THE GEOMETRY OF N = 2 MAXWELL/EINSTEIN SUPERGRAVITY AND JORDAN ALGEBRAS

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Abstract : We construct the general coupling of n N = 2 Maxwell supermultiplets to N = 2 supergravity in five spacetime dimensions. In the case that the scalar field manifold is symmetric we find a complete classification based on Jordan algebras. Apart from the generic case there are also four "exceptional" cases associated with the Jordan algebras $J_3^{\mathbb{A}}$ of 3x3 Hermitian matrices over the division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Similar results follow for four dimensions, by dimensional reduction.

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1. Introduction

It was recently shown that N=2 matter multiplets coupled to N=2 supergravity in spacetime dimension d=4 yield N=2 locally supersymmetric models for which the scalar manifold $\mathcal{M}$ is a non-compact quaternionic space [1] . This generalises the result of N=2 rigid supersymmetry where $\mathcal{M}$ is hyper-Kähler [2] . Unlike N=1 the scalar fields of N=2 supersymmetric theories can occur in both matter and vector multiplets. In this paper we study the nature of the scalar field manifold $\mathcal{M}$ for N=2 Maxwell (i.e. Abelian vector) supermultiplets coupled to N=2 supergravity in d=5 and d=4. Unlike the results for matter multiplets, those for vector multiplets depend on the spacetime dimension, because the number of scalar fields in the vector multiplet is d-dependent. In d=5 a vector multiplet has a single scalar and the scalar field geometry is real. In d=4 a vector multiplet has two scalars and the geometry is complex. We begin our investigations with the full construction in d=5 of the general coupling of n N=2 Maxwell multiplets to N=2 supergravity. Subsequently we derive some results for d=4 by dimensional reduction which, although partial, exhibit the principal features.

Previous constructions of N=2 vector multiplet couplings to d=4, N=2, supergravity have been undertaken by Luciani [3] , and more recently by de Wit et al. [4] . The scalar field manifolds of the models constructed by Luciani are $SU(n,1)/SU(n) \times U(1)$ for n Maxwell multiplets [5] . Those of de Wit et al. are Kähler manifolds of a special type for which the Kähler potential is determined by a single holomorphic function (as for the case of rigid supersymmetry [6]). There are also constructions of supergravity coupled to one vector multiplet in d=4 [7, 8] . We have made a direct comparison of our d=4 results only with those of ref (4); the two are consistent.

The results in d=5 are of course more restrictive than those of d=4 because not all d=4 models can be obtained by dimensional reduction. However, this restriction also has advantages. Firstly, the restrictions turn out to be very interesting from the mathematical point of view. In particular, one can reduce the problem of the characterisation of the allowed scalar field geometries to a purely algebraic problem. Secondly, given a model in d=5 one can arrange to break spontaneously the various symmetries, including supersymmetry, by non-trivial dimensional reduction [9] , so that for phenomenological purposes a construction in d=5 may in fact be more useful than a direct construction in d=4.

Our d=5 results can be summarized as follows : the n-dimensional space $\mathcal{M}$, parametrized by the scalar fields of n Maxwell multiplets coupled to supergravity, can be regarded as a hypersurface with vanishing second fundamental form of an $(n+1)$ - dimensional Riemannian space $\mathcal{E}$. The equation of the hypersurface is $\mathcal{N}(\xi)=1$ where $\mathcal{N}$ is a homogeneous cubic polynomial in the coordinates $\{\xi\}$ of $\mathcal{E}$. The fact that $\mathcal{N}$ is cubic is closely related to the appearance of the term ϵFFA in the action. The fact that the second fundamental form of $\mathcal{M}$, with respect to $\mathcal{E}$, vanishes means that the geometry of $\mathcal{M}$ is uniquely determined by that of $\mathcal{E}$. The allowed geometries of $\mathcal{M}$ can be classified according to whether $\mathcal{N}$ is factorizable into a linear times a quadratic polynomial, or not. If it is

factorizable it is either $\mathcal{N} \propto \xi_0^3$, $\xi_0 \in \mathbb{R}$, for n=o , which is the trivial case of pure d=5 supergravity, or it is of the form

$$\mathcal{N}(\xi) = \xi_0 \, Q(\vec{\xi}) \qquad ; \quad \xi = (\xi_0, \vec{\xi}) , \tag{1.1}$$

where Q is a quadratic form which, for positivity of the kinetic energy, must have Minkowski signature, $(+, -, -, \cdots\cdots, -)$. The factorizability of $\mathcal{N}$ implies the reducibility of $\mathcal{M}$ which is,

$$\mathcal{M} = \frac{SO(n-1,1)}{SO(n-1)} \times SO(1,1) \qquad , \; n \geqslant 1 \; . \tag{1.2}$$

This is the generic case, applicable for all n. In particular, for n=1 or n=2 $\mathcal{M}$ is flat. It is perhaps surprising that one of the Maxwell multiplets is on a different footing from the rest, but we suspect that this is a result of their origin in d=6.

If $\mathcal{N}$ is not factorizable then $\mathcal{M}$ may or may not be a symmetric space. We do not have a classification of non-symmetric spaces, but if it is a locally symmetric space[#1] then it is also homogeneous and is one of the following four possibilities, (allowed only for the special values of n=3 $(1+\dim \mathbb{A})-1$, where $\mathbb{A}$ is one of the four division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$) :

$$J_3^{\mathbb{R}} : \quad SL(3;\mathbb{R})/SO(3) \qquad (n=5)$$

$$J_3^{\mathbb{C}} : \quad SL(3;\mathbb{C})/SU(3) \qquad (n=8)$$

$$J_3^{\mathbb{H}} : \quad SU^*(6)/Sp(3) \qquad (n=14)$$

$$J_3^{\mathbb{O}} : \quad E_{6(-26)}/F_4 \qquad (n=26)$$

Because $\mathcal{N}$ is not factorizable, $\mathcal{M}$ is irreducible. These spaces are all non compact. The $J_3^{\mathbb{A}}$ notation indicates the connection of these spaces to Jordan algebras of 3x3 Hermitian matrices over the division algebras $\mathbb{A}$. This connection arises because the function $\mathcal{N}$ turns out to be interpretable as the norm form of a certain class of Jordan algebras. As we shall show in another article [10] the first three cases are obtainable by truncation of the d=5, N=8, maximal supergravity theory [11] . The last, octonionic, case is not obtainable in this way. It is the most interesting case because the associated Jordan algebra $J_3^{\mathbb{O}}$ is the unique "exceptional" Jordan algebra [12] , which has been proposed as the basis of the charge space of coloured quarks [13] , as well as leptons [14].

These results are described in section 2, 3, 4. In section 5 we proceed to the reduction to d=4. The reduction produces an extra Maxwell multiplet from the d=5 graviton multiplet and each of the resulting $(n+1)$ Maxwell multiplets contains two scalar fields. Hence the d=4 scalar field manifold obtained by reduction from d=5 is $2(n+1)$ dimensional.

In fact it belongs to a special class of Kähler manifolds known as bounded domains. In the case of a symmetric bounded domain it is a Hermitian symmetric space. The latter are classified by Jordan triple systems, rather than Jordan algebras. The full list of such spaces, as obtainable by dimensional reduction, is given in section 5. We bring to the reader's attention here only the most interesting case, which is that obtained by reduction of the $J_3^{\mathbb{O}}$ model. The scalar field manifold in d=4 becomes

$$\frac{E_{7(-25)}}{E_6 \times U(1)} \tag{1.3}$$

and the number of vector fields of the model is 28, just as for N=8 supergravity.

In d=6 a vector multiplet has no scalar fields, so that the geometry of d=6 vector multiplets coupled to d=6 supergravity is trivial. It might appear from this fact that at least some of our results could be obtained by an easy construction in d=6 followed by dimensional reduction to d=5. However, there are <u>two</u> d=6 supermultiplets that reduce to the d=5 vector multiplet. One is the d=6 vector multiplet and the other is the d=6 antisymmetric tensor multiplet [15] (and, in fact, <u>both</u> are needed to construct the N>2, d=6, supergravity theories). The antisymmetric tensor multiplet has one scalar so that one can have non-trivial geometry in N=2 d=6 theories even without matter multiplets. Hence d=6 is not really much simpler than d=5, although we expect that the scalar field manifold will be more severely restricted.
We review in two appendices some of the mathematics related to our work.

2. <u>Construction of the model</u>

The pure d=5 supergravity theory has been constructed by several authors [11, 8, 16] . The fields are the graviton $e_\mu{}^m$; the gravitini ψ_μ^i , which form a doublet of $SU(2)$, the automorphism group of the d=5 supersymmetry algebra; and the Abelian gauge field A_μ. All spinors, e.g. ψ_μ^i are "symplectic", i.e. the Dirac conjecgate $\bar\lambda$ of λ is given by

$$\bar\lambda^i \equiv (\lambda_i)^\dagger \Gamma_0 = \lambda^{iT} C \tag{2.1}$$

with C the charge conjugation matrix satisfying

$$C^T = -C = C^{-1} \quad , \quad C\Gamma^\mu C^{-1} = (\Gamma^\mu)^T \tag{2.2}$$

It follows that the matrices C , $C\Gamma^{\mu}$ are antisymmetric, whereas $C\Gamma^{\mu\nu}$ is symmetric. Our metric, Γ-matrix, and SU(2) conventions are

$$\eta_{mn} = \mathrm{diag.}(-,+,+,+,+)$$
$$\Gamma_{\mu_1\mu_2\cdots\mu_n} = \Gamma_{[\mu_1}\Gamma_{\mu_2}\cdots\Gamma_{\mu_n]}$$
$$V^i = \varepsilon^{ij}V_j \ , \ V_i = V^i\varepsilon_{ij} \ , \ \varepsilon_{12} = \varepsilon^{12} = 1 \qquad (2.3)$$

We use square brackets to denote antisymmetrization and round brackets to denote symmetrization, always with "strength one".

Each vector multiplet in d=5 contains one scalar ϕ , one SU(2) doublet symplectic spinor λ^i and a gauge field A_μ . For the coupling of n such multiplets to supergravity, we take the n scalar fields $\{\phi^x\}$, $x = 1,2,\cdots,n$ to parametrize an n-dimensional Riemannian space $\mathcal{M}$. The tangent space group of $\mathcal{M}$ is SO(n) and we take the n spinors $\{\lambda_i^a\}$ to transform as a vector of $SO(n)$. Indices of $\mathcal{M}$ are converted to those of $SO(n)$ by means of the vielbein f_x^a and its inverse f_a^x which satisfy

$$f_x^a f_y^b \delta_{ab} = g_{xy}$$
$$f_x^a f_y^b g^{xy} = \delta^{ab} , \qquad (2.4)$$

g_{xy} being the metric of $\mathcal{M}$. The "spin -connection " of $\mathcal{M}$, $\Omega_x{}^{ab}$ is given implicitly in terms of f_x^a through the usual formula

$$f^a_{[x,y]} + \Omega_{[y}{}^{ab} f^b_{x]} = 0 \qquad (2.5)$$

The n vectors of the vector multiplets can be put together with the single vector of the graviton multiplet to yield a set of (n + 1) vectors $\{A_\mu^I\}$, $I = 0,1,2,\ldots,n$ .Thus, the complete set of fields of the model are

$$\left\{ e_\mu{}^m , \psi_{\mu\alpha}^i , A_\mu^I , \lambda_\alpha^a , \phi^x \right\}$$

We have obtained the supersymmetry transformation laws of these fields and the action using standard methods. The most general set of transformation rules, and action, consistent with Lorentz invariance are written down, and this introduces various undetermined functions of the scalar fields ϕ^x, which can and do appear non-polynomially. The twin requirements of invariance of the action and closure of the transformation rules, up to equations of motion, imply various algebraic and differential constraints on these arbitrary functions.

The final results (with the gravitational constant, κ , set equal to 1) are as follows:

The supersymmetry transformation laws are

$$\delta e_\mu{}^m = \frac{1}{2}\bar{\epsilon}^i \Gamma^m \psi_{\mu i}$$
$$\delta\psi_{\mu i} = D_\mu(\hat{\omega})\epsilon_i + \frac{i}{4\sqrt{6}} h_I\left(\Gamma_\mu{}^{\nu\rho} - 4\delta_\mu{}^\nu\Gamma^\rho\right)\hat{F}_{\nu\rho}{}^I \epsilon_i$$
$$-\frac{1}{12}\Gamma_{\mu\nu}\epsilon^j\left(\bar{\lambda}^b_i\Gamma^\nu\lambda^b_j\right) + \frac{1}{48}\Gamma_{\mu\nu\rho}\epsilon^j\left(\bar{\lambda}^b_i\Gamma^{\nu\rho}\lambda^b_j\right)$$
$$+\frac{1}{6}\epsilon^j\left(\bar{\lambda}^b_i\Gamma_\mu\lambda^b_j\right) - \frac{1}{12}\Gamma^\nu\epsilon^j\left(\bar{\lambda}^b_i\Gamma_{\mu\nu}\lambda^b_j\right)$$

$$\delta A_\mu^{\ I} = -\tfrac{1}{2} h_a^{\ I}\, \bar\epsilon^i \Gamma_\mu \lambda_i^a + \frac{i\sqrt6}{4} h^I\, \bar\psi_\mu^{\ i} \epsilon_i$$

$$\delta \lambda_i^a = -\tfrac{i}{2} f_x^a (\not\partial \phi^x)\epsilon_i - \Omega_x^{\ ab}\, \delta\phi^x \lambda_i^b + \tfrac14 h_I^a\, \Gamma^{\mu\nu}\epsilon_i \hat F_{\mu\nu}^{\ I}$$

$$-\frac{i}{4\sqrt6}\Big(-3\,\epsilon^j(\bar\lambda_i^b \lambda_j^c) + \Gamma_\mu \epsilon^j(\bar\lambda_i^b \Gamma^\mu \lambda_j^c) + \tfrac12 \Gamma_{\mu\nu}\,\epsilon^j(\bar\lambda_i^b \Gamma^{\mu\nu}\lambda_j^c)\Big) T_{abc}$$

$$\delta\phi^x = \tfrac{i}{2} f_a^x\, \bar\epsilon^i \lambda_i^a \tag{1.6}$$

The action is $I = \int d^5x\, \mathcal{L}$ with

$$e^{-1}\mathcal{L} = -\tfrac12 R(\omega) - \tfrac12 \bar\psi_\mu^{\ i}\, \Gamma^{\mu\nu\rho} D_\nu \psi_{\rho i} - \tfrac14 \mathring{a}_{IJ}\, F_{\mu\nu}^{\ I} F^{\mu\nu J}$$

$$-\tfrac12 \bar\lambda^{ia}\big(\not\partial\, \delta^{ab} + \Omega_x^{\ ab}\, \not\partial\phi^x\big)\lambda_i^b - \tfrac12 g_{xy}(\partial_\mu\phi^x)(\partial^\mu\phi^y)$$

$$-\tfrac{i}{2}\bar\lambda^{ia}\Gamma^\mu\Gamma^\nu\psi_{\mu i}\, f_x^a\, \partial_\nu\phi^x + \tfrac14 h_I^a\, \bar\lambda^{ia}\Gamma^\mu\Gamma^{\lambda\rho}\psi_{\mu i}\, F_{\lambda\rho}^{\ I}$$

$$+\tfrac{i}{4}\Phi_{Iab}\, \bar\lambda^{ia}\Gamma^{\mu\nu}\lambda_i^b\, F_{\mu\nu}^{\ I} - \frac{3i}{8\sqrt6} h_I\Big[\bar\psi_\mu^{\ i}\, \Gamma^{\mu\nu\rho\sigma}\psi_{\nu i} F_{\rho\sigma}^{\ I} + 2\bar\psi^{\mu i}\psi_i^\nu F_{\mu\nu}^{\ I}\Big]$$

$$+\frac{e^{-1}}{6\sqrt6} C_{IJK}\, \varepsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}^{\ I} F_{\rho\sigma}^{\ J} A_\lambda^{\ K} \quad + \text{4-fermion terms} \tag{2.7}$$

The four fermion terms are given below; they are not important for an understanding of the geometry underlying the model. The various quantities appearing in (2.6) and (2.7) are as follows:

e is $\det e_\mu^{\ m}$; $\omega_{\mu mn}(e)$ is the usual spacetime spin connection

$$\omega_\mu^{\ mn}(e) = 2\, e^{[m\nu} e_{[\nu,\mu]}^{\ n]} + e^{m\lambda} e^{n\rho} e_{[\lambda,\rho]}^{\ \ \ell}\, e_{\mu\ell}\,, \tag{2.8}$$

in terms of which the Riemann tensor and its contractions are

$$R_{\mu\nu mn} = (\partial_\mu \omega_{\nu mn} + \omega_{\mu m}^{\ \ \ \ell}\, \omega_{\nu \ell n}) - (\mu \leftrightarrow \nu),$$

$$R_{\mu n} = e^{\nu m} R_{\mu\nu mn}\,, \quad R = e^{\mu n} R_{\mu n}\;; \tag{2.9}$$

$F_{\mu\nu}^{\ I}$ is the usual Maxwell field strength $2\partial_{[\mu}A_{\nu]}^{\ I}$; h_I and h^I, h_{Ia} and h_a^I, Φ_{Iab}, T_{abc} and $\mathring{a}_{IJ}$ are functions of ϕ^x, but without a geometrical interpretation as yet. $\Phi_{Iab} = \Phi_{Iba}$, $\mathring{a}_{IJ} = \mathring{a}_{JI}$, and T_{abc} is symmetric in all indices. C_{IJK} is symmetric in all indices and a <u>constant</u>, i.e. independent of ϕ^x, as required by gauge invariance of the action. The hatted quantities are the "supercovariantization" of the unhatted ones:

$$\hat F_{\mu\nu}^{\ I} = F_{\mu\nu}^{\ I} + h_a^I\, \bar\psi_{[\mu}^{\ j} \Gamma_{\nu]} \lambda_i^a + \frac{i\sqrt6}{4} h^I\, \bar\psi_{[\mu}^{\ i} \psi_{\nu]i}$$

$$(\hat\partial_\mu\phi)^x = \partial_\mu\phi^x - \tfrac{i}{2} f_b^x\, \bar\psi_\mu^{\ j} \lambda_j^b$$

$$\hat\omega_{\mu mn} = \omega_{\mu mn}(e) - \tfrac14\Big(\bar\psi_n^{\ i}\Gamma_\mu\psi_{mi} + 2\bar\psi_\mu^{\ i}\Gamma_{[n}\psi_{m]i}\Big)\,. \tag{2.10}$$

Supersymmetry, through invariance of the action and closure of the algebra, determines the functions $\Phi_{I\,xy} (= \Phi_{Iab} f_x^a f_y^b)$ and $\mathring{a}_{IJ}$ to be

$$\Phi_{Ixy} = \sqrt{\tfrac{2}{3}}\Big(\tfrac14 g_{xy} h_I + T_{xyz} h_I^z\Big)$$

$$\mathring{a}_{IJ} = h_I h_J + h_I^x h_J^y\, g_{xy} \tag{2.11}$$

In addition, the constants C_{IJK} are related to the h_I, h_I^x and T_{xyz} through

$$C_{IJK} = \frac{5}{2} h_I h_J h_K - \frac{3}{2} \mathring{a}_{(IJ} h_{K)} + T_{xyz} h_I^x h_J^y h_K^z \quad (2.12)$$

The h's are themselves subject to the algebraic constraints

$$h^I h_I = 1 \quad (2.13a)$$

$$h^I_x h_I = h_{Ix} h^I = 0 \quad (2.13b)$$

$$h^I_x h^J_y \mathring{a}_{IJ} = g_{xy} \quad (2.13c)$$

from which follows

$$h^I \mathring{a}_{IJ} = h_J$$
$$h^I_x \mathring{a}_{IJ} = h_{Jx} \; . \quad (2.14)$$

There are also differential constraints to be satisfied by the h's ;

$$h_{I,x} = \sqrt{\tfrac{2}{3}}\, h_{Ix}$$
$$h^I_{,x} = -\sqrt{\tfrac{2}{3}}\, h^I_x \; , \quad (2.15)$$

and

$$h_{Ix;y} = \sqrt{\tfrac{2}{3}} \left(g_{xy} h_I + T_{xyz} h_I^z \right)$$
$$h^I_{x;y} = -\sqrt{\tfrac{2}{3}} \left(g_{xy} h^I + T_{xyz} h^{Iz} \right) , \quad (2.16)$$

where the semi-colon indicates covariant differentiation with respect to the index x using the Christoffel connection calculated from the metric g_{xy} .

We have still to analyse the implications of the relation (2.12). Because C_{IJK} are constants, differentiation with respect to ϕ^x of this equation yields a further differential constraint which, since the derivatives of h_I and h_I^x are already fixed by (2.15) and (2.16), reduces to a differential constraint on T_{xyz} ;

$$T_{xyz;u} = \frac{\sqrt{6}}{2} \left(g_{(xy}\, g_{z)u} - 2\, T_{(xy}{}^{w}\, T_{z)uw} \right) . \quad (2.17)$$

By taking further derivatives of (2.17) we derive the integrability condition

$$K_{xyzu} = \frac{4}{3} \left(g_{x[u}\, g_{z]y} + T_{x[u}{}^{w}\, T_{z]yw} \right), \quad (2.18)$$

where $K_{xyzu} = f^a_z f^b_u K_{xyab}$ is the Riemann tensor of the scalar field manifold $\mathcal{M}$, defined as

$$K_{xyab} \equiv \left(\Omega_{yab,x} + \Omega_{xac} \Omega_{ycb} \right) - (x \leftrightarrow y) \quad (2.19)$$

Given (2.17) there is also an integrability condition for (2.16) which, remarkably, is again (2.18). Thus (2.18) is the final result for the restrictions placed on $\mathcal{M}$ by supersymmetry and gauge invariance. In the remainder of this paper we shall be concerned with, firstly, a geometrical interpretation of the results and, secondly, a classification of the allowed manifolds $\mathcal{M}$ for the special case for which T_{xyz} , and thereby K_{xyzu} , is covariantly constant.

To complete this section we give the four fermion terms in the action, and the local supersymmetry algebra. The former are

$$
\begin{aligned}
e^{-1}\mathcal{L}_{4\text{-fermion}} = \Big[& \frac{1}{48\sqrt{6}}\left(\bar\lambda^{ia}\Gamma_{\mu\nu}\lambda^{b}_{i}\right)\left(\bar\lambda^{jc}\Gamma^{\mu\nu}\lambda^{d}_{j}\right)T_{abc;d} \\
& + \frac{1}{24}K_{abcd}\left\{2\left(\bar\lambda^{ia}\lambda^{b}_{i}\right)\left(\bar\lambda^{jc}\lambda^{d}_{j}\right) + \left(\bar\lambda^{ia}\Gamma_{\mu}\lambda^{b}_{i}\right)\left(\bar\lambda^{jc}\Gamma^{\mu}\lambda^{d}_{j}\right)\right\} \\
& - \frac{1}{12}\left(\bar\lambda^{ia}\lambda^{ja}\right)\left(\bar\lambda^{b}_{j}\lambda^{b}_{i}\right) - \frac{1}{24}\left(\bar\lambda^{ia}\Gamma_{\mu}\lambda^{ja}\right)\left(\bar\lambda^{b}_{j}\Gamma^{\mu}\lambda^{b}_{i}\right) \\
& + \frac{1}{64}\left(\bar\lambda^{ia}\Gamma_{\mu\nu}\lambda^{ja}\right)\left(\bar\lambda^{b}_{j}\Gamma^{\mu\nu}\lambda^{b}_{i}\right)\Big] \\
& + \frac{2i}{3\sqrt{6}}T_{abc}\left[\left(\bar\lambda^{ia}\psi_{\mu}^{j}\right)\left(\bar\lambda^{b}_{i}\Gamma^{\mu}\lambda^{c}_{j}\right) - \frac{1}{2}\left(\bar\lambda^{ia}\Gamma^{\mu}\psi_{\mu}^{j}\right)\left(\bar\lambda^{b}_{i}\lambda^{c}_{j}\right)\right] \\
& + \Big[\frac{3}{32}\left(\bar\psi^{\mu}{}_{i}\psi_{\mu j}\right)\left(\bar\lambda^{ia}\lambda^{ja}\right) - \frac{1}{32}\left(\bar\psi_{\mu i}\Gamma_{\rho}\psi_{\nu j}\right)\bar\lambda^{ia}\left(3g^{\mu\nu}\Gamma^{\rho} + 2g^{\rho\mu}\Gamma^{\nu}\right)\lambda^{ja} \\
& - \frac{1}{64}\left(\bar\psi_{\mu i}\Gamma_{\rho\sigma}\psi_{\nu j}\right)\bar\lambda^{ia}\left(\Gamma^{\mu\nu\rho\sigma} + 2g^{\nu\sigma}g^{\mu\rho}\right)\lambda^{ja} \\
& + \frac{1}{64}\left(\bar\psi_{\mu}^{k}\psi_{\nu k}\right)\left(\bar\lambda^{ia}\Gamma^{\mu\nu}\lambda^{a}_{i}\right) - \frac{1}{64}\left(\bar\psi_{\mu}^{k}\Gamma_{\rho}\psi_{\nu k}\right)\left(\bar\lambda^{ia}\Gamma^{\mu\nu\rho}\lambda^{a}_{i}\right) \\
& + \frac{1}{32}\left(\bar\psi_{\mu}^{k}\Gamma_{\rho\sigma}\psi_{\nu k}\right)\bar\lambda^{ia}\left(g^{\rho\mu}\Gamma^{\nu\sigma} - \frac{1}{4}g^{\mu\nu}\Gamma^{\rho\sigma}\right)\lambda^{a}_{i}\Big] \\
& + \Big[\frac{3}{64}\left(\bar\psi_{\rho}^{i}\psi_{\sigma i}\right)\left(\bar\psi^{\rho j}\psi^{\sigma}_{j}\right) + \frac{1}{32}\left(\bar\psi_{\mu}^{i}\Gamma^{\mu\nu\rho\sigma}\psi_{\nu i}\right)\left(\bar\psi_{\rho}^{j}\psi_{\sigma j}\right) \\
& + \frac{1}{8}\left(\bar\psi^{i}\cdot\Gamma\psi_{\mu i}\right)\left(\bar\psi^{j}\cdot\Gamma\psi^{\mu}_{j}\right) + \frac{1}{32}\left(\bar\psi^{\nu i}\Gamma^{\mu}\psi^{\rho}_{i}\right)\left(\bar\psi^{j}_{\nu}\Gamma_{\mu}\psi_{\rho j}\right) \\
& - \frac{1}{8}\left(\bar\psi^{i(\nu}\Gamma^{\rho)}\psi^{\mu}_{i}\right)\left(\bar\psi^{j}{}_{\nu}\Gamma_{\rho}\psi_{\mu j}\right)\Big] .
\end{aligned}
\tag{2.20}
$$

The λ^4 terms in the action and the $\epsilon\lambda^2$ terms in $\delta\lambda$ and $\delta\psi$ were determined by requiring closure of the algebra on λ, modulo the λ field equation. This simultaneously yields the λ^3 terms in the λ field equation. The $\epsilon\psi\lambda$ terms in $\delta\lambda$ and the $\lambda^2\psi$, $\lambda\psi^2$ terms in the action are then fixed by supercovariance of $\delta\lambda$ and the λ field equation. The $\epsilon\psi^2$ terms in $\delta\psi$ and the ψ^4 terms in the action are independent of the matter coupling They are therefore the same as in pure d=5 supergravity and we have simply translated the results of ref (11) into our conventions. Note, however, that our results are expressed in the usual "2nd order" formalism in which the spin connection $\omega=\omega(e)$ is a <u>dependent</u> field. Thus no 4-fermion terms are implicit in (2.7); all are given explicitly in (2.23).

The supersymmetry algebra is

$$
[\delta_2, \delta_1] = \delta_{g.c.}(\xi^{\mu}) + \delta_{loc.L.}(\Lambda^{mn}) + \delta_{sup.}(\eta^{i}) + \delta_{gauge}(\alpha^{I})
\tag{2.21}
$$

where the composite parameters ξ^{μ}, Λ^{mn}, η^{i}, and α^{I} of general coordinate, local Lorentz, supersymmetry, and gauge transformations are given by

$$
\begin{aligned}
\xi^{\mu} &= \frac{1}{2}\bar\epsilon_1^{i}\Gamma^{\mu}\epsilon_{2i} \\
\Lambda^{mn} &= \xi^{\nu}\hat\omega_{\nu}{}^{mn} + \frac{i}{4\sqrt{6}}\bar\epsilon_1^{i}\left(\Gamma^{mnpq} - 4\eta^{mp}\eta^{nq}\right)\epsilon_{2i}\hat F_{pq}{}^{I}h_{I} \\
&\quad + \frac{1}{24}\left(\bar\epsilon_1^{i}\epsilon_{2i}\right)\left(\bar\lambda^{jb}\Gamma^{mn}\lambda^{b}_{j}\right) - \frac{1}{48}\left(\bar\epsilon_1^{i}\Gamma_{\mu}\epsilon_{2i}\right)\left(\bar\lambda^{jb}\Gamma^{\mu mn}\lambda^{b}_{j}\right) \\
&\quad + \frac{1}{24}\left(\bar\epsilon_1^{i}\Gamma_{\mu\nu}\epsilon_2^{j}\right)\left(\bar\lambda^{b}_{i}\Gamma^{\mu\nu mn}\lambda^{b}_{j}\right) \\
\eta^{i} &= -\xi^{\mu}\psi_{\mu}^{i} \\
\alpha^{I} &= -\frac{i\sqrt{6}}{4}h^{I}\bar\epsilon_1^{i}\epsilon_{2i} - \xi^{\nu}A_{\nu}^{I}
\end{aligned}
\tag{2.22}
$$

3. Geometrical interpretation

In order to gain an understanding of the results of the previous section it will be convenient to consider $\mathcal{M}$ as a hypersurface of an (n+1) - dimensional Riemannian space $\mathcal{C}$. The coordinates of $\mathcal{C}$ could be taken as $\{\phi^x\}$ plus one additional coordinate, $\mathcal{N}$, but we shall consider $\mathcal{C}$ as parametrized by coordinates $\{\xi^I\}$, which are functions of $\{\phi^x, \mathcal{N}\}$;

$$\xi^I = \xi^I(\phi^x, \mathcal{N}) \quad . \tag{3.1}$$

The equation

$$\ln \mathcal{N} = k , \tag{3.2}$$

with k a constant, defines a family of hypersurfaces of $\mathcal{C}$, $\mathcal{M}_k$, parametrized by k. The normal to one of these hypersurfaces is

$$n_I = \frac{\partial}{\partial \xi^I} \ln \mathcal{N} \equiv \partial_I \ln \mathcal{N} \quad . \tag{3.3}$$

The vectors $\xi^I{}_{,x}$ span the tangent space of the hypersurface, and their orthogonality to n_I is expressed by the relation

$$\xi^I{}_{,x} n_I = 0 \quad . \tag{3.4}$$

We shall show that $\mathcal{M}$ of section 2 can be identified with $\mathcal{M}_{k=0}$. The functions h_I and h^I of the previous section we take to be

$$\begin{aligned} h_I &= \alpha\, n_I \big|_{\mathcal{N}=1} \\ h^I &= \beta\, \xi^I \big|_{\mathcal{N}=1} \quad . \end{aligned} \tag{3.5}$$

With these identifications the orthogonality relation (3.4) is seen to be equivalent to that of (2.13b). At this stage the geometry of the spaces $\mathcal{C}$ and $\mathcal{M}$ are totally undetermined. But if we rewrite the condition $h^I h_I = 1$ of (2.13a) in terms of the coordinates ξ^I of $\mathcal{C}$ we find the equation

$$\xi^I \partial_I \ln \mathcal{N} = (\alpha\beta)^{-1} , \tag{3.6}$$

which restricts $\mathcal{N}$ to be a homogeneous function of degree $(\alpha\beta)^{-1}$.

Strictly speaking eq.(3.6) need be true only at $\mathcal{N}=1$, but we are obviously free to define the space $\mathcal{C}$ such that it holds everywhere. This is the natural choice, although not necessarily the only one. What is important is that $\mathcal{C}$ be defined in such a way that the Riemann tensor of the hypersurface $\mathcal{M}$ coincide with that given in eq.(2.18). We shall show in the following that this is the case, and from this it follows that our geometrical interpretation is a valid one.

Differentiation of (3.6) yields a relation between ξ^I and n_I which, if rewritten in terms of h^I and h_I , becomes

$$h_I = \left(-\frac{\alpha}{\beta}\, \partial_{IJ} \ln \mathcal{N}\right)\Big|_{\mathcal{N}=1} h^J \quad . \tag{3.7}$$

Comparing this with (2.14) we see that the function $\mathring{a}_{IJ}$ is simply $a_{IJ}|_{\mathcal{N}=1}$, with

$$a_{IJ} = -\frac{\alpha}{\beta}\, \partial_{IJ} \ln \mathcal{N} \tag{3.8}$$

being the metric of the space $\mathcal{C}$. Thus the geometry of $\mathcal{C}$ is determined by a single homogeneous function $\mathcal{N}$ of its coordinates. The line element $ds^2 = a_{IJ}\, d\xi^I d\xi^J$ of $\mathcal{C}$ when restricted to $\mathcal{M}$ is $ds^2 = \mathring{a}_{IJ}\, \xi^I{}_{,x} \xi^J{}_{,y}\, d\phi^x d\phi^y = g_{xy}\, d\phi^x d\phi^y$. Since $\xi^I{}_{,x} = \beta^{-1} h^I{}_{,x} = -\beta^{-1}(\sqrt{2/3})\, h^I{}_x$, from (3.5) and (2.15), the relation (2.13c) between $\mathring{a}_{IJ}$ and g_{xy} fixes β^2 to be

$$\beta^2 = \frac{2}{3} \quad . \tag{3.9}$$

From (3.8) we can derive the Christoffel connection of $\mathcal{C}$ as a function of $\mathcal{N}$;

$$\Gamma_{IJK} = -\frac{\alpha}{2\beta}\, \partial_{IJK} \ln \mathcal{N} \quad . \tag{3.10}$$

The Riemann tensor, in terms of Γ_{IJK} turns out to be

$$R_{IJ}{}^K{}_L = 2\, \Gamma^K{}_{M[J} \Gamma^M{}_{I]L} \quad , \tag{3.11}$$

where indices are lowered (raised) with a_{IJ} ($(a^{-1})^{IJ}$) . The reader may be surprised by this result because whereas R_{IJKL} is a tensor Γ_{IJK} is not. This means that (3.11) is valid only in a special class of coordinate systems. This class is defined as that for which eq.(3.6) is satisfied, i.e. for which $\mathcal{N}$ is a homogeneous function of ξ^I . Obviously $\mathcal{N}$ cannot remain homogeneous in an arbitrary coordinate system. Given a coordinate system for which (3.6) is satisfied, it will also be satisfied in any other that is related by a linear transformation. In other words, Γ_{IJK} is a tensor under linear transformations of the coordinates. The Riemann tensor has the property that

$$R_{IJ}{}^{K}{}_{L}\,\xi^L = 0 \quad ; \tag{3.12}$$

i.e. the holonomy group of $\mathcal{E}$ is a subgroup of the stability group; (this implies that $\mathcal{E}$ is locally reducible).

The Riemann tensor of the hypersurface $\mathcal{M}$ embedded in $\mathcal{E}$ is given by the Gauss equation as

$$K_{xyzu} = 2\beta^2\,\Omega_{z[x}\,\Omega_{y]u} + R_{IJKL}\,\xi^I{}_{,x}\,\xi^J{}_{,y}\,\xi^K{}_{,z}\,\xi^L{}_{,u}\Big|_{\mathcal{N}=1} , \tag{3.13}$$

where $\Omega_{xy} = \Omega_{yx}$ is the second fundamental form of $\mathcal{M}$. It is given by

$$\Omega_{xy} = \xi_I\left(\xi^I{}_{,x;y} + \Gamma^I{}_{JK}\,\xi^J{}_{,x}\,\xi^K{}_{,y}\right) , \tag{3.14}$$

and satisfies the Codazzi equation

$$\Omega_{x[y;z]} = \tfrac{1}{2} R_{IJKL}\,\xi^I{}_{,x}\,\xi^K{}_{,y}\,\xi^L{}_{,z}\,\xi^J \tag{3.15}$$

One sees immediately from (3.12) that $\Omega_{x[y;z]} = 0$ and, in fact, Ω_{xy} is zero as we now show. Firstly we derive an equation for Γ_{IJK} by differentiation of (3.7) and use of the differential constraints (2.15) on h_I and h^I ; this is

$$\beta^{-1}\,\Gamma_{IJK}\Big|_{\mathcal{N}=1} = 2h_I h_J h_K - 3\,\mathring{a}_{(IJ}h_{K)} - T_{xyz}\,h_I^{\,x}\,h_J^{\,y}\,h_K^{\,z} . \tag{3.16}$$

Use of this expression in (3.14) together with further use of the algebraic and differential restraints on h^I and $h^I{}_x$ yields the result

$$\Omega_{xy} = 0 . \tag{3.17}$$

Hence the Riemann tensor K_{xyzu} is given by the second term in (3.13). Using (3.11) and again (3.16) we deduce that

$$K_{xyzu} = \tfrac{4}{3}\left(g_{x[u}\,g_{z]y} + T_{x[u}{}^{w}\,T_{z]yw}\right) , \tag{3.18}$$

which is precisely that of (2.15). This establishes our claim that the space $\mathcal{M}$ of section 2 can be considered as the $\mathcal{N}=1$ hypersurface of $\mathcal{E}$. Moreover, as a consquence of the vanishing of the second fundamental form of $\mathcal{M}$ its geometry is entirely determined by that of $\mathcal{E}$, e.g. if $\mathcal{E}$ is flat so is $\mathcal{M}$.

To discover what further restrictions there are on the space $\mathcal{E}$ we must investigate the consequences of the fact that the C_{IJK} are constants. Comparing (3.16) with the similar expression (2.12) for C_{IJK} , we find a relation between C_{IJK} and Γ_{IJK} ;

$$C_{IJK} = -\beta^{-1}\,\Gamma_{IJK}\Big|_{\mathcal{N}=1} + \tfrac{9}{2}\,h_I h_J h_K - \tfrac{9}{2}\,\mathring{a}_{(IJ}h_{K)} . \tag{3.19}$$

This allows us to find an expression for C_{IJK} directly in terms of $\mathcal{N}$;

$$C_{IJK} = \Big[\frac{\alpha}{2\beta^2}\,\mathcal{N}_{,IJK} + \frac{9}{2}\,\frac{\alpha}{\beta^2}\left(\alpha\beta - \tfrac{1}{3}\right)\mathcal{N}_{,(IJ}\,\mathcal{N}_{,K)} + \frac{9\alpha}{2\beta^2}\left(\alpha\beta - \tfrac{1}{3}\right)\left(\alpha\beta - \tfrac{2}{3}\right)\mathcal{N}_{,I}\,\mathcal{N}_{,J}\,\mathcal{N}_{,K}\Big]\Big|_{\mathcal{N}=1} . \tag{3.20}$$

The homogeneity of $\mathcal{N}$ now implies that $C_{IJK,x} = 0$ can have solutions only for

$$(\alpha\beta)^{-1} = 3 \tag{3.21}$$

and then only if $\mathcal{N}$ is a polynomial, i.e. $\mathcal{N}$ <u>must be a homogeneous polynomial of degree three</u>. In fact,

$$\mathcal{N} = \beta^3 C_{IJK} \xi^I \xi^J \xi^K . \tag{3.22}$$

It also follows from (3.20), using the homogeneity of $\mathcal{N}$, that

$$C_{IJK} h^K = \tfrac{1}{2}\left(3 h_I h_J - \mathring{a}_{IJ}\right) . \tag{3.23}$$

On physical grounds $\mathring{a}_{IJ}$ must be positive definite (because it is the coefficient of the vector kinetic terms in the action), and we may therefore transform it locally to a multiple of δ_{IJ}. The multiple is irrelevant so, without loss of generality, we may choose a point $\xi^I = c^I$ in $\mathcal{E}$, lying on $\mathcal{M}$, such that

$$a_{IJ}\big|_c = \mathring{a}_{IJ}\big|_c = \delta_{IJ} . \tag{3.24}$$

From the fact that c is on $\mathcal{M}$ we have also that

$$\mathcal{N}(c) = 1 \tag{3.25}$$

From (3.23) we have

$$C_{IJK} c^K = \left(3\beta^2 c_I c_J - \delta_{IJ}\right) / 2\beta . \tag{3.26}$$

We now define an adjoint operation $\#$ as follows:

$$\xi^{\# I} = \beta C_{IJK} \xi^J \xi^K \tag{3.27}$$

From (3.8) we have

$$a_{IJ} = -\frac{\alpha}{\beta}\left[\frac{6\beta^3 C_{IJK}\xi^K}{\mathcal{N}} - \frac{9\beta^4}{\mathcal{N}^2}\, \xi^{\# I} \xi^{\# J}\right] \tag{3.28}$$

When evaluated at the point c we can use (3.26) to show that the condition (3.24) implies

$$c^{\# I} = c_I = c^I \tag{3.29}$$

i.e. c is "self adjoint".

The model of section 2 is now seen to depend only on the choice of constants C_{IJK}, but these must be chosen so as to satisfy (3.26) for some choice of the point c. This choice is not unique but, once made, it determines the general form of $\mathcal{N}$. For example, one choice of c is

$$c^I = \beta^{-1} \delta^{I0} , \tag{3.30}$$

(which, however, is not the most convenient in practice). This choice when substituted in (3.26) requires the norm $\mathcal{N}$ to be of the form

$$\mathcal{N} = \beta^3\left[(\xi^0)^3 - \tfrac{3}{2}\xi^0 \xi^i \xi^j \delta_{ij} + C_{ijk}\xi^i\xi^j\xi^k\right] \tag{3.31}$$

so that only the coefficients C_{ijk}, ($i,j,k = 1,2,\dots,n$) may be chosen at will.

At this juncture it seems worthwhile to summarize our interpretation of the model. Eqs. (2.11),(2.13b,c), and (2.14) are interpretable as embedding equations of $\mathcal{M}$ in $\mathcal{E}$. Eq.(2.13a) restricts $\mathcal{E}$ by requiring the function $\mathcal{N}$ that parametrizes the hypersurfaces $\mathcal{M}_k$ to be homogeneous. Eqs. (2.12),(2.15), and (2.16), together with the constancy of C_{IJK}, (which is required by invariance of the action under $\delta A^I_\mu = \partial_\mu \theta^I(x)$) further constrain $\mathcal{E}$ by determining the degree of homogeneity of $\mathcal{N}$ to be 3. As for the hypersurface $\mathcal{M}$

eqs. (2.15) and (2.16) can be interpreted as the condition of vanishing second fundamental form. On the one hand this means that the curvature of $\mathcal{M}$ is determined uniquely by that of $\mathcal{E}$ (i.e. (3.13) with $\Omega_{xy} = o$); on the other hand the integrability conditions of these equations directly restricts the curvature of $\mathcal{M}$ to be of a special form (i.e. (2.17)). These two methods for determining the curvature of $\mathcal{M}$ of course agree.

4. Symmetric spaces and Jordan algebras

Let us now return to the constraint (2.17) to be satisfied by T_{xyz} , A special case of interest is that for which

$$T_{xyz;\,u} = o \;, \tag{4.1}$$

from which follows the algebraic constraint

$$T_{(xy}{}^{w}\,T_{zu)w} = \tfrac{1}{2}\, g_{(xy}\, g_{zu)} \;. \tag{4.2}$$

It also follows, from the expression (2.18) for the Riemann tensor of $\mathcal{M}$, that

$$K_{xyzu;\,w} = o, \tag{4.3}$$

i.e. that $\mathcal{M}$ is a locally symmetric space.

As a direct consequence of (4.2) we have that

$$T_{xyz}T^{xyz} + \tfrac{1}{2}T_x T^x = \frac{n(n+2)}{4} \;, \tag{4.4}$$

where $T_x = T_{xyz}\, g^{yz}$. This implies that the scalar curvature[f2] $K = g^{xu} g^{yv} K_{xyuv}$ is given by ($n \geq 1$),

$$K = -\left[\frac{n(n-2)}{2} + T_x T^x\right] \;. \tag{4.5}$$

For n=1, for which $\mathcal{M}$ is necessarily flat, $T_x T^x = \frac{1}{2}$. For n=2 K will vanish if $T_x = 0$. In fact one can show that (4.2) implies $T_x = 0$ if $n = 2$, so that in this case, also, $\mathcal{M}$ is necessarily flat. For $n > 2$, $\mathcal{M}$ is a space of negative definite scalar curvature.

In order to say more about the $n > 2$ case we need a more sophisticated method of analysis. As we shall see, this is provided by the theory of Jordan algebras. We shall begin by recalling some basic facts about these algebras. A (commutative) Jordan algebra J is defined as an algebra with a symmetric product $\circ$,

$$x \circ y = y \circ x = z \quad , \quad x,y,z \in J \quad , \tag{4.6}$$

and for which the "Jordan identity"

$$x \circ (y \circ x^2) = (x \circ y) \circ x^2 \tag{4.7}$$

is satisfied. These algebras are commutative, but non-associative. The classic examples of simple Jordan algebras are the nxn Hermitian matrices over the associative division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, with the product $\circ$ being one half the anticommutator.. In their classic work, Jordan, von Neumann, and Wigner [12] showed that, but for one exceptional case, all finite dimmensional Jordan algebras have realizations in terms of associative matrices with the product $\circ$ being one half the anticommutator. The exception [17] is the algebra of 3x3 Hermitian matrices over the octonions, $\mathbb{O}$, with the product $\circ$ being again one half the anticommutator . This is the "exceptional" Jordan algebra $J_3^{\mathbb{O}}$, all the others being referred to as "Special".

There is another, so called "quadratic" formulation of Jordan algebras [18,19] based on the quadratic operator U_x associated to an element

x of J$^{(13)}$; U_x is given by

$$U_x y = \{x\, y\, x\} \quad , \tag{4.8}$$

where $\{x\, y\, z\}$ is the Jordan triple product

$$\begin{aligned} \{x\, y\, z\} &= x \circ (y \circ z) + z \circ (y \circ x) - (x \circ z) \circ y \\ &= \{z\, y\, x\} \end{aligned} \tag{4.9}$$

In this formulation the axioms of a unital Jordan algebra (i.e. one with a unit element e) are [19] :

Q1. $U_e = 1$

Q2. $U_{U_x y} = U_x U_y U_x$

Q3. $U_x \{y\, x\, z\} = \{(U_x y)\, z\, x\}$

In terms of the quadratic operator U the Jordan triple product is

$$\{x\, y\, z\} = \left[U_{(x+z)} - U_x - U_z\right] y \tag{4.10}$$

One of the remarkable properties of this formulation is that for every Jordan algebra one can define a norm $\mathcal{N}$ satisfying the composition property

$$\mathcal{N}(U_x y) = \mathcal{N}^2(x)\, \mathcal{N}(y) \ . \tag{4.11}$$

Conversely, McCrimmon has shown [21] that any normed algebra $\mathfrak{J}$ satisfying the above composition property is a "non commutative Jordan algebra" whose symmetrized algebra J is a (commutative) Jordan algebra. In fact one can reformulate the theory of quadratic Jordan algebras in terms of the norm form $\mathcal{N}$ and the differential calculus of rational mappings [21, 22]. It is this reformulation that renders manifest the connection between Jordan algebras and N=2 Maxwell/Einstein supergravity theories. As may be anticipated from the discussion in section 3, the relevant Jordan algebras are those whose norm form $\mathcal{N}$ is cubic. We shall now summarize McCrimmon's construction of this class of algebra (this unifies previous constructions of Freudenthal, Springer, and Tits [23]).

Assume that we are given (i) a vector space V over $\mathbb{R}$, on which is defined a cubic norm form $\mathcal{N}$ taking values in $\mathbb{R}$, (ii) a quadratic map $\# : x \to x^{\#}$ of V into itself, (iii) a "base point" $c \in V$; and that these are related by

(M1) : $\mathcal{N}(c) = 1$.

(M2) : $c^{\#} = c$.

(M3) : $T(x^{\#}, y) = y^{I} \partial_{I} \mathcal{N}(x)$

where

$$T(x,y) = -x^{I} y^{J} \left(\partial_{IJ} \ln \mathcal{N}(x)\right)\Big|_{c} \quad (4.12)$$

(M4) : $c \times y = T(y,c)\, c - y$

where

$$x \times y = (x+y)^{\#} - x^{\#} - y^{\#} \quad (4.13)$$

(M5) : $x^{\#\#} = \mathcal{N}(x)\, x$, (adjoint identity).

A vector space V with the above properties defines a unital Jordan algebra with Jordan product $\circ$ given by

$$x \circ y = \tfrac{1}{2}\left[T(c,x)y + T(c,y)x - T(c, x\times y)c + x \times y\right], \quad (4.14)$$

and quadratic operator U_x given by

$$U_x y = T(x,y)\,x - x^{\#} \times y \quad (4.15)$$

Referring to section 3 we can identify the relations (M1) through (M4) with the following equations

(M1) : eq. (3.25)

(M2) : eq. (3.29)

(M3) : eq. (3.27)

(M4) : eq. (3.26)

The first two identifications are obvious. The third, which defines the adjoint map $\#$, requires a little algebra, as does the fourth. The fifth relation, the "adjoint identity", has no counterpart in section 3. This relation implies that the constants C_{IJK} satisfy the algebraic constraint

$$C_{IJK}\, C_{J'(LM}\, C_{PQ)K'}\, \delta^{JJ'} \delta^{KK'} = \delta_{I(L}\, C_{MPQ)}. \quad (4.16)$$

This is just the relation

$$C^{IJK}\, C_{J(MN}\, C_{PQ)K} = \delta^{I}_{(M}\, C_{NPQ)} \quad (4.17)$$

evaluated at the point c, where $a_{IJ} = (a^{-1})^{IJ} = \delta_{IJ}$. If $\mathcal{M}$ is a <u>homogeneous</u> space (4.17) is implied by (4.16). By projection of this relation along the normal to $\mathcal{M}$ (with h_I) or in the tangent space (with h_{Ia}) one can show that it is equivalent to two algebraic constraints on T_{xyz} . These are (4.2) and

$$T^{xyz} T_{y(vw} T_{st)z} = \delta^{x}_{(v} T_{wst)} - \frac{1}{2} T^{x}_{(vw} g_{st)} \qquad (4.18)$$

But (4.18) is not independent of (4.2) , as one establishes after a considerable amount of algebra which we omit. Hence the constraint (4.17) is entirely equivalent to the requirement that T_{xyz} be covariantly constant.

If $T_{xyz;u}=0$ it follows that (MS) is satisfied. Conversely, if (MS) if satisfied it will follow that $T_{xyz;u}=0$ (and hence that $\mathcal{M}$ is a locally symmetric space), <u>if</u> $\mathcal{M}$ is homogeneous.

To establish the homogeneity of $\mathcal{M}$ we use the correspondence between elements $x \in J$ ($\equiv$ points x of $\mathcal{C}$) and the associated quadratic operator U_x . The U's satisfy

$$U_x U_y = U_z A \; ; \; x, y, z \in J , \qquad (4.19)$$

where A is an element of the automorphism group $Aut(J)$ of the Jordan algebra J . By the norm preserving property of the U operators, if $N(x) = N(y) = 1$ then $N(z) = 1$. In other words $(U_x : N(x)=1)$, belongs to the "reduced structure group", $Str_0(J)$ [24] of the Jordan algebra J ; the reduced structure group is defined as the invariance group of the norm N . This establishes that a point z on $\mathcal{M}$, i.e. a point z on $\mathcal{C}$ for which $N(z)=1$, is reached from a point y on $\mathcal{M}$ by the action of the reduced structure group of a Jordan algebra, modulo its automorphism group.

Hence $\mathcal{M}$ is the homogeneous space

$$\mathcal{M} = \frac{Str_0(J)}{Aut(J)} \qquad (4.20)$$

which is also a symmetric space. Aut(J) is the stability group of $\mathcal{M}$ because it leaves invariant the base point c,(which may be chosen arbitrarily on $\mathcal{M}$). This point can simúltaneously be regarded as the unit element of the Jordan algebra J.

We remark that the larger space $\mathcal{C}$, in which $\mathcal{M}$ is embedded, is the self-adjoint convex homogeneous cone of the associated Jordan algebra J. The Christoffel connection Γ_{IJK} of $\mathcal{C}$ evaluated at the base point, coincides with the structure constants of the Jordan algebra, i.e.

$$x^{I} \circ x^{J} = \delta^{II'} \delta^{JJ'} \Gamma_{I'J'K} x^{K} \qquad (4.21)$$

We leave to appendix A further discussion of self-adjoint convex homogeneous cones.

From the foregoing we see that the classification of locally symmetric spaces $\mathcal{M}$ for which the tensor T_{xyz} is covariantly constant(f1) reduces to the classification of Jordan algebras with cubic norm forms. According to Schafer [25] the possibilities are as follows :

1. $J = \mathbb{R}$, $N = a^3$, $a \in \mathbb{R}$. The base point may be chosen as c=1. This case applies to n=o, i.e. pure d=5 supergravity.

2. $J = \mathbb{R} + \Gamma$, where Γ is a simple algebra with identity $\vec{e}_1$ and quadratic form $Q(\vec{x})$, $\vec{x} \in \Gamma$, such that $Q(\vec{e}_1)=1$. The norm N is

$$\mathcal{N}(x) = a\, Q(\vec{x}) \tag{4.22}$$

with $x = (a, \vec{x})$. The base point may be chosen as $c = (1, \vec{e}_1)$. This includes the two special cases

(i) $\Gamma = \mathbb{R}$ and $Q = b^2$, $b \in \Gamma \Rightarrow \mathcal{N} = ab^2$. This is applicable for n=1,

(ii) $\Gamma = \mathbb{R} \oplus \mathbb{R}$ and $Q = bc$, $(b,c) \in \Gamma \Rightarrow \mathcal{N} = abc$ This is applicable for n=2

Notice that for these special cases the norm is completely factorized, so that the space $\mathcal{C}$, and therefore $\mathcal{M}$, is flat. For $n > 2$, $\mathcal{N}$ is still factorized into a linear and quadratic part, so that $\mathcal{M}$ is still reducible. The positive definiteness of the metric a_{IJ} of $\mathcal{C}$, which is required on physical grounds, requires that Q have "Minkowski" signature $(+, -, -, \cdots\cdots, -)$. The point $\vec{e}_1$ can be chosen as (1,0,0, ... 0). It is then obvious that the invariance group of the norm is

$$\mathrm{Str}_0(J) = SO(n-1,1) \times SO(1,1) \tag{4.23}$$

(where the SO(1,1) factor arises from the invariance under the dilatation $(a, \vec{x}) \to (e^{-2\lambda}a, e^{\lambda}\vec{x})$, $\lambda \in \mathbb{R}$), and that SO(n-1) is Aut(J). Hence

$$\mathcal{M} = \frac{SO(n-1,1)}{SO(n-1)} \times SO(1,1) \tag{4.24}$$

3. $J = J_3^{\mathbb{A}}$, $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. In this case an element x of J can be written as

$$x = \begin{bmatrix} \alpha_1 & a_3 & a_2^* \\ a_3^* & \alpha_2 & a_1 \\ a_2 & a_1^* & \alpha_3 \end{bmatrix}, \tag{4.25}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and $a_1, a_2, a_3 \in \mathbb{A}$. If the base point c is chosen as

$$c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.26}$$

the norm $\mathcal{N}$, following Freudenthal's construction [23], is

$$\mathcal{N}(x) = \alpha_1\alpha_2\alpha_3 - \alpha_1|a_1|^2 - \alpha_2|a_2|^2 - \alpha_3|a_3|^2 + 2\,\mathrm{Re}(a_1 a_2 a_3). \tag{4.27}$$

For $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$ this is just $\det(x)$. This norm is not factorizable into a linear and a quadratic factor, so that the spaces $\mathcal{M}$ are irreducible of dimension $3(1 + \dim \mathbb{A}) - 1$ and are

$J_3^{\mathbb{R}}$: $SL(3;\mathbb{R}) / SO(3)$

$J_3^{\mathbb{C}}$: $SL(3;\mathbb{C}) / SU(3)$

$J_3^{\mathbb{H}}$: $SU^*(6) / Sp(3)$

$J_3^{\mathbb{O}}$: $E_{6(-26)} / F_4$

The first three cases $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, but not the fourth, $\mathbb{O}$, are truncations of N=8 d=5 supergravity [10]. This suggests the possibility that there might exist some (unique) supergravity theory beyond N=8 with some octonionic structure, and which can be truncated to give the exceptional N=2 theory associated with $J_3^{\mathbb{O}}$. For example, one can imagine a new kind of supergravity as a theory of curved octonionic superspace [26].

5. Reduction to d=4

A large class of interesting Maxwell/Einstein models in d=4 can be obtained by reduction from d=5. In this article we shall discuss only the scalar field sector of these dimensionally reduced models, and establish the connection with the d=4 results of ref.(4). We defer to the future a full geometrical analysis of the d=4 case.

The general procedure for dimensional reduction of a supergravity theory is well known and we refer the reader to the review of Cremmer [27]. For the reduction from d=5 to d=4 the d=5 spacetime is assumed to be $M_4 \times S_1$, (where M_4 is d=4 Minkowski space) and the fünfbein $\hat{e}_{\hat{\mu}}{}^{\hat{m}}$ of d=5 ($\hat{\mu}, \hat{m}$ = 1,2,3,4,5) is taken to be

$$\hat{e}_{\hat{\mu}}{}^{\hat{m}} = \begin{bmatrix} e^{-\frac{1}{2}\sigma} e_{\mu}{}^{m} & 2 B_{\mu} e^{\sigma} \\ e_5{}^{m} = 0 & e^{\sigma} \end{bmatrix} \qquad (5.1)$$

where $e_{\mu}{}^{m}$ (μ, m =1,2,3,4) is the vierbein of d=4. The component $e_5{}^{m}$ may be chosen to vanish by a choice of local Lorentz gauge. Under the fifth component ξ^5 of d=5 general coordinate transformations, $e_{\mu}{}^{m}$ and the scalar field σ are invariant, but the vector field B_{μ} tranforms as

$$\delta_5 B_{\mu} = \tfrac{1}{2} \partial_{\mu} \xi^5 . \qquad (5.2)$$

Any other vector field in d=5, e.g. $A_{\hat{\mu}}^{I}$ breaks into a scalar, A^{I}, and a vector, A_{μ}^{I}, in d=4. However, the vector is not invariant under the ξ^5 transformation, and in fact

$$\delta_5 A_{\mu}^{I} = \xi^5{}_{,\mu} A^{I} \qquad (5.3)$$

Hence we define a new field $A_\mu^{\prime I}$ as

$$A_\mu^{\prime I} = A_\mu^I - 2A^I B_\mu \tag{5.4}$$

which is inert under ξ^5 transformations. Correspondingly we have a new field strength tensor

$$F_{\mu\nu}^{\prime I} = F_{\mu\nu}^I - 2B_{\mu\nu}A^I - 4B_{[\nu}\partial_{\mu]}A^I \tag{5.5}$$

where $B_{\mu\nu} = 2\partial_{[\mu}B_{\nu]}$ is the field strength of the vector field B_μ coming from the graviton.

It is now straightforward to derive the manifestly gauge invariant bosonic part of the d=4 action. We shall consider here only the scalar part of this action as this is sufficient for the characterization of the geometry of the theory. The contribution to the scalar action comes from three sources : (i) the Einstein action, through the dependence of $\hat{e}_\mu^{\hat{m}}$ on σ, (ii) the vector kinetic term, and (iii) the original scalar kinetic term. The Lagrangian for these three terms can be re-written as

$$e^{-1}\mathcal{L}(\text{scalars}) = -\tfrac{1}{2}\hat{a}_{IJ}(\hat{h})(\partial_\mu h^I)(\partial^\mu h^J) - \tfrac{1}{3}\hat{a}_{IJ}(\hat{h})(\partial_\mu A^I)(\partial^\mu A^J) \tag{5.6}$$

In this expression $\hat{h}^I$ is an object constructed from the $h^I(\phi^x)$ and the field σ of (5.1);

$$\hat{h}^I = e^\sigma h^I(\phi^x) \tag{5.7}$$

Observe that while h^I was constrained to satisfy $\mathcal{N}(h) = 1$, $\hat{h}^I$ satisfies

$$\mathcal{N}(\hat{h}) = e^{3\sigma} \tag{5.8}$$

because of the homogeneity of $\mathcal{N}$. The hypersurface $\sigma = 0$ is just the space $\mathcal{M}$ of the d=5 model. The hypersurfaces given by any other fixed value of σ are just rescalings of $\mathcal{M}$ and the full set of such hypersurfaces is by definition the space $\mathcal{C}$ of the d=5 model. As we have seen, a metric on this space is obtainable from the norm $\mathcal{N}$ as

$$\begin{aligned}\hat{a}_{IJ} &= -\tfrac{1}{2}\frac{\partial}{\partial\hat{h}^I}\frac{\partial}{\partial\hat{h}^J}\ln\mathcal{N}(\hat{h}) \\ &= \tfrac{3}{2}e^{-2\sigma}\,\mathring{a}_{IJ}(h)\end{aligned} \tag{5.9}$$

which is just a multiple of the metric a_{IJ} of section 3. The complex structure of the Lagrangian (5.7) becomes apparent if we introduce a new complex variable z^I as

$$z^I = \frac{\beta^{-1}A^I + ih^I}{\sqrt{2}}, \tag{5.10}$$

which (because $\mathcal{N}(\hat{h}) > 0$) satisfies

$$\mathcal{N}(\mathrm{Im}(z^I)) > 0 . \tag{5.11}$$

In terms of z^I the scalar field Lagrangian becomes

$$e^{-1}\mathcal{L}(\text{scalars}) = -g_{I\bar{J}}\,\partial_\mu z^I\,\partial^\mu\bar{z}^J \tag{5.12}$$

where

$$g_{I\bar{J}} = \hat{a}_{IJ}(z-\bar{z}) = -\tfrac{1}{2}\frac{\partial}{\partial z^I}\frac{\partial}{\partial\bar{z}^J}\ln\mathcal{N}(z-\bar{z}) \tag{5.13}$$

Thus the space parametrized by the 2(n+1) scalars of the d=4 theory, which we refer to as $\mathcal{D}$, is Kählerian with Kähler potential $F(z,\bar{z})$ $= \ln \mathcal{N}^{-1/2}(z-\bar{z})$. This is in agreement with the result of de Wit et al. [4]. These authors find a Kähler potential of the form

$$F(z,\bar z) = -\frac{1}{2}\ln\left\{ f(z) + \bar f(\bar z) - \frac{1}{2}\left(\frac{\partial f}{\partial z^I} z^I + \frac{\partial \bar f}{\partial \bar z^I}\bar z^I\right) + \frac{1}{2}\left(\frac{\partial f}{\partial z^I}\bar z^I + \frac{\partial \bar f}{\partial \bar z^I} z^I\right)\right\}, \qquad (5.14)$$

where $f(z)$ is a holomorphic function. If we take

$$f(x) = i\mathcal{N}(x) \qquad (5.15)$$

then we find that (5.14) reduces to

$$F(z,\bar z) = -\frac{1}{2}\ln \mathcal{N}(z-\bar z) + \text{constant} \qquad (5.16)$$

which yields a Kähler metric identical to that of (5.13). Observe that the Kähler potential F is real because $\mathcal{N}$ is positive as a result of the constraint (5.11) on the allowed range of the complex variable z^I.

We now specialise to the cases in which the space $\mathfrak{C}$, parametrized by the $\hat h^I$, is the cone $\mathfrak{C}(J)$ of a Jordan algebra J; i.e. we are considering the reduction to d=4 of those d=5 cases for which $\mathcal{M}$ is a symmetric space. The fields A^I can now be considered as the coordinates of an element of the Jordan algebra, so the space $\mathfrak{D}$ is

$$\mathfrak{D}(J) = J + i\,\mathfrak{C}(J), \qquad (5.17)$$

which is the "Koecher halfspace" of the Jordan algebra J [29]. These spaces are biholomorphically equivalent to bounded symmetric domains whose Bergman kernel, $B(z,\bar z)$, is just $\mathcal{N}(z-\bar z)$. We refer to Appendix B for a review of bounded symmetric domains.

The isometry group of $\mathfrak{D}(J)$ obviously contains the non-compact invariance group, $Str_0(J)$, of $\mathcal{N}$. It also contains the group of dilatations, D, of $\mathcal{N}$. The product $Str_0(J)\times D$ is the structure group of J, $Str(J)$. There are also further isometries of $\mathfrak{D}(J)$ generated by translations of z by an element of J and, less obviously, inversions of z, $z\to -z^{-1}$. The full set of transformations close into a group which goes by the various names of "special linear fractional group", "superstructure group", or "Möbius" group, and which we denote by $Mö(J)$ [30]. (The corresponding Lie algebra is the "Tits-Koecher" algebra of J [31]). The maximal compact subgroup of $Mö(J)$ is the compact form, $\widetilde{Str}(J)$, of the structure group, and this is also the stability group of $\mathfrak{D}(J)$. The latter is therefore the homogeneous space.

$$\mathfrak{D}(J) = \frac{Mö(J)}{\widetilde{Str}(J)} \qquad (5.18)$$

These spaces are all Hermitian symmetric. Those obtainable by dimensional reduction are as follows :

$$\frac{SO(n,2)}{SO(n)\times SO(2)} \times \frac{SO(2,1)}{SO(2)} \qquad (J=\Gamma)$$

$$\frac{Sp(6;\mathbb{R})}{U(3)} \qquad (J=J_3^{\mathbb{R}})$$

$$\frac{SU(3,3)}{S(U(3)\times U(3))} \qquad (J=J_3^{\mathbb{C}})$$

$$\frac{SO^*(12)}{U(6)} \qquad (J = J_3^{\mathbb{H}})$$

$$\frac{E_{7(-25)}}{E_6 \times U(1)} \qquad (J = J_3^{\mathbb{O}})$$

Observe that only those spaces associated with $J_3^{\mathbb{A}}$ are irreducible. As for the d=5 case those spaces $\mathfrak{D}(J)$ for $J = J_3^{\mathbb{R}}$, $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$ are obtainable by truncation of N=8 supergravity [6], whereas the $\mathfrak{D}(J_3^{\mathbb{O}})$ space is not.

We have observed that the spaces $\mathfrak{D}(J)$ are equivalent to bounded symmetric domains (which are again equivalent to Hermitian symmetric spaces). These spaces are in 1-1 correspondence with Hermitian Jordan triple systems [32] rather than Jordan algebras. We therefore expect a class of theories in d=4 whose scalar field manifold is classified by a subset of Hermitian Jordan triple systems, and for which there is no d=5 analogue(f+). For example the Hermitian Jordan triple of n×1 complex matrices x, y, z, with triple product $(xy^{\dagger}z + zy^{\dagger}x)$ is associated with the spaces SU(n,1)/U(n), which is the scalar field manifold of the models constructed by Luciani [3, 5]. We remark that in addition to the exceptional Jordan algebra $J_3^{\mathbb{O}}$, which yields an exceptional Hermitian Jordan triple system, there is another exceptional Hermitian Jordan triple system generated by 2 × 1 octonionic matrices, and for which the corresponding Hermitian symmetric space is

$$\frac{E_{6(-14)}}{SO(10) \times SO(2)} .$$

6. Comments

In this paper we have constructed the general N=2 Maxwell/Einstein supergravity theory in d=5, and analysed the possible geometries of the space $\mathcal{M}$ parametrized by the scalar fields. We have established a close connection between these spaces and the theory Jordan algebras. The most remarkable result of this analysis is that in addition to an infinite series of spaces , applicable for an arbitrary number, n, of Maxwell multiplets, there is a finite series of cases applicable only for special n=3 (1+ dim $\mathbb{A}$)-1, where $\mathbb{A}$ is one of the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. As we show elsewhere the $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ cases are all truncations of N=8, d=5, supergravity, whereas the octonionic case is not.

Of those d=4 theories obtained by reduction from d=5 the most interesting is that one associated with the exceptional Jordan algebra $J_3^{\mathbb{O}}$, and whose scalar manifold is $E_{7(-25)}/E_6 \times U(1)$ (this contains the other exceptional d=4 case $E_{6(-14)}/SO(10) \times SO(2)$). This raises the question of whether the exceptional N=2 Maxwell/Einstein supergravity, or its extension to include matter multiplets, may provide an adequate framework for the unification of all fundamental forces. We shall return to this question in a future article.

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Appendix A : Self-adjoint convex homogeneous cones, alias homogeneous domains of positivity

Domains of positivity were introduced by Koecher [31]. They constitute the special class of convex cones that are self-adjoint. In the case of homogeneous self-adjoint convex cones there is a one to one correspondence with formally real(35) Jordan algebras [31]. In this appendix we review the mathematics, following Koecher [29], Rothaus [36], and Vinberg [37], and demonstrate the connection with the geometrical interpretation of the N=2 Maxwell/Einstein supergravity theory of the text, viz. that the space $\mathcal{C}$ introduced in section (3) is a self-adjoint convex homogeneous cone if $\mathcal{M}$ is a symmetric space, (strictly if T_{xyz} is covariantly constant).

A domain of positivity D is an open subset of $\mathbb{R}^{n+1}$ equipped with a non-singular symmetric matrix S_{IJ}, called the characteristic of D, such that $x^I \in D$ iff $x^I S_{IJ} y^J > 0 \;\forall\, y^I \in \bar{D}$ (closure of D), $y^I \neq 0$. It follows that D is a convex cone, i.e. that $(x+y) \in D$ if $x, y \in D$ (convexity) and $e^\alpha x \in D$ if $x \in D$, $\alpha \in \mathbb{R}$ (cone). An example of such a space is the cone of positive definite Hermitian matrices [38]. Associated with D is a characteristic function $\omega(x)$ defined by

$$\omega^{-1}(x) = \int_D \exp\left(-x^I S_{IJ} y^J\right) d^{n+1}y \tag{A.1}$$

The automorphisms of D are given by the linear maps $x \to Ax$ such that $Ax \in D$ if $x \in D$. In this case

$$\omega(Ax) = (\det A)\, \omega(x) \tag{A.2}$$

In particular $A = e^\alpha \mathbb{1}$, $\alpha \in \mathbb{R}$, is always an automorphism, so that,

$$\omega(e^\alpha x) = e^{(n+1)\alpha}\, \omega(x) \;; \tag{A.3}$$

i.e. ω is a homogeneous function in x of degree $(n+1) = \dim D$. If the automorphism group of D acts transitively on D (i.e. if for any $x, y \in D$, $\exists\, A \in \mathrm{Aut}\, D$ such that $x = Ay$) then D is homogeneous. The domain D may be provided with a Riemannian metric g_{IJ} by means of the formula

$$g_{IJ} = -\partial_{IJ} \ln \omega \tag{A.4}$$

which can be shown to be positive definite. From (A.2) it is obvious that the automorphisms of D are its isometries with respect to this metric.

For any convex cone with characteristic function ω one can define a map $*: x \to x^*$, given by

$$x^{*I} = (S^{-1})^{IJ}\, \partial_J \ln \omega(x) \tag{A.5}$$

The adjoint cone is the set of points $\{x^*\}$. For the special case that the cone is a domain of positivity, D, one can show that $x^* \in D$ if $x \in D$, i.e. that D is a self-adjoint convex cone.

From now on we restrict ourselves to homogeneous self-adjoint convex cones. In this case one can show that D is a symmetric Riemannian space with non-positive curvature, and that ω^2 is a polynomial function. The correspondence with formally real Jordan algebras arises as follows :- to every formally real Jordan algebra is associated a positive cone $\mathfrak{D}(J)$ given by the set of all exponentials $e^x = \sum_0^\infty x^n/n!$ ($x \in J$ $x^n \equiv x^{n-1} \circ x \equiv x \circ x^{n-1}$; or using the U operators, $U_x x^{n-2} \equiv x^n$), which is an open subset of $\mathbb{R}^{n+1}$ (n+1 = dim J) and is a Riemannian symmetric space. At the identity of $\mathbb{1}$ of J the geodesic symmetric is the inversion $x \to x^{-1}$ and the exponential map is the ordinary algebraic exponential. Any other point p can be considered as the unit element of an isomorphic Jordan algebra $J^{(p)}$, (the "isotope" of J for which the U operator $U_x^{(p)}$ is $U_x U_p^{-1}$), so has geodesic symmetry $x \to x^{-1[p]}$ and the exponential map $e^{x[p]}$. The affine connection $\Gamma_{IJ}{}^K$ coincides with the structure constants of the Jordan algebra J [39] .

Observe the similarity of the formula (A.4) for the metric of D and the formula (3.8) for the metric a_{IJ} of $\mathfrak{C}$. The difference is that while ω is homogeneous of degree n+1, N is homogeneous of degree 3, so that we can identify N with ω only if n=2. However, for those Jordan algebras with cubic norm forms the charactristic function ω of $\mathfrak{D}(J)$ is either a power of N or a product of powers of the factors of N . For N of the form $N(x) = aQ(\vec{x})$, $x = (a, \vec{x})$ we have

$$\omega(x) = aQ^{n/2} \tag{A.6}$$

(Note that ω^2 is polynomial). For N irreducible we have

$$\omega(x) = N(x)^{\frac{n+1}{3}} \tag{A.7}$$

Again $\omega^2(x)$ (in fact $\omega(x)$)) is polynomial because for the 3x3 real Jordan algebras J , (n+1)/3 = 1 + dim $\mathbb{A}$.

For these cases the metrics g_{IJ} and a_{IJ} are not identical but they differ by unimportant scale factors. They describe isomorphic spaces. Hence we conclude, if $\mathfrak{M}$ is a symmetric space, that $\mathfrak{C}$ is a self-adjoint convex homogeneous cone. It is not clear to us whether or not $\mathfrak{C}$ is a convex cone in the general case.

Appendix B : Bounded domains and Jordan triple systems

We shall illustrate the remark of section 5 that the Koecher half spaces $\mathfrak{D}(J)$ are in 1-1 correspondence with bounded symmetric domains. We take as example that which follows from the choice $J = \mathbb{R}$. The norm function $N(x) = x^3$ and the cone $\mathfrak{C}(J)$ is $\mathbb{R}^+$. According to (5.13) $\mathfrak{D}(J)$ is simply

$$\mathfrak{D} = \mathbb{R} + i\mathbb{R}^+ \tag{B.1}$$

i.e. the upper half plane. This is the Koecher halfspace of $J = \mathbb{R}$. The line element ds^2 of $\mathfrak{D}(\mathbb{R})$ is, according to (5.12),

$$ds^2 = \frac{1}{(z-\bar{z})^2}\, dz\, d\bar{z} \quad , \operatorname{Im} z > 0 \tag{B.2}$$

The isometries of this metric are $Sl(2;\mathbb{R})$ which is the "conformal" group in one dimension, and is the closure of dilatations ($z \to e^a z$, $a \in \mathbb{R}$), translations ($z \to z+b$, $b \in \mathbb{R}$), and inversions ($z \to -z^{-1}$). We now make the following holomorphic transformation (Cayley transform)

$$z \to i(1+iz)(1-iz)^{-1} \tag{B.3}$$

The line element (B.2) is transformed into

$$ds^2 = \frac{1}{(1-z\bar{z})^2}\, dz\, d\bar{z} \quad , |z| < 1 \tag{B.4}$$

Thus the upper half plane is mapped into the interior of the unit circle. The latter is the simplest example of a bounded symmetric domain.

From (B.4) we see that, for this example

$$\mathfrak{D}(\mathbb{R}) = \frac{SU(1,1)}{U(1)} = \frac{M\ddot{o}(J=\mathbb{R})}{\mathfrak{Str}(J=\mathbb{R})} \tag{B.5}$$

which is an Hermitian symmetric space. Moreover every non-compact Hermitian symmetric space is isomorphic to a bounded symmetric domain [32, 33]. There is also a natural 1-1 correspondence between bounded symmetric domains and Hermitian Jordan triple systems.

A Jordan triple system [40] is a vector space $\mathbb{V}$ with a triple product $\{abc\} \in \mathbb{V}$ for $a,b,c \in \mathbb{V}$ such that,

$$\text{(i)} \quad \{abc\} = \{cba\}$$

$$\text{(ii)} \quad \{ab\{cde\}\} - \{cd\{abe\}\} + \{\{cda\}be\} - \{a\{dcb\}e\} = 0 \tag{B.6}$$

In the case that $\mathbb{V} = J$, $\{abc\}$ is the usual triple product, and the second defining relation is a linearization of the Jordan identity. However, the class of Jordan triple systems is larger than the class of Jordan algebras. Using the Jordan triple product one can define a bilinear operator S_{ab} acting on $\mathbb{V}$, via

$$S_{ab}\,c = \{abc\} \tag{B.7}$$

In the case that $\mathbb{V} = J$ this is the structure algebra of J. If the corresponding group contains a $U(1)$ factor, $\mathbb{V}$ is an Hermitian Jordan triple system.

The connection to bounded symmetric domains is as follows [33]. If $\mathfrak{D}$ is a domain with Bergman kernel $B(z,\bar{z})$, the Jordan triple structure is given by

$$\{abc\}_{\bar{l}} = C_{i\bar{j}k\bar{l}}\, a^i \bar{b}^j c^k, \tag{B.9}$$

where $C_{i\bar{j}k\bar{l}}$ are determined from the kernel B via

$$C_{i\bar{j}k\bar{l}} = \partial_i \partial_{\bar{j}} \partial_k \partial_{\bar{l}} \ln B(z,\bar{z}) \tag{B.10}$$

Footnotes :

(f1) Strictly speaking we study in this paper a special class of symmetric spaces ; those for which the tensor T_{xyz} of $\mathcal{M}$, to be defined later, is covariantly constant.

(f2) Observe that our convention for the scalar curvature, K, of $\mathcal{M}$ differs by a sign from our convention for the scalar curvature of spacetime. This is so that, for example, the scalar curvature of S^n (with the standard metric) is positive.

(f3) For a formulation of quantum mechanics in terms of quadratic Jordan algebras see ref.(20).

(f4) For the construction of the spaces associated with Hermitian Jordan triple systems see refs (32, 33, 34).

(f5) "Formally real" means that for any $a,b \in \mathcal{J}$, $a^2+b^2=0 \Rightarrow a=b=0$.

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