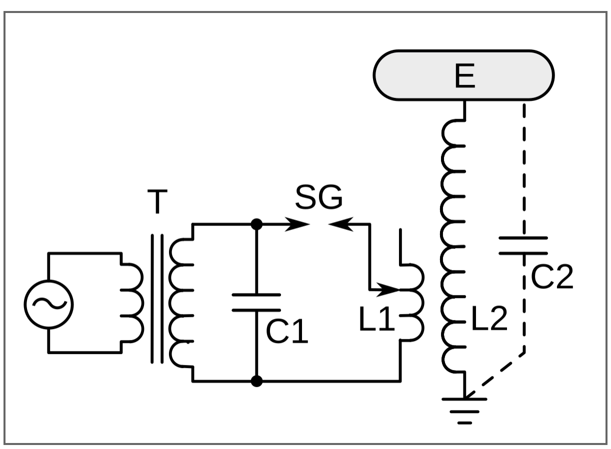
2.7. AC circuits & electromagnetic oscillations

How does a Tesla coil work? `ew10`

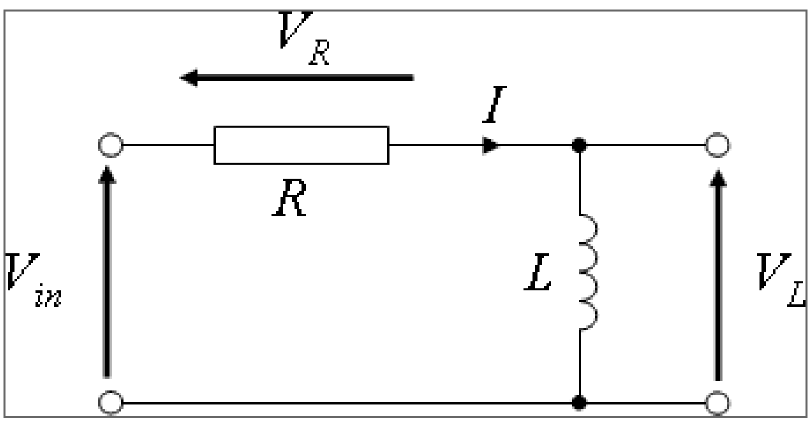
- electric resonant transformer circuit
- Tesla coil with primary and secondary winding, each forming its own LC-circuit
- spark gap (SG) acts as a switch
- need to understand: **LRC-circuits, resonance, and impedance**
- **plan for today**: investigate simply AC-circuits to derive deeper understanding of R, L, and C



LR-circuit

`em45`

- consider a simple DC circuit with the following elements in series:
    - ideal voltage source V_0
    - resistor R
    - inductor L



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LR-circuit: Switching DC supply on

- applying Kirchhoff's loop rule gives

$$V_0 = L\frac{dI}{dt} + IR$$

- rearrange and take integral ($0..I$ and $0..t$, i.e. current slowly builds up due to self-inductance):

$$\int_0^I \frac{dI}{V_0 - RI} = \int_0^t \frac{dt}{L}$$

- solving the differential equation yields

$$I(t) = \frac{V_0}{R}\left(1 - e^{-tR/L}\right)$$

LR-circuit: Switching DC supply on (cont')

- find V_L:

$$V_L = L\frac{dI}{dt} = \frac{d}{dt}\Big(\frac{V_0}{R}\Big(1 - e^{-tR/L}\Big)\Big)$$

$$V_L = L\Big(\frac{V_0}{R} \cdot \frac{-R}{L} \cdot e^{-tR/L}\Big)$$

$$V_L = -V_0 e^{-tR/L}$$

- **summary**: self-inductance causes:
  - voltage over L to be first opposed to V_0 (Lenz's rule) and decaying over times
  - current building up over time

LR-circuit: Switching DC supply off

- when the DC supply is switched off ($V_0 = 0$) the inductor resists (again) change in current
- Kirchhoff's loop rule becomes

$$L\frac{dI}{dt} + IR = 0$$

- solution for decaying current is:

$$I(t) = I_0 e^{-tR/L}$$

- voltage over inductor is:

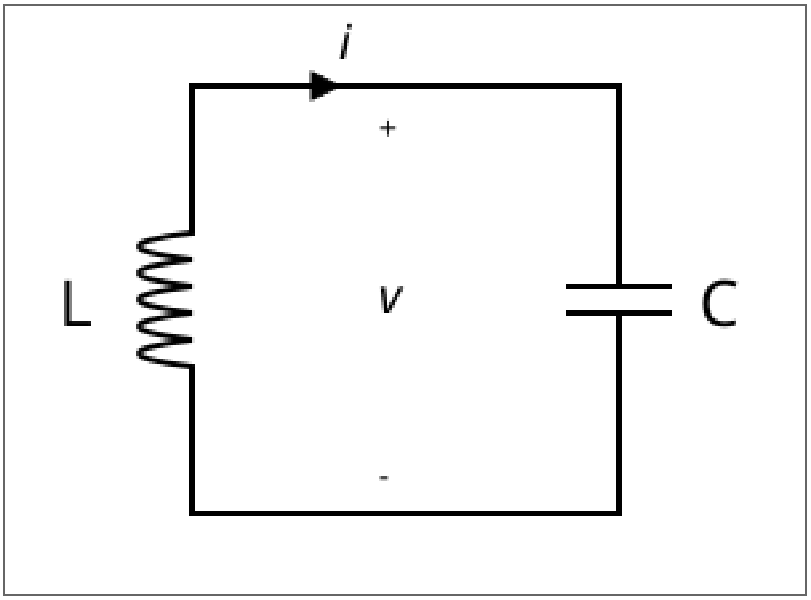
$$V_L = L\frac{dI}{dt} = \frac{d}{dt}\left(I_0 e^{-tR/L}\right) = \frac{-LR}{L} I_0 e^{-tR/L}$$

$$V_L = -V_0 e^{-tR/L}$$

LC-circuit: Fun with AC circuit

`ew04`

What happens if we put L and C in series in a AC circuit?



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LC-circuit: The equations

- applying Kirchhoff's rule leads to:

$$-L\frac{dI}{dt} + \frac{Q}{C} = 0$$

- using $I = -\frac{dQ}{dt}$ (minus because capacitor discharges)

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

- second order linear equation describing simple **harmonic oscillator** with general solution:

$$Q(t) = Q_0 \cos\Big(\omega t + \phi\Big) \quad \text{with the angular frequer}$$

- current $I(t)$ is time derivative of charge:

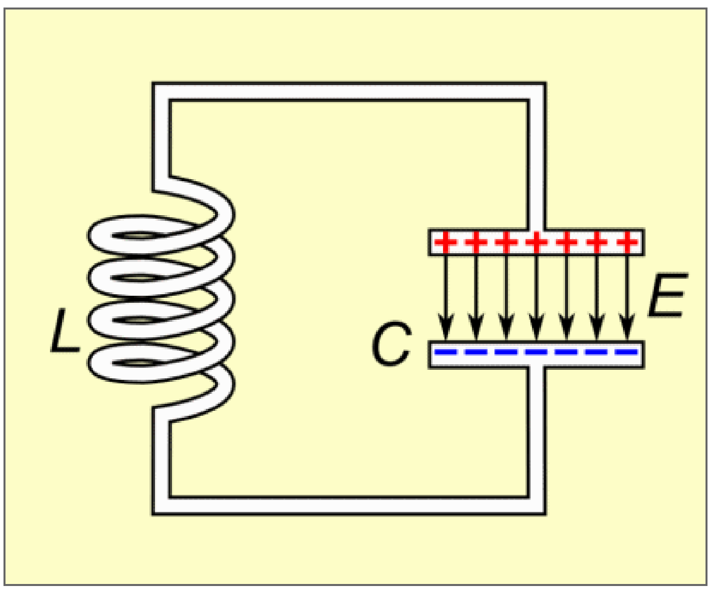
$$I(t) = -\frac{dQ}{dt} = \omega Q_0 \sin(\omega t + \phi) = I_0 \sin(\omega t + \phi)$$

LC-circuit: Electromagnetic oscillations

- the energy oscillates between capacitor (electric field) and inductor (magnetic field):

$$U = U_B + U_E = \frac{Q_0^2}{2C}\sin^2(\omega t + \phi) + \frac{Q_0^2}{2C}\cos^2(\omega t +$$

- losses are neglected



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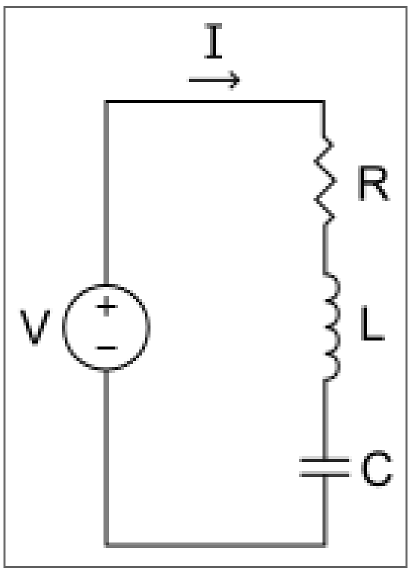
LRC-circuit

- adding a resistor
    - introduces damping into oscillations
    - models losses



LRC-circuit (cont')

`sim - damping LRC-circuit`

- apply Kirchhoff's loop rule and use $I = \frac{dQ}{dt}$:

$$-L\frac{dI}{dt} - IR + \frac{Q}{C} = 0$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

- second-order linear differential equation that describes **damped harmonic oscillator**:
    - $R^2 < \frac{4L}{C}$: Underdamped oscillation with exponential decay
    - $R^2 > \frac{4L}{C}$: Overdamped, i.e. the damping is too strong to allow any oscillations
    - $R^2 = \frac{4L}{C}$: Critically damped oscillation with the angular frequency $\omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

Resistance vs. reactance

`ew01 - phase shift`

- **LC-circuit**: energy conserved, only exchanged between L and C $\rightarrow$ oscillation $\rightarrow$ phase shift between V and I
- **LRC-circuit**: beyond oscillation, energy dissipated as heat in R

- **new concept**:
  - **resistance** R is independent of frequency and dissipates energy as heat
  - **reactance** (inductive X_L and capacitive X_C) depends on frequency and temporarily stores energy
  - the **phase difference** between voltage and current arises due to reactance

Resistor in AC circuit

- follows Ohm's law: $V = IR$
- if current is sinusoidal, i.e. $I = I_0 \cos(\omega t)$, the voltage is:

$$V = (I_0 \cos(\omega t))R = (I_0 R) \cos(\omega t) = V_0 \cos(\omega t)$$

- $\rightarrow$ voltage and current are **in phase**
- the average power is given by

$$\bar{P} = I_{rms}^2 R$$

Inductor in AC circuit

- the voltage across an inductor is: $V - L\frac{dI}{dt} = 0 \leftrightarrow V = L\frac{dI}{dt}$
- for sinusoidal current $I(t) = I_0 \cos(\omega t)$, the voltage is:

$$V(t) = L\frac{d(I_0 \cos(\omega t))}{dt} = -L\omega I_0 \sin(\omega t)$$

- with $\sin(t) = -\cos(t + \frac{\pi}{2})$

$$V(t) = \omega L I_0 \cos\left(\omega t + \frac{\pi}{2}\right) = X_L I_0 \cos\left(\omega t + \frac{\pi}{2}\right)$$

- the inductive reactance is $X_L = \omega L = 2\pi f L$
- the **voltage leads the current by** 90°
- since X_L **increases with frequency**, inductors resist high-frequency currents more than low-frequency ones

Capacitor in AC circuit

- for a sinusoidal current $I(t) = I_0 \cos(\omega t) = \frac{dQ}{dt}$, charge at capacitor is:

$$Q(t) = \int_0^t dQ = \int_0^t I_0 \cos(\omega t) dt = \frac{I_0}{\omega} \sin(\omega t)$$

- using $\sin\theta = \cos(\theta - \frac{\pi}{2})$, the voltage is:

$$V(t) = \frac{Q}{C} = \frac{1}{\omega C} I_0 \sin(\omega t) = \frac{I_0}{\omega C} \cos\left(\omega t - \frac{\pi}{2}\right) =$$

- the capacitive reactance is $X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$
- the **voltage lags the current by** 90°
- since X_C **decreases with frequency**, capacitors resist low-frequency currents more than high-frequency ones

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Summary of AC circuit components

- resistor:
    - no phase shift
    - dissipates energy as heat
- inductor:
    - voltage leads current
    - reactance $X_L = \omega L$
- capacitor:
    - voltage lags current
    - reactance $X_C = \frac{1}{\omega C}$

Impedance

- impedance determines the relationship between voltage and current in AC circuits
- impedance combines resistance and reactance into a complex quantity:

$$Z = R + j\left(X_L - X_C\right)$$

- its magnitude is $Z = \sqrt{R^2 + \left(X_L - X_C\right)^2}$ and its phase angle is $\tan\phi = \frac{X_L - X_C}{R}$
- impedance in series: $Z_{net} = \sum_i Z_i$
- impedance in parallel: $\frac{1}{Z_{net}} = \sum_i \frac{1}{Z_i}$

Revisiting AC LRC-circuit: Phasor diagrams

`sim - Phasor diagram`

- in a series LRC-circuit the sum of the voltage drops equals the source voltage: $V = V_R + V_L + V_C$
- in the phasor diagram:
    - V_R is drawn along the positive x-axis (in phase with the current)
    - V_L is drawn 90° ahead of V_R
    - V_C is drawn 90° behind V_R
- the resultant voltage is found by vector addition:

$$V_0 = I_0 Z \quad \text{with} \quad Z = \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

Impedance matching

`el35 - CASSY`

- maximum power transfer occurs when the source impedance matches the load impedance:

$$Z_1 = Z_2$$

- for purely resistive circuits, maximum power is delivered when $R_{\text{source}} = R_{\text{load}}$
- mismatched impedances lead to reduced power transfer efficiency and potential signal distortion

Revisiting AC LRC-circuit: Resonance

`sim - Resonance`

- resonance occurs when the inductive and capacitive reactances cancel: $X_L = X_C$
- the impedance is purely resistive: $Z = \sqrt{R^2 + \left(X_L - X_C\right)^2} = R$
- $\rightarrow$ the resonant (angular) frequency is

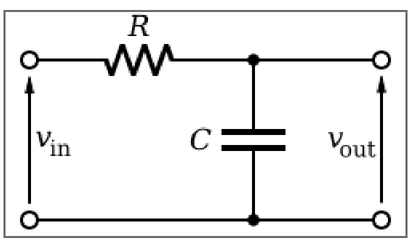
$$\omega_0 = \sqrt{\frac{1}{LC}} \leftrightarrow f_0 = \frac{1}{2\pi\sqrt{LC}}$$

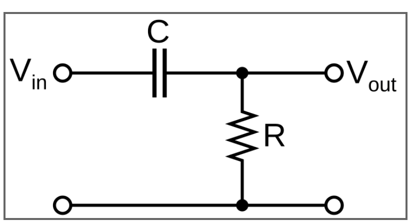
- $\rightarrow$ voltage and current are in phase: $\phi = 0$
- shape of resonance curve depends on the value of R

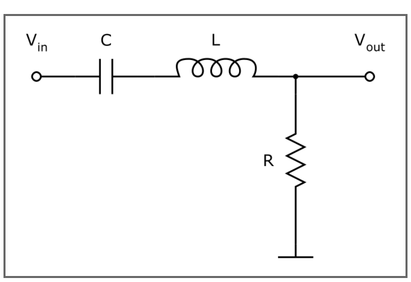
Filters

`ew07 - high/low pass`

- low-pass filters allow low-frequency signals to pass while attenuating high-frequency ones
- high-pass filters allow high-frequency signals to pass while attenuating low-frequency ones
- band-pass filters allow frequency band to pass while attenuating frequencies below and above the band
- these filters are common in signal processing and audio electronics







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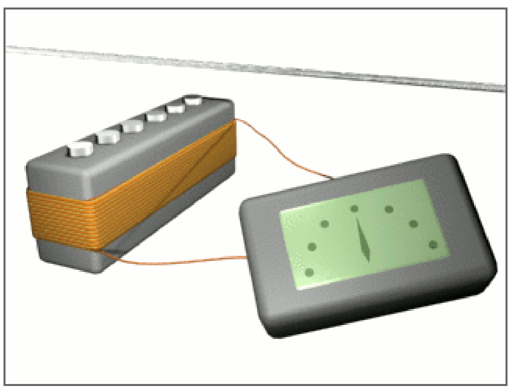
Demo induction, electromagnets, filters, and resonance



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YouTube - bandpass filter in action