De Sitter relativity and stereographic coordinates: introduction

1 de Sitter space time: stereographic coordinates

Let $E^{1,4}$ be the five-dimensional flat spacetime with Lorentzian signature. De Sitter spacetime is the hypersurface defined by

$$\eta_{AB}\chi^A\chi^B = \eta_{\mu\nu}\chi^\mu\chi^\nu + \mathfrak{s}(\chi^4)^2 = \mathfrak{s}l^2 \tag{1}$$

where $A = 0\dots 4$ and $\mu = 0\dots 3$.

In fact, this hypersurface both describes a de Sitter spacetime dS as well as an anti de Sitter spacetime AdS, depending on the value of $\mathfrak{s} \equiv \eta_{44}$. Table 1 gives a resumé for these values when different sign conventions are used. In the following we will denote these surfaces as a de Sitter spacetime, however the treatment will be equally valid for anti de Sitter spacetimes as well through the free parameter $\mathfrak{s}$.

	$\mathfrak{s} = -1$	$\mathfrak{s} = +1$
$\eta_{AB} = (+,-)$	dS	AdS
$\eta_{AB} = (-,+)$	AdS	dS

Table 1: Sign convention resumé

A useful coordinate system parametrizing $dS(1,4)$ is given by stereographic coordinates. This chart is defined in the following way. Consider the north pole (N) at $(0,0,0,0,l)$ and the hyperplane $\chi^4 = -l$, which contains the south pole of $dS(1,4)$. We project a point P of de Sitter spacetime onto a point p of the hyperplane by taking the line through N and P which gives the projected point p, there where the line meets the hyperplane.

Since N, P and p always lie on a straight line we have

$$\frac{\chi(P) - \chi(N)}{\chi(p) - \chi(N)} = n$$

where n will depend on the considered projection line. The projected point p lies in the subspace $E^{(1,3)}$ (i.e. $\chi^4 = -l$), with coordinates x^μ where $(\mu = 0\dots 3)$. Hence, we have

$$\frac{\chi^\mu - 0}{x^\mu - 0} = n, \quad \frac{\chi^4 - l}{-l - l} = n$$

such that we find

$$\chi^\mu = n x^\mu \tag{2}$$

$$\chi^4 = l(1 - 2n) \tag{3}$$

Remembering the fact that we consider points P on the $dS(1,4)$ we have to add the defining equation (1) for this hypersurface, namely

$$(\chi^4)^2 = l^2 - \mathfrak{s}n^2\eta_{\mu\nu}x^\mu x^\nu \tag{4}$$

Substituting (3) and (4) for χ^4 we find a defining equation for n in function of the other four coordinates, i.e.[1]

$$n^2(4l^2 + \mathfrak{s}\sigma^2) - 4nl^2 = 0 \tag{5}$$

which has two real solutions, given by $n = 0$ (for the projection of the north pole) and

$$n = \left(1 + \mathfrak{s}\frac{\sigma^2}{4l^2}\right)^{-1} =: \Omega(x) \tag{6}$$

which is the factor used in projecting all other points.

The stereographic coordinates x^μ are then given by

$$\chi^\mu = \Omega(x)x^\mu \tag{7}$$

$$\chi^4 = -l\Omega(x)\left(1 - \mathfrak{s}\frac{\sigma^2}{4l^2}\right) \tag{8}$$

which is defined for all points on $dS(1,4)$ but the north pole.

The inverse transformations are easily found to be

$$x^\mu = \Omega^{-1}\chi^\mu \quad \text{with} \quad \Omega = -\frac{1}{2}\left(\frac{\chi^4}{l} - 1\right) \tag{9}$$

2 Geometric quantities

In this section some objects will be derived, describing the geometric structure of de Sitter spacetime.

Metric tensor First we calculate the line element on dS in terms of stereographic coordinates. This line element is induced from the flat metric η_{AB} on $E^{(1,4)}$, hence

$$ds^2 = \eta_{AB}d\chi^A d\chi^B \quad \text{with} \quad \chi^4 = \pm(l^2 - \mathfrak{s}\eta_{\mu\nu}\chi^\mu\chi^\nu)^{\frac{1}{2}} \tag{10}$$

Our task now is to calculate the differentials appearing in this line element in terms of the stereographic coordinates x^μ. It will be useful to use the following relations between Ω, σ and its differentials

$$\sigma^2 = \mathfrak{s}4l^2\frac{1-\Omega}{\Omega}, \quad d(\sigma^2) = -\mathfrak{s}\frac{4l^2}{\Omega^2}d\Omega$$

substituting the Cartesian coordinates for the stereographic ones, we find

$$\eta_{\mu\nu}d\chi^\nu d\chi^\nu = \eta_{\mu\nu}d(\Omega(x)x^\mu)d(\Omega(x)x^\nu) \tag{11}$$

$$= d\Omega^2\sigma^2 + \Omega d\Omega d(\sigma)^2 + \Omega^2\sigma^2 \tag{12}$$

and

$$(d\chi^4)^2 = d(l - 2l\Omega)^2 = 4l^2 d\Omega^2 \tag{13}$$

Combining these results we find that the line element of de Sitter spacetime

$$ds^2 = \Omega^2(x)\eta_{\mu\nu}dx^\mu dx^\nu \tag{14}$$

which shows that the Sitter spacetime is conformally flat.

Although this line element is conformally flat, de Sitter spacetime does not have the same causal structure as Minkowski spacetime. [Question: What is the explanation for this apparent paradox?] [Possible answer: the transformation is singular at $\sigma^2 = 4l^2$. In fact this three-dimensional hypersurface is the image of past and future null infinity, since for these points Ω goes to infinity and we have that the χ^μ go to infinity as $\mathcal{O}(\Omega)$. From the equation of dS it follows that χ^4 goes to infinity, at the same rate. For these points l is essentially zero, such that its tangent space is the lightcone of the embedding Minkowski space. Hence, we are considering null-like infinity, which turns out to be three-dimensional. Note also that the transformation does not make a difference between such a point at infinity and its spacetime inversion, such that these two points are represented by the same stereographic coordinate.]

[1] We define $\sigma^2 = \eta_{\mu\nu}x^\mu x^\nu$.

Christoffel symbol Given the metric tensor, one obtains the corresponding Christoffel symbol

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_\mu \nu \right)$$

A rather short calculation shows that the symbol is given by

$$\Gamma^{\rho}{}_{\mu\nu} = \left(\delta^\rho_\nu \delta^\sigma_\mu + \delta^\rho_\mu \delta^\sigma_\nu - \eta^{\rho\sigma}\eta_{\mu\nu} \right) \partial_\sigma \ln |\Omega(x)| \tag{15}$$

Riemann tensor The Riemann curvature tensor can be defined through the Christoffel symbol as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\alpha\rho}\Gamma^\alpha{}_{\nu\sigma} + \Gamma^\mu{}_{\alpha\sigma}\Gamma^\alpha{}_{\nu\rho}$$

After a somewhat lengthier calculation one finds the familier curvature tensor for a maximally symmetric space

$$R^\mu{}_{\nu\rho\sigma} = \frac{\mathfrak{s}}{l^2} \left(\delta^\mu_\rho g_{\sigma\nu} - \delta^\mu_\sigma g_{\nu\rho} \right) \tag{16}$$

Ricci tensor Contracting the first with the third index of the Riemann tensor, one defines the Ricci tensor

$$R_{\mu\nu} = \frac{3\mathfrak{s}}{l^2} g_{\mu\nu} \tag{17}$$

Ricci scalar Finally, upon contracting the two indices of the Ricci tensor, the Ricci scalar is found, namely

$$R = \frac{12\mathfrak{s}}{l^2} \tag{18}$$

3 de Sitter space as a homogeneous space

De Sitter spacetime can be constructed as a pseudo-Riemannian symmetric space $dS = SO(1,4)/\mathcal{L}$.[2] This directly implies it is a homogeneous space, transitive under the elements generated by the Lie algebra $\mathfrak{p} = \mathfrak{so}(1,4) \bmod \mathfrak{so}(1,3)$. It is also a principal bundle $P(\mathcal{L}, dS)$ where the fibers are the Lorentz rotations.

This identification between de Sitter spacetime and its Lie group of isometries gives us some interesting tools to investigate its structure. For example, it is possible to derive geometric properties of the de Sitter spacetime from the Cartan-Killing form defined on the de Sitter Lie algebra. It also gives one a natural way to find the limiting spacetimes when varying the cosmological constant. Indeed, through an Inönü-Wigner contraction of the Lie algebra the corresponding pseudo-Riemannian spacetime can be constructed.

de Sitter spacetime - finite Λ Since de Sitter spacetime can be identified with the homogeneous space

$$dS(1,4) \equiv \frac{SO(1,4)}{\mathcal{L}} \tag{19}$$

it is transitive under the elements generated by (the basis elements of $\mathfrak{so}(1,4) \bmod \mathfrak{so}(1,3)$)

$$L_{\mu 4} = \mathfrak{s} l P_\mu + \frac{1}{4l} K_\mu \tag{20}$$

Minkowski spacetime - zero Λ ($l \to \infty$) Performing an adequate Inönü-Wigner contraction of the de Sitter group (through its algebra) one finds the Poincaré group $\mathcal{P}$, which is the semidirect product of the Lorentz group and ordinary translations. Hence, the de Sitter spacetime has been contracted to

$$M \equiv \frac{\mathcal{L} \rtimes \mathcal{T}}{\mathcal{L}} \tag{21}$$

a homogeneous space transitive under ordinary translations (generated by P_μ), Minkowski spacetime.

[2] [Doubt] The involution defining a Cartan decomposition is probably the spacetime inversion - algebraically the generators $L_{\mu 4}$ span the eigenspace corresponding to -1.

Conic spacetime - infinite Λ $(l \to 0)$ In this limit one contracts $SO(4,1)$ to the conformal Poincaré group $\mathcal{Q}$, the semidirect product of the Lorentz group and the special conformal group. The de Sitter space then becomes a (homogeneous) cone-spacetime

$$N \equiv \frac{\mathcal{L} \rtimes \mathcal{C}}{\mathcal{L}} \tag{22}$$

which is transitive under special conformal transformations (generated by K_μ).

Remark Transitivity on spacetime is not longer obtained through ordinary translations, but through a combination of translations and special conformal transformations (in stereographic coordinates). This will have deep implications. A direct consequence is that the notion of differentiation will change; indeed, ordinary differentiation of functions on manifolds is based on an underlying movement of translations, i.e.

$$\partial_\mu f(x^\mu) \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[f(x^\mu + \delta^\mu_{(\nu)} \varepsilon^\nu) - f(x^\mu) \right] \tag{23}$$

The tangent space at a point p of a manifold M is the space spanned by a maximal set of linearly independent differential operators at the given point. Usually these diferential operators are the ordinary derivatives, the generators of translations. If the manifold M is pseudo-Riemannian, the tangent space obtained in this way will be flat. [Through what argument? Work out. Probably it has to do with a submanifold generated by translations in the embedding manifold, which has the same signature as M.]

However, we could span the tangent space with a different set of linearly independent differential operators $\hat{\partial}$

$$\hat{\partial}_\mu f(x^\mu) \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[f(x^\mu + \xi^\mu_{(\nu)}(x) \varepsilon^\nu) - f(x^\mu) \right] \tag{24}$$

which would be a space transitive under the elements generated by $\hat{\partial}$.

If one takes the differential operators to be the de Sitter translations $L_{\mu 4}$, the tangent space at some manifold M, would be given by de Sitter spacetime dS. Note that a de Sitter spacetime, would be locally de Sitter - as a Minkowski spacetime is locally Minkowski. It gives a very precise meaning on how the equivalence principle changes in a de Sitter general relativity - and how it is the only change being made. General relativity still presumes that spacetime is a four-dimensional manifold with Lorentzian signature, where the metric is a solution to Einstein's equations. However, the modified equivalence principle states that one always can find a coordinate system in which the laws of physics are locally given by the laws of de Sitter special relativity. Indeed, the tangent space (the local structure) at a spacetime point is a de Sitter spacetime. ■

4 Spacetime inversion: duality between M and N

Let us begin with introducing the *spacetime inversion* on stereographic coordinates

$$x^\mu \to \bar{x}^\mu \equiv -\frac{x^\mu}{\sigma^2} \tag{25}$$

where as usual, $\sigma^2 = \eta_{\mu\nu} x^\mu x^\nu$. A first interesting question that may be posed is how this quantity behaves under the inversion (25). It is helpful to note that the notation $\eta_{\mu\nu} x^\mu x^\nu$ was introduced as a shorthand for $-(x^0)^2 + (x^i)^2$, rather than having a geometric meaning. For example, σ^2 in general is *not* the invariant length of x^μ. Therefore, it is plausible to assume the following transformation behaviour of σ^2 under the spacetime inversion, that is

$$\sigma^2 \to \bar{\sigma}^2 = \eta_{\mu\nu} \bar{x}^\mu \bar{x}^\nu = \frac{\eta_{\mu\nu} x^\mu x^\nu}{\sigma^4} = \frac{1}{\sigma^2} \tag{26}$$

[Hence, the $\eta_{\mu\nu}$ in σ^2 should *not* be thought of as the Minkowski tensor and considered constant under the inversion—the assumption is important for finding the inverse of the inversion and the consequences thereof]. The inverse transformation of (25) is then easily found to be

$$\bar{x}^\mu \to x^\mu = -\frac{\bar{x}^\mu}{\bar{\sigma}^2} \tag{27}$$

where, as discussed, $\bar{\sigma}^2 = \eta_{\mu\nu}\bar{x}^\mu\bar{x}^\nu$.

Given these facts, it is possible to find how the vector fields P_μ and K_μ transform. More specifically, invoking the chain rule for partial differentiation and (27) one calculates,

$$\begin{aligned}\bar{P}_\mu &= \frac{\partial}{\partial\bar{x}^\mu} = \frac{\partial x^\mu}{\partial\bar{x}^\mu}\frac{\partial}{\partial x^\nu} \\ &= -(\bar{\sigma}^{-2}\delta^\nu_\mu - \bar{x}^\nu\bar{\sigma}^4 2\eta_{\rho\sigma}\delta^\rho_\mu\bar{x}^\sigma)\frac{\partial}{\partial x^\nu} \\ &= (2\eta_{\mu\rho}x^\rho x^\nu - \sigma^2\delta^\nu_\mu)\frac{\partial}{\partial x^\nu} = K_\mu\end{aligned}$$

and

$$\begin{aligned}\bar{K}_\mu &= (2\eta_{\mu\rho}\bar{x}^\rho\bar{x}^\lambda - \bar{\sigma}^2\delta^\lambda_\mu)\bar{\partial}_\lambda \\ &= (2\eta_{\mu\rho}\frac{x^\rho x^\lambda}{\sigma^4} - \sigma^{-2}\delta^\lambda_\mu)(2\eta_{\lambda\sigma}x^\sigma x^\nu - \sigma^2\delta^\nu_\lambda)\partial_\nu \\ &= (4\eta_{\mu\rho}x^\rho x^\nu\sigma^{-2} - 2\eta_{\mu\rho}x^\rho x^\nu\sigma^{-2} - 2\eta_{\mu\sigma}x^\sigma x^\nu\sigma^{-2} + \delta^\nu_\mu)\partial_\nu \\ &= \frac{\partial}{\partial x^\mu} = P_\mu\end{aligned}$$

Furthermore, the vector fields $L_{\mu\nu}$ generating Lorentz transformations transform as

$$\begin{aligned}\bar{L}_{\mu\nu} &= \eta_{\mu\lambda}\bar{x}^\lambda\bar{\partial}_\nu - \eta_{\nu\lambda}\bar{x}^\lambda\bar{\partial}_\mu \\ &= -\eta_{\mu\lambda}x^\lambda\sigma^{-2}(2\eta_{\nu\sigma}x^\sigma x\rho - \sigma^2\delta^\rho_\nu)\partial_\nu - [\mu\leftrightarrow\nu] \\ &= \eta_{\mu\lambda}x^\lambda\partial_\nu - \eta_{\nu\lambda}x^\lambda\partial_\mu = L_{\mu\nu}\end{aligned}$$

One concludes that translations and special conformal transformations are interchanged under a spacetime inversion, while Lorentz rotations are unchanged, that is

$$\begin{aligned}P_\mu &\to \bar{P}_\mu = K_\mu \\ K_\mu &\to \bar{K}_\mu = P_\mu \\ L_{\mu\nu} &\to \bar{L}_{\mu\nu} = L_{\mu\nu}\end{aligned} \tag{28}$$

Because of the above alluded homogeneous character of M and N, these transformation properties directly imply the interchange of M and N under a spacetime inversion (27). In this sense, they are said to be dual to eachother and the duality relation is given through (28).

Given this duality we might wonder what happens with the Minkowski line element under a spacetime inversion. Since M goes into N under this transformation, the resulting line element may be interpreted as the invariant line element on N. Therefore,

$$dx^\mu \to d\bar{x}^\mu = d(-x^\mu\sigma^{-2}) = -\frac{dx^\mu}{\sigma^2} + \frac{x^\mu}{\sigma^4}2\eta_{\mu\nu}dx^\mu x^\nu$$

which we use for calculating

$$\begin{aligned}\eta_{\mu\nu}d\bar{x}^\mu d\bar{x}^\nu &= \eta_{\mu\nu}(-\sigma^{-2}dx^\mu + 2\sigma^{-4}x^\mu\eta_{\rho\sigma}dx^\rho x^\sigma)(-\sigma^{-2}dx^\nu + 2\sigma^{-4}x^\nu\eta_{\alpha\beta}dx^\alpha x^\beta) \\ &= \eta_{\mu\nu}\sigma^{-4}dx^\mu dx^\nu - 2\eta_{\mu\nu}\sigma^{-6}dx^\mu x^\nu\eta_{\alpha\beta}dx^\alpha x^\beta \\ &\quad - 2\eta_{\mu\nu}\sigma^{-6}dx^\nu x^\mu\eta_{\rho\sigma}dx^\rho x^\sigma + 4\sigma^{-8}x^\mu x^\nu\eta_{\rho\sigma}\eta_{\alpha\beta}dx^\rho x^\sigma dx^\alpha x^\beta \\ &= \eta_{\mu\nu}\sigma^{-4}dx^\mu x^\nu\end{aligned}$$

We thus have the following behaviour of the line element ds^2 under a spacetime inversion,

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \to d\bar{s}^2 = \eta_{\mu\nu}d\bar{x}^\mu d\bar{x}^\nu = \frac{\eta_{\mu\nu}}{\sigma^4}dx^\mu dx^\nu \tag{29}$$

A simpler [and equivalent?] derivation would be $\sigma^2 \to \sigma^{-2} = \sigma^{-4}\eta_{\mu\nu}x^\mu x^\nu$. Note that we did not consider any transformation on the Minkowski matrix $\eta_{\mu\nu}$. Of course, if one would consider the spacetime inversion as a diffeomorphism under which the Minkowski matrix transforms as a tensor, the line element would be a scalar.

Next we show that the above introduced matrix $\eta_{\mu\nu}\sigma^{-4}$ is invariant under special conformal transformations. A special conformal transformation (SCT) is composed of an inversion, followed by a translation and another inversion. Therefore, we first calculate the transformation behaviour of $\eta_{\mu\nu}$ under a spacetime inversion. To make it clear: now we do consider it as a tensor, i.e. the inversion acts once for each index. The reason is that the object $\eta_{\mu\nu}\sigma^{-4}$ is to be interpreted as the *metric tensor* on N. As will be shown, it is invariant under SCTs (and LTs) and therefore can be considered defining the conformal Poincaré group as the group leaving $\eta_{\mu\nu}\sigma^{-4}$ invariant. Because a second rank tensor transforms as

$$\eta_{\mu\nu} \to \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}\eta_{\alpha\beta}$$

and since $\partial x^\alpha / \partial \bar{x}^\mu = 2\eta_{\mu\rho}x^\rho x^\alpha - \sigma^2\delta^\alpha_\mu$, one finds

$$\begin{aligned}\bar{\eta}_{\mu\nu} &= (2\eta_{\mu\sigma}x^\sigma x^\alpha - \sigma^2\delta^\alpha_\mu)(2\eta_{\nu\rho}x^\rho x^\beta - \sigma^2\delta^\beta_\nu)\eta_{\alpha\beta} \\ &= 4\eta_{\mu\sigma}\eta_{\nu\rho}x^\sigma x^\rho - 2\sigma^2\eta_{\mu\sigma}\eta_{\alpha\nu}x^\sigma x^\alpha - 2\sigma^2\eta_{\mu\beta}\eta_{\nu\rho}x^\beta x^\rho + \sigma^4\eta_{\mu\nu}\end{aligned}$$

Interpreted as a second rank tensor, $\eta_{\mu\nu}$ thus transforms under a spacetime inversion according to

$$\eta_{\mu\nu} \to \bar{\eta}_{\mu\nu} = \sigma^4\eta_{\mu\nu} \tag{30}$$

Before calculating the behaviour of $\eta_{\mu\nu}\sigma^{-4}$ under special conformal transformations, let us remind that σ^2 does not behave as a scalar under spacetime inversions. By the argument given in the beginning of the section, it is mapped into σ^{-2}. Applying the first inversion, we have

$$\frac{\eta_{\mu\nu}}{\sigma^4} \to \sigma^8\eta_{\mu\nu}$$

The second operation in the special conformal transformation is an ordinary translation. Since the first inversion mapped the conic space into Minkowski space, the object $\sigma^8\eta_{\mu\nu}$ is a scalar under translations. Therefore, we can directly go to the third part of the special conformal transformation: another inversion. This maps us again into the conic spacetime and the metric tensor is mapped accordingly as

$$\sigma^8\eta_{\mu\nu} \to \frac{\eta_{\mu\nu}}{\sigma^4}$$

This shows that $\sigma^{-4}\eta_{\mu\nu}$ is invariant under arbitrary special conformal transformations. [The above discussion makes us guess that its invariance on N is in some sence "dual" to the invariance of the Minkowski metric on the dual space M.]