1.6. Rotational dynamics & angular momentum

`mk15`

Which egg is cooked? How to check?

Mass vs. mass distribution?

`mk14 - Abrollen versch. Körper`

Which body is the fastest & why?

- Study rotational motion
- Introduce **torque, moment of inertia,** & **angular momentum**
- New concepts of **rotational kinetic energy** & **angular momentum conservation**

Recap & Primer Rotational Motion

- The position on the circle is described by the **angular displacement** $\phi(t)$ (in radians) measured from the $+x$-axis
- **Angular velocity** ω is measured in **rad/s** and **angular acceleration** α in **rad/s²**:

$$\omega = \frac{d\phi}{dt}, \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\phi}{dt^2}$$

- Linear quantities from angular ones:

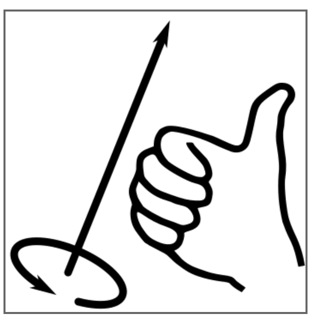
$$s = r\phi, \quad v = r\omega, \quad a_t = r\alpha, \quad a_r = \frac{v}{r^2} = r\omega$$

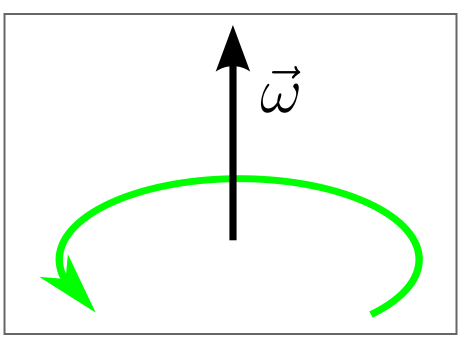
- **Key idea**: Angular motion is identical for all points,linear motion scales with radius r.

Angular motion with vectors

Question: Linear quantities are **vectors**? What about angular ones?

- $\vec{s}, \vec{v}, \vec{a}_t$ all tangential
- position vector $\vec{r}$ and $\vec{a}_r$ radially
- $\Rightarrow \vec{\phi}, \vec{\omega}, \vec{\alpha}$ oriented along rotation axis $\Leftrightarrow$ they **define the axis of rotation**
- **direction** (positive vs. negative) determined by **Right-Hand Rule** (a.k.a. convention)





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Angular motion with vectors (cont')

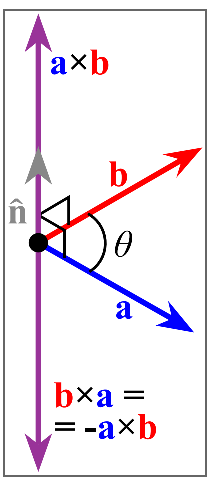
Cross product relates angular and linear quantities, e.g.

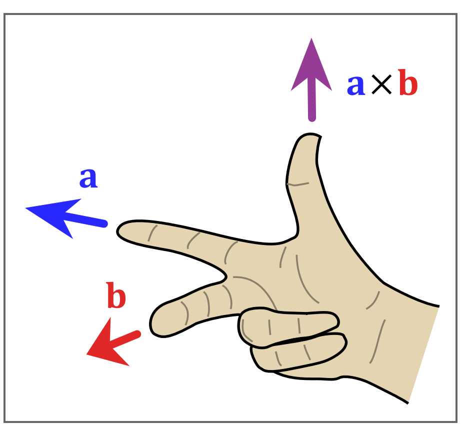
$$\vec{v} = \vec{\omega} \times \vec{r}$$

Key properties:

$$\vec{v} \perp (\vec{\omega} \times \vec{r})$$

$$\vec{\omega} \times \vec{r} = -\vec{r} \times \vec{\omega}$$





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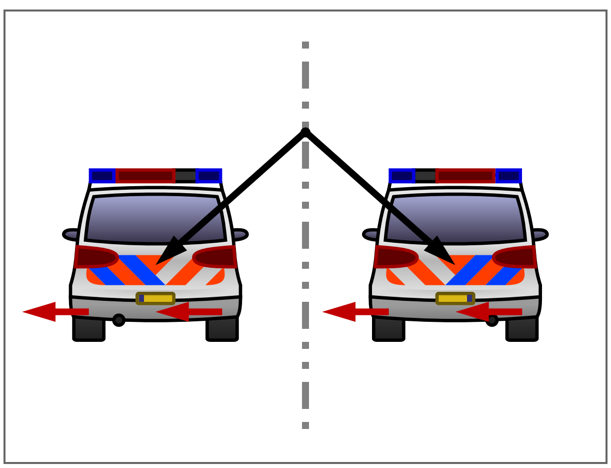
Side note: Pseudovectors

- $\vec{\phi}, \vec{\omega}, \vec{\alpha}$ are pseudovectors, also called axial vectors
- Under reflection, pseudovectors **do not flip sign**
- Imagine doing the right-hand-rule in front of a mirror
- Pseudovectors are relevant e.g. for symmetry-based solutions in physics



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Revisit acceleration $\vec{a}$ for circular motion

- To compute linear acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\vec{\omega} \times \vec{r})}{dt}$$

- Use **vector product rule**:

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

- Apply to $\vec{\omega}$ and $\vec{r}$:

$$\vec{a} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

Revisit acceleration $\vec{a}$ for circular motion (cont')

Substitute $\frac{d\vec{\omega}}{dt} = \vec{\alpha}$, & $\frac{d\vec{r}}{dt} = \vec{v} = \vec{\omega} \times \vec{r}$

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{a}_t + \vec{a_R}$$

Tangential:

$$\vec{a}_t = \vec{\alpha} \times \vec{r}$$

Radial (centripetal):

$$\vec{a_R} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

(note relation to magnitude of $a_R = \frac{v^2}{r} = \omega^2 r$)

Accelerated circular motion

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mk23 - accelerated circular motion
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What causes the angular acceleration?

Accelerated circular motion (cont’)

`mk02 - Drehmoment`

What (else) causes the angular acceleration?

Torque

- Linear motion: force causes acceleration ($F = ma$)
- Rotational motion: effectiveness of a force depends on **magnitude**, **angle**, and **where** it is applied
- Torque:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

- Rotational analogue of force and measured in $\mathrm{N \cdot m}$
- Direction:
  - Perpendicular to the plane of $\vec{r}$ and $\vec{F}$
  - Set by **right-hand rule**
  - CCW → out of page ⊙, CW → into page ⊗

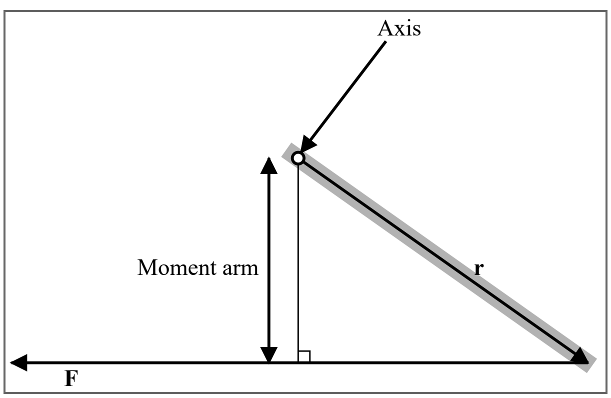
Torque Magnitude

`sim - Torque`

- Torque magnitude:

$$\tau = rF\sin\phi = r_{\perp}F = rF_{\perp}$$

  $\rightarrow$ Maximal when $\vec{F} \perp \vec{r}$; zero when $\vec{F} \parallel \vec{r}$
- Lever arm (moment arm) $r_{\perp} = r\sin\phi \rightarrow$ only the **perpendicular** component of the force causes rotation.



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Connection to Newton's second law:

$$\vec{F} = m\vec{a}, \qquad \vec{\tau} = \vec{r} \times \vec{F}$$

What component of $\vec{a}$ is relevant for torque?

$\rightarrow$ Tangential acceleration $a_t = r\alpha$
Because $\vec{a}_t \perp \vec{r}$, we obtain torque magnitude as:

$$\tau = rF_t = ra_t m = mr^2\alpha$$

Revisiting mass vs. mass distribution

`mk14 - Abrollen versch. Körper`

$$\tau = mr^2\alpha$$

$\Rightarrow$ **mass distribution captured by** mr^2 $\Rightarrow$ **moment of inertia** I

Net torque & moment of inertia

- Moment of inertia I is the **rotational analogue of mass** $\rightarrow$ measures **resistance to changes in rotational motion**.
- Larger I → smaller angular acceleration for the same torque
- For many particles in a rigid body:

$$\tau_{\text{net}} = \sum_i \tau_i = \sum_i m_i r_i^2 \alpha.$$

Since all particles share the same α, define

$$I = \sum_i m_i r_i^2, \qquad \tau_{\text{net}} = I\alpha.$$

- Continuous mass distribution:

$$I = \int r^2 \, dm.$$

r = perpendicular distance from axis to element dm.

Net torque & moment of inertia (cont’)

`mk23 - torque vs. moment of inertia`

$$I = \int r^2 \, dm, \qquad \tau_{\text{net}} = I\alpha$$

- Key idea:
  Mass farther from the axis contributes **much more** (because of r^2).
- Units of I: **kg·m²**.
- Rotational equilibrium:
  If $\vec{\tau}_{\text{net}} = 0$, then $\alpha = 0$ → body at rest or rotating at constant ω.

Moment of inertia: Example

Uniform rod of length L, mass M, with linear mass density $\lambda = \frac{M}{L}$, rotating about its center:

- Mass element:

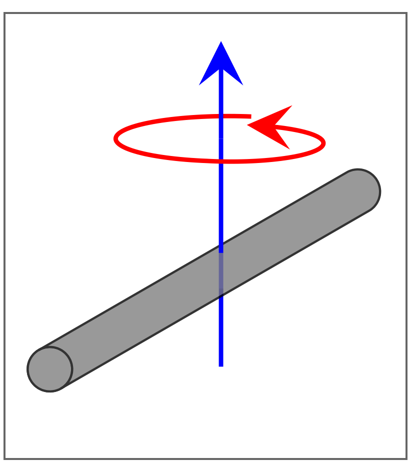
$$dm = \lambda\, dx = \frac{M}{L} dx.$$

- Compute I:

$$I = \frac{M}{L} \int_{-L/2}^{L/2} x^2 dx = \frac{1}{12} M L^2.$$

- Result for rod about its center:

$$I_{\text{center}} = \frac{1}{12} M L^2.$$



Moment of inertia: Example II

Uniform rod with L, M, rotating about one end:

$$I = \frac{M}{L} \int_0^L x^2 dx = \frac{1}{3} M L^2.$$

Interpretation: Moving the axis away from the center of mass increases I

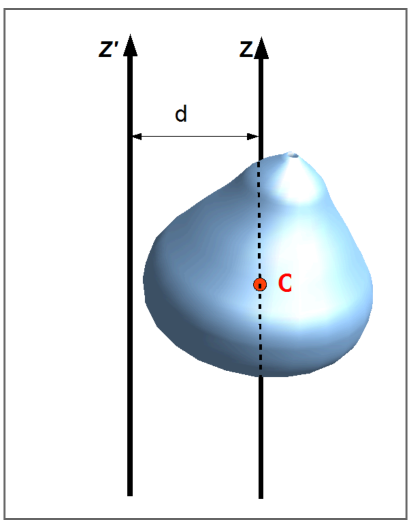
$\Rightarrow$ **Moment of inertia depends on rotation axis**

Parallel-axis Theorem (Steiner's theorem)

- Used when the rotation axis is **shifted** from the center-of-mass (CM) axis.
- Theorem:

$$I = I_{\mathrm{CM}} + Md^2,$$

with I_{CM} = moment of inertia about CM axis,d = distance between axes



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Moment of inertia: Revisit Example II

Uniform rod, length L, mass M, rotating about one end:

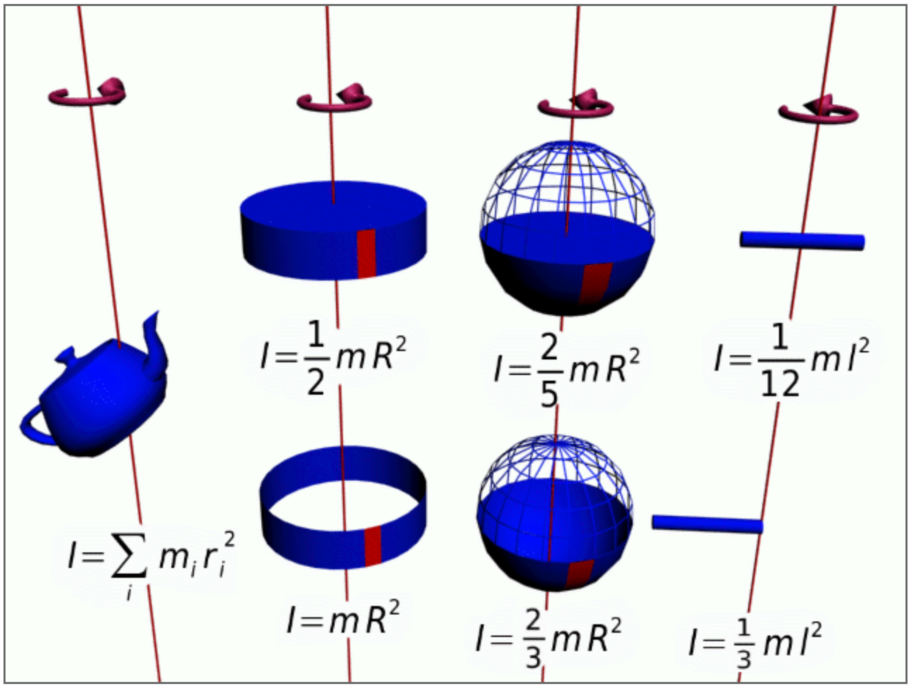
$$I_{\mathrm{CM}} = \frac{1}{12} M L^2, \qquad d = \frac{L}{2}.$$

Moment of inertia about end:

$$I_{\mathrm{end}} = \frac{1}{12} M L^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3} M L^2.$$

Practical note: Usually easier to start from a known **CM moment of inertia** and apply the **parallel-axis theorem** instead of re-integrating.

Common Moments of Inertia & the Tensor Perspective



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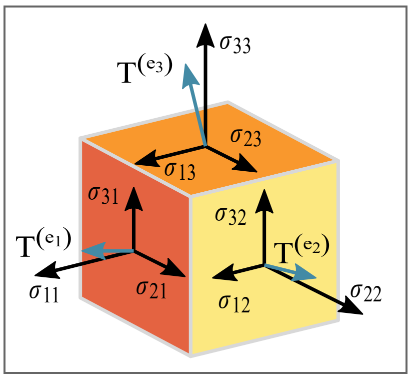
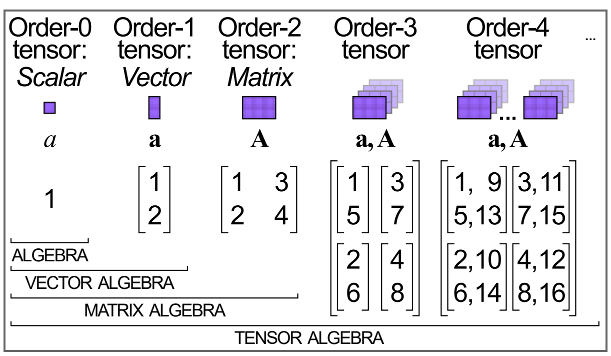
Moments of Inertia: Tensor Perspective

- Moment of inertia is an **inertia tensor** in 3D
- Different axes → different values of I
- In this course: rotation about **one symmetry axis**, so I behaves like a scalar





[left] ***Stress tensor example*** *from* ***wikipedia***, ***Attribution-Share Alike 3.0 Unported***; *[right] from* ***wikipedia***, ***Attribution-Share Alike 4.0 International***

Rotational Kinetic Energy

- Rotating rigid body:
  - Treat body as many small masses m_i at distances r_i with velocity $v_i = r_i\omega$
  - Kinetic energy of each element:

$$K_i = \tfrac{1}{2}m_i v_i^2 = \tfrac{1}{2}m_i r_i^2\omega^2.$$

- Total rotational kinetic energy:

$$K_{\text{rot}} = \sum_i K_i = \tfrac{1}{2}\omega^2 \sum_i m_i r_i^2.$$

- Recognize the moment of inertia: $I = \sum_i m_i r_i^2$
- Final result:

$$K_{\text{rot}} = \tfrac{1}{2}I\omega^2$$

- Direct analogue of $K_{\text{trans}} = \frac{1}{2}mv^2$ with $m \to I$ and $v \to \omega$.

Work Done by a Torque: Rotational Work–Energy Theorem

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta.$$

- Using $\tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\phi}\frac{d\phi}{dt} = I\frac{d\omega}{d\phi}\omega$, the net work becomes:

$$W = \int_{\theta_i}^{\theta_f} I\frac{d\omega}{d\phi}\,\omega d\phi = I\int_{\theta_i}^{\theta_f} \omega d\omega$$

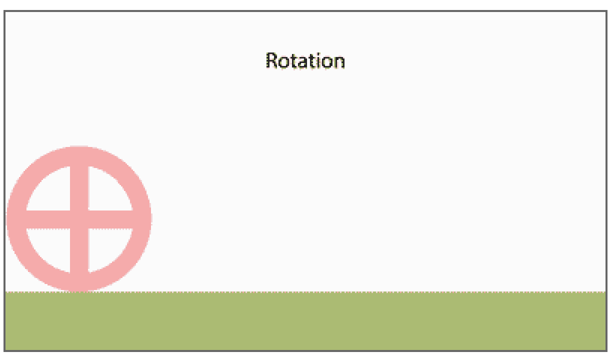
$$W_{\mathrm{net}} = \Delta K_{\mathrm{rot}} = \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2.$$

Rolling Without Slipping

- Rolling = **translation of the center of mass** + **rotation about an axis**.
- No-slip condition: **contact point is instantaneously at rest** relative to the surface

$$v_{\mathrm{CM}} - R\omega = 0 \ \leftrightarrow \ v_{\mathrm{CM}} = R\omega \ \leftrightarrow \ a_{\mathrm{CM}} = R\alpha$$

- Interpretation: one full revolution → object advances by $2\pi R$.
- If $v_{\mathrm{CM}} \neq R\omega$ → **slipping**



Flywheel: Storing rotational energy

stores rotational energy via conservation of angular momentum

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Angular momentum for a Particle

- **angular momentum** $\vec{L}$ (linear analogue $\vec{p} = m\vec{v}$):

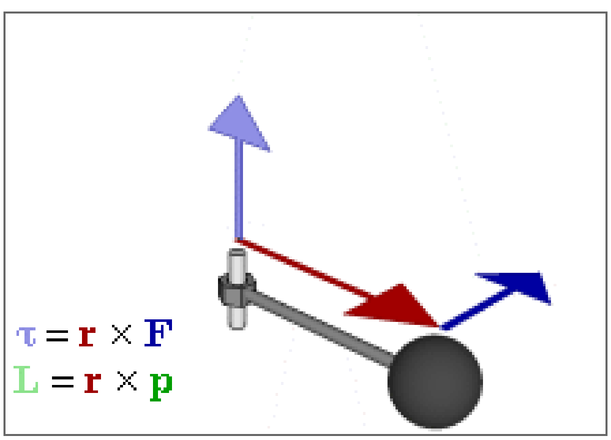
$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

- Angular momentum is a **vector** (axial/pseudovector) with direction from the **right-hand rule**
- Units: **kg·m²/s**
- Magnitude:

$$L = rp \sin\phi = rmv \sin\phi$$

- For particle in circular motion with $v = r\omega$ and $I = mr^2$:

$$L = mrv = mr(r\omega) = mr^2\omega = I\omega$$



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Angular momentum for a Particle (cont')

$$\vec{L} = \vec{r} \times \vec{p}$$

- Time derivative (product rule):

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

- Substitute $\frac{d\vec{r}}{dt} = \vec{v}$, $\vec{p} = m\vec{v}$, and $\vec{F} = \frac{d\vec{p}}{dt}$:
    - First term: $\vec{v} \times m\vec{v} = \vec{0}$ (parallel vectors).
    - Remaining term:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

- Analogue relation to translational dynamics:

$$\vec{\tau} = \frac{d\vec{L}}{dt} \;\leftrightarrow\; \vec{F} = \frac{d\vec{p}}{dt}$$

Angular momentum for a Rigid Object

$$\vec{L} = I\vec{\omega}$$

- Differentiate (rigid body → I constant):

$$\frac{d\vec{L}}{dt} = \frac{Id\vec{\omega}}{dt} = I\vec{\alpha}.$$

Therefore:

$$\vec{\tau}_{\text{net}} = I\vec{\alpha}.$$

Rotational analogue of $F = ma$.

Angular Momentum Summary:

- Particles: $\vec{L} = \vec{r} \times \vec{p}$
- Rigid bodies: $\vec{L} = I\vec{\omega}$ (for symmetry axes)
- Always:

$$\vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$$

Conservation of Angular Momentum - Theory

- If net external torque is zero:

$$\frac{d\vec{L}}{dt} = 0 \ \leftrightarrow \ \vec{L} = \text{const.}$$

- **If the net external torque is zero, angular momentum stays constant**
- Internal mass redistribution can change angular velocities, **but not total angular momentum**
- Applies to all scales: atoms, planets, stars, skaters, rotating machinery, etc

Conservation of Angular Momentum - Seeing is believing

`mk13 - Drehimpulserhaltung`

- For rotation about a fixed axis (direction of $\vec{L}$ unchanged):

$$L_1 = L_2, \qquad I_1\omega_1 = I_2\omega_2$$

Conservation of Angular Momentum - Key points

- Angular momentum is conserved when the **net external torque** is zero.
- Internal forces cannot change total $\vec{L}$, only its distribution.
- Reducing moment of inertia increases angular velocity if L is fixed.
- Conservation applies at all scales: atoms, machinery, stars, galaxies.
- When torque-free, both the **magnitude** and **direction** of $\vec{L}$ remain constant.
- Collisions or mass redistribution obey angular momentum conservation when isolated from external torques.

Solve initial question

`mk15`

Which egg is cooked?
How to check?

Bonus Puzzle

`mk11 - Maxwellsches Rad`

How can the scale show less weight?

$$\tau = Tr = I\alpha$$

$$ma = mg - T \;\leftrightarrow\; T = m(g - a)$$

$$\Rightarrow a < g$$