2.3. Capacitance, resistance & current

In this chapter, we will transition from static electricity to the flow of charges, i.e. *electrical current*. In the course, we will introduce important building blocks of electric circuits such as capacitors, batteries, & resistors. By doing that, we will learn about the associated physical concepts and laws such as *dielectrics*, *electric energy*, *electric power*, and *Ohm's law*.

Disclaimer: From here on we simplify the notation for voltage, i.e. difference in potential difference between point A and B, to $V = V_{BA} = V_B - V_A$

2.3.1. Capacitors & dielectrics

As seen in previous chapters, two plates with opposite charge produce an electric field and, therefore, a voltage. A *capacitor* (sometimes called condenser) is in essence a device that uses this principle to store electric charge. Commonly found in electronics, capacitors have various use cases such as storing charge and energy or protecting circuits.

The most simple capacitor configuration consists of two parallel plates of area A with distance d between them. By connecting a voltage, e.g. produced by a battery, the two plates become charged with opposite polarity quickly, i.e. the plate connected to the negative pole of the battery becomes negatively charge and the plate connected to the positive pole positively charge. The charge of both plates is equal in magnitude but opposite in sign. Further, the voltage provided by the battery is also found across the plates. The relation of charge Q and voltage V of a capacitor is:

$$Q = CV$$

with the proportionality constant C being the *capacitance* of the capacitor. In general, the capacitance of a capacitor does not depend on Q or V but the geometry (size, shape, relative position) of the two conducting plates and the material in between the plates. As seen in the equation, the capacitance is expressed as [C/V] which is called a *farad* [F]. Typically capacitors are on the order of picofarad to microfarad.

Determining capacitance analytically

While the capacitance can be determined experimentally for any capacitor, for simple geometries and idealized conditions we can compute the capacitance analytically. The general concept is to estimate the capacitance by first deriving an expression of charge Q in dependence of the voltage V and, subsequently use the relation

$$Q = CV \leftrightarrow C = \frac{Q}{V}$$

For a capacitor with *two parallel plates*, each with area A and both separated by distance d, we can reuse the equation for electric field E of such configuration:

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

with the charge density, i.e. charge per unit area σ.

In general, the voltage V dependence on the electric field E, but for the given geometry the path will be antiparallel to the field, i.e. pointing along the same axis but in opposite direction, it simplifies:

$$V = V_B - V_A = -\int_A^B \vec{E} d\vec{l} = +E \int_A^B l = \frac{Q}{\epsilon_0 A} d$$

The change in sign is because the angle between $\vec{E}$ and $d\vec{l}$ is 180° and, therefore, $El \cos(180°) = -El$.

Using this equation we derive at our solution of:

$$C = \frac{Q}{V} = \epsilon_0 \frac{A}{d} \quad \text{for a parallel-plate capacitor}$$

For a *single isolated conductor*, we define the potential at infinity to be zero, i.e. $V = 0$ at $r = \infty$. If we let r_A converge to infinity and use $A = 4\pi r$, the above equations can be adapted to:

$$V = V_B - V_A = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r_B} - \frac{1}{r_A} \right) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r_B}$$

Hence, the capacitance is:

$$C = \frac{Q}{V} = 4\pi\epsilon_0 r_B \quad \text{single conductor}$$

For a *coaxial cylindrical capacitor* with length l and inner and outer radius r_I, r_O, the capacitance is:

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 l}{\ln(r_I/r_O)} \quad \text{for a coaxial cylindrical capacitor}$$

For a *spherical cylindrical capacitor* with two thin spherical shells with an inner and outer radius r_I, r_O, the capacitance is:

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{r_I r_O}{r_O - r_I} \quad \text{for a spherical cylindrical capacitor}$$

Storing electric energy

Motivated by conservation, the work W required to charge the capacitor is equal to the electric energy stored in the capacitor U. Initially, a capacitor is uncharged. Connecting a battery will over time charge the capacitor but the work required to move a charge to the plate of same polarity will increase the more

charge is already on that plate. The work dW required to move a small amount of charge dq in presence of the potential difference V is $dW = V dq$. Integrating over the entire charge Q and leveraging $V = \frac{q}{C}$ yields:

$$W = \int_0^Q V dq = \frac{1}{C} \int_0^Q q dq = \frac{1}{2} \frac{Q^2}{C}$$

Hence, the energy U "stored" is:

$$U = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C V^2 = \frac{1}{2} Q V$$

Dielectrics

So far we considered the gap between the plates of the capacitor to be a vacuum, i.e. have the permittivity ϵ_0. In practice, this gap is filled with an insulating material such as paper of plastic, the so-called *dielectric*. The advantage is that an additional insulator allows for higher voltages compared with vacuum or air before the plates/conductors discharge across the gap. Further, the gap can be smaller, hence, the plates/conductors can be placed closer together which increases the capacitance. Experimentally, it was found that the dielectric between the two conductors increases the initial capacitance C_0 (vacuum in gap) by a factor K, the so-called *dielectric constant*:

$$C = K C_0$$

Consequently, the *permittivity* ϵ of any material is defined as

$$\epsilon = K \epsilon_0$$

For a parallel-plate capacitor, we can replace C_0 with the capacitors geometry relation and obtain:

$$C = K \epsilon_0 \frac{A}{d} = \epsilon \frac{A}{d} \quad \text{for a parallel-plate capacitor}$$

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2.3.2. Start moving: Electric current, resistance & Ohm's law

So far we investigated the static charges, i.e. electrostatics. We concluded that the electric field inside a conductor must be zero. Otherwise a force acts on the free electrons violating the static condition. It is time to look what happens in a conductor if an external source generates an electric potential difference, i.e. voltage is applied. The associated electric field will exert a force on the free electrons causing a "flow" of electrons, the so-called *electric current*. Note that everything we learned about the static condition is still valid, we just expand our understanding to non-static condition.

Electric battery

In order to keep charges moving, we need an electric field, hence, a difference in electric potential. This voltage can be provided by an *electric battery*. The history of the battery involves frog legs, dissimilar metals, and two great scientists, i.e. Luigi Galvani (1737-1798) and Alessandro Volta (1745-1827). While Galvani discovered that two dissimilar metals can be used to cause the muscles of a frog leg to contract, terming is "animal electricity" and suspecting it so be the "life-force", Volta was sceptic, but realized the potential (pun intended) of dissimilar metal. In 1800, Volta presented the first electric battery by combining two dissimilar metals, i.e. zinc and silver and soaking them in a salt solution. Two increase the overall electric potential, he combined several of these zinc-silver cells to form a *battery*.

Nowadays batteries follow the same principle and we call even a single cell a battery, although zinc-carbon is the most common combination. To understand how batteries work, we imagine the most simple battery, a single cell of a zinc and a carbon rod in a sulfuric acid. Without going into detail, the acid will dissolve the zinc, i.e. the atoms enters the acid while two electrons remain in the zinc rod. Hence, the zinc rod becomes negatively charged. The acid becomes positively charged, pulling electrodes from the carbon rod, causing the rod to become positively charged as well. Not that no charges are generated, merely separated, obeying the laws of conservation. If nothing is connected to the battery, this charge separation is eventually hampered as the negatively charged zinc rod will attract positive charges from the acid. Therefore, a constant voltage between the rods is reached. Only if connected, the free electrons can move which enables more zinc to be dissolved until, eventually, the battery becomes "dead".

The rods are called *electrodes* and their portion outside the solution (where we would connect wires), are called *terminals*. The negative electrode is called the *cathode* and the positive electrode is called the *anode*. The acid is called the *electrolyte*

The total charge available from a specific battery is expressed often in ampere-hours [A h].

Electric current

If a wire is connected to the battery, the voltage causes charges to move through the wire. We define the average *electric current* $\bar{I}$ as the number of charges ΔQ over a certain period of time Δt:

$$\bar{I} = \frac{\Delta Q}{\Delta t}$$

The instantaneous current I is defined as Q and t become infinitesimal small:

$$I = \frac{dQ}{dt}$$

The unit of current is therefore [C/s] which we define as an *ampere* [A] in recognition of André Ampère (1775-1836).

The conservation of charge states that for a steady current through a single wire, the current is the same throughout the wire. No charges can disappear or be generated. Even if the single wire would be connected to a battery of device such as a heat element, no charges are created or destroyed.

For the direction on current, we have to differentiate between:

- conventional current direction from positive to negative
- physical current directions of free electrons flowing from negative to positive

Both directions are (for most cases) equivalent (exception e.g. Hall effect) and the conventional current is a historical convention (see previous chapter; arbitrary choice). Note that in fluids and gases positively and negatively charged ions can move.

Ohm's law & resistors

Georg Simon Ohm (1787-1854) discovered that in a metal wire, the current is proportional to the potential difference. The proportionality constant is called the *resistance* R and we imaging it as how much the flow of free electrons is hindered by collisions with the atoms/lattice. If R is constant and independent of the voltage applied, we can interpret the current in a wire the response to a voltage scaled by the resistance:

$$V = IR \leftrightarrow I = \frac{V}{R}$$

While this is called *Ohm's law*, is is not a fundamental law of nature. In reality, not all *resistors* has a constant resistance and factors such as temperature can influence resistance. We call a resistor with constant resistance *ohmic*, i.e. metal conductors with temperature independent resistance, and, consequently, the others *non-ohmic* In recognition of his contribution, resistance is measure in *Ohm* [Ω] = [V/A] (capital greek omega) and all electric devices are associated with their own resistance (although we often neglect the resistance of wires in a circuit).

Beyond resistors and resistance, there is *resistivity*. Resistivity ρ is yet another experimentally found proportionality constant. In a simple conducting wire, the resistance depends on geometric properties, i.e. cross-sectional area A and wire length l, but also on the material used, i.e. the resistivity of the material [Ω m]:

$$R = \rho \frac{l}{A}$$

In addition to resistivity, conductivity σ is defined the reciprocal of the resistivity:

$$\sigma = \frac{1}{\rho}$$

The *temperature dependency* of a resistor is defined via its resistivity. For moderate temperature changes and metals, this dependency can be approximated as linear and with respect to the resistivity ρ_0 at some reference temperature T_0 :

$$\rho(T) = \rho_0(1 + \alpha(T - T_0))$$

with $alpha$ being the material-specific *temperature coefficient* of resistivity. Besides ohmic resistors, i.e. conductors with a temperature coefficient of 0, we can differentiate between:

- negative temperature coefficients (NTC), i.e. lower resistance when heated, such as semiconductor which have more free electrons available at higher temperatures
- positive temperature coefficients (PTC), i.e. higher resistance when heated, such as many metals as the higher temperature increases the likelihood of atom-electron collision

Electric power

Electric circuits are used to transmit electric energy to device which can convert this energy into e.g. thermal energy (obeying conservation of energy). The amount of electric power P delivered is therefore of interest (at the very least to you energy company/provider). In general, power is the energy U per unit time. Recalling work and electric energy, we know energy scales with charge and potential difference. Hence, for an infinitesimal small amount of charge dq moving through the potential difference V, the electric energy is:

$$dU = V \, dq$$

Combing these two ideas, we can derive:

$$P = \frac{dU}{dt} = \frac{dq}{dt} V$$

As we defined the current as $I = \frac{dq}{dt}$, we obtain:

$$P = VI$$

By applying Ohm's law $V = RI$ we can extend this to

$$P = VI = I^2 R = \frac{V^2}{R}$$

In general, the unit for power is Watt [W] = [J/s], but you will find (on your electric bill) kilowatt-hours (kWh) as well which is simply using $1000\text{W} \times 3600\text{s} = 3.6 \times 10^6 \text{J}$.

Current density & drift speed

So far we have used a macroscopic perspective to understand current. By investigating current from the microscopic perspective we can extent our understanding. Let's define the *current density* $\vec{\mathbf{j}}$ as the electric current per unit cross-sectional area A. Assuming a uniform electric current density yields:

$$j = \frac{I}{A} \leftrightarrow I = jA$$

For non-uniform current densities, we obtain:

$$I = \int \vec{\mathbf{j}} d\vec{\mathbf{A}}$$

with I as the current through the entire surface and $d\vec{\mathbf{A}}$ the infinitesimal small elements of that surface. Note that, I is a macroscopic quantity, defined for e.g. the entire cross-section of a wire, while $\vec{\mathbf{j}}$ is a

microscopic quantity, defined for each point.

The direction of electric field and current density align, i.e. $\vec{\mathbf{E}} \parallel \vec{\mathbf{j}}$. Therefore, electrons travel in the direction of $-\vec{\mathbf{E}} = -\vec{\mathbf{j}}$

Inside the wire, the electric field initially accelerated the free electrons before they reach a stead average speed (due to collisions) known as the *drift speed* v_d. Note that this average drift speed of electrons is relatively small (≈ 0.05 mm/s) but the electric field, hence electricity, propagates with the speed on light.

We can relate the drift speed to the macroscopic current I by the following:

- the current is charge over unit time $I = \frac{\Delta Q}{\Delta t}$
- the charge is number of electrons N times charge of an electron $\Delta Q = N(-e)$
- the number of electrons N is related to the number of free electron per unit volume n by $n = \frac{N}{Al} \leftrightarrow N = nAl$ with A as the cross-sectional area
- the average distance l an electron travels in a time Δt as $l = v_d \Delta l$.

Therefore we obtain:

$$I = \frac{\Delta Q}{\Delta t} = \frac{-eN}{\Delta t} = \frac{-enAl}{\Delta t} = \frac{-enAv_d\Delta t}{\Delta t} = -enAv_d$$

$$j = \frac{I}{A} = -env_d$$

The minus sign indicates that the average drift velocity of electrons and macroscopic current as opposite to each other. If we have different pools of charges, we can generalize these equations to

$$I = \sum_i q_i n_i A v_{d,i}$$

$$j = \sum_i q_i n_i v_{d,i}$$

It is also interesting to relate current density and electric field (in idealized conditions):

- resistances is related to the conductors geometry and resistivity by $R = \rho \frac{l}{A}$
- in a homogenous field, we have $V = \int E dl = El$
- the current I is $I = jA$
- Ohm's law states $V = RI$

Therefore, we have:

$$V = IR = jA\rho\frac{l}{A}$$

$$El = j\rho l$$

$$E = j\rho$$

This can be generalized to the so-called "microscopic" view of Ohm's law, relating the electric field (proxy for V) to the current density (proxy for I) and resistivity (proxy for R)):

$$\vec{\mathbf{E}} = \rho\vec{\mathbf{j}} \leftrightarrow \vec{\mathbf{j}} = \frac{1}{\rho}\vec{\mathbf{E}}$$

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