THE SPECIAL ORTHOGONAL ALGEBRAS AND GROUPS

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ABSTRACT. We bundle our conventions on the special orthogonal groups and algebras and on some of its special cases, e.g. the Lorentz algebra.

1. The general case

1.1. **Definition.** The special orthogonal group $SO(p,q)$ can be defined as a set of matrices acting on $\mathbb{R}^{p,q}$ that leave invariant the metric tensor η of signature (p,q). In this defining $n = p+q$-dimensional representation, elements Λ of $SO(p,q)$ are the non-singular matrices that satisfy

$$\eta_{cd}\Lambda^c{}_a\Lambda^d{}_b = \eta_{ab} \ ; \quad \det\Lambda = 1 \ . \tag{1.1}$$

Only the component connected to the identity element will be considered. This implies that any element of the group can be obtained by exponentiation of some element of the Lie algebra $\mathfrak{so}(p,q)$, the tangent vectors at the identity. Consider therefore the elements infinitesimally close to this identity, that is $\Lambda^a{}_b = \delta^a_b + \omega^a{}_b$. Since $\Lambda^a{}_b$ satisfies (1.1), one finds for the elements $\omega_{ab} \in \mathfrak{so}(p,q)$ that $\omega_{ab} = -\omega_{ba}$.* This implies that the algebra is of dimension $n(n-1)/2$.

Let us denote by M_{ab} the set of generators of $\mathfrak{so}(p,q)$ in some representation, i.e. they act as matrices $[M_{ab}]^A{}_B$ on some d-dimensional vector space.† Any element of the special orthogonal group can be obtained by exponentiation of some element of the algebra. For a given representation D of the group this means

$$D(\omega) = \exp(\tfrac{i}{2}\omega^{ab}M_{ab}) \equiv \exp(\tfrac{i}{2}\omega\cdot M) \ , \tag{1.2}$$

where one half has been introduced because of double counting (antisymmetry in ab). The numbers ω_{ab} are the group parameters and as mentioned before, the generators are the elements tangent to the identity

$$M_{ab} = -i\left.\frac{\partial}{\partial\omega^{ab}}\right|_0 D(\omega) \ .$$

An element of $\mathfrak{so}(p,q)$ is then given by $\frac{i}{2}\omega\cdot M$ for some choice of ω.

As an example, the n-dimensional or defining representation is recovered if the generators are chosen to be $[M_{cd}]^a{}_b = -i(\eta_{db}\delta^a_c - \eta_{cb}\delta^a_d)$. A generic element of the algebra is denoted by $\frac{i}{2}\omega^{cd}[M_{cd}]^a{}_b = \omega^a{}_b$, of course recovering our previous result. Furthermore, it explains the double use of the letter ω for both group parameters and elements of the Lie algebra in the defining representation.

1.2. **Commutation relations.** Characteristic for a Lie algebra are its commutations relations. In the following the latter are calculated for an arbitrary representation of $\mathfrak{so}(p,q)$. To do this one considers two arbitrary Lorentz transformations Λ and $\tilde{\Lambda}$ and the composition $\Lambda\circ\tilde{\Lambda}\circ\Lambda^{-1}\circ\tilde{\Lambda}^{-1}$. This group element equals the identity if and only if elements of $SO(p,q)$ commute. On the other hand, since the orthogonal group is not Abelian, such a sequence generally will not give the identity. The generating vector of this composition will be the definition of the commutator. Consider

$$D(\omega)\circ D(\tilde{\omega})\circ D(\omega)^{-1}\circ D(\tilde{\omega})^{-1} = D(\omega\,\tilde{\omega}\,\omega^{-1}\,\tilde{\omega}^{-1}) \tag{1.3}$$

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*Indices are lowered or raised by contraction with η.

†The indices ab a-priori are not representation indices; rather they enumerate the generators of the algebra.

for an arbitrary representation. To evaluate this, it is sufficient to consider group elements up to second order in the group parameters, that is

$$\begin{aligned}\Lambda &= \delta^a_b + \omega^a{}_b + \tfrac{1}{2}\omega^a{}_c\omega^c{}_b \\ D(\omega) &= \mathbb{1} + \tfrac{i}{2}\omega\cdot M + \tfrac{1}{2}(\tfrac{i}{2}\omega\cdot M)^2\end{aligned}$$

from which one is able to compute

$$D(\omega)\circ D(\tilde\omega)\circ D(\omega)^{-1}\circ D(\tilde\omega)^{-1} = \mathbb{1} + (\tfrac{i}{2})^2[\omega\cdot M, \tilde\omega\cdot M] + \mathcal{O}(3)$$

and

$$\Lambda\circ\tilde\Lambda\circ\Lambda^{-1}\circ\tilde\Lambda^{-1} = \delta^a_b + \omega^a{}_c\tilde\omega^c{}_b - \tilde\omega^a{}_c\omega^c{}_b + \mathcal{O}(3)\ .$$

Combing these results together with (1.3) we get

$$(\tfrac{i}{2})^2\omega^{ab}\tilde\omega^{cd}[M_{ab}, M_{cd}] = \tfrac{i}{2}(\omega^a{}_c\tilde\omega^{cb} - \tilde\omega^a{}_c\omega^{cb})M_{ab} = \tfrac{i}{2}\omega^{ab}\tilde\omega^{cd}(\eta_{bc}M_{ad} - \eta_{ad}M_{cb})\ .$$

This result does not depend yet on any information, typical for the orthogonal group. Recall that for the latter $M_{ab} = -M_{ba}$ so that

$$\tfrac{i}{2}\omega^{ab}\tilde\omega^{cd}[M_{ab}, M_{cd}] = \tfrac{1}{2}\omega^{ab}\tilde\omega^{cd}(\eta_{bc}M_{ad} - \eta_{bd}M_{ac} - \eta_{ad}M_{cb} + \eta_{ac}M_{db})\ .$$

Since (1.3) should be true for any group parameters considered, we find the commutation relations for $\mathfrak{so}(p,q)$, namely

$$-i[M_{ab}, M_{cd}] = \eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad}\ . \tag{1.4}$$

2. The de Sitter algebra

In this section the specific case where the orthogonal algebra is denoted by $\mathfrak{so}(1,4)$ is discussed in more detail. Naturally, its commutation relations are those described in (1.4).* The de Sitter algebra is symmetric, which means that there is a reductive splitting $\mathfrak{so}(1,4) = \mathfrak{so}(1,3)\oplus\mathfrak{p}$ so that the adjoint action of the Lorentz subalgebra on $\mathfrak{p}$ is an automorphism. Explicitely this goes as follows. Denote by M_{AB} a set of basis elements of the de Sitter algebra. Let then $\mathfrak{so}(1,3) = \mathrm{span}\{M_{ab}\}$ and $\mathfrak{p} = \mathrm{span}\{M_{a4}\}$. Introducing a length scale $P_a \equiv l^{-1}M_{a4}$, the commutation relations (1.4) are rewritten as

$$\begin{aligned}-i[M_{ab}, M_{cd}] &= \eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad} \\ -i[M_{ab}, P_c] &= \eta_{ac}P_b - \eta_{bc}P_a \\ -i[P_a, P_b] &= \frac{\mathfrak{s}}{l^2}M_{ab}\ ,\end{aligned} \tag{2.1}$$

from which it becomes manifest that the algebra is symmetric.

*We let $A = 0\ldots 4$ and $a = 0\ldots 3$.