DIFFERENTIAL GEOMETRY: A SURVEY

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ABSTRACT. In this file notes on differential geometry in general will be put; we will try to be sufficiently abstract and leave physical applications for other notes. However, it should be done in such a way that these applications are fairly easy obtained.

Books used: [3], [4], [2], [1].

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1. Preliminaries

1.1. Topological spaces.

Definition 1.1 (Topological space). *Let S be a set and denote by $\mathcal{J} = \{U_i | i \in I\}$ a collection of subsets of S. The pair $(S, \mathcal{J})$ is called a* topological space *if the following is true.*

(1) *Both the empty set and S are in $\mathcal{J}$, i.e. $\emptyset, S \in \mathcal{J}$.*
(2) *Any (possibly infinite) union of subsets of S which are in $\mathcal{J}$ is also in $\mathcal{J}$, i.e. $\cup_j U_j \in \mathcal{J}$, where $j \in J \subset I$.*
(3) *Any finite intersection of subsets of S which are in $\mathcal{J}$ is also in $\mathcal{J}$, i.e. $\cap_j U_j \in \mathcal{J}$, where $j \in J \subset I$.*

The U_i are called *open sets* and $\mathcal{J}$ gives a *topology* to S. If $\mathcal{J}$ is the collection of all subsets of S, it follows directly from the definition that it is a topology—the *discrete* topology. Another extreme example is the topology given by $\emptyset$ and S only, which is called the trivial topology.

Example 1.1 The open intervals (a, b) on $\mathbb{R}$ define a topology. Note that if one required an infinite intersection of open sets to be an open set, the open intervals would not give a topology to $\mathbb{R}$, indeed for example

$$\bigcap_n^{\infty} \left(-\frac{1}{n}, 1 - \frac{1}{n}\right) = [0, 1]$$

The closed intervals $[a, b]$ do not define a topology as an infinite union of them is generally not closed.

Let us call the elements of S points. A subset N of S is a *neighborhood* of a point x if it contains an open set U_i to which x belongs. If N is open it is an *open neighborhood.*

If for any arbitrary pair of points of a topological space* there exist disjoint neighborhoods, the topological space is a *Hausdorff space.*

*We will abuse terminology: let us refer to S also as the topological space, although strictly speaking the term should be reserved for the pair $(S, \mathcal{J})$.

Example 1.2 The real line with the usual topology is a Hausdorff space. Indeed for any two points x_1 and x_2, choose neighborhoods $(x_i - \delta, x_i + \delta)$ with $\delta \leqslant |x_1 - x_2|/2$.

1.2. **Homotopy.**

Definition 1.2 (Paths and loops). *Let S be a topological space and denote the interval $[0,1]$ by I. A* path *is a continuous map $\mu : I \to S : t \mapsto \mu_t$, with initial point $p = \mu_0$ and end point $q = \mu_1$. If $p = \mu_0 = \mu_1$, then μ is called a* loop *with* base point *p.*

A rather trivial but nonetheless important case is the *constant path*, i.e. $c_t = p$ $(0 \leq t \leq 1)$. One endows the set of paths in S with an algebraic structure by introducing the following operation.

Definition 1.3 (Product and inverse). *Let μ and τ be paths in S, for which $\mu_1 = \tau_0$. The* product *$\mu \cdot \tau$ (τ after μ) is the path in S defined by*

$$(\mu \cdot \tau)_t \equiv \begin{cases} \mu_{2t} & ; \quad 0 \leq t \leq \frac{1}{2} \ , \\ \tau_{2t-1} & ; \quad \frac{1}{2} \leq t \leq 1 \ . \end{cases} \tag{1.1}$$

The inverse path *μ^{-1} of μ is the path defined by*

$$\mu_t^{-1} \equiv \mu_{1-t} \ . \tag{1.2}$$

Note that the inverse path μ^{-1} begins at the end point and ends at the begin point of μ. It follows that $\mu \cdot \mu^{-1}$ is not a constant path.

Definition 1.4 (Homotopy). *Let μ and τ be loops at p in S. The loops μ and τ are said to be* homotopic *with each other if there exists a* continuous *mapping $f : I \times I \to S$ such that $f(t,0) = \mu_t$ and $f(t,1) = \tau_t$, while $f(0,s) = f(1,s) = p$. One calls f a homotopy between μ and τ and writes $\mu \sim \tau$.*

Intuitively, two loops are homotopic if they can be deformed smoothly into each other.

Proposition 1.1. *The homotopy relation is an equivalence relation.*

Proof. Reflectivity: $\mu \sim \mu$. Any loop is homotopic with itself; choose $f(t,s) \equiv \mu_t$. *Symmetry*: $\mu \sim \tau \Rightarrow \tau \sim \mu$. Let f be the homotopy $\mu \sim \tau$. Then $g : I \times I \to S$ defined by $g(t,s) \equiv f(t, s-1)$ is a homotopy $\tau \sim \mu$. *Transitivity*: $\mu \sim \tau$ and $\tau \sim \rho \Rightarrow \mu \sim \rho$. Let f be the homotopy between μ and τ, and g the homotopy between τ and ρ. Then

$$h(t,s) = \begin{cases} f(t,2s) & ; \quad 0 \le s \le \frac{1}{2} \\ g(t, 2s-1) & ; \quad \frac{1}{2} \le s \le 1 \end{cases}$$

is a homotopy $\mu \sim \rho$. □

Given this equivalence relation, one can define corresponding equivalence classes. Let μ be a loop at $p \in S$. The set of all loops at p homotopic with μ is denoted by $[\mu]$.

2. Manifolds

2.1. **Smooth manifolds.** Before introducing manifolds we define the concept of an *atlas*.

Definition 2.1 (Atlas). *A differentiable atlas $\mathcal{A}$ of dimension m of a topological space S is a family of pairs (U_i, ϕ_i)—the* charts*—such that*

(a) *Each U_i is an open set of S and $\cup_i U_i = S$*
(b) *Each ϕ_i is a homeomorphism from of U_i onto an open subset of $\mathbb{R}^m$*
(c) *Whenever $U_i \cap U_j$ is non-empty the map $\phi_j \circ \phi_i^{-1}$ from $\phi_i(U_i \cap U_j)$ to $\phi_j(U_i \cap U_j)$ is differentiable.**

The following then introduces *differentiable manifolds.*

Definition 2.2 (Manifold). *An m-dimensional differentiable manifold M is a Hausdorff space equipped with an m-dimensional differentiable atlas.*

*Differentiable or smooth: the mapping is infinitely differentiable.

Given two manifolds M and N, a mapping $f : M \to N$ is differentiable if for every chart (U_i, ϕ_i) of M and every chart (V_j, ψ_j) of N such that $f(U_i) \subset V_j$, the mapping $\psi_j \circ f \circ \phi_i^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable. If the inverse mapping f^{-1} exists and is differentiable, the mapping is called a *diffeomorphism* and M and N are said to be diffeomorphic. They are of the same dimension. If M and N are the same manifold, the diffeomorphism is also an *automorphism.*

A special case of a mapping is given by a curve $c : [a, b] \to M$, which maps a closed interval of $\mathbb{R}$ into a manifold M. Another interesting example are the functions $f : M \to \mathbb{R}$, which assign to each point of a manifold a real number. We will denote the algebra of smooth functions on M by $\mathfrak{F}(M)$.

Consider then a curve $c(t)$ on a manifold such that $c(t_0) = p$. A *tangent vector* X_p to $c(t)$ at the point p is the directional derivative of functions along the curve at p, i.e.

$$X_p f \equiv \left. \frac{df(c(t))}{dt} \right|_{t=t_0} \tag{2.1}$$

Definition 2.3. (Tangent vector) *Let p be a point on a manifold M. A tangent vector to M at p is a real-valued function $X_p : \mathfrak{F}(M) \to \mathbb{R}$ that is*

(1) $\mathbb{R}$*-linear, i.e.* $X_p(af + bg) = aX_p(f) + bX_p(g)$*, and*
(2) *Leibnizian, i.e.* $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$

for all $a, b \in \mathbb{R}$ *and* $f, g \in \mathfrak{F}(M)$.

The set of tangent vectors at the point p forms a vector space, the *tangent space* T_pM.

A *vector field* X on a manifold M is an assignment of a vector X_p to each point p on M.

$$(Xf)(p) = X_p(f) \quad \text{for all } p \in M \tag{2.2}$$

We call X differentiable if the function Xf is differentiable for every differentiable function f. Denote the set of all differentiable vector fields on M by $\mathfrak{X}(M)$. Then, $\mathfrak{X}(M)$ forms a module over the algebra $\mathfrak{F}(M)$. It follows immediately that it also is a vector space over the reals—as the latter are just constant elements in $\mathfrak{F}(M)$. This infinite dimensional vector space also is a Lie algebra over the reals, as the commutator of two vector fields is a vector field.

$$[X,Y]f = (X^j \partial_j Y^i - Y^j \partial_j X^i)\partial_i f \tag{2.3}$$

If one has a *frame* X_a over M, i.e. a set of n vector fields such that at each point p they form a basis for T_pM we can write

$$[X_a, X_b] = c_{ab}{}^c(p) X_c \tag{2.4}$$

Note that the *structure functions* depend on p since there is no reason to assume that the commutator of two frame fields is a linear combination of frame fields. The commutator will be a generic vector field, an element of an infinitedimensional vector space, for which the frame fields are *not* a basis.

A *1-form* ω on a manifold M is defined as an $\mathfrak{F}(M)$-linear mapping of the module $\mathfrak{X}(M)$ into $\mathfrak{F}(M)$. Hence, at each point p they are the duals of the tangent vectors and the space spanned by them is called the *cotangent space* T_p^*M. A 1-form ω is differentiable if $\omega(X)$ is differentiable for all $X \in \mathfrak{X}(M)$. The set of all 1-forms $\mathfrak{X}^*(M)$ then also is a module over the algebra $\mathfrak{F}(M)$. The *total differential* of $f \in \mathfrak{F}(M)$ at p is defined by the 1-form df_p for which

$$(df(X))_p = X_p(f) \quad \text{for all } X \in \mathfrak{X}(M) \tag{2.5}$$

Let $\varphi : M \to N$ be a mapping. It is possible to consider the following induced mappings. First let us consider the *pullback* of a generic function f, that is

$$\varphi^* : \mathfrak{F}(N) \to \mathfrak{F}(M) : f \mapsto \varphi^* f \quad \text{with} \quad \varphi^* f \equiv f \circ \varphi \ . \tag{2.6}$$

This allows one to define the *differential map* or *pushforward* of a vector field, the linear mapping

$$\varphi_* : TM \to TN : X \mapsto \varphi_* X \tag{2.7}$$

such that for every function $f \in \mathfrak{F}(N)$, we have

$$(\varphi_* X) f \equiv X(\varphi^* f) \tag{2.8}$$

Given the notion of a pushforward of a vectorfield, one defines the pullback of a 1-form ω by

$$\varphi^* : T^* N \to T^* M : \omega \mapsto \varphi^* \omega \tag{2.9}$$

such that for all vectors $X \in T_p M$

$$(\varphi^* \omega)(X) \equiv \omega(\varphi_* X) \tag{2.10}$$

These definitions are readily extended to incorporate the pushforward and pullback of respectively complete contravariant and covariant tensors. Indeed, we define

$$\varphi_* : T_{(r,0)} M \to T_{(r,0)} N : t \mapsto \varphi_* t \quad \text{with} \quad (\varphi_* t)(\omega_1, \ldots, \omega_r) \equiv t(\varphi^* \omega_1, \ldots, \varphi^* \omega_r) \tag{2.11}$$

and

$$\varphi^* : T_{(0,r)} N \to T_{(0,r)} M : t \mapsto \varphi^* t \quad \text{with} \quad (\varphi^* t)(X_1, \ldots, X_r) \equiv t(\varphi_* X_1, \ldots, \varphi_* X_r) \,. \tag{2.12}$$

Let us consider the effect of these mappings in some coordinate system $x^i(p)$ on M and $y^i(\varphi(p))$ on N. A vector field $X = X^i(x(p))\partial_i|_p$ transforms according to

$$\begin{aligned}(\varphi_* X)^i(y)\partial_i|_{\varphi(p)} f &\equiv X^i(x)\partial_i|_p f \circ \varphi \\ &= X^j(x)\frac{\partial y^i}{\partial x^j}\partial_i|_{\varphi(p)} f\end{aligned}$$

A 1-form $\omega = \omega_i(y(\varphi(p)))dy^i(\varphi(p))$ transforms as

$$(\varphi^*\omega)_i(x)dx^i(p)(X^j(x)\partial_j|_p) \equiv \omega_i(y)dy^i(\varphi(p))\underbrace{(X^j(x)\frac{\partial y^k}{\partial x^j}\partial_k|_{\varphi(p)})}_{\phi_* x}$$

$$(\phi^*\omega)_i(x)X^i(x) = \omega_j(y)\frac{\partial y^j}{\partial x^i}X^i(x)$$

These kind of calculations can be done straightforwardly for any completely contravariant or covariant tensor.

Note how the differential map works in the same direction as the mapping φ, while the pullback goes the other way around. Therefore, a mixed type tensor cannot be transported in the above defined manner. However, if the mapping φ is a diffeomorphism one can extend the definition of a pushforward (pullback) by noting that the inverse mapping exists. Indeed, given the fact that the pullback *pulls the tensor back*, we can push it along φ by considering the pullback of the inverse of φ. In other words, given the diffeomorphism ϕ and the pushforward φ_* of a vector, we map a form ω in the same direction as φ by invoking $\varphi^{-1*}\omega$.

Definition 2.4 (Differential map)**.** *Let $\varphi : M \to M$ be a diffeomorphism. The pushforward of an arbitrary tensorfield, induced by the diffeomorphism φ is the linear mapping*

$$\varphi_* : T_{(r,s)}M \to T_{(r,s)}M : t \mapsto \varphi_* t \tag{2.13}$$

such that for any set of 1-forms and vector fields

$$(\varphi_* t)(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) \equiv t(\varphi^*\omega_1, \ldots, \varphi^*\omega_r, \varphi_*^{-1}X_1, \ldots, \varphi_*^{-1}X_s) \ . \tag{2.14}$$

One could define also the pullback for a generic tensor field, but the above considerations show that $\phi^* = \phi_*^{-1}$ to be consistent.

Let $x^i(p)$ be a coordinate system on M. Under a the pushforward of a diffeomorphism, a generic tensor field transforms as (we omit the coordinate basis elements)

$$(\varphi_* t)^{i_1\cdots i_r}_{j_1\cdots j_s}(x(\varphi(p)) = t^{k_1\cdots k_r}_{l_1\cdots l_s}\frac{\partial x^{i_1}(\varphi(p))}{\partial x^{k_1}(p)}\cdots\frac{\partial x^{i_r}(\varphi(p))}{\partial x^{k_r}(p)}\frac{\partial x^{l_1}(p)}{\partial x^{j_1}(\varphi(p))}\cdots\frac{\partial x^{l_s}(p)}{\partial x^{j_s}(\varphi(p))} \tag{2.15}$$

The resemblence with the transformation law under general coordinate transformations is not a coincidence. Given a diffeomorphism φ and a coordinate system x^i, a coordinate transformation can be defined by pulling back the coordinate functions, i.e. $y^i(p) := (\varphi^* x^i)(p) = x^i(\varphi(p))$. Instead of performing the diffeomorphism on M—the *active* point of view—one can leave M as it is and perform the coordinate transformation by pulling back the coordinate functions—the *passive* point of view. The transformation laws of tensors under change of coordinates are then identified with the pushforwards/pullbacks of the given diffeomorphism. It is then a matter of taste if one thinks of it as an active or passive transformation.

Let M be an m-dimensional manifold with atlas $\{(U_i, \phi_i)\}$ and N an n-dimensional manifold with atlas $\{(V_i, \psi_i)\}$. The *product manifold* $M \times N$ is defined as the manifold with atlas $\{(U_i \times V_j), (\phi_i, \psi_j)\}$. A point (p, q) of $M \times N$ has coordinates $(\phi_i, \psi_j)(p, q) = (\phi_i(p), \psi_j(q)) \in \mathbb{R}^{m+n}$, from which it is clear that the dimension of $M \times N$ is $m + n$. For every $(p, q) \in M \times N$, the tangent space $T_{(p,q)}M \times N$ can be identified with the direct sum $T_pM + T_qN$ as follows. Let $X \in T_pM$ and $Y \in T_qN$ be the tangent vectors to the curves μ_t and ν_t, at $p = \mu_0$ and $q = \nu_0$ respectively. We identify $(X, Y) \in T_pM + T_qN$ with $Z \in T_{(p,q)}M \times N$, the tangent vector to (μ_t, ν_t) at $(p, q) = (\mu_0, \nu_0)$. Consider the mappings $\imath_q : M \to M \times N : p' \mapsto (p', q)$ and $\imath_p : N \to M \times N : q' \mapsto (p, q')$. It follows that $Z = \imath_{q*}X + \imath_{p*}Y$. To see this, consider

$$Zf = \tfrac{d}{dt}f(\mu_t, \nu_t)|_0 = \tfrac{d}{dt}f(\mu_t, q)|_0 + \tfrac{d}{dt}f(p, \nu_t)|_0 = (\imath_{q*}X)f + (\imath_{p*}Y)f\ .$$

Proposition 2.1 (Leibniz's formula). *Let $\varphi : M \times N \to K$ be a mapping and consider the compositions*

$$\varphi_1 \equiv \varphi \circ \imath_q : M \to K : p' \mapsto (p', q)$$
$$\varphi_2 \equiv \varphi \circ \imath_p : N \to K : q' \mapsto (p, q') \ .$$

The differential φ_* can be expressed in the following manner. If $Z \in T_{(p,q)}M \times N$ corresponds to $(X, Y) \in T_pM + T_qN$, then

$$\varphi_*(Z) = \varphi_{1*}(X) + \varphi_{2*}(Y) \ . \tag{2.16}$$

Proof. Since $Z = \imath_{q*}X + \imath_{p*}Y$, it follows that

$$\varphi_*(Z) = \varphi_*(\imath_{q*}X + \imath_{p*}Y) = \varphi_{1*}(X) + \varphi_{2*}(Y) \ .$$

□

2.2. **Differential forms.**

Definition 2.5 (Differential form). *A differential form ω of degree r is a $(0, r)$-type tensor field which is skew symmetric, that is*

$$\omega(X_{\pi(1)}, \ldots, X_{\pi(r)}) = \varepsilon(\pi) \cdot \omega(X_1, \ldots, X_r) \ , \tag{2.17}$$

where π is an element of the group of permutations of $(1, \ldots, r)$ and $\epsilon(\pi)$ the corresponding sign.

In other words, a differential form is a multilinear skew-symmetric mapping of $\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)$ into $\mathfrak{F}(M)$. We denote the module of differential r-forms on M by $\Omega^r(M)$.

Given an arbitrary tensor $(0, r)$-type tensor t, the *antisymmetrizer* A is defined as follows,

$$(At)(X_1, \ldots, X_r) := \frac{1}{r!} \sum_{\pi} \varepsilon(\pi) \cdot t(X_{\pi(1)}, \ldots, X_{\pi(r)}) \ . \tag{2.18}$$

From the definition of differential forms it follows that these are eigenstates of A, as can be seen from

$$\begin{aligned}(A\omega)(X_1, \ldots, X_r) &= \frac{1}{r!} \sum_{\pi} \varepsilon(\pi)^2 \cdot \omega(X_1, \ldots, X_r) \\ &= \omega(X_1, \ldots, X_r) \ .\end{aligned}$$

On the other hand, if $\omega = A\omega$ then it is a differential form–being completely skew-symmetric in its arguments. We conclude that a $(0, r)$-type tensor is a r-form if and only if it an eigenstate of the antisymmetrizer operator. Furthermore, since A is a projection operator, i.e. $A^2 = A$, any $(0, r)$-tensor is mapped into a differential form under A.

Given the antisymmetrizer operator A, the *exterior* or *wedge product* of two differential forms is defined as ($\omega \in \Omega^r(M)$, $\xi \in \Omega^s(M)$)

$$\omega \wedge \xi \equiv \tfrac{(r+s)!}{r!s!} A(\omega \otimes \xi) \tag{2.19}$$

which is an element of $\Omega^{r+s}(M)$.

Every differential r-form ω may be obtained from some $(0, r)$-tensor t by invoking A, that is $\omega = At$. Since, A is a linear operator and considering a local coordinate basis of M we have $\omega = t_{i_1 \cdots i_r} A(dx^{i_1} \otimes \cdots dx^{i_r})$. It follows that*

$$dx^{i_1} \wedge \cdots dx^{i_r} = r! A(dx^{i_1} \otimes \cdots dx^{i_r})$$

form a basis for the vector spaces $\Omega^r_p(M)$. Hence any differential form ω may be written as $\omega = \frac{1}{r!} \omega_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots dx^{i_r}$. Note that this is equivalent to

$$\omega = \tfrac{1}{r!} \omega_{i_1 \cdots i_r} \sum_{\pi} \varepsilon(\pi) \cdot dx^{\pi(i_1)} \otimes \cdots dx^{\pi(i_r)} = \omega_{[i_1 \cdots i_r]} dx^{i_1} \otimes \cdots dx^{i_r} \ . \tag{2.20}$$

Hence, the a differential form is just a tensor with antisymmetric component functions. This means also that the possible symmetric part in $\omega_{i_1 \ldots i_r}$ may be omitted. We state without derivation that the wedge product of

*Note that this is consistent with the definition of the wedge product, since $\omega \wedge \xi \wedge \eta = \frac{(r+s+q)!}{r!s!q!} A(\omega \otimes \xi \otimes \eta)$, for $\omega \in \Omega^r(M)$, $\xi \in \Omega^s(M)$ and $\eta \in \Omega^q(M)$.

two differential forms is expressed in local coordinates as

$$\omega \wedge \xi = \frac{1}{r!s!}\omega_{i_1\cdots i_r}\xi_{i_{r+1}\cdots\xi_{r+s}}dx^{i_1}\wedge\ldots dx^{i_{r+s}} \ . \tag{2.21}$$

This explicit construction reveals some interesting properties regarding differential forms. For example, the dimension of the vector space $\Omega^r_p(M)$ is given by $\binom{m}{r}$. Also, there are no r-forms for $r > m$. Given this information the reader may understand that the *exterior algebra**

$$\Omega(M) \equiv \Omega^0(M) \oplus \cdots \oplus \Omega^m(M) \tag{2.22}$$

of differential forms on M is closed under the exterior product.

Definition 2.6 (Exterior differentiation)**.** Exterior differentiation *is the unique linear mapping*

$$d : \Omega^r(M) \to \Omega^{r+1}(M)$$

satisfying the properties:

(1) *for any function $f \in \Omega^0(M)$, df is the total differential*
(2) $d \circ d \equiv 0$
(3) *for any $\omega \in \Omega^r(M)$ and $\xi \in \Omega^s(M)$,*

$$d(\omega \wedge \xi) = d\omega \wedge \xi + (-1)^r \omega \wedge d\xi$$

In local coordinates the action of this operator is given by

$$d\omega = \frac{1}{r!}\partial_i\omega_{i_1\cdots i_r}dx^i \wedge dx^{i_1} \wedge \cdots dx^{i_r} \ .$$

Definition 2.7 (Interior product)**.** *The* interior product *of a differential form with respect to a vector field X is the mapping*

$$i_X : \Omega^r(M) \to \Omega^{r-1}(M) \tag{2.23}$$

such that

(1) $i_X f = 0$ *where* $f \in \mathfrak{F}(M)$,
(2) $(i_X\omega)(X_1, \ldots, X_{r-1}) = \omega(X, X_1, \ldots, X_{r-1})$ *for* $\omega \in \Omega^r(M)$.

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*We remark that $\Omega^0(M) := \mathfrak{F}(M)$.

In local coordinates, $(i_X\omega)_{[i_2\cdots i_r]} = X^i\omega_{[ii_2\cdots i_r]}$ such that

$$i_X\omega = \frac{1}{(r-1)!}X^i\omega_{ii_2\cdots i_r}dx^{i_2}\wedge\ldots dx^{i_r} \tag{2.24}$$

This can be rewritten in a form that is useful for future reference. Namely, let $i_X\omega = \frac{1}{(r-1)!}X^{i_1}\omega_{i_1\cdots i_r}d\hat{x}^{i_1}\wedge dx^{i_2}\wedge\ldots dx^{i_r} = \frac{(-1)^{s-1}}{(r-1)!}X^{i_s}\omega_{i_1\cdots i_s\cdots i_r}dx^{i_1}\wedge \ldots d\hat{x}^{i_s}\wedge\ldots dx^{i_r}$, where a hat on top denotes that differential underneath is omitted and where we used the antisymmetry of ω. It then follows that

$$i_X\omega = \frac{1}{r!}\sum_s(-1)^{s-1}X^{i_s}\omega_{i_1\cdots i_s\cdots i_r}dx^{i_1}\wedge\ldots d\hat{x}^{i_s}\wedge\ldots dx^{i_r}\ . \tag{2.25}$$

2.3. **1-parameter groups of differentiable manifolds.** Let X be a vector field on a manifold M. An *integral curve* of X is a curve $x(t)$ on M such that for each t, $X_{x(t)}$ is tangent to the curve. This means that for each t one has

$$\frac{df(x(t))}{dt} = X_{x(t)}f \quad \text{for all } f\in\mathfrak{F}(M) \tag{2.26}$$

which can be rephrased in a local coordinate basis as

$$\frac{dx^i(t)}{dt}\frac{\partial f}{\partial x^i} = X^i(x(t))\frac{\partial f}{\partial x^i} \tag{2.27}$$

where we used the same notation for the coordinate system and the coordinates of the curve.*

Solving the system of ODEs (2.27) for given initial conditions $x^i_0 = x^i(0)$ gives one a unique integral curve on the manifold.

Example 2.1 Consider the vector field in $\mathbb{R}^2$

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad \text{i.e. } X^i = (-y, x). \tag{2.28}$$

The components of (2.27) then give two ODEs for the integral curve $(x(t), y(t))$,

$$\frac{dx(t)}{dt} = -y(t)\ , \quad \frac{dy(t)}{dt} = x(t)$$

*To make the distinction somewhat clearer, the integral curve coordinates keep an explicit dependence on the parameter in the notation.

and we denote initial values as x_0, y_0. This system has the unique solution

$$\begin{aligned} x(t) &= x_0 \cos t - y_0 \sin t \\ y(t) &= y_0 \cos t - x_0 \sin t \end{aligned} \tag{2.29}$$

which is a circle centered around the origin with radius $r = [(x_0)^2 + (y_0)^2]^{1/2}$. Hence, the infinite set of these circles are the integral curves of the considered vector field.

Definition 2.8 (1-parameter group of transformations). *A 1-parameter group of transformations of M is a mapping*

$$\mathbb{R} \times M \to M : (t, p) \mapsto \phi_t(p)$$

which satisfies the conditions

(a) *for each $t \in \mathbb{R}$, ϕ_t is a transformation of M*
(b) *for each $t, s \in \mathbb{R}$ and $p \in M$, $\phi_{t+s} = \phi_t(\phi_s)$*

This abelian group of transformations naturally induces a vector field X, i.e. the vector field for which $\phi_t(p)$ is the integral curve for some point $p = \phi_0(p)$.

One can also consider a local 1-parameter group of local diffeomorphisms, where t is in an open interval $(-\varepsilon, \varepsilon)$ and p lies in an open set of M. It can be proven that if a vector field X is given, there is for each point p a local 1-parameter group of local diffeomorphisms ϕ_t which induces X. See [2], pg. 13. The vector field X then is said to be the generator of ϕ_t in the neighborhood of p. In the remainder, it will be assumed that this correspondence holds for "big enough" regions.

This induced vector field will satisfy the ODE

$$\frac{d\phi_t^i(x)}{dt} = X^i(\phi_t(x)) \tag{2.30}$$

and the *flow*—the integral curve—asociated with the group $\phi_t(x)$ is generated by this vector field in the following manner.

$$\begin{aligned}
\phi^i_t(x) &= \sum_n^\infty \frac{t^n}{n!}\left(\frac{d}{dt}\right)^n \phi^i_t(x)|_{t=0} \\
&= \left[1 + \sum_{n=1}^\infty \frac{t^n}{n!}\left(\frac{d}{dt}\right)^n\right] \phi^i_t(x)|_{t=0} \\
&= \exp\left(t\frac{d}{dt}\right) \phi^i_t(x)|_{t=0}
\end{aligned}$$

This is of course the realization on the manifold of a finite translation in parameter space. We can rewrite it formally as

$$\phi^i_t(x) = \exp(tX)\, x^i \tag{2.31}$$

Let ϕ be a transformation of a manifold. If a vector field X generates the 1-parameter group ϕ_t, then the vector field $\phi_* X$ generates the 1-parameter group $\phi \circ \phi_t \circ \phi^{-1}$ (see [2], pg. 14). From this it follows that X is an ivariant vector field under ϕ if and only if $[\phi, \phi_t] = 0$, because X and $\phi_* X$ then satisfy the same ordinary differential equation (2.30) and the uniqueness of its solution.

2.4. **Lie derivative.** Let X and Y be vector fields on a manifold M and let φ_t be a 1-parameter group of transformations of M, generated by X.

Definition 2.9 (Lie derivative). *The Lie derivative of any tensor t with respect to X is given by*

$$\mathcal{L}_X t \equiv \lim_{\tau\to 0} \tfrac{1}{\tau}[t - \varphi_{\tau *} t] \ , \tag{2.32}$$

or more precisely

$$\mathcal{L}_X t = \lim_{\tau\to 0} \tfrac{1}{\tau}[t_p - (\varphi_{\tau *} t)_p] \ , \quad p \in M \ .$$

Note that for a differential form ω the Lie derivative is also given by

$$\mathcal{L}_X \omega = \lim_{\tau\to 0} \tfrac{1}{\tau}[\omega - \varphi^*_{-\tau}\omega] = \lim_{\tau\to 0} \tfrac{1}{\tau}[\varphi^*_\tau \omega - \omega] \tag{2.33}$$

It is clear that the Lie derivative acts linearly on the algebra of tensor fields. Let us then show that it is also a Leibnizian operator, that is

$$\mathcal{L}_X(t_1 \otimes t_2) = \mathcal{L}_X t_1 \otimes t_2 + t_1 \otimes \mathcal{L}_X t_2 \ . \tag{2.34}$$

Indeed, one finds

$$\begin{aligned}\mathcal{L}_X(t_1 \otimes t_2) &= \lim_{\tau\to 0} \tfrac{1}{\tau}[t_1 \otimes t_2 - \varphi_{\tau*}(t_1 \otimes t_2)] \\ &= \lim_{\tau\to 0} \tfrac{1}{\tau}[t_1 \otimes t_2 - \varphi_{\tau*}t_1 \otimes \varphi_{\tau*}t_2 \\ &= \lim_{\tau\to 0} \tfrac{1}{\tau}[t_1 - \varphi_{\tau*}t_1] \otimes t_2 + \lim_{\tau\to 0} \tfrac{1}{\tau}\varphi_{\tau*}t_1 \otimes [t_2 - \varphi_{\tau*}t_2] \\ &= \mathcal{L}_X t_1 \otimes t_2 + t_1 \otimes \mathcal{L}_X t_2 \ .\end{aligned}$$

3. Lie groups and Lie algebras

3.1. **Left invariant vector fields.**

Definition 3.1 (Lie group). *A Lie group G is a group which is at the same time a differentiable manifold such the group composition and the inverse operation are differentiable mappings. The dimension of the group is the dimension of the manifold.*

Let a and g then be elements of a Lie group G. Denote the *left translation* L_a of G by a as the automorphism

$$L_a : G \to G : g \mapsto ag \ , \quad g \in G \ . \tag{3.1}$$

The associated differential map is an invertible transformation of vector fields on G, i.e. $L_{a*} : T_gG \to T_{ag}G$.

This inherent automorphism acting on Lie group manifolds is a very interesting feature of them. Given a Lie group this automorphism singles out a particular class of vector fields on G, namely those ones which are invariant under left translations. We call them *left invariant vector fields* and they satisfy

$$L_{a*}X_g = X_{ag} \ . \tag{3.2}$$

In a local coordinate system this is rephrased as

$$(L_{a*}X)^i\partial_i|_{ag}f = X^i(g)\frac{\partial x^j(ag)}{\partial x^i(g)}\partial_j|_{ag}f \overset{LI}{=} X^i(ag)\partial_i|_{ag}f \tag{3.3}$$

Denote the set of left invariant vector fields by $\mathfrak{g}$. They form a subset of the infinite dimensional Lie algebra $\mathfrak{X}(G)$ with the usual addition and scalar multiplication. It inherits the bracket operation of $\mathfrak{X}(G)$, under which $\mathfrak{g}$ is closed as is implied by the following.

$$L_{a*}[X,Y]_g = [L_{a*}X_g, L_{a*}Y_g] = [X_{ag}, Y_{ag}] = [X,Y]_{ag} \tag{3.4}$$

This shows that the Lie bracket of two left invariant vector fields is a left invariant vector field. Hence, $\mathfrak{g}$ is closed and forms a subalgebra of $\mathfrak{X}(G)$.

As a vector space $\mathfrak{g}$ is isomorphic to T_eG. Denote a generic element of T_eG by X_e. These elements can be obtained by mapping a left invariant vector field X to its value at e. The other direction of the isomorphism then is defined by the transformation $X_e \mapsto X$, as in (3.2). Hence, the dimension of $\mathfrak{g}$ is $n = \dim G$ and one refers to it as *the Lie algebra of* G.

Example 3.1 Consider the general linear group $Gl(n,\mathbb{R})$, the group of all real non-singular $(n \times n)$-matrices. They form an open submanifold of $\mathbb{R}^{n\cdot n}$ and a generic element g will be specified by the coordinates $x^i{}_j(g)$. The unit element e has coordinates δ^i_j. The left translation is then given in the following terms

$$(L_a g)^i{}_j = x^i{}_j(ag) = x^i{}_k(a)x^k{}_j(g) \tag{3.5}$$

An element in the tangent space at the identity is a vector $V \in T_eG$

$$V = V^i_j \left.\frac{\partial}{\partial x^i{}_j}\right|_e . \tag{3.6}$$

We use (3.2) and (3.3) to construct a left invariant vector field X by left translating V over $Gl(n,\mathbb{R})$,

$$
\begin{aligned}
X_g = L_{g*}V &= (L_{g*}V)^i{}_j \partial_{x^i{}_j}|_g \\
&= V^i_j \frac{\partial x^k{}_l(ge)}{\partial x^i{}_j(e)} \partial_{x^k{}_l}|_g \\
&= V^i_j x^k{}_m(g) \frac{\partial x^m_l(e)}{\partial x^i{}_j(e)} \partial_{x^k{}_l}|_g \\
&= V^i_j x^k{}_m(g) \delta^m_i \delta^l_j \partial_{x^k{}_l}|_g \\
&= V^i_j x^k{}_i(g) \partial_{x^k{}_j}|_g = (gV)^i{}_j \partial_{x^i{}_j}|_g
\end{aligned}
$$

The vector field $X_g = (gV)^i{}_j \partial_{x^i{}_j}|_g$ is said to be generated by $V \equiv X_e \in T_eG$.* It is clearly left invariant since $L_{a*}X_g = (agV)^i{}_j \partial_{x^i{}_j}|_{ag} = X_{ag}$. These vector fields form the Lie algebra of $Gl(n,\mathbb{R})$. However, we will show that not only there is a Lie algebra isomorphism between these vector fields and their vectors at the identity, but that the *components* of the vectors at the identity also are isomorphic with $\mathfrak{g}$. Therefore, we first compute the commutator of two left invarant vector fields $X = (gV)^i{}_j \partial_{x^i{}_j}|_g$ and $Y = (gW)^i{}_j \partial_{x^i{}_j}|_g$.

$$
\begin{aligned}
[X,Y]|_g &= [x^i{}_j(g) V^j_k \partial_{x^i{}_k}|_g, x^a{}_b(g) W^b{}_d \partial_{x^a{}_d}|_g] \\
&= x^i{}_j(g) V^j_k \delta^a_i \delta^k_b W^b{}_d \partial_{x^a{}_d}|_g - [V \leftrightarrow W] \\
&= x^a{}_b(g)(V^b_j W^j{}_d - W^b_j V^j_d \partial_{x^a{}_d}|_g \\
&= x^a{}_b(g)[V,W]^b{}_d \partial_{x^a{}_d}|_g \\
&= (g[V,W])^i{}_j \partial_{x^i{}_j}|_g
\end{aligned}
$$

One concludes that the commutator of two left invariant vector fields is given by left translating the vector at the identity with components $V^i{}_k W^k_j - W^i{}_k V^k_j$. This is a very interesting result, as it means that the Lie algebra structure can be understood completely by considering the components only of the left invariant vector fields at the identity. These components are *arbitrary* $(n \times n)$-matrices.

---

*The notation $(gV)^i{}_j$ is to be understood as the matrix product of the coordinates of g with the components of V, i.e. $x^i{}_k(g)V^k_j$.

The Lie algebra of $Gl(n\mathbb{R})$ is then to be understood as these matrices together with the standard matrix commutator $[\cdot,\cdot]$ and is denoted by $\mathfrak{gl}(n,\mathbb{R})$.

We repeat that this is only true for matrix Lie groups. For generic Lie groups, the components of the left invariant vector fields at the identity do not have a structure to preserve the Lie algebra commutator and one needs the *vector* (field), not only its components.

In the remainder, we will both use V and $V^i_{\ j}$ to denote an element of $\mathfrak{gl}(n,\mathbb{R})$, although in the above introduced notation, V is a vector of $T_eGl(n,\mathbb{R})$ whereas only its components are considered an element of the matrix Lie algebra $\mathfrak{gl}(n,\mathbb{R})$.

3.2. **Frames and structure equations.** Let $E_a(a = 1\dots n)$ be a set of linearly independent left invariant vector fields on G. They form a basis for $\mathfrak{g}$ and at each point g they are a basis for the tangent space T_gG, hence a frame.

Since $\mathfrak{g}$ is closed under the Lie bracket operation, we can write (2.4) with constant structure functions

$$[E_a, E_b] = c_{ab}^{\ \ c}E_c \tag{3.7}$$

The c are then referred to as *the structure constants of the Lie algebra.*

Let us introduce a dual basis θ^a to the basis of $\mathfrak{g}$, i.e.

$$\theta^a(E_b) \equiv \delta^a_b \ , \quad \text{for all } g \in G \tag{3.8}$$

which are a basis for the set of left invariant 1-forms $\mathfrak{g}^*$. These are indeed left-invariant since

$$L_g^*\theta^a(E_b) = \theta^a(L_{g*}E_b) = \theta^a(E_b) \quad \text{ for all } E_b \in \mathfrak{g}\ .$$

We now derive the *Maurer-Cartan's structure equations*, see also [2], pg. 36.*

$$\begin{aligned} d\theta^a(E_b, E_c) &= [E_b(\theta^a(E_c)) - E_c(\theta^a(E_b)) - \theta^a([E_b, E_c])] \\ &= \theta^a([E_b, E_c]) \\ &= -c_{bc}{}^f \theta^a(E_f) \\ &= -\tfrac{1}{2} c_{de}{}^a 2\delta^d_{[b}\delta^e_{c]} \\ &= -\tfrac{1}{2} c_{de}{}^a \theta^d \wedge \theta^e(E_b, E_c) \end{aligned}$$

In the second to last step we used the fact that the structure constants are antisymmetric in the lower indices. Since the found result is true for arbitrary E_a we obtain the Maurer-Cartan's structure equations, the dual expression of (3.7), i.e.

$$d\theta^a = -\tfrac{1}{2} c_{bc}{}^a \theta^b \wedge \theta^c \tag{3.9}$$

The *canonical 1-form* ω is a left invariant Lie-algebra valued 1-form. This means it maps elements of $\mathfrak{g}$ into its vectors in the tangent space at e,

$$\omega(X) = L_{g^{-1}*}X \tag{3.10}$$

Hence, it can be expanded as $\omega_g = E_{e,a} \cdot \theta^a_g$, where θ^a_g is the dual basis of T^*_gG. Indeed, let $X = X^a E_a$ be an element of $\mathfrak{g}$. It follows that

$$\omega_g(X_g) = \omega_g(X^a E_{g,a}) = X^a E_{e,b} \cdot \theta^b_g(E_{g,a}) = E_{e,b} X^a \delta^b_a = X^a E_{e,a}$$

By noting that $\omega \wedge \omega \equiv \frac{1}{2}[E_{e,a}, E_{e,b}] \cdot \theta^a \wedge \theta^b$, it directly follows from (3.9) that

$$d\omega + \omega \wedge \omega = 0 \tag{3.11}$$

Example 3.2 Let us take a look what this means for the general linear group and algebra. The canonical 1-form is to be a map $T_g Gl(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$,

*We use another convention compared with [2], where an overal factor 1/2 is present. The result, i.e. the Maurer-Cartan equations are interestingly the same, due to another—or maybe consistent?—difference in conventions: the definition of the wedge product in the exterior algebra. To be complete, we adhere to $\theta^d \wedge \theta^e(E_b, E_c) = \theta^d(E_b)\theta^e(E_c) - \theta^d(E_d)\theta^e(E_c)$.

sending a left invariant vector field into the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$. Remember that a left invariant vector field on $Gl(n,\mathbb{R})$ is given by $X_g = (gV)^i{}_j \partial_{x^i{}_j}|_g$ and that the matrix of components V^i_j is the corresponding element of $\mathfrak{gl}(n,\mathbb{R})$. We show that

$$\omega^i{}_j \equiv x^i{}_k(g^{-1})dx^k{}_j(g) \tag{3.12}$$

is the (matrix valued) 1-form sought after and that it satisfies (3.11).

$$\begin{aligned}
\omega^i{}_j(X_g) &= x^i{}_k(g^{-1})dx^k{}_j(g)\left[(gV)^a{}_b\partial_{x^a{}_b}|_g\right] \\
&= x^i{}_k(g^{-1})x^a{}_c(g)V^c_b\delta^k_a\delta^b_j \\
&= x^i{}_a(g^{-1})x^a{}_c(g)V^c_j \\
&= \delta^i_c V^c_j = V^i_j
\end{aligned}$$

Furthermore does it satisfy (3.11). Since

$$\begin{aligned}
d\omega^i{}_j &= dx^i{}_k(g^{-1})dx^k{}_j(g) \\
&= -x^i{}_m(g^{-1})dx^m_l(g)x^l{}_k(g^{-1})dx^k{}_j(g)
\end{aligned}$$

and

$$\begin{aligned}
(\omega\wedge\omega)^i{}_j &= \tfrac{1}{2}2x^i{}_{[m}(g^{-1})x^l{}_{k]}(g^{-1})dx^m_l(g)\wedge dx^k{}_j(g) \\
&= x^i{}_m(g^{-1})x^l{}_k(g^{-1})dx^m_l(g)dx^k{}_j(g)
\end{aligned}$$

It then follows that

$$d\omega^i{}_j + (\omega\wedge\omega)^i{}_j = 0 \tag{3.13}$$

The matrix valued 1-form $\omega^i{}_j$ is often denoted by the shorthand "$g^{-1}dg$", which as a canonical 1-form does only make sense for matrix groups. The definition of ω for matrix groups maps a left invariant vector field into the *components* of the corresponding vector in $T_eGl(n,\mathbb{R})$. For a generic Lie group ω maps the left invariant vector field into the vector in the tangent space at the identity.

3.3. **1-parameter subgroups.**

Let ϕ_t be a 1-parameter group of transformations of G, generated by a left invariant vector field $X \in \mathfrak{g}$.

$$\frac{d\phi^i_t(g)}{dt} = X^i(\phi_t(g)) \tag{3.14}$$

Since X is left invariant we have $[L_a, \phi_t] = 0$ for any $a \in G$.

Set $a_t \equiv \phi_t(e) : \mathbb{R} \to G$. We have that $a_0 = e$ and

$$a_{t+s} = \phi_{t+s}(e) = \phi_{s+t}(e) = \phi_s \circ \phi_t(e) = \phi_s \circ L_{a_t}(e) = L_{a_t} \circ \phi_s(e) = L_{a_t}(a_s) = a_t a_s \ ,$$

$$a_t^{-1} = \phi_t^{-1}(e) = \phi_{-t}(e) = a_{-t}$$

The curve a_t is a *1-parameter subgroup of G generated by X*. The action of ϕ_t on G is given by the right action of a_t since

$$\phi_t(g) = \phi_t \circ L_g(e) = L_g \circ \phi_t(e) = L_g \circ a_t = g a_t$$

such that $\phi_t(g) = R_{a_t}(g)$. From this it follows that the Lie derivative $L_X Y = [X, Y]$ on G is given by

$$L_X Y = \lim_{t\to 0}[Y - (\phi_t)_* Y] = \lim_{t\to 0}[Y - R_{a_t *} Y] \tag{3.15}$$

To conclude this section the exponential map is defined. Set $\exp X \equiv a_1$. From this it follows that $a_t = \exp tX$. Indeed, consider first the 1-parameter subgroup a_{st} at the identity,

$$\left.\frac{da^i_{st}}{dt}\right|_{t=0} = s\ \left.\frac{da^i_u}{du}\right|_{u=0} = sX^i_e \tag{3.16}$$

which implies that a_{st} is generated by sX. But we also have that there exists a one-parameter group which is generated by sX,

$$\frac{db^i_t}{dt} = sX^i_{b_t} \tag{3.17}$$

By uniqueness of solutions this means that $b_t = a_{st}$. Hence, $\exp sX \equiv b_1 = a_s$. The desired result is found by replacing s by t.

3.4. **The adjoint action.** Every automorphism ϕ of a Lie group G induces an automorphism ϕ_* on its Lie algebra $\mathfrak{g}$. This can be seen as follows. Let X be an element of $\mathfrak{g}$.

$$L_{a*}(\phi_* X)(f) = L_{a*}X(f \circ \phi) = X(f \circ \phi) = (\phi_* X)(f)$$

Consider the mapping adj which relates to each element a of a Lie group an automorphism,

$$\mathrm{adj} : G \to \mathrm{Aut}(G) : a \mapsto \mathrm{adj}_a \ . \tag{3.18}$$

The automorphism adj_a introduced is called the *inner automorphism* of G and for a given element $a \in G$ it is defined as

$$\mathrm{adj}_a : G \to G : g \mapsto aga^{-1} \tag{3.19}$$

It is obvious that $\mathrm{adj}_a = L_a \circ R_{a^{-1}} = R_{a^{-1}} \circ L_a$. Furthermore, $\mathrm{adj}_a \circ \mathrm{adj}_b = \mathrm{adj}_{ab}$, hence adj is a group homomorphism.

The automorphism adj_a induces an automorphism adj_{a*} on the Lie algebra $\mathfrak{g}$. Define the mapping Ad_a then as the differential map adj_{a*} at the identity e of G, i.e.

$$\mathrm{Ad}_a : \mathfrak{g} \to \mathfrak{g} : X \mapsto \mathrm{Ad}_a X \tag{3.20}$$

What does $\mathrm{Ad}_a X$ looks like? For an element X of the Lie algebra we have that

$$\mathrm{Ad}_a X = (R_{a^{-1}} \circ L_a)_* X = R_{a^{-1}*} X$$

since X is left invariant. Let us give a concrete example for matrix groups.

Example 3.3 Let X be a left invariant vector field on $Gl(n, \mathbb{R})$, generated by the $V^i{}_j$. The adjoint action is given

$$\mathrm{Ad}_a X_g = (agVa^{-1})^i{}_j \partial_{x^i{}_j}|_g \quad (= R_{a^{-1}*} X_{ag}) \tag{3.21}$$

One way to find the corresponding generator in $\mathfrak{gl}(n, \mathbb{R})$ is to apply the Maurer-Cartan form to $\mathrm{Ad}_a X$, i.e.

$$\begin{aligned}
\omega(\mathrm{Ad}_a X)^i{}_j &= \omega((agVa^{-1})^i{}_j \partial_{x^i{}_j}|_g)^i{}_j \\
&= x^i{}_k([aga^{-1}]^{-1}) dx^k{}_j(aga^{-1}) \left[x^a{}_b(ag) V^b_c x^c{}_d(a^{-1}) \partial_{x^a{}_d}|_{aga^{-1}}\right] \\
&= x^i{}_l(a) x^l{}_k([ag]^{-1}) \delta^k_a x^a{}_b(ag) V^b_c x^c{}_d(a^{-1}) \delta^d_j \\
&= x^i{}_l(a) \delta^l_b V^b_c x^c{}_d(a^{-1}) \\
&= x^i{}_k(a) V^k_l x^l{}_j(a^{-1})
\end{aligned}$$

This implies that the adjoint representation of $Gl(n, \mathbb{R})$ on $\mathfrak{gl}(n, \mathbb{R})$ is given by

$$\mathrm{Ad}_a V = aVa^{-1} \tag{3.22}$$

Note that although the adjoint representation on left invariant vector fields is given by $R_{a^{-1}*}$ only, the corresponding action on the components at the identity is given by (3.22).

Note that Ad is a mapping of G into the automorphisms of its Lie algebra,

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) : a \mapsto \mathrm{Ad}_a \tag{3.23}$$

Since adj is a group homomorphism, Ad is a representation on $\mathfrak{g}$, called the *adjoint representation* of G.

Last but not least let us consider the differential map Ad_* at the identity and define this as ad,

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}) : X \mapsto \mathrm{ad}_X \tag{3.24}$$

We can now calculate how this operator works on an element Y of $\mathfrak{g}$. Let $a_t = \exp tX$ be the curve in G generated by X. By definition we have

$$\begin{aligned}
\mathrm{ad}_X Y &= (\mathrm{Ad}_{*e})_X Y \\
&= \lim_{t\to 0} \frac{1}{t}(\mathrm{Ad}_{a_t} Y - \mathrm{Ad}_e Y) \\
&= \lim_{t\to 0} \frac{1}{t}(R_{a_t^{-1}*} Y - Y) \\
&= L_X Y = [X, Y]
\end{aligned}$$

3.5. **Action of Lie groups on manifolds.**

Definition 3.2 (Lie transformation group).

Let G act on M on the right and consider a left invariant vector field $X \in \mathfrak{g}$. The *fundamental vector field* $X^\star$ on M corresponding to X is the vector field induced by the right action of the 1-parameter subgroup $a_t = \exp(tX)$ on M. In other words, $X^\star$ is the generator of R_{a_t} on M while X is the generator of a_t on G. Hence, we have defined a mapping

$$\rho : \mathfrak{g} \to \mathfrak{X}(M) : X \mapsto X^\star \tag{3.25}$$

Let us define the mapping ρ more carefully as follows. Consider first a mapping ρ_p for any $p \in M$ such that

$$\rho_p : G \to M : g \mapsto pg \tag{3.26}$$

The differential map $\rho_{p*} : T_g G \to T_{pg} M$ is at the identity given by the following. Remember that $X \in \mathfrak{g}$ is the generator of a_t.

$$\begin{aligned}
(\rho_{p*} X_e)_p f &= X_e(f \circ \rho_p) \\
&= \frac{d}{dt}(f \circ \rho_p(a_t))|_{t=0} \\
&= \frac{d}{dt}(f(R_{a_t}(p))|_{t=0} \\
&= A^\star_p f
\end{aligned}$$

An element of the Lie algebra of G is mapped onto the corresponding fundamental vector field at the point p on M. The desired result is then found in defining ρ implicitly by $(\rho X)_p \equiv \rho_{p*} X = X^\star_p$ for all $p \in M$.

The mapping ρ is a Lie algebra homomorphism, i.e. $\rho([X,Y]) = [\rho(X), \rho(Y)]$. Let us proof this proposition.

$$\begin{aligned}
[A^\star_p, B^\star_p] &= \lim_{t \to 0} \frac{1}{t} \left[B^\star_p - (R_{a_t *} B^\star)_p\right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[(\rho B)_p - R_{a_t *} (\rho B)_{p a_t^{-1}}\right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[\rho_{p*} B - R_{a_t *} \circ \rho_{p a_t^{-1} *} B\right]
\end{aligned}$$

To continue, note that $R_{a_t} \circ \rho_{p a_t^{-1}}(c) = R_{a_t}(p a_t^{-1} c) = p \ \mathrm{adj}_{a_t^{-1}}(c) = \rho_p \circ \mathrm{adj}_{a_t^{-1}}(c)$. Hence, we find

$$\begin{aligned}
[A^\star_p, B^\star_p] &= \lim_{t \to 0} \frac{1}{t} \left[\rho_{p*} B - (\rho_p \circ \mathrm{adj}_{a_t^{-1}})_* B\right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[\rho_{p*} B - \rho_{p*} R_{a_t *} B\right] \\
&= \rho_{p*} \lim_{t \to 0} \frac{1}{t} [B - R_{a_t *} B] \\
&= \rho_{p*} [A, B] = [A, B]^\star_p
\end{aligned}$$

Hence, ρ defines a Lie algebra homomorphism of $\mathfrak{g}$ into $\mathfrak{X}(M)$.

4. (Principal) Fibre bundles

4.1. Fibre bundles.

Definition 4.1 (Fibre bundle). *A differentiable fibre bundle* (E, π, M, F, G) *consists of the following elements;*

(a) *a manifold* E*, the* bundle space*; a manifold* M*, the* base space*; and a manifold* F*, the (typical)* fibre*.*

(b) *a surjection* $\pi : E \to M$ *which maps any point* u *of the bundle space into a point* $p = \pi(u)$ *on the base space. Note that information is lost under this* projection*. Its inverse image* $\pi^{-1}(p) = F_p$ *is diffeomorphic to* F *(see local trivialization) and is called the* fibre at p*.*

(c) *a Lie group* G *acting on* F *on the left. It is called the* structure group *of the bundle.*

(d) *a* local trivialization (U_i, ϕ_i) *where* $\{U_i\}$ *is an open covering of* M *and* ϕ_i *a diffeomorphism*

$$\phi_i : U_i \times F \to \pi^{-1}(U_i) \tag{4.1}$$

such that $\pi \circ \phi_i = \mathrm{id}_{U_i}$*, i.e.* $\pi\phi_i(p, f) = p$*.*

(e) *a set* $\{t_{ij}\}$ *of* transition functions*, defined as follows. Let* $\phi_{i,p}(f) \equiv \phi_i(p, f)$ *be the mapping* $\phi_{i,p} : F \to F_p$ *which sends an element* $f \in F$ *into an element of the fibre* $F_p \subset E$ *at* $p \in M$*. On an overlapping region* $U_i \cap U_j \neq \emptyset$*, require that*

$$t_{ij}(p) \equiv \phi_{i,p}^{-1}\phi_{j,p} : F \to F : f \mapsto t_{ij}(p)f \tag{4.2}$$

be an element of the structure group G*. This implies that the local trivializations* ϕ_i *and* ϕ_j *are related by the map* $t_{ij} : U_i \cap U_j \to G$ *in the sense that*

$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f) \ . \tag{4.3}$$

Note that ϕ_i^{-1} maps $\pi^{-1}(U_i)$ into a direct product structure, hence a fibre bundle is *locally* diffeomorphic to this direct product. For a given p this defines a diffeomorphism between F and F_p, which rectifies why both are being referred to as fibres. Although a fibre bundle is locally a direct product, this is not to be expected globally. Indeed, the concept of transition functions allows one to construct a rich variety of manifolds. Since overlapping charts of M will refer to a same element of the fibre bundle, using different elements of F in their direct product descriptions, reconstructing a global picture forces one to *glue*, i.e. identify the respective elements in F together by using the transition functions corresponding to the same element in the bundle. This spoils the direct product structure on a *global* level, which is exactly where the interest of fibre bundles has to be found. Of course, this glueing procedure will only make sense if the transition functions satisfy the consistency conditions

$$t_{ii}(p) = e \qquad p \in U_i \tag{4.4a}$$

$$t_{ij}(p) = t_{ji}^{-1}(p) \qquad p \in U_i \cap U_j \tag{4.4b}$$

$$t_{ij}(p) \circ t_{jk}(p) = t_{ik}(p) \qquad p \in U_i \cap U_j \cap U_k \tag{4.4c}$$

The trivial limit, where $t_{ij} = e$ for all overlapping regions, allows one to let the local direct product structure to be meaningful globally. This is nothing else then saying that the fibre bundle E is diffeomorphic with the direct product $M \times F$.

Let E be a fibre bundle. A *cross section* of E is mapping σ of the base space into bundle such that $\pi \circ \sigma = \mathrm{id}_M$, i.e.

$$\sigma : M \to E : p \mapsto \sigma(p) \in \pi^{-1}(p) \ . \tag{4.5}$$

Note that it is because $\pi \circ \sigma$ is the identity on M that a point gets mapped into a point on the bundle which is an element of its fibre F_p. A section may not be defined globally on the fibre bundle. A *local section* is a section defined on a subset $U_i \subset M$ of the base manifold.

4.2. **Principal bundles.** A principal bundle (P, π, M, G) is a fibre bundle where the fibre is identical with the structure group, i.e. $F \equiv G$. One defines the *right action* of G on the bundle P as follows. Consider a point $u \in P$ such that $\pi(u) = p$. A local trivialization then gives

$\phi_i^{-1}(u) = (p, g_i)$ where $g_i \in G$. The right action of G on $\pi^{-1}(U_i)$ is then defined by $\phi_i^{-1}(ua) \equiv (p, g_i a)$, or equivalently

$$ua \equiv \phi_i(p, g_i a) \tag{4.6}$$

for all $a \in G, u \in F_p$. It is easily shown that the definition is in fact independent of the local trivialization chosen. For consider a point $p \in U_i \cap U_j$ and let u be in the fibre at p:

$$ua = \phi_j(p, g_j a) = \phi_j(p, t_{ji}(p) g_i a) = \phi_i(p, g_i a) \ .$$

The right action is thus defined as a mapping $P \times G \to P : (u, a) \mapsto ua$, without the need of making reference to a specific local trivialization.

From its definition it is clear that the right action of G on any fibre $F_p \equiv \pi^{-1}(p)$ is transitive, since G acts transitively on itself on the right. Furthermore, the action is free. Let $ua = u$ for some $u \in P$ and $a \in G$. We then have that $ua = \phi_i(p, g_i a) = \phi_i(p, g_i)a = u = \phi_i(p, g_i)$, such that $\phi_i(p, g_i a) = \phi_i(p, g_i)$. Since ϕ_i is a bijection, $g_i a = g_i$ and it follows that $a = e$.

Given a local section $\sigma_i(p)$ over U_i, a *canonical local trivialization* $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ is defined as follows. Let u be a point in the fibre at $p \in U_i$. Because G acts transitively on $\pi^{-1}(p)$, there is a unique $g_u \in G$ such that $u = \sigma_i(p) g_u$. Define ϕ_i such that $\phi_i^{-1}(u) = (p, g_u)$. This is the canonical local trivialization. Note that $\phi_i^{-1}(\sigma_i(p)) = (p, e)$ and that $\phi_i(p, g) = \phi_i(p, e)g = \sigma_i(p)g$. If $p \in U_i \cap U_j$, two sections $\sigma_i(p)$ and $\sigma_j(p)$ are related by $\sigma_j(p) = \sigma_i(p) t_{ij}(p)$, since

$$\sigma_j(p) = \phi_j(p, e) = \phi_i(p, t_{ij}(p)e) = \phi_i(p, e) t_{ij}(p) = \sigma_i(p) t_{ij}(p)$$

Example 4.1 A widely used instance of principal bundles is the *bundle of linear frames*, which will be discussed in this example.

Let M be a manifold of dimension n. A linear frame u at a point $p \in M$ is an ordered basis $\{X_a\}$ $(a = 1 \dots n)$ of the tangent space at p, i.e. $T_p M$. Let $L_p M$ be the set of all frames at p. If x^i is a local coordinate system on $U_i \subset M$, the frame $u = \{X_a\}$ can be expressed as

$$X_a = X^i{}_a \left.\frac{\partial}{\partial x^i}\right|_p \ , \tag{4.7}$$

where $X^i{}_a$ is a non-singular matrix, since the elements of a frame are linearly independent. This implies that $X^i{}_a \in Gl(n, \mathbb{R})$ and $L_pM \simeq Gl(n, \mathbb{R})$.

Let LM be the set of all frames at all points $p \in M$ and define $\pi : LM \to M$ as the projection which maps a frame u at p into p. It is shown that this defines a principal bundle as follows. Define a local trivialization $\phi_i : U_i \times Gl(n, \mathbb{R}) \to \pi^{-1}(U_i)$ by $\phi_i^{-1}(u) = (p, X_a^i)$, i.e.

$$u = \phi_i(p, X^i{}_a) \ . \tag{4.8}$$

This implies that LM is a differentiable manifold (of dimension $m + m^2$), since M and $Gl(n, \mathbb{R})$ are smooth manifolds and ϕ_i is a smooth diffeomorphism. The right action of the fibre $Gl(n, \mathbb{R})$ on LM is given as follows. For $a \in Gl(n, \mathbb{R})$ and $u \in LM, \pi(u) = p$,

$$ua = \phi_i(p, X^i{}_b)a = \phi_i(p, X^i{}_b x^b{}_a(a)) = \phi_i(p, Y^i{}_a) \tag{4.9}$$

where the frame $\{X_i\}$ at p is transformed into a new frame $\{Y_a = X_b x^b{}_a(a)\}$ at p. Note that this action is free and transitive on the fibre. One concludes that $LM(M, Gl(n, \mathbb{R}))$ is a principal fibre bundle: the bundle of linear frames.

4.3. **Associated bundles.** Let $P(M, G)$ be a principal bundle and F a manifold on which G acts on the left. The manifolds G and F do not have to be of the same dimension. Consider the product manifold $P \times F$. The right action of G on $P \times F$ is defined as

$$R_a : P \times F \to P \times F : (u, f) \mapsto (ua, a^{-1}f) \ , \quad a \in G \tag{4.10}$$

We use this operation to define the space E of equivalence classes as

$$E \equiv P \times_G F \equiv \frac{P \times F}{G} \ . \tag{4.11}$$

Hence, the equivalence relation is given by the right action, i.e.

$$(u, f) \sim (ua, a^{-1}f) \ , \quad a \in G \ ,$$

and elements of E are denoted by $[u, f]_\sim$.

A projection π_E of E onto M is defined as

$$\pi_E : E \to M : (u, f)_\sim \mapsto \pi(u) \tag{4.12}$$

where $\pi(u)$ is the projection in P. Note that this definition is consistent with the equivalence relation since $\pi(ua) = \pi(u)$. For each $p \in M$, the set of points $\pi_E^{-1}(p) = \{(u, f)_\sim | \pi(u) = p\}$ is called the fibre of E over p.

Let us now define a differentiable structure in E such that it is turned into a smooth manifold. From the definition of a principal bundle it follows that every $p \in M$ has an open neighborhood U_i such that it is locally a direct product, i.e. $\pi^{-1}(U_i) \simeq U_i \times G$. Locally then, the right action of G on $P \times F$ goes as $(p, g, f) \mapsto (p, ga, a^{-1}f)$. Since this right action leaves the equivalent classes E invariant and because G is transitive under its own right action, the equivalence classes $[u, f]_\sim$ are locally described by a couple (p, f) $(\pi(u) = p)$. It follows that $\pi_E^{-1}(U_i) \simeq U_i \times F$. A differentiable structure in E is obtained by demanding that $\pi_E^{-1}(U_i)$ be an open submanifold of E.

Finally, we show that the above given local trivialization for E has the same transition functions as the the local trivialization for P. The local trivialization for E can be written down as (on $U_i \cap U_j \subset M$)

$$\psi_i^{-1}([u, f]_\sim) = (p, g, f)_\sim = (p, e, gf)_\sim$$
$$\psi_j^{-1}([u, f]_\sim) = (p, t_{ji}(p)g, f)_\sim = (p, e, t_{ji}(p)gf)_\sim$$

This implies that on any overlapping region

$$\psi_j(p, f) = \psi_i(p, t_{ij}(p)f) \tag{4.13}$$

As such, we have constructed a fibre bundle $(E, \pi_E, M, F, G; P)$ associated with the principal bundle P. In fact, it is just a fibre bundle (E, π, M, F, G) as in definition 4.1.

Given a principal bundle $P(M, G)$ and an associated bundle E with fibre F, one can define the following mapping. For each $u \in P$ and $f \in F$, denote by uf the image of $(u, f) \in P \times F$ under the natural projection $P \times F \to E$, i.e. $uf = [u, f]_\sim$. One has for each $u \in P$ a mapping $u : F \to F_p = \pi_E^{-1}(p)$, where $p = \pi(u)$. Since for any $a \in G$, $[ua, f]_\sim = [u, af]_\sim$, it follows that $(ua)f = u(af)$.

Let $F_p = \pi_E^{-1}(p)$ and $F_q = \pi_E^{-1}(q)$, $p, q \in M$. An isomorphism of a fibre F_p onto another fibre F_q is a diffeomorphism that can be written as $v \circ u^{-1}$, where $u \in \pi^{-1}(p)$ and $v \in \pi^{-1}(q)$ are considered mappings of F onto F_p and F_q respectively. An automorphism of F_p is a mapping of the form $v \circ u^{-1}$, where u and v are in the same fibre $\pi^{-1}(p)$. Since the fibre G_p is transitive under its own right action, $v = ua$ for some $a \in G$. It follows that any automorphism of F_p is of the form $u \circ a \circ u^{-1}$, for an arbitrarily *fixed* $u \in G_p$. Hence, the group of automorphisms of F_p is isomorphic with the structure group G.

Example 4.2 *Tangent bundle*

4.4. **Reduction of principal bundles.** A *homomorphism* $f : Q(M', H) \to P(M, G)$ between principal fibre bundles consists of a mapping $f' : Q \to P$ and a group homomorphism $f'' : H \to G$, such that $f'(qh) = f'(q)f''(h)$ for any $q \in Q$ and $h \in H$. In the remainder we denote these mappings all by the same letter f and assume that its meaning is clear from the element it is acting on. Every homomorphism f preserves the fibre structure of the bundles. For let $H_{p'} = \{qh \mid h \in H\}$ and $q \in H_{p'}$ arbitrary. Denote $u = f(q)$ where $u \in G_p$. It follows that $f(qh) = f(q)f(h)$ with $f(h) \in G$. Then $\pi(uf(h)) = \pi(u)$ so that $f(H_{p'}) \subset G_p$. Since f maps each fibre of Q into a fibre of P, this induces a mapping $f : M' \to M$.

The homomorphism f is called an *injection* if $f : Q \to P$ is an injection and $f : H \to G$ a group monomorphism. It follows that $f : M' \to M$ also is an injection. Given an injection, we can identify $Q \equiv f(Q)$, $H \equiv f(H)$ and $M' \equiv f(M')$. Then $Q(M', H)$ is said to be a *subbundle* of $P(M, G)$. If moreover $M' = M$ and $f : M' \to M$ is the identity transformation of M, $f : Q(M, H) \to P(M, G)$ is said to be a *reduction* of the structure group G of $P(M, G)$ to H. The subbundle $Q(M, H)$ is a *reduced bundle* of $P(M, G)$.

Let H be a closed subgroup of G and consider the coset space G/H. There is the natural left action $\tau_a : G/H \to G/H : gH \mapsto (ag)H$. Let then $E = P \times_G G/H$ be the associated bundle with standard fibre G/H. Furthermore, denote by P/H the space of equivalence classes with equivalence relation $u \sim uh$ for $h \in H$. We identify E with P/H by mapping each element $[u, gH] = [ug, eH] \in E$ into $[ug] \in P$. Lastly, we define the projection $\mu : P \to E : u \mapsto u(eH) = [u, eH]$, where we considered the mapping $u : G/H \to \pi_E^{-1}(p)$ with $\pi(u) = p$. Note that the projection preserves fibres and that it is a submersion by the transitivity of G on G/H. To conclude we summarize things in the following diagram:

$$\begin{tikzcd}
Q(M,H) \arrow[r, "f"] \arrow[dr, "\pi'"] & P(M,G) \arrow[r, "\mu"] \arrow[d, "\pi"] & E = P/H \arrow[dl, "\pi_E"] \\
& M &
\end{tikzcd}$$

Proposition 4.1. *The structure group G of $P(M, G)$ is reducible to a closed subgroup H if and only if the associated bundle $E = P \times_G G/H$ admits a globally defined section $\sigma : M \to E$.*

Proof. Let $f : Q(M, H) \to P(M, G)$ be a reduction of G to a closed subgroup H and let $\mu : P \to E$ be the projection $u \mapsto [u, eH]$. If q and q' are in the same fibre $\pi'^{-1}(p)$, then there exists some $h \in H$ so that $q = q'h$. One then has $\mu(f(q)) = \mu(f(q)f(h)) = \mu(f(q))$ so that $\mu \circ f$ is a constant function on each fibre of Q. This induces a mapping $\sigma : M \to E$ by $\sigma(p) = \mu(f(q))$ with $q \in \pi'^{-1}(p)$. As $\pi(f(q)) = \pi_E(\mu(f(q))) = p$, we have $\pi_E \circ \sigma = \mathrm{id}$ and σ is a section.

Conversely, let σ be a section of E. Denote by Q the subset of P such that $\mu(u) = \sigma(\pi(u))$. In other words, $Q = \mu^{-1}(\sigma(M))$. $\square$

5. Connections

5.1. **Ehresmann connections.** Let $P(M, G)$ be a principal bundle, $u \in P$ and $p = \pi(u) \in M$, such that the fibre at p is $G_p \equiv \pi^{-1}(p)$.

The *vertical subspace* V_uP at a point u is the space tangent to the fibre through u. It is a subspace of the tangent space T_uP at u. Note that given a $u \in P$ and $\pi(u) = p$, the fibre G_p through u is sweeped out by the right action of G on u (because G_p is transitive under this action). For any 1-parameter subgroup a_t of G, a curve $R_{a_t}(u)$ is traced out in the fibre G_p. The generating vector field for the latter curve is the fundamental vector field $A^\star$ corresponding to the generator $A \in \mathfrak{g}$ of a_t. It follows that V_uP is spanned by $n = \dim G$ linearly independent $A^\star_u$, at any $u \in P$. The vertical subspace is thus canonically given for a principal bundle. By definition $A^\star_{ug} = R_{g*}A^\star_u$ and one concludes that the vertical subspace at ug is given by right translating the vertical subspace at u, i.e. $V_{ug}P = R_{g*}V_uP$.* Furthermore, since a curve $R_{a_t}(u)$ in the fibre G_p will be projected into the same point $p \in M$, an element of V_uP will be anihilated under this projection: $\pi_* X^V_u(f) = X^V_u(f \circ \pi) = 0$ because $f \circ \pi$ is a constant function.

As mentioned in the last paragraph, a principal fibre bundle canonically picks out the vertical subspace of dimension $n = \dim G$ from the tangent space, at any point in the bundle. This implies directly that at any point a subspace of dimension $m = \dim P - \dim G$ is left over: the *horizontal subspace* H_uP. Unfortunately, the mere observation that at any point the horizontal subspace should be the linear complement of the vertical subspace does not give one a unique manner to smoothly assign one at each $u \in P$. Introducing extra structure into the picture by defining an *Ehresmann connection*† does allow one to have such an assignment.

*This does *not* imply that a generic vertical vector field X is right invariant, since such a vector field may not be a linear combination of the fundamental vector fields $A^\star$.

†An Ehresmann connection will be called just a connection

Definition 5.1 (Connection). *Let $P(M,G)$ be a principal bundle. A connection is a unique seperation of the tangent space T_uP at each $u \in P$ into the vertical subspace V_uP and the horizontal subspace H_uP, such that*

(a) $T_uP = V_uP \oplus H_uP$
(b) *A smooth vector field X on P is seperated into smooth vector fields X^V and X^H such that $X = X^V + X^H$ and for any $u \in P$, $X^V_u \in V_uP$ and $X^H_u \in H_uP$*
(c) $H_{ug}P = R_{g*}H_uP$ *for any* $u \in P, g \in G$

Consider the pushforward of the projection, i.e. $\pi_* : T_uP \to T_pM$. From item (a) it follows that $\pi_*T_uP = \pi_*V_uP \oplus \pi_*H_uP$. Since V_uP gets annihilated under this projection one concludes that $\dim \pi_*H_uP = \dim T_pM$. This means that once a connection is given, H_uP is projected isomorphically onto T_pM. Let us mention that $\dim H_uP = \dim T_pM$, since $\dim P = \dim G + \dim M$.

As we have seen, a connection basically splits up smoothly the tangent spaces along a principal bundle in vertical and horizontal subspaces. This splitting up effectively defines a projection of a tangent vector onto the vertical subspace. The latter is spanned by the fundamental vector fields corresponding to $\mathfrak{g}$, hence isomorphic to T_eG. A connection thus defines an operator which takes an element of T_uP as its input, mapping it into $T_eG \simeq \mathfrak{g}$. This operator is a $\mathfrak{g}$-valued 1-form on P and it is called the *connection 1-form.* Its definition is an equivalent starting point for discussing connections on principal bundles and will be used for applications.

Definition 5.2 (Connection 1-form). *A connection 1-form ω on a principal bundle P is $\mathfrak{g}$-valued 1-form, i.e. a projection of T_uP onto the vertical component $V_uP \simeq \mathfrak{g}$, for any $u \in P$; the projection property is summarized by the following requirements:*

(a) $\omega(A^\star) = A$, *where* $A \in \mathfrak{g}$

(b) $R_g^*\omega = \mathrm{Ad}_{g^{-1}}\omega$, *where* $\mathrm{Ad}_{g^{-1}}$ *acts on the* $\mathfrak{g}$ *in* $\omega \in \mathfrak{g}\otimes T^*P$.
The horizontal subspace H_uP *is the kernel of* ω*:*

$$H_uP \equiv \{X \in T_uP | \omega(X) = 0\} \tag{5.1}$$

At any point $u \in P$ and for every $X \in T_uP$, $\omega(X)$ is the unique $A \in \mathfrak{g}$ such that $A_u^\star$ is equal to the vertical component of X.

Consider requirement (b) again. Taking a closer look at the equation,

$$(R_g^*\omega)_u(X) = \omega_{ug}(R_{g*}X_u) \overset{def}{=} \mathrm{Ad}_{g^{-1}}\omega_u(X) = R_{g*}\omega_u(X)$$

shows that it basically tells the following. The projection of $R_{g*}X$ in $ug \in P$ gives the same result as right translating the projection of X in $u \in P$. This is the dual statement of item (c) in definition 5.1 which states that the horizontal space at ug is the right translation of the horizontal space at u. Indeed, the space defined in (5.1) satisfies $H_{ug}P = R_{g*}H_uP$ as we prove now. Let $X_u \in H_uP$, i.e. $\omega_u(X) = 0$. It then follows that $\omega_{ug}(R_{g*}X) = R_g^*\omega_{ug}(X) = R_{g*}\omega_u(X) = 0$, hence $R_{g*}X \in H_{ug}P$. In the other direction, since right translation in P is an invertible linear map, any element in $H_{ug}P$ can be written as $R_{g*}X$, for some $X \in H_uP$.

In the following, the above introduced connection will be expressed by a family of forms defined on open subsets of the base manifold M. Therefore, let $\{U_i\}$ be an open covering of M and consider a family of local sections $\sigma_i : U_i \to \pi^{-1}(U_i) \subset P$. The associated canonical local trivializations are isomorphisms ϕ_i such that $\phi_i(p, g) = \sigma_i(p)g$ and thus $\phi_i(p, e) = \sigma_i(p)$, for any $p \in M, g \in G$. Remember that $\sigma_j(p) = \sigma_i(p)t_{ij}(p)$, where $t_{ij} : U_i \cup U_j \to G$ are the transition functions. Let $\theta : T_gG \to T_eG$ be the Maurer-Cartan form on G.

Define on each non-empty intersection $U_i \cap U_j$ the $\mathfrak{g}$-valued 1-form

$$\theta_{ij} \equiv t_{ij}^*\theta : T_pM \to T_eG \simeq \mathfrak{g}\ . \tag{5.2}$$

To understand this definition, consider its action on an element X, i.e. $\theta_{ij}(X) = \theta(t_{ij*}X)$. Since $t_{ij*} : T_pM \to T_gG$ where $g = t_{ij}(p)$, X is a vector field on M. The Maurer-Cartan form then left translates the resulting vector into the tangent space at the identity, i.e. $\theta_{ij}(X) = L_{g^{-1}*}(t_{ij*}X_p)_g$, $(g = t_{ij}(p))$.

The *local connection forms* then are defined by pulling back the connection 1-form ω onto the open covering $\{U_i\}$,

$$\mathcal{A}_i \equiv \sigma_i^* \omega \tag{5.3}$$

Its action on an element X is given by $\mathcal{A}_i(X) = \omega(\sigma_{i*}X)$. Remember that $\sigma_{i*} : T_pU_i \to T_uP$ and $u = \sigma_i(p)$. It is clear that $\mathcal{A}_i : T_pU_i \to V_uP \simeq \mathfrak{g}$ is a $\mathfrak{g}$-valued 1-form on U_i, i.e. $\mathcal{A}_i \in \mathfrak{g} \otimes T^*U_i$. Note that ω maps $(\sigma_{i*}X)_u$ into an $A \in \mathfrak{g}$ such that $A_u^\star$ is the vertical component of $(\sigma_{i*}X)_u$.

On a region $U_i \cap U_j \neq \emptyset$, two local connection forms $\mathcal{A}_i$ and $\mathcal{A}_j$ are related by

$$\mathcal{A}_j = \mathrm{Ad}_{t_{ij}^{-1}} \mathcal{A}_i + \theta_{ij} \ , \tag{5.4}$$

a proof of which can be found in [2], pg. 66. We note without proof that although the connection forms $\mathcal{A}_i$ are defined only locally on regions of the base manifold, the whole set of them together with the transformation rule (5.4) do capture all the information related to the global definition of ω on the principal bundle, i.e. the separation of the tangent bundle TP in its vertical and horizontal components.

Example 5.1 Consider a principal bundle $P(M, Gl(n,\mathbb{R}))$. Since $\mathcal{A}_i \in \mathfrak{g}\otimes T^*U_i$ and for $Gl(n,\mathbb{R})$ the Lie algebra $\mathfrak{g} \simeq \mathfrak{gl}(n,\mathbb{R})$, the arbitrary $n \times n$-matrices, one concludes that the local connection forms will be of the form $\mathcal{A}^a_{i,b} = \mathcal{A}^a_{i,b\mu} dx^\mu$.* The transition functions $t_{ij}^{-1}(p)$ are elements of $Gl(n,\mathbb{R})$. Since $\mathrm{Ad}_{t_{ij}^{-1}}$ act on the Lie algebra one has that $\mathrm{Ad}_{t_{ij}^{-1}}(\mathcal{A}_i) \in \mathrm{Ad}_{t_{ij}^{-1}}\mathfrak{gl}(n,\mathbb{R}) \otimes T^*U_i$, hence (let $g(p) \equiv t_{ij}^{-1}(p)$),†

$$\begin{aligned} \mathrm{Ad}_g(\mathcal{A}^a_{i,b}) &= x^a{}_c(g) \mathcal{A}^c_{i,d\mu} x^d{}_b(g^{-1}) dx^\mu \\ &\approx g\mathcal{A}_i g^{-1} \end{aligned} \tag{5.5}$$

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*The "low" Latin indices a, b, c, ... are Lie-algebra indices. The Greek indices are indices refering to the base manifold coordinates.

†We will omit for simplicity the p-dependence. However, this fact should not be forgotten in the following.

Next, we compute $\theta_{ij} \equiv t_{ij}^{*}\theta \in \mathfrak{g}\otimes T^{\star}M$ explicitly. One has that $[\theta_{ij}(X_p)]^a{}_b = \theta(t_{ij*}X_p)^a{}_b$, where $X_p \in T_pM$. Since

$$t_{ij*}X^{\mu}\partial_{\mu}|_p f = X^{\mu}\partial_{\mu}|_p f(g^{-1}(p)) = X^{\mu}\frac{\partial x^c{}_d(g^{-1})}{\partial x^{\mu}(p)}\partial_{x^c{}_d}|_{g^{-1}} f$$

and $\theta^a{}_b = x^a{}_c(g)dx^c{}_b(g^{-1})$, one finds that

$$\begin{aligned}\theta_{ij}(X_p)^a{}_b &= x^a{}_c(g)dx^c{}_b(g^{-1})\left[X^{\mu}\partial_{\mu}x^e{}_d(g^{-1})\partial_{x^e{}_d}|_{g^{-1}}\right]\\ &= x^a{}_e(g)X^{\mu}\partial_{\mu}x^e{}_d(g^{-1})\delta^e_c\delta^d_b\\ &= x^a{}_c(g)\partial_{\mu}x^c{}_b(g^{-1})X^{\nu}dx^{\mu}(\partial_{\nu}|_p)\\ &= x^a{}_c(g)\partial_{\mu}x^c{}_b(g^{-1})dx^{\mu}(X_p)\end{aligned}$$

From this it is clear that

$$\begin{aligned}[\theta_{ij}]^a{}_b &= x^a{}_c(t_{ij}^{-1}(p))\partial_{\mu}x^c{}_b(t_{ij}(p))dx^{\mu}\\ &\approx t_{ij}^{-1}(p)dt_{ij}(p)\end{aligned} \tag{5.6}$$

To conclude, these results are used to rewrite eq. (5.4) for a general matrix group fibre bundle $P(M, Gl(n,\mathbb{R}))$,

$$\begin{aligned}\mathcal{A}^a_{i,b\mu}dx^{\mu} &= x^a{}_c(t_{ij}^{-1})\mathcal{A}^c_{i,d\mu}x^d{}_b(t_{ij})dx^{\mu} + x^a{}_c(t_{ij}^{-1})\partial_{\mu}x^c{}_b(t_{ij})dx^{\mu}\\ &\approx t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}dt_{ij}\end{aligned} \tag{5.7}$$

5.2. **Horizontal lift and parallelism.** Let $P(M,G)$ be a principal fibre bundle. *The horizontal lift* of a vector field X on M is a vector field X^h on P which is horizontal and projects onto X, i.e. $\omega_u(X^h) = 0$ and $\pi_*(X^h_u) = X_{\pi(u)}$ for all $u \in P$.

Given a connection in P, the horizontal lift exists and is unique, because the projection π gives a linear isomorphism between H_uP and T_uM. The horizontal lift X^h is right invariant, because (1) the horizontal subspace is right invariant and (2) the projection acts the same on the whole fibre ($\pi \circ R_a = \pi$). Conversely, given a right invariant horizontal vector field X^h on P, it is the lift of a vector field X on M.

Let γ_t $(0 \leq t \leq 1)$ be a curve in M. *A horizontal lift* of γ_t is a curve γ^h_t $(0 \leq t \leq 1)$ in P such that for any t it projects onto γ_t , i.e. $\pi(\gamma^h_t) = \gamma_t$, and its tangent vector field is horizontal.

The notions of the lift of a vector field and a lift of a curve are related to eachother. If X^h on P is the lift of X on M, then the integral curve u_t of X^h through a point u_0 is a lift of the integral curve γ_t of X through $\gamma_0 = \pi(\gamma_0^h)$. Indeed, let u_t be the integral curve of X^h through u_0. Then,

$$\pi_* X^h_{u_t} f = \pi_* \tfrac{d}{dt} f(u_t) = \tfrac{d}{dt} f(\pi(u_t)) \stackrel{!}{=} X_{\gamma_t} f = \tfrac{d}{dt} f(\gamma_t)$$

Since X^h is horizontal and because the last equality implies $\pi(u_t) = \gamma_t$, $\gamma_t^h \equiv u_t$ is a lift of γ_t. Note that the set of all horizontal lifts of a curve are the integral curves of the lift of the corresponding vector field.

Let γ_t $(0 \leq t \leq 1)$ be a curve in M and let $u_0 \in \pi^{-1}(\gamma_0)$. There exists a unique lift γ_t^h, such that $\gamma_0^h = u_0$. A proof can be found in [2], pg. 69. Because the lift is unique, there is a unique $u_1 = \gamma_1^h \in \pi^{-1}(\gamma_1)$. This point u_1 is called the *parallel displacement along the curve* γ^h of u_0. In this way, a mapping is defined

$$\Gamma(\gamma^h) : G_{\gamma_0} \to G_{\gamma_1} : u = \gamma_0^h \mapsto \gamma_1^h \tag{5.8}$$

Since the horizontal lift X^h is right invariant, a lifted curve is mapped into another one of the same vector field under the right action of G. Hence, one has that

$$\Gamma(\gamma^h) \circ R_g = R_g \circ \Gamma(\gamma^h) \ . \tag{5.9}$$

This then implies that the parallel displacement $\Gamma(\gamma^h)$ is an isomorphism between G_{γ_0} and G_{γ_1}. Indeed, if parallel displacement and right action would not commute, the inverse of $\Gamma(\gamma^h)$ would not be a well-defined mapping.

5.3. **Curvature and the Bianchi identity.** Let $P(M, G)$ be a principal bundle and ρ a representation of G on a vector space V.

Definition 5.3 (Pseudotensorial form). *A pseudotensorial form of degree r on P of type (ρ, V) is a V-valued r-form ϕ on P such that*

$$R_g^* \phi = \rho(g^{-1}) \phi \ , \quad g \in G \tag{5.10}$$

The form ϕ is called tensorial if it is horizontal, i.e.

$$\phi(X_1, \dots X_r) = 0 \tag{5.11}$$

whenever at least one of the X_i is vertical.

Example 5.2 Let φ be a tensorial form of degree r of type (ρ, V) on P and let E be the associated bundle $P \times_G V$. In this example we show that φ can be regarded as an assignment of a multilinear skew-symmetric mapping

$$\tilde{\varphi} : T_p^{(r,0)} M \to \pi_E^{-1}(p) \tag{5.12}$$

to each $p \in M$. Namely, define for $X_i \in T_p M$

$$\tilde{\varphi}(X_1, \dots, X_r) \equiv u(\varphi(X_1^*, \dots, X_r^*)) \tag{5.13}$$

where $u \in \pi^{-1}(p)$ and $X_i^* \in T_u P$ such that $\pi_*(X_i^*) = X_i$. Since φ takes its values in V and u maps elements of V into $\pi_E^{-1}(p)$, $\tilde{\varphi}(X_i)$ is indeed as proposed. We now show that this definition is independent of the choice $u \in \pi^{-1}(p)$ and $X_i^* \in T_u P$. Therefore, consider ua with $a \in G$. We then have $\tilde{\varphi}(X_i) = (ua)(\varphi_{ua}(Y_i^*))$. Since φ is a tensorial form we can forget about vertical components and assume $Y_i^* \in H_{ua} P$. Horizontal subspaces are invariant under the right G-action, so that there exists a $Z_i^* = R_{a^{-1}*} Y_i^*$ in $H_u P$. By construction $\pi_*(Z_i^*) = X_i$ and since π_* constitutes an isomorphism between $H_u P$ and $T_p M$, we have $Z_i^* = X_i^*$ or $Y_i^* = R_{a*} X_i^*$. This gives us

$$\tilde{\varphi}(X_i) = (ua)(\varphi(R_{a*} X_i^*) = (ua)\rho(a^{-1}) \cdot \varphi(X_i^*) = u(\phi(X_i^*))$$

where we used that $(ua)(V) = u(aV)$.

Conversely, given a skew-symmetric mapping (5.12) for each $p \in M$, a tensorial form φ of degree r of type (ρ, V) can be defined by

$$\varphi(X_i^*) \equiv u^{-1}(\tilde{\varphi}(\pi_*(X_i^*))) \tag{5.14}$$

It is easy to check that this is indeed the right tensorial form.

Denote by $h : T_u P \to H_u P : X_u \mapsto hX_u \equiv X_u^H$ the projection operator, which maps a vector field into its horizontal component. If ϕ is a pseudotensorial r-form on P of type (ρ, V), then one has the following.

(a) The form ϕh defined by

$$(\phi h)(X_i) \equiv \phi(hX_i) \tag{5.15}$$

is a tensorial r-form of type (ρ, V).

To prove this we first note that $R_g \circ h = h \circ R_g$. Therefore,

$$\begin{aligned} R_g^*(\phi h)(X_i) &= (\phi h)(R_{g*}X_i) \\ &= \phi(R_{g*}hX_i) \\ &= R_g^*\phi(hX_i) \\ &= \rho(g^{-1})\phi(hX_i) \\ &= \rho(g^{-1})(\phi h)(X_i) \end{aligned}$$

Furthermore, since $hX_i = 0$ whenever X_i vertical, one has the desired result.

(b) The form $d\phi$ is a pseudotensorial $(r+1)$-form of type (ρ, V).

This is shown to be true, by calculating the following.

$$\begin{aligned} R_g^*(d\phi)(X_i) &= d(R_g^*\phi)(X_i) \\ &= d(\rho(g^{-1})\phi)(X_i) \\ &= \rho(g^{-1})d\phi(X_i) \end{aligned}$$

where we used $f^* \circ d = d \circ f^*$.

(c) The *exterior covariant derivative* of ϕ is the form $D\phi$ defined by

$$D\phi(X_i) \equiv (d\phi h)(X_i) = d\phi(hX_i) \tag{5.16}$$

is a tensorial form of type (ρ, V). Hence, $R_g^* D\phi = \rho(g^{-1})D\phi$ and $D\phi(X_i) = 0$ whenever one of the X_i vertical. This result follows directly from (a) and (b).

Note that by construction, $\phi \in V \otimes \Omega^r(P)$ and $D\phi \in V \otimes \Omega^{r+1}(P)$ belong to the same representation of G: they transform covariantly.

From the definition of a connection 1-form $\omega \in \mathfrak{g} \otimes \Omega(P)$, it is clear that it is a pseudotensorial 1-form of type $(\mathrm{Ad}, \mathfrak{g})$, or simply of type Ad G. More precisely it is a consequence of $R_g^*\omega = \mathrm{Ad}_{g^{-1}}\omega$. It then follows that the exterior derivative $D\omega$ is a tensorial 2-form of type Ad G. It is called the *curvature* of ω and it is awarded the symbol $\Omega \equiv D\omega$.

$$\Omega \in \mathfrak{g} \otimes \Omega^2(P)$$
$$R_g^* \Omega = \mathrm{Ad}_{g^{-1}} \Omega$$

Let $X, Y \in TP$. The *Cartan structure equations* are given by*

$$d\omega(X,Y) + \tfrac{1}{2}[\omega,\omega](X,Y) = \Omega(X,Y) \tag{5.17}$$

which can be rewritten concisely as

$$d\omega + \omega \wedge \omega = \Omega \ . \tag{5.18}$$

Since the connection and curvature forms are $\mathfrak{g}$-valued differential forms they can be expanded in a frame E_a as $\omega = E_a \cdot \omega^a$ and $\Omega = E_a \cdot \Omega^a$. By noting that $[\omega,\omega] \equiv [E_a, E_b] \cdot \omega^a \wedge \omega^b$, the structure equations (5.17) are expanded in the same frame as

$$d\omega^a + \tfrac{1}{2} c_{bc}{}^a \omega^b \wedge \omega^b = \Omega^a \tag{5.19}$$

Taking the exterior derivative of the curvature form and using equation (5.19), one obtains

$$d\Omega = E_a \cdot \Omega^a = E_a \cdot \tfrac{1}{2} c_{bc}{}^a (d\omega^b \wedge \omega^c - \omega^b \wedge d\omega^c)$$

Since, $\omega(hX) = 0$ for any vector field X on P, we find the *Bianchi identity*

$$D\Omega(X,Y,Z) \equiv 0 \tag{5.20}$$

for any three vector fields X, Y, Z on P.

To conclude this section, the curvature form will be be pulled back onto the base manifold M. Remember that given an open covering $\{U_i\}$ of M, the local connection forms were given by $\mathcal{A}_i \equiv \sigma_i^* \omega$. Analogously, the local curvature forms are defined by

$$\mathcal{F}_i \equiv \sigma_i^* \Omega \ . \tag{5.21}$$

*For a proof see e.g. [2], pg. 77.

From Cartan's structure equations it follows that

$$\mathcal{F}_i = d\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i \ . \tag{5.22}$$

On overlapping regions $U_i \cap U_j \neq \emptyset$ two local curvature forms, pulled back by different section $\sigma_i(p)$ and $\sigma_j(p) = \sigma_i(p)t_{ij}(p)$, are related by

$$\mathcal{F}_j = \mathrm{Ad}_{t_{ij}^{-1}} \mathcal{F}_i \ . \tag{5.23}$$

This result follows basically from the proof for the transformation behaviour of the local connection forms, i.e. see [2], pg. 66 and by noting that the curvature form Ω is horizontal. The local version of the Bianchi identity is found by taking the exterior derivation of (5.22), i.e.

$$d\mathcal{F}_i = d\mathcal{A}_i \wedge \mathcal{A}_i - \mathcal{A}_i \wedge d\mathcal{A}_i$$

Since $d\mathcal{A}_i$ is a (Lie algebra valued) 2-form, the RHS is just the bracket $[d\mathcal{A}_i, \mathcal{A}_i]$ and it behaves as a commutator. Hence, $[d\mathcal{A}_i, \mathcal{A}_i] = -[\mathcal{A}_i, d\mathcal{A}_i] = [\mathcal{A}_i, \mathcal{A}_i \wedge \mathcal{A}_i] - [\mathcal{A}_i, \mathcal{F}_i]$. Since

$$[\mathcal{A}_i, \mathcal{A}_i \wedge \mathcal{A}_i] = \mathcal{A}_i \wedge \mathcal{A}_i \wedge \mathcal{A}_i - \mathcal{A}_i \wedge \mathcal{A}_i \wedge \mathcal{A}_i = 0$$

the well known form of the Bianchi identity is found, i.e.

$$d\mathcal{F}_i + [\mathcal{A}_i, \mathcal{F}_i] = 0 \ . \tag{5.24}$$

That this is in a way the pulled back version of the Bianchi identity introduced before for the curvature form on the principal bundle is not too difficult to understand. In deriving the Bianci identity for a principal bundle, the crucial ingredient was considering the exterior derivative of Cartan's structure equations. It is an easy exercise to check that pulling these back on the base manifold gives you the local Bianchi identities.

From now on, the index i will be omitted. Note that this index refers to the section σ_i which is used to pullback the connection and curvature form onto a region U_i of M. One changes index $i \to j$ with the given transformation rules (5.4) and (5.23). This is the transformation behaviour when a section is transformed into another section (using the transition functions) and it does not change the principal bundle at all.

This transformation is governed by the right action of the structure group on the fibre and we call them *gauge transformations*. The connection form $\mathcal{A}$ is called the gauge field and G the gauge group. So although we do "forget" about the index we do remember its existence: different but equivalent sections are related by gauge transformations and $\mathcal{A}$ and $\mathcal{F}$ transform as prescribed.

5.4. **Induced connection on associated bundles.** Let $P(M,G)$ be a principal bundle and Γ a connection on P. Denote by E the associated fibre bundle with fibre F, i.e. $E = P \times_G F$. Consider then the element $w = [u,\xi] \in E$ that lies in the fibre F_p, hence $\pi_E(w) = \pi(u) = p$. The vertical subspace $V_w E \subset T_w E$ is defined as the vector space that is tangent to the fibre F_p. Since at any point $w \in E$ the fibre F_p is a manifold of dimension $\dim F$, one has that $\dim V_w E = \dim F$.

Let us now define the horizontal subspaces of TE. Consider the natural projection $P \times F \to E$, i.e. $(u,\xi) \mapsto [u,\xi]$. Fixing $\xi \in F$ then defines a mapping

$$\xi : P \to E : u \mapsto [u,\xi] \tag{5.25}$$

Any two elements $v, u \in G_p$ can be connected by an element $a \in G$, i.e. $v = ua$. This implies that they will be mapped into the same fibre F_p; $\xi(G_p) \subset F_p$. On the other hand, for elements u, v belonging to different fibres in P there does not exist an $a \in G$ connecting them and they will get mapped in different fibres of E. The horizontal subspace $H_w E$ is then defined as the image of $H_u P$ under ξ, i.e. $H_w E \equiv \xi_* H_u P$ ($w = \xi(u)$). This definition is clearly independent of the choice $(u,\xi) \in P \times F$. Indeed, choose another pair (v,η), such that $\eta(v) = w$. This means that $[v,\eta] = [u,\xi]$, hence there exists some $a \in G$ such that $(v,\eta) = (ua, a^{-1}\xi)$. It follows that $\eta(u) = [u,\eta] = [ua^{-1},\xi] = \xi(ua^{-1})$ or $\eta(ua) = \xi(u)$. One then finds

$$T_w E \equiv \xi_* H_u P = \eta_* R_{a*} H_u P = \eta_* H_v P$$

which proves our assertion.

A curve in E is said to be horizontal if its generating vector field is horizontal at each point along the curve. Let γ_t be a curve in M. A curve γ_t^h in E is a horizontal lift of γ_t if it is horizontal and $\pi_E(\gamma_t^h) = \gamma_t$. Consider again the mapping $\xi : P \to E : u \mapsto [u, \xi]$ and let u_t^h be the horizontal lift of γ_t in P. By definition, ξ_* maps the generating vector field of u_t^h into a horizontal field in TE. It then follows that the curve $\gamma_t^h \equiv [u_t^h, \xi]$ is a horizontal lift of γ_t in E. Because ξ_* is an isomorphism between HP and HE, every horizontal lift in E can be obtained in this way.

As in the case of principal bundles, parallel transport in associated bundles is also governed by horizontal curves. Let γ_t be a curve in M and let $\gamma_t^h = [u_t^h, \xi]$ be the horizontal lift in E. Let $s(p)$ be a section in E such that $s(\gamma_0) = \gamma_0^h$. The parallel displacement of $s(\gamma_0) \in F_{\gamma_0}$ along γ_t is given by γ_t^h, for any t. Also, a section $s(\gamma_t) = [u_t^h, \xi(t)]$ is parallel transported if $\xi(t)$ is constant along γ_t, as this means that $s(\gamma_t)$ will be a horizontal lift of γ_t.

6. Linear connections

6.1. **Soldering.** Let $P(M, G)$ be a principal fibre bundle with G the general linear group, so that P is the bundle of linear frames LM. Let $TM = LM \times_G \mathbb{R}^n$. Remember that given a frame $u \in P$ and $\pi(u) = p$, $u : \mathbb{R}^n \to T_pM$ is a an isomorphism.

The *canonical form* or *solder form* θ on LM is the $\mathbb{R}^n$-valued 1-form defined by

$$\theta(X) \equiv u^{-1}(\pi(X)) \quad \text{for } X \in T_uLM \tag{6.1}$$

Note that $(R_a^*\theta)(X) = \theta(R_{a*}X) = a^{-1}u^{-1}(\pi(X)) = a^{-1}\theta(X)$. Since $\pi(X) = 0$ for X vertical, one concludes that the solder form is a tensorial 1-form of type $(Gl(n, \mathbb{R}), \mathbb{R}^n)$. Furthermore, in the sense of example 5.3, it is the identity transformation on T_pM. Indeed the associated mapping is given by $\tilde{\theta}(X) = u(u^{-1}(\pi(X^*)) = X$, for $X \in T_pM$.

6.2. **Covariant derivative on associated vector bundles.** In this subsection we focus on the cases where F is a vector space V and G a matrix group that acts on F through a representation ρ.

[The sections of E do form a vector space of infinite dimension]

Let φ be a section of E defined on $\gamma_t \in M$, so that $\pi_E \circ \varphi(\gamma_t) = \gamma_t$. Denote by $\dot\gamma_t$ the tangent vector field to γ_t. The *covariant derivative* of φ in the direction of $X \equiv \dot\gamma_t$ is given by

$$\nabla_X \varphi \equiv \lim_{h\to 0} \tfrac{1}{h}[\Gamma(\gamma^h)_t^{t+h}(\varphi(\gamma_{t+h})) - \varphi(\gamma_t)] \tag{6.2}$$

where $\Gamma(\gamma^h)_t^s : \pi_E^{-1}(\gamma_s) \to \pi_E^{-1}(\gamma_t)$ is the parallel displacement of the fibre at γ_s onto the fibre at γ_t. This can be rewritten in an equivalent form as follows. Let $\phi(\gamma_t) = [\sigma(t), \eta(t)] = [u_t^h a(t), \eta(t)] = [u_t^h, \xi(t)]$, where u_t^h is a horizontal lift of γ_t in E and $a(t) \in G$. Consider the horizontal lift of γ_t through the element $\varphi(\gamma_{t+h}) = [u_{t+h}^h, \xi(t+h)]$, i.e. $[u_t^h, \xi|_{t+h}]$. It then follows that

$$\nabla_X \varphi = \lim_{h\to 0} \tfrac{1}{h}([u_t^h, \xi|_{t+h}] - [u_t, \xi|_t]) = [u_t^h, \tfrac{d}{dt}\xi(t)]$$

The covariant derivative is ultimately a sum between two elements in the fibre F_{γ_t}, hence it is also an element of this same fibre. This happens smoothly along the curve γ_t and one concludes that given a section ϕ in a vector bundle, its covariant derivative is a section of the same type. Hence, given the fact that ϕ transforms through the representation ρ under G, its covariant derivative transforms through the same representation, i.e. *covariant.*

Note that the definition of a covariant derivative explicitely assumes the sum operation, which is defined in vector bundles.

The covariant derivative can be given a local expression as follows. Consider therefore a section $\sigma(t) : U \to P$, such that the horizontal lift of γ_t in P can be written as

$$u_t^h = \sigma(t) a_t$$

where a_t is a curve in G. Define a section $e_a(p)$ of E as

$$e_a(p) = [\sigma(p), \hat{e}_a]$$

where $\{\hat{e}_a\}$ is a basis for F, hence $(\hat{e}_a)^b = \delta^a_b$. Then this section along the curve γ_t is given by $e_a(t) = [\sigma(t), \hat{e}_a] = [u^h_t a_t^{-1}, \hat{e}_a] = [u^h_t, a_t^{-1}\hat{e}_a]$. The covariant derivative of e_a in the direction $X = \dot{\gamma}_t$ is given by (remember that G acts through a matrix representation on F)

$$\begin{aligned}\nabla_X e_a &= [u^h_t, \tfrac{d}{dt}(a_t \hat{e}_a)] \\ &= [u^h_t, -a_t^{-1}\dot{a}_t a_t^{-1}\hat{e}_a] \\ &= [u^h_t a_t^{-1}, -\dot{a}_t a_t^{-1}\hat{e}_a]\end{aligned}$$

Since the curve $\sigma(t)a_t$ is horizontal, one has that $\dot{a}_t a_t^{-1} = -\mathcal{A}(\dot{\gamma}_t)$.* This $\mathfrak{g}$-valued 1-form can be explicitely written out as $\mathcal{A}(\dot{\gamma}_t) = \mathcal{A}^a{}_{b\mu} dx^\mu(\dot{\gamma}^\nu \partial_\nu) = \mathcal{A}^a{}_{b\mu}\dot{\gamma}^\mu$. Pluging this information into the definition of the covariant derivative, one finds

$$\nabla_X e_a = [\sigma(t), \dot{\gamma}^\mu \mathcal{A}^b{}_{a\mu}\hat{e}_b] =: \dot{\gamma}^\mu \mathcal{A}^b{}_{a\mu} e_b \tag{6.3}$$

In the special case of $\gamma(t)$ being a coordinate curve, i.e. $\gamma^\mu_t = x^\mu_t$ with μ a fixed number, $X = \partial_\mu$ and covariant differentiation reduces to

$$\nabla_\mu e_a = \mathcal{A}^b{}_{a\mu} e_b$$

Then let us consider a generic section of E, i.e. $\varphi(p) = [\sigma(p), \xi^a(p)\hat{e}_a] = \xi^a(p)e_a$ with $\xi(p) = \xi^a(p)\hat{e}_a \in F$. The covariant derivative along the curve γ_t is then found to be

$$\nabla_X \varphi = \nabla_X(\xi^a(t)e_a) = X(\xi^a(t))e_a + \xi^a(t)\nabla_X e_a$$

where we used the chain rule for covariant differentiation and noted that $\xi^a(p)$ is a set of functions on M. Using the results obtained before, the covariant derivative is

$$\nabla_X \varphi = \dot{\gamma}^\mu(\partial_\mu \xi^a + \mathcal{A}^b{}_{a\mu}\xi^a)e_b \tag{6.4}$$

*See [3], pg. 69.

The result can be extended to find the covariant differentiation of tensor products of sections in E. Let $\psi(p) = [\sigma(p), \xi^{ab}\hat{e}_a \otimes \hat{e}_b] = \xi^{ab} e_a \otimes e_b$, an element of $E \otimes E$.* Then invoking the chain rule, one finds

$$\begin{aligned}
\nabla_X \psi &= \nabla_X (\xi^{ab} e_a \otimes e_b) \\
&= X(\xi^{ab}) e_a \otimes e_b + \xi^{ab} \nabla_X e_a \otimes e_b + \xi^{ab} e_a \otimes \nabla_X e_b \\
&= X^\mu \partial_\mu \xi^{ab} e_a \otimes e_b + \xi^{ab} X^\mu \mathcal{A}^c{}_{a\mu} e_c \otimes e_b + \xi^{ab} X^\mu \mathcal{A}^c{}_{b\mu} e_a \otimes e_c \ ,
\end{aligned}$$

so that

$$\nabla_X \psi = \dot{\gamma}^\mu (\partial_\mu \xi^{ab} + \mathcal{A}^a{}_{c\mu} \xi^{cb} + \mathcal{A}^b{}_{c\mu} \xi^{ac}) e_a \otimes e_b \ . \tag{6.5}$$

This is straightforwardly generalized for higher order tensor products.

Note that to obtain local expressions, one has to choose a section in P. From the definition of the covariant derivative however, it is clear that the latter is independent of the section chosen: it is an intrinsic notion, once a connection in P is chosen.

Example 6.1 Linear connection and spin connection

Appendix A. Vector valued differential forms

Let M be an m-dimensional manifold and V a vector space spanned by n basis elements E_a. Let ζ and η be respectively a vector valued p-form and q-form, i.e. $\zeta \in V \otimes \Omega^p(M)$ and $\eta \in V \otimes \Omega^q(M)$. This can be expanded as follows,

$$\begin{aligned}
\zeta &= E_a \cdot \zeta^a \ ; \quad \zeta^a \in \Omega^p(M) \\
\eta &= E_a \cdot \eta^a \ ; \quad \eta^a \in \Omega^q(M)
\end{aligned}$$

The wedge product of two vector valued differential forms is defined as†

$$\zeta \wedge \eta \equiv E_a \otimes E_b \cdot \zeta^a \wedge \eta^b \tag{A.1}$$

*The tensor product of two fibre bundles that have the same base manifold is constructed by considering at each base point the tensor product of the fibres. See for example [3].

†The tensor product $E_a \otimes E_b$ will also be denoted as just $E_a E_b$.

The exterior derivative is defined to be

$$d\zeta \equiv E_a \cdot d\zeta^a \tag{A.2}$$

From this it follows that

$$\begin{aligned} d(\zeta \wedge \eta) &= E_a \otimes E_b \cdot d\zeta^a \wedge \eta^b + (-1)^p \zeta^a \wedge d\eta^b \\ &= d\zeta \wedge \eta + (-1)^p \zeta \wedge d\eta \end{aligned} \tag{A.3}$$

Note that for vector valued differential forms, one does not have a relation between $\zeta \wedge \eta$ and $\eta \wedge \zeta$ because of the tensor product in $E_a \otimes E_b$.

For the following, we consider the case when V is a Lie algebra $\mathfrak{g}$, hence when ζ and η are $\mathfrak{g}$-valued differential forms. A Lie bracket is defined as

$$[\zeta, \eta] \equiv [E_a, E_b] \cdot \zeta^a \wedge \eta^b = c_{ab}{}^c E_c \cdot \zeta^a \wedge \eta^b \tag{A.4}$$

where $[E_a, E_b]$ denotes the usual Lie bracket on the algebra. From the definition of the wedge product one has that

$$[\zeta, \eta] = \zeta \wedge \eta - (-1)^{pq} \eta \wedge \zeta \tag{A.5}$$

This shows that the bracket $[\cdot, \cdot]$ is a graded commutator. It is an anti-commutator when pq is odd, being a commutator for pq even.

To conclude, consider the example of a Lie algebra valued 1-form ω. From (A.5) it follows that

$$[\omega, \omega] = 2\omega \wedge \omega \tag{A.6}$$

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