Cosmology

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Abstract

This is to be a rough survey of introductory concepts in Cosmology.

1 Kinematics due to the Cosmological Principle

From the Cosmological Principle, which is the assumption that space is homogeneous and isotropic on large scales, one is lead to the Friedmann-Lemaître-Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)d\Sigma_3 \tag{1}$$

where $d\Sigma_3$ is a three-dimensional space of constant curvature. It can be expressed as [Expand this by argumentating the form; fundamental observers, etc. See ...]

$$d\Sigma_3 = d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

$$S(\chi) = \begin{cases} \sin\chi & \text{positive spatial curvature} \\ \chi & \text{no spatial curvature} \\ \sinh\chi & \text{negative spatial curvature} \end{cases} \tag{3}$$

Note that the *comoving* coordinates—the coordinates of fundamental observers—χ, θ, ϕ are constants in time. These observers will move relative to eachother because their seperation distances change due to the time dependence of the scale factor $a(t)$, but their coordinate distance is independepent of time.

Due to the timedependent scale factor in the FRW–metric, identical photons which are emitted at different cosmic time will generally have different energies when observed on earth. This can be understood by noting that because of the changing scale factor, the distance between two wavecrests will increase. The relative change in wavelength is called the redshift factor z and is given by

$$z = \frac{\Delta\lambda}{\lambda_e} = \frac{a(t_o)}{a(t_e)} - 1 \tag{4}$$

Note that for an increasing scale factor photons are redshifted ($z > 0$), while for a decreasing factor photons are blueshifted ($z < 0$).

The scale factor $a(t)$ is not known at all cosmic times. One therefore starts with a Taylor series expansion around "now",[1] i.e. t_o. Doing so one finds,

$$\frac{a(t)}{a(t_o)} = 1 - \frac{\dot{a}(t_o)}{a(t_o)}(t_o - t) + \tfrac{1}{2}\ddot{a}(t_o)(t_o - t)^2 + O(3)$$

[1] Note that if photons are emitted very close to now, that this also implies that the emitting object is nearby in space. Of course what is considered close is to be defined with respect to the overall scale. This is also reflected in taking a taylor series w.r.t. a-priori dimensionful quantities. For example, when expanding around now, the quantity $\delta t = t - t_o$ is really $t - t_o/t_o$.

The Hubble parameter $H(t)$ and decelaration parameter $q(t)$ are then defined as

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \tag{5}$$

$$q(t) \equiv -\frac{\ddot{a}(t)a(t)}{\dot{a}^2(t)} \tag{6}$$

Hence, the scale parameter can be recast in the form

$$\frac{a(t)}{a(t_o)} = 1 - H_o(t_o - t) + \tfrac{1}{2}q_o H_o^2 (t_o - t)^2 + O(3) \tag{7}$$

It is a same exercise to find the redshift of a photon emitted at time t, as a Taylor expansion around t_0;

$$z(t) = H_o(t_o - t) + H_o^2(1 + \tfrac{1}{2}q_o)(t_o - t)^2 + O(3) \tag{8}$$

For nearby objects, the cosmological redshift is $z \approx H_o(t_o - t)$. Since for such objects $(t_o - t)$ is approximately equal to the proper distance between us and the object (t being the time the photon was emitted), one has *Hubble's Law*, that is

$$z \approx H_o d_P \tag{9}$$

Proper distance A photon travels along a null geodesic such that along its worldline, one always has $dt = \pm a(t)d\chi$. The proper distance between two fundamental observers at time t_o is given by

$$d_P \equiv \int_0^\chi a(t_o)d\chi \tag{10}$$

Let t_e be the time at which a photon was emitted at $\chi = \chi_e$ that has arrived now at the earth. One can rewrite the proper distance then as

$$d_P = a(t_o)\int_{t_e}^{t_o} \frac{dt}{a(t)} \tag{11}$$

Using the relation $dt = -H^{-1}(1+z)^{-1}dz$, this integral may be rewritten as

$$d_P = -\int_z^0 \frac{dz}{H(z)} = \int_0^z \frac{dz}{H(z)} \tag{12}$$

Which is the proper distance between us (as a fundamental observer) and another fundamental observer seen at redshift z.

Luminosity distance Let L be the *absolute luminosity* radiated by a fundamental observer (χ_e) and let f be the *observed flux*. The *luminosity distance* is the distance defined as

$$d_L = \left(\frac{L}{4\pi f}\right)^{1/2} \tag{13}$$

The observed flux will depend on the large scale geometry of spacetime. Indeed, it is the observed luminosity divided by the area of a sphere surrounding the emitting object. The observed luminosity will be redshifted twice: once because the photons are redshifted and once because the arrival rate will be affected by the same rate–the latter can be understood by imagining that the space *between* photons is scaled with a factor $1 + z$. Since the area of a sphere is given by $4\pi a^2(t_o)S^2(\chi_e)$, one finds that the observed flux is given by

$$f = \frac{L(1+z)^{-2}}{4\pi a^2(t_o)S^2(\chi_e)}$$

One concludes that the luminosity distance to an object at coordinate χ and redshift z is given by

$$d_L = a(t_o)S(\chi)(1+z) \tag{14}$$

Note that for flat spacelike sections $S(\chi) = \chi$ and one finds that

$$d_L = d_P(1+z) = (1+z)\int_0^z \frac{d\bar{z}}{H(\bar{z})}$$

2 Dynamics due to general relativity