

What is Special Kähler Geometry ?

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ABSTRACT

The scalars in vector multiplets of $N=2$ supersymmetric theories in 4 dimensions exhibit ‘special Kähler geometry’, related to duality symmetries, due to their coupling to the vectors. In the literature there is some confusion on the definition of special geometry. We show equivalences of some definitions and give examples which show that earlier definitions are not equivalent, and are not sufficient to restrict the Kähler metric to one that occurs in $N = 2$ supersymmetry. We treat the rigid as well as the local supersymmetry case. The connection is made to moduli spaces of Riemann surfaces and Calabi-Yau 3-folds. The conditions for the existence of a prepotential translate to a condition on the choice of canonical basis of cycles.

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1 Introduction

The geometry defined by the complex scalars in the coupling of $N = 2$ vector multiplets to supergravity [1] has been given the name [2] ‘special Kähler geometry’. A similar geometry in $N = 2$ *rigid* supersymmetry [3] has been called ‘rigid special Kähler’.

Whereas their origin and use in supersymmetric theories determines the concept of special Kähler manifolds, it is also of interest to have a description that does not need the full construction of a supergravity action to describe this class of manifolds. Among other merits, through capturing the essence, this description may enable one to find realisations of special geometry in a setting different from supergravity. Various authors [2, 4, 5, 6, 7, 8, 9] have proposed such a description, leading to different proposals for a general definition of the special Kähler geometries. Some of these were inspired by the specific setting of subvarieties of the moduli spaces of Riemann surfaces or Calabi-Yau manifolds, where these special Kähler geometries appear. These definitions are not completely equivalent. We will propose new definitions, and prove their equivalence. A short résumé of the results was given in [10], and a long version in [11].

The recent upsurge of interest in this question is due to duality. In four dimensions, duality transformations are transformations between the field strengths of spin-1 fields. In general, the kinetic terms of the vectors may depend on scalar fields. As we will be concerned finally mainly with the kinetic terms of the scalars, it is sufficient to consider abelian gauge groups. The generic form of the kinetic terms of the vector fields is then¹

$$\mathcal{L}_1 = \frac{1}{4}(\text{Im } \mathcal{N}_{IJ})\mathcal{F}_{\mu\nu}^I\mathcal{F}^{\mu\nu J} - \frac{i}{8}(\text{Re } \mathcal{N}_{IJ})\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}^I\mathcal{F}_{\rho\sigma}^J, \quad (1.1)$$

where $I, J = 1, \dots, m$, the symmetric matrix \mathcal{N}_{IJ} depends on the scalars, and $\mathcal{F}_{\mu\nu}^I$ are the field strengths of the vector fields. The imaginary part of the matrix \mathcal{N} should be negative definite in order to guarantee the positivity of the kinetic terms of the vectors. This matrix provides the bridge between the vector and the scalar sectors of the theory. The duality transformations of the vectors then impose on the scalar sector a structure which is central to special Kähler geometries. The most useful definitions of special geometry will use a formalism in which these duality transformations, which take the form of symplectic transformations [12, 1, 13, 14], are manifest. Such a relation between the scalars and the vectors arises naturally in $N = 2$ supersymmetry vector multiplets (or larger N extensions).

In section 2 we recall the general formalism of symplectic transformations for the vectors in 4 dimensions, clarifying in particular the role of the matrix \mathcal{N} , and recalling some of its properties. In section 3 the basic concepts of couplings of vector multiplets in rigid $N = 2$ supersymmetry are explained. This leads to a first definition of rigid special geometry, using straightforwardly the construction of the supersymmetric action, based on a prepotential². Our treatment of the symplectic structure in a superspace approach reveals that a compatibility condition between the symplectic and supersymmetric structure largely short-circuits the use of an action to obtain the field equations of the theory. A second definition takes the symplectic structure as central, and does not make use of a prepotential. We will show that the two definitions are equivalent. A third definition, also starting with the symplectic vector structure, prepares the stage for showing how certain subspaces of the moduli

¹We take metric signature $(-+++)$ and $\epsilon^{0123} = i$.

²We use the terminology ‘prepotential’ for the function which leads to a Kähler potential. The word ‘prepotential’ originally denoted an unconstrained superfield in terms of which $N = 2$ vector superfields can be constructed. In $N = 2$ supersymmetry these were found in [15].

spaces of Riemann surfaces provide examples of rigid special geometries: we will analyze the requirements on these subspaces and show how various examples in the literature fit in.

The same structure is repeated in section 4 for the local (supergravity) case. The first definition is close to the original construction of actions for vector multiplets coupled to $N = 2$ supergravity using the tensor calculus [1]. We will first repeat the essential steps of this construction of the supergravity action. For a review on another construction, using rheonomic methods [4], we refer to [8]. The second definition gives the symplectic structure the central role, and is based on the one by Strominger [2]. We point out that, whereas it is adequate within the setting of Calabi-Yau moduli spaces, in a more general setting it is incomplete. To show the equivalence between the two definitions we pay special attention to the formulations of these models that do not allow a prepotential [6]. These formulations were all obtained by starting from a formulation *with* a prepotential and then performing a symplectic transformation. A main result of this paper is in this section: the proof that all models without a prepotential can be obtained from a symplectically rotated formulation with a prepotential. Since the manifold defined by the scalars is invariant under such a symplectic transformation, the two definitions of special manifolds are then equivalent. Again we give a third definition, in terms of objects which have a geometric significance in Calabi-Yau moduli spaces (periods of 3-forms), and of course we then exhibit how this is realised in the Calabi-Yau context.

In section 5 we repeat and clarify the main conclusions, and give some final remarks. Many detailed proofs are given in appendices.

2 Symplectic transformations

In this section, we remind the reader of the connection between symplectic transformations and duality transformations [12]. Then we give some general properties of these transformations. It was realised recently that many aspects of this formalism can be written quite generally for any dimension and supersymmetry extension [14].

Consider a general action of the form \mathcal{L}_1 in (1.1) for abelian spin-1 fields. The field equations for the vectors are

$$0 = \frac{\partial \mathcal{L}}{\partial W_\mu^I} = 2\partial_\nu \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\mu\nu}^I} = 2\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\mu\nu}^{+I}} + \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\mu\nu}^{-I}} \right), \quad (2.1)$$

where $\mathcal{F}_{\mu\nu}^\pm = \frac{1}{2} (\mathcal{F}_{\mu\nu} \pm \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\rho\sigma})$. Defining

$$G_{+I}^{\mu\nu} \equiv 2i \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\mu\nu}^{+I}}; \quad G_{-I}^{\mu\nu} \equiv -2i \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\mu\nu}^{-I}}, \quad (2.2)$$

one finds, for \mathcal{L}_1 ,

$$G_{+I}^{\mu\nu} = \mathcal{N}_{IJ} \mathcal{F}^{+J\mu\nu}; \quad G_{-I}^{\mu\nu} = \bar{\mathcal{N}}_{IJ} \mathcal{F}^{-J\mu\nu}. \quad (2.3)$$

Observe that the symmetry of \mathcal{N} has been used in these relations. In the considerations of duality transformations, the spin-1 fields are represented by field strengths rather than potentials. This implies that the field equations are not the only conditions on the field strengths, but are to be supplemented with the equations which express that these field

strengths are in fact locally derivable from a vector, viz. the Bianchi identities. The complete set of equations for the field strengths can then be written as

$$\begin{aligned}\partial^\mu \text{Im } \mathcal{F}_{\mu\nu}^{+I} &= 0 & \text{Bianchi identities} \\ \partial_\mu \text{Im } G_{+I}^{\mu\nu} &= 0 & \text{Field equations .}\end{aligned}\tag{2.4}$$

This set of equations is invariant under $GL(2m, \mathbb{R})$ transformations:

$$\begin{pmatrix} \tilde{\mathcal{F}}^+ \\ \tilde{G}_+ \end{pmatrix} = \mathcal{S} \begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^+ \\ G_+ \end{pmatrix} .\tag{2.5}$$

The $G_{\mu\nu}$ are related to the $\mathcal{F}_{\mu\nu}$ as in (2.2). Now we limit the transformations to those that preserve such a relation. The relation between $\tilde{G}_{\mu\nu}$ and $\tilde{\mathcal{F}}_{\mu\nu}$ is given by the matrix

$$\tilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} .\tag{2.6}$$

As $\tilde{G}_{+\mu\nu}$ should be the derivative of a transformed action to describe the field equations, this requirement imposes that the matrix $\tilde{\mathcal{N}}$ should be symmetric. For a general \mathcal{N} this implies (using rescalings of the field strengths)

$$A^T C - C^T A = 0 \quad , \quad B^T D - D^T B = 0 \quad , \quad A^T D - C^T B = \mathbf{1} .\tag{2.7}$$

These equations express that

$$\mathcal{S} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}\tag{2.8}$$

is a symplectic matrix:

$$\mathcal{S} \in Sp(2m, \mathbb{R}) \quad : \quad \mathcal{S}^T \Omega \mathcal{S} = \Omega \quad \text{with} \quad \Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} .\tag{2.9}$$

A $2m$ component column V which transforms under the symplectic transformations as $\tilde{V} = \mathcal{S}V$ is called a symplectic vector. The invariant inner product of two symplectic vectors V and W is

$$\langle V, W \rangle \equiv V^T \Omega W .\tag{2.10}$$

The symplectic transformations thus transform solutions of (2.4) into solutions. However, they are not invariances of the action. Indeed, writing

$$\mathcal{L}_1 = \frac{1}{2} \text{Im} \left(\mathcal{N}_{IJ} \mathcal{F}_{\mu\nu}^{+I} \mathcal{F}^{+\mu\nu J} \right) = \frac{1}{2} \text{Im} \left(\mathcal{F}^{+I} G_{+I} \right) ,\tag{2.11}$$

we obtain

$$\text{Im } \tilde{\mathcal{F}}^{+I} \tilde{G}_{+I} = \text{Im} \left(\mathcal{F}^+ G_+ \right) + \text{Im} \left(2\mathcal{F}^+ (C^T B) G_+ + \mathcal{F}^+ (C^T A) \mathcal{F}^+ + G_+ (D^T B) G_+ \right) .\tag{2.12}$$

If $C \neq 0, B = 0$ the Lagrangian is invariant up to a four-divergence, as $\text{Im } \mathcal{F}^+ \mathcal{F}^+ = -\frac{i}{4} \epsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\mu\nu} \mathcal{F}_{\rho\sigma}$ and the matrices A and C are real. For $B \neq 0$ the Lagrangian is *not* invariant.

If sources are added to the equations (2.4), the transformations can be extended to the dyonic solutions of the field equations by letting the magnetic and electric charges $\begin{pmatrix} q_m^I \\ q_{eI} \end{pmatrix}$

transform as a symplectic vector too. The Schwinger-Zwanziger quantization condition restricts these charges to a lattice. Invariance of this lattice restricts the symplectic transformations to a discrete subgroup $Sp(2m, \mathbb{Z})$. We will not be concerned with this quantum aspect in this article.

To end this section, we recall some general properties of symplectic transformations that we will use in the following sections. For more details and proofs we refer to appendix A.

If a $2m \times m$ matrix

$$V = \begin{pmatrix} X^I \\ Y_I \end{pmatrix} \quad (2.13)$$

is of rank m , there exist (lemma A.1) symplectic transformations $\tilde{V} = \mathcal{S}V$ such that the upper half of \tilde{V} constitutes an invertible $m \times m$ matrix. For such a $2m \times m$ matrix V with invertible upper part X , one can define a square matrix

$$\mathcal{N} = Y X^{-1}. \quad (2.14)$$

If we take all m columns of V to transform as a symplectic vector, the matrix \mathcal{N} transforms under symplectic transformations into

$$\tilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} \quad (2.15)$$

with \mathcal{S} as in (2.8). The inverse of $A + B\mathcal{N}$ exists when \mathcal{N} is a symmetric matrix with a negative definite imaginary part; the negative definiteness and symmetry are preserved by the transformation in (2.15) (corollary A.4).

3 Rigid special geometry

3.1 The vector superfield

The superfield which is our starting point is the chiral superfield. The $N = 2$ superspace is built with anticommuting coordinates θ_α^i , and $\theta_{\alpha i}$, where the position of the index i indicates the chirality³. A *chiral* superfield is obtained by imposing the constraint $D^{\alpha i}\Phi = 0$, where $D^{\alpha i}$ indicates a (chiral) covariant derivative in superspace. The superfield Φ is complex, and can be expanded as

$$\Phi = A + \theta_\alpha^i \Psi_i^\alpha + \mathcal{C}^{\alpha\beta} \theta_\alpha^i \theta_\beta^j B_{ij} + \epsilon_{ij} \theta_\alpha^i \theta_\beta^j \mathcal{F}^{\alpha\beta} + \dots$$

One ends up with the set of fields

$$(A, \Psi_i, B_{ij}, \mathcal{F}^{\alpha\beta}, \Lambda_i, C) \quad (3.1)$$

Here, A and C are complex scalars, and Ψ_i and Λ_i are two doublet of spinors. The field B_{ij} is symmetric. Finally, as to $\mathcal{F}^{\alpha\beta}$, due to the chirality and the symmetry, it can be written as $\mathcal{F}^{\alpha\beta} = \sigma_{ab}^{\alpha\beta} \mathcal{F}^{ab-}$, where \mathcal{F}^{ab-} is an antisymmetric antiselfdual tensor. The complex conjugate superfield $\bar{\Phi}$ contains the corresponding selfdual part.

In $N = 2$ minimal off-shell multiplets have 8+8 real components. The chiral multiplet has 16+16 components and is a reducible multiplet. The *vector* multiplet [17] is an irreducible

³Recent reviews on several aspects of $N = 2$ supersymmetry are in [8] and [16]. We use the conventions explained in the appendix of the latter.

8+8 part of this chiral multiplet. The remaining components are combined into a *linear* multiplet, see below. The reduction is accomplished by an additional constraint, which in superspace reads

$$\left(\epsilon_{ij}\bar{D}^i\sigma_{ab}D^j\right)^2\bar{\Phi} = \mp 24\Box\Phi. \quad (3.2)$$

Both signs are possible as constraint; we will choose the upper one. In components, this is equivalent to the conditions

$$\begin{aligned} H &\equiv C + 2\partial_a\partial^a\bar{A} = 0 \\ L_{ij} &\equiv B_{ij} - \epsilon_{ik}\epsilon_{j\ell}\bar{B}^{k\ell} = 0 \\ \phi_i &\equiv \not{\partial}\Psi^i - \epsilon^{ij}\Lambda_j = 0 \\ E^a &\equiv \partial_b(\mathcal{F}^{+ab} - \mathcal{F}^{-ab}) = 0, \end{aligned} \quad (3.3)$$

up to some constants⁴.

The equation on B_{ij} is a reality condition. It leaves in B_{ij} only 3 free real components. Two of the other equations define Λ and C in terms of Ψ and A . The remaining one is a Bianchi identity for \mathcal{F}_{ab} , expressing that locally it is the derivative of a vector potential: hence the name vector multiplet for this constrained multiplet. To distinguish it from the general chiral superfields, we use a different notation for the independent fields of the vector multiplet, viz. $(X; \Omega_i; Y_{ij}; \mathcal{F}_{ab}^-)$.

3.2 Actions for $N = 2$ vector multiplets

A recipe for obtaining supersymmetric actions for these vector multiplets has been known for some time [3, 1]. For an arbitrary holomorphic function F of n constrained chiral multiplets Φ^A ($A = 1, \dots, n$), the superfield $F(\Phi^A)$ is again chiral. Upon integration over the chiral $N = 2$ superspace the action for the vector multiplets follows:

$$\int d^4x \int d^4\theta F(\Phi^A) + c.c. \quad (3.4)$$

Expanding, this gives rise to a scalar field Lagrangian

$$\mathcal{L}_0 = -g_{A\bar{B}}(X, \bar{X}) \partial_\mu X^A \partial^\mu \bar{X}^{\bar{B}}, \quad (3.5)$$

while the coupling of the scalars to the vectors is described as in (1.1) by the matrix

$$\mathcal{N}_{AB} = \bar{F}_{AB} \equiv \frac{\partial}{\partial \bar{X}^A} \frac{\partial}{\partial \bar{X}^B} \bar{F}, \quad (3.6)$$

where we introduced the notation

$$F_A \equiv \frac{\partial}{\partial X^A} F(X); \quad \bar{F}_A \equiv \frac{\partial}{\partial \bar{X}^A} \bar{F}(\bar{X}), \quad (3.7)$$

and so on for multiple indices. The metric in (3.5) is

$$\begin{aligned} g_{A\bar{B}}(X, \bar{X}) &= \partial_A \partial_{\bar{B}} K(X, \bar{X}) \\ K(X, \bar{X}) &= i(\bar{F}_A(\bar{X})X^A - F_A(X)\bar{X}^{\bar{A}}), \end{aligned} \quad (3.8)$$

⁴These constants are relevant for Fayet-Iliopoulos terms [18, 1], used recently for partial breaking of $N = 2$ supersymmetry [19].

the first equation expressing that it is of Kählerian type.

As is common for sigma models, the scalars can be interpreted as coordinates on a manifold, say \mathcal{M} , at least locally on some chart. The scalar Lagrangian provides the metric on \mathcal{M} , thus turning it into a Kähler manifold. Let z^α , for $\alpha = 1, \dots, n$, be complex coordinates in some patch such that the X^A depend on them holomorphically⁵. A more general parametrisation of the scalar degrees of freedom is obtained by using the z^α to describe them. The Kähler potential depends on the z^α but only through the holomorphic $X(z)$ dependence, so that the metric retains its hermitian form as in (3.5). For definiteness, we give the usual normalisation for the Kähler form:

$$\mathcal{K} \equiv \frac{i}{2\pi} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta = \frac{i}{2\pi} \partial_\alpha \partial_{\bar{\beta}} K dz^\alpha \wedge d\bar{z}^\beta = \frac{i}{2\pi} \partial \bar{\partial} K . \quad (3.9)$$

Note that the positivity condition for the Kähler metric,

$$g_{\alpha\bar{\beta}} = 2 \left(\partial_\alpha X^A \right) \text{Im} F_{AB} \left(\partial_{\bar{\beta}} \bar{X}^B \right) , \quad (3.10)$$

is the same as the requirement of negative definiteness of $\text{Im} \mathcal{N}_{AB}$, and thus guarantees the correct sign for the kinetic energies of the vectors. As already mentioned, this condition is preserved by symplectic transformations (2.6) (see corollary A.4). A further consequence of this positivity condition is that the following matrix is guaranteed to be invertible:

$$e_\alpha{}^A \equiv \partial_\alpha X^A . \quad (3.11)$$

One obvious choice of coordinates is then to identify the z^α with X^A , which reduces $e_\alpha{}^A$ to the unit matrix. These coordinates are called special coordinates. Whereas one can make general coordinate transformations on the z^α , the X^A have a meaning as lowest coordinates of superfields with constraints which are only invariant under *linear* transformations. That difference, which was stressed in [4], will play an important role below.

Different functions $F(X)$ may give rise to the same Lagrangian (up to a total divergence):

$$F \approx F + a + q_A X^A + c_{AB} X^A X^B , \quad (3.12)$$

where a and q_A can be complex, but c_{AB} should be real. These transformations are an invariance of the Kähler metric (3.8), and add to \mathcal{N} a real number, which leads in (1.1) only to an extra total divergence term.

3.3 Duality transformations in superspace

There is a remarkable interplay between the symplectic duality transformations and the supersymmetry.

The action of symplectic transformations on the vector fields, (2.5), requires that also the matrix \mathcal{N} transforms in a specific way, (2.6). The required transformation results if one assumes that

$$V = \begin{pmatrix} X^A \\ F_A \end{pmatrix} \quad (3.13)$$

⁵This implies the choice of a complex structure.

is a symplectic vector. However, X^A was taken to be the scalar of a $N = 2$ *vector* superfield, whereas F_A generically is a scalar of a *chiral* superfield. To make this situation more transparent, we adopt a superspace approach⁶.

Given a set of chiral superfields and a holomorphic function $F(X^A)$, the following superfields can be built

$$\begin{aligned}\Phi_X^A &= \Phi^A = X^A + \theta^i \Psi_i^A + \dots ; \\ \Phi_{F,A} &= \frac{\partial}{\partial \Phi^A} F(\Phi) = F_A(X) + \theta^i \Psi_{F,Ai} + \dots .\end{aligned}\quad (3.14)$$

Here Φ^A are independent chiral superfields, and the components of $\Phi_{F,A}$ are functions of the components of Φ^A , e.g.

$$\Psi_{F,Ai} = F_{AB} \Psi_i^B ; \quad \mathcal{F}_{F,A}^{-ab} = F_{AB} \mathcal{F}^{-abB} - \frac{1}{2} F_{ABC} \epsilon^{ij} \bar{\Psi}_i^B \sigma^{ab} \Psi_j^C . \quad (3.15)$$

Both sets are combined into a $2n$ component superfield which at the same time should be a symplectic vector:

$$\begin{pmatrix} \Phi_X^A \\ \Phi_F^A \end{pmatrix} \equiv \Phi_V = V + \theta^i \Psi_{V,i} + \dots . \quad (3.16)$$

It is therefore natural to impose (3.2) not just on the upper components of V , as we did in subsection 3.2 to reduce the chiral superfields to vector superfields, but on all components:

$$\left(\epsilon_{ij} \bar{D}^i \sigma_{ab} D^j \right)^2 \bar{\Phi}_V = -24 \square \Phi_V . \quad (3.17)$$

It turns out that the lower components of this constraint are the field equations of the action (3.4). So these equations contain *all* the dynamical constraints on the $2n$ chiral superfields. In particular one finds

$$\partial_b \begin{pmatrix} \mathcal{F}^{+ab,A} - \mathcal{F}^{-ab,A} \\ \mathcal{F}_{F,A}^{+ab} - \mathcal{F}_{F,A}^{-ab} \end{pmatrix} = 0 . \quad (3.18)$$

corresponding to E_a in (3.3). Inserting the explicit form for \mathcal{F}_F^{-ab} , (3.15), this becomes

$$\partial_b \begin{pmatrix} \text{Im } \mathcal{F}^{+ab,A} \\ \text{Im } \left(\mathcal{N}_{AB} \mathcal{F}^{+ab,B} + \frac{1}{2} F_{ABC} \epsilon^{ij} \Omega_i^B \sigma^{ab} \Omega_j^C \right) \end{pmatrix} = 0 . \quad (3.19)$$

We changed notation from Ψ to Ω , since now we are working with vector multiplets. The upper equations are the Bianchi identities that reflect this. The lower components are the field equations for the vector of the full action; for $\Omega = 0$, one recovers those that went with \mathcal{L}_1 of the previous section.

The set of constraints (3.17) is invariant under real linear transformations $GL(2n, \mathbb{R})$ on Φ_V . However, requiring that the lower superfields in the vector Φ_V are again derivatives of some function $F(\Phi)$ imposes the integrability condition $\partial_{[A} \tilde{F}_{B]} = 0$, which leads again to (2.7), restricting the invariances of the constraints to $Sp(2n, \mathbb{R})$. This could be expected from the vector part exhibit in the previous paragraph.

In conclusion, the full superfields Φ_V are now symplectic vectors, in particular the lowest components (3.13). The Kähler potential can now be written as a symplectic invariant:

$$K = i \langle V, \bar{V} \rangle , \quad (3.20)$$

⁶This way of presenting the duality transformations came up during a discussion with P. West.

making obvious the symplectic invariance of the scalar field action⁷.

Introducing arbitrary coordinates z^α on \mathcal{M} as in the previous section, the gradient of V defines the following $2n \times n$ matrix :

$$U_\alpha \equiv \begin{pmatrix} e^A_\alpha \\ h_{A\alpha} \end{pmatrix} \equiv \partial_\alpha V . \quad (3.21)$$

The metric,

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K = i \langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle , \quad (3.22)$$

is assumed to be positive. This implies that e^A_α is invertible (by a simplification of the argument in lemma B.5). Therefore we can use (2.14) to define a matrix which transforms appropriately under the symplectic transformations. The matrix \mathcal{N} introduced in (3.6) is in fact the complex conjugate of this,

$$\mathcal{N}_{AB} = \bar{h}_{A\bar{\alpha}} \bar{e}^{\bar{\alpha}}_B . \quad (3.23)$$

In particular, it will automatically be symmetric.

The Kähler potential is invariant under symplectic rotations. But as is well-known this invariance is not really required. If we combine different patches so as to cover a Kähler manifold, the following transformation of K preserves the Kähler property relating the local expressions for the metric and the potential:

$$K_{(i)}(z, \bar{z}) = K_{(j)}(z, \bar{z}) + f_{ij}(z) + \bar{f}_{ij}(\bar{z}) , \quad (3.24)$$

where f_{ij} is a holomorphic function on the overlap region between the patches. This is usually denoted as a Kähler transformation. In terms of the symplectic vector it translates into the freedom to perform a constant complex shift, $V' = V + b$, where b is some $2n$ -component complex vector. Another way to explain the origin of the inhomogeneous shift is that it reflects the linear part in the equivalence relation (3.12) relating different functions. The demand that the shift be constant originates in the fact that it should not affect \mathcal{N} . Likewise a constant phase shift of V neither changes K nor \mathcal{N} . In conclusion, the group of symplectic transformations can be extended with these constant shifts: we will denote⁸ this extension as $ISp(2n, \mathbb{R}) \times U(1)$, where the last factor reflects the phase rotations.

In this context it is clear that X^A and the z^α have a different role to play, even though the *values* of the X^A can be used as coordinates ('special coordinates') as mentioned before. In the viewpoint we adopt, the X^A are part of V which should be considered as a section of a specific vector bundle over \mathcal{M} , whereas z^α are coordinates in some chart of \mathcal{M} . In the bundle there are two groups acting *independently*. First, there are reparametrisations acting on the z^α , not transforming the sections. Second, one has the structure group (i.e. $ISp(2n, \mathbb{R}) \times U(1)$) acting on the fiber.

⁷Note that when one identifies the coordinates with the X^A then the symplectic transformation induces also a coordinate transformation $\tilde{X}^A = A^A_B X^B + B^{AB} F_B(X)$. This coordinate transformation, combined with the symplectic transformation, is only an invariance of the action if the function $\tilde{F}(X)$ is equal to $F(X)$, which is only true for a subset of the symplectic transformations, which are the *proper symmetries* (or hidden symmetries) discussed e.g. in [1, 20, 7, 21] (within supergravity).

⁸Warning on the notation: the restriction to \mathbb{R} applies to the homogeneous part only. The inhomogeneous part contains *complex* shifts.

3.4 Definitions of rigid special Kähler manifolds

Now we turn to our definitions of a rigid special Kähler manifold. The main criterion is that these manifolds should be obtainable from the scalar sector of (rigid) supersymmetric models. Historically as well, it was the analogue of this definition in the supergravity case to be discussed in the next section, that was the basis for various other proposed formulations.

In the first definition we use the construction from a chiral superfield F , as sketched above.

3.4.1 Definition 1

An n -dimensional Kähler⁹ manifold is said to be special Kähler if it satisfies the following conditions:

1. On every chart there are n independent holomorphic functions $X^A(z)$, where $A = 1, \dots, n$ and a holomorphic function $F(X)$ such that

$$K(z, \bar{z}) = i \left(X^A \frac{\partial}{\partial \bar{X}^A} \bar{F}(\bar{X}) - \bar{X}^A \frac{\partial}{\partial X^A} F(X) \right) ; \quad (3.25)$$

2. On overlaps of charts i and j there are transition functions of the following form:

$$\left(\frac{X}{\partial F} \right)_{(i)} = e^{ic_{ij}} M_{ij} \left(\frac{X}{\partial F} \right)_{(j)} + b_{ij} , \quad (3.26)$$

with

$$c_{ij} \in \mathbb{R} ; \quad M_{ij} \in Sp(2n, \mathbb{R}) ; \quad b_{ij} \in \mathbb{C}^{2n} ; \quad (3.27)$$

3. The transition functions satisfy the cocycle condition on overlaps of 3 charts.

From this definition it should be clear that the essential concept is some specific vector bundle over a Kähler manifold. To reveal the link with $N = 2$ supersymmetric models, first note that the X^A coincide with the complex scalar fields of the previous section. The holomorphy of the function F is then necessitated by the demand that $F(X)$ be the lowest component of a chiral superfield containing the lagrangian. F thus corresponds to the prepotential in the field theoretic picture. As has been discussed in section 3.3 the transition functions in (3.26) allow an interpretation as duality transformations.

It is clear that this definition applies to the manifold \mathcal{M} as constructed from the action in the previous subsections. Conversely, from the ingredients of definition 1, one has directly at one's disposal the Kähler manifold for the scalar field action, the matrix \mathcal{N} , defined by (3.6), for the spin-1 action, and also all other functions which enter in the construction of the rest of the action (as this is defined by (3.4)).

The previous definition heavily relies on the prepotential function $F(X)$, which is not a symplectically invariant object. From the supersymmetric field theory point of view it is the most natural definition, though. In a second definition, inspired by Strominger's definition [2] for local special geometry, we rest more heavily on the symplectic bundle structure.

⁹We always suppose here and below that the metric is positive definite.

3.4.2 Definition 2

A rigid special Kähler manifold is a Kähler manifold \mathcal{M} for which

1. there exists a $U(1) \times \text{Is}p(2n, \mathbb{R})$ vector bundle over \mathcal{M} with constant transition functions, in the sense of (3.26), i.e. with a complex inhomogeneous part. This bundle should have a holomorphic section V such that the Kähler form (3.9) is

$$\mathcal{K} = -\frac{1}{2\pi} \partial \bar{\partial} \langle V, \bar{V} \rangle , \quad (3.28)$$

2. and such that

$$\langle \partial_\alpha V, \partial_\beta V \rangle = 0 . \quad (3.29)$$

The brackets stand for the symplectic inproduct introduced in the previous section. To establish the equivalence of both formulations one first notes that the bundle properties 1 from definition 2 are made explicit in definition 1 both through (3.26) and the consistency condition 3. One further notices that (3.25) is the local equivalent of (3.28), after it has been established that V is of the form (3.13), i.e. that the lower components of V are the derivatives of a scalar function $F(X)$. Finally, we draw special attention to (3.29), which has no analogue in [2]. We will see that this additional condition is appropriate, since it leads to the local existence of a function F and guarantees the symmetry of the matrix \mathcal{N} which we will define.

We now prove the remaining issue of the local existence of a function F . As argued after (3.22), the matrix e^A_α appearing in (3.21), is invertible. For the lower components of V one can use the notation as in (3.13), but one should realise that the F_A are so far just functions of the coordinates z^α , and it remains to be shown that they are of the form (3.7). The invertibility of e^A_α allows us to define z^α as function of X^A , and therefore also the $F_A(z)$ become functions of X . We can then calculate

$$\frac{\partial}{\partial X^A} F_B = e^\alpha_A h_{B\alpha} = e^\alpha_A (h_{C\alpha} e^C_\beta) e^\beta_B \quad (3.30)$$

and this is symmetric due to (3.29). This is the integrability condition that proves the local existence of $F(X)$.

To complete the proof of the equivalence, by constructing all objects appearing in the action, we define \mathcal{N} with (3.23). The condition (3.29) then assures that the matrix \mathcal{N} is symmetric.

We would like to present yet another formulation of rigid special manifolds. As will be pointed out it will prove to be quite well-adapted to show that certain subspaces of Riemann surface moduli spaces exhibit special geometry.

3.4.3 Definition 3

A rigid special Kähler manifold is a complex manifold \mathcal{M} with on each chart $2n$ closed holomorphic 1-forms $U_\alpha dz^\alpha$:

$$\partial_{\bar{\alpha}} U_\beta = 0 ; \quad \partial_{[\alpha} U_{\beta]} = 0 , \quad (3.31)$$

such that

1.

$$\langle U_\alpha, U_\beta \rangle = 0 ; \quad (3.32)$$

$$g_{\alpha\bar{\beta}} = i\langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle \quad \text{is the (positive definite) Kähler metric ;} \quad (3.33)$$

2. the transition functions on overlap regions are expressed as

$$U_{\alpha,(i)} dz_{(i)}^\alpha = e^{ic_{ij}} M_{ij} U_{\alpha,(j)} dz_{(j)}^\alpha , \quad (3.34)$$

with $c_{ij} \in \mathbb{R}$ and $M_{ij} \in Sp(2n, \mathbb{R})$;

3. the cocycle conditions are satisfied on overlaps of 3 charts.

To establish the equivalence with the former definitions one should first notice that (3.31) imply the local existence of holomorphic vectors $V_{(i)}$ such that $U_{\beta,(i)} = \partial_\beta V_{(i)}$. Eq. (3.34) then guarantees that the $V_{(i)}$ combine globally into the section V of the product bundle in definition 2. Note the relevance of the transition functions in (3.34) being constant. With all this (3.29) coincides with (3.32). Finally, (3.33) is to be understood as defining the metric in this approach, which by (3.31) is guaranteed to be of Kählerian type, but which we should demand to be positive. In the transition functions, the inhomogeneous term (see (3.26)) is no longer present. It reappears in the integration from U_α to V by (3.31).

Matrix formulation. These constraints of this third definition can be concisely formulated in another way, which will also provide a link to previous formulations [22, 6, 9]. One starts from n symplectic vectors $U_\alpha(z, \bar{z})$ over a chart and their complex conjugates and defines the $2n \times 2n$ matrix field

$$\mathcal{V}(z, \bar{z}) \equiv \begin{pmatrix} U_\alpha^T \\ \bar{U}_{\bar{\alpha}}^T \end{pmatrix} . \quad (3.35)$$

One demands the condition

$$\mathcal{V} \Omega \mathcal{V}^T = \begin{pmatrix} 0 & -ig_{\alpha\bar{\beta}} \\ ig_{\beta\bar{\alpha}} & 0 \end{pmatrix} , \quad (3.36)$$

where Ω is the symplectic metric as in (2.9). This is thus the rewriting of (3.32) and (3.33). Then define

$$\hat{\mathcal{A}}_\alpha = \partial_\alpha \mathcal{V} \mathcal{V}^{-1} ; \quad \hat{\mathcal{A}}_{\bar{\alpha}} = \partial_{\bar{\alpha}} \mathcal{V} \mathcal{V}^{-1} , \quad (3.37)$$

and impose the constraints

$$\hat{\mathcal{A}}_\alpha = \begin{pmatrix} G_{(\alpha,\beta)}^\gamma & C_{(\alpha,\beta)}^{\bar{\gamma}} \\ 0 & 0 \end{pmatrix} ; \quad \hat{\mathcal{A}}_{\bar{\alpha}} = \begin{pmatrix} 0 & 0 \\ \bar{C}_{\bar{\alpha},\bar{\beta}}^{\gamma} & G_{\bar{\alpha},\bar{\beta}}^{\bar{\gamma}} \end{pmatrix} , \quad (3.38)$$

satisfying the indicated symmetry requirements (the form of $\hat{\mathcal{A}}_{\bar{\alpha}}$ follows by complex conjugation). These requirements are equivalent to the conditions (3.31).

One can further determine and simplify the matrix relations by using (3.36):

$$\begin{aligned} (\hat{\mathcal{A}})_\beta^\gamma \begin{pmatrix} 0 & -ig_{\gamma\bar{\delta}} \\ ig_{\delta\bar{\gamma}} & 0 \end{pmatrix} &= (\partial_\alpha \mathcal{V} \Omega \mathcal{V}^T)_{\beta\delta} \\ &= \partial_\alpha \begin{pmatrix} 0 & -ig_{\beta\bar{\delta}} \\ ig_{\delta\bar{\beta}} & 0 \end{pmatrix} - (\mathcal{V} \Omega \partial_\alpha \mathcal{V}^T)_{\beta\delta} \end{aligned} \quad (3.39)$$

The last term of the second line is the transpose of the left hand side of the first line. Furthermore, from (3.38) we have that

$$(\hat{\mathcal{A}})_{\beta}^{\gamma} \begin{pmatrix} 0 & -ig_{\gamma\bar{\delta}} \\ ig_{\delta\bar{\gamma}} & 0 \end{pmatrix} = \begin{pmatrix} iC_{(\alpha,\beta)\delta} & -iG_{(\alpha,\beta)\bar{\delta}} \\ 0 & 0 \end{pmatrix}, \quad (3.40)$$

(indices lowered by the metric $g_{\alpha\bar{\beta}}$), i.e. symmetric in $(\alpha\beta)$. Therefore, taking the second line of (3.39) minus the transpose of the first, we obtain

$$\begin{pmatrix} iC_{(\alpha,\beta)\delta} - iC_{(\alpha,\delta)\beta} & -iG_{(\alpha,\beta)\bar{\delta}} \\ iG_{(\alpha,\delta)\bar{\beta}} & 0 \end{pmatrix} = \partial_{\alpha} \begin{pmatrix} 0 & -ig_{\beta\bar{\delta}} \\ ig_{\delta\bar{\beta}} & 0 \end{pmatrix}. \quad (3.41)$$

It implies first that C is a 3-index symmetric tensor¹⁰. Furthermore it implies that $\partial_{[\alpha}g_{\beta]\bar{\gamma}} = 0$, i.e. that this is a Kähler metric. Therefore it is appropriate to define covariant derivatives with Levi-Civita connection:

$$\Gamma_{\alpha\beta}^{\gamma} = g^{\gamma\bar{\delta}}\partial_{\beta}g_{\alpha\bar{\delta}}; \quad \mathcal{D}_{\alpha}U_{\beta} = \partial_{\alpha}U_{\beta} - \Gamma_{\alpha\beta}^{\gamma}U_{\gamma}; \quad \mathcal{D}_{\bar{\alpha}}U_{\beta} = \partial_{\bar{\alpha}}U_{\beta}. \quad (3.42)$$

The tensors G are thus these connections, and the differential equations can be simplified to [22] :

$$\begin{aligned} \mathcal{D}_{\alpha}\mathcal{V} &= \mathcal{A}_{\alpha}\mathcal{V}; & \mathcal{D}_{\bar{\alpha}}\mathcal{V} &= \mathcal{A}_{\bar{\alpha}}\mathcal{V} \\ \mathcal{A}_{\alpha} &= \begin{pmatrix} 0 & C_{\alpha\beta}^{\bar{\gamma}} \\ 0 & 0 \end{pmatrix}; & \mathcal{A}_{\bar{\alpha}} &= \begin{pmatrix} 0 & 0 \\ \bar{C}_{\bar{\alpha}\bar{\beta}}^{\gamma} & 0 \end{pmatrix}. \end{aligned} \quad (3.43)$$

for a symmetric $C_{\alpha\beta\gamma}$. From the commutator of covariant derivatives, the following curvature formula is then easily derived

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -C_{\alpha\gamma\epsilon}\bar{C}_{\bar{\beta}\bar{\delta}\bar{\epsilon}}g^{\epsilon\bar{\epsilon}}. \quad (3.44)$$

3.5 Riemann surfaces moduli spaces

After the original formulation of *local* special geometry [1], a notion to be developed in the next sections, it was soon found out that the defining constraints are realised in Calabi-Yau moduli spaces [23]. In the study of *rigid* $N = 2$ super Yang-Mills models, Riemann surface moduli spaces turn out to be a candidate to take over the role of Calabi-Yau moduli spaces [24]. A relevant question therefore is the following. Do there exist r -dimensional subspaces of the full moduli space of genus g Riemann surfaces that obey the constraints dictated by rigid special geometry? A partial answer to this is the main subject of this section. Definition 3 is the formulation which contains the data by which such an identification is most easily established. Most of its constraints are automatically satisfied, and we will see that (3.31) is the crucial one to verify.

3.5.1 Useful concepts of Riemann surfaces

Let us first recall some facts from Riemann surface theory. For definiteness consider a genus g surface. As is well-known the first homology group then has $2g$ generators A^A, B_A with

¹⁰One can check that $C_{\alpha\beta\gamma} = ie_{\alpha}^A e_{\beta}^B e_{\gamma}^C F_{ABC}$.

index $A = 1, \dots, g$ which can be fixed to be canonical, i.e. with a particular intersection matrix defined by the following relations:

$$\begin{aligned} A^A \cap A^B &= 0 ; & B_A \cap B_B &= 0 ; \\ A^A \cap B_B &= -B_B \cap A^A = \delta_B^A . \end{aligned} \quad (3.45)$$

The $2g$ component vector consisting of integrals of a one-form ω along a canonical homology basis

$$\begin{pmatrix} \int_{A^A} \omega \\ \int_{B_A} \omega \end{pmatrix}$$

is referred to as the period vector of ω w.r.t that homology basis.

For any pair of closed one-forms ω and χ ,

$$\sum_A \left[\int_{A^A} \omega \cdot \int_{B_A} \chi - \int_{B_A} \omega \cdot \int_{A^A} \chi \right] = \int \int \omega \wedge \chi , \quad (3.46)$$

where the integral at the right hand side is over the Riemann surface.

Let $\{\lambda_\alpha\}$ with $\alpha = 1 \dots g$, be some basis for the one forms that are holomorphic w.r.t. the complex structure on the Riemann surface. The matrices

$$e^A_\alpha \equiv \int_{A^A} \lambda_\alpha ; \quad h_{A\alpha} \equiv \int_{B_A} \lambda_\alpha , \quad (3.47)$$

are both invertible. From these one defines the so-called period matrix Π :

$$\Pi_{AB} = h_{A\alpha} (e^{-1})^\alpha_B . \quad (3.48)$$

Using (3.46) with two holomorphic forms, λ_α and λ_β , the right hand side is zero, which is referred to as Riemann's first relation [25]. Therefore

$$0 = [e^A_\alpha h_{A\beta} - h_{A\alpha} e^A_\beta] = e^A_\alpha [h_{A\gamma} (e^{-1})^\gamma_B - (e^{-1})^\gamma_A h_{B\gamma}] e^B_\beta = e^A_\alpha [\Pi_{AB} - \Pi_{BA}] e^B_\beta , \quad (3.49)$$

so that Riemann's first relation expresses the symmetry of the period matrix.

Using (3.46) with the forms λ_α and their complex conjugates $\bar{\lambda}_\beta$, the right hand side is a positive matrix times $dx d\bar{x}$ (with x the holomorphic coordinate on the Riemann surface), and the surface can be oriented such that this is $-i$ times a positive quantity. Therefore we have

$$0 < i [e^A_\alpha \bar{h}_{A\beta} - h_{A\alpha} \bar{e}^A_\beta] = e^A_\alpha [i \bar{h}_{A\gamma} (\bar{e}^{-1})^\gamma_B - i (e^{-1})^\gamma_A h_{B\gamma}] \bar{e}^B_\beta = e^A_\alpha [i \bar{\Pi}_{AB} - i \Pi_{BA}] \bar{e}^B_\beta , \quad (3.50)$$

which is Riemann's second relation [25] :

$$\text{Im } \Pi > 0 . \quad (3.51)$$

It is to be noticed that both Riemann's relations (3.49) and (3.51) are invariant under a symplectic change of homology basis, taking a canonical basis into another one, preserving the intersection matrix. This follows from a general fact about symplectic transformations (see Corollary A.4).

3.5.2 Special moduli spaces of Riemann surfaces

Now we turn to the issue of special geometry in some moduli space of Riemann surfaces. It is clear that the properties of the period matrix Π make it extremely well suited to play the role of the matrix \mathcal{N} . In general however the dimensions do not agree: the period matrix has dimension g , whereas the dimension of the moduli space is generically larger. We will therefore consider an n -dimensional *subspace* of the moduli space. It will turn out that in the simplest case $n = g$ a special geometry structure is almost automatic, which motivated the consideration of these spaces for the construction of special geometries in the first place, but we will not limit ourselves to that case.

Let \mathcal{W} represent some family of genus g Riemann surfaces parametrized by n complex moduli, where $n \leq g$. Let γ_α , $\alpha = 1, \dots, n$ be a set of cohomologically independent 1-forms that are holomorphic with respect to the complex structure on the Riemann surfaces. For $n < g$ these span a subspace of the holomorphic 1-forms, $H^{(1,0)}(\mathcal{W})$. The number of 1-forms chosen is equal to the complex dimension of the considered moduli space.

Let A^A , B_A with $A = 1, \dots, n$ be some set of 1-cycles satisfying again (3.45), spanning a proper subspace of the first homology group if $n < g$. Identify now

$$U_\alpha = \begin{pmatrix} \int_{A^A} \gamma_\alpha \\ \int_{B_B} \gamma_\alpha \end{pmatrix}. \quad (3.52)$$

These 'periods' are then complex functions of the moduli, which are to be identified with the functions U_α introduced abstractly in definition 3.

The first constraint of (3.31) says that the moduli should be chosen such that these periods depend holomorphically on the moduli z^α . A convenient way to realize this automatically is to define the family of Riemann surfaces in a complex weighted projective space by an equation that is holomorphic in the variables of the projective space as well as in the moduli. The 1-forms can then be defined using the Griffiths residue map [26]. In that case this holomorphicity condition of U_α is immediate. We defer the discussion of the second condition of (3.31), and first take a look at the other elements of definition 3.

If $n = g$ then (3.32) is (3.49), Riemann's first relation. Then also (3.33), defining the metric, leads automatically to a positive definite metric, as it is (3.50), Riemann's second relation. The symplectic transformations, which appear in (3.26), and which in field theory embody duality transformations are now realised by canonical homology basis changes. Note that the restriction to integer symplectic matrices, $Sp(2n, \mathbb{Z})$, which occurs in the quantum theory, is also natural from this point of view. Finally, as already mentioned, $\tilde{\mathcal{N}}_{AB}$ is the period matrix Π_{AB} , (3.48). If $n < g$, we can not apply (3.46) but we will see some examples in the next subsection that show how Riemann's relations, combined with symmetries, may still suffice to prove (3.32) and (3.33). In the general case however, to which we now return, an explicit check of these conditions seems required. The symplectic group can still be realised by canonical changes of basis for the subspace of cycles one is considering, but the restriction of the period matrix Π_{AB} to the subset of forms and cycles, which should play the role of $\tilde{\mathcal{N}}_{AB}$, may not automatically have a symmetric and positive definite imaginary part. Again, in some examples, selecting the subspaces by some symmetry requirement suffices to remedy this.

Crucial for the metric to be Kählerian as well as for the existence of the prepotential $F(X)$ is the second constraint in (3.31). It is not known what the geometric significance is, of moduli spaces which satisfy such a constraint. If the family of Riemann surfaces is

defined in an embedding space (e.g. the weighted projective space mentioned above), and the cycles can be constructed in a way that is moduli independent (in some local patch of moduli space), then the condition translates into an integrability condition on the 1-forms:

$$\partial_{[\alpha}\gamma_{\beta]} = (d\eta)_{\alpha\beta} . \quad (3.53)$$

If the right hand side is zero, as in most examples we know, then locally there should exist a (meromorphic) 1-form λ whose derivatives give the holomorphic 1-forms. The form λ should have zero residues at its poles, such that its integrals over the 1-cycles, which are used to construct the symplectic vector V , are well-defined. Notice that we have used a coordinate-index α to label the n one forms γ_α , thus implicitly associating each of the one forms to a modulus. Apart from matching the dimensions one has to make sure that the (3.53) actually holds, for the chosen moduli subspace to be rigid special Kähler.

The conditions on the transition functions and cocycle conditions are also automatically satisfied if the cycles and forms are either a complete basis of those available on the Riemann surface, or if they are selected by some symmetry requirement.

At present it is not clear whether the outlined procedure may yield any rigid special Kähler subspace in the generic case for arbitrary g . This has to be contrasted with the case of Calabi-Yau moduli spaces, which always obey the local special geometry constraints. This will be discussed later.

3.5.3 Examples

As a first example, where $n = g$, we sketch briefly how the proposed $SU(n+1)$ curves [27] in $N = 2$ SYM fit into the scheme. The starting point is some genus n family of Riemann surfaces, corresponding to the rank of the gauge group. In general the full moduli space is $3g - 3$ (complex) dimensional, but it is reduced from the outset to $2g - 1$, taking only the class of hyperelliptic surfaces into consideration. A further reduction a posteriori, giving rise to the right number of moduli (i.e. n or equivalently, g) is performed through symmetry arguments (for a detailed discussion see [27]). One thus ends up with a g moduli dependent genus g family. The remaining integrability condition (3.53) is met as an explicit expression for the meromorphic 1-form λ is derived. As a consequence, the reduced moduli space is rigid special Kähler.

It is to be noticed how the notion of the symplectic group of Definition 3 arises in this context. First taking into account the identification (3.52), the period matrix Π of the Riemann surfaces is the kinetic vector coupling matrix $\tilde{\mathcal{N}}$. With this the symplectic transformations in the context of rigid special Kähler manifolds can be thought of as the canonical homology basis changes. Apart from the natural interpretation of the structure group ($ISp(2g, \mathbb{R})$ up to phase transformations) in the context of surfaces, the key point in matching the genus g and n , the dimension of the associated moduli space, resides in that the homology and cohomology groups have the right dimensionality to insure the validity of both Riemann's relations (3.49),(3.51).

In the above the main step was the isolation of the special Kähler moduli subspace within the full $3g - 3$ dimensional moduli space of genus g curves.

An extra complication arises when $n < g$, which we find e.g. in [28]. Specific families of surfaces are presented there as candidate curves for solving $N = 2$ SYM with gauge groups $SO(2n+1)$ and $SO(2n)$ respectively. One considers in both cases Riemann surfaces of genus

$g = 2n - 1$, represented by hyperelliptic curves whose defining equation $y^2 = P(x, z^\alpha)$ has a symmetry $P(-x, z^\alpha) = P(x, z^\alpha)$. There is a meromorphic form λ which is odd under the symmetry, and whose derivatives with respect to the moduli give n (odd) holomorphic 1-forms. Therefore the conditions (3.31) are already satisfied. The $2g = 2(2n - 1)$ one-cycles can be split in $2(n - 1)$ even and $2n$ odd under $x \leftrightarrow -x$. So one can restrict to the latter for the setup described above. The forms $\gamma_\alpha = \partial_\alpha \lambda$ can be completed to a basis of the g dimensional cohomology of $(1, 0)$ -forms by $(n - 1)$ forms which are even under the symmetry (all this was done explicitly in [29]). Therefore, e.g. the full matrix of integrals of $(1, 0)$ forms over the g A -cycles, is block diagonal, containing e_α^A as one of its blocks. The fact that this full matrix is invertible then implies the invertibility of e_α^A . The same block diagonal structure appears everywhere in the matrices which enter in Riemann's two identities, which therefore still imply (3.32) and the positivity of the metric in (3.33).

Also for other associations of the quantum theory of rigid $N = 2$ supersymmetry - Yang-Mills models with moduli spaces of Riemann surfaces [30] similar procedures can be followed to prove that they satisfy the conditions of definition 3.

4 (Local) Special geometry

4.1 Supergravity actions

We have seen that we have at our disposal quite a few equivalent formulations of rigid special geometry. In the remaining sections we would like to find out in how far these results are transferable to the supergravity or Calabi-Yau case.

The name special geometry [2] has been given to the manifold determined by the scalars of vector multiplets in $N = 2$ supergravity. The first construction [1] of these actions relied on superconformal tensor calculus, the principle of which is that one starts with defining multiplets of the superconformal group, and then at the very end breaks the symmetry down to the super Poincaré group¹¹. This, at first sight cumbersome, procedure in fact turns out to simplify life. The Poincaré supersymmetric action contains many terms of which the origin is not clear, unless one recognises how they arose naturally in the superconformal setup.

For $d = 4, N = 2$ the superconformal group is

$$SU(2, 2|N = 2) \supset SU(2, 2) \otimes U(1) \otimes SU(2) . \quad (4.1)$$

In the bosonic subgroup displayed, the conformal group is identified as the $SU(2, 2) = SO(4, 2)$ factor. On the fermionic side there are two supersymmetries Q^i and two special supersymmetries S^i . The multiplet gauging the superconformal group is called the Weyl multiplet, and has as independent fields

$$\{e_\mu^a, b_\mu, \psi_\mu^i, A_\mu, \mathcal{V}_\mu^i{}_j, T_{ab}^-, \chi^i, D\} . \quad (4.2)$$

These are the gauge fields for general coordinate transformations, dilatations, Q^i , $U(1)$ and $SU(2)$. The extra antiselfdual antisymmetric tensor, a doublet of fermions and a real scalar are included to close the algebra. In order to gauge fix the superfluous symmetries some vector multiplet and a second compensating multiplet (e.g. a hypermultiplet) are introduced. After the breakdown to the super Poincaré group, of the Weyl multiplet only the vierbein

¹¹For reviews, see e.g. [31], or [32] for a shorter recent one.

and the gravitinos will remain as physical fields. Likewise, the hypermultiplet disappears from the action by gauge fixing the $SU(2)$ and by the field equations of D and χ^i .

For a theory with n physical vector multiplets we introduce $n + 1$ vector multiplets to start with. One of these is to be identified as a compensating multiplet, of which the scalar disappears upon fixing the dilatational and $U(1)$ gauge, the fermion by the gauge fixing of S -supersymmetry, while its vector becomes the physical graviphoton. In the end one realises that n complex scalars, n doublets of spinors, and $n + 1$ vectors constitute the physical content originating from the vector multiplets.

These vector multiplets consist of the following fields:

$$\left(\begin{matrix} X^I & \Omega_i^I & \mathcal{A}_\mu^I & Y_{ij}^I \\ (1, -1), (\frac{3}{2}, -\frac{1}{2}), (0, 0), (2, 0) \end{matrix} \right) \quad \text{with} \quad I = 0, 1, \dots, n. \quad (4.3)$$

We have indicated for each component field the weights (w, c) defining their transformation under the dilatation and $U(1)$ transformations in the superconformal group: $\delta\phi = w\phi\Lambda_D + ic\phi\Lambda_{U(1)}$.

The action is then built exactly as in the rigid case, i.e. through the chiral superspace integral of a holomorphic function $F(X)$, called a *prepotential*. The scaling symmetry now imposes an extra requirement: $F(X)$ must be *homogeneous of degree two* in X^I , where $I = 0, 1, \dots, n$.

The X^I span a $(n + 1)$ -dimensional complex space, but as a result of the dilatation and $U(1)$ symmetry, the physical scalars parametrize an n -dimensional complex hypersurface which will turn out to be Kähler. We now proceed to fix these gauges, while building in the freedom to perform Kähler transformations on that hypersurface.

The gauge fixing can be performed in a way such that the symplectic structure remains manifest, and an arbitrariness in the gauge fixing of the phase transformations of X^I gives rise to the Kähler transformations. To perform such a gauge fixing, it is convenient to write the $n + 1$ complex fields X^I in terms of $n + 2$ complex fields a and Z^I :

$$X^I = aZ^I, \quad (4.4)$$

which implies the extra gauge invariance

$$a' = a e^{\Lambda_{aZ}}; \quad Z'^I = Z^I e^{-\Lambda_{aZ}}, \quad (4.5)$$

where Λ_{aZ} is an arbitrary complex gauge parameter. Since all X^I transform with the same weights under dilatations and $U(1)$ transformations, this allows us to separate out the action of these transformations by letting them act on a with the same weights, so that Z^I is invariant. Under the combined action of these gauge transformations and dilatations and $U(1)$ transformations we have then

$$X'^I = e^{\Lambda_D - i\Lambda_{U(1)}} X^I; \quad (4.6)$$

$$a' = e^{\Lambda_D - i\Lambda_{U(1)} + \Lambda_{aZ}} a; \quad (4.7)$$

$$Z'^I = e^{-\Lambda_{aZ}} Z^I. \quad (4.8)$$

Now we go on to fix these gauges.

In the action, the curvature is coupled to the scalars via the term

$$i(\bar{X}^I F_I - \bar{F}_I X^I) e R. \quad (4.9)$$

A standard Einstein action can be obtained by choosing as dilatational gauge fixing

$$i(\bar{X}^I F_I - \bar{F}_I X^I) = i\langle \bar{V}, V \rangle = 1 \quad \text{with} \quad V \equiv \begin{pmatrix} X^I \\ F_I \end{pmatrix}. \quad (4.10)$$

We define $K(Z, \bar{Z})$ and N_{IJ} through

$$e^{-K(Z, \bar{Z})} \equiv i \left(\bar{Z}^I F_I(Z) - \bar{F}_I(Z) Z^I \right) = -\bar{Z}^I N_{IJ} Z^J; \quad N_{IJ} \equiv 2 \operatorname{Im} F_{IJ}. \quad (4.11)$$

The homogeneity of F implies that $F_I(X) = a F_I(Z)$ and $N_{IJ}(X) = N_{IJ}(Z)$. The dilatational gauge fixing (4.10) then amounts to setting

$$|a|^2 = e^{K(Z, \bar{Z})}. \quad (4.12)$$

The $U(1)$ gauge invariance of the superconformal group is fixed by specifying the phase of a : we take a to be real and positive.

The remaining unfixed gauge invariance is now the transformation (4.5), accompanied by a $U(1)$ transformation, with parameter

$$2i\Lambda_{U(1)} = \Lambda_{aZ} - \bar{\Lambda}_{aZ}. \quad (4.13)$$

This gauge freedom can be fixed by specifying a single (complex) gauge condition. Generically, this could be done by specifying a rather arbitrary condition $F(Z^I, \bar{Z}^I) = 0$. Here, however, we impose that the gauge condition be an (inhomogeneous) equation that is *holomorphic* in Z^I . We do not need to make an explicit choice for this equation at this point. However, we have to discuss how to relate two such choices (both holomorphic) with each other. The relation is

$$Z'^I = Z^I e^{-\Lambda_K(Z)}. \quad (4.14)$$

Note that this transformation, in accordance with (4.13), implies a $U(1)$ rotation with parameter

$$2i\Lambda_{U(1)} = \Lambda_K(Z) - \bar{\Lambda}_K(\bar{Z}), \quad (4.15)$$

so that the total effect of this change of gauge is

$$X'^I = e^{-\frac{1}{2}(\Lambda_K(Z) - \bar{\Lambda}_K(\bar{Z}))} X^I; \quad (4.16)$$

$$a' = e^{\frac{1}{2}(\Lambda_K(Z) + \bar{\Lambda}_K(\bar{Z}))} a; \quad (4.17)$$

$$Z'^I = e^{-\Lambda_K(Z)} Z^I. \quad (4.18)$$

The corresponding transformation of K in (4.11) is

$$K'(Z, \bar{Z}) = K(Z, \bar{Z}) + \Lambda_K(Z) + \bar{\Lambda}_K(\bar{Z}). \quad (4.19)$$

This transformation will presently acquire the meaning of a Kähler transformation.

The action for the scalars is

$$\mathcal{L}_0 = -e N_{IJ} \mathcal{D}_\mu X^I \mathcal{D}^\mu \bar{X}^J \quad \text{with} \quad \mathcal{D}_\mu X^I = (\partial_\mu + i A_\mu) X^I, \quad (4.20)$$

with X^I given in (4.4) and a by the positive square root of (4.12). The auxiliary gauge field of the $U(1)$ transformation is eliminated by its field equation:

$$A_\mu = \frac{i}{2} N_{IJ} \left(X^I \partial_\mu \bar{X}^J - \partial_\mu X^I \bar{X}^J \right). \quad (4.21)$$

The spin-1 action is given by equation (1.1) with

$$\mathcal{N}_{IJ}(Z) = \bar{F}_{IJ}(Z) + 2i \frac{\text{Im } F_{IK}(Z) \text{Im } F_{JL}(Z) Z^K Z^L}{\text{Im } F_{KL}(Z) Z^K Z^L} . \quad (4.22)$$

The full action has been constructed in [33].

The scalar action describes a Kähler manifold with coordinates¹² Z^I and with Kähler potential

$$K(Z, \bar{Z}) = -\log \left[i \bar{Z}^I \frac{\partial}{\partial Z^I} F(Z) - i Z^I \frac{\partial}{\partial \bar{Z}^I} \bar{F}(\bar{Z}) \right] . \quad (4.23)$$

The transformation of coordinates Z^I in (4.14) induces the correct change of Kähler potential, (4.19). It is clear however that (4.14) is not the most general holomorphic change of coordinates that one may consider. Now we regard the Z^I ($I = 0, \dots, n$) as holomorphic functions of the 'physical scalars' z^α ($\alpha = 1, \dots, n$), which constitute a more general holomorphic coordinate system. As in the case of rigid supersymmetry these scalars are coordinates on (one patch of) a complex manifold. By restricting the Z^I to holomorphic functions of the coordinates z , the kinetic term of the scalars maintains its hermitian form.

Having adopted general holomorphic coordinates z , we can now consider arbitrary holomorphic coordinate changes, and the induced Kähler transformation. By letting Z^I transform as in (4.14) (but now with the parameter Λ_K depending on z), it becomes a section of a fibre bundle¹³. The complete *scalar manifold* can be viewed as a number of patches glued together, where in each patch the description above applies. In going from one patch to another the Z^I transform under a Kähler transformation. Note that with this interpretation, through the accompanying $U(1)$ transformation (4.15), the superfield components transform also. Therefore, they will have to be interpreted as sections of the appropriate bundles too.

In addition to Kähler transformations, there is another invariance in the theory, viz. the symplectic duality transformations. As in the rigid case, both classes of transformations preserve the Kähler metric, and the system of equations of motion and Bianchi identities for the vectors. Apart from one exception below, we will only have to consider the bosonic sector of the theory, but the terms with fermions have been shown to be invariant as well [1, 6].

There is an extra requirement that we want to impose, viz. the positivity of the kinetic energies. Let us start with the spin-2 sector. We can only make the dilatational gauge fixing (4.10) provided the left-hand side is positive from the start. This requirement restricts the domain where a particular function F can be used to construct an action. See [1] for an example. The spin-0 sector provides a second condition: the positivity of the metric of the scalar manifold. We take the positivity of the metric as part of the definition of a Kähler manifold. A third condition might be expected from the spin 1 sector, but it was proved in [20] that the vector kinetic energy is automatically positive if the other two conditions are satisfied.

There is one important global aspect, which as far as we know is the only instance where the fermion sector comes in. The fermions impose a quantisation condition on the Kähler form. For a compact gauge group, the cohomology class of the 2-form gauge field strength

¹²There are $n+1$ quantities Z^I , but after properly fixing the gauge only n of them are independent: these are good coordinates if the gauge fixing condition on the Z^I has been chosen properly.

¹³To be precise, we also allow the symplectic transformations discussed in the next paragraph.

is quantised: if fields transform as $\psi \rightarrow U\psi$, then the gauge field A_μ , normalised so that it transforms as $\partial_\mu + iA_\mu \rightarrow U^{-1}(\partial_\mu + iA_\mu)U$, has a field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, of which the integral over an arbitrary 2-cycle is

$$[F] \equiv \int F_{\mu\nu} dx^\mu dx^\nu = 2\pi i n ; \quad n \in \mathbb{Z} . \quad (4.24)$$

This is analogous to the quantisation of the magnetic charge and is in a similar setting nicely explained in [34]. In a more mathematical language this means that the transformation functions should define a complex $U(1)$ line bundle, whose first Chern class should be of integer cohomology [35]. If several line bundles exist, having different $U(1)$ charges that are multiples of a basic unit, then the most stringent condition is given by a bundle having this unit charge.

We now apply this consideration to the Kähler transformations¹⁴. By (4.15) they are accompanied with a $U(1)$ transformation on the fields X^I , and therefore of the whole supermultiplet. The fields with the lowest $U(1)$ weight in this multiplet are the fermions, see (4.3). Using (4.15), we get

$$\Omega_i \rightarrow e^{-\frac{1}{4}(\Lambda_K(z) - \bar{\Lambda}_K(\bar{z}))} \Omega_i . \quad (4.25)$$

This implies that the $U(1)$ connection in this bundle is normalised as¹⁵

$$A_\alpha = -\frac{1}{4}\partial_\alpha K ; \quad A_{\bar{\alpha}} = \frac{1}{4}\partial_{\bar{\alpha}} K , \quad (4.26)$$

and the curvature,

$$F_{\alpha\bar{\beta}} = -F_{\bar{\beta}\alpha} = \frac{1}{2}g_{\alpha\bar{\beta}} , \quad (4.27)$$

is proportional to the Kähler 2-form (3.9). The bundle's first Chern class, which is of integer cohomology, thus equals $\frac{1}{2}$ times the de Rham cohomology class of the Kähler form:

$$c_1 = \frac{1}{2}[\mathcal{K}] \in \mathbb{Z} . \quad (4.28)$$

We find that the Kähler form should be of even integer cohomology. If there had been no fermions present in the theory, the same argument applied just to the bosonic fields would have allowed an arbitrary integer cohomology. In the mathematical literature [35, 36] Kähler manifolds of which the Kähler form is of integer cohomology are called *Kähler manifolds of restricted type* or *Hodge manifolds*. With a slight abuse of language we will call *Hodge-Kähler manifold* a Kähler manifold with Kähler form of even integer cohomology.

4.2 Special Kähler manifolds

Two questions now arise. The first one is: which manifolds can be used as scalar manifold for vector multiplets coupled to $N = 2$ supergravity? These manifolds are called *special (Kähler) manifolds*. The second is: *how* can one construct a full supergravity action starting from a specific special Kähler manifold.

¹⁴Note that in this case the base manifold is the scalar manifold instead of spacetime.

¹⁵The connection is $(-i/2)$ times the gauge field of the superconformal $U(1)$, A_μ , as can be seen by the covariant derivatives in (4.20), which are written there for fields of chiral weight which is the double of that of the fermions.

In this paper we will be concerned mainly with the first question. We will propose three definitions of special manifolds and prove their equivalence. Depending on the context in which supergravity appears one of them will be more practical than the others.

The second question is briefly touched upon: we show how the spin-1 part of an action can be constructed.

The previous subsection leads us immediately to the first definition of a special Kähler manifold.

4.2.1 Definition 1

A special Kähler manifold¹⁶ is an n -dimensional Hodge-Kähler manifold¹⁷ with the following 3 properties.

1. On every chart there exist complex projective coordinate functions $Z^I(z)$, where $I = 0, \dots, n$ and a holomorphic function $F(Z^I)$ that is homogeneous of second degree, such that the Kähler potential is

$$K(z, \bar{z}) = -\log \left[i \bar{Z}^I \frac{\partial}{\partial Z^I} F(Z) - i Z^I \frac{\partial}{\partial \bar{Z}^I} \bar{F}(\bar{Z}) \right] ; \quad (4.29)$$

2. On overlaps of charts i and j , the corresponding functions in property 1 are connected by transition functions of the following form:

$$\begin{pmatrix} Z \\ \partial F \end{pmatrix}_{(i)} = e^{f_{ij}(z)} M_{ij} \begin{pmatrix} Z \\ \partial F \end{pmatrix}_{(j)} , \quad (4.30)$$

with f_{ij} holomorphic and $M_{ij} \in Sp(2n+2, \mathbb{R})$;

3. The transition functions satisfy the cocycle condition on overlap regions of three charts.

Comparing this definition with the corresponding one in the rigid case (see section 3.4.1), there are several differences. The $n+1$ coordinates Z^I are projective here (vs. n ordinary coordinates there), and the expression for the Kähler potential is different. Another difference is that local special geometry involves *local holomorphic* transition functions in the multiplication factor, vs. constant ones for the rigid case. This is related to the presence of the gauge field of the local $U(1)$ in the superconformal approach, as should be clear from section 4.1.

This definition of a special Kähler manifold clearly originates from the construction of an action starting from a prepotential. Now we would like to get rid of the prepotential in the definition. The main reason is that it is not invariant under symplectic (duality) transformations. Actions have been constructed [6] for which there does not even exist a prepotential¹⁸. The following example was given in [6]. Take $n = 1$ and start from the

¹⁶As in the rigid case, we will always assume a positive definite Kähler metric.

¹⁷i.e. with Kähler form of even integer cohomology, see subsection 4.1.

¹⁸ This case includes some physically important theories.

prepotential $F(Z) = -i Z^0 Z^1$. Then choose a coordinate z such that $Z^0 = 1$ and $Z^1 = z$. The symplectic vector in (4.30) is then

$$v = \begin{pmatrix} 1 \\ z \\ -iz \\ -i \end{pmatrix}, \quad (4.31)$$

leading to

$$e^{-K} = 2(z + \bar{z}) ; \quad \partial_z \partial_{\bar{z}} K = (z + \bar{z})^{-2}, \quad (4.32)$$

which is the $SU(1,1)/U(1)$ manifold with positivity domain $\text{Re } z > 0$. Now perform the symplectic mapping

$$v \rightarrow \tilde{v} = \mathcal{S}v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} v = \begin{pmatrix} 1 \\ i \\ -iz \\ z \end{pmatrix}. \quad (4.33)$$

After this mapping, the transformed vector \tilde{v} can no longer be written in the standard form with the function F , since the last two components clearly cannot be written as functions of the first two: no prepotential $\tilde{F}(\tilde{Z}^0, \tilde{Z}^1)$ exists. Therefore, after the duality rotation, the model can not be formulated directly in a superspace formulation with a function F . We still have the same Kähler manifold (4.32), but with different couplings of the scalars to the vectors.

In this case it is clear that, as stressed at the beginning of this subsection, there is no contradiction with definition 1: our construction made it explicit that on the same scalar manifold there is a supergravity action which *can* be obtained from a prepotential: it will be proved below that this is true in general. Our example only indicates that the first definition is not very handy in this case.

The second definition is, as in the rigid case, an intrinsic symplectic formulation.

4.2.2 Definition 2

A special Kähler manifold is an n -dimensional Hodge-Kähler manifold¹⁹ \mathcal{M} , with the following 2 properties.

1. There exists a holomorphic $Sp(2(n+1), \mathbb{R})$ vector bundle \mathcal{H} over \mathcal{M} , and a holomorphic section $v(z)$ of $\mathcal{L} \otimes \mathcal{H}$, such that the Kähler form (3.9) is

$$\mathcal{K} = -\frac{i}{2\pi} \partial \bar{\partial} \log [i \langle \bar{v}, v \rangle] \quad \text{or} \quad K = -\log [i \langle \bar{v}, v \rangle] ; \quad (4.34)$$

Here \mathcal{L} denotes the holomorphic line bundle over \mathcal{M} with transition functions as in (4.14), of which the first Chern class equals the cohomology class of the Kähler form.

2. The section $v(z)$ of property 1 satisfies

$$\langle v, \partial_\alpha v \rangle = 0. \quad (4.35)$$

¹⁹cf. footnotes 16 and 17 in definition 1.

This definition is essentially a rewriting of Strominger's [2] definition, where, however, the second condition, (4.35), is absent. Comparing with definition 2 of the rigid case in section 3.4.2, we notice that it differs also in that second condition. We now discuss these differences.

First, let us note that (4.35) is a proper equation in the \mathcal{L} bundle, since the Kähler covariant derivative of v is $\mathcal{D}_\alpha v \equiv \partial_\alpha v + (\partial_\alpha K)v$ and the symplectic inner product is antisymmetric. Second, the condition analogous to (3.29),

$$\langle \mathcal{D}_\alpha v, \mathcal{D}_\beta v \rangle = 0 . \quad (4.36)$$

follows from (4.35) by taking a (covariant) derivative and antisymmetrizing. Interestingly, using the invertibility of the Kähler metric, the converse is almost true as well: (4.36) *implies* (4.35) except for $n = 1$. This is shown[7] by taking a (Kähler covariant) antiholomorphic derivative, and noticing that the curvature is proportional to the invertible Kähler metric. Finally let us remark that in the rigid case it is certainly inappropriate to impose the analogue of (4.35): it would limit the Kähler potential to be homogeneous of second order, which in that context is an unnecessary restriction.

In the context of Calabi-Yau moduli spaces, in fact (4.36) is satisfied automatically (see section 4.3). To show the necessity of this condition in the general context we have constructed counterexamples, given in detail in appendix C. They show that without these conditions one could have manifolds which are not special according to definition 1, and to the known $N = 2$ supergravity constructions. We come back to the interpretation of the constraints at the end of this section.

Remark 1. The (local) constraints (4.34), (4.35) (and (4.36)) can be equivalently formulated in terms of a different bundle, with section $V \equiv e^{K/2}v$ where K is the Kähler potential. This V is the one appearing in (4.10). With

$$\begin{aligned} U_\alpha &\equiv \mathcal{D}_\alpha V \equiv \partial_\alpha V + \frac{1}{2}(\partial_\alpha K)V & \mathcal{D}_{\bar{\alpha}} V &\equiv \partial_{\bar{\alpha}} V - \frac{1}{2}(\partial_{\bar{\alpha}} K)V \\ \bar{U}_{\bar{\alpha}} &\equiv \mathcal{D}_{\bar{\alpha}} \bar{V} \equiv \partial_{\bar{\alpha}} \bar{V} + \frac{1}{2}(\partial_{\bar{\alpha}} K)\bar{V} & \mathcal{D}_\alpha \bar{V} &\equiv \partial_\alpha \bar{V} - \frac{1}{2}(\partial_\alpha K)\bar{V} , \end{aligned} \quad (4.37)$$

the constraints are:

1. $\langle V, \bar{V} \rangle = i$;
2. $\mathcal{D}_{\bar{\alpha}} V = 0$;
3. $\langle V, U_\alpha \rangle = 0$;
4. $\langle U_\alpha, U_\beta \rangle = 0$.

As explained above, given 1 and 2, constraint 3 implies 4, and 4 implies 3 unless $n = 1$ (in which case constraint 4 is empty).

We now turn to the proof of the equivalence of our two definitions. In this section we just give the outline in order to make clear the structure of the proof. The lemmas that constitute the different steps are proved in detail in appendix B.

We will formulate the proof mostly in terms of the section V , as in remark 1. Before embarking, let us make explicit the inner products of the symplectic vectors V , U_α and their complex conjugates. Taking a covariant derivative of condition 1 one immediately gets

$$\langle U_\alpha, \bar{V} \rangle = 0 ; \quad \langle \bar{U}_{\bar{\alpha}}, V \rangle = 0 . \quad (4.38)$$

Noticing that the curvature in the bundle is essentially the Kähler form:

$$[\mathcal{D}_\alpha, \mathcal{D}_{\bar{\beta}}] V = -g_{\alpha\bar{\beta}} V , \quad (4.39)$$

we can take a further covariant derivative of (4.38) to obtain

$$\langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle = -ig_{\alpha\bar{\beta}} . \quad (4.40)$$

Outline of the proof. The conditions of definition 1 clearly imply those of definition 2 (see [7]). So we only have to prove the converse.

The (global) section v in definition 2 is, in each patch, represented by a vector, and the transition between patches is made with transition functions that clearly satisfy the last two conditions of definition 1. For definition 1 it is in addition necessary that these vectors are of the form $\begin{pmatrix} Z \\ \partial F \end{pmatrix}$ for some function F . We will show how to construct, starting from an arbitrary vector v that satisfies condition 4.35, a vector of that form, using constant $Sp(2n+2, \mathbb{R})$ -transformations. The transition functions then keep the form required in (4.30).

It is thus *sufficient to prove* that given a section $V = \begin{pmatrix} X^I \\ F_I \end{pmatrix}$ satisfying the conditions of remark 1, there exists an $Sp(2n+2, \mathbb{R})$ -transformation which transforms V into a vector $\tilde{V} = \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix}$ with the property that the transformed \tilde{F}_I is the derivative of a second degree homogeneous holomorphic function: $\tilde{F}_I = \frac{\partial}{\partial \tilde{X}^I} \tilde{F}(\tilde{X})$.

Since the metric g is non-degenerate, lemma B.1 implies that

$$\text{rank} \begin{pmatrix} f_\alpha^I & h_{\alpha I} \\ X^I & F_I \end{pmatrix} = n+1 ,$$

where we have introduced notations for the components of U_α :

$$U_\alpha = \begin{pmatrix} f_\alpha^I \equiv \mathcal{D}_\alpha X^I \\ h_{\alpha I} \equiv \mathcal{D}_\alpha F_I \end{pmatrix} . \quad (4.41)$$

According to lemma A.1 there exists an $S \in Sp(2n+2, \mathbb{R})$ such that

$$SV \equiv \tilde{V} = \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} ,$$

with

$$\det \begin{pmatrix} \tilde{f}_\alpha^I \\ \tilde{X}^I \end{pmatrix} \neq 0 . \quad (4.42)$$

Using lemmas B.3 and B.4 we find a function $\tilde{F}(\tilde{X})$ such that $\tilde{F}_I = \frac{\partial \tilde{F}}{\partial \tilde{X}^I}$. Note that we have made use of the constraints (4), (3) and (2) of remark 1, which are $Sp(2n+2, \mathbb{R})$ invariant and are thus satisfied by \tilde{V} too. This completes the proof of the equivalence of both definitions.

The equivalence of the two definitions of a special Kähler manifold does *not* imply that the formulation of the corresponding supersymmetric theories without a prepotential may be

discarded. It only means that we can always describe the geometry of the scalars by means of a prepotential. It is not (necessarily) so that the symplectically rotated supergravity theory is equivalent to the original one. Especially for gauged (non-abelian) theories some interesting phenomena have been discovered:

- It is possible that symplectically related abelian theories allow different gauge groups.[6]
- In theories which have a prepotential it is impossible to partially break $N = 2$ supersymmetry to $N = 1$ by means of Fayet-Iliopoulos terms; this phenomenon does occur in some theories without a prepotential.[37]

The kinetic term of the vectors. Having investigated the scalar sector, we now come back to the question of how to construct the remaining part of the action of supergravity coupled to vector multiplets. The complete action constructed in [33] was rewritten in terms of these symplectic building blocks in [6, 9]. The tensor \mathcal{N} has a direct physical interpretation as its expectation value gives the coupling constants in the kinetic term of the vectors, see (1.1). Its symmetry was also the essential ingredient in restricting transformations between Bianchi identities and field equations to symplectic ones. Similarly, the extra constraints necessary for special Kähler manifolds can be seen to originate from this requirement of symmetry. Of course, if a prepotential exists for v we can construct \mathcal{N} via formula (4.22). This matrix is explicitly symmetric. Moreover it has been mentioned already that the negativeness of its imaginary part follows from the positivity of kinetic terms of scalars and vectors, as proven in [20]. These properties are preserved under symplectic transformations (corollary A.4) and as we already know that in any case we can find a symplectic transformation to a formulation where the prepotential exist, we could stop here. However, it is useful to have a definition which starts from the symplectic vector $v(z^\alpha)$, and see how constraints of special geometry can be understood as requirements on \mathcal{N} .

In order to define \mathcal{N} in the general case we need the invertibility of

$$\begin{pmatrix} f_\alpha^I & \bar{X}^I \end{pmatrix} . \quad (4.43)$$

This is proven in corollary B.6, whose conditions correspond again to the positivity of the spin 2 and spin 0 kinetic terms. Note that the lemma B.5 which led to that corollary was derived under a weaker condition than condition 3. We will come back to this remark below.

Then we can define as in equation (2.14):

$$\mathcal{N}_{IJ} \equiv \begin{pmatrix} \bar{h}_{\bar{\alpha}I} & F_I \end{pmatrix} \begin{pmatrix} \bar{f}_{\bar{\alpha}}^J & X^J \end{pmatrix}^{-1} , \quad (4.44)$$

which, as mentioned below (2.14), transforms under symplectic transformations as in (2.15). If a prepotential exists this definition coincides with (4.22) [7].

The equation

$$i \begin{pmatrix} \langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle & \langle U_\alpha, V \rangle \\ \langle \bar{V}, \bar{U}_{\bar{\beta}} \rangle & \langle \bar{V}, V \rangle \end{pmatrix} = \begin{pmatrix} f_\alpha^I \\ \bar{X}^I \end{pmatrix} i(\mathcal{N} - \mathcal{N}^\dagger)_{IJ} \begin{pmatrix} \bar{f}_{\bar{\beta}}^J & X^J \end{pmatrix} \quad (4.45)$$

follows from the definition of \mathcal{N} , and thus implies the negative definiteness of the antihermitian part of \mathcal{N} as a consequence of the positivity of the matrix on the left hand side. We can here make the same remark about the necessity or weakening of the condition 3 as after (4.43).

The symmetry of \mathcal{N} follows from a similar equation:

$$\begin{pmatrix} f_\alpha^I \\ \bar{X}^I \end{pmatrix} (\bar{\mathcal{N}} - \bar{\mathcal{N}}^T)_{IJ} \begin{pmatrix} f_\beta^J & \bar{X}^J \end{pmatrix} = \begin{pmatrix} \langle U_\alpha, U_\beta \rangle & \langle U_\alpha, \bar{V} \rangle \\ \langle \bar{V}, U_\beta \rangle & 0 \end{pmatrix}. \quad (4.46)$$

The off-diagonal elements of the right hand side are zero due to (4.38), a consequence of condition 1. The upper left entry is zero due to condition 4. The invertibility of (4.43) implies the symmetry of \mathcal{N} .

Interpretation of the constraints. Now we are in a position to interpret our extra constraints in the second definition of special geometry. The condition 2 is essential for the holomorphic structure. The condition 1 is a normalization which is possible if the spin-2 kinetic terms are positive definite. It corresponds to the conventional parametrization of the spin-2 action in the ‘Einstein frame’. Similarly the form of (4.37), or equivalently (4.40), corresponds to the scalar kinetic terms. For the spin-1 kinetic terms (i.e. first to be able to define \mathcal{N} uniquely, and also explicitly in the expression for $\text{Im } \mathcal{N}$ which should be negative definite for positive definite kinetic terms), we need the positivity of the matrix (4.45). Finally, the condition 4 is necessary for the symmetry of \mathcal{N} , as seen explicitly in (4.46).

The above arguments still did not lead to condition 3, but to the weaker condition of the positivity of (4.45). It is clear that if condition 3 is satisfied, that matrix is positive. The relaxation of the constraint is only relevant for $n = 1$. Indeed [7], from (4.37) (U has the same Kähler weight as V) one derives

$$\mathcal{D}_{\bar{\beta}} U_\alpha = g_{\alpha\bar{\beta}} V. \quad (4.47)$$

Then applying $g^{\alpha\bar{\gamma}} \mathcal{D}_{\bar{\gamma}}$ to condition 4 gives condition 4 if $n > 1$.

Remains $n = 1$. In appendix C we give two examples which prove that the extra constraint is relevant for $n = 1$, i.e. that relaxing this constraint effectively enlarges the allowed symplectic sections. In a first example, it is shown that having only (4.37) (or (4.40)) and conditions 1 and 2, would allow a larger class of Kähler manifolds, i.e. Kähler manifolds which are not allowed with condition 3. In that example the matrix (4.45) is not positive definite. A second example (related to the simple example we gave at the end of section 4.2.1) shows a symplectic section for which (4.45) is positive definite, but the condition 3 is violated. The Kähler manifold is the same as the one of (4.32), but the symplectic section is different and is *not* related to that in (4.31) by a symplectic rotation, as opposed to the one of (4.33). We discuss in appendix C.2 in how far this model violates the equations occurring in the constructions of $N = 2$ supergravity models.

In view of the fact that the constraint (4.36) has a direct interpretation in terms of \mathcal{N} , whereas (4.35) does not, one may wonder if it would be more appropriate to figure the latter together with the positivity of (4.45) in definition 2 of special Kähler manifolds. This would increase also the similarity with the rigid case. However, there exist at present no actions of $N = 2$ supergravity coupled with vector multiplets which give rise to symplectic sections which do not satisfy our constraints. So, the relaxation would not be in agreement with our general strategy that the sections satisfying the definitions should allow such an $N = 2$ supergravity action. Also, we would not be able to show the equivalence with the first definition, which was directly related to a supergravity action.

Invertibility of $(n+1) \times (n+1)$ matrices and existence of the prepotential. We draw attention to the different status of the invertibility on the matrices in (4.42) and (4.43). The latter is always invertible, while the invertibility of the former is a criterion for the existence of a prepotential in this symplectic basis. To illustrate this, consider the example (4.33). Taking the covariant derivatives, the matrix (4.43) is

$$\begin{pmatrix} f_\alpha^I & \bar{X}^I \end{pmatrix} = e^{K/2} \begin{pmatrix} -\frac{1}{z+\bar{z}} & 1 \\ -\frac{i}{z+\bar{z}} & -i \end{pmatrix}, \quad (4.48)$$

which is invertible. If the second column had been X rather than \bar{X} , the sign change in the last entry would have made this matrix non-invertible, reflecting the non-existence of the prepotential for this symplectic section.

4.2.3 Definition 3

There is a third definition (inspired by [5, 6, 7]), which is analogous to the third definition in the rigid case. Take a complex manifold \mathcal{M} . Suppose we have in every chart a $2(n+1)$ component vector $V(z^\alpha, \bar{z}^\alpha)$ such that on overlap regions there are transition functions of the form

$$e^{\frac{1}{2}(f(z^\alpha) - \bar{f}(\bar{z}^\alpha))} S,$$

with f a holomorphic function and S a constant $Sp(2(n+1), \mathbb{R})$ matrix. (These transition functions have to satisfy the cocycle condition.) Take a $U(1)$ connection of the form $\kappa_\alpha dz^\alpha + \kappa_{\bar{\alpha}} d\bar{z}^\alpha$ with

$$\kappa_{\bar{\alpha}} = -\overline{\kappa_\alpha}, \quad (4.49)$$

under which \bar{V} has opposite weight as V . Denote the covariant derivative by \mathcal{D} :

$$\begin{aligned} U_\alpha &\equiv \mathcal{D}_\alpha V \equiv \partial_\alpha V + \kappa_\alpha V; & \mathcal{D}_{\bar{\alpha}} V &\equiv \partial_{\bar{\alpha}} V + \kappa_{\bar{\alpha}} V \\ \bar{U}_{\bar{\alpha}} &\equiv \mathcal{D}_{\bar{\alpha}} \bar{V} \equiv \partial_{\bar{\alpha}} \bar{V} - \kappa_{\bar{\alpha}} \bar{V}; & \mathcal{D}_\alpha \bar{V} &\equiv \partial_\alpha \bar{V} - \kappa_\alpha \bar{V}. \end{aligned} \quad (4.50)$$

We impose the following conditions:

1. $\langle V, \bar{V} \rangle = i$;
2. $\mathcal{D}_{\bar{\alpha}} V = 0$;
3. $\mathcal{D}_{[\alpha} U_{\beta]} = 0$;
4. $\langle V, U_\alpha \rangle = 0$.

Define

$$g_{\alpha\bar{\beta}} \equiv i \langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle, \quad (4.51)$$

where $\bar{U}_{\bar{\beta}}$ denotes the complex conjugate of U_α . If this is a positive-definite metric, \mathcal{M} is called a special Kähler manifold.

We now prove the equivalence of this definition with definition 2.

Proof.

The second definition provides all the equations in remark 1, which appear here as conditions 1, 4 and 2. The values of κ_α and $\kappa_{\bar{\alpha}}$ can be obtained by comparing (4.37) and (4.50),

and they satisfy (4.49). Condition 3 is then also easily checked. This shows that a manifold satisfying definition 2 also satisfies definition 3.

We now turn to the converse. This requires a proof of the existence of the symplectic section mentioned in definition 2, from the constraints in definition 3. From condition 3 we get

$$0 = [\mathcal{D}_\alpha, \mathcal{D}_\beta]V = 2(\partial_{[\alpha}\kappa_{\beta]})V ,$$

and taking an inner product with \bar{V} implies that $\partial_{[\alpha}\kappa_{\beta]} = 0$, so a function K' exists for which

$$\kappa_\alpha = \frac{1}{2}\partial_\alpha K' . \quad (4.52)$$

With (4.49), this implies

$$\kappa_{\bar{\alpha}} = -\overline{\kappa_\alpha} = -\frac{1}{2}\partial_{\bar{\alpha}}\bar{K}' . \quad (4.53)$$

Applying derivatives on condition 1 we obtain (4.38). Also

$$\begin{aligned} 0 &= \partial_{\bar{\beta}}\langle\bar{V}, U_\alpha\rangle = \langle\bar{U}_{\bar{\beta}}, U_\alpha\rangle + \langle\bar{V}, \mathcal{D}_{\bar{\beta}}\mathcal{D}_\alpha V\rangle = ig_{\alpha\bar{\beta}} + 2i\partial_{[\alpha}\kappa_{\bar{\beta}]} \\ &\Rightarrow g_{\alpha\bar{\beta}} = -2\partial_{[\alpha}\kappa_{\bar{\beta}]} = \partial_\alpha\partial_{\bar{\beta}}(\text{Re } K') , \end{aligned} \quad (4.54)$$

and the real part of K' is thus the Kähler potential. Condition 2 and equation (4.53) lead us to the conclusion that the holomorphic v of definition 2 can be obtained from

$$V = e^{\bar{K}'/2}v = e^{-i\text{Im } K'/2}e^{\text{Re } K'/2}v . \quad (4.55)$$

This implies that \mathcal{M} is special Kähler according to the first two definitions. ■

Some further remarks:

- the vector

$$V' \equiv e^{i\text{Im } K'/2}V = e^{\text{Re } K'/2}v$$

satisfies the conditions 1 till 2 of remark 1, page 23.

- The remarks about possible replacement of the condition 4 by

$$\text{Condition 4'} : \langle U_\alpha, U_\beta \rangle = 0 , \quad (4.56)$$

together with, for $n = 1$ the positive definiteness of (4.45), apply here as well.

- Defining

$$C_{\alpha\beta\gamma} = -i\langle\mathcal{D}_\alpha U_\beta, U_\gamma\rangle , \quad (4.57)$$

we can see that it is completely symmetric with the help of (4.56) and condition 3.

$$\mathcal{D}_\alpha U_\beta = C_{\alpha\beta\gamma}\bar{U}^\gamma \quad \text{with} \quad \bar{U}^\alpha \equiv g^{\alpha\bar{\beta}}\bar{U}_{\bar{\beta}} . \quad (4.58)$$

- Rather than imposing condition 4, in the literature [5, 6, 9] one often imposes (4.58), together with the symmetry of $C_{\alpha\beta\gamma}$ ²⁰. This alternative is nearly equivalent to the conditions that we have imposed, but not quite. From the covariant derivatives of

²⁰One only has to require that $C_{\alpha\beta\gamma}$ be symmetric in the first two indices (the complete symmetry then follows later), which corresponds to our condition 3.

condition 1, using condition 2 one derives consecutively $\langle U_\alpha, \bar{V} \rangle = 0$, $\langle \mathcal{D}_\beta U_\alpha, \bar{V} \rangle = 0$, and $\langle \mathcal{D}_\beta U_\alpha, \bar{U}_{\bar{\gamma}} \rangle = 0$. Combining these with (4.58) leads to

$$C_{(\alpha\beta)\gamma} g^{\gamma\bar{\gamma}} \langle \bar{U}_{\bar{\gamma}}, \bar{V} \rangle = 0 \quad \text{and} \quad C_{(\alpha\beta)\gamma} g^{\gamma\bar{\gamma}} \langle \bar{U}_{\bar{\gamma}}, \bar{U}_{\bar{\delta}} \rangle = 0 . \quad (4.59)$$

If the $(n(n+1)/2) \times n$ matrix $C_{(\alpha\beta)\gamma}$ is of rank n , one can deduce from these equations that the symplectic inner products vanish. If this is the case, starting from (4.58) is equivalent to the condition 4 that we imposed ²¹. There are, however, manifolds for which this is not the case: the so-called 'minimal coupling models' [38], where F is a quadratic polynomial of the X , provide an example since they have $C_{(\alpha\beta)\gamma} = 0$. For $n = 1$, this is in fact the manifold that we used as an example before.

Matrix formulation. Just as in the rigid case, the constraints of definition 3 [4, 5, 6, 7, 9] can be summarised in a matrix form, which acquires a geometric significance in the realisation of these manifolds as moduli spaces of Calabi-Yau manifolds. We define the $2(n+1) \times 2(n+1)$ matrix

$$\mathcal{V} = \begin{pmatrix} \bar{V}^T \\ U_\alpha^T \\ V^T \\ \bar{U}^{\alpha T} \end{pmatrix} . \quad (4.60)$$

All the inner products and differential equations satisfied by these quantities can be summarized in the relations

$$\begin{aligned} \mathcal{V} \Omega \mathcal{V}^T &= i\Omega ; \\ \mathcal{D}_\alpha \mathcal{V} &= \mathcal{A}_\alpha \mathcal{V} ; \quad \mathcal{D}_{\bar{\alpha}} \mathcal{V} = \mathcal{A}_{\bar{\alpha}} \mathcal{V} . \end{aligned} \quad (4.61)$$

with

$$\mathcal{A}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{\alpha\beta\gamma} \\ 0 & \delta_\alpha^\gamma & 0 & 0 \\ \delta_\alpha^\beta & 0 & 0 & 0 \end{pmatrix} . \quad (4.62)$$

The covariant derivatives contain the Kähler connection, and when acting on U_α , also Levi-Civita connection as in (3.42). As for the rigid case one proves (see e.g. [7]) that C is a symmetric tensor, covariantly holomorphic, and using commutators of covariant derivatives, one obtains the curvature formula

$$R^\alpha_{\beta\gamma}{}^\delta = 2\delta_{(\beta}^\alpha \delta_{\gamma)}^\delta - C_{\beta\gamma\epsilon} \bar{C}^{\alpha\delta\epsilon} . \quad (4.63)$$

It is clear that the conditions (4.61) are an over-complete set of requirements, and therefore a definition based on them does not seem to be economical.

4.3 Calabi-Yau moduli spaces

It has been discovered [39, 13, 23, 2] that moduli spaces of Calabi-Yau threefolds present themselves as natural candidates for special Kähler manifolds. As will be argued below, the third formulation of special manifolds is tailored for making the connection.

²¹A similar remark could be made for the rigid case, replacing (3.32) with (4.58) if C satisfies the same non-degeneracy condition.

A Calabi-Yau threefold is a compact Kähler manifold of complex dimension 3 whose first Chern class vanishes. The Hodge diamond of a Calabi-Yau threefold exhibits a lot of symmetry. For example, if the Euler character is non-zero, the diamond has the following form:

$$\begin{array}{ccccc}
 & & h^{00} = 1 & & \\
 & 0 & & 0 & \\
 & & h^{11} = m & & 0 \\
 h^{30} = 1 & 0 & h^{21} = n & h^{12} = n & h^{03} = 1 \\
 & 0 & h^{22} = m & & 0 \\
 & 0 & & 0 & \\
 & & h^{33} = 1 & &
 \end{array}$$

The matrix (4.60) will be related to the period matrix of integrals of $(3,0)$, $(2,1)$, $(1,2)$ and $(0,3)$ forms (the rows of the matrix) over 3-cycles. There are $2(n+1)$ homologically different 3-cycles in the CY manifold. One chooses a canonical basis with intersection numbers, as for the 1-cycles on Riemann surfaces, (3.45). Again, symplectic rotations correspond to changes of this canonical homology basis, and the restriction to integers is natural.

E.g. take the unique (up to normalization) holomorphic threeform: $\Omega \in H^{(3,0)}$. We choose Ω to depend holomorphically on the moduli. We identify

$$v \equiv \begin{pmatrix} \int_{A^I} \Omega \\ \int_{B_I} \Omega \end{pmatrix}.$$

The vector v depends on the moduli of the Calabi-Yau manifold. At this point we could still multiply Ω by a holomorphic function of the moduli. Such a transformation will correspond to the Kähler transformations in special geometry.

On the Calabi-Yau manifold there is also the integral formula, similar to (3.46), now for 3-forms and 3-cycles. This allows us to identify the symplectic inner product of symplectic vectors, formed by periods of 3-forms, with the integral of the exterior product of the forms over the full Calabi-Yau space. With a suitable orientation $-i d^3x d^3\bar{x} > 0$ (where x are the holomorphic coordinates on the Calabi-Yau surface), and we have

$$-i \langle v, \bar{v} \rangle = -i \int_{CY} \Omega \wedge \bar{\Omega} > 0, \quad (4.64)$$

and we can thus identify the latter as $\exp(-K)$.

Small variations of a (p,q) form can give at most $(p \pm 1, q \mp 1)$ forms. Applied to the unique $(3,0)$ form this gives [23]

$$\partial_\alpha \Omega = \Omega_\alpha - k_\alpha \Omega, \quad (4.65)$$

where Ω_α are $(2,1)$ forms and k_α functions of the moduli. The integral formula implies that only inner products of periods of $(p, 3-p)$ with $(3-p, p)$ forms are non-vanishing. Therefore one can already see that (4.35) and (4.36) are satisfied trivially. (This observation can explain their absence in [2].) One then also obtains

$$k_\alpha = -\frac{\langle \partial_\alpha \Omega, \bar{\Omega} \rangle}{\langle \Omega, \bar{\Omega} \rangle} = \partial_\alpha K. \quad (4.66)$$

This allows us to identify (integrals over canonical basis of 3-cycles)

$$V = e^{K/2} \int \Omega; \quad U_\alpha = \left(\partial_\alpha + \frac{1}{2} (\partial_\alpha K) \right) V = e^{K/2} \int \Omega_\alpha. \quad (4.67)$$

If we then take $i\langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle$, we still have the order of the dx and $d\bar{x}$ in the integral that guarantees this to be a positive definite matrix, and thus all requirements for special geometry are fulfilled.

We thus obtain the following identifications:

$$V = e^{K/2} \int \Omega^{(3,0)} ; \quad U_\alpha = e^{K/2} \int \Omega_\alpha^{(2,1)} ; \quad \bar{U}_{\bar{\alpha}} = e^{K/2} \int \Omega_{\bar{\alpha}}^{(1,2)} ; \quad \bar{V} = e^{K/2} \int \Omega^{(0,3)} . \quad (4.68)$$

The relations satisfied by the derivatives of these periods, the so-called Picard-Fuchs equations, take the form of the defining equations of special geometry.

It is now interesting to translate our results on the invertibility of $(n+1) \times (n+1)$ matrices to these periods. First of all, the conditions of lemma B.5 are satisfied: the off-diagonal elements are zero because after using the integral formula one obtains products of $(3,0)$ and $(2,1)$. The diagonal elements are positive because of the order of dx and $d\bar{x}$ factors as mentioned above. Therefore the lemma leads now directly to the invertibility of the $(n+1) \times (n+1)$ matrix formed by integrating the $(3,0)$ and $(1,2)$ forms over the A -cycles, *for every choice of non-intersecting independent A -cycles*. Furthermore lemma B.4 implies that if we have chosen A -cycles such that the integrals of the $(3,0)$ and $(2,1)$ forms over the A -cycles gives an invertible matrix, then a prepotential exists. Lemma A.1 says that one can always choose such a basis of cycles. However, the statement often made for Calabi-Yau manifolds [23] that with

$$Z^I \equiv \int_{A^I} \Omega^{(3,0)} ; \quad \mathcal{G}_I \equiv \int_{B_I} \Omega^{(3,0)} , \quad (4.69)$$

the complex structure of the Calabi-Yau manifold is entirely determined by the z^I (as projective coordinates), so that $\mathcal{G}_I = \mathcal{G}_I(z)$ can thus be violated for a particular choice of the cycles. That is the content of the possibility of absence of prepotential, found in [6], translated to the Calabi-Yau context. The choice of cycles can get a physical meaning in the applications in string theory.

5 Remark and conclusions

We have given several equivalent definitions of special geometry. For the rigid as well as for the local case, one can define special geometry from a prepotential. However, a more useful definition starts from a symplectic bundle. The symplectic transformations, inherited from the dualities on vectors, are a crucial aspect for the scalar manifolds. A few extra constraints have to be imposed in order that this symplectic bundle can lead to a special manifold. It is in these extra constraints that earlier proposals for a 'coordinate free' definition were incomplete. The missing equations are mostly related to the condition of symmetry [7] of the matrix \mathcal{N}_{IJ} , which is a key ingredient in the discussion of symplectic transformations.

It is always immediate to go from the formulation with a prepotential to the symplectic one. In rigid special geometry, the existence of a prepotential can be seen for any symplectic bundle which satisfies the requirements in our definition. In the local case, this connection can be less straightforward. If the metric is positive definite it can in general be shown that the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} \mathcal{D}_\alpha X^I & \bar{X}^I \end{pmatrix} \quad (5.1)$$

is invertible. However, a prepotential only exists for the symplectic section if

$$\begin{pmatrix} \mathcal{D}_\alpha X^I & X^I \end{pmatrix} \quad (5.2)$$

is invertible²². The interesting supergravity models without prepotential are thus those for which the latter matrix has a zero mode. However, we have shown in general that one can then always make a symplectic transformation to a basis in which this matrix becomes invertible, and thus a prepotential does exist. As the scalar manifold is invariant under such a symplectic transformation, we can say that any special manifold can be obtained from a prepotential.

Special geometry is sometimes defined [5, 8, 9] by giving the curvature formulas (3.44) for the rigid case or (4.63) for the local case, for a symmetric tensor C , with $\mathcal{D}_{\bar{\alpha}} C_{\alpha\beta\gamma} = 0$ and $\mathcal{D}_{[\alpha} C_{\beta]\gamma\delta} = 0$. Whereas these equations are always valid, we have not investigated (and know of no proof elsewhere) in how far they constitute a sufficient condition.

We have also shown how certain moduli spaces of Riemann surfaces and moduli spaces of Calabi-Yau manifolds satisfy the definitions we have given. It is the reappearance of special geometry in these contexts which has led to recent new applications. As Riemann and Calabi-Yau surface theory and $N = 2$ field theories are related through the concept of special geometry, questions on one side can find an answer on the other side. We have seen that the connection of (local) special geometry to the moduli spaces of Calabi-Yau manifolds is rather straightforward. These spaces can always be endowed with a special Kähler metric. However, for moduli spaces of Riemann surfaces the constraint may necessitate a proper identification of some particular subspace in the full moduli space. We have shown in section 3.5 how our definition can be satisfied, but a better understanding of the geometrical significance of these subsets of moduli spaces would be welcome.

For Calabi-Yau moduli spaces, the statements about the matrices (5.1) and (5.2) imply, that for any choice of basis of non-intersecting (' A ') cycles, the matrix of periods of the $(3, 0)$ and $(1, 2)$ forms over these cycles is invertible. On the other hand, there is always a basis of cycles such that the integrals of the $(3, 0)$ and $(2, 1)$ forms over the A cycles is invertible. A prepotential, or even a dependence $\mathcal{G}_A(z)$ in the sense of (4.69), exists if and only if such a basis is chosen.

Acknowledgments.

We thank P. Candelas, B. de Wit, P. Frè and P. West for discussions. This work was supported by the European Commission TMR programme ERBFMRX-CT96-0045. Part of this work was performed during a visit of A.V.P. at the Newton Institute in Cambridge.

²²In rigid supersymmetry these are $n \times n$ matrices, the last column is not there and the index I is replaced by $A = 1, \dots, n$. Thus the two matrices are then identical.

A General theorems on symplectic transformations

First we will prove here some general properties of symplectic transformations. One important lemma (lemma A.1) says that for a rank m matrix (2.13) there always exist symplectic transformations $\tilde{V} = \mathcal{S}V$ such that the upper half of \tilde{V} becomes invertible. The proof leads immediately to property A.1, which says that all symplectic transformations can be written as a product of a finite number of symplectic matrices of the following form:

$$L = \begin{pmatrix} A & 0 \\ C & A^{-1T} \end{pmatrix} ; \quad S = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} . \quad (\text{A.1})$$

If X is invertible we can define \mathcal{N} as in (2.14), transforming as in (2.15). The invertibility of X , the symmetry of \mathcal{N} and the negative definiteness of its imaginary part are preserved by these symplectic transformations (theorem A.3). It is also to be remarked that any symmetric matrix with negative definite imaginary part is invertible (lemma A.2). Any such matrix \mathcal{N} can of course be written in a form (2.14), taking $X = \mathbf{1}$, and for such a matrix $(A + B\mathcal{N})$ is invertible and the matrix $\tilde{\mathcal{N}}$ in (2.15) is thus symmetric with negative definite imaginary part (corollary A.4). In fact,

$$\text{Im } \tilde{\mathcal{N}} = (A + B\tilde{\mathcal{N}})^{-1T} (\text{Im } \mathcal{N}) (A + B\mathcal{N})^{-1} \quad (\text{A.2})$$

already shows the preservation of the signature.

It is useful to realise first that for a general symplectic matrix

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \quad (\text{A.3})$$

the defining relations (2.7) imply that the following matrices are all symmetric A^TC , B^TD , CA^{-1} , BD^{-1} , $A^{-1}B$, $D^{-1}C$, AB^T , DC^T (if the mentioned inverses exist). The first four follow immediately. The fifth one follows from

$$\begin{aligned} B^TA^{-1T} &= B^TA^{-1T}(A^TD - C^TB) = B^TD - B^TCA^{-1}B \\ &= B^TD + (\mathbf{1} - D^TA)A^{-1}B = A^{-1}B , \end{aligned} \quad (\text{A.4})$$

after which the others are easy too.

Lemma A.1 *If the $2m \times m$ matrix $V = \begin{pmatrix} X^I \\ Y_I \end{pmatrix}$, where X^I and Y_I are m -vectors, is of rank m , then there exists a symplectic rotation $\mathcal{S} \in Sp(2m, \mathbb{R})$ such that $\tilde{V} = \mathcal{S}V = \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix}$ has the property that \tilde{X} is invertible.*

Proof.

We introduce the notation

$$X^r \sim [X^1, \dots, X^{r-1}] \quad (\text{A.5})$$

to indicate that X^r is a linear combination of the vectors X^1, \dots, X^{r-1} , and

$$0 \not\sim [X^1, \dots, X^{r-1}] \quad (\text{A.6})$$

means that these vectors are independent.

We will explicitly construct the matrix \mathcal{S} . Take the following hypothesis of induction.

$$0 \not\sim [X^1, \dots, X^{r-1}, Y_k] ; \quad (\text{A.7})$$

$$X^r \sim [X^1, \dots, X^{r-1}] . \quad (\text{A.8})$$

So V is already of rank $r-1$, and assuming $r-1 < m$, some vector Y_k should be independent of X^1, \dots, X^{r-1} . We now symplectically rotate this vector into the X part. The following cases are distinguished:

Case 1: $k = r$.

Consider the symplectic matrix

$$S_r = \begin{pmatrix} \mathbf{1} - E_{r,r} & E_{r,r} \\ -E_{r,r} & \mathbf{1} - E_{r,r} \end{pmatrix} , \quad (\text{A.9})$$

i.e.

$$\begin{aligned} \tilde{X}^I &= X^I , & I \neq r ; \\ \tilde{X}^r &= Y_k . \end{aligned}$$

This symplectic rotation takes the matrix into one of which the first r rows are independent.

Case 2: $k \neq r$.

In this case we use the symplectic matrix

$$S_{k,r} = \begin{pmatrix} \mathbf{1} - E_{r,r} - \frac{1}{\alpha} E_{k,r} & E_{r,k} + \alpha E_{r,r} \\ -\frac{1}{\alpha} E_{r,r} & \mathbf{1} - E_{r,r} \end{pmatrix} . \quad (\text{A.10})$$

In components this reads

$$\begin{aligned} \tilde{X}^I &= X^I & I \neq k, r ; \\ \tilde{X}^k &= X^k - \frac{1}{\alpha} X^r ; \\ \tilde{X}^r &= Y_k + \alpha Y_r ; \\ \tilde{Y}_I &= Y_I & I \neq r ; \\ \tilde{Y}_r &= -\frac{1}{\alpha} X^r . \end{aligned} \quad (\text{A.11})$$

Notice the introduction of the parameter α whose value can be taken to be arbitrary at this stage. Its relevance will be discussed below.

Next we find out about the conditions which will guarantee that $\tilde{X}^1, \dots, \tilde{X}^r$ are independent m -vectors. Let

$$\sum_{I=1}^r \alpha_I \tilde{X}^I = 0 . \quad (\text{A.12})$$

Substituting the transformation rules (A.11) gives

$$\sum_{I=1, I \neq k}^{r-1} \alpha_I X^I + \alpha_k (X^k - \frac{1}{\alpha} X^r) + \alpha_r (Y_k + \alpha Y_r) = 0 , \quad (\text{A.13})$$

with the convention $\alpha_k = 0$ for $k > r$. The hypothesis of induction (A.8) states that

$$X^r = \sum_{J=1, J \neq k}^{r-1} \lambda_J X^J + \mu X^k , \quad (\text{A.14})$$

for some λ_J, μ with once more the convention $\mu = 0$ for $k > r$. Upon substitution into (A.13) one finally finds

$$\sum_{I=1, I \neq k}^{r-1} (\alpha_I - \lambda_I \frac{\alpha_k}{\alpha}) X^I + \alpha_k (1 - \frac{\mu}{\alpha}) X^k + \alpha_r Y_k + \alpha_r \alpha Y_r = 0 . \quad (\text{A.15})$$

The following cases are distinguished:

1. $Y_r \not\sim [X^1, \dots, X^{r-1}, Y_k]$.

For α_k different from zero a priori (i.e. $k < r$) it suffices to pick $\alpha \neq \mu$ in order that all α_I vanish for $I = 1, \dots, r$. This is an immediate corollary from the assumed independence of the set X^I, Y_r (for $I = 1 \dots, r-1$) and Y_k in (A.15).

2. $Y_r \sim [X^1, \dots, X^{r-1}, Y_k]$.

Equivalently,

$$Y_r = \sum_{I=1, I \neq k}^{r-1} \beta_I X^I + \beta_k X^k + \nu Y_k , \quad (\text{A.16})$$

for appropriate values of the coefficients. In this way (A.15) reads alternatively

$$\begin{aligned} \sum_{I=1, I \neq k}^{r-1} (\alpha_I - \lambda_I \frac{\alpha_k}{\alpha} + \beta_I \alpha_r \alpha) X^I + (\alpha_k (1 - \frac{\mu}{\alpha}) + \alpha_r \alpha \beta_k) X^k \\ + \alpha_r (1 + \nu \alpha) Y_k = 0 . \end{aligned} \quad (\text{A.17})$$

All X^I and Y_k occuring in this equation are once more assumed to be independent quantities, implying the vanishing of the coefficients in front. Explicitly,

$$\begin{aligned} (\alpha_I - \lambda_I \frac{\alpha_k}{\alpha} + \beta_I \alpha_r \alpha) &= 0 \\ \text{for } I = 1, \dots, r-1; I \neq k ; \end{aligned} \quad (\text{A.18})$$

$$(\alpha_k (1 - \frac{\mu}{\alpha}) + \alpha_r \alpha \beta_k) = 0 ; \quad (\text{A.19})$$

$$\alpha_r (1 + \nu \alpha) = 0 . \quad (\text{A.20})$$

It is here that the relevance of the freedom to pick an arbitrary value for α emerges. For if we can choose α such that $1 + \alpha \nu \neq 0$, then (A.20) implies $\alpha_r = 0$. By the same token one concludes that the ability to choose $\alpha \neq \mu$ (in the case $k < r$) forces α_k to vanish by (A.19). Furthermore, all remaining α_I in the first equation become identically zero.

We conclude that with the hypothesis of induction (A.7) and (A.8) the following implication holds:

$$\sum_{I=1}^r \alpha_I (\tilde{X}^I) = 0 \Rightarrow \forall I : \alpha_I = 0 . \quad (\text{A.21})$$

This simply states the independence of the quantities $\tilde{X}^1, \dots, \tilde{X}^r$.

Some small comment is appropriate. In the above reasoning it was assumed that the constant parameter α could be chosen appropriately. At first sight this might conflict with

the fact that the m-vectors X^I (as well as the Y_I) are in general functions of z^α and $\bar{z}^{\bar{\alpha}}$. Recall the restrictions on α :

$$\alpha \neq \mu ; \quad (\text{A.22})$$

$$\alpha \neq -\frac{1}{\nu} ; \quad (\text{A.23})$$

These could possibly constrain the domain in which the above procedure is valid. Note however, that if the m-vectors depend continuously on z^α and $\bar{z}^{\bar{\alpha}}$ one can always find some value for α such that the constraints (A.22) and (A.23) are obeyed on some sufficiently small compact region $W \subset \mathcal{M}$. In any case some neighbourhood on which the algorithm is a valid procedure, is contained within this compact region. ■

Property A.1 *Every $Sp(2m, \mathbb{R})$ -matrix can be written as the product of a finite number of symplectic matrices of the following form:*

$$L = \begin{pmatrix} A & 0 \\ C & A^{-1T} \end{pmatrix} ; \quad S = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} . \quad (\text{A.24})$$

The property of symplecticity implies that $A^T C$ is symmetric.

Proof.

If A is invertible, the statement is proved by the decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & A^{-1T} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} -\mathbf{1} & 0 \\ A^{-1}B & -\mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} . \quad (\text{A.25})$$

The only non-trivial part of this equation is

$$CA^{-1}B + A^{-1T} = A^{-1T}(C^T B + \mathbf{1}) = A^{-1T}A^T D = D . \quad (\text{A.26})$$

If A is not invertible, the last of (2.7) still implies that

$$\begin{pmatrix} A \\ C \end{pmatrix}$$

has maximal rank. Therefore we can use lemma A.1 to transform it to the case of an invertible A . Reviewing the proof of the latter lemma, we see that we performed symplectic rotations of the form (A.9) or (A.10). Both of them can be written as the product of symplectic matrices with an invertible A -matrix, e.g. for $\alpha \neq 1$:

$$S_{k,r} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} + (\frac{1}{\alpha} - 1)E_{r,r} - \frac{1}{\alpha}E_{k,r} & E_{r,k} - \mathbf{1} + (1 + \alpha)E_{r,r} \\ -\frac{1}{\alpha}E_{r,r} & \mathbf{1} - E_{r,r} \end{pmatrix} . \quad (\text{A.27})$$

Therefore all symplectic matrices are written as products of matrices with an invertible A -part, which, using the first part of this proof, implies the statement. ■

Lemma A.2 *A symmetric matrix \mathcal{N} with negative-definite imaginary part is invertible.*

Proof.

The following identities hold for an arbitrary complex vector $z = x + iy$:

$$\begin{aligned} z^\dagger \mathcal{N} z &= (x^T - iy^T) \mathcal{N} (x + iy) \\ &= x^T \mathcal{N} x + y^T \mathcal{N} y - iy^T \mathcal{N} x + ix^T \mathcal{N} y . \end{aligned}$$

Because of the symmetry of \mathcal{N} the last two terms cancel. So

$$\text{Im} (z^\dagger \mathcal{N} z) = x^T (\text{Im } \mathcal{N}) x + y^T (\text{Im } \mathcal{N}) y ,$$

from which it follows that each nullvector z of \mathcal{N} equals zero. ■

Theorem A.3 *Take $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2m, \mathbb{R})$. Consider $(m \times m)$ -matrices X and Y such that*

1. X is invertible;
2. YX^{-1} is symmetric;
3. $\text{Im} (YX^{-1}) < 0$.

Define $\begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$, then \tilde{X} and \tilde{Y} satisfy the conditions 1 till 3.

Proof.

Note that proposition A.1 implies that we need only prove the theorem for the following cases:

1. $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$.
2. $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$.

• First we show that \tilde{X} is invertible in both cases:

1. $\tilde{X} = Y$. This is invertible because the items 2 and 3 and lemma A.2 imply that YX^{-1} is.
2. $\tilde{X} = AX$ is invertible since A and X are.

• $\tilde{Y}\tilde{X}^{-1}$ is symmetric:

$$\tilde{Y}\tilde{X}^{-1} - \tilde{X}^{-1T}\tilde{Y}^{-1T} = \tilde{X}^{-1T}(\tilde{X}^T\tilde{Y} - \tilde{Y}^T\tilde{X})\tilde{X}^{-1} .$$

Because of symplecticity the expression in brackets equals $(X^TY - Y^TX)$, which is zero since YX^{-1} is symmetric.

- $\text{Im} (\tilde{Y} \tilde{X}^{-1}) < 0$:

From the symmetry of $\tilde{Y} \tilde{X}^{-1}$ we get $(\tilde{Y} \tilde{X}^{-1})^* = (\tilde{Y} \tilde{X}^{-1})^\dagger$, so

$$\begin{aligned} \text{Im} (\tilde{Y} \tilde{X}^{-1}) &= \frac{1}{2i} [\tilde{Y} \tilde{X}^{-1} - (\tilde{Y} \tilde{X}^{-1})^\dagger] \\ &= \frac{1}{2i} \{ \tilde{X}^{-1+} [\tilde{X}^\dagger \tilde{Y} - \tilde{Y}^\dagger \tilde{X}] \tilde{X}^{-1} \} . \end{aligned}$$

The expression in brackets is the symplectic inner product of $(\tilde{X}^*, \tilde{Y}^*)$ and (\tilde{X}, \tilde{Y}) and thus equals $[X^\dagger Y - Y^\dagger X]$, so

$$\text{Im} (\tilde{Y} \tilde{X}^{-1}) = (X \tilde{X}^{-1})^\dagger \text{Im} (Y X^{-1}) (X \tilde{X}^{-1}) ,$$

which is negative-definite because of property 3 and the invertibility of $X \tilde{X}^{-1}$. ■

Corollary A.4 *If \mathcal{N} is a symmetric $(m \times m)$ -matrix with strictly negative imaginary part and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2m, \mathbb{R})$, then $A + B\mathcal{N}$ is invertible. Furthermore, $(C + D\mathcal{N})(A + B\mathcal{N})^{-1}$ is also a symmetric matrix with strictly negative imaginary part.*

Proof.

Take in proposition (A.3) $X = \mathbf{1}$ and $Y = \mathcal{N}$. ■

B Theorems on special geometry

Lemma B.1 *If V satisfies the conditions of remark 1, page 23, and $g_{\alpha\bar{\beta}}$ is non-degenerate then*

$$\mathcal{W} \equiv \begin{pmatrix} X^I & F_I \\ \mathcal{D}_\alpha X^I & \mathcal{D}_\alpha F_I \end{pmatrix}$$

(where $I = 0 \dots n$ and $\alpha = 1 \dots n$) has rank $n + 1$.

Proof.

Suppose $\text{rank}(\mathcal{W}) \leq n$, then there exist λ^α and λ^0 , not all zero, such that

$$\lambda^\alpha U_\alpha + \lambda^0 V = 0 . \tag{B.1}$$

Taking first an inner product with \bar{V} , and using (4.38) and condition 1, gives $\lambda^0 = 0$, and thus λ^α is not trivial. Then taking the inner product with $\bar{U}_{\bar{\beta}}$, and using (4.38) and (4.40), we get $\lambda^\alpha g_{\alpha\bar{\beta}} = 0$, which is in contradiction with our assumption that $g_{\alpha\bar{\beta}}$ is non-degenerate. ■

Lemma B.2

$$\det \begin{pmatrix} \mathcal{D}_\alpha X^B & \mathcal{D}_\alpha X^0 \\ X^B & X^0 \end{pmatrix} \neq 0 \Leftrightarrow \det \left[\partial_\alpha \left(\frac{X^B}{X^0} \right) \right] \neq 0$$

if $X^0 \neq 0$.

Proof.

The proof follows from the equality of the following determinants:

$$\begin{aligned} \det \begin{pmatrix} \mathcal{D}_\alpha X^B & \mathcal{D}_\alpha X^0 \\ X^B & X^0 \end{pmatrix} &= \det \begin{pmatrix} \partial_\alpha X^B & \partial_\alpha X^0 \\ X^B & X^0 \end{pmatrix} \\ &= \det \begin{pmatrix} \partial_\alpha X^B - \frac{X^B}{X^0} \partial_\alpha X^0 & \partial_\alpha X^0 \\ X^B - X^B & X^0 \end{pmatrix} = (X^0)^{n+1} \det \left[\partial_\alpha \left(\frac{X^B}{X^0} \right) \right] . \end{aligned} \quad (\text{B.2})$$

■

Lemma B.3 *If $\det \begin{pmatrix} \mathcal{D}_\alpha X^I \\ X^I \end{pmatrix} \neq 0$ and $\mathcal{D}_{\bar{\alpha}} \begin{pmatrix} X^I \\ F_I \end{pmatrix} = 0$, then some functions $\tilde{F}_I(X^J)$ exist such that*

$$\tilde{F}_I(X^J(z^\alpha, \bar{z}^\alpha)) = F_I(z^\alpha, \bar{z}^\alpha) , \quad (\text{B.3})$$

and

$$\tilde{F}_I = X^J \partial_J \tilde{F}_I . \quad (\text{B.4})$$

Proof.

Because of $\mathcal{D}_{\bar{\alpha}} \begin{pmatrix} X^I \\ F_I \end{pmatrix} = 0$, $\frac{X^B}{X^0}$ and $\frac{F_I}{X^0}$ are holomorphic functions of the z^α . Lemma B.2 guarantees that one can express the z^α in terms of the independent variables $\frac{X^B}{X^0}$. Now set e.g.

$$\tilde{F}_I(X^J) \equiv X^0 f_I \left(z^\alpha \left(\frac{X^B}{X^0} \right) \right) , \quad (\text{B.5})$$

with

$$f_I(z^\alpha) \equiv \frac{F_I(z^\alpha, \bar{z}^\alpha)}{X^0(z^\alpha, \bar{z}^\alpha)} . \quad (\text{B.6})$$

Equation (B.4) is immediate for \tilde{F}_I is homogeneous of first degree. ■

Lemma B.4 *If $\det \begin{pmatrix} \mathcal{D}_\alpha X^I \\ X^I \end{pmatrix} \neq 0$, conditions 4 and 3 of remark 1 (page 23) hold, and (B.3) and (B.4) hold, then some function $\tilde{F}(X^J)$ exists such that*

$$\tilde{F}_I = \partial_I \tilde{F}(X^I) . \quad (\text{B.7})$$

Proof.

The two conditions of remark 1 can, with the mentioned equations of lemma B.3, be written respectively as

$$\begin{aligned} 0 &= \mathcal{D}_\alpha X^I \mathcal{D}_\beta X^J \partial_{[I} \tilde{F}_{J]} \\ 0 &= X^I \partial_\alpha F_I - F_I \partial_\alpha X^I = \partial_\alpha X^I X^J \partial_{[I} \tilde{F}_{J]} = \mathcal{D}_\alpha X^I X^J \partial_{[I} \tilde{F}_{J]} . \end{aligned} \quad (\text{B.8})$$

Because of the invertibility of $\begin{pmatrix} \mathcal{D}_\alpha X^I \\ X^I \end{pmatrix}$, these equations imply that $\partial_{[I} \tilde{F}_{J]} = 0$, which is the integrability condition for the existence of $\tilde{F}(X)$. ■

Lemma B.5 *If the matrix*

$$i \begin{pmatrix} \langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle & \langle U_\alpha, V \rangle \\ \langle \bar{V}, \bar{U}_{\bar{\beta}} \rangle & \langle \bar{V}, V \rangle \end{pmatrix} \quad (\text{B.9})$$

is positive definite, then the matrix $\begin{pmatrix} \mathcal{D}_\alpha X^I & \bar{X}^I \end{pmatrix}$ is invertible.

Proof.

Suppose a linear combination of the columns of this matrix equals zero:

$$a^\alpha \mathcal{D}_\alpha X^I + b \bar{X}^I = 0 . \quad (\text{B.10})$$

Then the same statement applies to the complex conjugate equation. So it follows immediately that

$$\langle a^\alpha U_\alpha + b \bar{V}, \bar{a}^{\bar{\beta}} \bar{U}_{\bar{\beta}} + \bar{b} V \rangle = 0 . \quad (\text{B.11})$$

This can be written as follows:

$$-i \begin{pmatrix} a^\alpha & b \end{pmatrix} \begin{pmatrix} \langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle & \langle U_\alpha, V \rangle \\ \langle \bar{V}, \bar{U}_{\bar{\beta}} \rangle & \langle \bar{V}, V \rangle \end{pmatrix} \begin{pmatrix} \bar{a}^{\bar{\beta}} \\ \bar{b} \end{pmatrix} = 0 . \quad (\text{B.12})$$

The positivity of (B.9) implies that $a = b^\alpha = 0$. ■

Corollary B.6 *If the conditions $\langle V, \bar{V} \rangle = i$ and $\langle V, U_\alpha \rangle = 0$ are satisfied and the metric $g_{\alpha\bar{\beta}} \equiv i \langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle$ is positive definite then the matrix $\begin{pmatrix} \mathcal{D}_\alpha X^I & \bar{X}^I \end{pmatrix}$ is invertible.*

C Counterexamples

C.1 Example 1

We give an explicit example of a Kähler manifold which satisfies Strominger's definition of a special Kähler manifold, but which is not special according to our definitions. This example proves the non-triviality of the condition (4.35) for $n = 1$.

Our manifold is the following part of the complex plane: an open disk centered at 0 with sufficiently small radius (cf. infra). The Kähler metric is obtained from the Kähler potential

$$K = -\log \left(|1 + z^4 + z^5 + z^6 + z^7|^2 - |z|^2 + |z|^4 - |z|^6 \right) . \quad (\text{C.1})$$

The disk's radius should be small enough for the metric to be positive definite. Since $g_{z\bar{z}}(0) = 1$ a suitable radius exists.

First we prove that $g_{z\bar{z}}$ is special according to Strominger's definition, i.e. that there exists a holomorphic symplectic four-component vector $v(z)$ such that

$$e^{-K} = i \langle \bar{v}, v \rangle . \quad (\text{C.2})$$

Proof.

Note that

$$(x, y) \mapsto i \langle \bar{x}, y \rangle ; \quad x, y \in \mathbb{C}^4 \quad (\text{C.3})$$

defines a non-degenerate sesquilinear form. The following vectors constitute a basis of \mathbb{C}^4 which is orthonormal with respect to this form:

$$v_0 = \begin{pmatrix} 1 \\ 0 \\ -\frac{i}{2} \\ 0 \end{pmatrix} ; \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{i}{2} \\ 0 \end{pmatrix} ; \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{i}{2} \end{pmatrix} ; \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{i}{2} \end{pmatrix} . \quad (\text{C.4})$$

Indeed:

$$\langle v_k, \bar{v}_l \rangle = i\delta_{k,l}(-1)^k . \quad (\text{C.5})$$

Now define

$$v \equiv v_0(1 + z^4 + z^5 + z^6 + z^7) + v_1z + v_2z^2 + v_3z^3 . \quad (\text{C.6})$$

Then it follows from (C.5) that

$$\langle v, \bar{v} \rangle = i(1 + z^4 + z^5 + z^6 + z^7)(1 + \bar{z}^4 + \bar{z}^5 + \bar{z}^6 + \bar{z}^7) - iz\bar{z} + iz^2\bar{z}^2 - iz^3\bar{z}^3 , \quad (\text{C.7})$$

and thus (C.2). ■

Consider the point $z = 0$, which is in our domain. There $V = v = v_0$ and $U_z = \partial_z v = v_1$. So we have $\langle U_\alpha, V \rangle = \langle \partial_z v, v \rangle = -i$, so this violates (4.35) or the condition 3, page 23. The matrix (4.45) is at this point

$$i \begin{pmatrix} \langle U_z, \bar{U}_{\bar{z}} \rangle & \langle U_z, V \rangle \\ \langle \bar{V}, \bar{U}_{\bar{z}} \rangle & \langle \bar{V}, V \rangle \end{pmatrix} \Big|_{z=0} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} , \quad (\text{C.8})$$

which is thus not invertible. Also (4.43) is then not invertible, and we have no unique definition of \mathcal{N} . Therefore this symplectic section cannot be used for supergravity.

Finally we prove that $g_{z\bar{z}}$ is not special according to our second definition, i.e. that no holomorphic symplectic vector $w(z)$ exists which satisfies both

$$e^{-K} = i\langle \bar{w}, w \rangle \quad (\text{C.9})$$

and

$$\langle w, \partial_z w \rangle = 0 . \quad (\text{C.10})$$

[A priori we should impose instead of (C.9) the weaker condition

$$\exists f(z) : \quad i\langle \bar{w}, w \rangle = e^{-K+f(z)+\bar{f}(\bar{z})} , \quad (\text{C.11})$$

but then $e^{-f(z)}w$ would satisfy (C.9) and (C.10) whenever w satisfied (C.11) and (C.10). So what we will prove is sufficiently general.]

Proof.

Suppose there exists a holomorphic vector w satisfying equations (C.9) and (C.10). We show that this leads to a contradiction.

We can write w as a power series:

$$w = w_0 + w_1z + w_2z^2 + \dots \quad (\text{C.12})$$

with $w_k \in \mathbb{C}^4$ as yet arbitrary coefficients. Then

$$\langle w, \bar{w} \rangle = \sum_{k,l} \langle w_k, \bar{w}_l \rangle z_k \bar{z}_l . \quad (\text{C.13})$$

Identifying the coefficients $\langle w_k, \bar{w}_l \rangle$ and those of ie^{-K} , we find

$$\langle w_k, \bar{w}_l \rangle = i\delta_{k,l}(-1)^k \quad \text{voor } k, l = 0, 1, 2, 3 ; \quad (\text{C.14})$$

$$\langle w_k, \bar{w}_0 \rangle = i \quad \text{voor } 4 \leq k \leq 7 ; \quad (\text{C.15})$$

$$\langle w_k, \bar{w}_l \rangle = 0 \quad \text{voor } 4 \leq k \leq 7 , \quad l = 1, 2, 3 . \quad (\text{C.16})$$

Since $\{w_0, w_1, w_2, w_3\}$ is an orthonormal basis (see (C.14)), equation (C.16) implies that w_k is proportional to w_0 for $4 \leq k \leq 7$. Then it follows from equations (C.15) and (C.14) that

$$w_k = w_0 , \quad 4 \leq k \leq 7 . \quad (\text{C.17})$$

Summing up, equation (C.9) implies that $\{w_0, w_1, w_2, w_3\}$ is an orthonormal basis (with respect to (C.3)) and that $w_k = w_0$ for $4 \leq k \leq 7$.

Now we write out equation (C.10):

$$0 = \langle w, \partial_z w \rangle = \langle w_0 + w_1 z + w_2 z^2 + \dots, w_1 + 2w_2 z + 3w_3 z^2 + \dots \rangle . \quad (\text{C.18})$$

All terms of the power series should vanish. We put the coefficients of z^0 , z^1 and z^6 equal to zero and take equation (C.17) and the antisymmetry of $\langle \cdot, \cdot \rangle$ into account:

$$\langle w_0, w_1 \rangle = 0 ;$$

$$\langle w_0, w_2 \rangle = 0 ;$$

$$\langle w_3, w_0 \rangle = 0 .$$

These express that \bar{w}_0 is orthogonal to w_1 , w_2 and w_3 . Furthermore, \bar{w}_0 is clearly orthogonal to w_0 ($\langle w_0, w_0 \rangle = 0$). Thus is

$$\bar{w}_0 = 0 . \quad (\text{C.19})$$

This is in contradiction with $\langle w_0, \bar{w}_0 \rangle = i$. ■

C.2 Example 2

As a second example, consider

$$V = e^{K/2} \begin{pmatrix} 1 \\ z \\ -iz \\ -ia \end{pmatrix} , \quad (\text{C.20})$$

with $a \in \mathbb{R}$. It satisfies conditions 1 and 2 of remark 1. For $a = 1$ this is the symplectic section 4.31, originating from the prepotential $F = -iX^0 X^1$, and for all a it has the same Kähler potential up to a Kähler transformation:

$$e^{-K} = (1+a)(z + \bar{z}) ; \quad \partial_z \partial_{\bar{z}} K = (z + \bar{z})^{-2} , \quad (\text{C.21})$$

with a positivity domain if $a \neq -1$, for example $a > -1$ and $\text{Re } z > 0$. We find

$$U_z = \frac{1}{z + \bar{z}} e^{K/2} \begin{pmatrix} -1 \\ \bar{z} \\ -i\bar{z} \\ ia \end{pmatrix} \quad (\text{C.22})$$

and $\langle U_z, V \rangle = i(1-a)e^K$. So the constraint (4.35) selects $a = 1$. However, the matrix (4.45),

$$i \begin{pmatrix} \langle U_z, \bar{U}_{\bar{z}} \rangle & \langle U_z, V \rangle \\ \langle \bar{V}, \bar{U}_{\bar{z}} \rangle & \langle \bar{V}, V \rangle \end{pmatrix} = \begin{pmatrix} (z + \bar{z})^{-2} & \frac{a-1}{a+1}(z + \bar{z})^{-1} \\ \frac{a-1}{a+1}(z + \bar{z})^{-1} & 1 \end{pmatrix} \quad (\text{C.23})$$

remains positive definite for all $a > 0$. Therefore we can define the matrix \mathcal{N} , which is

$$\mathcal{N} = \begin{pmatrix} -iz & 0 \\ 0 & -\frac{ia}{z} \end{pmatrix}, \quad (\text{C.24})$$

i.e. symmetric and negative definite imaginary part for $a > 0$, as follows from the general arguments in section 4.2.2. In the present constructions of $N = 2$ supergravity couplings this matrix for $a \neq 1$ is never obtained because it violates

$$-\frac{1}{2}(\text{Im } \mathcal{N})^{-1} = \bar{f}_{\bar{\alpha}}^I g^{\bar{\alpha}\beta} f_{\beta}^J + X^I \bar{X}^J. \quad (\text{C.25})$$

In fact, this relation is a rewriting of (4.45), using in the left hand side the usual constraints of special geometry, including $\langle U_{\alpha}, V \rangle = 0$. This equation is used in proving the supersymmetry commutator on the vector field, or the invariance of the Pauli-terms in the action, where the two terms on the right hand side correspond to contributions via the spin- $\frac{1}{2}$ and spin- $\frac{3}{2}$ fermions respectively. However, we know no physical argument to exclude the possibility of a modification in case $\langle U_{\alpha}, V \rangle \neq 0$.

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