

Attractors and Black Rings

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Abstract

The attractor mechanism is usually thought of as the fixing of the near horizon moduli of a BPS black hole in terms of conserved charges measured at infinity. Recent progress in understanding BPS solutions in five dimensions indicates that this is an incomplete story. Moduli can instead be fixed in terms of dipole charges, and their corresponding values can be found by extremizing a certain attractor function built out of these charges. BPS black rings provide an example of this phenomenon. We give a general derivation of the attractor mechanism in five dimensions based on the recently developed classification of BPS solutions. This analysis shows when it is the dipole charges versus the conserved charges that fix the moduli. It also yields explicit expressions for the fixed moduli.

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1. Introduction

Our ability to make microscopic sense of the entropy of BPS black holes rests on, among other things, the attractor mechanism: the property that the scalar moduli fields at the horizon are fixed in terms of charges carried by the black hole.³ The point is that the moduli are continuous parameters which can be freely specified at infinity, raising the dangerous possibility that the entropy might depend on their values. Such a dependence presumably would lead to a violation of the second law of thermodynamics, since it would allow one to quasi-statically decrease the entropy by varying the moduli. What saves the day is that the entropy depends only on the values of the moduli at the horizon, and these turn out to be insensitive to the values at infinity. The black hole entropy thus ends being a function purely of the charges.

The existing literature on the attractor mechanism deals with spherically symmetric black holes in four dimensions, and spherically symmetric or rotating BMPV black holes in five dimensions [1,2,3].⁴ For these examples the attractor mechanism works in a simple way. The black holes carry conserved electric and magnetic charges, which can be measured at infinity in terms of flux integrals. The BPS mass formula is a particular combination of these charges and the values of the asymptotic moduli. If one computes the values of the moduli which minimize the BPS mass as a function of the charges, then it turns out that these are the same values as obtained by the moduli at the horizon. The derivation of this result uses the fact that the amount of supersymmetry preserved by the black hole solution is enhanced near the horizon. Vanishing of the corresponding Killing spinor equations leads to constraints on the moduli.

We now appreciate that the above class of BPS black hole solutions is far from the complete story. Recent work has provided a much better understanding of the structure of the BPS equations governing general solutions [5,6], and has led to interesting new examples, such as black rings in five dimensions [7,8,9,10,11,12,13,14,15]. Does the attractor mechanism function in this general context and, if so, how? This is the question that we address here.

In fact, a quick perusal of the black ring solution makes it evident that the attractor mechanism is functioning in a different way. First of all, the black ring entropy is not purely a function of the conserved charges, but also depends on the values of *dipole charges*, which are non-conserved quantities measured by flux integrals on surfaces linked with the ring. From this point of view it would not be a surprise if it turns out that the moduli at the

³ More precisely, it is the vectormultiplet moduli which are fixed. The hypermultiplet moduli are not fixed, but do not affect the black hole entropy.

⁴ Multi-centered black holes in four dimensions have also been considered, and lead to some of the same issues discussed in this paper [4].

horizon also depend on the dipole charges. Indeed, the moduli turn out to be determined *entirely* by the dipole charges. Obviously, the corresponding values cannot be ascertained by extremizing the BPS mass as above, since the BPS mass depends only on the conserved charges and not the dipole charges.

With this in mind, we will carefully rederive the details of the attractor mechanism in a general context. We will work in the five dimensional $N = 2$ supergravity corresponding to compactification of M-theory on a Calabi-Yau threefold. The five dimensional setting is advantageous since it leads to simpler formulas, and it is also the habitat of the black ring. It would of course be interesting to extend our results to the four dimensional context, noting that not every four dimensional case can be obtained via dimensional reduction from five dimensions. The four dimensional case has also been the subject of several interesting recent developments [16,17,18,19].

We will strive to be as general as possible, going as far as we can based just on the general BPS equations obeyed by any BPS solution. It will become apparent that there are two distinct cases to consider, depending on whether or not certain components of the field strengths (the *dipole field strengths*) are zero or nonzero. When they vanish, which is the case for the BMPV black hole, we will reproduce earlier results showing that the moduli are fixed in terms of the conserved charges. This can be phrased in terms of extremizing the central charge Z_e , defined as

$$Z_e = X^I Q_I , \quad (1.1)$$

where X^I are the vectormultiplet moduli. We also demonstrate that the flow of Z_e from infinity to the horizon is monotonically decreasing, in parallel to what is known for spherically symmetric solutions in four dimensions. At infinity Z_e gives the BPS mass, while at the horizon it fixes the central charge of the CFT dual to the near-horizon AdS geometry.

For nonvanishing dipole field strengths the story changes in an essential way; for instance Z_e no longer behaves monotonically, and indeed typically diverges at the horizon. Instead, a new attractor function Z_m takes over, defined as

$$Z_m = X_I q^I , \quad (1.2)$$

where q^I are the dipole charges. For sufficiently many nonzero q^I , extremization of Z_m yields the near horizon values of the moduli. Further, the value of Z_m at the horizon determines the central charge of the associated CFT.

In the case of the black ring solution, it turns out that Z_m is proportional to a certain combination of the angular momenta, $Z_m \propto J_\psi - J_\phi$, and so the near-horizon moduli can equivalently be determined by extremizing this quantity. This is analogous to extremizing the BPS mass in the case of the BMPV class of solutions. Extremization of Z_m also makes sense from another point of view. If the ring direction of the black ring was instead an

infinite line, then we would have a magnetic string solution whose BPS mass is proportional to Z_m .

The remainder of this paper is organized as follows. In section 2 we review the relevant aspects of real special geometry, which is the appropriate language for five dimensional $N = 2$ supergravity. The constraints of supersymmetry are reviewed in section 3. Section 4, which is the core of the paper, gives the general derivation of the attractor mechanism. We close with a brief discussion in section 5.

2. Review of Real Special Geometry

The general setting for our study is the five dimensional low energy supergravity theory corresponding to M-theory compactified on a Calabi-Yau threefold CY_3 . This subject goes by the name *real special geometry* or *very special geometry*. It is in several ways simpler than the special geometry employed in compactifications to four dimensions, albeit perhaps less familiar. Relevant references include [20,21,22,23,24,3]. Our notation will mainly follow [3]. In particular, we use a mostly plus signature metric, and the Clifford algebra reads $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$. The unit of length is the $D = 11$ Planck length l_p , which we set equal to 1: $l_p = (\pi/4G_5)^{1/3} = 1$.

We take our CY_3 to have Hodge numbers $h^{1,1}$ and $h^{2,1}$. Let J_I be a basis of $(1,1)$ forms, with $I = 1, 2, \dots, h^{1,1}$, and expand the Kähler form J on the CY_3 as

$$J = X^I J_I . \quad (2.1)$$

This defines the Kähler moduli X^I which are *real*⁵. In terms of homology the Kähler moduli correspond to the volumes of the 2-cycles Ω^I

$$X^I = \int_{\Omega^I} J . \quad (2.2)$$

The triple intersection numbers are defined as

$$C_{IJK} = \int_{CY} J_I \wedge J_J \wedge J_K . \quad (2.3)$$

The terminology arises because this quantity can equally well be defined in terms of homology and then the integral just counts the intersection points of three 4-cycles Ω_I , Ω_J , and Ω_K . The volumes of the 4-cycles Ω_I are given by⁶

$$X_I = \frac{1}{2} \int_{\Omega_I} J \wedge J = \frac{1}{2} \int_{CY} J \wedge J \wedge J_I = \frac{1}{2} C_{IJK} X^J X^K . \quad (2.4)$$

⁵ The notation $t^I \equiv 6^{-1/3} X^I$ is common in the literature, including [3].

⁶ Our convention for X_I differ from some papers (*e.g.* [3]) by $X_I^{\text{here}} = 3X_I^{\text{there}}$.

The Kähler moduli are the lower components of $N = 2$ vectormultiplets. The couplings of these are entirely determined by the prepotential

$$\mathcal{V} = \frac{1}{3!} \int_{CY} J \wedge J \wedge J = \frac{1}{6} C_{IJK} X^I X^J X^K . \quad (2.5)$$

This expression is interpreted geometrically as the overall volume of the CY_3 and is a component of a hypermultiplet. In this paper we will be ignoring the hypermultiplets in the sense that we consistently set them to fixed constant values. Therefore, we impose the condition

$$\mathcal{V} = \frac{1}{6} C_{IJK} X^I X^J X^K = 1 . \quad (2.6)$$

This is to be understood as a constraint on the Kähler moduli, to be imposed after varying the prepotential to derive the equations of motion. There are thus $n_v = h^{1,1} - 1$ independent vectormultiplets. We denote the n_v independent vectormultiplet scalars as ϕ^i , and the corresponding derivatives $\partial_i = \frac{\partial}{\partial \phi^i}$.

Reduction of the 3-form potential in M-theory gives rise to $h^{1,1}$ 1-form potentials A^I in $D = 5$:

$$\mathcal{A} = A^I \wedge J_I . \quad (2.7)$$

The linear combination $X_I A^I$ is identified as the graviphoton in the gravity multiplet. The remaining n_v gauge fields are the upper components of the $N = 2$ vectormultiplets. The kinetic terms for the gauge fields are governed by the metric

$$G_{IJ} = \frac{1}{2} \int_{CY} J_I \wedge \star J_J = -\frac{1}{2} (\partial_I \partial_J \ln \mathcal{V})_{\mathcal{V}=1} = -\frac{1}{2} (C_{IJK} X^K - X_I X_J) , \quad (2.8)$$

where we use the notation for derivatives: $\partial_I = \frac{\partial}{\partial X^I}$. The bosonic part of the $D = 5$ action is

$$\mathcal{L}^{(\text{bos})} = \frac{1}{16\pi G_5} \left\{ -R \star 1 - G_{IJ} dX^I \wedge \star dX^J - G_{IJ} F^I \wedge \star F^J - \frac{1}{6} C_{IJK} F^I \wedge F^J \wedge A^K \right\} . \quad (2.9)$$

Let us note some useful relations. First, (2.4) and (2.6) give

$$X_I X^I = 3 , \quad (2.10)$$

and so

$$X^I \partial_i X_I = \partial_i X^I X_I = 0 . \quad (2.11)$$

Next, combining these relations with (2.8) we find

$$X_I = 2G_{IJ} X^J , \quad \partial_i X_I = -2G_{IJ} \partial_i X^J . \quad (2.12)$$

Using the metric $2G_{IJ}$ to lower the indices I, J, \dots we can introduce the 2-cycle intersection numbers C^{IJK} normalized such that the volume condition on the 3-fold becomes $\frac{1}{6}C^{IJK}X_I X_J X_K = 1$. (Since 2-cycles do not in general intersect, the geometric interpretation of these numbers refers to the intersection of the dual 4-cycles.)

We will often consider the simplest case of T^6 , in which case we write the metric and 3-form as

$$\begin{aligned} ds^2 &= ds_5^2 + X^1 dz_1 d\bar{z}_1 + X^2 dz_2 d\bar{z}_2 + X^3 dz_3 d\bar{z}_3 \\ \mathcal{A} &= A^1 \wedge \left(\frac{i}{2} dz_1 \wedge d\bar{z}_1\right) + A^2 \wedge \left(\frac{i}{2} dz_2 \wedge d\bar{z}_2\right) + A^3 \wedge \left(\frac{i}{2} dz_3 \wedge d\bar{z}_3\right). \end{aligned} \quad (2.13)$$

In this case $C_{IJK} = 1$ if (IJK) is a permutation of (123) , and $C_{IJK} = 0$ otherwise. The metric G_{IJ} is

$$G_{IJ} = \frac{1}{2} \text{diag} \left((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2} \right). \quad (2.14)$$

We also have the relations

$$X^1 X^2 X^3 = 1, \quad X_I = \frac{1}{X^I}. \quad (2.15)$$

3. BPS Equations

We will be interested in solutions preserving some supersymmetry. For purely bosonic backgrounds we need to set to zero the supersymmetry variations of the gravitinos and the gauginos

$$\begin{aligned} \delta\psi_\mu &= \left[D_\mu(\omega) + \frac{i}{24} X_I (\Gamma_\mu^{\nu\rho} - 4\delta_\mu^\nu \Gamma^\rho) F_{\nu\rho}^I \right] \epsilon, \\ \delta\lambda_i &= -\frac{1}{4} G_{IJ} \partial_i X^I F_{\mu\nu}^J \Gamma^{\mu\nu} \epsilon - \frac{i}{2} G_{IJ} \partial_i X^I \Gamma^\mu \partial_\mu X^J \epsilon. \end{aligned} \quad (3.1)$$

Here $\Gamma^{\mu\nu} = \frac{1}{2}(\Gamma^\mu \Gamma^\nu - \Gamma^\nu \Gamma^\mu)$.

Preservation of some supersymmetry implies conditions on the bosonic fields, and these have been massaged into a compact and useable form in [5][6]. Supersymmetry implies the existence of a Killing vector, and assuming that it is time-like we first write the $D = 5$ metric in the form

$$ds_5^2 = -f^2(dt + \omega)^2 + f^{-1} h_{mn} dx^m dx^n, \quad (3.2)$$

where h_{mn} is a hyper-Kähler metric on the *base* \mathcal{B} , and ω is a 1-form on \mathcal{B} . We then define the *dipole field strengths* Θ^I by writing the field strengths $F^I = dA^I$ in the form

$$F^I = d \left[(f X^I (dt + \omega)) \right] + \Theta^I, \quad (3.3)$$

where the Θ^I are closed 2-forms on the base. The BPS equations then take the form⁷

$$\begin{aligned}\Theta^I &= \star_4 \Theta^I \\ \nabla^2(f^{-1}X_I) &= \frac{1}{4}C_{IJK}\Theta^J \cdot \Theta^K \\ d\omega + \star_4 d\omega &= -f^{-1}X_I\Theta^I ,\end{aligned}\tag{3.4}$$

where ∇^2 and \star_4 are defined with respect to the base \mathcal{B} , and for 2-forms α and β on \mathcal{B} we define $\alpha \cdot \beta = \alpha^{mn}\beta_{mn}$, with indices raised by h^{mn} . The BPS equations written in the above order are linear, as noted in [8].

4. Attractor Mechanism

4.1. The Flow Equation

The attractor equations ultimately follow from the gaugino variation in (3.1). In this subsection we derive an important intermediate result, a flow equation relating the flow of the moduli to changes in the gauge field.

We look for solutions of $\delta\lambda_i = 0$ with the spinor ϵ obeying

$$\Gamma^{\hat{t}}\epsilon = -i\epsilon .\tag{4.1}$$

Hatted indices are with respect to an orthonormal frame. In order that we preserve half the supersymmetries, we demand that ϵ be subject to no other projection equations. From (3.3) the components of the field strength are

$$\begin{aligned}F_{m\hat{t}}^I &= f^{-1}\partial_m(fX^I) , \\ F_{mn}^I &= fX^I(d\omega)_{mn} + \Theta_{mn}^I .\end{aligned}\tag{4.2}$$

The purely spatial components of F^I , displayed in the second line of (4.2), do not contribute to the gaugino variation equations: the first term can be eliminated using (2.11); and the self-duality of Θ^I and the projection (4.1) shows that the second term does not contribute either. The contribution of the remaining components $F_{m\hat{t}}^I$ to the gaugino variation implies the equation

$$G_{IJ}\partial_i X^I F_{m\hat{t}}^J = G_{IJ}\partial_i X^I \partial_m X^J .\tag{4.3}$$

We now proceed to manipulate (4.3). Define the electric field as

$$E_{mI} = G_{IJ}F_{m\hat{t}}^J ,\tag{4.4}$$

⁷ This form presumably remains valid after the assumption of a symmetric scalar manifold [6] is relaxed.

and take the following divergence with respect to the metric h_{mn} :

$$\begin{aligned}\nabla^m(f^{-1}X^I E_{mI}) &= (\nabla^m X^I)f^{-1}E_{mI} + X^I \nabla^m(f^{-1}E_{mI}) \\ &= f^{-1}G_{IJ}\nabla^m X^I \nabla_m X^J + X^I \nabla^m(f^{-1}E_{mI}) ,\end{aligned}\tag{4.5}$$

where we used (4.3) to arrive at the second line. Next, we write

$$\begin{aligned}\nabla^m[f^{-1}E_{mI}] &= \nabla^m[f^{-2}G_{IJ}\partial_m(fX^J)] \\ &= \frac{1}{2}\nabla^m[f^{-2}(\partial_m f)X_I - f^{-1}\partial_m X_I] \\ &= -\frac{1}{2}\nabla^m\partial_m[f^{-1}X_I] \\ &= -\frac{1}{8}C_{IJK}\Theta^J \cdot \Theta^K ,\end{aligned}\tag{4.6}$$

using first the definitions (4.2) and (4.4), then both equations in (2.12), and finally the BPS equation (3.4). Inserting back in to (4.5) we obtain

$$\nabla^m(f^{-1}X^I E_{mI}) = f^{-1}G_{IJ}\nabla^m X^I \nabla_m X^J - \frac{1}{8}C_{IJK}X^I \Theta^J \cdot \Theta^K .\tag{4.7}$$

This is the flow equation we wanted to derive.

4.2. Near Horizon Enhancement of SUSY

BPS black hole solutions typically exhibit a type of domain wall structure, interpolating between two maximally supersymmetric vacua of the theory: Minkowski space at infinity, and AdS at the horizon. From the point of view of gauge/gravity duality, this is interpreted as the renormalization group flow to an infrared fixed point CFT. The phenomenon of enhanced supersymmetry leads to strong constraints on the values of the vectormultiplet moduli at the horizon, as we now discuss.

We return to the supersymmetry variations (3.1), and now demand that we can set $\delta\psi_\mu = \delta\lambda_i = 0$ without imposing any projection conditions on ϵ analogous to (4.1). Vanishing of the gravitino variation implies that the metric is maximally symmetric, *e.g.* $\text{AdS}_2 \times S^3$ or $\text{AdS}_3 \times S^2$; see [3] for details. We focus instead on the gaugino variation. For general ϵ there is no possibility of a cancellation among the terms in $\delta\lambda_i$, and so we need to impose

$$\begin{aligned}G_{IJ}\partial_i X^I \partial_\mu X^J &= 0 , \\ G_{IJ}\partial_i X^I F_{\mu\nu}^J &= 0 , \quad \text{or equivalently} \quad \partial_i X_I F_{\mu\nu}^I = 0 .\end{aligned}\tag{4.8}$$

Multiplying the first equation by $\partial_\mu \phi^i$ and contracting the μ indices gives

$$G_{IJ}g^{\mu\nu}\partial_\mu X^I \partial_\nu X^J = 0 .\tag{4.9}$$

For a static configuration, this is positive semi-definite, and so implies constant moduli:

$$\partial_\mu X^I = 0 . \quad (4.10)$$

Then, assuming constant moduli, the field strength components (4.2) become

$$\begin{aligned} G_{IJ} F_{m\hat{t}}^I &= \frac{1}{2} X_I f^{-1} \partial_m f , \\ F_{mn}^I &= f X^I (d\omega)_{mn} + \Theta_{mn}^I . \end{aligned} \quad (4.11)$$

Using (2.11) we find that the second line of (4.8) is satisfied provided

$$\partial_i X_I \Theta_{mn}^I = 0 , \quad (4.12)$$

which in turn requires that Θ^I has the structure

$$\Theta_{mn}^I = X^I k_{mn} . \quad (4.13)$$

So to summarize, enhancement of supersymmetry implies the two conditions (4.10) and (4.13).

4.3. Charges

The power of the attractor mechanism is that it fixes moduli in terms of the charges carried by the black hole, whether of the conserved or dipole variety. We therefore need to give formulas for the charges, which we do in this subsection.

Let V be some bounded region in the base \mathcal{B} . We define the electric charge in V as

$$Q_I(V) = \frac{1}{2\pi^2} \int_{\partial V} dS f^{-1} n^m E_{mI} , \quad (4.14)$$

where n is the outward pointing unit normal vector. The conserved electric charge measured at infinity is obtained by taking $V = \mathcal{B}$. In general the value of the charge as we have defined it depends nontrivially on V ; indeed, from (4.6) we have

$$Q_I(V_1) - Q_I(V_2) = -\frac{1}{16\pi^2} \int_{V_1 - V_2} d^4x \sqrt{h} C_{IJK} \Theta^J \cdot \Theta^K . \quad (4.15)$$

Since it is the dressed field $X^I E_{mI}$ that appears in the flow equation (4.7), it is natural to define also

$$Z_e(V) = \frac{1}{2\pi^2} \int_{\partial V} dS f^{-1} X^I n^m E_{mI} , \quad (4.16)$$

which obeys

$$Z_e(V_1) - Z_e(V_2) = \int_{V_1-V_2} d^4x \sqrt{h} \left\{ \frac{1}{2\pi^2} f^{-1} G_{IJ} \nabla^m X^I \nabla_m X^J - \frac{1}{16\pi^2} C_{IJK} X^I \Theta^J \cdot \Theta^K \right\} . \quad (4.17)$$

Z_e is the electric charge corresponding to the graviphoton. As measured at infinity, it is also the central charge appearing in the BPS mass formula:

$$M = Z_e(\mathcal{B}) . \quad (4.18)$$

We next turn to the definition of the *dipole charges* q^I , which are defined as integrals of Θ^I over certain noncontractible 2-spheres in \mathcal{B} :

$$q^I = -\frac{1}{2\pi} \int_{S^2} \Theta^I . \quad (4.19)$$

These 2-spheres can arise in either of two ways. First, the base \mathcal{B} may be a smooth four manifold supporting such noncontractible spheres. Alternatively, \mathcal{B} could be topologically trivial, such as flat \mathbb{R}^4 . In this case, ω and Θ^I , viewed as differential forms on \mathcal{B} , may have singularities even though the full five-dimensional geometry is smooth. If these singularities lie along a closed curve, as is the case for the black ring solutions, then there will be noncontractible 2-spheres which surround the curve. In either case, we define the dipole charges as in (4.19). In analogy with (4.16) we also define

$$Z_m = -\frac{1}{2\pi} \int_{S^2} X_I \Theta^I . \quad (4.20)$$

One interpretation of Z_m is that if we take the singular curve described above to be an infinite straight line, then our solution will describe a magnetic string whose BPS mass formula is governed by $Z_m(\infty)$. From the M-theory point of view, such magnetic strings can be realized as M5-branes wrapping 4-cycles of the CY_3 .

In many considerations the string-like charges appear on more or less equal footing with the more familiar electric charges. For example, the Z_e and the Z_m at infinity both appear in the supersymmetry algebra when we allow for extended string solutions. Similarly, we will see that there are attractors controlled by the dipole charges, which are quite similar to the usual attractors dominated by point-like charges.

4.4. Attractor Flows With $\Theta^I = 0$

We now consider the special case in which the dipole field strengths vanish, $\Theta^I = 0$. This special case includes the BMPV black hole [25], and more generally, multi-centered versions of these.

For this case there is a monotonic flow of the central charge Z_e . In particular, from (4.17), we see that if V_2 is contained within V_1 then

$$Z_e(V_1) - Z_e(V_2) = \frac{1}{2\pi^2} \int_{V_1-V_2} d^4x \sqrt{h} f^{-1} G_{IJ} \nabla^m X^I \nabla_m X^J \geq 0 . \quad (4.21)$$

From (4.15) we also see that Q_I is independent of V .⁸

The typical situation is for there to be isolated pointlike singularities on \mathcal{B} , whose locations can intuitively be thought of as specifying the locations of branes. Take V to enclose a single singularity at P . The central charge decreases monotonically as the singularity is approached and, if the singularity is not too severe, it will reach a finite value in the limit. Furthermore, if there is a smooth black hole horizon at P then $f(P) = 0$, and the factor $\sqrt{h}f^{-1}$ diverges. Finiteness of $Z_e(P)$ then typically forces $\nabla_m X^I|_P = 0$, and so we are in the situation where supersymmetry is enhanced, as discussed in section 4.2. This is the attractor we want to study.

The essence of the attractor mechanism is that we can work out the values taken by the X^I at P in terms of the charges Q_I . To see this, we start contracting V around the point P . For sufficiently small V we have, from (4.4), (4.11), and (4.14),

$$Q_I = X_I \left(-\frac{1}{4\pi^2} \int_{\partial V} dS n^m \partial_m f^{-1} \right) . \quad (4.22)$$

The term in the bracket is just a constant of proportionality determined by the condition of unit volume for the CY₃. This gives the fixed point values

$$X_I = \frac{Q_I}{(\frac{1}{6} C^{JKL} Q_J Q_K Q_L)^{1/3}} . \quad (4.23)$$

An equivalent way of stating this is that we can find the fixed values of the moduli by extremizing the central charge Z_e . In particular, at P

$$Z_e = X^I(P) Q_I . \quad (4.24)$$

Extremizing the central charge with respect to the fixed moduli means that we impose

$$\partial_i Z_e = 0 . \quad (4.25)$$

The Q_I are held fixed, so the equation reads $\partial_i X^I(P) Q_I = 0$. But this equation implies that Q_I is proportional to X_I , which then leads to (4.23) in the same way that (4.22) led to (4.23).

⁸ We stress that this independence is only true provided no singularities pass into or out of V as we deform it.

We see that the moduli take values at P determined by the charges at P . For a single singularity these charges are the same as the conserved charges measured at infinity, and Z_e is the central charge appearing in the the BPS mass formula. More generally, for multiple singularities the charge at infinity is a sum of contributions from each singularity, and so it is not possible to read off the various fixed moduli directly from the charge at infinity.

As an example, let's consider the BMPV black hole. We take the compactification manifold to be T^6 , as in (2.13)-(2.15). In this case the base metric is flat

$$h_{mn}dx^M dx^N = dr^2 + r^2(d\theta^2 + \sin^2 \theta d\psi^2 + \cos^2 \theta d\phi^2) . \quad (4.26)$$

The five dimensional metric is given by (3.2) with

$$\begin{aligned} f^{-3} &= \frac{1}{6} C^{IJK} H_I H_J H_K , \quad H_I = 1 + \frac{Q_I}{r^2} , \\ \omega &= -\frac{4G_5}{\pi} J(\cos^2 \theta d\phi + \sin^2 \theta d\psi) . \end{aligned} \quad (4.27)$$

The solution carries angular momentum $J_\psi = J_\phi = J$. The gauge fields and moduli are

$$\begin{aligned} E_{rI} &= f \frac{Q_I}{r^3} , \\ X_I &= \frac{H_I}{(H_1 H_2 H_3)^{1/3}} . \end{aligned} \quad (4.28)$$

The horizon is at $r = 0$. At the horizon the moduli take values

$$X_I(r = 0) = \frac{Q_I}{(Q_1 Q_2 Q_3)^{1/3}} , \quad (4.29)$$

in agreement with (4.23). We define the attractor function Z_e as in (4.16) with V taken to be a 3-sphere of radius r . Z_e then takes the form

$$Z_e(r) = X^I Q_I = (H_1 H_2 H_3)^{1/3} H_I^{-1} Q_I , \quad (4.30)$$

and obeys $\frac{dZ_e}{dr} \geq 0$ in agreement with (4.21). Z_e interpolates between the following two values:

$$Z_e(r = 0) = 3(Q_1 Q_2 Q_3)^{1/3}, \quad Z_e(r = \infty) = Q_1 + Q_2 + Q_3 . \quad (4.31)$$

The value at the origin determines the radius of curvature of a near horizon $\text{AdS}_2 \times S^3$ geometry, while the value at infinity gives the mass according to (4.18).

4.5. Attractor Flows With $\Theta^I \neq 0$

We now consider the case of nonzero dipole field strengths Θ^I , and in particular, nonzero dipole charges q^I . In this case the moduli are *not* fixed in terms of the charges Q_I , and Z_e does *not* behave monotonically. Instead, the attractor flow is governed by the dipole charges q^I and the attractor function is Z_m . We first discuss this in general terms, and then illustrate using the example of the black ring solution.

We assume that there is a noncontractible S^2 in order to define the dipole charges q^I . The S^2 is taken to surround a closed curve on the base \mathcal{B} . We further assume that there are enough nonzero dipole charges so that $C_{IJK}q^Iq^Jq^K \neq 0$. Finally, we assume enhanced supersymmetry as we approach the curve. Under these assumption the near horizon geometry becomes $AdS_3 \times S^2$ (rather than $AdS_2 \times S^3$ for the case with no dipole charges). Typically, if the above conditions are not met then the moduli will not be stabilized and there will not be a regular horizon either. We should emphasize that this doesn't necessarily imply that the geometry is singular, just that there is not a regular black hole horizon. There might instead be a smooth horizon-free geometry. But in such a case there is no expectation that the attractor mechanism will be operative.

Let us now return to the conditions for enhanced supersymmetry. We first observe that (4.13) and (4.19) imply

$$q^I = \left(\frac{1}{2\pi} \int_{S^2} k \right) X^I . \quad (4.32)$$

Whenever $\int_{S^2} k \neq 0$, there is enough information to determine the near horizon values of X^I in terms of the dipole charges q^I . In particular, the solution is

$$X^I = \frac{q^I}{(\frac{1}{6}C_{IJK}q^Iq^Jq^K)^{1/3}} . \quad (4.33)$$

As before, an equivalent way of arriving at (4.33) is to consider $Z_m = X_I q^I$, and demand $\partial_i Z_m = 0$. This leads to (4.33) since the vanishing of $\partial_i Z_m = \partial_i X_I q^I$ implies that X^I is proportional to q^I , which is the content of (4.32).

We now illustrate the above with the black ring example [9]. The compactification manifold is T^6 as in (2.13)-(2.15). The base metric is flat,

$$\begin{aligned} h_{mn}dx^m dx^n &= dr^2 + r^2(d\theta^2 + \sin^2 \theta + \cos^2 \theta d\phi^2) \\ &= \frac{R^2}{(x-y)^2} \left[\frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2 \right] , \end{aligned} \quad (4.34)$$

where in the second coordinate system x and y have range: $-1 \leq x \leq 1$, $-\infty \leq y \leq -1$. The solution has

$$\begin{aligned} X_I &= \frac{H_I}{(H_1 H_2 H_3)^{1/3}} , \\ H_I &= 1 + \frac{Q_I - \frac{1}{2}C_{IJK}q^Jq^K}{2R^2}(x-y) - \frac{C_{IJK}q^Jq^K}{8R^2}(x^2 - y^2) . \end{aligned} \quad (4.35)$$

The field strengths take the form (3.3) with the dipole field strength given by

$$\Theta^I = -\frac{1}{2}q^I(dy \wedge d\psi + dx \wedge d\phi) . \quad (4.36)$$

We also have

$$f^{-3} = \frac{1}{6}C^{IJK}H_IH_JH_K . \quad (4.37)$$

Finally, the 1-form ω is

$$\begin{aligned} \omega_\psi &= -\frac{1}{8R^2}(1-x^2) \left[Q_I q^I - \frac{1}{6}C_{IJK}q^I q^J q^K (3+x+y) \right] , \\ \omega_\phi &= \frac{1}{2}(q^1 + q^2 + q^3)(1+y) + \omega_\psi . \end{aligned} \quad (4.38)$$

The horizon of the black ring is at $y = -\infty$ where the H_I diverge. ψ is the angular coordinate parameterizing the location along the ring. The noncontractible 2-spheres surrounding the ring are parameterized by x and ϕ . Given (4.36) it is indeed easy to see that integration on the 2-spheres gives

$$q^I = -\frac{1}{2\pi} \int_{S^2} \Theta^I , \quad (4.39)$$

as in (4.19). Also, in the limit $y \rightarrow -\infty$ we see from (4.35) that the moduli take on values in agreement with (4.33).

We now discuss the behavior of the attractor functions Z_e and Z_m in the full black ring geometry. Neither function is monotonic; rather, as the ring is approached from infinity there is a sort of crossover, with Z_e decreasing for large radius, and Z_m decreasing for small radius. We can think of this in terms of an RG flow, where we interpolate between a UV CFT controlled by the electric charges Q_I , and an IR CFT controlled by the dipole charges q^I .

Z_e is given by (4.16), where it is natural to take V to be an S^3 of radius r . A little manipulation yields

$$Z_e(r) = -\frac{3}{4\pi^2} \int dS \partial_r f^{-1} . \quad (4.40)$$

The explicit formula for f is

$$f = (H_1 H_2 H_3)^{-\frac{1}{3}} , \quad (4.41)$$

with

$$\begin{aligned} H_I &= 1 + \frac{Q_I - \frac{1}{2}C_{IJK}q^J q^K}{\Sigma} + \frac{1}{2}C_{IJK}q^J q^K \frac{r^2}{\Sigma^2} , \\ \Sigma &= \sqrt{(r^2 - R^2)^2 + 4R^2 r^2 \cos^2 \theta} . \end{aligned} \quad (4.42)$$

For large r the Q_I dominate and f behaves as in the BMPV case (4.28). But near the ring Z_e behaves as

$$Z_e(r) \sim 6 \frac{(q^1 q^2 q^3)^{2/3} R^6}{(r^2 - R^2)^3} \quad \text{as } r \rightarrow R. \quad (4.43)$$

So in the full geometry Z_e is neither monotonic nor bounded.

To define Z_m we instead work in the $x - y$ coordinates and integrate over 2-spheres of fixed y and ψ . From (4.20) and (4.36) this gives

$$Z_m(y) = \frac{1}{2} \int_{-1}^1 dx q^I X_I. \quad (4.44)$$

At the horizon $y \rightarrow -\infty$, and Z_m stabilizes at $Z_m = q^I X_I$ with X_I given by (4.33). In more detail, the near horizon behavior is

$$Z_m = 3(q^1 q^2 q^3)^{1/3} + \frac{2}{3} \frac{(q^1 Q_1 - q^2 Q_2)^2 + (q^2 Q_2 - q^3 Q_3)^2 + (q^3 Q_3 - q^1 Q_1)^2}{(q^1 q^2 q^3)^{5/3}} y^{-2} + O(y^{-3}). \quad (4.45)$$

So sufficiently near the ring Z_m is decreasing as it approaches its fixed values. However, analysis of the integral (4.44) reveals that farther from the ring Z_m is not monotonic.

While neither Z_e nor Z_m is monotonic throughout the full geometry, one might ask whether some other combination of charges and moduli does better in this regard. Our analysis does not reveal any obvious candidate, and in particular there is no natural family of surfaces on which to define such an expression.

4.6. Extremization Principles

As we have seen, the near horizon moduli are fixed by one of the two attractor functions: $Z_e = X^I Q_I$ or $Z_m = X_I q^I$. For $\Theta^I = 0$, the near moduli are fixed by extremizing Z_e ; *i.e.* solving $\partial_i Z_e = 0$. On the other hand, for sufficiently many nonzero dipole charges the moduli are instead found by extremizing Z_m . To be clear, we note that when we talk of extremizing Z_e and Z_m , we mean extremization with respect to the moduli X^I while holding the charges Q_I and dipole charges q^I fixed (G_5 is fixed throughout, *via* our choice of units).

It is also natural to rephrase the extremization procedure in terms of physical quantities measured at infinity. First consider the case of $\Theta^I = 0$. The general BPS mass formula is written in (4.18) (this is valid for the general case with nonzero Θ^I .) If we allow for arbitrary X^I at infinity, then we see that the BPS mass depends nontrivially on these X^I . We also see that the value of the X^I which extremizes the mass (while holding the charges fixed) are the same values as appear in the near horizon region of enhanced susy. Therefore, the near horizon moduli can be determined purely from considerations of the BPS mass measured at infinity. Recalling that Z_e is monotonic, we also learn that

if we choose moduli at infinity so as to minimize Z_e , then it must be the case that Z_e is constant throughout the flow to the horizon. This also implies that X^I are similarly constant throughout the flow in this case.

Now turn to the case of nonzero dipole charges such that the moduli are fixed by Z_m . Is it again possible to rephrase this in terms of extremizing some quantities of direct physical relevance? We will present two complimentary extremization principles based on the explicit black ring solutions.

First, we recall that in five dimensions there are two independent angular momenta, which we are calling J_ψ and J_ϕ . For the black ring the difference of these is

$$J_\psi - J_\phi = R^2 \overline{X}_I q^I = R^2 Z_m , \quad (4.46)$$

where R^2 is the ring radius, and the \overline{X}_I refer to the moduli at infinity. This combination of angular momenta can be interpreted as the intrinsic angular momentum of the ring (not including the surrounding field). Alternatively, it can be identified with the effective level of the dual CFT ⁹. Here we find that that extremizing Z_m is the same as extremizing $J_\psi - J_\phi$ while holding q^I and R fixed. It is not clear whether a version of this principle is true in general. In particular, for a well defined formulation one needs to give meaning to the ring radius R for a general solution, not necessarily of the black ring form. We leave this as an open question.

Another interesting extremization principle emerges from the black ring entropy formula [8,9]

$$S = 2\pi\sqrt{J_4} , \quad J_4 = J_4(Q_I, q^I, J_\psi - J_\phi) . \quad (4.47)$$

Here J_4 is the quartic $E_{7(7)}$ invariant; see [14,15] for the explicit formula. Therefore, we can obtain the fixed moduli by extremizing the entropy while holding fixed Q_I , q^I , and R . This suggests a thermodynamic interpretation of the attractor mechanism. Again, a general version of this requires formulating a general notion of R .

5. Discussion

We have obtained results on attractor flows in the context of general BPS solutions in five dimensions, generalizing earlier work on this subject. In particular, we saw that nonzero dipole charges, which are natural quantities to define in the context of real special geometry, lead to a new type of attractor flow governed by the attractor function Z_m . For

⁹ Heuristically, this is the momentum flowing along the string; more precisely, the effective level h_{eff} is the eigenvalue of $L_0 - \overline{L}_0$ when the supersymmetric sector of the (4,0) dual CFT is in its NS-sector ground state.

this class of flows, which includes the recently discovered BPS black rings, the near horizon moduli take values governed by the dipole charges, rather than by the conserved electric charges measured at infinity.

There are several issues worthy of better understanding. First, while Z_e is governed by the flow equation (4.17) we did not find any analogous equation for Z_m . Technically, this was because the \hat{t} index appearing in the projection equation (4.1) singled out the electric field as playing a special role. On the other hand, for the black rings we did observe that Z_m was decreasing as we got near the ring, and it would be nice to have a derivation of this from general principles.

A better understanding of the interplay between Z_e and Z_m might also shed light on general aspects of RG flows. For example, the standard near-horizon decoupling limit of the black ring solutions yields a solution describing an RG flow from a UV CFT with central charge set by Z_e to an IR CFT with central charge set by Z_m . From the explicit solution it always turns out that $c_{IR} \leq c_{UV}$. It would be interesting to establish that this is always the case.

Finally, it remains to be determined what, precisely, is the physical principle responsible for the attractor mechanism. There is some evidence that the attractor equations follows from the extremization of thermodynamic potentials such as the entropy, but the general formulation of such a physical principle is an open problem.

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