Equivalence Bianchi identities GR - TG

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Abstract

In this document we show that the Bianchi identities for Teleparallel Gravity, as they are derived in [1], i.e., from the Bianchi identities for General Relativity together with the dictionary $\omega = \mathring{\omega} + K$, are just the Bianchi identities for a Riemann-Cartan geometry for the connection ω.

A generic connection ω can be written as the sum (see e.g. [2])

$$\omega = \mathring{\omega} + K \ , \tag{0.1}$$

where $K \in \Omega^1(M, \mathfrak{so}(1,3))$ is called the *contortion* of ω. This connection fulfills the Bianchi identities

$$dR + [\omega, R] \equiv 0 \quad \text{or} \quad dR^{ab} + \omega^a{}_c \wedge R^{cb} + \omega^b{}_c \wedge R^{ac} \equiv 0 \ , \tag{0.2a}$$

$$dT + [\omega, T] + [e, R] \equiv 0 \quad \text{or} \quad dT^a + \omega^a{}_b \wedge T^b + e^c \wedge R_c{}^a \equiv 0 \ . \tag{0.2b}$$

We next show explicitely that the decomposition in Eq. (0.1) is consistent with the overal structure of the Cartan geometry and derive the geometry of $A = \omega + e$ from the geometry of $\mathring{A} = \mathring{\omega} + e$ by use of the dictionary $\mathring{\omega} = \omega - K$. Consider therefore first the curvature and torsion of $\mathring{\omega}$:

$$\begin{aligned} \mathring{R} &= d\mathring{\omega} + \frac{1}{2}[\mathring{\omega}, \mathring{\omega}] \\ &= d\omega - dK + \frac{1}{2}[\omega, \omega] - [\omega, K] + \frac{1}{2}[K, K] \\ &= R - Q \ , \end{aligned}$$

where we denoted $Q \coloneqq dK + [\omega, K] - \frac{1}{2}[K, K]$. For the torsion one finds

$$0 \equiv \mathring{T} = de + [\omega, e] - [K, e] \ ,$$

which implies that the torsion and the contortion of ω are related by

$$T = [K, e] \ . \tag{0.3}$$

Let us then turn attention to the Bianchi identities. The one for the curvature leads to

$$\begin{aligned}
0 &\equiv d\mathring{R} + [\mathring{\omega}, \mathring{R}] \\
&= dR - dQ + [\omega, R] - [\omega, Q] - [K, R] + [K, Q] \\
&= dR + [\omega, R] - [K, R] - [d\omega, K] + [\omega, dK] + [dK, K] - [\omega, dK] \\
&\quad - [\omega, [\omega, K]] + \tfrac{1}{2}[\omega, [K, K]] + [K, dK] + [K, [\omega, K]] - \tfrac{1}{2}[K, [K, K]] \\
&= dR + [\omega, R] - [K, R] - [d\omega, K] + \tfrac{1}{2}[K, [\omega, \omega]] + \tfrac{1}{2}[\omega, [K, K]] - \tfrac{1}{2}[\omega, [K, K]] \\
&= dR + [\omega, R] \ .
\end{aligned}$$

The Bianchi identity for the torsion gives result to

$$\begin{aligned}
0 &\equiv [e, \mathring{R}] \\
&= [e, R] - [e, Q] \\
&= [e, R] - [e, dK] - [e, [\omega, K]] + \tfrac{1}{2}[e, [K, K]] \\
&= [e, R] + d[e, K] - [de, K] - [[K, e], \omega] - [[e, \omega], K] + [[e, K], K] \\
&= [e, R] + dT - [T, K] - [T, \omega] + [T, K] \\
&= dT + [\omega, T] + [e, R] \ .
\end{aligned}$$

These results indeed are identical to the Bianchi identities of ω.

References

[1] R. Aldrovandi and J. G. Pereira. *Teleparallel Gravity: An Introduction*, volume 173 of *Fundamental Theories of Physics*. Springer Netherlands, 2012.

[2] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. Number vol. 1 in Wiley Classics Library. Wiley, 1996.