de Sitter space as a homogeneous space

Throughout this document, let G be a connected Lie group and H a closed subgroup.

1 Introduction: Klein geometries

In this section *Klein geometries* are introduced. We discuss the relation between geometry and a corresponding Lie group structure, after which the relation to homogeneous spaces becomes evident.

Felix Klein realized that a geometry can be understood as consisting of a connected manifold M together with a Lie group G of "motions" acting transitively on M and which leaves all the "properties of the figures" studied in the geometry invariant. Being the simplest example, Euclidean geometry is given by the manifold (as an affine space) $M = \mathbb{R}^n$ on which the group $G = \mathbb{R}^n \rtimes SO(n)$ acts effectively and transitively, that leaves invariant the properties angle and length of figures (sets of points). Since from Klein's point of view geometry is the study of the properties of the figures being invariant under the action of G, it is possible to shift emphasis from the space M to the group G in the following rigorous manner.

Fix a point p in M. Then one may define a map $\lambda_p : G \to M : g \mapsto gp$. Since G acts transitively on M, this mapping is surjective—however not necessarily one-to-one.[*] Furthermore, we assume in the remainder of the text that G acts effectively on M. Consider the subset $H_p \subset G$ that maps into p under λ_p, i.e.

$$H_p \equiv \lambda_p^{-1}(p) = \{g \in G \mid gp = p\} \tag{1}$$

and which is called the *stabilizer of* p. It is easily verified that H_p is a closed subgroup of G. Let $q \in M$ and consider $\lambda_p^{-1}(q) = \{g \in G \mid gp = q\}$. It follows that $\lambda_p^{-1}(q) = g_0 H_p$ where $g_0 \in \lambda_p^{-1}(q)$ is arbitrary. Different points in M are the image of elements in G that are not connected by H. Hence, $\lambda_p : G/H_p \to M$ is an isomorphism. If another origin $\tilde{p} = g_1 p$ were chosen, $H_{\tilde{p}} = \{g \in G \mid g g_1 p = g_1 p\}$ and $H_{\tilde{p}} = \mathrm{Ad}_{g_1} H_p$. Choosing different origins gives rise to conjugate stabilizers, implying they are isomorphic to eachother. Therefore it is common to just talk about *the* stabilizer H, although a calculation will force you to pick any of them. It becomes evident that the choice of origin is arbitrary which explains why all the points in a Klein geometry are said to be equivalent. The above considerations have established the isomorphism between G/H and M, where H is the maximal isotropy group of a point in M.

[*]This would require a free action.

Let us remark that while M has no preferred origin, since the space is transitive under the symmetry group G, G/H does have a preferred basepoint by picking out the identity coset. This is of course not a contradiction, since an element in M was promoted to origin in order to establish the isomorphism. As already mentioned, any other point could have been chosen which would correspond to a stabilizer conjugate to H. The knowledge that this choice is arbitrary restores this hidden symmetry in G/H.

To recapitulate, a Klein geometry is defined.

Definition 1.1 *A* Klein geometry *is a pair* (G, H)*, where* G *is a Lie group and* H *a closed subgroup such that* $G/H \equiv \{gH\}$ *is connected. The connected coset space* $M \equiv G/H$ *is called* the space of the Klein geometry*.*

A Klein geometry is called reductive if there is an $\mathrm{Ad}(H)$*-module decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$*, where* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are the Lie algebras of* G *and* H*, respectively.*

A metric Klein geometry is a reductive Klein geometry equipped with an $\mathrm{Ad}(H)$*-invariant metric* η *on* $\mathfrak{g}/\mathfrak{h}$*.*

The power of this definition is that the starting point is the Lie group structure, rather than a space. Given a Lie group G, a different Klein space is given for any closed subgroup H.* The *points* are the coset elements and their actual interpretation will depend on the subgroup H.

Since a homogeneous space (X, G) is a space X that is transitive under a symmetry group G, it is now clear that the Klein coset spaces are homogeneous.† The converse is not true in general, since X and G may have less structure than the objects defined above. In the remainder we will not make the distinction and assume a homogeneous space to have the mathematical structure of a Klein space, so that both terms may be used interchangeably.

2 Symmetric homogeneous spaces

3 Example: de Sitter space in four dimensions

3.1 dS as a Klein geometry

In this section, de Sitter space will be studied from the algebraic point of view.

Consider the $n+1$-dimensional vector space $\mathbb{R}^{n+1}$ with basis $(e_0, \ldots, e_n)$ and a metric structure, defined through the symmetric bilinear form

$$F(x, y) = x^{\tau} \eta y \quad \text{with } \eta = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix} \tag{2}$$

for x and y column vectors in $\mathbb{R}^{n+1}$. The orthogonal group $O(n, 1)$ leaving invariant

*Here we mean of course subgroups that are not isomorphic to eachother.

†Since the transitive action of G on a Klein space preserves the properties of the geometry, it is impossible to distinguish a point from any other by means of these properties of the geometry.

F is a subgroup of the general linear group, namely

$$O(n,1) = \{A \in Gl(n+1,\mathbb{R}) \mid A^\tau \eta A = \eta\} \tag{3}$$

The componenent connected to the identity has determinant 1, and is denoted by $SO(n,1)$. The elements of the Lie algebra $\mathfrak{o}(n,1)$ are the tangent vectors at the identity of $SO(n,1)$. Therefore, consider a curve $A(t)$ through the identity $\mathbb{I}$, such that $A(t) = \mathbb{I} + tX + O(t^2)$ where $X = \dot{A}(0)$. A-priori, $X \in \mathfrak{gl}(n+1,\mathbb{R})$ but given the restriction that $A \in SO(n,1)$, it is directly found that $X^\tau \eta + \eta X = 0$ and Tr $X = 0$.

Let us make the isomorphism between $SO(n,1)/SO(n-1,1)$ and n-dimensional de Sitter space dS_n manifest. The latter is the hypersurface in $\mathbb{R}^{n+1}$ fulfilling

$$dS_n : x^\tau \eta x = -l^2 \tag{4}$$

Consider $o = le_n \in dS_n$ and let H be the isotropy group of o, that is all matrices $A \in SO(n,1)$ of the form $(Ae_n = e_n)$

$$A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

and where $B \in SO(n-1,1)$ since we rotated around a spacelike axis. Let λ_o be the mapping such that for $A \in SO(n,1)$, $\lambda_o(\pi(A)) = Ae_n$, where $\pi : SO(n,1) \to SO(n,1)/SO(n-1,1)$ is the natural projection. Since $SO(n,1)$ acts transitively on dS_n, λ_o constitutes an isomorphism between $SO(n,1)/SO(n-1,1)$ and the n-dimensional de Sitter space dS_n.*

An involutive automorphism σ on $SO(n,1)$ is given by

$$\sigma(A) = SAS^{-1} \quad \text{with } S = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & -1 \end{pmatrix}$$

So $(SO(n,1), SO(n-1,1), \sigma)$ is a symmetric space and the canonical decomposition of the Lie algebra is given by $\mathfrak{o}(n,1) = \mathfrak{o}(n-1,1) \oplus \mathfrak{p}$.

In the following let us focus on the four-dimensional case $dS \simeq SO(4,1)/SO(3,1)$. A generic element of $\mathfrak{o}(4,1)$ in its fundamental representation is canonically decomposed with respect to σ as

$$\underbrace{\begin{pmatrix} 0 & x_{01} & x_{02} & x_{03} & x_{04} \\ x_{01} & 0 & x_{12} & x_{13} & x_{14} \\ x_{02} & -x_{12} & 0 & x_{23} & x_{24} \\ x_{03} & -x_{13} & -x_{23} & 0 & x_{34} \\ x_{04} & -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix}}_{\mathfrak{o}(4,1)} = \underbrace{\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}}_{\mathfrak{o}(3,1)} + \underbrace{\begin{pmatrix} 0 & \xi \\ \bar{\xi}^\tau & 0 \end{pmatrix}}_{\mathfrak{p}}$$

where $\xi^a \in \mathbb{R}^4$ and we used the shorthand notation $\bar{\xi} = \eta\xi$ with $\eta = (+,-,-,-)$.

*Note how it is the transitivity of dS_n under G which guarantees the isomorphism between G/H and dS_n: gH acts transitivily on dS_n. This does not seem to imply that "pure translations" act transitively on dS_n.

A generic element of $\mathfrak{p}$ is written as $\xi = \xi^a P_a$ with P_a the generators of de Sitter translations. We now use the Killing form on $\mathfrak{g}$ to define a non-degenerate bilinear symmetric metric on the Lie algebra. We then define a metric

$$\langle \xi, \eta \rangle := \tfrac{1}{2}\mathrm{tr}\,(\xi\eta)\;, \quad \xi, \eta \in \mathfrak{g}\;. \tag{5}$$

The restriction of this metric on the $\mathfrak{p}$-subspace gives a metric on $\mathfrak{p}$. A quick calculation shows that $\langle \xi, \eta \rangle = \xi^a \eta^b \eta_{ab}$, for $\xi, \eta \in \mathfrak{p}$.

The isomorphism $G/H \simeq dS$ induces an isomorphism between the corresponding tangent spaces, i.e. an isomorphism $\mathfrak{p} \simeq T_o dS$. Let us make this manifest. Consider de Sitter space dS with radius $\sqrt{3/\Lambda}$ and choose the origin $o = \sqrt{3/\Lambda}\, e_4 = (0, 0, 0, 0, \sqrt{3/\Lambda})$. Given this origin, the above introduced isomomorphism $\lambda_o : G/H \to dS$ was defined by $\lambda_o(\pi(A)) = Ao$ for any $A \in G$. Let $\frac{1}{l}\xi^a P_a \in \mathfrak{p}$ and $\exp(\frac{t}{l}\xi^a P_a)$ the corresponding one-parameter subgroup of G, where l is a yet to be fixed length scale [1]. An isomorphism between $\mathfrak{p}$ and $T_o dS$ is then induced by λ_o, namely by considering its differential

$$\lambda_{o*}\tfrac{d}{dt}f[\exp\left(\frac{t}{l}\xi^a P_a\right)]|_{t=0} = \tfrac{d}{dt}[\exp\left(\frac{t}{l}\xi^a P_a\right)o]^\mu \partial_\mu f = \frac{1}{l}\sqrt{\frac{3}{\Lambda}}\xi^\mu \partial_\mu f$$

for an arbitrary function f. Fixing the length scale [1]

$$l = \sqrt{\frac{3}{\Lambda}}$$

gives

$$\lambda_{o*} : \mathfrak{p} \to T_o dS : \frac{1}{l}\xi^a P_a \mapsto \xi^\mu \partial_\mu\;. \tag{6}$$

This mapping explains the choice for the length scale and not surpisingly is related to the fact that G/H is a Klein geometry. Given an element $\xi^a P_a \in \mathfrak{p}$, a length scale is introduced such that it may be identified with $\xi^\mu \partial_\mu$ on the tangent space of dS. In this case, the metric defined through the Killing form $\eta_{ab}\xi^a\xi^b$ equals the de Sitter metric applied to the corresponding elements in its tangent space at the origin, i.e. $\eta_{\mu\nu}\xi^\mu\xi^\nu$.

3.2 Geodesics: completeness but no connectedness

The geodesics with respect to the canonical connection through the point $p = -le_4$ are given by the orbits of p under $\exp \mathfrak{p}$. More precisely, considering the isomorphism between $\mathfrak{p}$ and $\mathbb{R}^4$ given by

$$X = \begin{pmatrix} 0 & \xi \\ \bar{\xi}^\tau & 0 \end{pmatrix} \leftrightarrow \xi$$

the geodesic through $o \in G/H$ in the direction $\xi \in T_o G/H$ is the curve $\exp(tX)o$.

We construct these geodesics for dS_4 through $-le_4$ as parametrized curves in $\mathbb{R}^5$. In fact, given $\xi \in T_0 G/H$, $\exp(tX)$ is a rotation in the plane P spanned by the the orthogonal basis $e_* = \xi^\mu e_\mu$ and e_4 ($\mu = 0 \ldots 3$). It is then clear that the orbit through p is the intersection of $P = \{ae_* + be_4 \mid a, b \in \mathbb{R}\}$ and dS_4. Since e_4

is spacelike ($F(e_4, e_4) = -1$), there are three cases possible. Namely, e_* timelike, spacelike or lightlike.

1. $F(e_*, e_*) = -\alpha^2$. In this case, all elements in P are spacelike. The intersection $P \cap dS_4$ is given by the elements in P that satisfy $F(ae_* + be_4, ae_* + be_4) = -l^2$, that is $\alpha^2 a^2 + b^2 = l^2$. It is clear that $a = \pm l\alpha^{-1} \sin t$ and $b = \pm l \cos t$. Given the fact that the curve goes through $p = -le_4$ we find the family of geodesics

$$\gamma(t) = \alpha^{-1} l \sin t \; e_* - l \cos t \; e_4 \;, \quad \forall t \in \mathbb{R} \tag{7}$$

   which are spacelike ellipses, since $F(\dot\gamma, \dot\gamma) = -l^2$.

2. $F(e_*, e_*) = \alpha^2$. In this case, part of the elements in P are timelike. The curves sought-after satisfy $F(ae_* + be_4, ae_* + be_4) = -l^2$, hence $-\alpha^2 a^2 + b^2 = l^2$. Given the hyperbolic identity $-\sinh^2 t + \cosh^2 t = 1$, it follows that $a = \pm l\alpha^{-1} \sinh t$ and $b = \pm l \cosh t$. The intersection are thus hyperbolae. The branch going through p is the geodesic that we looked for, i.e.

$$\gamma(t) = \alpha^{-1} l \sinh t \; e_* - l \cosh t \; e_4 \;, \quad \forall t \in \mathbb{R} \tag{8}$$

   It is directly verified that these are timelike everywhere, $F(\dot\gamma, \dot\gamma) = l^2$.

3. $F(e_*, e_*) = 0$. This case is the degenerate limit of the first two ($\alpha \to 0$). The intersection of P with dS_4 are the elements of P for which $0a^2 - b^2 = -l^2$. The solutions are $a \in \mathbb{R}$ and $b = \pm l$. Again choosing the ones through p singles out the family

$$\gamma(t) = t \; e_* - le_4 \;, \quad \forall t \in \mathbb{R} \tag{9}$$

These three cases exhaust all posibilities. Note also that these geodesics are manifestly complete. However, as we will prove now, dS_4 is not geodesically connected. Therefore, we consider again the orbits of "pure" de Sitter translations through $p = -le_4$ and check whether these connect with any point $q \in dS_4$. First, let us assume p and q are non-antipodal, that is $p \neq -q$. We discuss the mutually exclusive cases for which the vector $q - p$ is timelike, null-like or spacelike.

1. p, q timelike seperated. This means that $F(q - p, q - p) > 0$ or

$$F(p, q) < -l^2 \tag{10}$$

   In this case, only timelike geodesics through p could connect with q. Therefore consider a generic timelike geodesic (8) and the inner product $F(p, \gamma(t)) = -l^2 \cosh t$, $\forall t \in \mathbb{R}$. Hence, all timelike seperated points q can be connected by a timelike geodesic, for some choice $e_* \in \mathbb{R}^4$.

2. p, q lightlike seperated. In this case $F(q - p, q - p) = 0$ or

$$F(p, q) = -l^2 \tag{11}$$

   The inner product $F(p, \gamma(t)) = -l^2$ for lightlike geodesics (9) through p. Hence, they connect with all lightlike seperated points q.

3. p, q spacelike seperated. For $q - p$ to be spacelike one has that

$$F(p, q) > -l^2 \tag{12}$$

Again consider the inner product p and spacelike seperated points, connected by spacelike geodesics (7), that is $F(p, \gamma(t)) = -l^2 \cos t,\ \forall t \in \mathbb{R}$. It follows that these spacelike geodesics only will be able to connect with points q for which $-l^2 < F(p, q) < l^2$.

From this it follows that one can only connect points $p, q \in dS_4$ with pure de Sitter translations if $F(p, q) < l^2$. Note that this means that de Sitter space is not transitive under the exponentiation of $\mathfrak{p}$, $\Pi = \exp \mathfrak{p}$. It is transitive under a finite composition of such group elements Π. However, this in general will not be a pure de Sitter translation–and it will be certainly *not* a de Sitter translation if it connects to the above excluded region. Therefore, de Sitter space is transitive under elements of the full de Sitter group only.

3.3 Geodesics in stereographic coordinates

Let us project the above constructed geodesics on dS into the stereographic hyperplane $\chi^4 = -l$. Before deducing their explicit form, one may understand that these geodesics through the origin are straight lines for the stereographic observer. Remember that the orbits of pure de Sitter translations are the intersections of the two-planes through the origin and the south pole with the embedded de Sitter space in $\mathbb{R}^5$ and take notice that the stereographic projection of a point of dS is given by the intersection of the line through this point and the north pole with the stereographic hyperplane. Since all points of any given geodesic lie in a same plane that also goes through the north pole, the stereographic projection of such a geodesic is just the intersection of the respective plane with the stereographic hyperplane—that is, a straight line.

As we are to derive the explicit form of these geodesics through the origin $x^\mu = 0 \leftrightarrow \chi^4 = -l$, let us remind that the stereographic projection onto the hyperplane $\chi^4 = -l$ through the south pole is given by $x^\mu = \Omega^{-1}\chi^\mu$, where $\Omega = -\frac{1}{2}(\chi^4/l - 1)$.

The parametrized curves representing timelike geodesics containing the south pole were shown to be given by $\gamma(t) = \alpha^{-1} l \sinh t\ e_* - l \cosh t\ e_4$ or explicitly in Cartesian coordinates,

$$\chi^A(t) = l \begin{pmatrix} \alpha^{-1}\xi^\mu \sinh t \\ -\cosh t \end{pmatrix} \quad , \quad \alpha^2 = \xi^\mu \xi_\mu \equiv \eta_{\mu\nu}\xi^\mu \xi^\nu \ .$$

Projecting this family of geodesics onto $\chi^4 = -l$ gives their form in stereographic coordinates, that is

$$x^\mu(t) = 2l \frac{\xi^\mu}{\sqrt{\xi^\lambda \xi_\lambda}} \frac{\sinh t}{1 + \cosh t} \ , \tag{13}$$

which is clearly a straight line through the origin. Regarding these curves some remarks are in place. Note that the function

$$\frac{\sinh t}{1+\cosh t}$$

monotonically increases from $-1 \to 1$ for t going from $-\infty \to \infty$. Hence, the curves $x^\mu(t)$ are finite straight lines through the origin. More precisely, they start and stop at

$$x^\mu(\pm\infty) = \pm 2l \frac{\xi^\mu}{\sqrt{\xi^\lambda \xi_\lambda}}$$

Considering all possible timelike directions ξ^μ, the sets of start and end point gives the two-sheeted hyperboloid $\eta_{\mu\nu}x^\mu x^\nu = 4l^2$. Keeping in mind the singular character of the stereographic projection, this is consistent behaviour. Indeed, the infinite branches of the hyperbolae in dS get projected onto finite lines in stereographic coordinates as future and past null infinity get projected onto the given surfaces ($\sigma^2 = 4l^2$). Notwithstanding the finite coordinate range, these timelike geodesics have infinite length and it takes an infinite amount of proper time for the free particle to reach the above defined hyperboloid. To see this, let us consider the proper time along the geodesic

$$\tau(t) = \int_0^t \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} dt = lt$$

The result follows since $t \to \infty$ in approaching the given hyperboloid.

In the case of spacelike geodesics $\gamma(t) = \alpha^{-1} l \sin t \; e_* - l \cos t \; e_4$, the parametrization for Cartesian coordinates is given by

$$\chi^A(t) = l \begin{pmatrix} \alpha^{-1}\xi^\mu \sin t \\ -\cos t \end{pmatrix} \quad , \quad \alpha^2 = -\xi^\mu \xi_\mu \equiv -\eta_{\mu\nu}\xi^\mu \xi^\nu \; .$$

In stereographic coordinates these curves are represented by

$$x^\mu(t) = 2l \frac{\xi^\mu}{\sqrt{-\xi^\lambda \xi_\lambda}} \frac{\sin t}{1+\cos t} \; , \tag{14}$$

which again are straight lines through the origin. Here, singular behaviour seems to appear for $t = \pi$. However, a quick look to the Cartesian system shows that this corresponds to the north pole, the point discarded for the stereographic projection to be well-defined. In this case, we are confronted with a finite range of the parameter $t \in]-\pi, \pi[$ and an infinite range for $x^\mu(t)$, since

$$\lim_{t \to \pm\pi} \frac{\sin t}{1+\cos t} = \pm\infty \; .$$

As it turns out, the (finite) ellipses are projected onto infinite straight lines.* Let us emphasize that the proper length of these geodesics is finite. Indeed, since

$$\sigma(t) = \int_0^t \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} dt = lt$$

*In a sense, they are cut open at the north pole after which the conformal factor stretches them out infinitely.

the proper length of the curve originating at the origin ($t = 0$) and ending at infinity ($t = \pi$) is given by $\sigma(\pi) = l\pi$. Consistently, this is the arclength between the south pole and the north pole of dS embedded in $\mathbb{R}^5$.

Lightlike geodesics were found to be given by straight lines in $\mathbb{R}^5$, i.e. $\gamma(t) = \xi^\mu t\, e_\mu - le_4$. In Cartesian coordinates,

$$\chi^A(t) = \begin{pmatrix} \xi^\mu t \\ -l \end{pmatrix} \quad , \quad \xi^\mu \xi_\mu = \eta_{\mu\nu}\xi^\mu \xi^\nu = 0 \ .$$

In stereographic coordinates these are also a straight lines, because

$$x^\mu(t) = \xi^\mu t \ . \tag{15}$$

References

[1] Derek K. Wise. MacDowell-Mansouri gravity and Cartan geometry. *Class.Quant.Grav.*, 27:155010, 2010.