de Sitter Teleparallel Gravity

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Abstract

In this document we construct de Sitter Teleparallel Gravity.

1 de Sitter Teleparallel Gravity

1.1 Introduction

In this section we require that the $\mathfrak{h}$-valued part of the Cartan curvature $\bar{F}$ vanishes. In other words, the geometry outlined in the last section should at all times satisfy the following condition, namely

$$\bar{R} \equiv 0 \ . \tag{1.1}$$

From the discussion on transformation behavior in `nonlinear.pdf` it is clear that this condition is consistent with the geometry, i.e. invariant under G-gauge transformations. This construction may result in the mathemathical structure of Teleparallel Gravity in the corresponding limit, i.e. a diverging length scale $l(x) \to \infty$ at any point in spacetime. In that case, and because of the naturality of the given Cartan geometry together with a vanishing curvature (1.1), the thus obtained geometry could be seen as the generalization of Teleparallel Gravity, where the local kinematics are those governed by the de Sitter algebra.

Let us begin by taking a closer look at the condition of vanishing curvature, given in Eq.`~\eqref{eq:cond_Rnonlin0},nonlinear.odf`. Combining this requirement with Eq.`~\eqref{eq:nonlin_curv},nonlinear.pdf`, one finds that

$$R^{ab}{}_{\mu\nu} = \frac{\cosh z - 1}{\xi^2} \xi^c (\xi^a R^b{}_{c\mu\nu} - \xi^b R^a{}_{c\mu\nu}) + \frac{\sinh z}{l^2 z} (\xi^a T^b{}_{\mu\nu} - \xi^b T^a{}_{\mu\nu}) \ .$$

This expression can be contracted with ξ, which results in

$$\cosh z \, \xi^c R^a{}_{c\mu\nu} = \frac{\sinh z}{l^2 z} (\xi^a \xi_b T^b{}_{\mu\nu} - \xi^2 T^a{}_{\mu\nu}) \ . \tag{1.2}$$

Substituting this equations into the torsion $\bar{T}$, see Eq.`~\eqref{eq:nonlin_tors},nonlinear.pdf`, one obtains

$$\bar{T}^a{}_{\mu\nu} = \frac{1}{\cosh z} T^a{}_{\mu\nu} + \left(1 - \frac{1}{\cosh z}\right) \frac{\xi^a \xi_b T^b{}_{\mu\nu}}{\xi^2} \ . \tag{1.3}$$

Contracting both sides with ξ additionally shows that

$$\xi_a \bar{T}^a{}_{\mu\nu} = \xi_a T^a{}_{\mu\nu} \ .$$

Remark 1.1. It is interesting to have a look at the limiting situations for a vanishing, respectively diverging cosmological constant. In the case of $l(x) \to \infty$, z vanishes and from (1.3) it is found that

$$\lim_{\Lambda \to 0} \bar{T}^a{}_{\mu\nu} = T^a{}_{\mu\nu} \ ,$$

while on the other hand for $l(x) \to 0$, z diverges and

$$\lim_{\Lambda \to \infty} \bar{T}^a{}_{\mu\nu} = \frac{\xi^a \xi_b T^b{}_{\mu\nu}}{\xi^2} \ . \qquad \Diamond$$

Subsequently let us investigate the additional restriction of a vanishing torsion, i.e.*

$$\bar{T} = 0 \ . \tag{1.4}$$

From Eq. (1.3) and observing that $\xi \cdot \bar{T} = \xi \cdot T$ one infers that T vanishes. An obvious choice of gauge corresponding to such a geometry is $e = 0$, that is

$$e = 0 \quad \Rightarrow \quad \bar{T} = 0 \ .$$

On the other hand, a vanishing torsion does not necessarily imply that e is equal to zero. To find the most general e consistent with the condition (1.4), it is worthwhile to note that the latter is invariant under local gauge transformations and spacetime diffeomorphisms. These transformations are the most general at hand and their effect on e will exhaust its values, corresponding to a vanishing torsion. Since e transforms in a homogeneous way under both spacetime diffeomorphisms and local Lorentz transformations, these will leave $e = 0$ invariant. Therefore it is sufficient to consider the action on e due to de Sitter transvections $\exp(i\alpha \cdot P)$ solely. From the transformation rule~\eqref{eq:transvec_fin_Ah} for e one finds that $e^a = 0$ transforms into

$$e'^a = -\frac{\sinh z}{z}(d\alpha^a + \omega^a{}_b \alpha^b) + \frac{dl}{l}\alpha^a + \left(\frac{\sinh z}{z} - 1\right)\frac{\alpha_b d\alpha^b \alpha^a}{\alpha^2} \ . \tag{1.5}$$

The vierbein $\bar{e}'$ Lorentz rotates according to

*To be clear: the condition of a vanishing curvature is *not* relaxed.

$$\bar{e}'^a = \cosh z' e'^a - (\cosh z' - 1)\frac{\xi'_b e'^b \xi'^a}{\xi'^2} \\ + \frac{\sinh z'}{z'}(d\xi'^a + \omega'^a{}_b \xi'^b) - \frac{dl}{l}\xi'^a - \left(\frac{\sinh z'}{z'} - 1\right)\frac{\xi'_b d\xi'^b \xi'^a}{\xi'^2} \ ,$$

while the torsion $\bar{T}'^a$ remains zero. We call a vierbein $\bar{e}$ *trivial* if and only if e is of the form (1.5). In the case of a trivial vierbein, it is clear that some gauge transformation will render $e = 0$. But the relevant action is given by de Sitter transvections, which correspond to a shift in the section ξ. Since ξ is arbitrary, it is then without loss of generality to assume that a trivial vierbein is of the form

$$\bar{e}^a = \frac{\sinh z}{z}(d\xi^a + \omega^a{}_b \xi^b) - \frac{dl}{l}\xi^a - \left(\frac{\sinh z}{z} - 1\right)\frac{\xi_b d\xi^b \xi^a}{\xi^2} \ . \tag{1.6}$$

Hence, the vanishing of torsion entails the triviality of the vierbein. Conversely, in case the vielbein is trivial, the torsion will be equal to zero.

1.2 Equations of motion for a particle

Given the vielbein $\bar{e}$, it is possible to construct a line element on spacetime that is invariant under local de Sitter transformations. The quadratic line element is defined as

$$d\tau^2 = \bar{e}^a{}_\mu \bar{e}_{a\nu} dx^\mu dx^\nu \ ,$$

from which the square root is extracted, resulting in

$$d\tau = \bar{u}_a \bar{e}^a \ . \tag{1.7}$$

In the last expression the nonlinear four-velocity has been introduced, which is given by

$$\bar{u}^a = \bar{e}^a{}_\mu u^\mu \ .$$

The line element has the dimension of length, implying that a possible action for the worldline $x^\mu(\tau)$ of a particle with mass m equals

$$\mathcal{S} = -mc\int_{\tau_1}^{\tau_2} d\tau = -mc\int_{\tau_1}^{\tau_2} \bar{u}_a \bar{e}^a \ . \tag{1.8}$$

The action attains an extremum for the worldline being the physical one. This means that the equations of motion correspond to $\delta\mathcal{S} = 0$, where an infinitisemal variation of the worldline $x^\mu(\tau) \to x^\mu + \delta x^\mu(\tau)$ is considered. Under this deviation, the action (1.8) varies according to

$$\delta\mathcal{S} = -mc\int_{\tau_1}^{\tau_2} \delta\bar{u}_a \bar{e}^a + \bar{u}_a \delta\bar{e}^a = -mc\int_{\tau_1}^{\tau_2} \bar{u}_a \delta\bar{e}^a \ .$$

After a rather lengthy calculation, which we wrote down in Appendix (A.1), one finds

$$\delta\mathcal{S} = mc\int_{\tau_1}^{\tau_2} d\tau\delta x^\mu \left\{ \bar{e}^a{}_\mu \left(\frac{d\bar{u}_a}{d\tau} - \bar{\omega}^b{}_{a\rho}\bar{u}_b u^\rho + u^\rho \frac{\partial_\rho l}{l}\bar{u}_a \right) - \bar{T}^a{}_{\mu\rho}\,\bar{u}_a u^\rho - \frac{\partial_\mu l}{l} \right\} .$$

This quantity should vanish for an arbitrary variation, a condition that leads to the equations of motion:

$$u^\rho \bar{D}_\rho(l\bar{u}^a) = l\bar{e}^{a\mu}\left(\bar{T}^b{}_{\mu\rho}\,\bar{u}_b u^\rho + \frac{\partial_\mu l}{l} \right) ,$$

where $\bar{D} \equiv d + \bar{\omega}$ is the covariant derivative with respect to the spin connection $\bar{\omega}$. The equations of motion can be rewritten in the form

$$u^\rho \bar{D}_\rho \bar{u}^a = \bar{e}^{a\mu}\bar{T}^b{}_{\mu\rho}\,\bar{u}_b u^\rho + (\bar{e}^{a\mu} - \bar{u}^a u^\mu)\frac{\partial_\mu l}{l} . \tag{1.9}$$

It is interesting to have a closer look at this equation. First note that in the appropriate limit of a vanishing cosmological function ($l \to \infty$), the equation of motion of Teleparallel Gravity for a spinless particle in a gravitational field is recovered [1]. Similar to the equation there, we still have a force equation at hand in which both the terms on the right-hand side are indeed genuine relativistic forces, being orthogonal to the four-velocity $\bar{u}^a$. The first force is the obvious generalization to the given geometry of the gravitational force in ordinary Teleparallel gravity. The second force term however is new, and will be noticeable only in spacetime regions where the cosmological function varies relativily strong. Observe that the operator $\bar{e}^{a\mu} - \bar{u}^a u^\mu$ is a projector, since

$$(\bar{e}^{b\rho} - \bar{u}^b u^\rho)\bar{e}_{a\rho}(\bar{e}^{a\mu} - \bar{u}^a u^\mu) = \bar{e}^{b\mu} - \bar{u}^b u^\mu .$$

1.3 Field equations

In this subsection we look for the equations of motions that specify for the geometry in de Sitter Teleparallel Gravity. In a first attempt, the free action is the one given by replacing $T \to \bar{T}$ in the action that describes free Poincaré Teleparallel Gravity [Citations]. It is thus proposed that

$$\mathcal{S} = \frac{c^3}{16\pi G}\int \mathrm{Tr}\ \bar{T}\wedge\star\bar{T} = \frac{c^3}{16\pi G}\int d^4x\,\bar{e}\,\mathcal{L} , \tag{1.10}$$

where

$$\mathcal{L} = \tfrac{1}{4}\bar{T}^a{}_{\mu\nu}\,\bar{T}_a{}^{\mu\nu} + \tfrac{1}{2}\bar{T}^a{}_{\mu\nu}\,\bar{T}^{b\mu}{}_\lambda\,\bar{e}_a{}^\lambda\bar{e}_b{}^\nu - \bar{T}^a{}_{\mu\nu}\,\bar{T}^{b\mu}{}_\lambda\,\bar{e}_a{}^\nu\bar{e}_b{}^\lambda . \tag{1.11}$$

The corresponding field equations are found by extremizing (1.10) with respect to the vierbein $\bar{e}^a{}_\mu$, i.e.

$$
\begin{aligned}
0 = \delta\mathcal{S} &= \int d^4x\, \delta\bar{e}\,\mathcal{L} + \int d^4x\, \bar{e}\,\delta\mathcal{L} \\
&= \int d^4x\, \bar{e}\,\bar{e}_a{}^\mu \mathcal{L}\delta\bar{e}^a{}_\mu + \int d^4x\, \bar{e}\left(\frac{\partial\mathcal{L}}{\partial\bar{e}^a{}_\mu}\delta\bar{e}^a{}_\mu + \frac{\partial\mathcal{L}}{\partial\partial_\rho\bar{e}^a{}_\mu}\delta\partial_\rho\bar{e}^a{}_\mu\right) .
\end{aligned}
$$

After invoking Stokes' theorem together with the assumption that the fields go to zero when approaching infinity,* the equations of motion are

$$
\partial_\mu\left(\bar{e}\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{e}^a{}_\nu}\right) - \bar{e}\frac{\partial\mathcal{L}}{\partial\bar{e}^a{}_\nu} - \bar{e}\,\bar{e}_a{}^\nu\mathcal{L} = 0 . \tag{1.12}
$$

For the given Lagrangian it is shown in Appendix A.2 that these equations reduce to

$$
\partial_\mu(\bar{e}\,\bar{W}_a{}^{\mu\nu}) - \bar{e}\,\bar{\omega}^b{}_{a\mu}\bar{W}_b{}^{\mu\nu} + \bar{e}\frac{\partial_\mu l}{l}\bar{W}_a{}^{\mu\nu} + \bar{e}\,\bar{T}^b{}_{\mu a}\bar{W}_b{}^{\mu\nu} - \bar{e}\,\bar{e}_a{}^\nu\mathcal{L} = 0 ,
$$

where we introduced the notation

$$
\bar{W}_a{}^{\mu\nu} \equiv \bar{T}_a{}^{\mu\nu} + \bar{T}^{\nu\mu}{}_a - \bar{T}^{\mu\nu}{}_a - 2\bar{e}_a{}^\nu\bar{T}^{\lambda\mu}{}_\lambda + 2\bar{e}_a{}^\mu\bar{T}^{\lambda\nu}{}_\lambda . \tag{1.13}
$$

The field equations can be rewritten in a manifestly covariant form as

$$
\bar{D}_\mu(\bar{e}\,\bar{W}_a{}^{\mu\nu}) + \bar{e}\frac{\partial_\mu l}{l}\bar{W}_a{}^{\mu\nu} + \bar{e}\,\bar{t}_a{}^\nu = 0 , \tag{1.14}
$$

where we denoted the expression

$$
\bar{t}_a{}^\nu = \bar{T}^b{}_{\mu a}\bar{W}_b{}^{\mu\nu} - \bar{e}_a{}^\nu\mathcal{L} .
$$

Note further that

$$
\bar{D}_\nu\bar{D}_\mu(\bar{e}\,\bar{W}_a{}^{\mu\nu}) = \frac{1}{2}[\bar{D}_\nu,\bar{D}_\mu](\bar{e}\,\bar{W}_a{}^{\mu\nu}) = \frac{1}{2}\bar{e}\,\bar{B}_a{}^b{}_{\nu\mu}\bar{W}_b{}^{\mu\nu} .
$$

For the given geometry one has that $\bar{B}^{ab} = -l^{-2}\bar{e}^a\wedge\bar{e}^b$ so that

$$
\bar{D}_\nu\bar{D}_\mu(\bar{e}\,\bar{W}_a{}^{\mu\nu}) = \frac{1}{l^2}\bar{e}\,\bar{e}_{a\mu}\bar{e}^b{}_\nu\bar{W}_b{}^{\mu\nu} = \frac{1}{l^2}\bar{e}\,\bar{W}_{ba}{}^b .
$$

Since $\bar{W}_{ba}{}^b = -4\bar{T}_{ba}{}^b$ and because one can infer from the second Bianchi identity that the trace of $\bar{T}$ vanishes, one concludes that

$$
\bar{D}_\nu\bar{D}_\mu(\bar{e}\,\bar{W}_a{}^{\mu\nu}) = 0 . \tag{1.15}
$$

*Is it the fields that go to zero that legitimate the omitting of the boundary terms, or is it the vanishing of the variation?

It should be emphasized that this result is particular to the geometry at hand. More specifically is it a consequence of the condition $\bar{R} \equiv 0$. From the field equations (1.14) one observes that

$$\bar{D}_\mu(\bar{e}\,\bar{t}_a{}^\mu) = \frac{\partial_\mu l}{l}\bar{e}\,\bar{t}_a{}^\mu \ . \tag{1.16}$$

A de Sitter Teleparallel Gravity: intermediate results

In this section we work out to some extend, results related to de Sitter Teleparallel gravity that were used in the main body of the text.

A.1 Variation of $\int \bar{u}_a \bar{e}^a$

In this calculation the variation of $\int \bar{u}_a \bar{e}^a$ will be verified. To begin with let us rewrite the expression for a non-trivial vierbein, i.e.

$$\bar{e}^a = \cosh z\, e^a - (\cosh z - 1)\frac{\xi_b e^b \xi^a}{\xi^2} + \frac{\sinh z}{z}(d\xi^a + \omega^a{}_b \xi^b) - \frac{dl}{l}\xi^a - \Big(\frac{\sinh z}{z} - 1\Big)\frac{\xi_b d\xi^b \xi^a}{\xi^2} \ .$$

We compute:

$$\begin{aligned}
&\int \bar{u}_a \delta\bar{e}^a \\
&\quad = \int \bar{u}_a \Big\{ \delta\cosh z\, e^a + \cosh z\, \delta e^a{}_\rho dx^\rho + \cosh z\, e^a{}_\rho \delta dx^\rho - \delta\cosh z \frac{\xi_b e^b \xi^a}{\xi^2} \\
&\qquad + (\cosh z - 1)\Big[\frac{2\delta\xi}{\xi}\frac{\xi_b e^b \xi^a}{\xi^2} - \frac{\delta\xi_b e^b \xi^a}{\xi^2} - \frac{\xi_b \delta e^b{}_\rho \xi^a}{\xi^2} dx^\rho - \frac{\xi_b e^b \delta\xi^a}{\xi^2} \\
&\qquad - \frac{\xi_b e^b{}_\rho \xi^a}{\xi^2}\delta dx^\rho\Big] + \delta\Big(\frac{\sinh z}{z}\Big)(d\xi^a + \omega^a{}_b\xi^b) + \frac{\sinh z}{z}(\delta d\xi^a + \delta\omega^a{}_{b\rho}\xi^b dx^\rho \\
&\qquad + \omega^a{}_b \delta\xi^b + \omega^a{}_{b\rho}\xi^b \delta dx^\rho) + \frac{\delta l dl}{l^2}\xi^a - \frac{\delta dl}{l}\xi^a - \frac{1}{l}\delta\xi^a - \delta\Big(\frac{\sinh z}{z}\Big)\frac{\xi_b d\xi^b \xi^a}{\xi^2} \\
&\qquad + \Big(\frac{\sinh z}{z} - 1\Big)\Big[\frac{2\delta\xi}{\xi}\frac{\xi_b d\xi^b \xi^a}{\xi^2} - \frac{\delta\xi_b d\xi^b \xi^a}{\xi^2} - \frac{\xi_b \delta d\xi^b \xi^a}{\xi^2} - \frac{\xi_b d\xi^b \xi^a}{\xi^2}\Big]\Big\}
\end{aligned}$$

For any function on M, note that $df \to df + d\delta f$ so that $\delta(df) = d(\delta f)$, i.e.

$$[\delta, d]f = 0 \ . \tag{A.1}$$

The following variations also are useful:

$$\delta z = \delta(l^{-1}\xi) = -l^{-2}\delta l\, \xi + l^{-1}\delta\xi \ , \tag{A.2}$$

$$\delta\xi = \xi^{-1}\xi_a \delta\xi^a \ . \tag{A.3}$$

The variation is assumed to vanish at the endpoints of the particle's worldline, so that a total derivative over a term containing δx^ρ integrates to zero. One first integrates by parts the terms containing variations of the differentials $\delta dx^\rho = d\delta x^\rho$, after which the boundary integrals render zero. Doing so, one obtains

$$
\begin{aligned}
&\int \bar{u}_a \delta \bar{e}^a \\
&\quad = \int \Big[- d\bar{u}_a \Big\{ \cosh z\, {e^a}_\mu - (\cosh z - 1) \frac{\xi_b {e^b}_\mu \xi^a}{\xi^2} + \frac{\sinh z}{z} (\partial_\mu \xi^a + {\omega^a}_{b\mu} \xi^b) \\
&\quad - \frac{\partial_\mu l}{l} \xi^a - \big(\frac{\sinh z}{z} - 1 \big) \frac{\xi_b \partial_\mu \xi^b \xi^a}{\xi^2} \Big\} \delta x^\mu + \bar{u}_a \delta x^\mu dx^\rho \Big\{ \Big[\frac{\sinh z}{z} {\omega^a}_{b\rho} \Big(\partial_\mu \xi^b \\
&\quad - \xi^b \frac{\xi_c \partial_\mu \xi^c}{\xi^2} \Big) + \cosh z\, (\partial_\rho \xi^a + {\omega^a}_{b\rho} \xi^b) \frac{\xi_c \partial_\mu \xi^c}{\xi^2} - \partial_\rho \xi^a \frac{\xi_b \partial_\mu \xi^b}{\xi^2} - \cosh z \Big(\partial_\rho \xi^a \\
&\quad + {\omega^a}_{b\rho} \xi^b - \xi^a \frac{\xi_b \partial_\rho \xi^b}{\xi^2} \Big) \frac{\partial_\mu l}{l} + \partial_\rho \xi^a \frac{\partial_\mu l}{l} - 2(\cosh z - 1) \xi^a \frac{\xi_b \partial_\rho \xi^b}{\xi^2} \frac{\xi_c {e^c}_\mu}{\xi^2} \\
&\quad + (\cosh z - 1) \partial_\rho \xi^a \frac{\xi_b {e^b}_\mu}{\xi^2} + (\cosh z - 1) \xi^a \frac{\partial_\rho \xi_b {e^b}_\mu}{\xi^2} + z \sinh z \frac{\xi_c \partial_\mu \xi^c}{\xi^2} \Big({e^a}_\rho \\
&\quad - \xi^a \frac{\xi_b {e^b}_\rho}{\xi^2} \Big) - z \sinh z \Big({e^a}_\rho - \xi^a \frac{\xi_b {e^b}_\rho}{\xi^2} \Big) \frac{\partial_\mu l}{l} \Big] - \Big[\rho \leftrightarrow \mu \Big] \Big\} + \bar{u}_a \delta x^\mu dx^\rho \Big\{ \\
&\quad - \Big[\frac{\partial_\rho l}{l} \frac{\sinh z}{z} \Big(\partial_\mu \xi^a + {\omega^a}_{b\mu} \xi^b - \xi^a \frac{\xi_b \partial_\mu \xi^b}{\xi^2} \Big) + \frac{\partial_\rho l}{l} \frac{\partial_\mu l}{l} \xi^a + \cosh z\, \partial_\rho {e^a}_\mu \\
&\quad - (\cosh z - 1) \xi^a \frac{\xi_b \partial_\rho {e^b}_\mu}{\xi^2} + \frac{\sinh z}{z} \partial_\rho {\omega^a}_{b\rho} \xi^b \Big] + \Big[\rho \leftrightarrow \mu \Big] \Big\} \Big]
\end{aligned}
$$

The terms between the first pair of curly brackets is just the vierbein ${\bar{e}^a}_\mu$, while those between the second pair of curly brackets equal

$$
\begin{aligned}
&\Big[{\bar{\omega}^a}_{b\rho} {\bar{e}^b}_\mu - \frac{\sinh z}{z} {\omega^a}_{b\rho} {\omega^b}_{c\mu} \xi^c - \frac{\partial_\rho l}{l} \xi^a \frac{\xi_b \partial_\mu \xi^b}{\xi^2} - \cosh z\, {\omega^a}_{b\rho} {e^b}_\mu \\
&\qquad\qquad - (\cosh z - 1) \xi^a \frac{\omega_{bc\rho} \xi^c {e^b}_\mu}{\xi^2} - z \sinh z\, {e^a}_\rho \frac{\xi_c {e^c}_\mu}{\xi^2} \Big] - \Big[\rho \leftrightarrow \mu \Big] .
\end{aligned}
$$

This permits us to further work out

$$
\begin{aligned}
&\int \bar{u}_a \delta \bar{e}^a \\
&\quad = \int \Big[- d\bar{u}_a \bar{e}^a{}_\mu \delta x^\mu + \bar{u}_a \delta x^\mu dx^\rho \Big\{ \Big[\bar{\omega}^a{}_{b\rho} \bar{e}^b{}_\mu - \frac{\sinh z}{z} \xi^c \Big(\partial_\rho \omega^a{}_{c\rho} + \omega^a{}_{b\rho} \omega^b{}_{c\mu} \\
&\qquad + \frac{1}{l^2} e^a{}_\rho e_{c\mu} \Big) - \cosh z \Big(\partial_\rho e^a{}_\mu + \omega^a{}_{b\rho} e^b{}_\mu - \frac{\partial_\rho l}{l} e^a{}_\mu \Big) - (1 - \cosh z) \frac{\xi^a \xi_b}{\xi^2} \Big(\partial_\rho e^b{}_\mu \\
&\qquad + \omega^b{}_{c\rho} e^c{}_\mu - \frac{\partial_\rho l}{l} e^b{}_\mu \Big) - \frac{\partial_\rho l}{l} \Big(\cosh z \, e^a{}_\mu - (\cosh z - 1) \frac{\xi_b e^b{}_\mu \xi^a}{\xi^2} + \frac{\sinh z}{z} (\partial_\mu \xi^a \\
&\qquad + \omega^a{}_{b\mu} \xi^b) - \frac{\partial_\mu l}{l} \xi^a - \Big(\frac{\sinh z}{z} - 1 \Big) \frac{\xi_b \partial_\mu \xi^b \xi^a}{\xi^2} \Big) \Big] - \Big[\rho \leftrightarrow \mu \Big] \Big\} \Big] \\
&\quad = \int \delta x^\mu \Big[- d\bar{u}_a \bar{e}^a{}_\mu + \bar{u}_a \bar{\omega}^a{}_{b\rho} \bar{e}^b{}_\mu dx^\rho - \bar{u}_a \bar{\omega}^a{}_{b\mu} \bar{e}^b{}_\rho dx^\rho - \bar{u}_a dx^\rho \Big(\frac{\sinh z}{z} \xi^c R^a{}_{c\rho\mu} \\
&\qquad + \cosh z \, T^a{}_{\rho\mu} + (1 - \cosh z) \frac{\xi^a \xi_b T^b{}_{\rho\mu}}{\xi^2} \Big) - \frac{\partial_\rho l}{l} \bar{e}^a{}_\mu \bar{u}_a dx^\rho + \frac{\partial_\mu l}{l} \bar{e}^a{}_\rho \bar{u}_a dx^\rho \Big]
\end{aligned}
$$

In this expression one recognizes the torsion $\bar{T}$, as given in Eq.`~\eqref{eq:nonlin_tors}`. This leads to the end of this calculation:

$$
\int \bar{u}_a \delta \bar{e}^a = \int d\tau \delta x^\mu \Big\{ - \bar{e}^a{}_\mu \Big(\frac{d\bar{u}_a}{d\tau} - \bar{\omega}^b{}_{a\rho} \bar{u}_b u^\rho + u^\rho \frac{\partial_\rho l}{l} \bar{u}_a \Big) + \bar{T}^a{}_{\mu\rho} \bar{u}_a u^\rho + \frac{\partial_\mu l}{l} \Big\} \; . \quad \text{(A.4)}
$$

A.2 Variation of $\mathcal{L}$ with respect to $\bar{e}$.

In this subsection we work out some intermediary results that lead to the functional variation of the Lagrangian (1.11) with respect to the vierbein. More precisely, we calculate the derivatives of $\mathcal{L}$ with respect to $\bar{e}^c{}_\sigma$ and $\partial_\rho \bar{e}^c{}_\sigma$ in turn. From (1.11):

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \bar{e}^c{}_\sigma} &= \frac{1}{4} \frac{\partial \bar{T}^a{}_{\mu\nu}}{\partial \bar{e}^c{}_\sigma} \bar{T}_a{}^{\mu\nu} + \frac{1}{4} \bar{T}^a{}_{\mu\nu} \frac{\partial \bar{T}_a{}^{\mu\nu}}{\partial \bar{e}^c{}_\sigma} + \frac{1}{2} \frac{\partial \bar{T}^a{}_{\mu\nu}}{\partial \bar{e}^c{}_\sigma} \bar{T}^{b\mu}{}_\lambda \bar{e}_a{}^\lambda \bar{e}_b{}^\nu + \frac{1}{2} \bar{T}^a{}_{\mu\nu} \frac{\partial \bar{T}^{b\mu}{}_\lambda}{\partial \bar{e}^c{}_\sigma} \bar{e}_a{}^\lambda \bar{e}_b{}^\nu \\
&\quad + \frac{1}{2} \bar{T}^a{}_{\mu\nu} \bar{T}^{b\mu}{}_\lambda \frac{\partial (\bar{e}_a{}^\lambda \bar{e}_b{}^\nu)}{\partial \bar{e}^c{}_\sigma} - \frac{\partial \bar{T}^a{}_{\mu\nu}}{\partial \bar{e}^c{}_\sigma} \bar{T}^{b\mu}{}_\lambda \bar{e}_a{}^\nu \bar{e}_b{}^\lambda - \bar{T}^a{}_{\mu\nu} \frac{\partial \bar{T}^{b\mu}{}_\lambda}{\partial \bar{e}^c{}_\sigma} \bar{e}_a{}^\nu \bar{e}_b{}^\lambda \\
&\quad - \bar{T}^a{}_{\mu\nu} \bar{T}^{b\mu}{}_\lambda \frac{\partial (\bar{e}_a{}^\nu \bar{e}_b{}^\lambda)}{\partial \bar{e}^c{}_\sigma}
\end{aligned}
$$

It is thus useful to consider first the following equalities:

$$
\begin{aligned}
\frac{\partial \bar{T}^a{}_{\mu\nu}}{\partial \bar{e}^c{}_\sigma} &= [\bar{\omega}^a{}_{c\mu} \delta^\sigma_\nu - \frac{\partial_\mu l}{l} \delta^a_c \delta^\sigma_\nu] - [\mu \leftrightarrow \nu] \; , \\
\frac{\partial g_{\rho\lambda}}{\partial \bar{e}^c{}_\sigma} &= \frac{\partial (\bar{e}^a{}_\rho \bar{e}_{a\lambda})}{\partial \bar{e}^c{}_\sigma} = \bar{e}_{c\lambda} \delta^\sigma_\rho + \bar{e}_{c\rho} \delta^\sigma_\lambda \; , \\
\frac{\partial g^{\rho\lambda}}{\partial \bar{e}^c{}_\sigma} &= -g^{\sigma\rho} \bar{e}_c{}^\lambda - g^{\sigma\lambda} \bar{e}_c{}^\rho \; .
\end{aligned}
$$

Subsequently it is possible to obtain

$$\frac{\partial \bar{e}_a{}^\lambda}{\partial \bar{e}^c{}_\sigma} = -\bar{e}_a{}^\sigma \bar{e}_c{}^\lambda \ ,$$

$$\frac{\partial \bar{T}_a{}^{\mu\nu}}{\partial \bar{e}^c{}_\sigma} = [\eta_{ab}\bar{\omega}^b{}_{c\alpha} g^{\alpha\mu} g^{\sigma\nu} - \eta_{ac}\frac{\partial_\lambda l}{l} g^{\lambda\mu} g^{\sigma\nu} + \bar{T}_a{}^{\sigma\mu}\bar{e}_c{}^\nu + \bar{T}_{a\lambda}{}^\mu \bar{e}_c{}^\lambda g^{\sigma\nu}] - [\mu \leftrightarrow \nu] \ ,$$

$$\frac{\partial \bar{T}^{b\mu}{}_\lambda}{\partial \bar{e}^c{}_\sigma} = g^{\rho\mu}\bar{\omega}^b{}_{c\rho}\delta^\sigma_\lambda - g^{\sigma\mu}\bar{\omega}^b{}_{c\lambda} - g^{\rho\mu}\frac{\partial_\rho l}{l}\delta^b_c\delta^\sigma_\lambda + g^{\sigma\mu}\frac{\partial_\lambda l}{l}\delta^b_c - \bar{T}^b{}_{\rho\lambda}\,\bar{e}_c{}^\mu g^{\sigma\rho} - \bar{T}^b{}_{\rho\lambda}\,\bar{e}_c{}^\rho g^{\sigma\mu} \ .$$

Substituting these equations for $\partial\mathcal{L}/\partial\bar{e}^c{}_\sigma$, it takes some algebra to get the following:

$$\frac{\partial \mathcal{L}}{\partial \bar{e}^c{}_\sigma} = \bar{\omega}^a{}_{c\mu}\bar{W}_a{}^{\mu\sigma} + \bar{T}^a{}_{\mu c}\bar{W}_a{}^{\sigma\mu} - \frac{\partial_\mu l}{l}\bar{W}_c{}^{\mu\sigma} \ . \tag{A.5}$$

It is a simpler exercise to find the derivative of the Lagrangian with respect to the first order derivatives of the vierbein. One only needs the expression

$$\frac{\partial \bar{T}^a{}_{\mu\nu}}{\partial\partial_\rho \bar{e}^c{}_\sigma} = \delta^\rho_\mu\delta^\sigma_\nu\delta^a_c - \delta^\rho_\nu\delta^\sigma_\mu\delta^a_c \ .$$

This is sufficient since the derivative operator annihilates the metric $g_{\mu\nu} = \bar{e}^a{}_\mu\bar{e}_{a\nu}$ and we can freely raise and lower spacetime indices. Using this information, it is readily found that

$$\frac{\partial \mathcal{L}}{\partial\partial_\rho \bar{e}^c{}_\sigma} = \bar{W}_c{}^{\rho\sigma} \ . \tag{A.6}$$

References

[1] R. Aldrovandi and J. G. Pereira, *Teleparallel Gravity: An Introduction*, vol. 173 of *Fundamental Theories of Physics*. Springer Netherlands, 2012.