Cartan geometry of spacetimes with a nonconstant cosmological function Λ

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We present the geometry of spacetimes that are tangentially approximated by de Sitter spaces whose cosmological constants vary over spacetime. Cartan geometry provides one with the tools to describe manifolds that reduce to a homogeneous Klein space at the infinitesimal level. After briefly reviewing Cartan geometry, we discuss the case in which the underlying Klein space is at each point a de Sitter space, whose combined set of pseudo-radii forms a nonconstant function on spacetime. We show that the torsion of such a geometry receives a contribution, which is not present for a cosmological constant. The structure group of the obtained de Sitter-Cartan geometry is by construction the Lorentz group $SO(1,3)$. Invoking the theory of nonlinear realizations, we extend the class of symmetries to the enclosing de Sitter group $SO(1,4)$, and compute the corresponding spin connection, vierbein, curvature, and torsion.

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I. INTRODUCTION

In theories of gravity, the strong equivalence principle states that at any spacetime point in an arbitrary gravitational field the laws of classical physics reduce to the corresponding laws in the absence of gravity, namely, to their special relativistic form, when they are considered on a sufficiently small region around the point in question [1, 2]. This implies that in the presence of a gravitational field spacetime $\mathcal{M}$ is locally approximated by the spacetime underlying special relativity. Since the laws that govern special relativistic physics are covariant with respect to the Poincaré group $ISO(1,3)$, the corresponding spacetime is the affine Minkowski space M. In view of what follows, and with the risk of being overprecise, let us emphasize that the equivalence principle does not in the first place imply that the tangent spaces to spacetime are given by the affine Minkowski space, but rather that they are isomorphic to the infinitesimal structure of the latter. Although finite Poincaré translations are not defined for a generic spacetime, the equivalence principle indicates that locally they are in one-to-one correspondence with infinitesimal active diffeomorphisms, for both sets generate translations along spacetime [3][4]. Mathematically speaking, at any point x in spacetime there is a 1-form $T_x\mathcal{M} \to \mathfrak{t} = \mathfrak{iso}(1,3)/\mathfrak{so}(1,3)$, which is called the vierbein, and where $\mathfrak{t}$ is the algebra of Poincaré translations. The vierbein pulls back or *solders* the geometric and algebraic structure of $\mathfrak{t}$ to spacetime. For example, the Minkowski metric on $\mathfrak{t}$ gives way to a metric of the same signature on $\mathcal{M}$, from which it follows that the vierbein can be chosen to be an orthonormal frame—an idealized observer—on spacetime. Due to the equivalence principle, Lorentz transformations of these observers constitute a symmetry and are therefore elements of the structure group of the geometry, which in turn leads to the introduction of a spin connection. Since the vierbein is valued in $\mathfrak{p}$, it transforms as a four-vector under local Lorentz rotations, an observation that complies with its interpretation of representing an idealized observer at any given point.

The right mathematical framework for the setting just outlined is due to Elie Cartan [5], in which the $\mathfrak{so}(1,3)$-valued spin connection and the $\mathfrak{t}$-valued vierbein are combined into an $\mathfrak{iso}(1,3)$-valued Cartan connection, thereby defining a Riemann-Cartan geometry [4]. It is explained comprehensibly in [6] how the $\mathfrak{iso}(1,3)$-valued connection gives a prescription for *rolling without slipping* the affine Minkowski space along the integral curves of vector fields on spacetime. It is indeed the central idea behind Cartan geometry, to be reviewed in Sec. II, that a homogeneous model space is generalized to a nonhomogeneous space, for which the local structure is algebraically isomorphic to the one of the model space [7], and where the degree of nonhomogeneity is quantified by the presence of curvature and torsion. In the manner thus explained, the choice for a Riemann-Cartan geometry, i.e., a Cartan connection valued in the algebra $\mathfrak{iso}(1,3)$, to describe spacetimes underlying theories of gravity is implied by the equivalence principle, together with the assumption that the local kinematics are governed by the Poincaré group.

When the $\mathfrak{iso}(1,3)$-valued connection of a Riemann-Cartan geometry is replaced by a Cartan connection that is valued in the de Sitter algebra $\mathfrak{so}(1,4)$, spacetime is locally approximated by de Sitter space dS in place of the affine Minkowski space. Such a structure will be called a de Sitter-Cartan spacetime in the remainder of this work. In fact, since the vierbein is valued in the space of de Sitter transvections $\mathfrak{p} = \mathfrak{so}(1,4)/\mathfrak{so}(1,3)$, it solders the algebraic structure of dS to the tangent bundle of spacetime. The Cartan-Killing metric on $\mathfrak{p}$ is once again the diagonal Minkowski matrix, so that a Lorentzian metric is introduced on $\mathcal{M}$. Different from a Riemann-Cartan geometry, however, is that translations in spacetime are generated by de Sitter transvections, because the space of infinitesimal translations at any point x, i.e., $T_x\mathcal{M}$, is

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isomorphic to $\mathfrak{p}$. This implies that the commutator of infinitesimal translations is proportional to a Lorentz rotation. The constant of proportionality is essentially the cosmological constant of the tangent de Sitter spaces [6], see also Sec. III. It is then sensible to identify this geometric cosmological constant with the dark energy on spacetime. Such an interpretation is in concordance with the MacDowell-Mansouri model for gravity [8]. In this model, the fundamental field is indeed a $\mathfrak{so}(1,4)$-valued Cartan connection, for which the action is equivalent, up to topological terms, with the Palatini action for general relativity in the presence of a cosmological constant [6, 9]. Whatever may be its nature, the cosmological constant thus shows up in the kinematics of spacetime, where it is a measure for the noncommutativity of infinitesimal spacetime translations.

At any point in a de Sitter-Cartan spacetime, the cosmological constant is related to a length scale defined in the commutation relations of the de Sitter transvections. Therefore, it is rather straightforward to generalize to geometries in which this length scale becomes a nonconstant function on spacetime. In Sec. III, we claim some originality for constructing a de Sitter-Cartan geometry that provides spacetime with a *cosmological function* $\Lambda(x)$, which in general does not satisfy $d\Lambda = 0$. We shall see that a nonconstant Λ gives rise to a new term in the expression for the torsion of the de Sitter-Cartan geometry. The cosmological function could possibly serve to model dark energy that changes along space and time, in which way it might give an alternative description for—or interpretation of—one of the models for time-evolving dark energy [10, 11]. Naturally, an adequate action for gravity will have to be defined, which determines whether a spacetime-dependent cosmological function is possible and how its value at any point can be accounted for. In the present paper, we discuss the kinematics only and will not be concerned with the important issue of specifying for the function Λ dynamically. Of course, since Einstein's equations render a constant Λ, general relativity is not satisfactory and a generalization will have to be tried for. In view of this problem, we argue that it may be helpful to discuss the geometry of spacetimes with a generic cosmological function, for it reveals what quantities are available to construct possible geometric actions.

The organization of this paper is as follows. In Sec. II, an attempt to review Klein and Cartan geometry concisely is undertaken. This will clarify why Cartan connections are adequate to describe the geometry of theories of gravity, in favor of the better known Ehresmann connections. The specific case of a de Sitter-Cartan spacetime is reviewed in Sec. III, and is extended to incorporate a nonconstant cosmological function Λ. From our discussion in Sec. II, it furthermore will become clear that a $\mathfrak{so}(1,4)$-valued Cartan connection with structure group $SO(1,3)$ can be obtained from a $\mathfrak{so}(1,4)$-valued Ehresmann connection, for which the structure group $SO(1,4)$ is reduced to its Lorentz subgroup. This observation will clarify how the broken symmetries can be recovered, while preserving the well-defined existence of a spin connection and vierbein, if the Cartan connection is defined through an Ehresmann connection that is realized nonlinearly. It will be the subject of Sec. IV to construct such a $SO(1,4)$-invariant de Sitter-Cartan geometry, again for a nonconstant cosmological function. We conclude in Sec. V.

II. CARTAN CONNECTIONS AND THE INFINITESIMAL SYMMETRY OF SPACETIME

Let H and G be Lie groups and assume there is an injective group homomorphism between the same, namely,

$$i : H \to G, \tag{1}$$

for which $i(H)$ is a subgroup of G. Furthermore, let $i(H)$ be isomorphic to H, in which case it is said that $H \subset G$ is a subgroup. In general, the inclusion (1) is possible in different ways, hence the short exact sequence

$$e \longrightarrow H \xrightarrow{\;i\;} G \longrightarrow \frac{G}{i(H)} \longrightarrow e$$

is not canonically given. As any inclusion is supposed to be isomorphic to H, they are naturally isomorphic to each other and one usually denotes each of the inclusions $i(H)$ by H, a habit we shall follow. There is nevertheless a reason to have put emphasis on the existence of different, albeit isomorphic, inclusions (1).

These mathematically equivalent subgroups in G may have different interpretations from a concrete, physical point of view. An example of interest to us is found in the so-called Klein geometries [7, 12]. These geometries describe homogeneous spaces from a Lie group-theoretic perspective, roughly as follows. Let G be the symmetry group of a homogeneous space S and fix an arbitrary point $\xi \in S$. The subgroup of elements in G that leave ξ invariant is denoted by H_ξ and is called the isotropy group of the point in question. On account of the transitivity of S, there exists an isomorphism between S and G/H_ξ, by identifying a point $\zeta = g(\xi)$ in the former with an element gH_ξ in the latter. It is important to note that although it is necessary to choose a ξ in order to establish the isomorphism, the choice in itself is nonetheless completely arbitrary. If another point $\xi' = a(\xi)$ were chosen, S could similarly be identified with $G/H_{\xi'}$. The group $H_{\xi'}$ is related to the isotropy group of ξ through the adjoint action, i.e.,

$$H_{\xi'} = \mathrm{Ad}(a)(H_\xi) = aH_\xi a^{-1}.$$

This is true for any two points in S, so that the isotropy groups of different points are isomorphic to each other. In this way, they form different but equivalent inclusions (1) of a group H in G. Examples are given for the affine group in [13] and the de Sitter group in D dimensions in [14], of which in particular the latter for $D = 4$ is of interest for the sections below.

A *Klein geometry* is then unambiguously denoted by such a pair (G, H) for which the homogeneous space G/H is connected [7]. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Throughout the text it is assumed that $\mathfrak{g}$ is reductive, to wit, there is a direct sum decomposition as vector spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}, \tag{2}$$

which is $\mathrm{Ad}(H)$-invariant, thereby implying that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}$. Apart from the homogeneous space $S = G/H$, there are two more structures associated with a Klein geometry that are worth mentioning in view of what will follow [7]. The natural projection of G onto its right cosets gives rise to a principal bundle $\pi : G \to G/H$ with typical fibre H. Secondly, there is a $\mathfrak{g}$-valued differential form on the bundle space, that is to say, the Maurer-Cartan form ω_G of the Lie group G. Being a property of generic Maurer-Cartan forms, it follows that its exterior covariant derivative vanishes [15], i.e.,

$$d\omega_G + \tfrac{1}{2}[\omega_G, \omega_G] = 0. \tag{3}$$

The Lie-theoretic point of view on homogeneous spaces offered by Klein geometry is generalized to nonhomogeneous manifolds in Cartan geometry [5]. A given Cartan geometry is said to be modeled on some Klein geometry, since the base manifold associated with the former is tangentially approximated by the homogeneous space of the latter. Moreover, if the Cartan geometry is flat, the manifold is locally the same as the corresponding model space. In the following paragraphs, we give a short review of the subject. A detailed mathematical treatment can be found in [7, 16], while the articles [6, 9, 17] are very helpful in developing an intuition that goes along with the mathematics of Cartan geometry.

Let $P(\mathcal{M}, H)$ be a principal bundle $\pi : P \to \mathcal{M}$ with typical fibre H. A Cartan geometry modeled on the pair $(\mathfrak{g}, \mathfrak{h})$ is obtained by introducing a *Cartan connection* on P, which is defined as follows [7]:

Definition. *A Cartan connection A is a $\mathfrak{g}$-valued 1-form on P that satisfies the conditions:*

(i) for each $p \in P$, the linear map $A_p : T_pP \to \mathfrak{g}$ is an isomorphism;

(ii) $A(\zeta_X) = X$, for each fundamental vector field ζ_X corresponding to $X \in \mathfrak{h}$;

(iii) $R_h^* A = \mathrm{Ad}(h^{-1}) \cdot A$ for each $h \in H$.

From the first of these properties one is able to conclude that $\mathcal{M}$ and G/H are equal in dimension. This is an important observation, for it lies at the heart of the usefulness of Cartan connections to describe theories of gravity. The isomorphism hints at the existence of an object—the vielbein—that takes the geometry of the homogeneous space to the tangent structure of spacetime $\mathcal{M}$.

The Cartan curvature of the connection A is defined as its exterior covariant derivative, i.e., the $\mathfrak{g}$-valued 2-form

$$F = dA + \tfrac{1}{2}[A, A]. \tag{4}$$

The Cartan curvature is strictly horizontal in the sense that it annihilates vertical vector fields on the principal bundle. This is a consequence of the second defining property of the Cartan connection, which implies that A restricts to the Maurer-Cartan form ω_H along the fibres of P [7]. If the curvature of a Cartan connection vanishes, the corresponding geometry is said to be *flat*. An example of a flat Cartan geometry modeled on $(\mathfrak{g}, \mathfrak{h})$ is the Klein geometry (G, H). Indeed, it can be verified rather easily that the Maurer-Cartan form on the principal bundle $\pi : G \to G/H$ satisfies the defining properties of a Cartan connection. That it is a flat connection follows from its structural equation (3). Conversely, the spacetime $\mathcal{M}$ of a flat Cartan geometry modeled on $(\mathfrak{g}, \mathfrak{h})$ is in the neighborhood of any point of $\mathcal{M}$ isomorphic to the Klein geometry G/H, see [7]. The curvature F is thus a measure for the nonhomogeneity of $\mathcal{M}$, in comparison with the perfect homogeneity of the Klein space G/H.

The third defining property of a Cartan connection determines the H-equivariance of A and F under local H-transformations. Because the Cartan connection and its curvature are valued in a reductive Lie algebra, it is sensible to consider their projections according to the splitting (2), which we denote by

$$A = A_{\mathfrak{h}} + A_{\mathfrak{p}} \quad \text{and} \quad F = F_{\mathfrak{h}} + F_{\mathfrak{p}}. \tag{5}$$

Naturally, all projections are H-equivariant. The objects $A_{\mathfrak{p}}$, $F_{\mathfrak{h}}$ and $F_{\mathfrak{p}}$ are horizontal, while the $\mathfrak{h}$-valued differential form $A_{\mathfrak{h}}$ is an Ehresmann connection [16]. If the Lie algebra $\mathfrak{g}$ is symmetric, i.e., it is reductive and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$, the definition (4) allows one to express $F_{\mathfrak{h}}$ and $F_{\mathfrak{p}}$ in terms of $A_{\mathfrak{h}}$ and $A_{\mathfrak{p}}$, namely,

$$F_{\mathfrak{h}} = dA_{\mathfrak{h}} + \tfrac{1}{2}[A_{\mathfrak{h}}, A_{\mathfrak{h}}] + \tfrac{1}{2}[A_{\mathfrak{p}}, A_{\mathfrak{p}}], \tag{6a}$$

$$F_{\mathfrak{p}} = dA_{\mathfrak{p}} + [A_{\mathfrak{h}}, A_{\mathfrak{p}}]. \tag{6b}$$

The $\mathfrak{h}$-valued part $F_{\mathfrak{h}}$ is called the curvature of the geometry. Note that this in general is not the same as the exterior covariant derivative of the Ehresmann connection $A_{\mathfrak{h}}$, which is given by $dA_{\mathfrak{h}} + \frac{1}{2}[A_{\mathfrak{h}}, A_{\mathfrak{h}}]$ only. The $\mathfrak{p}$-component of the Cartan curvature is called the *torsion* of the geometry.

For the reason that the decompositions (5) are invariant under the action of local H-transformations, the corresponding projections of A and F are well-defined geometric objects. In the next section, for example, a Cartan geometry for the de Sitter algebra will be constructed, in which the spin connection and vierbein, and the curvature and torsion do not mix up under local Lorentz transformations. On the other hand, under local G-transformations the $\mathfrak{h}$- and $\mathfrak{p}$-valued parts of the Cartan connection and curvature will transform into each other. In case the local symmetry group is wished to be extended to G, the thus defined objects do not have a geometric meaning, as their construction is not G-invariant. The existence of a spin connection and vierbein is nonetheless mandatory to describe theories of gravity, so that it is necessary to maintain the decompositions (5) while extending the symmetry group from H to G.

To understand how these at first sight conflicting objectives can be reconciled, it is convenient to view a Cartan connection on a principal H-bundle as an Ehresmann connection on a principal G-bundle, for which the symmetry group G is broken to its subgroup H, see also [14, 18]. Mathematically speaking, this corresponds to a reduction of the principal G-bundle to the principal H-bundle. In view of this, it is useful to point to the existence of the following proposition, a proof of which can be found in, e.g., [19].

Proposition. *A principal G-bundle Q is reducible to a principal H-bundle P if and only if the associated bundle $Q[S] = Q \times_G S$ of homogeneous spaces $S \cong G/H$ admits a globally defined section.*

Locally, the section ξ alluded to in the proposition singles out a point $\xi(x)$ in each of the homogeneous spaces S_x over the points $x \in \mathcal{M}$. As a result, the symmetry group G of S_x is broken down to $H_{\xi(x)}$. This restriction of the structure group to $H_{\xi(x)} \cong H$ essentially constitutes the reduction of Q to P. Concomitantly, an Ehresmann connection on Q gives way to a Cartan connection on the reduced H-bundle P [7], for which the reductive splittings (5) make sense. The resulting geometry is once more the above considered Cartan geometry on a principal H-bundle. From the construction just discussed, however, it is obvious that the symmetries of G that are broken upon reduction are merely hidden. This is because the section ξ that has been singled out for the reduction is arbitrary, very much similar to the arbitrariness mentioned of in the above discussed Klein geometries. Different but equivalent sections are related by G-transformations that are not everywhere in H, i.e., they are connected through the broken symmetries. The latter can be incorporated in the principal H-bundle by nonlinearly realizing them as local H-transformations, see [20, 21] and Sec. IV. Since the broken symmetries are realized as elements of H, local G-invariance is restored, and at the same time there is a meaningful reductive decomposition of the nonlinear Cartan connection and curvature, i.e., the presence of a spin connection and vierbein, as well as curvature and torsion tensors. These objects will be constructed in Sec. IV on $SO(1,4)$-invariant de Sitter-Cartan geometry, in a way that is compatible with a nonconstant cosmological function on spacetime.

III. DE SITTER-CARTAN GEOMETRY WITH A COSMOLOGICAL FUNCTION

A de Sitter-Cartan geometry is the Cartan geometry that is modeled on $(\mathfrak{so}(1,4), SO(1,3))$ and thus consists of a principal Lorentz bundle $P(\mathcal{M}, SO(1,3))$ over spacetime, on which is defined a $\mathfrak{so}(1,4)$-valued Cartan connection A. This connection provides spacetime with the information that it is tangentially approximated by de Sitter spaces, see Sec. II. We shall construct a de Sitter-Cartan geometry, in which these tangent de Sitter spaces have cosmological constants that are not required to be the same along spacetime. As a consequence, the thus obtained geometry describes a spacetime on which a cosmological function $\Lambda(x)$ is defined from the onset. We shall see that the possibility for spacetime-dependent internal de Sitter pseudo-radii gives rise to a contribution to the torsion of the geometry, which disappears if Λ is constant.

In the preceding section, the Cartan connection was defined on the principal bundle P. As for the remainder of the text, on the other hand, it is understood that this connection is pulled back to a connection on spacetime by some local section $\sigma : U \subset \mathcal{M} \to P$. We prefer not to introduce new symbols and therefore recycle notation by denoting the pulled back connection $\sigma^* A$ also by A. The same is true for its exterior covariant derivative F.

Under the action of a local $SO(1,3)$-transformation, the de Sitter-Cartan connection transforms according to [7]

$$A \mapsto \mathrm{Ad}(h(x))(A + d), \tag{7}$$

which is the natural nonhomogeneous transformation behavior for connections. The 1-form A is valued in the de Sitter algebra $\mathfrak{so}(1,4)$, which is characterized by the commutation relations

$$\begin{aligned} -i[M_{ab}, M_{cd}] &= \eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad}, \\ -i[M_{ab}, P_c] &= \eta_{ac}P_b - \eta_{bc}P_a, \\ -i[P_a, P_b] &= -l^{-2}M_{ab}. \end{aligned} \tag{8}$$

The convention for the Minkowski metric is $\eta_{ab} = (+,-,-,-)$, while we parametrize an element of $\mathfrak{so}(1,4)$ by $\frac{i}{2}\lambda^{ab}M_{ab} + i\lambda^a P_a$. The generators of de Sitter transvections are defined by $P_a = M_{a4}/l$, where l is an *a-priori* arbitrary length scale that effectively determines the cosmological constant of the corresponding Klein geometry $dS = SO(1,4)/SO(1,3)$, namely, [6]

$$\Lambda = \frac{3}{l^2}. \tag{9}$$

At any point in spacetime, the Cartan connection is valued in a copy of the de Sitter algebra, thereby fixing the cosmological constant of the tangent de Sitter space. Because we wish the set of these cosmological constants to constitute a generic nonconstant function of spacetime, the length scales $l(x)$ should equally be considered spacetime-dependent. In the following paragraphs, we have a look at what this assumption implies for the geometry.

From the brackets (8) one concludes that the algebra is symmetric. The reductive splitting schematically reads as

$$\mathfrak{so}(1,4) = \mathfrak{so}(1,3) \oplus \mathfrak{p}, \tag{10}$$

where $\mathfrak{so}(1,3) = \mathrm{span}\{M_{ab}\}$ is the Lorentz subalgebra and $\mathfrak{p} = \mathrm{span}\{P_a\}$ the subspace of infinitesimal de Sitter transvections, or de Sitter translations. The corresponding decompositions (5) of the Cartan connection and curvature are denoted by

$$A = \tfrac{i}{2}A^{ab}M_{ab} + iA^aP_a \quad \text{and} \quad F = \tfrac{i}{2}F^{ab}M_{ab} + iF^aP_a, \tag{11}$$

from which one concludes that A^a and F^a have the dimension of length. The $\mathfrak{so}(1,3)$-valued 1-form A^{ab} is an Ehresmann connection for local Lorentz transformations [16], i.e., a spin connection, while the forms A^a constitute a vierbein. To any tangent vector X^μ at some event in spacetime, which singles out a direction at that point, it relates an infinitesimal de Sitter translation $A^a{}_\mu X^\mu$ in $\mathfrak{p}$. As we have mentioned in the introductory section, this gives a precise meaning to the statement that motion in a de Sitter-Cartan spacetime is generated by de Sitter translations, and that spacetime is locally approximated by a de Sitter space. Let us emphasize once more that the decompositions (11) are well defined, since local Lorentz transformations leave the reductive splittings invariant. Due to the presence of the spin connection and vierbein, it is possible to define local Lorentz and diffeomorphism covariant differentiation, as well as a metric structure on spacetime; see [22], for example.

Given the commutation relations (8), one is able to compute the curvature F^{ab} and the torsion F^a in terms of the spin connection and vierbein. According to the expressions (6), it follows that

$$\begin{aligned} F^{ab} &= dA^{ab} + A^a{}_c \wedge A^{cb} + \frac{1}{l^2}A^a \wedge A^b && \text{(12a)} \\ &= d_A A^{ab} + \frac{1}{l^2}A^a \wedge A^b, \\ F^a &= dA^a + A^a{}_b \wedge A^b - \frac{1}{l}dl \wedge A^a && \text{(12b)} \\ &= d_A A^a - \frac{1}{l}dl \wedge A^a. \end{aligned}$$

In these equations, the exterior covariant derivatives of the spin connection and vierbein with respect to the spin connection are denoted by $d_A A^{ab}$ and $d_A A^a$, respectively. Some further remarks concerning these results are in place. Note that in the limit of an everywhere diverging length scale l, or in other words, an everywhere vanishing cosmological constant, the expressions (12) reduce to the curvature $d_A A^{ab}$ and torsion $d_A A^a$ for a Riemann-Cartan geometry [22]. In the generic case, however, the curvature and torsion are not given by the exterior covariant derivatives of the spin connection and vierbein. The extra term in (12a) represents the curvature of the local de Sitter space. This contribution is present because the commutator of two infinitesimal de Sitter transvections equals an element of the Lorentz algebra. In addition, there is a new term in the expression (12b) for the torsion if the length scale is a nonconstant function. This term comes about as follows. Remember from equation (6b) that torsion is the $\mathfrak{p}$-valued 2-form $dA_\mathfrak{p} + [A_\mathfrak{h}, A_\mathfrak{p}]$. The first term in this expression really means

$$dA_\mathfrak{p} = d(iA^aP_a) = i\,dA^aP_a - i\left(\frac{dl}{l} \wedge A^a\right)P_a,$$

since $P_a = M_{a4}/l$ and l is generally not a constant function on $\mathcal{M}$. By use of the relation (9) between l and the cosmological function Λ, the last term of the torsion can be expressed as

$$-d\ln l \wedge A^a = \tfrac{1}{2}d\ln\Lambda \wedge A^a,$$

which shows that this contribution depends on the relative infinitesimal change of the cosmological function along spacetime, rather than on its absolute change.

Although the curvature and torsion have contributions that are not there for a Riemann-Cartan geometry, the Bianchi identities are unchanged, as they are given by

$$d_A \circ d_A A^{ab} \equiv 0, \tag{13a}$$
$$d_A \circ d_A A^a + A^b \wedge d_A A_b{}^a \equiv 0, \tag{13b}$$

where d_A is the exterior covariant derivative with respect to the spin connection.

The de Sitter-Cartan geometry hitherto discussed describes a spacetime that is approximated at any point x by a de Sitter space of cosmological constant $\Lambda(x)$. The transformations that are defined for this geometry are local Lorentz transformations and spacetime diffeomorphisms. In general, the latter are unphysical in the sense that they just relabel spacetime coordinates, whereas the former have physical significance in that they relate at any point the reference frames of different idealized observers, see also [23]. In contrast, generic local $SO(1,4)$-transformations are not in the structure group of symmetries for the de Sitter-Cartan geometry. This does not come as a surprise, since the Cartan connection A can be seen as a $SO(1,4)$ Ehresmann connection, for which the structure group is restricted to the subgroup $SO(1,3)$. This breaking of symmetry is necessary to render the decompositions (11) invariant under the action of the structure group $SO(1,3)$. Hence, the definitions of the spin connection, vierbein, curvature, and torsion are consistent with local Lorentz rotations, but not with local $SO(1,4)$-transformations. For example, a local infinitesimal *pure* de Sitter translation $1 + i\epsilon \cdot P$ leads to the following variations of the spin connection and vierbein,

$$\delta_\epsilon A^{ab} = \frac{1}{l^2}(\epsilon^aA^b - \epsilon^bA^a),$$
$$\delta_\epsilon A^a = -d\epsilon^a - A^a{}_b\epsilon^b + \frac{dl}{l}\epsilon^a,$$

while for the curvature and torsion it is found that

$$\delta_\epsilon F^{ab} = \frac{1}{l^2}(\epsilon^aF^b - \epsilon^bF^a) \quad \text{and} \quad \delta_\epsilon F^a = -\epsilon^bF^a{}_b\,.$$

It is manifest that the objects A^{ab}, A^a, F^{ab}, and F^a are well defined, only up to local Lorentz transformations.

Therefore, if a theory of gravity is to be invariant under local de Sitter transformations, the geometric objects defined through the decompositions (11) of the $SO(1,4)$ Ehresmann connection and curvature, corresponding to a restriction of the structure group to $SO(1,3)$, do not lead to the necessary structure. It is the subject of the next section to preserve the presence of these geometric objects, but whose definition will be consistent, not only with local $SO(1,3)$-transformations, but also with elements of the encompassing de Sitter group $SO(1,4)$.

IV. SO(1,4)-INVARIANT DE SITTER-CARTAN GEOMETRY WITH A COSMOLOGICAL FUNCTION

In the previous section, the definitions for the spin connection and vierbein, as well as for the curvature and torsion, are not consistent with theories of gravity in which local de Sitter invariance is demanded. We have pointed out repeatedly that this is due to $\mathfrak{so}(1,4)$'s being reductive with respect to the Lorentz group only. Nevertheless is it possible to extend the structure group from $SO(1,3)$ to $SO(1,4)$, while preserving a well-defined reductive splitting of the Cartan connection and its curvature, if they are realized nonlinearly.

The formalism of nonlinear realizations was developed to systematically study spontaneous symmetry breaking in phenomenological field theory [24–26], in which linearly transforming irreducible multiplets become nonlinear but reducible realizations, when the symmetry group is broken to one of its subgroups. The realization is reducible because the broken symmetries are realized through elements of the given subgroup, see below.

A Cartan connection on a principal Lorentz bundle may be thought of as an Ehresmann connection on a principal $SO(1,4)$-bundle that is reduced to the $SO(1,3)$-bundle. At the end of Sec. II it was argued that this is essentially a symmetry breaking process, for the reason that it corresponds to singling out a section ξ of the associated bundle of tangent de Sitter spaces, thereby reducing the structure group G to H_ξ [18]. Stelle and West [21] invoked nonlinear realizations to extend the symmetry group of the MacDowell-Mansouri action for general relativity from the Lorentz to the (anti-)de Sitter group. In their work, the internal de Sitter spaces were assumed to have the same curvature at all spacetime points. Here, this assumption is relaxed and we construct a geometry for which the tangent de Sitter spaces have pseudo-radii that vary along spacetime. To begin with, let us review the necessary facts on nonlinear realizations for the de Sitter group, see also [27, 28].

Within some neighborhood of the identity, an element g of $SO(1,4)$ can uniquely be represented in the form

$$g = \exp(i\xi \cdot P)\tilde{h},$$

with $\tilde{h} \in SO(1,3)$ and $\xi \cdot P = \xi^a P_a$. The ξ^a parametrize the coset space $SO(1,4)/SO(1,3)$ so that they constitute a coordinate system for de Sitter space. This parametrization allows us to define the action of $SO(1,4)$ on de Sitter space by

$$g_0 \exp(i\xi \cdot P) = \exp(i\xi' \cdot P)h' \ ; \quad h' = \tilde{h}'\tilde{h}^{-1},$$

where $\xi' = \xi'(g_0, \xi)$ and $h' = h'(g_0, \xi)$ are in general nonlinear functions of the indicated variables. In case $g_0 = h_0$ is an element of $SO(1,3)$, the action is linear since

$$\exp(i\xi' \cdot P) = h_0 \exp(i\xi \cdot P)h_0^{-1} \ ; \quad h' = h_0,$$

and the transformation of ξ is given explicitly by

$$h_0 : i\xi \cdot P \mapsto i\xi' \cdot P = i\xi \cdot \mathrm{Ad}(h_0)(P).$$

If on the other hand $g_0 = \exp(i\alpha \cdot P)$ is a pure de Sitter translation, the coordinates ξ change according to

$$\exp(i\alpha \cdot P)\exp(i\xi \cdot P) = \exp(i\xi' \cdot P)h'.$$

For an infinitesimal translation $1 + i\epsilon \cdot P$ this becomes

$$\begin{aligned} &\exp(-i\xi \cdot P)i\epsilon \cdot P \exp(i\xi \cdot P) \\ &\qquad - \exp(-i\xi \cdot P)\delta\exp(i\xi \cdot P) = \tfrac{i}{2}\delta h \cdot M, \end{aligned} \quad (14)$$

where $\frac{i}{2}\delta h \cdot M = h' - 1 \in \mathfrak{so}(1,3)$ and $\delta h \cdot M = \delta h^{ab} M_{ab}$. Equation (14) determines the variations $\delta\xi^a$ and δh^{ab} that are generated by the considered infinitesimal de Sitter translation. They are given by [21]

$$\delta\xi^a = \epsilon^a + \Big(\frac{z\cosh z}{\sinh z} - 1\Big)\Big(\epsilon^a - \frac{\xi^a \epsilon_b \xi^b}{\xi^2}\Big), \quad (15)$$

$$\delta h^{ab} = \frac{1}{l^2}\frac{\cosh z - 1}{z \sinh z}(\epsilon^a\xi^b - \epsilon^b\xi^a), \quad (16)$$

where we made use of the notation $z = l^{-1}\xi$ and $\xi = (\eta_{ab}\xi^a\xi^b)^{1/2}$.

Subsequently, let ψ be a field that belongs to some linear representation σ of $SO(1,4)$. Given a local section of the associated bundle of homogeneous de Sitter spaces, i.e., $\xi : U \subset \mathcal{M} \to U \times dS$, the corresponding nonlinear field is constructed pointwise as

$$\bar{\psi}(x) = \sigma(\exp(-i\xi(x) \cdot P))\psi(x). \quad (17)$$

Under a local de Sitter transformation g_0, it rotates only according to its $SO(1,3)$-indices, namely,

$$\bar{\psi}'(x) = \sigma(h'(\xi, g_0))\bar{\psi}(x). \quad (18)$$

It is manifest that the irreducible linear representation ψ has given way to a nonlinear and reducible realization $\bar{\psi}$. Note that upon restricting the action of $SO(1,4)$ to the Lorentz subgroup, the transformation (18) remains linear. Let us next make use of this general framework to construct a nonlinear de Sitter-Cartan geometry. More precisely, we realize a linear $SO(1,4)$ Ehresmann connection nonlinearly, which leads to a $\mathfrak{so}(1,4)$-valued Cartan connection that transforms with respect to its $SO(1,3)$-indices only, when acted upon by local de Sitter transformations.

In concordance with the prescription (17) to construct nonlinear realizations, the nonlinear $\mathfrak{so}(1,4)$-valued Cartan connection is defined as [21]

$$\bar{A} = \mathrm{Ad}(\exp(-i\xi \cdot P))(A + d). \tag{19}$$

Under local de Sitter transformations, the field $\bar{A}$ transforms according to

$$\bar{A} \mapsto \mathrm{Ad}(h'(\xi, g_0))(\bar{A} + d),$$

which is the correct compatibility law for a local de Sitter-Cartan connection on a principal Lorentz bundle [7]. Because elements of $SO(1,4)$ are nonlinearly realized as elements of $SO(1,3)$, the reductive decomposition $\bar{A}_{\mathfrak{h}} + \bar{A}_{\mathfrak{p}}$, with $\mathfrak{h} = \mathfrak{so}(1,3)$, is invariant under local de Sitter transformations. It is then sensible to define the spin connection and vierbein through these projections as $\omega = \bar{A}_{\mathfrak{h}}$ and $e = \bar{A}_{\mathfrak{p}}$, respectively, which form a reducible multiplet for the de Sitter group.

The spin connection ω and vierbein e can be expressed in terms of the section ξ and the projections $A_{\mathfrak{h}}$ and $A_{\mathfrak{p}}$ of the linear $SO(1,4)$ connection. These relations follow from (19), in which the different objects appear according to

$$\begin{aligned}&\tfrac{i}{2}\omega^{ab}M_{ab} + ie^a P_a \\ &\qquad = \mathrm{Ad}(\exp(-i\xi \cdot P))\Big(\tfrac{i}{2}A^{ab}M_{ab} + iA^a P_a + d\Big).\end{aligned}$$

The right-hand side of this equation has to be worked out and separated in two parts, where one is valued in the Lorentz algebra $\mathfrak{so}(1,3)$ and a second takes values in the subspace of transvections $\mathfrak{p}$. The former terms constitute the spin connection, while the latter realize the vierbein. To carry out the computation we utilize the techniques of [21, 27], explained in their appendices. Basically, one expands the adjoint action of the exponential as a power series in the adjoint action of its generating element $-i\xi \cdot P$. The latter is just the Lie commutator and is given explicitly in (8). We find

$$\omega^{ab} = A^{ab} - \frac{\cosh z - 1}{l^2 z^2}\left[\xi^a(d\xi^b + A^b{}_c\xi^c) - \xi^b(d\xi^a + A^a{}_c\xi^c)\right] - \frac{\sinh z}{l^2 z}(\xi^a A^b - \xi^b A^a), \tag{20a}$$

$$e^a = A^a + \frac{\sinh z}{z}(d\xi^a + A^a{}_b\xi^b) - \frac{dl}{l}\xi^a + (\cosh z - 1)\left(A^a - \frac{\xi^b A_b \xi^a}{\xi^2}\right) - \left(\frac{\sinh z}{z} - 1\right)\frac{\xi^b d\xi_b \xi^a}{\xi^2}. \tag{20b}$$

These expressions are almost identical to the corresponding objects found by Stelle and West [21]. The difference to note is that we have a new term in the expression (20b) for the vierbein, namely, $-l^{-1}dl\,\xi^a$. This term is present in the given geometry for it is possible that the internal de Sitter spaces are characterized by cosmological constants that are not necessarily equal along spacetime. More precisely, one has to take into account the possibility that the in $\mathfrak{p}$ defined length scale is a nonconstant function, see Sec. III. On the other hand, the results of [21] specialized for the case that the local de Sitter spaces have the same pseudo-radius at any point in spacetime. The extra contribution is proportional to the dimensionless factor $l^{-1}dl \sim \Lambda^{-1}d\Lambda$, which becomes relevant only if the variation is relatively vast. When l is a constant function, one naturally recovers the results of [21].

Upon the action of local de Sitter transformations, the linear curvature F rotates in the adjoint representation. Therefore, one deduces that the nonlinear Cartan curvature $\bar{F}$ is equal to the exterior covariant derivative of the nonlinear connection, i.e.,

$$\bar{F} = \mathrm{Ad}(\exp(-i\xi \cdot P))(F) = d\bar{A} + \tfrac{1}{2}[\bar{A}, \bar{A}], \tag{21}$$

which naturally complies with the structure of a Cartan geometry. The nonlinear Cartan curvature is a $\mathfrak{so}(1,4)$-valued 2-form on spacetime, which we decompose once again according to $\bar{F} = \bar{F}_{\mathfrak{h}} + \bar{F}_{\mathfrak{p}}$. Since $\bar{F}$ transforms—in general, nonlinearly—with elements of $SO(1,3)$, the reductive splitting is invariant under local de Sitter transformations. Similarly to our discussion on the nonlinear connection $\bar{A}$, the covariant nature of the decomposition suggests that $\bar{F}_{\mathfrak{h}}$ and $\bar{F}_{\mathfrak{p}}$ must be considered the genuine curvature and torsion of the Cartan geometry, which are denoted by R, respectively T. The definition (21) implies that

$$\begin{aligned}&\tfrac{i}{2}R^{ab}M_{ab} + iT^a P_a \\ &\qquad = \mathrm{Ad}(\exp(-i\xi \cdot P))\Big(\tfrac{i}{2}F^{ab}M_{ab} + iF^a P_a\Big),\end{aligned}$$

from which one is able to express the curvature and torsion in terms of ξ, $F_{\mathfrak{h}}$ and $F_{\mathfrak{p}}$. Indeed, by computation of the adjoint action as a series of nested commutators, we are led to the results

$$\begin{aligned}R^{ab} = F^{ab} &- \frac{\cosh z - 1}{l^2 z^2}\,\xi^c(\xi^a F^b{}_c - \xi^b F^a{}_c) \\ &- \frac{\sinh z}{l^2 z}(\xi^a F^b - \xi^b F^a),\end{aligned} \tag{22a}$$

$$T^a = \frac{\sinh z}{z}\xi^b F^a{}_b + \cosh z\, F^a + (1-\cosh z)\frac{\xi_b F^b \xi^a}{\xi^2}. \tag{22b}$$

From (21) it furthermore follows that

$$R^{ab} = d_\omega \omega^{ab} + \frac{1}{l^2} e^a \wedge e^b \quad \text{and} \quad T^a = d_\omega e^a - \frac{1}{l} dl \wedge e^a.$$

These equations, which express the curvature and torsion in terms of the spin connection and vierbein, are the ones expected for a Cartan geometry. Because the exterior covariant derivative of $\bar{F}$ is always zero, there are two Bianchi identities that are formally the same as those given by (13), i.e.,

$$d_\omega \circ d_\omega \omega^{ab} \equiv 0 \quad \text{and} \quad d_\omega \circ d_\omega e^a + e^b \wedge d_\omega \omega_b{}^a \equiv 0.$$

When the section ξ is gauge-fixed along spacetime, and for convenience at any point is chosen to be the origin of the tangent de Sitter spaces, i.e., $\xi^a(x) = 0$, all the expressions reduce to those of Sec. III. This is to be expected, because the broken symmetries are not considered, and the geometry is described simply by a $SO(1,4)$ Ehresmann connection for which only $SO(1,3)$-transformations—the isotropy group of $\xi^a = 0$—are taken into account. This has precisely been the way in which the de Sitter-Cartan geometry of Sec. III was set up. In this section, on the other hand, the section ξ has been left arbitrary, thereby recovering local $SO(1,4)$ invariance. The elements of $SO(1,4)/SO(1,3)$ act nonlinearly with elements of $SO(1,3)$, so that the $\mathfrak{so}(1,4)$-valued connection $\bar{A}$ is also a Cartan connection on a principal Lorentz bundle.

Finally, let us remark that if the fields A^{ab} and A^a can be made to vanish everywhere, so that also F^{ab} and F^a are equal to zero, it follows that

$$R^{ab} = 0 \quad \text{and} \quad T^a = 0.$$

This shows that the nonhomogeneity of $\mathcal{M}$ is encoded in A and F, and naturally independent of the section ξ.

V. CONCLUSIONS AND OUTLOOK

In this work we have generalized the geometric framework of de Sitter-Cartan spacetimes with a cosmological constant to the case of a nonconstant cosmological function Λ. A de Sitter-Cartan spacetime consists of a principal Lorentz bundle over spacetime, on which is defined a $\mathfrak{so}(1,4)$-valued Cartan connection. It accounts for a spin connection and vierbein, as well as for their curvature and torsion, whereas spacetime is locally approximated by de Sitter spaces. The cosmological constants of these tangent de Sitter spaces are determined by a length scale, defined in the translational part of $\mathfrak{so}(1,4)$. By letting this length scale depend arbitrarily on the spacetime point in Sec. III, we obtained a de Sitter-Cartan geometry that accommodates a cosmological function by construction. Most importantly, it was shown that a nonconstant Λ gives rise to an extra contribution in the expression for the torsion, in which the cosmological function appears through its logarithmic derivative. In the limit $\Lambda \to 0$ one recovers the well-known Riemann-Cartan spacetime with arbitrary curvature and torsion.

The structure group of this Cartan geometry is given by $SO(1,3)$. Moreover, the definitions for the different geometric objects are not consistent with the action of generic elements of the encompassing de Sitter group. However, by nonlinearly realizing a $SO(1,4)$-connection and its curvature in Sec. IV, elements of $SO(1,4)/SO(1,3)$ were realized nonlinearly as elements of the Lorentz group, and thus included in the structure group of the Cartan geometry. This allowed for $SO(1,4)$-covariant definitions for the spin connection and vierbein, and likewise for the curvature and torsion. Again, we generalized this framework to include a nonconstant cosmological function.

We reason that such a generalized framework with nonconstant Λ can be of use to construct theories of gravity, for it extends the geometric meaning of a cosmological constant to the case of a spacetime-dependent function. The cosmological function quantifies the lack of commutation of two infinitesimal spacetime translations, and therefore manifests itself in the local kinematics on spacetime. It is then an interesting question to pose and investigate, if and how it is possible to cook up actions for theories of gravity by making use of the geometric ingredients discussed in this work, and which can account for a spacetime-dependent dark energy. Doing so, one would obtain a link between the dynamical character of Λ and its kinematical implications.

Another point of interest comes about upon noting that, when the de Sitter algebra is contracted to the Poincaré algebra, namely, when $l \to \infty$ in the commutation relations (8), the geometric objects of Sec. IV reduce to those of teleparallel gravity [29, 30]. This observation suggests that the geometry of spacetime that underlies teleparallel gravity is described by a Riemann-Cartan geometry (with vanishing curvature), for which the Poincaré translations are realized nonlinearly as elements of $SO(1,3)$. In fact, from (16) one sees that the nonlinear element of the Lorentz algebra, which corresponds to an infinitesimal Poincaré translation with parameters ϵ^a, vanishes, for

$$\delta h^{ab} = \lim_{l\to\infty} \frac{1}{l^2} \frac{\cosh z - 1}{z \sinh z} (\epsilon^a \xi^b - \epsilon^b \xi^a) = 0.$$

One then concludes that any Poincaré translation is trivially realized by the identity transformation, a property that is relied upon in the interpretation of teleparallel gravity as a gauge theory for the Poincaré translations. Given the knowledge that the geometric structure of teleparallel gravity is such a Riemann-Cartan spacetime, the de Sitter-Cartan geometry of Sec. IV might be the right framework to generalize teleparallel gravity to a theory that is invariant under local $SO(1,4)$-transformations, in place of the elements of the Poincaré group.

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