Contracted Bianchi identities – Field equations

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1 Teleparallel Gravity

As discussed in the document `cartan_geo_ext.pdf` the geometric setting underlying Teleparallel Gravity is that of a Riemann-Cartan geometry with vanishing curvature. The Bianchi identities for a generic Riemann-Cartan* geometry are given by

$$\begin{aligned} dR + [\omega, R] &\equiv 0 \ , \\ dT + [\omega, T] + [e, R] &\equiv 0 \ , \end{aligned}$$

which in the case of Teleparallel Gravity reduce to

$$\begin{aligned} 0 &\equiv 0 \ , \\ dT + [\omega, T] &\equiv 0 \ . \end{aligned}$$

Apart from these Bianchi identities for the torsion T, there are two identities at hand for the corresponding contortion K. The latter is defined through the splitting $\omega = \mathring{\omega} + K$, where $\mathring{\omega}$ is the unique Levi-Civita spin connection, i.e., the spin connection without torsion. Introducing the $\mathfrak{so}(1,3)$-valued two form $Q := dK + [\omega, K] - \frac{1}{2}[K, K]$, these identities are of the form (for a generic RC geometry)

$$\begin{aligned} dQ + [\omega, Q] - [K, Q] + [K, R] &\equiv 0 \ , \\ [e, Q] - [e, R] &\equiv 0 \ . \end{aligned}$$

Specializing for Teleparallel Gravity, one finds that

$$dQ + [\omega, Q] - [K, Q] \equiv 0 \ , \tag{1.1a}$$

$$[e, Q] \equiv 0 \ . \tag{1.1b}$$

Of course, the two sets of Bianchi identities for ω and K are related by considering the identities for the Levi-Civita connection $\mathring{\omega}$.

*RC geometry.

Introducing the notation $\mathring{D} := d + \mathring{\omega}$, the first identity (1.1a) can be expanded as $\mathring{D}_{[\rho}Q^{ab}{}_{\mu\nu]} \equiv 0$. Contracting this equation twice with the vielbein results in

$$\mathring{D}_\rho \mathcal{Q} - 2\mathring{D}_\rho e_a{}^\mu Qi^a{}_\mu + 2e_a{}^\mu \mathring{D}_\mu Qi^a{}_\rho - 2e_a{}^\mu \mathring{D}_\mu e_b{}^\nu Q^{ab}{}_{\nu\rho} \equiv 0 \ ,$$

where the notation $\mathcal{Q} := Q^{ab}{}_{\mu\nu}e_a{}^\mu e_b{}^\nu$ and $Qi^a{}_\mu := Q^{ab}{}_{\mu\nu}e_b{}^\nu$ has been used.* The vielbein postulate $D_\rho e_a{}^\mu = -\Gamma^\mu{}_{\nu\rho}e_a{}^\nu$ allows us to eliminate the covariant derivatives on the tetrads, which renders

$$\begin{aligned} 0 &\equiv e_a{}^\mu \mathring{D}_\mu Qi^a{}_\rho - e_a{}^\mu \mathring{\Gamma}^\sigma{}_{\mu\rho} Qi^a{}_\sigma - \tfrac{1}{2}\partial_\rho \mathcal{Q} \\ &\equiv \mathring{\nabla}_\mu (Qi^\mu{}_\rho - \tfrac{1}{2}\delta^\mu_\rho \mathcal{Q}) \ . \end{aligned}$$

This equation can be rewritten as follows:

$$\begin{aligned} 0 &\equiv \mathring{\nabla}_\mu Qi^\mu{}_a e^a{}_\rho + Qi^\mu{}_a \mathring{\nabla}_\mu e^a{}_\rho - \tfrac{1}{2}\mathring{\nabla}_\mu e^a{}_\rho e_a{}^\mu \mathcal{Q} - \tfrac{1}{2}e^a{}_\rho \mathring{\nabla}_\mu e_a{}^\mu \mathcal{Q} - \tfrac{1}{2}e^a{}_\rho e_a{}^\mu \partial_\mu \mathcal{Q} \\ &= \partial_\mu Qi^\mu{}_a e^a{}_\rho + \partial_\mu \ln e \, Qi^\mu{}_a e^a{}_\rho - Qi^\mu{}_a \mathring{\omega}^a{}_{b\mu} e^b{}_\rho + \tfrac{1}{2}\mathring{\omega}^a{}_{b\mu} e^b{}_\rho e_a{}^\mu \mathcal{Q} \\ &\quad - \tfrac{1}{2}e^a{}_\rho \partial_\mu e_a{}^\mu \mathcal{Q} - \tfrac{1}{2}e^a{}_\rho \partial_\mu \ln e \, e_a{}^\mu \mathcal{Q} - \tfrac{1}{2}e^a{}_\rho e_a{}^\mu \partial_\mu \mathcal{Q} \\ &= e^a{}_\rho (\partial_\mu \ln e \, Qi^\mu{}_a + \mathring{D}_\mu Qi^\mu{}_a - \tfrac{1}{2}\partial_\mu \ln e \, e_a{}^\mu \mathcal{Q} - \tfrac{1}{2}\mathring{D}_\mu e_a{}^\mu \mathcal{Q} - \tfrac{1}{2}e_a{}^\mu \partial_\mu \mathcal{Q}) \ , \end{aligned}$$

which, given that the vierbein be invertible, is true if and only if

$$\mathring{D}_\mu (e\, Qi^\mu{}_a - \tfrac{1}{2} e\, e_a{}^\mu \mathcal{Q}) \equiv 0 \ . \tag{1.2}$$

To conclude what conditions the contracted Bianchi identity (1.2) puts on the field equations of Teleparallel Gravity, we further work out the tensors $Qi^\mu{}_a$ and $\mathcal{Q}$. These were defined as contractions of the Lorentz algebra valued two-form

$$Q^{ab}{}_{\mu\nu} = D_\mu K^{ab}{}_\nu - D_\nu K^{ab}{}_\mu - K^a{}_{c\mu} K^{cb}{}_\nu + K^a{}_{c\nu} K^{cb}{}_\mu \ .$$

Let us first calculate the scalar function $\mathcal{Q} = Q^{ab}{}_{\mu\nu} e_a{}^\mu e_b{}^\nu$:

$$\begin{aligned} \mathcal{Q} &= \partial_\mu K^{\mu\nu}{}_\nu - D_\mu e_a{}^\mu K^{a\nu}{}_\nu - D_\mu e_b{}^\nu K^{\mu b}{}_\nu - \partial_\nu K^{\mu\nu}{}_\mu + D_\nu e_a{}^\mu K^{a\nu}{}_\mu \\ &\quad + D_\nu e_b{}^\nu K^{\mu b}{}_\mu - K^\mu{}_{\rho\mu} K^{\rho\nu}{}_\nu + K^\mu{}_{\rho\nu} K^{\rho\nu}{}_\mu \\ &= 2\partial_\mu K^{\mu\nu}{}_\nu + \Gamma^\mu{}_{\rho\mu} e_a{}^\rho K^{a\nu}{}_\nu + \Gamma^\nu{}_{\rho\mu} e_b{}^\rho K^{\mu b}{}_\nu - \Gamma^\mu{}_{\rho\nu} e_a{}^\rho K^{a\nu}{}_\mu \\ &\quad - \Gamma^\nu{}_{\rho\nu} e_b{}^\rho K^{\mu b}{}_\mu - K^\mu{}_{\rho\mu} K^{\rho\nu}{}_\nu + K^\mu{}_{\rho\nu} K^{\rho\nu}{}_\mu \\ &= 2\partial_\mu K^{\mu\nu}{}_\nu + 2\mathring{\Gamma}^\mu{}_{\rho\mu} K^{\rho\nu}{}_\nu + 2K^\mu{}_{\rho\mu} K^{\rho\nu}{}_\nu + 2\mathring{\Gamma}^\nu{}_{\rho\mu} K^{\mu\rho}{}_\nu + 2K^\nu{}_{\rho\mu} K^{\mu\rho}{}_\nu \\ &\quad - K^\mu{}_{\rho\mu} K^{\rho\nu}{}_\nu + K^\mu{}_{\rho\nu} K^{\rho\nu}{}_\mu \\ &= \frac{2}{e}\partial_\mu (e K^{\mu\nu}{}_\nu) + K^\nu{}_{\rho\mu} K^{\mu\rho}{}_\nu - K^\mu{}_{\rho\mu} K^{\nu\rho}{}_\nu \ . \end{aligned}$$

*Let us note, *en passant*, that due to the identity (1.1b), $Qi_{\mu\nu}$ is a symmetric tensor.

Subsequently we take a better look at $Qi^{\mu}{}_{a} = Q^{cb}{}_{\rho\nu} e_{c}{}^{\mu} e_{a}{}^{\rho} e_{b}{}^{\nu}$:

$$
\begin{aligned}
Qi^{\mu}{}_{a} &= D_{\rho}(e_{a}{}^{\rho} K^{\mu\nu}{}_{\nu}) - D_{\rho} e_{c}{}^{\mu} e_{a}{}^{\rho} K^{c\nu}{}_{\nu} - D_{\rho} e_{a}{}^{\rho} K^{\mu\nu}{}_{\nu} - e_{a}{}^{\rho} D_{\rho} e_{b}{}^{\nu} K^{\mu b}{}_{\nu} - D_{\nu} K^{\mu\nu}{}_{a} \\
&\quad + D_{\nu} e_{c}{}^{\mu} K^{c\nu}{}_{a} + D_{\nu} e_{a}{}^{\rho} K^{\mu\nu}{}_{\rho} + D_{\nu} e_{b}{}^{\nu} K^{\mu b}{}_{a} - K^{\mu}{}_{\rho a} K^{\rho\nu}{}_{\nu} + K^{\mu}{}_{\rho\nu} K^{\rho\nu}{}_{a} \\
&= D_{\nu} K^{\nu\mu}{}_{a} - \Gamma^{\nu}{}_{\rho\nu} e_{b}{}^{\rho} K^{\mu b}{}_{a} + D_{\rho}(e_{a}{}^{\rho} K^{\mu\nu}{}_{\nu}) + \Gamma^{\rho}{}_{\sigma\rho} e_{a}{}^{\sigma} K^{\mu\nu}{}_{\nu} + \Gamma^{\mu}{}_{\sigma\rho} e_{c}{}^{\sigma} e_{a}{}^{\rho} K^{c\nu}{}_{\nu} \\
&\quad + e_{a}{}^{\rho} \Gamma^{\nu}{}_{\sigma\rho} e_{b}{}^{\sigma} K^{\mu b}{}_{\nu} - \Gamma^{\mu}{}_{\rho\nu} e_{c}{}^{\rho} K^{c\nu}{}_{a} - \Gamma^{\rho}{}_{\sigma\nu} e_{a}{}^{\sigma} K^{\mu\nu}{}_{\rho} - K^{\mu}{}_{\rho a} K^{\rho\nu}{}_{\nu} \\
&\quad + K^{\mu}{}_{\rho\nu} K^{\rho\nu}{}_{a} \\
&= e^{-1} D_{\nu}(e K^{\nu\mu}{}_{a}) + K^{\nu}{}_{\rho\nu} K^{\rho\mu}{}_{a} + e^{-1} D_{\nu}(e\, e_{a}{}^{\nu} K^{\mu\rho}{}_{\rho}) + K^{\rho}{}_{a\rho} K^{\mu\nu}{}_{\nu} \\
&\quad + e_{a}{}^{\rho} \Gamma^{\mu}{}_{\sigma\rho} K^{\sigma\nu}{}_{\nu} + e_{a}{}^{\rho} (\Gamma^{\nu}{}_{\sigma\rho} - \Gamma^{\nu}{}_{\rho\sigma}) K^{\mu\sigma}{}_{\nu} - K^{\mu}{}_{\rho\nu} K^{\rho\nu}{}_{a} - K^{\mu}{}_{\rho a} K^{\rho\nu}{}_{\nu} \\
&\quad + K^{\mu}{}_{\rho\nu} K^{\rho\nu}{}_{a} \\
&= e^{-1} D_{\nu}(e K^{\nu\mu}{}_{a} + e\, e_{a}{}^{\nu} K^{\mu\rho}{}_{\rho}) + T^{\nu}{}_{\sigma a} K^{\sigma\mu}{}_{\nu} + e_{a}{}^{\rho} \Gamma^{\mu}{}_{\sigma\rho} K^{\sigma\nu}{}_{\nu} + K^{\rho}{}_{a\rho} K^{\mu\nu}{}_{\nu} \ .
\end{aligned}
$$

One may consider the difference

$$
\begin{aligned}
Qi^{\mu}{}_{a} &- \tfrac{1}{2} e_{a}{}^{\mu} \mathcal{Q} \\
&= e^{-1} D_{\nu}(e K^{\nu\mu}{}_{a} + e\, e_{a}{}^{\nu} K^{\mu\rho}{}_{\rho}) + T^{\nu}{}_{\sigma a} K^{\sigma\mu}{}_{\nu} + e_{a}{}^{\rho} \Gamma^{\mu}{}_{\sigma\rho} K^{\sigma\nu}{}_{\nu} + K^{\rho}{}_{a\rho} K^{\mu\nu}{}_{\nu} \\
&\quad - e^{-1} \partial_{\nu}(e\, e_{a}{}^{\mu} K^{\nu\rho}{}_{\rho}) + \partial_{\nu} e_{a}{}^{\mu} K^{\nu\rho}{}_{\rho} - \tfrac{1}{2} e_{a}{}^{\mu} K^{\nu}{}_{\rho\sigma} K^{\sigma\rho}{}_{\nu} + \tfrac{1}{2} e_{a}{}^{\mu} K^{\sigma}{}_{\rho\sigma} K^{\nu\rho}{}_{\nu} \\
&= e^{-1} D_{\nu}(e K^{\nu\mu}{}_{a} + e\, e_{a}{}^{\nu} K^{\mu\rho}{}_{\rho} - e\, e_{a}{}^{\mu} K^{\nu\rho}{}_{\rho}) + T^{b}{}_{\sigma a} K^{\sigma\mu}{}_{b} + K^{\rho}{}_{a\rho} K^{\mu\nu}{}_{\nu} \\
&\quad + e_{a}{}^{\rho} (\Gamma^{\mu}{}_{\sigma\rho} - \Gamma^{\mu}{}_{\rho\sigma}) K^{\sigma\nu}{}_{\nu} - \tfrac{1}{2} e_{a}{}^{\mu} \mathcal{L}_{\rm tg} \\
&= \tfrac{1}{2} e^{-1} D_{\nu}(e\, W_{a}{}^{\nu\mu}) + T^{b}{}_{\sigma a} K^{\sigma\mu}{}_{b} - T^{\mu}{}_{\sigma a} K^{\sigma\nu}{}_{\nu} + K^{\rho}{}_{a\rho} K^{\mu\nu}{}_{\nu} - \tfrac{1}{2} e_{a}{}^{\mu} \mathcal{L}_{\rm tg} \ ,
\end{aligned}
$$

where the notation $W_{a}{}^{\nu\mu} = 2(K^{\nu\mu}{}_{a} + e_{a}{}^{\nu} K^{\mu\rho}{}_{\rho} - e_{a}{}^{\mu} K^{\nu\rho}{}_{\rho})$ is used in the last line. Since $K^{a\nu}{}_{\nu} = T^{\nu a}{}_{\nu}$, one further concludes that

$$
\begin{aligned}
Qi^{\mu}{}_{a} &- \tfrac{1}{2} e_{a}{}^{\mu} \mathcal{Q} \\
&= \tfrac{1}{2} e^{-1} D_{\nu}(e\, W_{a}{}^{\nu\mu}) + T^{b}{}_{\nu a} K^{\nu\mu}{}_{b} - T^{b}{}_{\nu a} e_{b}{}^{\mu} K^{\nu\rho}{}_{\rho} + T^{\rho}{}_{\rho a} K^{\mu\nu}{}_{\nu} - \tfrac{1}{2} e_{a}{}^{\mu} \mathcal{L}_{\rm tg} \\
&= \tfrac{1}{2} e^{-1} D_{\nu}(e\, W_{a}{}^{\nu\mu}) + T^{b}{}_{\nu a} (K^{\nu\mu}{}_{b} + e_{b}{}^{\nu} K^{\mu\rho}{}_{\rho} - e_{b}{}^{\mu} K^{\nu\rho}{}_{\rho}) - \tfrac{1}{2} e_{a}{}^{\mu} \mathcal{L}_{\rm tg} \\
&= \tfrac{1}{2} \left[e^{-1} D_{\nu}(e\, W_{a}{}^{\nu\mu}) + T^{b}{}_{\nu a} W_{b}{}^{\nu\mu} - e_{a}{}^{\mu} \mathcal{L}_{\rm tg} \right] \ .
\end{aligned}
$$

Finally, the contracted Bianchi identities (1.2) imply that

$$
\mathring{D}_{\mu} \Big[D_{\nu}(e\, W_{a}{}^{\nu\mu}) + e\, T^{b}{}_{\nu a} W_{b}{}^{\nu\mu} - e\, e_{a}{}^{\mu} \mathcal{L}_{\rm tg} \Big] \equiv 0 \ . \tag{1.3}
$$