Reduction of principal bundles

Hendrik

January 28, 2014

Abstract

We review the reduction of principal bundles and the relation between a Cartan and Ehresmann connection.

1 Reduction of a principal bundle

In this section the reduction process of a principal bundle is reviewed. Since our discussion is mostly based on the works [1, 2, 3], we would like to refer the reader to these in case a more complete and certainly rigorous treatment of the subject is desired.

Let H and G be Lie groups and consider an injective group homomorphism

$$i : H \to G\ .$$

It will be assumed that $i(H)$ is isomorphic to H, from which it follows that H is a subgroup of G. This inclusion of H into G is generally possible in different ways. In other words, the short exact sequence

$$e \longrightarrow H \xrightarrow{\ i\ } G \longrightarrow \frac{G}{i(H)} \longrightarrow e$$

is not canonically given. Of course, any differently chosen inclusion results in isomorphic subgroups of G. Therefore it might seem overprecise to refer to the specific inclusion by using the notation $i(H)$, rather than just denoting any of them by the letter H. We, however, adhere to this precision, since these isomorphic subgroups $i(H)$ may have quite a different physical meaning.

Indeed, consider the homogeneous space S that is symmetric under the left action of G and for which the isotropy subgroup of any point is isomorphic with H. The isomorphism $S \simeq G/H$ becomes manifest when choosing an origin $o \in S$ so that an element $gH_o \in G/H_o$ is identified with $\tau_g(o) \in S$. By denoting $H_o = i(H)$ this establishes the isomorphism $S \simeq G/i(H)$. If another origin $\xi = \tau_a(o)$ is chosen, S will be identified with G/H_ξ, where the isotropy group of ξ is related to the isotropy group of o through the adjoint action

$$H_\xi = aH_o a^{-1} = \mathrm{Ad}(a)(H_o) \ .$$

In both cases the origin singles out a subgroup $i(H)$ in G that are evidently isomorphic, nonetheless physically variant, being the isotropy subgroups of different points. It is also said that by prefering some point as the origin of S, the symmetry group G is *broken* to a subgroup $i(H)$.

Proposition 1.1. *Let $Q(M,G)$ be a principal G-bundle and let F be a left G-space. There is a one-to-one correspondence between sections of $Q[F] = Q \times_G F$ and maps $\varphi : Q \to F$ that are G-equivariant, i.e. they satisfy $R_g^* \varphi = g^{-1} \cdot \varphi$.*

Remark 1.1. A proof of this statement is given in [1], Sec. 4.8 on pg. 46. The bijective correspondence between such sections and G-equivariant mappings is as follows. In case φ is a map that satisfies $\varphi(qg) = g^{-1} \cdot \varphi(q)$, a section $M \to Q \times_G F$ is given by $\sigma(\pi(q)) \equiv [q, \varphi(q)]$ for any $q \in Q$. This is a well-defined construction because for any $g \in G$

$$\sigma(\pi(qg)) = [qg, \varphi(qg)] = [q, \varphi(q)] = \sigma(\pi(q)) \ .$$

Conversely, let σ be a section of $Q \times_G F$. There is a map $\varphi : Q \to F$ so that

$$\sigma(\pi(q)) = [q, \varphi(q)] = [qg, g^{-1} \cdot \varphi(q)] \ .$$

Since $\sigma(\pi(q)) = \sigma(\pi(qg)) = [qg, \varphi(qg)]$ for any $g \in G$, it follows that φ is G-equivariant. $\diamondsuit$

Denote by $Q/i(H)$ the space of equivalence classes with respect to the right action of $i(H) \subset G$ on Q. This quotient space can be identified with the associated G-bundle $Q[S] = Q \times_G S$, where S is the homogeneous space $G/i(H)$. More precisely, the correspondence is governed by the map[1]

$$Q/i(H) \to Q[S] : [q] \mapsto [q, \xi_a] = [q, a\xi_o] \ ,$$

where ξ_a is the origin of $G/i(H)$, hence $i(H) = H_a = aH_o a^{-1}$.

Definition 1.1. *Let $Q(M,G)$ and $P(M,H)$ be a principal G- and H-bundle, respectivily, and for which $i(H)$ is a closed subgroup of G. Let $\imath : P \to Q$ be an injection so that $\imath(ph) = \imath(p)i(h)$ for each $p \in P$ and $h \in H$. Then Q is an extension of P and P is a restriction of Q. The group G is said to be reduced to the group $H \simeq i(H)$.*

Given a principal H-bundle P with H being a subgroup of G, it is always possible to extend to a principal G-bundle. Since there is a natural left action of H on G, one also has the associated bundle $P[G] = P \times_H G$, whose elements are the equivalence

[1] This map is an isomorphism between fibre bundles; see e.g. [1], Sec. 6.1 on pg. 70.

classes

$$[p, g] = \{(ph, i(h)g) \mid h \in H\} \ .$$

There is a natural right G-action on this bundle, given by $[p, g]g' = [p, gg']$. Because the left and right actions commute, this is a well-defined principal G-bundle. We will write $Q(M, G) = P \times_{i(H)} G$. The extension of Q is then given by the natural injection

$$\imath : P \to Q : p \mapsto [p, e] \ . \tag{1.1}$$

On the other hand, it is not always possible to reduce a principal G-bundle to a principal H-bundle. This is the subject of discussion in the following proposition.

Proposition 1.2. *A principal G-bundle Q is reducible to a principal H-bundle P if and only if the associated bundle $Q[S] = Q \times_G S$, with $S \simeq G/H$, admits a globally defined section.*

Proof. Let $\imath : P \to Q$ be a reduction. The composition $\tilde{\sigma} \equiv \mu \circ \imath$, where $\mu : Q \to Q/i(H) \simeq Q[S]$ is the natural projection $q \mapsto [q] = [q, a\xi_o]$, is constant on the fibres of P;

$$\tilde{\sigma}(ph) = \mu(\imath(p)i(h)) = \tilde{\sigma}(p) \ , \quad h \in H \ .$$

Hence, $\tilde{\sigma}$ defines a section $M \to Q \times_G S$ by $\sigma(x) = \tilde{\sigma}(p)$ for any $p \in \pi^{-1}(x)$, since $\pi \circ \sigma = \mathrm{id}_M$.

Conversely, let σ be a section of $Q[S]$. From Proposition 1.1 it follows that there is a corresponding G-equivariant map $\varphi : Q \to S$. Let $\imath(P) \equiv \varphi^{-1}(\xi_a)$, where at each $x \in M$ we have that $i(H)(\xi_a) = \xi_a$. Consider the restriction of $\pi : Q \to M$ to $\imath(P)$ and let $\imath(p_1)$ and $\imath(p_2)$ be two elements in $\imath(P)$ for which

$$\pi|_{\imath(P)}(\imath(p_1)) = \pi|_{\imath(P)}(\imath(p_2)) \ .$$

There exists an element $g \in G$ for which $\imath(p_1) = \imath(p_2)g$, so that

$$\xi_a = \varphi(\imath(p_1)) = g^{-1}\varphi(\imath(p_2)) = g^{-1}\xi_a \ .$$

Hence, g must be an element of $i(H)$ and $\imath : P \to Q$ is a reduction from G to $i(H)$. $\square$

To conclude this section, let us summarize the reduction process in the following diagram.

$$
\begin{tikzcd}
 & & S \simeq G/H \\
P(M,H) \arrow[r, "\imath"] \arrow[ddr] & Q(M,G) \arrow[ur, "\varphi"] \arrow[dr, "\mu"] \arrow[dd] & \\
 & & Q \times_G S \arrow[dl] \\
 & M \arrow[ur, dashed, bend right, "\sigma"'] &
\end{tikzcd}
$$

2 Induced Cartan connection

Given a reduction $\imath : P \to Q$ one can wonder how Ehresmann connections on Q are related to Cartan connections on the reduced bundle P. In the following proposition we explain how an Ehresmann connection on Q may be interpreted as a Cartan connection on P, for a certain subclass of reductions $\imath$ [4]. It will be assumed that the dimension of P equals the dimension of G.

Proposition 2.1. *Let $\gamma \in \Omega(Q, \mathfrak{g})$ be an Ehresmann connection on Q. If*

$$\ker \gamma \cap \imath_*(TP) = 0 \ ,$$

then

$$\kappa \equiv \imath^* \gamma : TP \to \mathfrak{g} \tag{2.1}$$

is a Cartan connection on P.

Proof. Because $\ker \gamma \cap \imath_*(TP) = 0$, $\imath^*\gamma$ is a $\mathfrak{g}$-valued one-form on P that has no kernel. We verify the three defining properties of a Cartan connection for κ:

(i) Since $\dim P = \dim G$ and because $\imath$ is an injection, $\imath^*\gamma$ is an isomorphism.

(ii) Let ζ_X be the fundamental vector field on P corresponding to $X \in \mathfrak{h}$, i.e. $\zeta_X f = \dot{f}(p \exp(tX))$ for f a function on P. It follows that $\imath_* \zeta_X f = \dot{f}(\imath(p) i(\exp(tX)))$, which is a fundamental vector field on Q corresponding to $i(X) \in i(\mathfrak{h}) \simeq \mathfrak{h}$. It follows that $\kappa(\zeta_X) = i(X)$ for any $X \in \mathfrak{h}$.

(iii) Since for any $p \in P$

$$\imath \circ R_h(p) = [ph, e] = [p, i(h)] = R_{i(h)} \circ \imath(p) \ ,$$

it follows that

$$R_h^* \imath^* \gamma = \imath^* R_{i(h)}^* \gamma = \imath^* (\mathrm{Ad}(i(h^{-1})) \cdot \gamma) = \mathrm{Ad}(i(h^{-1})) \cdot \imath^* \gamma \ .$$

One concludes that $R_h^* \kappa = \mathrm{Ad}(i(h^{-1})) \cdot \kappa$.

The proof is completed when identifying H with $i(H)$. $\square$

References

[1] D. Husemöller. *Fibre Bundles*, volume 20 of *Graduate Texts in Mathematics*. Springer, New York, 1966.

[2] D. Husemöller, M. Joachim, B. Jurčo, and M. Schottenloher. *Basic Bundle Theory and K-Cohomology Invariants*, volume 726 of *Lecture Notes in Physics*. Springer, Berlin Heidelberg, 2008.

[3] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. Number vol. 1 in Wiley Classics Library. Wiley, 1996.

[4] R.W. Sharpe. *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*. Graduate Texts in Mathematics. Springer, 1997.