

Cartan geometry of spacetimes with a nonconstant cosmological function Λ

Hendrik Jennen*

Instituto de Física Teórica, UNESP-Universidade Estadual Paulista,

Rua Dr. Bento Teobaldo Ferraz, 271 ~~Bl.~~Block II, 01140-070, São Paulo, SP, Brazil

We present the geometry of spacetimes that are tangentially approximated by de Sitter spaces whose cosmological constants vary over spacetime. Cartan geometry provides one with the tools to describe manifolds that reduce to a homogeneous Klein space at the infinitesimal level. We consider a Cartan geometry in which the underlying Klein space is at each point a de Sitter space, for which the combined set of ~~pseudo-radii~~pseudoradii forms a nonconstant function on spacetime. We show that the torsion of such a geometry receives a contribution that is not present for a cosmological constant. The structure group of the obtained ~~de Sitter-Cartan~~Sitter-Cartan geometry is by construction the Lorentz group $SO(1, 3)$. Invoking the theory of nonlinear realizations, we extend the class of symmetries to the enclosing de Sitter group $SO(1, 4)$, and compute the corresponding spin connection, vierbein, curvature, and torsion.

PACS numbers: 04.20.Cv, 02.40.-k, 98.80.-k

I. INTRODUCTION

In theories of gravity, the strong equivalence principle implies that spacetime \mathcal{M} is locally approximated by the spacetime underlying special relativity [1]. Since the laws that govern special relativistic physics are covariant with respect to the Poincaré group $ISO(1, 3)$, the corresponding spacetime is the affine Minkowski space M . Although finite Poincaré translations are not defined for a generic spacetime, the equivalence principle indicates that locally they are in one-to-one correspondence with infinitesimal active diffeomorphisms, for both sets generate translations along spacetime [2]. Mathematically speaking, there is a 1-form, called the vierbein, which at any point is valued in the algebra of Poincaré translations $\mathfrak{t} = \mathfrak{iso}(1, 3)/\mathfrak{so}(1, 3)$. The vierbein pulls back or *solders* the geometric and algebraic structure of \mathfrak{t} to spacetime. For example, the Minkowski metric on \mathfrak{t} gives way to a metric of the same signature on \mathcal{M} , from which it follows that the vierbein can be chosen to be an orthonormal ~~frame — an idealized observer — along~~

*hjennen@ift.unesp.br

frame—an idealized observer—along spacetime. Due to the equivalence principle, Lorentz transformations of these observers constitute a symmetry and are therefore elements of the structure group of the geometry, which in turn leads to the introduction of a spin connection.

The right mathematical framework for the setting just outlined is due to Elie Cartan [3], in which the $\mathfrak{so}(1,3)$ -valued spin connection and the \mathfrak{t} -valued vierbein are combined into an $\mathfrak{iso}(1,3)$ -valued Cartan connection, thereby defining a Riemann-Cartan geometry [2]. It is explained comprehensively in [4] how the $\mathfrak{iso}(1,3)$ -valued connection gives a prescription for *rolling without slipping* the affine Minkowski space along the integral curves of vector fields on spacetime. It is indeed the central idea behind Cartan geometry that a homogeneous model space is generalized to a nonhomogeneous space, for which the local structure is algebraically isomorphic to the one of the model space [5], and where the degree of nonhomogeneity is quantified by the presence of curvature and torsion. In the manner thus explained, the choice for a Riemann-Cartan geometry to describe spacetimes underlying theories of gravity is implied by the equivalence principle, together with the assumption that the local kinematics are governed by the Poincaré group.

When the $\mathfrak{iso}(1,3)$ -valued Cartan connection is replaced by one that is valued in the de Sitter algebra $\mathfrak{so}(1,4)$, spacetime is locally approximated by a de Sitter space dS in place of the affine Minkowski space, a structure we shall call a de ~~Sitter-Cartan~~ Sitter-Cartan geometry. Since the vierbein is valued in the space of de Sitter transvections $\mathfrak{p} = \mathfrak{so}(1,4)/\mathfrak{so}(1,3)$, translations in a de ~~Sitter-Cartan~~ Sitter-Cartan spacetime are generated by elements of \mathfrak{p} . This implies that the commutator of infinitesimal translations is proportional to a Lorentz rotation. The constant of proportionality is essentially the cosmological constant of the tangent de Sitter spaces [4]. It is then sensible to identify this geometric cosmological constant with the dark energy on spacetime. Such an interpretation is in concordance with the MacDowell-Mansouri model for gravity [6]. In this model, the fundamental field is indeed a $\mathfrak{so}(1,4)$ -valued Cartan connection, for which the action is equivalent, up to topological terms, with the Palatini action for general relativity in the presence of a cosmological constant [4, 7].

At any point in a de ~~Sitter-Cartan~~ Sitter-Cartan spacetime, the cosmological constant is related to a length scale defined in the commutation relations of the de Sitter transvections. Therefore, it is rather straightforward to generalize to geometries in which this length scale becomes a nonconstant function on spacetime. In Sec. II, we claim some originality for constructing a de ~~Sitter-Cartan~~ Sitter-Cartan geometry that provides spacetime with a *cosmological function* Λ , which in general does not satisfy $d\Lambda = 0$. We shall see that a nonconstant Λ gives rise to a new term in the expression for the torsion of the de ~~Sitter-Cartan~~ Sitter-Cartan geometry.

II. DE ~~SITTER-CARTAN~~ SITTER-CARTAN GEOMETRY WITH A COSMOLOGICAL FUNCTION

A de Sitter-Cartan geometry is the Cartan geometry modeled on $(\mathfrak{so}(1,4), SO(1,3))$, which means it consists of a principal Lorentz bundle $P(\mathcal{M}, SO(1,3))$ over spacetime, on which is defined a $\mathfrak{so}(1,4)$ -valued Cartan connection A . For a rigorous discussion on Cartan geometry, see [5, 8], while the articles [4, 7, 9] are very helpful to develop an intuition that goes along with the mathematics. The connection A provides spacetime with the information that it is tangentially approximated by de Sitter space, the homogeneous Klein space with respect to which inhomogeneities are measured [10]. We shall construct a de ~~Sitter-Cartan~~ Sitter-Cartan geometry, in which these tangent de Sitter spaces have cosmological constants that are not required to be the same over spacetime. As a consequence, the thus obtained geometry describes a manifold with arbitrary curvature and torsion on which a nonconstant cosmological function Λ is defined from the onset.

Under the action of a local $SO(1,3)$ ~~transformation~~ transformation h , the de ~~Sitter-Cartan~~ Sitter-Cartan connection transforms according to [5]

$$A \mapsto \text{Ad}(h)(A + d). \quad (1)$$

The connection is valued in the de Sitter algebra $\mathfrak{so}(1,4)$, which is characterized by the commutation relations

$$\begin{aligned} -i[M_{ab}, M_{cd}] &= \eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad}, \\ -i[M_{ab}, P_c] &= \eta_{ac}P_b - \eta_{bc}P_a, \\ -i[P_a, P_b] &= -l^{-2}M_{ab}, \quad \text{where } \eta_{ab} = (+, -, -, -), \end{aligned}$$

while we parametrize an element of $\mathfrak{so}(1,4)$ by $i2\lambda^{ab}M_{ab} + i\lambda^a P_a$. The reductive nature of the algebra schematically reads as

$$\mathfrak{so}(1,4) = \mathfrak{so}(1,3) \oplus \mathfrak{p}, \quad (2)$$

where $\mathfrak{so}(1,3) = \text{span}\{M_{ab}\}$ is the Lorentz subalgebra and $\mathfrak{p} = \text{span}\{P_a\}$ the subspace of infinitesimal de Sitter transvections or translations. The latter are defined by $P_a = M_{a4}/l$, where l is an ~~a-priori~~ a priori arbitrary length scale that effectively determines the cosmological constant of the corresponding Klein geometry $dS = SO(1,4)/SO(1,3)$, namely, [4]

$$\Lambda = \frac{3}{l^2}. \quad (3)$$

Since the Cartan connection is at any point $x \in \mathcal{M}$ valued in a copy of $\mathfrak{so}(1,4)$, we may choose the set of length scales $l(x)$ to form a smooth function. Doing so, the cosmological constants of the corresponding tangent de Sitter

spaces also constitute a generic nonconstant cosmological function $\Lambda(x)$ on spacetime. In the following paragraphs we discuss the implications of a nonconstant Λ for the de ~~Sitter-Cartan~~ Sitter-Cartan geometry.

Corresponding to the reductive splitting $\text{eq:red}_{split} \mathfrak{so}(1,4)$, *we decompose the Cartan connection and its curvature* $F = dA + 12[A, A]_{as} A = i2A^{ab}M_{ab} + iA^a P_a$ and $F = i2F^{ab}M_{ab} + iF^a P_a$, (4) from which it follows that A^a and F^a have the dimension of length. The $\mathfrak{so}(1,3)$ -valued 1-form A^{ab} is an Ehresmann connection for local Lorentz transformations [8], i.e., a spin connection, while the forms A^a constitute a vierbein. Note that the decompositions *eq:red_{split} A F_d SC are well defined, since local Lorentz transformations leave the reductive splitting invariant. Due to the*

Given the commutation relations $\text{eq:comm}_{relations} \mathfrak{so}(1,4)$, *one computes the curvature* F^{ab} and torsion F^a in terms of the spin connection and vierbein: $F^{ab} = dA^{ab} + A_c^a \wedge A^{cb} + \frac{1}{l^2} A^a \wedge A^b$
 $= d_A A^{ab} + \frac{1}{l^2} A^a \wedge A^b$,
 $F^a = dA^a + A_b^a \wedge A^b - \frac{1}{l} dl \wedge A^a$
 $= d_A A^a - \frac{1}{l} dl \wedge A^a$. In the limit of an everywhere diverging length scale l , or equivalently, an everywhere vanishing cosmological constant, the expressions $\text{eqs:curvtors}_d SC$ *reduce to the curvature* $d_A A^{ab}$ and torsion $d_A A^a$ for a Riemann-Cartan geometry [11]. In the generic case, however, the curvature and torsion are not given by the exterior covariant derivatives of the spin connection and vierbein. The extra term in $\text{eq:curv}_d SC$ *represents the curvature of the local de Sitter space. This contribution is present because the commutator of two in* $dA_p + [A_{\mathfrak{h}}, A_p]$ [5], with $\mathfrak{h} = \mathfrak{so}(1,3)$. The first term in this expression is expanded as

$$dA_p = d(iA^a P_a) = i dA^a P_a - i \left(\frac{dl}{l} \wedge A^a \right) P_a,$$

since $P_a = M_{a4}/l$. By use of the relation eq:rel.Ll between l and the cosmological function Λ , the last term of the torsion can be rewritten as

$$-dl \ln l \wedge A^a = 12d \ln \Lambda \wedge A^a,$$

which shows that this contribution depends on the relative infinitesimal change of the cosmological function along spacetime, rather than on its absolute change.

Although the curvature and torsion have contributions that are not there for a Riemann-Cartan geometry, the Bianchi identities are unchanged: $d_A \circ d_A A^{ab} \equiv 0$,

$d_A \circ d_A A^a + A^b \wedge d_A A_b^a \equiv 0$, where d_A is the exterior covariant derivative with respect to the spin connection.

The transformations that are consistent with the given geometry are local Lorentz transformations and spacetime diffeomorphisms, the latter being unphysical as they merely rela-

bel spacetime coordinates [12]. In contrast, with respect to elements of $SO(1,4)$, we see from eq:red_{split} F_dSC that the spin connection and vierbein, and the torsion and curvature form irreducible multiplets. Due to the

III. $SO(1,4)$ -INVARIANT DE ~~SITTER-CARTAN~~SITTER-CARTAN GEOMETRY WITH A COSMOLOGICAL FUNCTION

In order to extend the structure group to $SO(1,4)$ and have geometric objects that are well defined through the decomposition of a Cartan connection and curvature according to the reductive splitting eq:red_{split} $so(1,4)$, we nonlinearly realize the de ~~Sitter-Cartan~~Sitter-Cartan connection of Sec II. To realize connection, 14], while its usefulness for theories of gravity has been pointed out in, e.g., [15–18]. The formalism of nonlinear realizations was –21], in which linearly transforming irreducible multiplets become nonlinear but reducible realizations, when the symmetry group

A Cartan connection on a principal Lorentz bundle P may be thought of as an Ehresmann connection on a principal $SO(1,4)$ ~~bundle~~bundle Q over \mathcal{M} that is reduced to P [5]. This is in essence a symmetry breaking process [15], for the reason that it corresponds to singling out a section ξ of the associated bundle $Q \times_{SO(1,4)} dS$ of tangent de Sitter spaces, thereby reducing the structure group $SO(1,4)$ pointwise to $SO(1,3)_\xi$, the isotropy group of the point $\xi(x)$ in the internal de Sitter space dS_x [22, 23]. Most importantly, the reduction is not canonical, i.e., the section ξ can be chosen arbitrarily, and the broken symmetries are nonmanifestly restored by realizing them nonlinearly through elements of the Lorentz group. Consequently, decomposing a nonlinear de ~~Sitter-Cartan~~Sitter-Cartan connection according to the reductive splitting of $\mathfrak{so}(1,4)$ gives way to true geometric objects, well defined with respect to all elements of $SO(1,4)$.

Before we construct a nonlinear de ~~Sitter-Cartan~~Sitter-Cartan geometry with a nonconstant cosmological function, we recall a handful of facts on nonlinear realizations for the de Sitter group ~~;~~; see also [24, 25]. Within some neighborhood of the identity, an element g of $SO(1,4)$ can uniquely be represented in the form

$$g = \exp(i\xi \cdot P)\tilde{h},$$

with $\tilde{h} \in SO(1,3)$ and $\xi \cdot P = \xi^a P_a$. The ξ^a parametrize the coset space $SO(1,4)/SO(1,3)$ so that they constitute a coordinate system for de Sitter space. This parametrization allows us to define the action of $SO(1,4) \ni g_0$ on de Sitter space by

$$g_0 \exp(i\xi \cdot P) = \exp(i\xi' \cdot P)h' ; \quad h' = \tilde{h}'\tilde{h}^{-1},$$

where $\xi' = \xi'(g_0, \xi)$ and $h' = h'(g_0, \xi)$ are in general nonlinear functions of the indicated variables. In case $g_0 = h_0$ is

an element of $SO(1, 3)$, the action is linear and the transformation of ξ is given explicitly by

$$h_0 : i\xi \cdot P \mapsto i\xi' \cdot P = i\xi \cdot \text{Ad}(h_0)(P).$$

If on the other hand $g_0 = 1 + i\epsilon \cdot P$ is an infinitesimal pure de Sitter translation, the variations $\delta\xi^a$ and δh^{ab} satisfy

$-\exp(-i\xi \cdot P)\delta\exp(i\xi \cdot P) = i2\delta h \cdot M$, where $i2\delta h \cdot M = h' - 1 \in \mathfrak{so}(1, 3)$ and $\delta h \cdot M = \delta h^{ab}M_{ab}$. This equation is

$$\text{solved by } \delta\xi^a = \epsilon^a + \left(\frac{z \cosh z}{\sinh z} - 1\right) \left(\epsilon^a - \frac{\xi^a \epsilon_b \xi^b}{\xi^2}\right),$$

$$\delta h^{ab} = \frac{1}{l^2} \frac{\cosh z - 1}{z \sinh z} (\epsilon^a \xi^b - \epsilon^b \xi^a), \text{ where we made use of the notation } z = l^{-1}\xi \text{ and } \xi = (\eta_{ab}\xi^a \xi^b)^{1/2}.$$

Subsequently, let ψ be a field that belongs to some linear representation σ of $SO(1, 4)$. Given a local section of the associated bundle of homogeneous de Sitter spaces, i.e., $\xi : U \subset \mathcal{M} \rightarrow U \times dS$, the corresponding nonlinear field is constructed pointwise as

$$\bar{\psi}(x) = \sigma(\exp(-i\xi(x) \cdot P))\psi(x). \quad (5)$$

Under a local de Sitter transformation g_0 , it rotates only according to its $SO(1, 3)$ ~~indices~~indices, namely, $\bar{\psi}'(x) = \sigma(h'(\xi, g_0))\bar{\psi}(x)$. It is manifest that the irreducible linear representation ψ has given way to a nonlinear and reducible realization $\bar{\psi}$.

In concordance with the prescription *eq:define nonlinear field to construct nonlinear realizations, then nonlinear $so(1, 4)$ -valued C*
 $\text{Ad}(\exp(-i\xi \cdot P))(A + d)$. (6) Under local de Sitter transformations, the field \bar{A} transforms according to

$$\bar{A} \mapsto \text{Ad}(h'(\xi, g_0))(\bar{A} + d).$$

Because elements of $SO(1, 4)$ are nonlinearly realized as elements of $SO(1, 3)$, the reductive decomposition $\bar{A}_{\mathfrak{h}} + \bar{A}_{\mathfrak{p}}$ is invariant under local de Sitter transformations. It is then sensible to define the spin connection and vierbein through these projections as $\omega = \bar{A}_{\mathfrak{h}}$ and $e = \bar{A}_{\mathfrak{p}}$, respectively.

The spin connection ω and vierbein e can be expressed in terms of the section ξ and the projections $A_{\mathfrak{h}}$ and $A_{\mathfrak{p}}$ of the linear $SO(1, 4)$ connection. These relations follow

from *eq: A_n online, in which the different objects appear according to $i2\omega^{ab}M_{ab} + ie^a P_a$*

$$= \text{Ad}(\exp(-i\xi \cdot P)) \left(i2A^{ab}M_{ab} + iA^a P_a + d \right). \text{To carry out the computation of the right-hand side we utilize the techniques of [14, 24], explained in their appendices. In short, one expands the adjoint action of the exponential}$$

$i\xi \cdot P$. The latter is just the Lie commutator and is given explicitly in *eq: comm relations $so(1, 4)$. We find*

$$\begin{aligned} \omega^{ab} &= A^{ab} - \frac{\cosh z - 1}{l^2 z^2} [\xi^a (d\xi^b + A_c^b \xi^c) - \xi^b (d\xi^a + A_c^a \xi^c)] - \frac{\sinh z}{l^2 z} (\xi^a A^b - \xi^b A^a), \\ e^a &= A^a + \frac{\sinh z}{z} (d\xi^a + A_b^a \xi^b) - \frac{dl}{l} \xi^a + (\cosh z - 1) \left(A^a - \frac{\xi^b A_b \xi^a}{\xi^2} \right) - \left(\frac{\sinh z}{z} - 1 \right) \frac{\xi^b d\xi_b \xi^a}{\xi^2}. \end{aligned}$$

These expressions are almost identical to the corresponding objects found by Stelle and West [14]. The difference to note is that we have a new term in the expression $\text{eq:nonlinearvierbeinforthevierbein, namely, } l^{-1} dl \xi^a$. This term is present because it is possible that the internal de Sitter spaces are characterized by cosmological constants that are not necessarily equal along spacetime. More precisely, one has to take into account the possibility that the in \mathfrak{p} defined length scale is a nonconstant function, ~~see~~ see Sec. II. On the other hand, the results of [14] specialize for the case that the local de Sitter spaces have the same ~~pseudo-radius~~ pseudoradius at any point in spacetime. When l is a constant function, one naturally recovers the results of [14].

Upon the action of local de Sitter transformations, the linear curvature F rotates in the adjoint representation. Therefore, one deduces that the nonlinear Cartan curvature \bar{F} is equal to the exterior covariant derivative of the nonlinear connection, i.e.,

$$\bar{F} = \text{Ad}(\exp(-i\xi \cdot P))(F) = d\bar{A} + 12[\bar{A}, \bar{A}], \quad (7)$$

which complies with the structure of a Cartan geometry. The nonlinear Cartan curvature is a $\mathfrak{so}(1,4)$ -valued 2-form on spacetime, which we decompose once again according to $\bar{F} = \bar{F}_{\mathfrak{h}} + \bar{F}_{\mathfrak{p}}$. Since \bar{F} ~~transforms in general,~~ transforms in general, nonlinearly with elements of $SO(1,3)$, the reductive splitting is invariant under local de Sitter transformations. This suggests that $\bar{F}_{\mathfrak{h}}$ and $\bar{F}_{\mathfrak{p}}$ must be considered the genuine curvature and torsion of the Cartan geometry, which are denoted by R ~~, respectively T .~~ and T , respectively. The definition $\text{eq:F}_n \text{online implies that } i2R^{ab}M_{ab} + iT^a P_a$

$= \text{Ad}(\exp(-i\xi \cdot P))(i2F^{ab}M_{ab} + iF^a P_a)$, from which one is able to express the curvature and torsion in terms of ξ , $F_{\mathfrak{h}}$ and

$$F_{\mathfrak{p}}: \quad R^{ab} = F^{ab} - \frac{\cosh z - 1}{l^2 z^2} \xi^c (\xi^a F_c^b - \xi^b F_c^a) \\ - \frac{\sinh z}{l^2 z} (\xi^a F^b - \xi^b F^a),$$

$T^a = \frac{\sinh z}{z} \xi^b F_b^a + \cosh z F^a + (1 - \cosh z) \frac{\xi_b F^b \xi^a}{\xi^2}$. From $\text{eq:F}_n \text{online it furthermore follows that } R^{ab} = d_\omega \omega^{ab} + \frac{1}{l^2} e^a \wedge e^b$ and $T^a = d_\omega e^a - \frac{1}{l} dl \wedge e^a$. These equations, which express the curvature and torsion in terms of the spin connection and vierbein

is always zero, there are two Bianchi identities that are formally the same as those given by eqs:lin Bianchi , i.e., $d_\omega \circ d_\omega \omega^{ab} \equiv 0$ and $d_\omega \circ d_\omega e^a + e^b \wedge d_\omega \omega_b^a \equiv 0$.

When the section ξ is ~~gauge fixed~~ gauge fixed along spacetime, and for convenience at any point ~~be is~~ is chosen the origin of the tangent de Sitter spaces, i.e., $\xi^a(x) = 0$, all the expressions reduce to those of Sec. II. This is to be expected, because the broken symmetries are not considered, and the geometry is described simply by a $SO(1,4)$ Ehresmann connection for which only $SO(1,3)$ ~~transformations the~~ transformations the isotropy group of $\xi^a = 0$ ~~are~~ are taken into account. This has precisely been the way in which the de ~~Sitter-Cartan~~ Sitter-Cartan

geometry of Sec. II was set up.

Finally, let us remark that if A^{ab} and A^a can be made to vanish everywhere, so that also F^{ab} and F^a are equal to zero, it follows that

$$R^{ab} = 0 \quad \text{and} \quad T^a = 0.$$

This shows that the nonhomogeneity of \mathcal{M} is encoded in A and F , and naturally independent of the section ξ .

IV. CONCLUSIONS AND OUTLOOK

In this work we have generalized the geometric framework of de ~~Sitter-Cartan~~ Sitter-Cartan spacetimes with a cosmological constant to the case of a nonconstant cosmological function Λ . A de ~~Sitter-Cartan~~ Sitter-Cartan spacetime consists of a principal Lorentz bundle over spacetime, on which is defined a $\mathfrak{so}(1,4)$ -valued Cartan connection. It accounts for a spin connection and vierbein, as well as for their curvature and torsion, whereas spacetime is locally approximated by de Sitter spaces. The cosmological constants of these tangent de Sitter spaces are determined by a length scale, defined in the translational part of $\mathfrak{so}(1,4)$. By letting this length scale depend arbitrarily on the spacetime point in Sec. II, we obtained a de ~~Sitter-Cartan~~ Sitter-Cartan geometry that accommodates a cosmological function by construction. Most importantly, it was shown that a nonconstant Λ gives rise to an extra contribution in the expression for the torsion, in which the cosmological function appears through its logarithmic derivative. In the limit $\Lambda \rightarrow 0$ one recovers the well-known Riemann-Cartan spacetime with arbitrary curvature and torsion. In Sec. III, the de ~~Sitter-Cartan~~ Sitter-Cartan connection has been realized nonlinearly in order to obtain $SO(1,4)$ -covariant definitions for the spin connection and vierbein, and likewise for the curvature and torsion. This generalized previous results to include a nonconstant Λ .

The cosmological function could be used to model dark energy that changes along space and time, in which way it might give an alternative description for one of the models for time-evolving dark energy [26, 27]. To determine the value of the cosmological function along spacetime, an adequate action for the gravitational field coupled to matter will have to be defined. By including invariants of the torsion tensor on the gravitational side, the first derivative of the cosmological function will automatically be present. Matter fields can be coupled both minimally to the gravitational field ~~or~~ and in another nonminimal way. The analysis of such models that make use of the framework outlined in this paper is an interesting and important subject of future research. Since the cosmological function quantifies the lack of commutation of two infinitesimal spacetime translations, the local kinematics on spacetime depend on Λ .

Consequently, there would be a link between the dynamical character of the cosmological function and its kinematical implications.

Another point of interest comes about upon noting that, when the de Sitter algebra is contracted to the Poincaré algebra, namely, when $l \rightarrow \infty$ in the commutation relations [eq:comm,relations_o\(1,4\),thegeometricobjectsofSec. III](#) *reduces to those of teleparallel gravity* [28, 29]. *This observation suggests that the geometry of spacetime that underlies teleparallel gravity is described by a Riemann–Cartan geometry (with vanishing curvature), for which the Poincaré translations are realized nonlinearly as elements of $SO(1,3)$ vanishes, for*

$$\delta h^{ab} = \lim_{l \rightarrow \infty} \frac{1}{l^2} \frac{\cosh z - 1}{z \sinh z} (\epsilon^a \xi^b - \epsilon^b \xi^a) = 0.$$

One then concludes that any Poincaré translation is trivially realized by the identity transformation, a property that is relied upon in the interpretation of teleparallel gravity as a gauge theory for the Poincaré translations. Given the knowledge that the geometric structure of teleparallel gravity is such a Riemann–Cartan spacetime, the ~~de Sitter–Cartan~~ [Sitter–Cartan](#) geometry of Sec. III might be the right framework to generalize teleparallel gravity to a theory that is invariant under local $SO(1,4)$ ~~transformations~~ [transformations](#), in place of the elements of the Poincaré group.

Acknowledgments

The author would like to thank J.-G. Pereira for [the](#) helpful discussions and suggestions. He also gratefully acknowledges financial support by CAPES.

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