a depends on x

Suppose the acceleration depends on **position**, i.e.

$$a = a(x) = \frac{dv}{dt}.$$

Both velocity and position depend on time:

$$v = v(t), \qquad x = x(t).$$

That means velocity can also be seen as a function of position: $v = v(x(t))$.
By the **chain rule** from calculus:

$$a(x) = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}\, v$$

Rearrange to collect v and x terms on opposite sides:

$$v\,dv = a(x)\,dx.$$

Integral Calculus for Kinematics: When a depends on x (cont')

Integrate from the **initial state** (x_0, v_0) to the **current state** (x, v):

$$\int_{v_0}^{v} v'\,dv' = \int_{x_0}^{x} a(x')\,dx'.$$

Evaluate the left-hand side:

$$\frac{1}{2}(v^2 - v_0^2) = \int_{x_0}^{x} a(x')\,dx'.$$

Example: If $a(x)$ is linear, e.g. $a(x) = \alpha x$:

$$\frac{1}{2}(v^2 - v_0^2) = \int_{x_0}^{x} a(x')\,dx' = \int_{x_0}^{x} \alpha x'\,dx' = \alpha[\frac{x'^2}{2}]_{x_0}^{x} = \frac{\alpha}{2}\,(x^2 - x_0^2).$$

Simplify:

$$v^2 - v_0^2 = \alpha\,(x^2 - x_0^2).$$

Integral Calculus for Kinematics: When a depends on v

Now suppose acceleration depends on **velocity**, i.e.

$$a = a(v) = \frac{dv}{dt}.$$

Rearrange to isolate v and t:

$$\frac{dv}{a(v)} = dt.$$

Integrate from v_0 at t_0 to v at t:

$$\int_{v_0}^{v} \frac{dv'}{a(v')} = \int_{t_0}^{t} dt' = t - t_0.$$

Integral Calculus for Kinematics: When a depends on v (cont')

Example: If acceleration depends linearly on velocity $a(v) = -kv$, $\&\, k > 0$, then:

$$a(v) = -kv = \frac{dv}{dt}.$$

Separate variables:

$$-k\,dt = \frac{dv}{v} \;\leftrightarrow\; \frac{1}{v}\,dv = -k\,dt.$$

Integrate from (t_0, v_0) to (t, v):

$$\int_{v_0}^{v} \frac{1}{v'}\,dv' = -k\int_{t_0}^{t} dt'.$$

Compute both sides:

$$\ln(\frac{v}{v_0}) = -k(t - t_0).$$

Exponentiate to solve for $v(t)$:

$$v(t) = v_0\, e^{-k(t-t_0)}.$$

$\rightarrow$ The velocity decays **exponentially** due to the drag force.

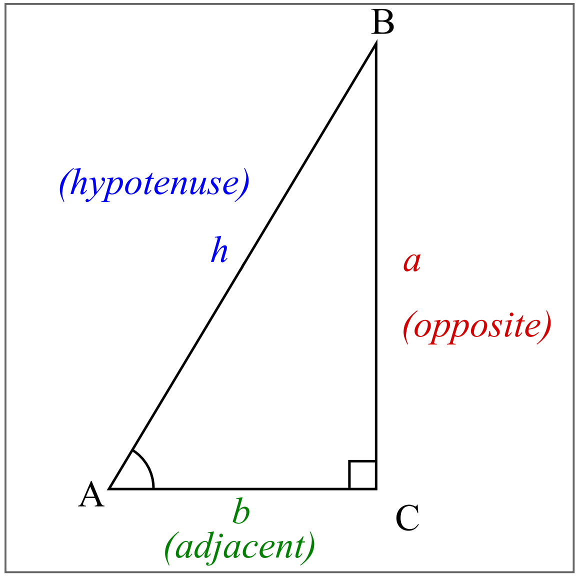
Trigonometry – basics

- Right triangle:

$$\sin\theta = \frac{opp}{hyp} \,\&\, \cos\theta = \frac{adj}{hyp} \,\&\, \tan\theta = \frac{opp}{adj}$$

- Identities:

$$\sin^2\theta + \cos^2\theta = 1 \,\&\, \tan\theta = \sin\theta/\cos\theta$$



Polar coordinates

- Represent point by (r, θ) instead of (x, y).
- Conversion:
    - $x = r\cos\theta$, $y = r\sin\theta$
- Inverse:
    - $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$