CARTAN GEOMETRY

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Abstract. In this document, Cartan geometry is reviewed and discussed explicitly for a geometry based on the Klein geometry of a de Sitter spacetime. Then, we try to generalize for a spacetime depending cosmological function, and check consistency with the Bianchi identities. This leads to postulating modified Einstein's equations.

1. Introduction

In de Sitter special relativity it is postulated that in the absence of gravitational fields, all physical laws are to be invariant under de Sitter transformations, that is the pseudo–orthogonal group $SO(4,1)$ [REF]. Of course, this is a generalization of ordinary special relativity in the sense that the vanishing cosmological constant is allowed to take a non-zero and positive value. Which value one should be using is a question to be answered outside special relativity, more precisely in an adequately adapted general relativistic theory of gravity.* Conceptually, this adaptation will find its expression in modifying the equivalence principle in such a way to be consistent with de Sitter special relativity [REF]. The equivalence principle may then be formulated as: *at every spacetime point in the presence of gravity it is possible to change reference frame, so that the laws of nature take the same form as in de Sitter special relativity.* This brings up the image of spacetime to be described locally by de Sitter spaces of possibly varying cosmological constant. Clearly, this image is not allowed by the standard formulation of general relativity in terms of a Riemannian manifold with a torsionless affine metric connection, where the local Minkowskian nature is exactly what the equivalence principle for ordinary general relativity prescribes. The question then arises, how to account for the modified equivalence principle?

An answer may be found in Cartan geometry [REF to original work]. This may be illustrated by the following commutative diagram, which is adapted from the generic one given in [2, 3].

Going from left to right

2. From Klein to Cartan geometry

2.1. Klein geometry.

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*At first sight it might seem contradictary to postulate de Sitter special relativity in the absence of gravity, while its cosmological constant is to be determined by the general relativistic theory. The resolution will be found in allowing for a cosmological function on spacetime in the presence of gravity. Note also that the special case $\Lambda = 0$ is still allowed.

2.2. **Cartan geometry.** Let us start by writing down the formal definition of a Cartan geometry as given in [2].

Definition 2.1 (Cartan geometry)**.** *A* ***Cartan geometry*** $(P, \tilde{\omega})$ ***modeled on*** $(\mathfrak{g}, \mathfrak{h})$ *consists of a principal bundle* $P(M, H)$ *together with a* $\mathfrak{g}$*-valued 1-form* $\tilde{\omega}$ *on* P*, satisfying the following properties:*

(i) *for each* $u \in P$*, the linear map* $\tilde{\omega}_u : T_u P \to \mathfrak{g}$ *is an isomorphism;*
(ii) $\tilde{\omega}(A^\dagger) = A$*, for each fundamental vector field* $A^\dagger$ *corresponding to* $A \in \mathfrak{h}$*;*
(iii) $R_h^* \tilde{\omega} = \mathrm{Ad}_{h^{-1}} \tilde{\omega}$ *for each* $h \in H$*.*

The 1-form $\tilde{\omega}$ *is called a* ***Cartan connection****.*

Since $\tilde{\omega} : TP \to \mathfrak{g}$ defines an isomorphism at any $u \in P$, it follows that $\dim M = \dim G/H$. Furthermore, the second property implies that the Cartan connection $\tilde{\omega}$ restricts to the Maurer-Cartan form ω_H on the fibers of P [2].

The **curvature** $\tilde{\Omega}$ of a Cartan connection is the $\mathfrak{g}$-valued 2-form on P defined by

$$\tilde{\Omega} = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] \ .$$

This 2-form is horizontal, in the sense that it vanishes when any of its arguments is tangent to the fiber. This is understood by noting that $\tilde{\omega}$ restricts to ω_H on the fibers for which the curvature is just the structure equation [1].

The **torsion** Θ of the Cartan connection is a $\mathfrak{g}/\mathfrak{h}$-valued 2-form on P obtained by composing the curvature form with the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, that is

By taking the exterior derivative of the curvature, one finds the **Bianchi identity**

$$d\tilde{\Omega} + [\tilde{\omega}, \tilde{\Omega}] = 0 \ .$$

2.3. **Reductive Cartan geometry.** A Cartan geometry $(P, \tilde{\omega})$ modeled on (G, H) is reductive if there is an H-module decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, i.e. a splitting of $\mathfrak{g}$ in $\mathrm{Ad}(H)$-invariant subspaces. Corresponding to this decomposition, one can break up the Cartan connection in an $\mathfrak{h}$-valued and a $\mathfrak{p}$-valued part

$$\tilde{\omega} = \omega + \theta \ ; \quad \omega \in \Omega^1(P, \mathfrak{h}), \ \theta \in \Omega^1(P, \mathfrak{p}) \ .$$

Since the splitting is reductive, the 1-form ω is an Ehresmann connection on $P(M, H)$. This follows from the definition* of a Cartan connection and because the reductive splitting implies $R_h^* \omega = \mathrm{Ad}_{h^{-1}} \omega$. The $\mathfrak{p}$-valued 1-form θ is called the **coframe field**.

In the same manner, the curvature form $\tilde{\Omega}$ can be written as

$$\tilde{\Omega} = \hat{\Omega} + \Theta \ ; \quad \hat{\Omega} \in \Omega^2(P, \mathfrak{h}), \ \Theta \in \Omega^2(P, \mathfrak{p}) \ .$$

Seperating the different contributions to $\tilde{\Omega}$ for a reductive Lie algebra $\mathfrak{g}$:

$$\tilde{\Omega} = \underbrace{d\omega + \tfrac{1}{2}[\omega, \omega]}_{\mathfrak{h}\text{-valued}} + \underbrace{d\theta + [\omega, \theta]}_{\mathfrak{p}\text{-valued}} + \underbrace{\tfrac{1}{2}[\theta, \theta]}_{\mathfrak{g}\text{-valued}}$$

In the case that $\mathfrak{g}$ is a *symmetric Lie algebra*, $[\theta, \theta] \in \Omega^2(P, \mathfrak{h})$ so that

$$\hat{\Omega} = \Omega + \tfrac{1}{2}[\theta, \theta] \ , \quad \Theta = d\theta + [\omega, \theta] \quad (\text{symmetric } \mathfrak{g}) \ .$$

*More precisely, from the second property.

Here, $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ is the curvature of the Ehresmann connection ω. A reductive Cartan geometry for which Θ vanishes is said to be **torsion-free**.

Going on in the same way, the Bianchi identity is decomposed in a $\mathfrak{h}$-valued and a $\mathfrak{p}$-valued part. This results in two Bianchi identities:

$$\begin{aligned} &d^2_\omega\theta + [\theta, \Omega] = 0 \qquad &&\text{(1st identity)} ,\\ &d_\omega\Omega = 0 &&\text{(2nd identity)} . \end{aligned}$$

Here, $d_\omega \cdot \equiv d \cdot + [\omega, \cdot]$ denotes the exterior covariant derivative.

3. de Sitter geometry

In this section, we apply to above formalism for the a Cartan geometry modeled on a de Sitter spacetime, which we will denote by a *de Sitter Cartan geometry.* In this case, the model Klein geometry is $(\mathfrak{o}(4,1)/\mathfrak{o}(3,1))$, which is well known to be symmetric:

$$\begin{aligned} [L_{ab}, L_{kl}] &= \eta_{al}L_{bk} + \eta_{bk}L_{al} - \eta_{ak}L_{bl} - \eta_{bl}L_{ak} \\ [L_{a4}, L_{kl}] &= \eta_{al}L_{4k} - \eta_{ak}L_{4l} \\ [L_{a4}, L_{b4}] &= \eta_{44}L_{ba} \end{aligned} \tag{3.1}$$

where indices run from 0 to 3. The generators L_{ab} span $\mathfrak{h} = \mathfrak{o}(3,1)$, while L_{a4} span $\mathfrak{p}$, the generators of de Sitter translations. Introducing some length scale l for the translations, that is let $P_a \equiv L_{4a}/l$, the commutation relations (3.1) can re-expressed as

$$\begin{aligned} [L_{ab}, L_{kl}] &= \eta_{al}L_{bk} + \eta_{bk}L_{al} - \eta_{ak}L_{bl} - \eta_{bl}L_{ak} \\ [L_{kl}, P_a] &= \eta_{al}P_k - \eta_{ak}P_l \\ [P_a, P_b] &= \frac{\eta_{44}}{l^2}L_{ba} \end{aligned} \tag{3.2}$$

A de Sitter Cartan geometry then consists of the bundle of orthonormal frames $OM(M, SO(3,1))$ over spacetime M and a Cartan connection of type $(\mathfrak{o}(4,1), \mathfrak{o}(3,1))$ on OM, that locally on M can be expanded as*

$$\tilde{\omega}_\mu dx^\mu = \tfrac{1}{2}\omega^{ab}{}_\mu L_{ab}dx^\mu + \theta^a{}_\mu P_a dx^\mu \ . \tag{3.3}$$

The curvature 2-form on the other hand, is expanded as†

$$\hat{R} = \tfrac{1}{4}\hat{R}^{ab}{}_{\mu\nu}L_{ab}\, dx^\mu \wedge dx^\nu \ .$$

Given the definition of $\hat{R} = d\omega + \frac{1}{2}[\omega, \omega] + \frac{1}{2}[\theta, \theta]$ and due to the commutation relations (3.2), the curvature of the Cartan connection is equal to

$$\begin{aligned} \hat{R}^{ab}{}_{\mu\nu} &= \partial_\mu\omega^{ab}{}_\nu - \partial_\nu\omega^{ab}{}_\mu + \omega^a{}_{c\mu}\omega^{cb}{}_\nu - \omega^a{}_{c\nu}\omega^{cb}{}_\mu - l^{-2}\eta_{44}(\theta^a{}_\mu\theta^b{}_\nu - \theta^a{}_\nu\theta^b{}_\mu) \\ &= R^{ab}{}_{\mu\nu} - l^{-2}\eta_{44}(\theta^a{}_\mu\theta^b{}_\nu - \theta^a{}_\nu\theta^b{}_\mu) \end{aligned}$$

where $R^{ab}{}_{\mu\nu}$ is the curvature of the spin connection. The second contribution to the curvature is due to the commutation relations of the translations. This term is a

*The same symbol is used for a Cartan connection on P and its pulled back version on M.

†The local versions of $\hat{\Omega}$ and Ω are given the symbols $\hat{R}$ and R, respectively.

crucial difference with the usual affine connection (modeled on Minkowski space). In the latter the corresponding commutator vanishes and the curvature of the metric affine connection equals the curvature of the spin connection. Note that a *flat* geometry ($\hat{R} = 0$) with respect to the Cartan connection implies a constantly curved geometry with respect to the spin connection. [Work out argument. In fact, maybe it is a better idea to first translate objects into spacetime language and then discuss this matter.]

The torsion 2-form is locally given by

$$T = \tfrac{1}{2} T^a{}_{\mu\nu} P_a dx^\mu \wedge dx^\nu \ .$$

Since the torsion equals $T = d\theta + [\omega, \theta]$, it follows that

$$T^a{}_{\mu\nu} = \partial_\mu \theta^a{}_\nu - \partial_\nu \theta^a{}_\mu + \omega^a{}_{b\mu}\theta^b{}_\nu - \omega^a{}_{b\nu}\theta^b{}_\mu \ .$$

As one would expect, the torsion equals the one associated with a metric affine connection—since the commutation relations involved are identical.

The Bianchi identities for this geometry are as follows. The first identity is locally expressed as*

$$D_\rho T^a_{\mu\nu} + D_\nu T^a_{\rho\mu} + D_\mu T^a_{\nu\rho} = \theta^b{}_\rho R^a{}_{b\mu\nu} + \theta^b{}_\nu R^a{}_{b\rho\mu} + \theta^b{}_\mu R^a{}_{b\nu\rho} \ ,$$

while the second identity is written out as

$$D_\rho R^a{}_{b\mu\nu} + D_\nu R^a{}_{b\rho\mu} + D_\mu R^a{}_{b\nu\rho} = 0 \ .$$

4. de Sitter geometry with spacetime dependent length scale

In this section we try to generalize de Sitter Cartan geometry where the length scale, which was introduced in the de Sitter translations, is allowed to vary over spacetime. The reason for this, is that this length scale fixes the local cosmological constant of the tangent de Sitter space [3]. A spacetime dependent length scale then induces a cosmological function. Of course, whether this can be made consistent with—possibly modified—Einstein's equations remains to be shown.

More precisely, the generalization is done by $l \to l(x)$ in (3.3), so that

$$\tilde{\omega}_\mu dx^\mu = \tfrac{1}{2}\omega^{ab}{}_\mu L_{ab} dx^\mu + l^{-1}(x)\theta^a{}_\mu L_{4a} dx^\mu \ . \tag{4.1}$$

Of course the commutation relations are not to be affected by this change, since de Sitter Klein geometry is still to be the model. What is affected by changing the length scale is the size—pseudo-radius—of the corresponding de Sitter space, see e.g. [3].

The resulting curvature and torsion tensors are then

$$\hat{R}^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu} - l^{-2}(x)\eta_{44}(\theta^a{}_\mu\theta^b{}_\nu - \theta^a{}_\nu\theta^b{}_\mu)$$

and†

$$l^{-1}(x)T^a{}_{\mu\nu} = l^{-1}(x)(\partial_\mu\theta^a{}_\nu - \partial_\nu\theta^a{}_\mu + \omega^a{}_{b\mu}\theta^b{}_\nu - \omega^a{}_{b\nu}\theta^b{}_\mu) \tag{4.2}$$

*The covariant derivative with respect to the spin connection ω is denoted by D.

†Note, $T = \frac{1}{2} l^{-1}(x) T^a_{\mu\nu} L_{4a} dx^\mu \wedge dx^\nu$.

$$- l^{-2}(x)(\partial_\mu l(x)\theta^a{}_\nu - \partial_\nu l(x)\theta^a{}_\mu) \ .$$

We conclude that the torsion is affected by a spacetime dependent length scale. [This does possibly make sense. Torsion is a measure for the amount the point of tangency of our model spacetime (dS) with spacetime M has been “translated", after moving along a closed path on M [3]. This amount will be influenced rather directly if this model spacetime has a changing cosmological constant!]. On the other hand, the curvature ($\hat{R}$) has changed in a rather passive way. It now shows that our spacetime is locally *flat* for a constant spin curvature (R) for the given length scale. Finally, the spin curvature is unchanged. [Again a possible interpretation is that the amount to which our frame has rotated after moving along a closed path is independent of the length scale of our model spacetime.]

5. Einstein's equations

In case we consider a de Sitter Cartan geometry with vanishing torsion, it seems we are led to the Einstein's equations, for which the left hand side would take the form [Check this!]

$$G_{\mu\nu} + \Lambda(x) g_{\mu\nu} \ . \tag{5.1}$$

References

[1] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. Number vol. 1 in Wiley Classics Library. Wiley, 1996.

[2] R.W. Sharpe. *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*. Graduate Texts in Mathematics. Springer, 1997.

[3] Derek K. Wise. MacDowell-Mansouri gravity and Cartan geometry. *Class.Quant.Grav.*, 27:155010, 2010.