Nonlinear realizations bis

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Abstract

Omitted appendix of article on cosmological function.

1 Equations

$$\exp(-i\xi\cdot P)i\epsilon\cdot P\exp(i\xi\cdot P)-\exp(-i\xi\cdot P)\delta\exp(i\xi\cdot P)=\tfrac{i}{2}\delta h\cdot M. \tag{1.1}$$

$$g_0\exp(i\xi\cdot P)=\exp(i\xi'\cdot P)h'\ ;\qquad h'=\tilde{h}'\tilde{h}^{-1}, \tag{1.2}$$

$$\begin{aligned}-i[M_{ab},M_{cd}]&=\eta_{ac}M_{bd}-\eta_{ad}M_{bc}+\eta_{bd}M_{ac}-\eta_{bc}M_{ad},\\ -i[M_{ab},P_c]&=\eta_{ac}P_b-\eta_{bc}P_a,\\ -i[P_a,P_b]&=-l^{-2}M_{ab}.\end{aligned} \tag{1.3}$$

$$\delta\xi^a=\epsilon^a+\Big(\frac{z\cosh z}{\sinh z}-1\Big)\Big(\epsilon^a-\frac{\xi^a\epsilon_b\xi^b}{\xi^2}\Big), \tag{1.4}$$

$$\delta h^{ab}=\frac{1}{l^2}\frac{\cosh z-1}{z\sinh z}(\epsilon^a\xi^b-\epsilon^b\xi^a), \tag{1.5}$$

$$\omega^{ab}=A^{ab}-\frac{\cosh z-1}{l^2z^2}\left[\xi^a(d\xi^b+A^b{}_c\xi^c)-\xi^b(d\xi^a+A^a{}_c\xi^c)\right]-\frac{\sinh z}{l^2z}(\xi^aA^b-\xi^bA^a), \tag{1.6a}$$

$$\begin{aligned}e^a=A^a&+\frac{\sinh z}{z}(d\xi^a+A^a{}_b\xi^b)-\frac{dl}{l}\xi^a\\ &+(\cosh z-1)\Big(A^a-\frac{\xi^bA_b\xi^a}{\xi^2}\Big)-\Big(\frac{\sinh z}{z}-1\Big)\frac{\xi^bd\xi_b\xi^a}{\xi^2}.\end{aligned} \tag{1.6b}$$

$$\bar{A}=\mathrm{Ad}(\exp(-i\xi\cdot P))(A+d). \tag{1.7}$$

2 Nonlinear realizations

In this appendix, some intermediate steps that are useful to verify the results of Sec. 4 are explained; see also [1].

For any two elements of a Lie algebra $\mathfrak{g}$, the adjoint action is denoted by

$$\wedge : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (X, Y) \mapsto X \wedge Y = \mathrm{ad}_X(Y) = [X, Y].$$

Note that the symbol $\wedge$ for the adjoint action of a Lie algebra is exclusive to this section, and should not be confused with the wedge product of differential forms, which is exclusive to the other sections throughout the text.

Moreover, we write

$$X^k \wedge Y = \mathrm{ad}_X^k(Y) = [X, [X, \ldots [X, Y] \ldots]],$$

so that for a power series $f(X) = \sum_k c_k X^k$

$$f(X) \wedge Y = \sum_k c_k X^k \wedge Y.$$

Given a second function $g(X) = \sum_l d_l X^l$, one obtains

$$g(X) \wedge f(X) \wedge X = \sum_{kl} c_k d_l \mathrm{ad}_X^l(\mathrm{ad}_X^k(Y)) = \sum_{kl} c_k d_l X^{k+l} \wedge Y = g(X) f(X) \wedge Y,$$

because of the linearity of the adjoint action. From this result it follows that the equation $f(X) \wedge Y = Z$ can be solved for $Y = f(X)^{-1} \wedge Z$. Note that the inverse function is supposed to be expressed as a power series. The following two identities are useful in carrying out the calculations of this section. The first is Hadamard's formula, namely,

$$\exp(X) Y \exp(-X) = \exp(X) \wedge Y. \tag{2.1}$$

The other is the Campbell-Poincaré fundamental identity, given by

$$\exp(-X)\delta\exp(X) = \frac{1 - \exp(-X)}{X} \wedge \delta X. \tag{2.2}$$

Let us then solve (1.1) for $\delta\xi \cdot P$ and $\delta h \cdot M$. Since $\mathfrak{g} = \mathfrak{so}(1,4) = \mathfrak{so}(1,3) \oplus \mathfrak{p}$ is symmetric, there is an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{so}(1,3)$ is an eigenspace with eigenvalue 1, while $\mathfrak{p} = \mathfrak{so}(1,4)/\mathfrak{so}(1,3)$ is an eigenspace with eigenvalue -1 [2]. This automorphism allows one to eliminate δh from (1.1), which leads to the expression

$$\begin{aligned} \frac{1 - \exp(-i\xi \cdot P)}{i\xi \cdot P} \wedge i\delta\xi \cdot P - \frac{1 - \exp(i\xi \cdot P)}{i\xi \cdot P} \wedge i\delta\xi \cdot P \\ = \exp(-i\xi \cdot P) \wedge i\epsilon \cdot P + \exp(i\xi \cdot P) \wedge i\epsilon \cdot P, \end{aligned}$$

and where we made use of the identities (2.1) and (2.2). This equation is solved for $\delta\xi^a$, which results in

$$i\delta\xi \cdot P = \frac{i\xi \cdot P \cosh(i\xi \cdot P)}{\sinh(i\xi \cdot P)} \wedge i\epsilon \cdot P. \tag{2.3}$$

Substituting (2.3) in (1.1), one subsequently solves for δh^{ab}:

$$\tfrac{i}{2}\delta h \cdot M = \frac{1 - \cosh(i\xi \cdot P)}{\sinh(i\xi \cdot P)} \wedge i\epsilon \cdot P, \tag{2.4}$$

which relates the infinitesimal element $h'(\xi, \epsilon) = 1 + \delta h$ in (1.2) to the transvection $g_0 = 1 + i\epsilon \cdot P$. In the expressions (2.3) and (2.4), the hyperbolic functions are given by the corresponding power series in $i\xi \cdot P$. They act on $i\epsilon \cdot P$ through the adjoint action, which can be worked out explicitly for the commutation relations (1.3). Doing so, the following intermediate results are useful ($z = l^{-1}\xi$ and $\xi = (\eta_{ab}\xi^a\xi^b)^{1/2}$):

$$\begin{aligned}
(i\xi \cdot P)^{2n} \wedge \epsilon \cdot P &= z^{2n}\left(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2}\right) \; ; \quad n \geqslant 1, \\
(i\xi \cdot P)^{2n+1} \wedge \epsilon \cdot P &= \tfrac{1}{2} l^{-2} z^{2n} (\xi^a \epsilon^b - \xi^b \epsilon^a) M_{ab} \; ; \quad n \geqslant 0, \\
(i\xi \cdot P)^{2n} \wedge \delta h \cdot M &= \delta h^{ab} l^{-2} z^{2n-2} \xi^c (\xi_b M_{ac} - \xi_a M_{bc}) \; ; \quad n \geqslant 1 \\
\text{and} \quad (i\xi \cdot P)^{2n+1} \wedge \delta h \cdot M &= \delta h^{ab} z^{2n} (\xi_a P_b - \xi_b P_a) \; ; \quad n \geqslant 0.
\end{aligned}$$

By means of these relations, the variations (2.3) and (2.4) are found to be given by

$$i\delta\xi \cdot P = i\epsilon \cdot P + \left(\frac{z \cosh z}{\sinh z} - 1\right)\left(i\epsilon \cdot P - \frac{\xi \cdot \epsilon \, i\xi \cdot P}{\xi^2}\right),$$

and

$$\tfrac{i}{2}\delta h \cdot M = \frac{i}{2l^2} \frac{\cosh z - 1}{z \sinh z} (\epsilon^a \xi^b - \epsilon^b \xi^a) M_{ab},$$

confirming the equalities (1.4) and (1.5).

Next we verify that the spin connection and vierbein in the nonlinear de Sitter-Cartan geometry are given by (1.6a) and (1.6b). Invoking Hadamard's formula (2.1) and the Campbell-Poincaré fundamental identity (2.2), the right-hand side of (1.7) can be rewritten as

$$\exp(-i\xi \cdot P) \wedge \left(\tfrac{i}{2} A^{ab} M_{ab} + i A^a P_a\right) + \frac{1 - \exp(-i\xi \cdot P)}{i\xi \cdot P} \wedge d(i\xi \cdot P).$$

Working out the different terms of this expression, it is found successively that

$$\begin{aligned}
\exp(-i\xi \cdot P) \wedge \tfrac{i}{2} A^{ab} M_{ab} = \tfrac{i}{2}\Big(A^{ab} &+ \frac{\cosh z - 1}{l^2 z^2} \xi_c (\xi^b A^{ac} - \xi^a A^{bc})\Big) M_{ab} \\
&+ i(z^{-1} \sinh z A^a{}_b \xi^b) P_a,
\end{aligned}$$

$$
\begin{aligned}
\exp(-i\xi\cdot P)\wedge iA^aP_a &= \tfrac{i}{2}\Big(\frac{\sinh z}{l^2 z}(A^a\xi^b - A^b\xi^a)\Big)M_{ab} \\
&\qquad + i\Big(A^a + (\cosh z - 1)\Big(A^a - \frac{\xi^b A_b \xi^a}{\xi^2}\Big)\Big)P_a, \\
\frac{1-\exp(-i\xi\cdot P)}{i\xi\cdot P}\wedge d(i\xi\cdot P) &= \tfrac{i}{2}\Big(\frac{\cosh z - 1}{l^2 z^2}(d\xi^a\xi^b - d\xi^b\xi^a)\Big)M_{ab} \\
&\qquad + i\Big(\frac{\sinh z}{z}\Big(d\xi^a - \frac{\xi^b d\xi_b \xi^a}{\xi^2}\Big) + \frac{\xi^b d\xi_b \xi^a}{\xi^2} - \frac{dl}{l}\xi^a\Big)P_a.
\end{aligned}
$$

Collecting these different contributions and separating terms according to whether they are valued in $\mathfrak{so}(1,3)$, respectively $\mathfrak{p}$, one recovers (1.6a) and (1.6b).

References

[1] B. Zumino, “Non-linear realization of supersymmetry in anti de Sitter space,” *Nucl. Phys.* **B127** (1977) 189–201.

[2] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 2. Wiley-Interscience, New York, 1996. Reprint of the 1969 original.