1.10. Waves

`ma07 - Schallwellen in Festkörpern`

Why do we get different pitches?

$\Rightarrow$ **Study waves**

- Waves describe transport in **matter, fields, and even spacetime**, with or without a medium
- **Energy, information, and structure** propagate without net matter flow
- The same wave concepts apply from **sound and light to seismic, and gravitational waves as well as biological (neural) dynamics**

What Is a Wave?

`ms25 - Gekoppelte Pendel & Julius`

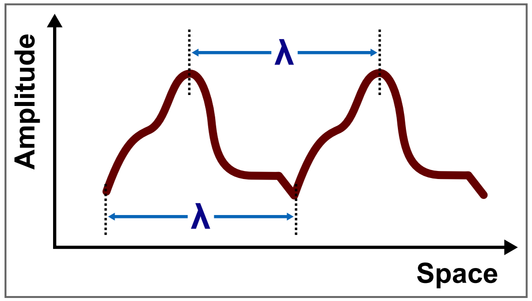
- A **wave** is a disturbance that propagates through **coupled oscillators**
- Each element oscillates about an **equilibrium position**
- Neighboring elements interact via **restoring forces**
- Energy is transferred **locally from oscillator to oscillator**
- **No net transport of matter** along the medium
- The wave emerges from **collective motion**, not a single particle

Important Characteristics of Waves

- **Amplitude** A: maximum displacement → energy content
- **Wavelength** λ: distance between points in the same phase
- **Frequency** f: oscillations per second (Hz)
- **Period** T: time for one oscillation, $T = 1/f$
- **Wave speed** v: speed of propagation of the disturbance
- Fundamental relation:

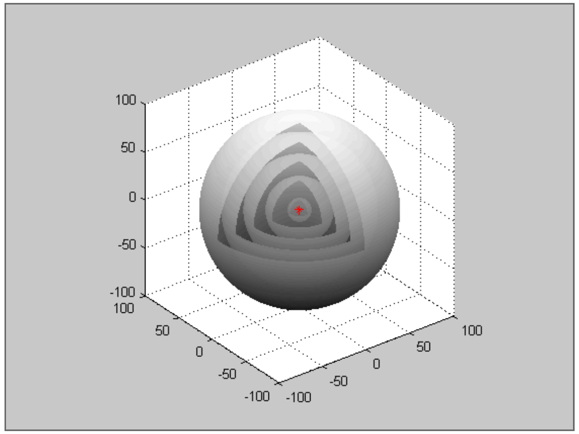
$$v = f\lambda = \frac{\lambda}{T}$$

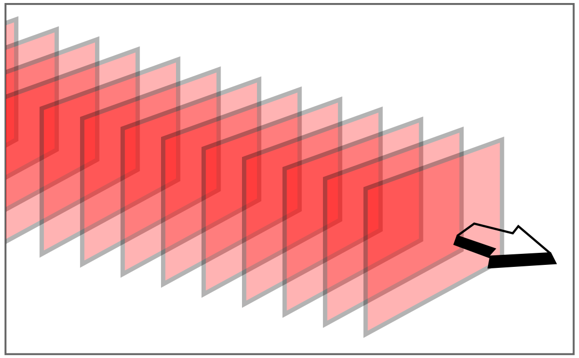
- **Frequency set by the source; wavelength adapts to the medium**



Beyond 1D: Plane Waves & Spherical Waves

- **Wavefront**: a surface connecting points that oscillate with the same phase
- **Spherical waves**:
    - Originate from a **point source**
    - Wavefronts are expanding spheres
- **Plane waves**:
    - Parallel wavefronts
    - Uniform phase and amplitude across each front
    - Good approximation far from the source





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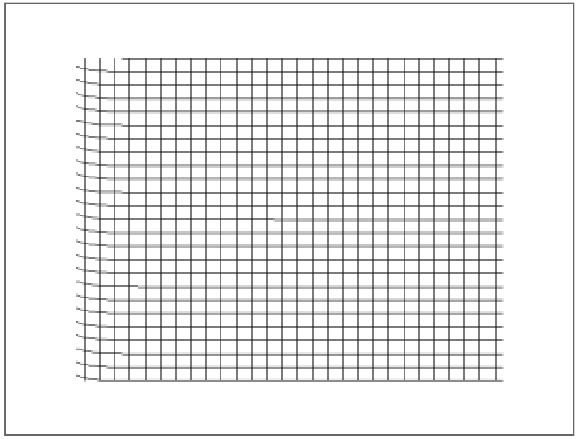
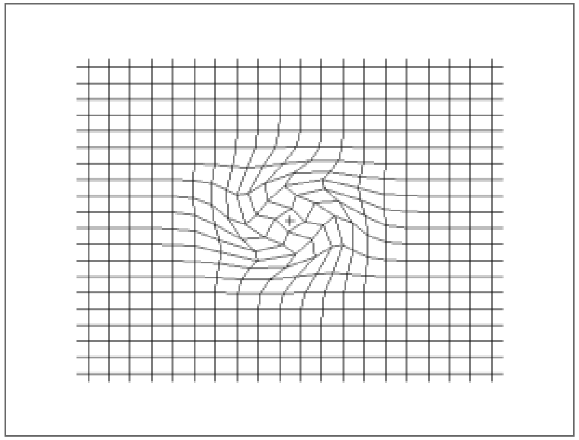
Transverse Waves

`ms22 - robe`

- Particle oscillation **perpendicular** to direction of propagation
- Characterized by **crests** and **troughs**
- Examples:
    - Waves on strings or ropes
    - Water surface waves
    - Electromagnetic waves





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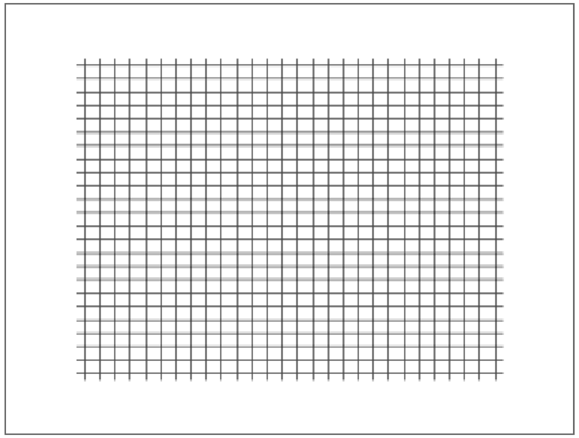
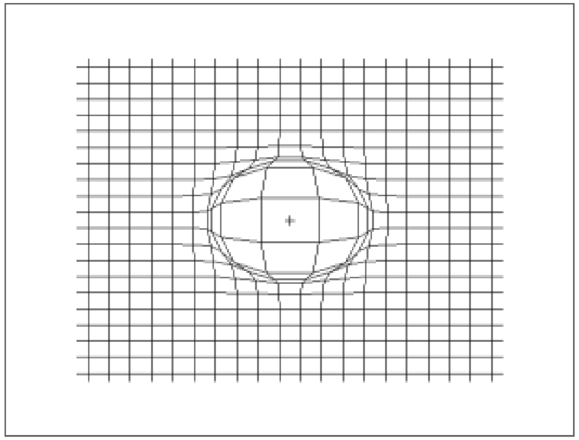
Longitudinal Waves

`ms36 - Demonstration einer Longitudinalwelle`

- Particle oscillation **parallel** to direction of propagation
- Wave consists of **compressions** and **rarefactions**, no crests or troughs
- Requires a **compressible medium**
- Examples:
    - Sound waves in air
    - Compression waves in springs
    - Seismic P-waves





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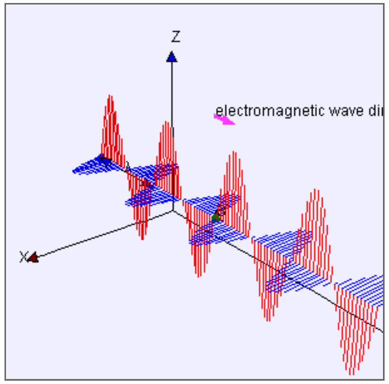
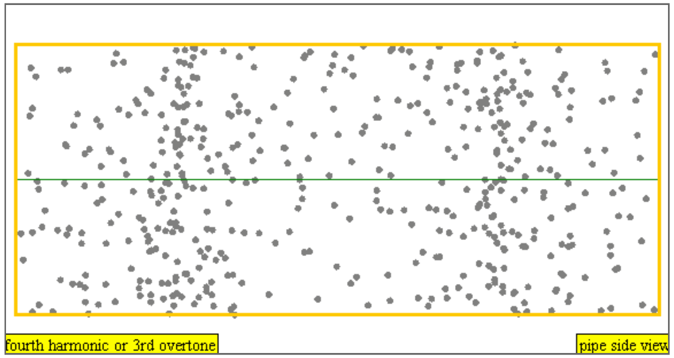
Transverse vs. Longitudinal Waves — Summary

`ms22 - slinky`

Wave type	Particle motion relative to propagation	Characteristic features	Medium requirement
Transverse wave	Perpendicular to propagation direction	Crests and troughs	Can occur with or without a medium
Longitudinal wave	Parallel to propagation direction	Compressions and rarefactions	Requires a compressible medium

In both cases, **energy propagates through the medium while matter oscillates locally** about equilibrium positions.





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Wave kinematics: Spatial and Temporal Structure

`ms24 - freeze Wellenmaschine`

- One-dimensional wave: $D = D(x,t)$
- Spatial profile at fixed time:

$$D(x) = A\sin\left(\frac{2\pi}{\lambda}x\right)$$

Wave kinematics: Spatial and Temporal Structure (cont')

`sim - Wave function`

- Wave translation with speed v:

$$D(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

- Relations for phase velocity v, wave number k & angular frequency ω

$$v = \lambda f, \qquad k = \frac{2\pi}{\lambda}, \qquad \omega = 2\pi f$$

- **Standard traveling-wave form**:

$$D(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - \lambda\frac{\omega}{2\pi}t)\right) = A\sin(kx - \omega t)$$

Wave kinematics: Wave Parameters and Direction

`ms04 - Alter Schwingungsapparat`

- **Standard traveling-wave form**:

$$D(x,t) = A\sin(kx - \omega t)$$

- Phase:

$$\phi = kx - \omega t$$

- Phase velocity:

$$v = \lambda f = \frac{2\pi}{k}\frac{\omega}{2\pi} = \frac{\omega}{k}$$

- Wave traveling in −x direction:

$$D(x,t) = A\sin(kx + \omega t)$$

The Wave Equation — Motivation and 1D Form

- From **wave function** $y(x,t)$ → to **dynamics**
- *Which equation governs wave propagation in time and space?* ⇒ **wave equation**
- Example: sinusoidal traveling wave

$$y(x,t) = A\sin(kx - \omega t), \qquad v = \frac{\omega}{k}$$

- Second derivatives:

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx - \omega t) = -k^2 y, \qquad \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx - \omega t) = -\omega^2 y$$

- Rearrange gives the **one-dimensional wave equation**

$$\frac{1}{-k^2}\frac{\partial^2 y}{\partial x^2} = \frac{1}{-\omega^2}\frac{\partial^2 y}{\partial t^2} \quad \Rightarrow \quad \boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

Generality, Physical Meaning, and 3D Form

- Valid for **any waveform**, not only sinusoidal:

$$y(x,t) = f(x - vt), \qquad y(x,t) = f(x + vt)$$

- **Physical meaning**
  - $\partial^2 y/\partial x^2$: spatial curvature
  - $\partial^2 y/\partial t^2$: particle acceleration
- **Three-dimensional wave equation**

$$\boxed{\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$$

- Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Governs sound, water waves, strings, and electromagnetic waves

Energy Transport by a Traveling Wave

- **Instantaneous energy (per string element)**:

$$dE = dK + dU = \mu A^2 \omega^2 \cos^2(kx - \omega t)\, dx$$

(energy oscillates between kinetic and potential forms)

- **Time-averaged energy per unit length**:

$$\frac{dE_{\text{avg}}}{dx} = \frac{1}{2} \mu A^2 \omega^2$$

(using $\langle \cos^2 \rangle = \frac{1}{2}$ over one period)

- **Average power transported by the wave**:

$$P = \frac{dE}{dt} = \frac{dE_{\text{avg}}}{dx}\,v = \frac{1}{2}\mu A^2\omega^2 v$$

(wave speed v converts energy per length into energy per time; $P \propto A^2$)

Intensity of a Spherical Wave

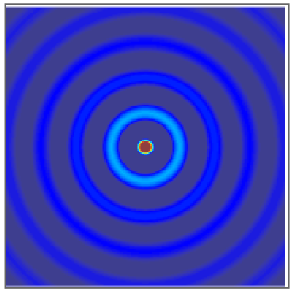
- Assume a **point source** $\rightarrow$ **spherical wave**
- Total power P spreads over spherical wavefront

$$A_{\text{cross}}(r) = 4\pi r^2$$

- **Intensity** (*power per area* or *energy per time per area*):

$$I(r) = \frac{P}{A_{\text{cross}}} = \frac{P}{4\pi r^2}$$

- Energy is conserved, **but** intensity decreases with distance due to **geometric spreading**



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Wave Reflection and Transmission at Boundaries

- **Boundaries** cause partial **reflection** and **transmission**
- Reflection and transmission depend on **medium properties** (wave speed, impedance)
- Boundaries impose **constraints** (fixed, free, or continuity conditions)
- Energy is **redistributed**, not lost: reflected + transmitted = incident
- Valid for **strings, sound, and light**
- ($\Rightarrow$ more details on wave phenomena in Physics II)

Reflection & Phase Change

`ms22 - Rope for fixed and free end`

Reflection from...	Equivalent situations	Reflected wave	Phase change	Physical interpretation
Higher impedance	Fixed end (string); $n_1 < n_2$ (light)	Inverted	180° phase shift	Boundary resists motion → displacement/field must reverse to satisfy constraints
Lower impedance	Free end (string); $n_1 > n_2$ (light)	Upright	No phase shift	Boundary conditions satisfied without sign reversal

Superposition

`ms17 - Wasserwellengerät`

`ms24 - Wellenmaschine`

$$\frac{\partial^2 D}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 D}{\partial t^2}$$

- **Superposition**: multiple waves overlap $\rightarrow$ total displacement = **sum of individual displacements**
- Wave equation's linearity implies that if D_1 and D_2 are solutions, then $D = aD_1 + bD_2$ is also a solution
- Overlapping waves do **not interact**: after overlap, each wave continues unchanged
- Superposition is the basis of **interference patterns**, **standing waves**, and **resonance**; it holds only for **small amplitudes** (linear restoring forces)

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Interference of Traveling Waves

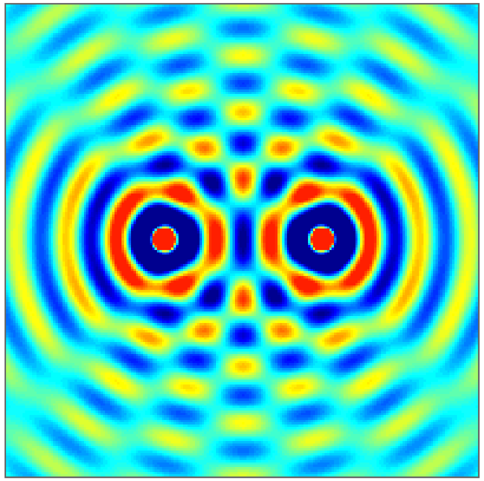
- **Interference** arises when **coherent** waves (i.e. constant phase relationship over time) overlap and superpose
- Resulting pattern depends on the **phase difference** ϕ
- For two equal-amplitude waves traveling in the same direction:

$$y_1 = A\sin(kx - \omega t), \qquad y_2 = A\sin(kx - \omega t + \phi)$$

- Using $\sin\alpha + \sin\beta = 2\sin\!\left(\frac{\alpha+\beta}{2}\right)\cos\!\left(\frac{\alpha-\beta}{2}\right)$:

$$y = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

- Amplitude depend on phase: $A_R = 2A\cos\left(\frac{\phi}{2}\right)$



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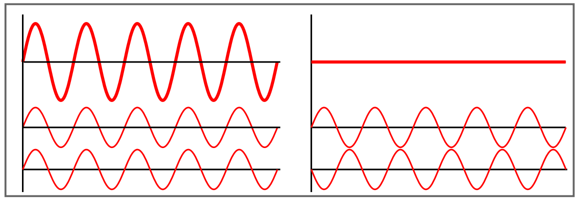
Constructive and Destructive Interference

`ms24 - Wellenmaschine`

- Stable interference patterns require **coherence** (same f, fixed ϕ)
- Energy is **redistributed**, not destroyed
- Phase difference often arises from path difference: $\phi = \frac{2\pi}{\lambda}\Delta x$

Interference type	Phase difference ϕ	Path difference Δx	Result
Constructive	$0,\ 2\pi,\ 4\pi, \ldots$	$\Delta x = n\lambda$	Maximum amplitude
Destructive	$\pi,\ 3\pi,\ 5\pi, \ldots$	$\Delta x = (n + \frac{1}{2})\lambda$	Cancellation



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Standing Waves: Mathematical Formation

`ms22 - rope`

- Arise from **superposition of two identical waves** traveling in opposite directions
- Incident and reflected waves:

$$y_1(x,t) = A\sin(kx - \omega t), \qquad y_2(x,t) = A\sin(kx + \omega t)$$

- Using $\sin\alpha + \sin\beta = 2\sin\!\left(\frac{\alpha+\beta}{2}\right)\cos\!\left(\frac{\alpha-\beta}{2}\right)$, resulting standing wave:

$$y(x,t) = 2A\sin(kx)\cos(\omega t)$$

- **Spatial factor** $\sin(kx)$ fixes the pattern, **temporal factor** $\cos(\omega t)$ sets the oscillation frequency

Standing Waves on a String and Resonance

- String of length L fixed at both ends:

$$y(0,t) = 0, \quad y(L,t) = 0$$

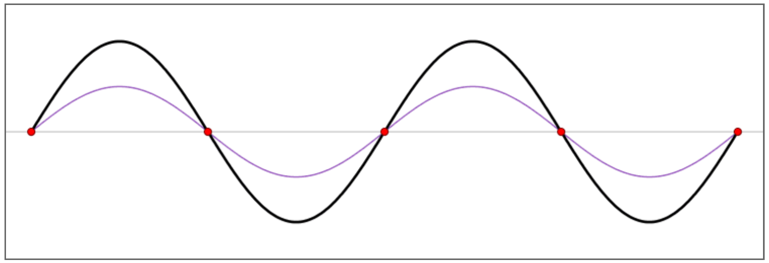
- Boundary condition:

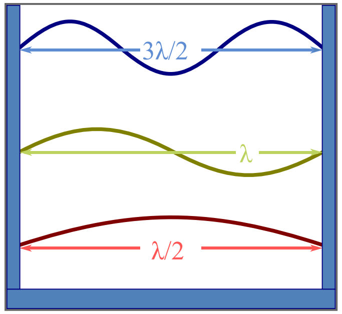
$$\sin(kL) = 0 \;\Rightarrow\; kL = n\pi$$

- Allowed wavelengths and frequencies with $k = \frac{2\pi}{\lambda}$ and $v = \lambda f$:

$$\lambda_n = \frac{2L}{n}, \qquad f_n = \frac{nv}{2L}$$

- Discrete **normal modes** (harmonics), lowest is the **fundamental**





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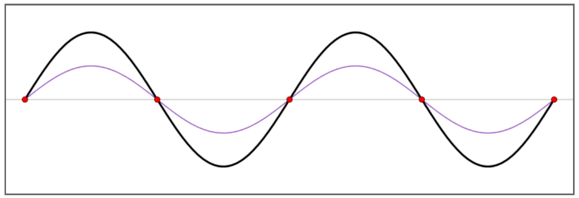
Nodes, Antinodes, and Energy

$$y(x,t) = 2A\sin(kx)\cos(\omega t)$$

- All points oscillate with the same angular frequency ω (SHM)
- Position-dependent amplitude: $A(x) = 2A\sin(kx)$
- **Nodes**: $\sin(kx) = 0 \;\Rightarrow\; x_n = n\frac{\lambda}{2}$
    - Zero displacement and zero kinetic energy
    - Potential energy maximal
- **Antinodes**: $|\sin(kx)| = 1$
    - Maximum displacement amplitude
- **No net energy transport**: energy oscillates locally between kinetic and potential forms



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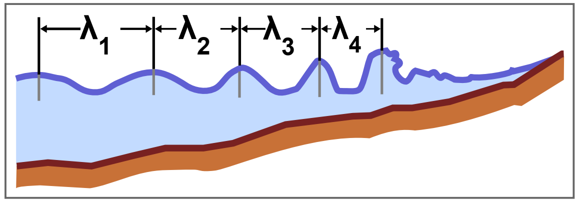
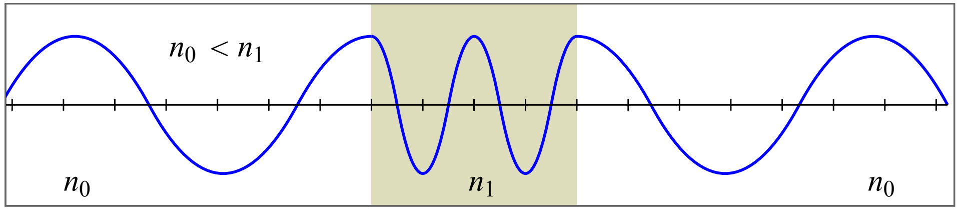
Wave Speed Variations: Refraction

- Wave behavior is controlled by the **wave speed** v, set by medium properties

$$v = \lambda f$$

- **Refraction** occurs when a wave enters a region with a different wave speed
- The **frequency remains constant** (set by the source)
- Changes in wave speed lead to changes in **wavelength** and **direction**
- Refraction occurs for mechanical waves, sound, seismic waves, and light





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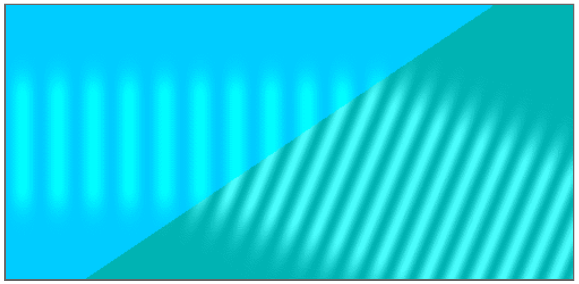
Wave Speed Variations: Refraction (cont')

- Wave speed, frequency, wavelength: $v = f\lambda$
- Frequency unchanged $\Rightarrow$ wavelength adjusts at the boundary:

$$\frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2}$$

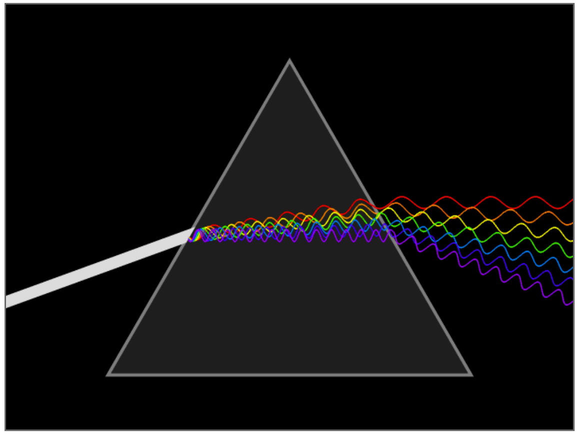
- Oblique incidence causes wavefront rotation
- Law of refraction for waves:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$



Wave Speed Variations: Dispersion

- A medium is **dispersive** if **wave speed depends on frequency**
- Different frequency components propagate at different speeds
- **Refraction vs. dispersion**:
    - Both effects arise from **variations in wave speed**
    - **Refraction**: wave speed varies **in space** $\rightarrow$ change in wavelength and propagation direction
    - **Dispersion**: wave speed varies **with frequency** $\rightarrow$ separation of spectral components and waveform distortion



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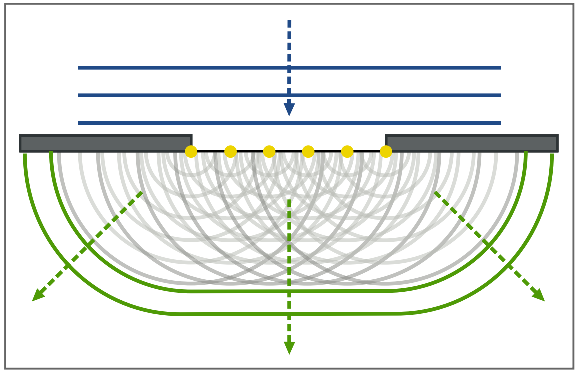
Diffraction and Huygens' Principle

`ms17 - Wasserwellengerät`

- **Diffraction**: spreading/bending of waves at edges, apertures, or obstacles
- Strength of diffraction depends on **wavelength λ vs object size a**
    - $a \gg \lambda$: weak diffraction → nearly straight-line propagation
    - $a \sim \lambda$: strong diffraction → pronounced spreading
    - $a < \lambda$: wave spreads broadly, often nearly spherical
- Explained by **Huygens' principle**:
    - Every point on a wavefront acts as a source of secondary wavelets
    - New wavefront = envelope of these wavelets
- Explains sound bending around corners, fuzzy shadows, limits of resolution in optics and imaging



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Revisit initial experiment: Escapement, oscillations, & clocks

`ma07 - Schallwellen in Festkörpern`

Why do we get different pitches?

- longitudinal and transversal wave generated depending on how *excited*
- $\rightarrow$ compression vs. vibration