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Extremal Black Holes in Supergravity

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Abstract

We present the main features of the physics of extremal black holes embedded in supersymmetric theories of gravitation, with a detailed analysis of the attractor mechanism for BPS and non-BPS black-hole solutions in four dimensions.

1 Introduction: Extremal Black Holes from Classical General Relativity to String Theory

The physics of black holes [1], with its theoretical and phenomenological implications, has a fertile impact on many branches of natural science, such as astrophysics, cosmology, particle physics and, more recently, mathematical physics [2] and quantum information theory [3]. This is not so astonishing in view of the fact that, owing to the singularity theorems of Penrose and Hawking [4], the existence of black holes seems to be an unavoidable consequence of Einstein's theory of general relativity and of its modern generalizations such as supergravity [5], superstrings and M-theory [6].

A fascinating aspect of black-hole physics is in their thermodynamic properties that seem to encode fundamental insights of a so far not established final theory of quantum gravity. In this context a central role is played by the Bekenstein–Hawking (in the following, B-H) entropy formula [7]:

$$S_{\text{B-H}} = \frac{k_B}{\ell_P^2}\,\frac{1}{4}\,\text{Area}_H\,, \tag{1}$$

where k_B is the Boltzman constant, $\ell_P^2 = G\hbar/c^3$ is the squared Planck length while Area_H denotes the area of the horizon surface (from now on we shall use the natural units $\hbar = c = G = k_B = 1$).

This relation between a thermodynamic quantity ($S_{\text{B-H}}$) and a geometric quantity (Area_H) is a puzzling aspect that motivated much theoretical work in the last decades. In fact a microscopic statistical explanation of the area/entropy formula, related to microstate counting, has been regarded as possible only within a consistent and satisfactory formulation of quantum gravity. Superstring theory is the most serious candidate for a theory of quantum gravity and, as such, should eventually provide such a microscopic explanation of the area law [8]. Since black holes are a typical non-perturbative phenomenon, perturbative string theory could say very little about their entropy: only non-perturbative string theory could have a handle on it. Progress in this direction came after 1995 [9], through the recognition of the role of string dualities. These dualities allow one to relate the strong coupling regime of one superstring model to the weak coupling regime of another. Interestingly enough, there is evidence that the (perturbative and non-perturbative) string dualities are all encoded in the global symmetry group (the U-duality group) of the low energy supergravity effective action [10].

Let us introduce a particular class of black-hole solutions, which will be particularly relevant to our discussion: the *extremal black holes*. The simplest instance of these solutions may be found within the class of the so-called Reissner–Nordström (R-N) space-time [11], whose metric describes a static, isotropic black hole of mass M and electric (or magnetic) charge Q:

$$ds^2 = dt^2 \left(1 - \frac{2M}{\rho} + \frac{Q^2}{\rho^2}\right) - d\rho^2 \left(1 - \frac{2M}{\rho} + \frac{Q^2}{\rho^2}\right)^{-1} - \rho^2 \, d\Omega^2 \,, \tag{2}$$

where $d\Omega^2 = (d\theta^2 + \sin^2\theta \, d\phi^2)$ is the metric on a 2-sphere. The metric (2) admits two Killing horizons, where the norm of the Killing vector $\frac{\partial}{\partial t}$ changes sign. The horizons are located at the two roots of the quadratic polynomial $\Delta \equiv -2M\rho + Q^2 + \rho^2$:

$$\rho_\pm = M \pm \sqrt{M^2 - Q^2} \,. \tag{3}$$

If $M < |Q|$ the two horizons disappear and we have a naked singularity. In classical general relativity people have postulated the so-called *cosmic censorship* conjecture [12, 5]: space-time singularities should always be hidden inside a horizon. This conjecture implies, in the R-N case, the bound:

$$M \geq |Q| \,. \tag{4}$$

Of particular interest are the states that saturate the bound (4). If

$$M = |Q| \,, \tag{5}$$

the two horizons coincide and, setting: $\rho = r + M$ (where $r^2 = \vec{x} \cdot \vec{x}$), the metric (2) can be rewritten as:

$$\begin{aligned} ds^2 &= dt^2 \left(1 + \frac{Q}{r}\right)^{-2} - \left(1 + \frac{Q}{r}\right)^2 \left(dr^2 + r^2 \, d\Omega^2\right) \\ &= H^{-2}(r) \, dt^2 - H^2(r) \, d\vec{x} \cdot d\vec{x} \end{aligned} \tag{6}$$

in terms of the harmonic function

$$H(r) = \left(1 + \frac{Q}{r}\right) \,. \tag{7}$$

As eq. (6) shows, the extremal R-N configuration may be regarded as a soliton of classical general relativity, interpolating between two vacua of the theory: the flat Minkowski space-time, asymptotically reached at spatial infinity $r \to \infty$, and the Bertotti–Robinson (B-R) metric [13], describing the conformally flat geometry $AdS_2 \times S^2$ near the horizon $r \to 0$ [5]:

$$ds^2_{\text{B-R}} = \frac{r^2}{M^2_{\text{B-R}}} dt^2 - \frac{M^2_{\text{B-R}}}{r^2} \left(dr^2 + r^2 d\Omega \right) . \tag{8}$$

Last, let us note that the condition $M = |Q|$ can be regarded as a no-force condition between the gravitational attraction $F_g = \frac{M}{r^2}$ and the electric repulsion $F_q = -\frac{Q}{r^2}$ on a unit mass carrying a unit charge.

Until now we have reviewed the concept of extremal black holes as it arises in classical general relativity. However, extremal black hole configurations are embedded in a natural way in supergravity theories. Indeed supergravity, being invariant under local super-Poincaré transformations, includes general relativity, i.e. it describes gravitation coupled to other fields in a supersymmetric framework. Therefore it admits black holes among its classical solutions. Moreover, as black holes describe a physical regime where the gravitational field is very strong, a complete understanding of their physics seems to require a theory of quantum gravity, like superstring theory is. In this respect, as anticipated above, extremal black holes have become objects of the utmost relevance in the context of superstrings after 1995 [8, 6, 5, 14]. This interest, which is just part of a more general interest in the p-brane classical solutions of supergravity theories in all dimensions $4 \leq D \leq 11$ [15, 16], stems from the interpretation of the classical solutions of supergravity that preserve a fraction of the original supersymmetries as non-perturbative states, necessary to complete the perturbative string spectrum and make it invariant under the many conjectured duality symmetries [17, 18, 10, 19, 20]. Extremal black holes and their parent p-branes in higher dimensions are then viewed as additional *particle-like* states that compose the spectrum of a fundamental quantum theory. As the monopoles in gauge theories, these non-perturbative quantum states originate from regular solutions of the classical field equations, the same Einstein equations one deals with in classical general relativity and astrophysics. The essential new ingredient, in this respect, is supersymmetry, which requires the presence of *vector fields* and *scalar fields* in appropriate proportions. Hence the black holes we are going to discuss are solutions of generalized Einstein–Maxwell-dilaton equations.

Within the superstring framework, supergravity provides an effective description that holds at lowest order in the string loop expansion and in the limit in which the space-time curvature is much smaller than the typical string scale (string tension). The supergravity description of extremal black holes is therefore reliable when the radius of the horizon is much larger than the string scale, and this corresponds to the limit of large charges. Superstring corrections induce higher derivative terms in the low energy action and therefore the B-H entropy formula is expected to be corrected as well by terms which are subleading in the small curvature limit. In this paper we will not consider these higher derivative effects.

Thinking of a black-hole configuration as a particular bosonic background of an N-extended locally supersymmetric theory gives a simple and natural understanding at the cosmic censorship conjecture. Indeed, in theories with extended supersymmetry ($N \geq 2$) the bound (4) is just a consequence of the supersymmetry algebra, and this ensures that in these theories the cosmic censorship conjecture is always verified, that is there are no naked singularities. When the black hole is embedded in extended supergravity, the model depends in general also on scalar fields. In this case, as we will see, the electric charge Q has to be replaced by the maximum eigenvalue of the central charge appearing in the supersymmetry algebra (depending on the expectation value of scalar fields and on the electric and magnetic charges). The R-N metric takes in general a more complicated form.

However, extremal black holes have a peculiar feature: even when the dynamics depends on scalar fields, the event horizon loses all information about the scalars; this is true independently of the fact that the solution preserves any supersymmetries or not. Then, as will be discussed extensively in section 4, also if the extremal black hole is coupled to scalar fields, the near-horizon geometry is still described by a conformally flat, B-R-type geometry, with a mass parameter $M_{\text{B-R}}$ depending on the given configuration of electric and magnetic charges, but not on the scalars. The horizon is in fact an attractor point [21, 22, 23]: scalar fields, independently of their boundary conditions at spatial infinity, when approaching the horizon flow to a fixed point given by a certain ratio of electric and magnetic charges. This may be understood in the context of Hawking theory. Indeed quantum black holes are not stable: they radiate a thermic radiation as a black body, and correspondingly lose their energy (mass). The only stable black-hole configurations are the extremal ones, because they have the minimal possible energy compatible with relation (4) and so they cannot radiate. Indeed, physically they represent the limit case in which the black-hole temperature, measured by the surface gravity at the horizon, is sent to zero.

Remembering now that the black-hole entropy is given by the area/entropy B-H relation (1), we see that the entropy of extremal black holes is a topological quantity, in the sense that it is fixed in terms of the quantized electric and magnetic charges, while it does not depend on continuous parameters such as scalars. The horizon mass parameter $M_{\text{B-R}}$ turns out to be given in this case (extremal configurations) by the maximum eigenvalue $Z_{\max}$ of the central charge appearing in the supersymmetry algebra, evaluated at the fixed point:

$$M_{\text{B-R}} = M_{\text{B-R}}(p,q) = |Z_{\max}(\phi_{\text{fix}}, p, q)| \tag{9}$$

this gives, for the B-H entropy:

$$S_{\text{B-H}} = \frac{A_{\text{B-R}}(p,q)}{4} = \pi |Z_{\max}(\phi_{\text{fix}}, p, q)|^2 \,. \tag{10}$$

A lot of efffort was made in the course of the years to give an explanation for the topological entropy of extremal black holes in the context of a quantum theory of gravity, such as string theory. A particularly interesting problem is finding a microscopic, statistical mechanics interpretation of this thermodynamic quantity. Although we will not deal with the microscopic point of view at all in this paper, it is important to mention that such an interpretation became possible after the introduction of D-branes in the context of string theory [24], [8]. Following this approach, extremal black holes are interpreted as bound states of D-branes in a space-time compactified to four or five dimensions, and the different microstates contributing to the B-H entropy are, for instance, related to the different ways of wrapping branes in the internal directions. Let us mention that all calculations made in particular cases using this approach provided values for the B-H entropy compatible with those obtained with the supergravity, macroscopic techniques. The entropy formula turns out to be in all cases a U-duality-invariant expression (homogeneous of degree 2) built out of electric and magnetic charges and as such it can be in fact also computed through certain (moduli-independent) topological quantities [25], which only depend on the nature of the U-duality groups and the appropriate representations of electric and magnetic charges [26]. We mention for completeness that, as previously pointed out, superstring corrections that take into account higher derivative effects determine a deviation from the area law for the entropy [27, 28]. Recently, a deeper insight into the microscopic description of black-hole entropy was gained, in this case, from the fruitful proposal in [29], describing the microscopic degrees of freedom of black holes in terms of topological strings.

Originally, the attention was mainly devoted to the so-called *BPS-extremal black holes*, i.e. to solutions which saturate the bound in (5). From an abstract viewpoint BPS saturated states are characterized by the fact that they preserve a fraction, 1/2 or 1/4 or 1/8, of the original supersymmetries. What this actually means is that there is a suitable projection operator $S^2 = S$ acting on the supersymmetry charge Q_{SUSY}, such that:

$$(S \cdot Q_{\mathrm{SUSY}}) \, |\, \mathrm{BPS\ state}\, \rangle \; = \; 0 \, . \tag{11}$$

Since the supersymmetry transformation rules of any supersymmetric field theory are linear in the first derivatives of the fields, eq. (11) is actually a system of first-order differential equations, to be combined with the second-order field equations of the theory. Translating eq. (11) into an explicit first-order differential system requires knowledge of the supersymmetry transformation rules of supergravity. The latter have a rich geometric structure whose analysis will be the subject of section 3. The BPS saturation condition transfers the geometric structure of supergravity, associated with its scalar sector, into the physics of extremal black holes. We note that first-order differential equations $\frac{d\Phi}{dr} = f(\Phi)$ have in general fixed points, corresponding to the values of r for which $f(\Phi) = 0$. For the BPS black holes, the fixed point is reached precisely at the black-hole horizon, and this is how the attractor behavior is realized for this class of extremal black holes.

For BPS configurations, non-renormalization theorems based on supersymmetry guarantee the validity of the (BPS) bound $M = |Q|$ beyond the perturbative regime: if the bound is saturated in the classical theory, the same must be true also when quantum corrections are taken into account and the theory is in a regime where the supergravity approximation breaks down. That it is actually an exact state of non-perturbative string theory follows from supersymmetry representation theory. The classical BPS state is by definition an element of a short supermultiplet and, if supersymmetry is unbroken, it cannot be renormalized to a long supermultiplet. For this class of extremal black holes an accurate agreement between the macroscopic and microscopic calculations was found. For example in the $N = 8$ theory the entropy was shown to correspond to the unique quartic $E_{7(7)}$-invariant built in terms of the 56 dimensional representation. Actually, topological U-invariants constructed in terms of the (moduli dependent) central charges and matter charges can be derived for all $N \geq 2$ theories; they can be shown, as expected, to coincide with the squared ADM mass at fixed scalars.

Quite recently it has been recognized that the attractor mechanism, which is responsible for the area/entropy relation, has a larger application [30, 31, 32, 33, 34, 35, 36, 37] beyond the BPS cases, being a peculiarity of all *extremal black-hole configurations*, BPS or not. The common feature is that extremal black-hole configurations always belong to some representation of supersymmetry, as will be surveyed in section 2 (this is not the case for non-extremal configurations, since the action of supersymmetry generators cannot be defined for non-zero temperature [38]). Extremal configurations that completely break supersymmetry will belong to long representations of supersymmetry.

Even for these more general cases, because of the topological nature of the extremality condition, the entropy formula turns out to be still given by a U-duality invariant expression built out of electric and magnetic charges. We will report in section 6 on the classification of all extremal solutions (BPS and non-BPS) of N-extended supergravity in four dimensions.

For all the N-extended theories in four dimensions, the general feature that allows us to find the B-H entropy as a topological invariant is the presence of vectors and scalars in the same representation of supersymmetry. This causes the electric/magnetic duality transformations on the vector field strengths (which for these theories are embedded into symplectic transformations) to also act as isometries on the scalar sectors [39] [1]. The symplectic structure of the various σ-models of N-extended supergravity in four dimensions and the relevant relations involving the charges obeyed by the scalars will be worked out in section 3.

As a final remark, let us observe that, since the aim of the present review is to calculate the B-H entropy of extremal black holes, we will only discuss solutions which have $S_{\text{B-H}} \neq 0$. For this class of solutions, known as *large black holes*, the classical area/entropy formula is valid, as it gives the dominant contribution to the black-hole entropy. For these configurations the area of the horizon is in fact proportional to a duality-invariant expression con-

[1] We note that symplectic transformations outside the U-duality group have a non-trivial action on the solutions, allowing one to bring a BPS configuration to a non-BPS one [40].

structed with the electric and magnetic charges, which for these states is not vanishing [41]. This will prove to be a powerful computational tool and will be the subject of section 5.2. As we will see in detail in the following sections, configurations with non-vanishing horizon area in supersymmetric theories preserve at most four supercharges ($N = 1$ supersymmetry) in the bulk of space-time. Black-hole solutions preserving more supercharges do exist, but they do not correspond to classical attractors since in that case the classical area/entropy formula vanishes. These configurations are named *small black holes* and require, for finding the entropy, a quantum attractor mechanism taking into account the presence of higher curvature terms [29, 42, 43].

The paper is organized as follows. Section 2 treats the supersymmetry structure of extremal black-hole solutions of supergravity theories, and the black-hole configurations are described as massive representations of supersymmetry. In section 3 we briefly review the properties of four-dimensional extended supergravity related to its global symmetries. A particular emphasis is given to the general symplectic structure characterizing the moduli spaces of these theories. The presence of this structure allows the global symmetries of extended supergravities to be realized as generalized electric-magnetic symplectic duality transformations acting on the electric and magnetic charges of dyonic solutions (as black holes). In section 4 we start reviewing extremal regular black-hole solutions embedded in supergravity and, for the BPS case, an explicit solution will be found by solving the Killing spinor equations. In section 5 we give a general overview of extremal and non-extremal solutions showing how the attractor mechanism comes about in the extremal case only. Then a general tool for calculating the Bekenstein–Hawking entropy for both BPS and non-BPS extremal black holes will be given, based on the observation that the black-hole potential takes a particularly simple form in the supergravity case, which is fixed in terms of the geometric properties of the moduli space of the given theory. Moreover, for theories based on moduli spaces given by symmetric manifolds G/H, which is the case of all supergravity theories with $N \geq 3$ extended supersymmetry, but also of several $N = 2$ models, the BPS and non-BPS black holes are classified by some U-duality invariant expressions, depending on the representation of the isometry group G under which the electric and magnetic charges are classified. Finally in section 6, by exploiting the supergravity machinery introduced in sections 3 and 4, we shall give a detailed analysis of the attractor solutions for the various theories of extended supergravity. Section 7 contains some concluding remarks.

Our discussion will be confined to four-dimensional black holes.

2 Extremal Black Holes as massive representations of supersymmetry

We are going to review in the present section the algebraic structure of the massive representations of supersymmetry, both for short and long multiplets, in order to pinpoint, for each supergravity theory, the extremal black-hole configurations corresponding to a given number of preserved supercharges. The condition of extremality is in fact independent on the supersymmetry preserved by the solution, the only difference between the supersymmetric and non-supersymmetic case being that the configurations preserving some supercharges correspond to short multiplets, while the configurations which completely break supersymmetry will instead belong to long representations of supersymmetry. The highest spin of the configuration [2] depends on the number of supercharges of the theory under consideration [44].

As a result of our analysis we find for example, as far as large BPS black-hole configurations are considered, that for $N = 2$ theories the highest spin of the configuration (which in this case is 1/2-BPS) is $J_{MAX} = 1/2$, for $N = 4$ theories (1/4-BPS) is $J_{MAX} = 3/2$, while for the $N = 8$ case (1/8-BPS) is $J_{MAX} = 7/2$. On the other hand, 1/2-BPS multiplets have maximum spin $J_{MAX} = N/4$ ($N = 2, 4, 8$) as for massless representations. They are given in Tables 2, 3, 4. The corresponding black holes (for $N > 2$) have vanishing classical entropy (small black holes) [25].

The long multiplets corresponding to non-BPS extremal black-hole configurations have $J_{MAX} = 1$ in the $N = 2$ theory, $J_{MAX} = 2$ in the $N = 4$ theory and $J_{MAX} = 4$ in the $N = 8$ theory. However, as we will see in detail in the following, for the non-BPS cases we may have solutions with vanishing or non-vanishing central charge. Since the central charge Z_{AB} is a complex matrix, it is not invariant under CPT symmetry, but transforms as $Z_{AB} \to \overline{Z}_{AB}$ [3]. The representation then depends on the charge of the configuration: if the solution has vanishing central charge the long-multiplet will be neutral (real), while if the solution has non-zero central charge the long multiplet will be charged (complex), with a doubled dimension as required

[2] We confine our analysis here to the *minimal* highest spin allowed for a given theory.

[3] We use here a different definition of central charge with respect to [44]: $Z_{AB} \to \mathrm{i} Z_{AB}$.

for CPT invariance [44].

We have listed in Tables 1, 2 and 3 all possible massive representations with highest spin $J_{MAX} \leq 3/2$ for $N \leq 8$. The occurence of long spin 3/2 multiplets is only possible for $N = 3, 2$ and of long spin 1 multiplets for $N = 2$. In $N = 1$ there is only one type of massive multiplet (long) since there are no central charges. Its structure is

$$[(J_0 + \frac{1}{2}), 2(J_0), (J_0 - \frac{1}{2})],$$

except for $J_0 = 0$ where we have $[(\frac{1}{2}), 2(0)]$.

In the tables we will denote the spin states by (J) and the number in front of them is their multiplicity. In the fundamental multiplet, with spin $J_0 = 0$ vacuum, the multiplicity of the spin $(N-q-k)/2$ is the dimension of the k-fold antisymmetric Ω-traceless representation of $USp(2(N-q))$. For multiplets with $J_0 \neq 0$ one has to make the tensor product of the fundamental multiplet with the representation of spin J_0. We also indicate if the multiplet is long or short.

2.1 Massive representations of the supersymmetry algebra

The $D = 4$ supersymmetry algebra is given by

$$\begin{aligned} \left\{\overline{Q}_{A\alpha}\,,\,\overline{Q}_{B\beta}\right\} \;&=\; -\,(C\,\gamma^{\mu})_{\alpha\beta}\;P_{\mu}\,\delta_{AB}\;+\mathrm{i}\,(C\,\mathbb{Z}_{AB})_{\alpha\beta} \\ &\quad (A,B = 1,\dots,2p) \qquad\qquad , \end{aligned} \tag{12}$$

where the SUSY charges $\overline{Q}_A \equiv Q_A^{\dagger}\gamma_0 = Q_A^T C$ are Majorana spinors, C is the charge conjugation matrix, P_μ is the 4-momentum operator and the antisymmetric tensor $\mathbb{Z}_{AB}$ is defined as:

$$\mathbb{Z}_{AB} \;=\; \mathrm{Re}(Z_{AB}) + \mathrm{i}\,\gamma^5\,\mathrm{Im}(Z_{AB})\,, \tag{13}$$

the complex matrix $Z_{AB} = -Z_{BA}$ being the central charge operator. For the sake of simplicity we shall suppress the spinorial indices in the formulae. Using the symmetries of the theory, it can always be reduced to normal form [45]. For N even it reads:

$$Z_{AB} \;=\; \begin{pmatrix} \epsilon Z_1 & 0 & \dots & 0 \\ 0 & \epsilon Z_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \epsilon Z_p \end{pmatrix}, \tag{14}$$

N	massive spin 3/2 multiplet	long	short
8	none		
6	$2\times[(\frac{3}{2}),6(1),14(\frac{1}{2}),14'(0)]$	no	$q=3$, $(\frac{1}{2}\text{BPS})$
5	$2\times[(\frac{3}{2}),6(1),14(\frac{1}{2}),14'(0)]$	no	$q=2$, $(\frac{2}{5}\text{BPS})$
4	$2\times[(\frac{3}{2}),6(1),14(\frac{1}{2}),14'(0)]$	no	$q=1$, $(\frac{1}{4}\text{BPS})$
	$2\times[(\frac{3}{2}),4(1),6(\frac{1}{2}),4(0)]$	no	$q=2$, $(\frac{1}{2}\text{BPS})$
3	$[(\frac{3}{2}),6(1),14(\frac{1}{2}),14'(0)]$	yes	no
	$2\times[(\frac{3}{2}),4(1),6(\frac{1}{2}),4(0)]$	no	$q=1$, $(\frac{1}{3}\text{BPS})$
2	$[(\frac{3}{2}),4(1),6(\frac{1}{2}),4(0)]$	yes	no
	$2\times[(\frac{3}{2}),2(1),(\frac{1}{2})]$	no	$q=1$, $(\frac{1}{2}\text{BPS})$
1	$[(\frac{3}{2}),2(1),(\frac{1}{2})]$	yes	no

Table 1: Massive spin 3/2 multiplets.

where ϵ is the 2×2 antisymmetric matrix, (every zero is a 2×2 zero matrix) and the p skew eigenvalues Z_m of Z_{AB} are the central charges. For N odd the central charge matrix has the same form as in (14) with $p=(N-1)/2$, except for one extra zero row and one extra zero column. Note that it is not always possible to reduce Z_{AB} to its normal form with real Z_m by means of symmetries of the theory [45]. This is the case in particular of $N=8$ supergravity where the $SU(8)$ R-symmetry does not affect the global phase of the skew-eigenvalues Z_m. Therefore we shall consider the general situation in which Z_m are complex and define for each of them the spinorial matrices which will enter the supersymmetry algebra:

$$\mathbb{Z}_m \quad = \quad \mathrm{Re}(Z_m) + \mathrm{i}\,\gamma^5\,\mathrm{Im}(Z_m)\,,$$

N	massive spin 1 multiplet	long	short
8,6,5	none		
4	$2\times[(1),4(\frac{1}{2}),5(0)]$	no	$q=2$, $(\frac{1}{2}\mathrm{BPS})$
3	$2\times[(1),4(\frac{1}{2}),5(0)]$	no	$q=1$, $(\frac{1}{3}\mathrm{BPS})$
2	$[(1),4(\frac{1}{2}),5(0)]$	yes	no
	$2\times[(1),2(\frac{1}{2}),(0)]$	no	$q=1$, $(\frac{1}{2}\mathrm{BPS})$
1	$[(1),2(\frac{1}{2}),(0)]$	yes	no

Table 2: Massive spin 1 multiplets.

$$\overline{\mathbb{Z}}_m \;=\; \mathrm{Re}(Z_m) - \mathrm{i}\,\gamma^5\,\mathrm{Im}(Z_m)\,, \quad n=1,\dots,p\,. \tag{15}$$

If we identify each index $A, B, \dots$ with the pair of indices

$$A=(a,m) \quad ; \quad a,b,\dots=1,2 \quad ; \quad m,n,\dots=1,\dots,p\,, \tag{16}$$

the matrix $\mathbb{Z}_{AB}$ in the normal frame will have the form:

$$\mathbb{Z}_{AB} \;=\; \mathbb{Z}_{am,\,bn} = \mathbb{Z}_m\,\delta_{mn}\,\epsilon_{ab}\,, \tag{17}$$

and the superalgebra (12) can be rewritten as:

$$\left\{\overline{Q}_{am}\,,\,\overline{Q}_{bn}\right\} \;=\; -\,(C\,\gamma^\mu)\;P_\mu\,\delta_{ab}\,\delta_{mn}\;+\mathrm{i}\,C\,\epsilon_{ab}\,\mathbb{Z}_m\,\delta_{mn} \tag{18}$$

where ϵ_{ab} is the two-dimensional Levi Civita symbol. Let us consider a generic unit time-like Killing vector ζ^μ ($\zeta^\mu\zeta_\mu=1$), in terms of which we define the following projectors acting on both the internal (a,m) and Lorentz indices (α,β) of the spinors:

$$\begin{aligned}
S^{(\pm)}_{am,\,bn} \;&=\; \frac{1}{2}\left(\delta_{ab}\,\delta_{mn}\pm\mathrm{i}\,\zeta_\mu\,\gamma^\mu\,\frac{\overline{\mathbb{Z}}_m}{|Z_m|}\,\delta_{mn}\,\epsilon_{ab}\right)\,,\\
\widetilde{S}^{(\pm)}_{am,\,bn} \;&=\; \frac{1}{2}\left(\delta_{ab}\,\delta_{mn}\pm\mathrm{i}\,\zeta_\mu\,\gamma^\mu\,\frac{\mathbb{Z}_m}{|Z_m|}\,\delta_{mn}\,\epsilon_{ab}\right)\,,
\end{aligned} \tag{19}$$

N	massive spin 1/2 multiplet	long	short
8,6,5,4,3	none		
2	$2\times[(\frac{1}{2}),2(0)]$	no	$q=1$, ($\frac{1}{2}$BPS)
1	$[(\frac{1}{2}),2(0)]$	yes	no

Table 3: Massive spin 1/2 multiplets.

and define the projected supersymmetry generators:

$$\overline{Q}^{(\pm)} \;=\; \overline{Q}\, S^{(\pm)}\,. \tag{20}$$

The anticommutation relation (18) can be rewritten in the following form:

$$\left\{Q^{(\pm)}_{am}\,,\,\overline{Q}^{(\pm)}_{bn}\right\} \;=\; \widetilde{S}^{(\pm)}_{am,\,bn}\,\zeta_\mu\,\gamma^\mu\,(\zeta_\nu\,P^\nu \mp |Z_m|)\,. \tag{21}$$

In the case in which $\zeta^\mu=(1,0,0,0)$ and we are in the rest frame ($P^0=M$) the above relation reads:

$$\left\{Q^{(\pm)}_{am}\,,\,Q^{(\pm)\,\dagger}_{bn}\right\} \;=\; \widetilde{S}^{(\pm)}_{am,\,bn}\,(M \mp |Z_m|)\,. \tag{22}$$

Since the left-hand side of (22) is non-negative definite, we deduce the BPS bound required by unitarity of the representations:

$$M \;\geq\; |\,Z_m| \qquad \forall Z_m\,,\; m=1,\dots,p\,. \tag{23}$$

It is an elementary consequence of the supersymmetry algebra and of the identification between central charges and topological charges [46].

Massive BPS multiplets

Suppose that on a given state $|BPS\rangle$ the BPS bound (23) is saturated by q of the p eigenvalues Z_m:

$$M \;=\; |Z_1| = |Z_2| = \ldots = |Z_q| \quad q\leq p\,, \tag{24}$$

then, from (22) we deduce that:

$$Q^{(+)}_{am}\,|BPS\rangle \;=\; 0\;,\;\; m=1,\dots,q\,, \tag{25}$$

namely q of the pairs of creation-annihilation operators, which have abelian anticommutation relations, annihilate the state. The multiplet obtained by acting on $|BPS\rangle$ with the remaining supersymmetry generators is said to be q/N BPS. Note that $q_{MAX} = N/2$ for N even and $q_{MAX} = (N-1)/2$ for N odd. The $USp(2N)$ symmetry is now reduced to $USp(2(N-q))$. The short multiplet has the same number of states as a long multiplet of the $N-q$ supersymmetry algebra. The fundamental multiplet, with $J = 0$ vacuum, contains $2\cdot 2^{2(N-q)}$ states with $J_{MAX} = (N-q)/2$. Note the doubling due to CPT invariance. Generic massive short multiplets can be obtained by making the tensor product with a spin J_0 representation of SU(2).

If we write the infinitesimal generator of a supersymmetry in the form:

$$\overline{Q}_A\,\epsilon_A \;=\; \overline{Q}^{(+)}_A\,\epsilon^{(+)}_A + \overline{Q}^{(-)}_A\,\epsilon^{(-)}_A\,, \tag{26}$$

the supersymmetries preserved by $|BPS\rangle$ are parametrized by $\epsilon^{(+)}_{am}$ with $m\le q$ and thus defined by the condition:

$$\begin{aligned}\epsilon^{(-)}_{am} &= S^{(-)}_{am,\,bn}\,\epsilon_{bn} = 0\;;\quad m,\,n\le q\,, && (27)\\ \epsilon_{am} &= 0\;;\;\; m>q\,, && (28)\end{aligned}$$

which can be written in terms of *Weyl* spinors ϵ_A, ϵ^A in the following form:

$$\begin{aligned}\epsilon_{am} &= \mathrm{i}\,\frac{Z_m}{|Z_m|}\,\zeta_\mu\,\gamma^\mu\,\epsilon_{ab}\,\epsilon^{bm} = \mathrm{i}\,\frac{Z_m}{|Z_m|}\,\epsilon_{ab}\,\gamma^0\,\epsilon^{bm}\;;\quad m\le q\,, && (29)\\ \epsilon_{am} &= 0\;;\;\; m>q\,. && (30)\end{aligned}$$

If, in a given supergravity theory, the state $|BPS\rangle$ corresponds to a background described by a certain configuration of fields, eq. (25) is translated into the request that the supersymmetry variations of all the fields are zero in the background. We consider extremal black-hole solutions for which the supersymmetry variations of the bosonic fields are identically zero. Then the condition (25) yields a set of first order differential equations for the bosonic fields, called "Killing spinor" equations, to be satisfied on the given configuration

$$0 = \delta\text{fermions} = \text{SUSY rule}\,(\text{bosons},\epsilon_{am})\;, \tag{31}$$

where the supersymmetry transformations are made with respect to the residual supersymmetry parameter $\epsilon^{(+)}_{am}$ defined by the conditions (30). These conditions are important in order to be able to recast equations (31) into differential equations involving only the bosonic fields of the solution.

Massive non-BPS multiplets

Massive multiplets with $Z_m = 0$ or $Z_m \neq 0$ but $M > |Z_m|$ are called long multiplets or non BPS states. They are qualitatively the same, the only difference being that in the first case the supermultiplets are real, while in the second one the representations must be doubled in order to have CPT invariance, since $Z_m \to \overline{Z}_m$ under CPT.

In both cases the supersymmetry algebra can be put in a form with $2N$ creation and $2N$ annihilation operators. It shows explicit invariance under $SU(2) \times USp(2N)$. The vacuum state is now labeled by the spin representation of SU(2), $|\Omega\rangle_J$. If $J = 0$ we have the fundamental massive multiplet with 2^{2N} states. These are organized in representations of SU(2) with $J_{MAX} = N/2$. With respect to $USp(2N)$ the states with fixed $0 < J < N/2$ are arranged in the $(N-2J)$-fold Ω-traceless antisymmetric representation, $[N-2J]$.

The general multiplet with a spin J vacuum can be obtained by tensoring the fundamental multiplet with spin J representation of SU(2). The total number of states is then $(2J+1) \cdot 2^{2N}$.

3 The general form of the supergravity action in four-dimensions and its BPS configurations

In this section we begin the study of extremal black-hole solutions of extended supergravity in four space-time dimensions. To this aim we first have to introduce the main features of four dimensional N-extended supergravities. These theories contain in the bosonic sector, besides the metric, a number n_V of vectors and m of (real) scalar fields. The relevant bosonic action is known to have the following general form:

$$\mathcal{S} \;=\; \int \sqrt{-g}\, d^4x \left(-\frac{1}{2}\, R + \mathrm{Im}\mathcal{N}_{\Lambda\Gamma} F^{\Lambda}_{\mu\nu} F^{\Gamma|\mu\nu} + \frac{1}{2\sqrt{-g}}\, \mathrm{Re}\mathcal{N}_{\Lambda\Gamma} \epsilon^{\mu\nu\rho\sigma}\, F^{\Lambda}_{\mu\nu} F^{\Gamma}_{\rho\sigma} + \right.$$

$$+ \quad \frac{1}{2} g_{rs}(\Phi)\partial_\mu \Phi^r \partial^\mu \Phi^s \Big) \ , \tag{32}$$

where $g_{rs}(\Phi)$ $(r, s, \cdots = 1, \cdots, m)$ is the scalar metric on the σ-model described by the scalar manifold $\mathcal{M}_{scalar}$ of real dimension m and the vectors kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(\Phi)$ is a complex, symmetric, $n_V \times n_V$ matrix depending on the scalar fields. The number of vectors and scalars, namely n_V and m, and the geometric properties of the scalar manifold $\mathcal{M}_{scalar}$ depend on the number N of supersymmetries and are resumed in Table 4. The imaginary part Im$\mathcal{N}$ of the vector kinetic matrix is negative definite and generalizes the inverse of the squared coupling constant appearing in ordinary gauge theories while its real part Re$\mathcal{N}$ is instead a generalization of the *theta*-angle of quantum chromodynamics. In supergravity theories, the kinetic matrix $\mathcal{N}$ is in general not a constant, its components being functions of the scalar fields. However, in extended supergravity $(N \geq 2)$ the relation between the scalar geometry and the kinetic matrix $\mathcal{N}$ has a very general and universal form. Indeed it is related to the solution of a general problem, namely how to lift the action of the scalar manifold isometries from the scalar to the vector fields. Such a lift is necessary because of supersymmetry since scalars and vectors generically belong to the same supermultiplet and must rotate coherently under symmetry operations. This problem has been solved in a general (non supersymmetric) framework in reference [39] by considering the possible extension of the Dirac electric-magnetic duality to more general theories involving scalars. In the next subsection we review this approach and in particular we show how enforcing covariance with respect to such duality rotations leads to a determination of the kinetic matrix $\mathcal{N}$. The structure of $\mathcal{N}$ enters the black-hole equations in a crucial way so that the topological invariant associated with the hole, that is its entropy, is an invariant of the group of electro-magnetic duality rotations, the U-duality group.

3.1 Duality Rotations and Symplectic Covariance

Let us review the general structure of an abelian theory of vectors and scalars displaying covariance under a group of duality rotations. The basic reference is the 1981 paper by Gaillard and Zumino [39]. A general presentation in $D = 2p$ dimensions can be found in [47]. Here we fix $D = 4$.

We consider a theory of n_V abelian gauge fields A^Λ_μ, in a $D = 4$ space-time with Lorentz signature (which we take to be mostly minus). They correspond

Table 4: *Scalar Manifolds of $N > 2$ Extended Supergravities*

N	Duality group G	isotropy H	$\mathcal{M}_{scalar}$	n_V	m
3	$SU(3,n)$	$SU(3)\times U(n)$	$\frac{SU(3,n)}{S(U(3)\times U(n))}$	$3+n$	$6n$
4	$SU(1,1)\otimes SO(6,n)$	$U(4)\times SO(n)$	$\frac{SU(1,1)}{U(1)}\otimes\frac{SO(6,n)}{SO(6)\times SO(n)}$	$6+n$	$6n+2$
5	$SU(1,5)$	$U(5)$	$\frac{SU(1,5)}{S(U(1)\times U(5))}$	10	10
6	$SO^\star(12)$	$U(6)$	$\frac{SO^\star(12)}{U(1)\times SU(6)}$	16	30
7, 8	$E_{7(7)}$	$SU(8)$	$\frac{E_{7(7)}}{SU(8)}$	28	70

In the table, n_V stands for the number of vectors and m for the number of real scalar fields. In all the cases the duality group G is embedded in $Sp(2\,n_V,\mathbb{R})$.

to a set of n_V differential 1-forms

$$A^\Lambda \;\equiv\; A^\Lambda_\mu\, dx^\mu \qquad (\Lambda = 1,\dots,n_V)\ . \tag{33}$$

The corresponding field strengths and their Hodge duals are defined by [4]:

$$\begin{aligned} F^\Lambda &\equiv d\,A^\Lambda \equiv F^\Lambda_{\mu\nu}\, dx^\mu \wedge dx^\nu \\ F^\Lambda_{\mu\nu} &\equiv \frac{1}{2}\left(\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu\right) \\ ({}^\star F^\Lambda)_{\mu\nu} &\equiv \frac{\sqrt{-g}}{2}\varepsilon_{\mu\nu\rho\sigma}\, F^{\Lambda|\rho\sigma}\ . \end{aligned} \tag{34}$$

The dynamics of a system of abelian gauge fields coupled to scalars in a gravity theory is encoded in the bosonic action (32).

Introducing self-dual and antiself-dual combinations

$$\begin{aligned} F^\pm &= \frac{1}{2}\left(F \pm \mathrm{i}\,{}^\star F\right)\ , \\ {}^\star F^\pm &= \mp\mathrm{i}F^\pm\ , \end{aligned} \tag{35}$$

the vector part of the Lagrangian defined by (32) can be rewritten in the form:

$$\mathcal{L}_{vec} = \mathrm{i}\left[F^{-T}\overline{\mathcal{N}}F^- - F^{+T}\mathcal{N}F^+\right]\ . \tag{36}$$

[4] We use, for the ϵ tensor, the convention: $\epsilon_{0123} = -1$.

Introducing further the new tensors

$$ {}^\star G_{\Lambda|\mu\nu} \equiv \frac{1}{2}\frac{\partial \mathcal{L}}{\partial F^\Lambda_{\mu\nu}} = \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\, F^\Sigma_{\mu\nu} + \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\, {}^\star F^\Sigma_{\mu\nu} \leftrightarrow G^\mp_{\Lambda|\mu\nu} \equiv \mp\frac{\mathrm{i}}{2}\frac{\partial \mathcal{L}}{\partial F^{\mp\Lambda}_{\mu\nu}}\,, \quad (37) $$

the Bianchi identities and field equations associated with the Lagrangian (32) can be written as

$$ \begin{aligned} \nabla^{\mu\star} F^\Lambda_{\mu\nu} &= 0 \\ \nabla^{\mu\star} G_{\Lambda|\mu\nu} &= 0 \end{aligned} \quad (38) $$

or equivalently

$$ \nabla^\mu \mathrm{Im} F^{\pm\Lambda}_{\mu\nu} = 0 \quad (39) $$

$$ \nabla^\mu \mathrm{Im} G^{\pm}_{\Lambda|\mu\nu} = 0\,. \quad (40) $$

This suggests that we introduce the $2n_V$ column vector

$$ \mathbf{V} \equiv \begin{pmatrix} {}^\star F \\ {}^\star G \end{pmatrix} \quad (41) $$

and that we consider general linear transformations on such a vector

$$ \begin{pmatrix} {}^\star F \\ {}^\star G \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} {}^\star F \\ {}^\star G \end{pmatrix}. \quad (42) $$

For any constant matrix $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2n_V, \mathbb{R})$ the new vector of magnetic and electric field-strengths $\mathbf{V}' = \mathcal{S} \cdot \mathbf{V}$ satisfies the same equations (38) as the old one. In a condensed notation we can write

$$ \partial\, \mathbf{V} = 0 \quad \Longleftrightarrow \quad \partial\, \mathbf{V}' = 0. \quad (43) $$

Separating the self-dual and antiself-dual parts

$$ F = \left(F^+ + F^-\right) \qquad ; \qquad G = \left(G^+ + G^-\right) \quad (44) $$

and taking into account that we have

$$ G^+ = \mathcal{N} F^+ \qquad ; \qquad G^- = \overline{\mathcal{N}} F^- \quad (45) $$

the duality rotation of eq. (42) can be rewritten as

$$\begin{pmatrix} F^+ \\ G^+ \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^+ \\ \mathcal{N} F^+ \end{pmatrix} \quad ; \quad \begin{pmatrix} F^- \\ G^- \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^- \\ \overline{\mathcal{N}} F^- \end{pmatrix} . \tag{46}$$

Now, let us note that, since in the system we are considering (eq. (32)) the gauge fields are coupled to the scalar sector via the scalar-dependent kinetic matrix $\mathcal{N}$, when a duality rotation is performed on the vector field strengths and their duals, we have to assume that the scalars get transformed correspondingly, through the action of some diffeomorphism on the scalar manifold $\mathcal{M}_{scalar}$. In particular, the kinetic matrix $\mathcal{N}(\Phi)$ transforms under a duality rotation. Then, a duality transformation ξ acts in the following way on the supersymmetric system:

$$\xi : \begin{cases} V & \rightarrow \quad V'^{\mp} = S_\xi V^{\mp} \\ \Phi & \rightarrow \quad \Phi' = \xi(\Phi) \\ \mathcal{N}(\Phi) & \rightarrow \quad \mathcal{N}' \left(\xi(\Phi)\right) \end{cases} \tag{47}$$

Thus, the transformation laws of the equations of motion and of $\mathcal{N}$, and so also the matrix $\mathcal{S}_\xi$, will be induced by a diffeomorphism of the scalar fields.

Focusing in particular on the first relation in (47), that explicitly reads:

$$\begin{pmatrix} F^{\pm\prime} \\ G^{\pm\prime} \end{pmatrix} = \begin{pmatrix} A_\xi F^{\pm} + B_\xi G^{\pm} \\ C_\xi F^{\pm} + D_\xi G^{\pm} \end{pmatrix} , \tag{48}$$

we note that it contains the magnetic field strength $G^{\mp}_\Lambda$ introduced in (37), which is defined as a variation of the kinetic lagrangian. Under the transformations (47) the lagrangian transforms in the following way:

$$\begin{aligned} \mathcal{L}' &= \mathrm{i}\Big[(A_\xi + B_\xi \mathcal{N})_\Gamma{}^\Lambda \, (A_\xi + B_\xi \mathcal{N})^\Sigma{}_\Delta \, \mathcal{N}'_{\Lambda\Sigma}(\Phi) F^{+\Gamma} F^{+\Delta} \\ &- \left(A_\xi + B_\xi \overline{\mathcal{N}}\right)_\Gamma{}^\Lambda \left(A_\xi + B_\xi \overline{\mathcal{N}}\right)^\Sigma{}_\Delta \overline{\mathcal{N}}'_{\Lambda\Sigma}(\Phi) F^{-\Gamma} F^{-\Delta} \Big] ; \end{aligned} \tag{49}$$

Equations (47) must be consistent with the definition of $G^{\mp}$ as a variation of the lagrangian (49):

$$G'^{+}_\Lambda = (C_\xi + D_\xi \mathcal{N})_{\Lambda\Sigma} \, F^{+\Sigma} \equiv -\frac{\mathrm{i}}{2} \frac{\partial \mathcal{L}'}{\partial F'^{+\Lambda}} = (A_\xi + B_\xi \mathcal{N})^\Delta{}_\Sigma \, \mathcal{N}'_{\Lambda\Delta} F^{+\Sigma} \tag{50}$$

that implies:

$$\mathcal{N}'_{\Lambda\Sigma}(\Phi') = \left[(C_\xi + D_\xi \mathcal{N}) \cdot (A_\xi + B_\xi \mathcal{N})^{-1} \right]_{\Lambda\Sigma} ; \tag{51}$$

The condition that the matrix $\mathcal{N}$ is symmetric, and that this property must be true also in the duality transformed system, gives the constraint:

$$\mathcal{S} \in Sp(2n_V, \mathbb{R}) \subset GL(2n_V, \mathbb{R}) \,, \tag{52}$$

that is:

$$\mathcal{S}^T \, \mathbb{C} \, \mathcal{S} \;\; = \;\; \mathbb{C} \,, \tag{53}$$

where $\mathbb{C}$ is the symplectic invariant $2n_V \times 2n_V$ matrix:

$$\mathbb{C} \;\; = \;\; \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \,. \tag{54}$$

It is useful to rewrite the symplectic condition (53) in terms of the $n_V \times n_V$ blocks defining $\mathcal{S}$:

$$A^T \, C - C^T \, A \;\; = \;\; B^T \, D - D^T \, B = 0 \;\; ; \;\; A^T \, D - C^T \, B = \mathbb{1} \,. \tag{55}$$

The above observation has important implications on the scalar manifold $\mathcal{M}_{scalar}$. Indeed, it implies that on the scalar manifold the following homomorphism is defined:

$$Diff(\mathcal{M}_{scalar}) \rightarrow Sp(2n, \mathbb{R}) \,. \tag{56}$$

In particular, the presence on the manifold of a function of scalars transforming with a fractional linear transformation under a duality rotation on the scalars, induces the existence on $\mathcal{M}_{scalar}$ of a linear structure (inherited from the vectors). As we are going to discuss in detail in section 3.2, this may be rephrased by saying that the scalar manifold is endowed with a symplectic bundle. As the transition functions of this bundle are given in terms of the *constant* matrix $\mathcal{S}$, the symplectic bundle is flat. In particular, as we will see in section 3.2, for the $N = 2$ four dimensional theory this implies that the scalar manifold be a *special manifold*, that is a Kähler–Hodge manifold endowed with a flat symplectic bundle.

If we are interested in the global symmetries of the theory (i.e. global symmetries of the field equations and Bianchi identities) we will need to restrict the duality transformations, namely the homomorphism in (56), to the isometries of the scalar manifold, which leave the scalar sector of the action invariant. The transformations (47), which are duality symmetries of the system field-equations/Bianchi-identities, cannot be extended in general to be symmetries of the lagrangian. The scalar part of the lagrangian (32) is invariant under the action of the isometry group of the metric g_{rs}, but the vector part is in general not invariant. The transformed lagrangian under the action of $\mathcal{S} \in Sp(2n_V, \mathbb{R})$ can be rewritten:

$$\begin{aligned}
\mathrm{I}m\left(F^{-\Lambda}G^-_\Lambda\right) &\rightarrow \mathrm{I}m\left(F'^{-\Lambda}G'^-_\Lambda\right) \\
&= \mathrm{I}m[F^{-\Lambda}G^-_\Lambda + 2(C^TB)_\Lambda{}^\Sigma F^{-\Lambda}G^-_\Sigma + \\
&+ (C^TA)_{\Lambda\Sigma}F^{-\Lambda}F^{-\Sigma} + (D^TB)^{\Lambda\Sigma}G^-_\Lambda G^-_\Sigma] \,. \qquad (57)
\end{aligned}$$

It is evident from (57) that only the transformations with $B = C = 0$ are symmetries.
If $C \neq 0$, $B = 0$ the lagrangian varies for a topological term:

$$(C^TA)_{\Lambda\Sigma}F^\Lambda_{\mu\nu}{}^\star F^{\Sigma|\mu\nu} \qquad (58)$$

corresponding to a redefinition of the function $\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}$; such a transformation being a total derivative it leaves classical physics invariant, but it is relevant in the quantum theory. It is a symmetry of the partition function only if $\Delta\mathrm{Re}\mathcal{N}_{\Lambda\Sigma} = \frac{1}{2}(C^TA)$ is an integer multiple of 2π, and this implies that $S \in Sp(2n_V, \mathbb{Z}) \subset Sp(2n_V, \mathbb{R})$.
For $B \neq 0$ neither the action nor the perturbative partition function are invariant. Let us observe that in this case the transformation law (51) of the kinetic matrix $\mathcal{N}$ contains the transformation $\mathcal{N} \rightarrow -\frac{1}{\mathcal{N}}$ that is it exchanges the weak and strong coupling regimes of the theory. One may then think of such a quantum field theory as being described by a collection of local lagrangians, each defined in a local patch. They are all equivalent once one defines for each of them what is *electric* and what is *magnetic*. Duality transformations map this set of lagrangians one into the other.

At this point we observe that the supergravity bosonic lagrangian (32) is exactly of the form considered in this section as far as the matter content is concerned, so that we may apply the above considerations about duality rotations to the supergravity case. In particular, the U-duality acts in all theories with $N \geq 2$ supersymmetries, where the vector supermultiplets contain both vectors and scalars. For $N = 1$ supergravity, instead, vectors and scalars are still present but they are not related by supersymmetry, and as a consequence they are not related by U-duality rotations, so that the previous formalism does not necessarily apply [5]. In the next subsection we will discuss in a geometric framework the structure of the supergravity theories for $N \geq 2$. In particular, for theories whose σ-model is a coset space (which includes all theories with $N > 2$) we will give the expression for the kinetic vector matrix $\mathcal{N}_{\Lambda\Sigma}$ in terms of the $Sp(2n_V)$ coset representatives embedding the U-duality group. Furthermore we will show that in the $N = 2$ case, although the σ-model of the scalars is not in general a coset space, yet it may be treated in a completely analogous way.

3.2 Duality symmetries and central charges

Let us restrict our attention to N-extended supersymmetric theories coupled to the gravitational field, that is to supergravity theories, whose bosonic action has been given in (32). For each theory we are going to analyze the group theoretical structure and to find the expression of the central charges, together with the properties they obey. As already mentioned, with the exception of the $N = 1$ and $N = 2$ cases, all supergravity theories in four dimensions contain scalar fields whose kinetic Lagrangians are described by σ-models of the form G/H (we have summarized these cases in Table 4). We will first examine the theories with $N > 2$, extending then the results to the $N = 2$ case. Here and in the following, G denotes a non compact group acting as isometry group on the scalar manifold while H, the isotropy subgroup, is of the form:

$$H = H_{Aut} \otimes H_{matter} \tag{59}$$

H_{Aut} being the automorphism group of the supersymmetry algebra while H_{matter} is related to the matter multiplets. (Of course $H_{matter} = \mathbb{1}$ in all cases where supersymmetric matter does not exist, namely $N > 4$).

We will see that in all the theories the fields are in some representation of the isometry group G of the scalar fields or of its maximal compact subgroup H. This is just a consequence of the Gaillard–Zumino duality acting on the 2-form field strengths and their duals, discussed in the preceding section.

The scalar manifolds and the automorphism groups of supergravity theories for any D and N can be found in the literature (see for instance [48],

[5] There are however $N = 1$ models where the scalar moduli space is given by a special-Kähler manifold. This is the case for example for the compactification of the heterotic theory on Calabi–Yau manifolds.

[49], [47], [50]). As it was discussed in the previous section, the group G acts linearly in a symplectic representation on the electric and magnetic field strengths appearing in the gravitational and matter multiplets. Here and in the following the index Λ runs over the dimensions of some representation of the duality group G. Since consistency of the quantum theory requires the electric and magnetic charges to satisfy a quantization condition, the true duality symmetry at the quantum level (U-duality), acting on quantized charges, is a suitable discrete version of the continuous group G [10]. The moduli space of these theories is $G(\mathbb{Z})\backslash G/H$.

All the properties of the given supergravity theories for $N \geq 3$ are completely fixed in terms of the geometry of G/H, namely in terms of the coset representatives L satisfying the relation:

$$L(\Phi') = gL(\Phi)h(g, \Phi) \tag{60}$$

where $g \in G$, $h \in H$ and $\Phi' = \Phi'(\Phi)$, Φ being the coordinates of G/H. Note that the scalar fields in G/H can be assigned, in the linearized theory, to linear representations R_H of the local isotropy group H so that dim R_H = dim G $-$ dim H (in the full theory, R_H is the representation which the vielbein of G/H belongs to).

With any field-strength F^Λ we may associate a magnetic charge p^Λ and an electric charge q_Λ given respectively by:

$$p^\Lambda = \frac{1}{4\pi}\int_{S^2} F^\Lambda \qquad\qquad q_\Lambda = \frac{1}{4\pi}\int_{S^2} G_\Lambda\,, \tag{61}$$

where S^2 is a spatial two-sphere in the space-time geometry of the dyonic solution (for instance, in Minkowski space-time the two-sphere at radial infinity S^2_∞). Clearly the presence of dyonic solutions requires the Maxwell equations (38) to be completed by adding corresponding electric and magnetic currents on the right hand side. These charges however are not the physical charges of the *interacting theory*; these latter can be computed by looking at the transformation laws of the fermion fields, where the physical field strengths appear dressed with the scalar fields [50],[51]. It is in terms of these interacting dressed field strengths that the field theory realization of the central charges occurring in the supersymmetry algebra (12) is given. Indeed, let us first introduce the central charges: they are associated with the dressed 2-form T_{AB} appearing in the supersymmetry transformation law of the gravitino 1-form. The physical graviphoton may be identified from the supersymmetry transformation law of the gravitino field in the interacting theory, namely:

$$\delta\psi_A = \nabla\epsilon_A + \alpha T_{AB|\mu\nu}\gamma^a\gamma^{\mu\nu}\epsilon^B V_a + \cdots \tag{62}$$

Here ∇ is the covariant derivative in terms of the space-time spin connection and the composite connection of the automorphism group H_{Aut}, α is a coefficient fixed by supersymmetry, V^a is the space-time vielbein, $A = 1, \cdots, N$ is the index acted on by the automorphism group. Here and in the following the dots denote trilinear fermion terms which are characteristic of any supersymmetric theory but do not play any role in the following discussion. The 2-form field strength T_{AB} will be constructed by dressing the bare field strengths F^Λ with the coset representative $L(\Phi)$ of G/H, Φ denoting a set of coordinates of G/H.

Note that the same field strength T_{AB} which appears in the gravitino transformation law is also present in the dilatino transformation law in the following way:

$$\delta\chi_{ABC} = P_{ABCD,\ell}\partial_\mu\phi^\ell\gamma^\mu\epsilon^D + \beta T_{[AB|\mu\nu}\gamma^{\mu\nu}\epsilon_{C]} + \cdots \tag{63}$$

Analogously, when vector multiplets are present, the matter vector field strengths T_I appearing in the transformation laws of the gaugino fields, which are named matter vector field strengths, are linear combinations of the field strengths dressed with a different combination of the scalars:

$$\delta\lambda_{IA} = \mathrm{i}P_{IAB,i}\partial_\mu\Phi^i\gamma^\mu\epsilon^B + \gamma T_{I|\mu\nu}\gamma^{\mu\nu}\epsilon_A + \cdots . \tag{64}$$

Here $P_{ABCD} = P_{ABCD,\ell}d\phi^\ell$ and $P^I_{AB} = P^I_{AB,i}d\Phi^i$ are the vielbein of the scalar manifolds spanned by the scalar fields of the gravitational and vector multiplets respectively (more precise definitions are given below), and β and γ are constants fixed by supersymmetry.

In order to give the explicit dependence on scalars of T_{AB}, T^I, it is necessary to recall from the previous subsection that, according to the Gaillard–Zumino construction, the isometry group G of the scalar manifold acts on the vector $(F^{-\Lambda}, G^-_\Lambda)$ (or its complex conjugate) as a subgroup of $Sp(2n_V, \mathbb{R})$ (n_V is the number of vector fields) with duality transformations interchanging electric and magnetic field strengths:

$$\mathcal{S}\begin{pmatrix} F^{-\Lambda} \\ G^-_\Lambda \end{pmatrix} = \begin{pmatrix} F^{-\Lambda} \\ G^-_\Lambda \end{pmatrix}' . \tag{65}$$

Let now $L(\Phi)$ be the coset representative of G in the symplectic representation, namely as a $2\,n_V \times 2\,n_V$ matrix belonging to $Sp(2n_V,\mathbb{R})$ and therefore, in each theory, it can be described in terms of $n_V \times n_V$ blocks A_L, B_L, C_L, D_L satisfying the same relations (55) as the corresponding blocks of the generic symplectic transformation $\mathcal{S}$.

Since the fermions of supergravity theories transform in a complex representation of the R-symmetry group $H_{Aut} \subset G$, it is useful to introduce a complex basis in the vector space of $Sp(2n_V,\mathbb{R})$, defined by the action of following unitary matrix:[6]

$$\mathcal{A} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & i\,\mathbb{1} \\ \mathbb{1} & -i\,\mathbb{1} \end{pmatrix}\,,$$

and to introduce a new matrix $\mathbf{V}(\Phi)$ obtained by complexifying the right index of the coset representative $L(\Phi)$, so as to make its transformation properties under right action of H manifest:

$$\mathbf{V}(\Phi) = \begin{pmatrix} \mathbf{f} & \overline{\mathbf{f}} \\ \mathbf{h} & \overline{\mathbf{h}} \end{pmatrix} = L(\Phi)\mathcal{A}^\dagger\,, \tag{66}$$

where:

$$\mathbf{f} \;=\; \frac{1}{\sqrt{2}}(A_L - \mathrm{i}B_L)\ ;\ \ \mathbf{h} = \frac{1}{\sqrt{2}}(C_L - \mathrm{i}D_L)\,,$$

From the properties of $L(\Phi)$ as a symplectic matrix, it is easy to derive the following properties for $\mathbf{V}$:

$$\mathbf{V}\,\eta\,\mathbf{V}^\dagger \;=\; -\mathrm{i}\mathbb{C}\ \ ;\ \ \mathbf{V}^\dagger\,\mathbb{C}\,\mathbf{V} = \mathrm{i}\eta\,, \tag{67}$$

where the symplectic invariant matrix $\mathbb{C}$ and η are defined as follows:

$$\mathbb{C} \;=\; \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\ ;\ \ \eta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\,, \tag{68}$$

[6]We adopt here and in the following a condensed notation where $\mathbb{1}$ denotes the n_V dimensional identity matrix $\mathbb{1}_N^M = \delta_N^M$. In supergravity calculations, the index M is often decomposed as $M = (AB, I)$, $AB = -BA$ labelling the two-times antisymmetric representation of the R-symmetry group H_{Aut} and I running over the H_{matter} representation of the matter fields. We use the convention that the sum over the antisymmetric couple AB be free and therefore supplemented by a factor 1/2 in order to avoid repetitions. In particular with these conventions, when restricted to the AB indices, the identity reads: $\mathbb{1}^{AB}_{CD} \equiv 2\,\delta^{AB}_{CD} = \delta^A_C\delta^B_D - \delta^A_D\delta^B_C$.

and, as usual, each block is an $n_V \times n_V$ matrix. The above relations imply on the matrices $\mathbf{f}$ and $\mathbf{h}$ the following properties:

$$\begin{cases} \mathrm{i}(\mathbf{f}^\dagger \mathbf{h} - \mathbf{h}^\dagger \mathbf{f}) & = & \mathbb{1} \\ (\mathbf{f}^t \mathbf{h} - \mathbf{h}^t \mathbf{f}) & = & 0 \end{cases} \tag{69}$$

The $n_V \times n_V$ blocks $\mathbf{f}$, $\mathbf{h}$ of $\mathbf{V}$ can be decomposed with respect to the isotropy group $H_{Aut} \times H_{matter}$ as:

$$\begin{aligned} \mathbf{f} &= (f^\Lambda_{AB}, \overline{f}^\Lambda_{\overline{I}}) \equiv (\mathbf{f}^\Lambda{}_M)\,, \\ \mathbf{h} &= (h_{\Lambda AB}, \overline{h}_{\Lambda \overline{I}}) \equiv (\mathbf{h}_{\Lambda\, M})\,, \end{aligned} \tag{70}$$

where AB are indices in the antisymmetric representation of $H_{Aut} = SU(N) \times U(1)$, I is an index of the fundamental representation of H_{matter} and $M = (AB, \overline{I})$. Upper $SU(N)$ indices label objects in the complex conjugate representation of $SU(N)$: $(f^\Lambda_{AB})^* = \overline{f}^{\Lambda AB}$, $(f^\Lambda_I)^* = \overline{f}^\Lambda_{\overline{I}} = \overline{f}^{\Lambda\, I}$ etc...

Let us remark that, in order to make contact with the notation used for the $N = 2$ case, in the definition (70) some of the entries ($\overline{f}^\Lambda_{\overline{I}}$ and $\overline{h}_{\Lambda \overline{I}}$) have been written as complex conjugates of other quantities (f^Λ_I and $h_{\Lambda I}$ respectively). In this way, f^Λ_{AB} and f^Λ_I are characterized by having Kähler weight of the same sign. Indeed, for all the matter coupled theories ($N = 2, 3, 4$) we have, as a general feature, that the entries of the blocks $\mathbf{f}$ and $\mathbf{h}$ carrying H_{matter} indices have a Kähler weight with an opposite sign with respect to the corresponding entries with H_{Aut} indices. This may be seen from the supersymmetry transformation rules of the supergravity fields, in virtue the fact that gravitinos and gauginos with the same chirality have opposite Kähler weight. We note however that this notation differs from the one in previous papers, where the upper and lower parts of the symplectic section were defined instead as $(f^\Lambda_{AB}, f^\Lambda_I)$, $(h_{\Lambda\, AB}, h_{\Lambda\, I})$.

It is useful to introduce the following quantities

$$\begin{aligned} \mathbf{V}_M &= (V_{AB}, \overline{V}_{\overline{I}})\,, \quad \text{where:} \\ V_{AB} &\equiv (f^\Lambda_{AB}, h_{\Lambda AB}) \;\; ; \;\; V_I \equiv (f^\Lambda_I,\, h_{\Lambda I})\,. \end{aligned} \tag{71}$$

The vectors $\mathbf{V}_M$ are (complex) symplectic sections of a $Sp(2n_V, \mathbb{R})$ bundle over G/H. As anticipated in the previous subsection, this bundle is actually flat. The real embedding given by $L(\Phi)$ is appropriate for duality transformations of $F^\pm$ and their duals $G^\pm$, according to equations (46), while the complex embedding in the matrix $\mathbf{V}$ is appropriate in writing down the fermion transformation laws and supercovariant field strengths. The kinetic matrix $\mathcal{N}$, according to Gaillard–Zumino [39], can be written in terms of the subblocks $\mathbf{f}$, $\mathbf{h}$, and turns out to be:

$$\mathcal{N} = \mathbf{h}\,\mathbf{f}^{-1}, \qquad \mathcal{N} = \mathcal{N}^t\,, \tag{72}$$

transforming projectively under $Sp(2n_V,\mathbb{R})$ duality rotations as already shown in the previous section. By using (69)and (72) we find that

$$(\mathbf{f}^t)^{-1} = \mathrm{i}(\mathcal{N} - \overline{\mathcal{N}})\overline{\mathbf{f}}\,, \tag{73}$$

that is

$$\begin{aligned} (\mathbf{f}^{-1})^{AB}{}_\Lambda &= \mathrm{i}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}\overline{f}^{\Sigma\,AB}\,, & (74)\\ (\mathbf{f}^{-1})^{\overline{I}}{}_\Lambda &= \mathrm{i}(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}f^{\Sigma\,\overline{I}}\,. & (75) \end{aligned}$$

It can be shown [50] that the dressed graviphotons and matter self-dual field strengths appearing in the transformation law of gravitino (62), dilatino (63) and gaugino (64) can be constructed as a symplectic invariant using the $\mathbf{f}$ and $\mathbf{h}$ matrices, as follows:

$$\begin{aligned} T^-_{AB} &= -\mathrm{i}(\overline{\mathbf{f}}^{-1})_{AB\Lambda}F^{-\Lambda} = f^\Lambda_{AB}\,(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}\,F^{-\Sigma} = h_{\Lambda AB}\,F^{-\Lambda} - f^\Lambda_{AB}\,G^-_\Lambda\,,\\ \overline{T}^-_{\overline{I}} &= -\mathrm{i}(\overline{\mathbf{f}}^{-1})_{\overline{I}\Lambda}\,F^{-\Lambda} = \overline{f}^\Lambda_{\overline{I}}\,(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma}\,F^{-\Sigma} = \overline{h}_{\Lambda\overline{I}}\,F^{-\Lambda} - \overline{f}^\Lambda_{\overline{I}}\,G^-_\Lambda\,,\\ \overline{T}^{+AB} &= (T^-_{AB})^*\,,\\ T^+_I &= (\overline{T}^-_{\overline{I}})^*\,, \end{aligned} \tag{76}$$

(for $N > 4$, supersymmetry does not allow matter multiplets and $f^\Lambda_I = 0 = T_I$). To construct the dressed charges one integrates $T_{AB} = T^+_{AB} + T^-_{AB}$ and (for $N = 3,4$) $\overline{T}_{\overline{I}} = \overline{T}^+_{\overline{I}} + \overline{T}^-_{\overline{I}}$ on a large 2-sphere. For this purpose we note that

$$\begin{aligned} T^+_{AB} &= h_{\Lambda AB}\,F^{+\Lambda} - f^\Lambda_{AB}\,G^+_\Lambda = 0\,, & (77)\\ \overline{T}^+_{\overline{I}} &= \overline{h}_{\Lambda\overline{I}}\,F^{+\Lambda} - \overline{f}^\Lambda_{\overline{I}}\,G^+_\Lambda = 0\,, & (78) \end{aligned}$$

as a consequence of eqs. (72), (45). Therefore we can introduce the central and matter charges as the dressed charges obtained by integrating the 2-forms

T_{AB} and $T_{\overline{I}}$:

$$\begin{aligned}
Z_{AB} &= -\frac{1}{4\pi}\int_{S^2} T_{AB} = -\frac{1}{4\pi}\int_{S^2}(T^+_{AB} + T^-_{AB}) = -\frac{1}{4\pi}\int_{S^2} T^-_{AB} = \\
&= f^\Lambda_{AB}\, q_\Lambda - h_{\Lambda AB}\, p^\Lambda\,, \qquad (79)\\
\overline{Z}_{\overline{I}} &= -\frac{1}{4\pi}\int_{S^2} \overline{T}_{\overline{I}} = -\frac{1}{4\pi}\int_{S^2}(\overline{T}^+_{\overline{I}} + \overline{T}^-_{\overline{I}}) = -\frac{1}{4\pi}\int_{S^2} \overline{T}^-_{\overline{I}} = \\
&= \overline{f}^\Lambda_{\overline{I}}\, q_\Lambda - \overline{h}_{\Lambda\overline{I}}\, p^\Lambda \quad (N \le 4)\,,
\end{aligned}$$
(80)

where p^Λ and q_Λ were defined in (61) and the sections (f^Λ, h_Λ) on the right hand side now depend on the v.e.v.'s $\Phi_\infty \equiv \Phi(r = \infty)$ of the scalar fields Φ^r. We see that because of the electric-magnetic duality, the central and matter charges are given in this case by symplectic invariant expressions.

The scalar field dependent combinations of fields strengths appearing in the fermion supersymmetry transformation rules have a profound meaning and, as we are going to see in the following, they play a key role in the physics of extremal black holes. The integral of the graviphoton $T_{AB\mu\nu}$ gives the value of the central charge Z_{AB} of the supersymmetry algebra, while by integration of the matter field strengths $T_{I|\mu\nu}$ one obtains the so called matter charges Z_I.

We are now able to derive some differential relations among the central and matter charges using the Maurer–Cartan equations obeyed by the scalars through the embedded coset representative $\mathbf{V}$. Indeed, let $\Gamma = \mathbf{V}^{-1}d\mathbf{V}$ be the $Sp(2n_V, \mathbb{R})$ Lie algebra left invariant one form satisfying:

$$d\Gamma + \Gamma\wedge\Gamma = 0\,. \qquad (81)$$

In terms of $(\mathbf{f}, \mathbf{h})$, Γ has the following form:

$$\Gamma \equiv \mathbf{V}^{-1}d\mathbf{V} = \begin{pmatrix} \mathrm{i}(\mathbf{f}^\dagger d\mathbf{h} - \mathbf{h}^\dagger d\mathbf{f}) & \mathrm{i}(\mathbf{f}^\dagger d\overline{\mathbf{h}} - \mathbf{h}^\dagger d\overline{\mathbf{f}}) \\ -\mathrm{i}(\mathbf{f}^t d\mathbf{h} - \mathbf{h}^t d\mathbf{f}) & -\mathrm{i}(\mathbf{f}^t d\overline{\mathbf{h}} - \mathbf{h}^t d\overline{\mathbf{f}}) \end{pmatrix} \equiv \begin{pmatrix} \Omega^{(H)} & \overline{\mathcal{P}} \\ \mathcal{P} & \overline{\Omega}^{(H)} \end{pmatrix}, \qquad (82)$$

where the $n_V \times n_V$ sub-blocks $\Omega^{(H)}$ and $\mathcal{P}$ embed the H connection and the vielbein of G/H respectively. This identification follows from the Cartan decomposition of the $Sp(2n_V, \mathbb{R})$ Lie algebra.

From (66) and (82), we obtain the $(n_V \times n_V)$ matrix equation:

$$\begin{aligned}
D(\Omega)\mathbf{f} &= \overline{\mathbf{f}}\,\mathcal{P}\,,\\
D(\Omega)\mathbf{h} &= \overline{\mathbf{h}}\,\mathcal{P}\,,
\end{aligned} \qquad (83)$$

together with their complex conjugates. Explicitly, if we define the $H_{Aut} \times H_{matter}$-covariant derivative of the $\mathbf{V}_M$ vectors, introduced in (71), as:

$$D\mathbf{V}_M = d\mathbf{V}_M - \mathbf{V}_N\,\omega^N{}_M, \quad \omega = \begin{pmatrix} \omega^{AB}{}_{CD} & 0 \\ 0 & \omega^I{}_J \end{pmatrix}, \tag{84}$$

we have:

$$\Omega^{(H)} = \mathrm{i}[\mathbf{f}^\dagger(D\mathbf{h} + \mathbf{h}\omega) - \mathbf{h}^\dagger(D\mathbf{f} + \mathbf{f}\omega)] = \omega\mathbb{1}\,, \tag{85}$$

where we have used:

$$D\mathbf{h} = \overline{\mathcal{N}}D\mathbf{f}; \quad \mathbf{h} = \mathcal{N}\mathbf{f}\,, \tag{86}$$

which follow from (83) and the fundamental identity (69). Furthermore, using the same relations, the embedded vielbein $\mathcal{P}$ can be written as follows:

$$\mathcal{P} = -\mathrm{i}(\mathbf{f}^t D\mathbf{h} - \mathbf{h}^t D\mathbf{f}) = \mathrm{i}\mathbf{f}^t(\mathcal{N} - \overline{\mathcal{N}})D\mathbf{f}\,. \tag{87}$$

Using further the definition (70) we have:

$$\begin{aligned} D(\omega)f^\Lambda_{AB} &= f^\Lambda_I P^I_{AB} + \frac{1}{2}\overline{f}^{\Lambda CD}P_{ABCD}\,, \\ D(\omega)\overline{f}^\Lambda_{\overline{I}} &= \frac{1}{2}\overline{f}^{\Lambda AB}P_{AB\overline{I}} + f^{\Lambda\overline{J}}P_{\overline{J}\overline{I}}\,, \end{aligned} \tag{88}$$

where we have decomposed the embedded vielbein $\mathcal{P}$ as follows:

$$\mathcal{P} = \begin{pmatrix} P_{ABCD} & P_{AB\overline{J}} \\ P_{\overline{I}CD} & P_{\overline{I}\overline{J}} \end{pmatrix}, \tag{89}$$

the subblocks being related to the vielbein of G/H, written in terms of the indices of $H_{Aut} \times H_{matter}$. In particular, the component P_{ABCD} is completely antisymmetric in its indices. Note that, since $\mathbf{f}$ belongs to the unitary matrix $\mathbf{V}$, we have: $\overline{\mathbf{V}}^M = (f^\Lambda_{AB}, \overline{f}^\Lambda_{\overline{I}})^\star = (\overline{f}^{\Lambda AB}, f^{\Lambda\overline{I}})$. Obviously, the same differential relations that we wrote for $\mathbf{f}$ hold true for the dual matrix $\mathbf{h}$ as well.

Using the definition of the charges (79), (80) we then get the following differential relations among charges:

$$\begin{aligned} D(\omega)Z_{AB} &= Z_I P^I_{AB} + \frac{1}{2}\overline{Z}^{CD}P_{ABCD}\,, \\ D(\omega)\overline{Z}_{\overline{I}} &= \frac{1}{2}\overline{Z}^{AB}P_{AB\overline{I}} + Z^{\overline{J}}P_{\overline{I}\overline{J}}\,. \end{aligned} \tag{90}$$

Depending on the coset manifold, some of the subblocks of (89) can be actually zero. For example in $N=3$ the vielbein of $G/H = \frac{SU(3,n)}{SU(3)\times SU(n)\times U(1)}$ [52] is $P_{\overline{I}AB}$ (AB antisymmetric), $I=1,\cdots,n$; $A,B=1,2,3$ and it turns out that $P_{ABCD}=P_{\overline{I}\overline{J}}=0$.

In $N=4$, $G/H=\frac{SU(1,1)}{U(1)}\times\frac{O(6,n)}{O(6)\times O(n)}$ [53], and we have $P_{ABCD}=\epsilon_{ABCD}P$, $P_{\overline{I}\overline{J}}=P\delta_{IJ}$, where P is the Kählerian vielbein of $\frac{SU(1,1)}{U(1)}$, ($A,\cdots,D$ $SU(4)$ indices and I,J $O(n)$ indices) and P_{IAB} is the vielbein of $\frac{O(6,n)}{O(6)\times O(n)}$.

For $N>4$ (no matter indices) we have that $\mathcal{P}$ coincides with the vielbein P_{ABCD} of the relevant G/H.

For the purpose of comparison of the previous formalism with the $N=2$ supergravity case, where the σ-model is in general not a coset, it is interesting to note that, if the connection $\Omega^{(H)}$ and the vielbein $\mathcal{P}$ are regarded as data of G/H, then the Maurer–Cartan equations (88) can be interpreted as an integrable system of differential equations for the section (V_{AB}, $\overline{V}_{\overline{I}}$, $\overline{V}^{AB}$, $V^{\overline{I}}$) of the symplectic fiber bundle constructed over G/H. Namely the integrable system (84) that we explicitly write in the following equivalent matrix form

$$D\begin{pmatrix} V_{AB} \\ \overline{V}_{\overline{I}} \\ \overline{V}^{AB} \\ V^{\overline{I}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2}P_{ABCD} & P_{AB\overline{J}} \\ 0 & 0 & \frac{1}{2}P_{\overline{I}CD} & P_{\overline{I}\overline{J}} \\ \frac{1}{2}\overline{P}^{ABCD} & \overline{P}^{AB\overline{J}} & 0 & 0 \\ \frac{1}{2}\overline{P}^{\overline{I}CD} & \overline{P}^{\overline{I}\overline{J}} & 0 & 0 \end{pmatrix} \begin{pmatrix} V_{CD} \\ \overline{V}_{\overline{J}} \\ \overline{V}^{CD} \\ V^{\overline{J}} \end{pmatrix}, \tag{91}$$

has $2n_V$ solutions given by $\mathbf{V}_M$. The integrability condition (81) means that Γ is a flat connection of the symplectic bundle. In terms of the geometry of G/H this in turn implies that the $\mathbb{H}$-curvature associated to the connection $\Omega^{(H)}$ (and hence, since the manifold is a symmetric space, also the Riemannian curvature) is constant, being proportional to the wedge product of two vielbein.

Furthermore, besides the differential relations (90) the charges also satisfy sum rules.

The sum rule has the following form:

$$\frac{1}{2}Z_{AB}\overline{Z}^{AB}+Z_I\overline{Z}^I=-\frac{1}{2}Q^t\mathcal{M}(\mathcal{N})Q\,, \tag{92}$$

where $\mathbb{C}$ is the symplectic metric while $\mathcal{M}(\mathcal{N})$ and Q are:

$$\mathcal{M}(\mathcal{N}) \;=\; \begin{pmatrix} \mathbb{1} & -\mathrm{Re}\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix}\cdot\begin{pmatrix} \mathrm{Im}\mathcal{N} & 0 \\ 0 & \mathrm{Im}\mathcal{N}^{-1} \end{pmatrix}\cdot\begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{N} & \mathbb{1} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}\mathrm{Im}\mathcal{N}^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}\mathrm{Im}\mathcal{N}^{-1} \\ -\mathrm{Im}\mathcal{N}^{-1}\mathrm{Re}\mathcal{N} & \mathrm{Im}\mathcal{N}^{-1} \end{pmatrix} = \mathbb{C}\,\mathbf{V}\,\mathbf{V}^\dagger\,\mathbb{C}\,, \tag{93}$$

and

$$Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} . \tag{94}$$

This result is obtained from the fundamental identities (69) and from the definition of $\mathbf{V}$ and of the kinetic matrix given in (66) and (72). Indeed one can verify that [50, 54]:

$$\begin{aligned} \mathbf{f}\,\mathbf{f}^\dagger &= -\mathrm{i}\left(\mathcal{N} - \overline{\mathcal{N}}\right)^{-1} , \\ \mathbf{h}\,\mathbf{h}^\dagger &= -\mathrm{i}\left(\overline{\mathcal{N}}^{-1} - \mathcal{N}^{-1}\right)^{-1} \equiv -\mathrm{i}\mathcal{N}\left(\mathcal{N} - \overline{\mathcal{N}}\right)^{-1}\overline{\mathcal{N}}\,, \\ \mathbf{h}\,\mathbf{f}^\dagger &= \mathcal{N}\mathbf{f}\,\mathbf{f}^\dagger\,, \\ \mathbf{f}\,\mathbf{h}^\dagger &= \mathbf{f}\,\mathbf{f}^\dagger\overline{\mathcal{N}}\,, \end{aligned} \tag{95}$$

so that, using the explicit expression for the charges in eqs. (79) and (80), eq. (92) is easily retrieved.

In the following, studying the applications of these formulas to extremal black holes, other relations coming from the same identities listed above will also be useful, in particular:

$$\begin{aligned} \frac{1}{2}\left(\mathcal{M} + \mathrm{i}\,\mathbb{C}\right) &= \begin{pmatrix} -\mathbf{h}\,\mathbf{h}^\dagger & \mathbf{h}\,\mathbf{f}^\dagger \\ \mathbf{f}\,\mathbf{h}^\dagger & -\mathbf{f}\,\mathbf{f}^\dagger \end{pmatrix} = \frac{1}{2}\mathbb{C}\,\mathbf{V}\left(\mathbb{1} + \eta\right)\mathbf{V}^\dagger\,\mathbb{C} = \\ &= -(\mathbb{C}\,\mathbf{V})_M\,(\mathbb{C}\,\overline{\mathbf{V}})^M\,, \end{aligned} \tag{96}$$

$$\frac{1}{2}\left(\mathcal{M} + \mathrm{i}\mathbb{C}\right)\,\mathbf{V}_M = \mathrm{i}\,\mathbb{C}\,\mathbf{V}_M\,, \tag{97}$$

$$\frac{1}{2}\left(\mathcal{M} - \mathrm{i}\mathbb{C}\right)\,\mathbf{V}_M = 0\,, \tag{98}$$

$$\mathcal{M}\,Q = \mathbb{C}\,\mathbf{V}\,\mathbf{V}^\dagger\,\mathbb{C}\,Q = -2\,\mathrm{Re}\left(\mathbb{C}\,\mathbf{V}_M\,< Q, \overline{\mathbf{V}}^M >\right) , \tag{99}$$

$$\mathbb{C}\,Q = -\mathrm{i}\,\mathbb{C}\,\mathbf{V}\,\eta\,\mathbf{V}^\dagger\mathbb{C}\,Q = -2\,\mathrm{Im}\left(\mathbb{C}\,\mathbf{V}_M\,< Q, \overline{\mathbf{V}}^M >\right) \tag{100}$$

The symplectic scalar product appearing in (99), (100) is defined as:

$$< V, W > \;\equiv\; V^t\,\mathbb{C}\,W\,, \tag{101}$$

moreover $\overline{\mathbf{V}}^M = (\mathbf{V}_M)^*$. Using eqs. (71), (79) and (80) we can use the following short-hand notation for the central charge vector:

$$Z_M = (Z_{AB}, \overline{Z}_{\overline{I}}) =< Q, \mathbf{V}_M > . \tag{102}$$

From the above expression and from eq. (96) equation (92) follows.

3.3 The $N = 2$ theory

The formalism we have developed so far for the $D = 4$, $N > 2$ theories is completely determined by the embedding of the coset representative of G/H in $Sp(2n, \mathbb{R})$ and by the embedded Maurer–Cartan equations (88). We want now to show that this formalism, and in particular the identities (69), the differential relations among charges (90) and the sum rules (92) of $N = 2$ matter-coupled supergravity [55],[56] can be obtained in a way completely analogous to the $N > 2$ cases discussed in the previous subsection, where the σ-model was a coset space. This follows essentially from the fact that, though the scalar manifold $\mathcal{M}_{scalar}$ of the $N = 2$ theory is not in general a coset manifold, nevertheless it has a symplectic structure identical to the $N > 2$ theories, as a consequence of the Gaillard–Zumino duality.

In the case of $N = 2$ supergravity the requirements imposed by supersymmetry on the scalar manifold $\mathcal{M}_{scalar}$ of the theory dictate that it should be the following direct product: $\mathcal{M}_{scalar} = \mathcal{M}^{SK} \otimes \mathcal{M}^{Q}$ where $\mathcal{M}^{SK}$ is a special Kähler manifold of complex dimension n and $\mathcal{M}^{Q}$ a quaternionic manifold of real dimension $4n_H$. Note that n and n_H are respectively the number of vector multiplets and hypermultiplets contained in the theory. The direct product structure imposed by supersymmetry precisely reflects the fact that the quaternionic and special Kähler scalars belong to different supermultiplets. In the construction of extremal black holes it turns out that the hyperscalars are spectators playing no dynamical role. Hence we do not discuss here the hypermultiplets any further and we confine our attention to an $N = 2$ supergravity where the graviton multiplet, containing besides the graviton $g_{\mu\nu}$ also a graviphoton A^0_μ, is coupled to n vector multiplets. Such a theory has an action of type (32) where the number of gauge fields is $n_V = 1 + n$ and the number of (real) scalar fields is $m = 2\,n$. We shall use capital Greek indices to label the vector fields: $\Lambda, \Sigma \ldots = 0, \ldots, n$. To make the action (32) fully explicit, we need to discuss the geometry of the manifold $\mathcal{M}^{SK}$ spanned by the vector-multiplet scalars, namely special

Kähler geometry. Since $\mathcal{M}^{SK}$ is in particular a complex manifold, we shall describe the corresponding scalars as complex fields: z^i, $\overline{z}^{\overline{\imath}}$, $i, \overline{\imath} = 1, \ldots, n$. We refer to [57] for a detailed analysis. A special Kähler manifold $\mathcal{M}^{SK}$ is a Kähler–Hodge manifold endowed with an extra symplectic structure. A Kähler manifold $\mathcal{M}$ is a Hodge manifold if and only if there exists a $U(1)$ bundle $\mathcal{L} \longrightarrow \mathcal{M}$ such that its first Chern class equals the cohomology class of the Kähler 2-form K:

$$c_1(\mathcal{L}) \;=\; [\,K\,]\;. \tag{103}$$

In local terms we can write

$$K \;=\; \mathrm{i}\, g_{i\overline{\jmath}}\, dz^i \,\wedge\, d\overline{z}^{\overline{\jmath}}\,, \tag{104}$$

where z^i are n holomorphic coordinates on $\mathcal{M}^{SK}$ and $g_{i\overline{\jmath}}$ its metric.
In this case the $U(1)$ Kähler connection is given by:

$$\mathcal{Q} = -\frac{\mathrm{i}}{2}\left(\partial_i\mathcal{K}dz^i - \partial_{\overline{\imath}}\mathcal{K}d\overline{z}^{\overline{\imath}}\right)\,, \tag{105}$$

where $\mathcal{K}$ is the Kähler potential, so that $K = d\mathcal{Q}$.

Let now $\Phi(z,\overline{z})$ be a section of the $U(1)$ bundle of weight p. By definition its covariant derivative is

$$D\Phi = (d + ip\mathcal{Q})\Phi\,, \tag{106}$$

or, in components,

$$D_i\Phi \;=\; (\partial_i + \tfrac{1}{2}p\partial_i\mathcal{K})\Phi \quad;\quad D_{\overline{\imath}}\Phi \;=\; (\partial_{\overline{\imath}} - \tfrac{1}{2}p\partial_{\overline{\imath}}\mathcal{K})\Phi\;. \tag{107}$$

A covariantly holomorphic section is defined by the equation: $D_{\overline{\imath}}\Phi = 0$. Setting:

$$\widetilde{\Phi} = e^{-p\mathcal{K}/2}\Phi\,, \tag{108}$$

we get:

$$D_i\widetilde{\Phi} \;=\; (\partial_i + p\partial_i\mathcal{K})\widetilde{\Phi} \quad;\quad D_{\overline{\imath}}\widetilde{\Phi} \;=\; \partial_{\overline{\imath}}\widetilde{\Phi}\;, \tag{109}$$

so that under this map covariantly holomorphic sections Φ become truly holomorphic sections.

There are several equivalent ways of defining what a special Kähler manifold is. An intrinsic definition is the following. A special Kähler manifold can be given by constructing a $2n+2$-dimensional flat symplectic bundle over the Kähler–Hodge manifold whose generic sections (with weight $p=1$)

$$V = (f^{\Lambda}, h_{\Lambda})\,, \tag{110}$$

are covariantly holomorphic

$$D_{\bar{\imath}} V = (\partial_{\bar{\imath}} - \frac{1}{2}\partial_{\bar{\imath}}\mathcal{K})V = 0\,, \tag{111}$$

and satisfy the further condition

$$\mathrm{i} < V, \overline{V} >= \mathrm{i}(\overline{f}^{\Lambda} h_{\Lambda} - \overline{h}_{\Lambda} f^{\Lambda}) = 1\ , \tag{112}$$

where the $<\,,\,>$ product was defined in (101). Defining

$$V_i = D_i V = (f^{\Lambda}_i, h_{\Lambda i}), \tag{113}$$

and introducing a symmetric three-tensor C_{ijk} by

$$D_i V_j = i C_{ijk} g^{k\bar{k}} \overline{V}_{\bar{k}}\ , \tag{114}$$

the set of differential equations

$$\begin{aligned} D_i V &= V_i\,, \\ D_i V_j &= i C_{ijk} g^{k\bar{k}} \overline{V}_{\bar{k}}\,, \\ D_i V_{\bar{\jmath}} &= g_{i\bar{\jmath}} \overline{V}\,, \\ D_i \overline{V} &= 0\,, \end{aligned} \tag{115}$$

defines a symplectic connection. Requiring that the differential system (115) is integrable is equivalent to requiring that the symplectic connection is flat. Since the integrability condition of (115) gives constraints on the base Kähler–Hodge manifold, we define special-Kähler a manifold whose associated symplectic connection is flat. At the end of this section we will give the restrictions on the manifold imposed by the flatness of the connection.

It must be noted that, for special Kähler manifolds, the Kähler potential can be computed as a symplectic invariant from eq. (112). Indeed, introducing also the holomorphic sections

$$\begin{aligned} \Omega &= e^{-\mathcal{K}/2} V = e^{-\mathcal{K}/2}(f^{\Lambda}, h_{\Lambda}) = (X^{\Lambda}, F_{\Lambda})\,, \\ \partial_{\bar{\imath}}\Omega &= 0\,, \end{aligned} \tag{116}$$

eq. (112) gives

$$\mathcal{K} = -\ln \mathrm{i} < \Omega, \overline{\Omega} >= -\ln \mathrm{i}(\overline{X}^{\Lambda} F_{\Lambda} - X^{\Lambda} \overline{F}_{\Lambda}) \ . \tag{117}$$

If we introduce the complex symmetric $(n+1)\times(n+1)$ matrix $\mathcal{N}_{\Lambda\Sigma}$ defined through the relations

$$h_{\Lambda} = \mathcal{N}_{\Lambda\Sigma} f^{\Sigma} \quad , \quad h_{\Lambda \bar{\imath}} = \mathcal{N}_{\Lambda\Sigma} \overline{f}^{\Sigma}_{\bar{\imath}} \ , \tag{118}$$

then we have:

$$< V, \overline{V} >= \left(\mathcal{N} - \overline{\mathcal{N}}\right)_{\Lambda\Sigma} f^{\Lambda} \overline{f}^{\Sigma} = -\mathrm{i} \, , \tag{119}$$

so that

$$\mathcal{K} = -\ln[\mathrm{i}(\overline{X}^{\Lambda} \left(\mathcal{N} - \overline{\mathcal{N}}\right)_{\Lambda\Sigma} X^{\Sigma})] \, , \tag{120}$$

and

$$\begin{aligned} g_{i\bar{\jmath}} &= -\mathrm{i} < V_i, V_{\bar{\jmath}} >= -2 f^{\Lambda}_i \mathrm{Im} \mathcal{N}_{\Lambda\Sigma} \overline{f}^{\Sigma}_{\bar{\jmath}} \ , && (121) \\ C_{ijk} &= < D_i V_j, V_k >= 2\mathrm{i} \mathrm{Im} \mathcal{N}_{\Lambda\Sigma} f^{\Lambda}_i D_j f^{\Sigma}_k \ . && (122) \end{aligned}$$

We shall also use the following identity which follows from the previous ones

$$f^{\Lambda}_i \, g^{i\bar{\jmath}} \, \overline{f}^{\Sigma}_{\bar{\jmath}} \;=\; -\frac{1}{2} \, (\mathrm{Im} \mathcal{N})^{-1\,\Lambda\Sigma} - \overline{L}^{\Lambda} \, L^{\Sigma} \, . \tag{123}$$

The matrix $\mathcal{N}_{\Lambda\Sigma}$ turns out to be the matrix appearing in the kinetic lagrangian of the vectors in $N = 2$ supergravity. Under coordinate transformations, the sections Ω transform as

$$\widetilde{\Omega} = e^{-f_S(z)} \mathcal{S} \Omega \ , \tag{124}$$

where $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an element of $Sp(2n_V, \mathbb{R})$ and the factor $e^{-f_S(z)}$ is a $U(1)$ Kähler transformation. We also note that, from the definition of $\mathcal{N}$, eq. (118):

$$\widetilde{\mathcal{N}}(\widetilde{X}, \widetilde{F}) = [C + D\mathcal{N}(X,F)][A + B\mathcal{N}(X,F)]^{-1} \ . \tag{125}$$

We can now define a matrix $\mathbf{V}$ as in (66) satisfying the relations (67), in terms of the quantities $(f^{\Lambda}$, $\overline{f}^{\Lambda}{}_{\bar{\imath}}$, h_{Λ}, $\overline{h}_{\Lambda\bar{\imath}})$ introduced in (110) and (113). In order to identify the blocks $\mathbf{f}$ and $\mathbf{h}$ of $\mathbf{V}$ in (66), we note that in $N = 2$

theories $H_{Aut} = SU(2)\times U(1)$, so that the f^Λ_{AB} and $h_{\Lambda\,AB}$ entries in (70) are actually $SU(2)$-singlets. We can therefore consistently write $\mathbf{f}$ and $\mathbf{h}$ as the following $n_V \times n_V$ matrices:

$$\mathbf{f} \equiv \left(f^\Lambda_{AB}, \overline{f}^\Lambda_{\overline{I}}\right) ; \quad \mathbf{h} \equiv \left(h_{\Lambda\,AB}, \overline{h}_{\Lambda\,\overline{I}}\right) , \tag{126}$$

where f^Λ_{AB}, $h_{\Lambda\,AB}$ and f^Λ_I, $h_{\Lambda\,I}$ are defined as follows:

$$\begin{aligned} f^\Lambda_{AB} &= f^\Lambda\,\epsilon_{AB} \ ; \ h_{\Lambda\,AB} = h_\Lambda\,\epsilon_{AB} \,, \\ f^\Lambda_I &= f^\Lambda_i\, P^i_I \ ; \ h_{\Lambda\,I} = h_{\Lambda\,i}\, P^i_I \,, \end{aligned} \tag{127}$$

P^i_I, $P^{\bar\imath}_{\overline{I}}$ being the inverse of the Kählerian vielbein P^I_i, $\overline{P}^{\overline{I}}_{\bar\imath}$ defined by the relation:

$$g_{i\bar\jmath} = P^I_i\,\overline{P}^{\overline{J}}_{\bar\jmath}\,\eta_{I\overline{J}} \,, \tag{128}$$

and $\eta_{I\overline{J}}$ is the flat metric. From the definition (126) and the properties (119), (121) it is straightforward to verify that the $\mathbf{f}$ and $\mathbf{h}$ blocks satisfy the relations (69), or equivalently that the matrix $\mathbf{V}$ satisfies the conditions (67). The relations (69) therefore encode the set of algebraic relations of special geometry.

Let us now consider the analogous of the embedded Maurer–Cartan equations of G/H. We introduce, as before, the matrix one-form $\Gamma = \mathbf{V}^{-1}d\mathbf{V}$ satisfying the relation $d\Gamma + \Gamma\wedge\Gamma = 0$. We further introduce the covariant derivative of the symplectic section $(f^\Lambda, \overline{f}^\Lambda_{\overline{I}}, \overline{f}^\Lambda, f^\Lambda_I)$ with respect to the $U(1)$-Kähler connection $\mathcal{Q}$ and the spin connection ω^{IJ} of $\mathcal{M}^{SK}$:

$$\begin{aligned} &D(f^\Lambda, \overline{f}^\Lambda_{\overline{I}}, \overline{f}^\Lambda, f^\Lambda_I) = \\ &d(f^\Lambda, \overline{f}^\Lambda_{\overline{I}}, \overline{f}^\Lambda, f^\Lambda_I) - (f^\Lambda, \overline{f}^\Lambda_{\overline{J}}, \overline{f}^\Lambda, f^\Lambda_J) \begin{pmatrix} -\mathrm{i}\mathcal{Q} & 0 & 0 & 0 \\ 0 & \mathrm{i}\mathcal{Q}\delta^{\overline{J}}_{\overline{I}} + \omega^{\overline{J}}{}_{\overline{I}} & 0 & 0 \\ 0 & 0 & \mathrm{i}\mathcal{Q} & 0 \\ 0 & 0 & 0 & -\mathrm{i}\mathcal{Q}\delta^J_I + \omega^J{}_I \end{pmatrix} \end{aligned} \tag{129}$$

the Kähler weight of $(f^\Lambda,\, f^\Lambda_I)$ and $(\overline{f}^\Lambda,\, \overline{f}^\Lambda_{\overline{I}})$ being $p = 1$ and $p = -1$ respectively. Using the same decomposition as in equation (82) and eq.s (84), (85) we have in the $N = 2$ case:

$$\begin{aligned} \Gamma &= \begin{pmatrix} \Omega & \overline{\mathcal{P}} \\ \mathcal{P} & \overline{\Omega} \end{pmatrix}, \\ \Omega &= \omega = \begin{pmatrix} -\mathrm{i}\mathcal{Q} & 0 \\ 0 & \mathrm{i}\mathcal{Q}\delta^I_J + \overline{\omega}^I{}_J \end{pmatrix} . \end{aligned} \tag{130}$$

For the subblock $\mathcal{P}$ we obtain:

$$\mathcal{P} = -\mathrm{i}(f^t Dh - h^t Df) = \mathrm{i} f^t(\mathcal{N} - \overline{\mathcal{N}})Df = \begin{pmatrix} 0 & P_{\overline{I}} \\ P^J & P^J_{\overline{I}} \end{pmatrix} , \tag{131}$$

where $P^J \equiv \eta^{J\overline{I}} P_{\overline{I}}$ is the $(1,0)$-form Kählerian vielbein while

$$P^J_{\overline{I}} \equiv \mathrm{i} \left(f^t(\mathcal{N} - \overline{\mathcal{N}})Df \right)^J_{\ \overline{I}} \tag{132}$$

is a one-form which in general, in the cases where the manifold is not a coset, represents a new geometric quantity on $\mathcal{M}^{SK}$. Note that we get zero in the first entry of equation (131) by virtue of the fact that the identity (69) implies $f^\Lambda(\mathcal{N} - \overline{\mathcal{N}})_{\Lambda\Sigma} f^\Sigma_I = 0$ and that f^Λ is covariantly holomorphic. If Ω and $\mathcal{P}$ are considered as data on $\mathcal{M}^{SK}$ then we may interpret $\Gamma = V^{-1}dV$ as an integrable system of differential equations, namely:

$$D(V, \overline{V}_{\overline{I}}, \overline{V}, V_I) = (V, \overline{V}_{\overline{J}}, \overline{V}, V_J) \begin{pmatrix} 0 & 0 & 0 & \overline{P}_I \\ 0 & 0 & \overline{P}^{\overline{J}} & \overline{P}^{\overline{J}}_{\ I} \\ 0 & P_{\overline{I}} & 0 & 0 \\ P^J & P^J_{\overline{I}} & 0 & 0 \end{pmatrix} , \tag{133}$$

where the flat Kähler indices $I, \overline{I}, \cdots$ are raised and lowered with the flat Kähler metric $\eta_{I\overline{J}}$. Note that the equation (133) coincides with the set of relations (115) if we trade world indices $i, \bar{\imath}$ with flat indices $I, \overline{I}$, provided we also identify:

$$\overline{P}^{\overline{J}}_{\ I} = \overline{P}^{\overline{J}}_{\ Ik} dz^k = P^{\overline{J},i} P_I^{\ j} C_{ijk} dz^k . \tag{134}$$

Then, the integrability condition $d\Gamma + \Gamma \wedge \Gamma = 0$ is equivalent to the flatness of the special Kähler symplectic connection and it gives the following three constraints on the Kähler base manifold:

$$\begin{aligned} d(\mathrm{i}\mathcal{Q}) + \overline{P}_I \wedge P^I &= 0 \to \partial_{\bar{\jmath}} \partial_i \mathcal{K} = P^I_{\ ,i} \overline{P}_{I,\bar{\jmath}} = g_{i\bar{\jmath}} \,, && (135) \\ (d\omega + \omega \wedge \omega)^{\overline{J}}_{\ \overline{I}} &= P_{\overline{I}} \wedge \overline{P}^{\overline{J}} - \mathrm{i} d\mathcal{Q} \delta^{\overline{J}}_{\overline{I}} - \overline{P}^{\overline{J}}_{\ L} \wedge P^L_{\overline{I}} \,, && (136) \\ DP^J_{\overline{I}} &= 0 \,, && (137) \\ \overline{P}_J \wedge P^J_{\overline{I}} &= 0 \,. && (138) \end{aligned}$$

Equation (135) implies that $\mathcal{M}^{SK}$ is a Kähler–Hodge manifold. Equation (136), written with holomorphic and antiholomorphic curved indices, gives:

$$R_{\bar{\imath}j\bar{k}l} = g_{\bar{\imath}l} g_{j\bar{k}} + g_{\bar{k}l} g_{\bar{\imath}j} - \overline{C}_{\bar{\imath}\bar{k}\overline{m}} C_{jln} g^{\overline{m}n} \,, \tag{139}$$

which is the usual constraint on the Riemann tensor of the special geometry. The further special geometry constraints on the three tensor C_{ijk} are then consequences of equations (137), (138), which imply:

$$\begin{aligned} D_{[l}C_{i]jk} &= 0\,, \\ D_{\bar{l}}C_{ijk} &= 0\,. \end{aligned} \qquad (140)$$

In particular, the first of eq. (140) also implies that C_{ijk} is a completely symmetric tensor.

In summary, we have seen that the $N = 2$ theory and the higher N theories have essentially the same symplectic structure, the only difference being that since the scalar manifold of $N = 2$ is not in general a coset manifold the symplectic structure allows the presence of a new geometric quantity which physically corresponds to the anomalous magnetic moments of the $N = 2$ theory. It goes without saying that, when $\mathcal{M}^{SK}$ is itself a coset manifold [58], then the anomalous magnetic moments C_{ijk} must be expressible in terms of the vielbein of G/H.

To complete the analogy between the $N = 2$ theory and the higher N theories in $D = 4$, we also give for completeness the supersymmetry transformation laws, the central and matter charges, the differential relations among them and the sum rules.

The transformation laws for the chiral gravitino ψ_A and gaugino λ^{iA} fields are:

$$\delta\psi_{A\mu} = \nabla_\mu\, \epsilon_A \; + \epsilon_{AB} T_{\mu\nu}\gamma^\nu \epsilon^B + \cdots\,, \qquad (141)$$

$$\delta\lambda^{iA} = \mathrm{i}\partial_\mu z^i \gamma^\mu \epsilon^A + \frac{\mathrm{i}}{2}\overline{T}_{\bar{\jmath}\mu\nu}\gamma^{\mu\nu} g^{i\bar{\jmath}}\epsilon^{AB}\epsilon_B + \cdots\,, \qquad (142)$$

where:

$$T \equiv h_\Lambda F^\Lambda - f^\Lambda G_\Lambda\,, \qquad (143)$$

$$\overline{T}_{\bar{\imath}} \equiv \overline{T}_{\bar{I}}\overline{P}^{\bar{I}}_{\bar{\imath}}\,, \text{ with: } \overline{T}_{\bar{I}} \equiv \overline{h}_{\Lambda\bar{I}}F^\Lambda - \overline{f}^\Lambda_{\bar{I}}G_\Lambda\,, \qquad (144)$$

are respectively the graviphoton and the matter-vectors, and the position of the $SU(2)$ automorphism index A (A,B=1,2) is related to chirality (namely (ψ_A, λ^{iA}) are chiral, $(\psi^A, \lambda^{\bar{\imath}}_A)$ antichiral). In principle only the (anti) self-dual part of F and G should appear in the transformation laws of the (anti)chiral fermi fields; however, exactly as in eqs. (77),(78) for $N > 2$ theories, from equations (115) it follows that :

$$\begin{aligned} T^+ &= h_\Lambda F^{+\Lambda} - f^\Lambda G^+_\Lambda = 0\,, \\ T^-_I &= h_{\Lambda\, I}\, F^{-\Lambda} - f^\Lambda_I\, G^-_\Lambda = 0\,, \end{aligned} \qquad (145)$$

so that $T = T^-$ and $T_I = T_I^+$ (i.e. $\overline{T} = \overline{T}^+$, $\overline{T}_{\overline{I}} = \overline{T}_{\overline{I}}^-$). Note that both the graviphoton and the matter vectors are symplectic invariant according to the fact that the fermions do not transform under the duality group (except for a possible R-symmetry phase). To define the physical charges let us recall the definition of the moduli-independent charges in (61). The central charges and the matter charges are now defined as the integrals over S^2 of the physical graviphoton and matter vectors:

$$\begin{aligned} Z &= -\frac{1}{4\pi}\int_{S^2} T = -\frac{1}{4\pi}\int_{S^2}(h_\Lambda F^\Lambda - f^\Lambda G_\Lambda) = f^\Lambda(z,\overline{z})q_\Lambda - h_\Lambda(z,\overline{z})p^\Lambda\,, \\ Z_I &= -\frac{1}{4\pi}\int_{S^2} T_I = -\frac{1}{4\pi}\int_{S^2}(h_{\Lambda\,I}\, F^\Lambda - f_I^\Lambda G_\Lambda) = f_I^\Lambda(z,\overline{z})q_\Lambda - h_{\Lambda\,I}(z,\overline{z})p^\Lambda\,. \end{aligned} \tag{146}$$

where $z^i, \overline{z}^{\overline{\imath}}$ denote the v.e.v. of the moduli fields in a given background. In virtue of eq. (115) we get immediately:

$$Z_I = P_I^i\, Z_i \;\; ; \quad Z_i \equiv D_i Z\,. \tag{147}$$

As a consequence of the symplectic structure, one can derive two sum rules for Z and Z_I:

$$|Z|^2 + |Z_I|^2 \equiv |Z|^2 + Z_i g^{i\overline{\jmath}}\overline{Z}_{\overline{\jmath}} = -\frac{1}{2}Q^t \mathcal{M} Q \tag{148}$$

where the symmetric matrix $\mathcal{M}$ was defined in (93) and Q is the symplectic vector of electric and magnetic charges defined in (94).

Equation (148) is obtained by using exactly the same procedure as in (92).

4 Supersymmetric black holes: General discussion

We are going to study in this section the peculiarities of extremal black holes that are solutions of extended supergravity theories.

As anticipated in the introduction, for black-hole configurations that are particular bosonic backgrounds of N-extended locally supersymmetric theories, the cosmic censorship conjecture (expressing the request that the space-time singularities are always hidden by event horizons) finds a simple and natural understanding. For the Reissner-Nordstrom black holes this is codified in the bound (4) on the mass M and charge Q of the solution, that we recall here

$$M \geq |Q|. \tag{149}$$

In extended supersymmetric theories this bound is just a consequence of the supersymmetry algebra (21), as a consequence of the fact that

$$\left\{Q^{(\pm)}_{am}\,,\,Q^{\dagger(\pm)}_{am}\right\} \geq 0 \tag{150}$$

so that the cosmic censorship conjecture is always verified.

Another general property of extremal black holes, that will be surveyed in section 5, is encoded in the so-called no-hair theorem. It states that the end point of the gravitational collapse of a black hole is independent of the initial conditions. Then, if one tries to perturb an extremal black hole with whatever additional hair (some slight mass anisotropy, or a long-range field, like a scalar) all these features disappear near the horizon, except for those associated with the conserved quantities of general relativity, namely, for a non-rotating black hole, its mass and charge. When the black hole is embedded in an N-extended supergravity theory, the solution depends in general also on scalar fields. In this case, the electric charge Q has to be replaced by the central charge appearing in the supersymmetry algebra (which is dressed with the expectation value of scalar fields). The black-hole metric takes a generalized form with respect to the Reissner–Nordström one. However, for the extremal case the event horizon looses all information about the scalar 'hair'. As for the Reissner–Nordström case, the near-horizon geometry is still described by a conformally flat, Bertotti–Robinson-type geometry, with a mass parameter $M_{\text{B-R}}$ which only depends on the distribution of charges and not on the scalar fields. As will be discussed extensively in section 5, this follows from the fact that the differential equations on the metric and scalars fields of the extremal black hole (200), (201) are solved under the condition that the horizon be an attractor point [2] (see equation (207)). Scalar fields, independently of their boundary conditions at spatial infinity, approaching the horizon flow to a fixed point given by a certain ratio of electric and magnetic charges. Since the dominant contribution to the black-hole entropy is given (at least for large black holes) by the area/entropy Bekenstein–Hawking relation (1), it follows that the entropy of extremal black holes is a topological quantity fixed in terms of the quantized electric and magnetic charges while it does not depend on continuous parameters like scalars.

It will be shown that the request that the scalars Φ^r be regular at the fixed point (reached at the horizon $\tau \to \infty$) implies two important conditions which have both to be satisfied:

$$\begin{aligned} \left(\frac{d\Phi^r}{d\tau}\right)_{hor} &= 0 & (151) \\ \left(\frac{\partial V_{\text{B-H}}(\Phi)}{\partial \Phi_i}\right)_{hor} &= 0. & (152) \end{aligned}$$

where the function $V_{\text{B-H}}(\Phi, p, q)$, called the black-hole potential, will be introduced in (203).

Exploiting (152), a decade ago a general rule was given [22] for finding the values of fixed scalars, and then the Bekenstein–Hawking entropy, in $N = 2$ theories, through an *extremum principle* in moduli space. This follows from the observation that, when the scalar fields are evaluated at spatial infinity ($\tau = 0$), $V_{\text{B-H}}$ coincides with the squared ADM mass of the black hole. Then, since equation (152) does not depend explicitly on the radial variable τ (as the extremization is done with respect to the scalar fields at any given point) the expectation values Φ_∞ may be chosen as independent variables. Equation (152) is then reformulated as the statement that the fixed scalars Φ_{fix} are the ones, among all the possible expectation values taken by scalar fields, that extremize the ADM mass of the black hole in moduli space:

$$\Phi_{\text{fix}} : \quad \frac{\partial M_{ADM}(\Phi_\infty)}{\partial \Phi^r_\infty}\Big|_{\Phi_{\text{fix}}} = 0 \tag{153}$$

Correspondingly, the Bekenstein–Hawking entropy is given in terms of that extremum among the possible ADM masses (given by all possible boundary conditions that one can impose on scalars at spatial infinity), this last being identified with the Bertotti–Robinson mass $M_{\text{B-R}}$:

$$M_{\text{B-R}} \equiv M_{ADM}(\Phi_{\text{fix}}). \tag{154}$$

The solutions with the scalar fields constant and everywhere equal to the fixed value Φ_{fix} are called *double extremal black holes.*

The approach outlined above will prove to be a very useful computational tool to calculate the B-H entropy since, as will be explained in section 5, in extended supergravity the explicit dependence of $V_{\text{B-H}}$ on the moduli is given.

4.1 BPS extremal black holes

For the case of BPS extremal black holes, the extremum principle (153) may be explained by means of the Killing spinor equations near the horizon and these are encoded in some relations on the scalars moduli spaces, discussed in detail in section 3.2 and 3.3, which express the embedding of the scalar geometry in a symplectic representation of the U-duality group [59]. For definiteness, to present the argument we will refer, for the sequel of this subsection, to the case $N = 2$, which is the model originally considered in [21, 22].

The Killing-spinor equations expressing the existence of unbroken supersymmetries are obtained, for the gauginos in the $N = 2$ case [57], by setting to zero the r.h.s. of equation (142) that is, using flat indices:

$$\delta\lambda^I_A = P^I_{,i}\partial_\mu z^i \gamma^\mu \varepsilon_{AB}\epsilon^B + \overline{T}^I_{\mu\nu}\gamma^{\mu\nu}\epsilon_A + \cdots = 0. \tag{155}$$

As we will see in detail in the next subsection, approaching the black-hole horizon the scalars z^i reach their fixed values z_{fix} [7] so that

$$\partial_\mu z^i = 0 \tag{156}$$

and equation (155) is satisfied for

$$T_I = 0 \tag{157}$$

which implies, using integrated quantities:

$$Z_I = Z_i P^i_I = -\frac{1}{4\pi}\int_{S^2} T_I = \left(f^\Lambda_I q_\Lambda - h_{\Lambda I} p^\Lambda\right)|_{\text{fix}} = 0. \tag{158}$$

What we have found is that the Killing spinor equation imposes the vanishing of the matter charges near the horizon. Then, remembering eq. (147), near the horizon we have:

$$Z_I = D_I Z = 0 \tag{159}$$

where Z is the central charge appearing in the $N = 2$ supersymmetry algebra, so that:

$$\partial_i |Z| = 0. \tag{160}$$

[7] A point x_{fix} where the phase velocity is vanishing is named *fixed point* and represents the system in equilibrium $v(x_{\text{fix}}) = 0$ [22, 23]. The fixed point is said to be an attractor if $\lim_{t\to\infty} x(t) = x_{\text{fix}}$.

For an extremal BPS black hole ($|Z| = M_{ADM}$), (160) coincides with eq. (153) giving the fixed scalars $\Phi_{\rm fix} \equiv z_{\rm fix}$ at the horizon. We then see that the entropy of the black hole is related to the central charge, namely to the integral of the graviphoton field strength evaluated for very special values of the scalar fields z^i. These special values, the *fixed scalars* $z^i_{\rm fix}$, are functions solely of the electric and magnetic charges $\{q_\Sigma, p^\Lambda\}$ of the black hole and are attained by the scalars $z^i(r)$ at the black hole horizon $r = 0$.

Let us discuss in detail the explicit solution of the Killing spinor equation and the general properties of $N = 2$ BPS saturated black holes [21, 60, 61, 62]. As our analysis will reveal, these properties are completely encoded in the special Kähler geometric structure of the mother theory.

Let us consider a black-hole ansatz for the metric[8], restricting the attention to static, spherically symmetric solutions:

$$ds^2 = e^{2U(r)}\, dt^2 - e^{-2U(r)} G_{ij}(r)\, dx^i dx^j; \quad \left(r^2 = G_{ij} x^i x^j\right), \quad i,j = 1,2,3 \tag{161}$$

and for the vector field strengths:

$$F^\Lambda = \frac{p^\Lambda}{2r^3} \epsilon_{abc} x^a dx^b \wedge dx^c - \frac{\ell^\Lambda(r)}{r^3} e^{2U} dt \wedge \vec{x} \cdot d\vec{x}. \tag{162}$$

Note that here r parametrizes the distance from the horizon.

It is convenient to rephrase the same ansatz in the complex formalism well-adapted to the $N = 2$ theory. To this effect we begin by constructing a 2-form which is anti-self-dual in the background of the metric (161) and whose integral on the 2-sphere at infinity S^2_∞ is normalized to 4π. A short calculation yields:

$$\begin{aligned} E^- &= \mathrm{i} \frac{e^{2U(r)}}{r^3}\, dt \wedge \vec{x} \cdot d\vec{x} + \frac{1}{2} \frac{x^a}{r^3}\, dx^b \wedge dx^c \epsilon_{abc}\,, \\ \int_{S^2_\infty} E^- &= 4\,\pi \end{aligned} \tag{163}$$

from which one obtains:

$$E^-_{\mu\nu}\, \gamma^{\mu\nu} = 2\,\mathrm{i} \frac{e^{2U(r)}}{r^3}\, \gamma_a x^a\, \gamma_0\, \frac{1}{2} \left[\mathbf{1} + \gamma_5\right] \tag{164}$$

[8]This ansatz is dictated by the general p-brane solution of supergravity bosonic equations in any dimensions [15].

which will simplify the unfolding of the supersymmetry transformation rules. Next, introducing the following complex combination:

$$t^\Lambda(r) \;=\; \frac{1}{2}\left(p^\Lambda + \mathrm{i}\ell^\Lambda(r)\right) \tag{165}$$

of the magnetic charges $p^\Lambda = \frac{1}{4\pi}\int_{S^2} F^\Lambda$ and of the functions $\ell^\Lambda(r) = -\frac{1}{4\pi}\int_{S^2} {}^\star F^\Lambda$ introduced in eq. (162), we can rewrite the ansatz (162) as:

$$F^{-|\Lambda} \;=\; t^\Lambda\, E^- \tag{166}$$

and we retrieve the original formulae from:

$$\begin{array}{rclcl} F^\Lambda & = & 2\mathrm{Re}F^{-|\Lambda} & = & \frac{p^\Lambda}{2r^3}\epsilon_{abc}x^a dx^b \wedge dx^c - \frac{\ell^\Lambda(r)}{r^3}e^{2U}dt \wedge \vec{x}\cdot d\vec{x} \\ {}^\star F^\Lambda & = & -2\mathrm{Im}F^{-|\Lambda} & = & -\frac{\ell^\Lambda(r)}{2r^3}\epsilon_{abc}x^a dx^b \wedge dx^c - \frac{p^\Lambda}{r^3}e^{2U}dt \wedge \vec{x}\cdot d\vec{x}. \end{array} \tag{167}$$

Before proceeding further it is convenient to define the electric and magnetic charges of the black hole as it is appropriate in any abelian gauge theory. Recalling the general form of the field equations and of the Bianchi identities as given in (38), we see that on-shell the field strengths $F_{\mu\nu}$ and $G_{\mu\nu}$ are both closed 2-forms, since their duals are divergenceless. Hence, for Gauss theorem, their integral on a closed space-like 2-sphere does not depend on the radius of the sphere. These integrals are the (constant) electric and magnetic charges of the black hole defined in (61) that, in a quantum theory, we expect to be quantized. Using the ansatze (167) and the definition (37), we find

$$q_\Lambda = \frac{1}{4\pi}\int_{S^2} G_\Lambda = \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\ell^\Sigma + \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}p^\Sigma = 2\mathrm{Re}\left(\mathcal{N}_{\Lambda\Sigma}\bar{t}^\Sigma\right). \tag{168}$$

From the above equation we can obtain the field dependence of the functions $\ell^\Lambda(r)$

$$\ell^\Lambda(r) \;=\; (\mathrm{Im}\mathcal{N})^{-1\,\Lambda\Sigma}\left(q_\Sigma - \mathrm{Re}\mathcal{N}_{\Sigma\Gamma}\,p^\Gamma\right). \tag{169}$$

Consider now the Killing spinor equations obtained from the supersymmetry transformations rules (141), (142):

$$0 \;=\; \nabla_\mu\,\xi_A + \epsilon_{AB}\,T^-_{\mu\nu}\,\gamma^\nu\xi^B\,, \tag{170}$$

$$0 \;=\; \mathrm{i}\,\nabla_\mu\, z^i\,\gamma^\mu\xi^A + \frac{\mathrm{i}}{2}\,g^{i\bar{\jmath}}\,\overline{T}^-_{\bar{\jmath}|\mu\nu}\,\gamma^{\mu\nu}\epsilon^{AB}\,\xi_B\,, \tag{171}$$

where the Killing spinor $\xi_A(r)$ is of the form of a single radial function times a constant spinor satisfying:

$$\begin{aligned} \xi_A(r) &= e^{f(r)}\chi_A \qquad\qquad \chi_A = \text{constant} \\ \gamma_0\chi_A &= \mathrm{i}\,\frac{Z}{|Z|}\,\epsilon_{AB}\chi^B \end{aligned} \tag{172}$$

We observe that the condition (172) halves the number of supercharges preserved by the solution. Inserting eq.s (143),(144),(172) into eq.s(170), (171) and using the result (164), with a little work we obtain the first order differential equations:

$$\begin{aligned} \frac{dz^i}{dr} &= -\left(\frac{e^{U(r)}}{r^2}\right)\frac{Z}{|Z|}\,g^{i\bar{\jmath}}\overline{f}^{\Lambda}_{\bar{\jmath}}(\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}t^{\Sigma} = \\ &= \left(\frac{e^{U(r)}}{r^2}\right)\frac{Z}{|Z|}\,g^{i\bar{\jmath}}D_{\bar{\jmath}}\overline{Z}(z,\overline{z},p,q) = 2\left(\frac{e^{U(r)}}{r^2}\right)g^{i\bar{\jmath}}\,\partial_{\bar{\jmath}}|Z(z,\overline{z},p,q)|\,, \end{aligned} \tag{173}$$

$$\frac{dU}{dr} = \left(\frac{e^{U(r)}}{r^2}\right)|h_{\Sigma}p^{\Sigma} - f^{\Lambda}q_{\Lambda}| = \left(\frac{e^{U(r)}}{r^2}\right)|Z(z,\overline{z},p,q)|\,, \tag{174}$$

where $\mathcal{N}_{\Lambda\Sigma}(z,\overline{z})$ is the kinetic matrix of special geometry defined by eq.(118), the vector $V = \big(f^{\Lambda}(z,\overline{z}), h_{\Sigma}(z,\overline{z})\big)$, according to eq. (110), is the covariantly holomorphic section of the symplectic bundle entering the definition of a Special Kähler manifold. Moreover, according to eq. (146),

$$Z(z,\overline{z},p,q) \equiv f^{\Lambda}q_{\Lambda} - h_{\Sigma}p^{\Sigma}\,, \tag{175}$$

is the local realization on the scalar manifold $\mathcal{SM}$ of the central charge of the $N = 2$ superalgebra,

$$\overline{Z}^{i}(z,\overline{z},p,q) \equiv g^{i\bar{\jmath}}D_{\bar{\jmath}}\overline{Z}(z,\overline{z},p,q)\,, \tag{176}$$

are the charges associated with the matter vectors, the so-called matter central charges, written with world indices of the special-Kähler manifold. In terms of the complex charge vector t^{Λ} introduced in (165), the central and matter charges have the following useful expressions

$$Z = -2\mathrm{i}\, f^{\Lambda}\,\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\,t^{\Sigma}\,, \tag{177}$$

$$\overline{Z}_{\bar{\imath}} = -2\mathrm{i}\,\overline{f}^{\Lambda}_{\bar{\imath}}\,\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\,t^{\Sigma}\,, \tag{178}$$

In summary, we have reduced the condition that the black hole should be a BPS saturated state to the pair of first order differential equations (173), (174) for the metric scale factor $U(r)$ and for the scalar fields $z^i(r)$. To obtain explicit solutions one should specify the special Kähler manifold one is working with, namely the specific Lagrangian model. There are, however, some very general and interesting conclusions that can be drawn in a model-independent way. They are just consequences of the fact that these BPS conditions are first order differential equations. Because of that there are fixed points (see footnote 7), namely values either of the metric or of the scalar fields which, once attained in the evolution parameter r (= the radial distance), will persist indefinitely. The fixed point values are just the zeros of the right hand side in either of the coupled eq.s (174) and (173). The fixed point for the metric equation (174) is $r = \infty$, which corresponds to its asymptotic flatness. The fixed point for the moduli equation (173) is $r = 0$. So, independently from the initial data at $r = \infty$ that determine the details of the evolution, the scalar fields flow into their fixed point values at $r = 0$, which, as we will show, turns out to be a horizon. Indeed in the vicinity of $r = 0$ also the metric takes the universal form of the Bertotti–Robinson $AdS_2 \times S^2$ metric.

Let us see this more closely. To begin with we consider the equations determining the fixed point values for the moduli and the universal form attained by the metric at the moduli fixed point. Using eq. (178), we find:

$$
\begin{aligned}
0 &= g^{i\bar{\jmath}}\,\overline{Z}_{\bar{\jmath}}\big|_{\text{fix}} = -2\mathrm{i}\; g^{i\bar{\jmath}}\,\overline{f}^{\Gamma}_{\bar{\jmath}}\,(\mathrm{Im}\mathcal{N})_{\Gamma\Lambda}\; t^{\Lambda}\Big|_{\text{fix}}\,, \qquad (179)\\
\left(\frac{dU}{dr}\right)\Big|_{\text{fix}} &= \left(\frac{e^{U(r)}}{r^2}\right)\, |Z\,(z,\overline{z},p,q)\,|\Big|_{\text{fix}}\,. \qquad (180)
\end{aligned}
$$

Multiplying eq.(179) by f_i^{Σ}, using the identity (123) and the definition (177) of the central charge we conclude that at the fixed point the following condition is true:

$$
0 = \left(t^{\Lambda} + \mathrm{i}\,\overline{f}^{\Lambda}\,Z\right)\Big|_{\text{fix}}\,. \qquad (181)
$$

In terms of the previously defined electric and magnetic charges (see eq.s (61), (168)) eq. (181) can be rewritten as:

$$
\begin{aligned}
p^{\Lambda} &= -\mathrm{i}\left(Z\,\overline{f}^{\Lambda} - \overline{Z}\,f^{\Lambda}\right)\Big|_{\text{fix}}\,, \qquad (182)\\
q_{\Sigma} &= -\mathrm{i}\left(Z\,\overline{h}_{\Lambda} - \overline{Z}\,h_{\Lambda}\right)\big|_{\text{fix}}\,. \qquad (183)
\end{aligned}
$$

Eq.s (179), or equivalently eq.s (182), (183), can be regarded as algebraic equations determining the value of the scalar fields at the fixed point as functions of the electric and magnetic charges p^Λ, q_Σ. Note therefore that, at the horizon, also the central charge depends only on the quantized charges: $Z(z,\overline{z},p,q)|_{\text{fix}} \equiv Z(p,q)$.

In the vicinity of the fixed point the differential equation for the metric becomes:

$$\frac{dU}{dr} = \frac{|Z(p,q)|}{r^2}\, e^{U(r)} \tag{184}$$

which has the approximate solution:

$$\exp[-U(r)] \stackrel{r\to 0}{\longrightarrow} \frac{|Z(p,q)|}{r} \tag{185}$$

Hence, near $r=0$ the metric (161) becomes of the Bertotti Robinson type (see eq.(8)) with Bertotti Robinson mass given by:

$$M^2_{\text{B-R}} = |Z(p,q)|^2 \tag{186}$$

In the metric (8) the surface $r=0$ is light-like and corresponds to a horizon since it is the locus where the Killing vector generating time translations $\frac{\partial}{\partial t}$, which is time-like at spatial infinity $r=\infty$, becomes light-like. The horizon $r=0$ has a finite area given by:

$$\text{Area}_H = \int_{r=0} \sqrt{g_{\theta\theta}\, g_{\phi\phi}}\, d\theta\, d\phi \;=\; 4\pi\, M^2_{\text{B-R}} \tag{187}$$

Hence, independently from the details of the considered model, the BPS saturated black holes in an N=2 theory have a Bekenstein–Hawking entropy given by the following horizon area:

$$\frac{\text{Area}_H}{4\pi} \;=\; |Z(p,q)|^2, \tag{188}$$

where (186) was used, the value of the central charge being determined by eq.s (182), (183). Such equations, as we shall see in the next secton, can also be seen as the variational equations for the minimization of the horizon area as given by (188), if the central charge is regarded as a function of both the scalar fields and the charges:

$$\begin{aligned} \text{Area}_H(z,\overline{z}) &= 4\pi\, |Z(z,\overline{z},p,q)|^2 \\ \frac{\delta \text{Area}_H}{\delta z} &= 0 \longrightarrow z = z_{\text{fix}}\,. \end{aligned} \tag{189}$$

5 BPS and non-BPS attractor mechanism: The geodesic potential

Quite recently it was noticed that the attractor behavior of extremal black holes in supersymmetric theories is not peculiar of BPS solutions preserving some supersymmetries [31], and examples of non-supersymmetric extremal black holes exhibiting the attractor phenomenon were found [34, 36, 63, 64, 65, 66].

It is then appropriate to introduce an alternative approach to extremality which does not rely on the existence of supersymmetry [67, 31, 36]. Let us start by writing the space-time metric of a black hole in terms of a new radial parameter τ:

$$ds^2 = e^{2U}\,dt^2 - e^{-2U}\left(\frac{c^4}{\sinh^4(c\tau)}\,d\tau^2 + \frac{c^2}{\sinh^2(c\tau)}\,d\Omega^2\right). \qquad (190)$$

The coordinate τ is related to the radial coordinate r by the following relation:

$$\frac{c^2}{\sinh^2(c\tau)} = (r-r_0)^2 - c^2 = (r-r^-)\,(r-r^+)\,. \qquad (191)$$

Here $c^2 \equiv 2ST$ is the extremality parameter of the solution, with S the entropy and T the temperature of the black hole. When c is non vanishing the black hole has two horizons located at $r^\pm = r_0 \pm c$. The outer horizon is located at $r_H = r^+$ corresponding to $\tau \to -\infty$. The extremality limit at which the two horizons coincide, $r_H = r^+ = r^- = r_0$, is $c \to 0$. In this case the metric (190) takes the simple form in the r coordinate

$$ds^2 = e^{2U}\,dt^2 - e^{-2U}\left(dr^2 + (r-r_H)^2\,d\Omega^2\right). \qquad (192)$$

In the general case, if we require the horizon to have a finite area A, the scale function U in the near-horizon limit should behave as follows

$$e^{-2U} \xrightarrow{\tau\to-\infty} \frac{A}{4\pi}\,\frac{\sinh^2(c\tau)}{c^2} = \frac{A}{4\pi}\,\frac{1}{(r-r^-)(r-r^+)}\,, \qquad (193)$$

so that the near–horizon metric reads

$$ds^2 = \frac{4\pi}{A}\,(r-r^-)(r-r^+)\,dt^2 - \frac{A}{4\pi}\,\frac{dr^2}{(r-r^-)(r-r^+)} - \frac{A}{4\pi}\,d\Omega^2\,. \qquad (194)$$

The above metric coincides with the near–horizon metric of a Reissner–Nordström solution with horizons located at $r^{\pm}$. It is useful to introduce the radial coordinate ρ defined as $\rho = 2\, e^{c\tau}$, in terms of which, in the near-horizon limit, we can write $e^{-2U} \sim \left(\frac{r_H}{\rho c}\right)^2$, where $r_H = \sqrt{A/4\pi}$ is the radius of the (outer) horizon, and the metric becomes

$$ds^2 = \left(\frac{\rho c}{r_H}\right)^2 dt^2 - (r_H)^2\,(d\rho^2 + d\Omega^2)\,. \tag{195}$$

The coordinate ρ measures the *physical distance* from the horizon, which is located at $\rho = 0$, in units of r_H. It is important to note that the distance of a point at some finite ρ_0 from the horizon is finite:

$$d = \int_0^{\rho_0} r_H\, d\rho = r_H\, \rho_0 < \infty\,. \tag{196}$$

Using this feature, in [36] an intuitive argument was given in order to justify the absence of a universal behavior for the scalar fields near the horizon of a non-extremal black hole: the distance from the horizon is not "long enough" in order for the scalar fields to "loose memory" of their initial values at infinity.

Let us now consider the extremal case $c = 0$. The relation between τ and r becomes $\tau = -1/(r - r_H)$. In order to have a finite horizon area, U should behave near the horizon as:

$$e^{-2U} \quad \sim \quad \left(\frac{r_H}{r - r_H}\right)^2\,, \tag{197}$$

The physical distance from the horizon is now measured in units r_H by the coordinate $\omega = \ln(r - r_H)$ in terms of which the near-horizon metric reads:

$$ds^2 = \frac{1}{(r_H)^2}\, e^{2\omega}\, dt^2 - (r_H)^2\,(d\omega^2 + d\Omega^2)\,. \tag{198}$$

Since now the horizon is located at $\omega \to -\infty$, the distance of a point at some finite ω_0 from the horizon is always infinite, as opposite to the non-extremal case:

$$d \quad = \quad \int_{-\infty}^{\omega_0} r_H\, d\omega = \infty\,. \tag{199}$$

Therefore, as observed in [36], the infinite distance from the horizon in the extremal case justifies the fact that the scalar fields at the horizon "loose memory" of their initial values at infinity and therefore exhibit a universality behavior. In order to simplify the notation, in the following we shall use the coordinate r to denote the distance from the horizon, consistently with our previous treatment of the BPS black hole solutions.

Let us consider the field equations for the metric components (see eq. (190)) and for the scalar fields Φ^r coming from the lagrangian (32). By eliminating the vector fields through their equations of motion, the resulting equations for the metric and the scalar fields, written in terms of the evolution parameter τ, take the following simple form [67]:

$$\frac{d^2U}{d\tau^2} = V_{\text{B-H}}(\Phi, p, q)e^{2U}\,, \tag{200}$$

$$\frac{D^2\Phi^r}{D\tau^2} = g^{rs}(\Phi)\,\frac{\partial V_{\text{B-H}}(\Phi, p, q)}{\partial\Phi^s}e^{2U}\,, \tag{201}$$

with the constraint

$$\left(\frac{dU}{d\tau}\right)^2 + \frac{1}{2}\,g_{rs}(\Phi)\,\frac{d\Phi^r}{d\tau}\frac{d\Phi^s}{d\tau} - V_{\text{B-H}}(\Phi, p, q)e^{2U} = c^2\,, \tag{202}$$

where $V_{\text{B-H}}(\Phi, p, q)$ is a function of the scalars and of the electric and magnetic charges of the theory defined by:

$$V_{\text{B-H}} = -\frac{1}{2}\,Q^t\mathcal{M}(\mathcal{N})Q\,, \tag{203}$$

where as usual Q is the symplectic vector of quantized electric and magnetic charges and $\mathcal{M}(\mathcal{N})$ is the symplectic matrix defined in (93) in terms of the matrix $\mathcal{N}_{\Lambda\Sigma}(\Phi)$. Let us note that the field equations (201) can be extracted from the effective one-dimensional lagrangian:

$$\mathcal{L}_{eff} = \left(\frac{dU}{d\tau}\right)^2 + \frac{1}{2}\,g_{rs}\frac{d\Phi^r}{d\tau}\frac{d\Phi^s}{d\tau} + V_{\text{B-H}}(\Phi, p, q)e^{2U}, \tag{204}$$

constrained with equation (202). The extremality condition is $c^2 \to 0$.

From equation (204) we see that the properties of extremal black holes are completely encoded in the metric of the scalar manifold g_{rs} and on the scalar effective potential $V_{\text{B-H}}$, known as black-hole potential or geodesic potential

[67, 31]. In particular, as it was shown in [67, 31, 36] and as we shall review below, the area of the event horizon is proportional to the value of $V_{\text{B-H}}$ at the horizon:

$$\frac{A}{4\pi} = V_{\text{B-H}}(\Phi_h, p, q) \tag{205}$$

where Φ_h denotes the value taken by the scalar fields at the horizon [9]. This follows from the property that there is an attractor mechanism at work in the extremal case. To see this let us consider the set of equations (201) at $c = 0$. Regularity of the scalar fields at the horizon, which is located, with respect to the physical distance parameter ω, at $\omega \to -\infty$, implies that at the horizon the first derivative of Φ^r with respect to ω vanishes: $\partial_\omega \Phi^r_{|h} = 0$. Near the horizon a solution to eqs. (201), under the hypothesis that $(\partial V_{\text{B-H}}/\partial \Phi^r)_h$ be finite, behaves as follows:

$$\Phi^r \sim \frac{1}{2\,(r_H)^2}\, g^{rs}(\Phi_h)\, \frac{\partial V_{\text{B-H}}}{\partial \Phi^s}\bigg|_{\Phi_h} \omega^2 + \Phi^r_h. \tag{206}$$

Regularity of Φ^r at $\omega \to -\infty$ then further requires that $(\partial V_{\text{B-H}}/\partial \Phi^r)_{|h} = 0$, implying that the horizon be an attractor point for the scalar fields. We conclude that in the extremal case the scalar fields tend in the near-horizon limit to some fixed values Φ^r_h which extremize the potential $V_{\text{B-H}}$:

$$\omega \to -\infty : \quad \Phi^r(\omega) \text{ regular } \Rightarrow \quad \frac{\partial V_{\text{B-H}}}{\partial \Phi^r}\bigg|_{\Phi_h} \to 0 \quad ; \quad \frac{d\Phi^r}{d\omega} \to 0. \tag{207}$$

These values are functions of the quantized electric and magnetic charges only: $\Phi^r_h = \Phi^r_h(p, q)$. Furthermore, let us consider eq. (202). In the extremal limit $c = 0$, near the horizon it becomes:

$$\left(\frac{dU}{d\tau}\right)^2 \sim V_{\text{B-H}}(\Phi_h(p, q), p, q) e^{2U} \tag{208}$$

from which it follows, for the metric components near the horizon:

$$e^{2U} \sim \frac{r^2}{V_{\text{B-H}}(\Phi_h)} = \left(\frac{r}{r_H}\right)^2 , \tag{209}$$

[9]For the sake of clarity in the comparison with equivalent formulas in [36], let us note that in [36] the definition $\Sigma^r = \frac{d\Phi^r}{d\tau}$ has been used.

that is:

$$ds^2_{hor} = \frac{r^2}{V_{\text{B-H}}(\Phi_h)}dt^2 - \frac{V_{\text{B-H}}(\Phi_h)}{r^2}\left(dr^2 + r^2 d\Omega\right). \tag{210}$$

From eqs. (208) and (210) we immediately see that the value of the potential at the horizon measures its area, as anticipated in eq. (205). The metric (210) describes a Bertotti–Robinson geometry $AdS_2 \times S^2$, with mass parameter $M^2_{\text{B-R}} = V_{\text{B-H}}(\Phi_h)$.

To summarize, we have just shown that the area of the event horizon of an extremal black hole (and hence its B-H entropy) is given by the black-hole potential evaluated at the horizon, where it gets an extremum. This justifies our assertion at the end of the previous section.

Let us briefly comment on the non-extremal case $c \neq 0$. For these solutions the physical distance is measured by the coordinate ρ introduced in eq. (193) and the horizon is located at $\rho = 0$. The requirement of regularity of the scalar fields at the horizon is less stringent. It just means that the scalars should admit a Taylor expansion in ρ around $\rho = 0$ and thus it poses no constraints, aside from finiteness, on their derivatives at the horizon:

$$\Phi^r \sim \Phi^r_h + \left.\frac{\partial \Phi^r}{\partial \rho}\right|_0 \rho + \frac{1}{2\,(r_H)^2}\, g^{rs}(\Phi_h)\left.\frac{\partial V_{\text{B-H}}}{\partial \Phi^s}\right|_{\Phi_h} \rho^2 + O(\rho^3). \tag{211}$$

The horizon is therefore not necessarily an attractor point, since at $\rho = 0$ $(\partial V_{\text{B-H}}/\partial \Phi^r)_{\Phi_h}$ can now be a non vanishing constant.

5.1 Extremal black holes in supergravity

For supergravity theories, supersymmetry fixes the black-hole potential $V_{\text{B-H}}$ defined in eqs. (203) to take a particular form that allows to find its extremum in an easy way. Indeed, an expression exactly coinciding with (203) has been found in section 3 in an apparently different context, as the result of a sum rule among central and matter charges in supergravity theories (93). So, in every supergravity theory, the black-hole potential has the general form:

$$V_{\text{B-H}} \equiv -\frac{1}{2}Q^t \mathcal{M}(\mathcal{N})Q = \frac{1}{2}Z_{AB}\overline{Z}^{AB} + Z_I \overline{Z}^I. \tag{212}$$

By making use of the geometric relations of section 3, the value of the charge vector $Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}$ in terms of the moduli Φ is given by equations (99), (100).

Then, to find the extremum of $V_{\text{B-H}}$ we can apply the differential relations (90) among central and matter charges found in Section 3.

Let us now analyze more in detail, for the case of supergravity theories, the extremality condition $c = 0$ as it comes from the constraint (202) which has to be imposed on the solution all over space-time. According to the discussion given in the previous section, the existence of solutions to equation (202) does not not rely on supersymmetry, therefore also non supersymmetric extremal black holes still exhibit an attractor behavior (207) (found at $c = 0$).

At spatial infinity $\tau \to 0$, where the macroscopic features of the black hole are well defined, we have $U \to M_{ADM}\tau$, as it follows from the general definition of ADM mass in General Relativity (see for example [1]). The metric (161) reduces to the Minkowski one and the constraint (202) becomes:

$$M_{ADM}^2 = |Z(\Phi_\infty, p, q)|^2 + |Z_I(\Phi_\infty, p, q)|^2 - \frac{1}{2}\, g_{rs}\frac{d\Phi_\infty^r}{d\tau}\frac{d\Phi_\infty^s}{d\tau}. \tag{213}$$

These solutions do not necessarily saturate the BPS bound, since in general, from (213), $M_{ADM}^2 \neq |Z(\Phi_\infty)|^2$. They then completely break supersymmetry. The behavior at the horizon may nevertheless be easily found thanks to the expression (212) that the black-hole potential takes in supergravity theories, by exploiting the condition (207) and in particular $\frac{\partial V_{\text{B-H}}}{\partial \Phi^r}|_{\Phi_h} \to 0$.

For the cases where the black-hole solution preserves some supersymmetries, we are going to find that the constraint (202) yields the BPS bound on the mass of the solution. Indeed in that case one may apply the results of section 4.1. Let us restrict to the case of $N = 2$ supergravity, where $V_{\text{B-H}} = |Z|^2 + |Z_I|^2$. The Killing-spinor equation $\delta_\epsilon \lambda = 0$ gives equation (173) that implies

$$\left|\frac{dz^i}{d\tau}\right|^2 = e^{2U}\, |g^{i\bar{\jmath}}\, D_{\bar{\jmath}} Z|^2. \tag{214}$$

By making use of (214), the constraint (202) reduces in the extremal limit $c = 0$ to the following equation, valid all over space-time

$$\left(\frac{dU}{d\tau}\right)^2 = e^{2U}\, |Z|^2. \tag{215}$$

At spatial infinity $\tau \to 0$, equations (214) and (215) become

$$M_{ADM}^2 = |Z(\Phi_\infty, p, q)|^2\,; \qquad |Z_I(\Phi_\infty, p, q)|^2 = g_{rs}\frac{d\Phi_\infty^r}{d\tau}\frac{d\Phi_\infty^s}{d\tau}\,. \tag{216}$$

The first equation in (216) may be recognised as the saturation of the BPS bound on the mass of the solution. On the other hand, near the horizon the attractor condition holds

$$\left.\frac{d\Phi^r}{d\tau}\right|_h = 0\,, \tag{217}$$

and from (214) it gives $Z_I|_h = 0$, which may be solved to find $\Phi_{\rm fix}(p,q)$ leaving, for the mass parameter at the horizon

$$\left(\frac{dU}{d\tau}\right)^2_h = M^2_{\text{B-R}}(p,q) = |Z(\Phi_{\rm fix},p,q)|^2. \tag{218}$$

Actually, the extrema of the black-hole potential may be systematically studied, both for the BPS and non-BPS case, by use of the geometric relations (90). One finds that the extrema are given by:

$$\begin{aligned} dV_{\text{B-H}} &= \frac{1}{2} DZ_{AB}\overline{Z}^{AB} + DZ_I\overline{Z}^I + c.c. = \\ &= \frac{1}{2}\left(\frac{1}{2}\overline{Z}^{AB}\overline{Z}^{CD}P_{ABCD} + \overline{Z}^{AB}\overline{Z}^I P_{ABI} + c.c.\right) \\ &+ \left(\frac{1}{2}\overline{Z}^{AB}\overline{Z}^I P_{ABI} + \overline{Z}^I\overline{Z}^J P_{IJ} + c.c.\right) = 0. \end{aligned} \tag{219}$$

Let us remark that the one introduced in (219) is a covariant procedure, not referring explicitly to the horizon properties for finding the entropy, so it is not necessary to specify explicitly horizon parameters (like the metric and the fixed values of scalars at that point), $V_{\text{B-H}}$ being a well defined quantity over all the space-time.

The conditions (219), defining the extremum of the black-hole potential and thus the fixed scalars, when restricted to the BPS case have the same content as, and are therefore completely equivalent to, the relations (173) and (174) found in the previous subsection from the Killing-spinor conditions. In particular, extremal black holes preserving one supersymmetry correspond to N-extended multiplets with

$$M_{ADM} = |Z_1| > |Z_2| \cdots > |Z_{[N/2]}| \tag{220}$$

where Z_m, $m = 1, \cdots, [N/2]$, are the skew-eigenvalues of the central charge antisymmetric matrix introduced in (14) [68],[69],[50],[51]: $Z_1 = Z_{12}$, $Z_2 = Z_{34}, \ldots$. At the attractor point, where M_{ADM} is extremized, supersymmetry requires the vanishing of each term on the right hand side of eq. (219). In particular we find $Z_I = 0$ (recall that Z_I do not exist for $N > 4$) and

$$\overline{Z}^{AB}\overline{Z}^{CD}P_{ABCD} \;=\; \Rightarrow\; \overline{Z}^{[AB}\overline{Z}^{CD]} = 0\,. \qquad (221)$$

The above condition is satisfied taking $Z_1 = Z_{12} \neq 0$ and $Z_m = 0$, $m > 1$. A general property of regular BPS black hole solutions is that supersymmetry doubles at the horizon. This is consistent with the fact that the near horizon geometry is a Bertotti–Robinson space–time of the form $AdS_2 \times S^2$ which is known to have an unbroken $N = 2$ supersymmetry [5]. Let us now give an argument for the vanishing of the supersymmetry variation along ϵ_1, ϵ_2 of the fermion fields at the horizon. As far as the dilatino fields are concerned, it is sufficient to remember that, since $(d\Phi^r/d\tau)_h = 0$, at the horizon the supersymmetry variation is proportional to $Z_{[AB}\epsilon_{C]}$. However this expression is also zero since the only non–vanishing central charge is $Z_1 \equiv Z_{12}$ and furthermore $Z_{[12}\epsilon_{1]} = Z_{[12}\epsilon_{2]} = 0$. As for the gaugini their supersymmetry variation at the horizon is automatically zero being $Z_I = 0$. Finally let us remark that the gravitino variation is not actually zero, however the variation of its field strength along ϵ_1, ϵ_2 vanishes because of the property of the Bertotti–Robinson solution of being conformally flat and the fact that the graviphoton field strength T_{AB} is Lorentz-covariantly constant at the horizon [22].

A case by case analysis of the BPS and non-BPS black holes in the various supergravity models, by inspection of the extrema of $V_{\text{B-H}}$, will be given in section 6. As an exemplification of the method, let us anticipate here the detailed study of the BPS solution of $D = 4$, $N = 4$ pure supergravity. The field content is given by the gravitational multiplet, that is by the graviton $g_{\mu\nu}$, four gravitini $\psi_{\mu A}$, $A = 1, \cdots, 4$, six vectors $A_\mu^{[AB]}$, four dilatini $\chi^{[ABC]}$ and a complex scalar $\phi = a + \mathrm{i}e^{\varphi}$ parametrizing the coset manifold $G/H = SU(1,1)/U(1)$. The symplectic $Sp(12)$-sections $(f^\Lambda_{AB}, h_{\Lambda AB})$ ($\Lambda \equiv [AB] = 1, \cdots, 6$) over the scalar manifold are given by:

$$\begin{aligned} f^\Lambda_{AB} &= e^{-\varphi/2}\delta^\Lambda_{AB} \\ h_{\Lambda AB} &= \phi e^{-\varphi/2}\delta_{\Lambda AB} \end{aligned} \qquad (222)$$

so that:

$$\mathcal{N}_{\Lambda\Sigma} = (\mathbf{h}\cdot\mathbf{f}^{-1})_{\Lambda\Sigma} = \phi\delta_{\Lambda\Sigma} \qquad (223)$$

The central charge matrix is then given by:

$$Z_{AB} = f^\Lambda_{AB}q_\Lambda - h_{\Lambda AB}p^\Lambda = -e^{-\varphi/2}(\phi p_{AB} - q_{AB})\,. \qquad (224)$$

The black-hole potential is therefore:

$$\begin{aligned}
V(\phi,p,q) &= \frac{1}{2}e^{-\varphi}(\phi p_{AB}-q_{AB})(\overline{\phi}p^{AB}-q^{AB}) \\
&= \frac{1}{2}(a^2e^{-\varphi}+e^{\varphi})p_{AB}q^{AB}+e^{-\varphi}q_{AB}q^{AB}-2ae^{-\varphi}q_{AB}p^{AB} \\
&\equiv \frac{1}{2}(p,q)\begin{pmatrix}1 & 0\\ -a & 1\end{pmatrix}\begin{pmatrix}e^{\varphi} & 0\\ 0 & e^{-\varphi}\end{pmatrix}\begin{pmatrix}1 & -a\\ 0 & 1\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix} \qquad (225)
\end{aligned}$$

By extremizing the potential in the moduli space we get:

$$\begin{aligned}
\frac{\partial V}{\partial a} = 0 \quad &\to \quad a_h = \frac{q_{AB}p^{AB}}{p_{AB}p^{AB}} \\
\frac{\partial V}{\partial \varphi} = 0 \quad &\to \quad e^{\varphi_h} = \frac{\sqrt{|q_{AB}q^{AB}p_{CD}p^{CD}-(q_{AB}p^{AB})^2|}}{p_{AB}p^{AB}} \qquad (226)
\end{aligned}$$

from which it follows that the entropy is:

$$S_{\text{B-H}} = 4\pi V(\phi_h,p,q) = 4\pi\sqrt{|q_{AB}q^{AB}p_{CD}p^{CD}-(q_{AB}p^{AB})^2|} \qquad (227)$$

As a final observation, let us note, following [31], that the extremum reached by the black-hole potential at the horizon is in particular a minimum, unless the metric of the scalar fields change sign, corresponding to some sort of phase transition, where the effective lagrangian description (204) of the theory breaks down. This can be seen from the properties of the Hessian of the black-hole potential. It was shown in [31] for the $N=2$, $D=4$ case that at the critical point $\Phi=\Phi_{\text{fix}}\equiv\Phi_h$, from the special geometry properties it follows:

$$(\partial_{\bar{\imath}}\partial_j|Z|)_{\text{fix}} = \frac{1}{2}g_{\bar{\imath}j}|Z|_{\text{fix}} \qquad (228)$$

and then, remembering, from the above discussion, that $V_{\text{fix}}=|Z_{\text{fix}}|^2$:

$$(\partial_{\bar{\imath}}\partial_j V)_{\text{fix}} = 2g_{\bar{\imath}j}|Z_{\text{fix}}|^2 \qquad (229)$$

From eq. (229) it follows, for the $N=2$ theory, that the minimum is unique. In the next section we will show one more technique for finding the entropy, exploiting the fact that it is a ‘topological quantity’ not depending on scalars. This last procedure is particularly interesting because it refers only to group theoretical properties of the coset manifolds spanned by scalars, and do not need the knowledge of any details of the black-hole horizon.

5.2 B-H entropy as a U-invariant for symmetric spaces

For theories based on moduli spaces given by symmetric manifolds G/H, which is the case of all supergravity theories with $N \geq 3$ extended supersymmetry, but also of several $N = 2$ models, the BPS and non-BPS black holes are classified by some U-duality invariant expressions depending on the representation of the group G of G/H under which the electric and magnetic charges are classified. In this respect, the classification of the $N = 2$ invariants is entirely similar to the $N > 2$ cases, where all scalar manifolds are symmetric spaces.

For theories that have a quartic invariant I_4 [70] (this includes all $N = 2$ symmetric spaces based on cubic prepotentials [71, 72] and $N = 4, 6, 8$ theories), the B-H entropy turns out to be proportional to its square root

$$S_{\text{B-H}} \propto \sqrt{|I_4|}. \tag{230}$$

The BPS solutions have $I_4 > 0$ while the non-BPS ones (with non vanishing central charge) have instead $I_4 < 0$. For all the above theories with the exception of the $N = 8$ case, there is also a second non-BPS solution with vanishing central charge and $I_4 > 0$.

For theories based on symmetric spaces with only a quadratic invariant I_2 (this includes $N = 2$ theories with quadratic prepotentials as well as $N = 3$ and $N = 5$ theories), the B-H entropy is

$$S_{\text{B-H}} \propto |I_2|. \tag{231}$$

In these cases, beyond the BPS solution which has $I_2 > 0$ there is only one non-BPS solution, with vanishing central charge and $I_2 < 0$.

All the solutions discussed here give $S_{\text{B-H}} \neq 0$ and then fall in the class of the so-called large black holes, for which the classical area/entropy formula is valid as it gives the dominant contribution to the black-hole entropy. Solutions with $I_4, I_2 = 0$ do exist but they do not correspond to classical attractors since in that case the classical area/entropy formula vanishes. In this case one deals with small black holes, and a quantum attractor mechanism, including higher curvature terms, has to be considered for finding the entropy.

The main purpose of this subsection is to provide particular expressions which give the entropy formula as a moduli-independent quantity in the entire moduli space and not just at the critical points. Namely, we are looking for quantities $S\left(Z_{AB}(\phi), \overline{Z}^{AB}(\phi), Z_I(\phi), \overline{Z}^I(\phi)\right)$ such that $\frac{\partial}{\partial\phi^i}S = 0$, ϕ^i being the moduli coordinates [10]. To this aim, let us first consider invariants I_α of the isotropy group H of the scalar manifold G/H, built with the central and matter charges. We will take all possible H-invariants up to quartic ones for four dimensional theories (except for the $N = 3$ case, where the invariants of order higher than quadratic are not irreducible). Then, let us consider a linear combination $S^2 = \sum_\alpha C_\alpha I_\alpha$ of the H-invariants, with arbitrary coefficients C_α. Now, let us extremize S in the moduli space $\frac{\partial S}{\partial\Phi^i} = 0$, for some set of $\{C_\alpha\}$. Since $\Phi^i \in G/H$, the quantity found in this way (which in all cases turns out to be unique) is a U-invariant, and is indeed proportional to the Bekenstein–Hawking entropy.

These formulae generalize the quartic $E_{7(-7)}$ invariant of $N = 8$ supergravity [70] to all other cases. [11]

Let us first consider the theories $N = 3, 4$, where matter can be present [52],[53].

The U-duality groups [12] are, in these cases, $SU(3,n)$ and $SU(1,1) \times SO(6,n)$ respectively. The central and matter charges Z_{AB}, Z_I transform in an obvious way under the isotropy groups

$$
\begin{aligned}
H &= SU(3) \times SU(n) \times U(1) \qquad (N = 3) && (232)\\
H &= SU(4) \times SO(n) \times U(1) \qquad (N = 4) && (233)
\end{aligned}
$$

Under the action of the elements of G/H the charges may get mixed with their complex conjugate. The infinitesimal transformation can be read from the differential relations satisfied by the charges (90) [50] .

For $N = 3$:

$$P^{ABCD} = P_{IJ} = 0, \quad P_{ABI} \equiv \epsilon_{ABC} P_I^C \quad Z_{AB} \equiv \epsilon_{ABC} Z^C \tag{234}$$

Then the variations are:

$$
\begin{aligned}
\delta Z^A &= \xi^{AI} Z_I && (235)\\
\delta Z_I &= \overline{\xi}_{AI} Z^A && (236)
\end{aligned}
$$

[10]The Bekenstein-Hawking entropy $S_{\text{B-H}} = \frac{A}{4}$ is actually πS in our notation.

[11]Our analysis is based on general properties of scalar coset manifolds. As a consequence, it can be applied straightforwardly also to the $N = 2$ cases, whenever one considers special coset manifolds.

[12]Here we denote by U-duality group the isometry group U acting on the scalars in a symplectic representation, although only a restriction of it to integers is the proper U-duality group [10].

where ξ^{AI} are infinitesimal parameters of $K = G/H$.

The possible quadratic H-invariants are:

$$\begin{aligned} I_1 &= Z^A \overline{Z}_A \\ I_2 &= Z_I \overline{Z}^I \end{aligned} \qquad (237)$$

So, the U-invariant expression is:

$$S = |Z^A \overline{Z}_A - Z_I \overline{Z}^I| \qquad (238)$$

In other words, $D_i S = \partial_i S = 0$, where the covariant derivative is defined in ref. [50].

Note that at the attractor point ($Z_I = 0$) it coincides with the moduli-dependent potential (212) computed at its extremum.

For $N = 4$

$$P_{ABCD} = \epsilon_{ABCD} P, \quad P_{IJ} = \eta_{IJ} P, \quad P_{ABI} = \frac{1}{2} \eta_{IJ} \epsilon_{ABCD} \overline{P}^{CDJ} \qquad (239)$$

and the transformations of $K = \frac{SU(1,1)}{U(1)} \times \frac{O(6,n)}{O(6)\times O(n)}$ are:

$$\delta Z_{AB} = \frac{1}{2} \epsilon_{ABCD} \, \xi \, \overline{Z}^{CD} + \xi^I_{AB} Z_I \qquad (240)$$

$$\delta Z_I = \overline{\xi} \, \eta_{IJ} \, \overline{Z}^J + \frac{1}{2} \overline{\xi}^{AB}_I \, Z_{AB} \qquad (241)$$

with $\overline{\xi}^{AB}_I = \frac{1}{2} \eta_{IJ} \epsilon^{ABCD} \xi^J_{CD}$.

The possible H-invariants are:

$$\begin{aligned} I_1 &= Z_{AB} \overline{Z}^{AB} \\ I_2 &= Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DA} \\ I_3 &= \epsilon^{ABCD} Z_{AB} Z_{CD} \\ I_4 &= Z_I Z^I \end{aligned} \qquad (242)$$

There are three $O(6,n)$ invariants given by S_1, S_2, $\overline{S}_2$ where:

$$S_1 = \frac{1}{2} Z_{AB} \overline{Z}^{AB} - Z_I \overline{Z}_I \qquad (243)$$

$$S_2 = \frac{1}{4} \epsilon^{ABCD} Z_{AB} Z_{CD} - Z_I Z_I \qquad (244)$$

and the unique $SU(1,1) \times O(6,n)$ invariant S, $DS = 0$, is given by:

$$S = \sqrt{|(S_1)^2 - |S_2|^2|} \tag{245}$$

At the attractor point $Z_I = 0$ and $\epsilon^{ABCD} Z_{AB} Z_{CD} = 0$ so that S reduces to the square of the BPS mass.
Note that, in absence of matter multiplets, one recovers the expression found in the previous subsecion by extremizing the black hole potential.

For $N = 5, 6, 8$ the U-duality invariant expression S is the square root of a unique invariant under the corresponding U-duality groups $SU(5,1)$, $O^*(12)$ and $E_{7(-7)}$. The strategy is to find a quartic expression S^2 in terms of Z_{AB} such that $DS = 0$, i.e. S is moduli-independent.

As before, this quantity is a particular combination of the H quartic invariants.

For $SU(5,1)$ there are only two $U(5)$ quartic invariants. In terms of the matrix $A_A^{\ B} = Z_{AC} \overline{Z}^{CB}$ they are: $(TrA)^2$, $Tr(A^2)$, where

$$\begin{aligned} TrA &= Z_{AB} \overline{Z}^{BA} && (246) \\ Tr(A^2) &= Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DA} && (247) \end{aligned}$$

As before, the relative coefficient is fixed by the transformation properties of Z_{AB} under $\frac{SU(5,1)}{U(5)}$ elements of infinitesimal parameter ξ^C:

$$\delta Z_{AB} = \frac{1}{2} \xi^C \epsilon_{CABPQ} \overline{Z}^{PQ} \tag{248}$$

It then follows that the required invariant is:

$$S = \frac{1}{2} \sqrt{|4Tr(A^2) - (TrA)^2|} \tag{249}$$

The $N = 6$ case is the more complicated because under $U(6)$ the left-handed spinor of $O^*(12)$ splits into:

$$32_L \rightarrow 15_1 + \overline{15}_{-1} + 1_{-3} + 1_3 \tag{250}$$

The transformations of $\frac{O^*(12)}{U(6)}$ are:

$$\begin{aligned} \delta Z_{AB} &= \frac{1}{4} \epsilon_{ABCDEF} \xi^{CD} \overline{Z}^{EF} + \xi_{AB} \overline{X} && (251) \\ \delta X &= \frac{1}{2} \xi_{AB} \overline{Z}^{AB} && (252) \end{aligned}$$

where we denote by X the $SU(6)$ singlet.

The quartic $U(6)$ invariants are:

$$\begin{aligned} I_1 &= (TrA)^2 && (253)\\ I_2 &= Tr(A^2) && (254)\\ I_3 &= Re(Pf\,ZX) = \frac{1}{2^3 3!} Re(\epsilon^{ABCDEF} Z_{AB} Z_{CD} Z_{EF} X) && (255)\\ I_4 &= (TrA) X \overline{X} && (256)\\ I_5 &= X^2 \overline{X}^2 && (257) \end{aligned}$$

where the matrix A is, as for the $N = 5$ case, $A_A^{\ B} = Z_{AC} \overline{Z}^{CB}$.

The unique $O^*(12)$ invariant is:

$$\begin{aligned} S &= \frac{1}{2}\sqrt{|4I_2 - I_1 + 32 I_3 + 4I_4 + 4I_5|} && (258)\\ DS &= 0 && (259) \end{aligned}$$

Note that at the BPS attractor point $Pf\,Z = 0$, $X = 0$ and S reduces to the square of the BPS mass.

For $N = 8$ the $SU(8)$ invariants are [13]:

$$\begin{aligned} I_1 &= (TrA)^2 && (260)\\ I_2 &= Tr(A^2) && (261)\\ I_3 &= Pf\,Z = \frac{1}{2^4 4!} \epsilon^{ABCDEFGH} Z_{AB} Z_{CD} Z_{EF} Z_{GH} && (262) \end{aligned}$$

The $\frac{E_{7(-7)}}{SU(8)}$ transformations are:

$$\delta Z_{AB} = \frac{1}{2} \xi_{ABCD} \overline{Z}^{CD} \tag{263}$$

where ξ_{ABCD} satisfies the reality constraint:

$$\xi_{ABCD} = \frac{1}{24} \epsilon_{ABCDEFGH} \overline{\xi}^{EFGH} \tag{264}$$

One finds the following $E_{7(-7)}$ invariant [70]:

$$S = \frac{1}{2}\sqrt{|4Tr(A^2) - (TrA)^2 + 32 Re(Pf\,Z)|} \tag{265}$$

[13] The Pfaffian of an $(n \times n)$ (n even) antisymmetric matrix is defined as $PfZ = \frac{1}{2^n n!} \epsilon^{A_1 \cdots A_n} Z_{A_1 A_2} \cdots Z_{A_{N-1} A_N}$, with the property: $|PfZ| = |detZ|^{1/2}$.

6 Detailed analysis of attractors in extended supergravities: BPS and non-BPS critical points

The extremum principle was found originally in the context of $N = 2$ four-dimensional black holes. However, as we have described in section 4, it has a more general validity, being true for all N-extended supergravities in four dimensions (in the cases where the Bekenstein–Hawking entropy is different from zero) [50]. Indeed, the general discussion of section 3.2 shows that the coset structure of extended supergravities in four dimensions (for $N > 2$) induces the existence, in every theory, of differential relations among central and matter charges that generalize the ones existing for the $N = 2$ case. Furthermore, as far as BPS solutions are considered, Killing-spinor equations for gauginos and dilatinos analogous to eq. (90) are obtained by setting to zero the supersymmetry transformation laws of the fermions. Correspondingly, at the fixed point $\partial_\mu \Phi^i = 0$, for any extended supergravity theories one gets some conditions that allow to find the value of fixed scalars and hence of the B-H entropy both for BPS and non-BPS black hole solutions.

We will first discuss in section 6.1 the case of $N = 2$ supergravity, then in section 6.2 the case of the other extended theories allowing matter couplings to the supergravity multiplet, that is $N = 3, 4$ extended supergravities, and finally we will pass to analyze in section 6.3 $N = 5, 6, 8$ theories, which are pure supergravity models.

For every theory, the strategy adopted to find the extrema will be to solve the equation $dV_{\text{B-H}} = 0$, as given in general in (219), by setting to zero all the independent components in the decomposition on a basis of vielbein of the moduli space [50].

We confine our analysis to large black holes, with finite horizon area.

6.1 $N = 2$ attractor equations

In the original paper [31] the $N = 2$ attractor conditions were introduced via an extremum condition on the black-hole potential (203)

$$V_{\text{B-H}} = -\frac{1}{2} Q^T \mathcal{M} Q = |Z|^2 + |D_i Z|^2 \tag{266}$$

discussed in section 5. Indeed, by making use of properties of $N = 2$ special geometry, the extremum condition was written in the form

$$\partial_i V_{\text{B-H}} = 2\overline{Z} D_i Z + \mathrm{i} C_{ijk} g^{j\overline{\jmath}} g^{k\overline{k}} D_{\overline{\jmath}} \overline{Z} D_{\overline{k}} \overline{Z} = 0 \,, \tag{267}$$

where use of the special geometry relations (115) was made.

Given (267), it is useful to write the attractor equations in a different form. Indeed, recalling equations (99), (100) [73, 74, 33] (which are true all over the moduli space) we may write:

$$Q - \mathrm{i}\,\mathbb{C}\,\mathcal{M}(\mathcal{N}) \cdot Q = -2\mathrm{i}\,\overline{\mathbf{V}}^M Z_M = -2\mathrm{i}\left(Z\overline{V} + g^{i\overline{\jmath}} D_{\overline{\jmath}} \overline{Z} D_i V\right) \,, \tag{268}$$

where V is the symplectic section introduced in (110); substituting the extremum condition from (267), eq. (268) gives the value of the charges in terms of the fixed scalars:

$$\begin{aligned} \left[Q - \mathrm{i}\,\mathbb{C}\,\mathcal{M}(\mathcal{N}) \cdot Q\right]\big|_{\text{fix}} &= -2\mathrm{i}\left(Z\overline{V} + \frac{\mathrm{i}}{2Z}\overline{C}^{ijk} D_i V\, D_j Z\, D_k Z\right)\bigg|_{\text{fix}} \quad \text{for } Z_{\text{fix}} \neq 0 \,, \\ \left[Q - \mathrm{i}\,\mathbb{C}\,\mathcal{M}(\mathcal{N}) \cdot Q\right]\big|_{\text{fix}} &= -2\mathrm{i}\,\left(g^{i\overline{\jmath}} D_{\overline{\jmath}} \overline{Z} D_i V\right)\big|_{\text{fix}} \quad \text{for } Z_{\text{fix}} = 0 \,. \end{aligned} \tag{269}$$

The BPS solution corresponds to set $D_i Z = 0$, in which case, for large black holes ($Z_{\text{fix}} \neq 0$), eq. (269) reduces to (168).

The attractive nature of the extremum was further seen to come from the fact that the mass matrix at that point is strictly positive since

$$\partial_i \partial_j V_{\text{B-H}}|_{(\partial_i V_{\text{B-H}}=0)} = 0 \,; \quad \partial_i \partial_{\overline{\jmath}} V_{\text{B-H}}|_{(\partial_i V_{\text{B-H}}=0)} = 2|Z|^2 g_{i\overline{\jmath}} \,. \tag{270}$$

Non supersymmetric extremal black holes with finite horizon area correspond to solutions of (267) with

$$D_i Z \neq 0 \,. \tag{271}$$

These solutions may be divided in two classes

- $D_i Z \neq 0, Z \neq 0$,
- $D_i Z \neq 0, Z = 0$.

For these more general cases, the horizon mass parameter $M_{\text{B-R}}$ which extremizes the ADM mass in moduli space is then given by

$$M^2_{\text{B-R}} = V_{\text{B-H}}|_{(\partial_i V_{\text{B-H}}=0)} = \left[|Z|^2 + |D_i Z|^2\right]_{(\partial_i V_{\text{B-H}}=0)} > |Z|^2_{(\partial_i V_{\text{B-H}}=0)} \,. \tag{272}$$

Equation (272) is nothing but the BPS bound on the mass.

If the central charge Z vanishes on the extremum, then $D_i Z$ have to satisfy

$$C_{ijk} g^{j\bar{\jmath}} g^{k\bar{k}} D_{\bar{\jmath}} \overline{Z} D_{\bar{k}} \overline{Z} = 0 \qquad \forall i \tag{273}$$

in order to fulfill (267). Solutions to the above equation, for the case of special geometries based on symmetric spaces, have been given in [75].

When $Z \neq 0, D_i Z \neq 0$, one may obtain some further consequences of (267). Let us define

$$Z^{\bar{\imath}} \equiv g^{i\bar{\imath}} D_i Z \,, \quad \overline{Z}^i \equiv g^{i\bar{\imath}} D_{\bar{\imath}} \overline{Z}. \tag{274}$$

From (267) we get, by multiplication with $g^{i\bar{\imath}}$

$$Z^{\bar{\imath}} = -\frac{\mathrm{i}}{2\overline{Z}} C^{\bar{\imath}}{}_{jk} \overline{Z}^j \overline{Z}^k \tag{275}$$

and, by multiplication with $\overline{Z}^i$

$$|D_i Z|^2 = -\frac{\mathrm{i}}{2\overline{Z}} N_3(\overline{Z}^k) = \frac{\mathrm{i}}{2Z} N_3(Z^{\bar{k}}) \tag{276}$$

where we have introduced the definition $N_3(\overline{Z}^k) \equiv C_{ijk} \overline{Z}^i \overline{Z}^j \overline{Z}^k$. Note that, if at the attractor point $N_3 = 0$, then $Z = 0$ (or $Z \neq 0$ but then $Z^{\bar{\imath}} = 0$).

The complex conjugate of (267) may be rewritten, using (275) as

$$2 Z D_{\bar{\imath}} \overline{Z} = -\frac{\mathrm{i}}{4\overline{Z}^2} C_{\bar{\imath}\bar{\jmath}\bar{k}} C^{\bar{\jmath}}{}_{\ell m} \overline{Z}^\ell \overline{Z}^m C^{\bar{k}}{}_{pq} \overline{Z}^p \overline{Z}^q. \tag{277}$$

By making use of the special geometry relation [76, 77, 75]

$$C_{\bar{\imath}\bar{\jmath}\bar{k}} C^{\bar{\jmath}}{}_{(\ell m} C^{\bar{k}}{}_{pq)} = \frac{4}{3} C_{(\ell m p} g_{q)\bar{\imath}} + \overline{E}_{\bar{\imath}\ell m p q}, \tag{278}$$

where the tensor $\overline{E}_{\bar{\imath}\ell mpq}$ defined by this relation is related to the covariant derivative of the Riemann tensor and it exactly vanishes for all symmetric spaces [14], we may finally rewrite (267) as

$$2\overline{Z} D_i Z = \frac{\mathrm{i}}{6Z^2} D_i Z C_{\bar{\jmath}\bar{k}\bar{\ell}} Z^{\bar{\jmath}} Z^{\bar{k}} Z^{\bar{\ell}} + \frac{\mathrm{i}}{8Z^2} E_{i\bar{\jmath}\bar{k}\bar{\ell}\bar{m}} Z^{\bar{\jmath}} Z^{\bar{k}} Z^{\bar{\ell}} Z^{\bar{m}}. \tag{279}$$

[14] In this case equation (278) is a consequence of the special geometry relation $D_i C_{jk\ell} = 0$.

Moreover, using also (276) we obtain

$$\left(|Z|^2 - \frac{1}{3}|D_i Z|^2\right) D_i Z = \frac{\mathrm{i}}{8Z} E_{i\overline{\jmath}\overline{k}\overline{\ell}\overline{m}} Z^{\overline{\jmath}} Z^{\overline{k}} Z^{\overline{\ell}} Z^{\overline{m}}. \tag{280}$$

For symmetric spaces eq. (280) gives

$$|D_i Z|^2 = 3|Z|^2 \tag{281}$$

implying that for these black holes: $M^2_{\text{B-R}} = 4|Z|^2_{(\partial_i V_{\text{B-H}}=0)}$.

This relation, for symmetric spaces, was obtained in [54] and then all the solutions of this type have been classified in [75]. In particular, solutions with $C_{ijk} \equiv 0$ correspond to the special series of symmetric special manifolds $\frac{SU(1,1+n)}{U(1)\times SU(1+n)}$ for which only non-BPS solutions with $Z = 0$ may exist.

Solutions of the type in (281) have also been found for non-symmetric spaces based on cubic prepotentials in [34].

However, because of (280), these cannot be the most general solutions. For the generic case of non-symmetric special manifolds, we have instead

$$|D_i Z|^2 = 3|Z|^2 + \Delta \tag{282}$$

where

$$\Delta = -\frac{3}{4} \frac{E_{i\overline{\jmath}\overline{k}\overline{\ell}\overline{m}} Z^{\overline{\jmath}} Z^{\overline{k}} Z^{\overline{\ell}} Z^{\overline{m}}}{N_3(Z^{\overline{k}})} \tag{283}$$

and the Bekenstein–Hawking entropy is

$$S_{\text{B-H}} = A/4 = \pi\left(4|Z|^2 + \Delta\right). \tag{284}$$

Note that, for these non-BPS black holes, at the attractor point Δ is real and, because of (282), it satisfies $-\Delta < 3|Z|^2$.

In all the cases, the attractive nature of the solution depends on the Hessian matrix, which however may have null directions.

6.2 $N > 2$ matter coupled attractors

6.2.1 The $N = 3$ case

The scalar manifold for this theory, as discussed in section 3.2, is the coset space

$$G/H = \frac{SU(3,n)}{SU(3)\times SU(n)\times U(1)} \tag{285}$$

and the relations among central and matter charges are (see (90))

$$\begin{aligned} D(\omega)Z_{AB} &= Z_I P^I_{AB}\,, \\ D(\omega)Z_I &= \frac{1}{2}Z_{AB}\overline{P}_I^{AB}\,. \end{aligned} \tag{286}$$

The extremum condition on the black-hole potential is then

$$\begin{aligned} dV_{\text{B-H}} &= \frac{1}{2}DZ_{AB}\overline{Z}^{AB} + \frac{1}{2}Z_{AB}D\overline{Z}^{AB} + DZ_I\overline{Z}^I + Z_I D\overline{Z}^I \\ &= P^I_{AB}\overline{Z}^{AB}Z_I + c.c. = 0 \end{aligned} \tag{287}$$

and allows two different solutions with non-zero area. This is expected from section 5.2 because the isometry group of the symmetric space (285) only has a quadratic invariant

$$I_2 = \frac{1}{2}|Z_{AB}|^2 - |Z_I|^2\,. \tag{288}$$

Then,

- either $Z_{AB} \neq 0$, $Z_I = 0$, in this case we have a BPS attractor and the black-hole potential becomes

$$V_{\text{B-H}}|_{\text{attr}} = I_2|_{\text{attr}} > 0\,, \tag{289}$$

- or $Z_I \neq 0$, $Z_{AB} = 0$, which gives a non-BPS attractor solution with black-hole potential

$$V_{\text{B-H}}|_{\text{attr}} = -I_2|_{\text{attr}} > 0\,. \tag{290}$$

6.2.2 The $N = 4$ case

In this case the scalar manifold is the coset space

$$G/H = \frac{SU(1,1)}{U(1)} \times \frac{SO(6,n)}{SO(6)\times SO(n)} \tag{291}$$

and the relations among central and matter charges are (see (90) and the discussion below)

$$\begin{aligned} D(\omega)Z_{AB} &= Z_I P^I_{AB} + \frac{1}{2}\overline{Z}^{CD}\epsilon_{ABCD}\,P\,, \\ D(\omega)\overline{Z}_I &= \frac{1}{2}\overline{Z}^{AB}P_{ABI} + Z_I\,P\,. \end{aligned} \tag{292}$$

We recall that for this theory the vielbein P_{ABI} satisfies the reality condition $\overline{P}^{ABI} \equiv (P_{ABI})^\star = \frac{1}{2}\epsilon^{ABCD}P^I_{CD}$.

The extremum condition on the black-hole potential is then

$$
\begin{aligned}
dV_{\text{B-H}} &= \frac{1}{2}DZ_{AB}\overline{Z}^{AB} + \frac{1}{2}Z_{AB}D\overline{Z}^{AB} + DZ_I\overline{Z}^I + Z_I D\overline{Z}^I = 0 \\
&= P_{ABI}\left(\overline{Z}^{AB}Z_I + \frac{1}{2}\epsilon^{ABCD}Z_{CD}\overline{Z}_I\right) + P\left(Z_I Z_I + \frac{1}{4}\epsilon_{ABCD}\overline{Z}^{AB}\overline{Z}^{CD}\right) + \\
&\quad + \overline{P}\left(\overline{Z}^I\overline{Z}^I + \frac{1}{4}\epsilon^{ABCD}\overline{Z}_{AB}\overline{Z}_{CD}\right) = 0\,.
\end{aligned}
\tag{293}
$$

Equation (293) is satisfied for

$$
\begin{cases}
\overline{Z}^{AB}Z^I + \frac{1}{2}\epsilon^{ABCD}Z_{CD}\overline{Z}^I &= 0 \\
Z^I Z^J \delta_{IJ} + \frac{1}{4}\epsilon_{ABCD}\overline{Z}^{AB}\overline{Z}^{CD} &= 0
\end{cases}.
\tag{294}
$$

Therefore we have, in terms of the proper values Z_1, Z_2 of the central charge antisymmetric matrix Z_{AB} (by means of a $U(1) \subset H$ transformation [45], they may always be chosen real and positive) and of the complex matter charges Z^I

$$
\begin{cases}
\overline{Z}_1 Z^I + Z_2\overline{Z}^I &= 0 \\
Z^I Z^I + 2\overline{Z}_1\overline{Z}_2 &= 0
\end{cases}.
\tag{295}
$$

- The BPS solution with finite area is found, as discussed in general in section 5, for

$$
Z_I = 0\,; \qquad Z_2 = 0 \quad (\text{for } Z_1 > Z_2)
\tag{296}
$$

and corresponds to the black-hole potential

$$
V_{\text{B-H}}|_{\text{attr}} = (Z_1)^2\,.
\tag{297}
$$

This solution partially breaks the symmetry of the moduli space, as

$$
\begin{cases}
SU(4) & \rightarrow \quad SU(2)\times SU(2)\times U(1) \\
SO(n) & \rightarrow \quad SO(n)
\end{cases}.
$$

There are also two non-BPS solutions:

- One is found by choosing $Z_I = (z, \vec{0})$

$$
\begin{cases}
Z_1 &= Z_2 = \rho \\
z &= \sqrt{2}\mathrm{i}\rho
\end{cases}
\tag{298}
$$

which gives, for the black-hole potential

$$V_{\text{B-H}}|_{\text{attr}} = (Z_1)^2 + (Z_2)^2 + |z|^2 = 4\rho^2 \,. \qquad (299)$$

In this case the isotropy symmetry then becomes

$$\begin{cases} SU(4) & \to & USp(4) \\ SO(n) & \to & SO(n-1) \end{cases} .$$

- The other is obtained by choosing instead $Z_I = (k_1, k_2, \vec{0})$ and $Z_{AB} = 0$. This solves (295) for $k_1^2 + k_2^2 = 0$, that is for $k_2 = \pm \mathrm{i} k_1 = \mathrm{i} k$, giving

$$V_{\text{B-H}}|_{\text{attr}} = |k_1|^2 + |k_2|^2 = 2|k|^2 \,. \qquad (300)$$

For this case, then, the isotropy symmetry preserved is

$$\begin{cases} SU(4) & \to & SU(4) \\ SO(n) & \to & SO(n-2) \end{cases} .$$

The analysis of this section is in accord with the discussion on U-invariants of section 5.2. Indeed, the isometry group of the scalar manifold (291) admits the quartic invariant (245)

$$I_4 = S_1^2 - |S_2|^2 \qquad (301)$$

where S_1 and S_2 are the $O(6, n)$ invariants introduced in (243), (244) and we have $S_{\text{B-H}} = \sqrt{|I_4|}$.

For the BPS case, $I_4 > 0$. For the non-BPS ones we have, in the first case $I_4 = -|S_2|^2 < 0$, in the second case $I_4 = S_1^2 > 0$.

The case of the pure $N = 4$ supergravity model anticipated as an example in section 5 falls in this classification and corresponds to the BPS solution (since in that case $Z_I \equiv 0$). It is however interesting to look at the $N = 2$ reduction of that model, where only 2 of the 6 vector fields survive, one as the graviphoton and one inside a vector multiplet whose scalars span the coset $\frac{SU(1,1)}{U(1)}$ (axion-dilaton system). Correspondingly, the two proper-values of the $N = 4$ central charge play now two different roles: one, say Z_1, is the $N = 2$ central charge, while the other, Z_2, is the matter charge. Equation (295) has now two distinct solutions (corresponding to the twice degenerate BPS solution in $N = 4$): the BPS one, for $Z_2 = 0$, $M_{ADM} = Z_1$, and a non-BPS one, for $Z_1 = 0$, $Z_2 \neq 0$. This is understood, in terms of invariants, from the fact that $SU(1, 1)$ does not have an independent quartic invariant, and in fact, in this case, one finds that I_4 reduces to $I_4 = [(Z_1)^2 - (Z_2)^2]^2$.

6.3 $N > 4$ pure supergravity attractors

We are going to discuss here the attractor solutions for the extended theories with $N > 4$, where no matter multiplets may be coupled. We will include a discussion of their relation to $N = 2$ BPS and non-BPS black holes, already presented in [71].

6.3.1 The $N = 5$ case

The moduli space of this model is

$$G/H = \frac{SU(1,5)}{U(5)}\,, \tag{302}$$

the theory contains 10 graviphotons and the relations among the central charges are

$$D(\omega)Z_{AB} \;=\; +\frac{1}{2}\overline{Z}^{CD}P_{ABCD}\,. \tag{303}$$

Correspondingly, the extremum condition on the black-hole potential is

$$\begin{aligned} dV_{\text{B-H}} &= \frac{1}{2}DZ_{AB}\overline{Z}^{AB} + \frac{1}{2}Z_{AB}D\overline{Z}^{AB} \\ &= \frac{1}{4}P_{ABCD}\overline{Z}^{AB}\overline{Z}^{CD} + c.c. = 0\,. \end{aligned} \tag{304}$$

This extremum condition allows only one solution with non-zero area, the BPS one. Indeed, in terms of the proper values Z_1, Z_2 of Z_{AB}, equation (304) becomes

$$Z_1Z_2 + \overline{Z}_1\overline{Z}_2 = 0\,. \tag{305}$$

However, by means of a $U(5)$ rotation Z_1, Z_2 may always be chosen real and non-negative [45], leaving as the only solution with non-zero area $Z_1 > 0$, $Z_2 = 0$ (or viceversa). The black-hole potential on this solution is

$$V_{\text{B-H}}|_{\text{attr}} = |Z_1|^2 \qquad (\text{or } 1 \leftrightarrow 2) \tag{306}$$

This solution is $\frac{1}{5}$-BPS and breaks the symmetry of the moduli space:

$$U(5) \rightarrow SU(2) \times SU(3) \times U(1).$$

However, if we truncate this model $N = 5 \to N = 2$, we have the following decomposition of the 10 vectors

$$\mathbf{10} \to \mathbf{1} + \overline{\mathbf{3}} + \mathbf{6}\,.$$

The singlet corresponds to the $N = 2$ graviphoton, while $\overline{\mathbf{3}}$ is the representation of the 3 vectors in the vector multiplets. The 6 extra vectors are projected out in the truncation. Correspondingly, the $N = 5$ central charge Z_{AB} reduces to:

$$Z_{AB} \to \begin{pmatrix} Z_{ab} = Z\delta_{ab} & 0 \\ 0 & Z_{IJ} = \epsilon_{IJK}\overline{Z}^K \end{pmatrix}, \quad a, b = 1, 2;\ I, J, K = 1, 2, 3 \tag{307}$$

The two solutions $Z_1 > 0, Z_2 = 0$ and $Z_1 = 0, Z_2 > 0$, which were BPS and degenerate in the $N = 5$ theory, in the $N = 2$ interpretation correspond the first to a BPS solution (if we set $Z_1 \equiv Z$) and the second to a non-BPS solution with $Z = 0$, as for the quadratic series discussed in section 6.1.

Let us inspect these results in terms of the discussion of section 5.2. The $SU(5, 1)$ invariant is (in terms of the $U(5)$ invariants introduced in section 5.2):

$$I_4 = 4Tr(A^2) - (TrA)^2 \tag{308}$$

that is, in terms of the proper-values of the central charge

$$I_4 = \left[(Z_1)^2 - (Z_2)^2\right]^2 . \tag{309}$$

The solutions $Z_1 \neq Z_2$ are separated by the solution $Z_1 = Z_2$, which corresponds to a small black hole, with $I_4 = 0$. This is the solution which preserves the maximal amount of supersymmetry ($\frac{2}{5}$ unbroken), but it does not come from the attractor equations.

6.3.2 The $N = 6$ case

The moduli space is

$$G/H = \frac{SO^*(12)}{U(6)}\,, \tag{310}$$

and the theory contains 16 graviphotons, 15 in the twice-antisymmetric representation of $U(6)$ plus a singlet. The attractor solutions for this theory have already been presented in [75].

The relations among the central charges are

$$\begin{aligned} D(\omega)Z_{AB} &= \frac{1}{2}\overline{Z}^{CD}P_{ABCD} + \frac{1}{4!}\overline{Z}\epsilon_{ABCDEF}\overline{P}^{CDEF}, \\ D(\omega)Z &= \frac{1}{2!4!}\overline{Z}^{AB}\epsilon_{ABCDEF}\overline{P}^{CDEF}. \end{aligned} \tag{311}$$

The black-hole potential for this theory is

$$V_{\text{B-H}} = \frac{1}{2}Z_{AB}\overline{Z}^{AB} + Z\overline{Z} \tag{312}$$

and the extremum condition is then

$$\begin{aligned} dV_{\text{B-H}} &= \frac{1}{2}DZ_{AB}\overline{Z}^{AB} + \frac{1}{2}Z_{AB}D\overline{Z}^{AB} + DZ\overline{Z} + ZD\overline{Z} = 0 \\ &= \frac{1}{4}P_{ABCD}\left(\overline{Z}^{AB}\overline{Z}^{CD} + \frac{1}{3!}\epsilon^{ABCDEF}Z_{EF}Z\right) + c.c. = 0\,. \end{aligned} \tag{313}$$

In terms of the proper-values Z_1, Z_2, Z_3 of Z_{AB}, which may always be chosen real and non negative by a $U(6)$ rotation, the condition to be satisfied on the extremum is:

$$Z_1Z_2 + ZZ_3 = 0 \qquad (1 \to 2 \to 3 \to 1 \text{ cyclically})\,. \tag{314}$$

This equation admits one solution $\frac{1}{6}$-BPS with $Z = 0$, and two independent non-BPS solutions, both with $Z \neq 0$.

- The BPS solution is found for

$$Z = 0 \qquad Z_2 = Z_3 = 0 \qquad Z_1 \neq 0\,, \tag{315}$$

  if we choose $Z_1 \geq Z_2 \geq Z_3$. In this case the black-hole potential becomes

$$V_{\text{B-H}}|_{\text{attr}} = |Z_1|^2 \qquad (\text{or } 1 \leftrightarrow 2 \leftrightarrow 3) \tag{316}$$

  and corresponds to $I_4 > 0$.

  This solution breaks the symmetry

$$U(6) \to SU(2) \times U(4)$$

  and corresponds to an $\frac{SO^*(12)}{SU(4,2)}$ orbit of the charge vector.

- One non-BPS solution is obtained for

$$Z \neq 0 \qquad Z_1 = Z_2 = Z_3 = 0 \,. \tag{317}$$

It gives for the black-hole potential

$$V_{\text{B-H}}|_{\text{attr}} = |Z|^2 \tag{318}$$

and preserves all the $U(6)$ symmetry of the moduli space. This solution corresponds to the orbit $\frac{SO^*(12)}{SU(6)}$. Also for this solution the quartic invariant is positive $I_4 > 0$.

- The third solution is found by setting

$$Z_1 = Z_2 = Z_3 = \rho \,, \qquad Z = -\rho \,. \tag{319}$$

In this case the black-hole potential becomes

$$V_{\text{B-H}}|_{\text{attr}} = 4\rho^2 \,. \tag{320}$$

This solution breaks the symmetry $U(6) \to USp(6)$, and corresponds to the charge orbit $\frac{SO^*(12)}{SU^*(6)}$. The quartic invariant for this solution is negative $I_4 < 0$.

It is interesting to note, as already observed in [50, 71, 75], that the bosonic sector of the $N = 6$ is exactly the same as the one of the $N = 2$ model coupled with 15 vector multiplets with scalar sector based on the same coset (310). In the $N = 2$ interpretation of this model, the singlet charge Z plays the role of central charge, while the 15 charges Z_{AB} are interpreted as matter charges.

The interpretation of the three attractor solutions is now different: the first one, which was $\frac{1}{6}$-BPS in the $N = 6$ model, is now non-BPS and breaks supersymmetry, while the second one in this model is $\frac{1}{2}$-BPS. The third solution, where all the proper forms of the dressed charges are different from zero, is non-BPS in both interpretations.

6.3.3 The $N = 8$ case

This model has been studied in detail in [54]. Its scalar manifold is the coset

$$G/H = \frac{E_{7(7)}}{SU(8)} \,. \tag{321}$$

The relations among the 28 central charges are

$$D(\omega)Z_{AB} = \frac{1}{2}\overline{Z}^{CD}P_{ABCD}\,, \tag{322}$$

where the vielbein P_{ABCD} satisfies the reality condition

$$\overline{P}^{ABCD} = \epsilon^{ABCDEFGH}P_{EFGH}\,. \tag{323}$$

The extremum condition is then

$$\begin{aligned} dV_{\text{B-H}} &= \frac{1}{2}DZ_{AB}\overline{Z}^{AB} + \frac{1}{2}Z_{AB}D\overline{Z}^{AB} = 0 \\ &= \frac{1}{4}P_{ABCD}\left(\overline{Z}^{AB}\overline{Z}^{CD} + \frac{1}{4!}\epsilon^{ABCDEFGH}Z_{EF}Z_{GH}\right) = 0\,. \end{aligned} \tag{324}$$

In terms of the central charge proper-values $Z_1, \cdots Z_4$ the condition for the extremum may be written

$$\begin{cases} Z_1Z_2 + \overline{Z}_3\overline{Z}_4 = 0 \\ Z_1Z_3 + \overline{Z}_2\overline{Z}_4 = 0 \\ Z_2Z_3 + \overline{Z}_1\overline{Z}_4 = 0 \end{cases} \tag{325}$$

and admits two independent attractor solutions:

- The BPS solution is found for

  $$Z_2 = Z_3 = Z_4 = 0 \qquad Z_1 \neq 0 \tag{326}$$

  if we choose $Z_1 \geq Z_2 \geq Z_3 \geq Z_4$. In this case the black hole potential becomes

  $$V_{\text{B-H}}|_{\text{attr}} = |Z_1|^2 \qquad (\text{or } 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4) \tag{327}$$

  and corresponds to $I_4 > 0$. This solution breaks the symmetry

  $$SU(8) \to SU(2) \times U(6)$$

  and corresponds to an $\frac{E_7}{E_{6(2)}}$ orbit of the charge vector.

- The non-BPS solution is obtained for

  $$Z_1 = Z_2 = Z_3 = Z_4 = e^{\mathrm{i}\frac{\pi}{4}}\rho \qquad \rho \in \mathbb{R}^+\,. \tag{328}$$

  It gives for the black-hole potential

  $$V_{\text{B-H}}|_{\text{attr}} = 4\rho^2\,. \tag{329}$$

  This solution breaks the symmetry $SU(8) \to USp(8)$, and corresponds to the charge orbit $\frac{E_7}{E_{6(6)}}$. The quartic invariant for this solution is negative $I_4 < 0$.

7 Conclusions

This survey has presented the main features of the physics of black holes embedded in supersymmetric theories of gravitation. They have an extremely rich structure and give an interplay between space-time singularities in solutions of Einstein-matter coupled equations and the solitonic, particle-like structure of these configurations such as mass, spin and charge.

The present analysis may be extended to rotating black holes and to geometries not necessarily asymptotically flat (such as, for example, asymptotically anti de Sitter solutions). Furthermore, the concept of entropy may be extended to theories which include higher curvature and higher derivative matter terms [27, 28, 42, 43]. This is important in order to make contact with superstring and M-theory where these terms unavoidably appear. In this context, a remarkable connection has been found between the entropy functional and the topological string partition function, an approach pioneered in [29].

Black hole attractors fall in the class of possible superstring vacua, which in a wide context have led to the study of the so-called landscape [78].

It is a challenging problem to see which new directions towards a fundamental theory of nature these investigations may suggest in the future.

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