Riemannian geometry

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1 Vector bundles

In these notes smooth will always mean C^∞, i.e., infinitely many times differentiable. In this section we will review the important notion of vector bundle. We will also define the notion of pull-back of a vector bundle.

Definition 1.1 A vector bundle is a smooth map $p : E \to M$ of smooth manifolds E and M with the following properties:

(a) p is surjective;

(b) for every $x \in M$ the fiber $E_x := p^{-1}(x)$ has the structure of a linear space;

(c) for every $a \in M$ there exists an open neighborhood U of a in M, a constant $k \in \mathbb{N}$ and a diffeomorphism τ from $E_U := p^{-1}(U)$ onto $U \times \mathbb{R}^k$, such that τ maps each fiber E_x, for $x \in U$, linearly isomorphically onto $\{x\} \times \mathbb{R}^k \simeq \mathbb{R}^k$.

The space E is called the total space of the vector bundle, M is called the base manifold. The map τ is said to be a local trivialization of the vector bundle E over U. The constant k is called the rank of the bundle over U.

Let $p : E \to M$ be a vector bundle. A smooth section of p is defined to be smooth map $s : M \to E$ such that $p \circ s = \mathrm{id}_M$. Equivalently, this means that $s(x) \in E_x$ for all $x \in M$. We note that for $U \subset M$ open the map $p|_{E_U} : E_U \to U$ defines a smooth vector bundle over U. The space of smooth sections of this bundle is denoted by $\Gamma^\infty(U, E)$. Note that this is the space of smooth maps $s : U \to E$ such that $p \circ s = \mathrm{id}_U$.

Let $\tau : E_U \to U \times \mathbb{R}^k$ be a trivialization of E over U. For $1 \le j \le k$ let e_j denote the j-th standard basis vector of $\mathbb{R}^k$. Then the map $s_j : U \to E$ defined by $\tau(s_j(x)) = (x, e_j)$, is a section. Clearly, the sections $s_1, \ldots, s_k$ have the property that $s_1(x), \ldots, s_k(x)$ is a basis of E_x, for every $x \in U$. Such an ordered k-tuple of sections is said to be a frame of E over U. Conversely, given a frame $(s_1, \ldots, s_k)$ of E over U, there exists a unique trivialization τ of E over U such that $\tau(s_j(x)) = (x, e_j)$, for $1 \le j \le k$.

Example 1.2 Let M be a manifold, then the tangent bundle $\pi : TM \to M$ is a vector bundle. This bundle has trivializations induced by coordinate charts. As a set, TM is the disjoint union of the sets T_xM. We will denote the elements of TM as pairs (x,ξ), with $x \in M$ and $\xi \in T_xM$. Thus, $\pi(x,\xi) = x$. Let $U \subset M$ be open and let $\chi : U \to \mathcal{O}$ be a diffeomorphism onto an open subset $\mathcal{O}$ of $\mathbb{R}^n$. Then the map $d\chi : (x,\xi) \mapsto (x, d\chi(x)(\xi))$. defines a trivialization of TM over U. Note that $\Gamma^\infty(U,TM)$ equals the space $\mathfrak{X}(U)$ of smooth vector fields on U. Let $x^1,\dots,x^n$ be the local coordinates associated with χ, then the local frame associated with the above trivialization is given by the vector fields $\partial_j = \frac{\partial}{\partial x^j}$.

Example 1.3 Let M be a manifold, then the cotangent bundle T^*M is a vector bundle over M. Sections of T^*M are just exterior 1-forms on M. A local coordinatization $(x^1,\dots,x^n)$ of a coordinate patch determines the local frame $dx^1,\dots,dx^n$ of T^*M. We note that this local frame is dual to $\partial_1,\dots,\partial_n$ in the sense that $dx^j(\partial_k) = \delta^j_k$ for all $1 \le j,k \le n$.

Example 1.4 More generally, for $p \in \mathbb{N}$, the bundle $\wedge^p T^*M$ of exterior p-forms is an example of a vector bundle. The fiber above x equals $\wedge^p T^*_xM$, the p-th exterior power of T^*_xM, which may be naturally identified with the space of alternating p-forms on T_xM. Its sections are the exterior p-forms on M. If $(x^1,\dots,x^n)$ is a local coordinate system on U, then a local frame of $\wedge^p T^*U$ over U is given by the forms

$$dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

with I running over the set of p-tuples $(i_1,\dots,i_p)$ with $1 \le i_1 < \cdots < i_p \le n$.

Example 1.5 If p is a natural number and V is a finite dimensional real linear space, we write

$$\otimes^p V = \overbrace{V \otimes \cdots \otimes V}^{p}.$$

If q is a second natural number, we write

$$\mathcal{T}^{p,q}V := \otimes^p V \otimes \otimes^q V^*$$

for the space of tensors of contravariance degree p and covariance degree q on V. If M is a manifold then the disjoint union

$$\mathcal{T}^{p,q}_M := \coprod_{x\in M} \mathcal{T}^{p,q}T_xM$$

carries a natural structure of smooth vector bundle over M whose sections are the tensorfields of type (p,q).

Given a system $x^1,\dots,x^n$ on an open subset U of M, a local frame of $\mathcal{T}^{p,q}_M$ over U is given by

$$T^J_I = \partial_{i_1}\dots\partial_{i_n} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_n}.$$

where I and J run over $\{1,\dots,n\}^p$ and $\{1,\dots,n\}^q$.

Let $p : E \to M$ be a vector bundle. Then the zero section defines an embedding of M onto a closed submanifold of E. We will sometimes use this zero section to view M as a submanifold of the bundle E.

A morphism from a vector bundles $p : E \to M$ to a vector bundle $p' : E' \to M'$ is a smooth map $f : E \to E'$ such that f maps each fiber of E linearly into a fiber of E'. In particular, this implies that f maps the zero section of E to the zero section of E'. Hence, there exists a unique smooth map $f_0 : M \to M'$ such that $p' \circ f = f_0 \circ p$. If $M = M'$ and $f_0 = \mathrm{id}_M$, we will say that f is the identity on M.

The notion of isomorphism of vector bundles is now clear. We note that a trivialization of a vector bundle $p : E \to M$ over an open subset U of M is just an isomorphism of E_U onto $U \times \mathbb{R}^k$, which is the identity on U.

Exercise 1.6 Let E, F be vector bundles over M. We denote by $\mathrm{Hom}(E, F)$ the space of morphisms from E to F which are identical on M. Show that $\mathrm{Hom}(E, F)$ may be seen as a vector bundle whose fiber over a point x equals the space $\mathrm{Hom}(E_x, F_x)$ of linear maps $E_x \to F_x$.

Let $p : E \to M$ be a vector bundle. Then by a subbundle of p we mean a morphism $j : F \to E$ of vector bundles over M, which is identical on M and which is injective on each fiber of F. This implies that $j(F_x)$ is a linear subspace of E_x for each $x \in M$.

Exercise 1.7 Let $a \in M$. Show that there exists an open neighborhood U of a and trivializations $\sigma : F_U \to U \times \mathbb{R}^m$ and $\tau : E_U \to U \times \mathbb{R}^k$ and a smooth map L from U to $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^k)$ such that

$$\tau \circ j(x, \xi) = L_x \circ \sigma(x, \xi)$$

for all $x \in U$ and $\xi \in F_x$.

Conversely, show that if j admits a representation of this form everywhere locally, then $j : F \to E$ is a subbundle of E.

We now come to the important notion of pull-back of a vector bundle. Let $p : E \to M$ be a vector bundle and $f : N \to M$ a smooth map. Then the pull-back $f^*p : f^*E \to N$ is a vector bundle over N, defined as follows.

As a set, f^*E is defined to be the collection of points $(y, \xi) \in N \times E$ such that $\xi \in E_{f(y)}$. The projection map f^*p is defined to be the restriction of $\mathrm{pr}_N : N \times E \to N$. To see that this defines a vector bundle, let $a \in N$. Let U be a trivializing neighborhood of $f(a)$ and let $V := f^{-1}(U)$. Under this trivialization, E_U corresponds to the trivial bundle $U \times \mathbb{R}^k$. Accordingly, f^*E_U corresponds to $\mathrm{graph}(f|_V) \times \mathbb{R}^k$ hence is a smooth submanifold of $N \times E_U$. It follows that f^*E is a smooth submanifold of $N \times E$. Moreover, $f^*p|_V$ corresponds to the projection map $(y, f(y), v) \mapsto y$, showing that $f^*p|_V$ is a vector bundle isomorphic to $V \times \mathbb{R}^k$.

Restriction of the projection map $N \times E \to E$ to f^*E defines a vector bundle morphism $\tilde{f} : f^*E \to E$. This vector bundle over N has the following universal property. Let $q : F \to N$ be a vector bundle and let $\varphi : F \to E$ be a vector bundle morphism which equals $f : N \to M$ on N. Thus, we have a commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ N & \stackrel{f}{\longrightarrow} & M. \end{array}$$

Then there exists a unique vector bundle morphism $\bar{\varphi} : F \to f^*E$ which equals the identity on N, such that $f \circ \bar{\varphi} = \varphi$. This universal property has a nice representation by arrows in the above diagram.

Exercise 1.8 Prove the above assertion. Show that the universal property property determines the bundle f^*E up to isomorphism.

2 Connections on a vector bundle

In this section, $p : E \to M$ will be a fixed vector bundle of rank k. We will define the notion of a connection on p. This notion will allow us to connect fibers of the vector bundle along curves on the base manifold. The connecting maps will be called parallel transport or holonomies, and will play a crucial role in the rest of this course.

At first we will define a connection in a geometric fashion. Later we will give a different definition in terms of a differential operator.

Let $x \in M$, then the inclusion map $i_x : E_x \to E$ induces the injective tangent map $di_x(\xi) : T_\xi E_x \to T_\xi E$, for each $\xi \in E_x$. As E_x has the structure of a linear space, $T_\xi E_x \simeq E_x$. Accordingly, we use $di_x(\xi)$ to identify E_x with a linear subspace of $T_\xi E$.

On the other hand, the projection map $p : E \to M$ induces a surjective linear map $dp(\xi) : T_\xi E \to T_x M$ for each $x \in M$ and $\xi \in E_x$. The kernel of this map, denoted V_ξ, is called the space of vertical tangent vectors to E in the point ξ. Note that $\dim V_x = \dim E - \dim M = k$.

Lemma 2.1 *With notation as above, let $x \in M$ and $\xi \in E_x$. Then $di_x(\xi)$ is a linear isomorphism from E_x onto V_ξ.*

Proof: From $p \circ i_x(\xi) = x$ for all $\xi \in E_x$, it follows that $dp(\xi) \circ di_x(\xi) = \mathrm{id}_{E_x}$. Hence $di_x(\xi)$ is a linear embedding of E_x into V_ξ. The surjectivity of this map follows for dimensional reasons. $\square$

In the sequel we shall use the map $di_x(\xi)$ to identify E_x with the subspace V_ξ of $T_\xi E$.

Let Z be a smooth manifold. By an n-dimensional tangent system (also called a distribution) of Z we mean a family of n-dimensional linear subspaces $H_z \subset T_z Z$ such that $z \mapsto H_z$ is smooth in the following sense. For each point of Z there should exist an open neighborhood U and smooth vector fields $v_1, \ldots, v_n \in \mathfrak{X}(U)$ such that $H_z = \mathrm{span}(v_1(z), \ldots, v_n(z))$, for all $z \in U$.

Definition 2.2 A connection on the vector bundle $p : E \to M$ is a smooth n-dimensional tangent system $H = (H_\xi)_{\xi \in E}$ ($n = \dim M$) such that $T_\xi E = E_{p(\xi)} \oplus H_\xi$ for all $\xi \in E$.

Thus, a connection may be viewed as a smooth family of spaces of 'horizontal' tangent vectors.

Example 2.3 If $E = M \times \mathbb{R}^k$ is the trivial bundle, then $T_\xi E \simeq T_{p(\xi)}M \times \mathbb{R}^k$, for all $\xi \in E$. The tangent system defined by

$$H_\xi = T_{p(\xi)}M \times \{0\}$$

is a connection, which is called the trivial connection.

Since $dp(\xi) : T_\xi E \to T_{p(\xi)}M$ is a surjective linear map, with kernel equal to $V_\xi = E_{p(\xi)}$, its restriction to H_ξ is a linear isomorphism onto $T_{p(\xi)}M$. It follows that there exists a unique injective linear map $\alpha_\xi : T_{p(\xi)}M \to T_\xi E$ with

$$dp(\xi) \circ \alpha_\xi = \mathrm{id}_{T_{p(\xi)}M} \quad \text{and} \quad \mathrm{im}\, \alpha_\xi = H_\xi. \tag{1}$$

Conversely, given a family of injective linear maps $\alpha_\xi : T_{p(\xi)}M \to T_\xi E$ depending smoothly on $\xi \in E$ and satisfying (1), the tangent system $H_\xi = \mathrm{image}(\alpha_\xi)$ defines a connection.

Given a connection as above, we use the direct sum decomposition to define the projection map $P_\xi : T_\xi E \to E_{p(\xi)}$, for $\xi \in E$. Note that

$$P_\xi = \mathrm{id}_{T_\xi E} - \alpha_\xi \circ dp(\xi).$$

Example 2.4 In the example of the trivial bundle, $\alpha_\xi : T_{p(\xi)}M \to T_\xi E \simeq T_{p(\xi)}M \times \mathbb{R}^k$ is given by $X \mapsto (X, L_\xi X)$, with $L_\xi : T_{p(\xi)}M \to \mathbb{R}^k$ a linear map depending smoothly on $\xi \in M \times \mathbb{R}^k$. Accordingly, P_ξ is given by $(X, v) \mapsto (0, v - L_\xi(X))$. Note that the linear map $L_\xi : T_{p(\xi)}M \to \mathbb{R}^k$ is uniquely determined by the requirement that

$$(X, L_\xi X) \in H_\xi, \qquad (X \in T_{p(\xi)}M). \tag{2}$$

We will now describe how to transport connections under diffeomorphisms. Let $q : F \to N$ be a second vector bundle, and let $\varphi : E \to F$ be an isomorphism of vector bundles. Let $x \in M$ and $\xi \in E_x$. Then the map $\varphi_x : E_x \to F_{\varphi(x)}$ is a linear isomorphism. Moreover, $d\varphi(\xi) : T_\xi E \to T_{\varphi(\xi)}F$ is a linear isomorphism which restricts to φ_x on $E_x \hookrightarrow T_\xi E$. Let H be a connection on $p : E \to M$. Then it follows that for each $\eta \in F$ the subspace

$$(\varphi_* H)_\eta := d\varphi(\varphi^{-1}(\eta))\, [\, H_{\varphi^{-1}(\eta)}\,].$$

is complementary to $F_{q(\eta)}$ in $T_\eta F$. We thus see that $\varphi_* H$ defines a connection on F.

Before proceeding, we note that connections behave well with respect to pull-back. Indeed, let H be a connection on $p : E \to M$ as above and let $f : N \to M$ be a smooth map. Let f^*E be the pull-back of E under f and let $\widetilde{f} : f^*E \to E$ be the associated natural vector bundle morphism. For each $\eta \in f^*E$ we define

$$(f^*H)_\eta := d\widetilde{f}(\eta)^{-1} H_{\widetilde{f}(\eta)}.$$

Lemma 2.5 *f^*H is a connection on f^*E.*

We call this connection the pull-back of H under f.

Proof: By localization we see that it suffices to prove this in case E is trivial. Thus, we may as well assume that $E = M \times \mathbb{R}^k$, so that $f^*E = \mathrm{graph}(f) \times \mathbb{R}^k$, with $f^*p : f^*E \to N$ given by $(y, f(y), v) \mapsto y$. Let $\eta \in f^*E$. Then $\eta = (y, x, v)$ with $y \in N$, $x = f(y)$ and $v \in \mathbb{R}^k$.

We note that $\xi := (x, v)$ is the image of η under $\widetilde{f}$. There exists a linear map $L_\xi : T_xM \to \mathbb{R}^k$ such that $H_\xi \subset T_xM \times \mathbb{R}^k$ equals the space consisting of the vectors $(X, L_\xi X)$, for $X \in T_xM$. The map $d\widetilde{f}(\eta) : T_\eta f^*E \to T_\xi E$ is given by the restriction of $\mathrm{pr}_{T_xM} \times \mathrm{id}_{\mathbb{R}^k}$ to $\mathrm{graph}(df(y)) \times \mathbb{R}^k$. It follows that

$$\begin{aligned}(f^*H)_\eta &= \{(Y, df(y)Y, v) \in T_\eta f^*E \mid (df(y)Y, v) \in H_\xi\} \\ &= \{(Y, df(y)Y, v) \in T_\eta f^*E \mid v = L_\xi \circ df(y)Y\}.\end{aligned}$$

Let $B_\eta : T_yN \to T_\eta f^*E$ be defined by $B_\eta(N) = (Y, df(y)Y, L_\xi df(y)Y)$. Then B_η is injective linear with image $(f^*H)_\eta$ and depends smoothly on η. Moreover, $dp(\eta) \circ B_\eta = \mathrm{id}_{T_yN}$. It follows that f^*H is a connection on the bundle f^*E. $\square$

Remark 2.6 If $f : N \to M$ is a diffeomorphism, then $\widetilde{f} : f^*E \to E$ is an isomorphism of vector bundles. It is readily seen that $(\widetilde{f})_* f^*H = H$ in this case.

By a curve in a manifold Z we mean a continuous map $c : I \to Z$, with $I \subset \mathbb{R}$ an interval.

Definition 2.7 Let $p : E \to M$ be a vector bundle equipped with a connection H. A differentiable curve $c : I \to E$ is said to be horizontal if

$$c'(t) \in H_{c(t)}$$

for all $t \in I$.

Definition 2.8 Let p, H be as in the previous definition. Let $c : I \to M$ be a differentiable curve. Then by a horizontal lift of c to E we mean a differentiable curve $\tilde{c} : I \to E$ such that

(a) $p \circ \tilde{c} = c$;

(b) $\tilde{c}$ is horizontal.

We shall now investigate the question whether a curve in M has horizontal lifts. Since this is essentially a local question we first assume that the bundle is trivial, i.e., $E = M \times \mathbb{R}^k$. Then the connection is given by a family of linear maps $L_\xi : T_{p(\xi)}M \to \mathbb{R}^k$, depending smoothly on $\xi \in M \times \mathbb{R}^k$. Let $c : I \to M$ be a curve. Then a differentiable lift of c has the form $\tilde{c}(t) = (c(t), d(t))$, with $d : I \to \mathbb{R}^k$ differentiable. The lift is horizontal if and only if $(c'(t), d'(t))$ is horizontal for all $t \in I$, which is equivalent to

$$d'(t) = L_{(c(t),d(t))}c'(t), \qquad (t \in I). \tag{3}$$

If c is continuously differentiable, the above equation has always local solutions. More precisely, let $t_0 \in I$ be a fixed point, and let $\xi_0 \in E_{c(t_0)}$. Then the equation

(3) has solution with $d(t_0) = \xi_0$ defined on an interval J open in I and containing t_0. Moreover, the uniqueness theorem for the initial value problem in ordinary differential equations guarantees that the solution is unique in the following sense. If J_1, J_2 are two intervals as above, and $d_s : J_s \to \mathbb{R}^k$ are solutions to (3) with $d_s(t_0) = \xi_0$ for $s = 1, 2$, then $d_1 = d_2$ on $J_1 \cap J_2$.

In the present generality there is no guarantee that the solution d extends to the full interval I. If we require the map $\xi \mapsto L_{x,\xi}$ to be linear for every $x \in M$, then the equation (3) becomes linear, and global existence of the solution follows.

Definition 2.9 Let $E = M \times \mathbb{R}^k$ be the trivial bundle, and let H be a connection on E. For each $\xi \in E$, let $L_\xi : T_{p(\xi)}M \to \mathbb{R}^k$ be the unique linear map determined by (2). The connection H is said to be affine if $\xi \mapsto L_\xi$ is linear on the fibers of E.

The following result implies that the notion of being affine can be extended to arbitrary connections.

Lemma 2.10 *Let H be a connection on the trivial bundle $E = M \times \mathbb{R}^k$. Let $\varphi : E \to E$ be an isomorphism of vector bundles which is identical on M. Then H is affine if and only if $\varphi_*(H)$ is affine.*

In the proof we shall use the notation $\mathrm{M}_k(\mathbb{R})$ for the space of $k \times k$-matrices with real entries, and $\mathrm{GL}(k, \mathbb{R})$ for the subset consisting of matrices $a \in \mathrm{M}_k(\mathbb{R})$ with $\det a \neq 0$. Then $\mathrm{M}_k(\mathbb{R})$ may be viewed as a real linear space of dimension k^2. The matrix entries induce linear coordinates on this space. Accordingly, the function $\det : \mathrm{M}_k(\mathbb{R}) \to \mathbb{R}$ is polynomial hence continuous, and we see that $\mathrm{GL}(k, \mathbb{R})$ is an open subset of $M_k(\mathbb{R})$. In particular, $\mathrm{GL}(k, \mathbb{R})$ has the structure of a smooth manifold.

Proof: Assume that H is affine, and let L be associated to H as in Example 2.4. The isomorphism φ has the form $\varphi(x, v) = (x, g(x)v)$, with $g : M \to \mathrm{GL}(k, \mathbb{R})$ a smooth map. The tangent map of φ at a fixed point $(x, v) \in M \times \mathbb{R}^k$ is given by

$$d\varphi(x, v)(X, V) = (X, (dg(x)X)v + g(x)V).$$

Here we note that $dg(x)$ is $\mathrm{M}_k(\mathbb{R})$-valued, since $\mathrm{GL}(k, \mathbb{R})$ is open in $\mathrm{M}_k(\mathbb{R})$. It follows that $\varphi_*(H)_{(x,g(x)v)}$ consists of all elements of the form

$$(X, (dg(x)X)v + g(x)L_{(x,v)}X), \qquad (X \in T_xM).$$

This shows that the linear map $L^\varphi_{(x,g(x)v)}$ associated to φ_*H is given by

$$L^\varphi_{(x,g(x)v)}X = (dg(x)X)v + g(x)L_{(x,v)}X.$$

This expression depends linearly on $v \in \mathbb{R}^k$ if and only $L_{(x,v)}X$ does. The result follows. $\square$

The above result guarantees correctness of the following definition.

Definition 2.11 Let H be a connection on a vector bundle $p : E \to M$. The connection H is said to be affine if for each $a \in M$ there exist an open neighborhood U and a trivialization $\tau : E_U \to U \times \mathbb{R}^k$ such that $\tau_* H$ is affine.

Theorem 2.12 *Let H be an affine connection on the vector bundle $p : E \to M$. Let $c : I \to M$ be a C^1-curve, $t_0 \in I$ and $\xi \in E_{c(t_0)}$. Then*

(a) *the curve c has a unique horizontal lift $\widetilde{c}_\xi : I \to E$ with $\widetilde{c}_\xi(t_0) = \xi$;*

(b) *the map $(t, \xi) \mapsto \tilde{c}_\xi(t)$, $I \times E_{c(t_0)} \to E$ is C^1 and linear in the second variable.*

Proof: For a trivial vector bundle, assertion (a) follows by our earlier discussion, motivating the definition of affine connection. The C^1-part of assertion (b) follows from the known regularity and parameter dependence results for ordinary differential equations. The statement about linearity follows from linearity of the ordinary differential equation (3) combined with uniqueness of the solution.

We will establish the general result by reduction to the case of a trivial bundle.

Let J be the subset of I consisting of points $t_1 \in I$ such that the assertion of the theorem is valid for I replaced by the interval $[t_0, t_1]$. (This notation stands for the interval of t contained between t_0, t_1, including the boundary points, and also makes sense for $t_1 \leq t_0$). Clearly, if $t_1 \in J$ then $[t_0, t_1] \subset J$, so that J is an interval. We will show that this interval is both open and closed in I. From this the result will follow by connectedness of I.

Let J' be any interval which is open in I and such that $c(J')$ is contained in a neighborhood on which E trivializes. We claim that if J and J' have a point in common, then $J' \subset J$. To see this, fix $t_1 \in J \cap J'$. Let γ_ξ be the horizontal lift of $c|_J$ with $\gamma(t_0) = \xi$. For $\eta \in E_{c(t_1)}$, let γ'_η be the horizontal lift of $c|_J$ with $\gamma(t_1) = \eta$. By hypothesis the map $\epsilon : \xi \mapsto \gamma_\xi(t_1)$ is linear from $E_{c(t_0)}$ to $E_{c(t_1)}$. By uniqueness of horizontal lift in the case of trivial bundles it follows that $\gamma_\xi = \gamma'_{\epsilon(\xi)}$ on $J \cap J'$, for all $\xi \in E_{c(t_0)}$. This implies that γ_ξ and $\gamma'_{\epsilon(\xi)}$ are the restrictions of a C^1-curve $d_\xi : J \cup J' \to E$ which is a horizontal lift of $c|_{J \cup J'}$ with initial value $d_\xi(t_0) = \xi$. It follows from the similar statements for the intervals J and J' that d_ξ is uniquely determined, and that the map $(t, \xi) \mapsto d_\xi(t)$ is C^1 and linear in the second variable. Hence, $J' \subset J$.

We can now show that J is open in I. Indeed, let $t_1 \in J$. Fix an open neighborhood U of $c(t_1)$ on which E trivializes. There exists an interval $J' \ni t_1$, open in I, and such that $c(J') \subset U$. It follows from the above assertion that $J' \subset J$. Hence, t_1 is an interior point of J and we see that J is open.

Similarly, we can show that J is closed in I. Let $t_2 \in I$ be an accumulation point of J. Then there exists a trivializing open neighborhood U of $c(t_2)$. Fix an interval J' open in I, containing t_2 and such that $c(J') \subset U$. Then $J \cap J' \neq \emptyset$, hence $J' \subset J$ by what we proved above. In particular it follows that J contains all of its accumulation points in I, hence is closed in I. $\square$

Let $p : E \to M$ be a vector bundle equipped with an affine connection H. Let $c : I \to M$ be a C^1-curve. Let $t_0, t_1 \in I$. We define the parallel transport $T_{c,t_1,t_0} = T_{t_1,t_0}$ from t_0 to t_1 along the curve c to be the linear map $E_{c(t_0)} \to E_{c(t_1)}$ given by

$$T_{t_1,t_0}\xi = \tilde{c}_\xi(t_1),$$

where $\tilde{c}_\xi$ denotes the unique horizontal lift of the curve c with initial value ξ at $t = t_0$.

Lemma 2.13 *With notation above, we have, for all $s, s', s'' \in I$,*

(a) $T_{s'',s'} \circ T_{s',s} = T_{s'',s}$;

(b) $T_{s,s} = \mathrm{id}_{E_{c(s)}}$;

(c) $T_{s',s}$ *is invertible with inverse* $T_{s,s'}$.

Proof: Easy and left to the reader. □

Exercise 2.14 The purpose of this exercise is to explain the name parallel transport.

First, let $E = M \times \mathbb{R}^k$ be a trivial bundle, and let H be the trivial connection on it. Show that for any C^1-curve $c : I \to M$ the associated parallel transport $T_{c,s',s}$ is given by

$$(c(s), v) \mapsto (c(s'), v).$$

We now take $M = U$ an open subset of $\mathbb{R}^n$. Then for each $x \in U$ the map $v \mapsto v_x := d/dt(x + tv)|_{t=0}$ defines a linear isomorphism from $\mathbb{R}^n$ to T_xU. Accordingly, $\varphi : (x, v) \mapsto v_x$ defines a vector bundle isomorphism from $U \times \mathbb{R}^n$ onto TU. Let H be the connection on TU obtained by transference of the trivial connection under φ. Show that the parallel transport in TU along a C^1 curve c in U is given by

$$T_{c,s',s}(v_{c(s)}) = v_{c(s')}$$

and explain the name parallel transport.

The following lemma asserts that parallel transport behaves well with respect to reparametrization.

Lemma 2.15 *With notation as in the previous lemma, let J be an interval and $\varphi : J \to I$ a C^1-map. Put $d = c \circ \varphi$. Then for all $s, s' \in J$,*

$$T_{d,s',s} = T_{c,\varphi(s'),\varphi(s)}.$$

Proof: Let $\tilde{c}$ be any horizontal lift of c. Put $\tilde{d} = \tilde{c} \circ \varphi$. Then for all $t \in J$, we have that

$$\tilde{d}'(t) = \varphi'(t)c'(\varphi(t)) \in \varphi'(s)H_{c(\varphi(t))} \subset H_{d(t)}.$$

This implies that $\widetilde{d}$ is a horizontal lift of d. Moreover, if $\widetilde{c}(\varphi(s)) = \xi$, then $\widetilde{d}(s) = \xi$, and we see that

$$T_{d,s',s}(\xi) = \widetilde{d}(s') = \widetilde{c}(\varphi(s')) = T_{c,\varphi(s'),\varphi(s)}(\xi).$$

□

Given a point $x \in M$ we denote by $\Omega(x)$ the space of C^1-curves $c : [0,1] \to M$ with $c(0) = c(1) = x$. Given $c \in \Omega(x)$ we define the holonomy of H along c by

$$\mathrm{hol}_x(c) = T_{c,1,0}.$$

Note that this holonomy is a linear transformation of E_x which by Lemma 2.13 (c) is invertible. Let $\mathrm{GL}(E_x)$ denote the group of invertible linear transformations of E_x.

Proposition 2.16 *The transformations* $\mathrm{hol}_x(c)$, *with* $c \in \Omega(x)$ *form a subgroup of* $\mathrm{GL}(E_x)$.

This subgroup is called the holonomy group of H at the point x.

Proof: Put $I = [0,1]$ and let $\varphi : I \to I$ be a fixed homeomorphism which is C^1 and satisfies $\varphi'(0) = \varphi'(1) = 0$. Then it follows from Lemma 2.15 that

$$\mathrm{hol}_x(c) = \mathrm{hol}_x(c \circ \varphi).$$

Given $c, d \in \Omega(x)$ we define $d * c : [0,1] \to M$ by

$$d * c(t) = \begin{cases} d(\varphi(2t)) & \text{if} \quad 0 \le t \le 1/2; \\ c(\varphi(2t-1)) & \text{if} \quad 1/2 \le t \le 1. \end{cases}$$

Then it is readily seen that $d * c \in \Omega(x)$. The reparametrization is needed to give c and d velocity zero at the enpoints, so that the composed loop is C^1. Moreover, by application of Lemmas 2.15 and 2.13 it follows that

$$\mathrm{hol}_x(d * c) = \mathrm{hol}_x(d) \circ \mathrm{hol}_x(c).$$

Given $c \in \Omega$ we define $c^{-1} \in \Omega(x)$ by $c^{-1}(t) = c(1-t)$. Applying Lemma 2.15 with $\varphi(t) = 1 - t$ we see that

$$\mathrm{hol}_x(c^{-1}) = T_{c^{-1},1,0} = T_{c,0,1} = T_{c,1,0}^{-1} = \mathrm{hol}_x(c)^{-1}.$$

Finally, the constant loop $\underline{x} : t \mapsto x$ has holonomy id_{E_x}. The result follows. $\square$

The rest of the course will focus on several aspects of holonomy, in particular in the context of a Riemannian manifold.

3 Covariant differentiation

Let $p : E \to M$ be a vector bundle equipped with an affine connection H. Let $x \in M$ and $\xi \in E_x$. Then the connection determines a unique projection $P_\xi : T_\xi E \to E_x$ with kernel H_ξ. The projection allows us to measure the deviation from horizontality of a section under infinitesimal displacement. Indeed, let s be a differentiable section of E defined in a neighborhood of x. Then $ds(x)$ is a linear map $T_x M \to E_{s(x)}$.

Definition 3.1 Let $X_x \in T_x M$ and s a section of E defined in a neighborhood of x. Then the covariant derivative of s at x in the direction X is defined by

$$\nabla_{X_x} s := P_{s(x)} \circ ds(x) X \in E_x.$$

Given a smooth vector field $X \in \mathfrak{X}(M)$ and a smooth section $s \in \Gamma^\infty(E)$ we define the section $\nabla_X s$ of E by $\nabla_X s(x) = \nabla_{X_x} s$ for every $x \in M$. Using trivializations of the vector bundle E it is readily seen that $\nabla_X s$ is a smooth section again.

Lemma 3.2 *Let $s, t \in \Gamma^\infty(E)$, $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Then*

(a) $\nabla_{fX} s = f \nabla_X s$;

(b) $\nabla_{X+Y} s = \nabla_X s + \nabla_Y s$;

(c) $\nabla_X(s+t) = \nabla_X s + \nabla_X t$;

(d) $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

Proof: (a) and (b) are straightforward from the definitions. We will derive (c) and (d) by using that H is affine. The assertions are of a local nature on M, so that we may well assume that $E = M \times \mathbb{R}^k$. Then above $a \in M$ the connection is given by linear maps $L_{a,v} : T_a M \to \mathbb{R}^k$, depending linearly on $v \in \mathbb{R}^k$. A section of E is given by $s(x) = (x, \varphi(x))$, with φ a smooth map $M \to \mathbb{R}^k$. Accordingly,

$$ds(a) = (I, d\varphi(a)) : T_a M \to T_a M \times \mathbb{R}^k.$$

The vertical projection at $s(a) = (a, \varphi(a))$ is given by

$$P_{s(a)}(X, v) = (0, v - L_{a,\varphi(a)} X).$$

It follows that

$$\nabla_{X_a} s = (0, d\varphi(a) X_a - L_{a,\varphi(a)} X_a),$$

viewed as an element of $T_\xi E \simeq T_x M \oplus \mathbb{R}^k$. Hence

$$\nabla_{X_a} s = (a, d\varphi(a) X_a - L_{a,\varphi(a)} X_a) \tag{4}$$

as an element of $E_a = \{a\} \times \mathbb{R}^k$. Assertions (c) and (d) follow from this formula, in view of the linear dependence of $L_{a,v}$ on v and the $C^\infty(M)$-linear dependence of φ_s on s. $\square$

The above rules for covariant differentiation allow a convenient expression of covariant differentiation in terms of a local frame.

Let U be an open subset of M, and let $e_1, \ldots, e_k$ be a frame for E on U. Thus, e_j are smooth sections of E on U and for each $x \in U$, the elements $e_1(x), \ldots, e_k(x)$ form a basis of E_x. We define the associated connection 1-form $A = A_e$ on U to be the $M_k(\mathbb{R})$-valued 1-form on U given by

$$\nabla_X e_i = \sum_j A(X)_i^j e_j.$$

Let s be a smooth section of E on U. Then with respect to the frame (e_j), the section may be represented by a smooth function $s_e : U \to \mathbb{R}^k$. More precisely,

$$s = \sum_{j=1}^k s^j e_j$$

for uniquely defined smooth functions $s^j = s^j_e \in C^\infty(U)$, and we put

$$s_e := (s^1, \dots, s^k)^{\mathrm{T}}.$$

Here T denotes that the transposed should be taken, in order to get a column vector. In terms of this representation of sections, covariant differentiation may now be described as follows.

Lemma 3.3 *Let $s \in \Gamma^\infty(U, E)$. Then for all $X \in \mathfrak{X}(U)$,*

$$(\nabla_X s)_e = ds_e \cdot X + A(X)s_e.$$

Proof: This follows by a straightforward application of Lemma 3.2. Indeed,

$$\begin{aligned}
\nabla_X s &= \sum_{j=1}^k \nabla_X (s^j e_j) \\
&= \sum_{j=1}^k (X s^j)\, e_j + s^j \nabla_X e_j \\
&= \sum_{j=1}^k ds^j(X) e_j + \sum_{i,j=1}^k s^j A(X)^i_j e_i \\
&= \sum_{j=1}^k ds^j(X) e_j + \sum_{i=1}^k (A(X)s)^i e_i
\end{aligned}$$

and the result follows. $\square$

We will now relate the connection H to the matrix A. The local frame $e_1, \dots, e_k$ of E on U determines the trivialization $\tau = \tau_e : E_U \to U \times \mathbb{R}^k$ given by

$$\tau(s^1 e_1(x) + \cdots + s^k e_k(x)) = (x, s),$$

where s denotes the column vector with entries s^j. The push-forward $\tau_* H$ of H under τ is a connection on the trivial bundle $U \times \mathbb{R}^k$ which may retrieved from A in a simple manner.

Lemma 3.4 *Let $p : E \to M$ be a rank k vector bundle equipped with an affine connection H. Let $e = (e_j)$ be a local frame of E over an open subset $U \subset M$. Let $A = A_e$ be the associated $\mathrm{M}_k(\mathbb{R})$-valued 1-form on U and let $\tau = \tau_e$ be the associated trivialization of E_U. Then for all $(x, v) \in U \times \mathbb{R}^k$,*

$$(\tau_* H)_{(x,v)} = \{(X, -A_x(X)v) \mid \ X \in T_x M\}.$$

Proof: Let s be any section of E over U and put $s_e = (s^1, \dots, s^k)^{\mathrm{T}}$. Then the section $\tau \circ s$ of the trivial bundle $U \times \mathbb{R}^k$ is given by

$$\tau \circ s(x) = (x, s_e(x)).$$

Then for each vector field X on U,

$$\tau \circ \nabla_X s(x) = (x, (\nabla_X s)_e) = (x, ds_e(x)X + A_x(X_x)s_e(x)).$$

Let L be associated with the connection $\tau_* H$ on $U \times \mathbb{R}^k$ as in (2). Then comparing with (4) we find that

$$A_x(X_x)s_e(x) = -L_{x,s_e(x)}(X_x),$$

for all $X \in \mathfrak{X}(U)$ and $s \in \Gamma^\infty(U)$. This implies that

$$A_x(X)v = -L_{x,v}(X)$$

for all $x \in U,\ X \in T_xX$ and $v \in \mathbb{R}^k$. The result now follows by using (2). $\square$

Proposition 3.5 *Let $\nabla : \mathfrak{X}(M) \times \Gamma^\infty(E) \to \Gamma^\infty(E),\ (X,s) \mapsto \nabla_X s$ be a bilinear map satisfying properties (a) - (d) of Lemma 3.2. Then there exists a unique affine connection H on E such that ∇ is the associated covariant differentiation. Moreover, let $x \in M$ and $\xi \in E_x$. Then H_ξ is given by*

$$H_\xi = \{ds(x)X_x - \nabla_X s(x) \mid s \in \Gamma^\infty(E),\ s(x) = \xi,\ X \in \mathfrak{X}(M)\}. \tag{5}$$

Proof: By using cut off functions we see that the statement is local on M. Thus, we may restrict ourselves to the case that the bundle is trivial, $E = M \times \mathbb{R}^k$. Write $\xi = (x, v)$. Let K_ξ denote the set on the right-hand side of (5). Let $e_1, \dots, e_k$ be the frame of E given by the standard basis of $\mathbb{R}^k$, and let A be the 1-form on M with values in $\mathrm{M}_k(\mathbb{R})$ given by $\nabla_X e_i = \sum_j A(X)_i^j e_j$. A section $s \in \Gamma^\infty(E)$ is any function of the form $s(x) = (x, \varphi(x))$, with $\varphi \in C^\infty(M, \mathbb{R}^k)$. As an element of $T_\xi E = T_x M \times \mathbb{R}^k$, the covariant derivative of such a section is given by

$$\nabla_X s(x) = (0, d\varphi(x)X_x + A_x(X_x)\varphi(x)).$$

On the other hand, $ds(x)X_x = (X_x, d\varphi(x)X_x)$. Let K_ξ denote the set on the right-hand side of (5). Then it follows that

$$K_\xi = \{(X_x, -A_x(X_x)v \mid X_x \in T_x M\}.$$

Clearly, this defines an affine connection on E. It is readily checked that the associated covariant derivative is ∇. If ∇ comes from a connection H, then from $P_{s(x)} ds(x)X = \nabla_X s(x)$ it follows that $ds(x)X - \nabla_X s(x) \in H_{s(x)}$. It follows that $K_\xi \subset H_\xi$. Since both spaces have the same dimension $\dim M$, it follows that $H_\xi = K_\xi$. $\square$

Remark 3.6 In view of the above proposition, an affine connection is often defined to be an operator $\nabla : \mathfrak{X}(M) \times \Gamma^\infty(E) \to \Gamma^\infty(E)$ such that the conditions of Lemma 3.2 are fulfilled.

We end this section with another useful characterization of covariant differentiation in terms of parallel transport.

Lemma 3.7 *Let $a \in M$ and let $X_a \in T_a M$. Let $c : (-1, 1) \to M$ be any C^1 curve with $c(0) = a$ and $c'(0) = X_a$. Moreover, let $T_t = T_{t,0}$ denote the parallel transport along c from 0 to t. Then for every smooth section s of E,*

$$\nabla_{X_a} s = \left.\frac{d}{dt}\right|_{t=0} T_t^{-1} s(c(t)).$$

Proof: We note that T_t is a linear map from E_a to $E_{c(t)}$. Thus $\varphi(t) = T_t^{-1}s(c(t))$ belongs to the finite dimensional vector space E_a for all $t \in (-1,1)$. We will show that φ is a C^1-function.

Let U be an open neighborhood of a on which the bundle E allows a trivialization $\tau : E_U \to U \times \mathbb{R}^k$. Let J be an open interval in $(-1,1)$ containing zero and such that $c(J) \subset U$. Given $x \in U$ we write τ_x for the linear isomorphism $E_x \to \mathbb{R}^k$ determined by $\tau(\xi) = (x, \tau_x(\xi))$. Then the map $t \mapsto S_t := t \mapsto \tau_{c(t)} \circ T_t \circ \tau_a^{-1}$ is a C^1-map from J to $\mathrm{GL}(k, \mathbb{R}^k)$. By Cramer's formula for the inverse of a matrix one sees that $t \mapsto S_t^{-1}$ is C^1 as well. Now $t \mapsto \tau_{c(t)}s(c(t))$ is C^1. It follows that

$$\tau_a \circ \varphi(t) = S_t^{-1} \circ \tau_{c(t)}s(c(t))$$

is a C^1-function of $t \in J$. Hence φ is C^1 on J. Other points of the interval $(-1,1)$ may be treated similarly, but we shall not need this.

We now observe that $s(c(t)) = T_t\varphi(t)$. This implies that

$$\begin{aligned}
\nabla_{X_a} s &= P_{s(a)} ds(a) c'(0) \\
&= P_{s(a)} \left.\frac{d}{dt}\right|_{t=0} s(c(t)) \\
&= P_{s(a)} \left.\frac{d}{dt}\right|_{t=0} T_t\varphi(t) \\
&= P_{s(a)} \left.\frac{d}{dt}\right|_{t=0} T_t\varphi(0) + P_{s(a)} T_0 \varphi'(0).
\end{aligned}$$

Now the curve $t \mapsto T_t\varphi(0)$ is horizontal by definition, so that its derivative at $t = 0$ is horizontal. Hence, the first term of the above sum is zero. Moreover, $T_0 = \mathrm{id}_{E_a}$ and $\varphi'(0)$ belongs to E_a, on which $P_{s(a)}$ equals the identity. It follows that

$$\nabla_{X_a} s = \varphi'(0)$$

as was to be shown. $\square$

Exercise 3.8 Let $p : E \to M$ be a vector bundle equipped with a connection H. Let $c : I \to M$ be C^1 curve, and let s be a smooth section of E such that

$$\nabla_{c'(t)} s = 0.$$

Show that s is parallel along c, i.e.,

$$T_{c,'t',t} s(c(t)) = s(c(t'))$$

for all $t, t' \in I$.

Exercise 3.9 Let $p : E \to M$ be a vector bundle equipped with a connection H. We assume that $F \subset E$ is a subbundle which has a complementary subbundle $F' \subset E$. By this we mean that $E_a = F_a \oplus F'_a$ for all $a \in M$. Let $\pi : E \to F$ denote the projection map along F'.

(a) Show that the map $(X, s) \mapsto \pi \circ \nabla_X s$ from $\mathfrak{X}(M) \times \Gamma^\infty(F)$ to $\Gamma^\infty(F)$ defines a covariant differential operator on F.

(b) Give a description of the connection H_F on F associated to the operator in (a) in terms of the connection H.

4 Curvature

Let $p : E \to M$ be a vector bundle equipped with a connection ∇. (From now on we will always assume connections to be affine.) We define the associated curvature map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma^\infty(M, E) \to \Gamma^\infty(M, E)$ by

$$R(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s,$$

for $X, Y \in \mathfrak{X}(M)$, $s \in \Gamma^\infty(M, E)$.

Lemma 4.1 *The map R is $C^\infty(M)$-trilinear.*

By this we mean of course that R is $C^\infty(M)$-linear in each of its variables.

Exercise 4.2 Prove the above lemma.

The following lemma will imply that R is essentially a vector bundle homomorphism $\otimes^2 TM \to \mathrm{End}(E)$. The first of these bundles is the vector bundle on M with fiber $T_xM \otimes T_xM$ and the second of these bundles is the vector bundle on M with fiber $\mathrm{End}(E_x)$, for $x \in M$.

Lemma 4.3 *Let $p : E \to M$ and $q : F \to M$ be vector bundles on M and let $L : \Gamma^\infty(M, E) \to \Gamma^\infty(M, F)$ be a C^∞-linear map. Then there exists a unique vector bundle homomorphism $\lambda : E \to F$, identical on M, such that*

$$Ls(x) = \lambda_x(s(x)), \qquad (\forall x \in M). \tag{6}$$

Proof: Fix $a \in M$ and consider the map $L_a : \Gamma^\infty(E) \to F_a$ given by $L_a s = (Ls)(a)$. Select a local frame $e_1, \ldots, e_k$ of E over an open neighborhood U of a in M. There exists a smooth function $\chi \in C^\infty(M)$ with compact support contained in U, such that $\chi(a) = 1$. For a section s of E, let $s^j \in C^\infty(U)$ denote the associated component functions with respect to the selected frame. Thus, $s = \sum_j s^j e_j$ on U. The functions χs^j may be viewed as smooth functions on M, through extension by zero outside U. Similarly, the local sections χe_j may be extended to smooth sections of E on U, vanishing outside the set U.

We now have, for any section $s : M \to E$,

$$\begin{aligned} L_a(s) &= \chi(a)^2 L_a(s) = (\chi^2 Ls)(a) \\ &= L(\chi^2 s)(a) = L(\sum_j (\chi s^j)(\chi e_j))(a) \\ &= \sum_j s^j(a) L_a(\chi e_j). \end{aligned}$$

Thus, if we define the linear map $\lambda_a : E_a \to F_a$ by $\lambda_a(e_j(a)) := L_a(\chi e_j)$, then

$$L_a s = \lambda_a(s(a)).$$

The evaluation map $s \mapsto s(a)$ is surjective from $\Gamma^\infty(M, E)$ onto E_a. Hence, λ_a is uniquely determined by the last displayed equality. Using local trivializations we see that $x \mapsto \lambda_x$ depends smoothly on x. Hence, λ defines a smooth section of $\mathrm{Hom}(E, F)$, or, equivalently, a vector bundle homomorphism $E \to F$, identical on M. $\square$

Corollary 4.4 *There exist unique bilinear maps $R_a : T_aM \times T_aM \to \mathrm{End}(E_a)$, depending smoothly on $a \in M$, such that*

$$R(X,Y)s = R_a(X_a, Y_a)s(a),$$

for all smooth vector fields X, Y on M and smooth sections s of E.

Proof: This follows from Lemma 4.1, by application of Lemma 4.3 to each of the variables separately. $\square$

Let $p : E \to M$ be a rank k vector bundle equipped with a connection H. The connection is said to be flat, or locally trivial, if for each point $a \in M$ there exists a trivialization τ of E in a neighborhood U of a in which H becomes the trivial connection, i.e., $\tau_* H$ is trivial. Equivalently, this means that the one form $A \in \Omega^1(U) \otimes \mathrm{M}_k(\mathbb{R})$ determined by τ vanishes.

Curvature measures the extent to which a connection is flat. More precisely, we have the following fundamental result.

Theorem 4.5 *Let $p : E \to M$ be a vector bundle equipped with a connection ∇. Then $R = 0$ if and only if the connection ∇ is flat.*

For the proof we need the notion of restriction of (E, H) to a submanifold N of M. Let N be such a submanifold, and let $j : N \to M$ denote the inclusion map. Then the restriction of E to N is defined to be the pull-back bundle $j^*(E)$. The restriction of H to E_N is defined to be the pull-back $f^*(H)$. These objects were previously defined in the more general context of pull-back under smooth maps between manifolds. The situation of a submanifold is particularly important, and easier to handle. Therefore, we shall give the definitions for this situation again and independently.

In the following $p : E \to M$ will be a vector bundle of rank k, H a connection on it and ∇ the associated covariant differentiation. Let N be a smooth submanifold of M. Then the restriction bundle $p_N : E_N \to N$ is defined as follows. The set E_N is defined by $E_N := p^{-1}(N)$. As p is a submersion, this is a smooth submanifold of E. The restriction $p_N := p|_{E_N} : E_N \to N$ is a smooth surjective map. Its fibers are $p_N^{-1}(x) = E_x$ for $x \in N$. Thus each of the fibers comes equipped with the structure of a linear space. Moreover, each trivialization of E in a neighborhood U of a point $a \in N$ restricts to a trivialization τ_N of E_N on $U \cap N$. Hence, E_N is a vector bundle. We note that a smooth section of E_N is given by a map $s : N \to E$ such that $p \circ s = \mathrm{id}_N$. For this reason, the space of sections of E_N is also denoted by $\Gamma^\infty(N, E)$, and we speak of sections of E over the submanifold N.

If $\xi \in E_N$, then the inclusion map $E_N \to E$ induces a linear embedding $T_\xi E_N \hookrightarrow T_\xi E$. Obviously, this embedding is compatible with the natural identifications $E_{p_N(\xi)} \simeq E_{p(\xi)}$ and with the natural embeddings $E_{p_N(\xi)} \hookrightarrow T_\xi E_N$ and $E_{p(\xi)} \hookrightarrow T_\xi E$. Identifying elements of all these spaces accordingly, we have $E_{p(\xi)} \subset T_\xi E_N \subset T_\xi E$. It follows that the linear projection map $P_\xi : T_\xi E \to E_{p(\xi)}$ restricts to a linear projection map $P_\xi^N : T_\xi E_N \to E_{p(\xi)}$. Put

$$H_\xi^N := H_\xi \cap T_\xi E_N.$$

Then H^N_ξ equals the kernel of P^N_ξ so that $T_\xi E_N = E_{p(\xi)} \oplus H^N_\xi$. It follows that H^N_ξ defines a connection on E_N, which is readily checked to be affine (use local trivializations). Let ∇^N denote the associated covariant differentiation.

Lemma 4.6 *Let $s \in \Gamma^\infty(N, E)$ and let $X_a \in T_aN$. Then*

$$\nabla^N_{X_a} s = P_{s(a)} \circ ds(a)X_a.$$

Proof: The equality is true with P^N in place of P. Now use that $ds(a)$ maps T_aN into $T_{s(a)}E_N$ and that $P_{s(a)}$ restricts to $P^N_{s(a)}$ on $T_{s(a)}E_N$. $\square$

Corollary 4.7 *Let X be a vector field on M which is tangent to N everywhere, so that $X|_N$ is a vector field on N. Then for all sections $s \in \Gamma^\infty(M, E)$,*

$$(\nabla_X s)|_N = \nabla^N_{X|_N}(s|_N).$$

Corollary 4.8 *Let $p : E \to M$ be a vector bundle, equipped with a connection ∇. Let $N \subset M$ be smooth manifold, $p|_N : E_N \to N$ the restricted bundle and ∇^N the restricted connection. Then*

$$R^N_a(X_a, Y_a) = R_a(X_a, Y_a)$$

for all $a \in N$ and $X_a, Y_a \in T_aN$.

Exercise 4.9 With notation as above, let $e_1, \ldots, e_k$ be a local frame of E on an open neighborhood U of a point $a \in M$. Assume that $a \in N$ and put $V := U \cap N$.

(a) Show that $e_1|_V, \ldots, e_k|_V$ is a local frame of E_N over U.

(b) Let A_e be the matrix valued 1-form on U associated with ∇ and the frame $e_1, \ldots, e_k$. Show that the matrix valued 1-form A^N_e on V associated with ∇^N and the local frame $e_j|_N$ is given by

$$A^N_e(x) = A_e(x)|_{T_xN},$$

for all $x \in V$.

We are now prepared for the proof of Theorem 4.5.

Proof of Thm. 4.5: If the connection is flat, then it is readily verified that $R = 0$. To prove the converse, assume $R = 0$. In view of the local nature of the result, we may as well assume that M is the n-fold Cartesian product I^n of the open interval $I = (-1, 1)$. In this situation it suffices to establish the existence of a frame $e_1, \ldots, e_n$ of E over M that is flat. By this we mean that $\nabla_X e_j = 0$ for all $X \in \mathfrak{X}(M)$. Let $x^1, \ldots, x^n$ denote the standard coordinate functions on M and $\partial_i = \partial/\partial x^i$ the associated standard vector fields. We will write $\nabla_i := \nabla_{\partial_i}$. Then by the rules of calculation with connections it suffices to show the existence of a frame $e_1, \ldots, e_k$ such that

$$\nabla_i e_j = 0$$

for all $1 \leq i \leq n$ and $1 \leq j \leq k$. We will achieve this by induction on n. For $n = 0$ the manifold M consists of a single point, and there is nothing to prove. Thus, let $n > 0$ and assume the result has been established for strictly smaller values of n.

Define the submanifold N of M by $N = I^{n-1} \times \{0\}$. Then the connection ∇^N on the restricted bundle E_N has curvature $R^N = 0$ by Corollary 4.8. By the inductive hypothesis, there exists a frame $e_1^N, \ldots, e_k^N$ of E over N such that

$$\nabla_i^N e_j^N = 0, \qquad (1 \leq i \leq n-1,\ 1 \leq j \leq k).$$

We now define the frame $e_1, \ldots, e_k$ of E over M as follows. For each $x \in N$ we consider the curve $c_x : I \to M$ given by $c_x(t) = (x, t)$. Let $T_{x,t} : E_x \to E_{(x,t)}$ denote the parallel transport along this curve for the connection ∇ and define

$$e_j = T_{x,t}(e_j^N(x)).$$

Then $e_1, \ldots, e_k$ defines a smooth frame of E over M. Moreover, in view of Lemma 3.7 it follows that

$$\nabla_n e_j = 0 \qquad (1 \leq j \leq k).$$

As the curvature tensor vanishes it therefore also follows that for $1 \leq i \leq n-1$ we have

$$\nabla_n \nabla_i e_j = \nabla_i \nabla_n e_j = 0, \qquad (1 \leq j \leq k).$$

This implies that

$$\nabla_i e_j(x,t) = T_{x,t}(\nabla_i e_j(x,0)) = 0,$$

for $1 \leq i \leq n-1$ and $1 \leq j \leq k$. The proof is complete. $\square$

Exercise 4.10 Let $p : E \to M$ be a vector bundle with a connection ∇. Let $e_1, \ldots, e_k$ be a local frame for E over an open subset U, and let $A = A_e$ be the associated $k \times k$-matrix of 1-forms.

(a) Given $X, Y \in \mathfrak{X}(U)$ we define $R(X,Y)_j^i = R_e(X,Y)_j^i$ by

$$R(X,Y)e_j = \sum_i R(X,Y)_j^i e_i.$$

Show that the $R(\,\cdot\,,\,\cdot\,)_j^i$ define differential 2-forms on U. We write R_e for the associated $\mathrm{M}_k(\mathbb{R})$-valued 2-form.

(b) Show that

$$R_e = dA_e + A_e \wedge A_e.$$

It is customary to omit the subscript e in this notation. The action of d on A is defined componentwise. Moreover,

$$(A \wedge A)_j^i := \sum_r A_r^i \wedge A_j^r.$$

5 Curvature and holonomy

In this section we will interpret curvature as infinitesimal holonomy. As preparation we need a result relating the flows of two vector fields to their Lie bracket. We start by recalling some basic definitions and results concerning this bracket.

Let X be a smooth vector field on the manifold M, and let $\alpha : (t,x) \mapsto \alpha_t(x)$ be its flow. We will write $\mathcal{D}_X$ for the domain of α; it is an open subset of $\mathbb{R} \times M$. For each $x \in M$ the set $I_{X,x}$ of $t \in \mathbb{R}$ with $(t,x) \in \mathcal{D}_X$ is an interval containing 0. In fact, it is the interval of definition of the maximal integral curve of X with starting point x. The integral curve is given by $t \mapsto \alpha_t(x)$, $I_{X,x} \to M$. Thus, $\alpha_0(x) = x$ and

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \alpha_t(x) = X(\alpha_t(x)), \qquad (t \in I_{X,x}).$$

We recall that $\alpha_0 = I$ and that for each compact set $C \subset M$ we have $\alpha_s \circ \alpha_t = \alpha_{s+t}$ on C, for s,t sufficiently close to zero.

Likewise, let Y be a smooth vector field on M and let β denote its flow, with domain $\mathcal{D}_Y$. For each $x \in M$, let $I_{Y,x}$ denote the interval of $t \in \mathbb{R}$ with $(t,x) \in \mathcal{D}_Y$. We recall that the Lie derivative of Y with respect to X is the vector field defined by

$$\mathcal{L}_X Y := \left.\frac{\partial}{\partial t}\right|_{t=0} \alpha_t^* Y.$$

Clearly this notion is local and coordinate invariant. If M is an open subset of $\mathbb{R}^n$, then X and Y may be indentified with smooth functions $M \to \mathbb{R}^n$ and then the Lie derivative can be computed as follows:

$$\begin{aligned}
\mathcal{L}_X Y(x) &= \left.\frac{\partial}{\partial t}\right|_{t=0} d(\alpha_{-t})(\alpha_t(x)) Y(\alpha_t(x)) \\
&= \left.\frac{\partial}{\partial t}\right|_{t=0} d\alpha_{-t}(x) Y(x) + \left.\frac{\partial}{\partial t}\right|_{t=0} Y(\alpha_t(x)) \\
&= -dX(x)\, Y(x) + dY(x) X(x).
\end{aligned} \tag{7}$$

Since the notion of Lie derivative is local and coordinate invariant, it follows from (7) that $\mathcal{L}_X Y = -\mathcal{L}_Y X$. Hence, the Lie bracket is anti-symmetric. From the local expression (7) it is also readily verified that the Lie bracket satisfies the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0,$$

for all smooth vector fields X, Y, Z on the manifold M. Accordingly, the real linear space $\mathfrak{X}(M)$ equipped with the Lie bracket is a Lie algebra.

We recall that a linear endomorphism $D \in \mathrm{End}(C^\infty(M))$ which satisfies the Leibniz rule $D(fg) = (Df)g + f(Dg)$ is called a derivation on $C^\infty(M)$. The space of such derivations is denoted by $\mathrm{Der}(M)$. It is readily verified that $\mathrm{Der}(M)$, equipped with the commutator bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is a Lie algebra. If X is a smooth vector field on M, we define

$$\partial_X f(x) := \left.\frac{\partial}{\partial t}\right|_{t=0} f(\alpha_t(x)) = df(x) X(x).$$

It is readily verified that ∂_X is a derivation on $C^\infty(M)$.

Exercise 5.1 The map $X \mapsto \partial_X$ defines an isomorphism of Lie algebas from $\mathfrak{X}(M)$ onto $\mathrm{Der}(M)$. Hint: use local coordinates and the local expression (7).

It is customary to abbreviate

$$Xf := \partial_X f, \qquad (f \in C^\infty(M), X \in \mathfrak{X}(M)).$$

With this notation the assertion about the map $X \mapsto \partial_X$ being a Lie algebra homomorphism may be rewritten as

$$X(Yf) - Y(Xf) = [X,Y]f, \qquad (f \in C^\infty(M),\; X, Y \in \mathfrak{X}(M)).$$

The following result expresses the Lie bracket $[X,Y]$ in terms of the flows of the vector fields X and Y.

Lemma 5.2 *Let X and Y be smooth vector fields on M with flows α and β, respectively. Then for each $x \in M$,*

$$[X,Y](x) = \left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} \beta_{-t}\alpha_{-s}\beta_t\alpha_s(x).$$

The formula of the lemma should be interpreted as follows. First, there exists a $\delta > 0$ such that $(s,t) \mapsto \beta_{-t}\alpha_{-s}\beta_t\alpha_s(x)$ is well-defined and smooth on $(-\delta,\delta) \times (-\delta,\delta)$. Moreover, for each $s \in (-\delta,\delta)$, the element

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \beta_{-t}\alpha_{-s}\beta_t\alpha_s(x)$$

belongs to T_xM and depends smoothly on s. Thus, the second differentiation is differentiation of a function with values in the fixed linear space T_xM. Its outcome is an element of T_xM.

Proof: By coordinate invariance and the local nature of the result, it suffices to prove the identity in case M is an open subset of $\mathbb{R}^n$. In this case the vector fields X and Y may be viewed as functions on M with values in a single space $\mathbb{R}^n$; this simplifies the computations. We have

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{t=0} \beta_{-t}\alpha_{-s}\beta_t\alpha_s(x) &= \left.\frac{\partial}{\partial t}\right|_{t=0} \beta_{-t}(x) + \left.\frac{\partial}{\partial t}\right|_{t=0} \alpha_{-s}\beta_t\alpha_s(x) \\
&= -Y(x) + d\alpha_{-s}(\alpha_s(x))Y(\alpha_s(x)) \\
&= -Y(x) + (\alpha_s^*Y)(x).
\end{aligned}$$

Differentiation with respect to s leads to

$$\left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} \beta_{-t}\alpha_{-s}\beta_t\alpha_s(x) = \left.\frac{\partial}{\partial s}\right|_{s=0} (\alpha_s^*Y)(x) = [X,Y].$$

□

We will now apply the above to the setting of a rank k vector bundle $p : E \to M$ equipped with a connection ∇. Let R be the associated curvature form. Let X, Y be two *commuting* vector fields on M, with flows α and β respectively. Let $a \in M$. It is a standard result that the flows of commuting vector fields locally commute. Hence, there exists a constant $\delta > 0$ such that for all $s, t \in (-\delta, \delta)$ we have

$$\beta_{-t}\alpha_{-s}\beta_t\alpha_s(a) = a. \tag{8}$$

We will define the so-called horizontal lifts of the vector fields X and Y. It will turn out that these vector fields on E need not commute.

For each $x \in M$ and $\xi \in E_x$ there exists a unique vector $\tilde{X}(\xi) \in T_\xi E$ such that

$$\tilde{X}(\xi) \in H_\xi \quad \text{and} \quad dp(\xi)\tilde{X}(\xi) = X(x).$$

By using local trivializations it is readily seen that $\xi \mapsto \tilde{X}(\xi)$ defines a smooth vector field on E, called the horizontal lift of X. Let $\tilde{\alpha}$ be the flow of $\tilde{X}$.

Lemma 5.3 *For every $\xi \in E$ the integral curve $s \mapsto \tilde{\alpha}_s(\xi)$ is the horizontal lift of the integral curve $s \mapsto \alpha_s(p(\xi))$.*

Proof: The tangent vector equals $\tilde{X}(\tilde{\alpha}_s(\xi))$ hence is horizontal, so that $s \mapsto \tilde{\alpha}_s(\xi)$ is a horizontal curve in E. Write $c(s) := p(\tilde{\alpha}_s(\xi))$ for its projection to M. Then

$$c'(s) = dp(\tilde{\alpha}_s(\xi))\tilde{X}(\tilde{\alpha}_s(\xi)) = X(p(\tilde{\alpha}_s(\xi)) = X(c(s)).$$

Thus, c is an integral curve for X. Since $c(0) = p(\xi)$ it follows that $c(s) = \alpha_s(p(\xi))$. $\square$

It follows from the above lemma that

$$\widetilde{\alpha}_s(\xi) = T_{c,s,0}(\xi),$$

where $T_{c,s,0}$ denotes parallel transport from 0 to s along the integral curve $\sigma \mapsto \alpha_\sigma(x)$ of the vector field X.

Likewise, let $\widetilde{Y}$ be the horizontal lift of Y, and let $\widetilde{\beta}$ denote its flow. Then $\widetilde{\beta}$ is the horizontal lift of the flow β.

In view of (8) it follows from the above that $\tilde{\beta}_{-t}\tilde{\alpha}_{-s}\tilde{\beta}_t\tilde{\alpha}_s(\xi)$ belongs to E_a for all $s, t \in (-\delta, \delta)$ and $\xi \in E_a$. In fact, the element may be viewed as the application of parallel transport along a loop based at a, composed of parts of integral curves of the vector fields X and Y. The following result expresses that curvature may be interpreted as infinitesimal holonomy.

Proposition 5.4 *With notation as above, let $a \in M$ and $\xi \in E_a$. Then*

$$R_a(X_a, Y_a)\xi = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\bigg|_{s=t=0} \tilde{\beta}_{-t}\tilde{\alpha}_{-s}\tilde{\beta}_t\tilde{\alpha}_s(\xi).$$

Proof: In view of the previous lemma, it suffices to show that

$$R_a(X_a, Y_a)\xi = [\tilde{X}, \tilde{Y}](\xi).$$

Let s be a smooth section of E with $s(a) = \xi$. We will show that

$$R(X,Y)s = [\tilde{X}, \tilde{Y}] \circ s$$

in a neighborhood of a. By the local nature of this assertion, and by invariance under diffeomorphism of manifolds and under isomorphism of vector bundles, we may assume that M is an open subset of $\mathbb{R}^n$ and that E is the trivial bundle $M \times \mathbb{R}^k$. Let A be the $k \times k$-matrix of 1-forms on M associated with the connection and the standard frame $e_1, \dots, e_k$. Through the standard frame, a section s of E is identified with a map $s_e : M \to \mathbb{R}^k$. This being understood, covariant differentiation is given by

$$(\nabla_Z s)_e(x) = ds_e(x)Z_x + A_x(Z_x)s_e(x) = (Zs_e)(x) + A(Z)(x)\, s_e(x).$$

This implies that

$$\begin{aligned} &(\nabla_X \nabla_Y s)_e \\ &\quad = \quad XYs_e + X[A(Y)s_e] + A(X)[Ys_e] + A(X)A(Y)s_e \\ &\quad = \quad XYs_e + X[A(Y)]s_e + A(Y)[Xs_e] + A(X)[Ys_e] + A(X)A(Y)s_e. \end{aligned}$$

Since X and Y commute, it follows from the definition of R that

$$(R(X,Y)s)_e = (X[A(Y)] - Y[A(X)])s_e + [A(X)A(Y) - A(Y)A(X)]s_e.$$

We now observe that the horizontal lift of the vector field X at the point $\xi = (x, v)$ is given by

$$\tilde{X}(\xi) = (X_x, A_x(X_x)v).$$

Similarly, the horizontal lift of Y is given by

$$\tilde{Y}(\xi) = (Y_x, A_x(Y_x)v).$$

It follows that

$$d\tilde{Y}(\xi)(Z, V) = (dY(x)Z\,,\; Z[A(Y)]_x\, v + A(Y)_x V)$$

so that

$$d\tilde{Y}(\xi)\tilde{X}(\xi) = (dY(x)X(x)\,,\; X[A(Y)]_x v + A(Y)_x A(X)_x v).$$

This implies that the Lie bracket of these horizontal lifts is given by

$$\begin{aligned} [\tilde{X}, \tilde{Y}]_\xi &= d(\tilde{Y})(\xi)\tilde{X}(\xi) - d(\tilde{X})(\xi)\tilde{Y}(\xi) \\ &= (dY(x)X(x) - dX(x)Y(x)\,,\; V(x)), \end{aligned}$$

where

$$V = (X[A(Y)] - Y[A(X)])v + [A(Y)A(X)v - A(X)A(Y)]v.$$

It follows that

$$\begin{aligned} [\tilde{X}, \tilde{Y}]_\xi &= ([X,Y]_x, R_x(X_x, Y_x)v)) \\ &= (0, R_x(X_x, Y_x)v). \end{aligned}$$

In view of the natural identification $\{0\} \times \mathbb{R}^k = T_\xi(E_x) \simeq E_x = \mathbb{R}^k$ this implies the result. $\square$

6 A useful derivative

In this section, we assume that $p : E \to M$ is a smooth vector bundle equipped with a connection H. We recall that at every $\xi \in E$ the tangent space decomposes as $T_\xi E = E_{p(\xi)} \oplus H_\xi$. The corresponding projection map $T_\xi E \to E_{p(\xi)}$ is denoted by P_ξ.

Let I be a real interval and $s : I \to E$ a smooth map. Then for each $t \in I$ the tangent vector $s'(t) = ds/dt$ belongs to $T_{s(t)}E$. We define

$$\frac{Ds}{dt}(t) := P_{s(t)}s'(t).$$

We note that $c := p \circ s$ is a differentiable curve in M. Since $s(t) \in E_{c(t)}$ for all $t \in I$ we say that s is a section of E over the curve c. We note that Ds/dt is again a section of E over the curve c.

Remark 6.1 Note that $s : I \to E$ is a section over the C^1-curve $c : I \to M$ if and only if s is a section of the pull-back bundle $c^*(E)$. Moreover, let ∇^c denote the pull-back of ∇ under c, then $Ds/dt = \nabla^c_1 s$.

Lemma 6.2 *Let I be a real interval and $s : I \to E$ a differentiable curve. Then s is horizontal if and only if $Ds/dt = 0$.*

Proof: The curve s in E is horizontal if and only if $s'(t) \in H_{s(t)}$ for all $t \in I$. This is equivalent to $P_{s(t)}s'(t) = 0$ for all $t \in I$, hence to $Ds/dt = 0$. $\square$

One would expect the derivative D/dt to be related to covariant differentiation. This is indeed the case.

Lemma 6.3 *Let $c : I \to M$ be a C^1-curve, and let $s : M \to E$ be a section. Then*

$$\frac{D}{dt}s(c(t)) = \nabla_{c'(t)}s, \qquad (t \in I).$$

Proof: By the chain rule,

$$\frac{D}{dt}s(c(t)) = P_{s(c(t))}\frac{d}{dt}\left[s(c(t))\right] = P_{s(c(t))}\, ds(c(t))\, c'(t) = \nabla_{c'(t)}s.$$

$\square$

This result is probably the reason that in the literature one also sees the notation $\nabla_{c'(t)}s$ for Ds/dt in case $s : I \to E$ is a general section over c.

We will now express the derivative Ds/dt in terms of a local frame. Let $e_1, \dots, e_k$ be a local frame, and let $A = A_e$ the $k \times k$ matrix of one forms associated with this frame. The local frame defines a local trivialization in which the vertical projection at a point $\xi = \sum_j v^j e_j(x)$, $x \in M$, is given by

$$(X, V) \mapsto V + A_x(X)v,\ T_aM \times \mathbb{R}^k \to \mathbb{R}^k.$$

Let $s_e : I \to \mathbb{R}^k$ be the function defined by

$$s(t) = \sum_{j=1}^{k} s_e^j(t) e_j(c(t)).$$

Then in the local trivialization s is given by $t \mapsto (c(t), s_e(t))$, so that $s'(t)$ is given by $(c'(t), s_e'(t))$. Hence, $(P_{s(t)} s'(t))_e = s_e'(t) + A_x(c'(t)) s_e(t)$ so that

$$(\frac{Ds}{dt}(t))_e = s_e'(t) + A_{c(t)}(c'(t)) s_e(t). \tag{9}$$

From this expression for Ds/dt relative to a frame, one readily deduces the following rules of calculation.

Lemma 6.4 *Let $c : I \to E$ be a C^1-curve, let $\sigma, \tau : I \to E$ be differentiable sections over c and let $f : I \to \mathbb{R}$ be a differentiable function. Then $f\sigma$ and $\sigma + \tau$ are differentiable sections over c and*

(a) $\frac{D}{dt}[\sigma + \tau] = \frac{D\sigma}{dt} + \frac{D\tau}{dt}$ *(additivity);*
(b) $\frac{D}{dt}[f\sigma] = f'\sigma + f\frac{D\sigma}{dt}$ *(Leibniz rule).*

Proof: With respect to a local frame $e_1, \ldots, e_k$ one has $[\sigma + \tau]_e = \sigma_e + \tau_e$ and $[f\sigma]_e = f\sigma_e$. Now use (9). This proof is essentially the same as the proof of Lemma 3.2 (c),(d). The affineness of H again plays a crucial role. $\square$

Exercise 6.5 Let $c : I \to M$ be a C^1-curve, and let s be a section of E over c. Show that for all $t_0 \in I$,

$$\frac{Ds}{dt}(t_0) = \frac{d}{dt}\Big|_{t=t_0} T_t^{-1}(s(t)),$$

where $T_t : E_{c(t_0)} \to E_{c(t)}$ denotes parallel transport along c. Hint: imitate the proof of Lemma 3.7.

7 Pseudo-Riemannian metrics

If V is a finite dimensional real linear space, then by a (not necessarily definite) inner product on V we mean a symmetric bilinear form $\beta : V \times V \to \mathbb{R}$ which is non-degenerate in the sense that $\beta(v, w) = 0$ for all $w \in V$ implies that $v = 0$. The inner product is said to be positive definite if $\beta(v, v) > 0$ for all non-zero $v \in V$.

An inner product β on V induces a linear map $V \to V^*$, $v \mapsto \beta(v, \cdot)$ which is also denoted by β. Non-degeneracy of the inner product means that $\beta : V \to V^*$ is injective. For dimensional reasons, β is also bijective. Thus we may use the linear isomorphism $\beta : V \to V^*$ to transfer the inner product β to an inner product β^* on V^*. In formula,

$$\beta^*(\beta(v), \beta(w)) = \beta(v, w), \qquad \forall v, w \in V.$$

The inner product β^* is called the dual inner product. If no confusion can arise from it, we will omit the star in the notation, and just write β for the dual inner product.

Exercise 7.1 Let $e_1, \ldots, e_n$ be a basis for V and let $e^1, \ldots, e^n$ be the dual basis for V^*, i.e., $e^i(e_j) = \delta^i_j$ for all i, j. Let $\beta_{ij} = \beta(e_i, e_j)$ and $\beta^{ij} = \beta(e^i, e^j)$. Show that the matrix β^{ij} is the inverse of the matrix β_{ij}. Hint: consider the matrix of the map $\beta : V \to V^*$ with respect to the basis and its dual.

Lemma 7.2 *Let β be an inner product on a finite dimensional real linear space V of dimension n. Then there exists a direct sum decomposition $V = V_+ \oplus V_-$, orthogonal with respect to β such that β is positive definite on V_+ and negative definite on V_-. The dimensions $p = \dim V_+$ and $q = \dim V_-$ are independent of the particular direct sum decomposition.*

The pair (p, q) in the above lemma is called the signature of the inner product β.

Proof: Let e_i be a basis of V and let e^i be the associated dual basis, as in the exercise preceding the lemma. Let $\varphi : V \to V^*$ be the linear isomorphism determined by $\varphi(e_i) = e^i$ for all $1 \leq i \leq n$. Then the linear map $h := \varphi^{-1}\beta : V \to V$ has matrix $(\beta(e_i, e_j)) = (\beta_{ij})$, which is symmetric. Hence h is symmetric with respect to the unique inner product on V for which the basis (e_i) is orthonormal. It follows that h diagonalizes with respect to an orthonormal basis. As h is invertible, non of the eigenvalues is zero. Let V_+ be the sum of the eigenspaces for the positive eigenvalues, and V_- the sum of the eigenspaces for the negative eigenvalues. Then $V = V_+ \oplus V_-$. Moreover, V_+ and V_- are orthogonal with respect to β.

Let P be any subspace of V on which β is positive definite. Then β is both positive definite and negative definite on $P \cap V_-$. This implies that $P \cap V_- = 0$ so that $\dim P \leq n - q = p$. It follows that p is the highest possible dimension of a subspace on which β is positive definite. Therefore, p is independent of the particular choice of decomposition. $\square$

In the following it will sometimes be used that the inner product β on V is uniquely determined by the quadratic form $f : v \mapsto \beta(v, v)$ on V. Indeed, by symmetry we have the identity

$$2\beta(v, w) = f(v + w) - f(v) - f(w).$$

By a quadratic form on V we mean a function $f : V \to \mathbb{R}$ which is homogeneous of degree 2, i.e., $f(\lambda v) = \lambda^2 v$ for all $v \in V$ and $\lambda \in \mathbb{R}$. Conversely, each quadratic form defines a symmetric bilinear form β by the above formula. The quadratic form is said to be non-degenerate if and only if the associated symmetric bilinear form is. We note that

$$f(v) = \beta(v, v) = \sum_{ij} \beta_{i,j} v^i v^j$$

which may also be expressed in terms of the dual basis as

$$f = \sum_{i,j} \beta_{ij} e^i e^j.$$

In this notation $e^i e^j$ stands for the product function $V \to \mathbb{R}$.

Definition 7.3 Let M be a smooth manifold. A (pseudo-)Riemannian metric on M is a family of inner products g_m on T_mM, for $m \in M$, depending smoothly on m in the sense that for all smooth vector fields $X, Y \in \mathfrak{X}(M)$ the scalar function $g(X,Y) : m \mapsto g_m(X_m, Y_m)$ is smooth. Equivalently, this means that g defines a smooth section of the tensor bundle $\otimes^2 T^*M$. The metric is said to be Riemannian if g_m is positive definite for all m. Otherwise, it is called pseudo-Riemannian. A manifold equipped with a (pseudo-)Riemannian metric is said to be a (pseudo-)Riemmanian manifold.

Let $x^1, \ldots, x^n$ be a local coordinate system, defined on an open subset U of M. Let $\partial_j = \partial/\partial x^j$ be the associated frame of vector fields over U and dx^j the associated frame of 1-forms. Let $g_{ij} = g(\partial_i, \partial_j)$, then viewing g as a section of the tensor bundle, we may write

$$g = \sum_{ij} g_{ij}\, dx^i \otimes dx^j \qquad \text{on } U.$$

In the literature one often encounters the local expression

$$g = \sum_{i,j} g_{ij}\, dx^i dx^j.$$

This expression makes sense if one interprets the symbol on the left-hand side as the quadratic form associated with the metric g.

Proposition 7.4 *Let M be a smooth manifold. Then there exists a Riemannian metric g on M.*

For the proof we will need the existence of a partition of unity. A partition of unity is a family of functions $\varphi_\alpha \in C_c^\infty(M)$, for α in some index set $\mathcal{A}$, such that

(a) $\varphi_\alpha \geq 0$ for each $\alpha \in \mathcal{A}$;
(b) the supports $\mathrm{supp}\varphi_\alpha$ form a locally finite family in the sense that for each compact $C \subset M$ the collection $\{\alpha \in \mathcal{A} \mid \mathrm{supp}\varphi_\alpha \cap C \neq \emptyset\}$ is finite;
(c) $\sum_{\alpha \in \mathcal{A}} \varphi_\alpha = 1$ (note that the sum is locally finite by (b)).

Let $\mathcal{U}$ be an open cover of M, i.e., a collection of open subsets of M such that the union $\bigcup \mathcal{U} = \cup_{U \in \mathcal{U}} U$ equals M. A partition of unity φ_α is said to be subordinate to the covering $\mathcal{U}$ if for each index α there exists an open subset $U_\alpha \in \mathcal{U}$ containing the support $\mathrm{supp}\,\varphi_\alpha$.

We need the following result for which we refer to the literature. For instance one may consult the book 'Differentiable manifolds' by S. Lang.

Proposition 7.5 *Let $\mathcal{U}$ be an open cover of M. Then there exists a partition of unity subordinate to $\mathcal{U}$.*

Proof of Proposition 7.4: Let $\mathcal{U}$ be a cover of M by coordinate charts. For $U \in \mathcal{U}$ select a system of coordinates $x^1, \ldots, x^n$ and write g_U for the associated standard metric $\sum_i dx^i dx^i$. Then $g_{U\,m}$ is positive definite at each $m \in U$.

Let φ_α, $\alpha \in \mathcal{A}$ be a partition of unity subordinate to the cover $\mathcal{U}$. For each $\alpha \in \mathcal{A}$ we select an open subset $U_\alpha \in \mathcal{U}$ such that $\mathrm{supp}\varphi_\alpha \subset U_\alpha$ and write $g_\alpha := g_{U_\alpha}$. We note that $\varphi_\alpha g_\alpha$ may be extended by zero outside U_α and thus becomes a smooth section of the tensor bundle $\otimes^2 T^*M$. Define the smooth section g of the same bundle by

$$g = \sum_{\alpha \in \mathcal{A}} \varphi_\alpha g_\alpha.$$

This definition makes sense, as the sum is locally finite. Note that g_m is a symmetric bilinear form at every point $m \in M$. We will finish the proof by showing that g_m is positive definite. Indeed, let $v \in T_m M$ be a non-zero vector, then

$$g(v, v) = \sum_{\alpha \in \mathcal{A}} \varphi_\alpha(m) g_{\alpha m}(v, v).$$

By positive definiteness of the $g_{\alpha m}$, all terms of this sum are non-negative. Moreover, from $\sum_\alpha \varphi_\alpha(m) = 1$ it follows that for some $\alpha \in \mathcal{A}$, $\varphi_\alpha(m) > 0$. For this α the associated term is positive. Therefore, $g(v, v) > 0$. $\square$

Remark 7.6 Note that the above proof uses positive definiteness in an essential way, and cannot be extended to give a similar result for pseudo-Riemannian metrics.

Let (M, g) be a Riemannian manifold, and let N be a smooth submanifold. For each $a \in N$ the restriction $g_{N,a}$ of g_a to $T_a N \subset T_a M$ is a positive definite inner product. Accordingly, g_N defines a Riemannian metric on the submanifold N, which is called the restriction of g to N. In general, for a pseudo-Riemannian metric g it may happen that for some $a \in N$ the restriction of g_a may be degenerate, hence does not define an inner product. The following example deals with a situation where the pseudo-Riemannian metric does restrict to a metric which turns out to be Riemannian.

Exercise 7.7 Let g be the standard pseudo-Riemannian metric of signature $(n, 1)$ on $\mathbb{R}^{n+1}$ given by

$$g = \sum_{i=1}^n dx^i dx^i - dx^{n+1} dx^{n+1}$$

Show that g restricts to a Riemannian metric g_N on the submanifold $N \subset \mathbb{R}^{n+1}$ given by the equation

$$\sum_{i=1}^n (x^i)^2 - (x^{n+1})^2 = -1, \qquad x^{n+1} > 0.$$

Show that the coordinate functions $x^1, \ldots, x^n$ restrict to a global coordinate system on N and express g_N in terms of the (restricted) one forms $dx^1, \ldots, dx^n$

on N. The manifold N equipped with this Riemannian metric is called n-dimensional hyperbolic space. Advice: first do the case $n = 2$.

Exercise 7.8 Let g be the standard pseudo-Riemannian metric of signature (p, q) on $\mathbb{R}^n$, $n = p + q$, $p \geq 2$, given by

$$g = \sum_{i=1}^{p} dx^i dx^i - \sum_{i=p+1}^{n} dx^i dx^i.$$

Let N be the submanifold of $\mathbb{R}^n$ determined by the equation

$$\sum_{i=1}^{p} (x^i)^2 - \sum_{i=p+1}^{n} (x^i)^2 = 1$$

Show that g restricts to a pseudo-Riemannian metric of signature $(p-1, q)$ on N. The resulting pseudo-Riemannian manifold is called pseudo-hyperbolic space of signature $(p-1, q)$.

Definition 7.9 Let g be a (pseudo-)Riemannian metric on M and let ∇ be a connection on the tangent bundle TM. The metric g is said to be covariantly constant, or flat, with respect to ∇ if

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Exercise 7.10 Let ∇ a connection (affine) on TM, and let g be a pseudo-Riemannian metric on M. Show that the following assertions are equivalent.

(a) The metric g is covariantly constant with respect to ∇;

(b) for any smooth curve $c : [0, 1] \to M$ the parallel transport $T_t = T_{c,t,0} : T_{c(0)}M \to T_{c(t)}M$ is isometric (relative to $g_{c(0)}$ and $g_{c(t)}$).

The following instructions should guide you through the exercise. We advise the reader to recall the results of Section 6.

(1) First assume (b). Let X, Y, Z be smooth vector fields on M and let $a \in M$. Let c be any smooth curve with $c'(0) = Z_a$ and use Lemma 3.7 to differentiate

$$g_{c(0)}(T_t^{-1} X(c(t)), T_t^{-1} Y(c(t)) - g_{c(t)}(X(c(t)), Y(c(t))$$

at $t = 0$.

Now assume (a). Let $c : [0, 1] \to M$ be any smooth curve. Our goal is to prove

$$\frac{d}{dt} g_{c(t)}(X_t, Y_t) = g_{c(t)}(\frac{D}{dt} X_t, Y_t) + g_{c(t)}(X_t, \frac{D}{dt} Y_t)$$

for all smooth vector fields $t \mapsto X_t, Y_t$ over c, i.e., $X_t, Y_t \in T_{c(t)}M$.

(2) First prove the identity for $X_t = X(c(t))$ and $Y_t = Y(c(t))$, with $X, Y \in \mathfrak{X}(M)$.

(3) Next prove the identity for $X_t = f(t)X(c(t))$, $Y_t = Y(c(t))$.

(4) By using a frame to decompose X_t, prove the identity without condition on $t \mapsto X_t$, and with $Y_t = Y(c(t))$.

(5) Prove the identity in general.

(6) Establish (b) by selecting $X_0, Y_0 \in T_{c(0)}M$ and applying the identity to the vector fields $X_t = T_t X_0$ and $Y_t = T_t Y_0$.

We will show that given a metric g, there always exists a connection for which the metric is flat. The connection will turn out to be unique if we require its so-called torsion to vanish.

Definition 7.11 Let ∇ be a connection on TM. Its torsion form is the map $T = T^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

for all $X, Y \in \mathfrak{X}(M)$. The connection ∇ is said to be torsion free (or symmetric) if the associated torsion form T^∇ equals zero.

Exercise 7.12 Show that the torsion form T defines a tensor (called the torsion tensor). By this we mean that for each $a \in T_aM$, the vector $T(X, Y)_a \in T_aM$ only depends on the vector fields X, Y through their values X_a, Y_a. Thus, T determines a bilinear map $T_a : T_aM \times T_aM \to T_aM$ depending smoothly on $a \in M$ and may be viewed as a section of the tensor bundle $\mathcal{T}_M^{1,2}$.

Theorem 7.13 *Let g be a (pseudo-)Riemannian metric on the manifold M. There exists a unique torsion free tensor ∇ for which g is covariantly constant.*

This connection is called the Levi-Civita connection associated with the metric g.

Proof: Let ∇ be a connection with the asserted properties. Then by flatness of the metric, it follows that, for smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned} &Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad = \; g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X). \end{aligned}$$

By symmetry of the connection it follows that the expression on the right-hand side equals

$$2g(\nabla_X Y, Z) + g([Y, X], Z) + g([X, Z], Y) + g([Y, Z], X),$$

so that

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & -g([Y, X], Z) - g([X, Z], Y) - g([Y, Z], X). \end{aligned} \tag{10}$$

It follows that ∇ is uniquely determined by the metric. Conversely, the latter formula may be used to define ∇ as a map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. It is a straightforward but somewhat tedious exercise to show that ∇ thus defined is a torsion free connection for which the metric is flat. $\square$

Let ∇ be any connection on the tangent bundle TM, and let $x^1, \dots, x^n$ be a system of local coordinates, defined on an open subset $U \subset M$. Then the smooth functions $\Gamma^k_{ij} \in C^\infty(U)$ determined by

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^n \Gamma^k_{ij}\, \partial_k$$

are called the Christoffel symbols of the connection with respect to the local coordinate system. Let X, Y be two smooth vector fields on U. Let X^i and Y^i be the component functions defined by

$$X = \sum_i X^i \partial_i, \qquad \text{and} \qquad Y = \sum_i Y^i \partial_i.$$

Then in terms of the Christoffel symbols the components of $\nabla_X Y$ are given by

$$(\nabla_X Y)^k = \sum_i X^i \partial_i Y^k + \sum_{i,j} \Gamma^k_{ij}\, X^i Y^j.$$

Exercise 7.14 Verify this.

Let us return to the situation of a (pseudo-)Riemannian metric g on M and the associated Levi-Civita connection ∇. It follows from (10) that the Christoffel symbols with respect to a local coordinate system $x^1, \dots, x^n$ may be expressed in terms of the components of the metric. Indeed, writing

$$g = \sum_{ij} g_{ij}\, dx^i \otimes dx^j$$

and applying (10) with $X = \partial_i, Y = \partial_j\ Z = \partial_l$, it follows that

$$2 \sum_k g_{kl} \Gamma^k_{ij} = \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}$$

(note that the vector fields $\partial_i, \partial_j, \partial_l$ commute). Writing (g^{lk}) for the inverse of the matrix (g_{kl}) (this is the matrix of the dual inner product), we find

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l g^{lk} [\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}].$$

Note that the expression on the right-hand side, and hence the Christoffel symbol on the left-hand side, is symmetric in i, j. This reflects the symmetry of the connection.

Exercise 7.15 Let g the standard metric of signature (p, q) on $\mathbb{R}^n$, $n = p + q$. Thus,

$$g = \sum_{i=1}^p dx^i dx^i - \sum_{i=p+1}^n dx^i dx^i.$$

Show that the Levi-Civita connection associated with g is given by

$$\nabla_X Y = \sum_{i,j} X^i \partial_i Y^j \partial_j = \sum_j X(Y^j) \partial_j.$$

The curvature form R associated with the Levi-Civita connection of the (pseudo-)Riemannian manifold (M, g) is also called the Riemannian curvature tensor of (M, g). The following For each $a \in M$ and all $X_a, Y_a \in T_aM$, the evaluation $R_a(X_a, Y_a)$ defines an element of $\mathrm{End}(T_aM)$, which depends on X_a, Y_a in an alternating fashion.

Lemma 7.16 *Let R be the curvature tensor for a (pseudo-)Riemannian manifold (M, g). Then for all smooth vector fields X, Y, Z, W on M,*

(a) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ *(Bianchi identity);*

(b) $g(R(X,Y)Z, W) = -g(Z, R(X,Y)W)$ *(anti-symmetry);*

(c) $g(R(X,Y)Z, W) = g(R(Z,W)X, Y)$.

Proof: Assertion (a) is a straightforward consequence of the symmetry of the Levi-Civita connection ∇. Assertion (b) may also be expressed as anti-symmetry of the form $(Z, W) \mapsto g(R(X,Y)Z, W)$. For this it suffices to establish the identity $g(R(X,Y)Z, Z) = 0$ for all Z. By flatness of the metric it follows that

$$XYg(Z,Z) = 2X(g(\nabla_Y Z, Z)) = 2g(\nabla_X\nabla_Y Z, Z) + 2g(\nabla_Y Z, \nabla_X Z)$$

and

$$YXg(Z,Z) = 2X(g(\nabla_X Z, Z)) = 2g(\nabla_Y\nabla_X Z, Z) + 2g(\nabla_X Z, \nabla_Y Z).$$

Subtracting these identities we obtain

$$[X,Y]g(Z,Z) = 2g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z).$$

By flatness of the metric the expression on the left-hand side may be rewritten as $2g(\nabla_{[X,Y]}Z, Z)$ and (b) follows.

Assertion (c) can be obtained from (a) and (b) in the following way. It follows from (a) that

$$g(R(X,Y)Z, W) = -g(R(Y,Z)X, W) - g(R(Z,X)Y, W).$$

Adding the similar equations with (X, Y, Z, W) replaced by $(-Z, X, Y, W)$, (W, Z, X, Y) and $(-X, W, Z, Y)$ and using the two anti-symmetry properties of R, we obtain

$$2g(R(X,Y)Z, W) + 2g(R(W,Z)X, Y) = 0.$$

Using $R(W,Z) = -R(Z,W)$ we obtain (c). $\square$

8 Isometries and curvature

Definition 8.1 Let (M, g) and (N, h) be (pseudo-)Riemannnian manifolds. An isometry from M onto N is defined to be diffeomorphism f from M onto N such that

$$h_{f(x)}(df(x)v, df(x)w) = g_x(v, w), \tag{11}$$

for all $x \in M$ and $v, w \in T_xM$.

Let now (M, g) be a pseudo-Riemannian manifold of signature (p, q), and let ∇ be the associated Levi-Civita connection on the tangent bundle TM. The associated curvature form now is the tensor $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$(X, Y, Z) \mapsto R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

For p, q nonnegative integers, let $\mathbb{E}^{p,q}$ be the space $\mathbb{R}^{p+q}$ equipped with the standard metric of signature (p, q) :

$$g = \sum_{i=1}^{p} dx^i dx^i - \sum_{i=p+1}^{p+q} dx^i dx^i.$$

In this case it follows from Exercise 7.15 that the associated curvature tensor vanishes identically. We will use Theorem 4.5 to show that R is the sole obstruction to M being locally isometric to the flat space $\mathbb{E}^{p,q}$.

Theorem 8.2 *Let (M, g) be a pseudo-Riemannian manifold of signature (p, q). Then the following assertions are equivalent.*

(a) *(M, g) is locally isometric to $\mathbb{E}^{p,q}$.*

(b) $R = 0$.

Proof: It remains to show that (b) implies (a). Assume (b) and let $a \in M$. The tangent space T_a admits a direct sum decomposition $T_a M = V_+ \oplus V_-$ such that V_+ and V_- are orthogonal, and such that g_a is positive definite on V_+ and negative definite on V_-. It follows from Lemma 7.2 that $\dim V_+ = p$ and $\dim V_- = q$. As g_a is positive definite on V_+ there exists a basis $v_1, \ldots, v_p$ of V_+ such that $g_a(v_i, v_j) = \delta_{ij}$ for all $1 \leq i, j \leq p$. Similarly, there exists a basis $v_{p+1}, \ldots, v_n$, $n = p + q$, such that $g_a(v_i, v_j) = -\delta_{ij}$ for all $q + 1 \leq i, j \leq n$. Put $\epsilon_i = 1$ for $1 \leq i \leq p$ and $\epsilon_i = -1$ for all $p + 1 \leq i \leq n$. Then it follows that

$$g_a(v_i, v_j) = \epsilon_i \delta_{ij}, \qquad (1 \leq i, j \leq n).$$

It follows from the proof of Theorem 4.5 that there exists a connected open neighborhood U of a in M and smooth vector fields $X_1, \ldots, X_n$ on U such that for each $1 \leq i \leq n$

(a) the vector field X_i is flat, i.e., $\nabla_Z X_i = 0$ for all $Z \in \mathfrak{X}(U)$;

(b) $X_i(a) = v_i$.

In fact, in view of parallel transport, the vector fields X_i are uniquely determined by these properties. By flatness of the metric, it follows that for all $Z \in \mathfrak{X}(U)$,

$$Zg(X_i, X_j) = g(\nabla_Z X_i, X_j) + g(X_i, \nabla_Z X_j) = 0.$$

Hence, the scalar function $g(X_i, X_j)$ is constant on U. By evaluation in (a), we see that

$$g(X_i, X_j) = \epsilon_i \delta_{ij}, \qquad (1 \leq i, j \leq n).$$

By symmetry of the connection, it follows that

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0, \qquad (1 \leq i, j \leq n).$$

Thus, the vector fields X_i commute with each other. Let $t \mapsto e^{tX_i}$ denote the flow of the vector field X_i. Then it is well known that there exists a $\delta > 0$ such that the map

$$\varphi : (t_1, \ldots, t_n) \mapsto e^{t_1 X_1} \circ \cdots \circ e^{t_n X_n}(a)$$

defines a diffeomorphism from $(-\delta, \delta)^n$ onto an open neighborhood of a in M. Adapting choices we may assume this neighborhood to be U. Obviously, $\varphi^*(X_j) = \partial_j$. Hence $\varphi^* g$ is the standard metric of signature (p, q) on $(-\delta, \delta)^n$. This that φ is an isometric isomorphism from U onto the open subset $(-\delta, \delta)^n$ of $\mathbb{E}^{p,q}$. Assertion (a) follows. $\square$

9 Geodesics and the exponential map

We assume that M is a manifold whose tangent bundle $p : TM \to M$ is equipped with a torsion free (or symmetric) connection ∇. Of course the Levi-Civita connection associated to a pseudo-Riemannian metric on M is a motivating example of such a connection.

Let now $c : I \to M$ be a C^2-curve. Then $c' : I \to TM$ may be viewed as a C^1-vector field over c. We define D/dt as in Section 6.

Definition 9.1 A geodesic relative to ∇ is defined to be a C^2-curve $\gamma : I \to M$ such that

$$\frac{D\gamma'}{dt} = 0. \tag{12}$$

Remark 9.2 Equivalently, this means that γ' is the horizontal lift of γ. Thus, if T_{t,t_0} denotes the parallel transport along γ from t_0 to t then it follows that $T_{t,t_0}\gamma'(t_0) = \gamma'(t)$. Since parallel transport is an isometry for the metrics $g_{\gamma(t_0)}$ and $g_{\gamma(t)}$, it follows that

$$g_{\gamma(t)}(\gamma'(t), \gamma'(t)) = \text{constant}, \qquad (t \in I).$$

Exercise 9.3 Let $\gamma : I \to M$ be a geodesic and let $\tau \in \mathbb{R}$. Show that the curve $c : \tau + I \to M$ given by $c(t) = \gamma(t - \tau)$ is a geodesic again.

Let $x^1, \ldots, x^n$ be a local coordinate system on a coordinate patch U of M. The associated vector fields $\partial_1, \ldots, \partial_n$ form a frame of TM over U. The associated one-form A on U with values in $M_n(\mathbb{R})$ is given by

$$\nabla_X(\partial_j) = \sum_{k=1}^{n} A(X)_j^k e_k.$$

It follows that the Christoffel symbols may be expressed in terms of A by

$$\Gamma_{ij}^k = A(\partial_i)_j^k.$$

Using that $dx^1, \ldots, dx^n$ is dual to $\partial_1, \ldots, \partial_n$, we see that conversely

$$A_j^k = \sum_i \Gamma_{ij}^k \, dx^i.$$

Put $\gamma^k := x^k \circ \gamma$. Then the k-th component of γ' with respect to the local frame $\partial_1, \ldots, \partial_n$ is given by the derivative $\dot{\gamma}^k = d\gamma^k/\gamma^k$. Accordingly, relative to the local frame, the components of $D\gamma'/dt$ are given by

$$\frac{D\gamma'}{dt}(t)^k = \ddot{\gamma}^k(t) + \sum_j A_{\gamma(t)}(\gamma'(t))_j^k \, \dot{\gamma}^j(t).$$

Accordingly, the geodesic equation (12) becomes

$$\ddot{\gamma}^k(t) + \sum_i \Gamma_{ij}^k(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0, \qquad (1 \leq k \leq n).$$

This is a system of n non-linear second order differential equations, which may be rewritten in the form

$$v'(t) = F(v(t)),$$

where $v(t)$ is the column vector with components γ^k and $\dot{\gamma}^k$, and where F is a smooth vector field on $x(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. It follows from the theory of ordinary differential equations that this system may be solved locally and uniquely, provided the initial data $\gamma(0)$ and $\dot{\gamma}(0)$ are given. Moreover, the local solution is smooth, and depends smoothly on the initial data. Applying this result everywhere locally, one obtains the following result.

Lemma 9.4 *Let $a \in M$ and $v \in T_aM$. Then there exists a unique smooth curve γ_v in M with domain an open interval I_v containing zero such that*

(a) *γ_v is a geodesic with $\gamma_v(0) = a$ and $\gamma_v'(0) = v$;*

(b) *if $\gamma : I \to M$ is any geodesic with $\gamma(0) = a$ and $\gamma'(0) = v$, then $I \subset I_v$ and $\gamma = \gamma_v|_I$.*

Any linear reparametrization of a geodesic is a geodesic again. More precisely, the following result is valid.

Lemma 9.5 *Let $v \in TM$ and $s \in \mathbb{R}$. Then $I_{sv} = s^{-1}I_v$. Moreover,*

$$\gamma_{sv}(t) = \gamma_v(st), \qquad (t \in s^{-1}I_v).$$

Proof: The result is trivial for $s = 0$, provided we agree that $s^{-1}I_v = \mathbb{R}$ in that case. Thus, assume $s \neq 0$.

Let $a = p(v)$. Then $v, sv \in T_aM$. The curve $\gamma(t) = \gamma_v(st)$ is well defined on $s^{-1}I_v$. Moreover, $\gamma'(t) = s\gamma_v(st)$ and $\gamma''(t) = s^2\gamma_v(st)$. From the local expression of the geodesic equation, which is linear in γ'' and quadratic in γ', one sees that γ is a geodesic. Moreover, $\gamma(0) = a$ and $\gamma'(0) = s\gamma_v'(0) = sv$. It follows that $s^{-1}I_v \subset I_{sv}$ and that $\gamma = \gamma_{sv}$ on $s^{-1}I_v$. The inclusion $s^{-1}I_v \supset I_{sv}$ is obtained by a similar reasoning, with sv, s^{-1} in place of v, s. $\square$

In view of the above, the smooth dependence on initial values leads to the following result.

Proposition 9.6 *The set $\Omega = \{v \in TM \mid 1 \in I_v\}$ is an open neighborhood of the zero section in TM. Moreover, the map $[0,1] \times \Omega \to M$, $(t,v) \mapsto \gamma_v(t)$ is smooth.*

Definition 9.7 With Ω defined as in the preceding proposition, we define the exponential map $\exp : \Omega \to M$ by $\exp(v) = \gamma_v(1)$. Given $a \in M$ we define $\exp_a$ to be the restriction of exp to $\Omega_a = \Omega \cap T_aM$.

The preceding discussion leads to the following result.

Lemma 9.8 *Let $a \in M$ and $v \in T_aM$. Then $I_v = \{t \in \mathbb{R} \mid tv \in \Omega\}$. Moreover,*

$$\gamma_v(t) = \exp(tv), \qquad (t \in I_v).$$

Proof: We note that $tv \in \Omega$ if and only if $1 \in I_{tv}$, which in turn is equivalent to $t \in I_v$, by Lemma 9.5. Moreover, by the same lemma, if the latter condition is fulfilled then $\exp(tv) = \gamma_{tv}(1) = \gamma_v(t)$. $\square$

Proposition 9.9 *Let $a \in M$. Then the exponential map $\exp_a : \Omega_a \to M$ restricts to a diffeomorphism from an open neighborhood of the origin in T_aM onto an open neighborhood of a in M. Viewed as a map from $T_aM \simeq T_0T_aM$ to a map T_aM the derivative $d\exp_a(0)$ equals the identity.*

Proof: The domain of $\exp_a$ is an open neighborhood of the origing in T_aM. Let $v \in T_aM$. Then $\gamma_v(t) = \exp_a(tv)$ for all $t \in \mathbb{R}$ sufficiently close to zero. By differentiation with respect to t at $t = 0$ it follows that

$$d\exp_a(0)v = \left.\frac{d}{dt}\right|_{t=0} \exp_a(tv) = \left.\frac{d}{dt}\right|_{t=0} \gamma_v(t) = v.$$

The result now follows by application of the inverse function theorem. $\square$

We end this section with a result on the possibility to locally connect points by geodesics.

Proposition 9.10 *Let (M,g) be a pseudo-Riemannian manifold and let $a \in M$. Let V be an open neighborhood of a equipped with a Riemannian metric h. There exist a constant $\delta > 0$ and an open neighborhood U of a in V such that:*

(a) *for every $p \in U$ the exponentional map $\exp_p$ maps the open h_p-ball of radius δ diffeomorphically onto an open neighborhood of p containing U;*

(b) *for each pair of distinct points $p, q \in U$ there exists a unique geodesic $\gamma : [0,b] \to M$ with $\gamma(0) = p$, $\gamma(b) = q$ and $h(\gamma'(0), \gamma'(0)) = 1$, $0 \leq b < \delta$.*

Proof: From the implicit function theorem it follows that there exist an open neighborhood U of a and an open neighborhood ω of V in TM such that for all $p \in \omega \cap M$ the exponential map exp maps $T_p \cap \omega$ diffeomorphically onto an open subset of M containing U. Let V_0 be a compact neighborhood of a in V. Then there exists a constant $\delta > 0$ such that for each $p \in V_0$ the h_p-ball $B_p(0;\delta)$ of radius δ in T_pM is contained in ω. Replacing U by a smaller neighborhood we may assume that $U \subset V_0$. Now (a) follows.

For (b), assume that $p, q \in U$ are distinct points. Then $q = \exp_p(v)$ for a unique vector $v \in B_p(0;\delta)$. There exists a unique vector $w \in T_pM$ of h_p-length 1 such that $bw = v$ for some $0 < b < \delta$. Then $q = \exp_p(v) = \gamma_v(1) = \gamma_w(b)$. Thus $\gamma = \gamma_w$ satisfies the requirement.

Conversely, let $\gamma : [0, r] \to M$ be a geodesic as stated. Then $\gamma(t) = \exp_p(tw)$, for all $0 \leq t < \delta$, where $w = \gamma'(0)$. By injectivity of $\exp_p$ on $B_p(0;\delta)$, the equality $\exp_p(bw) = \gamma(b) = q = \exp_p(v)$ implies that $bw = v$. Hence, $\gamma = \gamma_v$ and uniqueness of γ follows. $\square$

10 Riemannian distance

We now assume that (M, g) is a Riemannian manifold. Then g_x is a positive definite inner product on T_xM, for every $x \in M$. The associated norm is denoted by

$$\|v\|_x := \sqrt{g_x(v, v)}, \qquad (v \in T_xM).$$

A curve $c : [a, b] \to M$ is said to be piecewise C^1 if c is continuous and there exists a partition $a = t_0 < t_1 < \cdots < t_k = b$ of the interval such that for each $1 \leq j \leq k$ the restriction $c|_{(t_{j-1}, t_j)}$ extends to a C^1-curve $[t_{j-1}, t_j] \to M$.

If $c : [a, b] \to M$ is a piecewise C^1-curve, we define its length $L(c)$ by

$$L(c) = \int_a^b \|c'(t)\|_{c(t)}\, dt.$$

If $\varphi : [\alpha, \beta] \to [a, b]$ is a surjective piecewise C^1-map with $\varphi' \geq 0$ (or with $\varphi \leq 0$) in all but finitely many points, then by the substitution rule for integration, it follows that

$$L(c) = L(c \circ \varphi).$$

In particular, it follows that L is invariant under a piecewise C^1-reparametrization.

If $\varphi : (M, g) \to (N, h)$ is an isometric isomorphism of Riemannian manifolds, then for all $a \in M$ and $v \in T_aM$ we have $\|d\varphi(a)v\|_{\varphi(a)} = \|v\|_a$. From this it readily follows that $L(\varphi \circ c) = L(c)$ for any C^1-curve in M.

Given two points $p, q \in M$ we define the distance $d(p, q)$ from p to q to be the infimum

$$d(p, q) = \inf_c L(c),$$

as c varies over all curves as above with $c(a) = p$ and $c(b) = q$. If $\varphi : (M, g) \to (N, h)$ is an isometric isomorphism, it is readily verified that $d_N(\varphi(p), \varphi(q)) = d_M(p, q)$, for all $p, q \in M$.

Lemma 10.1 *(M, d) is a metric space whose underlying topology coincides with that of M.*

Proof: First, since $L(c) \geq 0$ for all curves, it follows that $d \geq 0$. The function d is clearly symmetric, and it is readily seen to verify the triangle inequality. To see that it is a metric in the sense of topology, we must show that $d(p, q) > 0$ when p and q are distinct. For this we select a chart (κ, U) with U an open neighborhood of p in M and κ a diffeomorphism of U onto an open ball of center 0 in $\mathbb{R}^n$ such that $\kappa(p) = 0$. Let h denote the Riemannian metric on $\kappa(U)$ for which κ becomes an isometry. For $x \in U$, put

$$m(x) = \min_{\|v\|=1} h_{\kappa(x)}(v, v), \quad M(x) = \max_{\|v\|=1} h_{\kappa(x)}(v, v).$$

Replacing U by a smaller neighborhood if necessary, we may assume that there exists a constant $C > 0$ such that $C^2 < m(p)$ and $C^{-2} > M(p)$ for all $p \in U$. By homogeneity it follows that

$$C\|v\| \leq \|v\|_{h_{\kappa(x)}} \leq C^{-1}\|v\|$$

for all $x \in U$ and $v \in \mathbb{R}^n$. Given a piecewise C^1-curve c in $\mathbb{R}^n$ let $l(c)$ denote its Euclidean length. Then it follows from the above that for any piecewise C^1-curve c in U we have

$$Cl(\kappa \circ c) \leq L_h(\kappa \circ c) = L(c) \leq C^{-1} l(\kappa \circ c). \tag{13}$$

Here L_h denotes the length relative to the metric h, and L denotes the length relative to g. Let $r > 0$ be such that the closed ball $\bar{B} := \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ is contained in $\kappa(U)$. Let S be its boundary, the sphere of center 0 and radius r. Replacing r by a smaller constant if necessary, we may assume that $q \notin \kappa^{-1}(U)$. If $c : [a, b] \to M$ is a piecewise C^1-curve connecting p and q, let t_0 be the infimum of $t \in [a, b]$ such that $c(t) \notin \kappa^{-1}(\bar{B})$. Then it follows that $c(t_0) \in \kappa^{-1}(S)$. Since $\kappa \circ c|_{[a,t_0]}$ is a curve connecting 0 and a point of S we conclude that its Euclidean length is at least r. It follows that

$$L(c) \geq L(c|_{[a,t_0]}) = L_h(\kappa \circ c|_{[a,t_0]}) \geq Cr.$$

Since this is true for any curve connecting p and q, we conclude that $d(p, q) \geq Cr > 0$. It follows that d is a metric in the sense of topology.

For any $\delta > 0$ we denote by $B(p; \delta)$ the set of $x \in M$ with $d(p, x) < \delta$. By $B'(p, \delta)$ we denote the set of $x \in U$ with $\|\kappa(x)\| < \delta$. We will complete the proof by showing that the system of neighborhoods $B(p, \delta)$, $\delta > 0$, is equivalent to the system of neighborhoods $B'(p, \delta)$, $\delta > 0$.

Let $\epsilon < r$ and let $x \in B(p; C\epsilon)$. Then there exists a curve c in M with initial point p and end point x, such that $L(c) < C\epsilon$. By what we showed earlier, the curve c must be contained entirely in $\bar{B}$. It follows that

$$L(c) = L_h(\kappa \circ c) \geq Cl(\kappa \circ c) \geq C\|\kappa(x)\|,$$

so that $x \in B(p; \epsilon)$. This establishes the inclusion $B(p; C\epsilon) \subset B'(p; \epsilon)$ for all $\epsilon < r$.

Conversely, let $\epsilon > 0$ and let $x \in B'(p; C\epsilon)$. Let c be the curve in U such that $\kappa \circ c$ is the straight line connecting $\kappa(p) = 0$ and $\kappa(x)$. Then

$$\|\kappa(x)\| = l(\kappa \circ c) \geq CL_h(\kappa \circ c) = CL(c) \geq Cd(p, x).$$

It follows that $x \in B(p; \epsilon)$. This establishes the inclusion $B'(p; C\epsilon) \subset B(p; \epsilon)$ for all $\epsilon > 0$. $\square$

Lemma 10.2 (Gauß lemma) *Let M be a Riemannian manifold and $p \in M$. Let $\delta > 0$ be such that $\exp_p$ is a diffeomorphism from $B_{g_p}(0; \delta)$ onto an open neighborhood of p in M. Let S be the unit sphere in T_pM and let $\varphi : [0, \delta) \times S \to M$ be defined by $\varphi(r, \sigma) = \exp_p(r\sigma)$. Then*

$$\varphi^*(g) = dr^2 + r^2 h_r,$$

where h_r is a Riemannian metric on S and $\lim_{r \downarrow 0} h_r = g_p|_S$.

Proof: We observe that the map φ restricts to a diffeomorphism from $D := (0, \delta) \times S$ onto a punctured neighborhood of p in M. Let $\partial/\partial r$ be the vector field on $(0, \delta) \times S$ for which the associated derivation is partial differentiation with respect to the first variable. Let v be a vector field on S and let $\tilde{v} := (0, v)$ denote the associated vector field on $(0, \delta) \times S$. Then $\partial/\partial r$ and $\tilde{v}$ commute, hence so do their images R and V under φ. By symmetry of the Levi-Civita connection associated to g it follows that $\nabla_R V = \nabla_V R$.

We note that for $(r, \sigma) \in D$,

$$R(r, \sigma) = d\varphi(r, s) \left(\frac{\partial}{\partial r} \right)_{r, \sigma} = \frac{\partial}{\partial r} \varphi(r\sigma) = \dot{\gamma}_\sigma(r),$$

so that $\nabla_R R = 0$ on $\varphi(D)$. We see that

$$\begin{aligned} \varphi^*(g)(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})_{(r,\sigma)} &= g(R, R)_{\varphi(r,\sigma)} \\ &= g_{\gamma_\sigma(r)}(\dot{\gamma}_\sigma(r), \dot{\gamma}_\sigma(r)) \\ &= g_p(\sigma, \sigma) = 1. \end{aligned} \tag{14}$$

Also, for $(r, \sigma) \in D$,

$$V(r, \sigma) = d\varphi(r, s)(0, v) = rd(\exp_p)(r\sigma)v,$$

so that

$$\varphi^*(g)(\tilde{v}, \tilde{v})_{(r,\sigma)} = g(V, V)_{\varphi(r,\sigma)} = r^2\, h_r(v, v)_\sigma, \tag{15}$$

where

$$h_r(v, v)_\sigma = g_{\varphi(r,\sigma)}(d\exp_p(r\sigma)v, d\exp_p(r\sigma)v).$$

It remains to consider the cross term

$$(\varphi^* g)(\partial/\partial r, \tilde{v}) = \varphi^*(g(R, V)). \tag{16}$$

For this we note that

$$\begin{aligned} Rg(R,V) &= g(\nabla_R R, V) + g(R, \nabla_R V) \\ &= 0 + g(R, \nabla_V R) \\ &= \frac{1}{2} V g(R,R) = 0. \end{aligned}$$

This implies that $g(R,V)_{\varphi(r,\sigma)}$ is constant as a function of r. Now

$$g(R,V)_{\varphi(r,\sigma)} = r g_{\varphi(r,\sigma)}(\dot{\gamma}_\sigma(r), d\exp_p(r\sigma)) \to 0$$

as $r \downarrow 0$. Hence,

$$g(R,V) = 0. \tag{17}$$

The result follows from combining (14), (15), (16) and (17). $\square$

Corollary 10.3 *Let M, p, δ, S be as above. Then for every $(r,\sigma) \in [0,\delta) \times S$,*

$$d(p, \exp_p(r\sigma)) = r. \tag{18}$$

Let c be any piecewise C^1-curve of length r with endpoints p and $\exp_p(r\sigma)$. Then c is a piecewise C^1-reparametrization of the geodesic $\gamma_\sigma : [0,r] \to M$.

Proof: The geodesic $\gamma_\sigma : [0,r] \to M$ has tangent vector of length

$$g_p(\dot{\gamma}_\sigma(0), \dot{\gamma}_\sigma(0)) = g_p(\sigma,\sigma) = 1$$

(see Remark 9.2). It follows that $L(\gamma_\sigma) = r$, so that $d(p, \exp_p(r\sigma)) \leq r$.

On the other hand, let $c : [\tau_1, \tau_2] \to M$ be any piecewise C^1-curve connecting p and $q := \exp_p(r\sigma)$. Let a be the supremum of $\tau \in [\tau_1, \tau_2]$ such that $c(\tau) = p$. Then $c(a) = p$ and $c(\tau) \neq p$ for $\tau > a$. Let b be the supremum of $\tau \in [a, \tau_2]$ such that $c(\tau) \in \exp_p(B(0,r))$. Then $c(b) = \exp_p(r\sigma')$ for a $\sigma' \in S$. There exist unique piecewise C^1-functions $r : (a,b] \to (0,\delta)$ and $\sigma : (a,b] \to S$, such that $c(t) = \exp_p(r(t)\sigma(t))$ for $t \in (a,b]$. Hence

$$\begin{aligned} L(c) &= \int_{\tau_0}^{\tau_1} \|c'(t)\|_{c(t)} \, dt \\ &\geq \int_a^b \|c'(t)\|_{c(t)} \, dt \\ &= \int_a^b \sqrt{|r'(t)|^2 + r(t)^2 h_r(\sigma'(t), \sigma'(t))} \, dt \\ &\geq \int_a^b |r'(t)| \, dt \geq \int_a^b r'(t) \, dt \\ &= r(b) - r(a). \end{aligned}$$

The latter expression equals $r - 0 = r$. Hence, $d(p,q) \geq L(c) \geq r$. This establishes (18). Now assume that the length of c above equals r. Then all inequalities must be equalities. Hence (except for possibly finitely many points) $c' = 0$ outside $[a,b]$, $\sigma' = 0$, and $r'(t) \geq 0$, so that $c(t) = \exp_p(r(t)\sigma) = \gamma_\sigma(r(t))$ is a piecewise C^1-reparametrization of $\gamma_\sigma : [0,r] \to M$. $\square$

A piecewise C^1-curve $c : I \to M$ is said to be parametrized by arc length if $I = [0, r]$ for some $r \geq 0$ and if $L(c|_{[0,t]}) = t$ for all $t \in [0, r]$. Note that the last condition is equivalent to $\|c'(t)\|_{c(t)} = 1$ for all $t \in [0, r]$.

Let $\gamma : [a, b] \to M$ be a (non-constant) geodesic. Then $\|\gamma'(t)\|_{\gamma(t)}$ equals a constant $r > 0$ (see Remark 9.2). The reparametrization $\tilde{\gamma} : [0, r(b-a)] \to M$ defined by

$$\tilde{\gamma}(t) = \gamma(a + r^{-1}t)$$

is again a geodesic. As this geodesic is parametrized by arc length, it is said to be the reparametrization of γ by arc length.

Corollary 10.4 *Let (M, g) be a Riemannian manifold and let $a \in M$. There exist a constant $\delta > 0$ and an open neighborhood U of a in M such that:*

(a) *for every $p \in U$ the exponentional map $\exp_p$ maps the open ball of radius δ in T_pM diffeomorphically onto an open neighborhood of p containing U;*

(b) *for each pair of distinct points $p, q \in U$ there exists a unique geodesic $\gamma_{p,q}$, parametrized by arc length and of length less than δ, with initial point p and end point q;*

(c) *the length of $\gamma_{p,q}$ equals $d(p, q)$.*

Proof: Applying Proposition 9.10 with $V = M$ and $h = g$, we obtain U such that (a) and (b) are valid. Finally, (c) follows from Corollary 10.3. $\square$

Corollary 10.5 *Let $c : [a, b] \to M$ be piecewise C^1-curve such that $L(c) = d(c(a), c(b))$. Then the reparametrization of c by arc length is a geodesic.*

Proof: Let $a < \alpha < \beta < b$. Then by the triangle inequality for d,

$$\begin{aligned} L(c) &= d(c(a), c(b)) \leq d(c(a), c(\alpha)) + d(c(\alpha), c(\beta)) + d(c(\beta), c(b)) \\ &= L(c|_{[a,\alpha]}) + L(c|_{[\alpha,\beta]}) + L(c|_{[\beta,b]}) \\ &= L(c). \end{aligned}$$

It follows that the length of $c|_{[\alpha,\beta]}$ equals $d(c(\alpha), c(\beta))$. In view of the previous corollary, it now follows that for each $t \in [a, b]$ there exists a $\epsilon > 0$ such that the restriction of c to $[a, b] \cap (t - \epsilon, t + \epsilon)$ has a reparametrization by arc length which is a geodesic. By compactness of $[a, b]$, the result follows. $\square$

A Riemannian manifold (M, g) is said to be geodesically complete if each geodesic extends to a geodesic with domain $\mathbb{R}$. Equivalently, this means that the exponential map exp has domain TM. Not every Riemannian manifold is geodesically complete. Indeed, let $\gamma : \mathbb{R} \to M$ be a geodesic in M, let $b > 0$ be such that γ is injective on $[0, b]$. Then $M' = M \setminus \{\gamma(b)\}$ is a Riemannian manifold and $\gamma_{[0,b)}$ is a geodesic in M', whose maximal extension to a geodesic in M' equals $\gamma|_I$, where I is the connected component of $\mathbb{R} \setminus \gamma^{-1}(b)$ which contains 0.

It is easily seen that the metric of M' is the restriction of the metric of M. Let (t_n) be an increasing sequence in $[0, b)$ with limit b. Then the sequence $\gamma(t_n)$

is Cauchy in M' but has no limit in M'. Therefore, M' is not complete as a metric space.

We end this section with the important Hopf-Rinow theorem, which gives the relation between geodesic completeness and metric completeness.

Theorem 10.6 (Hopf-Rinow) *The following assertions are equivalent:*

(c) *M is complete as a metric space.*

(a) *there exists a $p \in M$ such that* $\exp_p$ *has domain* T_pM;

(b) $\exp$ *has domain* TM;

We will not give the proof of this theorem here. The interested reader is referred to the lecture notes by E.J.N. Looijenga, which can be found on the website for the course.

11 Gauss curvature of surfaces

In this section we assume that (M, g) is a Riemannian manifold, and that N is a smooth submanifold of M. The induced Riemannian metric on N is denoted by h. Thus, for all $x \in N$ the positive definite inner product h_x is the restriction of the positive definite inner product g_x to the subspace T_xN of T_xM. We will denote the Levi-Civita connection associated to g by ∇, and the one associated to h by ∇^h.

In Section 4, text below Theorem 4.5, we defined the restriction of a vector bundle to a smooth submanifold. The restriction $TM|_N$ of the tangent bundle of M to N is a particular example. It is a vector bundle of rank equal to the dimension of M; its fiber above a point $x \in N$ equals T_xM. The inclusion map $N \to M$ induces a natural injective morphism of vector bundles $TN \to TM|_N$, over N. Fiberwise this morphism gives the natural injection $T_xN \hookrightarrow T_xM$, for $x \in N$.

We define $T_xN^\perp$ to be the g_x-orthocomplement of T_xN in T_xM, for $x \in N$. The union $TN^\perp$ of the spaces $T_xN^\perp$ form a subbundle of $TM|_N$, called the normal bundle of N relative to M. The restricted tangent bundle $TM|_N$ decomposes as the following direct sum of vector bundles:

$$TM|_N = TN \oplus TN^\perp.$$

The associate projection map $\pi : TM|_N \to TN$ is a morphism of vector bundles. For each $x \in N$ the map $\pi_x : T_xM \to T_xN$ is the orthogonal projection relative to the inner product g_x.

In Section 4 we also defined the restriction of a connection to a submanifold. In particular, the Levi-Civita connection ∇ of (M, g) may be restricted to the to the restricted bundle $TM|_N$. The restricted connection is denoted ∇^N. If X, Y are smooth vector fields on N which extend to smooth vector fields $\tilde{X}$ and $\tilde{Y}$ on M, then by Corollary 4.7 it follows that

$$\nabla^N_X Y = \nabla_{\tilde{X}} \tilde{Y}|_N;$$

note however that the restriction on the right is a section of $TM|_N$, but not necessarily a section of TN.

Lemma 11.1 *The Levi-Civita connection ∇^h is given by*

$$\nabla^h_X Y = \pi \circ \nabla^N_X Y,$$

for $X, Y \in \mathfrak{X}(N)$.

Proof: Since the assertion is of a local nature, we may assume that M, N are such that every vector field X on N extends to a smooth vector field $\tilde{X}$ on M. We must show that $\pi \circ \nabla^N$ is a torsion free affine connection, for which h is covariantly constant. That $\pi \circ \nabla^N$ is an affine connection is readily verified.

Let $X, Y \in \mathfrak{X}(N)$ be smooth vector fields on N with smooth extensions $\tilde{X}, \tilde{Y}$ to M. Then

$$\begin{aligned}
\pi \circ \nabla^N_X Y - \pi \circ \nabla^h_Y X - [X,Y] &= \pi \circ (\nabla^N_X Y - \nabla^N_Y X - [X,Y]) \\
&= \pi \circ (\nabla_{\tilde{X}} \tilde{Y} - \nabla_{\tilde{Y}} \tilde{X} - [\tilde{X}, \tilde{Y}])|_N \\
&= \pi \circ T^{\nabla}(\tilde{X}, \tilde{Y})|_N = 0.
\end{aligned}$$

This shows that $\pi \circ \nabla^N$ is torsion free.

To see that h is covariantly constant for $\pi \circ \nabla^N$, let X, Y, X be smooth vector fields on N, and let $\tilde{X}, \tilde{Y}, \tilde{Z}$ be extensions to smooth vector fields on M. Then

$$\begin{aligned}
Zh(X,Y) = \tilde{Z} g(\tilde{X}, \tilde{Y})|_N &= g(\nabla_{\tilde{Z}} \tilde{X}, \tilde{Y})|_N + g(\tilde{X}, \nabla_{\tilde{Z}} \tilde{Y})|_N \\
&= h(\nabla^N_Z X, Y) + h(X, \nabla^N_Z Y) \\
&= h(\pi \circ \nabla^N_Z X, Y) + h(X, \pi \circ \nabla^N_Z Y).
\end{aligned}$$

□

We will now describe the relation between the curvature tensor R of (M, g) and the curvature tensor R^h of (N, h). The key ingredient for the comparison is the so-called second fundamental form.

Given $X, Y \in \mathfrak{X}(N)$ we define

$$H(X,Y) := \nabla^N_X Y - \nabla^h_X Y = \nabla^N_X Y - \pi \circ \nabla^N_X Y.$$

Then $H(X,Y)$ is a section of the normal bundle $TN^\perp$. The map $H : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \Gamma(TN^\perp)$ is called the second fundamental form. It is readily verified to be bilinear over the ring $C^\infty(N)$, so that in fact $H(X,Y)_x$ only depends on the values X_x, Y_x, for $x \in N$. Thus, H defines vector bundle morphism $TN \otimes TN \to TN^\perp$.

Lemma 11.2 *H is symmetric.*

Proof: Since the assertion is of a local nature, we may assume that every $X \in \mathfrak{X}(N)$ extends to a vector field $\tilde{X} \in \mathfrak{X}(M)$. Then

$$\begin{aligned}
H(X,Y) - H(Y,X) &= \nabla^N_X Y - \nabla^N_Y X - (\nabla^h_X Y - \nabla^h_X Y) \\
&= (\nabla_{\tilde{X}} \tilde{Y} - \nabla_{\tilde{Y}} \tilde{X})|_N - \pi \circ (\nabla_{\tilde{X}} \tilde{Y} - \nabla_{\tilde{Y}} \tilde{X})|_N \\
&= [\tilde{X}, \tilde{Y}]|_N - \pi \circ [\tilde{X}, \tilde{Y}]|_N \\
&= [X,Y] - \pi \circ [X,Y] \\
&= [X,Y] - [X,Y] = 0.
\end{aligned}$$

□

In the following we will denote the curvature associated to (M, g) by R and the curvature associated to (N, h) by R^h. The relation between the two may be expressed by means of the second fundamental form.

Lemma 11.3 *Let $X, Y, V, W \in \mathfrak{X}(N)$. Then*

$$\begin{aligned} & h(R^h(X,Y)V, W) \\ & \quad = \quad g(R(X,Y)V,W) - g(H(X,V), H(Y,W)) + g(H(Y,V), H(X,W)). \end{aligned}$$

Proof: From $\nabla^h_Y V = \nabla^N_Y V - H(Y,V)$ it follows that

$$\nabla^h_X \nabla^h_Y V = \pi \circ \nabla^N_X (\nabla^N_Y V - H(Y,V)),$$

hence

$$\begin{aligned} h(\nabla^h_X \nabla^h_Y V, W) &= g(\nabla^N_X \nabla^N_Y V, W) - g(\nabla^N_X H(Y,V)), W) \\ &= g(\nabla^N_X \nabla^N_Y V, W) + g(H(Y,V)), \nabla^N_X W) \\ &= g(\nabla^N_X \nabla^N_Y V, W) + g(H(Y,V)), H(X,W)). \end{aligned}$$

Here the second equality follows from the fact that $g(H(Y,V), W) = 0$ and that the restriction of g to $TM|_N$ is covariantly constant with respect to ∇^N. From the equality obtained we may subtract the similar equality with X and Y interchanged. This gives

$$\begin{aligned} & h(R^h(X,Y)V,W) + h(\nabla^h_{[X,Y]}V, W) = \\ & \quad = \quad g(R(X,Y)V,W) + g(\nabla^N_{[X,Y]}V, W) + \\ & \qquad\quad + g(H(Y,V)), H(X,W)) - g(H(X,V)), H(Y,W)). \end{aligned}$$

We now use that

$$h(\nabla^h_{[X,Y]}V, W) = g(\nabla^h_{[X,Y]}V, W) = g(\nabla^N_{[X,Y]}V, W)$$

to complete the proof. □

We will now apply the above to the special case that N is a smooth two dimensional submanifold (surface) of three dimensional Euclidean space $\mathbb{E}^3$, and that h is the restriction of the Euclidean metric to N.

Corollary 11.4 *Let N be a smooth submanifold of Euclidean space $\mathbb{E}^3$, equipped with the restricted metric h. Then for all $X, Y, V, W \in \mathfrak{X}(N)$,*

$$-h(R^h(X,Y)V,W) = \langle H(X,V)\,,\, H(Y,W)\rangle - \langle H(Y,V)\,,\, H(X,W)\rangle.$$

Proof: This is an immediate consequence of the fact that $R = 0$ and the fact g is given by the standard inner product $\langle \cdot\,,\,\cdot \rangle$. □

Let $a \in N$ be given and let U be an open neighborhood of a in N diffeomorphic to a disc. Then there exists a smooth local section $\nu : U \to TU^{\perp}$ with $\|\nu\| = 1$. Of course, ν is uniquely determined up to sign. We agree to define

$$\mathcal{H}_x(X_x, Y_x) := \langle H_x(X_x, Y_x)\, ,\, \nu(x)\rangle,$$

for $x \in U$ and $X_x, Y_x \in T_xN$.

The unit normal ν may be viewed as a smooth map from U to the unit sphere S in $\mathbb{E}^3$. Accordingly, for every $x \in U$, the derivative $d\nu(x)$ is a linear map $T_xN \to T_{\nu(x)}S$. Now $T_xN \simeq \nu(x)^{\perp} \simeq T_{\nu(x)}S$, so that $d\nu(x)$ may be viewed as a linear map $T_xN \to T_xN$. The determinant of this map is called the Gauß curvature of N at the point x, and denoted by $K(x)$. Here we note that $K(x)$ is independent of the sign of $\nu(x)$. This implies that the local definitions of K are compatible and the restrictions of a uniquely defined smooth map $N \to \mathbb{R}$, called the Gauß curvature of N.

Lemma 11.5 *Let $x \in N$ and $v, w \in T_xN$. Then*

$$\mathcal{H}_x(v, w) = -\langle d\nu(x)v\, ,\, w\rangle.$$

Proof: Let V, W be vector fields on N such that $V_x = v$ and $W_x = w$. Then it follows from the fact that $\langle W\, ,\, \nu\rangle = 0$ that

$$0 = \langle \nabla^N_V W\, ,\, \nu\rangle + \langle W\, ,\, \nabla^N_V \nu\rangle = \langle H(V,W)\, ,\, \nu\rangle + \langle W\, ,\, d\nu(V)\rangle.$$

Evaluation in x yields the desired identity. $\square$

We note that it follows from this lemma that $d\nu(x) : T_xN \to T_xN$ is symmetric with respect to the restriction h_x of the inner product of $\mathbb{E}^3$. Hence $d\nu(x)$ diagonalizes with real eigenvalues. If these are different and non-zero, they are called the principal curvatures of N at x; the corresponding eigenspaces are called the principal axes. Note that these are perpendicular.

The Gauß curvature may be expressed in terms of the curvature tensor as follows.

Corollary 11.6 *Let $x \in N$ and let v, w be an orthonormal basis of T_xN. Then*

$$K(x) = -h_x(R^h(v, w)v, w). \tag{19}$$

Proof: The determinant of $d\nu(x)$ equals

$$\begin{aligned} K(x) &= \langle d\nu(x)v\, ,\, v\rangle\langle d\nu(x)w\, ,\, w\rangle - \langle d\nu(v)\, ,\, w\rangle\langle d\nu(w)\, ,\, v\rangle \\ &= \mathcal{H}_x(v,v)\mathcal{H}_x(w,w) - \mathcal{H}_x(v,w)\mathcal{H}_x(w,v) \\ &= \langle H_x(v,v)\, ,\, H_x(w,w)\rangle - \langle H_x(v,w)\, ,\, H_x(w,v)\rangle \\ &= -h_x(R^h_x(v,w)v, w). \end{aligned}$$

It follows from the above result that the Gauß curvature, although defined by using the isometric embedding of (N, h) in $\mathbb{E}^3$, is really an intrinsic notion of the Riemannian manifold (M, h). In particular, it does not depend on the embedding used. This was discovered by Gauß who found the result so striking he named it the Theorema Egregium.

In view of the Theorema Egregium, we may extend the definition of Gauß curvature to any two dimensional manifold by using the formula (19.

12 Several notions of curvature

In this section we assume that (M, g) is a pseudo-Riemannian manifold. Let $x \in M$, then $R_x : T_xM \otimes T_xM \to \mathrm{End}(T_xM)$ is anti-symmetric in its argument, hence induces a map $R_x : \wedge^2 T_x \to \mathrm{End}(T_xM)$. It follows from Lemma 7.16 (b) that

$$(X \wedge Y, V \wedge W) \mapsto g_x(R_x(X, Y)V, W)$$

extends to a well-defined bilinear map $\wedge^2 T_xM \times \wedge^2 T_xM \to \mathbb{R}$. Moreover, from (c) of the mentioned lemma it follows that this bilinear map is symmetric. Finally, in view of the Bianchi identity (a), it satisfies

$$\beta(v_1 \wedge v_2, v_3 \wedge v_4) + \beta(v_2 \wedge v_3, v_1 \wedge v_4) = \beta(v_1 \wedge v_3, v_2 \wedge v_4) \tag{20}$$

for all $v_1, v_2, v_3, v_4 \in T_xM$.

Lemma 12.1 *Let V be a finite dimensional real linear space, and let $\beta : \wedge^2 V \times \wedge^2 V \to \mathbb{R}$ be a symmetric bilinear form such that (20) holds for all $v_1, \ldots, v_4 \in V$. Then β is completely determined by the values $\beta(v \wedge w, v \wedge w)$, for $v, w \in V$.*

Exercise 12.2 The goal of this exercise is to prove Lemma 12.1. Let β be a form as above. Assume that $\beta(v \wedge w, v \wedge w)$ is known for all $v, w \in V$. Then it suffices to show that β is known.

(a) Let $v_1, v_2, v_3 \in V$. Show that $\beta(v_1 \wedge v_2, v_2 \wedge v_3)$ can be written as a linear combination of $\beta((v_1 + v_3) \wedge v_2, (v_1 + v_3) \wedge v_2)$, $\beta(v_1 \wedge v_2, v_1 \wedge v_2)$ and $\beta(v_3 \wedge v_2, v_3 \wedge v_2)$, with rational coefficients independent of the particular choice of v_1, v_2, v_3.

(b) Given a vector $\lambda \in \mathbb{Z}^4$ we define the linear map $\lambda : V^4 \to V$ by $\lambda(v_1, \ldots, v_4) = \sum_i \lambda v_i$. Show that there exists a finite sequence of integers $m^1, \ldots, m^r$ and for every $1 \leq j \leq r$ vectors $\lambda_j, \mu_j \in \mathbb{Z}^4$ such that for all $v_1, v_2, v_3, v_4 \in V$ we have

$$\beta(v_1 \wedge v_2, v_3 \wedge v_4) = \frac{1}{4} \sum_j m^j \, \beta(\lambda_j(v) \wedge \mu_j(v), \lambda_j(v) \wedge \mu_j(v)).$$

Hint: expand $\beta((v_1 + v_2) \wedge (v_3 + v_4))$.

(c) Complete the proof of the lemma.

It follows from the lemma above that the curvature tensor at $x \in M$ is completely determined by the values $g_x(R_x(v, w)v, w)$, for $v, w \in T_xM$.

Sectional curvature We now assume that (M, g) is Riemannian, i.e., g is positive definite. If σ is a two-dimensional linear subspace of T_xM, for $x \in M$, then $\wedge^2\sigma$ is 1-dimensional, hence has two elements of length 1 for the metric induced by g_x. If v, w form an orthonormal basis of σ, then $v \wedge w$ has length 1 in $\wedge^2\sigma$, so that $v \wedge w$ and $w \wedge v$ form the elements of length 1 in $\wedge^2\sigma$. It follows from this that the endomorphism $R_x(v,w) \in \mathrm{End}(T_xM)$ is uniquely determined by σ, up to sign. Moreover, the scalar

$$K(\sigma) = -g_x(R_x(v,w)v,w)$$

does not depend on the particular choice of orthonormal basis, and is uniquely determined by σ. In case M is two-dimensional, the scalar $K(T_xM)$ equals the Gauß curvature of M at x. In general, $K(\sigma)$ is called the sectional curvature of M along σ.

Remark 12.3 It follows from Lemma 12.1 that the curvature tensor $R_x : \wedge^2 T_xM \to \mathrm{End}(T_xM)$ is completely determined by the sectional curvatures $K(\sigma)$, with σ a two-dimensional linear subspace of T_xM.

Exercise 12.4 Let (M,g) be a Riemannian manifold, let $x \in M$ and let σ be a two dimensional subspace of T_xM. Show that for any basis v,w of σ we have

$$K(\sigma) = -\frac{g_x(R_x(v,w)v,w)}{g_x(v,v)g_x(w,w) - g_x(v,w)^2}.$$

Components Let $x^1, \ldots, x^n$ be a system of local coordinates on an open set $U \subset M$. Let $\partial_1, \ldots, \partial_n$ be the associated vector fields, and $dx^1, \ldots, dx^n$ the associated one forms over U. Then $\partial_1, \ldots, \partial_n$ is a frame for TU and $dx^1, \ldots, dx^n$ is the dual frame of T^*U. We agree to write

$$R^l_{ijk} = dx^l(R(\partial_i, \partial_j)\partial_k),$$

so that

$$R = R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \partial_l.$$

Here we agree to use the Einstein summation convention. Each time a symbol occurs both as a superscript and as a subscript, we sum over its full range. Thus, with summation symbols the above may be written as

$$R = \sum_{1 \leq i,j,k,l \leq n} R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \partial_l.$$

In terms of components, the anti-symmetry of the curvature tensor becomes

$$R^l_{ijk} = -R^l_{jik}.$$

The Bianchi identity (see Lemma 7.16 (a)) may be formulated as

$$R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0.$$

The tensor $\tilde{R} : (X,Y,V,W) \mapsto g(R(X,Y)V,W)$ has components

$$\tilde{R}_{ijkm} = g_{ml} R^l_{ijk}.$$

Its symmetry and anti-symmetry properties are now described by

$$R_{ijkm} = -R_{jikm} = -R_{ijmk} \quad \text{and} \quad R_{ijkm} = R_{kmij}.$$

In terms of the component notation, the symmetry and anti-symmetry properties of the curvature tensor may be reformulated as

$$R^l_{ijk} = -R^l_{jik}$$

Ricci curvature We return to the situation that (M, g) is a pseudo-Riemannian manifold. The Ricci curvature of M at a point $x \in M$ is defined to be the bilinear form $Ric_x : T_xM \times T_xM \to \mathbb{R}$ given by

$$Ric_x(X, Y) = \mathrm{tr}\,[\, R_x(X, \,\cdot\,)Y \,].$$

In terms of components of the curvature tensor, the Ricci tensor is given by

$$Ric_{ik} = R^j_{ijk},$$

where again the Einstein convention has been used to suppress the summation symbol. We note that the Ricci tensor is symmetric. Indeed, if $X, Y \in T_xM$, and if $e_1, \ldots, e_n$ is a basis of T_xM such that $g_x(e_i, e_j) = \epsilon_i\delta_{ij}$, with $\epsilon_i = \pm 1$, then

$$\begin{aligned} Ric(X, Y) &= \sum_j [R(X, e_j)Y]^j = \sum_j \epsilon_j g(R(X, e_j)Y, e_j) \\ &= \sum_j \epsilon_j (R(Y, e_j)X, e_j) = Ric(Y, X). \end{aligned}$$

in view of Lemma 7.16 (c).

If g is positive definite, and $e \in T_xM$ a vector of unit length, let $e = e_1, \ldots, e_n$ be an extension to an orthonormal basis of T_xM. Then

$$Ric_x(e, e) = \sum_i g(R(e, e_i)e, e_i) = \sum_i K(\sigma_{1i}),$$

where $\sigma_{ij} = \mathrm{span}\{e_i, e_j\}$. Thus, $Ric_x(e, e)$ is the sum of the sectional curvatures along the coordinate planes which contain e.

Scalar curvature In general, if (M, g) is pseudo-Riemannian, we define the linear map $\widetilde{Ric}_x : T_xM \to T_xM$ by $\widetilde{Ric}_x = g_x^{-1}Ric_x$, or,

$$g_x(\widetilde{Ric}_x(X), Y) = Ric_x(X, Y),$$

for $X, Y \in T_xM$. The components of this tensor are given by

$$\widetilde{Ric}^l_i = g^{lk}Ric_{ik}.$$

The scalar curvature $K(x)$ at the point $x \in M$ is defined by contraction:

$$K(x) = \mathrm{tr}\,(\widetilde{Ric}_x).$$

In components:

$$K = g^{ik}Ric_{ik} = g^{ik}R^j_{ijk}.$$

If g is positive definite, and $x \in M$, let $e_1, \ldots, e_n$ be an orthonormal basis for $T_x M$. Then it follows that

$$K(x) = \sum_{ij} g_x(R_x(e_i, e_j)e_i, e_j) = 2 \sum_{i<j} K_x(\sigma_{ij}),$$

so that the scalar curvature equals twice the sum of the sectional curvatures along the coordinate hyperplanes in $T_x M$.

Exercise 12.5 Let (M, g) be a 3-dimensional Riemannian manifold. The purpose of this exercise is to show that the sectional curvature at a point $x \in M$ can be retrieved from the Ricci form $Ric_x : T_x M \times T_x M \to \mathbb{R}$ (and the metric g_x). (Thus, also the full curvature tensor can be retrieved from the Ricci form.) The three dimensionality of M is a crucial assumption.

(a) The metric g_x on $T_x M$ induces a linear isomorphism $T_x M \to T_x M^*$. The dual metric g_x^* on $T_x M^*$ is defined to be the unique inner product for which this isomorphism becomes an isometry. Let f_1, f_2, f_3 be a basis for $T_x M$ and let f^1, f^2, f^3 be the dual basis for $T_x^* M$. Show that the following assertions are equivalent:

- f_1, f_2, f_3 is orthonormal for g;
- $g(f_j) = f^j$ for alle j;
- f^1, f^2, f^3 is orthonormal for g^*.

(b) Equip $T_x M$ with an orientation, and let $\omega \in \wedge^3 T_x M$ be the associated unit element. This means that $\omega = f_1 \wedge f_2 \wedge f_3$ for any positively oriented orthonormal basis. We define the pairing $\gamma : \wedge^2 T_x M \times T_x M \to \mathbb{R}$ by $\lambda \wedge X = \gamma(\lambda, X)\omega$. Show that γ is non-degenerate, hence induces an isomorphism $\wedge^2 T_x M \to T_x M^*$. Show that for f_1, f_2, f_3 a positively oriented orthonormal basis of $T_x M^*$ we have

$$\gamma(f_1 \wedge f_2) = f^3.$$

Determine $\gamma(f_2 \wedge f_3), \gamma(f_3 \wedge f_1)$.

(c) Via the isomorphism γ the form β may be transfered to a bilinear form

$$\beta^* : T_x^* M \times T_x^* M \to \mathbb{R}.$$

Show that β^* diagonalizes with respect to a g^*-orthonormal basis e^1, e^2, e^3 of $T_x M^*$. Let e_1, e_2, e_3 be the associated dual basis of $T_x M$. Show that β diagonalizes with respect to the basis $e_1 \wedge e_2$, $e_2 \wedge e_3$, and $e_3 \wedge e_1$ of $\wedge^2 T_x M$.

(d) Show that for each $i \neq j$ the value $\beta(e_i \wedge e_j, e_i \wedge e_j)$ is a linear combination of $Ric_x(e_k, e_k)$, $k = 1, 2, 3$. Hint: solve a system of three linear equations with three unknowns.

(e) Show that all sectional curvature at x can be retrieved from the Ricci form.

Exercise 12.6 We assume that M is a three dimensional connected Riemannian manifold, whose Ricci form is proportional to the metric g, i.e., there exists a smooth function $f : M \to \mathbb{R}$ such that $Ric_x = f(x) \cdot g$ for all $x \in M$. The purpose of this exercise is to show that f is constant.

(a) Let e_1, e_2, e_3 be an orthonormal basis of T_xM. Show that for every $X \in T_xM$,

$$R_x(X, e_1)e_1 + R_x(X, e_2)e_2 + R_x(X, e_3)e_3 = f(x)X.$$

(b) Write $A = R_x(e_1, e_2)$, $B = R_x(e_2, e_3)$, $C = R_x(e_3, e_1)$. Argue that the matrices of A, B, C with respect to the basis e_1, e_2, e_3 are anti-symmetric. Use this in combination with the previous item to conclude that $A_{13} = A_{23} = B_{12} = B_{13} = C_{12} = C_{23} = 0$ and that $A_{12} = B_{23} = C_{31} = \frac{1}{2}f(x)$.

(c) Show that for all $U, V, X \in T_xM$,

$$R_x(U, V)X = \frac{1}{2}f(x)\,[g_x(V, X)U - g_x(U, X)V].$$

Hint: use that both sides are bilinear and anti-symmetric with respect to U, V and test the identity at basis elements.

(d) As an intermezzo, conclude that for each plane $\sigma \in T_xM$ the sectional curvature $K(\sigma)$ equals $-\frac{1}{2}f(x)$.

The rest of the exercise is more complicated, and relies on the so-called second Bianchi identity, which we shall not prove here. It says that for all vector fields U, V, W, one has

$$\nabla_U R(V, W) + \nabla_V R(W, U) + \nabla_W R(U, V) = 0.$$

(e) Let R° be the tensor given by $R^\circ(U, V) = g(V, \cdot\,)U - g(U, \cdot\,)V$, for vector fields U, V. Then the conclusion of item (e) may be rephrased as

$$R(U, V) = \frac{1}{2}f \cdot R^\circ(U, V).$$

Show that the tensor R° satisfies the analogue of the second Bianchi identity for all commuting vector fields U, V, W. Conclude that

$$Uf \cdot R^\circ(V, W) + Vf \cdot R^\circ(W, U) + Wf \cdot R^\circ(U, V) = 0$$

for all commuting vector fields U, V, W. Conclude that the same identity holds at a fixed point x for any choice of vectors $U, V, W \in T_xM$.

(f) Conclude that $df(x) = 0$. Hint: choose an orthonormal basis U, V, W for T_xM and apply the above identity to U. Finally, conclude that f is constant.

13 Exercises

Exercise 13.1 Let $p : E \to M$ be a rank k vector bundle over a smooth manifold M. Let $q : E^* \to M$ be the dual bundle. Thus, for each $x \in M$ the fiber $q^{-1}(x)$ equals the dual E_x^* of E_x. The vector bundle structure of E^* is determined by the requirement that for each local frame $e_1, \dots, e_k$ of E over an open set $U \subset M$, the maps $e^j : U \to E^*$ determined by $e^j(x)(e_i(x)) = \delta_i^j$ form a local frame of E^* over U.

(a) Given $s \in \Gamma(E)$ and $s^* \in \Gamma(E^*)$ we define the function $\langle s\,,\, s^* \rangle : M \to \mathbb{R}$ by $\langle s\,,\, s^* \rangle(x) = s^*(x)(s(x))$. Show that $\langle s\,,\, s^* \rangle$ is smooth.

(b) Let ∇ be a connection on E. Show that there exists a unique connection ∇^* on E^* such that

$$\langle \nabla_X s\,,\, s^* \rangle + \langle s\,,\, \nabla_X^* s^* \rangle = X \langle s\,,\, s^* \rangle$$

for all $s \in \Gamma(E)$, $s^* \in \Gamma(E^*)$ and $X \in \mathfrak{X}(M)$.

Let $p' : E' \to M$ be a rank l-vector bundle over M. The tensor product $E \otimes E'$ is a rank kl-vector bundle whose fiber over $x \in M$ equals $E_x \otimes E'_x$. If U is an open subset of M and if $e_1, \dots, e_k$ and $f_1, \dots, f_l$ are local frames of E and E' over U, then $e_i \otimes f_j$, $(1 \leq i \leq k, 1 \leq j \leq l)$ is a local frame of $E \otimes E'$ over U.

(c) Let ∇ be a connection on E and let ∇' be a connection on E'. Show that there exists a unique connection ∇'' on $E'' = E \otimes E'$ such that

$$\nabla''_X (s \otimes s') = \nabla_X s \otimes s' + s \otimes \nabla_X s'$$

for all $s \in \Gamma(E)$, $s' \in \Gamma(E')$ and $X \in \mathfrak{X}(M)$.

(d) Let M be a manifold whose tangent bundle is equipped with a connection ∇. Let ∇^* be the induced connection on T^*M and let ∇'' be the induced connection on $T^*M \otimes T^*M$. Let g be a pseudo-Riemannian metric on M. Then g may be viewed as a section of $T^*M \otimes T^*M$. Show that g is flat if and only if

$$\nabla''_X g = 0, \qquad (\forall X \in \mathfrak{X}(M)).$$

Exercise 13.2 Let M be a smooth surface in Euclidean space $\mathbb{E}^3$ ($\mathbb{R}^3$ equipped with the standard inner product). Let g be the restricted Riemannian metric on M. Thus, g_x is the restriction of $\langle \cdot\,,\, \cdot \rangle$ to $T_xM \subset \mathbb{E}^3$, for every $x \in M$. For each $x \in M$, let $\pi_x : \mathbb{E}^3 \to T_xM$ denote the orthogonal projection.

(a) Show that the Levi-Civita connection on TM is given by the formula

$$\nabla_X Y(x) = \pi_x dY(x) X(x),$$

for $X, Y \in \mathfrak{X}(M)$. Here $dY(x)$ denotes the derivative of Y viewed as a map $M \to \mathbb{E}^3$.

(b) Let $c : I \to M$ be a C^1-curve. Show that for every vector field $t \mapsto X(t)$ along c,

$$\frac{DX}{dt}(t) = \pi_{c(t)}(X'(t)).$$

(c) Show that a C^2-curve $\gamma : J \to M$ is a geodesic if and only if

$$\gamma''(t) \perp T_{\gamma(t)}M, \qquad (\forall t \in J).$$

Exercise 13.3 The purpose of this exercise is to exhibit a different realisation of two dimensional hyperbolic space $\mathbb{H}_2$. Let D be the unit disk in $\mathbb{R}^2$, consisting of the points (x, y) with $x^2 + y^2 < 1$. Put $r = r(x, y) = \sqrt{x^2 + y^2}$ and define the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(x, y) = \frac{1}{1 - r(x, y)^2}\,(2x, 2y, 1 + r(x, y)^2).$$

(a) Show that the pull-back of the metric $dx_1^2 + dx_2^2 - dx_3^2$ on $\mathbb{R}^3$ under f equals

$$f^*(dx_1^2 + dx_2^2 - dx_3^2) = 4\,\frac{dx^2 + dy^2}{(1 - r(x, y)^2)^2} \qquad (*)$$

To keep the amount of calculation limited, it is a good idea to introduce the function $s = 1 - r^2$, and to keep working with s and the form ds as long as possible.

(b) Show that f is an isometry from D, equipped with the Riemannian metric on the right-hand side of (*), onto $\mathbb{H}_2$.

The disk D, equipped with the given Riemannian metric, is called the Poincaré disk.

Exercise 13.4 Let (M, g) be a compact Riemannian manifold.

(a) Assume that $r > 0$ and that $\gamma : [0, r) \to M$ is a geodesic. Show that the sequence $x_n := \gamma(r - \frac{1}{n})$ is a Cauchy sequence for the metric d determined by g.

(b) Show that $\lim_{t \uparrow r} \gamma(t)$ exists.

(c) Show that every geodesic $\gamma : I \to M$ extends to a geodesic $\mathbb{R} \to M$.

Exercise 13.5 In this exercise we assume that (M, g) and (N, h) are Riemannian manifolds, and that $\varphi : M \to N$ is an isometry.

(a) Let $a \in M$ and $v \in T_aM$. Let $\gamma_v; I_v \to M$ be the associated maximal geodesic with initial point a and initial tangent vector v. Show that $\varphi \circ \gamma_v$ is a geodesic in N. In fact, let $w = d\varphi(a)v$. Show that $\varphi \circ \gamma_v = \gamma_w$.

(b) Let $a \in M$ and put $b = \varphi(a)$. Show that $d\varphi(a)$ maps the domain Ω_a of $\exp_a$ onto the domain Ω_b of $\exp_b$. Moreover, show that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \exp_a \uparrow & & \uparrow \exp_b \\ \Omega_a & \xrightarrow{d\varphi(a)} & T_bN. \end{array}$$

Let now σ be an isometry of M onto itself. Let $S := \{x \in M \mid \sigma(x) = x\}$ be the set of fixed points for σ.

(c) For $a \in S$, let $d_a = \dim \ker(d\sigma(a) - I)$. Show that at the point a the set S is a submanifold of M of dimension d_a.

(d) Let $a \in S$ and $v \in T_a S$. Show that the geodesic $\gamma_v : I_v \to M$ is contained in S. A submanifold of M with this property is called totally geodesic in M.

(e) Let now M be the unit sphere in $\mathbb{E}^3$. Show that the geodesics of M are precisely the curves traversing big circles with constant velocity.

Exercise 13.6 We assume that $p : E \to M$ is a rank k vector bundle over the manifold M, equipped with a connection. Given an open subset $U \subset M$ we agree to write $\Omega^p(U, E)$ for the space of E-valued p-forms on U. By this we mean the space of smooth sections of the bundle $E \otimes \wedge^p T^*M$. In particular, we write $\Omega^0(U, E)$ for the space of smooth sections of E over U. We agree to write $\Omega^p(E)$ for $\Omega^p(M, E)$.

(a) Show that the connection may be viewed as a map $D : \Omega^0(E) \to \Omega^1(E)$ which (1) is locally defined, (2) is $\mathbb{R}$-linear, and (3) satisfies the Leibniz rule

$$D(fs) = fDs + s \otimes df$$

for all $s \in \Gamma(E)$ and $f \in C^\infty(M)$.

We agree to write $\Omega(E)$ for the direct sum of the spaces $\Omega^p(E)$, for $p \geq 0$. Then D may be viewed as a map $\Omega^0(E) \to \Omega(E)$.

(b) Show that D has a unique extension to a map $\Omega(E) \to \Omega(E)$ which (1) is locally defined, (2) is $\mathbb{R}$-linear, and (3) satisfies the generalized Leibniz rule

$$D(s \otimes \alpha) = Ds \wedge \alpha + s \otimes d\alpha,$$

for $s \in \mathcal{E}(E)$ and $\alpha \in \Omega^p(M)$. Hint: use a local frame for E.

(c) Show that there exists a unique bilinear map $\wedge : \Omega(\mathrm{End}(E)) \times \Omega(E) \to \Omega(E)$ such that

$$(A \otimes \alpha) \wedge (b \otimes \beta) = Ab \otimes (\alpha \wedge \beta)$$

for all $A \in \Gamma(\mathrm{End}(E)), b \in \Gamma(E)$, and $\alpha, \beta \in \Omega(M)$.

(d) We recall from Exercise 13.1 that the connection of E induces a connection on $\mathrm{End}(E) \simeq E \otimes E^*$ such that the associated operator D satisfies $D(Aa) = DA \wedge a + A \wedge Da$, for all $A \in \Gamma(\mathrm{End}(E))$ and $a \in \Gamma(E)$. Show that

$$D(\Phi \wedge \lambda) = D\Phi \wedge \lambda + (-1)^p \Phi \wedge D\lambda$$

for $\Phi \in \Omega^p(\mathrm{End}(E))$ and $\lambda \in \Omega(E)$.

(e) The curvature form R of the connection on E may be viewed as an element of $\Omega^2(\mathrm{End}(E))$. Show that for every $\omega \in \Omega(E)$ we have

$$DD(\omega) = R \wedge \omega.$$

(d) Show that $D(R) = 0$. (This is called the Bianchi identity.)

Exercise 13.7 Let (M, g) be a Riemannian manifold. A submanifold $N \subset M$ is said to be totally geodesic if for any $a \in N$ and $v \in T_a N$ the maximal geodesic γ_v is entirely contained in N. Let h be the restriction metric on N. Show that the following three assertions are equivalent.

(a) N is totally geodesic.

(b) For every $v \in TN$, the maximal geodesic γ_v^N in N relative to h equals the maximal geodesic γ_v in M.

(c) Every geodesic in N relative to h is a geodesic in M.

Exercise 13.8 Let (M, g) be a Riemannian manifold, and let N be a submanifold of M. Let h be the restriction metric on N. Let $c : I \to N$ be a C^1-curve, and let $X : I \to TN$ be a vector field over c. We denote by DX/dt the covariant derivative of X along c with respect to the Levi-Civita connection associated with g. Moreover, by $D^h X/dt$ we denote the covariant derivative of X along c with respect to the Levi-Civita connection associated with h.

(a) Show that
$$\frac{D^h X}{dt}(t) = \pi_{c(t)} \frac{DX}{dt}(t),$$
where π_x denotes the orthogonal projection $T_x M \to T_x N$, for $x \in N$. Hint: first prove this for a vector field of the form $X(t) = Y(c(t))$, with $Y \in \mathfrak{X}(N)$. Then prove it for vector fields of the form $X(t) = f(t)Y(c(t))$, with $f \in C^\infty(N)$.

(b) Show that
$$H(c'(t), X(t)) = \frac{DX}{dt}(t) - \frac{D^h X}{dt}(t).$$

(c) Show that $c : I \to N$ is a geodesic in N if and only if
$$\frac{Dc'(t)}{dt} \perp T_{c(t)} N$$
for all $t \in I$.

(d) The submanifold N is said to be totally geodesic if and only if for all $v \in TN$ the maximal geodesic $\gamma_v : I_v \to M$ is entirely contained in N. Show that the following assertions are equivalent

  (1) N is totally geodesic;

  (2) the second fundamental form H is identically zero.

Exercise 13.9 Let S be the sphere of center 0 and radius $R > 0$ in $\mathbb{E}^3$. Determine the Gauß curvature of S. Give an a priori reason why the Gauß curvature is constant.

Exercise 13.10 Let $R > 0$. We consider the cilinder C in $\mathbb{E}^3$ given by the equation $x^2 + y^2 = R^2$.

(a) Show that the Gauß curvature of C is zero.

(b) Show that the curvature form of C is zero.

(c) Specify an isometry from $\mathbb{R} \times (-\pi, \pi)$ onto an open subset of C and prove the correctness of your answer.