1.10. Waves

`ma07 - Schallwellen in Festkörpern`

Why do we get different pitches?

$\Rightarrow$ **Study waves**

- Waves describe transport in **matter, fields, and even spacetime**, with or without a medium
- **Energy, information, and structure** propagate without net matter flow
- The same wave concepts apply from **sound and light to seismic, and gravitational waves as well as biological (neural) dynamics**

What Is a Wave?

`ms25 - Gekoppelte Pendel & Julius`

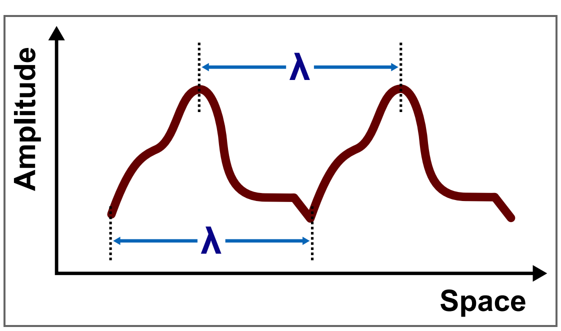
- A **wave** is a disturbance that propagates through **coupled oscillators**
- Each element oscillates about an **equilibrium position**
- Neighboring elements interact via **restoring forces**
- Energy is transferred **locally from oscillator to oscillator**
- **No net transport of matter** along the medium
- The wave emerges from **collective motion**, not a single particle

Important Characteristics of Waves

- **Amplitude** A: maximum displacement → energy content
- **Wavelength** λ: distance between points in the same phase
- **Frequency** f: oscillations per second (Hz)
- **Period** T: time for one oscillation, $T = 1/f$
- **Wave speed** v: speed of propagation of the disturbance
- Fundamental relation:

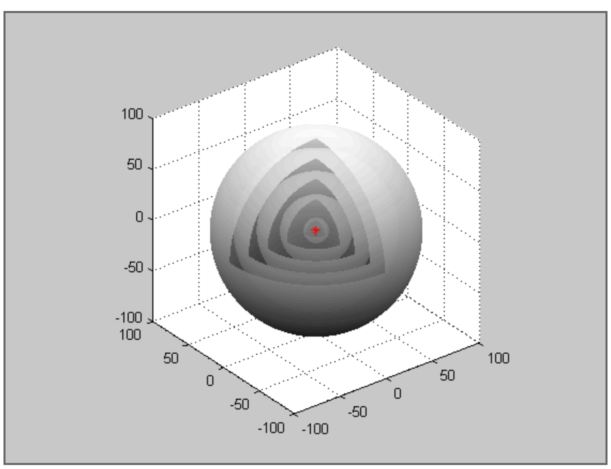
$$v = f\lambda = \frac{\lambda}{T}$$

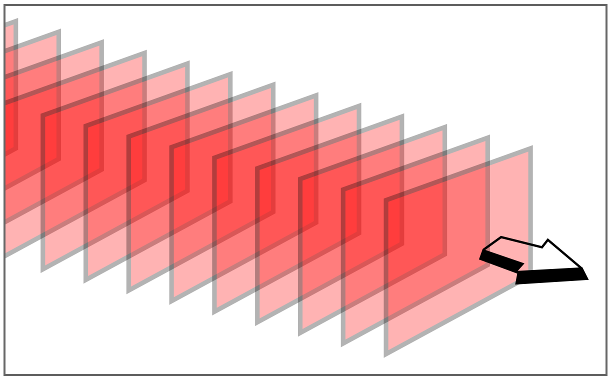
- **Frequency set by the source; wavelength adapts to the medium**



Beyond 1D: Plane Waves & Spherical Waves

- **Wavefront**: a surface connecting points that oscillate with the same phase
- **Spherical waves**:
  - Originate from a **point source**
  - Wavefronts are expanding spheres
- **Plane waves**:
  - Parallel wavefronts
  - Uniform phase and amplitude across each front
  - Good approximation far from the source





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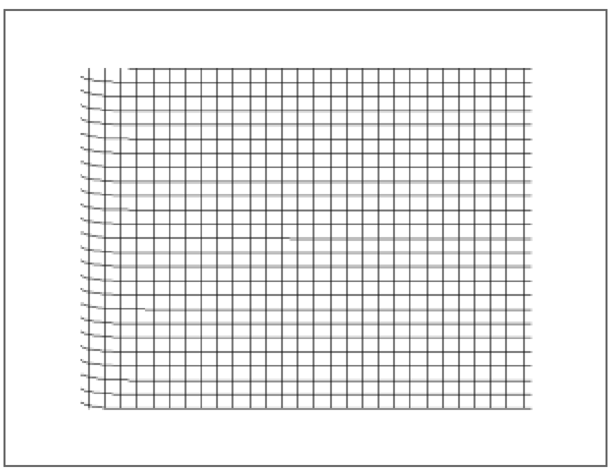
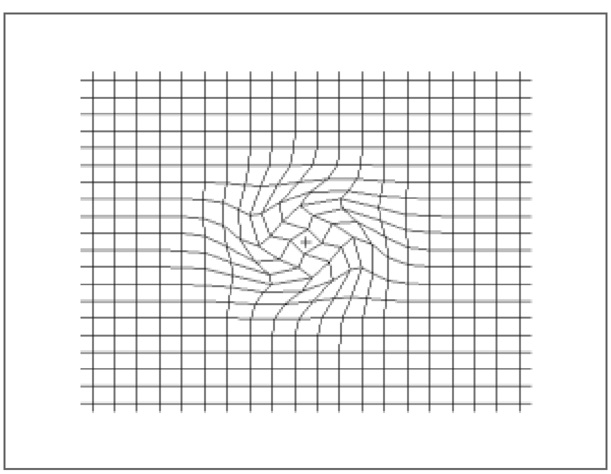
Transverse Waves

`ms22 - robe`

- Particle oscillation **perpendicular** to direction of propagation
- Characterized by **crests** and **troughs**
- Examples:
    - Waves on strings or ropes
    - Water surface waves
    - Electromagnetic waves

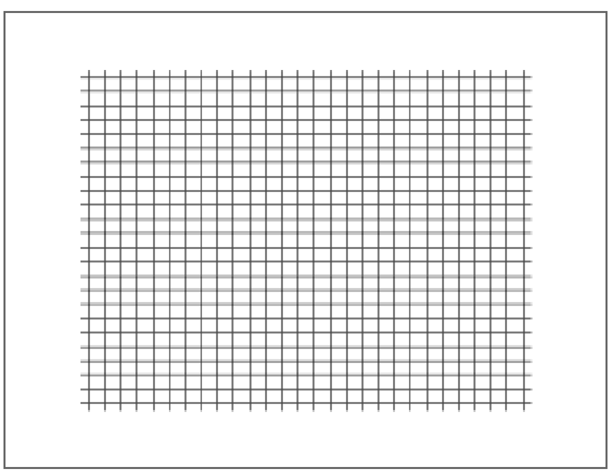


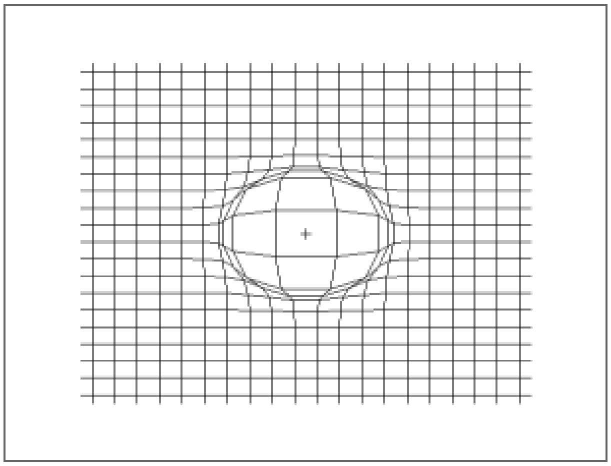


Longitudinal Waves

`ms36 - Demonstration einer Longitudinalwelle`

- Particle oscillation **parallel** to direction of propagation
- Wave consists of **compressions** and **rarefactions**, no crests or troughs
- Requires a **compressible medium**
- Examples:
    - Sound waves in air
    - Compression waves in springs
    - Seismic P-waves





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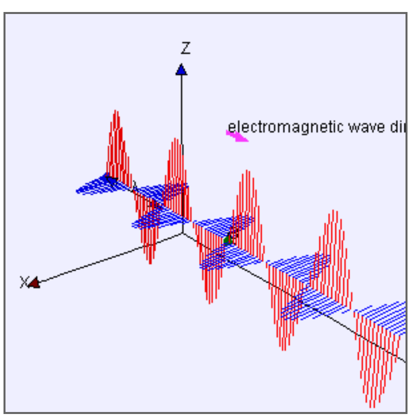
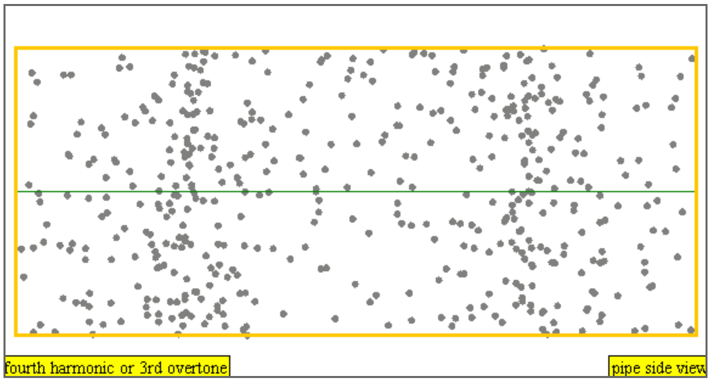
Transverse vs. Longitudinal Waves — Summary

`ms22 - slinky`

Wave type	Particle motion relative to propagation	Characteristic features	Medi requi
Transverse wave	Perpendicular to propagation direction	Crests and troughs	Can with with mec
Longitudinal wave	Parallel to propagation direction	Compressions and rarefactions	Req com mec

In both cases, **energy propagates through the medium while matter oscillates locally** about equilibrium positions.





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Wave kinematics: Spatial and Temporal Structure

`ms24 - freeze Wellenmaschine`

- One-dimensional wave: $D = D(x,t)$
- Spatial profile at fixed time:

$$D(x) = A\sin\left(\frac{2\pi}{\lambda}x\right)$$

Wave kinematics: Spatial and Temporal Structure (cont')

`sim - Wave function`

- Wave translation with speed v:

$$D(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

- Relations for phase velocity v, wave number k & angular frequency ω

$$v = \lambda f, \qquad k = \frac{2\pi}{\lambda}, \qquad \omega = 2\pi f$$

- **Standard traveling-wave form**:

$$D(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - \lambda\frac{\omega}{2\pi}t)\right) = A\sin(kx - \omega$$

Wave kinematics: Wave Parameters and Direction

`ms04 - Alter Schwingungsapparat`

- **Standard traveling-wave form**:

$$D(x,t) = A\sin(kx - \omega t)$$

- Phase:

$$\phi = kx - \omega t$$

- Phase velocity:

$$v = \lambda f = \frac{2\pi}{k}\frac{\omega}{2\pi} = \frac{\omega}{k}$$

- Wave traveling in −x direction:

$$D(x,t) = A\sin(kx + \omega t)$$

The Wave Equation — Motivation and 1D Form

- From **wave function** $y(x,t)$ → to **dynamics**
- *Which equation governs wave propagation in time and space?* $\Rightarrow$ **wave equation**
- Example: sinusoidal traveling wave

$$y(x,t) = A\sin(kx - \omega t), \qquad v = \frac{\omega}{k}$$

- Second derivatives:

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx - \omega t) = -k^2 y, \qquad \frac{\partial^2 y}{\partial t^2} = -$$

- Rearrange gives the **one-dimensional wave equation**

$$\frac{1}{-k^2}\frac{\partial^2 y}{\partial x^2} = \frac{1}{-\omega^2}\frac{\partial^2 y}{\partial t^2} \quad \Rightarrow \quad \boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

Generality, Physical Meaning, and 3D Form

- Valid for **any waveform**, not only sinusoidal:

$$y(x,t) = f(x - vt), \qquad y(x,t) = f(x + vt)$$

- **Physical meaning**
  - $\partial^2 y/\partial x^2$: spatial curvature
  - $\partial^2 y/\partial t^2$: particle acceleration
- **Three-dimensional wave equation**

$$\boxed{\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$$

- Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Governs sound, water waves, strings, and electromagnetic waves

Energy Transport by a Traveling Wave

- **Instantaneous energy (per string element)**:

  $$dE = dK + dU = \mu A^2 \omega^2 \cos^2(kx - \omega t)\, dx$$

  (energy oscillates between kinetic and potential forms)
- **Time-averaged energy per unit length**:

  $$\frac{dE_{\text{avg}}}{dx} = \frac{1}{2}\mu A^2 \omega^2$$

  (using $\langle \cos^2 \rangle = \frac{1}{2}$ over one period)
- **Average power transported by the wave**:

  $$P = \frac{dE}{dt} = \frac{dE_{\text{avg}}}{dx}\, v = \frac{1}{2}\mu A^2 \omega^2 v$$

  (wave speed v converts energy per length into energy per time; $P \propto A^2$)

Intensity of a Spherical Wave

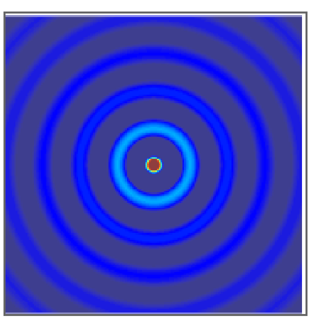
- Assume a **point source** $\rightarrow$ **spherical wave**
- Total power P spreads over spherical wavefront

$$A_{\text{cross}}(r) = 4\pi r^2$$

- **Intensity** (*power per area* or *energy per time per area*):

$$I(r) = \frac{P}{A_{\text{cross}}} = \frac{P}{4\pi r^2}$$

- Energy is conserved, **but** intensity decreases with distance due to **geometric spreading**



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Wave Reflection and Transmission at Boundaries

- **Boundaries** cause partial **reflection** and **transmission**
- Reflection and transmission depend on **medium properties** (wave speed, impedance)
- Boundaries impose **constraints** (fixed, free, or continuity conditions)
- Energy is **redistributed**, not lost: reflected + transmitted = incident
- Valid for **strings, sound, and light**
- ($\Rightarrow$ more details on wave phenomena in Physics II)

Reflection & Phase Change

`ms22 - Rope for fixed and free end`

Reflection from...	Equivalent situations	Reflected wave	Phase change	Physical in
Higher impedance	Fixed end (string); $n_1 < n_2$ (light)	Inverted	180° phase shift	Boundar motion displace must rev satisfy co
Lower impedance	Free end (string); $n_1 > n_2$ (light)	Upright	No phase shift	Boundar conditio satisfied sign reve

Superposition

`ms17 - Wasserwellengerät`
`ms24 - Wellenmaschine`

$$\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2}$$

- **Superposition**: multiple waves overlap $\rightarrow$ total displacement = **sum of individual displacements**
- Wave equation's linearity implies that if D_1 and D_2 are solutions, then $D = aD_1 + bD_2$ is also a solution
- Overlapping waves do **not interact**: after overlap, each wave continues unchanged
- Superposition is the basis of **interference patterns**, **standing waves**, and **resonance**; it holds only for **small amplitudes** (linear restoring forces)

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Interference of Traveling Waves

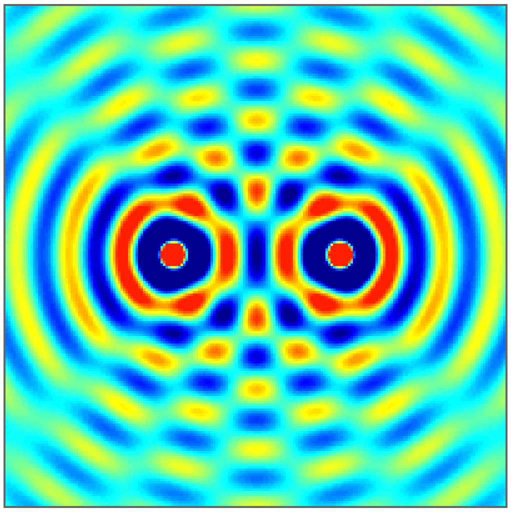
- **Interference** arises when **coherent** waves (i.e. constant phase relationship over time) overlap and superpose
- Resulting pattern depends on the **phase difference** ϕ
- For two equal-amplitude waves traveling in the same direction:

$$y_1 = A\sin(kx - \omega t), \qquad y_2 = A\sin(kx - \omega t + \phi)$$

- Using $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$:

$$y = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

- Amplitude depend on phase: $A_R = 2A\cos(\frac{\phi}{2})$



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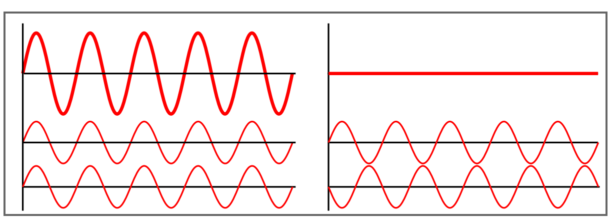
Constructive and Destructive Interference

`ms24 - Wellenmaschine`

- Stable interference patterns require **coherence** (same f, fixed ϕ)
- Energy is **redistributed**, not destroyed
- Phase difference often arises from path difference: $\phi = \frac{2\pi}{\lambda}\Delta x$

Interference type	Phase difference ϕ	Path difference Δx	Resul
Constructive	$0,\ 2\pi,\ 4\pi, \ldots$	$\Delta x = n\lambda$	Max amp
Destructive	$\pi,\ 3\pi,\ 5\pi, \ldots$	$\Delta x = (n + \frac{1}{2})\lambda$	Can



Standing Waves: Mathematical Formation

`ms22 - rope`

- Arise from **superposition of two identical waves** traveling in opposite directions
- Incident and reflected waves:

$$y_1(x,t) = A\sin(kx - \omega t), \qquad y_2(x,t) = A\sin(kx$$

- Using $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$, resulting standing wave:

$$y(x,t) = 2A\sin(kx)\cos(\omega t)$$

- **Spatial factor** $\sin(kx)$ fixes the pattern, **temporal factor** $\cos(\omega t)$ sets the oscillation frequency

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Standing Waves on a String and Resonance

- String of length L fixed at both ends:

$$y(0,t) = 0, \quad y(L,t) = 0$$

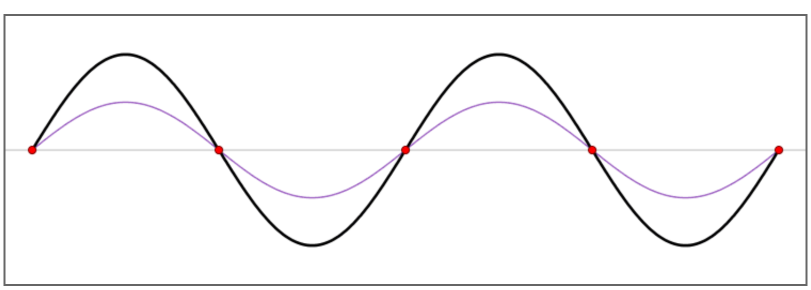
- Boundary condition:

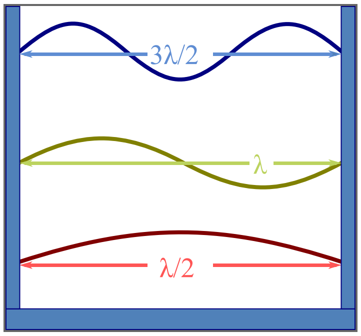
$$\sin(kL) = 0 \;\Rightarrow\; kL = n\pi$$

- Allowed wavelengths and frequencies with $k = \frac{2\pi}{\lambda}$ and $v = \lambda f$:

$$\lambda_n = \frac{2L}{n}, \qquad f_n = \frac{nv}{2L}$$

- Discrete **normal modes** (harmonics), lowest is the **fundamental**





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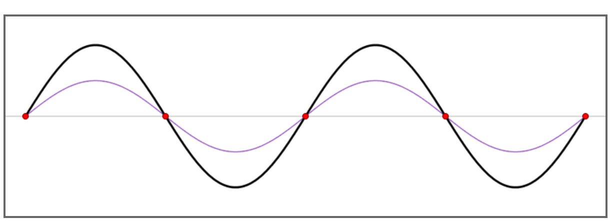
Nodes, Antinodes, and Energy

$$y(x,t) = 2A\sin(kx)\cos(\omega t)$$

- All points oscillate with the same angular frequency ω (SHM)
- Position-dependent amplitude: $A(x) = 2A\sin(kx)$
- **Nodes**: $\sin(kx) = 0 \;\Rightarrow\; x_n = n\frac{\lambda}{2}$
  - Zero displacement and zero kinetic energy
  - Potential energy maximal
- **Antinodes**: $|\sin(kx)| = 1$
  - Maximum displacement amplitude
- **No net energy transport**: energy oscillates locally between kinetic and potential forms



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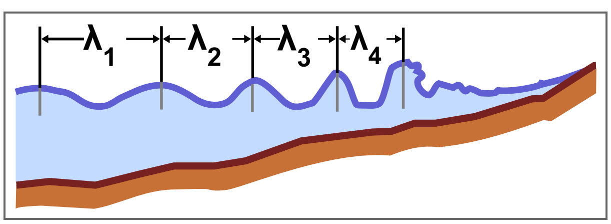
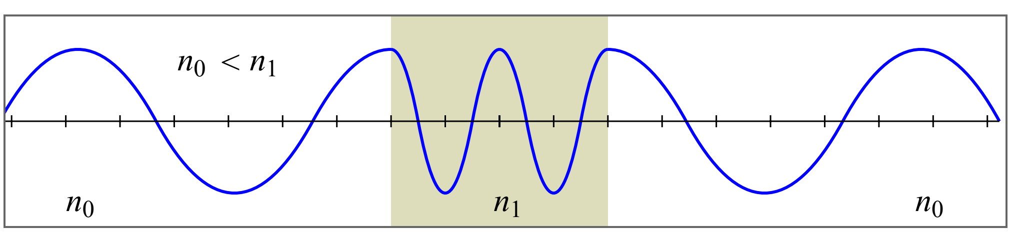
Wave Speed Variations: Refraction

- Wave behavior is controlled by the **wave speed** v, set by medium properties

$$v = \lambda f$$

- **Refraction** occurs when a wave enters a region with a different wave speed
- The **frequency remains constant** (set by the source)
- Changes in wave speed lead to changes in **wavelength** and **direction**
- Refraction occurs for mechanical waves, sound, seismic waves, and light





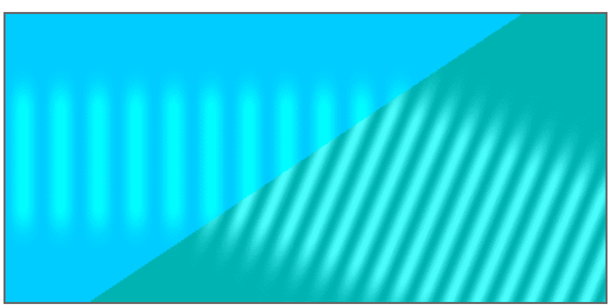
Wave Speed Variations: Refraction (cont')

- Wave speed, frequency, wavelength: $v = f\lambda$
- Frequency unchanged $\Rightarrow$ wavelength adjusts at the boundary:

$$\frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2}$$

- Oblique incidence causes wavefront rotation
- Law of refraction for waves:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$



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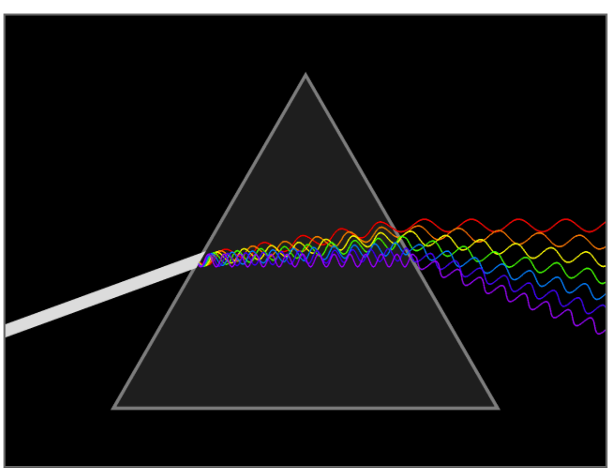
Wave Speed Variations: Dispersion

- A medium is **dispersive** if **wave speed depends on frequency**
- Different frequency components propagate at different speeds
- **Refraction vs. dispersion**:
    - Both effects arise from **variations in wave speed**
    - **Refraction**: wave speed varies **in space** $\rightarrow$ change in wavelength and propagation direction
    - **Dispersion**: wave speed varies **with frequency** $\rightarrow$ separation of spectral components and waveform distortion



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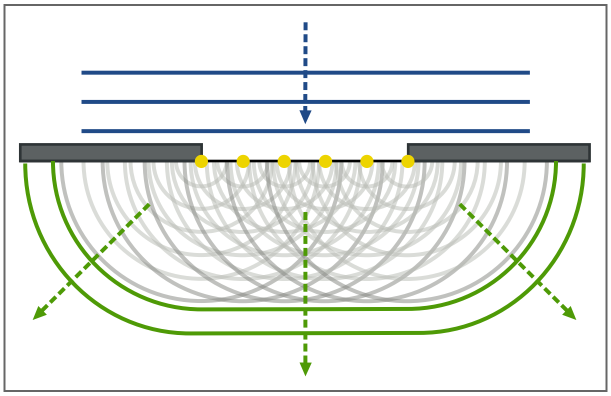
Diffraction and Huygens' Principle

`ms17 - Wasserwellengerät`

- **Diffraction**: spreading/bending of waves at edges, apertures, or obstacles
- Strength of diffraction depends on **wavelength λ vs object size a**
  - $a \gg \lambda$: weak diffraction → nearly straight-line propagation
  - $a \sim \lambda$: strong diffraction → pronounced spreading
  - $a < \lambda$: wave spreads broadly, often nearly spherical
- Explained by **Huygens' principle**:
  - Every point on a wavefront acts as a source of secondary wavelets
  - New wavefront = envelope of these wavelets
- Explains sound bending around corners, fuzzy shadows, limits of resolution in optics and imaging



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Revisit initial experiment: Escapement, oscillations, & clocks

`ma07 - Schallwellen in Festkörpern`

Why do we get different pitches?

- longitudinal and transversal wave generated depending on how *excited*
- $\rightarrow$ compression vs. vibration