Inönü-Wigner contraction

1 Inönü-Wigner contraction of a Lie algebra

Consider an n-dimensional Lie algebra $\mathfrak{g} = (V, [\cdot,\cdot])$, where V is the underlying vector space over some field $\mathbb{F}$ and $[\cdot,\cdot]$ is the Lie bracket, which expressed in a particular basis X_i, $i = 0 \ldots n$ is characterized by the structure constants ${c_{ij}}^k$,

$$[X_i, X_j] = {c_{ij}}^k X_k \tag{1}$$

Consider next the continuous function $U : (0, \varepsilon] \to GL(V)$. Hence $U(\varepsilon)$ is a non-singular linear operator on V, which transforms the basis X_i into a new basis Y_i, with corresponding Lie bracket

$$[Y_i, Y_j] = {\tilde{c}_{ij}}^k Y_k \ . \tag{2}$$

It is direct consequence of the transformation properties of the $Y_i = {U(\varepsilon)_i}^j X_j$ that the structure constants transform as a $(1,2)$-tensor, i.e.

$${\tilde{c}_{ij}}^k = {U(\varepsilon)_i}^s {U(\varepsilon)_j}^t {c_{st}}^r {U^{-1}(\varepsilon)_r}^k \tag{3}$$

[QUESTION: The Lie bracket implies that ${c_{ij}}^k X_k$ is a vector, then how is this to be consistent with the transformation properties of c, as these seem to imply that the Lie bracket gives a (1,1)-tensor.]

It is clear that the Lie algebras defined by (1) and (2) are isomorphic, since they are related by a non-singular linear transformation. However, if we consider a mapping $U(\varepsilon)$ which becomes singular for ε going to zero, a new algebra might be the result. Since an algebra is charecterized completely by its structure constants, one has to investigate their behaviour in the limit of a singular transformation. More precisely, if the limit

$$\lim_{\varepsilon \to 0} {\tilde{c}_{ij}}^k \tag{4}$$

exists, the result is a new Lie algebra $\mathfrak{g}_0 = (V, [\cdot,\cdot]_0)$ which may or may not be isomorphic with the original algebra. This process of obtaining a new Lie algebra through the limiting procedure (4) is referred to as *contraction*. The dimension of the contracted algebra will be equal to the dimension of the original algebra, as we are assuming that the basis elements are well defined for the limiting singular transformation.

There are of course different possibilities to have $U(\varepsilon)$ singular when the parameter approaches zero. However, we will limit our interest to the case where the function is of the form

$$U(\varepsilon) = \mathrm{diag}(1, \ldots, 1, \varepsilon, \ldots, \varepsilon) \tag{5}$$

and the contracting procedure is called *Inönü-Wigner contraction*.

Now, the diagonal form of U implies that the underlying vector space (the algebra) has the direct sum structure

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \tag{6}$$

where $\mathfrak{h}$ is spanned by X_α and $\mathfrak{p}$ is spanned by X_a, for which we evidently have

$$U(\varepsilon) X_\alpha = X_\alpha$$
$$U(\varepsilon) X_a = \varepsilon X_a$$

The structure constants, on the other hand, transform as

$$\tilde{c}_{ij}{}^{k} = \varepsilon^{p} c_{ij}{}^{k} \tag{7}$$

where p is the difference in covariant and contravariant Latin indexes of the structure constant tensor.[1] In contracting the algebra, there will be a problem of convergence if $p = -1$, i.e. if there are two covariant Greek indices and one contravariant Latin index. Hence, the contraction procedure will only give a well-defined structure constant tensor if for the original algebra $c_{\alpha\beta}{}^{c} = 0$, which means that the commutation relations for $\mathfrak{g}$ are given by

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \tag{8}$$

$$[\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{g} \tag{9}$$

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{g} \tag{10}$$

where the first relation, necessary to have the limit well-defined, means that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Let us now finish the contraction procedure and have a look at the resulting algebra $\mathfrak{g}_0$. Since

$$\begin{bmatrix} \mathfrak{h}_0 \\ \mathfrak{p}_0 \end{bmatrix} = \lim_{\varepsilon \to 0} \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} \mathfrak{h} \\ \mathfrak{p} \end{bmatrix} \tag{11}$$

we find the algebra $\mathfrak{g}_0 = \mathfrak{h} \oplus_s \mathfrak{p}_0$ with commutation relations

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \tag{12}$$

$$[\mathfrak{h}, \mathfrak{p}_0] \subseteq \mathfrak{p}_0 \tag{13}$$

$$[\mathfrak{p}_0, \mathfrak{p}_0] = 0 \tag{14}$$

where we used that $\lim_{\varepsilon \to 0} \varepsilon \mathfrak{g} = \mathfrak{p}_0$ and $\lim_{\varepsilon \to 0} \varepsilon^2 \mathfrak{g} = 0$. The algebra $\mathfrak{g}$ has been contracted with respect to the subalgebra $\mathfrak{h}$ into the algebra $\mathfrak{g}_0$ which has the same subalgebra $\mathfrak{h}$, whereas the complement $\mathfrak{p}$ of $\mathfrak{h}$ in $\mathfrak{g}$ has become an Abelian ideal $\mathfrak{p}_0$ of $\mathfrak{g}_0$, such that the contracted algebra therefore always is non-semisimple. Since the contraction procedure replaces $\mathfrak{p}$ by $\mathfrak{p}_0$, the Lie bracket (13) is inherited from the original algebra (for the $\mathfrak{p}_0$-terms).

2 Example: from de Sitter to Poincaré

In this section, the algebra $\mathfrak{so}(1,4)$ will be contracted into the Poincaré algebra $\mathfrak{iso}(1,3)$. Therefore, we recall the commutation relations for the generators L_{ij}, spanning $\mathfrak{so}(1,4)$ $(i = 0 \dots 4)$

$$[L_{ij}, L_{kl}] = \eta_{il} L_{jk} + \eta_{jk} L_{il} - \eta_{ik} L_{jl} - \eta_{jl} L_{ik} \tag{15}$$

This algebra can be contracted with respect to its subalgebra $\mathfrak{so}(1,3)$. Hence we consider the direct sum structure $\mathfrak{so}(1,4) = \mathfrak{so}(1,3) \oplus \mathfrak{p}$, which are spanned respectively by $L_{\alpha\beta}$ and $L_{\alpha 4}$ $(\alpha = 0 \dots 3)$ and for which we rewrite the commutation relations (15) explicitly as

$$\begin{aligned}
[L_{\alpha\beta}, L_{\kappa\lambda}] &= \eta_{\alpha\lambda} L_{\beta\kappa} + \eta_{\beta\kappa} L_{\alpha\lambda} - \eta_{\alpha\kappa} L_{\beta\lambda} - \eta_{\beta\lambda} L_{\alpha\kappa} \\
[L_{\alpha 4}, L_{\kappa\lambda}] &= \eta_{\alpha\lambda} L_{4\kappa} - \eta_{\alpha\kappa} L_{4\lambda} \\
[L_{\alpha 4}, L_{\beta 4}] &= \eta_{44} L_{\beta\alpha}
\end{aligned} \tag{16}$$

Consider the following non-singular basis transformation

$$L_{\alpha\beta} \to L_{\alpha\beta} \quad \text{and} \quad L_{\alpha 4} \to \Pi_\alpha \equiv \varepsilon L_{\alpha 4} \tag{17}$$

We now contract the algebra $\mathfrak{so}(1,4)$ with respect to $\mathfrak{so}(1,3)$. Defining $P_\alpha \equiv lim_{\varepsilon \to 0} \Pi_\alpha$ we find

$$\begin{aligned}
[L_{\alpha\beta}, L_{\kappa\lambda}] &= \eta_{\alpha\lambda} L_{\beta\kappa} + \eta_{\beta\kappa} L_{\alpha\lambda} - \eta_{\alpha\kappa} L_{\beta\lambda} - \eta_{\beta\lambda} L_{\alpha\kappa} \\
[P_\alpha, L_{\kappa\lambda}] &= \eta_{\alpha\kappa} P_\lambda - \eta_{\alpha\lambda} P_\kappa \\
[P_\alpha, P_\beta] &= 0
\end{aligned} \tag{18}$$

[1] This number p follows automatically from the diagonal structure of U and the transformation properties of the structure constants.

which are commutation relations, defining the Poincaré algebra which has the semidirect sum structure $\mathfrak{iso}(1,3) = \mathfrak{so}(1,3) \oplus_s \mathfrak{t}$.

The above discussion is equally valid for the case where we start from the anti de Sitter algebra $\mathfrak{so}(2,3)$, since this algebra also has a Lorentz subalgebra. However, this would not be the case for $\mathfrak{so}(0,5)$.

3 From de Sitter to Poincaré algebra(s): stereographic coordinates

In this section we will explicitly consider the IW-contraction of the de Sitter into the Poincaré algebra. More precisely, we will consider the realization of the de Sitter algebra as differential operators in stereographic coordinates, after which we will perform an IW-contraction. A first possibility, which gives well defined basis elements in the singular limit, is the contraction from de Sitter rotations into ordinary translations. Interestingly enough, there will be a second possibility for a consistent contraction into a different set of differential operators, namely the special conformal transformations.

To begin with, consider the basis elements of $\mathfrak{so}(1,4)$ realized as differential operators on function space, which are given by (in terms of the Cartesian coordinates labeling the embedding pseudo-Euclidean space)

$$L_{\mu\nu} = \eta_{\mu\lambda}\chi^\lambda \partial_\nu - \eta_{\nu\lambda}\chi^\lambda \partial_\mu \tag{19}$$

$$L_{\mu 4} = \eta_{\mu\lambda}\chi^\lambda \partial_4 - \eta_{44}\chi^4 \partial_\mu \tag{20}$$

and which of course solve the usual commutation relations.

Before contractng the algebra it will be useful to express the differential operators in terms of stereographic coordinates x^μ given by

$$x^\mu = \Omega^{-1}\chi^\mu \quad \text{with} \quad \Omega = -\frac{1}{2}\left(\frac{\chi^4}{l} - 1\right) \tag{21}$$

such that upon invoking the chain rule for partial differentiation, one finds

$$\frac{\partial}{\partial \chi^\mu} = \Omega^{-1}(x)\frac{\partial}{\partial x^\mu} \tag{22}$$

$$\frac{\partial}{\partial \chi^4} = \frac{\Omega^{-1}(x)}{2l} x^\lambda \frac{\partial}{\partial x^\lambda} \tag{23}$$

Hence, substituting these expressions for (19) and (20), the generators of the de Sitter group are found to be

$$L_{\mu\nu} = \eta_{\mu\lambda} x^\lambda \partial_\nu - \eta_{\nu\lambda} x^\lambda \partial_\mu \tag{24}$$

$$L_{\mu 4} = \mathfrak{s} l \partial_\mu + \frac{1}{4l}\left(2\eta_{\mu\rho}x^\rho x^\lambda - \sigma^2 \delta^\lambda_\mu\right)\partial_\lambda \tag{25}$$

Remember that the generators for ordinary translations P_μ and special conformal transformations K_μ are given by

$$P_\mu = \frac{\partial}{\partial x^\mu} \quad \text{and} \quad K_\mu = \left(2\eta_{\mu\rho}x^\rho x^\lambda - \sigma^2 \delta^\lambda_\mu\right)\frac{\partial}{\partial x^\lambda} \tag{26}$$

such that we can express the generators of the de Sitter group as

$$L_{\mu\nu} = \eta_{\mu\lambda} x^\lambda P_\nu - \eta_{\nu\lambda} x^\lambda P_\mu \tag{27}$$

$$L_{\mu 4} = \mathfrak{s} l P_\mu + \frac{1}{4l} K_\mu \tag{28}$$

which solve the Lie brackets (16).

Note that the operators $L_{\mu\nu}$ are still readily interpreted as the generators for Lorentz rotations in stereographic coordinates, which leave the origin $x^\mu = 0$ fixed, and hence span the subalgebra $\mathfrak{so}(1,3)$. On the other hand, the generators $L_{\mu 4}$ will move the origin and define the transitivity on de Sitter spacetime. In stereographic coordinates these operators are a combination of ordinary translations and special conformal transformations, although they essantially are rotations as this follows from their Lie brackets.

Let us now consider the Inönu-Wigner contraction of $\mathfrak{so}(1,4)$ with respect to $\mathfrak{so}(1,3)$, i.e. we will consider an adequate non-singular transformation of the base element $L_{\alpha 4}$ and take a well-defined singular limit to end up with a new and non-equivalent algebra. As it turns out there are two possibilies.

Poincaré algebra Consider the transformation

$$L_{\mu 4} \to \Pi_\mu \equiv \frac{L_{4\mu}}{l} = -\mathfrak{s} P_\mu - \frac{1}{4l^2} K_\mu \tag{29}$$

where the Π_μ are called *de Sitter translations*. Now we can take the limit where the radius l goes to infinity such that we find the well-defined differential operators

$$\lim_{l\to\infty} \Pi_\mu = -\mathfrak{s} P_\mu \tag{30}$$

which together with the unchanged $L_{\mu\nu}$ and the commutation relations (18) span the Poincaré algebra $\mathfrak{so}(1,3) \oplus_s \mathfrak{t}$.

Conic algebra As a second possibility transform the $L_{\mu 4}$ according to

$$L_{\mu 4} \to \kappa_\mu \equiv l L_{\alpha 4} = \mathfrak{s} l^2 P_\mu + \frac{1}{4} K_\mu \tag{31}$$

It is clear that again we can take a limit such that the transformation becomes singular, now however letting the radius l go to zero. Then one finds the differential operators

$$\lim_{l\to 0} \kappa_\mu = \frac{1}{4} K_\mu \tag{32}$$

which again with $L_{\mu\nu}$ and the Lie brackets (18) span the algebra $\mathfrak{so}(1,3) \oplus_s \mathfrak{q}$.

It is interesting to see how two physically very different limits give rise to an identical algebraic structure. From the theoretical discussion on Inönü-Wigner contractions however it is clear that these algebras should have the same commutation relations. The latter are fixed once the subalgebra is chosen to which the algebra is contracted, whilst the basis elements of the complement of the subalgebra are assumed merely to exist in the singular limit. Which possibilities for these basis elements are available—or also, which limits can be taken, after a suitable basis transformation—can be discussed only given a representation (realization). Nonetheless every choice will lead to the same Lie bracket.