2.8. Maxwell’s equations & electromagnetic waves

The development of electromagnetic theory in the 19th century was a collaborative effort involving key figures such as Oersted, Ampère, and Faraday. Initially, the understanding of electricity and magnetism was fragmented, with these phenomena being studied in isolation. *James Clerk Maxwell* (1831–1879) unified electricity and magnetism through four fundamental equations, known as Maxwell’s equations. Furthermore, he used the concept of fields, introduced by Faraday, and predicted the existence of electromagnetic waves, which were later experimentally confirmed by *Heinrich Hertz* (1857-1894), providing a cornerstone for the development of modern communication technologies. The Maxwell's equations are as fundamental to electromagnetism as Newton's laws are to mechanics. Moreover, Maxwell's equations are consistent with the theory of relativity, making them even more foundational in the context of modern physics.

However, before we dive into Maxwell's equation, we start by revisiting Ampère's law and Gauss's law in the context of magnetism.

2.8.1. Revisiting Ampère's law: Changing electric fields produce magnetic fields

Physicists are suckers for symmetry. We already know that:

1. time-constant electric currents generate time-constant magnetic fields (first observed by Ørsted)
2. changing magnetic fields generate electric fields (Faraday's law)

Hence, it seems only intuitive that changing electric fields produce magnetic fields. It was actually one of Maxwell's key contributions to realize that a *changing electric field produces a magnetic field*, leading to the concept of *displacement current*.

Maxwell's correction of Ampère’s law: The need for displacement current

From earlier discussions on electromagnetism, we recall Ampère’s law, which states that the circulation of the magnetic field $\vec{B}$ around a closed loop is proportional to the total current enclosed by that loop:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I_{\text{enc}}$$

with μ_0 as the permeability of free space and I_{enc} as the enclosed current.

Maxwell realized that Ampère’s law runs into contradiction when observing the case of a charging capacitor. Let's start with two observations:

1. Charges do not pass the gap between the plates of the capacitor.
2. During (dis)charing, the current in the wire generates a magnetic field

Imagine the surface used to evaluate Ampère’s law is not a circle but a open bag, like a garbage bag. We "pull" the bag over one plate of the capacitor and up to the wire without closing the bag. Thus, the bag, a.k.a. surface "senses" a magnetic field, i.e. $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} \neq 0$, but there is no current flowing through our surface because we do not cross the wire and inside the capacitor gap no current is flowing. Since the (dis)charging means the electric field changes, Maxwell proposed that the change in electric field induces a magnetic field. Further, this change in electric field is represented by the (somewhat misleadingly named) *displacement current* I_D. There is no actual flow of charges, it is only the equivalent produced by the change in electric field. For further differentiation, conventional flow of charges is called *conduction current*.

Let's describe the displacement current mathematically, by assuming we have a changing electric field inside a capacitor. First, we combine three equations which we already know to describe charge Q, capacitance C, and voltage V in presence of an uniform electrical field E in between the capacitor's plates with distance d:

$$Q = CV = (\epsilon_0 \frac{A}{d})(Ed) = \epsilon_0 AE$$

The current of the (dis)charging capacitor can be described as:

$$I = \frac{dQ}{dt} = \epsilon_0 A \frac{dE}{dt} = \epsilon_0 \frac{d\Phi_E}{dt}$$

with the electric flux $\Phi_E = AE$ through the surface (our "garbage bag"). Hence, this current through the garbage bag, which is actually just a changing electric flux over time, is our displacement current:

$$I_D = \epsilon_0 \frac{\Phi_E}{dt}$$

Now we combine provide Maxwell's correction, i.e. the contribution due to displacement currents, to Ampère's law:

$$\oint \vec{B} \cdot d\vec{L} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

Note that, the addition of the displacement current is the basis for electromagnetic waves (see later section) as it shows that electric fields can generate magnetic fields.

Magnetic Field Outside the Plates

Outside the plates, the field satisfies the equation:

$$B = \frac{\mu_0 I}{2\pi r}$$

which matches the field of a current-carrying wire, showing that the displacement current generates the same magnetic field as a conduction current.

2.8.2. Gauss's law for magnetism

Before we introduce Maxwell's equation, we need to revisit Gauss's law which describes the electric flux Φ_E through a closed surface:

$$\Phi_E = \oint \vec{\mathbf{E}} d\vec{\mathbf{A}} = \frac{Q_{enc}}{\epsilon_0}$$

In essence, the electric flux through a closed surface is proportional to the enclosed charge. To adapt Gauss's law to magnetism, we simply recap that no magnetic monopole have been observed so far. Unlike for electrical fields, there are no sources or sinks of the magnetic fields. Hence, magnetic field lines are continuous, a.k.a. loops, and, for a close surface, the same number of lines enter and leave the surface. Therefore, the surface integral should be zero which is the motivation for *Gauss's law for magnetism*:

$$\oint \vec{\mathbf{B}} d\vec{\mathbf{A}} = 0$$

```
interactive(children=(RadioButtons(description='Mode:', options=('Electricity', 'Magnetism'), value='Electrici…
<function __main__.combined_gauss_law_simulation(mode, surface_x, surface_y, surface_radius, q_mag=0, magnet_magnitude=1)>
```

2.8.3. Maxwell’s Equations

The four Maxwell’s equations in differential form are:

Gauss's law for electricity:

$$\oint \vec{\mathbf{E}} d\vec{\mathbf{A}} = \frac{Q_{enc}}{\epsilon_0}$$

This states that electric charges produce electric fields, and the total flux of $\mathbf{E}$ through a closed surface is proportional to the charge enclosed.

`Note: We take the integral over a closed surface.`

Gauss’s law for magnetism:

$$\oint \vec{\mathbf{B}} d\vec{\mathbf{A}} = 0$$

This implies that magnetic field lines have no beginning or end, meaning no magnetic monopoles exist.

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Note: We take the integral over a closed surface.
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Faraday’s law of induction:

$$\oint \vec{\mathbf{E}} d\vec{\mathbf{l}} = -\frac{d\Phi_B}{dt}$$

This describes how a changing magnetic field induces an electric field, which is the principle behind electrical generators.

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Note: We take the integral around a closed path.
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Ampère's law with Maxwell correction:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

This states that an electric current and/or a changing electric field generates a magnetic field.

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Note: We take the integral around a closed path.
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Summary

We can express Maxwell's equations in integral or differential form. In the following table, the suffix "enc"/enclosed was dropped for simplicity:

Equation Name	Integral Form	Differential Form
Gauss's law for electricity	$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\epsilon_0}$	$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0}$
Gauss's law for magnetism	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$	$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$
Faraday's law of induction	$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d\Phi_B}{dt}$	$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$
Ampère's law with Maxwell's correction	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$	$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{j}} + \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$

with $\vec{\nabla} = \hat{\mathbf{i}} \frac{\delta}{\delta x} + \hat{\mathbf{j}} \frac{\delta}{\delta y} + \hat{\mathbf{k}} \frac{\delta}{\delta z}$, ρ as the total volume charge density, and $\vec{\mathbf{j}}$ as the current density.

Significance of Maxwell's Equations

Maxwell's Equations are much more than just a set of equations describing electricity and magnetism. Their significance is profound and far-reaching:

1. **Unification of Electricity and Magnetism:** They revealed that electricity and magnetism are not separate entities. They are intrinsically linked.

2. **Prediction of Electromagnetic Waves and Light:**
   Maxwell's equations predicted the existence of electromagnetic waves that propagate at a speed calculated from ϵ_0 and μ_0. This speed turned out to be the speed of light (see later section). This was a revolutionary discovery, showing that light itself is an electromagnetic wave. This unified optics with electromagnetism.

3. **Foundation of Classical Electromagnetism:** Maxwell's equations form the complete and consistent foundation of classical electromagnetism. They describe all classical electromagnetic phenomena. They are used in countless applications across engineering and physics, from electrical circuits and antennas to motors, generators, and telecommunications.

4. **Basis for Special Relativity:**
   While formulated within classical physics, Maxwell's equations played a crucial role in the development of Einstein's theory of Special Relativity.
   The fact that the speed of light is constant for all observers, a postulate of special relativity, is deeply connected to Maxwell's equations and the properties of electromagnetic waves.

2.8.4. Electromagnetic Waves

For simplicity, we assume empty space, thus, no charges are present. Nevertheless, a changing electric field (due to the concept of the displacement current) would produce and changing magnetic field, which again would produce a changing electric field. Hence, the fields interact with each other and in doing so would propagate through space even without any matter being present. This gives rise to *electromagnetic (EM) waves* which were proposed by Maxwell and later experimentally proven by Hertz.

Basic concept & generation

Consider two conducting rods that will serve as an "antenna" and are connected to the opposite terminals of a battery. Right after connecting the uncharged rods, one rod quickly becomes positively charged while the other one negatively charged. Hence, an electric field and a magnetic field are produced near the antenna and they are perpendicular to each other. While the static field would go extent outwards indefinitely far (although with decreasing intensity), their propagation, i.e. transport of energy, is not instantaneous but bound to a finite speed (see later section).

Now let us use an AC source instead of a battery. While alternating, the current will constantly produce changing fields. Interestingly, we can image the field lines from the previous cycle and current one to form closed loops while propagating through space. While magnetic field lines are inherently closed, in this scenario this also applies to the electric field lines (see script 6, electric fields from magnetic fields, i.e. no charges enclosed, have closed field lines). While complex to describe near the antenna, i.e. the *near field*, far away from the antenna, i.e. the *far field* or also called *radiation field*, the resulting EM waves are reasonably flat and can be approximated as *plane waves*. The EM waves are further characterized by the fact that electric $\vec{\mathbf{E}}$ and magnetic field $\vec{\mathbf{B}}$ are always *perpendicular* to each other and *waves travel perpendicular to the* $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ with the wave velocity ν (greek lower-case nu). Hence, *EM waves are transverse waves*. Further, the electric and magnetic field are *in phase*, i.e. are at the same point in space zero or at their maximum. The intriguing aspect of these EM waves is that they are pure waves of fields and not matter (they still carry energy and have an associated momentum, see later). Last but not least, not just antennas or other AC-sources can produce EM waves. *Any accelerating electrical charge gives rise to an EM wave.*

Mathematical description and wave velocity

In general, Maxwell's equation simplify in empty space, i.e. with zero charge and zero current enclosed, to:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 0$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

While these equation show an intriguing degree of symmetry and highlight again the importance of the displacement current, we can also describe the sinusoidal waves using the wavelength λ and frequency f :

$$E = E_y = E_0 \sin(kx - \omega t)$$

$$B = B_z = B_0 \sin(kx - \omega t)$$

with the wave number $k = \frac{2\pi}{\lambda}$, angular frequency $\omega = 2\pi f$, and using the wave velocity's relation $\nu = f\lambda = \frac{\omega}{k}$

Note that we arbitrarily set the direction of velocity in x, the electric field oscillation in y, and the magnetic field oscillation in z direction. Further, we defined the waves as moving to the right, i.e. $(kx - \omega t)$, instead of moving to the left, i.e. $(kx + \omega t)$.

The velocity ν can be described by:

$$\nu = \frac{E_0}{B_0} = \frac{\omega}{k}$$

$$\nu = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c$$

Hence, the EM wave's propagation from empty space depends not on the fields itself but the permittivity and permeability of free space. Further, the velocity is equal to the speed of light which therefore means:

$$c = \frac{E}{B}$$

Energy & Poynting vector

While the EM wave is not a wave of matter but pure fields, it still caries energy. From previous chapters we know:

- the energy density u_E (J/m^3) of an electric field E is $u_E = \frac{1}{2}\epsilon_0 E^2$
- the energy density u_B (J/m^3) of an magnetic field B is $u_B = \frac{1}{2}\frac{B^2}{\mu_0}$

Thus, the total energy density u is:

$$u = u_E + u_B = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}$$

With $B = \frac{E}{c}$ and $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$, we can reformulate the previous equation into multiple version:

$$u = \epsilon_0 E^2 = \epsilon_0 c^2 B^2 = \frac{B^2}{\mu_0} = \epsilon_0 cEB = \sqrt{\frac{\epsilon_0}{\mu_0}}EB$$

Let's determine what *energy is transported per unit time and unit area* by the wave. This is characterized by the Poynting vector $\vec{\mathbf{S}}$ (named after John Henry Poynting (1852-1914)) which points in the direction the energy is transported, i.e. in the direction of the wave's velocity:

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$$

$\vec{\mathbf{S}}$ is reported in units of W/m^2 and its average can be computed as:

$$\overline{S} = \frac{1}{2}\epsilon_0 c E_0^2 = \frac{1}{2}\frac{c}{\mu_0}B_0^2 = \frac{E_0 B_0}{2\mu_0} = \frac{E_{rms}B_{rms}}{\mu_0}$$

Radiation pressure

As the EM waves carries energy, it will be associated with a linear momentum. Following Newtons second law, force is defined as the rate of change of momentum:

$$F = ma = m\frac{dv}{dt} = \frac{d(mv)}{dt} = \frac{dp}{dt}$$

Thus, if the EM wave carries energy ΔU at the speed of light c, it has a momentum Δp, which will be transferred to the surface of an object upon "collision".

$$\Delta p = \frac{\Delta U}{c} \quad \text{radiation fully absorbed}$$

$$\Delta p = \frac{2\Delta U}{c} \quad \text{radiation fully reflected}$$

Let's investigate the pressure P due to the force F on the area A for fully absorbed radiation:

$$P = \frac{F}{A} = \frac{1}{A}\frac{dp}{dt} = \frac{1}{Ac}\frac{dU}{dt} = \frac{\overline{S}}{c}$$

as the Poynting vector is energy per unit time and unit area.

For fully reflected radiation the equation becomes

$$P = \frac{\overline{2S}}{c}$$

Radiation pressure from electromagnetic waves influences astrophysical phenomena like comet tails and stellar dynamics, and enables technologies such as solar sails for spacecraft propulsion. Additionally, it forms the basis for precise laser manipulation techniques, like optical tweezers, used in microscopic research.

Electromagnetic spectrum & and light

As predicted by Maxwell, all EM waves move at the speed of light, independent of their wavelength:

$$c = \frac{E}{B} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 300.000.000 \text{ m/s} = 300 \times 10^6 \text{ m/s}$$

Nowadays, we accept that light is part of the electromagnetic spectrum, but this idea was revolutionary and, initially, controversial in the 19th century. Light occupies the frequencies from 4.0×10^14 Hz to 7.5×10^14 Hz which corresponds to wavelengths between 400 nm to 750 nm. The relation of frequency f and wavelength λ is governed by:

$$c = \lambda f$$

Beyond our eye, we can "detect" EM waves with our skin, i.e. infrared (IR) radiation. Besides visible light, the sun emits the bulk of its radiation in the IR as well as ultraviolet (UV) frequency band. The molecule in our skin "resonate" at the IR frequency spectrum, thus, the absorb these IR waves which dissipate energy as heat. While the human body is beside these three frequency bands rather insensitive to other parts of the EM spectrum, there are plenty of technical applications and natural phenomena to be found such as radio waves, microwaves, x-rays, and gamma waves.

This concludes our studies of electromagnetism. In the next chapter we will continue with light, first as a ray, then again as a wave.