Geodesics: an attempt

1 Introduction

Consider a homogeneous manifold $M = G/H = \{gH | g \in G\}$. Assume that M is reductive, i.e. there exists a subspace $\mathfrak{p} \simeq T_pM \in \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. Let $o = H$ be the origin of M. The left action of G on itself induces a left action of G on M, that is

$$\lambda_g : M \to M : g_0H \mapsto gg_0H$$

This action is transitive since the left action of G on itself is transitive. Lastly, consider the set of right invariant vector fields $\{X_a\}(a = 1 \ldots \dim G)$, which span the Lie algebra $\mathfrak{g}$. They generate the left action of G on itself, i.e., if a_t is the integral curve of $\epsilon^a X_a$ through e, $a_t = \exp(t\epsilon^a X_a)$.

Then let us define the geodesics $\gamma(t)$ through $o \in M$ as the curves obtained by considering the left action G on o,

$$\gamma(t) = a_t(o) = \exp(t\epsilon^a X_a)o \tag{1.1}$$

The tangent vector field to $\gamma(t)$ is left invariant under the 1-parameter group a_t, as we show now. Consider this tangent vector field $\dot{\gamma}$

$$\dot{\gamma}(t)f = \tfrac{d}{dt} f(\gamma(t))$$

where f is an arbitrary function along $\gamma(t)$. Left translating $\dot{\gamma}$ then results in

$$(\lambda_{a_s *}\dot{\gamma}(t))f = \dot{\gamma}(t)(f \circ \lambda_{a_s}) = \tfrac{d}{dt} f(a_{s+t}(o)) = \dot{\gamma}(s+t)$$

2 An attempt: de Sitter spacetime

Inspired by the introduction, we define a geodesic through p as a curve which is of the form $\gamma(t) = a_t(p)$. This implies that parallel transport is defined through left translation, canonically given for de Sitter spacetime.

Let $Y = Y^\mu(x)\partial_\mu$ be a vector field defined on some open subset of dS. We translate it from x to $x' = g(x)$ where $g = \exp(\epsilon^a \xi_a)$. For f an arbitrary function on dS, one has

$$Y^\mu(x)\partial_\mu \mapsto (\lambda_g Y)^\mu(x')\partial'_\mu f = Y^\nu(x)\frac{\partial x'^\mu}{\partial x^\nu}\partial'_\mu f$$

We are interested in finding differential equations to be fulfilled by geodesics, hence we consider infinitesimal left translations. i.e.

$$(\lambda_g Y)^\mu(x + \Delta x) = Y^\lambda \partial_\lambda (x^\mu + \Delta x^\rho \delta^a_\rho \xi^\mu_a(x)) \tag{2.1}$$

A covariant derivative is then defined in the usual way, that is

$$\nabla_\nu Y = \lim_{\Delta x^\nu} \tfrac{1}{\Delta x^\nu}(Y^\mu(x+\Delta x) - (\lambda_g Y)^\mu(x+\Delta x))\partial_\mu$$

which is found to be equal to

$$\nabla_\nu Y = (\partial_\nu Y^\mu(x) - Y^\lambda \partial_\lambda \xi^\mu_\nu(x))\partial_\mu \tag{2.2}$$

It is easily verified that the covariant derivative of Y in the direction of Z is given by

$$\nabla_Z Y = Z^\nu \nabla_\nu Y = Z^\nu(\partial_\nu Y^\mu(x) - Y^\lambda \partial_\lambda \xi^\mu_\nu(x))\partial_\mu \tag{2.3}$$

This leaves us in a position where we can define geodesics: *geodesics are curves $\gamma(t)$ such that its tangent vector field is parallel transported along the curve, under left de Sitter translations.* Let $x^\mu(t)$ be the coordinates of $\gamma(t)$. Since the defining differential equation for a geodesic is $\nabla_{\dot\gamma(t)}\dot\gamma(t) \equiv 0$, we have in components that

$$\ddot x^\mu - \dot x^\nu \dot x^\lambda \partial_\lambda \xi^\mu_a \delta^a_\nu = 0 \tag{2.4}$$

Before we compare this result with the usual geodesic equation, i.e. when parallel transport is defined w.r.t. the Levi-Civita connection, we make (2.4) explicit in stereographic coordinates. Therefore we calculate the derivative of $\xi^\mu_a = -\mathfrak{s}\delta^\mu_a - (4l^2)^{-2}(2\eta_{a\rho}x^\rho x^\mu - \sigma^2\delta^\mu_a)$ w.r.t. the coordinates, that is

$$\partial_\lambda \xi^\mu_a = -\tfrac{1}{4l^2}(2\eta_{a\lambda}x^\mu + 2\eta_{a\rho}x^\rho\delta^\mu_\lambda - 2\eta_{\lambda\sigma}x^\sigma\delta^\mu_a)$$

Substituting for (2.4), then gives

$$\ddot x^\mu + \tfrac{1}{4l^2}\dot x^\nu \dot x^\lambda \eta_{\nu\lambda}x^\mu = 0 \tag{2.5}$$

According to our definition of parallel transport through (left) de Sitter translations, these are the differential equations in stereographic coordinates, whose solutions are geodesics on $(A)dS$.

3 Comparison with the standard defintion

To compare with the usual definition of geodesics in pseudo-Riemannian spacetimes, let us remind that the defining differential equations are given by

$$\ddot x^\mu + \dot x^\nu \dot x^\lambda \Gamma^\mu_{\lambda\nu} = 0 \tag{3.1}$$

where, for $(A)dS$ and using stereographic coordinates, the Levi-Civita connection is locally expressed as

$$\Gamma^\mu_{\lambda\nu} = (\delta^\mu_\nu\delta^\sigma_\lambda + \delta^\mu_\lambda\delta^\sigma_\nu - \eta^{\mu\sigma}\eta_{\lambda\nu})\partial_\sigma \ln|\Omega|$$

and $\Omega(x) = (1 + \frac{\mathfrak{s}\sigma^2}{4l^2})^{-1}$.

Some algebra then gives us the explicit differential equations,

$$\ddot x^\mu - \tfrac{\mathfrak{s}\Omega}{4l^2}(4\dot x^\mu \dot x^\sigma \eta_{\sigma\kappa}x^\kappa - \dot x^\nu \dot x^\lambda \eta_{\lambda\nu}x^\mu) = 0 \tag{3.2}$$