Nonlinear realizations

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1 Introduction

In the quest for a theory of gravity whose local kinematical structure is invariant under the de Sitter group, we try to find an appropriate geometrical language. Indeed, if one is of the opinion that kinematics are described by geometry and dynamics by Lagrangians, *a priori* it is the former that may have to be given another underlying algebraic structure. In more precise terms, the desired geometry describes a manifold—spacetime—that locally reduces to a de Sitter space whose cosmological constant is allowed to vary along the manifold. Loosely stated, geometry is the description of an Ehresmann connection on a principal G bundle.[*] The structure group G contains the symmetry transformations of the internal (associated) fibres. Consequently, for the case at hand it makes sense to let $G = SO(1,4)$ and to consider the associated bundle of homogeneous de Sitter spaces, i.e. $E = P \times_G G/H$. A nonvanishing curvature of the gauge field reflects the nonhomogeneity of the underlying base manifold, i.e. spacetime. However, this is not the end of the story. Gravitational theories live in the tangent structure of spacetime M, which naturally is lacking any information about the geometry of the principal bundle. The structure that is needed to be able to see the geometry in the tangent bundle is a vielbein. Such a well defined field would solder the internal de Sitter fibres to spacetime, but is missing in the general case of Ehresmann connections. This makes it clear that these type of connections are not appropriate for the kind of geometry to be described in theories of gravity. Rather, one has to specify for *Cartan connections*. These objects are $\mathfrak{g}$-valued connections on a principal bundle $P(M,H)$ such that the dimension of $\mathfrak{g}/\mathfrak{h}$ equals the dimension of M. Given this equality in dimension, one understands that a vielbein directly is at hand.

Nonetheless, Cartans connections are Ehresmann connections—but not *vice versa*, so that one should be able to reconstruct a useful Cartan connection from the right Ehresmann connection. This is the place where symmetry breaking and nonlinear realizations come into play.[†] It is a standard result in differential geometry that sections ξ of $P \times_G G/H$ are in one–to–one correspondence to reductions $\imath$ of $P(M,G)$ to $P(M,H)$, schematically expressed as[‡]

$$\begin{tikzcd}
P(M,H) \arrow[r, "\imath"] \arrow[dr] & P(M,G) \arrow[r] \arrow[d] & P \times_G G/H \arrow[dl] \\
& M \arrow[ur, dashed, bend right, "\sigma"'] &
\end{tikzcd}$$

The reduction can be understood as a symmetry breaking process. At any point

[*]For a nuanced discussion on this statement, see the introduction to [1].
[†]For a concise review on the role of symmetry breaking in gravity, see [2].
[‡]A proof of this statement is given in [3].

p of spacetime, the section singles out some point of the homogeneous de Sitter spaces. In this manner the symmetry group G of the homogeneous space is broken to the isotropy group H_ξ of the point $\xi(p)$. The broken symmetries translate the section $\xi(p) \mapsto g(p)\xi(p)$ for which the isotropy groups are related by the adjoint action, i.e. $H_{g\xi} = \mathrm{Ad}(g)\, H_\xi$. Going along the same line, a $\mathfrak{g}$-valued Ehresmann connection gives way to a $\mathfrak{g}$-valued Cartan connection on the reduced bundle [1]. If the algebra $\mathfrak{g}$ is reductive, the broken directions $\mathfrak{g}/\mathfrak{h}_\xi \simeq T_\xi dS$ can identified with the tangent structure at p, effectively introducing a vielbein. Since the algebra is reductive, the identification remains consistent under local gauge transformations. In the remainder of this text, we will explicitly construct such a Cartan geometry on a principal $SO(3,1)$ bundle starting from an Ehresmann connection on a principal $SO(4,1)$ bundle. To this purpose we use the techniques of [4, 5], however slightly generalizing for a spacetime dependent de Sitter length scale.

2 Nonlinear realizations

The theory of nonlinear realizations of Lie groups on homogeneous manifolds was introduced in the context of spontaneous symmetry breaking [6, 7, 8] and has been applied in the context of supersymmetry and supergravity [citations].

2.1 Nonlinear realizations

Consider a Lie group G of dimension n for which H is a d-dimensional closed subgroup. It is assumed that there is a reductive splitting on the level of the Lie algebras, that is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ so that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. Given a homogeneous space S that is symmetric under the left action of G,

$$\tau_g : S \to S : p \mapsto gp \ , \tag{2.1}$$

and for which the isotropy group of a given point o is given by $H_o \simeq H$, there is an isomorphism between $S \simeq G/H_o$ due to $g \in G \leftrightarrow \tau_g(o) \in S$.

Let $\mathfrak{p} = \mathrm{span}\{P_a\}$ $(a = 1 \ldots n-d)$. Within some neighborhood of the identity, a group element of G can be represented (uniquely(?)) in the form

$$g = \exp(\xi \cdot P)\tilde{h} \ ,$$

where $\tilde{h}$ is an element of the stability subgroup H and $\xi \cdot P = \xi^a P_a$. The ξ^a parametrize the coset space G/H—at least in some neigborhood of the identity—and can be considered a coordinate system of the manifold M. It follows that the left action of G on itself can be written as

$$g_0 g = \exp(\xi' \cdot P)\tilde{h}'$$

or

$$g_0 \exp(\xi \cdot P) = \exp(\xi' \cdot P) h' \ ; \qquad h' := \tilde{h}' \tilde{h}^{-1} \ , \tag{2.2}$$

where $\xi' = \xi'(g_0, \xi)$ and $h' = h'(g_0, \xi)$ depend on the indicated variables. Given a linear representation of H,

$$h : \psi \mapsto D(h)\psi \ ,$$

a nonlinear realization of G can be constructed through

$$g_0 : \ \xi \mapsto \xi' \ , \ \psi \mapsto D(h')\psi \ . \tag{2.3}$$

To show that it is a realization let us consider

$$\begin{aligned} g_0 \exp(\xi \cdot P) &= \exp(\xi' \cdot P) h' \\ g_1 \exp(\xi' \cdot P) &= \exp(\xi'' \cdot P) h'' \\ (g_1 g_0) \exp(\xi \cdot P) &= \exp(\xi''' \cdot P) h''' \end{aligned}$$

Since the first two equations also imply that $g_1 g_0 \exp(\xi \cdot P) = \exp(\xi'' \cdot P) h'' h'$, we see that $h''' = h'' h'$ and hence

$$D(h''') = D(h'') D(h') \ ,$$

because D is a representation of H. Furthermore, one concludes that $\xi''' = \xi''$ so that $(g_1 g_0)\xi = g_1(g_0 \xi)$, which implies that the transformation of G on ξ can also be considered a group realization. Remark that the composition $h'' h'$ depends on the transformation of ξ so that the realization (2.3) only is meaningful together with the transformation properties of ξ. The latter transform in a nonlinear way under the G action. Therefore, the realization on ψ is also said to be nonlinear and the $D(h')$ constitute a nonlinear realization of G.

Consider the case for which $g_0 = h_0$ is an element of the isotropy subgroup H. From the general action of G as given in (2.2) one obtains

$$h_0 \exp(\xi \cdot P) h_0^{-1} h_0 = \exp(\xi' \cdot P) h' \ .$$

On the other hand, it is well-known that the derivative of h_0 at the identity of G/H is a linear automorphism given by the adjoint representation,*

$$h_{0*} : \xi \cdot P \mapsto \xi \cdot \mathrm{Ad}(h_0) P =: \xi' \cdot P \ ,$$

so that $h_0 : \xi \mapsto \xi'$ is a linear transformation. For matrix groups this amounts to conjugation so that we also have

$$\exp(\xi' \cdot P) = h_0 \exp(\xi \cdot P) h_0^{-1} \quad \text{and} \quad h' = h_0 \ .$$

*This result is valid only for reductive algebras, the ones considered here.

It is manifest that h' does not depend on ξ. Consequently, for these elements the realization (2.3) is a linear representation. One concludes that the nonlinear action of G reduces to a linear action when restricted to the isotropy subgroup H.

On the other hand, consider the case where the elements are of the form $g_0 = \exp(\xi_0 \cdot P)$. Then (2.2) becomes

$$\exp(\xi_0 \cdot P) \exp(\xi \cdot P) = \exp(\xi' \cdot P) h' \ . \tag{2.4}$$

2.2 Infinitesimal transformations

Let us focus on the case where an element of G lies infinitesimally close to the identity, namely $g_0 = e + \delta g_0$ with $\delta g_0 \in \mathfrak{g}$. Then to first order in δg_0 we have

$$\exp(\xi' \cdot P) = \exp(\xi \cdot P) + \delta \exp(\xi \cdot P) \ , \quad h' = (\tilde{h} + \delta\tilde{h})\tilde{h}^{-1} = e + \delta h \ .$$

Substituting this into (2.2) and retaining terms up to first order in δg_0, we get

$$\exp(-\xi \cdot P)\delta g_0 \exp(\xi \cdot P) - \exp(-\xi \cdot P)\delta \exp(\xi \cdot P) = \delta h \ .$$

If the elements are pure translations, i.e. of the form $g_0 = \exp(\xi_0 \cdot P)$, the transformation parameters satisfy the equation (let $\delta g_0 = \epsilon \cdot P$)

$$\exp(-\xi \cdot P)\epsilon \cdot P \exp(\xi \cdot P) - \exp(-\xi \cdot P)\delta \exp(\xi \cdot P) = \delta h \ . \tag{2.5}$$

2.3 Symmetric Lie algebra

Assume the Lie algebra $\mathfrak{g}$ is not only reductive but also symmetric. This means that there is an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{h}$ is an eigenspace with eigenvalue 1, while $\mathfrak{p}$ is an eigenspace with eigenvalue -1. Group elements of H that are obtained by exponentiation of elements in $\mathfrak{h}$ are invariant under σ, while elements generated by elements of $\mathfrak{p}$ are mapped into their inverse. The automorphism directly implies a third restriction on the commutation relations of $\mathfrak{g}$, i.e. $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$.

We can consider the automorphism σ acting on (2.2) together with the original equation, after which $h' = \sigma(h')$ can be eliminated. Doing so, this yields

$$g_0 \exp(2\xi \cdot P)\sigma(g_0^{-1}) = \exp(2\xi' \cdot P) \ . \tag{2.6}$$

Written this way, it is manifest that $g_0 : \xi \mapsto \xi'$ is a group realization and, when restricted to H, this realization becomes a linear representation.

In the following, we concentrate on infinitisimal transformations under pure translations, namely those that are of the form $g_0 = e + \epsilon \cdot P$. Therefore, we first use the information that $\mathfrak{g}$ is symmetric to eliminate δh from eq. (2.5), which leads to

$$\exp(-\xi \cdot P)\delta \exp(\xi \cdot P) - \exp(\xi \cdot P)\delta \exp(-\xi \cdot P)$$

$$= \exp(-\xi \cdot P)\epsilon \cdot P \exp(\xi \cdot P) + \exp(\xi \cdot P)\epsilon \cdot P \exp(-\xi \cdot P) \ .$$

Using eqs. (A.1) and (A.2) one finds

$$\frac{1 - \exp(-\xi \cdot P)}{\xi \cdot P} \wedge \delta\xi \cdot P - \frac{1 - \exp(\xi \cdot P)}{\xi \cdot P} \wedge \delta\xi \cdot P \\ = \exp(-\xi \cdot P) \wedge \epsilon \cdot P + \exp(\xi \cdot P) \wedge \epsilon \cdot P \ .$$

The expression can be solved for $\delta\xi \cdot P$, leading to

$$\delta\xi \cdot P = \frac{\xi \cdot P \cosh(\xi \cdot P)}{\sinh(\xi \cdot P)} \wedge \epsilon \cdot P \ . \tag{2.7}$$

This result gives the infinitesimal change in coset parameters due to an infinitesimal pure translation $\epsilon \cdot P$. Remember that it is only valid for symmetric spaces. To explicitly solve for $\delta\xi^a$, one needs the specific commutation relations of the underlying algebra.

Given the nonlinear transformation (2.7), one can work out the corresponding transformation (2.5) for δh. In other words, for symmetric algebras we may specify further the transformation $h'(\xi, \epsilon)$ under a pure translation $\delta g_0 = \epsilon \cdot P$.

Using eqs. (A.1) and (A.2), we find from (2.5) that $\delta h \equiv \delta h \cdot M \in \mathfrak{h}$ is equal to*

$$\delta h \cdot M = \exp(-\xi \cdot P) \wedge \epsilon \cdot P - \frac{1 - \exp(-\xi \cdot P)}{\xi \cdot P} \wedge \delta\xi \cdot P \ . \tag{2.8}$$

In the case of symmetric Lie algebras one may substitute for (2.7). After some algebra it is found that

$$\delta h \cdot M = \frac{1 - \cosh(\xi \cdot P)}{\sinh(\xi \cdot P)} \wedge \epsilon \cdot P \ . \tag{2.9}$$

2.4 From linear to nonlinear realizations

Let ψ be a field belonging to some linear representation σ of G, that is

$$g : \psi(x) \mapsto \psi'(x) = \sigma(g)\psi \ . \tag{2.10}$$

Given a section ξ of $P \times_G G/H$, as considered above, a nonlinear realization of G is constructed by

$$\bar{\psi}(x) \equiv \sigma(\exp(-\xi \cdot P))\psi \ . \tag{2.11}$$

*This result is true for all reductive algebras, that is symmetric *and* non-symmetric.

This field transforms nonlinearly *and only with respect to its H-indices* under the action of a generic element g_0 of G:

$$\begin{aligned}\bar{\psi}'(x) &= \sigma(\exp(-\xi' \cdot P))\psi'(x) \\ &= \sigma(\exp(-\xi' \cdot P)g_0)\psi(x) \\ &= \sigma(\exp(-\xi' \cdot P)g_0 \exp(\xi \cdot P))\bar{\psi}(x) \\ &= \sigma(h'(\xi, g_0))\bar{\psi}(x) \ .\end{aligned}$$

It follows that a linear irreducible representation of G becomes a nonlinear and reducible representation. The price to be paid for getting irreducible H-representations is that they transform in a nonlinear way. Nonetheless, when restricted to the isotropy group H, the field (2.11) transforms according to a linear representation.

3 An example: de Sitter space

3.1 Transformation of group parameters

In this subsection, the change of the group parameters ξ^a and δh^{ab} due to infinitesimal de Sitter translations are calculated. Remember that the coordinates ξ^a are defined by the exponentiation of elements of $\mathfrak{p}$. They are also referred to as Goldstone fields, because of the resemblance of their role in the scheme of spontaneous symmetry breaking in field theory. As we have reviewed in the last section, these coordinates transform according to a nonlinear realization of the full symmetry group G. On the other hand, they transform linearly when the action is restricted to the subgroup H of unbroken symmetries. One understands that the pure translations are the set of transformations that are responsible for the nonlinear behaviour.[*]

Let us begin by recalling the de Sitter commutation relations that involve translations, i.e.[†]

$$\begin{aligned}-i[M_{ab}, P_c] &= \eta_{ac}P_b - \eta_{bc}P_a \\ -i[P_a, P_b] &= \mathfrak{s} l^{-2} M_{ab}\end{aligned} \tag{3.1}$$

with $\mathfrak{s} \equiv \eta_{44}$. The de Sitter translations were introduced as $P_a \equiv l^{-1}(x)M_{a4}$, whilst the M_{ab} span the Lorentz subalgebra $\mathfrak{h} = \mathfrak{so}(3,1)$. It is manifest that the de Sitter algebra is symmetric. In what follows we adhere to the convention $\mathfrak{s} = -1$ so that $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$.

The transformation of the coset parameters ξ^a under an infinitesimal de Sitter translation $\epsilon \cdot P$ is given by (2.7). In the parametrization used in this section, this can be rewritten as

$$\delta\xi \cdot P = \frac{i\xi \cdot P \, \cosh(i\xi \cdot P)}{\sinh(i\xi \cdot P)} \wedge \epsilon \cdot P \ . \tag{3.2}$$

[*]In general these elements do not form a group.

[†]An element of $SO(4,1)$ is given by $\exp(\frac{i}{2}\omega^{ab}M_{ab} + i\xi^a P_a)$.

Recall that the left hand side should be understood as a power series in the adjoint action (see also Appendix A.1). The power series of the relevant hyperbolic functions have the form*

$$\cosh(i\xi\cdot P)=\sum_{n=0}^{\infty}\frac{(i\xi\cdot P)^{2n}}{(2n)!}\ ,$$
$$\operatorname{csch}(i\xi\cdot P)=(i\xi\cdot P)^{-1}+\sum_{n=1}^{\infty}\frac{c_{2n}}{(2n)!}(i\xi\cdot P)^{2n-1}\ .$$

Invoking the identity (A.3), one is able to work out the cosinus hyperbolicus, i.e.

$$\begin{aligned}\cosh(i\xi\cdot P)\wedge\epsilon\cdot P&=\epsilon\cdot P+\sum_{n=1}^{\infty}\frac{(l^{-1}\xi)^{2n}}{(2n)!}\wedge\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\\&=\cosh(l^{-1}\xi)\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)+\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\ .\end{aligned}\tag{3.3}$$

By equal means, the right hand side of (3.2) is found to be

$$\begin{aligned}&i\xi\cdot P\operatorname{csch}(i\xi\cdot P)\wedge\cosh(i\xi\cdot P)\wedge\epsilon\cdot P\\&\quad=\left(\mathbb{1}+\sum_{n=1}^{\infty}\frac{c_{2n}}{(2n)!}(i\xi\cdot P)^{2n}\right)\wedge\left[\cosh(l^{-1}\xi)\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)+\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right]\\&\quad=\cosh(l^{-1}\xi)\left(1+\sum_{n=1}^{\infty}\frac{c_{2n}}{(2n)!}(l^{-1}\xi)^{2n}\right)\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)+\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\\&\quad=\cosh(l^{-1}\xi)(l^{-1}\xi)\operatorname{csch}(l^{-1}\xi)\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)+\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\\&\quad=\epsilon\cdot P+\frac{l^{-1}\xi\cosh(l^{-1}\xi)}{\sinh(l^{-1}\xi)}\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)+\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}-\epsilon\cdot P\ .\end{aligned}$$

The introduction of the extra $\epsilon\cdot P$ terms in the last line is just a matter of convention, which allows one to write eq. (3.2) as

$$\delta\xi\cdot P=\epsilon\cdot P+\Big(\frac{l^{-1}\xi\cosh(l^{-1}\xi)}{\sinh(l^{-1}\xi)}-1\Big)\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\ .\tag{3.4}$$

This implies that the infinitesimal change of the coset parameters is given by

$$\delta\xi^a=\epsilon^a+\Big(\frac{z\cosh z}{\sinh z}-1\Big)\left(\epsilon^a-\frac{\xi^a\epsilon_b\xi^b}{\xi^2}\right)\ ,\tag{3.5}$$

where $z=l^{-1}\xi$.

A comment on the derivation of eq. (3.4) is in place. The solution was found after use of the power series for the hyperbolic cosecant. In the case of real numbers, the series is only defined for values between $-\pi$ and $+\pi$. One could thus wonder if

*The coefficients in the power series for the hyperbolic cosecant are $c_{2n}=2(1-2^{2n-1})B_{2n}$ with B_n the n-th Bernouilli number.

this convergence issue inhibits us of trusting the solution found above. Remember that eq. (3.2) can be rewritten as

$$(i\xi \cdot P)^{-1} \sinh(i\xi \cdot P) \wedge \delta\xi \cdot P = \cosh(i\xi \cdot P) \wedge \epsilon \cdot P \ ,$$

which reduces to

$$z^{-1} \sinh z \Big(\delta\xi \cdot P - \frac{\xi \cdot \delta\xi \xi \cdot P}{\xi^2} \Big) + \frac{\xi \cdot \delta\xi \xi \cdot P}{\xi^2} = \cosh z \Big(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2} \Big) + \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2} \ .$$

This result relies on the power series expansion of the hyperbolic sine, which is convergent for all values of its argument. It is readily checked that the solution (3.4) satisfies the above equation. Therefore, we may conclude that (3.4) is the right solution.

Given the de Sitter algebra, it is also possible to compute $\delta h = \frac{i}{2} \delta h^{ab} M_{ab}$ explicitly. From (2.9) it follows that the element of $\mathfrak{h}$, corresponding to an infinitesimal de Sitter translation, is a solution of

$$\tfrac{1}{2} \sinh(i\xi \cdot P) \wedge \delta h \cdot M = (\mathbb{1} - \cosh(i\xi \cdot P)) \wedge \epsilon \cdot P \ . \tag{3.6}$$

The right hand side is readily found by reconsidering (3.3), implying that

$$(\mathbb{1} - \cosh(i\xi \cdot P)) \wedge \epsilon \cdot P = (1 - \cosh z) \left(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2} \right) \ .$$

From the power series expansion of the hyperbolic sine,

$$\sinh(i\xi \cdot P) = \sum_{n=0}^{\infty} \frac{(i\xi \cdot P)^{2n+1}}{(2n+1)!} \ ,$$

and from (A.6), it follows that

$$\begin{aligned} \sinh(i\xi \cdot P) \wedge \delta h \cdot M &= \delta h^{ab} \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n+1)!} (\xi_a P_b - \xi_b P_a) \\ &= z^{-1} \sinh z \, \delta h^{ab} (\xi_a P_b - \xi_b P_a) \\ &= 2 z^{-1} \sinh z \, \delta h^{ab} \xi_a P_b \ . \end{aligned}$$

Putting these equations together, (3.6) is rewritten as

$$\begin{aligned} \delta h^{ab} \xi_a P_b &= \frac{z(1 - \cosh z)}{\sinh z} \left(\epsilon^b - \frac{\xi^a \epsilon_a \xi^b}{\xi^2} \right) P_b \\ &= (l\xi)^{-1} \frac{1 - \cosh z}{\sinh z} (\epsilon^b \xi^a - \epsilon^a \xi^b) \xi_a P_b \ , \end{aligned}$$

from which it can be concluded that the sought-after quantities are

$$\delta h = \tfrac{i}{2}\delta h^{ab} M_{ab} = \frac{i}{2l^2}\frac{\cosh z - 1}{z \sinh z}(\epsilon^a \xi^b - \epsilon^b \xi^a) M_{ab} \ . \tag{3.7}$$

3.2 Towards a Cartan geometry

Invoking the theory of nonlinear realizations, an Ehresmann connection A and its corresponding curvature $F = dA + \frac{1}{2}[A, A]$ on $P(M, G)$ can be pulled back to an associated Cartan geometry on the reduced bundle $P(M, H)$.* In what follows such a Cartan connection will be constructed explicitly.

Under local gauge transformations g_0 on some associated vector bundle (the fibre of the principal bundle transforms with the inverse), the connection and its curvature transform as

$$A \mapsto g_0 A g_0^{-1} + g_0 d g_0^{-1} = \mathrm{Ad}(g_0) \cdot (A + d) \tag{3.8a}$$

$$F \mapsto g_0 F g_0^{-1} = \mathrm{Ad}(g_0) \cdot F \ . \tag{3.8b}$$

These $\mathfrak{g}$-valued differential forms can be decomposed with respect to their $\mathfrak{h}$- and $\mathfrak{p}$-valued parts, i.e.

$$\begin{aligned} A_\mu &= \tfrac{i}{2}\omega^{ab}{}_\mu M_{ab} + i e^a{}_\mu P_a \\ F_{\mu\nu} &= \tfrac{i}{2} R^{ab}{}_{\mu\nu} M_{ab} + i T^a{}_{\mu\nu} P_a \ , \end{aligned}$$

with $P_a = M_{a4}/l(x)$ and

$$\begin{aligned} R^{ab}{}_{\mu\nu} &= \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^a{}_{c\mu}\omega^{cb}{}_\nu - \omega^a{}_{c\nu}\omega^{cb}{}_\mu + l^{-2}(e^a{}_\mu e^b{}_\nu - e^a{}_\nu e^b{}_\mu) && (3.9) \\ &\equiv B^{ab}{}_{\mu\nu} + l^{-2}(e^a{}_\mu e^b{}_\nu - e^a{}_\nu e^b{}_\mu) \\ T^a{}_{\mu\nu} &= \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu - l^{-1}(\partial_\mu l \, e^a{}_\nu - \partial_\nu l \, e^a{}_\mu) \ . && (3.10) \end{aligned}$$

Although it is not a coincidence that the above notation may remind of the curvature and the torsion tensors, it must be emphasized that these quantities are by no means the curvature or torsion of some geometric object. First of all and for the moment being, there is only a curvature F of the connection A in play, while for the latter torsion is not defined. Secondly, remark that the decomposition is not canonical in the sense that local gauge transformations mix up the $\mathfrak{h}$- and $\mathfrak{p}$-valued parts. Stated equivalently, A and F transform irreducibly under G. As a result, the decomposition is not respected by the symmetries of the geometry, from which it would be difficult to atribute it any physical meaning. It is here where symmetry breaking assumes an important role: by means of a section ξ of the associated bundle of homogeneous spaces, one reduces the principal G bundle to a principal H bundle. From the general prescription (2.11) and its particular transformation behaviour (3.8a), the

*$G = SO(1,4)$; $H = SO(1,3)$.

corresponding nonlinear de Sitter connection is defined as

$$
\begin{aligned}
\bar{A} &= \mathrm{Ad}(\exp(-i\xi\cdot P))\cdot(A+d) \\
\tfrac{i}{2}\bar{\omega}^{ab}{}_{\mu}M_{ab} + i\bar{e}^{a}{}_{\mu}P_a &\equiv \exp(-i\xi\cdot P)(\partial_\mu + \tfrac{i}{2}\omega^{ab}{}_{\mu}M_{ab} + ie^{a}{}_{\mu}P_a)\exp(i\xi\cdot P)
\end{aligned}
\tag{3.11}
$$

The left hand side can be calculated using the techniques of nested commutators. Carefully doing so, one is led to the expressions

$$
\begin{aligned}
\bar{\omega}^{ab}{}_{\mu} = \omega^{ab}{}_{\mu} &- \frac{\cosh z - 1}{l^2 z^2}\left(\xi^a(\partial_\mu \xi^b + \omega^{b}{}_{c\mu}\xi^c) - \xi^b(\partial_\mu\xi^a + \omega^{a}{}_{c\mu}\xi^c)\right) \\
&- \frac{\sinh z}{l^2 z}(\xi^a e^{b}{}_{\mu} - \xi^b e^{a}{}_{\mu})
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
\bar{e}^{a}{}_{\mu} &= \cosh z\, e^{a}{}_{\mu} - (\cosh z - 1)\frac{\xi_b e^{b}{}_{\mu}\xi^a}{\xi^2} \\
&\quad + \frac{\sinh z}{z}(\partial_\mu \xi^a + \omega^{a}{}_{b\mu}\xi^b) - \frac{\partial_\mu l}{l}\xi^a - \left(\frac{\sinh z}{z} - 1\right)\frac{\xi_b \partial_\mu \xi^b \xi^a}{\xi^2}\ ,
\end{aligned}
\tag{3.13}
$$

These expressions coincide with those found by Stelle and West [5], aside from the term $l^{-1}\partial_\mu l \xi^a$ in Eq. (3.13). The latter is new for the geometry constructed here, since it is allowed for the tangent de Sitter spaces to have a varying cosmological constant. More specifically, one has to take in account that the length scale defined for the elements of $\mathfrak{p}$ may change along spacetime. On the other hand, the derivation of the result found in [5] relies on a constant de Sitter radius at any basepoint. This extra contribution is proportional to the dimensionless factor $l^{-1}\partial_\mu l$, which will be noticeable only if the variation is relatively vast. As will be made manifest in the following section, these barred objects transform with respect to their H-indices only, although nonlinearly for elements not belonging to the stabilizer of G.

From the definition of F and its transformation behaviour (3.8b) under local G transformations, it follows that the nonlinear curvature equals the curvature of the nonlinear connection, i.e.

$$
\bar{F} = d\bar{A} + \tfrac{1}{2}[\bar{A},\bar{A}]\ . \tag{3.14}
$$

Decomposing the curvature of $\bar{A}$ according to the reductive splitting of $\mathfrak{g}$

$$
\bar{F}_{\mu\nu} = \tfrac{i}{2}\bar{R}^{ab}{}_{\mu\nu}M_{ab} + i\bar{T}^{a}{}_{\mu\nu}P_a\ ,
$$

while remembering that $\bar{F} = \mathrm{Ad}(\exp(-i\xi\cdot P))\cdot F$, it is a matter of algebra to conclude that

$$
\begin{aligned}
\bar{R}^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu} &- \frac{\cosh z - 1}{l^2 z^2}\xi^c(\xi^a R^{b}{}_{c\mu\nu} - \xi^b R^{a}{}_{c\mu\nu}) \\
&- \frac{\sinh z}{l^2 z}(\xi^a T^{b}{}_{\mu\nu} - \xi^b T^{a}{}_{\mu\nu})\ ,
\end{aligned}
\tag{3.15a}
$$

$$\bar{T}^a{}_{\mu\nu} = \frac{\sinh z}{z}\xi^c R^a{}_{c\mu\nu} + \cosh z\, T^a{}_{\mu\nu} + (1-\cosh z)\frac{\xi_b T^b{}_{\mu\nu}\xi^a}{\xi^2} \ . \tag{3.15b}$$

Of course, one could also obtain this result by calculating $d\bar{A} + \frac{1}{2}[\bar{A}, \bar{A}]$ directly from $\bar{\omega}$ and $\bar{e}$. The quantities $\bar{R}$ and $\bar{T}$ are the ($\mathfrak{h}$-)curvature and torsion, respectively, of the Cartan connection $\bar{A}$.

3.3 Transformation behavior

In this subsection we discuss in more detail, the way in which the different fields introduced above transform. The general relation between linear and nonlinear fields indicates that the latter belong to the same representation space as the former, although the nonlinear fields become reducible under the action of the full gauge group G. Let us first review how the linear fields transform under elements of G, after which we turn attention to their nonlinear counterparts.

Under G gauge transformations A transforms as in Eq. (3.8a). When restricted to the subgroup H, the reductive splitting $A = \omega + e$ remains intact. It is easily inferred that ω and e transform as

$$\omega \mapsto \mathrm{Ad}(h_0)\cdot(\omega + d) \quad \text{and} \quad e \mapsto \mathrm{Ad}(h_0)\cdot e \ .$$

As has been mentioned before, this symmetric splitting is not respected under a generic G-transformation. Consider for example an infinitesimal pure de Sitter translation $e + i\epsilon\cdot P$, under which to first order in the transformation parameter ϵ

$$\begin{aligned}\delta_\epsilon A &= i[\epsilon\cdot P, A] - id(\epsilon\cdot P)\\ &= i[\epsilon\cdot P, A] - id\epsilon\cdot P + \frac{dl}{l}i\epsilon\cdot P \ .\end{aligned}$$

After applying the commutation relations for the de Sitter algebra, one discovers the variations of ω and e, namely

$$\delta_\epsilon\omega^{ab} = \frac{1}{l^2}(\epsilon^a e^b - \epsilon^b e^a) \ , \tag{3.16a}$$

$$\delta_\epsilon e^a = -d\epsilon^a - \omega^a{}_b\epsilon^b + \frac{dl}{l}\epsilon^a \ . \tag{3.16b}$$

It is manifest how ω and e form an irreducible multiplet for the full de Sitter group.

To obtain the variations of ω and e, we could have saved energy by reconsidering the definition of $\bar{A}$. The latter equals $\mathrm{Ad}(\exp(-i\xi\cdot P))\cdot(A+d)$, which is equal to a transformation of A under a pure de Sitter translation with transformation parameter $\alpha = -\xi$. One thus concludes, that under a *finite* translation $\exp(i\alpha\cdot P)$

the fields ω and e transform as ($z \equiv l^{-1}(\alpha\cdot\alpha)^{1/2}$)

$$\omega^{ab} \mapsto \omega^{ab} - \frac{\cosh z - 1}{l^2 z^2}\Big[\alpha^a(d\alpha^b + \omega^b{}_c\alpha^c) \\ - \alpha^b(d\alpha^a + \omega^a{}_c\alpha^c)\Big] + \frac{\sinh z}{l^2 z}(\alpha^a e^b - \alpha^b e^a) \ , \tag{3.17a}$$

$$e^a \mapsto e^a - d\alpha^a - \frac{\sinh z}{z}\omega^a{}_b\alpha^b + \frac{dl}{l}\alpha^a \\ + (\cosh z - 1)\Big(e^a - \frac{\alpha_b e^b \alpha^a}{\alpha^2}\Big) - \Big(\frac{\sinh z}{z} - 1\Big)\Big(d\alpha^a - \frac{\alpha_b d\alpha^b \alpha^a}{\alpha^2}\Big) \ . \tag{3.17b}$$

The infinitesimal transformations are recovered by taking the limit $\alpha \to \epsilon$.

The curvature F transforms in a covariant way under gauge transformations, as was written down in Eq. (3.8b). Upon restriction to the subgroup H of Lorentz rotations, the reductive splitting $F = R + T$ is invariant and R and T transform according to

$$R \mapsto \mathrm{Ad}(h_0)\cdot R \quad \text{and} \quad T \mapsto \mathrm{Ad}(h_0)\cdot T \ .$$

On the other hand, under a pure de Sitter translation $\exp(i\alpha\cdot P)$ they form an irreducible multiplet:

$$R^{ab}{}_{\mu\nu} \mapsto R^{ab}{}_{\mu\nu} - \frac{\cosh z - 1}{l^2 z^2}\alpha^c(\alpha^a R^b{}_{c\mu\nu} - \alpha^b R^a{}_{c\mu\nu}) + \frac{\sinh z}{l^2 z}(\alpha^a T^b{}_{\mu\nu} - \alpha^b T^a{}_{\mu\nu}) \ , \tag{3.18a}$$

$$T^a{}_{\mu\nu} \mapsto -\frac{\sinh z}{z}\alpha^c R^a{}_{c\mu\nu} + \cosh z\, T^a{}_{\mu\nu} + (1 - \cosh z)\frac{\alpha_b T^b{}_{\mu\nu}\alpha^a}{\alpha^2} \ . \tag{3.18b}$$

For an infinitesimal translation $e + i\epsilon\cdot P$, the variations reduce to

$$\delta_\epsilon R^{ab}{}_{\mu\nu} = \frac{1}{l^2}(\epsilon^a T^b{}_{\mu\nu} - \epsilon^b T^a{}_{\mu\nu}) \ , \tag{3.19a}$$

$$\delta_\epsilon T^a{}_{\mu\nu} = -\epsilon^b R^a{}_{b\mu\nu} \ . \tag{3.19b}$$

Under the action of G-transformations the nonlinear fields $\bar{A}$ and $\bar{F}$ behave in a similar manner as the linear fields A and F, respectivily. More precisely, the linear and the nonlinear field belongs to the same representation space. The crucial difference between the two is that while A and F are irreducible under their respective action of G, the barred fields $\bar{A}$ and $\bar{F}$ are reducible with respect to the full gauge group G. As we have seen, any element of G acts on the nonlinear fields through an element of H, so that only the H-components of these fields get mixed up under local G-transformations. Although a consequence of the general scheme, we discuss this in more detail for the nonlinear fields $\bar{A}$ and $\bar{F}$. In doing this, it may become obvious that the thus obtained construction is natural for the kind of geometry we are wishing to describe.

To verify the transformation behavior of $\bar{A}$, note first that its definition (3.11) indicates that $\mathrm{Ad}(e)(A+d) = \mathrm{Ad}(\exp(i\xi \cdot P))(\bar{A}+d)$. It follows that under the action of $g_0 \in G$

$$\begin{aligned}\bar{A} \mapsto \bar{A}' &= \mathrm{Ad}(\exp(-i\xi' \cdot P))\mathrm{Ad}(g_0)(A+d) \\ &= \mathrm{Ad}(\exp(-i\xi' \cdot P)g_0)\mathrm{Ad}(\exp(i\xi' \cdot P))(\bar{A}+d) \\ &= \mathrm{Ad}(h'(\xi, g_0))(\bar{A}+d) \ .\end{aligned}$$

This confirms that $\bar{A}$ transforms only with elements belonging to H and that it is a $\mathfrak{g}$-valued connection. It seems very plausible that it is (the pull-back of) a Cartan connection on $P(M,H)$[Check this: see the base definition of the latter as given in [1]]. In case g_0 is en element of H, the transformation becomes linear.* Remember that the reductive decomposition $A = \omega + e$ is not preserved under G gauge transformations, which made us conclude that the splitting does not make any sense with respect to a local de Sitter geometry. For the nonlinear connection the story changes completely. Since any element of G acts through an element of H, the reductive splitting $\bar{A} = \bar{\omega} + \bar{e}$ will be invariant under G-transformations. Indeed, writing out the reductive splitting explicitely, $\bar{A}$ transforms as

$$\tfrac{i}{2}\bar{\omega}'^{ab}M_{ab} + i\bar{e}^a P_a = \mathrm{Ad}(h'(\xi, g_0))(\tfrac{i}{2}\bar{\omega}^{ab}M_{ab} + i\bar{e}^a P_a + d) \ ,$$

Since *both* $\mathfrak{h}$ *and* $\mathfrak{p}$ *form a representation of* H, it follows that

$$\bar{e}' = \mathrm{Ad}(h'(\xi, g_0)) \cdot \bar{e} \tag{3.20a}$$

$$\bar{\omega}' = \mathrm{Ad}(h'(\xi, g_0)) \cdot (\bar{\omega} + d) \ . \tag{3.20b}$$

It is manifest that $\bar{\omega}$ and $\bar{e}$ do not mix under local G transformations. Note that $\bar{\omega}$ is an $\mathfrak{h}$-valued spin connection, which is (the pull-back of) an Ehresmann connection on $P(M,H)$. When the gauge transfromations are restricted to the subgroup H, the nonlinear fields transform the same as their linear counterparts.

These conclusions are equally drawn for the $\mathfrak{g}$-curvature $\bar{F}$ and its projections $\bar{R}$ and $\bar{T}$. They all transform covariantly in the adjoint representation of the group, with respect to their H-indices under generic elements of G. Therefore it becomes manifest that $\bar{R}$ and $\bar{T}$ are true geometric objects, for the splitting is a gauge independent construction. They are referred to as the curvature and torsion of the geometry.

3.4 Interpretation of the vielbein $\bar{e}$

Stelle and West [4, 5] claim that the vierbein $\bar{e}$ is a smooth mapping between the tangent space to spacetime at any $p \in M$ and the tangent space to the internal de Sitter space at $\xi(p)$. Unfortunately, a concrete argument did not seem to be included

*Note that a Cartan connection is defined on a principal H-bundle, and that is only demanded that the connection transforms in a certain way under the action of H gauge transformations. Whether the elements of H form a linear or nonlinear realization is not relevant.

following this statement. Furthermore, under local H gauge transformations the vierbein $\bar{e}$ transforms as a vector with an element $h \in H_o$, as can be seen from (3.20a). This indicates that its $SO(3,1)$-indices belong to the tangent space at the origin ($\xi = 0$) of dS. To verify its transformation behavior under local G transformations, let us explicitly reconsider its construction.

The vierbein is defined as the $\mathfrak{p}$-valued part of the Cartan connection $\bar{A} \in \Omega^1(M, \mathfrak{g})$ on $P(M,H)$. To give this statement a precise notation, we consider the natural projection $\pi : G \to G/H_o : g \mapsto gH_o$. The differential of this mapping is a projection of $T_eG = \mathfrak{g}$ onto $\mathfrak{p} \simeq T_odS$. The vierbein is obtained from the connection by invoking this projection, i.e. $\bar{e} = \pi_*\bar{A}$. This shows clearly that the vierbein is a 1-form on M with values in T_odS. Nonetheless, let us also concentrate on the definition of $\bar{A}$ itself to understand what happens with a tangent vector to spacetime under the action of $\bar{e}$, before it ends up in T_odS. The definition was given in Eq. (3.11), which we rewrite here for $g = \exp(-\xi \cdot P)$, i.e.

$$\bar{A} \equiv \mathrm{Ad}(g) \cdot A + (g^{-1})^*\theta \ .$$

It should be understood that the adjoint action acts on the algebra $\mathfrak{g}$, that θ is the Maurer-Cartan form on G and that g^* is the pullback that comes from the mapping $g : M \to G : p \mapsto g$. Consider next a vector $X \in T_pM$. One then finds,

$$\begin{aligned}\bar{A}(X) &= \mathrm{Ad}(g) \cdot A(X) + \theta(g_*^{-1}X) \\ &= L_{g*}\Big(R_{g^{-1}*} \cdot A(X) + g_*^{-1}X\Big) \ .\end{aligned}$$

Denote by $X^\star$ the left invariant vector field on G so that

$$X^\star_{g^{-1}} = R_{g^{-1}*} \cdot A(X) + g_*^{-1}X \ .$$

It follows directly that $\bar{A}(X) = X^\star_e$. Since $\pi \circ L_g = \tau_g \circ \pi$, one also has

$$\bar{e}(X) = \pi_* L_{g*} X^\star_{g^{-1}} = \tau_{g*}\pi_* X^\star_{g^{-1}} \ .$$

Recall that $g = \exp(-\xi \cdot P)$ so that $g^{-1} = \exp(\xi \cdot P)$. This implies that $\pi_* X^\star_{g^{-1}} \in T_\xi dS$, since $\exp(\xi \cdot P)o = \xi \in dS$. The element $\bar{e}(X) \in T_odS$ is the parallel transported vector of $\pi_* X^\star_{g^{-1}}$, with respect to the canonical connection on G/H_o (See Ch. X in [3]). Therefore, it is understandable that one may interpret $\bar{e}$ to be a mapping from the tangent space to M at p onto the tangent space to dS at ξ, confirming the interpretation given by Stelle and West.

3.5 Discussion

To conclude let us retrace our steps and try to understand what has been going on. We started by introducing a G-connection on a principal G-bundle $P(M,G)$. This contains information about a geometry for which the internal symmetry group is G. Since one is interested in describing a spacetime, whose local geometry is invariant under the action of the de Sitter group, this seems a good starting point. However, one does not have a canonical—i.e. consistent with the geometry—spin connection and vielbein. This is a crucial shortcoming, as it will not be possible to relate the local geometry of the gauge field to the geometry of spacetime (no soldering). By means of a section ξ, which takes its values in the associated bundle $P \times_G G/H$, the principle bundle $P(M, G)$ is reduced to a bundle $P(M, H)$.[*] Choosing a section breaks the symmetry from G to H. As shown in the previous section, ξ can be used to construct a Cartan connection $\bar{A}$ on $P(M, H)$ from the Ehresmann connection A on $P(M, G)$. This is shown schematically in the following diagram:

$$
\begin{tikzcd}
 & \omega & & & \bar{\omega} \\
A \arrow[ur, "\mathfrak{h}"] \arrow[dr, "\mathfrak{p}"'] \arrow[ddd] & & & \bar{A} \arrow[ur, "\mathfrak{h}"] \arrow[dr, "\mathfrak{p}"'] \arrow[ddd] & \\
 & e & \arrow[r, "\xi"] & {} & \bar{e} \\
 & R = d\omega + \frac{1}{2}[\omega, \omega] + \frac{1}{2}[e, e] & & & \bar{R} = d\bar{\omega} + \frac{1}{2}[\bar{\omega}, \bar{\omega}] + \frac{1}{2}[\bar{e}, \bar{e}] \\
F \arrow[ur, "\mathfrak{h}"] \arrow[dr, "\mathfrak{p}"'] & & & \bar{F} \arrow[ur, "\mathfrak{h}"] \arrow[dr, "\mathfrak{p}"'] & \\
 & T = de + [\omega, e] & & & \bar{T} = d\bar{e} + [\bar{\omega}, \bar{e}]
\end{tikzcd}
$$

The broken symmetries act through a nonlinear realization with the elements $h'(\xi, g_0)$, and merely change the point of tangency between the local de Sitter fibres and spacetime. On the other hand, the unbroken symmetries (H) leave the point of tangency fixed and act through a linear representation. Note that the Cartan connection gives rise to a well defined spin connection and vierbein, i.e. they do *not* form an irreducible multiplet under the action of G. Due to the existence of a vierbein $\bar{e}$, spacetime is soldered to the de Sitter fibres and one is able to pull back all geometric information onto the tangent bundle of spacetime—the arena in which takes place gravity. Crucially, one had to make the realization nonlinear to have soldering.

4 de Sitter Teleparallel Gravity

4.1 Introduction

In this section we require that the $\mathfrak{h}$-valued part of the Cartan curvature $\bar{F}$ vanishes. In other words, the geometry outlined in the last section should at all times satisfy

[*] For an enlightning proof, see [3].

the following condition, namely

$$\bar{R} \equiv 0 \ . \tag{4.1}$$

From the discussion on transformation behavior in Subsection 3.3 it is clear that this condition is consistent with the geometry, i.e. invariant under G-gauge transformations. This construction may result in the mathemathical structure of Teleparallel Gravity in the corresponding limit, i.e. a diverging length scale $l(x) \to \infty$ at any point in spacetime. In that case, and because of the naturality of the given Cartan geometry together with a vanishing curvature (4.1), the thus obtained geometry could be seen as the generalization of Teleparallel Gravity, where the local kinematics are those governed by the de Sitter algebra.

Let us begin by taking a closer look at the condition of vanishing curvature, given in Eq. (4.1). Combining this requirement with Eq. (3.15a), one finds that

$$R^{ab}{}_{\mu\nu} = \frac{\cosh z - 1}{\xi^2} \xi^c (\xi^a R^b{}_{c\mu\nu} - \xi^b R^a{}_{c\mu\nu}) + \frac{\sinh z}{l^2 z} (\xi^a T^b{}_{\mu\nu} - \xi^b T^a{}_{\mu\nu}) \ .$$

This expression can be contracted with ξ, which results in

$$\cosh z \, \xi^c R^a{}_{c\mu\nu} = \frac{\sinh z}{l^2 z} (\xi^a \xi_b T^b{}_{\mu\nu} - \xi^2 T^a{}_{\mu\nu}) \ . \tag{4.2}$$

Substituting this equations into the torsion $\bar{T}$, see Eq. (3.15b), one obtains

$$\bar{T}^a{}_{\mu\nu} = \frac{1}{\cosh z} T^a{}_{\mu\nu} + \left(1 - \frac{1}{\cosh z}\right) \frac{\xi^a \xi_b T^b{}_{\mu\nu}}{\xi^2} \ . \tag{4.3}$$

Contracting both sides with ξ additionally shows that*

$$\xi_a \bar{T}^a{}_{\mu\nu} = \xi_a T^a{}_{\mu\nu} \ .$$

Remark 4.1. It is interesting to have a look at the limiting situations for a vanishing, respectively diverging cosmological constant. In the case of $l(x) \to \infty$, z vanishes and from (4.3) it is found that

$$\lim_{\Lambda \to 0} \bar{T}^a{}_{\mu\nu} = T^a{}_{\mu\nu} \ ,$$

while on the other hand for $l(x) \to 0$, z diverges and

$$\lim_{\Lambda \to \infty} \bar{T}^a{}_{\mu\nu} = \frac{\xi^a \xi_b T^b{}_{\mu\nu}}{\xi^2} \ . \qquad \Diamond$$

Subsequently let us investigate the additional restriction of a vanishing torsion,

*This equality holds also for a non-vanishing curvature $\bar{R}$, as can be seen from Eq. (3.15b).

i.e.*

$$\bar{T} = 0 \ . \tag{4.4}$$

From Eq. (4.3) and observing that $\xi \cdot \bar{T} = \xi \cdot T$ one infers that T vanishes. An obvious choice of gauge corresponding to such a geometry is $e = 0$, that is

$$e = 0 \quad \Rightarrow \quad \bar{T} = 0 \ .$$

On the other hand, a vanishing torsion does not necessarily imply that e is equal to zero. To find the most general e consistent with the condition (4.4), it is worthwhile to note that the latter is invariant under local gauge transformations and spacetime diffeomorphisms. These transformations are the most general at hand and their effect on e will exhaust its values, corresponding to a vanishing torsion. Since e transforms in a homogeneous way under both spacetime diffeomorphisms and local Lorentz transformations, these will leave $e = 0$ invariant. Therefore it is sufficient to consider the action on e due to de Sitter transvections $\exp(i\alpha \cdot P)$ solely. From the transformation rule (3.17b) for e one finds that $e^a = 0$ transforms into

$$e'^a = -\frac{\sinh z}{z}(d\alpha^a + \omega^a{}_b \alpha^b) + \frac{dl}{l}\alpha^a + \left(\frac{\sinh z}{z} - 1\right)\frac{\alpha_b d\alpha^b \alpha^a}{\alpha^2} \ . \tag{4.5}$$

The vierbein $\bar{e}'$ Lorentz rotates according to

$$\begin{aligned}\bar{e}'^a = \cosh z' e'^a - (\cosh z' - 1)\frac{\xi'_b e'^b \xi'^a}{\xi'^2} \\ + \frac{\sinh z'}{z'}(d\xi'^a + \omega'^a{}_b \xi'^b) - \frac{dl}{l}\xi'^a - \left(\frac{\sinh z'}{z'} - 1\right)\frac{\xi'_b d\xi'^b \xi'^a}{\xi'^2} \ ,\end{aligned}$$

while the torsion $\bar{T}'^a$ remains zero. We call a vierbein $\bar{e}$ *trivial* if and only if e is of the form (4.5). In the case of a trivial vierbein, it is clear that some gauge transformation will render $e = 0$. But the relevant action is given by de Sitter transvections, which correspond to a shift in the section ξ. Since ξ is arbitrary, it is then without loss of generality to assume that a trivial vierbein is of the form

$$\bar{e}^a = \frac{\sinh z}{z}(d\xi^a + \omega^a{}_b \xi^b) - \frac{dl}{l}\xi^a - \left(\frac{\sinh z}{z} - 1\right)\frac{\xi_b d\xi^b \xi^a}{\xi^2} \ . \tag{4.6}$$

Hence, the vanishing of torsion entails the triviality of the vierbein. Conversely, in case the vielbein is trivial, the torsion will be equal to zero.

4.2 Equations of motion for a particle

Given the vielbein $\bar{e}$, it is possible to construct a line element on spacetime that is invariant under local de Sitter transformations. The quadratic line element is defined

*To be clear: the condition of a vanishing curvature is *not* relaxed.

as

$$d\tau^2 = \bar{e}^a{}_\mu \bar{e}_{a\nu} dx^\mu dx^\nu \ ,$$

from which the square root is extracted, resulting in

$$d\tau = \bar{u}_a \bar{e}^a \ . \tag{4.7}$$

In the last expression the nonlinear four-velocity has been introduced, which is given by

$$\bar{u}^a = \bar{e}^a{}_\mu u^\mu \ .$$

The line element has the dimension of length, implying that a possible action for the worldline $x^\mu(\tau)$ of a particle with mass m equals

$$\mathcal{S} = -mc \int_{\tau_1}^{\tau_2} d\tau = -mc \int_{\tau_1}^{\tau_2} \bar{u}_a \bar{e}^a \ . \tag{4.8}$$

The action attains an extremum for the worldline being the physical one. This means that the equations of motion correspond to $\delta\mathcal{S} = 0$, where an infinitisemal variation of the worldline $x^\mu(\tau) \to x^\mu + \delta x^\mu(\tau)$ is considered. Under this deviation, the action (4.8) varies according to

$$\delta\mathcal{S} = -mc \int_{\tau_1}^{\tau_2} \delta\bar{u}_a \bar{e}^a + \bar{u}_a \delta\bar{e}^a = -mc \int_{\tau_1}^{\tau_2} \bar{u}_a \delta\bar{e}^a \ .$$

After a rather lengthy calculation, which we wrote down in Appendix (B.1), one finds

$$\delta\mathcal{S} = mc \int_{\tau_1}^{\tau_2} d\tau \delta x^\mu \left\{ \bar{e}^a{}_\mu \left(\frac{d\bar{u}_a}{d\tau} - \bar{\omega}^b{}_{a\rho} \bar{u}_b u^\rho + u^\rho \frac{\partial_\rho l}{l} \bar{u}_a \right) - \bar{T}^a{}_{\mu\rho} \bar{u}_a u^\rho - \frac{\partial_\mu l}{l} \right\} \ .$$

This quantity should vanish for an arbitrary variation, a condition that leads to the equations of motion:

$$u^\rho \bar{D}_\rho (l \bar{u}^a) = l \bar{e}^{a\mu} \left(\bar{T}^b{}_{\mu\rho} \bar{u}_b u^\rho + \frac{\partial_\mu l}{l} \right) \ ,$$

where $\bar{D} \equiv d + \bar{\omega}$ is the covariant derivative with respect to the spin connection $\bar{\omega}$. The equations of motion can be rewritten in the form

$$u^\rho \bar{D}_\rho \bar{u}^a = \bar{e}^{a\mu} \bar{T}^b{}_{\mu\rho} \bar{u}_b u^\rho + (\bar{e}^{a\mu} - \bar{u}^a u^\mu) \frac{\partial_\mu l}{l} \ . \tag{4.9}$$

It is interesting to have a closer look at this equation. First note that in the appropriate limit of a vanishing cosmological function ($l \to \infty$), the equation of motion of Teleparallel Gravity for a spinless particle in a gravitational field is recovered [9]. Similar to the equation there, we still have a force equation at hand in which both the terms on the right-hand side are indeed genuine relativistic forces, being orthogonal to the four-velocity $\bar{u}^a$. The first force is the obvious generalization to the given geometry of the gravitational force in ordinary Teleparallel gravity. The second force term however is new, and will be noticeable only in spacetime regions where the cosmological function varies relativily strong. Observe that the operator $\bar{e}^{a\mu} - \bar{u}^a u^\mu$ is a projector, since

$$(\bar{e}^{b\rho} - \bar{u}^b u^\rho)\bar{e}_{a\rho}(\bar{e}^{a\mu} - \bar{u}^a u^\mu) = \bar{e}^{b\mu} - \bar{u}^b u^\mu \ .$$

4.3 Field equations

In this subsection we look for the equations of motions that specify for the geometry in de Sitter Teleparallel Gravity. In a first attempt, the free action is the one given by replacing $T \to \bar{T}$ in the action that describes free Poincaré Teleparallel Gravity [Citations]. It is thus proposed that

$$\mathcal{S} = \frac{c^3}{16\pi G}\int \mathrm{Tr}\ \bar{T} \wedge \star\bar{T} = \frac{c^3}{16\pi G}\int d^4x\, \bar{e}\, \mathcal{L} \ , \tag{4.10}$$

where

$$\mathcal{L} = \tfrac{1}{4}\bar{T}^a{}_{\mu\nu}\bar{T}_a{}^{\mu\nu} + \tfrac{1}{2}\bar{T}^a{}_{\mu\nu}\bar{T}^{b\mu}{}_\lambda\,\bar{e}_a{}^\lambda\bar{e}_b{}^\nu - \bar{T}^a{}_{\mu\nu}\bar{T}^{b\mu}{}_\lambda\,\bar{e}_a{}^\nu\bar{e}_b{}^\lambda \ . \tag{4.11}$$

The corresponding field equations are found by extremizing (4.10) with respect to the vierbein $\bar{e}^a{}_\mu$, i.e.

$$\begin{aligned} 0 = \delta\mathcal{S} &= \int d^4x\, \delta\bar{e}\, \mathcal{L} + \int d^4x\, \bar{e}\, \delta\mathcal{L} \\ &= \int d^4x\, \bar{e}\, \bar{e}_a{}^\mu \mathcal{L}\delta\bar{e}^a{}_\mu + \int d^4x\, \bar{e}\Big(\frac{\partial\mathcal{L}}{\partial\bar{e}^a{}_\mu}\delta\bar{e}^a{}_\mu + \frac{\partial\mathcal{L}}{\partial\partial_\rho\bar{e}^a{}_\mu}\delta\partial_\rho\bar{e}^a{}_\mu\Big) \ . \end{aligned}$$

After invoking Stokes' theorem together with the assumption that the fields go to zero when approaching infinity,* the equations of motion are

$$\partial_\mu\Big(\bar{e}\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{e}^a{}_\nu}\Big) - \bar{e}\frac{\partial\mathcal{L}}{\partial\bar{e}^a{}_\nu} - \bar{e}\,\bar{e}_a{}^\nu\mathcal{L} = 0 \ . \tag{4.12}$$

For the given Lagrangian it is shown in Appendix B.2 that these equations reduce to

$$\partial_\mu(\bar{e}\,\bar{W}_a{}^{\mu\nu}) - \bar{e}\,\bar{\omega}^b{}_{a\mu}\bar{W}_b{}^{\mu\nu} + \bar{e}\frac{\partial_\mu l}{l}\bar{W}_a{}^{\mu\nu} + \bar{e}\,\bar{T}^b{}_{\mu a}\bar{W}_b{}^{\mu\nu} - \bar{e}\,\bar{e}_a{}^\nu\mathcal{L} = 0 \ ,$$

where we introduced the notation

$$\bar{W}_a{}^{\mu\nu} \equiv \bar{T}_a{}^{\mu\nu} + \bar{T}^{\nu\mu}{}_a - \bar{T}^{\mu\nu}{}_a - 2\bar{e}_a{}^\nu\bar{T}^{\lambda\mu}{}_\lambda + 2\bar{e}_a{}^\mu\bar{T}^{\lambda\nu}{}_\lambda \ . \tag{4.13}$$

*Is it the fields that go to zero that legitimate the omitting of the boundary terms, or is it the vanishing of the variation?

The field equations can be rewritten in a manifestly covariant form as

$$\bar{D}_\mu(\bar{e}\,\bar{W}_a{}^{\mu\nu}) + \bar{e}\frac{\partial_\mu l}{l}\bar{W}_a{}^{\mu\nu} + \bar{e}\,\bar{t}_a{}^{\nu} = 0 \ , \tag{4.14}$$

where we denoted the expression

$$\bar{t}_a{}^{\nu} = \bar{T}^b{}_{\mu a}\bar{W}_b{}^{\mu\nu} - \bar{e}_a{}^{\nu}\mathcal{L} \ .$$

Note further that

$$\bar{D}_\nu\bar{D}_\mu(\bar{e}\,\bar{W}_a{}^{\mu\nu}) = \frac{1}{2}[\bar{D}_\nu,\bar{D}_\mu](\bar{e}\,\bar{W}_a{}^{\mu\nu}) = \frac{1}{2}\bar{e}\,\bar{B}_a{}^b{}_{\nu\mu}\bar{W}_b{}^{\mu\nu} \ .$$

For the given geometry one has that $\bar{B}^{ab} = -l^{-2}\bar{e}^a\wedge\bar{e}^b$ so that

$$\bar{D}_\nu\bar{D}_\mu(\bar{e}\,\bar{W}_a{}^{\mu\nu}) = \frac{1}{l^2}\bar{e}\,\bar{e}_{a\mu}\bar{e}^b{}_\nu\bar{W}_b{}^{\mu\nu} = \frac{1}{l^2}\bar{e}\,\bar{W}_{ba}{}^b \ .$$

Since $\bar{W}_{ba}{}^b = -4\bar{T}_{ba}{}^b$ and because one can infer from the second Bianchi identity that the trace of $\bar{T}$ vanishes, one concludes that

$$\bar{D}_\nu\bar{D}_\mu(\bar{e}\,\bar{W}_a{}^{\mu\nu}) = 0 \ . \tag{4.15}$$

It should be emphasized that this result is particular to the geometry at hand. More specifically is it a consequence of the condition $\bar{R}\equiv 0$. From the field equations (4.14) one observes that

$$\bar{D}_\mu(\bar{e}\,\bar{t}_a{}^{\mu}) = \frac{\partial_\mu l}{l}\bar{e}\,\bar{t}_a{}^{\mu} \ . \tag{4.16}$$

A Nested commutators

A.1 Notation

For any two elements X and Y of a Lie algebra we define

$$X \wedge Y \equiv \mathrm{ad}_X(Y) = [X, Y]$$

and consequently

$$X^k \wedge Y \equiv \mathrm{ad}_X^k(Y) = [X, [X, \ldots [X, Y] \ldots]] \ .$$

This can be extended to arbitrary functions, where a function is considered a power series in X, that is

$$f(X) \wedge Y = \sum_k c_k X^k \wedge Y \ .$$

Consider a second function $g(X) = \sum_l d_l X^l$. One obtains

$$g(X) \wedge f(X) \wedge X = \sum_{kl} c_k d_l \mathrm{ad}_X^l(\mathrm{ad}_X^k(Y)) = \sum_{kl} c_k d_l X^{k+l} \wedge Y = g(X) f(X) \wedge Y \ ,$$

where we used the linearity of the adjoint action. From this result it follows that the equation $f(X) \wedge Y = Z$ can be solved for $Y = f(X)^{-1} \wedge Z$. Note that the inverse function also is supposed to be expressed as a power series.

To conclude we write down two identities, using the introduced notation. The first is Hadamard's formula

$$\exp(X) Y \exp(-X) = \exp(X) \wedge Y \ , \tag{A.1}$$

the other is the Campbell-Poincaré fundamental identity,

$$\exp(-X) \delta \exp(X) = \frac{1 - \exp(-X)}{X} \wedge \delta X \ . \tag{A.2}$$

A.2 de Sitter algebra: some results

In this subsection, we compute some intermediary results that are used troughout the text. The commutatation relations considered are those given by (3.1), for the convention $\mathfrak{s} = -1$.

The first identity to be verified is

$$(i\xi \cdot P)^{2n} \wedge \epsilon \cdot P = z^{2n} \left(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2} \right) \ ; \quad n \geqslant 1 \ . \tag{A.3}$$

Therefore, we compute the sequence

$$i\xi \cdot P \wedge \epsilon \cdot P = i\xi^a \epsilon^b [P_a, P_b] = l^{-2} \xi^a \epsilon^b M_{ab} \ ;$$

$$
\begin{aligned}
(i\xi\cdot P)^2\wedge\epsilon\cdot P &= i\xi^c P_c\wedge l^{-2}\xi^a\epsilon^b M_{ab}\\
&= -il^{-2}\xi^a\epsilon^b\xi^c[M_{ab},P_c]\\
&= l^{-2}\xi^a\epsilon^b\xi^c(\eta_{ac}P_b-\eta_{bc}P_a)\\
&= l^{-2}\xi^2\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\ ;\\
(i\xi\cdot P)^4\wedge\epsilon\cdot P &= l^{-2}\xi^2(i\xi\cdot P)^2\wedge\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\\
&= l^{-2}\xi^2(i\xi\cdot P)^2\wedge\epsilon\cdot P\\
&= (l^{-2}\xi^2)^2\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\ ;\\
&\ \ \vdots\\
(i\xi\cdot P)^{2n}\wedge\epsilon\cdot P &= (l^{-2}\xi^2)^n\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\ .
\end{aligned}
$$

The identity follows by letting $z\equiv l^{-1}(\xi^a\xi_a)^{1/2}$.

From (A.3) it follows that

$$
\begin{aligned}
(i\xi\cdot P)^{2n+1}\wedge\epsilon\cdot P &= (i\xi\cdot P)\wedge z^{2n}\left(\epsilon\cdot P-\frac{\xi\cdot\epsilon\xi\cdot P}{\xi^2}\right)\\
&= l^{-2}z^{2n}\xi^a\epsilon^b M_{ab}\ ,
\end{aligned}
$$

hence, another useful identity is given by

$$
(i\xi\cdot P)^{2n+1}\wedge\epsilon\cdot P=\tfrac{1}{2}l^{-2}z^{2n}(\xi^a\epsilon^b-\xi^b\epsilon^a)M_{ab}\ ;\quad n\geqslant 0\ . \tag{A.4}
$$

Finally, the following two identities are derived*

$$
(i\xi\cdot P)^{2n}\wedge\delta h\cdot M=\delta h^{ab}l^{-2}z^{2n-2}\xi^c(\xi_b M_{ac}-\xi_a M_{bc})\ ;\quad n\geqslant 1\ , \tag{A.5}
$$
$$
(i\xi\cdot P)^{2n+1}\wedge\delta h\cdot M=\delta h^{ab}z^{2n}(\xi_a P_b-\xi_b P_a)\ ;\quad n\geqslant 0\ . \tag{A.6}
$$

To verify them consider the following series of equations.

$$
\begin{aligned}
(i\xi\cdot P)\wedge\delta h\cdot M &= \delta h^{ab}\xi^c(-i)[M_{ab},P_c]=\delta h^{ab}(\xi_a P_b-\xi_b P_a)\ ;\\
(i\xi\cdot P)^2\wedge\delta h\cdot M &= 2\delta h^{ab}i\xi\cdot P\wedge\xi_a P_b\\
&= 2\delta h^{ab}\xi_a\xi^c(-i)[P_b,P_c]\\
&= 2\delta h^{ab}l^{-2}\xi_a\xi^c M_{cb}\\
&= \delta h^{ab}l^{-2}\xi^c(\xi_b M_{ac}-\xi_a M_{bc})\ ;\\
(i\xi\cdot P)^4\wedge\delta h\cdot M &= 2\delta h^{ab}l^{-2}\xi_b\xi^c(i\xi\cdot P)^2\wedge M_{ac}\\
&= 2\delta h^{ab}l^{-2}\xi_b\xi^c l^{-2}\xi^d(\xi_c M_{ad}-\xi_a M_{cd})\\
&= 2\delta h^{ab}l^{-2}z^2\xi^d\xi_b M_{ad}
\end{aligned}
$$

*Note that $\delta h\cdot M=\delta h^{ab}M_{ab}$.

$$
\begin{aligned}
&= \delta h^{ab} l^{-2} z^2 \xi^c (\xi_b M_{ac} - \xi_a M_{bc}) \\
&\vdots \\
(i\xi \cdot P)^{2n} \wedge \delta h \cdot M &= \delta h^{ab} l^{-2} z^{2n-2} \xi^c (\xi_b M_{ac} - \xi_a M_{bc}) \\
(i\xi \cdot P)^{2n+1} \wedge \delta h \cdot M &= 2\delta h^{ab} l^{-2} z^{2n-2} \xi^c \xi_b (-i) [M_{ac}, P_d] \\
&= 2\delta h^{ab} l^{-2} z^{2n-2} (\xi_b \xi_a \xi \cdot P - \xi^2 \xi_b P_a) \\
&= \delta h^{ab} z^{2n} (\xi_a P_b - \xi_b P_a) \ .
\end{aligned}
$$

B de Sitter Teleparallel Gravity: intermediate results

In this section we work out to some extend, results related to de Sitter Teleparallel gravity that were used in the main body of the text.

B.1 Variation of $\int \bar{u}_a \bar{e}^a$

In this calculation the variation of $\int \bar{u}_a \bar{e}^a$ will be verified. To begin with let us rewrite the expression for a non-trivial vierbein, i.e.

$$
\begin{aligned}
\bar{e}^a = \cosh z\, e^a - (\cosh z - 1) \frac{\xi_b e^b \xi^a}{\xi^2} \\
+ \frac{\sinh z}{z} (d\xi^a + \omega^a{}_b \xi^b) - \frac{dl}{l} \xi^a - \Big(\frac{\sinh z}{z} - 1 \Big) \frac{\xi_b d\xi^b \xi^a}{\xi^2} \ .
\end{aligned}
$$

We compute:

$$
\begin{aligned}
&\int \bar{u}_a \delta \bar{e}^a \\
&\quad = \int \bar{u}_a \Big\{ \delta \cosh z\, e^a + \cosh z\, \delta e^a{}_\rho dx^\rho + \cosh z\, e^a{}_\rho \delta dx^\rho - \delta \cosh z \frac{\xi_b e^b \xi^a}{\xi^2} \\
&\quad + (\cosh z - 1) \Big[\frac{2\delta\xi}{\xi} \frac{\xi_b e^b \xi^a}{\xi^2} - \frac{\delta\xi_b e^b \xi^a}{\xi^2} - \frac{\xi_b \delta e^b{}_\rho \xi^a}{\xi^2} dx^\rho - \frac{\xi_b e^b \delta\xi^a}{\xi^2} \\
&\quad - \frac{\xi_b e^b{}_\rho \xi^a}{\xi^2} \delta dx^\rho \Big] + \delta \Big(\frac{\sinh z}{z} \Big) (d\xi^a + \omega^a{}_b \xi^b) + \frac{\sinh z}{z} (\delta d\xi^a + \delta\omega^a{}_{b\rho} \xi^b dx^\rho \\
&\quad + \omega^a{}_b \delta\xi^b + \omega^a{}_{b\rho} \xi^b \delta dx^\rho) + \frac{\delta l dl}{l^2} \xi^a - \frac{\delta dl}{l} \xi^a - \frac{1}{l} \delta\xi^a - \delta \Big(\frac{\sinh z}{z} \Big) \frac{\xi_b d\xi^b \xi^a}{\xi^2} \\
&\quad + \Big(\frac{\sinh z}{z} - 1 \Big) \Big[\frac{2\delta\xi}{\xi} \frac{\xi_b d\xi^b \xi^a}{\xi^2} - \frac{\delta\xi_b d\xi^b \xi^a}{\xi^2} - \frac{\xi_b \delta d\xi^b \xi^a}{\xi^2} - \frac{\xi_b d\xi^b \xi^a}{\xi^2} \Big] \Big\}
\end{aligned}
$$

For any function on M, note that $df \to df + d\delta f$ so that $\delta(df) = d(\delta f)$, i.e.

$$
[\delta, d] f = 0 \ . \tag{B.1}
$$

The following variations also are useful:

$$\delta z = \delta(l^{-1}\xi) = -l^{-2}\delta l\,\xi + l^{-1}\delta\xi\ , \tag{B.2}$$

$$\delta\xi = \xi^{-1}\xi_a\delta\xi^a\ . \tag{B.3}$$

The variation is assumed to vanish at the endpoints of the particle's worldline, so that a total derivative over a term containing δx^ρ integrates to zero. One first integrates by parts the terms containing variations of the differentials $\delta dx^\rho = d\delta x^\rho$, after which the boundary integrals render zero. Doing so, one obtains

$$\begin{aligned}
&\int \bar{u}_a\delta\bar{e}^a \\
&\quad= \int \Big[- d\bar{u}_a\Big\{ \cosh z\, e^a{}_\mu - (\cosh z - 1)\frac{\xi_b e^b{}_\mu \xi^a}{\xi^2} + \frac{\sinh z}{z}(\partial_\mu\xi^a + \omega^a{}_{b\mu}\xi^b) \\
&\qquad - \frac{\partial_\mu l}{l}\xi^a - \Big(\frac{\sinh z}{z} - 1\Big)\frac{\xi_b\partial_\mu\xi^b\xi^a}{\xi^2}\Big\}\delta x^\mu + \bar{u}_a\delta x^\mu dx^\rho\Big\{\Big[\frac{\sinh z}{z}\omega^a{}_{b\rho}\Big(\partial_\mu\xi^b \\
&\qquad - \xi^b\frac{\xi_c\partial_\mu\xi^c}{\xi^2}\Big) + \cosh z\,(\partial_\rho\xi^a + \omega^a{}_{b\rho}\xi^b)\frac{\xi_c\partial_\mu\xi^c}{\xi^2} - \partial_\rho\xi^a\frac{\xi_b\partial_\mu\xi^b}{\xi^2} - \cosh z\Big(\partial_\rho\xi^a \\
&\qquad + \omega^a{}_{b\rho}\xi^b - \xi^a\frac{\xi_b\partial_\rho\xi^b}{\xi^2}\Big)\frac{\partial_\mu l}{l} + \partial_\rho\xi^a\frac{\partial_\mu l}{l} - 2(\cosh z - 1)\xi^a\frac{\xi_b\partial_\rho\xi^b}{\xi^2}\frac{\xi_c e^c{}_\mu}{\xi^2} \\
&\qquad + (\cosh z - 1)\partial_\rho\xi^a\frac{\xi_b e^b{}_\mu}{\xi^2} + (\cosh z - 1)\xi^a\frac{\partial_\rho\xi_b e^b{}_\mu}{\xi^2} + z\sinh z\frac{\xi_c\partial_\mu\xi^c}{\xi^2}\Big(e^a{}_\rho \\
&\qquad - \xi^a\frac{\xi_b e^b{}_\rho}{\xi^2}\Big) - z\sinh z\Big(e^a{}_\rho - \xi^a\frac{\xi_b e^b{}_\rho}{\xi^2}\Big)\frac{\partial_\mu l}{l}\Big] - \Big[\rho\leftrightarrow\mu\Big]\Big\} + \bar{u}_a\delta x^\mu dx^\rho\Big\{ \\
&\qquad - \Big[\frac{\partial_\rho l}{l}\frac{\sinh z}{z}\Big(\partial_\mu\xi^a + \omega^a{}_{b\mu}\xi^b - \xi^a\frac{\xi_b\partial_\mu\xi^b}{\xi^2}\Big) + \frac{\partial_\rho l}{l}\frac{\partial_\mu l}{l}\xi^a + \cosh z\,\partial_\rho e^a{}_\mu \\
&\qquad - (\cosh z - 1)\xi^a\frac{\xi_b\partial_\rho e^b{}_\mu}{\xi^2} + \frac{\sinh z}{z}\partial_\rho\omega^a{}_{b\rho}\xi^b\Big] + \Big[\rho\leftrightarrow\mu\Big]\Big\}\Big]
\end{aligned}$$

The terms between the first pair of curly brackets is just the vierbein $\bar{e}^a{}_\mu$, while those between the second pair of curly brackets equal

$$\begin{aligned}
&\Big[\bar{\omega}^a{}_{b\rho}\bar{e}^b{}_\mu - \frac{\sinh z}{z}\omega^a{}_{b\rho}\omega^b{}_{c\mu}\xi^c - \frac{\partial_\rho l}{l}\xi^a\frac{\xi_b\partial_\mu\xi^b}{\xi^2} - \cosh z\,\omega^a{}_{b\rho}e^b{}_\mu \\
&\qquad\qquad - (\cosh z - 1)\xi^a\frac{\omega_{bc\rho}\xi^c e^b{}_\mu}{\xi^2} - z\sinh z\, e^a{}_\rho\frac{\xi_c e^c{}_\mu}{\xi^2}\Big] - \Big[\rho\leftrightarrow\mu\Big]\ .
\end{aligned}$$

This permits us to further work out

$$
\begin{aligned}
&\int \bar{u}_a \delta \bar{e}^a \\
&\quad = \int \Big[- d\bar{u}_a \bar{e}^a{}_\mu \delta x^\mu + \bar{u}_a \delta x^\mu dx^\rho \Big\{ \Big[\bar{\omega}^a{}_{b\rho} \bar{e}^b{}_\mu - \frac{\sinh z}{z} \xi^c \Big(\partial_\rho \omega^a{}_{c\rho} + \omega^a{}_{b\rho} \omega^b{}_{c\mu} \\
&\qquad + \frac{1}{l^2} e^a{}_\rho e_{c\mu} \Big) - \cosh z \Big(\partial_\rho e^a{}_\mu + \omega^a{}_{b\rho} e^b{}_\mu - \frac{\partial_\rho l}{l} e^a{}_\mu \Big) - (1 - \cosh z) \frac{\xi^a \xi_b}{\xi^2} \Big(\partial_\rho e^b{}_\mu \\
&\qquad + \omega^b{}_{c\rho} e^c{}_\mu - \frac{\partial_\rho l}{l} e^b{}_\mu \Big) - \frac{\partial_\rho l}{l} \Big(\cosh z \, e^a{}_\mu - (\cosh z - 1) \frac{\xi_b e^b{}_\mu \xi^a}{\xi^2} + \frac{\sinh z}{z} (\partial_\mu \xi^a \\
&\qquad + \omega^a{}_{b\mu} \xi^b) - \frac{\partial_\mu l}{l} \xi^a - \Big(\frac{\sinh z}{z} - 1 \Big) \frac{\xi_b \partial_\mu \xi^b \xi^a}{\xi^2} \Big) \Big] - \Big[\rho \leftrightarrow \mu \Big] \Big\} \Big] \\
&\quad = \int \delta x^\mu \Big[- d\bar{u}_a \bar{e}^a{}_\mu + \bar{u}_a \bar{\omega}^a{}_{b\rho} \bar{e}^b{}_\mu dx^\rho - \bar{u}_a \bar{\omega}^a{}_{b\mu} \bar{e}^b{}_\rho dx^\rho - \bar{u}_a dx^\rho \Big(\frac{\sinh z}{z} \xi^c R^a{}_{c\rho\mu} \\
&\qquad + \cosh z \, T^a{}_{\rho\mu} + (1 - \cosh z) \frac{\xi^a \xi_b T^b{}_{\rho\mu}}{\xi^2} \Big) - \frac{\partial_\rho l}{l} \bar{e}^a{}_\mu \bar{u}_a dx^\rho + \frac{\partial_\mu l}{l} \bar{e}^a{}_\rho \bar{u}_a dx^\rho \Big]
\end{aligned}
$$

In this expression one recognizes the torsion $\bar{T}$, as given in Eq. (3.15b). This leads to the end of this calculation:

$$
\int \bar{u}_a \delta \bar{e}^a = \int d\tau \delta x^\mu \Big\{ - \bar{e}^a{}_\mu \Big(\frac{d\bar{u}_a}{d\tau} - \bar{\omega}^b{}_{a\rho} \bar{u}_b u^\rho + u^\rho \frac{\partial_\rho l}{l} \bar{u}_a \Big) + \bar{T}^a{}_{\mu\rho} \bar{u}_a u^\rho + \frac{\partial_\mu l}{l} \Big\} \, . \tag{B.4}
$$

B.2 Variation of $\mathcal{L}$ with respect to $\bar{e}$.

In this subsection we work out some intermediary results that lead to the functional variation of the Lagrangian (4.11) with respect to the vierbein. More precisely, we calculate the derivatives of $\mathcal{L}$ with respect to $\bar{e}^c{}_\sigma$ and $\partial_\rho \bar{e}^c{}_\sigma$ in turn. From (4.11):

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \bar{e}^c{}_\sigma} = &\frac{1}{4} \frac{\partial \bar{T}^a{}_{\mu\nu}}{\partial \bar{e}^c{}_\sigma} \bar{T}_a{}^{\mu\nu} + \frac{1}{4} \bar{T}^a{}_{\mu\nu} \frac{\partial \bar{T}_a{}^{\mu\nu}}{\partial \bar{e}^c{}_\sigma} + \frac{1}{2} \frac{\partial \bar{T}^a{}_{\mu\nu}}{\partial \bar{e}^c{}_\sigma} \bar{T}^{b\mu}{}_\lambda \, \bar{e}_a{}^\lambda \bar{e}_b{}^\nu + \frac{1}{2} \bar{T}^a{}_{\mu\nu} \frac{\partial \bar{T}^{b\mu}{}_\lambda}{\partial \bar{e}^c{}_\sigma} \bar{e}_a{}^\lambda \bar{e}_b{}^\nu \\
&+ \frac{1}{2} \bar{T}^a{}_{\mu\nu} \bar{T}^{b\mu}{}_\lambda \frac{\partial (\bar{e}_a{}^\lambda \bar{e}_b{}^\nu)}{\partial \bar{e}^c{}_\sigma} - \frac{\partial \bar{T}^a{}_{\mu\nu}}{\partial \bar{e}^c{}_\sigma} \bar{T}^{b\mu}{}_\lambda \, \bar{e}_a{}^\nu \bar{e}_b{}^\lambda - \bar{T}^a{}_{\mu\nu} \frac{\partial \bar{T}^{b\mu}{}_\lambda}{\partial \bar{e}^c{}_\sigma} \bar{e}_a{}^\nu \bar{e}_b{}^\lambda \\
&- \bar{T}^a{}_{\mu\nu} \bar{T}^{b\mu}{}_\lambda \frac{\partial (\bar{e}_a{}^\nu \bar{e}_b{}^\lambda)}{\partial \bar{e}^c{}_\sigma}
\end{aligned}
$$

It is thus useful to consider first the following equalities:

$$
\begin{aligned}
\frac{\partial \bar{T}^a{}_{\mu\nu}}{\partial \bar{e}^c{}_\sigma} &= [\bar{\omega}^a{}_{c\mu} \delta^\sigma_\nu - \frac{\partial_\mu l}{l} \delta^a_c \delta^\sigma_\nu] - [\mu \leftrightarrow \nu] \ , \\
\frac{\partial g_{\rho\lambda}}{\partial \bar{e}^c{}_\sigma} &= \frac{\partial (\bar{e}^a{}_\rho \bar{e}_{a\lambda})}{\partial \bar{e}^c{}_\sigma} = \bar{e}_{c\lambda} \delta^\sigma_\rho + \bar{e}_{c\rho} \delta^\sigma_\lambda \ , \\
\frac{\partial g^{\rho\lambda}}{\partial \bar{e}^c{}_\sigma} &= -g^{\sigma\rho} \bar{e}_c{}^\lambda - g^{\sigma\lambda} \bar{e}_c{}^\rho \ .
\end{aligned}
$$

Subsequently it is possible to obtain

$$
\begin{aligned}
\frac{\partial \bar{e}_a{}^\lambda}{\partial \bar{e}^c{}_\sigma} &= -\bar{e}_a{}^\sigma \bar{e}_c{}^\lambda \ , \\
\frac{\partial \bar{T}_a{}^{\mu\nu}}{\partial \bar{e}^c{}_\sigma} &= [\eta_{ab}\bar{\omega}^b{}_{c\alpha} g^{\alpha\mu} g^{\sigma\nu} - \eta_{ac}\frac{\partial_\lambda l}{l} g^{\lambda\mu} g^{\sigma\nu} + \bar{T}_a{}^{\sigma\mu}\bar{e}_c{}^\nu \\
&\quad + \bar{T}_{a\lambda}{}^\mu \bar{e}_c{}^\lambda g^{\sigma\nu}] - [\mu \leftrightarrow \nu] \ , \\
\frac{\partial \bar{T}^{b\mu}{}_\lambda}{\partial \bar{e}^c{}_\sigma} &= g^{\rho\mu}\bar{\omega}^b{}_{c\rho}\delta^\sigma_\lambda - g^{\sigma\mu}\bar{\omega}^b{}_{c\lambda} - g^{\rho\mu}\frac{\partial_\rho l}{l}\delta^b_c\delta^\sigma_\lambda + g^{\sigma\mu}\frac{\partial_\lambda l}{l}\delta^b_c \\
&\quad - \bar{T}^b{}_{\rho\lambda}\,\bar{e}_c{}^\mu g^{\sigma\rho} - \bar{T}^b{}_{\rho\lambda}\,\bar{e}_c{}^\rho g^{\sigma\mu} \ .
\end{aligned}
$$

Substituting these equations for $\partial\mathcal{L}/\partial\bar{e}^c{}_\sigma$, it takes some algebra to get the following:

$$
\frac{\partial \mathcal{L}}{\partial \bar{e}^c{}_\sigma} = \bar{\omega}^a{}_{c\mu}\bar{W}_a{}^{\mu\sigma} + \bar{T}^a{}_{\mu c}\bar{W}_a{}^{\sigma\mu} - \frac{\partial_\mu l}{l}\bar{W}_c{}^{\mu\sigma} \ . \tag{B.5}
$$

It is a simpler exercise to find the derivative of the Lagrangian with respect to the first order derivatives of the vierbein. One only needs the expression

$$
\frac{\partial \bar{T}^a{}_{\mu\nu}}{\partial\partial_\rho \bar{e}^c{}_\sigma} = \delta^\rho_\mu\delta^\sigma_\nu\delta^a_c - \delta^\rho_\nu\delta^\sigma_\mu\delta^a_c \ .
$$

This is sufficient since the derivative operator annihilates the metric $g_{\mu\nu} = \bar{e}^a{}_\mu \bar{e}_{a\nu}$ and we can freely raise and lower spacetime indices. Using this information, it is readily found that

$$
\frac{\partial \mathcal{L}}{\partial\partial_\rho \bar{e}^c{}_\sigma} = \bar{W}_c{}^{\rho\sigma} \ . \tag{B.6}
$$

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