Cartan geometry

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February 26, 2014

Abstract

We give an extended discussion on Cartan geometry.

1 Introduction and definition

[

Introduction has to be expanded and must include

- A note on the relation to Klein geometries, which should make our motivation for using Cartan geometries clear. The reduction of the latter to the former must be pointed out in the definitions.
- References used or interesting for further readings: including Sharpe's book, Michor's and Wise's papers.
- A short note on history: Cartan's original articles. This is related to the first point.

]

Let $G \supset H$ be a Lie group with H a closed subgroup and let $P(M, H)$ denote the principal bundle $\pi : P \to M$ with typical fibre H. We begin this short review by writing down the formal definition of a Cartan geometry as given in [4].

1.1 Definition

Definition 1.1 (Cartan geometry)**.** *A **Cartan geometry** (P, κ) **modeled on** $(\mathfrak{g}, H)$ consists of a principal bundle $P(M, H)$ together with a $\mathfrak{g}$-valued 1-form κ on P, satisfying the following properties:*

(i) for each $p \in P$, the linear map $\kappa_p : T_pP \to \mathfrak{g}$ is an isomorphism;

(ii) $\kappa(\zeta_X) = X$, for each fundamental vector field ζ_X corresponding to $X \in \mathfrak{h}$;

(iii) $R_h^*\kappa = \mathrm{Ad}(h^{-1}) \cdot \kappa$ for each $h \in H$.

*The 1-form κ is called the **Cartan connection** of the geometry.*

Since $\kappa : TP \to \mathfrak{g}$ defines an isomorphism at any point in P, it follows that $\dim P = \dim G$ and one obtains that

$$\dim M = \dim G/H \ .$$

The second property in the definition implies that a Cartan connection κ restricts to the Maurer-Cartan form on the fibers of P, i.e. $\kappa|_H = \kappa_H$, a proof of which can be found in [4]. The bijective mapping κ_p can be inverted, i.e.

$$\kappa_p^{-1} : \mathfrak{g} \to T_pP \ ,$$

and for which the equivariance property of κ implies that

$$\kappa_{uh}^{-1}(X) = R_{h*}\, \kappa_u^{-1}(\mathrm{Ad}(h)X) \ .$$

This can be understood by considering the commuting diagram that represents the equivariance property of κ, namely

$$\begin{tikzcd}
T_uP \arrow[r, "R_{h*}"] \arrow[d, "\kappa_u"] & T_{uh}P \arrow[d, "\kappa_{uh}"] \\
\mathfrak{g} \arrow[r, "\mathrm{Ad}(h^{-1})"] & \mathfrak{g}
\end{tikzcd} \ ,$$

which is easily inverted to obtain the equivariance of κ^{-1}.

The **Cartan curvature** K of a Cartan connection is the $\mathfrak{g}$-valued 2-form on P defined by

$$K = d\kappa + \frac{1}{2}[\kappa, \kappa] \ . \tag{1.1}$$

This curvature form is horizontal, which means that it vanishes when any of its arguments is tangent to the fibers of P. This may be understood by noting that κ restricts to κ_H in the direction of the fibers, for which the curvature is nothing but the structure equation [2]. Explicitly this can be checked as follows. Because $K(\xi, \eta)$ is antisymmetric in its arguments, it is sufficient to show that $K(\xi, \eta) = 0$ where ξ is a vertical and η a vector field on P. Being vertical, at any given point one has that $\xi = \kappa^{-1}(X) = \zeta_X$ for some $X \in \mathfrak{h}$. One is able to verify that

$$\begin{aligned}
K(\kappa^{-1}(X), \eta) &= d\kappa(\kappa^{-1}(X), \eta) + [\kappa(\kappa^{-1}(X)), \kappa(\eta)] \\
&= i_{\kappa^{-1}(X)} \circ d\kappa(\eta) + [X, \kappa(\eta)] \ ,
\end{aligned}$$

where $i_{\kappa^{-1}(X)} \circ d\kappa(\eta) = d \circ i_{\kappa^{-1}(X)}\kappa(\eta) - \mathcal{L}_{\kappa^{-1}(X)}\kappa(\eta) = -\mathcal{L}_{\kappa^{-1}(X)}\kappa(\eta)$. The last equality is due to the constancy of the function $i_{\kappa^{-1}(X)}\kappa$. Next consider the equalities

$$\begin{aligned}\mathcal{L}_{\kappa^{-1}(X)}\kappa(\eta) &= \lim_{t\to 0}\frac{1}{t}(R^*_{x_t}\kappa - \kappa)\\ &= \lim_{t\to 0}\frac{1}{t}(\mathrm{Ad}(x_t^{-1})\cdot\kappa - \kappa)\\ &= \mathrm{ad}_X\kappa \ ,\end{aligned}$$

where $x_t = \exp(tX)$ is the one-parameter subgroup of H that is generated by X. Gathering this information we find that the curvature is indeed horizontal, that is

$$K(\xi,\eta) = -\mathrm{ad}_X\kappa(\eta) + \mathrm{ad}_X\kappa(\eta) = 0 \ .$$

The **torsion** Θ of the Cartan connection is a $\mathfrak{g}/\mathfrak{h}$-valued 2-form on P obtained by composing the curvature form with the canonical projection $\mathfrak{g}\to\mathfrak{g}/\mathfrak{h}$, that is

$$\begin{tikzcd} T^{(2,0)}P \arrow[r, "K"] \arrow[rr, bend right, "\Theta"'] & \mathfrak{g} \arrow[r] & \mathfrak{g}/\mathfrak{h} \end{tikzcd}$$

By taking the exterior derivative of the curvature, one finds the **Bianchi identity**

$$dK + [\kappa, K] = 0 \ . \tag{1.2}$$

1.2 Reductive Cartan geometry

A Cartan geometry (P,κ) modeled on $(\mathfrak{g}, H)$ is reductive if there is an H-module decomposition $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p}$, i.e. a splitting of the algebra $\mathfrak{g}$ in $\mathrm{Ad}(H)$-invariant subspaces. Corresponding to the reductive splitting, the Cartan connection is expanded as a sum of an $\mathfrak{h}$-valued and a $\mathfrak{p}$-valued part:

$$\kappa = \omega + e \ ; \quad \text{with} \quad \begin{cases} \omega := \mathrm{pr}_{\mathfrak{h}}\circ\kappa \in \Omega^1(P,\mathfrak{h}) \ , \\ e := \mathrm{pr}_{\mathfrak{p}}\circ\kappa \in \Omega^1(P,\mathfrak{p}) \ . \end{cases}$$

These projections make sense, since by definition the reductive splitting is preserved under gauge transformations. The issue is further discussed in the following proposition.

Proposition 1.1. *Let (P,κ) be a reductive Cartan geometry with a corresponding splitting of the connection $\kappa = \omega + e$. The one-form ω is an Ehresmann connection on P, while e is a displacement form, i.e. H-equivariant and strictly horizontal.*

Proof. By the H-equivariance of κ, i.e. $R_h^*\kappa = \mathrm{Ad}(h^{-1})\cdot\kappa$, it follows that

$$R_h^*\omega + R_h^*e = \mathrm{Ad}(h^{-1})\cdot\omega + \mathrm{Ad}(h^{-1})\cdot e \ ,$$

since the splitting is reductive. Rearranging this equation,

$$R_h^*\omega - \mathrm{Ad}(h^{-1})\cdot\omega = -R_h^*e + \mathrm{Ad}(h^{-1})\cdot e\ ,$$

the left-hand side is $\mathfrak{h}$-valued, while the right-hand side is $\mathfrak{p}$-valued. The equality makes sense only when both sides equal zero. This proves the equivariance of both ω and e.

Next let ζ_X be the fundamental vector field corresponding to $X \in \mathfrak{h}$. Then

$$X = \kappa(\zeta_X) = \omega(\zeta_X) + e(\zeta_X)\ ,$$

or

$$X - \omega(\zeta_X) = e(\zeta_X)\ .$$

The left-hand side is valued in $\mathfrak{h}$, while the right-hand side is valued in $\mathfrak{p}$. This implies that $\omega(\zeta_X) = X$ and $e(\zeta_X) = 0$ for any $X \in \mathfrak{h}$. □

Note that a vector field X on P, which at every point maps into $\mathfrak{p}$ under κ, is horizontal with respect to the Ehresmann connection ω. This is easily shown to be true by considering $\omega(X) = \mathrm{pr}_{\mathfrak{h}} \circ \kappa(X) = 0$.

The Cartan curvature can likewise consistently be written as the sum of an $\mathfrak{h}$-, respectively $\mathfrak{p}$-valued differential form:

$$K = R + T\ ; \quad \text{with} \quad \begin{cases} R := \mathrm{pr}_{\mathfrak{h}} \circ K \in \Omega^2(P, \mathfrak{h})\ , \\ T := \mathrm{pr}_{\mathfrak{p}} \circ K \in \Omega^2(P, \mathfrak{p})\ . \end{cases}$$

The definition for the Cartan curvature allows us to find explicit expressions for R and T in terms of ω and e. Therefore we substitute κ in Eq. (1.1) for the latter two and take into account the reductive nature of the geometry, which results in

$$K = \underbrace{d\omega + \frac{1}{2}[\omega,\omega]}_{\mathfrak{h}\text{-valued}} + \underbrace{de + [\omega, e]}_{\mathfrak{p}\text{-valued}} + \underbrace{\frac{1}{2}[e,e]}_{\mathfrak{g}\text{-valued}}\ .$$

For a reductive algebra two elements in $\mathfrak{p}$ commute into a generic element of $\mathfrak{g}$, so that the last term generally contributes in part to both R and T. However, in case $\mathfrak{g}$ is not only a reductive but also a *symmetric* Lie algebra, elements of the form $[e, e]$ are $\mathfrak{h}$-valued and it follows that

$$\begin{aligned} R &= d\omega + \frac{1}{2}[\omega,\omega] + \frac{1}{2}[e,e] = B + \tfrac{1}{2}[e,e]\ , \\ T &= de + [\omega, e]\ . \end{aligned}$$

Here we recognized $B \equiv d\omega + \frac{1}{2}[\omega,\omega]$ as the exterior covariant derivative of the Ehresmann connection ω. One therefore concludes that a flat Cartan geometry

($K = 0$) does not necessarily imply that the curvature of ω vanishes ($B = 0$), but rather that $B = -\frac{1}{2}[e, e]$. A Cartan geometry for which T vanishes is said to be **torsion-free**.

Given a reductive Cartan connection, the corresponding Bianchi identity (1.2) may also be seperated in an $\mathfrak{h}$-valued and a $\mathfrak{p}$-valued part. This results in two Bianchi identities, namely

$$dB + [\omega, B] \equiv 0 \qquad \text{(1st identity)} \ , \tag{1.3a}$$

$$dT + [\omega, T] + [e, B] \equiv 0 \qquad \text{(2nd identity)} \ . \tag{1.3b}$$

The first identity is $\mathfrak{p}$-valued, while the second is $\mathfrak{h}$-valued. Note that the first Bianchi identity is just the usual identity for the Ehresmann connection ω.

2 The coframe field. . . just the Vielbein!

[This section must be reviewed. Especially, notation should be made consistent with other parts of the documents.]

The above introduced solder form or coframe field $\theta : TP \to \mathfrak{p}$ seems to be a quiet different mapping then the coframe or *vielbein* generally used by physicists. Given an associated bundle $E = P \times_H \mathfrak{p}$, the Vielbein is a E-valued 1-form on M, as in the following diagram

$$\begin{tikzcd} TM \arrow[rr, "e"] \arrow[dr] & & E \arrow[dl, "\pi_E"] \\ & M & \end{tikzcd}$$

Locally, e maps a vector of T_pM into $\pi_E^{-1}(p) \simeq \mathfrak{p}$, such that it gives a local isomorphism $e : T_pM \to \mathfrak{p}$, for any $p \in M$. In the following it is shown that θ and e are in fact two different descriptions of the same mapping [5].

Suppose a tetrad $e : TM \to E$ is given so that a corresponding solder form $\theta : TP \to \mathfrak{p}$ is constructed as follows. For any $v \in T_uP$, $e(\pi(v)) \in \pi_E^{-1}(p)$, where $\pi(u) = p$. Hence, one can write $e(\pi(v)) = [u', X]$ for $u' \in \pi^{-1}(p)$ and $X \in \mathfrak{p}$.[1] Then define $\theta(v)$ to be the unique element in $\mathfrak{p}$ such that

$$e(\pi(v)) = [u, \theta(v)] \ .$$

This construction is well-defined if θ is H-equivariant. Indeed, for

$$e(\pi(v)) = [u, \theta(v)] = [uh, \mathrm{Ad}_{h^{-1}}\theta(v)] \ ,$$

[1] $[\cdot, \cdot]$ an element of the equivalence classes E, that is $(u, X) \sim (uh, Ad_{h^{-1}}X)$.

but also

$$e(\pi(v)) = e(\pi(R_h(v)) = [uh, \theta(R_h(v))] \ ,$$

so that consistency requires $R_h^*\theta = \mathrm{Ad}_{h^{-1}}\theta$. Furthermore, θ is strictly horizontal because if $v \in V_uP$, $e(\pi(v)) = 0 = [u, \theta(v)]$, hence $\theta(v) = 0$. Since the thus constructed form $\theta : TP \to \mathfrak{p}$ is a H-equivariant horizontal form, it is a legitimate solder form.

Next we start with a given solder form θ and define a corresponding frame field e as follows. Remember also that a reductive Cartan geometry canonically incorporates a connection ω on P. Let $\pi(v) \in T_pM$ and for any $u \in \pi^{-1}(p)$ let $v \in T_uP$ be its horizontal lift with respect to ω. Define then e trough (set $w = \pi(v)$)

$$e(w) = \pi(v)) = [u, \theta(v)] \ ,$$

which is clearly an element of $\pi_E^{-1}(p)$. Note how it is manifestly the inverse construction of the one before, but that now we had to make use of a connection to unambiguously pick out a $v \in H_uP$. We check again consistency of this construction. If uh were chosen then $v_{uh} = R_h v_u$, due to the right invariance of a horizontal lift, so that

$$e(w) = [uh, \theta(R_h v_u)] = [uh, \mathrm{Ad}_{h^{-1}}\theta(v)] = [u, \theta(v)] \ .$$

These results are nicely summarized in the following commuting diagram:

$$\begin{tikzcd}
T_uP \arrow[r, "\theta"] \arrow[d, "\pi"] & \mathfrak{p} \arrow[d, "{[u,\cdot]}"] \\
T_pM \arrow[r, "e"] & \pi_E^{-1}(p)
\end{tikzcd}$$

3 Riemann-Cartan geometry

In this section we discuss in some detail Riemann-Cartan geometry. This geometry is the mathematical framework that underlies Einstein-Cartan theory, General Relativity in the Palatini formalism and Teleparellel Gravity, although the latter two only use part of its available structure. It does not include the description of the so-called metric-affine theories of gravity, since Riemann-Cartan geometry is automatically metric, as will become clear in the following.

The relevant Cartan geometry[1] is modeled on $(\mathfrak{iso}(1,3), SO(1,3))$, where

$$\mathfrak{iso}(1,3) = \mathfrak{so}(1,3) \oplus \mathbb{R}^{3,1}$$

[1]Let us make clear that in what follows we will work directly on the base manifold M. More precisely, given some section $\sigma : M \to P$, the Cartan connection is a 1-form $A = \sigma^*\kappa$ on M, while we will denote the pulled back Cartan curvature by $F = \sigma^*K = dA + \frac{1}{2}[A, A]$.

is the Poincaré algebra,[1] whose commutation relations are given by (4.1) for the contraction $l \to \infty$. The corresponding Cartan connection is a 1-form A on spacetime that is valued in the Poincaré algebra,[2] which we split according to the symmetric nature of the algebra as

$$A = \omega + e = \tfrac{i}{2}\omega^{ab} M_{ab} + i e^a P_a \ .$$

Since the Poincaré algebra is reductive, the 1-form ω^{ab} is an Ehresmann connection for the Lorentz algebra, i.e. a *spin connection.* A corresponding covariant derivative $D := d + \omega$ is readily defined. Furthermore, the 1-forms e^a define a symmetric 2-form g on M—the metric—given by

$$g_{\mu\nu} := e^a{}_\mu e^b{}_\nu \eta_{ab} \ . \tag{3.1}$$

The forms $e^a{}_\mu dx^\mu$ are called the coframe fields—there is one for each a. It is then useful to define a set of dual vector fields $\vartheta_a{}^\mu \partial_\mu$ through $\vartheta_a \rfloor e^b = \delta^b_a$, which generally are given the name of a *vierbein* or tetrad. From the definition of the metric it follows that $g_{\mu\nu}\vartheta_a{}^\nu = e_{a\mu}$. Therefore in the case of the matrix $e^a{}_\mu$ being non-degenerate, an inverse of the metric exists and the coframe field and the vielbein are related by raising or lowering spacetime indices. The use of ϑ_a to denote the vielbein is then obsolete and will be referred to in the following by e_a. It then follows directly that $e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu$ and that $g^{\mu\nu} = e^{a\mu} e_a{}^\nu$.

Furthermore, given the spin covariant derivative D and the vierbein, it is possible to define covariant differentiation of world tensors. The corresponding derivative $\nabla = d + \Gamma$ is defined such that $\nabla_\mu V^\rho = e_a{}^\rho D_\mu V^a$. This implies that the relation between the spin and linear connections is given by

$$\Gamma^\mu{}_{\nu\rho} = e_a{}^\mu D_\rho e^a{}_\nu \quad \text{and} \quad \omega^a{}_{b\rho} = e^a{}_\mu \nabla_\rho e_b{}^\mu \ .$$

and that the vierbein postulate is true, i.e. $D_\rho e^a{}_\mu - \Gamma^\nu{}_{\mu\rho} e^a{}_\nu \equiv 0$. Furthermore, for such a geometry one has that

$$\begin{aligned} \nabla_\rho g_{\mu\nu} &= -\omega^a{}_{b\rho} e^b{}_\mu e_{a\nu} - \omega_{ab\rho} e^b{}_\nu e_{a\mu} \\ &= (\omega_{ba\rho} - \omega_{ab\rho}) e^b{}_\mu e^a{}_\nu \equiv 0 \ , \end{aligned}$$

since ω is valued in the Lorentz algebra. One concludes that Riemann-Cartan geometry is *metric*, as would be any reductive Cartan geometry where the subalgebra $\mathfrak{h}$ is the Lorentz algebra.

[1] Therefore, a correcter nomenclature for the Riemann-Cartan geometry could be *Poinaré-Cartan geometry.*

[2] Note that the relevant Cartan connection for metric-affine theories of gravity is valued in the affine algebra, which directly results in loosing the metricity of the geometry.

The Cartan curvature can equally be written as a sum of a Lorentz-, respectively translation-valued part, i.e.

$$F = R + T = \tfrac{i}{2} R^{ab} M_{ab} + i T^a P_a \ .$$

The curvature and torsion of the geometry are given by

$$R = d\omega + \tfrac{1}{2}[\omega, \omega] \quad \text{or} \quad R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \ , \tag{3.2a}$$

$$T = de + [\omega, e] \quad \text{or} \quad T^a = de^a + \omega^a{}_b \wedge e^b \ , \tag{3.2b}$$

while the Bianchi idendities for the Cartan connection are expressed as functions of the curvature and torsion:

$$dR + [\omega, R] \equiv 0 \quad \text{or} \quad dR^{ab} + \omega^a{}_c \wedge R^{cb} + \omega^b{}_c \wedge R^{ac} \equiv 0 \ , \tag{3.3a}$$

$$dT + [\omega, T] + [e, R] \equiv 0 \quad \text{or} \quad dT^a + \omega^a{}_b \wedge T^b + e^c \wedge R_c{}^a \equiv 0 \ . \tag{3.3b}$$

The expression (3.2b) for the torsion can be solved for the spin connection, which yields

$$\omega^a{}_{b\mu} = \tfrac{1}{2} e_{c\mu} (\Omega_b{}^{ca} + \Omega_b{}^{ac} - \Omega^{ac}{}_b) + K^a{}_{b\mu} = \mathring{\omega}^a{}_{b\mu} + K^a{}_{b\mu} \ . \tag{3.4}$$

In this expression $\mathring{\omega}$ is the torsionless Levi-Civita connection, while we introduced the coefficients of anholonomy $\Omega_{abc} := e_b \rfloor e_a \rfloor de_c = e_a{}^\mu e_b{}^\nu (\partial_\mu e_{c\nu} - \partial_\nu e_{c\mu})$ and the *contortion* K of ω:

$$K^a{}_{b\mu} := \tfrac{1}{2} (T^a{}_{\mu b} + T_\mu{}^a{}_b + T_b{}^a{}_\mu) \ . \tag{3.5}$$

Remark 3.1. Cartan's beautiful language to describe a Riemann-Cartan geometry makes the distinction between local Lorentz transformations and diffeomorphisms manifest. Mathematically, the former are given by sections of the principal Lorentz bundle over spacetime, while the latter are 1-to-1 mappings of spacetime to itself. Physically, only local Lorentz transformations have a non-trivial meaning. At any point in spacetime they relate the reference frames of observers that have a relative velocity. Diffeomorphisms, on the other hand, represent coordinate transformations of spacetime. They merely relate different ways of labeling space and time, and as such are lacking any deeper meaning. $\Diamond$

Remark 3.2. The geometric setting used to describe the Palatini formalism of General Relativity singles out the zero-torsion Levi-Civita spin connection $\mathring{\omega}$. The contortion vanishes naturally and the connection is determined fully by the vierbein, i.e. the gravitational field. The presence of a gravitational interaction is quantified by the curvature $\mathring{R}^{ab}$. This is what is meant when saying that the gravitational interaction is geometrized. There is a price that has to be paid, however. The spin connection does not only represent the gauge freedom of local Lorentz transformations but also contains information whether there is a gravitational field present or not, so that it is impossible to seperate inertial effects from gravitation. There are arguments to see this as a shortcoming from a conceptual point of view, since they might be quite different *physically*. Inertial forces certainly can be created and get rid of by considering local Lorentz transformations. On the contrary, the presence of a gravitational field, encoded in the covariant object $\mathring{R}^{ab}$, is objective.

It is usually argued that choosing a reference frame for which the Levi-Civita connection vanishes at some point yields a *local inertial reference frame*. The reason for such nomenclature is of course the reduction of covariant derivatives to ordinary derivatives at that point, so that the laws of physics recover their special relativistic form. Since inertial and gravitational motion is identical for infinitesimal objects, the gravitational interaction is said to be inertial, and as such the gravitational field determines inertial motion.[1] Having a look at processes that take place on a finite region in spacetime however, the freely falling observer will notice the consequences of tidal effects, present when $\mathring{R}^{ab} \neq 0$ and which cannot be accounted for by changing reference frame. Mathematically, the spin connection cannot be gauged away over a finite region in spacetime, so that this notion of inertial frames in General Relativity can at most be a *local* concept.

[[In this manner, one has to give up the idea of inertial effects being a non-local manifestation of the observer's motion. Otherwise, the identification of gravitational and inertial forces is not tenable, since curvature cannot be gauged away and tidal effects will be present for any class of observers. One can wonder about the usefulness of this modified notion for inertial motion and desire to understand inertial forces as a result of one's motion, which can be eliminated everywhere by a suitable Lorentz rotation. These fictious forces are very different in nature of the covariant gravitational interaction. As mentioned already, such point of view cannot be taken in General Relativity, since the spin connection includes both types of ineraction at any point in spacetime. In fact, the possibility to seperate them consistently at the mathematical level lies at the heart of Teleparallel Gravity.]]

To conclude, let us specify the Bianchi identities for the geometry underlying General Relativity, i.e.

$$\begin{aligned} d\mathring{R} + [\mathring{\omega}, \mathring{R}] &\equiv 0 \ , \\ [e, \mathring{R}] &\equiv 0 \ . \end{aligned} \qquad \diamondsuit$$

Remark 3.3. In Teleparallel Gravity the connection considered is flat and is related to the Levi-Civita spin connection through Eq. (3.4). This relation together with the geometry of General Relativity gives rise to the geometric structure of Teleparallel Gravity, as it is derived in [1]. From the discussion in `equiv_bianchi_GR_TG.pdf` it follows that this geometric framework is a special case of Riemann-Cartan geometry, with vanishing curvature but non-zero torsion.

The spin connection is flat, i.e. $d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = 0$, a condition very different

[1] For a clear discussion on this and related points, see Chapter 2 in [3]

in nature of the one for vanishing torsion in General Relativity. By the fundamental theorem of calculus, it follows that the spin connection is pure gauge, which means that $\omega^{ab} = {\Lambda_c}^a d\Lambda^{cb}$ in which ${\Lambda^a}_b \in \Omega^0(M, \mathfrak{so}(1,3))$ is a local Lorentz transformation. The spin connection then accounts only for inertial effects and one can choose a class of inertial observers, so that $\omega^{ab} = 0$ everywhere. In this theory, the presence of a gravitational field is encoded in a non-vanishing torsion. The obvious advantage is that inertial and gravitational effects are not mixed up in the spin connection so that both the physically very important concepts of inertial forces and the gravitational interaction have a clear well-distincted description. $\diamondsuit$

Finally, the geometry of Einstein-Cartan theory is *a-priori* a generic Riemann-Cartan geometry, i.e. having non-vanishing curvature and torsion. Also note that the geometry only fixes the kinematics of the respective theories, and that their ultimate content is determined further once an action describing their dynamics is postulated.

4 de Sitter-Cartan geometry

The Cartan geometry that is modeled on $(\mathfrak{so}(1,4), SO(1,3))$, will be given the name of a *de Sitter-Cartan geometry.* It is a slight generalization of Riemann-Cartan geometry, in the sense that in some well-defined limit the latter can be recovered from it. The Cartan connection for this geometry is a 1-form valued in the de Sitter algebra $\mathfrak{so}(1,4)$, which is characterized by its commutation relations[1]

$$\begin{aligned} -i[M_{ab}, M_{cd}] &= \eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad} \\ -i[M_{ab}, P_c] &= \eta_{ac}P_b - \eta_{bc}P_a \\ -i[P_a, P_b] &= -l^{-2}M_{ab} \ , \end{aligned} \tag{4.1}$$

The de Sitter transvections are defined by $P_a \equiv M_{a4}/l$, where it should be emphasized that the length scale $l(x)$ is *a-priori* allowed to be a function on M. From these brackets it is manifest that the algebra is symmetric, hence reductive. The corresponding reductive splitting reads as

$$\mathfrak{so}(1,4) = \mathfrak{so}(1,3) \oplus \mathfrak{p} \ ,$$

where $\mathfrak{so}(1,3) = \mathrm{span}\{M_{ab}\}$ is the Lorentz subalgebra and $\mathfrak{p} = \mathrm{span}\{P_a\}$ the subspace of de Sitter transvections, or de Sitter translations. This Cartan decomposition can be carried through to the connection and curvature forms, denoted by

$$A = \omega + e = \tfrac{i}{2}\omega^{ab}M_{ab} + ie^a P_a \ , \tag{4.2a}$$

$$F = R + T = \tfrac{i}{2}R^{ab}M_{ab} + iT^a P_a \ . \tag{4.2b}$$

[1] We adhere to the convention $\eta_{ab} = (+,-,-,-)$.

Due to the reductive nature of the de Sitter algebra, ω^{ab} is a spin connection. Identical to the discussion in Section 3, $SO(1,3)$-covariant differentiation can be defined and a metric structure constructed. We would like to refer to the discourse there, which is equally valid here.

Given the commutation relations (4.1), one is able to compute the curvature R^{ab} and the torsion T^a for de Sitter-Cartan geometry, namely

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} + \frac{1}{l^2} e^a \wedge e^b =: B^{ab} + \frac{1}{l^2} e^a \wedge e^b \ , \tag{4.3a}$$

$$T^a = de^a + \omega^a{}_b \wedge e^b - \frac{1}{l} dl \wedge e^a =: G^a - \frac{1}{l} dl \wedge e^a \ . \tag{4.3b}$$

In these equations the exterior covariant derivative of the spin connection and vierbein are denoted by B^{ab} and G^a, respectively. Some further remarks concerning these results are in place. Note that in the limit of a diverging length scale l, the expressions reduce to the curvature B^{ab} and torsion G^a for a Riemann-Cartan geometry. In the generic case however, the curvature and torsion are not given by the exterior covariant derivative of the spin connection and vierbein. The extra term in the expression (4.3a) represents the curvature of the local de Sitter space. This contribution is present because the commutator of two de Sitter transvections equals a Lorentz rotation. In the expression (4.3b) for the torsion, there is a new term when the length scale is not a constant function. This term comes about as follows. Remember that torsion is the $\mathfrak{p}$-valued 2-form $de + [\omega, e]$. The first contribution to this expression really means

$$de = d(ie^a P_a) = d(il^{-1} e^a M_{a4}) = ide^a P_a - i(l^{-1} dl \wedge e^a) P_a \ ,$$

since l is allowed to change along M.

As in the case of a Riemann-Cartan geometry, expression (4.3b) can be solved for the spin connection. It is found that $\omega^a{}_{b\mu} = \mathring{\omega}^a{}_{b\mu} + K^a{}_{b\mu}$, where $\mathring{\omega}^{ab}$ is the Levi-Civita spin connection for which the covariant exterior derivative $\mathring{G}^a$ vanishes, while the contortion is given by

$$K^a{}_{b\mu} := \tfrac{1}{2}(G^a{}_{\mu b} + G_\mu{}^a{}_b + G_b{}^a{}_\mu) \ .$$

These results are identical to the corresponding in a Riemann-Cartan geometry, since the torsion there is just the exterior covariant derivative of the vierbein, which here is denoted by G^a. Note that the torsion $\mathring{T}^a$ does not vanish for the Levi-Civita connection in a de Sitter-Cartan geometry, rather is its torsion equal to $\mathring{T}^a = -l^{-1} dl \wedge e^a$.

The Bianchi identities for the given de Sitter-Cartan geometry are of the form

$$\begin{aligned} dR^{ab} + \omega^a{}_c \wedge R^{cb} + \omega^b{}_c \wedge R^{ac} + \frac{1}{l^2} e^{[a} \wedge T^{b]} &\equiv 0 \\ \text{or} \quad dB^{ab} + \omega^a{}_c \wedge B^{cb} + \omega^b{}_c \wedge B^{ac} &\equiv 0 \ , \end{aligned} \tag{4.4a}$$

$$dT^a + \omega^a{}_b \wedge T^b + e^c \wedge R_c{}^a - \frac{1}{l} dl \wedge T^a \equiv 0 \ . \tag{4.4b}$$

The first Bianchi identity (4.4a) is the usual identity for the Ehresmann connection ω.

Remark 4.1. Consider for a moment the gauge invariant condition

$$R \equiv 0 \quad \text{or} \quad B \equiv -\frac{1}{2}[e, e] \ .$$

The Bianchi identity for the torsion (4.4b) clearly reduces to

$$dT^a + \omega^a{}_b \wedge T^b - \frac{1}{l} dl \wedge T^a = 0 \ .$$

From the first Bianchi identity (4.4a) one concludes that $[e, T] \equiv 0$ or

$$\frac{1}{l^2} e^{[a} \wedge T^{b]} = 0 \ . \tag{4.5}$$

Note that this restriction does carry through upon taking the contraction limit to the Riemann-Cartan geometry, since there $[e, T] = 0$ so that (4.4a) would be trivially satisfied.

Next consider the interior product[1] of the left-hand side of Eq. (4.5):

$$\begin{aligned} e_a \rfloor \left(e^{[a} \wedge T^{b]} \right) &= 4T^b - e^a \wedge e_a \rfloor T^b - T^b + e^b \wedge e_a \rfloor T^a \\ &= T^b + e^b \wedge e_a \rfloor T^a \ . \end{aligned}$$

Here we made use of the Leibniz rule $e_a \rfloor (\alpha \wedge \beta) = e_a \rfloor \alpha \wedge \beta + (-)^p \alpha \wedge e_a \rfloor \beta$, where α is assumed to be a p-form. It is also implied that $e^a \wedge e_a \rfloor T^b = 2T^b$. A second contraction gives

$$e_b \rfloor e_a \rfloor \left(e^{[a} \wedge T^{b]} \right) = e_b \rfloor T^b + 4 e_a \rfloor T^a - e^b \wedge e_b \rfloor e_a \rfloor T^a = 4 e_a \rfloor T^a \ .$$

In this result one should consider that $e_a \rfloor T^a = e^b \wedge e_b \rfloor e_a \rfloor T^a$, which can be verified by a direct calculation. Gathering these results one finds that the second Bianchi identity (4.5) leads to

$$l^{-2} e_a \rfloor T^a = l^{-2} T^a{}_{a\mu} \, dx^\mu = 0 \ ,$$

[1] For any vector field X on M the interior product of a differential form with respect to X will be denoted by

$$X \rfloor \omega := i_X \omega \ .$$

It then follows that $e_a \rfloor e^b = \delta_a^b$.

hence that the torsion is *traceless*. Furthermore, from the first contraction it followed that

$$T^b = -e^b \wedge e_a \rfloor T^a = 0 \ .$$

It is a quite remarkable result, but setting the curvature R to zero forces the geometry to have vanishing torsion. The resulting geometry describes a Klein geometry $SO(1,4)/SO(1,3)$. ♢

Remark 4.2. It is equally possible to consider a de Sitter-Cartan geometry for which the spin curvature B^{ab} is constrained to be zero, i.e.

$$B^{ab} \equiv 0 \ .$$

This means that the exterior covariant derivative of ω^{ab} is zero, a condition well known to render the Weitzenböck connection. Note that in de Sitter-Cartan geometry the Weitzenböck connection has a non-vanishing curvature $R^{ab} = l^{-2} e^a \wedge e^b$. This is similar to the conclusion that the torsion of the Levi-Civita connection is non-zero in a de Sitter-Cartan geometry. Nonetheless, as explained above, both connection are identical to those considered in a Riemann-Cartan geometry.

Note that the condition that the spin curvature should be equal to zero is gauge invariant, since B^{ab} transform covariantly under local Lorentz transformations. The first Bianchi identity (4.4a) is trivially satisfied, while the second reduces to (4.4b) for $e^b \wedge {R_b}^a = e^b \wedge {B_b}^a = 0$. It can be seen from these expression that they do not put any further restriction on the given geometry.

This kind of geometry will be used in formulating de Sitter Teleparallel Gravity. ♢

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