Noether's theorem in curved spaces

We consider a scalar field ϕ on a curved spacetime—that is, coupled to gravity—described by an action where the Lagrangian depends on ϕ, its first order derivatives and on the inverse metric g^{-1},*

$$S[\phi] = \int_M dg\, \mathcal{L}(\phi, d\phi, g^{-1}) \tag{1}$$

The invariant volume element dg is given by $dg = \sqrt{|g|}dx^0 \wedge \dots dx^3$ and $\mathcal{L}$ is a scalar (0-form), such that the integral is well defined. The explicit indication of the inverse of the metric in the Lagrangian may seem exagerated. However, since $\mathcal{L}$ depends on ϕ and $d\phi$, it can *only* depend on g through the inverse if $\mathcal{L}$ were to be a function.

1 Euler-Lagrange equations

It is a standard procedure in modern physics—and interestingly enough, a technique finding its roots in classical mechanics—that the equations of motion can be obtained from an action (1), more precisely by invoking the *action principle.*

> "The actual evolution of physical variables is so that the action, describing the corresponding system, attains an extremal value."

In other words, the equations of motions are given such that its solutions extremize the system's action.

Therefore, assuming the physical fields are given by ϕ the equations of motion are equivalent to the condition

$$\tfrac{d}{d\varepsilon} S[\phi + \varepsilon\eta]\big|_{\varepsilon=0} = 0 \ , \tag{2}$$

where $\eta : M \to \mathbb{R}$ is an arbitrary function that vanishes on ∂M. Then,

$$\begin{aligned} \tfrac{d}{d\varepsilon} S[\phi + \varepsilon\eta]\big|_{\varepsilon=0} &= \int_M dg \left(\frac{\partial \mathcal{L}}{\partial(\phi + \varepsilon\eta)} \tfrac{d}{d\varepsilon}(\phi + \varepsilon\eta)\big|_{\varepsilon=0} + \frac{\partial \mathcal{L}}{\partial\partial_i(\phi + \varepsilon\eta)} \tfrac{d}{d\varepsilon}\partial_i(\phi + \varepsilon\eta)\big|_{\varepsilon=0} \right) \\ &= \int_M dg \left(\frac{\partial \mathcal{L}}{\partial\phi}\eta + \frac{\partial \mathcal{L}}{\partial\partial_i\phi}\partial_i\eta \right) \end{aligned}$$

*The kinetic regime for gravity itself is not included for the moment. We assume the metric as "given", while only focusing on the matter content in the theory.

The second term can be further worked out as

$$\int_M d^4x\sqrt{g}\frac{\partial\mathcal{L}}{\partial\partial_i\phi}\partial_i\eta = \int_M d^4x\partial_i\Big(\sqrt{g}\frac{\partial\mathcal{L}}{\partial\partial_i\phi}\eta\Big) - \int_M d^4x\partial_i\Big(\sqrt{g}\frac{\partial\mathcal{L}}{\partial\partial_i\phi}\Big)\eta$$

or, more elegantly written

$$\int_M dg\,\frac{\partial\mathcal{L}}{\partial d\phi}d\eta = \underbrace{\int_M dg\,\mathrm{div}\Big(\frac{\partial\mathcal{L}}{\partial d\phi}\eta\Big)}_{①} - \int_M dg\,\mathrm{div}\Big(\frac{\partial\mathcal{L}}{\partial d\phi}\Big)\eta \tag{3}$$

where $\frac{\partial\mathcal{L}}{\partial d\phi}$ is the vector field with components $\frac{\partial\mathcal{L}}{\partial\partial_i\phi}$ and the divergence of a vector field is defined through*

$$\mathcal{L}_X dg = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}X^i)dg =: \mathrm{div}(X)dg. \tag{4}$$

Remember a version of Stokes' theorem for Riemannian spacetimes, that is

$$\int_M dg\,\mathrm{div}(X) = \int_{\partial M} d\tilde{g}\,g(X,\hat{n}) \tag{5}$$

where $d\tilde{g}$ is the volume element on ∂M and $\hat{n}$ is the outward pointing unit vector field along ∂M. Since $\eta_{\partial M} = 0$, the first term ① in (3) vanishes. Then the condition (2) is found to be

$$\tfrac{d}{d\varepsilon}S[\phi+\varepsilon\eta]\big|_{\varepsilon=0} = \int_M dg\,\Big(\frac{\partial\mathcal{L}}{\partial\phi} - \mathrm{div}\Big(\frac{\partial\mathcal{L}}{\partial d\phi}\Big)\Big)\eta = 0$$

Since η was chosen to be arbitrary in the bulk of M, we find the Euler-Lagrange equation corresponding to the system (1), namely

$$\frac{\partial\mathcal{L}}{\partial\phi} - \mathrm{div}\Big(\frac{\partial\mathcal{L}}{\partial d\phi}\Big) = 0, \text{ or} \tag{6a}$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{1}{\sqrt{g}}\partial_i\Big(\sqrt{g}\frac{\partial\mathcal{L}}{\partial\partial_i\phi}\Big) = 0\ . \tag{6b}$$

2 Noether's theorem: diffeomorphism invariance

In this section we will consider the diffeomorphism invariance of (1) and investigate what currents are conserved due to this symmetry.

Let $\varphi_t : M \to M$ be a one-parameter group of diffeomorphism. By construction of the action (1), we have a one-parameter group of symmetries of the theory, i.e.

$$\int_M dg\,\mathcal{L}(\phi, d\phi, g^{-1}) = \int_M \varphi_t^*\big(dg\,\mathcal{L}(\phi, d\phi, g^{-1})\big)$$

*Note that $\mathrm{div}(X) = \nabla_i X^i$, where covariant differentiation is understood w.r.t. the Levi-Civita connection.

Dividing this equation by t and considering the limit $t \to 0$ implies that

$$\int_M \mathcal{L}_X \big(dg\, \mathcal{L}(\phi, d\phi, g^{-1})\big) = 0 \ , \tag{7}$$

where $X = \dot{\varphi}_t$ compactly supported on M. This is the infinitesimal version of the fact that our theory be diffeomorphism covariant.

The LHS of (7) is now computed. Invoking the Leibniz and chain rule for Lie differentiation, one finds*

$$\int_M dg\,\Big(\frac{\partial \mathcal{L}}{\partial \phi}\mathcal{L}_X \phi + \underbrace{\frac{\partial \mathcal{L}}{\partial d\phi}\mathcal{L}_X d\phi}_{①} + \frac{\partial \mathcal{L}}{\partial g^{-1}}\mathcal{L}_X g^{-1}\Big) + \int_M \mathcal{L}\mathcal{L}_X dg$$

Let us first consider ①. Since Lie and exterior differentiation commutes we have

$$① = \int d^4x\sqrt{g}\frac{\partial \mathcal{L}}{\partial \partial_i \phi}\partial_i \mathcal{L}_X \phi = \underbrace{\int_M dg\, \mathrm{div}\Big(\frac{\partial \mathcal{L}}{\partial d\phi}\mathcal{L}_X \phi\Big)}_{②} - \int_M dg\, \mathrm{div}\Big(\frac{\partial \mathcal{L}}{\partial d\phi}\Big)\mathcal{L}_X \phi \ .$$

Invoking Stokes' theorem for ② learns us that this term is zero (due to the fact that X is compactly supported on M). Also making use of the definition for the divergence of a vector field (4), (7) becomes

$$\int_M dg\, \Big\{\Big(\frac{\partial \mathcal{L}}{\partial \phi} - \mathrm{div}\Big(\frac{\partial \mathcal{L}}{\partial d\phi}\Big)\Big)\mathcal{L}_X \phi + \underbrace{\frac{\partial \mathcal{L}}{\partial g^{-1}}\mathcal{L}_X g^{-1} + \mathcal{L}\,\mathrm{div}(X)}_{③}\Big\} = 0 \ .$$

To go on we take a closer look at ③. The tensor g^{-1} has components $(g^{-1})^{ij} = g^{ij}$. Furthermore, let the contraction $C(g \otimes g^{-1}) := g_{ik}g^{kj} = \delta_i^j$. Then it is clear that

$$0 = \mathcal{L}_X C(g \otimes g^{-1}) = C(\mathcal{L}_X g \otimes g^{-1}) + C(g \otimes \mathcal{L}_X g^{-1}) \ ,$$

so that $(\mathcal{L}_X g)_{ik}g^{kj} = -g_{ik}(\mathcal{L}_X g^{-1})^{kj}$ and

$$(\mathcal{L}_X g)^{ij} = -(\mathcal{L}_X g^{-1})^{ij} \ . \tag{8}$$

It is a straightforward computation that

$$\mathrm{div}(X) = \tfrac{1}{2}g_{ij}(\mathcal{L}_X g)^{ij} \ . \tag{9}$$

Then substituting ③ for (8) and (9), one finds

$$③ = \Big(\tfrac{1}{2}g_{ij}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial g^{ij}}\Big)(\mathcal{L}_X g)^{ij}.$$

*Remember that dg is a top-form, so that any other combinations of tensors and differential forms are assumed to be contracted. For example $dg\,\omega\, t \equiv dg\, t_{i_1\cdots i_r}\omega^{i_1\cdots i_r}$.

At this point, one usually defines the *energy-momentum tensor*

$$T_{ij} := \frac{\partial \mathcal{L}}{\partial g^{ij}} - \tfrac{1}{2} g_{ij} \mathcal{L} \ , \tag{10}$$

so that (7) becomes

$$\int_M dg \left\{ \Big(\frac{\partial \mathcal{L}}{\partial \phi} - \mathrm{div}\Big(\frac{\partial \mathcal{L}}{\partial d\phi} \Big) \Big) \mathcal{L}_X \phi - \underbrace{T_{ij} (\mathcal{L}_X g)^{ij}}_{④} \right\} = 0 \ .$$

Finally, let us exploit some last mathematical identities to rewrite ④. Therefore, we first show that $(\mathcal{L}_X g)^{ij} = 2\nabla^{(i} X^{j)}$, where covariant differentiation is w.r.t. the Levi-Civita connection. To do this the LHS and RHS are calculated consecutively and are shown to be equal.

$$(\mathcal{L}_X g)^{ij} = -(\mathcal{L}_X g^{-1})^{ij} = -X^l \partial_l g^{ij} + g^{il} \partial_l X^j + g^{lj} \partial_l X^i$$

On the other side of the equation we have

$$\begin{aligned} \nabla^i X^j + \nabla^j X^i &= g^{il} (\partial_l X^j + \Gamma^j_{lk} X^k) + [i \leftrightarrow j] \\ &= g^{il} (\partial_l X^j + \tfrac{1}{2} g^{jm} (\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{lk}) X^k) + [i \leftrightarrow j] \\ &= g^{il} \partial_l X^j + g^{jl} \partial_l X^i + 2 g^{i(l} g^{m)j} \partial_{[l} g_{m]k} X^k + g^{i(l} g^{m)j} \partial_k g_{ml} X^k \\ &= g^{il} \partial_l X^j + g^{jl} \partial_l X^i + g^{il} g^{mj} \partial_k g_{ml} X^k \\ &= g^{il} \partial_l X^j + g^{jl} \partial_l X^i - \partial_k g^{ij} X^k \end{aligned}$$

So we find the sought after identity,

$$(\mathcal{L}_X g)^{ij} = 2\nabla^{(i} X^{j)} \ . \tag{11}$$

Inserting this in ④ and using the symmetry of T_{ij}, it follows that

$$④ = \int_M dg \, 2 T^i{}_j \nabla_i X^j = \int_M dg \left\{ 2 \mathrm{div}\big(T^i{}_j X^j \big) - \nabla^i T_{ij} X^j \right\}$$

Once again invoking Stokes' theorem for the first term in the last integral, invariance of the action under infinitesimal diffeomorphism (7) translates into

$$\int_M dg \left\{ \Big(\frac{\partial \mathcal{L}}{\partial \phi} - \mathrm{div}\Big(\frac{\partial \mathcal{L}}{\partial d\phi} \Big) \Big) \mathcal{L}_X \phi + \nabla^i T_{ij} X^j \right\} = 0 \ . \tag{12}$$

Since the vector field X considered is arbitrary, the energy-momentum tensor T_{ij} is conserved

$$\nabla^i T_{ij} = 0 \tag{13}$$

if and only if the equations of motion (6a) are satisfied, that is the matter fields are *on-shell*. This shows that the covariant conservation of the Noether current

associated with diffeomorphism invariance is a dynamical conservation equation.*
However, this covariant conservation does not necessarily lead to a conservation law.

The fact that the energy momentum tensor is divergenceless *only with the matter fields on-shell* is an important point to keep in mind. We could have added the Einstein-Hilbert term to the action (1) and solve for the equations of motion for the metric components. Then the energy-momentum tensor of the matter fields will appear as a source in Einstein's equations. The latter are divergenceless because of the Bianchi-identity—a geometric identity. Then Einstein's equations only make sense for all the fields on-shell, since for the matter fields *off-shell* the right hand side (the energy-momentum tensor) has non-vanishing divergence, contradicting the Bianchi identity on the left hand side.

*Dynamical here means that the covariant conservation is only true for the fields on-shell.