Nonlinear de Sitter-Cartan geometry

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Contents

1 Motivation

[Should contain main motivation for using nonlinear realization of Cartan connection: we want reductive splitting (gravity) + local de Sitter invariance]

The geometric structure encoded in the above *(SEC XXX)* constructed de Sitter-Cartan connection is useful, since it describes a generic spacetime with non-vanishing curvature and torsion that locally reduces to de Sitter spaces. In particular did the reductive splitting of the de Sitter-Cartan connection result in a spin connection and vierbein, mandatory for any theory of gravity. Unfortunately is the situation at this point not yet satisfactory. The shortcoming of the given setting is that it incorporates only local Lorentz invariance. This follows directly from the definition, where the $\mathfrak{g}$-valued de Sitter-Cartan connection lives on a principal H-bundle. Intuitively this can be understood by reconsidering the reduction of an Ehresmann G-connection on Q to a corresponding Cartan connection on P. The reduction process consists in singling out a section ξ of the associated G-bundle of homogeneous de Sitter spaces dS, namely

$$\xi : M \to Q[dS] : x \mapsto \xi(x) \ .$$

Upon picking out such a global section one breaks the symmetry of the de Sitter spaces at any point in spacetime M to the isotropy group of the singled out section over these points, i.e. H_ξ.

Local Lorentz invariance is necessary but not good enough, for the reason that we wish to construct theories that have the larger de Sitter group as local symmetries. At first sight we arrived at a seeming dead end: fixing a section ξ implied the existence of the required spin connection and vierbein but at the same time necessarily broke the de Sitter group down to its Lorentz subgroup. We would like to retain the existence of a spin connection and vierbein but recover the full symmetry group of de Sitter rotations. The way out of the apparent impasse is by noting that the broken symmetries upon reduction are merely hidden. This is so because the section has been chosen completely *arbitrary*, so that any section is as good as any other. Different but equivalent sections are related by elements in $\Omega^0(M, G)$ that are not in $\Omega^0(M, H_\xi)$, so that it is readily seen that those local de Sitter transformations that relate different section are just the broken symmetries.

These transformations can be incorporated in the principal Lorentz bundle by nonlinearly realizing them as local Lorentz transformations. In this manner, local de Sitter invariance is restored while retaining a well-defined reductive splitting of the Cartan connection, i.e. the presence of a spin connection and vierbein. Nonlinearly realizing the broken symmetries yields a nonlinear de Sitter-Cartan geometry. Why such a nonlinear Cartan geometry is preferred over the geometry in the original $SO(1,4)$-bundle will become clear later on in the text. As mentioned in this introduction, the crucial reason is that nonlinearly realizing the Cartan connection reconciles the presence of local de Sitter invariance together with the continuance of a well-defined spin connection and vierbein, as well as a curvature and torsion.

2 Introducing nonlinear realizations

The theory of nonlinear realizations of Lie groups on homogeneous manifolds was introduced in the context of spontaneous symmetry breaking, see [1, 2, 3]. A succinct review on the matter is also exposed in the appendices of [4]. In the following subsections we base our discussion on these works to introduce the subject of nonlinear realization.

Expand this review of literature

2.1 Reductive Lie algebra

In the following let G be a Lie group of dimension n and denote by H a m-dimensional closed subgroup. It is assumed that there is a reductive splitting on the level of the Lie algebras, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ so that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. Let S be a homogeneous space S that is symmetric under the left action of G,

$$\tau_g : S \to S : p \mapsto gp \ , \tag{2.1}$$

and for which the isotropy group of a given point o is given by $H_o \simeq H$. Then there is an isomorphism between S and G/H_o, which comes from identifying $gH_o \in G/H_o$ with $\tau_g(o) \in S$. Let the elements P_a form a basis for the $(d = n - m)$-dimensional vector space $\mathfrak{p}$.[1] In some neighborhood of the identity, a group element of G can be decomposed uniquely in the form[2]

$$g = \exp(i\xi \cdot P)\tilde{h} \ ,$$

where $\tilde{h}$ is an element of the stability subgroup H and $\xi \cdot P = \xi^a P_a$. The ξ^a parametrize the coset space G/H—at least in some neigborhood of the identity—so that it is sensible to interpret them constituting a coordinate system on the homogeneous space S. Next let g_0 be some element in G. Because a Lie group is naturally closed under its own action, the left action of G on itself may equally be written as

$$g_0 g = \exp(i\xi' \cdot P)\tilde{h}'$$

or

$$g_0 \exp(i\xi \cdot P) = \exp(i\xi' \cdot P)h' \ ; \qquad h' := \tilde{h}'\tilde{h}^{-1} \ , \tag{2.2}$$

where $\xi' = \xi'(g_0, \xi)$ and $h' = h'(g_0, \xi)$ depend on the indicated variables. In case λ is a linear representation of the subgroup H, as in

$$h : \psi \mapsto \lambda(h)\psi \ ,$$

a realization of G is constructed through the joint transformation law

$$g_0 : \ \xi \mapsto \xi' \ , \ \psi \mapsto \lambda(h')\psi \ . \tag{2.3}$$

That this gives way to a realization of G can be seen from the consecutive action of two elements g_0 and g_1. The transformation of ξ and h is given implicitely by (2.2), namely

$$g_0 \exp(i\xi \cdot P) = \exp(i\xi' \cdot P)h' \quad \text{and} \quad g_1 \exp(i\xi' \cdot P) = \exp(i\xi'' \cdot P)h'' \ .$$

[1] The index a runs from 0 to $d - 1$.

[2] This is so if a set of generators for H and the P_a are chosen orthonormal with respect to the Cartan inner product.

These transformations are then composed to yield $g_1 g_0 \exp(i\xi \cdot P) = \exp(i\xi'' \cdot P) h'' h'$. But because $g_1 \circ g_0$ also lies in G one resultantly has

$$(g_1 g_0) \exp(i\xi \cdot P) = \exp(i\xi''' \cdot P) h''' \ .$$

Combining these considerations readily leads us to conclude that $h''' = h'' h'$ and consequently that

$$\lambda(h''') = \lambda(h'')\lambda(h') \ ,$$

because λ is a representation of H. Moreover has it become clear that $\xi''' = \xi''$, from which we deduce the equality $(g_1 g_0)\xi = g_1(g_0 \xi)$. This proves that the transformation of G on ξ is a group realization as well. Remark that the composition $h'' h'$ depends on the transformation of ξ so that the realization (2.3) only is meaningful together with the transformation properties of ξ. Accordingly does the transformation of ψ depends on ξ. For these reason is the realization (2.3) called *nonlinear*. In section 2.3 we will show how any representation of G can be nonlinearly realized in the form of (2.3).

Let us for a moment consider the subcase for which the left action is given by an element of the isotropy group H, say h_0. The general transformation prescription (2.2) is rewritten in a trivial way in the form

$$h_0 \exp(i\xi \cdot P) h_0^{-1} h_0 = \exp(i\xi' \cdot P) h' \ .$$

It is a well-known result for Lie groups that the exponential map commutes with the adjoint action, by which it is meant that

$$h_0 \exp(i\xi \cdot P) h_0^{-1} = \exp(\mathrm{Ad}(h_0)(i\xi \cdot P)) \ .$$

Owing to the reductive nature of the Lie algebra the adjoint action leaves $\mathfrak{p}$ invariant so that

$$\exp(i\xi' \cdot P) = h_0 \exp(i\xi \cdot P) h_0^{-1} \quad \text{and} \quad h' = h_0 \ ,$$

and where the transformation of ξ is explicitely given by

$$h_0 : i\xi \cdot P \mapsto i\xi \cdot \mathrm{Ad}(h_0) P =: i\xi' \cdot P \ ,$$

The adjoint action being a linear automorphism, one easily understands that the coset parameters transform linearly. In addition to this we remark that h' does not depend on these parameters. Upon restriction to the subgroup H, the realization (2.3) consequently reduces to a linear representation.

There is another subclass of transformations that will be of much interest to us in the following subsections. These are the pure de Sitter translations, namely elements that are of the form $g_0 = \exp(i\alpha \cdot P)$. Acted upon by such translations, the coordinates ξ change according to

$$\exp(i\alpha \cdot P)\exp(i\xi \cdot P) = \exp(i\xi' \cdot P)h' \ . \tag{2.4}$$

To conclude this review of nonlinear realizations on reductive Klein geometries, we take a look at the case for which the transformation considered in Eq. (2.2) is infinitesimal. Thence let g_0 be an element in G that lies infinitesimally close to the identity, so that $g_0 = e + \delta g_0$ and $\delta g_0 \in \mathfrak{g}$. The coset coordinates ξ transform into

$$\xi'(g_0) = \xi'(e) + \partial_g \xi'(g)|_e \delta g_0 + \mathcal{O}((\delta g_0)^2) \simeq \xi + \delta\xi \ .$$

Similarly, it is found up to first order in δg_0 that

$$\exp(i\xi' \cdot P) = \exp(i\xi \cdot P) + \delta \exp(i\xi \cdot P) \ , \quad h' = (\tilde{h} + \delta\tilde{h})\tilde{h}^{-1} = e + \delta h \ .$$

Note that the variation on the exponential comes from the variation $\delta\xi$. Substituting these expansions into (2.2) and retaining terms up to first order, one finds the equation that determines the infinitesimal variations $\delta\xi$ and δh:

$$\exp(-i\xi \cdot P)\delta g_0 \exp(i\xi \cdot P) - \exp(-i\xi \cdot P)\delta \exp(i\xi \cdot P) = \delta h \ .$$

If the elements are pure translations, hence of the form $g_0 = e + i\epsilon \cdot P$, the transformation parameters satisfy the equation

$$\exp(-i\xi \cdot P)i\epsilon \cdot P \exp(i\xi \cdot P) - \exp(-i\xi \cdot P)\delta \exp(i\xi \cdot P) = \delta h \ . \tag{2.5}$$

2.2 Symmetric Lie algebra

In this subsection it will be assumed that the Lie algebra $\mathfrak{g}$ is not only reductive but also symmetric. This means that there is an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{h}$ is an eigenspace with eigenvalue 1, while $\mathfrak{p}$ is an eigenspace with eigenvalue -1. Group elements of H that are obtained by exponentiation of elements in $\mathfrak{h}$ are invariant under σ, while elements generated by elements of $\mathfrak{p}$ are mapped into their inverse. The automorphism directly implies a third restriction on the commutation relations of $\mathfrak{g}$, which is $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$. Then Eq. (2.2) implies

$$\sigma(g_0)\exp(-i\xi \cdot P) = \exp(-i\xi' \cdot P)\sigma(h') \ ,$$

after which $h' = \sigma(h')$ can be eliminated from Eq. (2.2). One obtains in consequence

$$g_0 \exp(2i\xi \cdot P)\sigma(g_0^{-1}) = \exp(2i\xi' \cdot P) \ . \tag{2.6}$$

Written this way, it is manifest that $g_0 : \xi \mapsto \xi'$ is a group realization and, when restricted to H, this realization becomes a linear representation.

Then let us concentrate on the variation of the coset parameters caused by infinitisimal transvections, that is to say elements of the form $g_0 = e + i\epsilon \cdot P$. Such a variation $\delta\xi$ is a solution to Eq. (2.5). By use of the involutive automorphism one eliminates δh, so that

$$\begin{aligned}
&\exp(-i\xi \cdot P)\delta \exp(i\xi \cdot P) - \exp(i\xi \cdot P)\delta \exp(-i\xi \cdot P) \\
&\qquad = \exp(-i\xi \cdot P) i\epsilon \cdot P \exp(i\xi \cdot P) + \exp(i\xi \cdot P) i\epsilon \cdot P \exp(-i\xi \cdot P) \ .
\end{aligned}$$

Using eqs. (A.1) and (A.2), this is rewritten as

$$\begin{aligned}
&\frac{1 - \exp(-i\xi \cdot P)}{i\xi \cdot P} \wedge i\delta\xi \cdot P - \frac{1 - \exp(i\xi \cdot P)}{i\xi \cdot P} \wedge i\delta\xi \cdot P \\
&\qquad = \exp(-i\xi \cdot P) \wedge i\epsilon \cdot P + \exp(i\xi \cdot P) \wedge i\epsilon \cdot P \ .
\end{aligned}$$

The expression can be solved for $i\delta\xi \cdot P$, leading to

$$i\delta\xi \cdot P = \frac{i\xi \cdot P \cosh(i\xi \cdot P)}{\sinh(i\xi \cdot P)} \wedge i\epsilon \cdot P \ . \tag{2.7}$$

This result gives the variation of coset parameters due to an infinitesimal pure translation $i\epsilon \cdot P$. Remember that this is only valid for symmetric Klein geometries. To explicitly solve for $\delta\xi^a$, one needs the specific commutation relations of the underlying Lie algebra. In the next section such a calculation will be worked for the de Sitter space.

One is equally able to find an expression that relates the infinitesimal element $h'(\xi, \epsilon) = e + \delta h$ to the corresponding transvection $g_0 = e + i\epsilon \cdot P$. The element $\delta h = \frac{i}{2}\delta h \cdot M \in \mathfrak{h}$ is given by Eq. (2.5). Invoking the identities (A.1) and (A.2) one obtains[1]

$$\tfrac{i}{2}\delta h \cdot M = \exp(-i\xi \cdot P) \wedge i\epsilon \cdot P - \frac{1 - \exp(-i\xi \cdot P)}{i\xi \cdot P} \wedge i\delta\xi \cdot P \ . \tag{2.8}$$

For symmetric Klein geometries we have already found an expression for $\delta\xi$. Let us substitute for (2.7), after which one gets sought-after relation

$$\tfrac{i}{2}\delta h \cdot M = \frac{1 - \cosh(i\xi \cdot P)}{\sinh(i\xi \cdot P)} \wedge i\epsilon \cdot P \ . \tag{2.9}$$

Upon use of an explicit set of cummutation relations, this equation renders the nonlinear realization of an infinitesimal transvection.

[1]This result is not only true for symmetric Lie algebras, but for reductive algebras also.

2.3 Construction of nonlinear realizations

In conclusion of this section on nonlinear realizations, the construction of such a realization out of a generic linear representation of G is established. Therefore let V be a representation space of G so that the action of G is linear and given by

$$\sigma(g) : V \to V \ , \quad \forall g \in G \ .$$

Then consider a section of the associated bundle of V-spaces, namely

$$\psi : M \to Q[V] = Q \times_G V \ .$$

At any given point $x \in M$ the field ψ transform according to the representation σ,

$$g : \psi(x) \mapsto \psi'(x) = \sigma(g)\psi(x) \ .$$

Next let ξ be a section of the associated bundle of homogeneous spaces $Q[G/H] = Q \times_G G/H$. The nonlinear realization of ψ is pointwise defined as

$$\bar{\psi}(x) \equiv \sigma(\exp(-i\xi \cdot P))\psi(x) \ . \tag{2.10}$$

That the field $\bar{\psi}$ indeed transforms nonlinearly in the sense of Eq. (2.3) *and only with respect to its H-indices* under the action of a generic element g_0 of G, is verified as follows:

$$\begin{aligned}
\bar{\psi}'(x) &= \sigma(\exp(-i\xi' \cdot P))\psi'(x) \\
&= \sigma(\exp(-i\xi' \cdot P)g_0)\psi(x) \\
&= \sigma(\exp(-i\xi' \cdot P)g_0 \exp(i\xi \cdot P))\bar{\psi}(x) \\
&= \sigma(h'(\xi, g_0))\bar{\psi}(x) \ .
\end{aligned}$$

It follows that a linear irreducible representation of G becomes a nonlinear and reducible representation. The price to be paid for getting irreducible H-representations is that they transform in a nonlinear way. Nonetheless, when restricted to the isotropy group H, the field (2.10) transforms according to a linear representation.

3 An example: de Sitter space

3.1 Transformation of group parameters

This subsection has to be reviewed and updated.

In this subsection, the change of the group parameters ξ^a and δh^{ab} due to infinitesimal de Sitter translations are calculated. Remember that the coordinates ξ^a are defined by the exponentiation of elements of $\mathfrak{p}$. They are also referred to as Goldstone fields, because of the resemblance of their role in the scheme of spontaneous symmetry breaking in field theory. As we have reviewed in the last section, these coordinates transform according to a nonlinear realization of the full symmetry group G. On the other hand, they transform linearly when the action is restricted to the subgroup H of unbroken symmetries. One understands that the pure translations are the set of transformations that are responsible for the nonlinear behaviour.[1]

Let us begin by recalling the de Sitter commutation relations that involve translations, i.e.[2]

$$\begin{aligned} -i[M_{ab}, P_c] &= \eta_{ac}P_b - \eta_{bc}P_a \\ -i[P_a, P_b] &= \mathfrak{s}l^{-2}M_{ab} \end{aligned} \tag{3.1}$$

with $\mathfrak{s} \equiv \eta_{44}$. The de Sitter translations were introduced as $P_a \equiv l^{-1}(x)M_{a4}$, whilst the M_{ab} span the Lorentz subalgebra $\mathfrak{h} = \mathfrak{so}(3,1)$. It is manifest that the de Sitter algebra is symmetric. In what follows we adhere to the convention $\mathfrak{s} = -1$ so that $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$.

The transformation of the coset parameters ξ^a under an infinitesimal de Sitter translation $\epsilon \cdot P$ is given by (2.7). In the parametrization used in this section, this can be rewritten as

$$\delta\xi \cdot P = \frac{i\xi \cdot P \cosh(i\xi \cdot P)}{\sinh(i\xi \cdot P)} \wedge \epsilon \cdot P \ . \tag{3.2}$$

Recall that the left hand side should be understood as a power series in the adjoint action (see also Appendix A.1). The power series of the relevant hyperbolic functions have the form[3]

$$\begin{aligned} \cosh(i\xi \cdot P) &= \sum_{n=0}^{\infty} \frac{(i\xi \cdot P)^{2n}}{(2n)!} \ , \\ \mathrm{csch}(i\xi \cdot P) &= (i\xi \cdot P)^{-1} + \sum_{n=1}^{\infty} \frac{c_{2n}}{(2n)!}(i\xi \cdot P)^{2n-1} \ . \end{aligned}$$

Invoking the identity (A.3), one is able to work out the cosinus hyperbolicus, i.e.

$$\begin{aligned} \cosh(i\xi \cdot P) \wedge \epsilon \cdot P &= \epsilon \cdot P + \sum_{n=1}^{\infty} \frac{(l^{-1}\xi)^{2n}}{(2n)!} \wedge \left(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2}\right) \\ &= \cosh(l^{-1}\xi)\left(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2}\right) + \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2} \ . \end{aligned} \tag{3.3}$$

[1] In general these elements do not form a group.

[2] An element of $SO(4,1)$ is given by $\exp(\frac{i}{2}\omega^{ab}M_{ab} + i\xi^a P_a)$.

[3] The coefficients in the power series for the hyperbolic cosecant are $c_{2n} = 2(1 - 2^{2n-1})B_{2n}$ with B_n the n-th Bernouilli number.

By equal means, the right hand side of (3.2) is found to be

$$
\begin{aligned}
&i\xi\cdot P\,\mathrm{csch}(i\xi\cdot P)\wedge\cosh(i\xi\cdot P)\wedge\epsilon\cdot P\\
&\qquad=\Big(\mathbb{1}+\sum_{n=1}^{\infty}\frac{c_{2n}}{(2n)!}(i\xi\cdot P)^{2n}\Big)\wedge\Big[\cosh(l^{-1}\xi)\Big(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\Big)+\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\Big]\\
&\qquad=\cosh(l^{-1}\xi)\Big(1+\sum_{n=1}^{\infty}\frac{c_{2n}}{(2n)!}(l^{-1}\xi)^{2n}\Big)\Big(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\Big)+\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\\
&\qquad=\cosh(l^{-1}\xi)(l^{-1}\xi)\,\mathrm{csch}(l^{-1}\xi)\Big(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\Big)+\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\\
&\qquad=\epsilon\cdot P+\frac{l^{-1}\xi\cosh(l^{-1}\xi)}{\sinh(l^{-1}\xi)}\Big(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\Big)+\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}-\epsilon\cdot P\ .
\end{aligned}
$$

The introduction of the extra $\epsilon\cdot P$ terms in the last line is just a matter of convention, which allows one to write eq. (3.2) as

$$
\delta\xi\cdot P=\epsilon\cdot P+\Big(\frac{l^{-1}\xi\cosh(l^{-1}\xi)}{\sinh(l^{-1}\xi)}-1\Big)\Big(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\Big)\ . \tag{3.4}
$$

This implies that the infinitesimal change of the coset parameters is given by

$$
\delta\xi^a=\epsilon^a+\Big(\frac{z\cosh z}{\sinh z}-1\Big)\Big(\epsilon^a-\frac{\xi^a\epsilon_b\xi^b}{\xi^2}\Big)\ , \tag{3.5}
$$

where $z=l^{-1}\xi$.

A comment on the derivation of eq. (3.4) is in place. The solution was found after use of the power series for the hyperbolic cosecant. In the case of real numbers, the series is only defined for values between $-\pi$ and $+\pi$. One could thus wonder if this convergence issue inhibits us of trusting the solution found above. Remember that eq. (3.2) can be rewritten as

$$
(i\xi\cdot P)^{-1}\sinh(i\xi\cdot P)\wedge\delta\xi\cdot P=\cosh(i\xi\cdot P)\wedge\epsilon\cdot P\ ,
$$

which reduces to

$$
z^{-1}\sinh z\Big(\delta\xi\cdot P-\frac{\xi\cdot\delta\xi\xi\cdot P}{\xi^2}\Big)+\frac{\xi\cdot\delta\xi\xi\cdot P}{\xi^2}=\cosh z\Big(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\Big)+\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\ .
$$

This result relies on the power series expansion of the hyperbolic sine, which is convergent for all values of its argument. It is readily checked that the solution (3.4) satisfies the above equation. Therefore, we may conclude that (3.4) is the right solution.

Given the de Sitter algebra, it is also possible to compute $\delta h=\frac{i}{2}\delta h^{ab}M_{ab}$ explicitly. From (2.9) it follows that the element of $\mathfrak{h}$, corresponding to an infinitesimal de Sitter translation, is a solution of

$$\tfrac{1}{2}\sinh(i\xi\cdot P)\wedge\delta h\cdot M=(\mathbb{1}-\cosh(i\xi\cdot P))\wedge\epsilon\cdot P\ . \tag{3.6}$$

The right hand side is readily found by reconsidering (3.3), implying that

$$(\mathbb{1}-\cosh(i\xi\cdot P))\wedge\epsilon\cdot P=(1-\cosh z)\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\ .$$

From the power series expansion of the hyperbolic sine,

$$\sinh(i\xi\cdot P)=\sum_{n=0}^{\infty}\frac{(i\xi\cdot P)^{2n+1}}{(2n+1)!}\ ,$$

and from (A.6), it follows that

$$\begin{aligned}\sinh(i\xi\cdot P)\wedge\delta h\cdot M&=\delta h^{ab}\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n+1)!}(\xi_aP_b-\xi_bP_a)\\&=z^{-1}\sinh z\,\delta h^{ab}(\xi_aP_b-\xi_bP_a)\\&=2z^{-1}\sinh z\,\delta h^{ab}\xi_aP_b\ .\end{aligned}$$

Putting these equations together, (3.6) is rewritten as

$$\begin{aligned}\delta h^{ab}\xi_aP_b&=\frac{z(1-\cosh z)}{\sinh z}\left(\epsilon^b-\frac{\xi^a\epsilon_a\xi^b}{\xi^2}\right)P_b\\&=(l\xi)^{-1}\frac{1-\cosh z}{\sinh z}(\epsilon^b\xi^a-\epsilon^a\xi^b)\xi_aP_b\ ,\end{aligned}$$

from which it can be concluded that the sought-after quantities are

$$\delta h=\tfrac{i}{2}\delta h^{ab}M_{ab}=\frac{i}{2l^2}\frac{\cosh z-1}{z\sinh z}(\epsilon^a\xi^b-\epsilon^b\xi^a)M_{ab}\ . \tag{3.7}$$

3.2 Nonlinear de Sitter-Cartan geometry

As motivated in the introductory section of this chapter, we will use the theory of nonlinear realizations to construct a nonlinear de Sitter-Cartan geometry. In doing so, it will become clear that local de Sitter invariance is restored while preserving the spin connection and vierbein. This treatment closely follows the original work of Stelle and West, see [5, 6]. In this discussion however, the cosmological constant of the local de Sitter spaces is not assumed to be the same at any point in spacetime. Rather, we allow for a spacetime varying *cosmological function* $\Lambda(x)$.

Remember that a de Sitter-Cartan connection on a principal H-bundle is induced from a G-connection on a principal G-bundle, by reducing the G-bundle upon choosing a section ξ in the associated bunlde of homogeneous de Sitter spaces G/H. This induced Cartan connection is to be realized nonlinearly. Accordingly let us begin by considering a principal de Sitter bundle $Q(M,G)$ over spacetime M. Additionaly a geometry is introduced by in the form of a local Ehresmann connection $A \in \Omega^1(M, \mathfrak{g})$, that is directly on spacetime. The connection is further characterized by its curvature

$$F = dA + \tfrac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g}) \ .$$

These $\mathfrak{g}$-valued differential forms can be decomposed with respect to their $\mathfrak{h}$- and $\mathfrak{p}$-valued parts, i.e.

$$A = A_{\mathfrak{h}} + A_{\mathfrak{p}} = \tfrac{i}{2} A^{ab} M_{ab} + i A^a P_a \quad \text{and} \quad F = F_{\mathfrak{h}} + F_{\mathfrak{p}} = \tfrac{i}{2} F^{ab} M_{ab} + i F^a P_a \ ,$$

where as usual $P_a = M_{a4}/l(x)$. This relation implies that A^a and F^a have dimensions of length. Due to the symmetric nature of $SO(1,4)$, one may express F^{ab} and F^a in terms of A^{ab} and A^a, which leads to the following equations:

$$F^{ab} = dA^{ab} + A^a{}_c \wedge A^{cb} + \frac{1}{l^2} A^a \wedge A^b \ , \tag{3.8a}$$

$$F^a = dA^a + A^a{}_b \wedge A^b - \frac{1}{l} dl \wedge A^a \ . \tag{3.8b}$$

Although it is not a coincidence that the above expressions remind one of the corresponding expressions for curvature and the torsion of a Cartan connection, it must be emphasized that the quantities (3.8) are by no means the curvature or torsion of some geometric object. At this point there is only a curvature F of the Ehresmann connection A in play, while for the latter torsion is not defined. Furthermore, remark that the decomposition of F (A) into $F_{\mathfrak{h}}$ $(A_{\mathfrak{h}})$ and $F_{\mathfrak{p}}$ $(A_{\mathfrak{p}})$ is not well-defined with respect to the geometry, in the sense that local gauge transformations mix up the $\mathfrak{h}$- and $\mathfrak{p}$-valued parts. Stated equivalently, A and F each transform irreducibly under G. As a result the decompositions are not respected by the symmetries of the geometry, so that it would be difficult to atribute it any physical meaning.

This observation is very important, since it is the main motivation to make use of nonlinear realizations. The symmetric splitting of A and F leads directly to the structure of a Cartan geometry on a principal Lorentz bundle. This splitting is invariant under local Lorentz transformations and the corresponding projections make up true geometric objects. We, however, want to retain the full symmetry of local de Sitter transformations *together* with the symmetric splitting of A and F. The solution to this situation is using nonlinear realizations. After we have constructed the nonlinear versions of A and F, we will come back once more to the physical reasons of why it is necessary to have both the symmetric splitting and local de Sitter invariance. *[The motivation for nonlinear realization and the role of symmetry breaking (choosing a section ξ) and (vs.?) nonlinear realization should be carefully explained.]*

Remember from our previous discussion on nonlinear realizations that, in order to construct such objects, it is necessary to single out an arbitrary section $\xi : M \to P[G/H]$ of the associated bundle of de Sitter spaces. Given a linearly tranforming quantity and such a section ξ, the general prescription to obtain a nonlinearly transforming object from the linear one is summarized in Eq. (2.10). Because we are interested in finding the nonlinear versions of A and F, it is useful to write down their transformation behaviour under the action of the de Sitter group. For a local gauge transformation $g_0 \in \Omega^0(M,G)$ of an associated vector bundle, it is well-known that this behaviour is given by[1]

$$A \mapsto g_0 A g_0^{-1} + g_0 d g_0^{-1} = \mathrm{Ad}(g_0) \cdot (A + d) \ , \tag{3.9a}$$

respectevily

$$F \mapsto g_0 F g_0^{-1} = \mathrm{Ad}(g_0) \cdot F \ . \tag{3.9b}$$

These transformation laws together with the prescription (2.10) allow us to find their nonlinear counterparts relatively straightforward.

First let us construct the nonlinear version of the G-connection A. Its nonhomogeneous way of tranforming (3.9a) leads one to define the corresponding nonlinear de Sitter connection as

$$\bar{A} := \mathrm{Ad}(\exp(-i\xi \cdot P)) \cdot (A + d) \ . \tag{3.10}$$

The resulting object is once again a $\mathfrak{g}$-valued differential form on M, for which the symmetric splitting takes the form $\bar{A} = \bar{A}_{\mathfrak{h}} + \bar{A}_{\mathfrak{p}}$.[2] The crucial difference with the splitting of the linear connection A is that the decomposition for $\bar{A}$ is invariant under the whole de Sitter group. This will be the subject of a more thorough discussion in the next section, where the importance of this fact will be further explained. Especially, it will become clear that $\bar{A}_{\mathfrak{h}}$ and $\bar{A}_{\mathfrak{p}}$ are the genuine spin connection, respectively vierbein, for a locally de Sitter invariant theory of gravity. Anticipating on this interpretation we introduce the notation $\omega := \bar{A}_{\mathfrak{h}}$ and $e := \bar{A}_{\mathfrak{p}}$.

Next we use the definition of the nonlinear connection $\bar{A}$ to explicitely calculate the spin connection ω and e in terms of ξ, $A_{\mathfrak{h}}$ and $A_{\mathfrak{p}}$. It follows from Eq. (3.10) that

$$\tfrac{i}{2}\omega^{ab} M_{ab} + i e^a P_a = \mathrm{Ad}(\exp(-i\xi \cdot P)) \Big(\tfrac{i}{2} A^{ab} M_{ab} + i A^a P_a + d \Big) \ .$$

Invoking Hadamard's formula (A.1) and the Campbell-Poincaré fundamental iden-

[1] The fibres of the principal bundle transforms with the inverse.

[2] It is worthwhile to emphasize that this denotes the splitting of $\bar{A}$, rather than the sum of the nonlinear versions of $A_{\mathfrak{h}}$ and $A_{\mathfrak{p}}$. The latter are not even defined, since they separately do not form a linear representation of the de Sitter group.

tity (A.2) the right-hand side can be rewritten as[1]

$$\exp(-i\xi\cdot P)\wedge\left(\tfrac{i}{2}A^{ab}M_{ab}+iA^aP_a\right)+\frac{1-\exp(-i\xi\cdot P)}{i\xi\cdot P}\wedge d(i\xi\cdot P)\ .$$

This expression has to be worked out and separated in two parts, one valued in the Lorentz algebra $\mathfrak{h}$ and a second taking values in the subspace of transvections $\mathfrak{p}$. That such a decomposition can be done explicitely follows from the symmetric nature of the de Sitter algebra: for any two elements X and Y in $\mathfrak{h}$ or $\mathfrak{p}$, the element $X\wedge Y$ is in $\mathfrak{h}$ or $\mathfrak{p}$. In order to carry out the calculation, one must use the results of App. A.2. Then it is found successively:

$$\begin{aligned}
\exp(-i\xi\cdot P)\wedge\tfrac{i}{2}A^{ab}M_{ab} &= \tfrac{i}{2}\Big(A^{ab}+\frac{\cosh z-1}{l^2z^2}\xi_c(\xi^bA^{ac}-\xi^aA^{bc})\Big)M_{ab}\\
&\qquad\qquad + i\big(z^{-1}\sinh z A^a{}_b\xi^b\big)P_a\ ,\\
\exp(-i\xi\cdot P)\wedge iA^aP_a &= \tfrac{i}{2}\Big(\frac{\sinh z}{l^2z}(A^a\xi^b-A^b\xi^a)\Big)M_{ab}\\
&\qquad + i\Big(A^a+(\cosh z-1)\Big(A^a-\frac{\xi^bA_b\xi^a}{\xi^2}\Big)\Big)P_a\ ,\\
\frac{1-\exp(-i\xi\cdot P)}{i\xi\cdot P}\wedge d(i\xi\cdot P) &= \tfrac{i}{2}\Big(\frac{\cosh z-1}{l^2z^2}(d\xi^a\xi^b-d\xi^b\xi^a)\Big)M_{ab}\\
&\quad + i\Big(\frac{\sinh z}{z}\Big(d\xi^a-\frac{\xi^bd\xi_b\xi^a}{\xi^2}\Big)+\frac{\xi^bd\xi_b\xi^a}{\xi^2}-\frac{dl}{l}\xi^a\Big)P_a\ .
\end{aligned}$$

Collecting these different contributions and seperating terms according to whether they are valued in $\mathfrak{h}$, respectively $\mathfrak{p}$, one gets the explicit expressions for the spin connection ω^{ab} and vierbein e^a:

$$\begin{aligned}
\omega^{ab} &= A^{ab}-\frac{\cosh z-1}{l^2z^2}\big(\xi^a(d\xi^b+A^b{}_c\xi^c)-\xi^b(d\xi^a+A^a{}_c\xi^c)\big)\\
&\qquad\qquad -\frac{\sinh z}{l^2z}(\xi^aA^b-\xi^bA^a)\ , &\text{(3.11a)}\\
e^a &= A^a+\frac{\sinh z}{z}(d\xi^a+A^a{}_b\xi^b)-\frac{dl}{l}\xi^a\\
&\qquad + (\cosh z-1)\Big(A^a-\frac{\xi^bA_b\xi^a}{\xi^2}\Big)-\Big(\frac{\sinh z}{z}-1\Big)\frac{\xi^bd\xi_b\xi^a}{\xi^2}\ . &\text{(3.11b)}
\end{aligned}$$

These expressions are almost equal to the corresponding objects found by Stelle and West [6]. The crucial difference is that we have a new term $l^{-1}dl\,\xi^a$ in the expression (3.11b) for the vierbein. This term is present in the given geometry, since we have allowed the tangent de Sitter spaces to have cosmological constants that are not necessarily equal along spacetime. More specifically, one has to take in account that the length scale defined for the elements in $\mathfrak{p}$ may vary. On the other hand, the

[1] The operation $\wedge:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}:(X,Y)\mapsto\mathrm{ad}_X(Y)$ should not be confused with the wedge product for differential forms. See also App. A.1.

results of [6] rely on the assumption that the local de Sitter spaces have the same pseudo-radius at any point in spacetime. The extra contribution is proportional to the dimensionless factor $l^{-1}dl$, which will be noticeable only if the variation is relatively vast. In case l is a constant function, one of course recovers the results of [6]. As mentioned already before, the main reason for their relevance is that ω^{ab} and e^a transform irreducibily under local de Sitter transformations, an argument that will be underpinned further in the next section.

After having constructed the nonlinear connection $\bar{A}$ and consequently the spin connection and vierbein of the geometry, we turn attention to the nonlinear curvature $\bar{F}$. From the definition of F and its transformation behaviour (3.9b) under local de Sitter transformations, it follows that the nonlinear curvature equals the curvature of the nonlinear connection, i.e.

$$\bar{F} := \mathrm{Ad}(\exp(-i\xi \cdot P)) \cdot F = d\bar{A} + \tfrac{1}{2}[\bar{A}, \bar{A}] \ . \tag{3.12}$$

This is a $\mathfrak{g}$-valued 2-form on spacetime, which we decompose once more according to $\bar{F} = \bar{F}_{\mathfrak{h}} + \bar{F}_{\mathfrak{p}}$. Due to the nonlinear nature of $\bar{F}$, the reductive split is invariant under local gauge transformations. Being similar to the discussion on the nonlinear connection $\bar{A}$, the covariant nature of the decomposition suggests that $F_{\mathfrak{h}}$ and $F_{\mathfrak{p}}$ must be considered the curvature and torsion of the given geometry. Therefore, let us make use of the suggestive notation $R := F_{\mathfrak{h}}$ and $T := F_{\mathfrak{p}}$.

Let us then look for expressions that give the curvature R and torsion T in terms of ξ, $F_{\mathfrak{h}}$ and $F_{\mathfrak{p}}$. Equivalently to the derivation outlined above in finding the spin connection and vierbein, one considers the definition for the nonlinear curvature $\bar{F}$, i.e.

$$\tfrac{i}{2}R^{ab}M_{ab} + iT^aP_a = \mathrm{Ad}(\exp(-i\xi \cdot P))\Big(\tfrac{i}{2}F^{ab}M_{ab} + iF^aP_a\Big) \ ,$$

after which the right-hand side of this equation must be written as the sum of an $\mathfrak{h}$-valued and a $\mathfrak{p}$-valued part. This calculation is to a large extend identical to the one done for $\bar{A}$ and leads to the following quantities:

$$R^{ab} = F^{ab} - \frac{\cosh z - 1}{l^2 z^2}\xi^c(\xi^a F^b{}_c - \xi^b F^a{}_c) - \frac{\sinh z}{l^2 z}(\xi^a F^b - \xi^b F^a) \ , \tag{3.13a}$$

$$T^a = \frac{\sinh z}{z}\xi^b F^a{}_b + \cosh z\, F^a + (1 - \cosh z)\frac{\xi_b F^b \xi^a}{\xi^2} \ . \tag{3.13b}$$

Note that Eq. (3.12) implies that these expressions can equally be obtained by calculating $d\bar{A} + \frac{1}{2}[\bar{A}, \bar{A}]$ directly. More precisely, decomposing Eq. (3.12) according to the reductive splitting of the de Sitter algebra, it follows that

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} + \frac{1}{l^2}e^a \wedge e^b \ ,$$

$$T^a = de^a + \omega^a{}_b \wedge e^b - \frac{1}{l}dl \wedge e^a \ ,$$

after which one can substitute for Eqs. (3.11a) and (3.11b) to obtain R^{ab} and T^a. These equations, which express the curvature and torsion in terms of the spin connection and vierbein, are the ones expected for a Cartan geometry. In this manner, we constructed a de Sitter-Cartan geometry on a principal Lorentz bundle. Because elements of $SO(1,4)$ that are not in $SO(1,3)$ are nonlinearly realized, we actually have $SO(1,4)$ invariance of the geometry, while having a well-defined spin connection and vierbein at hand. This will be studied in more detail in the next section.

To conclude this section, Table 1 summarizes the various symbols used to denote the linear and nonlinear quantities introduced above.

Table 1: Notation for the linear and nonlinear Cartan connection and curvature and their reductive decomposition.

linear	nonlinear
$A = \frac{i}{2}A^{ab}M_{ab} + iA^aP_a$	$\bar{A} = \frac{i}{2}\omega^{ab}M_{ab} + ie^aP_a$
$F = \frac{i}{2}F^{ab}M_{ab} + iF^aP_a$	$\bar{F} = \frac{i}{2}R^{ab}M_{ab} + iT^aP_a$

3.3 Transformation behaviour: linear vs. nonlinear

In the last subsection a nonlinear Cartan geometry for the de Sitter algebra has been introduced. We claimed that the nonlinear de Sitter-Cartan connection gives rise to a genuine spin connection and vierbein, while its Cartan curvature breaks up in a well-defined Lorentz curvature and torsion tensor. This led us to introduce the suggestive notation for the respective objects as it is summarized in Table 1. The reason for having to use the nonlinear fields is that their reductive decomposition is invariant under local de Sitter transformations, which should be compared with the merely local Lorentz invariance of their linear counterparts. Since we wish local de Sitter invariance to be at the heart of any theory of gravity, ω^{ab} and e^a are objects that are well-defined in light of such a principle. In the following paragraphs and at risk of being overprecise, we therefore review the transformation properties of the linear and nonlinear fields and confirm their particular behaviour under the action of the Lorentz and de Sitter groups.

From the general relation between a linear field and its nonlinear realization, see Eq. (2.10), one deduces that they belong to the same representation space. The difference between them is that the nonlinear field becomes reducible under the action of the gauge group G, even if the linear field transforms irreducibly. More precisely, the nonlinear field is acted upon only with respect to its H-indices, since elements in G that are not in the subgroup H are nonlinearly realized by elements of the latter. Naturally the same is true for the fields at interest, namely the Cartan connection and its curvature. Notwithstanding being a special case of the situation just described, let us take a closer look at the transformation rules of the linear connection and curvature, after which we will turn attention to the nonlinear ones.

Under local de Sitter transformations the connection A changes according to Eq. (3.9a). If the action is restricted to be an element of the Lorentz subgroup H, the reductive splitting $A = A_{\mathfrak{h}} + A_{\mathfrak{p}}$ is invariant. Due to the symmetric nature of $\mathfrak{so}(1,4)$ it is readily inferred that $A_{\mathfrak{h}}$ and $A_{\mathfrak{p}}$ transform according to

$$A_{\mathfrak{h}} \mapsto \mathrm{Ad}(h_0) \cdot (A_{\mathfrak{h}} + d) \quad \text{and} \quad A_{\mathfrak{p}} \mapsto \mathrm{Ad}(h_0) \cdot A_{\mathfrak{p}} \ ,$$

where $h_0 \in H$. If only local Lorentz invariance were required, these objects would be adequate as a spin connection and vierbein. Since we additionally desire local de Sitter invariance, such a decomposition will not be satisfying. This is so because the symmetric splitting is not respected under a generic G-transformation. To see this explicitely, consider an infinitesimal pure de Sitter translation $e + i\epsilon \cdot P$. To first order in the transformation parameter ϵ, one finds that

$$\delta_\epsilon A = i[\epsilon \cdot P, A] - id(\epsilon \cdot P) = i[\epsilon \cdot P, A] - id\epsilon \cdot P + \frac{dl}{l} i\epsilon \cdot P \ .$$

To find the variations of A^{ab} and A^a, one further works out the right-hand side and separates $\mathfrak{h}$- and $\mathfrak{p}$-valued parts. The variations due to the action of an infinitesimal de Sitter translation are then concluded to be

$$\delta_\epsilon A^{ab} = \frac{1}{l^2}(\epsilon^a A^b - \epsilon^b A^a) \ , \tag{3.14a}$$

$$\delta_\epsilon A^a = -d\epsilon^a - A^a{}_b \epsilon^b + \frac{dl}{l}\epsilon^a \ . \tag{3.14b}$$

These results show manifestly how A^{ab} and A^a form an irreducible multiplet for the de Sitter group.

The variations (3.14) could equally have been found by reconsidering the calculation of $\bar{A}$ in section 3.2. Remember that the definition of the nonlinear connection is given by $\mathrm{Ad}(\exp(-i\xi \cdot P)) \cdot (A + d)$. This is nothing but the transformation of A under a pure de Sitter translation with transformation parameter $\alpha = -\xi$. The calculation of $\bar{A}$ can hence be used here to find the transformations of A^{ab} and A^a under a finite pure de Sitter translation $\exp(i\alpha \cdot P)$. More precisely and according to the logic just explained, we may copy the results of Eqs. (3.11) together with the substitution $\xi \to -\alpha$. This leads to the finite transformations laws:

$$A^{ab} \mapsto A^{ab} + \frac{1 - \cosh z}{l^2 z^2}(\alpha^a(d\alpha^b + A^b{}_c\alpha^c) - \alpha^b(d\alpha^a + A^a{}_c\alpha^c)) + \frac{\sinh z}{l^2 z}(\alpha^a A^b - \alpha^b A^a) \ , \tag{3.15a}$$

$$A^a \mapsto A^a - \frac{\sinh z}{z}(d\alpha^a + A^a{}_b \alpha^b) + \frac{dl}{l}\alpha^a$$
$$+ (\cosh z - 1)\left(A^a - \frac{\alpha^b A_b \alpha^a}{\alpha^2}\right) + \left(\frac{\sinh z}{z} - 1\right)\frac{\alpha^b d\alpha_b \alpha^a}{\alpha^2} \ . \qquad (3.15b)$$

The infinitesimal variations (3.14) are recovered straightforwardly after taking the limit $\alpha \to \epsilon$.

The discussion can be extended to the linear curvature F in an evident way. This object transforms in a covariant way under gauge transformations, as written down in Eq. (3.9b). In case we only consider elements h_0 that belong to the subgroup of Lorentz rotations, the reductive splitting $F = F_{\mathfrak{h}} + F_{\mathfrak{p}}$ is invariant and both projections rotate independently according to

$$F_{\mathfrak{h}} \mapsto \mathrm{Ad}(h_0) \cdot F_{\mathfrak{h}} \quad \text{and} \quad F_{\mathfrak{p}} \mapsto \mathrm{Ad}(h_0) \cdot F_{\mathfrak{p}} \ .$$

On the other hand, this reducible behaviour is not present if the action is governed by a generic element of the de Sitter group. To see this in an explicit way, let us have a look at the transformations of $F_{\mathfrak{h}}$ and $F_{\mathfrak{p}}$ under a pure de Sitter translation $\exp(i\alpha \cdot P)$. Again it is possible to recycle the calculation for the nonlinear $\bar{F}$. Indeed, the transformed linear curvature $\mathrm{Ad}(\exp(\alpha \cdot P)) \cdot F$ is equal to the nonlinear $\bar{F}$ with ξ replaced by $-\alpha$, which can be seen from Eq. (3.12). The transformation of the $\mathfrak{h}$- and $\mathfrak{p}$-valued parts of F are then given by considering this replacement in the Eqs. (3.13), yielding

$$F^{ab} \mapsto F^{ab} - \frac{\cosh z - 1}{l^2 z^2}\alpha^c(\alpha^a F^b{}_c - \alpha^b F^a{}_c) + \frac{\sinh z}{l^2 z}(\alpha^a F^b - \alpha^b F^a) \ , \qquad (3.16a)$$
$$F^a \mapsto -\frac{\sinh z}{z}\alpha^b F^a{}_b + \cosh z\, F^a + (1 - \cosh z)\frac{\alpha_b F^b \alpha^a}{\alpha^2} \ . \qquad (3.16b)$$

To be complete, the infinitesimal variations of $F_{\mathfrak{h}}$ and $F_{\mathfrak{p}}$ are also written down. These are obtained from the finite transformations after taking $\alpha \to \epsilon$ and retaining terms up to first order. One directly concludes:

$$\delta_\epsilon R^{ab}{}_{\mu\nu} = \frac{1}{l^2}(\epsilon^a T^b{}_{\mu\nu} - \epsilon^b T^a{}_{\mu\nu}) \ , \qquad (3.17a)$$
$$\delta_\epsilon T^a{}_{\mu\nu} = -\epsilon^b R^a{}_{b\mu\nu} \ . \qquad (3.17b)$$

Consequent to the way in which the reductive projections of A and F transform under the de Sitter group, they cannot be given well-defined meaning consistent with a principle of local de Sitter invariance. Repeatedly mentioned in this text, this is the place where the theory of nonlinear realizations shows its usefulness. As we will make plain in the following paragraphs, the nonlinear fields $\bar{A}$ and $\bar{F}$ possess a reductive splitting that is invariant under local de Sitter transformations, and which allows us to pin down the real geometric objects that can be used in a theory of gravity that is locally de Sitter invariant.

Being a generic property of nonlinear realizations, the nonlinear connection $\bar{A}$ and curvature $\bar{F}$ belong to the same representation space as A and F, respectivily. Notwithstanding this similarity in the way of transforming, the nonlinear fields are reducible, whereas their linear versions obviously are not. This is so because any element of the de Sitter group G acting on the nonlinear fields is realized through an element of its subgroup H of Lorentz transformations. As a consequence, the nonlinear connection and curvature are acted upon only through their H-components, even if the element considered belongs to the enclosing de Sitter group. To verify the transformation behavior of $\bar{A}$, note first that its definition (3.10) indicates that $\mathrm{Ad}(e)(A+d) = \mathrm{Ad}(\exp(i\xi \cdot P))(\bar{A}+d)$. It follows that under the action of an element g_0 of the de Sitter group:

$$\begin{aligned}\bar{A} \mapsto\ & \mathrm{Ad}(\exp(-i\xi' \cdot P))\mathrm{Ad}(g_0)(A+d) \\ &= \mathrm{Ad}(\exp(-i\xi' \cdot P)g_0)\mathrm{Ad}(\exp(i\xi' \cdot P))(\bar{A}+d) \\ &= \mathrm{Ad}(h'(\xi, g_0))(\bar{A}+d)\ .\end{aligned}$$

This reconfirms the way a nonlinear field transforms, where a generic de Sitter transformation is realized by a Lorentz rotation. In particular do we conclude that the $\mathfrak{g}$-valued 1-form $\bar{A}$ behaves as a Cartan connection on a principal Lorentz bundle.[1] If the element g_0 belongs to the Lorentz group H, the transformation becomes linear.[2] Because any element of the de Sitter group is realized as a Lorentz transformation, the reductive splitting $\bar{A} = \omega + e$ is preserved under local de Sitter transformations. Indeed reconsidering the transformation of $\bar{A}$ explicitely for its reductive decompositions, namely

$$\omega + e \mapsto \mathrm{Ad}(h'(\xi, g_0)) \cdot (\omega + e + d)\ ,$$

it directly follows that

$$\omega \mapsto \mathrm{Ad}(h'(\xi, g_0)) \cdot (\omega + d) \quad \text{and} \quad e \mapsto \mathrm{Ad}(h'(\xi, g_0)) \cdot e\ , \tag{3.18}$$

as a result of the symmetric nature of $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$. It is manifest that ω and e do not mix under local de Sitter transformations. Note that ω is an $\mathfrak{h}$-valued spin connection, while e is a $\mathfrak{p}$-valued 1-form that transforms covariantly. In case the gauge transfromations are restricted to be elements of the Lorentz subgroup, the nonlinear fields transform identical to their linear counterparts.

These conclusions are equally drawn for the $\mathfrak{g}$-curvature $\bar{F}$ and its projections R and T. Both 2-forms transform covariantly in the adjoint representation of the

[1][*It seems very plausible that it is (the pull-back of) a Cartan connection on $P(M,H)$ Check this: see the base definition of the latter as given in [7]].*

[2]Note that a Cartan connection is defined on a principal H-bundle, and that is only demanded that the connection transforms in a certain way under the action of H gauge transformations. Whether the elements of H form a linear or nonlinear realization is not relevant.

group, and do not mix up in this process. Therefore it is clear that R and T are true geometric objects, for the splitting is a gauge independent construction. They are referred to as the curvature and torsion of the geometry.

3.4 Interpretation of the vielbein $\bar{e}$

Stelle and West [5, 6] claim that the vierbein $\bar{e}$ is a smooth mapping between the tangent space to spacetime at any $p \in M$ and the tangent space to the internal de Sitter space at $\xi(p)$. Unfortunately, a concrete argument did not seem to be included following this statement. Furthermore, under local H gauge transformations the vierbein $\bar{e}$ transforms as a vector with an element $h \in H_o$, as can be seen from (3.18). This indicates that its $SO(3,1)$-indices belong to the tangent space at the origin ($\xi = 0$) of dS. To verify its transformation behavior under local G transformations, let us explicitly reconsider its construction.

The vierbein is defined as the $\mathfrak{p}$-valued part of the Cartan connection $\bar{A} \in \Omega^1(M, \mathfrak{g})$ on $P(M,H)$. To give this statement a precise notation, we consider the natural projection $\pi : G \to G/H_o : g \mapsto gH_o$. The differential of this mapping is a projection of $T_eG = \mathfrak{g}$ onto $\mathfrak{p} \simeq T_o dS$. The vierbein is obtained from the connection by invoking this projection, i.e. $\bar{e} = \pi_* \bar{A}$. This shows clearly that the vierbein is a 1-form on M with values in $T_o dS$. Nonetheless, let us also concentrate on the definition of $\bar{A}$ itself to understand what happens with a tangent vector to spacetime under the action of $\bar{e}$, before it ends up in $T_o dS$. The definition was given in Eq. (3.10), which we rewrite here for $g = \exp(-\xi \cdot P)$, i.e.

$$\bar{A} \equiv \mathrm{Ad}(g) \cdot A + (g^{-1})^* \theta \ .$$

It should be understood that the adjoint action acts on the algebra $\mathfrak{g}$, that θ is the Maurer-Cartan form on G and that g^* is the pullback that comes from the mapping $g : M \to G : p \mapsto g$. Consider next a vector $X \in T_p M$. One then finds,

$$\begin{aligned} \bar{A}(X) &= \mathrm{Ad}(g) \cdot A(X) + \theta(g_*^{-1} X) \\ &= L_{g*}\Big(R_{g^{-1}*} \cdot A(X) + g_*^{-1} X\Big) \ . \end{aligned}$$

Denote by $X^\star$ the left invariant vector field on G so that

$$X^\star_{g^{-1}} = R_{g^{-1}*} \cdot A(X) + g_*^{-1} X \ .$$

It follows directly that $\bar{A}(X) = X^\star_e$. Since $\pi \circ L_g = \tau_g \circ \pi$, one also has

$$\bar{e}(X) = \pi_* L_{g*} X^\star_{g^{-1}} = \tau_{g*} \pi_* X^\star_{g^{-1}} \ .$$

Recall that $g = \exp(-\xi \cdot P)$ so that $g^{-1} = \exp(\xi \cdot P)$. This implies that $\pi_* X^\star_{g^{-1}} \in T_\xi dS$, since $\exp(\xi \cdot P) o = \xi \in dS$. The element $\bar{e}(X) \in T_o dS$ is the parallel transported vector of $\pi_* X^\star_{g^{-1}}$, with respect to the canonical connection on G/H_o (See Ch. X in [8]). Therefore, it is understandable that one may interpret $\bar{e}$ to be a mapping from the tangent space to M at p onto the tangent space to dS at ξ, confirming the interpretation given by Stelle and West.

3.5 Discussion

To conclude let us retrace our steps and try to understand what has been going on. We started by introducing a G-connection on a principal G-bundle $P(M, G)$. This contains information about a geometry for which the internal symmetry group is G. Since one is interested in describing a spacetime, whose local geometry is invariant under the action of the de Sitter group, this seems a good starting point. However, one does not have a canonical—i.e. consistent with the geometry—spin connection and vielbein. This is a crucial shortcoming, as it will not be possible to relate the local geometry of the gauge field to the geometry of spacetime (no soldering). By means of a section ξ, which takes its values in the associated bundle $P \times_G G/H$, the principle bundle $P(M, G)$ is reduced to a bundle $P(M, H)$.[1] Choosing a section breaks the symmetry from G to H. As shown in the previous section, ξ can be used to construct a Cartan connection $\bar{A}$ on $P(M, H)$ from the Ehresmann connection A on $P(M, G)$. This is shown schematically in the following diagram:

$$\begin{tikzcd}
 & \omega & & & \bar{\omega} \\
A \arrow[ur, "\mathfrak{h}"] \arrow[dr, "\mathfrak{p}"'] \arrow[ddd] & & & \bar{A} \arrow[ur, "\mathfrak{h}"] \arrow[dr, "\mathfrak{p}"'] \arrow[ddd] & \\
 & e & \arrow[r, "\xi"] & {} & \bar{e} \\
 & R = d\omega + \frac{1}{2}[\omega,\omega] + \frac{1}{2}[e,e] & & & \bar{R} = d\bar{\omega} + \frac{1}{2}[\bar{\omega},\bar{\omega}] + \frac{1}{2}[\bar{e},\bar{e}] \\
F \arrow[ur, "\mathfrak{h}"] \arrow[dr, "\mathfrak{p}"'] & & & \bar{F} \arrow[ur, "\mathfrak{h}"] \arrow[dr, "\mathfrak{p}"'] & \\
 & T = de + [\omega, e] & & & \bar{T} = d\bar{e} + [\bar{\omega}, \bar{e}]
\end{tikzcd}$$

The broken symmetries act through a nonlinear realization with the elements $h'(\xi, g_0)$, and merely change the point of tangency between the local de Sitter fibres and spacetime. On the other hand, the unbroken symmetries (H) leave the point of tangency fixed and act through a linear representation. Note that the Cartan connection gives rise to a well defined spin connection and vierbein, i.e. they do *not* form an irreducible multiplet under the action of G. Due to the existence of a vierbein $\bar{e}$, spacetime is soldered to the de Sitter fibres and one is able to pull back all geometric information onto the tangent bundle of spacetime—the arena in which takes place gravity. Crucially, one had to make the realization nonlinear to have

[1] For an enlightning proof, see [8].

soldering.

A Nested commutators

A.1 Notation

For any two elements X and Y of a Lie algebra we define

$$X \wedge Y \equiv \mathrm{ad}_X(Y) = [X, Y]$$

and consequently

$$X^k \wedge Y \equiv \mathrm{ad}_X^k(Y) = [X, [X, \ldots [X, Y] \ldots]] \ .$$

This can be extended to arbitrary functions, where a function is considered a power series in X, that is

$$f(X) \wedge Y = \sum_k c_k X^k \wedge Y \ .$$

Consider a second function $g(X) = \sum_l d_l X^l$. One obtains

$$g(X) \wedge f(X) \wedge X = \sum_{kl} c_k d_l \mathrm{ad}_X^l(\mathrm{ad}_X^k(Y)) = \sum_{kl} c_k d_l X^{k+l} \wedge Y = g(X) f(X) \wedge Y \ ,$$

where we used the linearity of the adjoint action. From this result it follows that the equation $f(X) \wedge Y = Z$ can be solved for $Y = f(X)^{-1} \wedge Z$. Note that the inverse function also is supposed to be expressed as a power series.

To conclude we write down two identities, using the introduced notation. The first is Hadamard's formula

$$\exp(X) Y \exp(-X) = \exp(X) \wedge Y \ , \tag{A.1}$$

the other is the Campbell-Poincaré fundamental identity,

$$\exp(-X) \delta \exp(X) = \frac{1 - \exp(-X)}{X} \wedge \delta X \ . \tag{A.2}$$

A.2 de Sitter algebra: some results

In this subsection, we compute some intermediary results that are used troughout the text. The commutatation relations considered are those given by (3.1), for the convention $\mathfrak{s} = -1$.

The first identity to be verified is

$$(i\xi \cdot P)^{2n} \wedge \epsilon \cdot P = z^{2n} \left(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2} \right) \ ; \quad n \geqslant 1 \ . \tag{A.3}$$

Therefore, we compute the sequence

$$i\xi \cdot P \wedge \epsilon \cdot P = i\xi^a \epsilon^b [P_a, P_b] = l^{-2} \xi^a \epsilon^b M_{ab} \ ;$$

$$
\begin{aligned}
(i\xi\cdot P)^2\wedge\epsilon\cdot P &= i\xi^c P_c\wedge l^{-2}\xi^a\epsilon^b M_{ab}\\
&= -il^{-2}\xi^a\epsilon^b\xi^c[M_{ab},P_c]\\
&= l^{-2}\xi^a\epsilon^b\xi^c(\eta_{ac}P_b-\eta_{bc}P_a)\\
&= l^{-2}\xi^2\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\ ;\\
(i\xi\cdot P)^4\wedge\epsilon\cdot P &= l^{-2}\xi^2(i\xi\cdot P)^2\wedge\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\\
&= l^{-2}\xi^2(i\xi\cdot P)^2\wedge\epsilon\cdot P\\
&= (l^{-2}\xi^2)^2\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\ ;\\
&\ \ \vdots\\
(i\xi\cdot P)^{2n}\wedge\epsilon\cdot P &= (l^{-2}\xi^2)^n\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\ .
\end{aligned}
$$

The identity follows by letting $z\equiv l^{-1}(\xi^a\xi_a)^{1/2}$.

From (A.3) it follows that

$$
\begin{aligned}
(i\xi\cdot P)^{2n+1}\wedge\epsilon\cdot P &= (i\xi\cdot P)\wedge z^{2n}\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\\
&= l^{-2}z^{2n}\xi^a\epsilon^b M_{ab}\ ,
\end{aligned}
$$

hence, another useful identity is given by

$$
(i\xi\cdot P)^{2n+1}\wedge\epsilon\cdot P=\tfrac{1}{2}l^{-2}z^{2n}(\xi^a\epsilon^b-\xi^b\epsilon^a)M_{ab}\ ;\quad n\geqslant 0\ . \tag{A.4}
$$

Finally, the following two identities are derived[1]

$$
\begin{aligned}
(i\xi\cdot P)^{2n}\wedge\delta h\cdot M &= \delta h^{ab}l^{-2}z^{2n-2}\xi^c(\xi_b M_{ac}-\xi_a M_{bc})\ ;\quad n\geqslant 1\ , && \text{(A.5)}\\
(i\xi\cdot P)^{2n+1}\wedge\delta h\cdot M &= \delta h^{ab}z^{2n}(\xi_a P_b-\xi_b P_a)\ ;\quad n\geqslant 0\ . && \text{(A.6)}
\end{aligned}
$$

To verify them consider the following series of equations.

$$
\begin{aligned}
(i\xi\cdot P)\wedge\delta h\cdot M &= \delta h^{ab}\xi^c(-i)[M_{ab},P_c]=\delta h^{ab}(\xi_a P_b-\xi_b P_a)\ ;\\
(i\xi\cdot P)^2\wedge\delta h\cdot M &= 2\delta h^{ab}i\xi\cdot P\wedge\xi_a P_b\\
&= 2\delta h^{ab}\xi_a\xi^c(-i)[P_b,P_c]\\
&= 2\delta h^{ab}l^{-2}\xi_a\xi^c M_{cb}\\
&= \delta h^{ab}l^{-2}\xi^c(\xi_b M_{ac}-\xi_a M_{bc})\ ;\\
(i\xi\cdot P)^4\wedge\delta h\cdot M &= 2\delta h^{ab}l^{-2}\xi_b\xi^c(i\xi\cdot P)^2\wedge M_{ac}\\
&= 2\delta h^{ab}l^{-2}\xi_b\xi^c l^{-2}\xi^d(\xi_c M_{ad}-\xi_a M_{cd})\\
&= 2\delta h^{ab}l^{-2}z^2\xi^d\xi_b M_{ad}
\end{aligned}
$$

[1]Note that $\delta h\cdot M=\delta h^{ab}M_{ab}$.

$$
\begin{aligned}
&= \delta h^{ab} l^{-2} z^2 \xi^c (\xi_b M_{ac} - \xi_a M_{bc}) \\
&\vdots \\
(i\xi \cdot P)^{2n} \wedge \delta h \cdot M &= \delta h^{ab} l^{-2} z^{2n-2} \xi^c (\xi_b M_{ac} - \xi_a M_{bc}) \\
(i\xi \cdot P)^{2n+1} \wedge \delta h \cdot M &= 2\delta h^{ab} l^{-2} z^{2n-2} \xi^c \xi_b (-i)[M_{ac}, P_d] \\
&= 2\delta h^{ab} l^{-2} z^{2n-2} (\xi_b \xi_a \xi \cdot P - \xi^2 \xi_b P_a) \\
&= \delta h^{ab} z^{2n} (\xi_a P_b - \xi_b P_a) \ .
\end{aligned}
$$

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