Cartan geometry

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1 Lie algebra-valued differential forms

Let $\mathcal{M}$ be an m-dimensional smooth manifold and let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. The space of $\mathfrak{g}$-valued differential p-forms on $\mathcal{M}$, i.e., $\mathfrak{g} \otimes \Omega^p(\mathcal{M})$, is denoted by $\Omega^p(\mathcal{M}, \mathfrak{g})$. An element $\eta \in \Omega^p(\mathcal{M}, \mathfrak{g})$ may be expanded in a basis E_a for $\mathfrak{g}$ as $\eta = \eta^a \otimes E_a$, also written as $\eta^a E_a$, and where each $\eta^a \in \Omega^p(\mathcal{M})$.

For any two elements $\eta \in \Omega^p(\mathcal{M}, \mathfrak{g})$ and $\theta \in \Omega^q(\mathcal{M}, \mathfrak{g})$ with $p + q \leq m$, a Lie bracket is defined by

$$[.,.] : \Omega^p(\mathcal{M}, \mathfrak{g}) \times \Omega^q(\mathcal{M}, \mathfrak{g}) \to \Omega^{p+q}(\mathcal{M}, \mathfrak{g}) : (\eta, \theta) \mapsto [\eta, \theta] = \eta^a \wedge \theta^b \otimes [E_a, E_b],$$

where the bracket in the last term is of course the ordinary Lie bracket of $\mathfrak{g}$. It is easily verified that this operation is a graded commutator, namely,

$$[\eta, \theta] = (-1)^{pq+1}[\theta, \eta].$$

The Jacobi identity for $\mathfrak{g}$ consequently generalizes to a graded Jacobi identity (let $\omega \in \Omega^r(\mathcal{M}, \mathfrak{g})$:

$$(-1)^{rp}[[\eta, \theta], \omega] + (-1)^{pq}[[\theta, \omega], \eta] + (-1)^{qr}[[\omega, \eta], \theta] \equiv 0.$$

The exterior derivative $d : \Omega^p(\mathcal{M}) \to \Omega^{p+1}(\mathcal{M})$ of differential forms can equally be defined an operation on the space of $\mathfrak{g}$-valued differential forms, by limiting its action on the form parts, i.e.,

$$d : \Omega^p(\mathcal{M}, \mathfrak{g}) \to \Omega^{p+1}(\mathcal{M}, \mathfrak{g}) : \eta \mapsto d\eta = d\eta^a \otimes E_a.$$

This derivative respects a graded Leibniz rule:

$$d[\eta, \theta] = [d\eta, \theta] + (-1)^p[\eta, d\theta].$$