

BLACK HOLES OF D=5 SUPERGRAVITY

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Abstract

We discuss some general features of black holes of five-dimensional supergravity, such as the first law of black hole mechanics. We also discuss some special features of rotating supersymmetric black holes. In particular, we show that the horizon is a non-singular, and *non-rotating*, null hypersurface whose intersection with a Cauchy surface is a squashed 3-sphere. We find the Killing spinors of the near-horizon geometry and thereby determine the near-horizon isometry supergroup.

1 Introduction

Stationary, charged, asymptotically-flat black hole solutions of the vacuum Einstein-Maxwell equations exist for all spacetime dimensions $D \geq 4$ [1, 2]. They are characterized by their mass M , charge Q , and a number of angular momenta \mathbf{J} equal to the rank of the rotation group $SO(D-1)$. They have an event horizon with surface gravity κ and $(D-2)$ -volume \mathcal{A} , electric potential Φ_H and angular velocities $\boldsymbol{\Omega}_H$. These quantities are related by the first law of black hole mechanics

$$dM = \frac{\kappa}{8\pi G_D} d\mathcal{A} + \Phi_H dQ + \boldsymbol{\Omega}_H \cdot d\mathbf{J}. \quad (1)$$

where G_D is the D -dimensional Newton constant. This law was first established in the context of $D = 4$ Einstein-Maxwell theory [3]. It has been generalized to arbitrary D for uncharged black holes [2, 4]. One purpose of this paper is to provide the further generalization to charged black holes.

We shall be interested principally in *supersymmetric* charged black holes. These are naturally viewed as (special) solutions of D -dimensional supergravity theories, but they can be defined independently of supergravity as black hole solutions admitting at least one Killing spinor. Remarkably, such solutions exist *only* for $D = 4$ and $D = 5$. The $D = 4$ case is already well-understood, so the main focus of this paper will be on the $D = 5$ case. We should emphasize that by ‘black hole’ we mean here an asymptotically flat spacetime that is non-singular on and outside an event horizon. There are supersymmetric particle-like solutions of $D > 5$ supergravity theories that are sometimes called black holes, but these are always singular. There are also supersymmetric black holes in $D = 3$, but the spacetime in that case is asymptotically anti-de Sitter rather than asymptotically flat. Of course, there are non-singular supersymmetric black brane solutions in various $D \geq 4$ supergravity theories but these are neither ‘particle-like’ nor, strictly speaking, asymptotically flat.

The bosonic sector of $N = 2$ $D = 4$ supergravity is Einstein-Maxwell theory. For any asymptotically flat solution of Einstein-Maxwell theory that is non-singular on and outside the event horizon the mass M is bounded below by the charge Q [5]. In geometrized units of charge the bound is $M \geq |Q|$. A quantum version of this bound follows directly from the supersymmetry algebra [6], and configurations that saturate it, i.e. those with

$M = |Q|$, preserve half the supersymmetry of the vacuum. These configurations are the extreme Reissner-Nordström (RN) (multi) black holes, and the asymptotic values of the Killing spinor fields that they admit can be identified with the zero-eigenvalue eigenspinors of the anticommutator of supersymmetry charges. Although the bound $M \geq |Q|$ makes no mention of the angular momentum J , solutions that saturate it are singular¹ unless $J = 0$. Another way to understand why the angular momentum must vanish for supersymmetric $D = 4$ black holes is to observe that the unique Killing vector field k with normalization $k^2 = -1$ at infinity is the ‘square’ of a non-vanishing Killing spinor that is non-singular everywhere outside the horizon. This implies that k is timelike everywhere outside the horizon and hence that there is no ergoregion, but a black hole for which the horizon has a non-vanishing angular velocity necessarily has an ergoregion. We thereby deduce that a supersymmetric black hole must be ‘non-rotating’ in the sense that its horizon cannot rotate. It then follows that $J = 0$, e.g. from the theorem that *every stationary, non-rotating Einstein-Maxwell black hole must be static* [8].

When considering how these results about $D = 4$ Einstein-Maxwell theory may generalize to $D = 5$ it is important to appreciate that in $D = 5$ it is possible to include an additional ‘ $AF\bar{F}$ ’ Chern-Simons (CS) term. This makes no difference to the class of static solutions but it does affect the class of stationary solutions. Here we shall take the $D = 5$ ‘Einstein-Maxwell’ theory to be the bosonic sector of $D = 5$ supergravity for which the CS term is present with a particular coefficient. Specifically, the Lagrangian is [9]

$$L = \frac{1}{16\pi G_5} \left[\sqrt{-g}(R - F^2) - \frac{2}{3\sqrt{3}} \varepsilon^{mnpqr} A_m F_{np} F_{qr} \right] \quad (2)$$

The mass M of any asymptotically flat solution satisfies the bound [10]

$$M \geq \frac{\sqrt{3}}{2} |Q| \quad (3)$$

and solutions that saturate this bound admit Killing spinors.

The general stationary black hole solution of $D = 5$ supergravity will depend on four parameters, the mass M , the electric charge Q , and two angular momenta, J_1 and J_2 . It is implicit in the 7-parameter string solution of a $D = 6$ model discussed in [11], since the

¹The singularity may be resolved in Kaluza-Klein theory [7].

$D = 5$ Lagrangian (2) is a truncation of an S^1 compactification of the $D = 6$ Lagrangian used there, but we shall not need the explicit solution of [11] here. The bound (3) makes no reference to the two angular momenta J_1 and J_2 but, as in the $D = 4$ case, the requirement of non-singularity constrains these parameters. When the bound is saturated the constraint is such that one linear combination of J_1 and J_2 must vanish. Thus, the general *supersymmetric* $D = 5$ black hole [12, 13] (see also [14]) is parameterized by its mass M and one angular momentum, which we shall call J . The $J = 0$ case is just a straightforward generalization to $D = 5$ of the extreme RN solution, and it is sometimes called the ‘Tangherlini’ black hole [1]; in the context of $D = 5$ supergravity it preserves half the supersymmetry of the vacuum [10].

The fact that there exist *rotating* supersymmetric black holes is quite surprising in view of the previous argument that the horizon must be non-rotating. That argument was presented in the context of $D = 4$ black holes but it is equally valid for $D = 5$. Thus, it is *not* true that every stationary, non-rotating $D = 5$ Einstein-Maxwell black hole is static, if by ‘non-rotating’ one means vanishing angular velocity of the horizon, and by ‘Einstein-Maxwell’ one means the bosonic sector of $D = 5$ supergravity. The supersymmetric rotating black holes are counter-examples to this would-be theorem². We should stress that these are solutions of the ‘Einstein-Maxwell’ equations only if the latter are understood to include the contribution of the CS term with the precise coefficient required by $D = 5$ supergravity. It is quite possible that this particular theory is exceptional in this regard among the class of Einstein-Maxwell theories parameterized by the CS coefficient.

These observations indicate that a thorough investigation of the properties of black hole solutions of $D = 5$ supergravity is warranted, supersymmetric black holes in particular. Here we begin this investigation with a derivation of the first law of black hole mechanics (generalizing the result of [2] for uncharged black holes) and a study of the local and global properties of supersymmetric black holes. In particular, we confirm that

²An apparent further counter-example in $D \geq 6$ is provided by a class of rotating black holes for which the angular velocity of the horizon, but not the total angular momentum, vanishes in an extremal limit [15]. But the $(D - 2)$ -volume of the horizon of these solutions vanishes in the same limit, so a non-zero angular velocity would have been paradoxical.

a rotating supersymmetric black hole has a non-singular and non-rotating horizon. The effect of rotation on the horizon is not to make it rotate but to deform it from a round 3-sphere to a squashed 3-sphere! This implies that the angular momentum is stored in the Maxwell field, which it is, but another surprise is that a negative fraction of the total is stored *behind* the horizon.

It has been appreciated for some time that the extreme RN black hole interpolates between D=4 Minkowski spacetime (near infinity) and the Robinson-Bertotti $adS_2 \times S^2$ spacetime (near the horizon) [16] and that the latter is, like D=4 Minkowski, a maximally supersymmetric vacuum solution of N=2 D=4 supergravity [17]. In fact, it admits an $SU(1, 1|2)$ isometry supergroup [18]. There is therefore a restoration of full supersymmetry in either of the two asymptotic regions. These results have a direct generalization to the Tangherlini black hole. This solution interpolates between maximally supersymmetric vacua of $D = 5$ supergravity, in this case between $D = 5$ Minkowski spacetime (near spatial infinity) and $adS_2 \times S^3$ (near the horizon) [10]. There is again a full restoration of supersymmetry in either asymptotic region [19]. Here we are able to identify the isometry supergroup in the near-horizon limit as $SU(1, 1|2) \times SU(2)_R$ where $SU(2)_R$ is the group generated by the left-invariant vector fields on $S^3 \cong SU(2)$ and $SU(1, 1|2)$ contains the $SU(2)_L$ subgroup generated by the right-invariant vector fields. Apart from the $SU(2)_R$ factor this is the same as the D=4 case³.

Like the (non-rotating) Tangherlini black hole, the rotating supersymmetric $D = 5$ black hole also preserves half the supersymmetry of the vacuum. Moreover, it was shown in [20] that there is again a full restoration of supersymmetry near the horizon. We confirm this result here by a direct determination of the Killing spinors. We are then able to show, using a method introduced in [21], that the isometry supergroup of the near-horizon limit is

$$SU(1, 1|2) \times U(1)_R. \quad (4)$$

In other words, the rotation breaks $SU(2)_R$ to $U(1)_R$ without affecting the supersymme-

³This might seem surprising but it is a reflection of the fact that the $SU(2)$ gauge fields in the effective D=2 supergravity theory arising from compactification of $D = 5$ supergravity on S^3 belong not to the graviton supermultiplet but to an $SU(2)$ super-Yang-Mills multiplet.

tries. The near-horizon spacetime is a homogeneous spacetime of the form

$$[SO(2, 1) \times SU(2)_L \times U(1)_R]/[U(1) \times U(1)] . \quad (5)$$

One such space is the direct product of adS_2 with a squashed 3-sphere. The horizon (or rather, its intersection with a Cauchy surface) is indeed a squashed 3-sphere, with a squashing parameter simply related to J , but the full five-dimensional spacetime is *not* a direct product.

The organization of this paper is as follows. We begin with a derivation of the first law of black hole mechanics for black holes of $D = 5$ supergravity. We then solve the Killing spinor equations to recover the explicit rotating supersymmetric black hole solutions of [13]. We then determine various properties of the horizon, of the global structure behind the horizon, and we study the distribution of the angular momentum in the Maxwell field. We follow this with a detailed analysis of the near-horizon limit, its Killing spinors and its isometry supergroup. We conclude with some speculations about how the physics of $D = 5$ black holes in general Einstein-Maxwell-CS theories might depend on the CS coefficient.

2 The First Law

The bosonic fields of minimal D=5 supergravity are the 5-metric g and a Maxwell 1-form A with 2-form field strength $F = dA$. The field equations include Einstein's equations (for coordinates x^m , $m = 0, 1, \dots, 4$)

$$G_{mn} = 2T_{mn} \quad (6)$$

with the Maxwell stress-energy tensor

$$T_{mn} = F_{mp}F_n{}^p - \frac{1}{4}g_{mn}F^2 , \quad (7)$$

and Maxwell's equations modified by the CS contribution

$$D_m F^{mn} = \frac{1}{2\sqrt{3}\sqrt{-g}}\varepsilon^{npqrs}F_{pq}F_{rs} . \quad (8)$$

Note that the stress-energy tensor is covariantly conserved (i.e. $D_m T^{mn} = 0$) for solutions of the Maxwell/CS field equations by virtue of the five-dimensional identity

$$F_{[mn}F_{pq}F_{r]s} \equiv 0. \quad (9)$$

Our main aim in this section will be to derive a Smarr-type formula for (stationary and asymptotically flat) black hole solutions of the above equations. We will then use this to derive the first law (1). In attacking this problem, however, it will be useful to consider first the pure Einstein-Maxwell theory in which (8) is replaced by the simpler Maxwell-equation

$$D_m F^{mn} = 0. \quad (10)$$

In this case the extension to arbitrary spacetime dimension D is straightforward, and will facilitate the comparison of the $D = 4$ and $D = 5$ cases. We will then extend the analysis to the case of odd dimensions $D = 2n + 1$ in which the Einstein-Maxwell action may be supplemented by an ‘ $A F^n$ ’ CS term. This will include the $D = 5$ case of interest, which will turn out to be a rather special case among the class of odd-dimensional Einstein-Maxwell-CS theories.

Let us begin with the general analysis of the Einstein-Maxwell system. We shall be concerned with stationary asymptotically flat solutions, for which there exists a unique (timelike) Killing vector field k normalized such that $k^2 = -1$ at spatial infinity. For D spacetime dimensions (with $D \geq 4$) the total mass is given by [2]

$$M = -\frac{(D-2)}{(D-3)16\pi G_D} \oint_{\infty} dS_{mn} D^m k^n, \quad (11)$$

where the integral is taken over the $(D-2)$ -sphere at spatial infinity.

We shall further restrict our attention to solutions admitting an additional $[(D-1)/2]$ commuting spacelike vector fields \mathbf{m} with closed orbits; these are associated with the angular momenta \mathbf{J} [2]; there will be two such Killing vector fields for $D = 5$ corresponding to the two angular momenta J_1 and J_2 . We can choose coordinates such that $m_i = \partial/\partial\varphi^i$ and since the orbits are closed we can normalize \mathbf{m} by requiring the coordinates φ^i to be identified modulo 2π . The associated angular momenta are then [2]

$$\mathbf{J} = \frac{1}{16\pi G_D} \oint_{\infty} dS_{mn} D^m \mathbf{m}^n. \quad (12)$$

Finally, the total electric charge, in geometrized units, is

$$Q = \frac{1}{8\pi G_D} \oint_{\infty} dS_{mn} F^{mn}. \quad (13)$$

Our derivation of the first law relies on the theorem that the event horizon of a stationary black hole is a Killing horizon of some linear combination of k and \mathbf{m} [22]. Let

$$\xi = k + \boldsymbol{\Omega}_H \cdot \mathbf{m} \quad (14)$$

be this Killing vector field. The constant coefficients $\boldsymbol{\Omega}_H$ are the angular velocities of the horizon. Let Σ be a spacelike hypersurface with boundaries at spatial infinity and on the horizon. Using Gauss' law and (14), we can then rewrite each of the formulae for M and \mathbf{J} as the sum of an integral over Σ and a surface integral over the boundary H of Σ on the horizon. This leads to the formula

$$\begin{aligned} M = & -\frac{(D-2)}{(D-3)4\pi G_D} \int_{\Sigma} dS_m R^m{}_n \xi^n \\ & + \frac{(D-2)}{(D-3)} \boldsymbol{\Omega}_H \cdot \mathbf{J} - \frac{(D-2)}{(D-3)16\pi G_D} \oint_H dS_{mn} D^m \xi^n \end{aligned} \quad (15)$$

We may write $dS_{mn} = 2d\mathcal{A} \xi_{[m} n_{n]}$ for some null vector field n such that $\xi \cdot n = -1$. Then, using the fact that $(\xi \cdot D)\xi = \kappa\xi$ on the horizon, where κ is the horizon surface gravity, which is constant by the zeroth law, we may express the final integral in terms of κ and the $(D-2)$ -volume of the horizon \mathcal{A} . If in addition we use the Einstein equation in the form

$$R^m{}_n = F^{mp} F_{np} - \frac{1}{2(D-2)} \delta_n^m F^2, \quad (16)$$

then we arrive at the formula

$$\begin{aligned} M = & -\frac{(D-2)}{(D-3)4\pi G_D} \int_{\Sigma} dS_m [F^{mp} (\xi^n F_{np}) - \frac{1}{2(D-2)} \xi^m F^2] \\ & + \frac{(D-2)}{(D-3)} \boldsymbol{\Omega}_H \cdot \mathbf{J} + \frac{(D-2)}{(D-3)8\pi G_D} \kappa \mathcal{A} \end{aligned} \quad (17)$$

We shall assume that $\mathcal{L}_{\xi} F = 0$, where \mathcal{L}_{ξ} is the Lie derivative with respect to the vector field ξ . By a choice of gauge we can then arrange that $\mathcal{L}_{\xi} A = 0$, from which it follows that

$$\xi^n F_{np} = -\partial_p (\xi \cdot A), \quad (18)$$

and hence that

$$F^{mp}(\xi^n F_{np}) = -D_p [(\xi \cdot A)F^{mp}] - (\xi \cdot A)(D_p F^{pn}). \quad (19)$$

It also follows that

$$\xi^m F^2 = 4D_p [\xi^{[m} F^{p]n} A_n] - 2\xi^m A_n (D_p F^{pn}). \quad (20)$$

Using these relations we find that

$$\begin{aligned} M &= \frac{(D-2)}{(D-3)4\pi G_D} \int_{\Sigma} dS_m D_p \left[(\xi \cdot A)F^{mp} + \frac{2}{(D-2)} \xi^{[m} F^{p]n} A_n \right] \\ &\quad + \frac{1}{(D-3)4\pi G_D} \int_{\Sigma} dS_m [(D-2)(\xi \cdot A)\delta_n^m - \xi^m A_n] (D_p F^{pn}) \\ &\quad + \frac{(D-2)}{(D-3)} \Omega_H \cdot \mathbf{J} + \frac{(D-2)}{(D-3)8\pi G_D} \kappa \mathcal{A} \end{aligned} \quad (21)$$

The second of the above integrals vanishes for solutions of the Maxwell equations (10); we keep it here because it will not vanish when CS interactions are added. The first integral can be rewritten as the difference of surface integrals at spatial infinity and on the horizon. We shall suppose that A falls off sufficiently fast near spatial infinity that the surface integral at spatial infinity vanishes. We are then left with the surface integral

$$- \frac{(D-2)}{(D-3)8\pi G_D} \oint_H dS_{mp} \left[(\xi \cdot A)F^{mp} + \frac{2}{(D-2)} \xi^m F^{pn} A_n \right]. \quad (22)$$

Now, Raychaudhuri's equation for geodesic deviation implies that the scalar $R_{mn}\xi^m\xi^n$ vanishes on a Killing horizon of ξ . Using the Einstein equation and the fact that ξ is null on the horizon we deduce that the 1-form $V = i_{\xi}F$ is also null on the horizon. But, $\xi \cdot V \equiv 0$ so $V^m \partial_m$ is tangent to the horizon. Any tangent to the horizon that is null must be proportional to ξ , so we deduce that, on the horizon, $\xi^m F_{mn}$ is some function times ξ_n . It then follows (by the use of the relation $dS_{mn} = 2d\mathcal{A}\xi_{[m}n_{n]}$) that

$$\oint_H dS_{mp} \xi^m F^{pn} A_n = -\frac{1}{2} \oint_H dS_{mp} (\xi \cdot A) F^{mp}, \quad (23)$$

which allows us to reduce (22) to

$$- \frac{1}{8\pi G_D} \oint_H dS_{mn} (\xi \cdot A) F^{mn} \quad (24)$$

Now $(\xi \cdot A)$ is constant on the horizon⁴. The constant is, by definition, minus the co-rotating electric potential Φ_H , i.e. $(\xi \cdot A)_H = -\Phi_H$. Taking this constant outside the integral, an application of Gauss' theorem then yields

$$-\frac{1}{8\pi G_D} \oint_H dS_{mn} (\xi \cdot A) F^{mn} = \Phi_H \left[Q + \frac{1}{4\pi G_D} \int_\Sigma dS_m D_p F^{pm} \right]. \quad (25)$$

If we now assume the validity of the Maxwell equations (10) then we may rewrite the expression (21) for M as

$$M = \Phi_H Q + \frac{(D-2)}{(D-3)} \Omega_H \cdot \mathbf{J} + \frac{(D-2)}{(D-3)8\pi G_D} \kappa \mathcal{A}. \quad (26)$$

We have thus arrived at a Smarr-type formula for D -dimensional Einstein-Maxwell black holes from which the first law can be deduced. We shall begin by assuming that black hole solutions are uniquely determined by their mass, charge, and angular momenta. This is what one would expect if the uniqueness theorems proved for $D = 4$ can be extended to $D = 5$. In this case the mass is a function $M(Q, \mathbf{J}, \mathcal{A})$ of the charge, angular momenta, and $(D-2)$ -volume of the horizon cross-section. In units for which $G_D = 1$ the independent variables of this function have dimensions as follows:

$$[Q] = M, \quad [J] = [\mathcal{A}] = M^{(D-2)/(D-3)}. \quad (27)$$

There can be no dependence of the function M on any other dimensionful quantity because the Einstein-Maxwell Lagrangian involves no dimensional constants other than the overall factor of $1/16\pi G_D$. This is because every term in the Lagrangian has precisely two derivatives (and we take the metric and gauge potential to be dimensionless). It follows that $M(Q, \mathbf{J}, \mathcal{A})$ must be a weighted homogeneous function, to which an application of Euler's theorem yields

$$Q \frac{\partial M}{\partial Q} + \frac{(D-2)}{(D-3)} \mathbf{J} \cdot \frac{\partial M}{\partial \mathbf{J}} + \frac{(D-2)}{(D-3)} \mathcal{A} \frac{\partial M}{\partial \mathcal{A}} = M \quad (28)$$

Substituting (26) on the right hand side, and using the independence of the variables $(Q, \mathbf{J}, \mathcal{A})$, we deduce (reinstating the G_D -dependence) that

$$\frac{\partial M}{\partial Q} = \Phi_H, \quad \frac{\partial M}{\partial \mathbf{J}} = \Omega_H, \quad \frac{\partial M}{\partial \mathcal{A}} = \frac{\kappa}{8\pi G_D} \quad (29)$$

⁴From (18) we have $d(\xi \cdot A) = -i_\xi F$. As explained above, $i_\xi F \propto dx^m \xi_m$ on the horizon. It follows that $i_t d(\xi \cdot A)_H \propto t \cdot \xi$, which vanishes if t is tangent to the horizon.

and hence the first law (1).

Of course, the preceding analysis does not apply to the five-dimensional theory in which we were originally interested because of the CS term in (2). More generally, it fails to apply in any odd spacetime dimension $D = 2n + 1$ for which the Lagrangian includes a CS term, which we may write as

$$L_{\text{int}} = -\frac{1}{16\pi G_D} \frac{4\lambda}{n+1} \varepsilon^{mnp\cdots qr} A_m F_{np} \cdots F_{qr} \quad (30)$$

for some coupling constant λ . With the addition of this term the equation of motion for A becomes

$$D_m F^{mn} = \frac{\lambda}{\sqrt{-g}} \varepsilon^{npq\cdots rs} F_{pq} \cdots F_{rs} . \quad (31)$$

It will be useful to rewrite this equation as

$$D_m (F^{mn} + 2\lambda J^{mn}) = 0 \quad (32)$$

where

$$J^{mn} = \frac{1}{\sqrt{-g}} \varepsilon^{mnpqr\cdots st} A_p F_{qr} \cdots F_{st} . \quad (33)$$

We also note that the five-dimensional identity (9), which ensured stress-energy conservation, is extended to

$$F_{[mn} F_{pq} \cdots F_{r]s} \equiv 0 \quad (34)$$

with $n + 1$ factors of the field strength in $D = 2n + 1$ dimensions.

Most of the previous analysis remains unchanged, but we must now reconsider the integral

$$\frac{1}{(D-3)4\pi G_D} \int_{\Sigma} dS_m [(D-2)(\xi \cdot A) \delta_n^m - \xi^m A_n] (D_p F^{pn}) \quad (35)$$

in (21). This contribution was previously zero. Now, as a result of (32), it equals

$$- \frac{\lambda}{(D-3)2\pi G_D} \int_{\Sigma} dS_m [(D-2)(\xi \cdot A) \delta_n^m - \xi^m A_n] (D_p J^{pn}) \quad (36)$$

Using the identity $A_n J^{np} \equiv 0$, we may simplify this to

$$- \frac{\lambda}{(D-3)4\pi G_D} \int_{\Sigma} dS_m [2(D-2)(\xi \cdot A) D_p J^{pm} + \xi^m F_{pn} J^{pn}] \quad (37)$$

Now, the identity $\xi^{[m} \varepsilon^{npq\cdots rs]} \equiv 0$ can be used to show that

$$\xi^m F_{pn} J^{pn} = -2(\xi \cdot A) D_p J^{pm} - (D-1) \partial_n (\xi \cdot A) J^{mn} \quad (38)$$

which allows us to rewrite the expression (37) as

$$- \frac{\lambda}{(D-3)2\pi G_D} \int_{\Sigma} dS_m [2(D-3)D_p(\xi \cdot A J^{pm}) - (D-5)\partial_p(\xi \cdot A)J^{pm}] \quad (39)$$

Under the same assumptions as before regarding the behaviour of A near spatial infinity, and again using (18), this equals

$$- \frac{\lambda}{4\pi G_D} \oint_H dS_{mp}(\xi \cdot A)J^{mp} - \frac{(D-5)\lambda}{(D-3)4\pi G_D} \int_{\Sigma} dS_m \xi^n F_{np} J^{pm} \quad (40)$$

The surface term may be combined with that in (24) to yield

$$- \frac{1}{8\pi G_D} \oint_H dS_{mn}(\xi \cdot A) [F^{mn} + 2\lambda J^{mn}] \quad (41)$$

As before $(\xi \cdot A)$ is constant on the horizon and can be taken outside the integral. Using Gauss' theorem and the modified equation of motion (32) we may reduce this surface term to $\Phi_H Q$, so our expression for M becomes

$$M = \Phi_H Q + \frac{(D-2)}{(D-3)} \boldsymbol{\Omega}_H \cdot \mathbf{J} + \frac{(D-2)}{(D-3)8\pi G_D} \kappa \mathcal{A} + \frac{(D-5)}{(D-3)} \lambda I. \quad (42)$$

where

$$I = - \frac{1}{4\pi G_D} \int_{\Sigma} dS_m \xi^n F_{np} J^{pm}. \quad (43)$$

The case of principal interest here is $D = 5$ for which the coefficient of the last term in (42) vanishes and we therefore recover the Smarr-type formula used previously to derive the first law. The $D = 5$ case is also special in one other regard: the CS term is quadratic in derivatives and therefore has the same dimension as the other terms in the Lagrangian, which again involves no dimensionful parameters other than the overall factor of $1/16\pi G_D$. The previous proof of the first law therefore goes through without change. In the case of supersymmetric black holes for which (as we shall confirm in the following section) κ and $\boldsymbol{\Omega}_H$ vanish, we have $M = \Phi_H Q$. Because supersymmetric solutions saturate the bound (3) we deduce that $\Phi_H = \sqrt{3}/2$. We shall shortly confirm this prediction.

For all odd spacetime dimensions higher than five, the coefficient of the λI term in (42) does not vanish, with the consequence that this formula is no longer equivalent to (26). This is not unexpected because the proof of the first law via Euler's theorem

also requires modification. This is because the CS term is now more than quadratic in derivatives and the coupling constant λ is therefore a new dimensionful parameter. Specifically, the dimension of λ , for $G_D = 1$, is $[\lambda] = (D - 5)/(D - 3)$. Euler's theorem is still applicable provided that M is now considered to be a function of λ (as well as Q , \mathbf{J} and \mathcal{A}), but this leads to the addition of $(D - 5)/(D - 3) \lambda (\partial M / \partial \lambda)$ to the left hand side of (28). Comparing this new equation with (42) we deduce that

$$I = \frac{\partial M}{\partial \lambda} \quad (44)$$

and hence that

$$dM = \frac{\kappa}{8\pi G_D} d\mathcal{A} + \Phi_H dQ + \boldsymbol{\Omega}_H \cdot d\mathbf{J} + I d\lambda. \quad (45)$$

This is a version of the first law that is reminiscent of its extension in [23] to include scalar vacuum expectation values. In our case, however, the parameter λ labels *theories* rather than solutions, so for any given theory λ is fixed, the last term vanishes, and we recover the first law as it appears in (1). We thus see, not unexpectedly, that the first law is quite robust and does not depend on special features of D=5 supergravity. Nevertheless, the black holes of D=5 supergravity do have many special features, and we shall explore some of them in the remainder of this article.

3 Supersymmetric black holes

Supersymmetric solutions of the field equations (6) are those for which there exist non-vanishing solutions for ζ of the Killing spinor equation [10]

$$\left[d + \frac{1}{4} \omega_{ab} \Gamma^{ab} + \frac{i}{4\sqrt{3}} \left(e^a \Gamma^{bc} F_{bc} - 4e^a \Gamma^b F_{ab} \right) \right] \zeta = 0 \quad (46)$$

where e^a are the frame 1-forms and ω_{ab} is the spin-connection 1-form⁵. The spinor ζ is necessarily complex. The Dirac matrices are also complex but the product of all five is $\pm i$. A choice of the sign amounts to a choice of one of two inequivalent 4×4 representations. We shall choose the Dirac matrices such that

$$\Gamma^{01234} = i. \quad (47)$$

⁵In contrast to [10] we use here the ‘mostly plus’ metric convention, which accounts for the factor of i .

We shall consider configurations of the form

$$ds^2 = -(e^0)^2 + e^i e^j \delta_{ij} \quad , \quad F = \frac{\sqrt{3}}{2} de^0 \quad , \quad (48)$$

where

$$e^0 = H^{-1}(dt + a) \quad , \quad e^i = H^{\frac{1}{2}} dx^i \quad (i = 1, 2, 3, 4). \quad (49)$$

with time-independent function H and 1-form a . A calculation yields

$$\omega_{0i} = e^0 H^{-3/2} \partial_i H - \frac{1}{2} e^j H^{-2} f_{ij} \quad (50)$$

$$\omega_{ij} = e^k H^{-3/2} \delta_{k[i} \partial_{j]} H + \frac{1}{2} e^0 H^{-2} f_{ij} \quad (51)$$

$$F = -\frac{\sqrt{3}}{2} H^{-3/2} \partial_i H e^i \wedge e^0 + \frac{\sqrt{3}}{4} H^{-2} f_{ij} e^i \wedge e^j \quad (52)$$

where

$$f_{ij} = \partial_i a_j - \partial_j a_i. \quad (53)$$

Since

$$d = e^0 H \partial_t + e^i H^{-1/2} (\partial_i - a_i \partial_t) \quad (54)$$

we find that the Killing spinor equations become

$$\begin{aligned} \partial_t \zeta &= \frac{1}{2} H^{-3} \left(i H^{1/2} \partial_i H \Gamma^i - \frac{1}{4} f_{ij} \Gamma^{ij} \right) (1 - i \Gamma^0) \zeta \\ (\partial_i - a_i \partial_t) \zeta &= -\frac{1}{4} H^{-3/2} \partial_j H \Gamma^{ij} (1 - i \Gamma^0) \zeta + i \partial_i (\log H^{-1/2}) \Gamma^0 \zeta \\ &\quad + \frac{1}{4} (f_{ij} + \tilde{f}_{ij}) \Gamma^j \Gamma^0 \zeta \end{aligned} \quad (55)$$

where

$$\tilde{f}_{ij} = \frac{1}{2} \epsilon_{ijkl} f_{kl}. \quad (56)$$

The integrability conditions are consistent with non-zero ζ only if

$$i \Gamma^0 \zeta = \zeta, \quad (57)$$

and

$$f_{ij} + \tilde{f}_{ij} = 0. \quad (58)$$

The Killing spinor equations are then solved by

$$\zeta = H^{-1/2} \zeta_0 \quad (59)$$

where ζ_0 is a constant eigenspinor of $i\Gamma^0$ with eigenvalue 1. The Maxwell field equation implies that H is harmonic. Allowing for point singularities, we have

$$H = 1 + \sum_i \mu_i / |x - x_i|^2 \quad (60)$$

where x is a displacement vector in \mathbb{E}^4 and μ_i are a set of positive constants.

The anti-self-duality equation (58) for $f = da$ is solved by

$$a = dx^m J_m{}^n \partial_n K \quad (61)$$

where K is a harmonic function on $\mathbb{C}^2 \cong \mathbb{E}^4$ with complex structure J . If we require that a vanish at spatial infinity then any non-constant K must have singularities, but these can be chosen to be point singularities that coincide with the singularities of H . Thus each singularity of H is associated with some residue of the function K . Under circumstances to be spelled out below these singularities are merely coordinate singularities at Killing horizons of the metric, the union of which is the event horizon. For A to vanish at infinity as assumed in our discussion of the first law we must choose the Maxwell gauge such that

$$A = \frac{\sqrt{3}}{2} [H^{-1}(dt + a) - dt] . \quad (62)$$

Since H^{-1} vanishes on the event horizon we deduce that Φ_H takes the same constant value, $\sqrt{3}/2$, on each connected component of the horizon. As pointed out in our earlier discussion this value is required by any black hole solution that saturates the bound (3).

If we now assume a connected event horizon then the harmonic function H has a single point singularity in \mathbb{E}^4 , which we may choose to be the origin of spherical polar coordinates. Thus, the \mathbb{E}^4 metric is

$$ds^2(\mathbb{E}^4) = dr^2 + r^2 d\Omega_3^2 \quad (63)$$

where $d\Omega_3^2$ is the $SO(4)$ invariant metric on the 3-sphere. In these coordinates we have

$$H = 1 + \frac{\mu}{r^2} \quad (64)$$

where μ is a positive constant related to the mass M by

$$M = \frac{3\pi\mu}{4G_5} . \quad (65)$$

We may write the 3-sphere metric as

$$d\Omega_3 = \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \quad (66)$$

where σ_i ($i = 1, 2, 3$) are the three left-invariant one-forms satisfying

$$d\sigma_1 = \sigma_2 \wedge \sigma_3 \quad \text{and cyclic} \quad (67)$$

A useful explicit choice of coordinates on S^3 is

$$d\Omega_3^2 = \frac{1}{4}[d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\psi d\phi], \quad (68)$$

where

$$0 \leq \theta < \pi, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \psi < 4\pi. \quad (69)$$

With this choice, the three one-forms σ_i may be written as

$$\begin{aligned} \sigma_1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi \\ \sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi \\ \sigma_3 &= d\psi + \cos\theta d\phi \end{aligned} \quad (70)$$

If the vector fields $\partial/\partial\psi$ and $\partial/\partial\phi$ are assumed to be Killing then the 1-form a can be assumed, without loss of generality, to be

$$a = \frac{j}{2r^2}\sigma_3 \quad (71)$$

where j is a parameter related to the angular momentum J associated with the Killing vector field $2\partial/\partial\psi$ (which has the normalization assumed earlier)⁶. Specifically,

$$J = -\frac{j\pi}{2G_5}. \quad (72)$$

The angular momentum associated with ∂_ϕ vanishes. Thus, as stated earlier, supersymmetry imposes one constraint on the two angular momenta of the general black hole solution.

⁶Note that in the ‘Cartesian’ coordinates of eqs. (48,49), this angular momentum is related to simultaneous rotations in two orthogonal planes.

Having specified both H and a , we now have the explicit metric

$$ds^2 = - \left(1 + \frac{\mu}{r^2}\right)^{-2} \left(dt + \frac{j\sigma_3}{2r^2}\right)^2 + \left(1 + \frac{\mu}{r^2}\right) (dr^2 + r^2 d\Omega_3^2) \quad (73)$$

Note that ∂_ψ can become timelike if $|j| > \mu^{3/2}$, which would imply the existence of closed time-like curves (outside the horizon). We shall therefore assume in what follows that $|j| < \mu^{3/2}$, in which case we can write

$$j = \mu^{3/2} \sin \beta \quad (74)$$

for *real* β . We now turn to a study of the properties of the horizon and the region behind the horizon.

4 The squashed horizon and closed timelike curves

The metric (73) is singular at $r = 0$, but this is only a coordinate singularity. To establish this we introduce the new coordinates \tilde{t} and λ by

$$t = \mu^{1/2} \tilde{t} \quad , \quad r^2 = \mu(\lambda^2 - 1) \quad . \quad (75)$$

The metric in these coordinates is given by $ds^2 = \mu d\tilde{s}^2$, where

$$\begin{aligned} d\tilde{s}^2 = & - \left[\left(1 - \frac{1}{\lambda^2}\right) d\tilde{t} + \frac{\sin \beta}{2\lambda^2} \sigma_3 \right]^2 + \left(1 - \frac{1}{\lambda^2}\right)^{-2} d\lambda^2 \\ & + \frac{1}{4} \lambda^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \end{aligned} \quad (76)$$

This form of the metric is essentially the same as that originally given in [13].

Clearly $\lambda = 0, 1$ are singular hypersurfaces because g_{tt} blows up at $\lambda = 0$ and $g_{\lambda\lambda}$ blows up at $\lambda = 1$. Everywhere else the metric is finite, and invertible provided $\theta \neq 0$ since

$$\det \tilde{g} = - \left(\frac{\lambda^2}{4}\right)^3 \sin^2 \theta \quad (77)$$

The singularity at $\theta = 0$ is the usual coordinate singularity on the ‘axis’ of symmetry (actually a 2-plane) of polar coordinates. The singularity at $\lambda = 1$ (i.e. at $r = 0$) may be removed as follows. We introduce new coordinates u, ψ' such that

$$\begin{aligned} du &= d\tilde{t} - \left(1 - \frac{1}{\lambda^2}\right)^{-2} F(\lambda^2) d\lambda \\ d\psi' &= d\psi - 2 \left(1 - \frac{1}{\lambda^2}\right)^{-1} \lambda^2 G(\lambda^2) d\lambda \end{aligned} \quad (78)$$

where F and G are functions of λ to be specified later. We then find that

$$\begin{aligned}
d\tilde{s}^2 = & -\left(1 - \frac{1}{\lambda^2}\right)^2 du^2 - 2(F + \sin \beta G) du d\lambda - \left(1 - \frac{1}{\lambda^2}\right) \frac{\sin \beta}{\lambda^2} du \sigma'_3 \\
& - (\lambda^2 - 1)^{-1} [\sin \beta F - (\lambda^6 - \sin^2 \beta) G] d\lambda \sigma'_3 \\
& + \frac{1}{4} \lambda^2 (\sigma_1^2 + \sigma_2^2) + \frac{1}{4\lambda^4} (\lambda^6 - \sin^2 \beta) (\sigma'_3)^2 \\
& + \left(1 - \frac{1}{\lambda^2}\right)^{-2} [1 - F^2 - 2 \sin \beta F G + (\lambda^6 - \sin^2 \beta) G^2] d\lambda^2
\end{aligned} \tag{79}$$

To ensure non-singularity at $\lambda = 1$ we must choose $F(\lambda)$ and $G(\lambda)$ such that

$$F(1) = \pm \cos \beta \quad G(1) = \pm \tan \beta \quad F'(1) = \pm 3 \sin \beta \tan \beta \tag{80}$$

where the minus sign must be chosen if we wish the future horizon to be non-singular. A simple choice is

$$F = -\left(1 - \frac{\sin^2 \beta}{\lambda^6}\right)^{1/2} \quad G = -\sin \beta [\lambda^6 (\lambda^6 - \sin^2 \beta)]^{-1/2}, \tag{81}$$

for which the metric is then

$$\begin{aligned}
\mu^{-1} ds^2 = & -\left(1 - \frac{1}{\lambda^2}\right)^2 du^2 + \frac{2\lambda^3}{(\lambda^6 - \sin^2 \beta)^{1/2}} du d\lambda - \left(1 - \frac{1}{\lambda^2}\right) \frac{\sin \beta}{\lambda^2} du \sigma'_3 \\
& + \frac{1}{4} \lambda^2 (\sigma_1^2 + \sigma_2^2) + \frac{1}{4\lambda^4} (\lambda^6 - \sin^2 \beta) (\sigma'_3)^2
\end{aligned} \tag{82}$$

The limit $\lambda \rightarrow 1$ yields the near-horizon solution

$$\begin{aligned}
d\tilde{s}^2 & \sim -4\rho^2 du^2 + 2\sec \beta du d\rho - 2\rho \sin \beta du \sigma'_3 + \frac{1}{4} (\sigma_1^2 + \sigma_2^2 + \cos^2 \beta (\sigma'_3)^2) \\
\mu^{-1/2} F & \sim \sqrt{3} d\rho \wedge du + \frac{\sqrt{3} \sin \beta}{4} \sigma_1 \wedge \sigma_2
\end{aligned} \tag{83}$$

where $\rho = \lambda - 1$. The metric is non-singular provided that $\cos \beta \neq 0$, which we henceforth assume. Note that this solution is invariant under the isometry

$$\rho \rightarrow -\rho \quad u \rightarrow -u \tag{84}$$

which exchanges the region behind the Killing horizon of k with the region outside it. The maximal analytic continuation of this metric is therefore singularity free (in contrast to the full metric) because of the absence of singularities in the exterior region. This fact

is obvious when $\sin \beta = 0$ because the maximal analytic extension of the near-horizon solution is then the direct product of S^3 with the covering space of adS_2 .

When $\sin \beta \neq 0$ the intersection of the $\rho = 0$ hypersurface with a hypersurface of constant u is a squashed 3-sphere with squashing parameter $\sin \beta$. The hypersurface $\rho = 0$ is the event horizon. Its normal is

$$\ell = g^{mn} \partial_n \rho|_{\rho=0} \partial_m = -\cos \beta \partial_u \quad (85)$$

which is clearly null. The vector field ∂_u is the timelike Killing vector field k expressed in the new coordinates. The null hypersurface $\rho = 0$ is therefore a Killing horizon of k and it follows that the angular velocity of the horizon vanishes. In other words, *the rotation at infinity corresponds to a squashed horizon rather than a rotating one*. In addition, a calculation yields

$$[(k \cdot D)k^m]|_{\rho=0} = 0 \quad (86)$$

so the event horizon has vanishing surface gravity; it is a degenerate Killing horizon of the timelike vector field k . We have now verified that supersymmetric black holes have a horizon with both vanishing surface gravity and vanishing angular velocities, as claimed earlier.

The coordinates leading to (82) have the disadvantage that there is a spurious coordinate singularity at $\lambda^3 = \sin \beta$. To see the significance of this coordinate singularity, and to continue the metric through it, we instead choose

$$F = \cos \beta + \frac{3}{2} \sin \beta \tan \beta (\lambda^2 - 1) \quad G = \tan \beta \quad (87)$$

The metric is now

$$\begin{aligned} d\tilde{s}^2 = & -\left(1 - \frac{1}{\lambda^2}\right)^2 du^2 - 2 \cos \beta \left[1 + \frac{1}{2} \tan^2 \beta (3\lambda^2 - 1)\right] du d\lambda - \left(1 - \frac{1}{\lambda^2}\right) \frac{\sin \beta}{\lambda^2} du d\sigma'_3 \\ & + \tan \beta \left[1 + \lambda^2 + \lambda^4 - \frac{3}{2} \sin^2 \beta\right] d\lambda d\sigma'_3 + \tan^2 \beta \lambda^4 \left(\lambda^2 + 2 - \frac{9}{4} \sin^2 \beta\right) d\lambda^2 \\ & + \frac{1}{4} \lambda^2 (\sigma_1^2 + \sigma_2^2) + \frac{1}{4\lambda^4} (\lambda^6 - \sin^2 \beta) (\sigma'_3)^2, \end{aligned} \quad (88)$$

which has no singularities other than the one at $\lambda = 0$. That $\lambda = 0$ is a curvature singularity can be seen from the fact that

$$R = \frac{2}{\lambda^8} (2 \sin^2 \beta - \lambda^2). \quad (89)$$

Having passed through the horizon by one of the above two changes of coordinates we may now change back to the original coordinates to recover the metric (76) but now with the restriction to $\lambda < 1$, i.e.,

$$d\tilde{s}^2 = -\frac{1}{\lambda^4} \left[(1 - \lambda^2) d\tilde{t} + \sin \beta \sigma_3 \right]^2 + \frac{\lambda^4 d\lambda^2}{(1 - \lambda^2)^2} + \frac{1}{4} \lambda^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \quad (90)$$

It may appear at first sight that the global structure of this spinning black hole coincides with that of the case $J = 0$ ($\sin \beta = 0$), with an extremal horizon surrounding a time-like and *point-like* singularity. This is certainly true if one only considers radial motions, *i.e.*, trajectories in u (or \tilde{t}) and λ with fixed angles. In particular, eq. (89) shows that any radial geodesic approaching $\lambda = 0$ encounters a curvature singularity independent of the angles. One may note though that the angular momentum changes the nature of the singularity, in that it increases the rate of the divergence as well as the overall sign. Similar behavior, in particular the same angular independence, can be seen in other curvature invariants, such as $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$.⁷ This is to be contrasted with the four-dimensional Kerr-Newman metric, where there is a ring-like singularity which trajectories may pass through to enter a new asymptotically flat region (with negative mass and no horizon).

However, the simple picture above does not withstand closer scrutiny. Firstly, the Killing vector field $m = 2\partial_\psi$ becomes null at $\lambda^3 = \sin^2 \beta$ and is timelike for $\lambda^3 < \sin^2 \beta$. Since the orbits of m are closed, it is clear that the region around the singularity contains closed timelike curves. In fact, in this region, the future light-cone tips over to encompass vectors of the form $v(b) = -\partial_\psi + b \partial_u$ with $b > -(\sin \beta - \lambda^3)/2(1 - \lambda^2)$ (here we assume that $\sin \beta > 0$). This can be seen to imply the existence of a closed time-like curve passing through *any* point inside the horizon⁸, as follows: from any point inside the horizon an observer can follow a timelike trajectory into the region $\lambda^3 < \sin^2 \beta$. Once there, she can continue her (non-geodesic) timelike trajectory on an orbit of a vector field of the type $v(b)$, travelling backward in u while winding around ψ . Having travelled sufficiently

⁷This leads one to wonder what it is about the *spacetime* geometry that is ‘squashed’. The ‘squashing’ appears not to manifest itself in scalar curvature invariants, but it presumably appears in tidal forces.

⁸A similar result was established in [24] for the Kerr-Newman geometry.

far back in u , our observer can then follow a timelike path that moves radially outward and intersects the initial point on the trajectory. An alternative characterization of the hypersurface $\lambda^3 = \sin^2 \beta$ is that the normals to the hypersurfaces of constant t become null at the intersection with $\lambda^3 = \sin^2 \beta$ and are spacelike for $\lambda^3 < \sin^2 \beta$. In other words the hypersurfaces of constant t are not globally spacelike, but become timelike in the region $\lambda < (\sin \beta)^{2/3}$. Thus one may conclude that the full causal structure for the present spacetime and the structure of the singularity is far more complicated than revealed by only radial motions. Of course, the geometry of the region with $\lambda > (\sin \beta)^{2/3}$ is similar to that of the non-rotating case.

5 Distribution of angular momentum

Because the horizon of a supersymmetric $D = 5$ black holes is non-rotating, any angular momentum must be stored in the Maxwell field. Let Σ be the same spacelike hypersurface as before with boundaries at spatial infinity and on the horizon. An application of Gauss' law to the formula (12) for the total angular momentum associated with the Killing vector field $m = 2\partial_\psi$ yields

$$J = \frac{1}{4\pi G_D} \int_\Sigma dS_m R^m{}_n m^n + J_H \quad (91)$$

where

$$J_H = \frac{1}{16\pi G_5} \oint_H dS_{mn} D^m m^n. \quad (92)$$

It would be natural to associate the surface integral J_H with the angular momentum due to rotation of the horizon, in which case it would have to vanish for a non-rotating horizon. However, if there is any contribution to the bulk integral of (91) just outside the horizon (and there is) the existence of the isometry of the near-horizon solution that exchanges the interior and exterior regions implies that there must be angular momentum in the Maxwell field *inside* the horizon. This being the case it would be surprising if J_H did vanish. In fact, it does not, as shall now verify by a direct calculation using the near-horizon metric (83). Note first that in these coordinates

$$k = \mu^{-1/2} \partial_u \quad m = 2\partial_\psi. \quad (93)$$

We begin with the formula

$$dS_{mn} = d\mathcal{A}(k_m n_n - k_n n_m) \quad (94)$$

where n is any null vector field with $k \cdot n = -1$ on the horizon. A suitable choice is

$$n = \mu^{-1/2} \cos \beta \partial_\rho. \quad (95)$$

Because k is Killing and commutes with m we have

$$(k_m n_n - k_n n_m) D^m m^n = -n \cdot \partial(k \cdot m) = 2 \cos \beta \sin \beta \quad (96)$$

and hence

$$J_H = -\frac{1}{8\pi G_5} \cos \beta \sin \beta \mathcal{A}. \quad (97)$$

Now the 3-volume of the squashed 3-sphere horizon is

$$\mathcal{A} = 2\pi^2 \mu^{3/2} \cos \beta, \quad (98)$$

so, using $j = \sin \beta \mu^{3/2}$ and the formula (72) for the total angular momentum J we see that

$$J_H = -\frac{1}{2} \cos^2 \beta J \quad (99)$$

Not only is this non-zero, it is also a *negative* fraction of the total angular momentum.

As a check on this result we shall now calculate the bulk contribution on the spacelike hypersurface Σ ,

$$J_\Sigma = \frac{1}{4\pi G_D} \int_\Sigma dS_m R^m{}_n m^n. \quad (100)$$

To do so we observe that the value of J_H is independent of the value of u at which Σ meets the horizon, because ∂_u is Killing. We may therefore drag the boundary H of Σ back to $u = -\infty$ along the orbits of ∂_u on the horizon, which are its null geodesic generators, without affecting the value of J_H . In this limit the surface Σ can be chosen to be the hypersurface of constant t in the metric (73). Using the Einstein equation we then have

$$J_\Sigma = \frac{1}{2G_5} \int d^4x \sqrt{-g} g^{0p} e_p{}^a e_\psi{}^c F_{ab} F^b{}_c \quad (101)$$

From the explicit form of the metric (73), and choosing the basis one-forms to be

$$e^0 = H^{-1} \left(dt + \frac{j\sigma_3}{2r^2} \right), \quad e^r = H^{1/2} dr, \quad e^I = \frac{1}{2} r H^{1/2} \sigma_I \quad (I = 1, 2, 3), \quad (102)$$

we find that

$$J_\Sigma = \frac{1}{2G_5} \int d^4x \left[\frac{H^2 \sin \theta}{8r^3} (r^6 - j^2 H^{-3}) F_{r0} (F_{r0} e_\psi^0 + F_{r3} e_\psi^3) + \frac{j \sin \theta}{4r} (e_\psi^0 F_{r0} + e_\psi^3 F_{r3})^2 \right] \quad (103)$$

Using

$$F = \sqrt{3} e^r \wedge \left[e^0 \frac{\mu}{r^3} H^{-3/2} - e^3 \frac{j}{r^3} H^{-2} \right] + \sqrt{3} e^1 \wedge e^2 \left[\frac{j}{r^2} H^{-2} \right] \quad (104)$$

we compute

$$\begin{aligned} F_{r0} (F_{r0} e_\psi^0 + F_{r3} e_\psi^3) &= \frac{3j\mu H^{-4}}{2r^6} \\ (e_\psi^0 F_{r0} + e_\psi^3 F_{r3})^2 &= \frac{3j^2 H^{-5}}{4r^6} \end{aligned} \quad (105)$$

and hence

$$J_\Sigma = \frac{1}{2G_5} \int d^4x \frac{3}{16} \sin \theta \left[\frac{j^3}{r^7 H^4} - \frac{j\mu}{r^3 H^2} \right] \quad (106)$$

The integrals are now easily done. Using $\sin \beta = j/\mu^{3/2}$, we find the result

$$J_\Sigma = \frac{\pi j}{4G_5} (\sin^2 \beta - 3) = \left(1 + \frac{1}{2} \cos^2 \beta \right) J \quad (107)$$

This is larger than the total angular momentum J , but is precisely such that $J_\Sigma + J_H = J$, as required.

The fact that J_H vanishes as $\cos \beta \rightarrow 0$ suggests that the angular momentum behind the horizon, on a surface of constant t in the metric (90), is distributed in the region between the horizon and the intersection of the constant t hypersurface with the hypersurface $\lambda^3 = \sin^2 \beta$, since this region vanishes in the same limit. We have not verified this, but since the hypersurface of constant t becomes timelike for $\lambda^3 < \sin^2 \beta$ it is not clear how it would make sense to extend the bulk integration into this region.

6 Near-horizon supersymmetry

The isometry group of the general stationary black hole solution of D=5 supergravity is $\mathbb{R} \times U(1) \times U(1)$, corresponding to time translations and rotational invariance within two independent 2-planes. Black holes for which the only non-zero angular momentum is that associated to the vector field $m = 2\partial_\psi$ have a larger $\mathbb{R} \times SU(2) \times U(1)$ isometry group. This

includes the supersymmetric rotating black holes, but in this case the time translations are enhanced to a supergroup generated by the hamiltonian and four supercharges. This corresponds to a 1/2 breaking of the supersymmetry of the Minkowski vacuum solution, but it is known that there is a full restoration of supersymmetry near the horizon [20]. Our aim here is to determine the isometry supergroup of this near-horizon solution, which is obtained from the full solution (73) by dropping the constant term from H . For convenience we set $\mu = 1$ in this section to get the near-horizon metric

$$ds^2 = - \left(r^2 dt^2 + \frac{j}{2} \sigma_3 \right)^2 + \frac{dr^2}{r^2} + d\Omega_3^2. \quad (108)$$

We have already shown that the singularity at $r = 0$ is a coordinate singularity at a Killing horizon of $k = \partial_t$, and that the metric can be continued through it. However, the metric (108) will suffice present purposes. Our first task will be to find the Killing spinors admitted by the near-horizon solution. These are solutions of (46).

The near-horizon solution (108) has the same form as the full solution (48) but with

$$\begin{aligned} e^0 &= r^2 dt + \frac{j}{2} \sigma_3 \\ e^r &= r^{-1} dr \\ e^I &= \frac{1}{2} \sigma_I \quad (I = 1, 2, 3) \end{aligned} \quad (109)$$

The Maxwell 2-form F continues to be given by $(\sqrt{3}/2)de^0$. We compute that

$$\begin{aligned} \frac{1}{4} \omega_{ab} \Gamma^{ab} &= e^0 \left(\Gamma^{r0} + \frac{j}{2} \Gamma^{12} - \frac{j}{2} \Gamma^{r3} \right) + e^r \left(-\frac{j}{2} \Gamma^{03} \right) + e^1 \left(\frac{1}{2} \Gamma^{23} + \frac{j}{2} \Gamma^{02} \right) \\ &\quad + e^2 \left(\frac{1}{2} \Gamma^{31} - \frac{j}{2} \Gamma^{01} \right) + e^3 \left(\frac{1}{2} \Gamma^{12} - \frac{j}{2} \Gamma^{r0} \right) \end{aligned} \quad (110)$$

and that

$$\begin{aligned} \frac{1}{4\sqrt{3}} \left(e^a \Gamma^{bc} {}_a F_{bc} - 4e^a \Gamma^b F_{ab} \right) &= e^0 \left(\Gamma^r + \frac{j}{2} \Gamma^{0r3} - \frac{j}{2} \Gamma^{012} \right) + e^r \left(-\Gamma^0 + j\Gamma^3 + \frac{j}{2} \Gamma^{r12} \right) \\ &\quad + e^1 \left(\frac{1}{2} \Gamma^{1r0} - \frac{j}{2} \Gamma^{1r3} - j\Gamma^2 \right) + e^2 \left(\frac{1}{2} \Gamma^{2r0} - \frac{j}{2} \Gamma^{2r3} + j\Gamma^1 \right) \\ &\quad + e^3 \left(\frac{1}{2} \Gamma^{3r0} - j\Gamma^r + \frac{j}{2} \Gamma^{123} \right) \end{aligned} \quad (111)$$

From these results we obtain the following Killing spinor equations:

$$0 = [\partial_t + r^2(\Gamma^{r0} + i\Gamma^r)]\zeta$$

$$\begin{aligned}
0 &= [\partial_r - r^{-1}(i\Gamma^0 + j\Gamma^{03} - ij\Gamma^3)]\zeta \\
0 &= [\partial_\theta - \frac{1}{2}\sin\psi\hat{M}_1 + \frac{1}{2}\cos\psi\hat{M}_2]\zeta \\
0 &= [\partial_\phi + \frac{1}{2}\cos\psi\sin\theta\hat{M}_1 + \frac{1}{2}\sin\psi\sin\theta\hat{M}_2 + \frac{1}{2}\Gamma^{12}\cos\theta]\zeta \\
0 &= [\partial_\psi + \frac{1}{2}\Gamma^{12}]\zeta
\end{aligned} \tag{112}$$

where

$$\hat{M}_1 = \Gamma^{23} + j\Gamma^{02} - ij\Gamma^2, \quad \hat{M}_2 = \Gamma^{31} - j\Gamma^{01} + ij\Gamma^1 \tag{113}$$

All integrability conditions are satisfied, in agreement with [20]. The Killing spinors are

$$\begin{aligned}
\zeta^+ &= r\Omega\eta^+ \\
\zeta^- &= \left[\frac{1}{r}(1 - j\Gamma^{03}) - 2rt\Gamma^{r0} \right] \Omega\eta^-
\end{aligned} \tag{114}$$

where

$$\Omega = e^{\frac{1}{2}\Gamma^{21}\psi} e^{\frac{1}{2}\Gamma^{13}\theta} e^{\frac{1}{2}\Gamma^{21}\phi} \tag{115}$$

and η^\pm are constant spinors satisfying⁹

$$i\Gamma^0\eta^\pm = \pm\eta^\pm. \tag{116}$$

A modification of an argument in [21] shows that if ζ and ζ' are Killing spinors then the vector field

$$v = \bar{\zeta}\Gamma^a\zeta'\tilde{e}_a \tag{117}$$

is Killing, where \tilde{e}_a ($a = 0, 1, 2, 3, r$) are the basis vector fields dual to e^a . The Killing vector fields found this way are linear combinations of the vector fields that generate the bosonic subgroup of the isometry supergroup. Moreover, as shown in [21], the linear combination associated with any pair of Killing spinors is the same as the linear combination of bosonic charges appearing in the anticommutator of spinor charges. The isometry supergroup can therefore be determined (modulo purely bosonic factors) from a knowledge of the Killing spinors.

⁹Note that ζ^- is not an eigenspinor of $i\Gamma^0$.

To proceed, we note that

$$\tilde{e}^a = (r^{-2}\partial_t, \quad r\partial_r, \quad 2\xi_1^R, \quad 2\xi_2^R, \quad 2\xi_3^R - jr^{-2}\partial_t) \quad (118)$$

where

$$\begin{aligned} \xi_1^R &= -\sin\psi\partial_\theta + \cos\psi\operatorname{cosec}\theta\partial_\phi - \cot\theta\cos\psi\partial_\psi \\ \xi_2^R &= \cos\psi\partial_\theta + \sin\psi\operatorname{cosec}\theta\partial_\phi - \cot\theta\sin\psi\partial_\psi \\ \xi_3^R &= \partial_\psi. \end{aligned} \quad (119)$$

These are the three left-invariant vector fields ξ_I^R ($I = 1, 2, 3$) generating right translations on $SU(2)$; they are dual to the basis of left-invariant 1-forms σ_I in the sense that $(\xi_I^R, \sigma_J) = \delta_{IJ}$. We will also need the right-invariant vector fields

$$\begin{aligned} \xi_1^L &= \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi - \cos\phi\operatorname{cosec}\theta\partial_\psi \\ \xi_2^L &= \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi + \sin\phi\operatorname{cosec}\theta\partial_\psi \\ \xi_3^L &= \partial_\phi, \end{aligned} \quad (120)$$

which commute with ξ_I^R and generate left translations on $SU(2)$. The non-vanishing commutation relations among all six vector fields are

$$[\xi_i^R, \xi_j^R] = -\epsilon_{ijk}\xi_k^R, \quad [\xi_i^L, \xi_j^L] = \epsilon_{ijk}\xi_k^L \quad (121)$$

Now, we know that the following vectors fields are Killing:

$$\begin{aligned} v^{++} &= (\bar{\zeta}^+\Gamma^a\zeta^+)\tilde{e}_a \\ v^{+-} &= (\bar{\zeta}^+\Gamma^a\zeta^-)\tilde{e}_a \\ v^{--} &= (\bar{\zeta}^-\Gamma^a\zeta^-)\tilde{e}_a. \end{aligned} \quad (122)$$

A computation yields the result

$$\begin{aligned} v^{++} &= (\bar{\eta}_+\Gamma^0\eta_+)r^2\tilde{e}_0 \\ v^{+-} &= (\bar{\eta}_+\Gamma^r\eta_-)(\tilde{e}_r - 2r^2t\tilde{e}_0) + (\bar{\eta}_-\Omega^{-1}\Gamma^1\Omega\eta_-)\tilde{e}_1 \\ &\quad + (\bar{\eta}_-\Omega^{-1}\Gamma^2\Omega\eta_-)\tilde{e}_2 + (\bar{\eta}_-\Omega^{-1}\Gamma^3\Omega\eta_-)(\tilde{e}_3 + j\tilde{e}_0) \\ v^{--} &= (\bar{\eta}_-\Gamma^0\eta_-)[r^{-2}(1 + j^2 + 4r^4t^2)\tilde{e}_0 - 4t\tilde{e}_r + 2jr^{-2}\tilde{e}_3]. \end{aligned} \quad (123)$$

But

$$\Omega^{-1}\Gamma^i\Omega \equiv \Gamma^j R_j^i(\Omega) \quad (124)$$

where $R(\Omega)$ is a matrix such that

$$R_j^i(\Omega)\xi_i^R = \xi_j^L. \quad (125)$$

It follows that

$$\begin{aligned} v_{++} &= (\bar{\eta}_+\Gamma^0\eta_+)k \\ v_{+-} &= (\bar{\eta}_+\Gamma^r\eta_-)\ell + 2(\bar{\eta}_-\Gamma^1\eta_-)\xi_1^L + 2(\bar{\eta}_-\Gamma^2\eta_-)\xi_2^L + 2(\bar{\eta}_-\Gamma^3\eta_-)\xi_3^L \\ v_{--} &= (\bar{\eta}_-\Gamma^0\eta_-)m \end{aligned} \quad (126)$$

where

$$\begin{aligned} k &= \partial_t \\ \ell &= r\partial_r - 2t\partial_t \\ m &= r^{-4}(1-j^2)\partial_t + 4t^2\partial_t - 4tr\partial_r + 4jr^{-2}\partial_\psi \end{aligned} \quad (127)$$

We thus deduce that the vector fields ξ^L are Killing, as must be k, ℓ, m . The latter commute with ξ^L , and also with ξ^R . Their commutation relations with each other are

$$[\ell, k] = 2k \quad [\ell, m] = -2m \quad [k, m] = -4\ell \quad (128)$$

which are those of $sl(2; \mathbb{R})$. We conclude that the anticommutator of supersymmetry charges closes on a set of bosonic charges that generate $Sl(2; \mathbb{R}) \times SU(2)$. The vector field ξ_3^R is also Killing, as are ξ_1^R and ξ_2^R when $J = 0$, but these vector fields are *not* constructible from Killing spinors and so are not associated with charges in the supersymmetry algebra. This explains why it is possible to break rotational symmetry without losing supersymmetry.

From the above information we deduce that the isometry supergroup of the near-horizon limit of a $D = 5$ supersymmetric black hole is $SU(1, 1|2) \times SU(2)$ for $J = 0$ and $SU(1, 1|2) \times U(1)$ for $J \neq 0$.

7 Discussion

The bosonic equations of $D = 5$ supergravity can be viewed as the special $\nu = 1$ case of the general $D = 5$ Einstein-Maxwell-CS equations derived from the Lagrangian

$$L = \frac{1}{16\pi G_5} \left[\sqrt{-g}(R - F^2) - \frac{2\nu}{3\sqrt{3}} \varepsilon^{mnpqr} A_m F_{np} F_{qr} \right] \quad (129)$$

where ν is a CS coupling constant. For $\nu = 1$ there is a class of supersymmetric black hole solutions parameterized by the charge Q and an angular momentum J , with the mass M being fixed in terms of the charge as a consequence of the fact that it must saturate the bound (3). Here we have uncovered a number of surprising features of these solutions. The horizon is non-rotating even when the angular momentum is non-zero. This is possible because angular momentum can be stored in the Maxwell field, but another surprise is that a negative fraction of the total is stored behind the horizon. Physically, one might think that these two results are related. The fact that the horizon is non-rotating is a statement that the null generators of the horizon experience no angular frame dragging with respect to infinity. One might be able to regard this as the result of a cancellation of the individual dragging effects which the angular momentum distributions inside and outside of the horizon would have produced.

The derivation in [10] of the bound (3) on the mass of black holes of $D = 5$ supergravity made crucial use of the fact that $\nu = 1$. However, the (non-rotating) Tangherlini black hole is a solution of the Einstein-Maxwell-CS equations for any value of ν , and its extremal mass is still given by the formula $M = (\sqrt{3}/2)|Q|$. It also has vanishing surface gravity, and so cannot decrease its mass by Hawking radiation. It might therefore appear that the bound $M \geq (\sqrt{3}/2)|Q|$ must be valid for any value of ν . It is possible that the methods of [25] may serve to derive it for $\nu < 1$, but there are some reasons to think that it may fail when $\nu > 1$. The way in which the proof in [10] fails when $\nu \neq 1$ shows that any violation must involve field configurations for which $F \wedge F$ is non-zero. This condition can be satisfied for perturbations which redistribute the angular momentum in the Maxwell field, suggesting that any instability of the (spherically symmetric) Tangherlini black hole may involve rotation. It might seem unlikely that a black hole solution of zero Hawking temperature and spherical symmetry could be unstable, but a $\nu = 1$ Tangherlini

black hole is only marginally stable because there is a rotating black hole with precisely the same total mass.

As far we are aware, the analogue of the Kerr-Newman solution for $D = 5$ Einstein-Maxwell ($\nu = 0$) is not yet known, but it is likely to be a straightforward extension of the $D = 5$ analogue of the Kerr solution found in [2]. If so, the mass of black hole solutions with vanishing surface gravity can be expected to be a strictly increasing function of either of their two angular momenta. Given the known solutions for $\nu = 1$ it then seems likely that the increase in the energy with at least one angular momentum J decreases as ν increases from zero to 1, so as to become independent of J at $\nu = 1$. If this is the case then an extrapolation to $\nu > 1$ might lead to an energy that decreases with increasing J , creating an instability of the Tangherlini black hole against a perturbation in which photons carry away both energy and angular momentum to infinity.

It is interesting to frame this discussion in terms of the first law, which we derived in section 2. As we are discussing extremal black holes ($\kappa = 0$) and processes which leave the charge unchanged ($dQ = 0$) the first law reduces to $dM = \boldsymbol{\Omega}_H \cdot d\mathbf{J}$. As long as the angular velocities of the horizon are nonvanishing and positive (i.e., the rotation is in the same sense as the angular momentum) when \mathbf{J} are nonvanishing, a further increase of the angular momenta produces an increase in the mass. This corresponds to the situation which we expect to hold for $\nu = 0$, and in fact $\nu < 1$. We know that at precisely $\nu = 1$, the angular velocity of the horizon can vanish even though the angular momentum is nonvanishing. This produces the marginal situation where variations of the angular momentum leave the mass unchanged. The instability that is speculated to arise for $\nu > 1$ would require that at least one of the $\boldsymbol{\Omega}_H$ is *negative* when $\nu > 1$. That is, the horizon would be rotating in the opposite sense to the angular momentum. In this case, a further increase of the angular momentum could be used to reduce the black hole mass.

Pursuing the speculation on frame dragging above, we would have the following physical picture. Imagine we start with the $\nu = 0$ solution with, for simplicity, only the angular momentum (72) associated with $2\partial_\psi$ nonvanishing. We expect that both the corresponding angular velocity Ω_H and the interior angular momentum, measured by J_H , will be nonvanishing and positive. As the CS coupling is increased, the solution will change,

presumably such that both J_H and Ω_H are reduced. For some $\nu < 1$, J_H would reach zero and become negative, but with Ω_H still positive. The latter would only reach zero at $\nu = 1$. A further increase in the CS coefficient to $\nu > 1$ would then, according to our conjecture, produce an unstable solution for which both J_H and Ω_H are negative. The rearrangement in the distribution of the angular momentum would then be the essential difference in the solutions for different values of the CS coupling.

The instability conjectured to exist here for a certain range of the five-dimensional Chern-Simons coupling is in some respects reminiscent of the instability in four-dimensional theories with massive charged vectors [26]. There a magnetically charged Reissner-Nordstrom black hole becomes unstable to developing massive vector hair which is not spherically symmetric when the vector mass reaches a critical value¹⁰. While these instabilities appear to violate the ‘standard’ no-hair theorems, the latter are evaded by the ‘nonstandard’ matter content of the theories in question [27]. Of course, the instability in the present case could be ruled out by an extension of the uniqueness theorems to $D = 5$, but there is no evidence that these theorems apply in $D = 5$ for arbitrary ν . One is free to surmise that uniqueness also applies only for $\nu \leq 1$. It would not be surprising to learn once again that supersymmetry is associated with a borderline between stability and instability.

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References

¹⁰In this case, the critical vector mass is correlated to the black hole size.

- [1] F.R. Tangherlini, *Schwarzschild field in n dimensions and the dimensionality of space problem*, Nuovo Cim. **27** (1963) 636.
- [2] R.C. Myers and M.J. Perry, *Black holes in higher dimensional space-times*, Ann. Phys. (NY) **172** (1986) 304.
- [3] J. Bardeen, B. Carter and S.W. Hawking, *The four laws of black hole mechanics*, Commun. Math. Phys. **31** (1973) 161.
- [4] R.M. Wald, *Black hole entropy is Noether charge*, Phys. Rev. **D48** (1993) 3427.
- [5] G.W. Gibbons and C.M. Hull, *A Bogomolnyi bound for General Relativity and solitons in $N=2$ supergravity*, Phys. Lett. **109B** (1982) 190.
- [6] E. Witten and D. Olive, *Supersymmetry algebras that include topological charges*, Phys. Lett. **78B** (1978) 97.
- [7] J.P. Gauntlett and T. Tada, *Entropy of four-dimensional rotating BPS dyons*, Phys. Rev. **D55** (1997) 1707.
- [8] R.M. Wald, *The first law of black hole mechanics*, gr-qc/9305022.
- [9] E. Cremmer, *Supergravities in 5 dimensions*, in *Superspace and Supergravity*, eds. S.W. Hawking and M. Roček (CUP 1980).
- [10] G.W. Gibbons, D. Kastor, L.A.J. London, P.K. Townsend and J. Traschen, *Supersymmetric self-gravitating solitons*, Nucl. Phys. **B416** (1994) 850.
- [11] J.C. Breckenridge, D.A. Lowe, R.C. Myers, A.W. Peet, A. Strominger and C. Vafa, *Macroscopic and microscopic entropy of near-extremal spinning black holes*, Phys. Lett. **381B** (1996) 423.
- [12] A.A. Tseytlin, *Extreme dyonic black holes in string theory*, Mod. Phys. Lett. **A11** (1996) 689.
- [13] J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa, *D-branes and spinning black holes*, Phys. Lett. **391B** (1997) 93.

- [14] A.H. Chamseddine and W.A. Sabra, *Metrics admitting Killing spinors in five dimensions*, Phys. Lett. **426B** (1998) 36.
- [15] G. Horowitz and A. Sen, *Rotating black holes which saturate a Bogomolnyi bound*, Phys. Rev. **D53** (1996) 808.
- [16] B. Carter, in *Black Holes*, eds. C. and B. DeWitt, Gordon and Breach, New York, 1973 (1972 Les Houches Summer School Lectures)
- [17] G.W. Gibbons, *Aspects of supergravity theories*, in *Supersymmetry, Supergravity and Related Topics*, eds. F. Del Aguila, J.A. de Azcárraga and L.E. Ibañez, World Scientific, Singapore, 1985.
- [18] P. Claus, M. Derix, R. Kallosh, J. Kumar, P.K. Townsend and A. Van Proeyen, *Superconformal mechanics and black holes*, hep-th/9804177.
- [19] A. Chamseddine, S. Ferrara, G.W. Gibbons and R. Kallosh, *Enhancement of supersymmetry near the 5D black hole horizon*, Phys. Rev. **D55** (1997) 3647.
- [20] R. Kallosh, A. Rajaraman and W.K. Wong, *Supersymmetric rotating black holes and attractors*, Phys. Rev. **D55** (1997) 3246.
- [21] J.P. Gauntlett, R.C. Myers and P.K. Townsend, *Supersymmetry of rotating branes*, hep-th/9809065, to appear in Physical Review D.
- [22] S.W. Hawking, *Black Holes in General Relativity*, Commun. Math. Phys. **25** (1972) 152.
- [23] G.W. Gibbons, R. Kallosh and B. Kol, *Moduli, scalar charges, and the first law of black hole thermodynamics*, Phys. Rev. Lett. **77** (1996) 4992.
- [24] B. Carter, *Global structure of the Kerr family of gravitational fields*, Phys. Rev. **D174** (1968) 1559; *Domains of Stationary Communications in Spacetime*, Gen. Rel. Grav. **9** (1978) 437.
- [25] O.M. Moreschi and G.A.J. Sparling, *On the positive energy theorem involving mass and electromagnetic charges*, Commun. Math. Phys. **95** (1984) 113.

- [26] S.A. Ridgway and E.J. Weinberg, *Are All Static Black Hole Solutions Spherically Symmetric?*, Gen. Rel. Grav. **27** (1995) 1017 [gr-qc/9504003]; *Static Black Hole Solutions Without Rotational Symmetry*, Phys. Rev. **D52** (1995) 3440 [gr-qc/9503035].
- [27] D. Kastor and J. Traschen, *Horizons Inside Classical Lumps*, Phys. Rev. **D46** (1992) 5399 [hep-th/9207070].