

# 1.9. Oscillations

ms06

## **What is the purpose of the mechanism?**

- ⇒ Escapement with pendulum as found in mechanical clocks
- ⇒ To understand the mechanism we need to **study oscillations and simple harmonic motion**



# Simple Harmonic Motion — Motivation & Notation

## ms07 - Verschiedene Swinger

- Many oscillatory systems (springs, pendulums, etc.) share the same mathematical structure  $\Rightarrow$  **simple harmonic motion (SHM)**
- Displacement from equilibrium:  $x(t)$
- Velocity:  $\dot{x} = \frac{dx}{dt}$
- Acceleration:  $\ddot{x} = \frac{d^2x}{dt^2}$

## Horizontal Mass–Spring System — Model and Equation

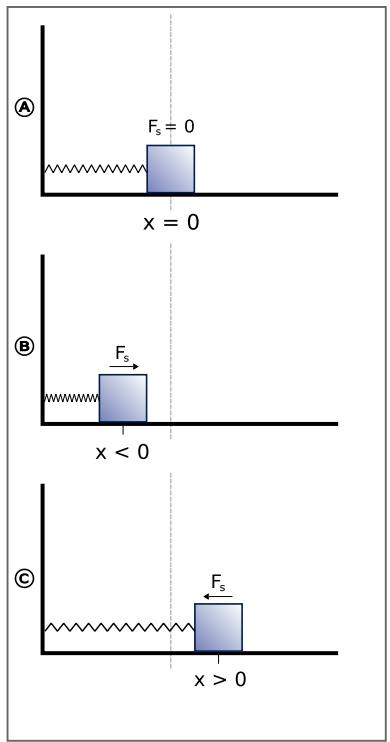
- Ideal mass–spring system on a frictionless horizontal surface
- Spring force obeys Hooke's law:  $F = -kx$
- Newton's second law:

$$ma = m\ddot{x} = -kx$$

- Equation of motion:

$$\ddot{x} = -\frac{k}{m}x \quad \Leftrightarrow \quad \ddot{x} + \frac{k}{m}x = 0$$

- The equation already shows the key SHM feature:
  - acceleration proportional to displacement
  - acceleration opposite in direction



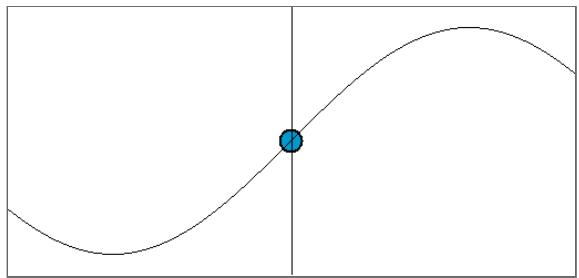
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## Horizontal Mass–Spring System — What a Solution Must Do & Guessing it

- We seek a function  $x(t)$  that satisfies the equation  $\ddot{x} + \frac{k}{m}x = 0$  for **all times**
  - ⇒ Any valid solution must cancel the explicit time dependence when inserted
- Motion swings repeatedly between two extreme positions
  - ⇒ **Trigonometric** functions naturally describe oscillations
  - ⇒ Mass release at maximum displacement  $x_0 \rightarrow$  use **cosine**
- **Solution Guess:**

$$x(t) = x_0 \cos(\omega t)$$

- **Angular frequency**  $\omega$  controls how fast the motion evolves in time
- Since cos repeats every  $2\pi$ , period  $T$  defined as  $T = \frac{2\pi}{\omega}$



from [wikipedia](#), public domain

## Horizontal Mass–Spring System — Time Derivatives

- Solution guess (displacement):

$$x(t) = x_0 \cos(\omega t)$$

- First derivative (velocity):

$$\dot{x}(t) = -x_0 \omega \sin(\omega t)$$

- Second derivative (acceleration):

$$\ddot{x}(t) = -x_0 \omega^2 \cos(\omega t)$$

## Horizontal Mass–Spring System — Inserting the Guess

- Substitute into

$$\ddot{x} + \frac{k}{m}x = 0$$

- This gives

$$-x_0\omega^2 \cos(\omega t) + \frac{k}{m}x_0 \cos(\omega t) = 0$$

- Common factor  $\cos(\omega t)$  cancels
- Remaining condition:

$$-\omega^2 + \frac{k}{m} = 0$$

## Horizontal Mass–Spring System — Result

- Cancellation for all times requires:

$$\omega^2 = \frac{k}{m}$$

- Solution:

$$x(t) = x_0 \cos(\omega t), \quad \omega = \sqrt{\frac{k}{m}}$$

- Period of oscillation:

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$

- Period depends only on mass and spring constant, not on amplitude

## Vertical Mass–Spring System — Equilibrium Shift

### ms07 - Verschiedene Schwingungen

- Forces are weight  $mg$  and spring force  $-kx$ , thus Newton's second law:

$$m\ddot{x} = mg - kx$$

- Equilibrium **after** mass placed on spring ( $\ddot{x} = 0$ ):

$$mg = kx_0 \Leftrightarrow x_0 = \frac{mg}{k}$$

$\Rightarrow x_0$  is the static extension due to the mass's weight

- Total extension of spring  $x_0 + x$  gives

$$m\ddot{x} = mg - k(x_0 + x) = mg - mg - kx = -kx$$

⇒ Redefine  $x(t)$  as displacement **from equilibrium** removes the constant gravity term from the motion

## Vertical Mass–Spring System — Oscillation Equation

$$m\ddot{x} = -kx \quad \Rightarrow \quad \ddot{x} + \frac{k}{m}x = 0$$

- Same equation as the horizontal system
- Cosine solution:

$$x(t) = x_0 \cos(\omega t), \quad \omega^2 = \frac{k}{m}$$

- Period:

$$T = 2\pi \sqrt{\frac{m}{k}}$$

- Gravity shifts equilibrium but does **not** change the period

## Simple Harmonic Motion (SHM) — General Form

- **Simple harmonic motion (SHM)** not limited to spring but applied whenever

$$\ddot{x} + \omega^2 x = 0 \quad \Leftrightarrow \quad \ddot{x} = -\omega^2 x$$

- Acceleration is proportional to displacement and always directed toward equilibrium
- **Harmonic** means: sinusoidal, single-frequency motion from a linear restoring mechanism
- General solution:

$$x(t) = A \cos(\omega t + \phi)$$

- with  $A$  as the amplitude
- with **phase constant**  $\phi$  to capture different initial positions and velocities
- with  $\omega$  as the **angular eigenfrequency** (natural frequency)

- with period  $T = \frac{2\pi}{\omega}$
- Note: equivalently  $A \sin(\omega t + \phi)$ ; sine and cosine differ only by a phase shift

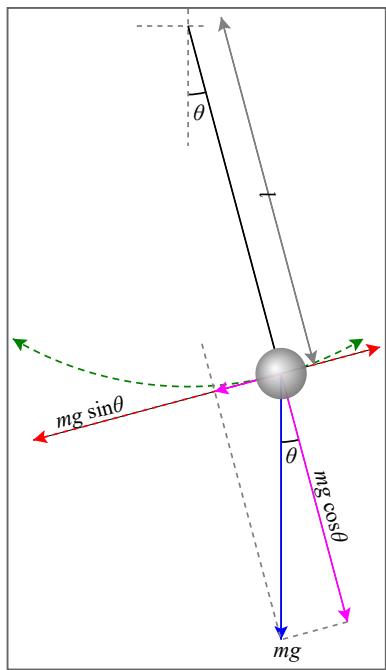
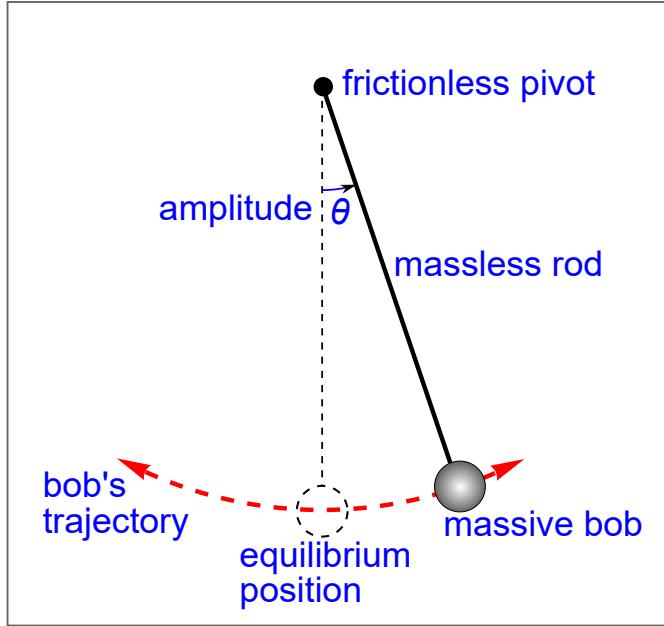
## Mathematical Pendulum — Assumptions & Restoring Force

ms18 - Schwingungsdauer bestimmen - Lichtschranke

- **Assumptions:** Point mass on a massless, inextensible string of length  $L$ , only small angular displacements, no friction
- Restoring force (tangential component of weight  $mg$ ):

$$F_r = -mg \sin \theta$$

⇒ Minus sign: force always points toward equilibrium



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# Mathematical Pendulum — Small-Angle Approximation & Newton's Law

- For small angles (in radians):

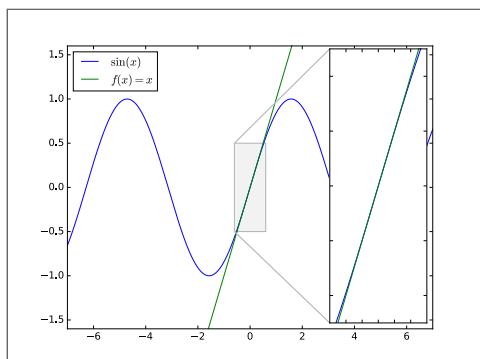
$$\sin \theta \approx \theta$$

- Restoring force becomes linear:

$$F_r \approx -mg\theta$$

- Newton's second law (tangential direction) with  $a_t = L\alpha = L\ddot{\theta}$ :

$$ma_t = mL\ddot{\theta} = -mg\theta = F_r$$



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## Mathematical Pendulum — Equation of Motion & Initial Guess

$$mL\ddot{\theta} = -mg\theta$$

$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

⇒ This is the standard equation of **simple harmonic motion**

Solution guess with  $\theta_0$  as maximum angular displacement:

$$\theta(t) = \theta_0 \cos(\omega t + \phi)$$

## Mathematical Pendulum — Determining $\omega$ and the Period

- Derivatives:

$$\dot{\theta} = -\theta_0 \omega \sin(\omega t + \phi), \quad \ddot{\theta} = -\theta_0 \omega^2 \cos(\omega t + \phi)$$

- Insert into the equation of motion:

$$-\theta_0 \omega^2 \cos(\omega t + \phi) + \frac{g}{L} \theta_0 \cos(\omega t + \phi) = 0$$

- Time dependence cancels for all  $t$  if

$$\omega^2 = \frac{g}{L}$$

- Angular eigenfrequency, period and frequency ( $f = \frac{1}{T} = \frac{\omega}{2\pi}$ ):

$$\omega = \sqrt{\frac{g}{L}}, \quad T = 2\pi \sqrt{\frac{L}{g}}, \quad f = \frac{1}{2\pi} \sqrt{\frac{g}{L}}$$

⇒ Period depends only on  $L$  and  $g$ , **not on mass or amplitude**

# Other Pendulums — Examples

## ms07 - Verschiedene Swinger

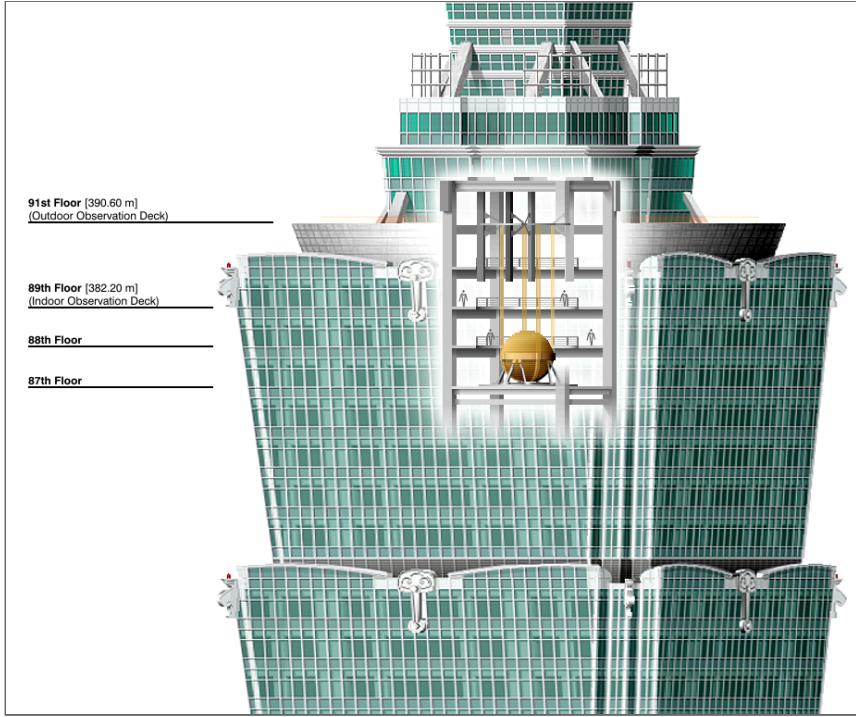
Feature	Physical Pendulum	Torsion Pendulum
Key difference to mathematical pendulum	Mass <b>distributed</b> in a rigid body	No gravity-driven swing, <b>twisting</b> motion
Restoring mechanism	Gravity acting on center of mass	Elastic torsion of a wire/fiber
Newton's law used	Rotational: $I\ddot{\theta} = \tau \approx -mgd\theta$	Rotational: $I\ddot{\theta} = \tau = -\kappa\theta$
Equation of motion	$\ddot{\theta} + \frac{mgd}{I}\theta = 0$	$\ddot{\theta} + \frac{\kappa}{I}\theta = 0$
Angular frequency & period	$\omega = \sqrt{\frac{mgd}{I}}$ & $T = 2\pi\sqrt{\frac{I}{mgd}}$	$\omega = \sqrt{\frac{\kappa}{I}}$ & $T = 2\pi\sqrt{\frac{I}{\kappa}}$

<b>Feature</b>	<b>Physical Pendulum</b>	<b>Torsion Pendulum</b>
Reduced / equivalent length	$L_{\text{eq}} = \frac{I}{md}$	not applicable

## Other Pendulums — Examples (cont')

ms07 - Swinger mit Fluid





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# SHM as Projection of Uniform Circular Motion

## ms35 - Projektion einer Kreisbewegung

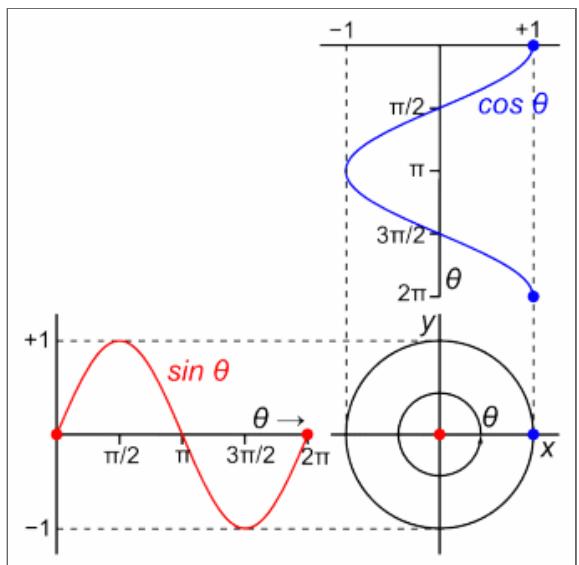
- Simple harmonic motion can be viewed as the **projection of uniform circular motion** onto a straight line
- This geometric picture explains amplitude, phase, angular frequency, and phase relations
- Angular position for a point moving on a circle of radius  $r_0$  with constant angular speed  $\omega$

$$\theta(t) = \omega t + \phi$$

- Cartesian coordinates of the circular motion:

$$x(t) = r_0 \cos(\omega t + \phi), \quad y(t) = r_0 \sin(\omega t + \phi)$$

⇒ These are the SHM solutions for projections on the x- and y-axis, respectively



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## Damped Harmonic Motion — Motivation & Model

### ms07 - Verschiedene Swinger

- Real oscillators lose energy due to friction, air resistance, internal dissipation, etc.
- Amplitude decreases with time → **damped harmonic motion**
- Linear damping model:

$$F_{\text{damp}} = -b \dot{x}$$

- Mass–spring system:

$$F = -kx - b \dot{x}$$

- Newton's second law:

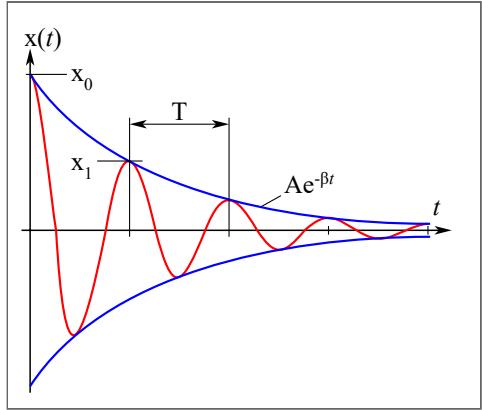
$$m\ddot{x} + b\dot{x} + kx = 0$$

## Damped Harmonic Motion — Motivated Ansatz

- Undamped case: cosine motion between two turning points
- Weak damping should:
  - keep oscillatory behavior
  - slowly reduce amplitude
- Simplest model: oscillation  $\times$  exponential decay
- **Ansatz:**

$$x(t) = x_0 e^{-\gamma t} \cos(\omega t)$$

- $\gamma$  controls amplitude decay,  $\omega$  is modified oscillation frequency



from [wikipedia](#) by Jahobr, gemeinfrei

## Damped Harmonic Motion — Derivatives (Result)

- First derivative:

$$\dot{x} = x_0 e^{-\gamma t} \left( -\gamma \cos(\omega t) - \omega \sin(\omega t) \right)$$

- Second derivative:

$$\ddot{x} = x_0 e^{-\gamma t} \left( (\gamma^2 - \omega^2) \cos(\omega t) + 2\gamma\omega \sin(\omega t) \right)$$

## Damped Harmonic Motion — Inserting into the ODE

- Insert  $\dot{x}$  and  $\ddot{x}$  into

$$m\ddot{x} + b\dot{x} + kx = 0$$

- After collecting terms:

$$x_0 e^{-\gamma t} \left[ (m(\gamma^2 - \omega^2) - b\gamma + k) \cos(\omega t) + (2m\gamma\omega - b\omega) \sin(\omega t) \right] = 0$$

- Time dependence cancels only if **both coefficients vanish**

## Damped Harmonic Motion — Damping Constant

- Time dependence cancels only if **both coefficients vanish**

$$x_0 e^{-\gamma t} \left[ (m(\gamma^2 - \omega^2) - b\gamma + k) \cos(\omega t) + (2m\gamma\omega - b\omega) \sin(\omega t) \right] = 0$$

- From the sine term:

$$2m\gamma\omega - b\omega = 0$$

- Non-trivial motion requires  $\omega \neq 0$
- Result:

$$\gamma = \frac{b}{2m}$$

## Damped Harmonic Motion — Frequency Shift

- Time dependence cancels only if **both coefficients vanish**

$$x_0 e^{-\gamma t} \left[ (m(\gamma^2 - \omega^2) - b\gamma + k) \cos(\omega t) + (2m\gamma\omega - b\omega) \sin(\omega t) \right] = 0$$

- From the cosine term:

$$m(\gamma^2 - \omega^2) - b\gamma + k = 0$$

- Using  $\gamma = \frac{b}{2m} \leftrightarrow \frac{b}{m} = 2\gamma$ :

$$(\gamma^2 - \omega^2) - \frac{b}{m}\gamma + \frac{k}{m} = 0$$

$$\omega^2 = \frac{k}{m} - \gamma^2$$

- With undamped eigenfrequency  $\omega_0^2 = \frac{k}{m}$ , final result:

$$\omega = \sqrt{\omega_0^2 - \gamma^2}$$

## Damped Harmonic Motion — Underdamped Case

- Condition:

$$\gamma < \omega_0$$

- Oscillatory motion with decaying amplitude:

$$x(t) = x_0 e^{-\gamma t} \cos(\omega t)$$

- Energy decay:

$$E(t) = E_0 e^{-2\gamma t}$$

## Damped Harmonic Motion — Critical Damping

- Boundary case:

$$\gamma = \omega_0$$

- Fastest return to equilibrium **without oscillation**
- Solution form ( $C_1$  and  $C_2$  determined from initial conditions):

$$x(t) = (C_1 + C_2 t)e^{-\gamma t}$$

- Important in e.g. analogue measurement instruments and shock absorbers

## Damped Harmonic Motion — Overdamped Case

- Strong damping:

$$\gamma > \omega_0$$

- No oscillations
- Two exponential decay modes:

$$x(t) = C_1 e^{-(\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + C_2 e^{-(\gamma - \sqrt{\gamma^2 - \omega_0^2})t}$$

- Typically: fast initial decay + slow long-time relaxation

## Damped Harmonic Motion — Summary

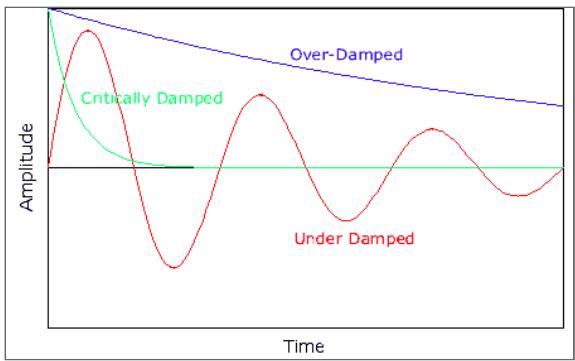
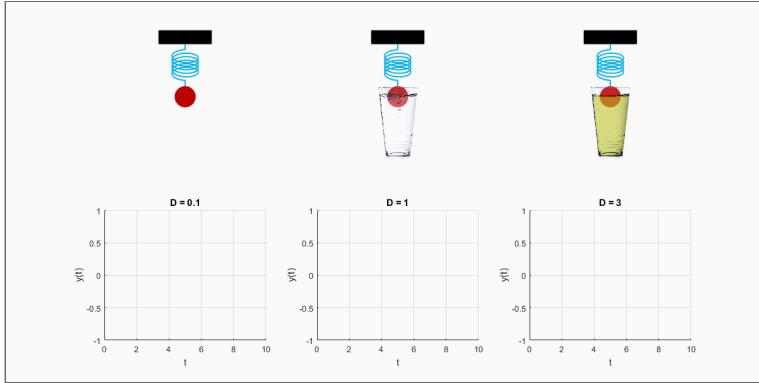
- Damping introduces energy loss and amplitude decay
- Equation:

$$m\ddot{x} + b\dot{x} + kx = 0$$

- Key parameters:

$$\gamma = \frac{b}{2m}, \quad \omega = \sqrt{\omega_0^2 - \gamma^2}$$

- Three regimes: underdamped (oscillatory), critically damped (fastest non-oscillatory), & overdamped (slow, monotonic)
- Realistic extension of ideal SHM



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## Forced Oscillations — Motivation and Model

### ms15 - Schwingender Motor

- Real systems are often driven by a **periodic external force**
- Energy input competes with damping → **forced oscillations** and **resonance**
- Sinusoidal driving force:

$$F(t) = F_0 \cos(\omega_d t)$$

- Equation of motion:

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega_d t)$$

## Forced Oscillations — Steady-State Response

- Initial motion depends on initial conditions and decays with damping
- Long-term motion is maintained by the driving force
- **Steady-state ansatz** (same frequency as driving force):

$$x(t) = A \cos(\omega_d t - \delta)$$

⇒ System oscillates at the **driving frequency**  $\omega_d$  and with phase offset  $\delta$

## Forced Oscillations — Amplitude Response

- Solving the equation yields the amplitude:

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2}}$$

with

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \gamma = \frac{b}{2m}$$

- Amplitude depends strongly on the driving frequency

## Resonance — Resonance

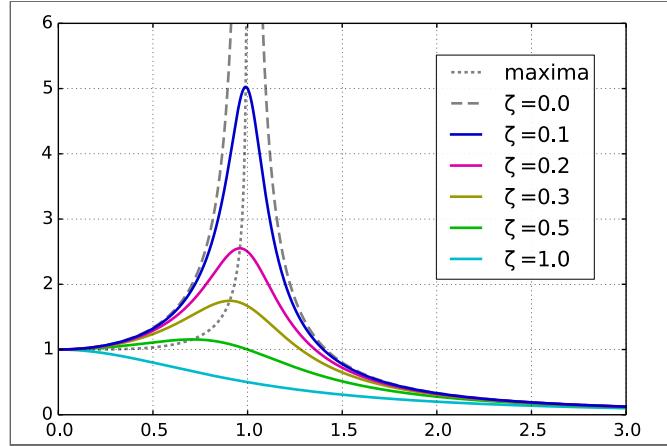
ms20 - Resonanz zwischen Motor und Pendeln verschiedener Länge

- Energy input balances dissipation → large amplitudes for weak damping
- Maximum amplitude at resonance:

$$A(\omega_r) \approx \frac{F_0/m}{2\gamma\omega_0}$$

- **Quality factor** measures resonance sharpness:

$$Q = \frac{\omega_0}{2\gamma}$$



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Revisit initial experiment: Escapement,  
oscillations, & clocks



ms06

- weight provides (potential) energy to drive machine
- via escapement, a defined impulse to the pendulum → **forced oscillation**
- pendulum length determines period, i.e.  $T = 1\text{s}$ , and releases escapement for next step (lowering of mass and subsequent new impulse)
- add gears to build proper clock