1.4. Work, Energy, & Power

`md28 - Pendel`

What is the min. height required for the object to do a looping?

$\Rightarrow$ Solution at the end

New concepts required:

- Introduce Hooke's law, work, & power
- Derive kinetic energy & potential energy from work
- Conservative vs. nonconservative force
- Fundamental concept of **conservation of energy**

Definition of Work

- In physics, work has a strict quantitative meaning (not “effort” as in everyday language)
- **Work describes energy transfer caused by a force acting on an object along the direction of displacement**

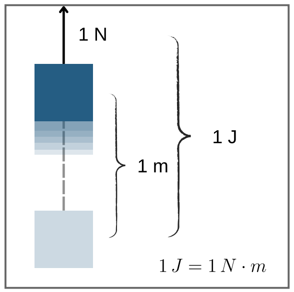
Work for Straight-Line Motion under a Constant Force

- Work is defined by the dot product of $\vec{F}$ and the displacement $\vec{d}$:

$$W = \vec{F} \cdot \vec{d}$$

- Dot product $\rightarrow$ work is a **scalar**
- Inputs: force vector + displacement vector
- Units:

$$1\ \mathrm{J} = 1\ \mathrm{N} \cdot \mathrm{m} = 1\ \mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2$$



Work for Straight-Line Motion under a Constant Force (cont')

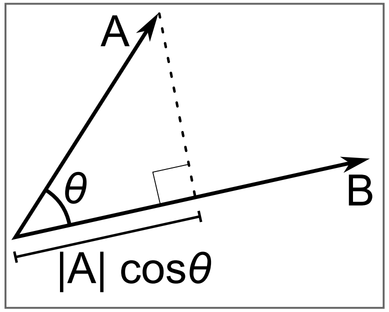
- With θ as the angle between $\vec{F}$ & $\vec{d}$

$$W = \vec{F} \cdot \vec{d} = Fd\cos\theta$$

- Projection interpretation:

$$\vec{F} \cdot \vec{d} = F_{\parallel} d$$

$\rightarrow$ Only component of force parallel to displacement does work.



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Work and constant Force: Examples

Example	Situation / Key Idea	Work Result
Pushing against a wall		
Lifting an object		
Lowering an object slowly		
Carrying an object		
Pulling a box at an angle		
Sliding a box with friction		
Circular motion		

Work and constant Force: Solutions

Example	Situation / Key Idea	Work Result
Pushing against a wall	Force applied, but **displacement = 0**	**no work**
Lifting an object up	Force ↑, displacement ↑	**positive work**
Lowering an object slowly	Force ↑, displacement ↓	**negative work**
Carrying an object	Force ↑, displacement → (horizontal), angle = 90°	**no work**
Pulling a box at an angle	Only horizontal component $F_\parallel$ contributes	**positive work**
Sliding a box with friction	Applied force → positive; friction → negative	**net work = sum of works**

Example	Situation / Key Idea	Work Result
Circular motion	Centripetal force $\perp$ velocity	**no work**

Work and constant Force: Summary

- Work from constant force:

$$W = \vec{F}\vec{d} = Fd\cos\theta$$

- Sign from angle:
    - $0^\circ \leq \theta < 90^\circ \rightarrow$ **positive work**
    - $\theta = 90^\circ \rightarrow$ **zero work**
    - $90^\circ < \theta \leq 180^\circ \rightarrow$ **negative work**
- Only the **parallel component** of force transfers energy
- Net work: $W_{\text{net}} = \sum_i W_i = \sum_i \vec{F}_i\vec{d} = \vec{F}_{\text{net}}\,\vec{d}$

Work Done by a Variable Force

- Force may vary in magnitude and/or direction
- Split path into infinitesimal segments along the position vector $\vec{r}$:

$$\mathrm{d}W = \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r}$$

- Total work = sum of all infinitesimal contributions:

$$W = \int \mathrm{d}W = \int \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r}$$

- Dot product ensures only the **parallel component** contributes

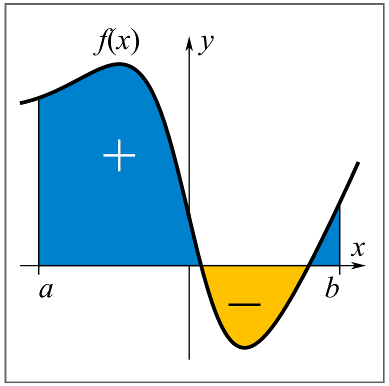
Work as a Line Integral

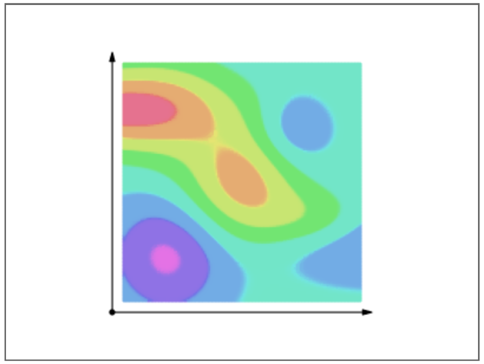
- General form between $\vec{r}_1$ and $\vec{r}_2$: $W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r} \Rightarrow$ **signed area under the $F(\vec{r})$ curve**
- 1D case:

$$W = \int_{x_1}^{x_2} F(x)\,\mathrm{d}x$$

- 3D case:

$$W = \int_{x_1}^{x_2} F_x\,\mathrm{d}x + \int_{y_1}^{y_2} F_y\,\mathrm{d}y + \int_{z_1}^{z_2} F_z\,\mathrm{d}z$$





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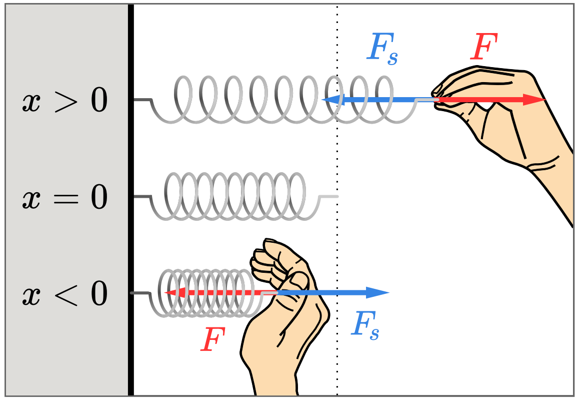
Hooke's Law: Spring Force

`mi17 - Feder`

- Ideal spring force with spring constant (stiffness) k & displacement from equilibrium x:

$$F_S = -kx$$

- Negative sign $\to$ restoring force (points opposite to x)
- Force increases **linearly** with displacement
  → straight line in the F–x diagram



Work Done by a Spring

- Work done **by the spring** (from x_1 to x_2):

$$W_S = \int_{x_1}^{x_2} F_S \, \mathrm{d}x = \int_{x_1}^{x_2} (-kx) \, \mathrm{d}x = -\tfrac{1}{2} k(x_2^2 - x_1^2)$$

- Work done **on the spring** by external force to stretch from 0 to x:

$$F_{\text{ext}} = +kx$$

$$W_{\text{ext}} = \int_{0}^{x} (+kx) \, \mathrm{d}x = \tfrac{1}{2} kx^2$$

- $\rightarrow$ equal to **area under the F–x graph** (triangle)

Deriving Kinetic Energy from Work

`md36 - Schuss`

- Infinitesimal work:

$$\mathrm{d}W = \vec{F}_{\text{net}} \cdot \mathrm{d}\vec{l} = m\vec{a} \cdot \mathrm{d}\vec{l}$$

- With $\vec{a} = \dfrac{d\vec{v}}{dt}$ and $\mathrm{d}\vec{l} = \vec{v}\,\mathrm{d}t$:

$$\mathrm{d}W = m\,\frac{d\vec{v}}{dt} \cdot \vec{v}\,\mathrm{d}t = m\,\vec{v} \cdot \mathrm{d}\vec{v}$$

- Integrate from $\vec{v}_1$ to $\vec{v}_2$:

$$W_{\text{net}} = \int_{\vec{v}_1}^{\vec{v}_2} m\,\vec{v} \cdot \mathrm{d}\vec{v} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

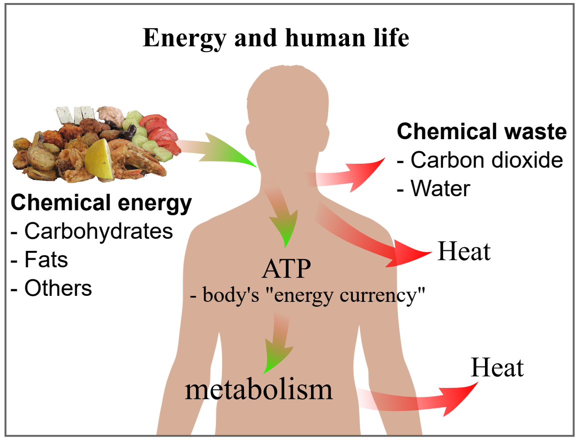
- Define work–energy relation & kinetic energy:

$$W_{\mathrm{net}} = \Delta K, \quad K = \frac{1}{2}mv^2$$

Energy

> *Energy is a scalar quantity that measures the ability of a system to perform work or to produce changes in the system or its environment.*

$\rightarrow$ Energy also measure in joule [J]
$\rightarrow$ Several forms of energy, e.g. kinetic energy describing motion, or potential energy representing capacity to do work due to position/configuration



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Deriving Potential Energy from Work

- Total work from A to B:

$$W_{AB} = \int \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{l}$$

- For **conservative forces** (gravity, spring), **work depends only on start/end points**
- Define potential energy change:

$$\Delta U = U_B - U_A = -W_{AB} = -\int \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{l}$$

Gravitational Potential Energy

`md25 - Anheben Kette`

- Gravitational force (uniform field) wih $\hat{y}$ pointing upwards:

$$\vec{F}_G = -mg\,\hat{y}$$

- Work done by gravity on the object:

$$W_G = \int_{y_1}^{y_2} (-mg)\,\mathrm{d}y = -mg(y_2 - y_1)$$

- Define potential energy with $U = 0$ at $y = 0$:

$$U_{\text{grav}} = -W_G = mgy$$

Elastic Potential Energy (Spring)

- Spring force obeying Hooke's law:

$$F_S = -kx$$

- Work done by the spring on the object:

$$W_S = \int_{x_1}^{x_2} (-kx)\,\mathrm{d}x = -\tfrac{1}{2}k(x_2^2 - x_1^2)$$

- Define potential energy with $U(x = 0) = 0$:

$$U_{\text{elastic}} = -W_S = \tfrac{1}{2}kx^2$$

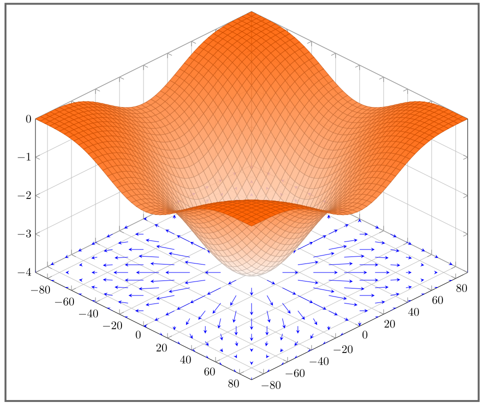
Reference Point & Force–Potential Relation

`sim - reference height`

- Zero of potential energy is **arbitrary**
  $\rightarrow$ only **changes** in U matter
- Force–potential connection (1D):

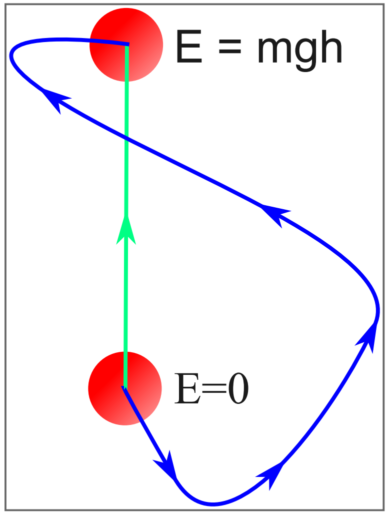
$$U(x) = -\int F(x)\,\mathrm{d}x + C \quad \Leftrightarrow \quad F(x) = -\frac{dU}{dx}$$

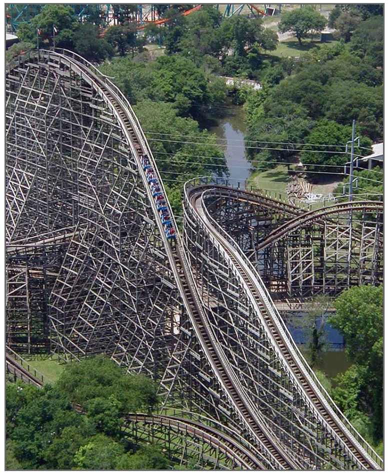
- Generalization: $\vec{F} = -\nabla U$
- $\rightarrow$ Force points toward **lower** potential energy (work by force: $W = -\Delta U$)



Conservative vs. Nonconservative Forces

- **Conservative forces**:
  - **Work independent of path**, e.g. gravity, springs, electrostatic force
  - Closed-path work: $\oint \vec{F} \cdot \mathrm{d}\vec{l} = 0$
- **Nonconservative forces**:
  - **Work depends on path**, e.g. friction, air resistance, drag
  - Mechanical energy not fully recoverable





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Conservation of Energy

`md35 - Vase`

`md29 - Looping`

> ***Law of Conservation of Energy:***
> *The total energy of an isolated system remains constant.*
> *Energy may be transferred or transformed, but cannot be created or destroyed.*

- Total energy includes all forms:
  mechanical, thermal, chemical, electrical, nuclear, radiant
- Dissipative forces (friction, drag):
  convert mechanical energy $\rightarrow$ thermal energy

- Energy balance (isolated system):

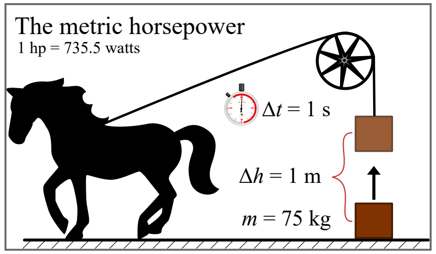
$$K + U + E_{\text{other}} = \text{const} \;\Leftrightarrow\; \Delta(K + U + E_{\text{other}}) = 0$$

Power & Efficiency

- **Power** is the **rate of doing work**:

$$P_{\text{avg}} = \frac{W}{\Delta t}, \quad \& \quad P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\vec{F}\mathrm{d}\vec{l}}{\mathrm{d}t} = \vec{F} \cdot \vec{v}$$

- **Units**: watt (W = J/s)
- **Efficiency** (useful output vs. input): $\eta = \dfrac{E_{\text{useful}}}{E_{\text{input}}}$
- Express as decimal or percentage;
  ideal $\eta = 1$ not achievable due to losses



Revisit first experiment

`md28`

What height required for the object to do a looping?

Bonus I

`md30`

Which track has the most energy?
Which track is the fastest?

Bonus II

`mi17`

At what height does the object reach its maximum velocity?