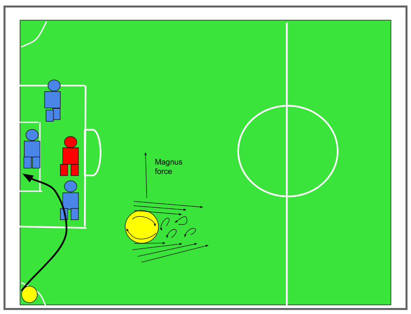
1.8. Fluid dynamics

`mf23 - Magnus effect`

Witchcraft?

⇒ Study **fluids in motion** and their effects



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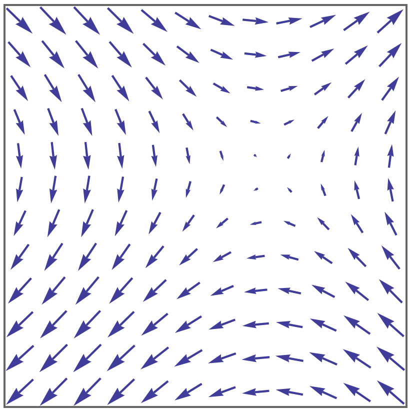
Types of Fluid Flow

Fluid motion is described by a **velocity vector field**:

$$\vec{v}(x, y, z, t) = v_x(x, y, z, t)\,\hat{i} + v_y(x, y, z, t)\,\hat{j} + v_z$$

$\rightarrow$ Every point in the fluid has a local velocity vector that may **vary in space and time**
Flow types:

- Steady vs. Unsteady: **steady (stationary)** $\rightarrow$ velocity field does not change with time; **unsteady (time-dependent)** $\rightarrow$ velocity field evolves in time
- Laminar vs. Turbulent: **laminar** $\rightarrow$ smooth, ordered motion, no mixing of layers; **turbulent** $\rightarrow$ chaotic motion with eddies and fluctuations



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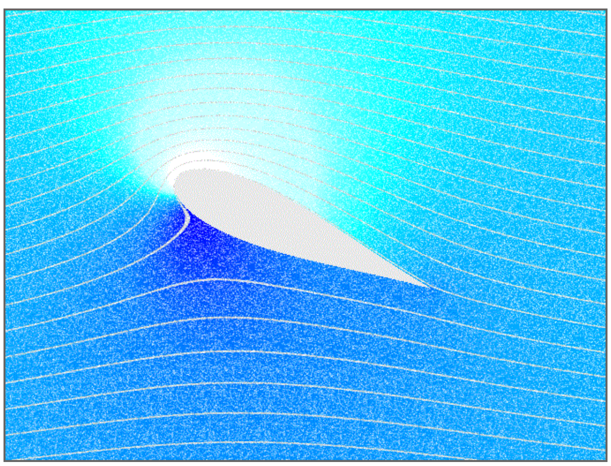
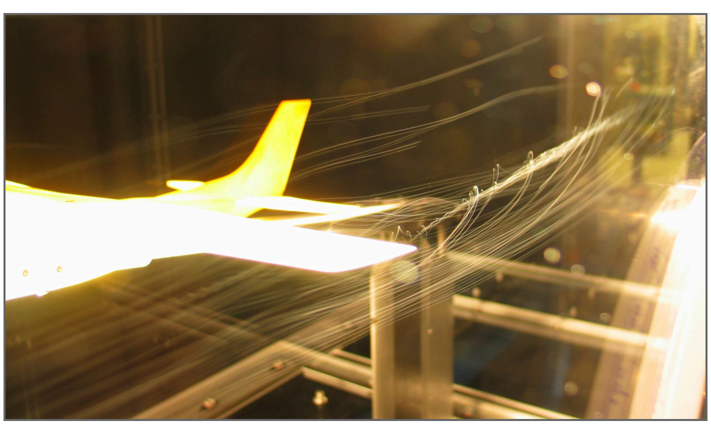
Streamlines

`mf28 - Wellenwanne`

A **streamline** is a curve everywhere **tangent** to the instantaneous velocity field:

- Streamlines never intersect.
- Dense spacing indicates higher speed.
- In **steady** flow, streamlines coincide with particle paths; in **unsteady** or turbulent flow, streamlines exist but change with time.





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Volume & Mass Flow Rate

Given A the cross-sectional area, v the average speed, and ρ the density, we define:

Volume flow rate

$$Q = \frac{dV}{dt} = Av$$

Mass flow rate

$$\frac{dm}{dt} = \rho \frac{dV}{dt} = \rho Av$$

$$\dot{m} = \rho Q$$

For **incompressible fluids** $(\rho = \text{constant}) \to Q$ and $\dot{m}$ **remain constant**

Continuity Equation

Mass conservation in steady flow requires

$$\dot{m} = \rho A v = \text{constant}$$

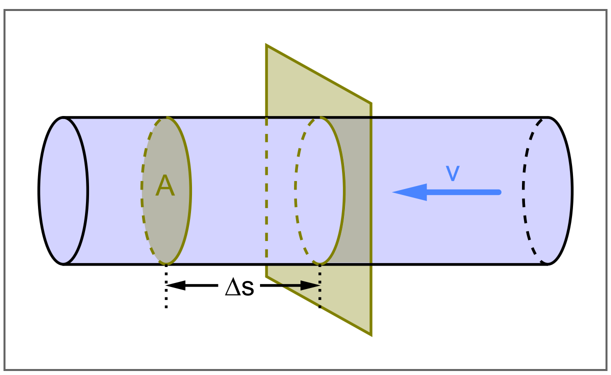
Between two cross-sections:

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

For **incompressible flow** ($\rho_1 = \rho_2$):

$$A_1 v_1 = A_2 v_2 = Q = \text{constant}$$

$\rightarrow$ Narrow sections (small A) $\rightarrow$ higher velocity



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Bernoulli's Equation: Assumptions & Energy of a Fluid Element

Assumptions

- Steady flow (no time dependence)
- Incompressible fluid ($\rho = \text{const}$)
- Non-viscous (no friction losses $\rightarrow$ **energy conserved**)
- Motion along a streamline

Energy of a small volume element (dV, $m = \rho dV$)

- Kinetic energy: $\frac{1}{2}\rho v^2\,dV$
- Potential energy: $\rho g h\,dV$
- Pressure energy (work done **on** dV): $P\,dV$

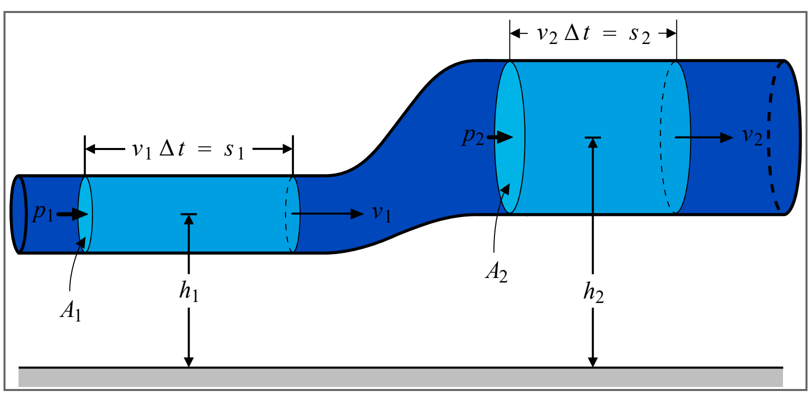
Bernoulli's Equation: From Energy to Formula

Energy conservation for volume element dV in an ideal fluid flow along a streamline:
Total mechanical energy per unit volume is constant:

$$P\,dV + \frac{1}{2}\rho v^2 dV + \rho g h\,dV = \text{const.}$$

Divide by dV (incompressible → dV constant) gives **Bernoulli's equation**:

$$\boxed{\underbrace{P}_{\text{pressure energy density}} + \underbrace{\frac{1}{2}\rho v^2}_{\text{kinetic energy density}} + \underbrace{\rho g h}_{\text{gravitational ener}}}$$



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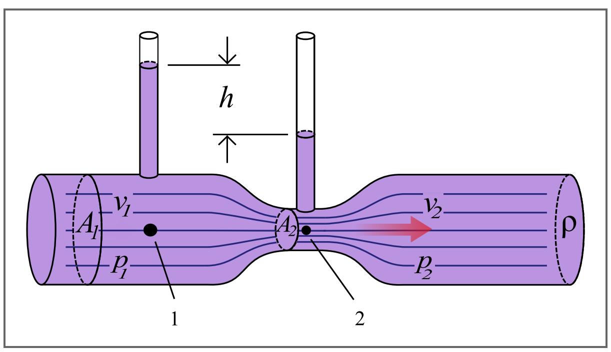
Venturi Effect (Horizontal Flow)

`mf35 - Wasserstrahlpumpe`

- Horizontal flow: $h_1 = h_2$
- Bernoulli: $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$
- Continuity: $A_1 v_1 = A_2 v_2 \rightarrow v_2 > v_1$ in constriction
- Pressure drop:

$$\Delta P = \frac{1}{2}\rho(v_2^2 - v_1^2)$$

- Narrow section → higher speed → lower pressure



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Torricelli's Theorem (Efflux Speed)

`mf06 - Schweredruck vs. ausströmende Flüssigkeit`

- Apply Bernoulli between the liquid surface and the hole:

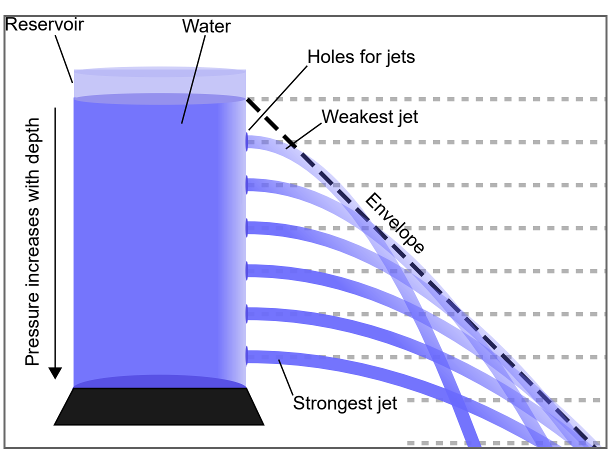
$$P_{\text{atm}} + \rho g h = P_{\text{atm}} + \frac{1}{2}\rho v^2$$

$$\rho g h = \frac{1}{2}\rho v^2$$

- Efflux speed: **Speed of the outflowing jet equals the speed a body would acquire when falling freely through the height h.**

$$v = \sqrt{2gh}$$

(Note: h is the height of liquid above the hole, not the height above ground.)



Hydrodynamic Paradox

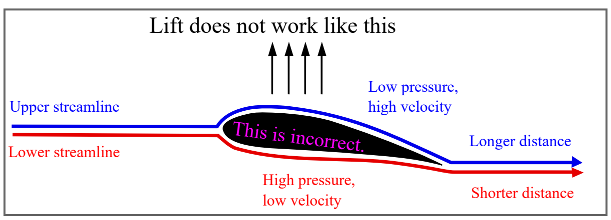
`mf30 - Hydrodynamisches Paradoxon`

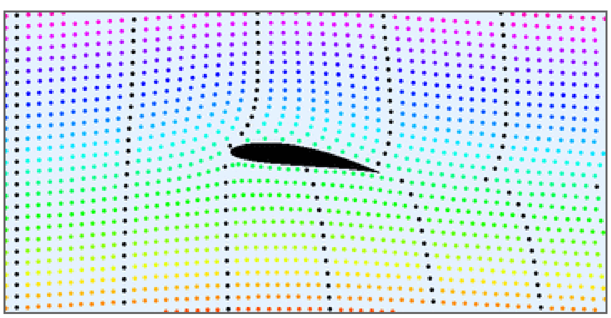
- Faster flow → lower pressure (Bernoulli).
- A fast jet can **pull** objects toward it instead of pushing them away.
- Thus, paper lifted toward a blowing stream, balls “sticking” in an air jet.

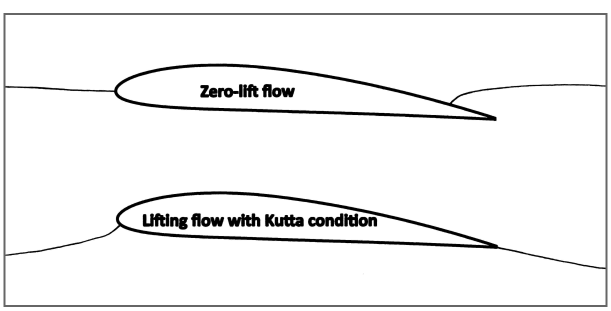
Aside: Why Bernoulli Cannot Explain Lift

`mf42 - Stromlinien auf Velourpapier`

- Bernoulli's “faster flow → lower pressure → lift” is **not** the full explanation
- Real lift requires viscosity & boundary layer, circulation (Kutta condition), asymmetric flow due to wing shape / angle of attack







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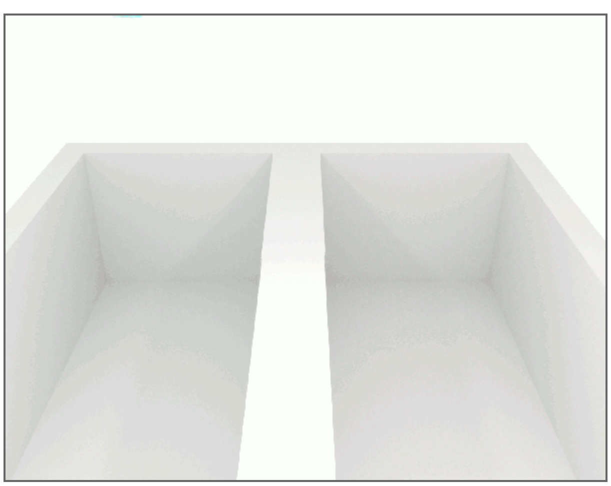
Viscosity

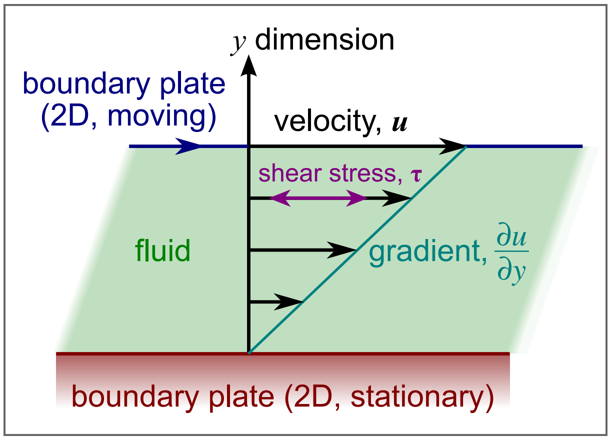
`mf22 - Viskosität`

- Fluid between lower stationary plate and upper plate moving with v under tangential force $F_{\parallel}$
- **Viscosity = internal friction** between fluid layers
- Shear stress is proportional to the velocity gradient: $\frac{F_{\parallel}}{A} = \eta \frac{dv}{dy}$
- Required force to keep the plate moving:

$$F_{\parallel} = \eta A \, \frac{v}{d}$$

- Units of viscosity: $[\eta] = \mathrm{Pa} \cdot \mathrm{s}$ (Poise: $1\,\mathrm{P} = 0.1\,\mathrm{Pa} \cdot \mathrm{s}$)





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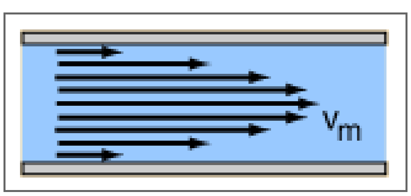
Parabolic Laminar Flow (Poiseuille flow)

- Tube wall is stationary $\rightarrow$ **no-slip condition**: $v = 0$ at $r = R$
- Fluid at the center moves fastest $\rightarrow$ velocity decreases toward the wall
- **Viscosity couples layers**: inner layers drag outer layers $\rightarrow$ smooth velocity variation
- Cylindrical geometry + viscous shear → **parabolic profile**

$$v(r) = v_{\max}\left(1 - \frac{r^2}{R^2}\right)$$



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Poiseuille's Equation (1): Setup, Assumptions, and Pressure Force

- Steady, incompressible, Newtonian fluid flow in rigid cylindrical tube of radius R
- **Viscosity causes energy loss** $\Rightarrow$ **pressure must drop along the tube**

Pressure acts on the cross-section $A_{\perp} = \pi r^2$ giving use the differential pressure force:

$$dF_P = -\,dP\,\pi r^2$$

Per unit length:

$$\frac{dF_P}{dx} = -\,\frac{dP}{dx}\,\pi r^2$$

Note: $-\frac{dP}{dx} > 0$ because pressure decreases downstream

Poiseuille's Equation (2): Viscous Force on the Cylindrical Shell

Viscous shear acts on the lateral surface
$dA_{\parallel} = 2\pi r \cdot dx$ (circumference · length)
Differential viscous force:

$$dF_v = \eta \, dA_{\parallel} \, \frac{dv}{dr} = 2\pi r \, dx \, \eta \, \frac{dv}{dr}$$

Per unit length:

$$\frac{dF_v}{dx} = 2\pi r \, \eta \, \frac{dv}{dr}$$

Poiseuille's Equation (3): Force Balance and Velocity Profile

Set pressure and viscous forces equal (steady flow):

$$-\frac{dP}{dx}\,\pi r^2 = 2\pi r\,\eta\,\frac{dv}{dr}$$

Solve for dv by integrating:

$$\int_{v(r)}^{0} dv = -\frac{1}{2\eta}\,\frac{dP}{dx}\,\int_{r}^{R} r\,dr$$

$$v(r) = \frac{1}{4\eta}\,\frac{dP}{dx}\,(R^2 - r^2)$$

With constant pressure gradient $\frac{dP}{dx} = -\frac{\Delta P}{L}$, final velocity profile:

$$v(r) = \frac{\Delta P}{4\eta L}(R^2 - r^2)$$

Poiseuille's Equation (4): Volume Flow Rate

Sum contributions from all concentric shells with area element $dA = 2\pi r\,dr$:

$$Q = \int_0^R v(r)dA = \int_0^R v(r)\,2\pi r\,dr.$$

Insert $v(r)$ give **Poiseuille's equation** for laminar flow through a circular tube:

$$Q = \frac{\pi R^4 \Delta P}{8\eta L}.$$

Poiseuille's Equation (5): Demo

`mf24 - Gesetz von Bernoulli und Hagen – Poiseuille`

$$v(r) = \frac{\Delta P}{4\eta L}(R^2 - r^2)$$

$$Q = \frac{\pi R^4 \Delta P}{8\eta L}.$$

$\rightarrow$ Poiseuille's equation links **pressure**, **radius**, **viscosity**, and **flow rate** in laminar flow.
$\rightarrow$ Flow depends very strongly on tube radius $\propto R^4$.
$\rightarrow$ Velocity profile is parabolic; $v_{\max} = 2v_{\text{avg}}$.
$\rightarrow$ Viscosity causes linear pressure drop along the tube.
$\rightarrow$ Breaks down for turbulent or non-Newtonian flow. **Application**: Blood flow regulation relies strongly on vessel diameter.
Reminder: Bernoulli's equation
$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$

Navier–Stokes Equation: Motivation

- Real fluid flow can be **unsteady**, **accelerating**, or even **turbulent**
- Described by **incompressible Navier–Stokes equation**:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v} + \vec{g}$$

- Instead of deriving it, we will **build intuition** using an **analogy from Prof. René Matzdorf**
- Imagine: **you are a person inside a dense moving crowd** → a small **fluid element** whose motion is influenced by surroundings.

Navier–Stokes Equation: Local Acceleration

- You can change your own speed $\Rightarrow$ **Velocity changes at a point because time passes, not because you moved elsewhere**

$$\frac{\partial \vec{v}}{\partial t}$$

Navier–Stokes Equation: Convection (Advection)

- Speed changes because you move into a region where the crowd flows differently
- Convective acceleration from spatial changes in the velocity field
- Nabla ∇ collects all spatial derivatives of a vector field

$$\nabla\vec{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{pmatrix} \quad \Rightarrow \quad (\vec{v}\cdot\nabla)\vec{v} = \begin{pmatrix} v_x \frac{\partial v_x}{\partial x} \\ \\ v_x \frac{\partial v_y}{\partial x} \\ \\ v_x \frac{\partial v_z}{\partial x} \end{pmatrix}$$

- $(\vec{v}\cdot\nabla)\vec{v}$ = how $\vec{v}$ changes **in the direction of** $\vec{v}$
- **Change in velocity (i.e. acceleration) caused by moving through the flow**

Navier–Stokes Equation: Example of $(\vec{v}\cdot\nabla)\vec{v}$

- Example field $\vec{v} = (x,\ 2y,\ 0)$
- Insert $v_x = x$, $v_y = 2y$, $v_z = 0$ and all derivatives directly into the component formula

$$(\vec{v}\cdot\nabla)\vec{v} = \begin{pmatrix} v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} \\ v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z} \\ v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z} \end{pmatrix} = \left(\begin{matrix} x\,\frac{\partial x}{\partial x} + \\ x\,\frac{\partial(2y)}{\partial x} + \\ x\,\frac{\partial 0}{\partial x} + \end{matrix} \right.$$

- Evaluate derivatives

$$(\vec{v}\cdot\nabla)\vec{v} = \begin{pmatrix} x\cdot 1 + 2y\cdot 0 + 0\cdot 0 \\ x\cdot 0 + 2y\cdot 2 + 0\cdot 0 \\ x\cdot 0 + 2y\cdot 0 + 0\cdot 0 \end{pmatrix} = \begin{pmatrix} x \\ 4y \\ 0 \end{pmatrix}$$

Navier–Stokes Equation: Gravity

- Crowd pulled downhill on a slope
- **Gravity acts as a body force everywhere in the fluid**

$$\vec{g}$$

Navier–Stokes Equation: Pressure Gradient

- Crowd denser on one side and looser on the other
- Net shove toward the emptier low-pressure region
- **Pressure force drives fluid from high to low pressure**
- ∇p uses the same spatial derivatives as before

$$-\frac{1}{\rho}\nabla p$$

Navier–Stokes Equation: Viscous Forces

- Shoulder-to-shoulder interactions pull or brake you $\Rightarrow$ **Faster regions drag slower ones, slower regions resist faster ones**
- Laplacian detects where a component of $\vec{v}$ is **higher or lower than its neighborhood**
- High relative values decrease, low relative values increase $\Rightarrow$ momentum diffusion
- Viscous forces **smooth out** velocity differences $\Rightarrow$ **viscous diffusion** term with $\nu = \eta/\rho$ as the kinematic viscosity:

$$\nu \nabla^2 \vec{v}$$

- Laplacian of a vector field $\vec{v} = (v_x, v_y, v_z)$ (each component uses the scalar Laplacian)

$$\nabla^2 \vec{v} = \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}, \ \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \right.$$

Navier–Stokes Equation: Summary

$$\underbrace{\frac{\partial \vec{v}}{\partial t}}_{\text{local acceleration}} + \underbrace{(\vec{v} \cdot \nabla)\vec{v}}_{\text{convective acceleration}} = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{pressure force}} + \quad \text{vi}$$

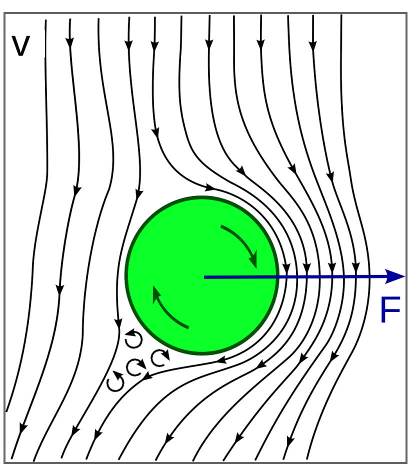
with incompressibility condition $\nabla \cdot \vec{v} = 0$

- **Local acceleration:** you change your own speed
- **Convective acceleration:** you move into a region with a different flow
- **Pressure force:** crowd pushes you from dense to less dense regions
- **Viscous diffusion:** neighbors slow you down or pull you forward
- **Gravity:** downhill pull on the whole crowd

Revisit initial experiment: The Magnus Effect

`mf23 - Magnus effect`

- A spinning object **drags fluid** in its boundary layer (viscosity)
- One side: faster airflow $\rightarrow$ **lower pressure**
- Other side: slower airflow $\rightarrow$ **higher pressure**
- Pressure difference creates a **sideways force**
- Explains curved paths of spinning balls; **requires viscosity**



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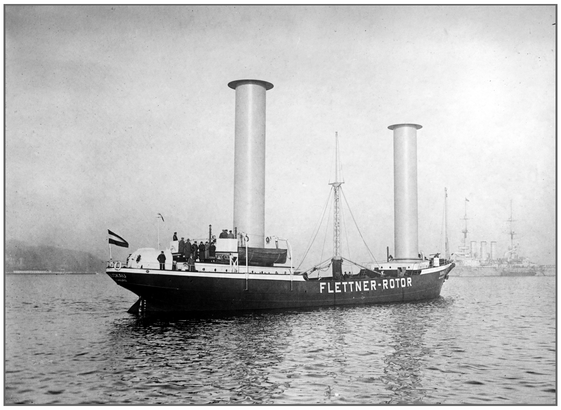
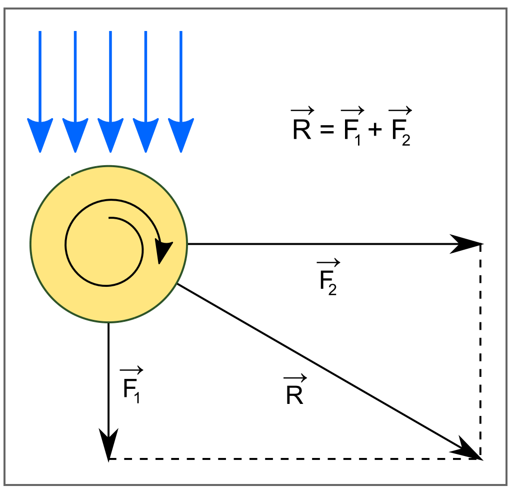
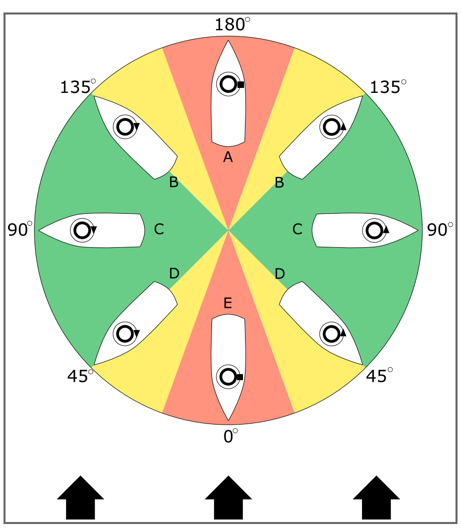
Magnus Effect in Action: Flettner Rotor

`mf32 - Flettner`







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