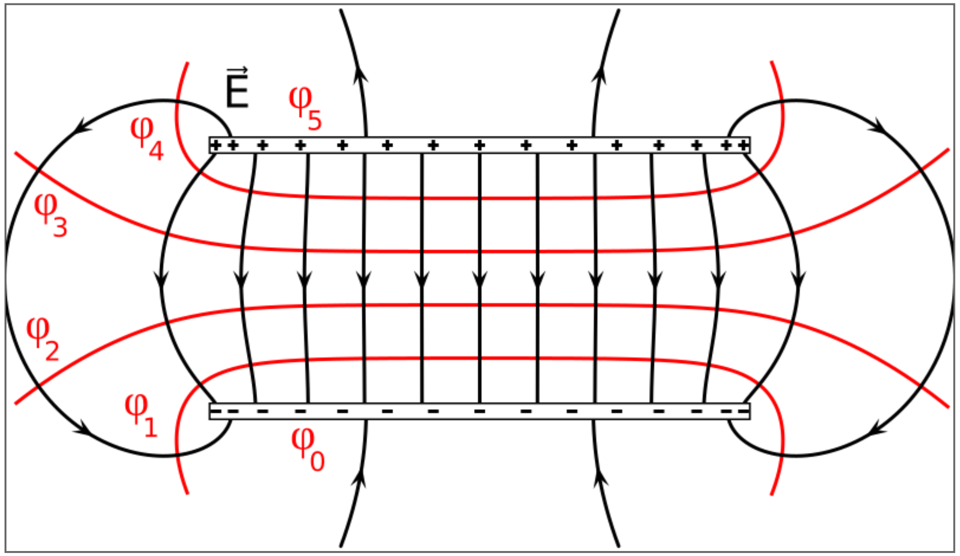
2.3. Capacitance, resistance & current

How can we store electric energy?

- using circuit elements such as:
  - capacitors (separated charges)
  - electric batteries (electrochemical voltage source)
- transition from **static electricity to flow of charges**
- associated physical concepts: dielectrics, electric power, current, resistance, & Ohm's law
- **disclaimer**: simplify notation for voltage to $V = V_{BA} = V_B - V_A$

Capacitors - Basic concept

- two plates with opposite charge produce an electric field



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Capacitors - Relation of V and d

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- simple configuration:
  - two parallel plates of area A separated by distance d
  - voltage source connected
  - $\rightarrow$ plates accumulate charge Q with equal magnitude but opposite sign

What happens if we change the distance? (if $Q = \text{const}$)

- **voltage increases with distance** (until breakdown voltage reached)

$$V = V_B - V_A = -\int_A^B \vec{\mathbf{E}} d\vec{\mathbf{l}} = +E \int_A^B dl = Ed = \frac{Q}{\epsilon_0 A} d$$

Notes:

- electric field for two parallel planes: $E = \frac{Q}{\epsilon_0 A}$

- angle between $\vec{\mathbf{E}}$ and $d\vec{\mathbf{l}}$ is 180° $\rightarrow El\cos(180^\circ) = -El$.

Capacitance C

- **fundamental relation**: $Q = CV$
- **capacitance**:
  - proportionality constant $C = \frac{Q}{V}$
  - unit: **farad** [F] = [C/V], typically capacitors in picofarad to microfarad range
  - capacitance determined by **geometry**: size, shape, relative position of plates
- determining capacitance **analytically** for uniform $\vec{\mathbf{E}}$:

$$V = Ed = \frac{Qd}{\epsilon_0 A}$$

$$\frac{V}{Q} = \frac{d}{\epsilon_0 A}$$

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}$$

Storing electric energy

- **conservation of energy**: the work W required to charge the capacitor is equal to the electric energy stored in the capacitor U
- work required to move small amount of charge in presence of potential difference: $dW = -Vdq$
- integrating over the entire charge Q and with $V = \frac{q}{C}$, we get:

$$W = -\int_0^Q V dq = -\frac{1}{C}\int_0^Q q dq = -\frac{1}{2}\frac{Q^2}{C}$$

- $\rightarrow$ energy U "stored" is:

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

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script simulation: C, Q, E, U
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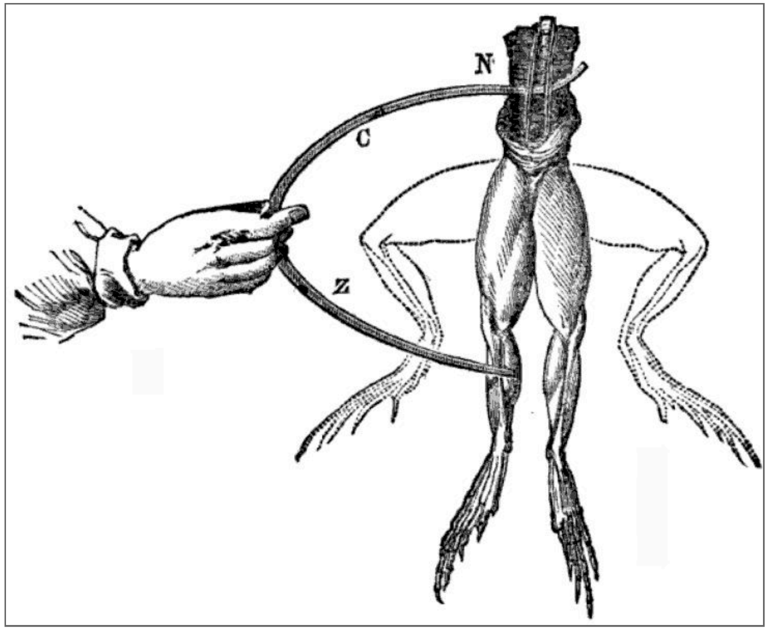
Dielectrics

`es21`

- dielectrics are **insulating material**
- dielectric increases capacitance by a factor K: $C = K\,C_0$
- material permittivity is defined as $\epsilon = K\,\epsilon_0$
- for a parallel-plate capacitor with a dielectric: $C = \frac{\epsilon A}{d}$
- inserting a dielectric increases the breakdown voltage and allows smaller gaps between plates

History of electric battery: Galvani vs. Volta

- Luigi Galvani (1737-1798) connected a cupper and iron wire to a frog leg and saw muscle contraction
- $\rightarrow$ interpreted as *life-force*
- Alessandro Volta (1745-1827) disagreed and realized the potential (pun intended) of dissimilar metal
- $\rightarrow$ combined cells of zinc & silver soaked in salt solution to form a *battery* of cells



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Concept of electric battery

`el23 + el25`

- battery consists of:
    - **electrodes** - metal rods connected to terminals:
        - **cathode** - negative electrode e.g. Pb (lead)
        - **anode** - positive electrode e.g. Pb02 (lead dioxide)
    - **electrolyte** e.g. sulfuric acid, apple, or frog leg
- oversimplified reaction: Pb + acid $\rightarrow$ net effect of accumulation of electrons in cathode
- if connected, battery provides a voltage that drives current in a circuit
- **no charges are generated, merely separated, obeying the laws of conservation**

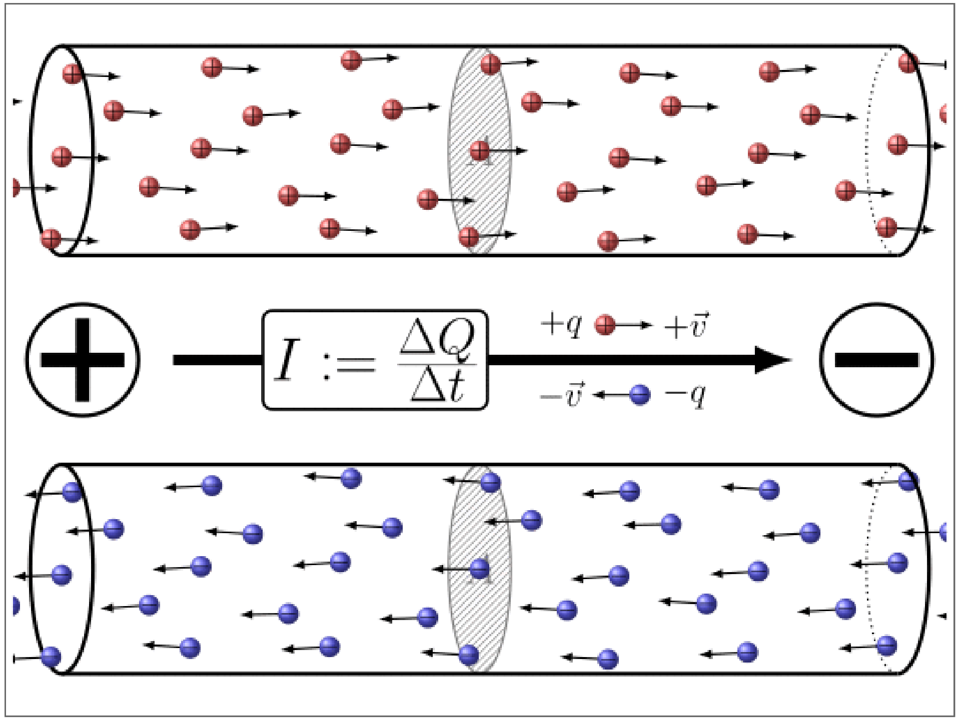
Start moving: electric current, resistance & ohm's law

`es15`

Time to leave electrostatics, i.e. resting charges, and consider moving charges

Electric current

- average current defined as $\bar{I} = \frac{\Delta Q}{\Delta t}$
- instantaneous current defined as $I = \frac{dQ}{dt}$
- **unit** *ampere* [A]=[C/s] in recognition of André Ampère (1775-1836)
- **conservation of charge**: current is constant throughout a continuous conductor
- **flow direction**:
  - conventional current flows from positive to negative (Franklin), while electrons move from negative to positive (physics)
  - directionality (usually) non-critical/yield equal results (exception e.g. Hall effect)


$I := \frac{\Delta Q}{\Delta t}$
$+q$
$+\vec{v}$
$-\vec{v}$
$-q$
A
A

Ohm’s law & resistors

- **Ohm's law** relates voltage, current, and resistance: $V = I\,R \leftrightarrow I = \frac{V}{R}$
- **resistance:**
  - proportionality constant which quantifies the **hindrance of electron flow**
  - unit of resistance: **Ohm** ($\Omega = \frac{V}{A}$) in recognition of Georg Simon Ohm (1787-1854)
- Ohmic resistors have a constant R, while non-ohmic resistors change with conditions such as temperature

Resistivity & conductivity

`el32 + el04`

- **resistance** in a uniform wire **depends on**:
    - cross-sectional area A
    - wire length l,
    - material used, i.e. resistivity ρ in [Ω m]:

$$R = \rho \frac{l}{A}$$

- alternative material property: conductivity: $\sigma = \frac{1}{\rho}$

Temperature dependency of resistivity

`el03`

- **temperature dependence of resistivity** can be approximated as

$$\rho(T) = \rho_0 \left(1 + \alpha \left(T - T_0\right)\right)$$

    - α being the material-specific **temperature coefficient** of resistivity
    - **negative temperature coefficients (NTC)**, i.e. lower resistance when heated, such as semiconductor which have more free electrons available at higher temperatures
    - **positive temperature coefficients (PTC)**, i.e. higher resistance when heated, such as many metals as the higher temperature increases the likelihood of atom-electron collision

Electric power

- **electric circuits transmit electric energy** $\rightarrow$ amount of electric power P delivered is therefore of interest (at the very least to you energy provider)
- **electric power is the energy per unit time**: $P = \frac{dU}{dt}$
- unit: **Watt** [W]=[J/s]
- using $V = \frac{U}{q} \rightarrow dU = V dq$ as well as $I = \frac{dq}{dt}$

$$P = \frac{dU}{dt} = \frac{dq}{dt} V$$

$$P = VI$$

- applying Ohm's law $V = RI$, we can extend this to:

$$P = VI = I^2 R = \frac{V^2}{R}$$

Current density

`el07`

- switch from macro- to microscopic perspective
- **current density** j: current per unit cross-sectional area: $j = \frac{I}{A} \leftrightarrow I = jA$
- for non-uniform current: $I = \int \vec{\mathbf{j}} \cdot d\vec{\mathbf{A}}$
- I is a macroscopic quantity, defined for e.g. the entire cross-section of a wire, while $\vec{\mathbf{j}}$ is a microscopic quantity, defined for each point.

Drift speed

- macroscopic perspective: electricity moves with the speed of light
- microscopic perspective:
  - electrons collide with lattice
  - electrons move with an **average drift speed** $v_d \approx 0.05$ mm/s
- relate drift speed to (macroscopic) current via number of free electron per unit volume $n = \frac{N}{Al}$:

$$I = \frac{\Delta Q}{\Delta t} = \frac{-eN}{\Delta t} = \frac{-enAl}{\Delta t} = \frac{-enAv_d\Delta t}{\Delta t} = -enAv_d$$

$$j = \frac{I}{A} = -env_d$$

- minus sign indicates that electrons drift in opposite direction to macroscopic current

Microscopic view on Ohm’s law

- relate current density and electric field in idealized conditions:
  - resistances is related to the conductors geometry and resistivity: $R = \rho \frac{l}{A}$
  - in a uniform field: $V = \int E dl = El$
  - current I is: $I = jA$
  - Ohm's law states: $V = RI$

$$V = IR$$

$$El = jA\rho \frac{l}{A}$$

$$El = j\rho l$$

$$E = j\rho$$

- **generalization to microscopic view of Ohm's law**

- electric field as proxy for V
- current density as proxy for I
- resistivity as proxy for R

$$\vec{\mathbf{E}} = \rho \vec{\mathbf{j}} \quad \leftrightarrow \quad \vec{\mathbf{j}} = \frac{1}{\rho} \vec{\mathbf{E}}$$