Supergravity

D.Z. Freedman and A. Van Proeyen
M.I.T.
Cambridge, Massachusetts

Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven,
Celestijnenlaan 200D B-3001 Leuven, Belgium.

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Contents

1 Scalar Field Theory and its Symmetries **13**
1.1 The scalar field system 13
1.2 Symmetries of the system 14
1.3 Noether currents and charges 24
1.4 Symmetries in the canonical formalism 27
1.5 The Lorentz group for $D = 4$ 30

2 The Dirac Field **31**
2.1 The homomorphism of $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,1)$. 31
2.2 The Dirac Equation 34
2.3 The spinors $u(\vec{p},s)$ and $v(\vec{p},s)$ for $D = 4$. 37
2.4 Dirac adjoint and bilinear form 39
2.5 Dirac Action 41
2.6 Weyl spinor fields in even spacetime dimension. 41
2.7 Conserved currents 43

3 Clifford algebras and spinors **45**
3.1 The Clifford Algebra in General Dimension 45
3.2 Spinors in general dimensions 55
3.3 Majorana spinors 61
3.4 Majorana spinors in physical theories 65
3.A Appendix: details on the Clifford algebras for $D = 2m$. 68

4 The Maxwell and Yang-Mills Gauge Fields. **75**
4.1 The abelian gauge field $A_\mu(x)$ 76
4.2 Electromagnetic duality 85
4.3 Non-abelian Gauge Symmetry 93
4.4 Internal Symmetry for Majorana Spinors 100

5 The free Rarita-Schwinger field 102
5.1 The Initial Value Problem 104
5.2 Sources and Green's Function 106
5.3 Dimensional Reduction on Minkowski$_D \times S^1$. 109

6 Differential Geometry 115
6.1 Manifolds 115
6.2 Scalars, vectors, tensors, etc. 117
6.3 The algebra and calculus of differential forms 120
6.4 The metric and frame field on a manifold 123
6.5 Volume forms and integration 127
6.6 Hodge duality of forms 129
6.7 p-form gauge fields 131
6.8 Connections and covariant derivatives 132
6.9 The second structure equation and the curvature tensor 139
6.10 The nonlinear σ-model 142
6.11 Symmetries and Killing vectors 144

7 The first and second order formulations of general relativity 150
7.1 Second order formalism for gravity and bosonic matter 151
7.2 Gravitational fluctuations of flat spacetime 152
7.3 Second order formalism for gravity and fermions 154
7.4 The first order formalism for gravity and fermions 157

8 $\mathcal{N} = 1$ Global Supersymmetry in $D = 4$ 159
8.1 Basic SUSY Field Theory 161
8.2 SUSY field theories of the chiral multiplet 165
8.3 SUSY Gauge Theories 171
8.4 Massless supersymmetry multiplets 175

9 $\mathcal{N} = 1$ pure supergravity in 4 dimensions 177
9.1 The universal part of supergravity 178
9.2 Supergravity in the First Order Formalism 180
9.3 The 1.5 order formalism 181
9.4 Local supersymmetry of $\mathcal{N} = 1$, $D = 4$ supergravity 182
9.5 The algebra of local supersymmetry 185

10 $D = 11$ supergravity 188
10.1 $D \leq 11$ from dimensional reduction 188
10.2 The field content of $D = 11$ supergravity 190
10.3 Construction of the action and transformation rules 190
10.4 The algebra of $D = 11$ supergravity 197

11 General gauge theory 198
11.1 Symmetries 198
11.2 Covariant quantities 208
11.3 Gauged spacetime translations 211
11.A Appendix: manipulating covariant derivatives. 218

12 Survey of supergravities 222
12.1 Minimal and extended superalgebras 222
12.2 The R-symmetry group 225
12.3 Multiplets 226
12.4 Supergravity theories: towards a catalogue 232
12.5 General characteristics of the actions 235
12.6 Scalars and geometry 235

13 Complex Manifolds 240
13.1 The local description of complex and Kähler manifolds 240
13.2 Mathematical structure of Kähler manifolds 244
13.3 The Kähler manifolds CP^n 246
13.4 Symmetries of Kähler Metrics 249

14 General actions with $\mathcal{N} = 1$ supersymmetry 254
14.1 Multiplets 254
14.2 Generalized actions by multiplet calculus 258
14.3 Kähler geometry from chiral multiplets 260
14.4 General couplings of chiral multiplets and gauge multiplets 262
14.5 The physical theory 271
14.A Appendix: Superspace 280

15 Gravity as a conformal gauge theory 311
15.1 The strategy 312
15.2 The conformal algebra 313
15.3 Conformal transformations on fields 314
15.4 The gauge fields and constraints 317
15.5 The action 319
15.6 Recapitulation 320
15.A Appendix: Homothetic Killing vectors 321

16 Pure $\mathcal{N} = 1$ supergravity from conformal methods 325
16.1 Ingredients 325
16.2 The action 336

17 Matter-coupled $\mathcal{N}=1$ supergravity 342
17.1 Superconformal tensor calculus 343
17.2 Construction of the action 348
17.3 Projective Kähler manifolds 356
17.4 The Poincaré matter-coupled supergravity 371
17.A Appendix: Kähler-Hodge manifolds 379
17.B Appendix: Steps in the derivation of (17.7) 383

18 Applications of $\mathcal{N}=1$ supergravity 385
18.1 Supersymmetry breaking and the super-BEH effect 386
18.2 Supersymmetry and anti-de Sitter space 391
18.3 R-symmetry and Fayet-Iliopoulos terms 393
18.4 Phenomenology 397
18.5 Cosmology 397

19 $\mathcal{N}=2$ supergravity 398
19.1 Short overview of matter couplings in $\mathcal{N}=1$ and $\mathcal{N}=2$ supergravity 398
19.2 Global $\mathcal{N}=2$ supersymmetry 399
19.3 Special Kähler geometry 402
19.4 Hyper-Kähler and quaternionic-Kähler manifolds 405
19.5 Calabi-Yau manifolds and special geometry 406
19.6 $D=5$ supergravity 406

20 $D=10$ supergravities 407
20.1 IIA supergravity as dimensional reduction from $D=11$ 407
20.2 IIB supergravity 407
20.3 A unified formulation 407
20.4 IIB on $AdS_5 \times S^5$ backgrounds 407

21 Classical Solutions of Gravity and Supergravity 408
21.1 Some solutions of the field equations 409
21.2 Killing Spinors and BPS Solutions 424
21.3 Killing spinors for anti-de Sitter space 429
21.4 Extremal Reissner-Nordstrom spacetimes as BPS solutions 431
21.5 Attractor behaviour 433
21.6 D3 brane solutions 434
21.7 Final remarks on solutions 434

Appendix A: Comparison of notations 436
A.1 Spacetime and gravity 436
A.2 Spinor conventions 438
A.3 From forms to components 439
A.4 Covariant derivatives 439

Appendix B: Simple algebras and superalgebras **441**
B.1 Groups and representations 441
B.2 Bosonic algebras 442
B.3 Superalgebras 446
B.4 OSp algebras as an example 448

References 450

Index 462

Preface

An overview of supergravity

History of supergravity. Two developments in the late 60's and early 70's set the stage for supergravity. First the standard model took shape and was decisively confirmed by experiments. The key theoretical concept underlying this progress was non-abelian 'gauge symmetry,' the idea that symmetry transformations act independently at each point of spacetime. In the standard model these are 'internal symmetries', whose parameters are Lorentz scalars $\theta^A(x)$ which are arbitrary functions of the spacetime point x^μ. These parameters are coordinates of a compact Lie group, $\mathrm{SU}(3)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1)$ for the standard model. Scalar, spinor, and vector fields of the theory are each classified in representations of this group, and the Lagrangian is invariant under group transformations. The special dynamics associated with the non-abelian gauge principle allows different realizations of the symmetry in the particle spectrum and interactions that would be observed in experiments. For example, part of the gauge symmetry may be 'spontaneously broken'. In the standard model this produces the 'unification' of weak and electromagnetic interactions, The observed strengths of these forces are very different, yet the gauge symmetry gives them a common origin.

The other development was global (also called rigid) supersymmetry. It is the unique framework that allows fields and particles of different spin to be unified in representations of an algebraic system called a superalgebra. The symmetry parameters are spinors ϵ_α that are *constant*, independent of x^μ. The simplest $\mathcal{N}=1$ superalgebra contains a spinor supercharge Q_α and the energy-momentum operator P_a. The anti-commutator of two supercharges is a translations in spacetime. The $\mathcal{N}=1$ supersymmetry algebra has representations containing massless particles of spins $(s,s-1/2)$ for $s=1/2,1,\ldots$ and slightly more complicated massive representations. Thus supersymmetry always unites bosons, integer spin, with fermions, 1/2-integer spin. The focus of early work was interacting field theories of the $(1/2,0)$ and $(1,1/2)$ multiplets, and it was found that the ultraviolet divergences of supersymmetric theories are less severe than in the standard model.

Global supersymmetry models got much attention for phenomenology. The R-symmetry, plays then an important part for eliminating unwanted couplings. Advantages of supersymmetric models with respect to ordinary standard model or unification theories include:

- Divergences of loop diagrams are milder due to cancellations of contributions of fermion and boson loops.
- Supersymmetry avoids some fine-tuning problems related to the hierarchy of scales.
- A main success is the unification of coupling constants taking their renormalization behaviour into account.

- Supersymmetry provides natural candidates for dark matter that known to dominate our universe from cosmology.

Another important application of global symmetry is their role in AdS/CFT dualities. There are many good reviews for these developments, and we will not treat them here. We refer for the applications of global supersymmetry to the well-known review [1].

The role of gauge symmetry in the standard model suggested that a gauged form of supersymmetry would be interesting and perhaps more powerful than the global form. Such a theory would contain gauge fields for both spacetime translations P_a and SUSY transformations generated by Q_α. Thus gauged supersymmetry was expected to be an extension of general relativity in which the graviton would have a fermionic partner called the gravitino. The name supergravity is certainly appropriate and was used even before the theory was actually found. It was reasonable to think that the gauge fields of the theory would be the vierbein, $e_\mu^a(x)$, needed to describe gravity coupled to fermions, and a vector-spinor field, $\psi_{\mu\alpha}(x)$, for the gravitino. The graviton and gravitino belong to the $(2, 3/2)$ representation of the algebra. A Lagrangian field theory of supergravity was formulated in the spring of 1976 in [2]. The approach taken was to modify the known free field Lagrangian for $\psi_{\mu\alpha}$ to agree with gravitational gauge symmetry and then find, by a systematic procedure, the additional terms necessary for invariance under supersymmetry transformations with arbitrary $\epsilon_\alpha(x)$. Soon an alternate approach appeared [3] in which the most complicated calculation required in [2] could be avoided.

In the following years, research in supergravity became very intense. Several versions, depending on the amount of supersymmetry (simple $\mathcal{N} = 1$ to a maximum of $\mathcal{N} = 8$), were discovered. Further, a lot of research was done in the structure of supergravity theories, and e.g. 'gauged supergravities' were found, where the symmetries acting between the different supersymmetries (later called 'R-symmetries') were gauged. Furthermore, it turned out to be convenient to consider supergravity theories in spacetimes of dimension greater than 4. In this direction, it was shown that field theory imposes a maximum: 11 dimensions. However, maximal theories did not have much success at that time because they could not include the symmetries of the standard model. Phenomenologists rather preferred to exploit the many possibilities offered by various matter fields coupled to minimal supergravity in a supersymmetric way. Supergravity meanwhile lead to an improved quantum behavior with respect to other quantum gravity theories, but the renormalizability problem was not solved.

In 1984, important results in *superstring theory* were found, leading to the general idea that a consistent quantum theory of gravity should be a superstring theory. Such a theory is not a field theory for physical particles, it describes them instead as the vibration modes of a string. However, when one takes the limit that the string is so small that it looks like a particle, the theory reduces to a supergravity (field) theory. Since the main string theory results were not found from such an approximation, though, research in supergravity diminished. Moreover, it was annoying that only 10 and not 11 dimensions appeared in string theories, and that there seemed to be no place at first for the gauged supergravities.

Superstring theories, first believed to be nearly unique, were shown to also exist in many varieties. In 1994 and 1995, it was proven that all the known superstring theories can be related to each other by the concept of *duality*. First, Hull and Townsend [4] found relations between string theories, and Witten [5] completed the picture with the concept of 'M-theory'. This unification made again contact with 11-dimensional supergravity. Later on, other supergravity theories became useful to understand different aspects of the full picture. The whole concept of dualities was first clarified in the context of $\mathcal{N} = 2$ supersymmetry by Seiberg and Witten [6, 7]. This discovery revived the research activity in various supergravity theories. A new understanding of the physics of black holes, arising as solutions of supergravity theories, and of their thermodynamical properties further stimulated this revival, eventually leading, a few years later, to the understanding of black hole thermodynamics from the point of view of string microstates [8].

In 1997, a duality between conformal field theory and anti-de Sitter theories was discovered. The latter are solutions of supergravity, which can be treated classically to get information on the quantum aspects of the dual field theory. The applications of this so-called AdS/CFT duality and its extensions became in the following years more and more successful and led recently to e.g. the study of the quark-gluon plasma and of non-perturbative aspects of QCD using supergravity duals.

Meanwhile, phenomenological models were built using compactifications of the original 10-dimensional string theories on a 6-dimensional 'Calabi-Yau' manifold. This procedure leads to different 4-dimensional theories, characterized by 'special geometries' [9]. Furthermore, extensions have been built such that all sort of other supergravities also play a role now for constructing realistic models.

In the last 8 years cosmologists also showed some interest for various stringy models, using mostly the supergravity limit. Many successful models, using data from the cosmic microwave background, have now been built starting from supergravity. Furthermore, cosmic string solutions, domain walls, Randall-Sundrum scenarios, and other similar ideas made use of supergravity.

There are still many open problems in supergravity, independent of the applications. There are many versions of gauged or deformed supergravities, but a complete catalogue does not yet exist. Even for those that are known to exist, there are many versions that have not yet been completely constructed. Furthermore, we still do not know the final word about the quantum properties. Recent surprising results revived the conjecture that $\mathcal{N} = 8$ supergravity in 4 dimensions is an UV finite theory of quantum gravity. Finally, of course there is the research in classical solutions of supergravity. There are attempts to come to classifications of all such solutions.

In conclusion, various supergravity theories have been and are still being used for many applications. This is the case for phenomenological applications as well as for the duality-based treatment of gauge theories. They are often inspired (or derived in some limit) from superstring theory. But there are also supergravity theories that cannot (at least currently) be related to any string theory. Despite more than 30 years of research there are still many open problems. The relations between high-energy experiments and strings and between cosmology and strings will have to use the supergravity language.

Supergravity in experiments In the coming years applications of supergravity might be expected to get heavily boosted by the upcoming new-generation experiments. At CERN, the newly-built accelerator LHC will open a new energy range and the search for supersymmetry is one of the main first goals of its colliding experiments. Moreover, in cosmology the precision measurements of the cosmic microwave background will be further improved and gravitational wave experiments will reach a crucial stage.

If LHC discovers supersymmetry (SUSY) partners of known particles, it is very probable that general relativity is also included in a supersymmetric model, and some form of supergravity should be effective. Therefore, for new model building, researchers will very likely use supergravity. Supergravity effects may or may not be important at LHC scales, depending on how SUSY is broken.

The phenomenology of supergravity is based for a large part on $\mathcal{N} = 1$ supergravity, where the minimal multiplet is coupled to chiral and gauge multiplets. Some fields appear then in representations of $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, while others are in a hidden sector. SUSY breaking is an important item as masses of bosons and fermions are different in nature. Supersymmetry breaking would leave a large non-zero vacuum energy that is not consistent with gravity. The cosmological constant that is left would be many orders of magnitude too high for a realistic model. Supergravity offers in fact mechanisms of breaking that are not available in SUSY, and that lead to more realistic phenomenology. Supergravity allows negative contributions to the vacuum energy avoiding the mentioned main problem. It thus solves the acute problem of magnitude of the cosmological constant, but it does not naturally produce the small observed positive value. A lot of fine-tuning is required.

In the last 20 years, many models have been studied in this framework in the hope of agreement with LHC experiments. Models from string theory use a supergravity framework to obtain predictions for 4-dimensional physics.

For cosmological applications, there is much research to obtain inflation, using the scalars of supergravity models. Another 'fine-tuning problem' occurs then to match the parameters of slow-roll inflation. Final inspiration and naturalness of the fine-tunings is expected from the context in superstring theories.

A spin-3/2 particle is the key prediction of supergravity. How can such a particle be found? SUSY breaking should give it a mass whose magnitude depends on the mechanism of 'mediation', i.e. how SUSY breaking in the hidden sector is communicated to particles on LHC mass scales.

- Gravity mediation gives a rather heavy gravitino, in the range 200 GeV to

1 TeV. It is very difficult to find this particle because it couples to known particles with feeble strength of the Planck coupling constant $\kappa = 1/M_P$. It would be unstable and decay to gluon plus gluino and more.

- Gauge mediation gives a very light gravitino with mass less than 10 MeV (in recent paper of Vafa) . A light gravitino can be stable and can be a dark matter candidate.

Gravitino production rate at the LHC will be tiny, but it can be copiously produced at the ultra-high temperatures at or near the big bang. This gives constraints on both high mass or low mass gravitinos. Further, the results are different if you assume non-inflationary or inflationary cosmology.

1. If there is no inflation, then primordial production occurs at temperatures of the order of the Planck mass (work in early 1980's by Pagels and Primack, Weinberg). A heavy unstable gravitino must decay before the final stage of nucleosynthesis. Since the lifetime shortens as gravitino mass increases, this gives a lower bound for the gravitino mass of the order of 10 TeV. A light mass stable gravitino can be so copiously produced that it over-closes the universe, which would lead to an upper bound of a few KeV.

2. Inflation wipes out the primordial processes, so that particle production takes place at the reheating temperature $T_{\mathrm{rh}} \ll M_P$ (work in 1990's by Murayama et al). Both upper and lower bounds are changed markedly; how much depends on T_{rh}. Very approximate results for the lower bound on heavy gravitinos indicate 100GeV, and an upper bound on light gravitinos would be 1 to 10 MeV.

It is interesting that inflation reopens the possibility for phenomenological supergravity.

If LHC fails to find SUSY there is still an 'escape hatch' to higher dimensions, with e.g.

$D = 5$: The AdS/CFT correspondence uses supergravity to obtain non-perturbative results in field theory. The 4-dimensional applications to condensed matter physics are very intriguing.

$D = 10$ supergravity is the low-energy limit of superstring theory.

$D = 11$ supergravity is at the core of M-theory. The latter is in fact the theory of 11-dimensional supergravity with supermembranes.

Supergravity is thus connected to fundamental ideas in theoretical physics. It has several theoretical applications and supergravity effects in LHC physics may turn out to be observable. Furthermore, there is an important input from cosmology in the subject. This real side of the subject is far from confirmation, but it is to be taken seriously.

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1
Scalar Field Theory and its Symmetries

The major purpose of the early chapters of this book is to review the basic notions of relativistic field theory that underly our treatment of supergravity. In this chapter we discuss the implementation of internal and spacetime symmetries using the model of a system of free scalar fields as an example. The general Noether formalism for symmetries is also discussed. Our book largely involves classical field theory. However, we adopt conventions for symmetries that are compatible with implementation at the quantum level.

Our treatment is not designed to teach the material to readers who are encountering it for the first time. Rather we try to gather the ideas (and the formulas!) that are useful background for later chapters. Supersymmetry and supergravity are based on symmetries such as the spacetime symmetry of the Poincaré group and much more!

As in much of this book, we assume general spacetime dimension D, with special emphasis on the case $D = 4$.

1.1 The scalar field system

We consider a system of n real scalar fields $\phi^i(x),\ i = 1, \ldots, n$ that propagate in a flat spacetime whose metric tensor

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-, +, \ldots, +)\,, \tag{1.1}$$

describes one time and $D - 1$ space dimensions. This is Minkowski spacetime, in which we use Cartesian coordinates x^μ, $\mu = 0, 1, \ldots D - 1$ with time coordinate $x^0 = t$ (with $c = 1$).

Practicing physicists and mathematicians are largely concerned with fields that satisfy nonlinear equations. However, linear wave equations, which describe free relativistic particles, have much to teach about the basic ideas. We therefore assume that our fields satisfy the Klein-Gordon equation

$$\boxed{\Box\phi^i(x) = m^2\phi^i(x)\,,} \tag{1.2}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the Lorentz invariant d'Alembertian wave operator.

The equation has plane-wave solutions $\mathrm{e}^{\pm\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}$, which provide the wave functions for particles of spatial momentum $\vec{p}$, with spatial components p^i, and energy $E = p^0 = \sqrt{\vec{p}^2+m^2}$. The general solution of the equation is the sum of a positive frequency part, which can be expressed as the $(D-1)$-dimensional Fourier transform in the plane-waves $\mathrm{e}^{\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}$ plus a negative frequency part, which is the Fourier transform in the $\mathrm{e}^{-\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}$

$$\begin{aligned}
\phi^i(x) &= \phi^i_+(x) + \phi^i_-(x)\,,\\
\phi^i_+(x) &= \int \frac{\mathrm{d}^{D-1}\vec{p}}{(2\pi)^{(D-1)}2E}\mathrm{e}^{\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}a^i(\vec{p})\,,\\
\phi^i_-(x) &= \int \frac{\mathrm{d}^{D-1}\vec{p}}{(2\pi)^{(D-1)}2E}\mathrm{e}^{-\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}a^{i*}(\vec{p})\,.
\end{aligned} \tag{1.3}$$

In the classical theory the quantities $a^i(\vec{p}),\ a(\vec{p})^{i*}$ are simply complex valued functions of the spatial momentum $\vec{p}$. After quantization one arrives at the true quantum field theory[1] in which $\mathbf{a}^i(\vec{p}),\ \mathbf{a}^{i*}(\vec{p})$ are annihilation and creation operators[2] for the particles described by the field operator $\phi^i(\vec{x})$.

The Klein-Gordon equation (1.2) is the variational derivative $\delta S/\delta\phi^i(x)$ of the action

$$S = \int \mathrm{d}^D x\,\mathcal{L}(x) = -\tfrac{1}{2}\int \mathrm{d}^D x\left[\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^i + m^2\phi^i\phi^i\right]\,. \tag{1.4}$$

The repeated index i is summed. The action is a *functional* of the fields $\phi^i(x)$. It is a real number that depends on the configuration of the fields throughout spacetime.

1.2 Symmetries of the system

Consider a set of fields such as the $\phi^i(x)$ that satisfy equations of motion such as (1.2). A general symmetry of the system is a mapping of the configuration space, $\phi^i(x) \to \phi'^i(x)$, with the property that if the original field configuration $\phi^i(x)$ is a solution of the equations of motion, then the transformed configuration $\phi'^i(x)$ is also a solution. For scalar fields and for most other systems of interest in this book,

[1] When desirable for clarity we use bold-face to indicate the operator in the quantum theory that corresponds to a given classical quantity.

[2] In the conventions above, creation and annihilation operators are normalized in the quantum theory by
$[\mathbf{a}(\vec{p}),\ \mathbf{a}^*(\vec{p}')] = (2\pi)^{D-1}2E\,\delta(\vec{p}-\vec{p}')$.

we can restrict to symmetry transformations that leave the action invariant. Thus we require that the mapping has the property[3]

$$S[\phi^i] = S[\phi'^i] \,. \tag{1.5}$$

Here is an example:

Ex. 1.1 *Verify that the map $\phi^i(x) \to \phi'^i(x) = \phi^i(x+a)$ satisfies (1.5) if a^μ is a constant vector. This symmetry is called a global spacetime translation.*

We consider both spacetime symmetries, which involve a motion in Minkowski spacetime such as the global translation of the exercise, and internal symmetries, which do not. Internal symmetries are simpler to describe. So we start with them.

1.2.1 SO(n) *internal symmetry*

Let $R^i{}_j$ be a matrix of the orthogonal group SO(n). This means that it is an $n \times n$ matrix that satisfies

$$R^i{}_k \delta_{ij} R^j{}_\ell = \delta_{k\ell} \,, \qquad \det R = 1 \,. \tag{1.6}$$

It is quite obvious that the linear map,

$$\phi^i(x) \to \phi'^i(x) = R^i{}_j \phi^j(x) \,, \tag{1.7}$$

satisfies (1.5) and is an internal symmetry of the action. This symmetry is called a continuous symmetry because a matrix of SO(n) depends continuously on $\frac{1}{2}n(n-1)$ independent group parameters. We will discuss one useful choice of parameters shortly. We also call the symmetry a global symmetry because the parameters are constants. In Ch. 4 we will consider local or gauged internal symmetries in which the group parameters are arbitrary functions of x^μ.

It is worth stating the intuitive picture of this symmetry. One may consider the field ϕ^i as an n-dimensional vector, that is an element of $\mathbb{R}^n$. The transformation $\phi^i \to R^i{}_j \phi^j$ is a rotation in this internal space. Such a rotation preserves the usual norm $\phi^i \delta_{ij} \phi^j$.

We now introduce the Lie algebra of the group SO(n). To first order in the small parameter ϵ, we write the infinitesimal transformation

$$R^i{}_j \;=\; \delta^i_j - \epsilon r^i{}_j \,. \tag{1.8}$$

This satisfies (1.6) if $r^i{}_j = -r^j{}_i$. Any antisymmetric matrix $r^i{}_j$ is called a generator of SO(n). The *Lie algebra* is the linear space spanned by the $\frac{1}{2}n(n-1)$ independent generators, with the commutator product

$$[r, r'] = r\, r' - r' r \,. \tag{1.9}$$

[3] Such mappings should also respect the boundary conditions. This requirement can be non-trivial, e.g. Neumann and Dirichlet boundary conditions for the bosonic string lead to different symmetry groups. We will mostly assume that field configurations vanish at large spacetime distances.

Note that matrices are multiplied[4] as $r^i{}_k r'^k{}_j$.

A useful basis for the Lie algebra is to choose generators that act in each of the $\frac{1}{2}n(n-1)$ independent 2-planes of $\mathbb{R}^n$. For the 2-plane in the directions $\hat{i}\hat{j}$ this generator is given by

$$r_{[\hat{i}\hat{j}]}{}^i{}_j \equiv \delta^i_{\hat{i}}\delta_{\hat{j}j} - \delta^i_{\hat{j}}\delta_{\hat{i}j} = -r_{[\hat{j}\hat{i}]}{}^i{}_j \,. \tag{1.10}$$

Note the distinction between the coordinate plane labels in brackets with hatted indices and the row and column indices. The commutators of the generators defined in (1.10) are

$$[r_{[\hat{i}\hat{j}]}, r_{[\hat{k}\hat{l}]}] = \delta_{jk} r_{[\hat{i}\hat{l}]} - \delta_{ik} r_{[\hat{j}\hat{l}]} - \delta_{jl} r_{[\hat{i}\hat{k}]} + \delta_{il} r_{[\hat{j}\hat{k}]} \,. \tag{1.11}$$

The row and column indices are suppressed in this equation, and this will be our practice when it causes no ambiguity. The equation implicitly specifies the structure constants of the Lie algebra in the basis of (1.10).

In this basis, a finite transformation of SO(n) is determined by a set of $\frac{1}{2}n(n-1)$ real parameters $\theta^{\hat{i}\hat{j}}$ which specify the angles of rotation in the independent 2-planes. A general element of (the connected component) of the group can be written as an exponential

$$R = \mathrm{e}^{-\frac{1}{2}\theta^{\hat{i}\hat{j}} r_{[\hat{i}\hat{j}]}} \,. \tag{1.12}$$

1.2.2 General internal symmetry

It will be useful to establish the notation for the general situation of linearly realized internal symmetry under an arbitrary connected Lie group G, usually a compact group, of dimension $\dim G$. We will be interested in an n-dimensional representation of G in which the generators of its Lie algebra are a set of $n \times n$ matrices $(t_A)^i{}_j$, $A = 1, 2, \ldots, \dim G$. Their commutation relations are[5]

$$[t_A, t_B] = f_{AB}{}^C t_C \,, \tag{1.13}$$

and the $f_{AB}{}^C$ are structure constants of the Lie algebra. The representative of a general element of the Lie algebra is a matrix Θ which is a superposition of the generators with real parameters θ^A, i.e.

$$\Theta = \theta^A t_A. \tag{1.14}$$

[4] Some mathematical readers may initially be perturbed by the indices used to express many equations in this book. We will follow the standard conventions used in physics. Unless ambiguity arises we use the Einstein summation convention for repeated indices, one downstairs and one upstairs. In this chapter, the components of a column vector are written with indices upstairs, while those of row vectors are written with indices downstairs. A matrix acting on a column vector thus has its rows labelled by an initial raised index and its columns labelled by the subsequent lowered index. The summation convention then incorporates the standard rules of matrix multiplication.

[5] Although it is common in the physics literature to insert the imaginary i in the commutation rule, we do not do this in order to eliminate i's in most of the formulas of the book. This means that compact generators t_A in this book are anti-hermitian matrices.

An element of the group is represented by the matrix exponential

$$U(\Theta) = \mathrm{e}^{-\Theta} = \mathrm{e}^{-\theta^A t_A} \,. \tag{1.15}$$

We consider a set of scalar fields $\phi^i(x)$ which transforms in the representation just described. The fields may be real or complex. If complex, the complex conjugate of every element is also included in the set. A group transformation acts by matrix multiplication on the fields:

$$\phi^i(x) \to \phi'^i(x) \equiv U(\Theta)^i{}_j \phi^j(x) \,. \tag{1.16}$$

If the system is complex, the representation of G will typically be reducible, and the full set $\{\phi^i\}$ splits into equal numbers of fields and conjugates on which $U(\Theta)$ acts independently. It is not necessary to distinguish the real and complex cases explicitly in our notation.

We assume that the equations of motion of the system are obtained from an action

$$S[\phi^i] = \int \mathrm{d}^D x\, \mathcal{L}(\phi^i, \partial_\mu \phi^i) \,, \tag{1.17}$$

which is invariant under (1.16). Internal symmetries are characterized by the stronger property that the Lagrangian density is invariant, that is

$$\mathcal{L}(\phi^i, \partial_\mu \phi^i) = \mathcal{L}(\phi'^i, \partial_\mu \phi'^i) \,. \tag{1.18}$$

Ex. 1.2 *Verify that Lagrangian density of (1.4) is invariant under the* SO(n) *symmetry of Sec. 1.2.1, but not under the spacetime translation of Ex. 1.1.*

An infinitesimal transformation of the group is defined as the truncation of the exponential power series in (1.15) to first order in Θ. This gives the field variation (matrix and vector indices suppressed)

$$\delta\phi = -\Theta\phi \,, \tag{1.19}$$

which defines the action of a Lie algebra element on the fields.

It is important to define iterated Lie algebra variations carefully. The definition we make below may seem unfamiliar. However, we show that it does give a representation of the algebra. Later, in Sec. 1.4, we will see that the definition is compatible with implementation of symmetry transformations by Poisson brackets of their conserved Noether charges at the classical level and by unitary transformation after quantization.

The action of a transformation δ_2 with Lie algebra element Θ_2 followed by a transformation δ_1 with element Θ_1 is defined by

$$\delta_1\delta_2\phi \equiv -\Theta_2\delta_1\phi = \Theta_2\Theta_1\phi \,. \tag{1.20}$$

The second variation acts only on the dynamical variables of the system, the fields ϕ^i, and is not affected by the matrix Θ_2 which multiplies ϕ^i. In detail,

$$\delta_1\delta_2\phi = \theta_1^A\theta_2^B t_B t_A\phi \,. \tag{1.21}$$

The commutator of two symmetry variations is then

$$\begin{aligned} [\delta_1,\delta_2]\phi &= -[\Theta_1,\Theta_2]\phi \equiv \delta_3\phi\,, \\ \Theta_3 &= [\Theta_1,\Theta_2] = f_{AB}{}^C\theta_1^A\theta_2^B t_C\,. \end{aligned} \tag{1.22}$$

The commutator of two Lie algebra transformations is again an algebra transformation by the element $\Theta_3 = [\Theta_1,\Theta_2]$.

It follows that finite group transformations compose as

$$\phi \xrightarrow{\theta_2} \phi' = U(\Theta_2)\phi \xrightarrow{\theta_1} \phi'' = U(\Theta_2)U(\Theta_1)\phi\,. \tag{1.23}$$

We show in Sec. 1.4 that this agrees with the composition of the unitary transformations which implement the symmetry in the quantum theory.

1.2.3 Spacetime symmetries – the Lorentz and Poincaré groups

The Lorentz group is defined as the group of homogeneous linear transformations of coordinates in D-dimensional Minkowski spacetime that preserve the Minkowski norm of any vector. We write

$$x^\mu \;=\; \Lambda^\mu{}_\nu x'^\nu \qquad \text{or} \qquad x'^\mu \;=\; \Lambda^{-1\mu}{}_\nu x^\nu\,, \tag{1.24}$$

and require that

$$x^\mu\eta_{\mu\nu}x^\nu = x'^\mu\eta_{\mu\nu}x'^\nu\,. \tag{1.25}$$

The Poincaré group is defined by adjoining global translations and considering

$$x'^\mu \;=\; \Lambda^{-1\mu}{}_\nu\left(x^\nu \,-\, a^\nu\right)\,. \tag{1.26}$$

In this section we review the properties of these groups, their Lie algebras and the group action on fields such as $\phi^i(x)$.

If (1.25) holds for any vector x^μ, it follows that

$$\boxed{\Lambda^\mu{}_\rho\eta_{\mu\nu}\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}\,.} \tag{1.27}$$

This is the defining property of the Λ matrices. If the Minkowski metric were replaced by the Kronecker delta, $\delta_{\mu\nu}$, these conditions would define the orthogonal group $O(D)$, but here we have the pseudo-orthogonal group $O(D-1,1)$. For most purposes we need only the connected component of this group, which we call the proper Lorentz group.

The metric tensor (and its inverse) are used to lower (or raise) vector indices. Thus one has, for example, $x_\mu = \eta_{\mu\rho}x^\rho$ and $\Lambda_{\mu\nu} = \eta_{\mu\rho}\Lambda^\rho{}_\nu$. Upper or lower indices are called contravariant or covariant, respectively.

Ex. 1.3 *Show that (1.24),(1.27) imply*

$$\begin{aligned} \Lambda_{\mu\nu} &= (\Lambda^{-1})_{\nu\mu}\,, \qquad \Lambda^{\mu}{}_{\nu} = (\Lambda^{-1})_{\nu}{}^{\mu}\,, \\ x'_{\mu} &= (\Lambda^{-1})_{\mu}{}^{\nu} x_{\nu} \;=\; x_{\nu}\Lambda^{\nu}{}_{\mu}\,. \end{aligned} \tag{1.28}$$

The first relation of the exercise resembles the standard matrix orthogonality property, but it holds for Lorentz when both indices are down (or both up), which is not the correct position for their action as linear transformations. Indeed, Lorentz matrices must be multiplied with indices in up-down position, viz. $\Lambda^{\mu}{}_{\rho}\Lambda'^{\rho}{}_{\nu}$.

We now introduce the Lie algebra of the Lorentz group, proceeding in parallel[6] to the discussion of the group SO(n) in Sec. 1.2.1. For a small parameter ϵ, we write the infinitesimal transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \epsilon m^{\mu}{}_{\nu} + \cdots\,. \tag{1.29}$$

It is straightforward to see that (1.29) satisfies (1.27) to first order in ϵ as long as the generator (m with two lower indices) is anti-symmetric, viz.

$$m_{\mu\nu} \equiv \eta_{\mu\rho} m^{\rho}{}_{\nu} = -m_{\nu\mu}\,. \tag{1.30}$$

The *Lie algebra* is the real linear space spanned by the $\frac{1}{2}D(D-1)$ independent generators, with the commutator product $[m, m'] = m\,m' - m'm$. These matrices must also be multiplied as $m^{\mu}{}_{\rho}m'^{\rho}{}_{\nu}$, but the forms with both indices down, as in (1.30), (or both up) are often convenient.

A useful basis for the Lie algebra is to choose generators that act in each of the $\frac{1}{2}D(D-1)$ coordinate 2-planes. For the 2-plane in the directions ρ, σ this generator is given by

$$m_{[\rho\sigma]}{}^{\mu}{}_{\nu} \equiv \delta^{\mu}_{\rho}\eta_{\nu\sigma} - \delta^{\mu}_{\sigma}\eta_{\rho\nu} \;= -m_{[\sigma\rho]}{}^{\mu}{}_{\nu}\,. \tag{1.31}$$

Note the distinction between the coordinate plane labels in brackets and the row and column indices. In this basis, a finite proper Lorentz transformation is specified by a set of $\frac{1}{2}D(D-1)$ real parameters $\lambda^{\rho\sigma} = -\lambda^{\sigma\rho}$ and takes exponential form

$$\Lambda = \mathrm{e}^{\frac{1}{2}\lambda^{\rho\sigma} m_{[\rho\sigma]}}\,. \tag{1.32}$$

When matrix indices are restored, we have, with the representation (1.31), the series[7]

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \lambda^{\mu}{}_{\nu} + \tfrac{1}{2}\lambda^{\mu}{}_{\rho}\lambda^{\rho}{}_{\nu} + \ldots\,. \tag{1.33}$$

The commutators of the generators defined in (1.31) are

$$\boxed{[m_{[\mu\nu]}, m_{[\rho\sigma]}] = \eta_{\nu\rho} m_{[\mu\sigma]} - \eta_{\mu\rho} m_{[\nu\sigma]} - \eta_{\nu\sigma} m_{[\mu\rho]} + \eta_{\mu\sigma} m_{[\nu\rho]}\,.} \tag{1.34}$$

---

[6] Specifically, it is Λ^{-1} which is the analogue of R in (1.12) and of $U(\theta)$ in (1.15).

[7] Thus, we now replace in (1.29) $\epsilon m^{\mu}{}_{\nu}$ by $\frac{1}{2}\lambda^{\rho\sigma} m_{[\rho\sigma]}{}^{\mu}{}_{\nu}$.

These equations specify the structure constants of the Lie algebra which may be written as

$$f_{[\mu\nu][\rho\sigma]}{}^{[\kappa\tau]} = 8\eta_{[\rho[\nu}\delta_{\mu]}^{[\kappa}\delta_{\sigma]}^{\tau]} \,. \tag{1.35}$$

Note that antisymmetrization is always done with 'weight 1', see (A.9), such that the right hand side can be written as 8 terms with coefficients ± 1.

Ex. 1.4 *Check that (1.34) leads to (1.35). To compare with (1.13), you have to replace each of the indices A, B, C by antisymmetric combinations, e.g. $A \to [\mu\nu]$. Moreover, you have to insert a factor $\frac{1}{2}$ each time that you sum over such a combined index to avoid double counting, as e.g. the factor $\frac{1}{2}$ in (1.32). Therefore we rewrite (1.34) as*

$$[m_A, m_B] = f_{AB}{}^C m_C \to [m_{[\mu\nu]}, m_{[\rho\sigma]}] = \tfrac{1}{2} f_{[\mu\nu][\rho\sigma]}{}^{[\kappa\tau]} m_{[\kappa\tau]} \,. \tag{1.36}$$

Under a symmetry, each group element is mapped to a transformation of the configuration space of the dynamical fields . This map must give a group homomorphism. For the Lorentz matrix Λ, the transformation of the scalar fields is defined as

$$\phi^i(x) \xrightarrow{\Lambda} \phi'^i(x) \equiv \phi^i(\Lambda x) \,. \tag{1.37}$$

Using (1.24), we find that $\phi'^i(x') = \phi^i(x)$.

Ex. 1.5 *Show that the action (1.4) is invariant under the transformation (1.37).*

We now define differential operators which implement the coordinate change due to an infinitesimal transformation. A transformation in the $[\rho\sigma]$ 2-plane is generated by

$$L_{[\rho\sigma]} \equiv x_\rho \partial_\sigma - x_\sigma \partial_\rho \,. \tag{1.38}$$

The commutator algebra of these operators is isomorphic to (1.34), and we thus have a realization of the Lie algebra acting as differential operators on functions.

Ex. 1.6 *Compute the commutators $[L_{[\mu\nu]}, L_{[\rho\sigma]}]$ and show that they agree with that of (1.34) for matrix generators. Show that to first order in $\lambda^{\rho\sigma}$*

$$\phi^i(x^\mu) - \tfrac{1}{2}\lambda^{\rho\sigma} L_{[\rho\sigma]} \phi^i(x^\mu) = \phi^i(x^\mu + \lambda^{\mu\nu} x_\nu) \,. \tag{1.39}$$

We then define the differential operator

$$U(\Lambda) \equiv \mathrm{e}^{-\frac{1}{2}\lambda^{\rho\sigma} L_{[\rho\sigma]}} \,. \tag{1.40}$$

Using this operator, the mapping (1.37) which defines the action of finite Lorentz transformations on scalar fields can then be written as

$$\phi^i(x) \to \phi'^i(x) = U(\Lambda)\phi^i(x) = \phi^i(\Lambda x). \tag{1.41}$$

We see that both Lorentz and internal symmetries (see (1.16)) are implemented by linear operators acting on the classical fields, a differential operator for Lorentz and a matrix operator for internal. Both operators depend on group parameters in the same way.

By expanding the exponential, this defines the infinitesimal Lorentz transformation of the scalars:

$$\delta(\lambda)\phi^i(x) = \phi'^i(x) - \phi^i(x) = \phi^i(\Lambda x) - \phi^i(x) = \lambda^\mu{}_\nu x^\nu \partial_\mu \phi^i(x) = -\tfrac{1}{2}\lambda^{\mu\nu} L_{[\mu\nu]}\phi^i(x)\,. \tag{1.42}$$

This definition does give a homomorphism of the group; namely the product of maps, first with Λ_2, then with Λ_1, produces the compound map associated with the product $\Lambda_1\Lambda_2$. This can be seen from the sequence of steps

$$\begin{aligned} \phi^i(x) \xrightarrow{\Lambda_2} \phi'^i(x) = U(\Lambda_2)\phi^i(x) \xrightarrow{\Lambda_1} \phi''^i(x) &= U(\Lambda_2)U(\Lambda_1)\phi^i(x) \\ &= U(\Lambda_2)\phi^i(\Lambda_1 x) \\ &= \phi^i(\Lambda_1\Lambda_2 x)\,. \end{aligned} \tag{1.43}$$

As mentioned above, a symmetry transformation acts *directly on the fields.* This convention determines the order of operations in the first line. In the second and third line, we use the action of the differential operators to arrive at a transformation with matrix product $\Lambda_1\Lambda_2$. This is exactly the same for internal symmetries in (1.23), where the action of U is obtained by matrix multiplication.

Ex. 1.7 *It is instructive to check (1.43) for Lorentz transformations which are close to the identity. Specifically, use the definition (1.40) to show that the order $\lambda_1\,\lambda_2$ terms in the product $U(\Lambda_2)U(\Lambda_1)\phi^i$ of differential operators acting on ϕ^i agrees with terms of the same order in $\phi^i(\Lambda_1\Lambda_2 x)$.*

Perform now the calculation in the form of commutators of infinitesimal transformations. The second transformation acts on $\phi^i(x)$, and not on the x-dependent factor in (1.42). The commutator of two transformations is a Lorentz transformation with parameter-matrix $[\lambda_1, \lambda_2]$.

It is important to extend the Lorentz transformation rules to covariant and contravariant vector fields, which are, respectively, sections of the cotangent and tangent bundles of Minkowski spacetime. The transformation of a general covariant vector field $W_\mu(x)$ can be modelled on that of the gradient of a scalar $\partial_\mu\phi(x)$. From (1.37) we find

$$\partial_\mu\phi(x) \to \partial_\mu\phi'(x) = \frac{\partial}{\partial x^\mu}\phi(\Lambda x) = \Lambda^{-1}{}_\mu{}^\nu(\partial_\nu\phi)(\Lambda x)\,, \tag{1.44}$$

where we have used the chain rule and (1.28) in the last step. Thus we define the transformation of a general covariant field as

$$W_\mu(x) \to W'_\mu(x) \equiv \Lambda^{-1}{}_\mu{}^\nu W_\nu(\Lambda x)\,. \tag{1.45}$$

For contravariant vectors we assume a transformation of the form

$$V^{\mu}(x) \to B^{\mu}{}_{\sigma} V^{\sigma}(\Lambda x)\,. \tag{1.46}$$

The matrix can determined by requiring that the inner product $V^{\mu}(x)W_{\mu}(x)$ transforms as a scalar. This fixes $B^{\mu}{}_{\sigma} = \Lambda^{-1\mu}{}_{\sigma}$, and we have the transformation rule

$$V^{\mu}(x) \to V'^{\mu}(x) \equiv \Lambda^{-1\mu}{}_{\nu} V^{\nu}(\Lambda x)\,. \tag{1.47}$$

Next we define generators of Lorentz transformations appropriate to the covariant and contravariant vector representations. They are combined differential/matrix operators given by

$$\begin{aligned} J_{[\rho\sigma]} V^{\mu}(x) &\equiv (L_{[\rho\sigma]} \delta^{\mu}{}_{\nu} + m_{[\rho\sigma]}{}^{\mu}{}_{\nu}) V^{\nu}(x)\,, \\ J_{[\rho\sigma]} W_{\nu}(x) &\equiv (L_{[\rho\sigma]} \delta_{\nu}{}^{\mu} + m_{[\rho\sigma]\nu}{}^{\mu}) W_{\mu}(x)\,. \end{aligned} \tag{1.48}$$

We avoid an overly decorative notation by suppressing row and column indices on $J_{[\rho\sigma]}$. For each case a finite Lorentz transformation is implemented by the operator

$$U(\Lambda) = \mathrm{e}^{-\frac{1}{2}\lambda^{\rho\sigma} J_{[\rho\sigma]}}\,. \tag{1.49}$$

Ex. 1.8 *Verify that* $J_{[\rho\sigma]}(V^{\mu}W_{\mu}) = L_{[\rho\sigma]}(V^{\mu}W_{\mu})$.

The Lorentz group has many representations, both higher rank tensor and spinor representations (to be discussed in Ch. 2) and combinations thereof. Let $\psi^{i}(x)$ denote a set of fields with i an index of the components of a general representation. There is a corresponding Lie algebra representation with matrices $m_{[\rho\sigma]}$ which act on the indices and differential/matrix generators

$$J_{[\rho\sigma]} = L_{[\rho\sigma]} \mathbb{1} + m_{[\rho\sigma]}\,, \tag{1.50}$$

in which $\mathbb{1}^{i}{}_{j} = \delta^{i}_{j}$ is the unit matrix. A finite Lorentz transformation is then the mapping (suppressing the index i for simplicity of the notation)

$$\psi(x) \to \psi'(x) = U(\Lambda)\psi(x) = \mathrm{e}^{-\frac{1}{2}\lambda^{\rho\sigma} m_{[\rho\sigma]}} \psi(\Lambda x)\,. \tag{1.51}$$

The precise form of the operators $m_{[\rho\sigma]}$, $J_{[\rho\sigma]}$, and $U(\Lambda)$ is determined by the representation.

Spacetime translations $x^{\mu} \to x'^{\mu} = x^{\mu} - a^{\mu}$ are much simpler because they are implemented in the same way in all representations of the Lorentz group, namely as the mapping

$$\psi^{i}(x) \to \psi'^{i}(x) = \psi^{i}(x+a) = U(a)\psi^{i}(x)\,, \tag{1.52}$$

$$U(a) = \mathrm{e}^{a^{\mu} P_{\mu}}\,, \tag{1.53}$$

$$P_{\mu} = \partial_{\mu} = \frac{\partial}{\partial x^{\mu}}\,. \tag{1.54}$$

In (1.52) we have defined the generator P_μ and the finite translation operator $U(a)$ which are differential operators. Finite transformations of the Poincaré group are implemented by the operator $U(a,\Lambda) \equiv U(\Lambda)U(a)$, which acts as follows:

$$\begin{aligned} \psi(x) \;\rightarrow\; \psi'(x) &\equiv U(a,\Lambda)\psi(x) = U(\Lambda)U(a)\psi(x) \\ &= \mathrm{e}^{-\frac{1}{2}\lambda^{\rho\sigma}m_{[\rho\sigma]}}\psi(\Lambda x + a)\,. \end{aligned} \tag{1.55}$$

Ex. 1.9 *Prove that* $U((\Lambda)^{-1}a)U(\Lambda) = U(\Lambda)U(a)$. *Verify for operators which are close to the identity that*

$$\begin{aligned} U(a)\phi(\Lambda' x + b) &= \phi(\Lambda' x + \Lambda' a + b)\,, \\ U(\Lambda)\phi(\Lambda' x + b) &= \phi(\Lambda'\Lambda x + b)\,. \end{aligned} \tag{1.56}$$

The Lie algebra of the Poincaré group contains the $D(D+1)/2$ generators $J_{[\mu\nu]}$, P_μ. The following commutation rules complete the specification of the Lie algebra:

$$[J_{[\mu\nu]}, J_{[\rho\sigma]}] = \eta_{\nu\rho}J_{[\mu\sigma]} - \eta_{\mu\rho}J_{[\nu\sigma]} - \eta_{\nu\sigma}J_{[\mu\rho]} + \eta_{\mu\sigma}J_{[\nu\rho]}\,, \tag{1.57}$$

$$[J_{[\rho\sigma]}, P_\mu] = P_\rho\eta_{\sigma\mu} - P_\sigma\eta_{\rho\mu}\,, \tag{1.58}$$

$$[P_\mu, P_\nu] = 0\,. \tag{1.59}$$

Ex. 1.10 *Verify (1.57), (1.58).*

We now discuss the implementation of this Lie algebra on fields. The treatment is parallel to the discussion of internal symmetry at the end of Sec. 1.2.2. The infinitesimal variation of the fields $\psi^i(x)$ is defined as the first order truncation of the exponential in (1.55):

$$\delta\psi = [a^\mu P_\mu - \tfrac{1}{2}\lambda^{\rho\sigma}J_{[\rho\sigma]}]\psi\,. \tag{1.60}$$

The $J_{[\rho\sigma]}$ operator appropriate to the representation is used, and representation indices are suppressed for simplicity. Consider now the transformation δ_2 with group parameters (a_2, λ_2) followed by transformation δ_1 with parameters (a_1, λ_1). The result is defined as

$$\begin{aligned} \delta_1\delta_2\psi &= \delta_1[a_2^\mu P_\mu - \tfrac{1}{2}\lambda_2^{\rho\sigma}J_{[\rho\sigma]}]\psi \\ &= [a_2^\mu P_\mu - \tfrac{1}{2}\lambda_2^{\rho\sigma}J_{[\rho\sigma]}][a_1^\nu P_\nu - \tfrac{1}{2}\lambda_1^{\kappa\tau}J_{[\kappa\tau]}]\psi\,. \end{aligned} \tag{1.61}$$

The second transformation acts only on the field variable. With some care one can calculate the commutator of two variations, which yields a third transformation with parameters $(a_3,\, l_3)$, that is

$$\begin{aligned}
[\delta_1,\, \delta_2]\psi &= \delta_3\psi\,, \\
\lambda_3^{\rho\sigma} &= [\lambda_1,\, \lambda_2]^{\rho\sigma} = \tfrac{1}{4} f_{[\mu\nu][\kappa\tau]}{}^{[\rho\sigma]}\lambda_1^{\mu\nu}\lambda_2^{\kappa\tau}\,, \\
a_3^\mu &= a_{1\rho}\lambda_2^{\rho\mu} - a_{2\rho}\lambda_1^{\rho\mu} = \tfrac{1}{2} f_{\nu[\rho\sigma]}{}^\mu \left(a_1^\nu\lambda_2^{\rho\sigma} - a_2^\nu\lambda_1^{\rho\sigma}\right)\,, \qquad (1.62)
\end{aligned}$$

(remember Ex. 1.4). Note that the parameters of infinitesimal transformations compose with the structure constants of the Lie algebra, exactly as in the case of internal symmetry.

Ex. 1.11 *Verify the rule of combination of parameters in (1.62).*

As in the discussion of internal symmetry at the end of Sec. 1.2.2, the product of two finite transformations of the Poincaré group, the first with parameters $(a_2,\, \lambda_2)$ followed by the second with parameters $(a_1,\, \lambda_1)$ is given by

$$\psi(x) \to \psi''(x) = U(a_2,\, \lambda_2)U(a_1,\, \lambda_1)\psi(x)\,. \qquad (1.63)$$

The compound map is a representation, as we discuss at the end of Sec. 1.4 below.

1.3 Noether currents and charges

It is well known that, using the Noether method, one can construct a conserved current for every continuous global symmetry of the action of a classical field theory. Integrals of the time component of the currents are conserved charges, and an infinitesimal symmetry transformation is implemented on fields by the Poisson bracket of charge and field. Poisson brackets and the correspondence principle provide a useful bridge to the quantum theory in which finite group transformation are implemented through unitary transformations. In this section and the next we review this formalism in order to show that the conventions for symmetries used are compatible with readers' previous study of symmetries in quantum field theory. The Noether formalism has other important applications in supersymmetry and supergravity which enter in later chapters of this book.

We assume that we are dealing with a system of scalar fields $\phi^i(x)$, $i = 1, \ldots, n$ whose Lagrangian density is a function of the fields and their first derivatives, as described by the action (1.17). The Euler-Lagrange equations of motion are

$$\frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i(x)} - \frac{\delta\mathcal{L}}{\delta\phi^i(x)} = 0\,. \qquad (1.64)$$

We define a generic infinitesimal symmetry variation of the fields by

$$\delta\phi^i(x) \equiv \epsilon^A\Delta_A\phi^i(x)\,, \qquad (1.65)$$

in which the ϵ^A are constant parameters. This formula includes as special cases the various internal and spacetime symmetries discussed in Sec. 1.2. In these cases each $\Delta_A\phi^i(x)$ is linear in ϕ^i and given by a matrix or differential operator applied to ϕ^i. Specifically,

$$\text{internal} \qquad \epsilon^A\Delta_A\phi^i \to -\theta^A(t_A)^i{}_j\phi^j\,, \tag{1.66}$$

$$\text{spacetime} \qquad \epsilon^A\Delta_A\phi^i \to \left[a^\mu\partial_\mu - \tfrac{1}{2}\lambda^{\rho\sigma}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\right]\phi^i\,. \tag{1.67}$$

If (1.65) is a symmetry of the theory, then the action is invariant, and the variation of the Lagrangian density is an explicit total derivative, i.e. $\delta\mathcal{L} = \epsilon^A\partial_\mu K^\mu_A$. This must hold for *all* field configurations, not merely those which satisfy the equations of motion (1.64). In detail, the variation of the Lagrangian density is

$$\delta\mathcal{L} \equiv \epsilon^A\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu\Delta_A\phi^i + \frac{\delta\mathcal{L}}{\delta\phi^i}\Delta_A\phi^i\right] = \epsilon^A\partial_\mu K^\mu_A\,. \tag{1.68}$$

Using (1.64) we can rearrange (1.68) to read $\partial_\mu J^\mu_A = 0$, where J^μ_A is the Noether current

$$J^\mu{}_A = -\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\Delta_A\phi^i + K^\mu_A\,. \tag{1.69}$$

This is a conserved current, by which we mean that $\partial_\mu J^\mu_A = 0$ for all solutions of the equations of motion of the system.

We temporarily assume that the symmetry parameters are arbitrary functions $\epsilon^A(x)$. In this case the variation of the action is

$$\begin{aligned}
\delta S &= \int \mathrm{d}^D x \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu(\epsilon^A\Delta_A\phi^i) + \frac{\delta\mathcal{L}}{\delta\phi^i}\epsilon^A\Delta_A\phi^i\right] \\
&= \int \mathrm{d}^D x \left[\epsilon^A\partial_\mu K^\mu_A + (\partial_\mu\epsilon^A)\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\Delta_A\phi^i\right] \\
&= -\int \mathrm{d}^D x\,(\partial_\mu\epsilon^A)J^\mu_A\,.
\end{aligned} \tag{1.70}$$

The use of varying parameters $\epsilon^A(x)$ is usually an efficient way to obtain the conserved Noether current. Note that surface terms from partial integrations in the manipulations above have been neglected because the field configurations are assumed to vanish at large spacetime distances.

For each conserved current one can define an integrated Noether charge, which is a constant of the motion, i.e., independent of time. Suppose that we have a foliation of Minkowski space-time by a family of space-like $(D-1)$-dimensional surfaces $\Sigma(\tau)$. A space-like surface has a time-like normal vector n^μ at every point[8] . Then for each Noether current there is an integrated charge

$$Q_A = \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu\, J^\mu{}_A(x)\,, \tag{1.71}$$

[8] The Minkowski space norm of any vector v^μ is $v^\mu\eta_{\mu\nu}v^\nu$. A vector is called space-like if its norm is positive, time-like if the norm is negative, and null for vanishing norm. $\mathrm{d}\Sigma_\mu$ is a vector proportional to n_μ which represents a surface element orthogonal to $\Sigma(\tau)$.

which is conserved, that is independent of τ for all solutions that are suitably damped at infinity. The simplest foliation is given by the family of equal time surfaces $\Sigma(t)$, which are flat $(D-1)$-dimensional hyperplanes with fixed $x^0 = t$. In this case

$$Q_A = \int \mathrm{d}^{D-1}\vec{x}\, J^0{}_A(\vec{x}, t)\,. \tag{1.72}$$

We now discuss the specific Noether currents for the linear internal and spacetime transformations of this chapter. To simplify the discussion we restrict to systems with conventional scalar kinetic term, so the Lagrangian density is

$$\mathcal{L} \;=\; -\tfrac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - V(\phi^i)\,. \tag{1.73}$$

For internal symmetry, substitution of the first line of (1.66) in (1.69) gives

$$J^\mu{}_A = -\partial^\mu\phi t_A\phi\,. \tag{1.74}$$

For spacetime translations, the index A of the generic current (1.69) is replaced by the vector index ν, and the Noether current obtained from (1.67) and (1.69) is the conventional stress tensor (or energy-momentum tensor)

$$T^\mu{}_\nu = \partial^\mu\phi\,\partial_\nu\phi + \delta^\mu_\nu\mathcal{L}\,. \tag{1.75}$$

For Lorentz transformations, the index A becomes the antisymmetric pair $\rho\sigma$, and (1.67), (1.69) give the current

$$J^\mu{}_{[\rho\sigma]} = -x_\rho T^\mu{}_\sigma + x_\sigma T^\mu{}_\rho\,. \tag{1.76}$$

The conserved charges for internal, translations and Lorentz transformations are denoted by T_A, P_μ, and $J_{[\rho\sigma]}$, respectively. They are given by

$$\begin{aligned} T_A &= -\int \mathrm{d}^{D-1}\vec{x}\,\partial^0\phi t_A\phi\,,\\ P_\mu &= \int \mathrm{d}^{D-1}\vec{x}\,T^0{}_\mu\,,\\ J_{[\rho\sigma]} &= \int \mathrm{d}^{D-1}\vec{x}\,J^0{}_{[\rho\sigma]}\,. \end{aligned} \tag{1.77}$$

Although we use the same notation, the distinction between these Noether charges and the differential operators in (1.50) and (1.54) will be clear from context.

Note that one does not need the detailed form of the stress tensor (1.75) to show that the current (1.76) is conserved. The situation is indeed simpler. A current of the form (1.76) is conserved if $T_{\mu\nu}$ is both conserved and symmetric, $T_{\mu\nu} = T_{\nu\mu}$. For many systems of fields, such as the Dirac field discussed in the next chapter, the stress tensor given by the Noether procedure is conserved but not symmetric. In all cases one can modify $T_{\mu\nu}$ to restore symmetry.

In general the symmetry currents obtained by the Noether procedure are not unique. They can be modified by adding terms of the form $\Delta J^\mu{}_A \equiv \partial_\rho S^{\rho\mu}{}_A$, where $S^{\rho\mu}{}_A = -S^{\mu\rho}{}_A$. The added term is identically conserved, and the Noether charges are not changed by the addition since $\Delta J^0{}_A$ involves total spatial derivatives. It is frequently the case that the Noether currents of space-time symmetries need to be "improved" by adding such terms in order to satisfy all desiderata, such as symmetry of $T_{\mu\nu}$. Another example is the stress tensor of the electromagnetic field which we discuss in Chapter 4. The Noether stress tensor is conserved but neither gauge invariant nor symmetric. It can be made gauge invariant and symmetric by improvement, and the improved stress tensor is naturally selected by the coupling of the electromagnetic field to gravity.

1.4 Symmetries in the canonical formalism

In this section we discuss the implementation of symmetries in the canonical formalism at the classical and quantum level. In much modern work in quantum field theory the canonical formalism has been superseded by the use of Feynman path integrals, but canonical methods provide a quick pedagogical treatment of the issues of immediate concern. For scalar fields ϕ^i, the canonical coordinates at fixed time $t = 0$ are the field variables $\phi(\vec{x}, 0)$ at each point $\vec{x}$ of space, and the canonical momenta are given by $\pi(\vec{x}, 0) = \delta S/\delta\partial_t\phi(\vec{x}, 0)$. For the action (1.73), the canonical momentum is $\pi_i = \partial_0\phi^i = -\partial^0\phi^i$.

We consider explicitly the special cases of internal symmetry, space translations and rotations in which the vector K^μ_A of (1.68) has vanishing time component. In these cases the formula (1.72) for the Noether charge simplifies to

$$\begin{aligned} Q_A &= -\int \mathrm{d}^{D-1}\vec{x}\,\frac{\delta\mathcal{L}}{\delta\partial_0\phi^i}\Delta_A\phi^i \\ &= -\int \mathrm{d}^{D-1}\vec{x}\,\pi_i\Delta_A\phi^i\,. \end{aligned} \tag{1.78}$$

We work in this generic notation and ask readers to verify the results using (1.66) for internal symmetry and (1.67) for space translations and rotations. The following results are also valid for time translations and Lorentz boosts, although the manipulations are a little more complicated.

We remind readers that the basic (equal time) Poisson bracket is $\{\phi^i(\vec{x}), \pi_j(\vec{y})\} = \delta^i_j\delta^{D-1}(\vec{x}-\vec{y})$. The Poisson bracket of two observables $A(\phi,\pi)$, $B(\phi,\pi)$ is

$$\{A, B\} \equiv \int \mathrm{d}^{D-1}\vec{x}\left(\frac{\delta A}{\delta\phi^i(\vec{x})}\frac{\delta B}{\delta\pi_i(\vec{x})} - \frac{\delta A}{\delta\pi_i(\vec{x})}\frac{\delta B}{\delta\phi^i(\vec{x})}\right). \tag{1.79}$$

Poisson brackets $\{A, \{B, C\}\}$ obey the Jacobi identity

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0\,. \tag{1.80}$$

In the quantum theory each classical observable becomes an operator[9] in Hilbert space, which we denote by bold-faced type, e.g. $A(\phi,\pi) \to \mathbf{A}(\mathbf{\Phi},\mathbf{\Pi})$. The correspondence principle states that, if the Poisson bracket of two observables gives a third observable, i,e. $\{A,B\}=C$, then the commutator of the corresponding operators is $[\mathbf{A},\mathbf{B}]_{\mathrm{qu}} = \mathrm{i}\mathbf{C}$. Note that we use $\hbar = 1$.

It is now easy to see that the infinitesimal symmetry transformation $\Delta_A\phi^i$ is generated by its Poisson bracket with the Noether charge Q_A. In detail

$$\Delta_A\phi^i(x) \;=\; \{Q_A,\phi^i(x)\} = -\int \mathrm{d}^{D-1}\vec{y}\{\pi_j(\vec{y})\Delta_A\phi^j(\vec{y}),\phi^i(x)\}. \tag{1.81}$$

For the example of spatial translations one has

$$\delta\phi = a^\mu\partial_\mu\phi = a^\mu\,\{P_\mu,\phi\}\,. \tag{1.82}$$

Further, Poisson brackets of the conserved charges obey the Lie algebra of the symmetry group,

$$\{Q_A\,,\,Q_B\} \;=\; f_{AB}{}^C Q_C\,. \tag{1.83}$$

Ex. 1.12 . *Readers are invited to verify (1.83) for the Noether charges of internal symmetry, spatial translations and rotations using the Noether charges given in (1.72) and the structure constants of the subalgebra of Poincaré transformations that do not change the time coordinate.*

After quantization these relations become the operator commutators

$$\begin{aligned}\Delta_A\mathbf{\Phi}^i &= -\mathrm{i}\left[\mathbf{Q}_A,\mathbf{\Phi}^i\right]_{\mathrm{qu}}\,,\\ [\mathbf{Q}_A\,,\,\mathbf{Q}_B]_{\mathrm{qu}} &= \mathrm{i}\,f_{AB}{}^C\mathbf{Q}_C\,.\end{aligned} \tag{1.84}$$

The first relation implies that a finite group transformation with parameters ϵ^A is implemented by the unitary transformation

$$\mathbf{\Phi}^i(x) \to \mathrm{e}^{-\mathrm{i}\epsilon^A\mathbf{Q}_A}\mathbf{\Phi}^i(x)\mathrm{e}^{\mathrm{i}\epsilon^A\mathbf{Q}_A} = U(\epsilon)\mathbf{\Phi}^i(x)\,. \tag{1.85}$$

Here $U(\epsilon)$ is a generic notation for a finite group transformation. More specifically, for internal symmetry $U(\epsilon) \to U(\Theta)$ of (1.15), for translations $U(\epsilon) \to U(a)$ of (1.53), and for Lorentz $U(\epsilon) \to U(\Lambda)$ of (1.49). For finite transformations of an internal symmetry group G or the Poincaré group, (1.85) reads

$$\mathrm{e}^{-\mathrm{i}\theta^A\mathbf{T}_A}\mathbf{\Phi}^i(x)\mathrm{e}^{\mathrm{i}\theta^A\mathbf{T}_A} \;=\; \mathrm{e}^{-\Theta}\mathbf{\Phi}^i(x)\,, \tag{1.86}$$

$$\mathrm{e}^{-\mathrm{i}[a^\mu\mathbf{P}_\mu-\frac{1}{2}\lambda^{\rho\sigma}\mathbf{J}_{[\rho\sigma]}]}\mathbf{\Phi}^i(x)\mathrm{e}^{\mathrm{i}[a^\mu\mathbf{P}_\mu-\frac{1}{2}\lambda^{\rho\sigma}\mathbf{J}_{[\rho\sigma]}]} \;=\; \mathbf{\Phi}^i(\Lambda x+a)\,. \tag{1.87}$$

[9] Subtleties such as operator ordering in the definition of $\mathbf{A}(\mathbf{\Phi},\mathbf{\Pi})$ are ignored because they are not relevant for the questions of interest to us.

These relations agree with texts in quantum field theory.[10]

In the Poisson bracket formalism an iterated symmetry variation, δ_2 with parameters ϵ_2^B followed by δ_1 with parameters ϵ_1^A, is given by

$$\delta_1\delta_2\phi^i = \epsilon_1^A\epsilon_2^B\{Q_A,\{Q_B,\phi^i\}\}\,. \tag{1.88}$$

Using the Jacobi identity (1.80) one easily obtains the commutator

$$[\delta_1,\delta_2]\,\phi^i = f_{AB}{}^C\epsilon_1^A\epsilon_2^B\{Q_C,\phi^i\}\,. \tag{1.89}$$

Note that the symmetry parameters compose exactly as in (1.22) and (1.62).

Ex. 1.13 *Verify the corresponding quantum operator relation*

$$[\delta_1,\delta_2]\,\mathbf{\Phi}^i = -\mathrm{i} f_{AB}{}^C\epsilon_1^A\epsilon_2^B\left[\mathbf{Q}_C,\mathbf{\Phi}^i\right]_{\mathrm{qu}}\,. \tag{1.90}$$

It is also useful to verify the composition of finite group transformations. A transformation with parameters ϵ_2^A followed by another one with parameters ϵ_1^A is is found by applying (1.85) twice. One obtains

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}\epsilon_1^A\mathbf{Q}_A}\mathrm{e}^{-\mathrm{i}\epsilon_2^B\mathbf{Q}_B}\mathbf{\Phi}^i(x)\mathrm{e}^{\mathrm{i}\epsilon_2^B\mathbf{Q}_B}\mathrm{e}^{\mathrm{i}\epsilon_1^A\mathbf{Q}_A} &= U(\epsilon_2)\mathrm{e}^{-\mathrm{i}\epsilon_1^A\mathbf{Q}_A}\mathbf{\Phi}^i(x)\mathrm{e}^{\mathrm{i}\epsilon_1^A\mathbf{Q}_A}\\
&= U(\epsilon_2)U(\epsilon_1)\mathbf{\Phi}^i(x)\,.
\end{aligned} \tag{1.91}$$

This agrees with (1.23) for internal symmetry and its analogue for spacetime transformations. Furthermore the group composition law for the product $\mathrm{e}^{\mathrm{i}\epsilon_2^B\mathbf{Q}_B}\mathrm{e}^{\mathrm{i}\epsilon_1^A\mathbf{Q}_A}$ of unitary operators is the same as for the classical operators $U(\epsilon_2)U(\epsilon_1)$, so we do get a consistent representation of the symmetry group.

Ex. 1.14 *Obtain from (1.77) and (1.75) the Hamiltonian $H = P^0$ in terms of the canonical momenta and coordinates*

$$H = \tfrac{1}{2}\int \mathrm{d}^{D-1}\vec{x}\left[\pi^2 + (\vec{\partial}\phi)^2\right]\,. \tag{1.92}$$

Check that this leads, using (1.3), to the quantum commutation relation

$$[H,\mathbf{\Phi}]_{\mathrm{qu}} = -\mathrm{i}\pi = -\mathrm{i}\partial_0\phi = \int\frac{\mathrm{d}^{D-1}\vec{p}}{(2\pi)^{(D-1)}2E}E\left(-\mathrm{e}^{\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}a(\vec{p}) + \mathrm{e}^{-\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}a^*(\vec{p})\right)\,. \tag{1.93}$$

Express the Hamiltonian in terms of $a(\vec{p})$ and $a^*(\vec{p})$ (you can use the books of Weinberg [10] (7.1.25) or the full calculation in the same notations in Srednicki [11] (3.26) if this is not obvious)

$$H = \tfrac{1}{2}\int\frac{\mathrm{d}^{D-1}\vec{p}}{(2\pi)^{(D-1)}2E}\left[a^*(\vec{p})a(\vec{p}) + a(\vec{p})a^*(\vec{p})\right]\,. \tag{1.94}$$

[10] For example, readers can compare (1.87) with [10]. The formulas (2.4.3), (5.1.6) and (5.1.7) of [10] are relevant as well as Sec. 7.3.

Using

$$\Big[a(\vec{p}), a^*(\vec{p}')\Big]_{\rm qu} = (2\pi)^3 2E(\vec{p})\delta^3(\vec{p}-\vec{p}')\,. \tag{1.95}$$

you can then reobtain (1.93).

1.5 The Lorentz group for $D = 4$

In Sec. 1.2.3 we introduced fields transforming in a general finite dimensional representation of the D-dimensional Lorentz group $\mathrm{SO}(D-1,1)$ without giving a detailed description of these representations. There are several special features of the case $D = 4$, which reduce the description of the finite dimensional representations of $\mathrm{SO}(3,1)$ to those of the familiar representations of $\mathrm{SU}(2)$, as we now review.

First we note that the proper subgroup of $\mathrm{O}(3,1)$ is characterized by the two conditions $\det(\Lambda) = 1$ and $\Lambda^0{}_0 \geq 1$. The latter means that the sign of the time coordinate of any point is preserved. There are 3 disconnected components, which contain the product of the discrete transformations P, T, and PT, describing inversion in space and/or time, with a proper transformation. Lorentz transformations in the disconnected components satisfy either $\det(\Lambda) = -1$ or $\Lambda^0{}_0 \leq -1$ or both.

Let $m_{[\mu\nu]}$ denote the matrices of a representation of the Lie algebra (1.34) for $D = 4$. The 6 independent matrices consist of 3 spatial rotations $I_i = -\frac{1}{2}\varepsilon_{ijk}m_{[jk]}$ (where ε_{ijk} is the alternating symbol with $\varepsilon_{123} = 1$) and 3 boosts $K_i = m_{[0i]}$. It is a straightforward and important exercise to show, using (1.34), that the linear combinations

$$\begin{aligned} J_k &= \tfrac{1}{2}(I_k - \mathrm{i}K_k)\,, \qquad k = 1,2,3\,,\\ J'_k &= \tfrac{1}{2}(I_k + \mathrm{i}K_k)\,, \end{aligned} \tag{1.96}$$

satisfy the commutation relations of two independent copies of the Lie algebra $\mathfrak{su}(2)$, viz.

$$\begin{aligned} [J_i, J_j] &= \varepsilon_{ijk}J_k\,,\\ [J'_i, J'_j] &= \varepsilon_{ijk}J'_k\,,\\ [J_i, J'_j] &= 0\,. \end{aligned} \tag{1.97}$$

Note that the operators (1.96) are defined for the complexified algebra. The complexified Lie algebra of $\mathfrak{so}(3,1)$ is thus related to $\mathfrak{su}(2)\otimes\mathfrak{su}(2)$, and as such all finite dimensional irreducible representations of $\mathfrak{so}(3,1)$ are obtained from products of two representations of $\mathfrak{su}(2)$ and thus classified by the pair of non-negative $\frac{1}{2}$-integers (j,j'). The (j,j') representation has dimension $(2j+1)(2j'+1)$. The representations (j,j') and (j',j) are inequivalent representations when $j\neq j'$. Rather, (j',j) is equivalent to the complex conjugate of (j,j'). The four-dimensional vector representation of (1.31) for $D = 4$ is denoted by $(\frac{1}{2},\frac{1}{2})$.

Ex. 1.15 *Verify (1.97).*

2
The Dirac Field

The Dirac equation is based on special representations of the Lorentz group, called spinor representations. They were discovered by Élie Cartan in 1913 [12, 13]. These representations are very different from the D-dimensional defining representation discussed in Ch. 1 (and from tensor products of the defining representation). It is remarkable and profound that spinor representations are realized in nature and not just mathematical curiosities. They describe fermionic particles such as the electron and quarks, and they are required for supersymmetry.

Our treatment of the Dirac equation should be viewed as part of our review of the basic notions of relativistic field theory needed to move ahead in this book. Many readers will already be familiar with the Dirac field. We advise them to skim this chapter to learn our conventions and then move on to Ch. 3 in which the Clifford algebra and Majorana spinors are discussed. It is this material that is really essential in applications to SUSY and supergravity later in the book.

2.1 The homomorphism of $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,1)$.

Spinor representations exist for all spacetime dimensions D. We introduce them for $D = 4$. In the notation of Sec. 1.5, a general spinor representation is labeled $(j\,,\,j')$ with j a $\frac{1}{2}$-integer and j' an integer, or vice versa. The fundamental spinor representations are $(\frac{1}{2}, 0)$ and its complex conjugate $(0, \frac{1}{2})$. We now discuss the important $2:1$ homomorphism of the group $\mathrm{SL}(2,\mathbb{C})$ of unimodular 2×2 complex matrices onto the connected component of $O(3,1)$. It will lead to an explicit description of the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations, and it is central to the treatment of fermions in quantum field theory.

First we note that a general 2×2 hermitian matrix can be parameterized as

$$\mathbf{x} = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix} . \tag{2.1}$$

and that $\det \mathbf{x} = -x^\mu \eta_{\mu\nu} x^\nu$, which is the negative of the Minkowski norm of the 4-vector x^μ. This suggests a close relation between the linear space of hermitian 2×2 matrices and 4-dimensional Minkowski space. Indeed, there is an isomorphism between these spaces, which we now elucidate.

For this purpose we introduce two complete sets of 2×2 matrices

$$\sigma_\mu = (\,-\mathbb{1}, \sigma_i\,)\;, \qquad \bar{\sigma}_\mu = \sigma^\mu = (\,\mathbb{1}, \sigma_i\,) \tag{2.2}$$

where $\mathbb{1}$ is the unit matrix, and the 3 Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\,, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.3}$$

The index μ on the matrices of (2.2) is a Lorentz vector index, suggesting that the matrices are 4-vectors. We will make this precise shortly, and, in anticipation, we will raise and lower these indices using the Minkowski metric.

Ex. 2.1 *Show that*

$$\begin{aligned} \sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu &= 2\eta_{\mu\nu}\mathbb{1}\,, && (2.4)\\ \operatorname{tr}(\sigma^\mu\, \bar{\sigma}_\nu) &= 2\delta^\mu{}_\nu\,. && (2.5) \end{aligned}$$

Using the matrices σ_μ and $\bar{\sigma}_\mu$ and (2.5), we easily find

$$\mathbf{x} = \bar{\sigma}_\mu x^\mu\,, \qquad x^\mu = \tfrac{1}{2}\operatorname{tr}(\sigma^\mu \mathbf{x})\,, \tag{2.6}$$

which gives the explicit form of the isomorphism. Given the 4-vector x^μ, one can construct the associated matrix $\mathbf{x}$ from the first equation of (2.6), and one can obtain x^μ from $\mathbf{x}$ using the second equation.

Let A be a matrix of $\mathrm{SL}(2,\mathbb{C})$, and consider the linear map

$$\mathbf{x} \to \mathbf{x}' \equiv A\mathbf{x}A^\dagger\,. \tag{2.7}$$

The associated 4-vectors are also linearly related, i.e. $x'^\mu \equiv \phi(A)^\mu{}_\nu x^\nu$, and (2.6) can be used to obtain the explicit form of the matrix $\phi(A)$,

$$\phi(A)^\mu{}_\nu = \tfrac{1}{2}\mathrm{tr}(\sigma^\mu \mathrm{A}\bar{\sigma}_\nu \mathrm{A}^\dagger)\,. \tag{2.8}$$

Since the transformation (2.7) preserves $\det \mathbf{x}$, the Minkowski norm $x^\mu\eta_{\mu\nu}x^\nu$ is invariant. This means that the matrix $\phi(A)$ satisfies (1.27); it must be a Lorentz transformation, and we can write, using (1.24),

$$\Lambda^{-1\,\mu}{}_\nu = \phi(A)^\mu{}_\nu\,. \tag{2.9}$$

Since the group $\mathrm{SL}(2,\mathbb{C})$ is connected [14], Λ lies in the connected component of $O(3,1)$, i.e. the proper Lorentz group.

Here are some exercises to help familiarize readers with this important homomorphism.

Ex. 2.2 *Verify that (2.8) is a group homomorphism by showing that* $\phi(AB) = \phi(A)\phi(B)$.

Ex. 2.3 *Show that the kernel of the homomorphism consists of the matrices* $(\mathbb{1}, -\mathbb{1})$.

Ex. 2.4 *Show that* $A\bar{\sigma}_\mu A^\dagger = \bar{\sigma}_\nu {\Lambda^{-1}}^\nu{}_\mu$ *and* $A^\dagger \sigma_\mu A = \sigma_\nu \Lambda^\nu{}_\mu$. *This gives precise meaning to the statement that the matrices* $\bar{\sigma}_\mu$, σ_ν *are 4-vectors.*

Let us introduce two sets of matrices that will turn out to be generators of the Lie algebra $\mathfrak{so}(3,1)$ in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations. We define them in terms of the matrices $\sigma_\mu, \bar{\sigma}_\nu$ as

$$\begin{aligned} \sigma_{\mu\nu} &= \tfrac{1}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)\,, \\ \bar{\sigma}_{\mu\nu} &= \tfrac{1}{4}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu)\,. \end{aligned} \tag{2.10}$$

Note that $\sigma^{\mu\nu\dagger} = -\bar{\sigma}^{\mu\nu}$. The finite Lorentz transformation (1.32) is then represented as

$$L(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu\nu}\sigma_{\mu\nu}}\,, \tag{2.11}$$

$$\bar{L}(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu\nu}\bar{\sigma}_{\mu\nu}}\,. \tag{2.12}$$

Ex. 2.5 *Show that*

$$L(\lambda)^\dagger = \bar{L}(-\lambda) = \bar{L}(\lambda)^{-1}\,. \tag{2.13}$$

Ex. 2.6 *Use (2.4) to show that the commutator algebras of* $\sigma^{[\mu\nu]}$ *and* $\bar{\sigma}^{[\mu\nu]}$ *are isomorphic to (1.34). According to (1.96-1.97), the commutators of the representatives* $J_k = -\frac{1}{2}(\varepsilon_{ijk}\sigma_{ij} + \mathrm{i}\sigma_{0k})$ *should satisfy (1.97), while the matrices* $J'_k = -\frac{1}{2}(\varepsilon_{ijk}\sigma_{ij} - \mathrm{i}\sigma_{0k})$ *should vanish, since we are in the* $(\frac{1}{2}, 0)$ *representation. Verify these properties. Note that* ε_{ijk} *is the 3-dimensional Levi-Civita symbol.*

The representation matrices L, $\bar{L}$ of (2.11), (2.12) are directly related to the $\mathrm{SL}(2,\mathbb{C})$ map (2.7). For a Lorentz transformation Λ^{-1} related to $\mathrm{SL}(2,\mathbb{C})$ matrix A by (2.9), we can identify $A = L^{-1}$ and $A^\dagger = \bar{L}$.

We now argue that one can reach any proper Lorentz transformation from $\mathrm{SL}(2,\mathbb{C})$ via the homomorphism (2.7). We show explicitly that the $\mathrm{SL}(2,\mathbb{C})$ matrix $A = L(\lambda)^{-1}$ with $\lambda_{03} = -\lambda_{30} = -\rho$ and other $\lambda_{\mu\nu} = 0$ maps to a Lorentz boost in the 3-direction under (2.7). The parameter ρ is conventionally called the rapidity. We use $A\sigma_\mu A^\dagger = \sigma_\nu \Lambda^\nu{}_\mu$, which was proven in Ex. 2.4, to obtain the corresponding Lorentz transformation. Using $A = \mathrm{e}^{-\frac{1}{2}\rho\sigma_3}$ we compute

$$\begin{aligned} A\mathbb{1}A^\dagger &= \mathbb{1}\cosh\rho - \sigma_3 \sinh\rho\,, \\ A\sigma_3 A^\dagger &= \mathbb{1} - \sinh\rho + \sigma_3 \cosh\rho\,, \\ A\sigma_{1,2}A^\dagger &= \sigma_{1,2}\,. \end{aligned} \tag{2.14}$$

On the right side we recognize the matrix of the Lorentz boost

$$\Lambda^{-1} = \begin{pmatrix} \cosh\rho & 0 & 0 & -\sinh\rho \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\rho & 0 & 0 & \cosh\rho \end{pmatrix} . \tag{2.15}$$

Similarly, one can consider the $\mathrm{SL}(2,\mathbb{C})$ matrices $A = L(\lambda)$ with non-vanishing λ_{ij} only, and show that (2.7) maps them to rotations of the spatial σ_i with any desired rotation angle.

Ex. 2.7 *Show that (2.7) works as claimed for spatial rotations about the* $1, 2, 3$ *axes.*

Since any proper Lorentz transformation can be expressed as a product of rotations and a boost in the 3-direction, our discussion shows that the homomorphism is 'onto.'

In any even spacetime dimension $D = 2m$ the situation is similar to what we have described in detail in 4 dimensions. The Lie algebra $\mathfrak{so}(2m-1,1)$ has a pair of inequivalent fundamental spinor representations of dimension 2^{m-1}. The products of exponentials of matrices of either representation satisfy the composition rules of a Lie group that is called $\mathrm{Spin}(2m-1,1)$. There is a 2:1 homomorphism of $\mathrm{Spin}(2m-1,1)$ onto the connected component of $O(2m-1,1)$, and $\mathrm{Spin}(2m-1,1)$ is the universal covering group of the proper subgroup of $\mathrm{O}(2m-1,1)$. For odd $D = 2m+1$ the situation of spinor representations is somewhat different and will be described in Ch. 3.

2.2 The Dirac Equation

We follow, at least roughly, the historical development, and introduce the Dirac equation as a relativistic wave equation describing a particle with internal structure. Such a particle is described by a multi-component field, e.g. $\Phi^M(x)$. The indices M label the components of a column vector that transforms under some finite-dimensional representation of the Lorentz group. For the particular case of the Dirac field the representation is closely related to the fundamental spinor representations we have just discussed. We again work for general D, with special emphasis on $D = 4$.

Dirac postulated that the electron is described by a complex valued multi-component field $\Psi(x)$ called a spinor field, which satisfies the first order wave equation

$$\boxed{\partial\!\!\!/\Psi(x) \equiv \gamma^\mu \partial_\mu \Psi(x) = m\Psi(x)\,.} \tag{2.16}$$

The quantities γ^μ, $\mu = 0, 1, \ldots, D-1$ are a set of square matrices, which act on the indices of the spinor field Ψ . Applying the Dirac operator again, one finds

$$\begin{aligned} \partial\!\!\!/^2\Psi &= m^2\Psi\,, \\ \tfrac{1}{2}\{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\}\partial_\mu\partial_\nu\Psi &= m^2\Psi\,. \end{aligned} \tag{2.17}$$

If we require that the second order differential operator on the left is equal to the d'Alembertian, then we fulfill the physical requirement of plane-wave solutions discussed in Ch. 1. This means that the matrices must satisfy

$$\boxed{\{\gamma^\mu,\gamma^\nu\}\equiv\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2\,\eta^{\mu\nu}\,\mathbb{1}\,,} \tag{2.18}$$

where $\mathbb{1}$ is the identity matrix in the spinor indices.

The condition (2.18) is the defining relation of the Clifford algebra associated with the Lorentz group. The D matrices γ^μ are the generating elements of the Clifford algebra, and a basis of the algebra consists of the unit matrix and all independent products of the generators. The structure of the Clifford algebra is important, and it is discussed systematically for general D in Sec. 3.1. For immediate purposes, we note that there is an irreducible representation by square matrices of dimension $2^{[\frac{D}{2}]}$, where $[x]$ is the largest integer less than or equal to x. The representation is unique up to equivalence for even $D=2m$, and there are two inequivalent representations for odd dimensions. It is always possible to choose a representation in which the spatial gamma matrices γ^i, $i=1,2,\ldots,D-1$ are hermitian and γ^0 is anti-hermitian. We will always work in such a representation, which we call a hermitian representation.

For generic D, the matrices γ^μ are necessarily complex, so the spinor field must have complex components. After Dirac's work, Majorana discovered that there are real representations in $D=4$ dimensions, and it is now known that such Majorana representations exist in dimensions $D=2,\,3,\,4 \bmod 8$. In these dimensions, one may impose the constraint that the field $\Psi(x)$ is real. The special case of Majorana spinors is very important for supersymmetry and supergravity, and it will be discussed in Sec. 3.3. In this chapter we assume that $\Psi(x)$ is complex.

There are various levels of interpretation of the components of $\Psi(x)$. In the first quantized formalism and in many classical applications, they are simply complex numbers. However, when second quantization is introduced through the fermionic path integral, the components of $\Psi(x)$ are anti-commuting Grassmann variables, which satisfy $\{\Psi_\alpha(x),\Psi_\beta(y)\}=0$. Finally in the second quantized operator formalism, they are operators in Hilbert space. The equations we write are valid in both of the first two situations. Although the quantized theory appears rarely in this book, our basic formulas are compatible with the canonical formalism, positive Hilbert space metric and positive energy.

Equivalent representations of the Clifford algebra describe equivalent physics. Therefore the physical implications of the Dirac formalism should be independent of the choice of representation. Indeed much of the physics can be deduced in a representation independent fashion, but an explicit representation is convenient for some purposes. We display one useful representation for $D=4$, namely a Weyl representation in which the 4×4 γ^μ have the 2×2 Weyl matrices of (2.2) in off-diagonal blocks

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} . \tag{2.19}$$

There are 'block off-diagonal' representations of this type in all even dimensions. This is shown in Ex. 3.10 of Ch. 3.

One important fact about the Clifford algebra in general dimension D is that the commutators

$$\Sigma^{\mu\nu} \equiv \tfrac{1}{4} [\gamma^\mu, \gamma^\nu] , \tag{2.20}$$

are generators of a $2^{[\frac{D}{2}]}$-dimensional representation of the Lie algebra of $\mathrm{SO}(D-1,1)$. It is a straightforward exercise to show, using only the Clifford property (2.18), that the commutator algebra of the matrices $\Sigma^{\mu\nu}$ is isomorphic to (1.34). An explicit representation is not needed.

Ex. 2.8 *Do this straightforward exercise.*

Ex. 2.9 *Show, using only (2.18), that* $[\Sigma^{\mu\nu}, \gamma^\rho] = 2\gamma^{[\mu}\eta^{\nu]\rho} = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\mu\rho}$.

In the Weyl representation given above, one sees that the matrices $\Sigma^{\mu\nu}$ are block-diagonal with the 2-component $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ of (2.11) (2.12) as the diagonal entries. The 4-dimensional spinor representation of $\mathfrak{so}(3,1)$ is therefore reducible. Indeed, it is the direct sum of the irreducible $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations discussed in Sec. 2.1. Of course, reducibility of the representation $\Sigma^{\mu\nu}$ holds for any choice of the γ^μ, since the matrices $\Sigma^{\mu\nu}$ are equivalent to those of the Weyl representation. In all even dimensions, there is an analogous reduction to the direct sum of a $\frac{1}{2}2^{[\frac{D}{2}]}$-dimensional irreducible representation of $\mathfrak{so}(D-1,1)$ plus the conjugate representation. (See Sec. 3.1.6.) In odd space-time dimension the representation given by the $\Sigma^{\mu\nu}$ is irreducible.

Finite proper Lorentz transformations are represented by the matrices

$$L(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}} , \tag{2.21}$$

and we have

$$L(\lambda)\gamma^\rho L(\lambda)^{-1} = \gamma^\sigma \Lambda(\lambda)_\sigma{}^\rho . \tag{2.22}$$

From this one can deduce the important Lorentz covariance property of the Dirac equation; given *any solution* $\Psi(x)$, then

$$\Psi'(x) = L(\lambda)^{-1}\Psi(\Lambda(\lambda)x) \tag{2.23}$$

is also a solution.

Ex. 2.10 *Use the result of exercise 2.9 and (2.21) to prove (2.22) and (2.23).*

Since the Dirac field is also a solution of the Klein-Gordon equation, the general solution is also the sum of positive and negative frequency parts in analogy with (1.3). This Fourier expansion reads

$$\begin{aligned}\Psi(x) &= \Psi_+(x) + \Psi_-(x)\,,\\ \Psi_+(x) &= \int \frac{\mathrm{d}^{(D-1)}\vec{p}}{(2\pi)^{D-1}2E}\mathrm{e}^{\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}\sum_s u(\vec{p},s)c(\vec{p},s)\,,\\ \Psi_-(x) &= \int \frac{\mathrm{d}^{(D-1)}\vec{p}}{(2\pi)^{D-1}2E}\mathrm{e}^{-\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}\sum_s v(\vec{p},s)d(\vec{p},s)^*\,.\end{aligned} \tag{2.24}$$

The * indicates complex conjugation in the classical theory and an operator adjoint after quantization.

The new features in the spinor case are the momentum space wave functions[1] $u(\vec{p},s)$, $v(\vec{p},s)$, which are column vectors (with the same number of components as the field Ψ, namely $2^{[\frac{D}{2}]}$ components), and s is a discrete label with $\frac{1}{2}2^{[\frac{D}{2}]}$ values. These describe the various 'spin-states' of the Dirac particle. For $m \neq 0$ these states transform is an irreducible representation of $\mathrm{SO}(D-1)$, which is the subgroup of $\mathrm{SO}(D-1,1)$ that fixes the timelike energy-momentum vector $(E,\vec{p})$.

The analogous expansions of the Dirac adjoint field $\bar{\Psi}$, defined in the next section, involve the conjugate quantities c^*, d. In the second quantized theory of the complex spinor field, $c(\vec{p},s)$, $c(\vec{p},s)^*$, $d(\vec{p},s)$, $d(\vec{p},s)^*$ are the annihilation and creation operators for particles and anti-particles of momentum $\vec{p}$ and spin quantum number s. For a Majorana spinor field, anti-particles are not distinct from particles and we replace d, $d^* \to c$, c^* in these expansions. Furthermore, v is related to u^*, which the reader can check later in Ex. 3.35.

2.3 The spinors $u(\vec{p},s)$ and $v(\vec{p},s)$ for $D=4$.

In this section we construct the $\vec{p}$-space spinors $u(\vec{p},s)$ and $v(\vec{p},s)$, which appear in the plane wave expansion (2.24) of the free Dirac field. We treat the case $D=4$ explicitly, but the ideas involved are similar in all spacetime dimensions. The details are not essential to the understanding of supergravity, but it is useful to complete the story of the expansion (2.24). Furthermore, the spinors we find for the Dirac field are used to form the momentum space wave functions for the gravitino in Ch. 5.

Since $\Psi_\pm(x)$ in (2.24) satisfy the Dirac equation (2.16) and the plane waves $\mathrm{e}^{\mathrm{i}p\cdot x}$ for different 4-vectors p^μ are linearly independent, it follows that the spinors satisfy the algebraic equations

$$\begin{aligned}\gamma\cdot p\, u(\vec{p},s) &= -\mathrm{i}m\, u(\vec{p},s)\,,\\ \gamma\cdot p\, v(\vec{p},s) &= +\mathrm{i}m\, v(\vec{p},s)\,.\end{aligned} \tag{2.25}$$

[1] For $D=4$, these wave functions will be discussed in detail in Sec. 2.3.

We will solve these equations using the Weyl representation (2.19) in which the equation for $u(\vec{p}, s)$ in (2.25) becomes

$$\begin{pmatrix} 0 & \sigma \cdot p \\ \bar{\sigma} \cdot p & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -\mathrm{i}m \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} . \tag{2.26}$$

Here w_1 and w_2 are a temporary notation for the upper and lower components of $u(\vec{p}, s)$. The equation for $v(\vec{p}, s)$ is similar, differing only in the sign of the right-hand side.

We assume that $m > 0$ and take a direct approach to the solution of (2.26), suggested by the treatment of Sec. 3.3 of [15]. We note that the matrices $-\sigma \cdot p$ and $\bar{\sigma} \cdot p$ each have the two positive eigenvalues $E \pm |\vec{p}|$, and they therefore have matrix square roots $\sqrt{-\sigma \cdot p}$ and $\sqrt{\bar{\sigma} \cdot p}$ defined by taking the positive root in each eigenspace. We also have the relations $-\sigma \cdot p\, \bar{\sigma} \cdot p = -\bar{\sigma} \cdot p\, \sigma \cdot p = m^2$. Using this information it is easy to see that

$$u(p) = \begin{pmatrix} \sqrt{-\sigma \cdot p}\, \xi \\ \mathrm{i}\sqrt{\bar{\sigma} \cdot p}\, \xi \end{pmatrix} \tag{2.27}$$

is a solution of (2.26) for any 2-component spinor ξ. Similarly

$$v(p) = \begin{pmatrix} \sqrt{-\sigma \cdot p}\, \eta \\ -\mathrm{i}\sqrt{\bar{\sigma} \cdot p}\, \eta \end{pmatrix} \tag{2.28}$$

is a solution of the corresponding equation for $v(p)$ for any 2-component η. Note that we have omitted the index s, which describes the spin state of the particle because that information is determined by the choice of ξ and η, to which we now turn.

It is convenient to choose spin states which are eigenstates of the helicity, the component of angular momentum in the direction of motion of the particle. Therefore we define spinors $\xi(\vec{p}, \pm)$ that satisfy

$$\vec{\sigma} \cdot \vec{p}\, \xi(\vec{p}, \pm) \; = \; \pm|\vec{p}|\, \xi(\vec{p}, \pm) \, . \tag{2.29}$$

Note that $\vec{\sigma} \cdot \vec{p} \equiv \sigma^i p^i$ is summed over the spatial components only. We assume these spinors to be normalized:

$$\xi(\vec{p}, \pm)^\dagger \xi(\vec{p}, \pm) = 1 \, , \qquad \xi(\vec{p}, \pm)^\dagger \xi(\vec{p}, \mp) = 0 \, . \tag{2.30}$$

Since the angular momentum operator is $\vec{J} = \frac{1}{2}\vec{\sigma}$, the spinor $\xi(\vec{p}, \pm)$ is an eigenstate of $\vec{p} \cdot \vec{J}/|\vec{p}|$ with eigenvalue $\pm 1/2$. We also choose

$$\eta(\vec{p}, \pm) = -\sigma_2\, \xi(\vec{p}, \pm)^* \, , \tag{2.31}$$

which is then normalized in the same way as ξ in (2.30).

Ex. 2.11 *Assume that* $(p^1, p^2, p^3) = |\vec{p}|(\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$. *Find* $\xi(\vec{p}, \pm)$ *explicitly.*

Ex. 2.12 *Show that* $\vec{\sigma}\cdot\vec{p}\,\eta(\vec{p},\pm) = \mp|\vec{p}|\,\eta(\vec{p},\pm)$.

We may now specify the precise spinors to be inserted in the Fourier expansion (2.24) as

$$\begin{aligned} u(\vec{p},\pm) &= \begin{pmatrix} \sqrt{E\mp|\vec{p}|}\,\xi(\vec{p},\pm) \\ \mathrm{i}\sqrt{E\pm|\vec{p}|}\,\xi(\vec{p},\pm) \end{pmatrix}, && (2.32) \\ v(\vec{p},\pm) &= \begin{pmatrix} \sqrt{E\pm|\vec{p}|}\,\eta(\vec{p},\pm) \\ -\mathrm{i}\sqrt{E\mp|\vec{p}|}\,\eta(\vec{p},\pm) \end{pmatrix}. && (2.33) \end{aligned}$$

Let us note that the spinors (2.32, 2.33) have a smooth massless limit and satisfy the *massless* Dirac equation in this limit. The limit simplifies because of the relations $(E\pm\vec{\sigma}\cdot\vec{p})\xi(\vec{p},\pm) = 2E\xi(\vec{p},\pm)$, and $(E\pm\vec{\sigma}\cdot\vec{p})\xi(\vec{p},\mp) = 0$, where $E = |\vec{p}|$ for a massless particle. Thus the massless u-spinors take the simple form[2]

$$u(\vec{p},-) = \sqrt{2E}\begin{pmatrix} \xi(\vec{p},-) \\ 0 \end{pmatrix}, \qquad u(\vec{p},+) = \sqrt{2E}\begin{pmatrix} 0 \\ \mathrm{i}\xi(\vec{p},+) \end{pmatrix}. \tag{2.34}$$

Similarly, one finds for massless v-spinors

$$v(\vec{p},-) = \sqrt{2E}\begin{pmatrix} 0 \\ -\mathrm{i}\eta(\vec{p},-) \end{pmatrix}, \qquad v(\vec{p},+) = \sqrt{2E}\begin{pmatrix} \eta(\vec{p},+) \\ 0 \end{pmatrix}. \tag{2.35}$$

Ex. 2.13 *The following properties of bilinears of the* u, v *spinors are frequently useful. Derive them.*

$$\begin{aligned} \bar{u}(\vec{p},s)\,u(\vec{p},s') &= -\bar{v}(\vec{p},s)v(\vec{p},s') = -2m\delta_{ss'}\,, \\ \bar{u}(\vec{p},s)v(\vec{p},s') &= \bar{v}(\vec{p},s)u(\vec{p},s') = 0\,, \\ \bar{u}(\vec{p},s)\gamma^\mu u(\vec{p},s) &= \bar{v}(\vec{p},s)\gamma^\mu v(\vec{p},s) = -2\mathrm{i}p^\mu\,. \end{aligned} \tag{2.36}$$

2.4 Dirac adjoint and bilinear form

Our task in this section is to find a suitable Lorentz invariant bilinear form for the Dirac field that can be used to construct the Lagrangian density and other fundamental quantities. Under an infinitesimal Lorentz transformation in the $[\mu\nu]$-plane, the variations of Ψ and its adjoint $\Psi^\dagger$ are

$$\begin{aligned} \delta\Psi(x) &= -\tfrac{1}{2}\lambda^{\mu\nu}(\Sigma_{\mu\nu} + L_{[\mu\nu]})\Psi(x) = -\tfrac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\Psi(x) + \lambda^\mu{}_\nu x^\nu\partial_\mu\Psi(x)\,, \\ \delta\Psi^\dagger(x) &= -\tfrac{1}{2}\lambda^{\mu\nu}\Psi^\dagger{\Sigma_{\mu\nu}}^\dagger + \lambda^\mu{}_\nu x^\nu\partial_\mu\Psi(x)^\dagger\,. \end{aligned} \tag{2.37}$$

[2] Since Lorentz transformations cannot change the helicity of a massless particle, the relative phase of +ve and -ve helicity spinors is arbitrary, the relative phase of the corresponding spinors is arbitrary. One can redefine $u(\vec{p},+) \to \alpha u(\vec{p},+)$, $v(\vec{p},+) \to \alpha^* v(\vec{p},+)$ where α is an arbitrary phase; $\alpha = -\mathrm{i}$ is particularly convenient.

This can be compared with (1.60) with $J_{[\mu\nu]} = \Sigma_{[\mu\nu]} + L_{[\mu\nu]}$. We suppose that our Lorentz invariant non-degenerate bilinear form may be written as

$$\Psi^\dagger \beta \Psi \,, \tag{2.38}$$

where β is some square, invertible matrix that we want to find. Lorentz invariance requires that

$$\Sigma^\dagger_{\mu\nu}\beta + \beta\Sigma_{\mu\nu} = 0\,. \tag{2.39}$$

We look for a real bilinear form[3], so we will choose an hermitian matrix β.

The generators of spatial rotations Σ_{ij} are anti-hermitian but those of boosts Σ_{0i} are hermitian. Thus we cannot just choose β to be the identity. This may be understood as follows: if the Lorentz group were compact, it would have finite dimensional unitary representations, in which the representatives of its Lie algebra would be anti-hermitian, satisfying $\Sigma^\dagger_{\mu\nu} = -\Sigma_{\mu\nu}$ and $\Psi^\dagger\Psi$ would be the desired Lorentz scalar. However, the Lorentz group is non-compact, and it has no finite dimensional unitary representations and the required anti-hermitian property holds for only spatial rotations but not for boosts.

Nevertheless, it is easy to check that (2.39) is satisfied if we take β to be any multiple of γ^0. This form is valid in any hermitian representation and is unique. It gives

$$\begin{aligned} \beta\gamma_\mu\beta^{-1} &= -\gamma^\dagger_\mu \,, \\ \beta\Sigma_{\mu\nu}\beta^{-1} &= -\Sigma_{\mu\nu}{}^\dagger \,. \end{aligned} \tag{2.40}$$

The last relation is a rewriting of (2.39), which shows that the Dirac spinor representation of the Lorentz group is equivalent to the transposed, complex conjugate representation.

It is convenient to make the specific choice

$$\beta = \mathrm{i}\gamma^0 \,. \tag{2.41}$$

We then define the Dirac adjoint, a row vector, by

$$\bar\Psi = \Psi^\dagger\beta = \Psi^\dagger \mathrm{i}\gamma^0 \,, \tag{2.42}$$

and write our invariant bilinear form as

$$\bar\Psi\Psi \,. \tag{2.43}$$

Ex. 2.14 *Show that the bilinear form $\bar\Psi\Psi$ has signature $(2,2)$ in 4 dimensions.*

Ex. 2.15 *Show that $(\bar\Psi_1\Psi_2)^\dagger = \bar\Psi_2\Psi_1$ for any pair of Dirac spinors and valid for both commuting and anti-commuting (Grassmann-valued) components.*

---

[3] In this book the adjoint operation on a pair of quantities is defined to include both conjugation and reversal of order, i.e. $(AB)^\dagger \equiv B^\dagger A^\dagger$

2.5 Dirac Action

We can now define the action of the free Dirac field,

$$S[\bar{\Psi}, \Psi] = -\int \mathrm{d}^D x \bar{\Psi}[\gamma^\mu \partial_\mu - m]\Psi(x) \,. \tag{2.44}$$

The condition that it is stationary reads (including integration by parts in the second term)

$$\delta S[\bar{\Psi}, \Psi] = -\int \mathrm{d}^D x \{\overline{\delta\Psi}[\gamma^\mu \partial_\mu - m]\Psi \; - \; \bar{\Psi}[\gamma^\mu \overleftarrow{\partial}_\mu + m]\delta\Psi\} = 0 \,. \tag{2.45}$$

The variations $\delta\Psi$ and $\overline{\delta\Psi}$ are arbitrary infinitesimal quantities related by conjugation as in (2.42). Given one choice, one may consider another with $\delta'\Psi = \mathrm{i}\delta\Psi$ and $\overline{\delta'\Psi} = -\mathrm{i}\overline{\delta\Psi}$. With the second choice, the relative sign between the two terms in (2.45) changes. Thus the coefficients of $\delta\Psi$ and $\overline{\delta\Psi}$ in (2.45) must vanish independently. The Euler-Lagrange variational process therefore yields the Dirac equation (2.16) and its conjugate

$$\bar{\Psi}[\gamma^\mu \overleftarrow{\partial}_\mu + m] = 0 \,. \tag{2.46}$$

Note that, in the discussion above, we assumed that the components of Ψ were ordinary complex numbers. In later work we will want to take them to be either anti-commuting Grassmann numbers or as operators on Hilbert space. Since the manipulations we carried out did not require a change of the order of Ψ and $\delta\Psi$ they remain valid in that more general situation.

2.6 Weyl spinor fields in even spacetime dimension.

Let's return to the case of even dimension $D = 2m$. We saw in Sec. 2.2 that the Dirac representation of the Lorentz group is reducible if $D = 4$. Since there is a "block off-diagonal" representation for any even dimension, the same is true for all $D = 2m$ with two irreducible subrepresentations, each of dimension $2^{(m-1)}$. This suggests that a Dirac spinor $\Psi(x)$ is not the simplest Lorentz covariant field. In a sense this is correct. One can define a Weyl field $\psi(x)$ with $2^{(m-1)}$ components, which is defined to transform as

$$\psi(x) \to \psi'(x) = L(\lambda)^{-1}\psi(\Lambda(\lambda)x) \,, \tag{2.47}$$

where $L(\lambda)$ is defined as in (2.11), but now for any even dimension. One can also define a field $\bar{\chi}(x)$ that transforms in the conjugate representation, namely as

$$\bar{\chi}(x) \to \bar{\chi}'(x) = \bar{L}(\lambda)^{-1}\bar{\chi}(\Lambda(\lambda)x) \,, \tag{2.48}$$

with $\bar{L}(\lambda)$ defined as in (2.12). There are Lorentz invariant wave equations for these fields:

$$\bar{\sigma}^\mu \partial_\mu \psi(x) = 0 \,, \tag{2.49}$$

$$\sigma^\mu \partial_\mu \bar{\chi}(x) = 0 \,. \tag{2.50}$$

Ex. 2.16 *Extend the discussion of Lorentz transformations in Sec. 2.1 to any even dimension and show that the wave equations are indeed Lorentz invariant. Show that (2.49, 2.50) imply that* $\Box\psi(x) = 0$ *and* $\Box\bar{\chi}(x) = 0$.

The Weyl equations thus admit plane wave solutions $\mathrm{e}^{\pm \mathrm{i}p\cdot x}$ with $p^0 = |\vec{p}|$ and therefore describe *massless* particles. The fields ψ and $\bar{\chi}$ each describe particles of a single helicity value, namely negative for ψ and positive for $\bar{\chi}$, and their anti-particles of opposite helicity. The plane wave expansion of $\psi(x)$ is

$$\psi(x) = \int \frac{\mathrm{d}^{(D-1)}\vec{p}}{(2\pi)^{\frac{1}{2}(D-1)}\sqrt{2E}} \left[\mathrm{e}^{\mathrm{i}p\cdot x} c(\vec{p},-) + \mathrm{e}^{-\mathrm{i}p\cdot x} d(\vec{p},+)^*\right] \xi(\vec{p},-)\,. \tag{2.51}$$

It follows from (2.49) that one can choose the same momentum space spinor $\xi(\vec{p},-)$ in the positive and negative frequency terms. The expansion for $\bar{\chi}(x)$ is similar.

For $D = 2m$ the expansion contains a sum over 2^{m-2} values of the 'spin label' s, and the spin states of the particle transform in an irreducible spinor representation of $\mathrm{SO}(D-2)$.

To write a kinetic action for the Weyl field we need both $\psi(x)$ and its adjoint $\psi(x)^\dagger$. From these we can construct a bilinear form that transforms as a vector.

Ex. 2.17 *Use (2.47) and the results of 2.4 and 2.5 to show that under the Lorentz transformation* Λ,

$$\psi(x)^\dagger \bar{\sigma}^\mu \psi(x) \to \Lambda^{-1\,\mu}{}_\nu \psi(\Lambda x)^\dagger \bar{\sigma}^\nu \psi(\Lambda x)\,. \tag{2.52}$$

The Lorentz invariant hermitian action is then

$$S[\psi,\bar{\psi}] = -\int \mathrm{d}^D x\, \mathrm{i}\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi\,. \tag{2.53}$$

There is an analogous action for the field $\bar{\chi}(x)$ and its adjoint.

It is intriguing that with a single Weyl field (either ψ or $\bar{\chi}$), there is no way to introduce a mass. The candidate wave equation

$$\bar{\sigma}^\mu \partial_\mu \psi(x) = m\psi(x)\,, \tag{2.54}$$

is *not* Lorentz invariant.

Ex. 2.18 *Why not?*

One can describe massive particles using both $\psi(x)$ and $\bar{\chi}(x)$. In fact this is the secret content of a single Dirac field in even dimension, and this can be exhibited using a Weyl representation of the γ-matrices.

Ex. 2.19 *Show this! Write the Dirac field as the column*

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \bar{\chi}(x) \end{pmatrix}\,. \tag{2.55}$$

and show that the Dirac equation (2.16) in the representation (2.19) is equivalent to the pair of equations

$$\bar{\sigma}^\mu \partial_\mu \psi(x) = m\bar{\chi}(x)\,, \qquad \sigma^\mu \partial_\mu \bar{\chi}(x) = m\psi(x)\,. \tag{2.56}$$

Ex. 2.20 *Show that the Dirac Lagrangian in (2.44) can be rewritten in terms of ψ, $\bar{\chi}$ and their adjoints as*

$$\mathcal{L} = \mathrm{i}\left[-\psi^\dagger\,\bar{\sigma}\cdot\partial\,\psi + \bar{\chi}^\dagger\,\sigma\cdot\partial\,\bar{\chi} - m\,\bar{\chi}^\dagger\,\psi + m\,\psi^\dagger\,\bar{\chi}\right]\,. \tag{2.57}$$

Show that each of the four terms is a Lorentz scalar. Note that the result in (2.13) holds for all even $D = 2m$.

2.7 Conserved currents

2.7.1 Conserved U(1) *current*

In this section we discuss the global U(1) symmetry property of the Dirac field in any spacetime dimension. The free Dirac action (2.44) is invariant under the global U(1) phase transformation $\Psi(x) \to \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}\theta}\Psi(x)$. The conserved Noether current for this symmetry is

$$J^\mu = \mathrm{i}\bar{\Psi}\gamma^\mu\Psi\,. \tag{2.58}$$

The time component is positive in all Lorentz frames for commuting complex spinors,

$$J^0 = \Psi^\dagger\Psi > 0\,. \tag{2.59}$$

Thus, the vector J^μ is generically future-directed timelike.

One of Dirac's original motivations for his famous equation was that, unlike the Klein-Gordon equation, the quantity J^0 could be regarded as a *positive* probability density. This is important in the first quantized version of the theory that Dirac considered, but it ceases to be an issue in the second quantized quantum field theory.

2.7.2 Energy-momentum tensors for the Dirac field

The action (2.44) of a complex spinor field is also invariant under spacetime translations and Lorentz transformations. For translations the Noether current (1.69) obtained from the action (2.44) is

$$T_{\mu\nu} = \bar{\Psi}\gamma_\mu\partial_\nu\Psi + \eta_{\mu\nu}\mathcal{L}\,, \tag{2.60}$$

where the Lagrangian density $\mathcal{L}$ is the integrand of (2.44). This tensor, which is sometimes called the canonical energy-momentum tensor, is conserved on the first index only and non-symmetric, which is different from the scalar stress tensor in (1.74). Since symmetry fails, the simple form (1.76) of the Noether current for Lorentz transformations does not hold if $T_{\mu\nu}$ is used. For these reasons the canonical stress tensor needs to be improved, and we now guide the reader through some exercises that accomplish this.

Ex. 2.21 *It is well known that the Lagrangian density of a field theory can be changed by adding a total divergence $\partial_\mu B^\mu$, since the Euler-Lagrange equations are unaffected. Show that the addition of $\frac{1}{2}\partial_\mu(\bar\Psi\gamma^\mu\Psi)$ brings the action to the form*

$$S' = -\int \mathrm{d}^D x \left[\tfrac{1}{2}\bar\Psi\gamma^\mu \overleftrightarrow{\partial}_\mu\Psi - m\bar\Psi\Psi\right]. \tag{2.61}$$

Note the anti-symmetric derivative, defined as

$$A\overleftrightarrow{\partial}_\mu B \equiv A(\partial_\mu B) - (\partial_\mu A)B\,. \tag{2.62}$$

The advantage of the form (2.61) of the Dirac theory is that the Lagrangian density $\mathcal{L}'$ is hermitian as an operator in Hilbert space.

Ex. 2.22 *Show that the Noether stress tensor obtained using $\mathcal{L}'$ is*

$$T'_{\mu\nu} = \tfrac{1}{2}\bar\Psi\gamma_\mu\overleftrightarrow{\partial}_\nu\Psi + \eta_{\mu\nu}\mathcal{L}'\,. \tag{2.63}$$

Show that $T'_{\mu\nu} - T_{\mu\nu} = \partial^\rho S_{\rho\mu\nu}$ where the tensor $S_{\rho\mu\nu}$ satisfies $S_{\rho\mu\nu} = -S_{\mu\rho\nu}$, as in the discussion of improved Noether currents in Sec. 1.3.

Ex. 2.23 *Show that the addition of $\Delta T_{\mu\nu} = \frac{1}{4}\partial^\rho\left(\bar\Psi\{\Sigma_{\rho\mu},\gamma_\nu\}\Psi\right)$ to $T'_{\mu\nu}$ produces the symmetric energy-momentum tensor*

$$\Theta_{\mu\nu} = \tfrac{1}{4}\bar\Psi(\gamma_\mu\overleftrightarrow{\partial}_\nu + \gamma_\nu\overleftrightarrow{\partial}_\mu)\Psi + \eta_{\mu\nu}\mathcal{L}'. \tag{2.64}$$

Note that symmetry currents are evaluated 'on-shell', i.e. one should assume that Ψ and $\bar\Psi$ satisfy the Dirac equation.

Ex. 2.24 *Consider the variation $\delta(\lambda)\Psi = -\frac{1}{2}\lambda^{\rho\sigma}(\Sigma_{\rho\sigma} + L_{[\rho\sigma]})\Psi$, where $L_{[\rho\sigma]}$ was defined in (1.38), under infinitesimal Lorentz transformations and the analogue for $\delta\bar\Psi$. These variations may be obtained using (2.21),(2.23). Show that the Noether current and Lorentz generators $J_{[\rho\sigma]}$ can be written in the form of (1.76), but using the symmetric stress tensor (2.64).*

3
Clifford algebras and spinors

The Dirac equation is a relativistic wave equation which is first order in space and time derivatives. The key to this remarkable property is the set of γ-matrices, which satisfy the anti-commutation relations (2.18):

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\,\mathbb{1}\,. \tag{3.1}$$

These matrices are the generating elements of a Clifford algebra which plays an important role in supersymmetric and supergravity theories. In the first part of this chapter we discuss the structure of this Clifford algebra for general spacetime dimension D. For a generic value of D, the γ-matrices are necessarily complex. This is why we assumed that the Dirac field is complex in the previous chapter. In certain spacetime dimensions representation of the Clifford algebra is real which means that the γ-matrices are conjugate to real matrices. In this case the basic spinor field may be taken to be real and is called a Majorana spinor field. Since the Majorana field has a smaller number of independent components, it is fair to say that, when it exists, it is more fundamental than the Dirac field. For this reason Majorana spinors are selected in supersymmetry and supergravity. We study the special properties of Majorana spinors in the second part of this chapter.

In the body of the chapter we take a practical approach, intended as a guide to the applications we will need later in the book. More theoretical material is collected in the Appendix 3.A.

3.1 The Clifford Algebra in General Dimension

3.1.1 The generating γ matrices

The main purpose of this section is to discuss the Clifford algebra associated with the Lorentz group in D dimensions. But to be concrete, we start with a general and explicit construction of the generating γ-matrices. It is simplest first to construct Euclidean γ-matrices, which satisfy (3.1) with Minkowski metric $\eta_{\mu\nu}$ replaced by

$\delta_{\mu\nu}$:

$$\begin{aligned}
\gamma^1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots \\
\gamma^2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots \\
\gamma^3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes \ldots \\
\gamma^4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes \ldots \\
\gamma^5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \ldots \\
\ldots &= \ldots .
\end{aligned} \tag{3.2}$$

These matrices are all hermitian with squares equal to $\mathbb{1}$, and they mutually anticommute. Suppose that $D = 2m$ is even. Then we need m factors in the construction (3.2) to obtain γ^μ, $1 \leq \mu \leq D = 2m$. Thus we obtain a representation of dimension $2^{D/2}$. For odd $D = 2m+1$ we need one additional matrix, and we take γ^{2m+1} from the list above, but we keep only the first m factors, i.e. deleting a σ_1. Thus there is no increase in the dimension of the representation in going from $D = 2m$ to $D = 2m+1$, and we can say in general that the construction (3.2) gives a representation of dimension $2^{[D/2]}$, where $[D/2]$ means the integer part of $D/2$.

Euclidean γ-matrices do have physical applications, but we need Lorentzian γ's for the subject matter of this book. To obtain these, all we need to do is pick any single matrix from the Euclidean construction, multiply it by i and label it γ^0 for the time-like direction. This matrix is anti-hermitian and satisfies $(\gamma^0)^2 = -\mathbb{1}$. We then relabel the remaining $D-1$ matrices to obtain the Lorentzian set γ^μ, $0 \leq \mu \leq D-1$. The hermiticity properties of the Lorentzian γ's are summarized by

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 . \tag{3.3}$$

In Appendix 3.A, we give an elegant argument using basic properties of representations of finite groups that, up to equivalence, there is a unique irreducible representation (irrep) of the Clifford algebra by $2^m \times 2^m$ matrices for even dimension $D = 2m$. Any another representation is reducible and equivalent to a direct sum of copies of the irrep above. One can always choose an hermitian irrep, defined as one which satisfies (3.3). In odd dimensions there are two mathematically inequivalent irreps, which differ only in the sign of the 'final' $\pm\gamma^{2m+1}$. In this book we will always use a hermitian irrep of the γ-matrices. Physical consequences are independent of the particular representation chosen.

3.1.2 The complete Clifford algebra

The full Clifford algebra consists of the identity $\mathbb{1}$, the D generating elements γ^μ, plus all independent matrices formed from products of the generators. Since symmetric products reduce to a product containing fewer γ-matrices by (3.1), the new elements must be antisymmetric products. We thus define

$$\gamma^{\mu_1\ldots\mu_r} = \gamma^{[\mu_1}\ldots\gamma^{\mu_r]} , \qquad \text{e.g.} \qquad \gamma^{\mu\nu} = \tfrac{1}{2}\gamma^\mu\gamma^\nu - \tfrac{1}{2}\gamma^\nu\gamma^\mu , \tag{3.4}$$

where the antisymmetrization indicated with $[\ldots]$ is always with total weight 1. Thus the right side of (3.4) contains the overall factor $1/r!$ times a sum of $r!$ signed permutations of the indices. Non-vanishing tensor components have distinct index values and can be written as the products

$$\gamma^{\mu_1\mu_2\cdots\mu_r} = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_r} \qquad \text{for } \mu_1 \neq \mu_2 \neq \cdots \neq \mu_r\,. \tag{3.5}$$

All these matrices are traceless (a proof can be found in Appendix 3.A), except for the lowest rank $r = 0$, which is the unit matrix, and the highest rank matrix with $r = D$ which is traceless only for even D as we will see below.

There are C_r^D (binomial coefficients) independent index choices at rank r. We use the notation Γ^A to denote a generic matrix of this set where A is a 'multi-index' indicating rank r and a set of distinct index values of $\mu_1, \ldots \mu_r$. For even spacetime dimension the matrices are linearly independent, so that the Clifford algebra is an algebra of dimension 2^D.

Ex. 3.1 *Show that the higher rank γ-matrices can be defined as the alternate commutators or anti-commutators*

$$\begin{aligned}
\gamma^{\mu\nu} &= \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]\,,\\
\gamma^{\mu_1\mu_2\mu_3} &= \tfrac{1}{2}\{\gamma^{\mu_1}, \gamma^{\mu_2\mu_3}\}\,,\\
\gamma^{\mu_1\mu_2\mu_3\mu_4} &= \tfrac{1}{2}[\gamma^{\mu_1}, \gamma^{\mu_2\mu_3\mu_4}]\,,\\
&\text{etc.}
\end{aligned} \tag{3.6}$$

3.1.3 Levi-Civita tensor

We need a short technical digression to introduce the Levi-Civita tensor density and derive some of its properties. In every dimension D this is defined as the totally antisymmetric rank D tensor $\varepsilon_{\mu_1\mu_2\cdots\mu_D}$ or $\varepsilon^{\mu_1\mu_2\cdots\mu_D}$ with

$$\varepsilon_{012(D-1)} = 1\,, \qquad \varepsilon^{012(D-1)} = -1\,. \tag{3.7}$$

Indices are raised using the Minkowski metric which leads to the difference in sign above (due to the single timelike direction).

Ex. 3.2 *Prove the contraction identity for these tensors*

$$\varepsilon_{\mu_1\ldots\mu_n\nu_1\ldots\nu_p}\varepsilon^{\mu_1\ldots\mu_n\rho_1\ldots\rho_p} = -\,p!\,n!\,\delta^{\rho_1\cdots\rho_p}_{\nu_1\cdots\nu_p}\,, \qquad p = D - n\,. \tag{3.8}$$

The anti-symmetric p-index Kronecker δ is in turn defined by

$$\delta^{\nu_1\ldots\nu_q}_{\rho_1\cdots\rho_p} \equiv \delta^{\nu_1}_{[\rho_1}\delta^{\nu_2}_{\rho_2}\cdots\delta^{\nu_p}_{\rho_p]}\,, \tag{3.9}$$

which includes a signed sum over $p!$ permutations of the lower indices, each with a coefficient $1/p!$, such that the 'total weight' is 1 (as in (A.9)) .

In 4 dimensions the totally antisymmetric Levi-Civita tensor density is written as $\varepsilon^{\mu\nu\rho\sigma}$. Because an antisymmetric tensor of rank 5 necessarily vanishes when $D = 4$, this satisfies the Schouten identity

$$0 = 5\delta_\mu{}^{[\nu}\varepsilon^{\rho\sigma\tau\lambda]} \equiv \delta_\mu{}^\nu\varepsilon^{\rho\sigma\tau\lambda} + \delta_\mu{}^\rho\varepsilon^{\sigma\tau\lambda\nu} + \delta_\mu{}^\sigma\varepsilon^{\tau\lambda\nu\rho} + \delta_\mu{}^\tau\varepsilon^{\lambda\nu\rho\sigma} + \delta_\mu{}^\lambda\varepsilon^{\nu\rho\sigma\tau}\,. \quad (3.10)$$

3.1.4 Practical gamma matrix manipulation

Supersymmetry and supergravity theories emerge from the concept of fermion spin. It should be no surprise that intricate features of the Clifford algebra are needed to establish and explore the physical properties of these field theories. In this section we explain some useful tricks to multiply γ matrices and higher rank Clifford elements of (3.4). The results are valid for both even and odd D.

Consider first products with index contractions such as

$$\gamma^{\mu\nu}\gamma_\nu = (D-1)\gamma^\mu\,. \quad (3.11)$$

You can memorize this rule, but it is easier to recall the simple logic behind it: ν runs over all values except μ, so there are $(D-1)$ terms in the sum. Similar logic explains the result

$$\gamma^{\mu\nu\rho}\gamma_\rho = (D-2)\gamma^{\mu\nu}\,, \quad (3.12)$$

or even more generally

$$\gamma^{\mu_1\ldots\mu_r\nu_1\ldots\nu_s}\gamma_{\nu_s\ldots\nu_1} = \frac{(D-r)!}{(D-r-s)!}\gamma^{\mu_1\ldots\mu_r}\,. \quad (3.13)$$

Indeed, first we can write $\gamma_{\nu_s\ldots\nu_1}$ as the product $\gamma_{\nu_s}\ldots\gamma_{\nu_1}$ as the antisymmetry is guaranteed by the first factor. Then the index ν_s has $(D-(r+s-1))$ values, while ν_{s-1} has $(D-(r+s-2)$ values, and this pattern continues to $(D-r)$ values for the last one. Note that the second γ on the left-hand side has its indices in opposite order, so that no signs appear when contracting the indices. It is useful to remember the general order reversal symmetry as in

$$\gamma^{\nu_1\ldots\nu_r} = (-)^{r(r-1)/2}\gamma^{\nu_r\ldots\nu_1}\,. \quad (3.14)$$

The sign factor $(-)^{r(r-1)/2}$ is negative for $r = 2, 3 \bmod 4$.

Even if one does not sum over indices, similar combinatorial tricks can be used. For example, when calculating

$$\gamma^{\mu_1\mu_2}\gamma_{\nu_1\ldots\nu_D}\,, \quad (3.15)$$

one knows that the index values μ_1 and μ_2 appear in the set of $\{\nu_i\}$. There are D possibilities for μ_2, and since μ_1 should be different, there remain $D-1$ possibilities for μ_1. Hence the result is

$$\gamma^{\mu_1\mu_2}\gamma_{\nu_1\ldots\nu_D} = D(D-1)\delta^{\mu_2\mu_1}_{[\nu_1\nu_2}\gamma_{\nu_3\ldots\nu_D]}\,. \quad (3.16)$$

Note that such generalized δ functions are always normalized with 'weight one', i.e.

$$\delta^{\mu_2\mu_1}_{\nu_1\nu_2} = \tfrac{1}{2}\left(\delta^{\mu_2}_{\nu_1}\delta^{\mu_1}_{\nu_2} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\right) . \tag{3.17}$$

This makes contractions easy; e.g. we obtain from (3.16)

$$\gamma^{\mu_1\mu_2}\gamma_{\nu_1\ldots\nu_D}\varepsilon^{\nu_1\ldots\nu_D} = D(D-1)\varepsilon^{\mu_2\mu_1\nu_3\ldots\nu_D}\gamma_{\nu_3\ldots\nu_D} . \tag{3.18}$$

We now consider products of gamma matrices without index contractions. The very simplest case is

$$\gamma^\mu\gamma^\nu = \gamma^{\mu\nu} + \eta^{\mu\nu} . \tag{3.19}$$

This follows directly from the definitions: the antisymmetric part of the product is defined in (3.4) to be $\gamma^{\mu\nu}$, while the symmetric part of the product is $\eta^{\mu\nu}$, by virtue of (3.1). This already illustrates the general approach: one first writes the totally anti-symmetric Clifford matrix which contains all the indices and then adds terms for all possible index pairings.

Here is another example:

$$\gamma^{\mu\nu\rho}\gamma_{\sigma\tau} = \gamma^{\mu\nu\rho}{}_{\sigma\tau} + 6\gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]} + 6\gamma^{[\mu}\delta^{\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]} . \tag{3.20}$$

This follows the same pattern. We write the indices $\sigma\tau$ in down position to make it easier to indicate the antisymmetry. The second term contains one contraction. One can choose three indices from the first factor and two indices from the second one, which gives the factor 6. For the third term there are also 6 ways to make two contractions. The delta functions contract indices that were adjacent, or separated by already contracted indices, so that no minus signs appear.

Ex. 3.3 *As a similar exercise, derive*

$$\gamma^{\mu_1\cdots\mu_4}\gamma_{\nu_1\nu_2} = \gamma^{\mu_1\cdots\mu_4}{}_{\nu_1\nu_2} + 8\gamma^{[\mu_1\cdots\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{\nu_1]} + 12\gamma^{[\mu_1\mu_2}\delta^{\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{\nu_1]} . \tag{3.21}$$

Finally, we consider products with both contracted and uncontracted indices. Consider $\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho\nu_1\nu_2}$. The result should contain terms similar to (3.21), but each term has an extra numerical factor reflecting the number of values that ρ can take in this sum. E.g. in the second term there is one contraction between an upper and lower index, and therefore ρ can run over all D values except the 4 values $\mu_1,\ldots\mu_4$, and the remaining ν that is different. This counting thus gives

$$\begin{aligned}\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho\nu_1\nu_2} &= (D-6)\gamma^{\mu_1\cdots\mu_4}{}_{\nu_1\nu_2} + 8(D-5)\gamma^{[\mu_1\cdots\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{\nu_1]} \\ &\quad +12(D-4)\gamma^{[\mu_1\mu_2}\delta^{\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{\nu_1]} .\end{aligned} \tag{3.22}$$

Ex. 3.4 *Show that*

$$\begin{aligned}\gamma_\nu\gamma^\mu\gamma^\nu &= (2-D)\gamma^\mu ,\\ \gamma_\rho\gamma^{\mu\nu}\gamma^\rho &= (D-4)\gamma^{\mu\nu} .\end{aligned} \tag{3.23}$$

Find a general expression for $\gamma_\rho\gamma^{\mu_1\mu_2\cdots\mu_r}\gamma^\rho$.

3.1.5 Basis of the algebra for even dimension $D = 2m$.

To continue our study we restrict to even dimensional spacetime and construct an orthogonal basis of the Clifford algebra. It will be easy to extend the results to odd D later.

The basis is denoted by the following list $\{\Gamma^A\}$ of matrices chosen from those defined in Sec. 3.1.2:

$$\{\Gamma^A = \mathbb{1}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \cdots, \gamma^{\mu_1,\cdots\mu_D}\}\,. \tag{3.24}$$

Index values satisfy the conditions $\mu_1 < \mu_2 <, \ldots < \mu_r$. There are C_r^D distinct index choices at each rank r and a total of 2^D matrices. To see that this is a basis, it is convenient to define the reverse order list

$$\{\Gamma_A \ = \mathbb{1}, \gamma_\mu, \gamma_{\mu_2\mu_1}, \gamma_{\mu_3\mu_2\mu_1}, \cdots, \gamma_{\mu_D,\cdots\mu_1}\}\,. \tag{3.25}$$

By (3.14) the matrices of this list differ from those of (3.24) by sign factors only.

Ex. 3.5 *Show that $\Gamma^A\Gamma^B = \pm\Gamma^C$, where Γ^C is the basis element whose indices are those of A and B with common indices excluded. Derive the trace orthogonality property*

$$\mathrm{Tr}(\Gamma^A\Gamma_B) = 2^m\,\delta^A_B\,. \tag{3.26}$$

The list (3.24) contains 2^D trace orthogonal matrices in an algebra of total dimension 2^D. Therefore it is a basis of the space of matrices M of dimension $2^m \times 2^m$.

Ex. 3.6 *Show that any matrix M can be expanded in the basis $\{\Gamma^A\}$ as*

$$M \ = \sum_A m_A\Gamma^A\,, \qquad m_A \ = \frac{1}{2^m}\,\mathrm{Tr}(M\Gamma_A)\,. \tag{3.27}$$

Readers may already have noted that the signature of spacetime has played little role in the discussion above. The basic conclusion that there is a unique representation of the Clifford algebra of dimension 2^m is true for pseudo-Euclidean metrics of any signature (p, q). Another general fact is that the second rank Clifford elements $\gamma^{\mu\nu}$ are the generators of a representation of the Lie algebra $\mathfrak{so}(p, q)$, with $p + q = D = 2m$, see (1.34) with the metric signature (p,q). Only the hermiticity properties depend on the signature in an obvious fashion.

Ex. 3.7 *Show that*

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 2^m[\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}]\,. \tag{3.28}$$

Ex. 3.8 *Count the number of elements in the basis 3.24 for odd dimensions, and see that it contains twice the number of independent matrices in spinor space. Check that we already have enough matrices if we consider the matrices up to $\gamma_{\mu_1\ldots\mu_{(D-1)/2}}$. Therefore the results of this section hold only for even dimensions and will have to be modified for odd dimensions, see Sec. 3.1.7.*

3.1.6 The highest rank Clifford algebra element.

For several reasons it is useful to study the highest rank tensor element of the Clifford algebra. It provides the link between even and odd dimensions and it is closely related to the chirality of fermions, an important physical property. We define

$$\gamma_* \equiv (-\mathrm{i})^{m+1}\gamma_0\gamma_1 \dots \gamma_{D-1}\,, \tag{3.29}$$

which satisfies $\gamma_*^2 = \mathbb{1}$ in every even dimension and is hermitian. For space-time dimension $D = 2m$, the matrix γ_* is frequently called γ_{D+1} in the physics literature, as e.g. in 4 dimensions where it is called γ_5.

This matrix occurs as the unique highest rank element in (3.24). For any order of components μ_i, one can write

$$\gamma_{\mu_1\mu_2\cdots\mu_D} = \mathrm{i}^{m+1}\varepsilon_{\mu_1\mu_2\cdots\mu_D}\,\gamma_*\,, \tag{3.30}$$

where the Levi-Civita tensor introduced in Sec. 3.1.3 is used.

Ex. 3.9 *Show that γ_* commutes with all even rank elements of the Clifford algebra and anti-commute with all odd rank. Thus, for example*

$$\begin{aligned} \{\gamma_*, \gamma^\mu\} &= 0\,, & (3.31)\\ [\gamma_*, \gamma^{\mu\nu}] &= 0\,. & (3.32) \end{aligned}$$

Since $\gamma_*^2 = \mathbb{1}$ and $\mathrm{Tr}\,\gamma_* = 0$, it follows that one can choose a representation in which

$$\gamma_* = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{3.33}$$

Some exercises follow, which illustrate the properties of a representation of the full Clifford algebra in which γ_* takes the form in (3.33).

Ex. 3.10 *Assume a general block form,*

$$\gamma^\mu = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.34}$$

for the generating elements in a basis where (3.33) holds. Show that (3.31) implies the block off-diagonal form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \tag{3.35}$$

in which the matrices σ^μ and $\bar\sigma^\mu$ are $2^{m-1} \times 2^{m-1}$ generalizations of the explicit Weyl matrices of (2.2).

Ex. 3.11 *Show that the matrices σ^μ and $\bar\sigma^\mu$ satisfy (2.4) and that $\mathrm{Tr}(\sigma^\mu\bar\sigma_\nu) = 2^{(m-1)}\delta^\mu_\nu$.*

Ex. 3.12 *Show similarly that (3.32) implies that the second rank matrices take the block diagonal form*

$$\Sigma^{\mu\nu} = \frac{1}{2}\gamma^{\mu\nu} = \frac{1}{4}\begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\mu\sigma^\nu \end{pmatrix} . \tag{3.36}$$

This exercise shows explicitly that the Dirac representation of $\mathfrak{so}(D-1,1)$, which is generated by $\Sigma^{\mu\nu}$, is reducible (for even D). The matrices of the upper and lower blocks in (3.36) are generators of two subrepresentations, which are inequivalent and irreducible. Indeed they are related to the two fundamental spinor representations of D_m denoted by Dynkin integers $(0,0,\ldots,1,0)$ and $(0,0,\ldots,0,1)$.

Ex. 3.13 *Show that all requirements are satisfied by generalized Weyl matrices in which the spatial matrices are $\sigma^i = \bar\sigma^i$, where the σ^i are hermitian generators of the Clifford algebra in odd dimension $2m-1$ Euclidean space, and the time matrices are $\sigma^0 = -\bar\sigma^0 = \mathbb{1}$. Thus the form of the Weyl matrices in $D=2m$ dimensions is the same as in $D=4$.*

It is frequently useful to note that the Weyl fields ψ, χ can be obtained from a Dirac Ψ field by applying the chiral projectors

$$P_L = \tfrac{1}{2}(\mathbb{1}+\gamma_*) , \qquad P_R = \tfrac{1}{2}(\mathbb{1}-\gamma_*) . \tag{3.37}$$

Thus

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} \equiv P_L\Psi , \qquad \begin{pmatrix} 0 \\ \bar\chi \end{pmatrix} \equiv P_R\Psi . \tag{3.38}$$

The specific Weyl representation (3.35) will not be used in the rest of this book. However, we will use the projectors P_L, P_R to define the chiral parts of Dirac (and Majorana) spinors in a general representation of the γ-matrices.

Ex. 3.14 *Show that the matrices (3.37) project to orthogonal subspaces, i.e. $P_LP_L = P_L$, $P_RP_R = P_R$ and $P_LP_R = 0$. No specific choice of the Clifford algebra representation is needed.*

3.1.7 Odd spacetime dimension $D = 2m+1$.

The basic idea we need is that the Clifford algebra for dimension $D=2m+1$ can be obtained by reorganizing the matrices in the Clifford algebra for dimension $D=2m$. In particular we can define *two sets* of $2m+1$ generating elements by adjoining the highest rank γ_* as follows

$$\gamma_\pm{}^\mu = (\gamma^0, \gamma^1, \ldots, \gamma^{(2m-1)}, \gamma^{2m} = \pm\gamma_*) . \tag{3.39}$$

This gives us two sets of matrices, which each satisfy (2.18) for dimension $D=2m+1$. The two sets $\{\gamma_\pm^\mu\}$ are not equivalent, but they lead to equivalent representations of the Lorentz group, see Appendix 3.A.3. (In the product construction of Euclidean γ-matrices in Sec. 3.1.1, the top matrix γ^{2m+1} for odd D is also (a phase factor times) the product of all γ^μ, $\mu \leq 2m$.)

The main difference with the case of even dimensions is that the matrices in the list (3.24) are not all independent and are thus an over-complete set. Indeed, the highest element of that list, which is the product of all gamma matrices, is, due to (3.39), a phase factor times the unit matrix. More generally, the rank r and rank $D-r$ sectors are related by the duality relations

$$\gamma_\pm^{\mu_1\ldots\mu_r} = \pm\mathrm{i}^{m+1}\frac{1}{(D-r)!}\varepsilon^{\mu_1\ldots\mu_D}\gamma_{\pm\,\mu_D\ldots\mu_{r+1}}\,. \tag{3.40}$$

Note that the order of the indices in the γ matrix in the right-hand side is inverted. Otherwise there would be different sign factors.

Ex. 3.15 *Prove the relation (3.40) and the analogous but different relation for* even *dimension*

$$\gamma^{\mu_1\mu_2\ldots\mu_r}\gamma_* = -(-\mathrm{i})^{m+1}\frac{1}{(D-r)!}\varepsilon^{\mu_r\mu_{r-1}\ldots\mu_1\nu_1\nu_2\ldots\nu_{D-r}}\gamma_{\nu_1\nu_2\ldots\nu_{D-r}}\,. \tag{3.41}$$

You can use the tricks explained in Sec. 3.1.4. See that this leads in 4 dimensions to

$$\gamma_{\mu\nu\rho} = \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_*\,. \tag{3.42}$$

Thus, a basis of the Clifford algebra in $D=2m+1$ dimensions contains the matrices in (3.24) only up to rank m. This agrees with the counting argument in Ex. 3.8. For example, the set $\{\mathbb{1},\gamma^\mu,\gamma^{\mu\nu}\}$ of $1+5+10=16$ matrices is a basis of the Clifford algebra for $D=5$. Ex. 3.15 shows that it is a rearrangement of the basis $\{\Gamma^A\}$ for $D=4$.

3.1.8 Symmetries of gamma matrices

In the Clifford algebra of the $2^m\times 2^m$ matrices, for both $D=2m$ and $D=2m+1$, one can distinguish between the symmetric and the antisymmetric matrices where the symmetry property is defined in the following way. There exists a unitary matrix, C, called the charge conjugation matrix, such that all the matrices $C\Gamma^A$ are each either symmetric or antisymmetric. Symmetry depends only on the rank r of the matrix Γ^A, so we can write:

$$(C\Gamma^{(r)})^T = -t_r C\Gamma^{(r)}\,,\qquad t_r=\pm 1\,, \tag{3.43}$$

where $\Gamma^{(r)}$ is a matrix in the set (3.24) of rank r. (The $-$ sign in (3.43) is convenient make in later manipulations.) For rank $r=0$ and 1, one obtains from (3.43)

$$C^T = -t_0 C\,,\qquad \gamma^{\mu T} = t_0t_1\,C\gamma^\mu C^{-1}\,. \tag{3.44}$$

These relations suffice to determine the symmetries of all $C\gamma^{\mu_1\ldots\mu_r}$ and thus all coefficients t_r: i.e. $t_2=-t_0$ and $t_3=-t_1$.

Ex. 3.16 *A formal proof of the existence of C can be found in [16, 17], but you can check that the following two matrices work as claimed for even D. They are given in the product representation of (3.2).*[1]

$$\begin{array}{rclr} C_+ &=& \sigma_1\otimes\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\ldots & t_0t_1=1\,,\\ C_- &=& \sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_1\otimes\ldots & t_0t_1=-1\,. \end{array} \qquad (3.45)$$

The possible sign factors depend on the spacetime dimension D modulo 8 and on r modulo 4, and are given in table 3.1. A proof is given in Appendix 3.A.5, and an exercise for the simple case $D=5$ follows below. For even dimension there are two possibilities. One can go from one to the other by replacing the charge conjugation matrix to C by $C\gamma_*$ (up to a normalizing phase factor). For applications in supersymmetry we use the ones indicated in boldface. For odd dimension, C is unique (again up to a phase factor).

Table 3.1. *Symmetries of gamma matrices. The entries contain the numbers r mod 4 for which $t_r=\pm 1$. For even dimensions, in boldface are the choices that are most convenient for supersymmetry.*

D (mod 8)	$t_r=-1$	$t_r=+1$
0	0, 3	2, 1
	0, 1	**2, 3**
1	0, 1	2, 3
2	0, 1	2, 3
	1, 2	**0, 3**
3	1, 2	0, 3
4	**1, 2**	**0, 3**
	2, 3	0, 1
5	2, 3	0, 1
6	2, 3	0, 1
	0, 3	**1, 2**
7	0, 3	1, 2

Ex. 3.17 *Check that in 5 dimensions, where the Clifford algebra basis contains only matrices of rank 0, 1 and 2, the numbers in the table are fixed by counting the number of matrices of each rank, and fitting this by the requirement that there should be 10 symmetric and 6 antisymmetric matrices in a basis of 4×4 matrices.*

Since we use hermitian representations, which satisfy (3.3), the symmetry property of a γ-matrix determines also its complex conjugation property. To see this, we

[1] We consider here only the Minkowski signature of spacetime. A full treatment is in [18], for which you should set $\epsilon=t_0$ and $\eta=-t_0t_1$.

define the unitary matrix

$$B = \mathrm{i}t_0 C\gamma^0 \,. \tag{3.46}$$

Ex. 3.18 *Derive*

$$\gamma^{\mu *} = -t_0 t_1 \, B\gamma^\mu B^{-1} \,. \tag{3.47}$$

Ex. 3.19 *Prove that* $B^* B = -t_1 \mathbb{1}$.

Ex. 3.20 *Show that in the Weyl representation (2.19), one can choose* $B = \gamma^0\gamma^1\gamma^3$, *which is real, symmetric, and satisfies* $B^2 = \mathbb{1}$. *Then* $C = \mathrm{i}\gamma^3\gamma^1$.

3.2 Spinors in general dimensions

In Ch. 2 we used complex spinors. We defined the Dirac adjoint (2.42), which involves the complex conjugate spinor, and used it to obtain a Lorentz invariant bilinear form. In this section we start rather differently. We define the 'conjugate' of any spinor λ using its transpose and the charge conjugation matrix, viz.

$$\bar\lambda \equiv \lambda^T C \,. \tag{3.48}$$

The bilinear form $\bar\lambda\chi$ is Lorentz invariant as readers will show in Ex. 3.22 below. It is appropriate to use (3.48) in supersymmetry and supergravity in which the symmetry properties of γ-matrices and of spinor bilinears are very important and these properties are determined by C. For Majorana spinors, to be defined in Sec. 3.3, the definitions (3.48) and (2.42) are equivalent.

Unless otherwise stated, we assume in this book that spinor components are anticommuting Grassmann numbers. This reflects the important physical relation between spin and statistics.

3.2.1 Spinors and spinor bilinears

Using the definition (3.48) and the property (3.43), we obtain

$$\bar\lambda\gamma_{\mu_1\ldots\mu_r}\chi = t_r \bar\chi\gamma_{\mu_1\ldots\mu_r}\lambda \,. \tag{3.49}$$

The minus sign obtained by changing the order of Grassmann valued spinor components has been incorporated. The symmetry property (3.49) is valid for Dirac spinors, but its main application for us will be to Majorana spinors. For this reason we use the term 'Majorana flip relations' to refer to (3.49).

We now give some further relations that are useful for spinor manipulations. In fact, the same sign factors can be used for a longer chain of Clifford matrices:

$$\bar\lambda\Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\chi = t_0^{p-1} t_{r_1} t_{r_2}\cdots t_{r_p}\, \bar\chi\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)}\lambda \,, \tag{3.50}$$

where $\Gamma^{(r)}$ stands for any rank r matrix $\gamma_{\mu_1\ldots\mu_r}$. Note that the prefactor t_0^{p-1} is not relevant in 4 dimensions, where $t_0 = 1$.

Ex. 3.21 *One often encounters the special case that the bilinear contains the product of individual γ^μ-matrices. Prove that for the Majorana dimensions $D = 2, 3, 4$ mod 8,*

$$\bar\lambda\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_p}\chi = (-)^p\bar\chi\gamma^{\mu_p}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\lambda . \tag{3.51}$$

The previous relations imply also the following rule. For any relation between spinors that includes gamma matrices, there is a corresponding relation between the barred spinors:

$$\chi_{\mu_1\ldots\mu_r} = \gamma_{\mu_1\ldots\mu_r}\lambda \implies \bar\chi_{\mu_1\ldots\mu_r} = t_0 t_r\bar\lambda\gamma_{\mu_1\ldots\mu_r}\,, \tag{3.52}$$

and similar for longer chains:

$$\chi = \Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\lambda \implies \bar\chi = t_0^p t_{r_1}t_{r_2}\cdots t_{r_p}\,\bar\lambda\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)} . \tag{3.53}$$

In even dimensions we define left-handed and right-handed parts of spinors using the projection matrices (3.37). The definition (3.48) implies that the chirality of the conjugate spinor is dependent on $t_0 t_D$, and we obtain[2]

$$\chi = P_L\lambda \;\rightarrow\; \bar\chi = \begin{cases} \bar\lambda P_L & \text{for } D = 0, 4, 8, \ldots, \\ \bar\lambda P_R & \text{for } D = 2, 6, 10, \ldots . \end{cases} \tag{3.54}$$

Ex. 3.22 *Using the 'spin part' of the infinitesimal Lorentz transformation (2.37),*

$$\delta\chi = -\tfrac14\lambda^{\mu\nu}\gamma_{\mu\nu}\chi\,, \tag{3.55}$$

prove that the spinor bilinear $\bar\lambda\chi$ is a Lorentz scalar.

3.2.2 Spinor indices

For most of this book we do not need spinor indices because they appear contracted within Lorentz covariant expressions. However, in some cases indices are necessary, for example, to write (anti)commutation relations of supersymmetry generators. The components of the basic spinor λ are indicated as λ_α. The components of the barred spinor defined in (3.48) are indicated with upper indices: λ^α. Sometimes we write $\bar\lambda^\alpha$ to stress that these are the components of the barred spinor, but in fact the bar can be omitted. We introduce the raising matrix $\mathcal{C}^{\alpha\beta}$ such that

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta . \tag{3.56}$$

Comparing with (3.48) we see that $\mathcal{C}^{\alpha\beta}$ are the components of the matrix C^T. Note that the summation index β in (3.56) appears in a NW-SE line in the equation when the quantities are written in the order such that contracted indices appear

[2] The definition (2.42) would always lead to $\bar\chi = \bar\lambda P_R$.

next to each other. Therefore, this convention is frequently called the NW-SE spinor convention. This is relevant when the raising matrix is antisymmetric ($t_0 = 1$ in the terminology of table 3.1). Most applications in the book are for dimensions in which this is the case, e.g. $D = 2, 3, 4, 5, 10, 11$.

We also introduce a lowering matrix such that (again NW-SE contraction)

$$\lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha} \,. \tag{3.57}$$

In order for these two equations to be consistent, we must require

$$\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\beta} = \delta_\gamma{}^\alpha \,, \qquad \mathcal{C}_{\beta\alpha}\mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma \,. \tag{3.58}$$

Hence $\mathcal{C}_{\alpha\beta}$ are the components of C^{-1}.

If we translate a covariant expression to spinor matrix notation, the gamma matrices have components $(\gamma_\mu)_\alpha{}^\beta$. E.g. for the simplest case:

$$\bar{\chi}\gamma_\mu\lambda = \chi^\alpha(\gamma_\mu)_\alpha{}^\beta\lambda_\beta \,, \tag{3.59}$$

where again all contractions are NW-SE.

One can now raise or lower indices consistently. E.g. one can define

$$(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha{}^\gamma\mathcal{C}_{\gamma\beta} \,. \tag{3.60}$$

These γ-matrices with indices at the 'same level' have a definite symmetry or antisymmetry property, which follows from (3.43):

$$(\gamma_{\mu_1\ldots\mu_r})_{\alpha\beta} = -t_r(\gamma_{\mu_1\ldots\mu_r})_{\beta\alpha} \,. \tag{3.61}$$

An interesting property is that

$$\lambda^\alpha\chi_\alpha = -t_0\lambda_\alpha\chi^\alpha \,, \tag{3.62}$$

Thus, in 4 dimensions, raising and lowering a contracted index produces a minus sign. The same property can be used when the contracted indices involved are on γ matrices, e.g. $\gamma_{\mu\alpha}{}^\beta\gamma_{\nu\beta}{}^\gamma = -t_0\gamma_{\mu\alpha\beta}\gamma_\nu{}^{\beta\gamma}$.

Ex. 3.23 *Using this property and (3.61) prove the relation (3.50). Do not forget the sign due to interchange of two (anticommuting) spinors.*

Ex. 3.24 *Show that using the index raising and lowering conventions, $\mathcal{C}_\alpha{}^\beta = \delta_\alpha^\beta$, and for $D = 4$ that $\mathcal{C}^\alpha{}_\beta = -\delta_\beta^\alpha$.*

3.2.3 Fierz rearrangement

In this subsection we study an important consequence of the completeness of the Clifford algebra basis $\{\Gamma^A\}$ in (3.24). As we saw in Ex. 3.6 completeness means that any matrix M has a unique expansion in the basis with coefficients obtained using trace orthogonality. The expansion was derived for even $D = 2m$ in Ex. 3.6, but it is also valid for odd $D = 2m + 1$ provided that the sum is restricted to rank $r \leq m$. We saw at the end of Sec. 3.1.7 that the list of (3.24) is complete for odd D when so restricted. The rearrangement properties we derive using completeness are frequently need in supergravity. These involve changing the pairing of spinors in products of spinor bilinears, which is called a 'Fierz rearrangement'.

Let's proceed to derive the basic Fierz identity. Using spinor indices, we can regard the quantity $\delta_\alpha{}^\beta \delta_\gamma{}^\delta$ as a matrix in the indices $\gamma\,\beta$ with the indices $\alpha\,\delta$ as inert 'spectators'. We thus use (3.27) with M a matrix in indices $\gamma\,\beta$, that is different for any value of $\alpha\,\delta$. Hence the coefficients m_A depend on the latter, and are $(m_A)_\alpha^\delta = 2^{-m}\delta_\alpha{}^\beta\delta_\gamma{}^\delta(\Gamma_A)_\beta{}^\gamma = 2^{-m}(\Gamma_A)_\alpha{}^\delta$. Therefore, we obtain the basic rearrangement lemma

$$\delta_\alpha{}^\beta \delta_\gamma{}^\delta = \frac{1}{2^m}\sum_A (\Gamma_A)_\alpha{}^\delta (\Gamma^A)_\gamma{}^\beta \,. \tag{3.63}$$

Note that the 'column indices' on the left and right sides have been exchanged.

Ex. 3.25 *Derive the following result including the explicit values of the expansion coefficients v_A:*

$$(\gamma^\mu)_\alpha{}^\beta (\gamma_\mu)_\gamma{}^\delta = \frac{1}{2^m}\sum_A v_A (\Gamma_A)_\alpha{}^\delta (\Gamma^A)_\gamma{}^\beta \,. \tag{3.64}$$

Ex. 3.26 *Lower the β and δ index in the result of the previous exercise and consider the completely symmetric part in $(\beta\gamma\delta)$. The left-hand side is only non-vanishing for dimensions where $t_1 = -1$. Consider now the right-hand side and use Table 3.1 and the result for v_A to prove that for $D = 3$ and $D = 4$ the non-vanishing terms in the left-hand side are only those for $r_A = 1$. See that for $D = 4$ you have to use the boldface row in the Table to arrive at that result. You can also check that there are no other dimensions where this occurs. You can use this to prove that for these dimensions*

$$(\gamma_\mu)_{\alpha(\beta} (\gamma^\mu)_{\gamma\delta)} = 0 \,. \tag{3.65}$$

This is called the cyclic identity and is important in the context of string and brane actions. It can be extended to some other dimensions under further restrictions. As we will see below, for $D = 2$ and $D = 10$ chiral spinors can be introduced, and due to (3.54) this will imply that only odd rank can occur in the sum over A. See that this is sufficient to extend the result (3.65) to these cases. With the same restrictions of chirality (i.e. only odd r_A in the right-hand side of (3.64), check that for $D = 6$ there is an analogous identity for the completely antisymmetric part in $[\beta\gamma\delta]$. Other similar identities exist, which are crucial for the existence of brane actions, and can be obtained from the basic identity (3.63).

The following application of Fierz rearrangement is valid for any anti-commuting spinor fields. Given any set of 4 spinor fields, the basic Fierz identity (3.63) immediately gives

$$\bar\lambda_1\lambda_2\bar\lambda_3\lambda_4 = -\frac{1}{2^m}\sum_A \bar\lambda_1\Gamma^A\lambda_4\bar\lambda_3\Gamma_A\lambda_2\,. \qquad (3.66)$$

This can be generalized to include general matrices M, M' of the Clifford algebra.

Ex. 3.27 *Show that*

$$\begin{aligned}\bar\lambda_1 M\lambda_2\bar\lambda_3 M'\lambda_4 &= -\frac{1}{2^m}\sum_A \bar\lambda_1 M\Gamma_A M'\lambda_4\bar\lambda_3\Gamma^A\lambda_2\\ &= -\frac{1}{2^m}\sum_A \bar\lambda_1\Gamma_A M'\lambda_4\bar\lambda_3\Gamma^A M\lambda_2\,. \qquad (3.67)\end{aligned}$$

When $\lambda_{1,2,3,4}$ are not all independent, it is frequently the case that some terms in the rearranged sum vanish due to symmetry relations such as (3.49).

One can write the Fierz relation (3.63) in the alternate form:

$$M = \sum_{k=0}^{[D]} \tfrac{1}{k!}\Gamma_{\mu_1\ldots\mu_k}\ \mathrm{Tr}\,(\Gamma^{\mu_k\cdots\mu_1}M) \quad \text{where} \quad \left\{\begin{array}{ll}[D]=D & \text{for even } D\,,\\ [D]=(D-1)/2 & \text{for odd } D\,.\end{array}\right. \qquad (3.68)$$

The factor $1/k!$ compensates for the fact that in the sum over $\mu_1\ldots\mu_k$ each matrix of the basis appears $k!$ times.

Ex. 3.28 *Prove the following chiral Fierz identities for $D=4$:*

$$\begin{aligned}P_L\chi\bar\lambda P_L &= -\tfrac12 P_L\left(\bar\lambda P_L\chi\right) + \tfrac18 P_L\gamma^{\mu\nu}\left(\bar\lambda\gamma_{\mu\nu}P_L\chi\right)\,,\\ P_L\chi\bar\lambda P_R &= -\tfrac12 P_L\gamma^\mu\left(\bar\lambda\gamma_\mu P_L\chi\right)\,. \qquad (3.69)\end{aligned}$$

You will need (3.41) to combine terms in (3.68).

Ex. 3.29 *Prove that for $D=5$ the matrix $\chi\bar\lambda-\lambda\bar\chi$ can be written as*

$$\chi\bar\lambda-\lambda\bar\chi = \gamma_{\mu\nu}(\bar\lambda\gamma^{\mu\nu}\chi)\,. \qquad (3.70)$$

Readers who understand the Majorana flip properties and Fierz rearrangement are well equipped for supersymmetry and supergravity!

3.2.4 Reality

So far in this section, we did not discuss the complex conjugation of any of the spinor fields that we worked with. Yet complex conjugation is necessary for some purposes, such as the verification that a term in the Lagrangian involving spinor bilinears is hermitian. At first glance the complex conjugation of a bilinear[3] seems to be a

[3] We use the convention that we interchange fermion fields in the process of complex conjugation. See (A.16) with $\beta=-1$.

complicated operation since the hermiticity of both the charge conjugation matrix in (3.48) and γ-matrices are involved. However, in practice there is an easier method in which complex conjugation is replaced by charge conjugation. For a scalar, charge conjugation and complex conjugation are the same. Since the Lagrangian is a scalar, charge conjugation can be used in intermediate manipulations.

First we define the charge conjugate of any spinor as

$$\lambda^C \equiv B^{-1}\lambda^* \,. \tag{3.71}$$

The barred charge conjugate spinor is then

$$\overline{\lambda^C} = (-t_0 t_1)\mathrm{i}\lambda^\dagger\gamma^0 \,. \tag{3.72}$$

Note that this is the Dirac conjugate that was defined in (2.42) except for the numerical factor $(-t_0t_1)$. The meaning of this will become clear below when we discuss Majorana spinors. The number $(-t_0t_1)$ is e.g. $+1$ in 2,3,4, 10 or 11 dimensions.[4]

The charge conjugate of any $2^m \times 2^m$ matrix M, is defined as

$$M^C \equiv B^{-1}M^*B \,. \tag{3.73}$$

Charge conjugation does not change the order of matrices: $(MN)^C = M^C N^C$. Therefore, in practice we need only the charge conjugation property of the generating γ-matrices, which is

$$(\gamma_\mu)^C \equiv B^{-1}\gamma_\mu^* B \;=\; (-t_0t_1)\gamma_\mu \,. \tag{3.74}$$

Ex. 3.30 *Start from (3.74) and note that charge conjugation on any number is just complex conjugation. Prove that*

$$(\gamma_*)^C = (-)^{D/2+1}\gamma_* \,. \tag{3.75}$$

With these ingredients, we can give the rule for complex conjugation of a spinor bilinear involving an arbitrary matrix M:

$$(\bar\chi M\lambda)^* \equiv (\bar\chi M\lambda)^C = (-t_0t_1)\overline{\chi^C}M^C\lambda^C \,. \tag{3.76}$$

Note that we have defined complex conjugation to include interchange of order of the fermions. However, the corresponding signs are already taken into account in (3.76) so that for charge conjugation we do not have to change any order.

Ex. 3.31 *It is important that any spinor λ and its conjugate λ^C transform in the same way under a Lorentz transformation. Prove this using (3.55) and the rules above.*

Ex. 3.32 *Show that for any spinor $(\lambda^C)^C = -t_1\lambda$, and for any matrix $(M^C)^C = M$.*

[4] In these cases the spinor bilinears of Chapters 2 and 3 are related by $(\bar\lambda\chi)_{\rm Ch.2} = (\overline{\lambda^C}\chi)_{\rm Ch.3}$.

3.3 Majorana spinors

The concept of supersymmetry is closely tied to the relativistic treatment of particle spin. Indeed the transformation parameters are spinors ϵ_α. It is reasonable to suppose that the simplest supersymmetric field theories in each spacetime dimension D are based on the simplest spinors that are compatible with invariance under the Lorentz group $\mathrm{SO}(D-1,1)$. In even dimension $D = 2m$ we already know that Weyl fields, rather than Dirac fields, transform irreducibly under Lorentz transformations. Weyl fields were first discussed in Sec. 2.6. They have 2^{m-1} complex components while a Dirac field has 2^m complex components. Weyl fields can be obtained by applying the chiral projector P_L or P_R to a Dirac field.

In this section we introduce Majorana fields, which are Dirac fields that satisfy an additional 'reality condition', which reduces the number of independent components by factor of two. Thus, like Weyl fields, a Majorana spinor field has half the degrees of freedom and can be viewed as more fundamental than a complex Dirac field. Physically the properties of particles described by a Majorana field are similar to Dirac particles, *except that particles and anti-particles are identical.* The spin states of massive and massless Majorana spinors transform in representations of $\mathrm{SO}(D-1)$ and $\mathrm{SO}(D-2)$, respectively.

The spirit of the discussion which now follows is to present the bare, useful facts about Majorana spinors with detailed proofs deferred to Appendix 3.A.4.

3.3.1 Definition and properties

In view of the results in Sec. 3.2.4, the reader might expect that the reality condition that defines a Majorana field should be

$$\psi = \psi^C = B^{-1}\psi^* \,, \qquad \text{i.e.} \qquad \psi^* = B\psi \,. \tag{3.77}$$

In Ex. 3.31 readers showed that both sides transform in the same way under Lorentz transformations in any dimension D, so the constraint is compatible with Lorentz symmetry. However there is an important consistency condition which we now derive, which restricts the spacetime dimension in which Majorana spinors can exist. It is easy to see that the reality condition (3.77) is not automatically consistent. Take the complex conjugate of the second form of the condition and use it again to obtain $\psi = B^*B\,\psi$. Thus the reality condition is mathematically consistent only if $B^*B = \mathbb{1}$. Using Ex. 3.19, we see that this requires[5] $t_1 = -1$.

We can still distinguish the two cases $t_0 = \pm 1$. Consulting Table 3.1, we see that $t_0 = +1$ holds for spacetime dimension $D = 2, 3, 4$, mod 8. In this case we call the spinors which satisfy (3.77) *Majorana spinors*. It is clear from (3.72) that if $t_0 = 1$ and $t_1 = -1$, the barred (3.48) and Dirac adjoint spinors (2.42) agree for Majorana spinors. In fact, this gives an alternate definition of a Majorana spinor.

[5] This manipulation is the same as working out the exercise 3.32, and this thus leads to the same result.

Another fact about the Majorana case is that there are representations of the γ-matrices that are explicitly real and may be called really real representations. Here is a really real representation for $D = 4$:

$$\begin{aligned}
\gamma_0 &= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \mathrm{i}\sigma_2 \otimes \mathbb{1}\,, \\
\gamma_1 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \sigma_3 \otimes \mathbb{1}\,, \\
\gamma_2 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_1\,, \\
\gamma_3 &= \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3\,.
\end{aligned} \tag{3.78}$$

Note that the γ_i are symmetric, while γ_0 is antisymmetric. This is required by hermiticity in any real representation. We construct really real representations in all allowed dimensions $D = 2$, 3, 4 mod 8 in Appendix 3.A.6.

In such representations (3.47) implies that $B = \mathbb{1}$ (up to a phase). The relation (3.46) then gives $C = \mathrm{i}\gamma^0$. Further, a Majorana spinor field is really real since (3.77) reduces to $\Psi^* = \Psi$.

Really real representations are sometimes convenient, but we emphasize that the physics of Majorana spinors is the same in, and can be explored in any representation of the Clifford algebra, replacing complex conjugation with charge conjugation. Therefore, we will for convenience often write 'complex conjugation' when in fact it is 'charge conjugation'. E.g. the complex conjugate of $\bar\chi\gamma_{\mu_1\ldots\mu_r}\psi$, where χ and ψ are Majorana, is computed as follows. We write according to Sec. 3.2.4:

$$(\bar\chi\gamma_{\mu_1\ldots\mu_r}\psi)^* = (\bar\chi\gamma_{\mu_1\ldots\mu_r}\psi)^C = \bar\chi(\gamma_{\mu_1\ldots\mu_r})^C\psi = \bar\chi\gamma_{\mu_1\ldots\mu_r}\psi\,. \tag{3.79}$$

In the second equation, we considered the Majorana spinors as real, and in the last step we also treat γ-matrices as real according to (3.74). Hence, bilinears such as $\bar\chi\psi$ and $\bar\chi\gamma_{\mu_1\ldots\mu_r}\psi$ are real.

In the case $t_0 = -1$ (and still $t_1 = -1$) such spinors are denoted as *pseudo-Majorana* spinors, mostly relevant for $D = 8$ or 9. There are no really real representations in these dimensions; instead there are representations of the Clifford algebra in which the generating γ-matrices are imaginary, $(\gamma^\mu)^* = -\gamma^\mu$. In any representation (3.76) and (3.74) hold with $t_0 = t_1 = -1$. This implies that the reality properties of bilinears are different from those of Majorana spinors. However, taking these signs into account, the essential property that a complex spinor can be reduced to a real spinor still holds, and therefore we will not always distinguish between Majorana and pseudo-Majorana spinors.

We now consider (pseudo-)Majorana spinors in even dimensions $D = 0, 2, 4$ mod 8. We can quickly show using (3.75) that these cases are somewhat different. For $D = 2$ mod 8 we have $(\gamma_*\psi)^C = \gamma_*\psi^C$. Thus the two constraints

$$\text{Majorana:}\qquad \psi^C = \psi\,,\qquad \text{Weyl:}\qquad P_{L,R}\psi = \psi\,, \tag{3.80}$$

are compatible. It is equivalent to observe that the chiral projections of a Majorana spinor ψ satisfy

$$(P_L\psi)^C = P_L\psi\,, \qquad (P_R\psi)^C = P_R\psi\,. \tag{3.81}$$

Thus the chiral projections of a Majorana spinor are also Majorana spinors. A spinor of either form satisfies both constraints in (3.80) and is called a *Majorana-Weyl spinor* . They have 2^{m-1} independent 'real' components in dimension $D = 2m = 2$ mod 8 and are the 'most fundamental' spinors available in these dimensions. It is not surprising that supergravity and superstring theories in $D = 10$ dimensions are based on Majorana-Weyl spinors.

For $D = 4$ mod 4 dimensions we have $(\gamma_*\psi)^C = -\gamma_*\psi^C$, so that the equations of (3.81) are replaced by

$$(P_L\psi)^C = P_R\psi\,, \qquad (P_R\psi)^C = P_L\psi\,. \tag{3.82}$$

These equations states that the 'left' and 'right' components of a Majorana spinor are related by charge conjugation. One important consequence is that when one has an expression for a left-handed spinor $P_L\psi$, where ψ is a Majorana spinor, then the expression for $P_R\psi$ follows by complex conjugation. There exist chiral spinors built from a Dirac spinor ψ, i.e. such that $P_L\psi = 0$ and $P_R\psi \neq 0$. Such spinors are called *Weyl spinors*. However, this is not consistent when ψ is Majorana, as complex conjugation of the first equation, then implies $P_R\psi = 0$, and hence $\psi = (P_L + P_R)\psi = 0$.

3.3.2 Symplectic Majorana spinors

When $t_1 = 1$ we cannot define Majorana spinors, but we can define 'symplectic Majorana spinors'. These consist of an even number of spinors χ^i, with $i = 1, \ldots, 2k$, which satisfy a 'reality condition' containing the matrix B and a non-singular anti-symmetric matrix ε^{ij}. The inverse matrix ε_{ij} satisfies $\varepsilon^{ij}\varepsilon_{kj} = \delta^i_k$. Symplectic Majorana spinors satisfy the condition

$$\chi^i = \varepsilon^{ij}B^{-1}(\chi^j)^*\,. \tag{3.83}$$

The consistency check discussed after (3.77) now works for $t_1 = 1$ because of the anti-symmetric ε^{ij}.

Ex. 3.33 *Check that in 5 dimensions with symplectic Majorana spinors, $\bar\psi^i\chi_i \equiv \bar\psi^i\chi^j\varepsilon_{ji}$ is pure imaginary while $\bar\psi^i\gamma_\mu\chi_i$ is real.*

For dimensions $D = 6$ mod 8, one can use (3.75) to show that the symplectic Majorana constraint is compatible with chirality. We can therefore define the *symplectic Majorana-Weyl spinors* $P_L\chi^i$ or $P_R\chi^i$.

3.3.3 Dimensions of minimal spinors

The various types of spinors we have discussed are linked to the signs of t_0 and t_1 as follows:

$$\begin{array}{lll} t_1=-1\,, & t_0=1\ : & \text{Majorana} \\ & t_0=-1\ : & \text{pseudo-Majorana} \\ t_1=1 & & \text{symplectic Majorana}\,. \end{array} \tag{3.84}$$

The difference between pseudo-Majorana and Majorana is not relevant further, and we will not distinguish them. In any even dimension one can define Weyl spinors, while in dimensions $D = 2 \bmod 4$, one can combine the (symplectic) Majorana condition and Weyl conditions. This leads to the overview in table 3.2. For each

Table 3.2. *Irreducible spinors, number of components and symmetry properties.*

Dim	Spinor	min # components	antisymmetric
2	MW	1	1
3	M	2	1,2
4	M	4	1,2
5	S	8	2,3
6	SW	8	3
7	S	16	0,3
8	M	16	0,1
9	M	16	0,1
10	MW	16	1
11	M	32	1,2

spacetime dimension it is indicated whether Majorana (M), Majorana-Weyl (MW), symplectic (S) or symplectic-Weyl (SW) spinors can be defined as the 'minimal spinor'. The number of components of this minimal spinor is given. The table is for Minkowski signature and has a periodicity of 8 in dimension. When D is changed to $D + 8$, the number of spinor components is multiplied by 16. The final column indicates the ranks of the anti-symmetric spinor bilinears, e.g. a 0 indicates that $\bar\epsilon_2\epsilon_1 = -\bar\epsilon_1\epsilon_2$, and a 2 indicates that $\bar\epsilon_2\gamma_{\mu\nu}\epsilon_1 = -\bar\epsilon_1\gamma_{\mu\nu}\epsilon_2$. This entry is modulo 4, i.e. if rank 0 is anti-symmetric, then so are rank 4 and 8 bilinears. For the even dimensions with Weyl-like spinors ($D = 2 \bmod 4$), one can only define a symmetry if both spinors have the same chirality. The property (3.54) implies that the bilinears should then contain an odd number of gamma matrices. In the other even dimensions, $D = 4 \bmod 4$, there are always two possibilities for reality conditions and we give here the one that includes the '1' in the column 'antisymmetric' since we will need this property for the supersymmetry algebra.

3.4 Majorana spinors in physical theories

3.4.1 Variation of a Majorana Lagrangian

In this section we consider a prototype action for a Majorana spinor field in dimension $D = 2,\ 3,\ 4 \bmod 8$. Majorana and Dirac fields transform the same way under Lorentz transformations, but Majorana spinors have half as many degrees of freedom, so we write

$$S[\Psi] = -\tfrac{1}{2}\int \mathrm{d}^D x\, \bar{\Psi}[\gamma^\mu\partial_\mu - m]\Psi(x)\,. \tag{3.85}$$

There is an immediate and curious subtlety due to the symmetries of the matrices C and $C\gamma^\mu$. Using (3.48), we see that the mass and kinetic terms are proportional to $\Psi^T C\Psi$ and $\Psi^T C\gamma^\mu\partial_\mu\Psi$. Suppose that the field components Ψ are conventional commuting numbers. Since C is anti-symmetric, the mass term vanishes. Since $C\gamma^\mu$ is symmetric, the kinetic term is a total derivative and thus vanishes when integrated in the action. For commuting field components, there is no dynamics! To restore the dynamics we must assume that Majorana fields are anti-commuting Grassmann variables, which we have always assumed in Sec. 3.2.

Let's derive the Euler-Lagrange equation for Ψ. Field variations must satisfy the Majorana conditions (3.77), which shows that $\delta\Psi$ and $\delta\bar{\Psi}$ are related following Sec. 3.2.1. So $\delta S[\Psi]$ contains two terms. However, after a Majorana flip and partial integration, one can see that the two terms are equal, so that $S[\Psi]$ can be written as the single term

$$\delta S[\Psi] = -\int \mathrm{d}^D x\, \delta\bar{\Psi}[\gamma^\mu\partial_\mu - m]\Psi(x)\,. \tag{3.86}$$

Thus a Majorana field satisfies the conventional Dirac equation.

This fact is no surprise, but it is an example of a more general and simplifying rule for the variation of Majorana spinor actions. If integration by parts is valid, it is sufficient to vary $\bar{\Psi}$ and multiply by 2 to account for the variation of Ψ.

Ex. 3.34 *Derive (3.86) in full detail.*

Ex. 3.35 *A Majorana field is simply a Dirac field subject to the reality condition (3.77). Let's impose that constraint on the plane wave expansion (2.24) for $D = 4$ using the relation $v = u^C = Bu^*$, which holds for the u and v spinors defined in (2.27) and (2.28). In this way one derives $d(\vec{p}, s) = c(\vec{p}, s)$ which proves that a Majorana particle is its own anti-particle. Readers should derive this fact!*

Ex. 3.36 *Show that*

$$v(\vec{p}, s) = u(\vec{p}, s)^C\,. \tag{3.87}$$

holds for the u and v spinors defined for the Weyl representation in Sec. 2.3. This was the motivation for the choice (2.31).

3.4.2 Relation of Majorana and Weyl spinor theories

In even dimensions $D = 0, 2, 4 \bmod 8$, both Majorana and Weyl fields exist and both have legitimate claims to be more fundamental than a Dirac fermion. In fact both fields describe equivalent physics. It is simplest to see this for $D = 4$. We can rewrite the action (3.85) as

$$\begin{aligned} S[\psi] &= -\tfrac{1}{2}\int \mathrm{d}^4x\, \left[\bar{\Psi}\gamma^\mu\partial_\mu - m\right](P_L + P_R)\Psi \\ &= -\int \mathrm{d}^4x\, \left[\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi - \tfrac{1}{2}m\bar{\Psi}P_L\Psi - \tfrac{1}{2}m\bar{\Psi}P_R\Psi\right] . \end{aligned} \tag{3.88}$$

We obtained the second line by a Majorana flip and partial integration. In the second form of the action, the Majorana field is replaced by its chiral projections. In our treatment of chiral multiplets in supersymmetry, we will exercise the option to write Majorana fermion actions in this way.

Ex. 3.37 *Show that the Euler Lagrange equations that follow from the variation of the second form of the action in (3.88) are*

$$\partial\!\!\!/ P_L\Psi = mP_R\Psi\,, \qquad \partial\!\!\!/ P_R\Psi = mP_L\Psi\,. \tag{3.89}$$

Derive $\Box P_{L,R}\Psi = m^2 P_{L,R}\Psi$ *from the equations above.*

Let's return to the Weyl representation (2.19) for the final step in the argument to show that the equation of motion for a Majorana field can be reexpressed in terms of a Weyl field and its adjoint. The Majorana condition $\Psi = B^{-1}\Psi^* = \gamma^0\gamma^1\gamma^3\Psi^*$ requires that Ψ take the form

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_2^* \\ -\psi_1^* \end{pmatrix} . \tag{3.90}$$

With (3.90) and (2.55) in view we define the 2-component Weyl fields

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \qquad \tilde{\psi} = \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix} . \tag{3.91}$$

Using the form of γ^μ (2.19) and γ_* (3.33) in the Weyl representation, we see that we can identify

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} = P_L\Psi\,, \qquad \begin{pmatrix} 0 \\ \tilde{\psi} \end{pmatrix} = (P_L\Psi)^C = P_R\Psi\,. \tag{3.92}$$

The equations of motion (3.89) can then be rewritten as

$$\bar{\sigma}^\mu\partial_\mu\psi = m\tilde{\psi}\,, \qquad \sigma^\mu\partial_\mu\tilde{\psi} = m\psi\,. \tag{3.93}$$

These are equivalent to the pair of Weyl equations in (2.56) with the restriction $\tilde{\psi} = \bar{\chi}$ which comes because we started in this section with a Majorana rather than a Dirac field.

Let's briefly discuss the relation between interacting Majorana and Weyl fields (independent of the Clifford algebra representation). Recall that interacting fields have nonlinear equations of motion. Any field theory of a Majorana spinor field Ψ can be rewritten in terms of a Weyl field $P_L\Psi$ and its complex conjugate. Conversely, any theory involving the chiral field $\chi = P_L\chi$ and its conjugate $\chi^C = P_R\chi^C$ can be rephrased as a Majorana equation if one defines the Majorana field $\Psi = P_L\chi + P_R\chi^C$. Supersymmetry theories in $D = 4$ are formulated in both descriptions in the physics literature.

3.4.3 U(1) *symmetries of a Majorana field*

In Sec. 2.7.1 we considered the U(1) symmetry operation $\Psi \to \Psi' = \mathrm{e}^{\mathrm{i}\theta}\Psi$. This symmetry is obviously *incompatible* with the Majorana condition (3.77). Thus the simplest internal symmetry of a Dirac fermion cannot be defined in a field theory of a (single) Majorana field. However, it is easy to see that $(\mathrm{i}\gamma_*)^C = \mathrm{i}\gamma_*$, so the chiral transformation $\Psi \to \Psi' = \mathrm{e}^{\mathrm{i}\gamma_*\theta}\Psi$ preserves the Majorana condition. Let's ask whether the infinitesimal limit of this transformation is a symmetry of the free massive Majorana action (3.85).

Ex. 3.38 *Use $\delta\bar{\Psi} = \mathrm{i}\theta\bar{\Psi}\gamma_*$ and partial integration to derive the variation*

$$\delta S[\Psi] = \mathrm{i}\theta m \int \mathrm{d}^4x\, \bar{\Psi}\gamma_*\Psi\,, \tag{3.94}$$

which vanishes only for a massless *Majorana field.*

Thus we have learned that

- The conventional vector U(1) symmetry is incompatible with the Majorana condition.
- The axial transformation above is compatible and is a symmetry of the action for a massless Majorana field only.

Ex. 3.39 *Show that the axial current*

$$J_*^\mu = \tfrac{1}{2}\mathrm{i}\bar{\Psi}\gamma^\mu\gamma_*\Psi \tag{3.95}$$

is the Noether current for the chiral symmetry defined above. Use the equations of motion to show that

$$\partial_\mu J_*^\mu = -\mathrm{i}m\,\bar{\Psi}\gamma_*\Psi\,. \tag{3.96}$$

The current is conserved only for massless Majorana fermions.

The dynamics of a Majorana field Ψ can be expressed in terms of its chiral projections $P_{L,R}\Psi$. So can the chiral transformation, which becomes $P_{L,R}\Psi \to P_{L,R}\Psi' = \mathrm{e}^{\pm\mathrm{i}\theta}\Psi$.

Throughout this section we used the simple dynamics of a free massive fermion to illustrate the relation between Majorana and Weyl fields and to explore their U(1) symmetries. It is straightforward to extend these ideas to interacting field theories with non-linear equations of motion.

3.A Appendix: details on the Clifford algebras for $D = 2m$.

3.A.1 Traces and the basis of the Clifford algebra

Let us start with the following facts established in Sec. 3.1. The Clifford algebra in even dimension $D = 2m$ has a basis of 2^m linearly independent, trace orthogonal matrices, given in (3.24). Any representation by matrices of dimension 2^m is irreducible.

The trace properties of the matrices are important for these proofs. The matrices Γ^A for tensor rank $1 \le r \le D-1$ are traceless. One simple way to see this is to use the Lorentz transformations (2.22) and its extension to general rank

$$L(\lambda)\gamma^{\mu_1\mu_2\ldots\mu_r}L(\lambda)^{-1} = \gamma^{\nu_1\nu_2\ldots\nu_r}\Lambda_{\nu_1}{}^{\mu_1}\ldots\Lambda_{\nu_r}{}^{\mu_r}\,. \tag{3.97}$$

Traces then satisfy the Lorentz transformation law as suggested by their free indices:

$$\operatorname{Tr}\gamma^{\mu_1\mu_2\ldots\mu_r} = \operatorname{Tr}\gamma^{\nu_1\nu_2\ldots\nu_r}\Lambda_{\nu_1}{}^{\mu_1}\ldots\Lambda_{\nu_r}{}^{\mu_r}\,. \tag{3.98}$$

This means that the traces must be *totally anti-symmetric Lorentz invariant* tensors. However the only invariant tensors available are the Minkowski metric $\eta^{\mu\nu}$ and the Levi-Civita tensor $\varepsilon^{\mu_1\mu_2\ldots\mu_D}$ introduced in Sec. 3.1.3. No totally anti-symmetric tensor can be formed from products of $\eta^{\mu\nu}$. This proves that $\operatorname{Tr}\Gamma^A = 0$ for all elements of rank $1 \le r \le D-1$.

The argument does not apply to the highest rank element. However, one can see from the pattern of alternation in (3.6) that this is given by a commutator for even $D = 2m$ and and anti-commutator for odd $D = 2m+1$. Thus the trace of the highest rank element vanishes for $D = 2m$ but need not (and does not) vanish for $D = 2m+1$. This is actually a fundamental distinction between the Clifford algebras for even and odd dimensions. It might have been expected since the second rank elements (See Ex. 2.8) give a representation of the Lorentz algebras $\mathfrak{so}(D-1,1)$ which are real forms of different Lie algebras in the Cartan classification, namely D_m for even $D = 2m$ and B_m for $D = 2m+1$.

There is another way to prove the traceless property, which does not require information concerning invariant tensors. For rank 1, we simply take the trace of the formula derived in exercise 2.9. Contraction with $\eta_{\nu\rho}$ immediately gives $\operatorname{Tr}\gamma^\mu = 0$. As an exercise, the reader can extend this argument to higher rank.

The trace property leads also to the proof of independence of the elements of the basis (3.24) for even spacetime dimensions. One uses the 'reverse-order' basis of (3.25) and the trace orthogonality property (3.26). We suppose that there is a set of coefficients x_A such that

$$\sum_A x_A \Gamma^A = 0\,. \tag{3.99}$$

Multiply by Γ_B from the right. Take the trace and use the trace orthogonality to obtain

$$\sum_A x_A \operatorname{Tr} \Gamma^A \Gamma^B = \pm x_B \operatorname{Tr} \mathbb{1} = 0\,. \tag{3.100}$$

Hence all $x_A = 0$ and linear independence is proven.

Furthermore, since we have a linearly independent, indeed trace orthogonal, basis of the algebra, the Γ^A are a complete set in the space of $2^m \times 2^m$ matrices.

It now follows that, in any representation of the Clifford algebra for $D = 2m$ spacetime dimensions, the dimension of the $N \times N$ matrices satisfies $N \geq 2^m$. The reason is that no linear independent set of matrices of any smaller dimension exists. It also follows that any representation of dimension 2^m is irreducible. It can have no non-trivial invariant subspace, since a set of linearly independent matrices of smaller dimension would be realized by projection to this subspace.

3.A.2 Uniqueness of the γ-matrix representation

We must now show that there is exactly one irreducible representation up to equivalence. We use the basic properties of representations of finite groups. However, the Clifford algebra is not quite a group because the minus signs that necessarily occur in the set of products $\Gamma^A \Gamma^B = \pm \Gamma^C$ are not allowed by the definition of a group. This problem is solved by doubling the basis in (3.24) to the larger set $\{\Gamma^A, -\Gamma^A\}$. This set is a group of order 2^{2m+1} since all products are contained within the larger set. For $m = 1$, the group obtained is isomorphic to the quaternions, so the groups defined by doubling the Clifford algebras are called generalized quaternionic groups.

Every representation of the Clifford algebra by a set of matrices $D(\Gamma^A)$ extends to a representation of the group if we define $D(-\Gamma^A) = -D(\Gamma^A)$. It is not true that every group representation gives a representation of the algebra. For example, in a one-dimensional group representation, the matrices $D(\gamma^\mu)$ of the Clifford generators cannot satisfy $\{D(\gamma^\mu), D(\gamma^\nu)\} = 2\eta^{\mu\nu}$.

The three basic facts that we need are discussed in many mathematical texts such as [19, 20]. Consider the set of all finite dimensional irreducible representations and choose one representative within each class of equivalent representations. The set so formed, which may be called the set of all inequivalent irreducible representations, has the properties:

1. the sum of the squares of the dimensions of these representations is equal to the order of the group.

2. the number of inequivalent irreducible representations is equal to the number of conjugacy classes in the group.
3. the number of inequivalent 1-dimensional representations is equal to the index of the commutator subgroup G_c. The index of a subgroup is the ratio of the order of the group divided by the order of the subgroup.

The conjugacy classes of the group are sets of products $\pm\Gamma^B\Gamma^A(\Gamma^B)^{-1}$ (with no sum on B).

Ex. 3.40 *Show that for rank $r \geq 1$ there is a conjugacy class containing the pair $(\Gamma^A, -\Gamma^A)$ for each distinct Γ^A, and that $\mathbb{1}$ and $-\mathbb{1}$ belong to different conjugacy classes.*

Thus there are a total of $2^{2m}+1$ conjugacy classes.

The commutator subgroup is generated by all products of the form $\pm\Gamma^B\Gamma^A(\Gamma^B)^{-1}(\Gamma^A)^{-1}$. But in our case this subgroup contains only $\pm\mathbb{1}$, so the order of the subgroup is 2 and its index is 2^{2m}.

These facts establish that the group has exactly one irreducible representation of dimension 2^m plus 2^{2m} inequivalent one-dimensional representations. We must now show that the 2^m-dimensional representation of the group is also a representation of the algebra. We use the fact that any finite dimensional algebra has a (reducible) representation called the *regular representation* for which the algebra itself is the carrier space. The dimension is thus the dimension of the algebra, 2^{2m} in our case. The representation $\Gamma^A \to T(\Gamma^A)$ is defined by $T(\Gamma^A)\Gamma^B \equiv \Gamma^A\Gamma^B$. This algebra representation, in which $T(-\Gamma^A) = -T(\Gamma^A)$ is necessarily satisfied is also a group representation. Its decomposition into irreducible components thus cannot contain any one-dimensional group representations in which $D(-\Gamma^A) = +D(\Gamma^A)$. Thus the only possibility is that the regular representation decomposes into 2^m copies of the 2^m dimensional irreducible representation. This proves the essential fact that there is exactly one irreducible representation of the Clifford algebra for even spacetime dimension. For dimension $D = 2m$, the dimension of the Clifford representation is 2^m.

Another fact from finite group theory is helpful at this point. Any representation of a finite group is equivalent to a representation by unitary matrices. We can and therefore will choose a representation in which the spatial gamma matrices γ^i, $i = 1, \ldots, D-1$, which satisfy $(\gamma^i)^2 = \mathbb{1}$ are hermitian, and γ^0, which satisfies $(\gamma^0)^2 = -\mathbb{1}$, is anti-hermitian.

3.A.3 The Clifford algebra for odd spacetime dimensions

We gave in (3.39) two different sets of γ-matrices for odd dimensions. They are inequivalent as representations of the generating elements. Indeed it is easily seen that $S\gamma_+{}^\mu S^{-1} = \gamma_-{}^\mu$ cannot be satisfied. This requires $S\gamma^\mu S^{-1} = \gamma^\mu$ for the first $2m$ components. But then, from the product form in (3.5),(3.29), we obtain $S\gamma^{2m}S^{-1} = +\gamma^{2m}$, rather than the opposite sign needed.

It follows from Ex. 2.8 that the two sets of second rank elements constructed from the generating elements above, namely

$$\begin{aligned}
\Sigma_\pm{}^{\mu\nu} &= \tfrac{1}{4}[\gamma^\mu, \gamma^\nu]\,, \qquad \mu,\nu = 0,\ldots,2m-1\,,\\
&= \tfrac{1}{4}[\gamma^\mu, \pm\gamma_*]\,, \qquad \mu = 0,\ldots,2m-1\,; \quad \nu = 2m\,,
\end{aligned} \tag{3.101}$$

are each representations of the Lie algebra $\mathfrak{so}(2m,1)$. The two representations are equivalent, however, since $\gamma_*\Sigma_+{}^{\mu\nu}\gamma_* = \Sigma_-{}^{\mu\nu}$. This representation is irreducible; indeed it is a copy of the unique 2^{2m}-dimensional fundamental irreducible representation with Dynkin designation $(0,0,\ldots,0,1)$. It is associated with the short simple root of the Dynkin diagram for B_m.

3.A.4 Proofs of Majorana properties

A Majorana spinor representation of the Dirac algebra is a representation in which the generating γ^μ are explicitly real. For Minkowski signature spacetime such representations exist only for dimensions $2,3,4$ mod 8. In addition there are pseudo-Majorana representations for dimensions $2,8,9$ mod 8 in which the γ^μ are pure imaginary. In both these cases it is a consistent option to impose the constraint that the spinor field is real.

A detailed derivation of the Majorana representations is quite involved, but the practical manipulations one uses in their applications are straightforward. In this section we outline the derivation, with further details. We can also refer to[6] [16, 17, 21, 18, 22].

We already know that the physics of a spinor field is the same in all equivalent representations. Thus we are really concerned with classes of representations related by conjugacy, i.e.

$$\gamma'^\mu = S\gamma^\mu S^{-1}\,, \tag{3.102}$$

which contain representations by purely real matrices. Since we consider only hermitian representations, in which (3.3) holds, the matrix S must be unitary. In this more general viewpoint certain matrices S can be used to characterize Majorana representations. Given one set of matrices γ^μ which satisfy the anti-commutation relations (2.18), their complex conjugates $\gamma^{\mu *}$ and their positive or negative transposes $\pm\gamma^{\mu T}$ also satisfy (2.18). In any even dimension $D = 2m$, to which restrict until further notice, there is a unique representation up to equivalence. Thus there are unitary matrices B, C such that

$$\gamma^{\mu *} = B\gamma^\mu B^{-1}\,, \tag{3.103}$$

$$-\gamma^{\mu T} = C\gamma^\mu C^{-1}\,. \tag{3.104}$$

We analyze the negative transpose condition (i.e. $t_0 t_1 = -1$ in the language of the main text), because it leads to the most commonly used Majorana spinors. The

[6] In [21], the discussion of Majorana spinors is in section 4, pages 843-851.

positive transpose leads to different results, which are summarised in section 3.2.1. One can show in general that B has definite transposition symmetry. We start from the unitarity of B, i.e. $B^{-1} = B^\dagger$. Take the complex conjugate of (3.103) and use (3.103) again to show that B^*B commutes with all Γ_A. By Schur's Lemma, $B^*B = -t_1\mathbb{1}$, where t_1 is so far undetermined, but we use the notation that we know from the main text, see Ex. 3.19. By unitarity, this becomes $(B^\dagger)^T B = (B^{-1})^T B = -t_1\mathbb{1}$, and finally $B = -t_1 B^T$. Take the transpose of this last relation and use it again to obtain $B = (t_1)^2 B$, so that $t_1 = \pm 1$.

It follows then that C has opposite transposition symmetry, i.e.

$$B^T = -t_1 B\,, \qquad C^T = -t_0 C\,, \qquad t_0 = -t_1 \pm 1\,, . \tag{3.105}$$

Ex. 3.41 *Show that $C = B\gamma^0$ up to multiplication by an arbitrary phase factor. There are other equivalent expressions.*

Further, in any given representation, B and C are unique up to multiplication by a complex number of modulus one. We will choose that they relate as in (3.46). These equations imply that

$$B^*B = -C^*C = \epsilon\mathbb{1}\,. \tag{3.106}$$

3.A.5 Determination of symmetries of gamma matrices

We will determine the possible symmetries of gamma matrices for each spacetime dimension $D = 2m$ by showing that each matrix $C\Gamma_A$ formed from the basis (3.24) has a definite symmetry that depends only on the tensor rank r. Then we will count the number of symmetric and anti-symmetric matrices in the list $\{C\Gamma_A\}$, which must be equal to $2^{m-1}(2^m \pm 1)$ for $D = 2m$. For a given value of t_0 and t_1, the number of antisymmetric matrices in the list $\{C\Gamma_A\}$ is given, using (3.43), by

$$\begin{aligned} N_- &= \sum_{r=0}^{2m} \tfrac{1}{2}\left[1 + t_r\right] C_r^{2m} \\ &= 2^{2m-1} + \tfrac{1}{2} t_0 \sum_{s=0}^{m} (-)^s C_{2s}^{2m} + \tfrac{1}{2} t_1 \sum_{s=0}^{m-1} (-)^s C_{2s+1}^{2m} \\ &= 2^{2m-1} + t_0\, 2^{m-1} \cos\frac{m\pi}{2} + t_1\, 2^{m-1} \sin\frac{m\pi}{2} \\ &= 2^{m-1}(2^m - 1)\,. \end{aligned} \tag{3.107}$$

We thus find

$$t_0 \cos\frac{m\pi}{2} + t_1\, 2^{m-1} \sin\frac{m\pi}{2} = -1\,, \tag{3.108}$$

which leads to the solutions that are in table 3.1 for even dimensions.

To understand the situation in odd $D = 2m + 1$ we note that the highest rank Clifford element γ_* in (3.29) has the symmetry determined by t_{2m}. Since we attach $\pm\gamma_* = \gamma^{2m}$ as the last generating element in (3.39) we must require it to have the same symmetry as the other generating γ^μ, and thus t_{2m} should be equal to t_1. This determines which of the two possibilities for even dimensions in table 3.1 is valid in the next odd dimension.

3.A.6 Friendly representations.

General construction. In this section we present an explicit recursive construction of the generating γ^μ for any even dimension $D = 2m$. In this representation each generating matrix will be either pure real or pure imaginary. A representation of this type will be called a friendly representation.[7] Using this representation it also possible to prove the existence of Majorana (and pseudo-Majorana) spinors in a quite simple way [23, 18] (see Appendix B in the former).

We already know that the γ matrices in dimension $D = 2m$ are $2^m \times 2^m$ matrices. In the recursive construction the generating matrices γ^μ for dimension $D = 2m$ will be written as direct products of the $\tilde{\gamma}^\mu$ and $\tilde{\gamma}_*$ for dimension $D = 2m - 2$ with the Pauli matrices σ_i.

We start in $D = 2$ and write

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \mathrm{i}\sigma_2\,, \qquad \gamma_1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \sigma_3\,, \tag{3.109}$$

which is a really real, hermitian, and friendly representation. The matrix γ_* is also real:

$$\gamma_* = -\gamma_0\gamma_1 = \sigma_1\,. \tag{3.110}$$

Adding it to (3.109) as γ_2 gives a real representation in $D = 3$.

The recursion relation for moving from a $D = 2m - 2$ representation with $\tilde{\gamma}$ to $D = 2m$ is

$$\begin{aligned} &\gamma_\mu = \tilde{\gamma}_\mu \otimes \mathbb{1}\,, \qquad \mu = 0, \ldots, 2m-3\,, \\ &\gamma_{2m-2} = \tilde{\gamma}_* \otimes \sigma_1\,, \qquad \gamma_{2m-1} = \tilde{\gamma}_* \otimes \sigma_3\,. \end{aligned} \tag{3.111}$$

This gives

$$\gamma_* = -\tilde{\gamma}_* \otimes \sigma_2\,. \tag{3.112}$$

This matrix γ_* can be used as γ_{2m} to define a representation in $D = 2m + 1$ dimensions.

This construction gives a real representation in 4 dimensions, which is explicitly given in (3.78). This one has an imaginary γ_* and hence this construction will not give real representations for higher dimensions. The matrix B is obtained as the product of all the imaginary γ-matrices.

[7] All of our friends use friendly representations.

We thus obtained representations for all dimensions, and really real for $D =$ 2, 3, 4. The latter can be extended to any $D =$ 10, 11, 12 or any other dimension that differs by it modulo 8. To see this, consider the following 16×16 matrices:

$$\begin{aligned}
E_1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}\,,\\
E_2 &= \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}\,,\\
E_3 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \otimes \mathbb{1}\,,\\
E_4 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \mathbb{1}\,,\\
E_5 &= \sigma_2 \otimes \sigma_1 \otimes \mathbb{1} \otimes \sigma_2\,,\\
E_6 &= \sigma_2 \otimes \sigma_3 \otimes \mathbb{1} \otimes \sigma_2\,,\\
E_7 &= \sigma_2 \otimes \mathbb{1} \otimes \sigma_2 \otimes \sigma_1\,,\\
E_8 &= \sigma_2 \otimes \mathbb{1} \otimes \sigma_2 \otimes \sigma_3\,,\\
E_* &= E_1 \dots E_8 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2\,.
\end{aligned} \tag{3.113}$$

This is a real representation for Euclidean gamma matrices in $D = 8$ (or $D = 9$ if one includes E_*). Using this and a representation $\tilde{\gamma}_\mu$ in any D, one can construct a representation γ_μ in $D + 8$ dimensions by

$$\begin{aligned}
\gamma_\mu &= \tilde{\gamma}_\mu \otimes E_*\,, \qquad \mu = 0, \dots, D-1\,,\\
\gamma_{D-1+i} &= \mathbb{1} \otimes E_i\,, \qquad i = 1, \dots, 8\,.
\end{aligned} \tag{3.114}$$

When the $\tilde{\gamma}_\mu$ are real, the gamma matrices in $D + 8$ are also real. Hence this gives explicitly real representations in $D =$ 2, 3, 4 mod 8. For even dimensions, one obtains

$$\gamma_* = \tilde{\gamma}_* \otimes E_*\,. \tag{3.115}$$

Hence this is real if $\tilde{\gamma}_*$ is real. For the real representations we saw that it is real for $D = 2$ and not in $D = 4$. This shows explicitly that we can define real projections P_L and P_R on real spinors if and only if $D = 2$ mod 8. These are called *Majorana-Weyl* representations.

Ex. 3.42 *We denote a Clifford algebra in s spacelike and t timelike directions as $\mathcal{C}(s,t)$ (the one discussed above are thus of the form $\mathcal{C}(D-1,1)$, apart from the E_i that correspond to $\mathcal{C}(8,0)$. See that the above construction proves that the reality properties of $\mathcal{C}(s+8,t)$ are the same as $\mathcal{C}(s,t)$. Further, show that the analogous construction starting with (3.109) shows that also $\mathcal{C}(s+1,t+1)$ has the same properties as $\mathcal{C}(s,t)$.*

4
The Maxwell and Yang-Mills Gauge Fields.

In this chapter we discuss the classical abelian and non-abelian gauge fields. Although our treatment is self-contained, it is best taken as a review for readers who have previously studied the role of vector potential as the gauge field in Maxwell's electromagnetism and also have some acquaintance with Yang-Mills theory.

We will again take a general dimensional viewpoint, but let's begin the discussion in 4 dimensions with some remarks about the particle representations of the Poincaré group and the fields usually used to describe elementary particles. A particle is classified by its mass m and spin s, and a massive particle of spin s has $2s+1$ helicity states. Massless particles of spin $s=0$ or $s=1/2$ have one or two helicity states, respectively, in agreement with the counting for massive particles. However, massless particles of spin $s \geq 1/2$ have 2 helicity states, for all values of s.

Helicity is defined as the eigenvalue of the component of angular momentum in the direction of motion. For a massless particle of spin s, the two helicity states have eigenvalues $\pm s$. For a massive particle of spin s the helicity eigenvalues, called λ, range in integer steps from $\lambda = s$ to $\lambda = -s$.

Let us compare the count of the helicity states with the number of independent functions that must be specified as initial data for the Cauchy initial value problem of the associated field. The first number can be considered to be the '*number of on-shell degrees of freedom*', or number of quantum degrees of freedom, while the second is the number of classical degrees of freedom.

Let's do the counting for massless particles that are identified with their anti-particles. The associated fields are real for bosons and satisfy the Majorana condition for fermions. The counting is similar for complex fields. We assume that the equations of motion are second order in time for bosons and first order for fermions. A unique solution of the Cauchy problem for the scalar $\phi(x)$ requires the initial data $\phi(\vec{x},0)$ and $\dot{\phi}(\vec{x},0)$, the time derivative. For $\Psi_\alpha(x)$, we must specify the initial values $\Psi_\alpha(\vec{x},0)$ of all 4 components, and the first-order Dirac equation then determines the future evolution of $\Psi_\alpha(\vec{x},t)$ and thus the time derivatives $\dot{\Psi}_\alpha(\vec{x},0)$. The number of helicity states (number of on-shell degrees of freedom) is 1 for $\phi(x)$

and 2 for $\Psi_\alpha(x)$. The number of classical degrees of freedom is twice the number of helicity states.

We continue this counting, in a naive fashion, for vector $A_\mu(x)$, vector-spinor $\Psi_{\mu\alpha}(x)$, and symmetric tensor $h_{\mu\nu}(x)$ fields, the latter describing gravitons in Minkowski space. Following the earlier pattern we would expect to need 8, 16, and 20 functions, respectively, as initial data. These numbers greatly exceed the two helicity states for spin 1, spin $\frac{3}{2}$ and spin 2 particles. Something new is required to resolve this mismatch.

The lessons from QED, Yang-Mills theory, general relativity and supergravity teach us that the only way to proceed is to use very special field equations with gauge invariance. Gauge invariance accomplishes the following goals:
a) Relativistic covariance is maintained;
b) The field equations do not determine certain "longitudinal" field components (such as $\partial^\mu A_\mu$ for vector fields);
c) A subset of the field equations are constraints on the initial data rather than time evolution equations. The independent initial data is contained in 4 real functions, thus again 2 for each helicity state.
d) The field describes a pure spin s particle with no lower spin admixtures. Otherwise there would be some negative metric ghosts.
e) Most important, for $s = 1,\ \frac{3}{2},\ 2$, gauge invariant interactions can be introduced.[1] Classical dynamics is consistent at the nonlinear level and the theories can be quantized (although power-counting renormalizability is expected to fail except for spin 1).

Before embarking on the technical discussion we reiterate that the classical degrees of freedom we are concerned with are the independent functions required as initial data for the Cauchy problem of hyperbolic equations. In the analysis we also find that certain field components satisfy elliptic equations, such as the basic Laplace equation $\nabla^2\phi = 0$. The specification of a solution requires supplying data on a boundary 2-surface rather than the 3-dimensional Cauchy surface. Thus a field component satisfying an elliptic equation does not contain any degrees of freedom.[2]

4.1 The abelian gauge field $A_\mu(x)$

We now review the elementary features of gauge invariance for spin 1. One purpose is to set the stage for spin $\frac{3}{2}$ in the next chapter.

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[1] There are gauge invariant *free* fields for massless particles of any spin, (see [24], for example). It appears to be impossible to introduce consistent interactions for any finite subset of these, but remarkably one can make progress for certain infinite sets of fields or using different backgrounds than Minkowski spaces [25, 26, 27].

[2] It is instructive to note that the Fourier transform of the harmonic condition $\nabla^2\phi(x) = 0$ is $\vec{k}^2\hat{\phi}(\vec{k}) = 0$ whose only smooth solution vanishes identically.

4.1.1 Gauge invariance and fields with electric charge

In Chapters 1 and 2 we discussed the global U(1) symmetry of complex scalar and spinor fields. The abelian gauge symmetry of quantum electrodynamics is an extension in which the phase parameter $\theta(x)$ becomes an arbitrary function in Minkowski spacetime. We generalize the previous discussion slightly and assign an electric charge q, an arbitrary real number at this stage, to each complex field in the system. For a Dirac spinor field of charge q, the gauge transformation, a local change of the phase of the complex field, is[3]

$$\Psi(x) \to \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}q\theta(x)}\Psi(x). \tag{4.1}$$

The goal is to formulate field equations that transform covariantly under the gauge transformation. This requires the introduction of a new field, the vector gauge field or vector potential $A_\mu(x)$, which is defined to transform as

$$A_\mu(x) \to A'_\mu(x) \equiv A_\mu(x) + \partial_\mu\theta(x). \tag{4.2}$$

One then defines the covariant derivative

$$D_\mu\Psi(x) \equiv (\partial_\mu - \mathrm{i}qA_\mu(x))\Psi(x), \tag{4.3}$$

which transforms with the same phase factor as $\Psi(x)$, namely $D_\mu\Psi(x) \to \mathrm{e}^{\mathrm{i}q\theta(x)}D_\mu\Psi(x)$. The desired field equation is obtained by replacing $\partial_\mu\Psi \to D_\mu\Psi$ in the free Dirac equation (2.16), viz.

$$[\gamma^\mu D_\mu - m]\Psi \equiv [\gamma^\mu(\partial_\mu - \mathrm{i}qA_\mu) - m]\Psi = 0\,. \tag{4.4}$$

This equation is gauge covariant; if $\Psi(x)$ satisfies (4.4) with gauge potential $A_\mu(x)$, then $\Psi'(x)$ satisfies the same equation with gauge potential $A'_\mu(x)$.

The same procedure can be applied to a complex scalar field $\phi(x)$, to which we assign an electric charge q (which may differ from the charge of Ψ). We extend the global U(1) symmetry discussed in Chapter 1 to the local gauge symmetry $\phi(x) \to \phi'(x) = \mathrm{e}^{\mathrm{i}q\theta(x)}\phi(x)$ by defining the covariant derivative $D_\mu\phi = (\partial_\mu - \mathrm{i}qA_\mu)\phi$ and modifying the Klein-Gordon equation to the form

$$[D^\mu D_\mu - m^2]\phi = 0\,. \tag{4.5}$$

The procedure of promoting the global U(1) symmetry of the Dirac or Klein-Gordon equation to a local or gauge symmetry through the introduction of the vector potential in the covariant derivative is called the principle of minimal coupling. Another part of standard vocabulary is to say that fields with electric charge, such as ϕ or Ψ, are charged 'matter fields', which are minimally coupled to the gauge field A_μ.

---

[3] In the notation of Ch. 1, the "matrix" generator is $t = -\mathrm{i}q$. The U(1) transformation in Sec. 2.7.1 corresponds to the choice $q = 1$.

4.1.2 The free gauge field

It is quite remarkable that the promotion of global to gauge symmetry requires a new field $A_\mu(x)$. In some cases one may wish to consider (4.4) or (4.5) in a fixed external background gauge potential, but it is far more interesting to think of $A_\mu(x)$ as a field that is itself determined dynamically by its coupling to charged matter in a gauge invariant fashion. The resulting theory is quantum electrodynamics, the quantum and Lorentz covariant version of Maxwell's theory of electromagnetism. The predictions of this theory, both classical and quantum, are well confirmed by experiment. There can be no doubt that Nature knows about gauge principles.

Although we expect that readers are familiar with classical electromagnetism, we review the construction because there are similar patterns in Yang-Mills theory, gravity, and supergravity. The first step is the observation that the anti-symmetric derivative of the gauge potential, called the field strength

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x), \tag{4.6}$$

is invariant under the gauge transformation, a fact that is trivial to verify. In 4 dimensions $F_{\mu\nu}$ has 6 components, which split into the the electric $E_i = F_{i0}$ and magnetic $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$ fields.

Since A_μ is a bosonic field, we expect it to satisfy a second order wave equation. The only Lorentz covariant and gauge invariant quantity available is $\partial^\mu F_{\mu\nu}$, so the free electromagnetic field satisfies

$$\partial^\mu F_{\mu\nu} \;=\; 0\,. \tag{4.7}$$

Since $\partial^\nu\partial^\mu F_{\mu\nu}$ vanishes *identically*[4], (4.7) comprises $D-1$ independent components in D-dimensional Minkowski spacetime. This is not enough to determine the D components of A_μ, which is not surprising because of the gauge symmetry. One can change $A_\mu \to A_\mu + \partial_\mu\theta$ without affecting $F_{\mu\nu}$. So far, we did not yet use the field equations. Therefore, we will call this number, i.e. $(D-1)$ for the gauge vectors, the '*number of off-shell degrees of freedom*'.

One deals with this situation by 'fixing the gauge'. This means that one imposes one condition on the D components of A_μ, which eliminates the freedom to change gauge. Different gauge conditions illuminate different physical features of the theory. We will look first at the condition $\partial^i A_i(\vec{x},t) = 0$, which is called the Coulomb gauge condition. Although non-covariant it is a useful gauge to explore the structure of the initial value problem and determine the true degrees of freedom of the system. Note that the time-space split implicit in the initial-value problem is also non-covariant.

First let's show that this condition does eliminate the gauge freedom. We check whether there are gauge functions $\theta(x)$ with the property that $\partial^i A'_i = \partial^i(A_i + \partial_i\theta) = 0$ when $\partial^i A_i = 0$. This requires that $\nabla^2\theta = 0$. As explained above, the only smooth solution is $\theta(x) \equiv 0$, so the gauge freedom has been completely fixed.

---

[4] This is the 'Noether identity', an relation between the field equations that always be found as a consequence of the presence of gauge symmetries.

Let's write out the time ($\mu \to 0$) and space ($\mu \to i$) components of the Maxwell equation (4.7). Using (4.6) and lowering all indices with the Minkowski metric, one finds

$$\begin{aligned} \nabla^2 A_0 - \partial_0(\partial^i A_i) &= 0\,, \\ \Box A_i - \partial_i \partial^0 A_0 - \partial_i(\partial^j A_j) &= 0\,. \end{aligned} \tag{4.8}$$

In the Coulomb gauge, the first equation simplifies to $\nabla^2 A_0 = 0$, and we see that[5] A_0 does not contain any degrees of freedom. The second equation in (4.8) then becomes $\Box A_i = 0$, so the spatial components A_i satisfy the massless scalar wave equation.

We can now count the classical degrees of freedom, which are the initial data $A_i(\vec{x},0)$, $\dot{A}_i(\vec{x},0)$ required for a unique solution of $\Box A_i = 0$. There are a total of $2(D-2)$ independent degrees of freedom, because the initial data must be constrained to obey the Coulomb gauge condition.

This number thus agrees for $D = 4$ with the rule that the classical degrees of freedom are twice the number of on-shell degrees of freedom counted as helicity states. In general, we find for the gauge vectors $(D-1)$ off-shell degrees of freedom and $(D-2)$ on-shell degrees of freedom. On the other hand, the number of off-shell degrees of freedom is the dimension of a representation of $\mathrm{SO}(D-1)$. For the gauge field, we find for both the vector representation.

It is instructive to write the solution of $\Box A_i = 0$ as the Fourier transform

$$A_i(x) = \int \frac{\mathrm{d}^{(D-1)}k}{(2\pi)^{(D-1)}2k^0} \sum_\lambda [\mathrm{e}^{\mathrm{i}k\cdot x}\epsilon_i(\vec{k},\lambda)a(\vec{k},\lambda) + \mathrm{e}^{-\mathrm{i}k\cdot x}\epsilon_i^*(\vec{k},\lambda)a^*(\vec{k},\lambda)], \tag{4.9}$$

where $\vec{k}$, $k^0 = |\vec{k}|$ is the on-shell energy-momentum vector. The $\epsilon_i(\vec{k},\lambda)$ are called polarization vectors, which are constrained by the Coulomb gauge condition to satisfy $k^i\epsilon_i(\vec{k},\lambda) = 0$. So there are $(D-2)$ independent polarization vectors, indexed by λ, and there are $2(D-2)$ independent real degrees of freedom contained in the complex quantities $a(\vec{k},\lambda)$. As in the case of the plane-wave expansions of the free Klein-Gordon and Dirac fields, $a(\vec{k},\lambda)$ and $a^*(\vec{k},\lambda)$ are interpreted as Fourier amplitudes in the classical theory and as annihilation and creation operators for particle states after quantization.

To understand these particle states better, we discuss the case of $D = 4$ and assume that the spatial momentum is in the 3-direction, i.e. $\vec{k} = (0,0,k)$ with $k > 0$. The two polarization vectors may be taken to be $\epsilon_i((0,0,k),\pm) = \frac{1}{\sqrt{2}}(1,\pm\mathrm{i},0)$. We formally add the 0-component $\epsilon_0 = 0$ to form 4-vectors $\epsilon_\mu((0,0,k),\pm)$, which are eigenvectors of the rotation generator $I_3 = -m_{[12]}$, about the 3-axis, (see text above (1.96)), with angular momentum $\lambda = \pm 1$. For general spatial momentum $\vec{k} =$

[5] When a source current is added to the Maxwell equation (4.7), A_0 no longer vanishes, but it is determined by the source. Thus it is not a degree of freedom of the gauge field system.

$k(\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$, we define polarization vectors $\epsilon_\mu(\vec{k}, \pm)$ by applying the spatial rotation with Euler angles α, β, which rotates the 3-axis to the direction of $\vec{k}$. The associated particle states are photons with helicity ± 1.

The same ideas determine the properties of particle states of the gauge field for $D \geq 5$. For spatial momentum in the direction $D-1$, i.e. $\vec{k} = (0, 0, \ldots, k)$, there are $D-2$ independent polarization vectors. We need not specify these in detail; the important point to note is that these vectors are a basis of the vector representation of the Lie group $\mathrm{SO}(D-2)$, which is the group that 'fixes' the vector $\vec{k}$. The associated particle states also transform in this representation. On the other hand it is clear from (4.9) that the Coulomb gauge vector potential transforms in the vector representation of $\mathrm{SO}(D-1)$.

It should be noted that the equations of the *free* electromagnetic field can be formulated as conditions involving only the field strength components $F_{\mu\nu}$, with the gauge potential A_μ appearing as a derived quantity. In this form of the theory one has the pair of equations

$$\partial^\mu F_{\mu\nu} = 0\,, \tag{4.10}$$

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0\,. \tag{4.11}$$

The second equation is called the Bianchi identity. It is easy to see that it is automatically satisfied if $F_{\mu\nu}$ is expressed in terms of A_μ as in (4.6). In a topologically trivial spacetime such as Minkowski space, this is the general solution. This is a consequence of the Poincaré lemma in the theory of differential forms discussed in Ch. 6. Although the manifestly gauge invariant formalism (4.10),(4.11) is available for the free gauge field, the vector potential is required ab initio to describe the minimal coupling to charged matter fields.

This chapter has progressed too far without exercises for readers, so we must now try to remedy this deficiency.

Ex. 4.1 *Derive from (4.3) that*

$$[D_\mu, D_\nu]\Psi \equiv (D_\mu D_\nu - D_\nu D_\mu)\Psi = -\mathrm{i}qF_{\mu\nu}\Psi\,. \tag{4.12}$$

Derive from (4.4) that the charged Dirac field also satisfies the second order equation

$$\left[D^\mu D_\mu - \tfrac{1}{2}\mathrm{i}q\gamma^{\mu\nu}F_{\mu\nu} - m^2\right]\Psi = 0\,. \tag{4.13}$$

Ex. 4.2 *Using only (4.10),(4.11), show that the field strength tensor satisfies the equation $\Box F_{\mu\nu} = 0$. This is a gauge invariant derivation of the fact that the free electromagnetic field describes massless particles.*

4.1.3 Sources and Green's function

Let us now discuss sources for the electromagnetic field. Conventionally one takes an electric source that appears only in (4.10), which is modified to read

$$\partial^\mu F_{\mu\nu} = -J_\nu\,, \tag{4.14}$$

while the Bianchi identity (4.11) is unchanged, so that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Lorentz covariance requires that the source is a vector, which is called the electric current vector. Since $\partial^\nu \partial^\mu F_{\mu\nu}$ vanishes *identically*, the current must be conserved. The theory is inconsistent unless the current satisfies $\partial^\nu J_\nu = 0$. This condition simply reflects the conventional idea that electric charge cannot be created or destroyed. It is also possible to include sources that carry magnetic charge and appear on the right side of (4.11). However this requires quite sophisticated consideration of the topology of the gauge group and field quantization, so we will confine our attention to electric sources.

Ex. 4.3 *Repeat 4.2 when there is an electric source. Show that*

$$\Box F_{\nu\rho} = -\partial_\nu J_\rho + \partial_\rho J_\nu \,. \tag{4.15}$$

Consider the analogous problem of the scalar field coupled to a source $J(x)$

$$(\Box - m^2)\phi(x) \;=\; -J(x)\,. \tag{4.16}$$

The response to the source is determined by the Green's function $G(x-y)$, which satisfies the equation

$$(\Box - m^2)G(x-y) = -\delta(x-y)\,. \tag{4.17}$$

The translation symmetry of Minkowski space-time implies that the Green's function depends only on the coordinate difference $x^\mu - y^\mu$ between observation point x^μ and source point y^μ. Lorentz symmetry implies that it depends only on the invariant quantities $(x-y)^2 = \eta^{\mu\nu}(x-y)_\mu(x-y)_\nu$ and $\mathrm{sgn}(x^0-y^0)$. In Euclidean space $\mathcal{R}^D$, there is a unique solution of the equation analogous to (4.17), which is damped in the limit of large separation of observation and source points. In Lorentzian signature Minkowski space, there are several choices, which differ in their causal structure, that is in the dependence on $\mathrm{sgn}(x^0-y^0)$. Many texts on quantum field theory, such as [28, 29, 15], discuss these choices.

The Euclidean Green's function is simplest and sufficient for the purposes of this book. The solution of (4.17) can be written as the Fourier transform

$$G(x-y) = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\mathrm{e}^{\mathrm{i}k\cdot(x-y)}}{k^2+m^2}\,. \tag{4.18}$$

The integral can be expressed in terms of modified Bessel functions. In the massless case the result simplifies to the power law [30]

$$G(x-y) = \frac{\Gamma(\frac{1}{2}(D-2))}{4\pi^{\frac{1}{2}D}(x-y)^{(D-2)}}\,. \tag{4.19}$$

Here $(x-y)^2 = \delta_{\mu\nu}(x-y)^\mu(x-y)^\nu$ is the Euclidean distance between source point y and observation point x. Given $G(x-y)$, the solution of (4.16) can be expressed as the integral

$$\phi(x) = \int \mathrm{d}^D y\, G(x-y)\, J(y). \tag{4.20}$$

One may note that the Green's function is formally the inverse of the wave operator, i.e. $G = -(\Box - m^2)^{-1}$. In Euclidean space $\Box = \nabla^2$ which is the D-dimensional Laplacian.

Let's return to Minkowski space and find the Green's function for the gauge field. We might expect to solve (4.14) using a Green's function $G_{\mu\nu}(x-y)$, which is a tensor. However, we run into the immediate difficulty that there is no solution to the equation

$$(\eta^{\mu\rho}\Box - \partial^\mu\partial^\rho)G_{\rho\nu}(x-y) = -\delta^\mu_\nu\delta(x-y). \tag{4.21}$$

The Maxwell wave operator $\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu$ is not invertible since any pure gradient $\partial_\nu f(x)$ is a zero mode. This problem is easily resolved. Since the source J_ν is conserved, we can replace (4.21) by the weaker condition

$$(\eta^{\mu\rho}\Box - \partial^\mu\partial^\rho)G_{\rho\nu}(x,y) = -\delta^\mu_\nu\delta(x-y) + \frac{\partial}{\partial y^\nu}\Omega^\mu(x,y)\,, \tag{4.22}$$

where $\Omega^\mu(x,y)$ is an arbitrary vector function. If $\Omega^\mu(x,y)$ and $J_\nu(y)$ are suitably damped at large distance, the effect of the second term in (4.22) cancels (after partial integration) in the formula

$$A_\mu(x) = \int \mathrm{d}^D y\, G_{\mu\nu}(x,y) J^\nu(y)\,, \tag{4.23}$$

which is the analogue of (4.20).

We now derive the precise form of $G_{\mu\nu}(x,y)$. By Poincaré symmetry, we can assume the tensor form

$$G_{\mu\nu}(x,y) = \eta_{\mu\nu}F(\sigma) + (x-y)_\mu(x-y)_\nu\hat{S}(\sigma)\,, \tag{4.24}$$

where $\sigma = \frac{1}{2}(x-y)^2$. It is more useful, but equivalent, to take advantage of gauge invariance and rewrite this ansatz as

$$G_{\mu\nu}(x,y) = \eta_{\mu\nu}F(\sigma) + \partial_\mu\partial_\nu S(\sigma)\,, \tag{4.25}$$

because the pure gauge term involving $S(\sigma)$ has no effect in (4.23) and cancels in (4.22). We may also assume that the gauge term in (4.22) has the Poincaré invariant form $\partial^\mu\partial_\nu\Omega(\sigma)$. Substituting (4.25) in (4.22) we find the two independent tensors δ^μ_ν and $(x-y)^\mu(x-y)_\nu$ and thus two independent differential equations involving F and Ω, namely

$$\begin{aligned} 2\sigma F''(\sigma) + (D-1)F'(\sigma) &= \Omega'(\sigma) \\ F''(\sigma) &= -\Omega''(\sigma)\,. \end{aligned} \tag{4.26}$$

Note that $F'(\sigma) = \mathrm{d}F(\sigma)/\mathrm{d}\sigma$, etc. We have dropped the δ-function term in (4.22), because we will first solve these equations for $\sigma \neq 0$. The second equation in (4.26) may be integrated immediately, giving $F'(\sigma) = -\Omega'(\sigma)$; a possible integration constant is chosen to vanish, so that $F'(\sigma)$ vanishes at large distance. The first equation then becomes $2\sigma F''(\sigma) + DF'(\sigma) = 0$, which has the power law solution $F(\sigma) \sim \sigma^{1-\frac{1}{2}D}$. However, on any function of σ, the d'Alembertian acts as $\Box F(\sigma) = 2\sigma F''(\sigma) + DF'(\sigma)$. Therefore the effect of the delta function in (4.22) is automatically incorporated if we take $F(\sigma) = G((x-y)^2)$ where G is the massless scalar Green's function for Minkowski space. The result of this analysis is the gauge field Green's function

$$G_{\mu\nu}(x,y) = \eta_{\mu\nu} G(x-y) + \partial_\mu \partial_\nu S(\sigma)\,. \tag{4.27}$$

The gauge function $S(\sigma)$ is arbitrary and may be taken to vanish. Then (4.27) becomes the usual Feynman gauge propagator.

The gauge field propagators found above are usually derived after gauge fixing in the path integral formalism in quantum field theory texts. The derivation here is purely classical, as appropriate since the response of the gauge field to a conserved current source is a purely classical phenomenon.

It may not be obvious why this method works. To see why, apply $\partial/\partial x^\mu$ to both sides of (4.22). obtaining

$$0 \;=\; -\partial_\nu \delta(x-y) - \partial_\nu \Box \Omega(\sigma)\,, \tag{4.28}$$

in which $\partial_\nu = \partial/\partial x^\nu$. This consistency condition is satisfied because the analysis above led us the result $\Omega(\sigma) = -F(\sigma) = -G((x-y)^2)$.

4.1.4 Quantum electrodynamics

The current vector J_ν in (4.14) may describe a piece of laboratory apparatus, such as a magnetic solenoid. However, we are more interested in the case where the source is the field of a charged elementary particle, such as the Dirac spinor Ψ. This is the theory of Quantum Electrodynamics, which contains equations that determine both the electromagnetic field A_μ, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and Ψ. In dealing with coupled fields it is generally best to package the dynamics in a Lorentz invariant action. The equations of motion then emerge as the condition for a critical point of the action functional and are guaranteed to be mutually consistent.

It is also advantageous to change notation from that of Sec. 4.1.1 by scaling the vector potential, $A_\mu \to eA_\mu$, where e is the conventional coupling constant of the electromagnetic field to charged fields; $\frac{e^2}{4\pi} \approx \frac{1}{137}$ is called the fine structure constant. In this notation the relevant equations of Sec. 4.1 read:

$$F_{\mu\nu} \;\equiv\; \partial_\mu A_\nu - \partial_\nu A_\mu\,,$$

$$
\begin{aligned}
A_\mu \to A'_\mu &\equiv A_\mu + \frac{1}{e}\partial_\mu\theta\,, \\
D_\mu\Psi &\equiv (\partial_\mu - \mathrm{i}eqA_\mu)\Psi\,, \\
[D_\mu, D_\nu]\Psi &= -\mathrm{i}eqF_{\mu\nu}\Psi\,.
\end{aligned}
\tag{4.29}
$$

The electric charges q of the various charged fields are then simple rational numbers, for example $q = 1$ for the electron.[6]

The action functional for the electromagnetic field interacting with a field of charge q, which we take to be a massive Dirac field, is the sum of two terms, each gauge invariant,

$$
S[A_\mu, \bar{\Psi}, \Psi] = \int \mathrm{d}^D x \left[-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \bar{\Psi}(\gamma^\mu D_\mu - m)\Psi\right]\,. \tag{4.30}
$$

The Euler variation of (4.30) with respect to the gauge potential A_ν is

$$
\frac{\delta\mathcal{L}}{\delta A^\nu} = \partial^\mu F_{\mu\nu} + \mathrm{i}eq\bar{\Psi}\gamma_\nu\Psi = 0\,. \tag{4.31}
$$

This is equivalent to (4.14) with the electric current a multiple of the Noether current of the global U(1) phase symmetry discussed in Sec. 2.7.1. It is a typical feature of the various fundamental gauge symmetries in physics that the Noether current of a system with global symmetry becomes the source for the gauge field introduced when the symmetry is gauged. The Euler variation with respect to $\bar{\Psi}$ gives the gauge covariant Dirac equation (4.4), with $D_\mu\Psi$ given in (4.29).

4.1.5 The stress tensor and gauge covariant translations

The situation of the stress tensor of this system is quite curious. The canonical stress tensor, calculated from the Noether formula (1.69) with $\Delta\phi^i_\alpha \to \partial_\nu A_\rho,\ \partial_\nu\bar{\Psi},\ \partial_\nu\Psi$ for the three independent fields, is

$$
T^\mu{}_\nu = F^{\mu\rho}\partial_\nu A_\rho + \bar{\Psi}\gamma^\mu\partial_\nu\Psi + \delta^\mu_\nu\mathcal{L}\,. \tag{4.32}
$$

It is conserved on the index μ, but not on ν, not symmetric and not gauge invariant. The situation can be improved by treating fermion terms as in Sec. 2.7.2 and then adding $\Delta T^\mu{}_\nu = -\partial_\rho(F^{\mu\rho}A_\nu)$ in accord with the discussion in Sec. 1.3. The final result is the gauge invariant symmetric stress tensor

$$
\Theta_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho + \tfrac{1}{4}\bar{\Psi}(\gamma_\mu\overleftrightarrow{D}_\nu + \gamma_\nu\overleftrightarrow{D}_\mu)\Psi + \eta_{\mu\nu}\mathcal{L}\,. \tag{4.33}
$$

---

[6] It is an interesting question why the electric charges of elementary particles in Nature are quantized; that they appear to be integer multiples of a lowest fundamental charge. Two reasons have been found. The first is that quantum theory requires quantization of electric charge if magnetic monopoles exist. Second, electric charge can emerge as an unbroken U(1) generator of a larger non-abelian gauge theory with spontaneous gauge symmetry breaking. These reasons are not independent since monopoles solutions exist when gauge symmetry is broken with residual U(1) symmetry. See [31] for discussion of these ideas.

Ex. 4.4 *Consider the gauge covariant translation, defined by $\delta A_\mu = a^\nu F_{\nu\mu}$ and $\delta\Psi = a^\nu D_\nu \Psi$. Show that they differ from a conventional translation by a gauge transformation with gauge-dependent parameter $\theta = -e\, a^\nu A_\nu$. Gauge covariant translations are a symmetry of the action (4.30). What is the Noether current for this symmetry? How is it related to the stress tensor (4.33)?*

4.2 Electromagnetic duality

The subject of electromagnetic duality has several interesting applications in supergravity theories. For example, the symmetry group of black hole solutions of matter-coupled supergravity theories generally contains duality transformations. We recommend that all readers study Sec. 4.2.1. However, because the applications of duality are somewhat advanced, the rest of the section can be omitted in the first reading of the book.

4.2.1 Dual tensors

We begin by discussing the duality property of second rank anti-symmetric tensors $H_{\mu\nu}$ in 4-dimensional Minkowski spacetime. We use the Levi-Civita tensor introduced in Sec. 3.1.3 to define the dual tensor

$$\tilde{H}^{\mu\nu} \equiv -\tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma} H_{\rho\sigma}\,. \tag{4.34}$$

In our conventions the dual tensor is imaginary. The indices of $\tilde{H}$ can be raised and lowered with the Minkowski[7] metric $\eta_{\mu\nu}$. It is also useful to define the linear combinations

$$H^{\pm}_{\mu\nu} = \tfrac{1}{2}\left(H_{\mu\nu} \pm \tilde{H}_{\mu\nu}\right), \qquad H^{\pm}_{\mu\nu} = \left(H^{\mp}_{\mu\nu}\right)^*\,. \tag{4.35}$$

Ex. 4.5 *Prove that the dual of the dual is the identity, specifically that*

$$-\tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\tilde{H}_{\rho\sigma} = H^{\mu\nu}\,. \tag{4.36}$$

You will need (3.8). The validity of this property is the reason for the i *in the definition (4.34).*

Show that $H^+_{\mu\nu}$ and $H^-_{\mu\nu}$ are, respectively, self-dual and anti-self-dual, i.e.

$$-\tfrac{1}{2}\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma} H^{\pm}_{\rho\sigma} = \pm H^{\pm}_{\mu\nu}\,. \tag{4.37}$$

Let $G_{\mu\nu}$ be another anti-symmetric tensor with $G^{\pm}_{\mu\nu}$ defined as in (4.35). Prove the following relations (where $(\mu\nu)$ means symmetrization between the indices)

$$G^{+\mu\nu}H^-{}_{\mu\nu} = 0\,, \qquad G^{\pm\rho(\mu}H^{\pm\nu)}{}_\rho = -\tfrac{1}{4}\eta^{\mu\nu}G^{\pm\rho\sigma}H^{\pm}{}_{\rho\sigma}\,, \qquad G^+{}_{\rho[\mu}H^-{}_{\nu]}{}^\rho = 0\,. \tag{4.38}$$

[7] The definition (4.34) is valid in Minkowski space, but must be modified in curved spacetimes as we will discuss in Ch. 6.

Ex. 4.6 *The duality operation can also be applied to matrices of the Clifford algebra. Define the quantity* $L_{\mu\nu} = \gamma_{\mu\nu} P_L$. *Show that this is anti-self-dual. Hint: check first that* $\gamma_{\mu\nu}\gamma_* = \frac{1}{2}\mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}$.

4.2.2 Duality for one free electromagnetic field

Duality operates as an interesting symmetry of field theories containing one or more abelian gauge fields which may interact with other fields, principally scalars. In this section we discuss the simplest case, namely a single free gauge field. First note that, after contraction with the ε-tensor, the Bianchi identity (4.11) can be expressed as $\partial_\mu \tilde{F}^{\mu\nu} = 0$. So we can temporarily ignore the vector potential and regard $F_{\mu\nu}$ as the basic field variable which must satisfy both the Maxwell and Bianchi equations:

$$\partial_\mu F^{\mu\nu} = 0\,, \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0\,. \tag{4.39}$$

We can now consider the change of variables (the i is included to make the transformation real):

$$F^{\mu\nu} \to F'^{\mu\nu} = \mathrm{i}\tilde{F}^{\mu\nu}\,. \tag{4.40}$$

Since $F'^{\mu\nu}$ also obeys both equations of (4.39) we have defined a symmetry of the free electromagnetic field.

Ex. 4.7 *Show that the symmetry (4.40) exchanges the electric and magnetic fields:* $E_i \to E'_i = -B_i$, *and* $B_i \to B'_i = E_i$.

It is not useful to extend the symmetry to the vector potentials $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$ because A_μ and A'_μ are not related by any *local* transformation.

Here are some basic exercises involving the duality transform of the field strength tensor $F_{\mu\nu}$.

Ex. 4.8 *Show that the self-dual combinations* $F^{\pm}_{\mu\nu}$ *contain only photons of one polarization in their plane wave expansions:*

$$F^{\pm}_{\mu\nu} = 2\mathrm{i} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k^0} \left[\mathrm{e}^{\mathrm{i}k\cdot x} k_{[\mu}\epsilon_{\nu]}(\vec{k},\pm) a(\vec{k},\pm) - \mathrm{e}^{-\mathrm{i}k\cdot x} k_{[\mu}\epsilon^*_{\nu]}(\vec{k},\mp) a^*(\vec{k},\mp) \right]. \tag{4.41}$$

To perform this exercise, check first that with the polarization vectors given in Sec. 4.1.2, one has

$$-\tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma} k_\rho \epsilon_\sigma(\vec{k},\pm) = \pm k^{[\mu}\epsilon^{\nu]}(\vec{k},\pm)\,. \tag{4.42}$$

Ex. 4.9 *Show that the quantity* $F_{\mu\nu}\tilde{F}^{\mu\nu}$ *is a total derivative, i.e.*

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -\mathrm{i}\partial_\mu (\varepsilon^{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma})\,. \tag{4.43}$$

Show, using (1.45), that under a Lorentz transformation

$$\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)(x) \to \det\Lambda^{-1}\,\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)(\Lambda x)\,. \tag{4.44}$$

Thus $F_{\mu\nu}\tilde{F}^{\mu\nu}$ transforms as a scalar under proper Lorentz transformations but changes sign under space or time reflections. Use the Schouten identity (3.10) to prove that

$$F_{\mu\rho}\tilde{F}_{\nu}{}^{\rho} = \tfrac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}\tilde{F}^{\rho\sigma}\,. \tag{4.45}$$

4.2.3 Duality for gauge field and complex scalar

The simplest case of electromagnetic duality in an interacting field theory occurs with one abelian gauge field $A_\mu(x)$ and a complex scalar field $Z(x)$. The electromagnetic part of the Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}(\operatorname{Re}Z)F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{8}(\operatorname{Im}Z)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,. \tag{4.46}$$

Actions in which the gauge field kinetic term is multiplied by a function of complex scalar fields are quite common in supersymmetry and supergravity. In this theory we can define an extension of the duality transformation (4.40) which gives a non-abelian global $\mathrm{SL}(2,\mathbb{R})$ symmetry of the gauge field equations of this theory. Later we will add a generalized scalar kinetic term to the action (4.46) and extend the symmetry to include the scalar equation of motion.

The gauge Bianchi identity and equation of motion of our theory are

$$\partial_\mu\tilde{F}^{\mu\nu} = 0\,, \qquad \partial_\mu\left[(\operatorname{Re}Z)\,F^{\mu\nu} - \mathrm{i}(\operatorname{Im}Z)\,\tilde{F}^{\mu\nu}\right] = 0\,. \tag{4.47}$$

It is convenient to define the tensor

$$G^{\mu\nu} \equiv \varepsilon^{\mu\nu\rho\sigma}\frac{\delta S}{\delta F^{\rho\sigma}} = -\mathrm{i}(\operatorname{Re}Z)\tilde{F}^{\mu\nu} - (\operatorname{Im}Z)F^{\mu\nu}\,, \tag{4.48}$$

and to consider the self-dual combinations $F^{\mu\nu\,\pm}$ and $G^{\mu\nu\,\pm}$. Note that these are related by

$$G^{\mu\nu-} = \mathrm{i}ZF^{\mu\nu-}\,, \qquad G^{\mu\nu+} = -\mathrm{i}\bar{Z}F^{\mu\nu+}\,. \tag{4.49}$$

The information in (4.47) can then be reexpressed as

$$\partial_\mu\operatorname{Im}F^{\mu\nu\,-} = 0\,, \qquad \partial_\mu\operatorname{Im}G^{\mu\nu\,-} = 0\,. \tag{4.50}$$

We define a matrix of the group $\mathrm{SL}(2,\mathbb{R})$ by

$$\mathcal{S} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1\,. \tag{4.51}$$

The group $\mathrm{SL}(2,\mathbb{R})$ acts on the tensors F^- and G^- as follows:

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^- \\ G^- \end{pmatrix} . \tag{4.52}$$

Since $\mathcal{S}$ is real, the conjugate tensors F^+ and G^+ also transform in the same way.

Ex. 4.10 *Assume that* $\operatorname{Im} F^-$ *and* $\operatorname{Im} G^-$ *satisfy (4.50), and show that* $\operatorname{Im} F'^-$ *and* $\operatorname{Im} G'^-$ *also obey the same equations. Show that* G'^- *and a transformed scalar* Z' *satisfy* $G'^{\mu\nu\,-} = \mathrm{i}Z'F'^{\mu\nu\,-}$*, if* Z' *is defined as the following nonlinear transform of* Z*:*

$$\mathrm{i}Z' = \frac{c + \mathrm{i}dZ}{a + \mathrm{i}bZ} . \tag{4.53}$$

The two equations (4.52) and (4.53) specify the $\mathrm{SL}(2,\mathbb{R})$ duality transformation on the field strength and complex scalar of our system. The exercise shows that the Bianchi identity and generalized Maxwell equations are duality invariant. In general the duality transform is *not* a symmetry of the Lagrangian or the action integral. The following exercise illustrates this.

Ex. 4.11 *Show that the Lagrangian (4.46) can be rewritten as*

$$\mathcal{L}(F,Z) \;=\; -\tfrac{1}{2}\operatorname{Re}(ZF^-_{\mu\nu}F^{\mu\nu\,-}) . \tag{4.54}$$

Consider the $\mathrm{SL}(2,\mathbb{R})$ *transformation with parameters* $a = d = 1$ *and* $c = 0$*. Show that*

$$\mathcal{L}(F',Z') = -\tfrac{1}{2}\operatorname{Re}\left(Z(1 + \mathrm{i}b\,Z)F^-_{\mu\nu}F^{\mu\nu-}\right) \neq \mathcal{L}(F,Z) . \tag{4.55}$$

The symmetric gauge invariant stress tensor of this theory is

$$\Theta^{\mu\nu} = (\operatorname{Re} Z)\left(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) . \tag{4.56}$$

As we will see in Ch. 7, when the theory is coupled to gravity, it is this stress tensor which is the source of the gravitational field, see (7.4). It is an important fact that the stress tensor is *invariant* under the duality transformations (4.52) and (4.53). This is the reason for the duality symmetry of many black hole solutions of supergravity,

Ex. 4.12 *Prove that the energy-momentum tensor (4.56) is invariant under duality. Here are some helpful relations which you will need:*

$$\operatorname{Re} Z' \;=\; \frac{\operatorname{Re} Z}{(a + \mathrm{i}bZ)(a - \mathrm{i}b\bar{Z})} . \tag{4.57}$$

Further you need again (4.45) and a similar identity (proven by contracting ε *tensors)*

$$\tilde{F}_{\mu\rho}\tilde{F}_{\nu}{}^{\rho} = -F_{\mu\rho}F_{\nu}{}^{\rho} + \tfrac{1}{2}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} . \tag{4.58}$$

This leads to

$$F'_{\mu\rho}F'^{\ \rho}_{\nu} - \tfrac{1}{4}\eta_{\mu\nu}F'_{\rho\sigma}F'^{\rho\sigma} = |a+\mathrm{i}b|^2\left[F_{\mu\rho}F_{\nu}{}^{\rho} - \tfrac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right]. \tag{4.59}$$

When the $\mathrm{SL}(2,\mathbb{R})$ duality transformation appears in supergravity, an additional term appears in the Lagrangian which introduces a dynamical role for the scalar field $Z(x)$. This term must be duality invariant, i.e. invariant under (4.53). The prototype for this term is the non-linear σ-model whose target space is the Poincaré plane. This model and its $\mathrm{SL}(2,\mathbb{R})$ symmetry group will be discussed in Sec. 6.11, see (6.137) and (6.138). The scalar field Z of this chapter and that of Sec. 6.11 are related by $Z = -\mathrm{i}Z_{\mathrm{Ch.6}}$. It is then important to note that the Poincaré plane is the *upper half-plane* in the variable $Z_{\mathrm{Ch.6}}$. Thus $\mathrm{Re}\,Z = \mathrm{Im}\,Z_{\mathrm{Ch.6}} > 0$. The positive sign is preserved by $\mathrm{SL}(2,\mathbb{R})$ transformations. This shows that the energy density obtained from the stress tensor $\Theta_{\mu\nu}$ above will be *positive* when the dynamics of the scalar field is appropriately specified!

In many applications of electromagnetic duality, magnetic and electric charges appear as sources for the Bianchi 'identity' and generalized Maxwell equations of (4.47). There is then a vector of charges $\begin{pmatrix} p \\ -q \end{pmatrix}$, which transforms under $\mathrm{SL}(2,\mathbb{R})$ in the same way as the tensors F^- and G^- in (4.52). Particles that carry both electric and magnetic charge are called dyons. In quantum mechanics, dyon charges must obey the Schwinger-Zwanziger quantization condition. If a theory contains two dyons with charges (p_1,q_1) and (p_2,q_2), these charges must satisfy $p_1q_2 - p_2q_1 = 2\pi n$, where n is an integer.[8] This condition is invariant under $\mathrm{SL}(2,\mathbb{R})$ transformations of the charges. However, one can show [32] that there is a lowest non-zero value of the electric charge and that all allowed charges are restricted to an infinite discrete set of points called the charge lattice. The allowed $\mathrm{SL}(2)$ transformations must take one lattice point to another, and this restricts the group parameters in (4.51) to be integers. This restriction defines the subgroup $\mathrm{SL}(2,\mathbb{Z})$, often called the modular group. One can show that this subgroup is generated by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$Z' = Z - \mathrm{i} \quad , \quad Z' = \frac{1}{Z}. \tag{4.60}$$

This means that one can express any element of $\mathrm{SL}(2,\mathbb{Z})$ as products of (finitely many) factors of the two generators above and their inverses.

Ex. 4.13 *Suppose that Z is replaced by a real coupling constant $1/g^2$ rather than a field. Then the first transformation of (4.60) does not preserve this reality restriction. However, the second one does. Prove that this transformation interchanges*

[8] For the case $(p_1,q_1) = (p,0)$ and $(p_2,q_2) = (0,q)$, this reduces to condition $pq = 2\pi n$ found by Dirac in 1933.

the electric and magnetic fields in the simplest electromagnetic theory. It transforms g to its inverse, and thus relates the strong and weak coupling descriptions of the theory.

In the sections 4.1 and 4.2.2 we considered $Z = g = 1$. Check that general duality transformations in this case are of the form

$$F'^{-}_{\mu\nu} = (a + \mathrm{i}b)F^{-}_{\mu\nu}\,, \qquad i.e. \qquad F'_{\mu\nu} = aF_{\mu\nu} - \mathrm{i}b\tilde{F}_{\mu\nu}\,. \tag{4.61}$$

4.2.4 Electromagnetic duality for coupled Maxwell fields

In this section we explore how the duality symmetry is extended to systems containing a set of Abelian gauge fields $A^A_\mu(x)$, indexed by $A = 1, 2, \ldots, m$ together with scalar fields ϕ^i. Scalars enter the theory through complex functions $f_{AB}(\phi) = f_{BA}(\phi)$. We consider the action

$$S = \int \mathrm{d}^4x\,\mathcal{L}\,, \qquad \mathcal{L} = -\tfrac{1}{4}(\mathrm{Re}\,f_{AB})F^A_{\mu\nu}F^{\mu\nu\,B} + \tfrac{1}{4}\mathrm{i}(\mathrm{Im}\,f_{AB})F^A_{\mu\nu}\tilde{F}^{\mu\nu\,B}\,, \tag{4.62}$$

which is real since $\tilde{F}^{\mu\nu}$ is pure imaginary, as defined in (4.34). The first term is a generalized kinetic Lagrangian for the gauge fields, so we usually require that $\mathrm{Re}\,f_{AB}$ is a positive definite matrix. This ensures that gauge field kinetic energies are positive. Although $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative, the second term does contribute to the equations of motion when $\mathrm{Im}\,f_{AB}$ is a function of the scalars ϕ^i. Our discussion will not involve the scalars directly. However, as in Sec. 4.2.3, additional terms to specify the scalar dynamics will appear when theories of this type are encountered in extended $D = 4$ supergravity. The treatment which now follows is closely modelled on Sec. 4.2.3.

Using the self-dual tensors of (4.35), we then rewrite the Lagrangian (4.62) as

$$\begin{aligned}\mathcal{L}(F^+, F^-) &= -\tfrac{1}{2}\,\mathrm{Re}\left(f_{AB}F^{-\,A}_{\mu\nu}F^{\mu\nu\,-\,B}\right)\\ &= -\tfrac{1}{4}\left(f_{AB}F^{-\,A}_{\mu\nu}F^{\mu\nu\,-\,B} + f^*_{AB}F^{+\,A}_{\mu\nu}F^{\mu\nu\,+\,B}\right)\,,\end{aligned} \tag{4.63}$$

and define the new tensors

$$\begin{aligned}G^{\mu\nu}_A &= \varepsilon^{\mu\nu\rho\sigma}\frac{\delta S}{\delta F^{\rho\sigma\,A}} = -(\mathrm{Im}\,f_{AB})F^{\mu\nu\,B} - \mathrm{i}(\mathrm{Re}\,f_{AB})\tilde{F}^{\mu\nu\,B} = G^{\mu\nu\,+}_A + G^{\mu\nu\,-}_A\,,\\ G^{\mu\nu\,-}_A &= -2\mathrm{i}\frac{\delta S(F^+, F^-)}{\delta F^{-\,A}_{\mu\nu}} = \mathrm{i}f_{AB}F^{\mu\nu\,-\,B}\,,\\ G^{\mu\nu\,+}_A &= 2\mathrm{i}\frac{\delta S(F^+, F^-)}{\delta F^{+\,A}_{\mu\nu}} = -\mathrm{i}f^*_{AB}F^{\mu\nu\,+\,B}\,.\end{aligned} \tag{4.64}$$

Since the field equation for the action containing (4.63) is

$$0 = \frac{\delta S}{\delta A^A_\nu} = -2\partial_\mu\frac{\delta S}{\delta F^A_{\mu\nu}}\,, \tag{4.65}$$

the Bianchi identity and the equation of motion can be expressed in the concise form

$$\begin{aligned} \partial^\mu \operatorname{Im} F^{A\,-}_{\mu\nu} &= 0 \quad : \quad \text{Bianchi identities,} \\ \partial_\mu \operatorname{Im} G_A^{\mu\nu\,-} &= 0 \quad : \quad \text{Equations of motion.} \end{aligned} \tag{4.66}$$

(The same equations hold for $\operatorname{Im} F^{A\,+}$ and $\operatorname{Im} G_A{}^+$.)

Duality transformations are linear transformations of the $2m$ tensors $F^{A\,\mu\nu}$ and $G_A^{\mu\nu}$ (accompanied by transformations of the f_{AB}) which mix Bianchi identities and equations of motion, but preserve the structure which led to (4.66). Since the equations (4.66) are real, we can mix them by a real $2m \times 2m$ matrix. We extend these transformations to the (anti)self-dual tensors, and consider

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^- \\ G^- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^- \\ G^- \end{pmatrix}, \tag{4.67}$$

with real $m \times m$ sub-matrices A, B, C, D. Due to the reality of these matrices, the same relations hold for the self-dual tensors F^+ and G^+.

We require that the transformed field tensors F'^A, G'_A are also related by the definitions (4.64), with appropriately transformed f_{AB}. We work out this requirement in the following steps:

$$\begin{aligned} & G'^- = (C + \mathrm{i}Df)F^- = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1} F'^- \\ \rightarrow\ & \boxed{\mathrm{i}f' = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}} \quad . \end{aligned} \tag{4.68}$$

The last equation gives the symmetry transformation relating f'_{AB} to f_{AB}. If $G'^-_{\mu\nu}$ is to be the variational derivative of a transformed action, as (4.64) requires, then the matrix f' must be symmetric. For a generic[9] symmetric f, this requires that the matrices A, B, C, D satisfy

$$A^T C - C^T A = 0\,, \qquad B^T D - D^T B = 0\,, \qquad A^T D - C^T B = \mathbb{1}\,. \tag{4.69}$$

These relations among A, B, C, D are the defining conditions of a matrix of the symplectic group in dimension $2m$ so we reach the conclusion that

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2m, \mathbb{R})\,. \tag{4.70}$$

The conditions (4.69) may be summarized as

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \qquad \text{where} \qquad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{4.71}$$

[9] If the initial f_{AB} is non-generic, then the matrix $\mathbb{1}$ in the last equation can be replaced by any matrix which commutes with f_{AB}. For generic f_{AB}, this must be a constant multiple of the unit matrix. The constant, which should be positive to preserve the sign of the kinetic energy of the vectors, can be absorbed by rescaling the matrices A, B, C, D.

The matrix Ω is often called the symplectic metric, and the transformations (4.67) are then called symplectic transformations. This is the main result originally derived in [33]. Duality transformations in 4 spacetime dimensions are transformations of the group $\mathrm{Sp}(2m,\mathbb{R})$, which is a non-compact group.

Ex. 4.14 *The dimension of the group* $\mathrm{Sp}(2m,\mathbb{R})$ *is the number of elements of the matrix* $\mathcal{S}$, *namely* $4m^2$ *minus the number of independent conditions contained in (4.71). Show that the dimension is* $m(2m+1)$.

Duality transformations have two types of applications: they can describe symmetries of one theory and they can describe transformations from one theory to another one. In the first case, this concerns then symmetries of the theory that are embedded in the group $\mathrm{Sp}(2m,\mathbb{R})$. The transformations (4.68) of $f_{AB}(\phi^i)$ are then determined by the symmetry transformations of the elementary scalars ϕ^i, which must also be symmetries of other parts of the Lagrangian. This is e.g. the case in the example of Sec. 4.2.3, where the transformation of Z as given in (4.53), could be part of a symmetry group of the kinetic terms of that field. In extended supergravities it turns out that all the symmetry transformations that act on the scalars are of this type. In this framework the duality group of the theory is a subgroup of the 'maximal' group $\mathrm{Sp}(2m,\mathbb{R})$ discussed above.

However, another application is of the type as we encountered in Ex. 4.13. In that case constants that specify the theory under consideration change under the duality transformations. These constants that transform are sometimes called 'spurionic quantities'. The transformations thus relate two different theories. Solutions of one theory are mapped in solutions of the other one. This is the basic idea of dualities in M-theory. The use of duality transformations in this sense is the basis of the 'embedding tensor formalism', which provides a framework to discuss general gauge theories [34, 35].

Symplectic transformations always transform solutions of (4.66) into other solutions. However, they are not always invariances of the action. Indeed, writing

$$\mathcal{L} = -\tfrac{1}{2}\,\mathrm{Re}\left(f_{AB}F_{\mu\nu}^{-\,A}F^{\mu\nu\,-\,B}\right) = -\tfrac{1}{2}\,\mathrm{Im}\left(F_{\mu\nu}^{-\,A}G_A^{\mu\nu\,-}\right)\,, \tag{4.72}$$

we obtain

$$\mathrm{Im}\,F'^{-}G'^{-} = \mathrm{Im}\left(F^-G^-\right) + \mathrm{Im}\left[2F^-(C^TB)G^- + F^-(C^TA)F^- + G^-(D^TB)G^-\right]\,. \tag{4.73}$$

If $C\neq 0, B=0$ the Lagrangian is invariant up to a four-divergence, since $\mathrm{Im}\,F^-F^- = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ and the matrices A and C are real constants. For $B\neq 0$ neither the Lagrangian nor the action is invariant.

Electromagnetic duality has important applications to black hole solutions of extended supergravity theories. Supergravity is also very relevant to the analysis of black hole solutions of string theory. Many black holes are dyons; they carry both magnetic and electric charges. The general situation is a generalization of what was discussed at the end of Sec. 4.2.3. The charges form a symplectic vector $\begin{pmatrix} q_m^A \\ -q_{e\,A} \end{pmatrix}$ which must transform as in (4.67). The Dirac-Schwinger-Zwanziger quantization condition restricts these charges to a lattice. Invariance of this lattice restricts the symplectic transformations of (4.67) to a discrete subgroup $\mathrm{Sp}(2m, \mathbb{Z})$, which is analogous to the $\mathrm{SL}(2, \mathbb{Z})$ group discussed previously.

Finally, we comment that symplectic transformations with $B \neq 0$ should be considered as non–perturbative for the following reasons. A system with no magnetic charges as in classical electromagnetism is transformed to a system with magnetic charges. The elements of f_{AB} may be regarded as coupling constants (see Ex. 4.13), and a system with weak coupling is transformed to one with strong coupling. A duality transformation which mixes electric and magnetic fields cannot be realized by transformation of the vector potential A_μ. One would need a 'magnetic' partner of A_μ to re-express the $F'_{\mu\nu}$ and $G'_{\mu\nu}$ in terms of potentials.

The important properties of the matrix f_{AB} are that it is symmetric and that $\mathrm{Re}\, f_{AB}$ define a positive definite quadratic form in order to have positive gauge field energy. These properties are preserved under symplectic transformations defined by (4.68).

4.3 Non-abelian Gauge Symmetry

Yang-Mills theory is based on a non-abelian generalisation of the U(1) gauge symmetry. It is the fundamental idea underlying the standard model of elementary particle interactions. We follow the pattern of Sec. 4.1.1, starting with the global symmetry and then gauging it. The focus of our discussion is the derivation of the basic formulas of the classical gauge theory. Readers may need more information on the underlying geometric ideas and the structure and stunning applications of the quantized theory. They are referred to a modern text in quantum field theory.[10]

4.3.1 Global internal symmetry

Suppose that G is a compact, simple Lie group of dimension $\dim_G$. Closely associated with the group is its Lie algebra, denoted by $\mathfrak{g}$, which is a real algebra of dimension $\dim_G$. The theory of Lie algebras and Lie groups is an important subject of mathematics with many applications to physics. With some oversimplification we review only the most essential features required by Yang-Mills theory for compact simple groups.

Each compact simple Lie algebra has an infinite number of inequivalent finite dimensional irreducible representations R of dimension $\dim_R$. In each representation, there is a basis of matrix generators t_A, $A = 1, \ldots, \dim_G$, which are anti-hermitian

[10] See, for example, Chapter 15 of [15]. This text also reviews aspects of group theory needed in physical applications.

in the case of a compact gauge group. The commutator of the generators determines the local geometrical structure of the group:

$$[t_A, t_B] = f_{AB}{}^C t_C \,. \tag{4.74}$$

The array of real numbers $f_{AB}{}^C$ are structure constants of the algebra (the same in all representations). They obey the Jacobi identity

$$f_{AD}{}^E f_{BC}{}^D + f_{BD}{}^E f_{CA}{}^D + f_{CD}{}^E f_{AB}{}^D = 0 \,. \tag{4.75}$$

The indices can be lowered by the Cartan-Killing metric, defined in the Appendix B, see (B.5) , and then the f_{ABC} are totally antisymmetric. For simple algebras, the generators can be chosen to be trace orthogonal, $\mathrm{Tr}(t_A t_B) = -c\delta_{AB}$, with c positive for compact groups, and the Cartan-Killing metric is then proportional to this expression.

One important representation is the adjoint representation of dimension $\dim_R = \dim_G$, in which the representation matrices are closely related to the structure constants by $(t_A)^D{}_E = f_{AE}{}^D$. Note that the labels DE denote row and column indices of the matrix t_A. The adjoint representation is a real representation; the representation matrices are real and antisymmetric for compact algebras. For complex representations we will use the notation $(t_A)^\alpha{}_\beta$. Anti-hermiticity then requires $(t^*_A)^\alpha{}_\beta = -(t_A)^\beta{}_\alpha$. The row and column indices will often be suppressed when no ambiguity arises.

Ex. 4.15 *Use (4.75) to show that the matrices $(t_A)^D{}_E = f_{AE}{}^D$ satisfy (4.74) and therefore give a representation.*

The general element of $\mathfrak{g}$ is represented by a superposition of generators $\theta^A t_A$ where the θ^A are $\dim_G$ real parameters. The relation between G and $\mathfrak{g}$ is given by exponentiation, namely $\mathrm{e}^{-\theta^A t_A}$ is an element of G in the representation R.

A theory with global non-abelian internal symmetry contains scalar and spinor fields, each of which transforms in an irreducible representation R. For example, there may be a Dirac spinor[11] field $\Psi^\alpha(x)$, $\alpha = 1, \ldots, \dim_R$ that transforms in the complex representation R as

$$\Psi^\alpha(x) \to (\mathrm{e}^{-\theta^A t_A})^\alpha{}_\beta \Psi^\beta(x) \,, \tag{4.76}$$

The conjugate spinor[12] is denoted by $\bar\Psi_\alpha$ and transforms as

$$\bar\Psi_\alpha \to \bar\Psi_\beta (\mathrm{e}^{\theta^A t_A})^\beta{}_\alpha \,. \tag{4.77}$$

[11] Note that we use here indices $\alpha, \ldots$ for the representation of the gauge group. They should not be confused with spinor indices, which we usually omit.

[12] We use here the Dirac conjugate (2.42) rather than the Majorana conjugate.

For most of our discussion it is sufficient to restrict attention to the infinitesimal transformations,

$$\begin{aligned} \delta\Psi &= -\theta^A t_A \Psi\,, \\ \delta\bar{\Psi} &= \bar{\Psi}\theta^A t_A\,, \\ \delta\phi^A &= \theta^C f_{BC}{}^A \phi^B\,. \end{aligned} \tag{4.78}$$

The first two relations are just the terms of (4.76),(4.77) that are first order in θ^A. The last relation is the infinitesimal transformation of a field in the adjoint representation, taken here as the set of $\dim_G$ real scalars ϕ^A. Of course, scalars could be assigned to any representation R.

Actions, such as the kinetic action for massive fermion fields,

$$S[\bar{\Psi},\Psi] = -\int \mathrm{d}^D x \bar{\Psi}[\gamma^\mu\partial_\mu - m]\Psi\,, \tag{4.79}$$

are required to be invariant under (4.76).

Ex. 4.16 *Show that (4.79) is invariant under the transformation (4.76) and (4.77). Consider an infinitesimal transformation and derive the conserved current*

$$J_{A\mu} = -\bar{\Psi} t_A \gamma_\mu \Psi\,, \qquad A = 1,\ldots,\dim_G\,. \tag{4.80}$$

Show that the current transforms as a field in the adjoint representation, i.e.

$$\delta J_{A\mu} = \theta^C f_{CA}{}^B J_{B\mu}\,. \tag{4.81}$$

4.3.2 Gauging the symmetry

In gauged non-abelian internal symmetry, the group parameter $\theta^A(x)$ is promoted to an arbitrary function of x^μ. The first step in the systematic formulation of gauge invariant field equations is to introduce the gauge potentials, namely a set of vectors $A_\mu^A(x)$ whose infinitesimal transformation rule is

$$\delta A_\mu^A(x) = \frac{1}{g}\partial_\mu\theta^A + \theta^C(x)A_\mu^B(x) f_{BC}{}^A\,. \tag{4.82}$$

The first term is the gradient term similar to that for the abelian gauge field in (4.2), and the second is exactly the transformation of a field in the adjoint representation, as one can see from the third equation in (4.78). The constant g is the Yang-Mills coupling, which replaces the electromagnetic coupling e of Sec. 4.1.4.

Following the pattern of Sec. 4.1.1, we next define the covariant derivative of a field in the representation R with matrix generators t_A. For the fields Ψ^α, $\bar{\Psi}_\alpha$, and ϕ^A of (4.78) we write

$$\begin{aligned} D_\mu\Psi &= (\partial_\mu + g t_A A_\mu^A)\Psi\,, \\ D_\mu\bar{\Psi} &= \partial_\mu\bar{\Psi} - g\bar{\Psi} t_A A_\mu^A\,, \\ D_\mu\phi^A &= \partial_\mu\phi^A + g f_{BC}{}^A A_\mu^B \phi^C\,. \end{aligned} \tag{4.83}$$

Note that the gauge transformation (4.82) can be written as $\delta A^A_\mu(x) = \frac{1}{g} D_\mu \theta^A$ using the covariant derivative for the adjoint representation.

Ex. 4.17 *Show that the covariant derivatives of the three fields in (4.83) transform in the same way as the fields themselves, and with no derivatives of the gauge parameters. For example* $\delta D_\mu \Psi = -\theta^A t_A D_\mu \Psi$.

Given this result it is easy to see that any globally symmetric action for scalar and spinor matter fields becomes gauge invariant if one replaces $\partial_\mu \to D_\mu$ for all fields. If this is done in (4.79), one obtains the equation of motion

$$\frac{\delta S}{\delta \bar{\Psi}_\alpha} = -[\gamma^\mu D_\mu - m]\Psi^\alpha = 0\,. \tag{4.84}$$

4.3.3 Yang-Mills field strength and action

The next step in the development is to define the quantities that determine the dynamics of the gauge field itself. The simplest way to proceed is to compute the commutator of two covariant derivatives acting on a field in the representation R. We would get the same information, no matter which representation, so we will study just the case $[D_\mu, D_\nu]\Psi \equiv (D_\mu D_\nu - D_\nu D_\mu)\Psi$. A careful computation gives

$$[D_\mu, D_\nu]\Psi = g F^A_{\mu\nu} t_A \Psi\,, \tag{4.85}$$

where

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g f_{BC}{}^A A^B_\mu A^C_\nu\,. \tag{4.86}$$

The properties of the covariant derivative guarantee that the right side of (4.85) transforms as a field in the same representation as Ψ. Thus $F^A_{\mu\nu}$ should have simple transformation properties. Indeed, one can derive

$$\delta F^A_{\mu\nu} = \theta^C F^B_{\mu\nu} f_{BC}{}^A\,. \tag{4.87}$$

We see that $F^A_{\mu\nu}$ is an anti-symmetric tensor in spacetime, which transforms as a field in the adjoint representation of $\mathfrak{g}$. $F^A_{\mu\nu}$ is the non-abelian generalization of the electromagnetic field strength (4.6). The principle differences between abelian and non-abelian gauge symmetry are that the non-abelian field strength is not gauge invariant, but transforms in the adjoint representation and that it is *nonlinear* in the gauge potential A^A_μ.

Ex. 4.18 *Derive (4.87).*

Despite these significant differences, it is quite straightforward to formulate the Yang-Mills equations by following the ideas of the electromagnetic case. Since both the current and field strength transform in the adjoint representation, and the covariant derivative does not change the transformation properties, the equation

$$D^\mu F^A_{\mu\nu} = -J^A_\nu \tag{4.88}$$

is both gauge and Lorentz covariant. It is the basic dynamical equation of classical Yang-Mills theory and the analogue of (4.14) for electromagnetism. One important difference, however, is that in the absence of matter sources, when the right side of (4.88) vanishes, that equation is still a (much studied!) non-linear equation for A^A_μ.

There is also a non-abelian analogue of the Bianchi identity (4.11), which takes the form

$$D_\mu F^A_{\nu\rho} + D_\nu F^A_{\rho\mu} + D_\rho F^A_{\mu\nu} = 0\,, \tag{4.89}$$

where $D_\mu F^A_{\nu\rho} = \partial_\mu F^A_{\nu\rho} + g f_{BC}{}^A A^B_\mu F^A_{\nu\rho}$.

Ex. 4.19 *Show that (4.89) is satisfied identically if $F^A_{\nu\rho}$ is written in the form (4.86).*

Ex. 4.20 *Show that $D^\nu D^\mu F^A_{\mu\nu}$ vanishes* identically *(despite the non-linearity). This is again a Noether identity: a relation between field equations that follows from the gauge symmetry.*

As in the electromagnetic case, this means that the equation of motion (4.88) is consistent only if the current is covariantly conserved, i.e. only if $D^\nu J^A_\nu = 0$. It also means that (4.88) contains $(D-1)\mathrm{dim}_G$ independent equations, which is enough to determine the $D\mathrm{dim}_G$ components of A^A_μ up to a gauge transformation. It is usually convenient to "fix the gauge" by specifying dim_G conditions on the components of A^A_μ.

Note that in the limit $g \to 0$, the equations (4.86), (4.88), (4.89) reduce to linear equations, which are dim_G copies of the corresponding equations for the free electromagnetic field. The count of degrees of freedom of Sec. 4.1.2 can be repeated in the Coulomb gauge $\partial^i A^A_i(\vec{x},t) = 0$. For each component $A = 1, \ldots, dim_G$, $2(D-2)$ functions must be specified as initial data, and each $A^A_i(x)$ has a Fourier transform identical to (4.9). In this free limit, the gauge field thus describes a particle with $D-2$ polarization states transforming in the adjoint representation of $\mathfrak{g}$.

The equations of motion of the Yang-Mills field A^A coupled to the Dirac field Ψ^α can be obtained from an action functional that is a natural generalization of (4.30)

$$S[A^A_\mu, \bar\Psi_\alpha, \Psi^\alpha] = \int \mathrm{d}^D x \,\left[-\tfrac{1}{4} F^{A\mu\nu} F^A_{\mu\nu} - \bar\Psi_\alpha(\gamma^\mu D_\mu - m)\Psi^\alpha \right]\,. \tag{4.90}$$

The action is gauge invariant. The Euler variation with respect to A^A_ν gives (4.88) with current source (4.80), and the variation with respect to $\bar\Psi_\alpha$ gives (4.84).

4.3.4 Yang-Mills theory for $G = \mathrm{SU}(N)$

The most commonly studied gauge group for Yang-Mills theory is $\mathrm{SU}(N)$. The generators of the fundamental representation of its Lie algebra are a set of $N^2 - 1$ traceless anti-hermitian $N \times N$ matrices t_A, which are normalized by the bilinear trace relation

$$\mathrm{Tr}(t_A t_B) = -\frac{1}{2}\delta_{AB}\,. \tag{4.91}$$

In this section we discuss the special notation that has been developed for this case and is frequently used in the literature. In this notation gauge transformations are explicitly realized at the level of the group $\mathrm{SU}(N)$ rather than just at the level of its Lie algebra $\mathfrak{su}(N)$ as in the previous sections.

We will use the notation $U(x) = \mathrm{e}^{-\Theta(x)}$, with $\Theta(x) = \theta^A(x)t_A$, to denote an element of the gauge group in the fundamental representation. This may be viewed as a map $x^\mu \to U(x^\mu)$ from Minkowski spacetime into the group $\mathrm{SU}(N)$. In this notation the gauge transformation of a spinor field Ψ in the fundamental representation can be written (see (4.76))

$$\Psi(x) \to U(x)\Psi(x). \tag{4.92}$$

Row and column indices of the fundamental representation are consistently omitted in this notation. Usually we will omit the spacetime argument x^μ also, unless useful for special emphasis.

Given any matrix generator t_A, the unitary transformation $U(x)t_AU(x)^{-1}$ gives another traceless anti-hermitian matrix, which must then be a linear combination of the t_B. Therefore we can write

$$U(x)t_AU(x)^{-1} = t_B R(x)^B{}_A\,, \tag{4.93}$$

where $R(x)^B{}_A$ is a real $(N^2-1)\times(N^2-1)$ matrix.

Ex. 4.21 *Consider the product of two gauge group elements U_1 and U_2, which gives a third via $U_1U_2 = U_3$. For each element U_i, there is an associated matrix $(R(x)_i)^B{}_A$, defined by $U_it_AU_i^{-1} = t_B(R(x)_i)^B{}_A$. Prove that $(R_3)^B{}_A = (R_1)^B{}_C(R_2)^C{}_A$, which shows that the matrices $R^B{}_A$ defined by (4.93) are the matrices of an $(N^2-1)-$dimensional representation of* $\mathrm{SU}(N)$*. Use (4.93) to show that, to first order in the gauge parameters θ^C, $R^B{}_A = \delta^B_A + \theta^C f_{AC}{}^B + \ldots$. This shows that the matrices $R^B{}_A$ are exactly those of the adjoint representation.*[13]

Given any set of N^2-1 real quantities X^A, that is any element of the vector space $\mathbb{R}^{N^2-1}$, we can form the matrix $\mathbf{X} = t_AX^A$. For any group element U, we

[13] The equation (4.93) is true for the generators t_A of *any* representation of *any* Lie algebra $\mathfrak{g}$ and the associated group element $U = \mathrm{e}^{-\theta^C t_C}$. It follows that the matrices R^{BA} are those of the adjoint representation of G. A matrix description of Yang-Mills theory in this general context can then be constructed by following the procedure discussed below for the fundamental representation of $\mathrm{SU}(N)$.

have $U\mathbf{X}U^{-1} = t_B R^B{}_A X^A$. Thus the unitary transformation of the matrix $\mathbf{X}$, i.e. demanding that $\mathbf{X} \to U\mathbf{X}U^{-1}$, contains the information that the quantities $X^A = -2\delta^{AB}\,\mathrm{Tr}(t_B\mathbf{X})$ transform in the adjoint representation, that is as $X^A \to R^A{}_B X^B$. Thus, given any field in the adjoint representation, such as $\phi^A(x)$, we can form the matrix $\mathbf{\Phi}(x) = t_A\phi^A(x)$. Gauge transformations can then be implemented as

$$\mathbf{\Phi}(x) \to U(x)\mathbf{\Phi}(x)U(x)^{-1}. \tag{4.94}$$

One can also form the matrix $\mathbf{A}_\mu(x) = t_A A_\mu^A(x)$ for the gauge potential. Quite remarkably, the gauge transformation of the potential can be expressed in matrix form if we define the transformation by

$$\mathbf{A}_\mu(x) \to \mathbf{A}'_\mu(x) \equiv \frac{1}{g}U(x)\partial_\mu U(x)^{-1} + U(x)\mathbf{A}_\mu(x)U(x)^{-1}. \tag{4.95}$$

For infinitesimal transformations this becomes

$$\delta A_\mu(x) = \frac{1}{g}\partial_\mu\Theta(x) + \left[A_\mu(x), \Theta(x)\right], \tag{4.96}$$

which agrees with (4.82).

Ex. 4.22 *Suppose that $\mathbf{A}_\mu \to \mathbf{A}'_\mu$ by the gauge transformation $U_2(x)$ followed by $\mathbf{A}'_\mu \to \mathbf{A}''_\mu$ by the gauge transformation $U_1(x)$. Show that the combined transformation $\mathbf{A}_\mu \to \mathbf{A}''_\mu$ is correctly described by the definition (4.95) for the product matrix $U_2(x)U_1(x)$. This result is compatible with (1.23) and with the implementation of gauge transformations by unitary operators in the quantum theory.*

It is easy to define covariant derivatives in which the gauge potential appears in matrix form. For fields Ψ in the fundamental and $\bar\Psi$ in the anti-fundamental representation (and transforming as $\bar\Psi \to \bar\Psi U^{-1}$), the previous definitions in (4.83) can simply be rewritten as

$$\begin{aligned} D_\mu\Psi &\equiv (\partial_\mu + g\mathbf{A}_\mu)\Psi\,, \\ D_\mu\bar\Psi &\equiv \partial_\mu\bar\Psi - g\bar\Psi\mathbf{A}_\mu\,. \end{aligned} \tag{4.97}$$

For a field in the adjoint representation, such as $\mathbf{\Phi}$ we define

$$D_\mu\mathbf{\Phi} = \partial_\mu\mathbf{\Phi} + g[\mathbf{A}_\mu, \mathbf{\Phi}]\,, \tag{4.98}$$

which involves the matrix commutator.

Ex. 4.23 *Demonstrate that these covariant derivatives transform correctly, specifically that*

$$D_\mu\Psi \to U(x)D_\mu\Psi\,, \quad D_\mu\bar\Psi \to D_\mu\bar\Psi U(x)^{-1}\,, \quad D_\mu\mathbf{\Phi} \to U(x)D_\mu\mathbf{\Phi}U(x)^{-1}\,. \tag{4.99}$$

The non-abelian field strength can also be converted to matrix form as

$$\mathbf{F}_{\mu\nu} = t_A F^A_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g[\mathbf{A}_\mu, \mathbf{A}_\nu]\,. \tag{4.100}$$

We have used the general rule for quantities in the adjoint representation in the first equality and then expressed, using (4.6), the result in terms of $\mathbf{A}_\mu$ in the second equality.

The matrix formalism is a convenient way to express quantities of interest in the theory. For example the Yang-Mills action (4.90) can be written as

$$S[\mathbf{A}_\mu, \bar{\Psi}, \Psi] = \int \mathrm{d}^D x \left[\frac{1}{2}\mathrm{Tr}(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}) - \bar{\Psi}(\gamma^\mu \mathrm{D}_\mu - \mathrm{m})\Psi \right]. \tag{4.101}$$

The $N^2 - 1$ matrix generators $(t_A)^\alpha{}_\beta$ of the fundamental representation, normalized as in (4.91), together with the matrix $\mathrm{i}\delta^\alpha_\beta$ form a complete set of $N \times N$ anti-hermitian matrices, which are orthogonal in the trace norm. Therefore one can expand any $N \times N$ anti-hermitian matrix $H^\alpha{}_\beta$ in this set as

$$\begin{aligned} H^\alpha{}_\beta &= \mathrm{i}h_0 \delta^\alpha{}_\beta + h^A (t_A)^\alpha{}_\beta\,, \\ h_0 = -\frac{\mathrm{i}}{N}\,\mathrm{Tr}\,H \qquad &, \qquad h^A = -2\delta^{AB}\,\mathrm{Tr}(H t_B)\,. \end{aligned} \tag{4.102}$$

Note that there is a sum over the $N^2 - 1$ values of the repeated indices A, B in (4.102) and in the exercise below.

Ex. 4.24 *Use the completeness property (perhaps with Sec. 3.2.3 as a guide) to derive the rearrangement relation*

$$\delta^\alpha_\beta\,\delta^\gamma_\delta = \frac{1}{N}\delta^\alpha_\delta \delta^\gamma_\beta - 2(t_A)^\alpha{}_\delta \delta^{AB} (t_B)^\gamma{}_\beta\,. \tag{4.103}$$

4.4 Internal Symmetry for Majorana Spinors

Majorana spinors play a central role in supersymmetric field theories. In many applications they transform in a representation of a non-abelian internal symmetry group. For example, the spinor fields of super-Yang-Mills theory are denoted as λ^A and transform in the adjoint representation of the gauge group. In the notation of Sec. 4.3.4, we have $\lambda^A \to \lambda'^A = R^A{}_B \lambda^B$. Since the matrix $R^A{}_B$ is real, this transformation rule is consistent with the fact that Majorana spinors obey a reality constraint. Indeed, as shown in Sec. 3.A.6, there are really real representations of the Clifford algebra in which the spinors are explicitly real. One can consider the more general situation of a set of Majorana spinors Ψ^α transforming as $\Psi^\alpha \to \Psi'^\alpha = (\mathrm{e}^{-\theta^A t_A})^\alpha{}_\beta\, \Psi^\beta$. The transformed Ψ'^α must also satisfy the Majorana condition, and this requires that the matrices $\mathrm{e}^{-\theta^A t_A}$ are those of a really real representation of the group G. (Obviously there is a similar requirement on the symmetry transformation of a set of real scalars, such as the ϕ^A of Sec. 4.3.1).

In $D=4$ dimensions, the requirement that Majorana spinors transform in a real representation of the gauge group can be bypassed because internal symmetries can include chiral transformations, which involve the highest rank element $\gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$ of the Clifford algebra discussed in Sec. 3.1.6. This matrix is imaginary in a Majorana representation, or in general under the C-operation, see (3.75). We use the chiral projectors P_L and P_R as in (3.37). Suppose that the matrices t_A are generators of a complex representation of the Lie algebra. Then the complex conjugate matrices t_A^* are generators of the conjugate representation. Let χ^α denote a set of Majorana spinors to which we assign the group transformation rule

$$\chi^\alpha \to \chi'^\alpha \equiv (\mathrm{e}^{-\theta^A(t_A P_L + t_A^* P_R)})^\alpha{}_\beta \chi^\beta \,. \tag{4.104}$$

The matrices $t_A P_L + t_A^* P_R$ are generators of a representation of an explicitly real representation of the Lie algebra, so the transformed spinors χ'^α also satisfy the Majorana condition. This is the transformation rule used for Majorana spinors in supersymmetric gauge theories in Ch. 8.

By applying the projectors to (4.104), one can see that the chiral and anti-chiral components of χ^α transform as

$$\begin{aligned} P_L\chi^\alpha \quad &\to \quad P_L\chi'^\alpha \equiv (\mathrm{e}^{-\theta^A t_A})^\alpha{}_\beta P_L \chi^\beta \,, \\ P_R\chi^\alpha \quad &\to \quad P_R\chi'^\alpha \equiv (\mathrm{e}^{-\theta^A t_A^*})^\alpha{}_\beta P_R \chi^\beta \,. \end{aligned} \tag{4.105}$$

Ex. 4.25 *What is the covariant derivative $D_\mu\chi$? Show that the kinetic Lagrangian density $\bar\chi\gamma^\mu D_\mu\chi$ is invariant under the infinitesimal limit of the transformation (4.104) for anti-hermitian t_A, and that the variation of the mass term is*

$$\delta(\bar\chi\chi) = -\theta^A \bar\chi (t_A + t_A^T)\gamma_*\chi \,. \tag{4.106}$$

The mass term is invariant only for the subset of generators that are antisymmetric, and thus real. This condition defines a subalgebra of the original Lie algebra $\mathfrak{g}$ of the theory, specifically the subalgebra which contains only parity conserving vector-like gauge transformations. For the case $\mathfrak{g} = \mathfrak{su}(N)$, the subalgebra is isomorphic to $\mathfrak{so}(N)$. Non-invariance of the Majorana mass term is a special case of the general idea that chiral symmetry requires massless fermions.

5
The free Rarita-Schwinger field

In this chapter we begin to assemble the ingredients of supergravity by studying the free spin $\frac{3}{2}$ field. Supergravity is the gauge theory of global supersymmetry, which we will usually abbreviate as SUSY. The key feature is that the symmetry parameter of global SUSY transformations is a constant spinor ϵ_α. In supergravity it becomes a general function in spacetime, $\epsilon_\alpha(x)$. The associated gauge field is a vector-spinor $\Psi_{\mu\alpha}(x)$. This field and the corresponding particle have acquired the name 'gravitino'.

Supergravity theories necessarily contain the gauge multiplet, the set of fields required to gauge the symmetry in a consistent interacting theory, and may contain matter multiplets, sets of fields on which global SUSY is realized. The gauge multiplet contains the gravitational field, one or more vector-spinors, and sometimes other fields. This structure is derived from representations of the SUSY algebras in Sec. 8.4. In this chapter we are concerned with the free limit, in which the various fields do not interact, and we can consider them separately. In particular we consider $\Psi_\mu(x)$ (omitting the spinor index α) as a free field, subject to the gauge transformation

$$\Psi_\mu(x) \to \Psi_\mu(x) + \partial_\mu \epsilon(x)\,. \tag{5.1}$$

Furthermore we will assume that Ψ_μ and ϵ are complex spinors with $2^{[D/2]}$ spinor components for spacetime dimension D. This is fine for the *free* theory in any dimension D, but interacting supergravity theories are more restrictive as to the spinor type permitted in a given spacetime dimension (and such theories exist only for $D \leq 11$). We will need to use the required Majorana and/or Weyl spinors when we study these theories in later chapters (and the number $2^{[D/2]}$ must be adjusted to agree with the number of components of each type of spinor).

It is consistent with the pattern set in the previous chapter that the gauge field $\Psi_\mu(x)$ is a field with one more vector index than the gauge parameter $\epsilon(x)$. Furthermore, as in the case of electromagnetism, the anti-symmetric derivative $\partial_\mu\Psi_\nu - \partial_\nu\Psi_\mu$ is gauge invariant. An important difference arises because we now seek a gauge invariant *first* order wave equation for the fermion field. It is advantageous to start with the action, which must be a) Lorentz invariant, b) first order in spacetime derivatives, c) invariant under the gauge transformation (5.1) and the simultaneous conjugate transformation of $\bar{\Psi}_\mu$, and d) Hermitian, so that the Euler-Lagrange equation for $\bar{\Psi}_\mu$ is the Dirac conjugate of that for Ψ_μ. It is easy to see that the expression

$$S = -\int \mathrm{d}^D x\, \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho\,, \tag{5.2}$$

which contains the third rank Clifford algebra element $\gamma^{\mu\nu\rho}$, has all these properties. Note that the action is gauge invariant but the Lagrangian density is not. Instead its variation is the total derivative $\delta\mathcal{L} = -\partial_\mu(\bar{\epsilon}\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho)$. The reason is that the fermionic gauge symmetry is the remnant of supersymmetry, and the anticommutator of two SUSY transformations is a spacetime symmetry.

It should be noted that a physically equivalent theory can be obtained by rewriting (5.2) in terms of the new field variable $\Psi'_\mu \equiv \Psi_\mu + a\gamma_\mu\gamma\cdot\Psi$ where a is an arbitrary parameter.[1] The gauge transformation is modified accordingly. The presentation in (5.1),(5.2) is universally used in the modern literature, because the gauge transformation is simplest and closely resembles that of electromagnetism. Historically, Rarita and Schwinger invented a wave equation for a *massive* spin $\frac{3}{2}$ particle in 1941. The massless limit of the action is a transformed version of (5.2), and Rarita and Schwinger simply noted that it possesses a fermionic gauge symmetry.[2]

The equation of motion obtained from (5.2) reads

$$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = 0\,. \tag{5.3}$$

One can immediately see that it shares some of the properties of the analogous electromagnetic equation (4.7), which is $\partial^\mu F_{\mu\nu} = 0$. Gauge invariance is manifest, and the left side vanishes *identically* when the derivative ∂_μ is applied. Thus (5.3) comprises $2^{[\frac{D}{2}]}(D-1)$ independent equations, which is enough to determine the $2^{[\frac{D}{2}]}D$ components of Ψ_ρ up to the freedom of a gauge transformation. The difference between the number of components of the gauge field and those of the gauge parameter, in this case $2^{[\frac{D}{2}]}(D-1)$, is called the number of off-shell degrees of freedom.

Ex. 5.1 *Show directly that for $D = 3$, the field equation (5.3) implies that $\partial_\nu\Psi_\rho - \partial_\rho\Psi_\nu = 0$. This means that the field has no gauge invariant degrees of freedom and thus no propagating particle modes. This is the supersymmetric counterpart of the situation in gravity for $D = 3$, where the field equation $R_{\mu\nu} = 0$ implies that the full curvature tensor $R_{\mu\nu\rho\sigma} = 0$. Hence no degrees of freedom.*

[1] The case $a = -\frac{1}{D}$ requires special treatment since $\gamma\cdot\Psi' = 0$.

[2] One of the present authors met Prof. Schwinger at a cocktail party in the early 1980's. Supergravity came up in the conversation, and Schwinger remarked lightheartedly "I should have discovered supergravity."

We notice that (5.3) can be rewritten in an equivalent but simpler form. For this purpose, we use the γ-matrix relation $\gamma_\mu \gamma^{\mu\nu\rho} = (D-2)\gamma^{\nu\rho}$, which implies that $\gamma^{\nu\rho}\partial_\nu \Psi_\rho = 0$ in spacetime dimension $D > 2$. We also note that $\gamma^{\mu\nu\rho} = \gamma^\mu \gamma^{\nu\rho} - 2\eta^{\mu[\nu}\gamma^{\rho]}$. Using this information, it is easy to see that (5.3) implies that

$$\gamma^\mu(\partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu) = 0\,. \tag{5.4}$$

This an alternate form of the equation of motion, equivalent to (5.3), but which cannot be obtained *directly* from an action. To see that (5.4) is equivalent, note that one can apply γ^ν and obtain $\gamma^{\nu\rho}\partial_\nu \Psi_\rho = 0$. The previous steps can then be reversed to obtain (5.3) from (5.4). One can also show that the left side of (5.4) vanishes *identically* if $\gamma^\nu \not\partial$ is applied. Finally, let's apply ∂_ρ to (5.4) and anti-symmetrize in $\rho\nu$ to obtain:

$$\not\partial(\partial_\rho \Psi_\nu - \partial_\nu \Psi_\rho) = 0\,. \tag{5.5}$$

This is a gauge invariant derivation of the fact that the wave equations, either (5.3) or (5.4), describe *massless* particles.

Ex. 5.2 *Do all the manipulations in the preceding paragraph. Do them backwards and forwards.*

5.1 The Initial Value Problem

Let's now study the initial value problem for (5.3) and thus count the number of on-shell degrees of freedom. We must untangle constraints on the initial data from time evolution equations. For this purpose we need to fix the gauge, so we impose the non-covariant condition

$$\gamma^i \Psi_i \;=\; 0\,, \tag{5.6}$$

which will play the same role as the Coulomb gauge condition we used in Sec. 4.1.2.

Ex. 5.3 *Show by an argument analogous to that in Sec. 4.1.2 that this condition does fix the gauge uniquely.*

We use the equivalent form (5.4) of the field equations. The $\nu = 0$ and $\nu \to i$ components are

$$\begin{aligned} \gamma^i \partial_i \Psi_0 - \partial_0 \gamma^i \Psi_i &= 0\,, \\ \gamma \cdot \partial \Psi_i - \partial_i \gamma \cdot \Psi &= 0\,. \end{aligned} \tag{5.7}$$

Using the gauge condition one can see that $\nabla^2 \Psi_0 = 0$, so $\Psi_0 = 0$ according to the discussion on page 76. The spatial components Ψ_i then satisfy the Dirac equation

$$\gamma \cdot \partial \Psi_i = 0\,, \tag{5.8}$$

which is a time evolution equation. However, there is an additional constraint, $\partial^i \Psi_i = 0$, obtained by contracting (5.8) with γ^i. Thus from the gauge condition and the equation of motion, we find $3 \times 2^{[\frac{D}{2}]}$ independent constraints on the initial data, namely

$$\begin{aligned} \gamma^i \Psi_i(\vec{x}, 0) &= 0\,, && (5.9)\\ \Psi_0(\vec{x}, 0) &= 0\,, && (5.10)\\ \partial^i \Psi_i(\vec{x}, 0) &= 0\,. && (5.11) \end{aligned}$$

Therefore, as initial conditions there are only $2^{[\frac{D}{2}]}(D-3)$ initial components of Ψ_i to be specified. The time derivatives are already determined by the Dirac equation (5.8). Hence there are $2^{[\frac{D}{2}]}(D-3)$ classical degrees of freedom for the Rarita-Schwinger gauge field in D dimensional Minkowski space. The on-shell degrees of freedom are one half of this number. In dimension $D = 4$, with Majorana conditions, we find the two states expected for a massless particle for any spin $s > 0$. We will show below that these states carry helicity $\pm 3/2$. In general dimension, it should be a representation of $\mathrm{SO}(D-2)$ as we mentioned already in Sec. 4.1.2. Indeed, a vector-spinor representation is an irreducible representation when one subtracts the γ-trace, giving again as number of components $\frac{1}{2}(D-3)2^{[\frac{D}{2}]}$.

Ex. 5.4 *Analyze the degrees of freedom using the original equation of motion (5.3).*

According to the discussion for $D = 4$ at the beginning of Chapter 4, we would expect the Fourier expansion of the field to contain annihilation and creation operators for states of helicity $\lambda = \pm\frac{3}{2}$. Let's derive this fact starting from the plane-wave

$$\Psi_i(x) = \mathrm{e}^{\mathrm{i}p\cdot x} v_i(\vec{p}) u(\vec{p})\,, \tag{5.12}$$

for a positive null energy-momentum vector $p^\mu = (|\vec{p}|, \vec{p})$. Since $\Psi_i(x)$ satisfies the Dirac equation (5.8), the 4-component spinor $u(\vec{p})$ must be a superposition of the massless helicity spinors $u(\vec{p}, \pm)$ given in (2.34). Thus we use the Weyl representation (2.19) of the γ-matrices. The vector $v_i(\vec{p})$ may be expanded in the complete set

$$v_i(\vec{p}) = a p_i + b\epsilon_i(\vec{p}, +) + c\epsilon_i(\vec{p}, -)\,, \tag{5.13}$$

where $\epsilon_i(\vec{p}, \pm)$ are the transverse polarization vectors of Sec. 4.1.2, i.e. they satisfy $p^i \epsilon_i(\vec{p}, \pm) = 0$. The constraint (5.11) requires that $a = 0$. Thus (5.12) is reduced to the form:

$$\begin{aligned} \Psi_i(x) \;=\; \mathrm{e}^{\mathrm{i}p\cdot x} &\left[b_+ \epsilon_i(\vec{p}, +) u(\vec{p}, +) + c_+ \epsilon_i(\vec{p}, -) u(\vec{p}, +) \right. \\ &\left. + b_- \epsilon_i(\vec{p}, +) u(\vec{p}, -) + c_- \epsilon_i(\vec{p}, -) u(\vec{p}, -) \right]. \end{aligned} \tag{5.14}$$

We must still enforce the constraint $\gamma^i \Psi_i = 0$. Some detailed algebra is needed, which we leave to the reader; the result is that $c_+ = b_- = 0$, while b_+ and c_- are arbitrary. Thus there are two independent physical wave functions $\epsilon_i(\vec{p}, \pm) u(\vec{p}, \pm)$ for each p^μ.

Ex. 5.5 *Do the algebra that was just left for the reader. Show that the resulting vector-spinor wave functions $\epsilon_i(\vec{p},\pm)u(\vec{p},\pm)$ carry helicity $\pm\frac{3}{2}$. Show that the spinor wave functions for the conjugate plane-wave are $\epsilon_i^*(\vec{p},\pm)v(\vec{p},\pm)$, where $v(\vec{p},\pm) = B^{-1}u(\vec{p},\pm)^*$ are the massless v-spinors of (2.35).*

The net result of this analysis is that the Rarita-Schwinger field that satisfies the equation of motion and constraints above has the Fourier expansion (we add the trivial 0-components $\epsilon_0 = 0$ to polarization vectors)

$$\Psi_\mu(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0} \sum_\lambda [\mathrm{e}^{\mathrm{i}p\cdot x}\epsilon_\mu(\vec{p},\lambda)u(\vec{p},\lambda)c(\vec{p},\lambda) + \mathrm{e}^{-\mathrm{i}p\cdot x}\epsilon_\mu^*(\vec{p},\lambda)v(\vec{p},\lambda)d^*(\vec{p},\lambda)]\,. \tag{5.15}$$

The sum extends over the two physical wave functions of helicity $\pm\frac{3}{2}$. In the quantum theory the Fourier amplitude $c(\vec{p},\lambda)$ becomes the annihilation operator for helicity $\pm\frac{3}{2}$ particles, and $d^*(\vec{p},\lambda)$ becomes the creation operator for antiparticles. The situation is similar to that for the Dirac field in (2.24). A Majorana gravitino has the same expansion, with $d^*(\vec{p},\lambda) = c^*(\vec{p},\lambda)$, since there is no distinction between particles and anti-particles.

In dimension $D > 4$ the allowed gravitino modes are obtained by starting with products of the $D-2$ transverse polarization vectors $\epsilon_i(\vec{p},j)$ and the $\frac{1}{2}2^{[\frac{D}{2}]}$ massless Dirac spinors $u(\vec{p},s)$. The gauge-fixing constraint $\gamma^i\Psi_i = 0$ must then be enforced on linear combinations of these products as was done in (5.14). This leads to $\frac{1}{2}2^{[\frac{D}{2}]}(D-3)$ independent wave functions, which describe the on-shell states of the gravitino.

The canonical stress tensor obtained from (5.2) is

$$T_{\mu\nu} = \bar{\Psi}_\rho\gamma^{\rho\sigma}{}_\mu\partial_\nu\Psi_\sigma - \eta_{\mu\nu}\mathcal{L}\,. \tag{5.16}$$

It is neither symmetric nor gauge invariant under (5.1) (and its Dirac conjugate). It can be made symmetric, see [36], but gauge non-invariance is intrinsic and cannot be restored by adding terms of the form $\partial_\sigma S^{\sigma\mu\nu}$. The reason is that the gravitino must be joined with gravity in the gauge multiplet of SUSY. In a gravitational theory there is no well defined energy *density*.

Ex. 5.6 *Show that the total energy momentum $P^\nu = \int \mathrm{d}^3\vec{x}\,T^{0\nu}(\vec{x},t)$ is gauge invariant and given (for $D = 4$) by*

$$P^\nu = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0} p^\nu \sum_\lambda [c^*(\vec{p},\lambda)c(\vec{p},\lambda) - d(\vec{p},\lambda)d^*(\vec{p},\lambda)]\,. \tag{5.17}$$

5.2 Sources and Green's Function

Let's follow the pattern of Sec. 4.1.3 and couple the Rarita-Schwinger field to a vector spinor-source via

$$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = J^\mu\,. \tag{5.18}$$

The contraction of ∂_μ with the left side vanishes identically, which indicates that (5.18) is a consistent equation only if the source current is conserved, i.e., $\partial_\mu J^\mu = 0$. This is the exact analogue of what happens in electromagnetism and Yang-Mills theory. In those theories, the gauge field was later coupled to matter systems, and the source was the Noether current of the global symmetry. Supergravity theories are more complicated. The same phenomenon occurs, but only as an approximation valid to lowest order in the gravitational coupling. The current J^μ is the Noether super-current of the matter multiplets in the theory.

Let's now apply the method of Sec. 4.1.3 to find the Green's function that determines the response of the field to the source. We first solve the simpler problem for the Dirac field,

$$(\partial\!\!\!/ - m)\Psi(x) = J(x)\,. \tag{5.19}$$

Given a Green's function $S(x-y)$ that satisfies

$$(\partial\!\!\!/_x - m)S(x-y) = -\delta(x-y)\,, \tag{5.20}$$

the solution of (5.19) is given by

$$\Psi(x) = -\int \mathrm{d}^D y\, \mathrm{e}^{\mathrm{i}p\cdot(x-y)} S(x-y)J(y)\,. \tag{5.21}$$

Let's solve this problem using the Fourier transform. The symmetries of Minkowski spacetime allow us to assume the Fourier representation

$$S(x-y) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \mathrm{e}^{\mathrm{i}p\cdot x} S(p)\,. \tag{5.22}$$

In momentum space, (5.20) reads

$$(\mathrm{i}p\!\!\!/ - m)S(p) = -1, \tag{5.23}$$

and the solution (with Feynman's causal structure) is

$$S(p) = -\frac{1}{\mathrm{i}\,p\!\!\!/ - m} = \frac{\mathrm{i}\,p\!\!\!/ + m}{p^2 + m^2 - \mathrm{i}\epsilon}\,. \tag{5.24}$$

Comparing with (4.18), we see that we can express $S(x-y)$ in terms of the scalar Green's function as

$$S(x-y) = (\partial\!\!\!/_x + m)G(x-y)\,. \tag{5.25}$$

This result satisfies (5.20) by inspection and could have been guessed at the start. However the Fourier transform method is useful as a warmup for the more complicated case of the Rarita-Schwinger field.

We expect the Green's function solution of (5.18) to take the form

$$\Psi_\mu(x) = -\int \mathrm{d}^D y\, S_{\mu\nu}(x-y)J^\nu(y)\,, \tag{5.26}$$

where $S_{\mu\nu}(x-y)$ is a tensor bispinor. A bispinor has two spinor indices, which are suppressed in our notation, and it can be regarded as a matrix of the Clifford algebra. As in the electromagnetic case, the Rarita-Schwinger operator is not invertible, but we can assume that the Green's function satisfies

$$\gamma^{\mu\sigma\rho}\frac{\partial}{\partial x^\sigma}S_{\rho\nu}(x-y) = -\delta^\mu_\nu\delta(x-y) + \frac{\partial}{\partial y^\nu}\Omega^\mu(x-y)\,. \tag{5.27}$$

The last term on the right is a 'pure gauge' in the source point index. In momentum space (5.27) reads

$$\mathrm{i}\gamma^{\mu\sigma\rho}p_\sigma S_{\rho\nu}(p) = -\delta^\mu_\nu - \mathrm{i}p_\nu\Omega^\mu(p)\,. \tag{5.28}$$

We will solve (5.28) by writing an appropriate ansatz for $S_{\rho\nu}(p)$ and then find the unknown functions in the ansatz. The matrix $\gamma^{\mu\sigma\rho}p_\sigma$ in (5.28) contains an odd rank element of the Clifford algebra and it is odd under the reflection $p_\sigma \to -p_\sigma$. It is reasonable to guess that the ansatz we need should also involve odd rank Clifford elements and be odd under the reflection. We would also expect that terms that contain the momentum vectors p_ρ or p_ν are 'pure gauges' and thus arbitrary additions to the propagator, which would not be determined by the equation (5.28). So we omit such terms and postulate the ansatz

$$\mathrm{i}\,S_{\rho\nu}(p) = A(p^2)\,\eta_{\rho\nu}\,\not{p} + B(p^2)\,\gamma_\rho\,\not{p}\,\gamma_\nu\,. \tag{5.29}$$

The next step is to substitute the ansatz in (5.28) and simplify the products of γ-matrices that appear. This process yields

$$\begin{aligned}
\mathrm{i}\gamma^{\mu\sigma\rho}p_\sigma S_{\rho\nu}(p) &= A\gamma^{\mu\sigma}{}_\nu\not{p}p_\sigma + (D-2)B\gamma^{\mu\sigma}\not{p}\gamma_\nu p_\sigma \\
&= A\left(p^\mu\gamma^\sigma{}_\nu - p^\sigma\gamma^\mu{}_\nu\right)p_\sigma + (D-2)B\left(-p^\mu\gamma^\sigma + p^\sigma\gamma^\mu\right)\gamma_\nu p_\sigma \\
&\quad + \ldots \\
&= \left[A-(D-2)B\right]\left(p^\mu\gamma^\sigma{}_\nu - p^\sigma\gamma^\mu{}_\nu\right)p_\sigma + (D-2)Bp^2\delta^\mu_\nu \\
&\quad + \ldots\,.
\end{aligned} \tag{5.30}$$

We have omitted terms ... which are proportional to the vector p_ν, because such terms will be 'matched' in (5.28) by $\Omega^\mu(p)$ rather than by δ^μ_ν. It is now easy to see that the δ^μ_ν term in (5.28) determines the values $A = -1/p^2$ and $B = -1/((D-2)p^2)$. Thus we have found the gravitino propagator

$$S_{\mu\nu}(p) = \mathrm{i}\frac{1}{p^2}\left[\eta_{\mu\nu}\not{p} + \frac{1}{D-2}\gamma_\mu\not{p}\gamma_\nu + C\,p_\mu\gamma_\nu + E\,\gamma_\mu p_\nu + F\,p_\mu\not{p}p_\nu\right]\,, \tag{5.31}$$

in which we have added possible gauge terms that are not determined by this procedure. In position space the propagator is

$$S_{\mu\nu}(x-y) = \left[\eta_{\mu\nu}\not{\partial} + \frac{1}{D-2}\gamma_\mu\not{\partial}\gamma_\nu + C\,\partial_\mu\gamma_\nu + E\,\gamma_\mu\partial_\nu - F\partial_\mu\not{\partial}\partial_\nu\right]G(x-y)\,, \tag{5.32}$$

where $G(x-y)$ is the massless scalar propagator (4.19), and all derivatives are with respect to x.

Ex. 5.7 *Include the omitted p_ν terms in (5.30) and $\Omega(p)$ in the analysis and verify that the gauge terms in the propagator are arbitrary. Show that for the choice $E = -\frac{1}{D-2}$, and arbitrary C and F, the propagator satisfies*

$$\mathrm{i}\gamma^{\mu\sigma\rho} p_\sigma S_{\rho\nu}(p) = -\left(\delta^\mu_\nu - \frac{p^\mu p_\nu}{p^2}\right) . \tag{5.33}$$

Show that for $D = 4$, the propagator, with $C = -1$, takes the 'reverse-index' form $S_{\mu\nu}(p) = -\mathrm{i}\frac{1}{2}\gamma_\nu \not{p} \gamma_\mu$ that is the form used in most of the literature on perturbative studies in supergravity [37].

5.3 Dimensional Reduction on Minkowski$_D \times S^1$.

Our aim in this section is quite narrow, but the approach will be broad. The narrow goal is to extend the Rarita-Schwinger equation to describe *massive* gravitinos, but we wish to do it by introducing the important technique of dimensional reduction, which is also called Kaluza-Klein theory. The main idea is that a fundamental theory, perhaps supergravity or string theory, that is formulated in D' spacetime dimensions can lead to an observable spacetime of dimension $D < D'$. In the most common variant of this scenario, there is a stable solution of the equations of the fundamental theory that describes a manifold of the structure $M_{D'} = M_D \times X_d$ with $d = D' - D$. The factor M_D is the spacetime in which we might live, thus non-compact with small curvature, while X_d is a tiny compact manifold of spatial extent L. The compact space X_d can be thought of as hidden dimensions of spacetime that are not accessible to direct observation because of basic properties of wave physics that are coded in quantum mechanics as the uncertainty principle. This principle asserts that it would take wave excitations of energy $E \approx 1/L$ to explore structures of spatial scale L. If L is sufficiently small, this energy scale cannot be achieved by available apparatus. Nevertheless, the dimensional reduction might be confirmed since the presence of X_d has important indirect effects on physics in M_D.

In this section we study an elementary version of dimensional reduction, which still has interesting physics to teach. Instead of obtaining the structure $M_D \times X_d$ from a fundamental theory including gravity, we will simply explore the physics of the various *free* fields we have studied, *assuming* that the $(D+1)$-dimensional spacetime is Minkowski$_D \otimes S^1$. The main feature is that Fourier modes of fields on S^1 are observed as infinite 'towers' of *massive* particles by an observer in Minkowski$_D$. The reduction of the free massless gravitino equation in $D+1$ dimensions will then tell us the correct description of massive gravitinos. Massive gravitinos appear in the physical spectrum of $D = 4$ supergravity when SUSY is spontaneously broken.

5.3.1 Dimensional reduction for scalar fields.

Let's change to a more convenient notation and rename the coordinates of the $(D+1)$-dimensional product spacetime $x^0 = t, x^1, \ldots, x^{D-1}, y$, where y is the coordinate of S^1 with range $0 \leq y \leq 2\pi L$. We consider a massive complex scalar field $\phi(x^\mu, y)$ that obeys the Klein-Gordon equation

$$[\Box_{D+1} - m^2]\phi = \left[\Box_D + \left(\frac{\partial}{\partial y}\right)^2 - m^2\right]\phi = 0\,. \tag{5.34}$$

Acceptable solutions must be single-valued on S^1 and thus have a Fourier series expansion

$$\phi(x^\mu, y) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{\frac{\mathrm{i}ky}{L}} \phi_k(x^\mu). \tag{5.35}$$

It is immediate that the spacetime function associated with the kth Fourier mode, namely $\phi_k(x^\mu)$, satisfies

$$\left[\Box_D - \left(\frac{k}{L}\right)^2 - m^2\right]\phi_k = 0\,. \tag{5.36}$$

Thus it describes a particle of mass ${m_k}^2 = \left(\frac{k}{L}\right)^2 + m^2$. So the spectrum of the theory, as viewed in Minkowski$_D$, contains an infinite tower of massive scalars!

There is an even simpler way to find the mass spectrum. Just substitute the plane wave $\mathrm{e}^{\mathrm{i}p^\mu x_\mu}\,\mathrm{e}^{\frac{\mathrm{i}ky}{L}}$ directly in the $(D+1)$-dimensional equation (5.34). The D-component energy-momentum vector must satisfy $p^\mu p_\mu = (k/L)^2 + m^2$. The mass shift due to the Fourier wave on S^1 is immediately visible.

5.3.2 Dimensional reduction for spinor fields.

We will consider the dimensional reduction process for a complex spinor $\Psi(x^\mu, y)$ for even $D = 2m$ (so that the spinors in $D+1$ dimensions have the same number of components). Two new ideas enter the game. The first just involves the Dirac equation in D dimensions. We remark that if $\Psi(x)$ satisfies

$$[\partial\!\!\!/_D - m]\Psi(x) = 0\,, \tag{5.37}$$

then the new field $\tilde\Psi \equiv \mathrm{e}^{-\mathrm{i}\gamma_*\beta}\Psi$, obtained by applying a chiral phase factor, satisfies

$$[\partial\!\!\!/_D - m(\cos 2\beta + \mathrm{i}\gamma_* \sin 2\beta)]\tilde\Psi = 0\,. \tag{5.38}$$

Physical quantities are unchanged by the field redefinition, so both equations describe particles of mass m. One simple implication is that the sign of m in (5.37) has no physical significance, since it can be changed by field redefinition with $\beta = \pi/2$.

The second new idea is that a fermion field can be either periodic or anti-periodic $\Psi(x^\mu, y) = \pm\Psi(x^\mu, y+2\pi L))$. Anti-periodic behavior is permitted because a fermion field is not observable. Rather, bilinear quantities such as the energy density $T^{00} =$

$-\bar{\Psi}\gamma^0\partial^0\Psi$ are observables and they are periodic even when Ψ is anti-periodic. Thus we consider the Fourier series

$$\Psi(x^\mu, y) = \sum_k \mathrm{e}^{\frac{\mathrm{i}ky}{L}}\Psi_k(x^\mu), \tag{5.39}$$

where the mode number k is integer or half-integer for periodic or anti- periodic fields, respectively. In either case when we substitute (5.39) in the $(D+1)$-dimensional Dirac equation $[\not{\partial}_{D+1} - m]\Psi((x^\mu, y) = 0$, we find that $\Psi_k(x^\mu)$ satisfies[3]

$$\left[\not{\partial}_D - \left(m + \mathrm{i}\gamma_* \frac{k}{L}\right)\right]\Psi_k(x^\mu) = 0\,. \tag{5.40}$$

By applying a chiral transformation with phase $\tan 2\beta = k/(mL)$, we see that $\Psi_k(x^\mu)$ describes particles of mass ${m_k}^2 = (\frac{k}{L})^2 + m^2$. Again we would observe an infinite tower of massive spinor particles with distinct spectra for the periodic and anti-periodic cases.

5.3.3 Dimensional reduction for the vector gauge field.

We now apply circular dimensional reduction to Maxwell's equation

$$\partial^\nu F_{\nu\mu} = \Box_{D+1}A_\mu - \partial_\mu(\partial^\nu A_\nu) = 0 \tag{5.41}$$

in $D+1$ dimensions, and we assume a periodic Fourier series representation

$$A_\mu(x,y) = \sum_k \mathrm{e}^{\frac{\mathrm{i}ky}{L}} A_{\mu k}(x)\,, \qquad A_D(x,y) = \sum_k \mathrm{e}^{\frac{\mathrm{i}ky}{L}} A_{Dk}(x)\,, \tag{5.42}$$

with k an integer. The analysis simplifies greatly if we assume the gauge conditions $A_{Dk}(x) = 0$ for $k \neq 0$ and vector component D tangent to S^1. It is easy to see that this gauge can be achieved and uniquely fixes the Fourier modes $\theta_k(x)$, $k \neq 0$, of the gauge function. The gauge invariant Fourier mode $A_{D0}(x)$ remains a physical field in the dimensionally reduced theory. A quick examination of the $\mu \to D$ component of (5.41) shows that it reduces to

$$\begin{aligned} k = 0 \quad &: \quad \Box_{D+1}A_{D0} = \Box_D A_{D0} \;=\; 0\,, \\ k \neq 0 \quad &: \quad \partial^\mu A_{\mu k} \;=\; 0\,, \end{aligned} \tag{5.43}$$

so the mode $A_{D0}(x)$ simply describes a massless scalar in D dimensions. For $\mu \leq D-1$, the wave equation (5.41) implies that the vector modes $A_{\mu k}(x)$ satisfy

$$\left[\Box_D - \frac{k^2}{L^2}\right] A_{\mu k} - \partial_\mu(\partial^\nu A_{\nu k}) = 0\,. \tag{5.44}$$

[3] Recall from Ch. 3 that for odd spacetime dimension $D = 2m+1$, $\gamma^D = \pm\gamma_*$ where γ_* is the highest rank Clifford element in $D = 2m$ dimensions.

For mode number $k = 0$ this is just the Maxwell equation in D dimensions with its gauge symmetry under $A_{\mu 0} \to A_{\mu 0} + \partial_\mu \theta_0$ intact, since the Fourier mode $\theta_0(x)$ remained unfixed in the process above. For mode number $k \neq 0$, (5.44) is the standard equation[4] for a massive vector field with mass $m_k^2 = k^2/L^2$, namely the equation of motion of the action

$$S = \int \mathrm{d}^D x \, \left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}m^2 A_\mu A^\mu\right] . \tag{5.45}$$

A counting argument similar to that for the massless case in Ch. 4 shows that we have the D component field $A_{\mu k}$ subject to the single constraint (5.43) and thus giving $D-1$ quantum degrees of freedom for each Fourier mode $k \neq 0$. The $D-1$ particle states for each fixed energy- momentum p^μ transform in the vector representation of $\mathrm{SO}(D-1)$ as appropriate for a massive particle. Note that there are 3 states for $D = 4$, which agrees with $2s+1$ for spin $s = 1$. The count of states is the same in the massless $k = 0$ sector also, where we have the gauge vector $A_{\mu 0}$ plus the scalar A_{D0} with $D-2$ plus 1 on-shell degrees of freedom.

5.3.4 Finally $\Psi_\mu(x, y)$.

Let's apply dimensional reduction to the massless Rarita-Schwinger field in $D+1$ dimensions with $D = 2m$. We will assume that the field $\Psi_\mu(x, y)$ is anti-periodic in y so that its Fourier series involves modes $\exp(\mathrm{i}ky/L)\Psi_{\mu k}(x)$ with half-integral k. This assumption simplifies the analysis, since only $k \neq 0$ occurs, and all modes will be massive.

We would like to start with (5.3) in dimension $D+1$ and derive the wave equation of a massive gravitino in Minkowski$_D$. A gauge choice makes this task much easier. All Fourier modes have $k \neq 0$, so we can impose the gauge condition $\Psi_{Dk}(x) = 0$ on all modes and completely eliminate the field component $\Psi_D(x, y)$.

Let's write out the $\mu = D$ and $\mu \leq D-1$ components of (5.3) with $\Psi_D = 0$ (using $\gamma^D = \gamma_*$):

$$\begin{aligned} \gamma^{\nu\rho}\partial_\nu \Psi_{\rho k} &= 0\,, \\ \left[\gamma^{\mu\nu\rho}\partial_\nu - \mathrm{i}\frac{k}{L}\gamma_*\gamma^{\mu\rho}\right]\Psi_{\rho k} &= 0\,. \end{aligned} \tag{5.46}$$

Note that the first equation of (5.46) follows by application of ∂^μ to the second one.

Ex. 5.8 *Show that the chiral transformation $\Psi_{\rho k} = \mathrm{e}^{(-\mathrm{i}\pi\gamma_*/4)}\Psi'_{\rho k}$ leads, after replacing $\Psi' \to \Psi$, to the equation of motion*

$$\left(\gamma^{\mu\nu\rho}\partial_\nu - m\gamma^{\mu\rho}\right)\Psi_\rho = 0\,. \tag{5.47}$$

[4] Note that the result (5.43) can be obtained by applying ∂^μ to the equation (5.44) and is thus consistent with that equation.

The last equation is the Euler-Lagrange equation of the action

$$S = -\int \mathrm{d}^D x\, \bar{\Psi}_\mu \left[\gamma^{\mu\nu\rho}\partial_\nu - m\gamma^{\mu\rho}\right]\Psi_\rho\,. \tag{5.48}$$

Ex. 5.9 *The equation of motion (5.47) also contains constraints on the initial data. Obtain $\gamma^{\mu\nu}\partial_\mu\Psi_\nu = 0$, which is not a constraint, by contracting the equation with ∂_μ. Then find the constraint $\gamma^\mu\Psi_\mu = 0$ by contracting with γ_μ. Show that the $\mu = 0$ component of the equation of motion gives the constraint*
$(\gamma^{ij}\partial_i - m\gamma^j)\Psi_j = 0$.

Ex. 5.10 *By analysis similar to that which led from (5.3) to (5.4) in the massless case, derive $(\not{\partial} + m)\Psi_\mu = 0$, which closely resembles the Dirac equation. The constraints of Ex. 5.9 must still be applied to the initial data, but the new equation clearly shows that the field has definite mass m.*

It is useful to recapitulate the equations that we obtained during the analysis (or directly from (5.47)) that determine the counting of the number of the initial data and thus the number of degrees of freedom.

$$\begin{aligned}
&\gamma^\mu\Psi_\mu = 0\,,\\
&(\gamma^{ij}\partial_i - m\gamma^j)\Psi_j = 0\,,\\
&\left[\not{\partial} + m\right]\Psi_\mu = 0\,.
\end{aligned} \tag{5.49}$$

As in the massless case, the time derivatives are determined by the Dirac equation (last equation of (5.49)). The initial data are thus the values at $t = 0$ of the Ψ_μ restricted by the first two equations of (5.49). Hence, the complex field $\Psi_\mu(x)$ with $D \times 2^{[D/2]}$ degrees of freedom contains $(D-2)\times 2^{[D/2]}$ independent classical degrees of freedom and thus $\frac{1}{2}(D-2)\times 2^{[D/2]}$ on-shell physical states. For $D = 4$ these are the 4 helicity states required for a massive particle of spin $s = 3/2$. In the situation of dimensional reduction, there is a massive gravitino with $m = |k|/L$ for every Fourier mode k, each with $\frac{1}{2}(D-2)2^{[D/2]}$ states. Note that this is the same as the number of states of a *massless* gravitino in $D+1$ dimensions.

Ex. 5.11 *Study the Kaluza-Klein reduction for the Rarita-Schwinger field assuming periodicity $\Psi_\mu(x, y + 2\pi) = \Psi_\mu(x, y)$ in y. Show that the spectrum seen in Minkowski$_D$ consists of a massive gravitino for each Fourier mode $k \neq 0$ plus a massless gravitino and massless Dirac particle for the zero mode.*

The dimensional reduction process has thus taught us the correct action for a massive gravitino. In particular the mass term is $m\bar{\Psi}_\mu\gamma^{\mu\nu}\Psi_\nu$. There is a more general action, namely

$$S = -\int \mathrm{d}^D x\, \bar{\Psi}_\mu \left[\gamma^{\mu\nu\rho}\partial_\nu - m\gamma^{\mu\rho} - m'\eta^{\mu\rho}\right]\Psi_\rho\,, \tag{5.50}$$

which contains an additional Lorentz invariant term with a coefficient m' with the dimension of mass. It is curious that this does not give the correct description of a massive gravitino, because it contains too many degrees of freedom. In the following exercise we ask readers to verify this.

Ex. 5.12 *Derive the equation of motion for the action (5.50). Analyze this equation as in Ex. 5.9, and show that the previous constraint* $\gamma \cdot \Psi = 0$ does not *hold if* $m' \neq 0$. *The field components* $\gamma \cdot \Psi$ *then describe additional degrees of freedom (which propagate as negative Hilbert space metric 'ghosts'). See [37] for an analysis in terms of projection operators.*

6
Differential Geometry

In this chapter we collect the ideas of differential geometry that are required to formulate general relativity and supergravity. There are several books, written for physicists, which explore this subject at greater length and greater depth [38, 39, 40, 41, 42].

In general relativity spacetime is viewed as a differentiable manifold of dimension $D \geq 4$ with a metric of Lorentzian signature $(-, +, +, \ldots +)$ indicating one time dimension and $D - 1$ space dimensions. We assume that readers of this book are not intimidated by the idea of $D - 4$ hidden dimensions that are not directly observed. We will also need to consider manifolds of purely Euclidean signature $(+, +, \ldots, +)$, which may appear in the hidden extra dimensions and as the target space of nonlinear σ-models.

We will give a reasonably rigorous definition of a manifold and then introduce the various quantities that 'live on it' in a less formal manner, emphasizing the way that the quantities transform under changes of coordinates. Invariance under coordinate transformations is one of the key principles that underlie general relativity. The most important structures we need are the metric, connection, and curvature. But other quantities such as vector and tensor fields and differential forms are also very useful. We will discuss them first since they require only the manifold structure.

It would be good if readers have already encountered some of the more elementary ideas before, perhaps in a course on general relativity. Our primary purpose is to collect the necessary ideas and explain them, hopefully clearly albeit non-rigorously, and thus to prepare readers for later chapters where the ideas are applied. Readers who do the suggested exercises will achieve the most thorough preparation.

6.1 Manifolds

A D-dimensional manifold is a topological space M together with a family of open sets M_i that cover it, i.e., $M = \cup_i M_i$. The M_i are called coordinate patches. On each patch there is a $1:1$ map ϕ_i, called a chart, from $M_i \to \mathbb{R}^D$. In more concrete language a point $p \in M_i \subset M$ is mapped to $\phi_i(p) = (x^1, x^2, \ldots, x^D)$. We say that the set $(x^1, x^2, \ldots, x^D)$ are the local coordinates of the point p in the patch M_i. If $p \in M_i \cap M_j$, then the map $\phi_j(p) = (x'^1, x'^2, \ldots, x'^D)$ specifies a second set of coordinates for the point p. The compound map $\phi_j \circ \phi_i^{-1}$ from $\mathbb{R}^D \to \mathbb{R}^D$ is then specified by the set of functions $x'^\mu(x^\nu)$. These functions, and their inverses $x^\nu(x'^\mu)$ are required to be smooth, usually C^∞. See Fig. 6.1for an illustration of the ideas just discussed.

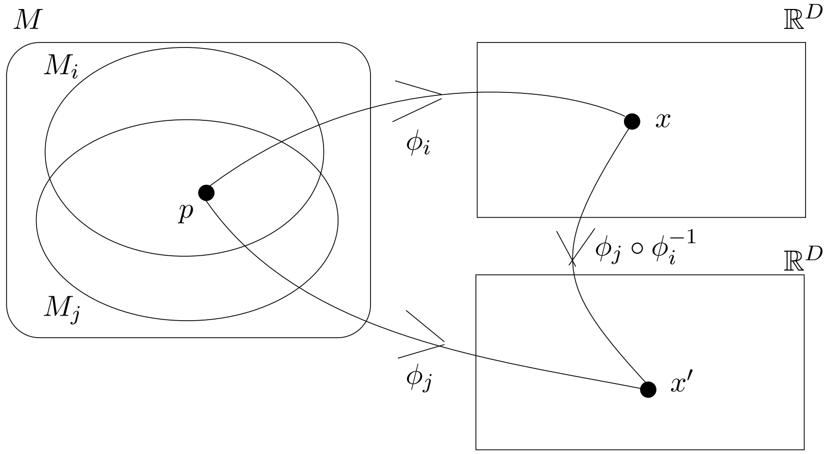

Fig. 6.1. *Two charts in $\mathbb{R}^D$ for subsets M_i and M_j of the space M, and the compound map.*

We now describe the unit two-sphere S^2 as an interesting and useful example of a manifold. Initially it may be defined as the surface $x^2+y^2+z^2 = 1$ embedded in $\mathbb{R}^3$. It is common to use the usual spherical polar coordinates θ, ϕ with $z = \cos\theta$, $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$. This is fine for some purposes, but it does not define a good coordinate chart at the poles $\theta = 0, \pi$, since these points have no unique values of ϕ.

There are many ways to introduce coordinate charts to define a manifold structure. One useful way is to use the stereographic projection illustrated in Fig. 6.2. There are two patches whose union covers the sphere, namely M_1, consisting of the sphere with south pole deleted, and M_2, which is the sphere with north pole deleted. From the plane geometry of the triangles in Fig. 6.2, one defines the maps ϕ_1 and ϕ_2 to the central plane in the figure. These maps take the point with polar coordinates θ, ϕ to points X, Y and X', Y' respectively. The maps are given by

$$\begin{aligned} \phi_1 : \qquad X + \mathrm{i}Y &= \mathrm{e}^{\mathrm{i}\phi}\tan(\theta/2)\,, \\ \phi_2 : \qquad X' + \mathrm{i}Y' &= \mathrm{e}^{\mathrm{i}\phi}\cot(\theta/2)\,. \end{aligned} \tag{6.1}$$

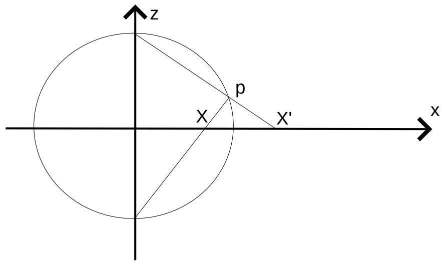

Fig. 6.2. *Stereographic projection.*

On the overlap, we see that

$$\phi_2 \circ \phi_1^{-1}(X, Y) = X' + \mathrm{i}Y' = 1/(X - \mathrm{i}Y)\,. \tag{6.2}$$

Ex. 6.1 *Derive (6.1) and (6.2).*

6.2 Scalars, vectors, tensors, etc.

The simplest objects to define on a manifold M are scalar functions f that map $M \to \mathbb{R}$. We say that the point p maps to $f(p) = z \in \mathbb{R}$. On each coordinate patch M_i we can define the compound map $f \circ \phi_i^{-1}$ from $\mathbb{R}^D \to \mathbb{R}$ as $f_i(x) \equiv f \circ \phi_i^{-1}(x) = z$, where x stands for $\{x^\mu\}$. On the overlap $M_i \cap M_j$ of two patches with local coordinates x^μ and x'^ν of the point p, the two descriptions of f must agree. Thus $f_i(x) = f_j(x')$.

We now define the properties of scalar functions in the less formal way we will use for most of the objects that live on M. We no longer refer to a covering by coordinate patches. Instead we conceive of the manifold as a set whose points may be described by many different coordinate systems, say $(x^0, x^1, \ldots, x^{D-1})$ and $(x'^0, x'^1, \ldots, x'^{D-1})$. Any two sets of coordinates are related by a set of C^∞ functions, e.g. $x'^\mu(x^\nu)$ with non-singular Jacobean $\partial x'^\mu / \partial x^\nu$. We refer to such a change of coordinates as a general coordinate transformation. A scalar function, also called a scalar field, is described by $f(x)$ in one set of coordinates and $f'(x')$ in the second set. The two functions must be pointwise equal, i.e.

$$f'(x') = f(x)\,. \tag{6.3}$$

Locally, at least, the informal definition agrees with the more formal one above.

In the same fashion, a contravariant vector field is described by D functions $V^\mu(x)$ in one coordinate system and D functions $V'^\mu(x')$ in the second. They are related by

$$V'^\mu(x') \;=\; \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x)\,, \tag{6.4}$$

with a summation convention on the repeated index ν. We go on to define covariant vector fields $\omega_\mu(x)$ and (mixed) tensors $T^\mu_\nu(x)$ by their behavior under coordinate transformations, namely

$$\begin{aligned} \omega'_\mu(x') &= \frac{\partial x^\nu}{\partial x'^\mu}\omega_\nu(x)\,, \\ T'^\mu_\nu(x') &= \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\nu}T^\sigma_\rho(x)\,. \end{aligned} \tag{6.5}$$

We leave it to the reader to devise the analogous definitions of higher rank tensors such as $T^{\mu\nu}(x)$, $S_{\mu\nu\rho}$ etc. A tensor field with p contravariant and q covariant indices is called a tensor of type (p,q) and rank $p+q$.

At this point in the development, contravariant and covariant quantities are unrelated objects, which transform differently. However a contravariant and covariant index can be contracted (i.e. summed) to define tensorial quantities of lower rank.

Ex. 6.2 *Given $V^\mu(x)$, $\omega_\mu(x)$, $T^\mu_\nu(x)$, show that $V^\mu(x)\omega_\mu(x)$ transforms as a scalar field and that $T^\mu_\nu(x)V^\nu(x)$ transforms as a contravariant vector.*

One can proceed with concrete local definitions of this type to obtain a physically satisfactory formulation of general relativity. However there is much richness to be gained, and considerable practical advantage, if we develop the ideas further and incorporate some of the concepts of a more mathematical treatment of differential geometry.

Given a contravariant vector field $V^\mu(x)$, one can consider the system of differential equations

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = V^\mu(x)\,. \tag{6.6}$$

A solution $x^\mu(\lambda)$ is a map from $\mathbb{R} \to M$, which is a curve on M, called an integral curve of the vector field. There is an integral curve through every point of any open subset of M in which the vector field does not vanish. If the manifold is $\mathbb{R}^D$, then we know that the vector $\mathrm{d}x^\mu/\mathrm{d}\lambda$ is tangent to the curve $x^\mu(\lambda)$, and we make the same interpretation for a general manifold.

Let $x^\mu(\lambda)$ be the integral curve through the point p of M with coordinates $x^\mu(\lambda_0)$. Then $\mathrm{d}x^\mu/\mathrm{d}\lambda|_{\lambda_0} = V^\mu(x(\lambda_0))$ is the tangent vector to the curve $x^\mu(\lambda)$ at p. We can now consider $D-1$ other vector fields $\tilde{V}^\mu(x)$ whose values $\tilde{V}^\mu(x(\lambda_0))$, together with the first $V^\mu(x(\lambda_0))$, fill out a basis of $\mathbb{R}^D$. Each $\tilde{V}^\mu(x(\lambda_0))$ is the tangent vector of an integral curve $\tilde{x}(\lambda)$ through p. Thus the vector fields evaluated at p determine the D-dimensional vector space $T_p(M)$, the tangent space to the manifold at point p. A vector field $V^\mu(x)$ may then be thought of as a smooth assignment of a tangent vector in each $T_p(M)$ as p varies over M. We shall use the notation $T(M)$ to denote the space of contravariant vector fields on M

One important structure that one can form using the components $V^\mu(x)$ of a contravariant vector field is the differential operator $V = V^\mu(x)\partial/\partial x^\mu$. It follows from the transformation property (6.4) and the chain rule that V is constructed in the same way in all coordinate system, e.g. $V = V'^{\mu}(x')\partial/\partial x'^{\mu}$. In this sense it is invariant under coordinate transformations. The differential operator V acts naturally on a scalar field $f(x)$, yielding another scalar field

$$\mathcal{L}_V f(x) \equiv V^{\mu}(x)\frac{\partial f}{\partial x^{\mu}}\,. \tag{6.7}$$

On the manifold $\mathbb{R}^D$, this operation is just the directional derivative $V \cdot \nabla f$, and it has the same interpretation on a general manifold M. At each point p with coordinates x^m, $\mathcal{L}_V f(x)$ is the derivative of $f(x)$ in the direction of the tangent of the integral curve of $V^{\mu}(x)$ through p.

Locally, there is a 1:1 correspondence between contravariant vector fields $V^{\mu}(x)$ and differential operators. In mathematical treatments a vector field is viewed as a smooth assignment of a differential operator at each point p. The set of elementary operators $\{\partial/\partial x^{\mu},\ \mu = 1,\ldots,D\}$ are a basis in this view of the tangent space $T_p(M)$. This is consistent with our discussion since $\partial f/\partial x^{\mu}$ for a given value of μ is the derivative in the direction of the tangent to the curve on which the single coordinate x^{μ} changes, but the other coordinates x^{ν} for $\nu \neq \mu$ are constant. The basis $\{\partial/\partial x^{\mu}, \mu = 1,\ldots,D\}$ is called a coordinate basis because these operators differentiate along such coordinate curves at each p.

The derivative $\mathcal{L}_V f(x)$ defined in (6.7) may be extended to vector and tensor fields of any type (p,q), always yielding another tensor of the same type. For the vectors and tensors in (6.4)-(6.5), the precise definition is

$$\begin{aligned}
\mathcal{L}_V U^{\mu} &= V^{\rho}\partial_{\rho}U^{\mu} - (\partial_{\rho}V^{\mu})U^{\rho}\,,\\
\mathcal{L}_V \omega_{\mu} &= V^{\rho}\partial_{\rho}\omega_{\mu} + (\partial_{\mu}V^{\rho})\omega_{\rho}\,,\\
\mathcal{L}_V T^{\mu}_{\nu} &= V^{\rho}\partial_{\rho}T^{\mu}_{\nu} - (\partial_{\rho}V^{\mu})T^{\rho}_{\nu} + (\partial_{\nu}V^{\rho})T^{\mu}_{\rho}\,.
\end{aligned} \tag{6.8}$$

The derivative defined in this way is called the Lie derivative. Its definition requires a vector field, but not a connection; yet it preserves the tensor transformation property.

Ex. 6.3 *Show explicitly that $\mathcal{L}_V U^{\mu}$, $\mathcal{L}_V \omega_{\mu}$, and $\mathcal{L}_V T^{\mu}_{\nu}$ defined in (6.8) do transform under coordinate transformations as required by (6.4)-(6.5).*

The Lie derivative of a contravariant vector field has special significance because it occurs in the commutator of the corresponding differential operators $U = U^{\mu}(x)\partial/\partial x^{\mu}$ and $V = V^{\mu}(x)\partial/\partial x^{\mu}$. An elementary calculation gives

$$[U,V] = W = W^{\mu}(x)\frac{\partial}{\partial x^{\mu}}\,, \tag{6.9}$$

with $W^{\mu} = \mathcal{L}_U V^{\mu} = -\mathcal{L}_V U^{\mu}$. The new vector field W^{μ} is called the Lie bracket of U^{μ} and V^{μ}. This discussion also shows that the contravariant tensor fields on M naturally form a Lie algebra.

Let us consider the transformation properties of (6.3)-(6.5) for infinitesimal coordinate transformations, namely those for which $x'^\mu = x^\mu - \xi^\mu(x)$. We work to first order in $\xi^\mu(x)$.

Ex. 6.4 *Show to first order in $\xi^\mu(x)$ that the previous transformation rules can be expressed in terms of Lie derivatives as*

$$\begin{aligned}
\delta\phi(x) &\equiv \phi'(x) - \phi(x) = \mathcal{L}_\xi \phi\,, \\
\delta V^\mu(x) &\equiv V'^\mu(x) - V^\mu(x) = \mathcal{L}_\xi V^\mu\,, \\
\delta\omega_\mu(x) &\equiv \omega'_\mu(x) - \omega_\mu(x) = \mathcal{L}_\xi \omega_\mu\,, \\
\delta T^\mu_\nu(x) &\equiv T'^\mu_\nu(x) - T^\mu_\nu(x) = \mathcal{L}_\xi T^\mu_\nu\,.
\end{aligned} \tag{6.10}$$

Thus one of the useful roles of Lie derivatives is in the description of infinitesimal coordinate transformations.

Next we focus attention on covariant vector fields, such as $\omega_\mu(x)$. We already noted in Ex. 6.2 that the contraction $\omega_\mu(x)V^\mu(x)$ with any contravariant vector field gives a scalar field. Thus at any point p with coordinates x^ν, $\omega_\mu(x)$ can be regarded as an element of the dual space $T^*_p(M)$, a linear functional that maps $T_p(M) \to \mathbb{R}$. The space $T^*_p(M)$ is usually called the cotangent space at p.

In parallel to the way in which we associated contravariant vector fields $V^\mu(x)$ with differential operators $V = V^\mu(x)\partial/\partial x^\mu$, we use the coordinate differentials $\mathrm{d}x^\mu$ to write $\Omega = \omega_\mu(x)\mathrm{d}x^\mu$. Note that both $\omega_\mu(x)$ and $\mathrm{d}x^\mu$ transform under coordinate transformations, but $\Omega = \omega'_\mu(x')\mathrm{d}x'^\mu$ is constructed in the same way in any coordinate system. Ω is called a differential 1-form on M. Note that the gradient $\partial_\mu\phi(x)$ of any scalar transforms as a covariant vector and that the associated differential 1-form $d\phi = \partial_\mu\phi\mathrm{d}x^\mu$ is just the differential of calculus. We can think of the set of coordinate differentials $\{\mathrm{d}x^\mu,\ \mu = 1,\ldots,D\}$ as a basis of the space of 1-forms.

The notion of the cotangent space $T^*_p(M)$ of linear functionals on $T_p(M)$ is naturally extended to the level of 1-forms and differential operators. We define the pairing of basis elements as $\langle \mathrm{d}x^\mu | \partial/\partial x^\nu\rangle \equiv \delta^\mu_\nu$. This is extended using linearity to any general 1-form Ω and differential operator V, so that we then have $\langle\Omega|V\rangle = \omega_\mu(x)V^\mu(x)$. This agrees with the initial definition as the contraction of component indices.

6.3 The algebra and calculus of differential forms

Among the various fields defined on M, the scalars ϕ, covariant vectors ω_μ, and totally antisymmetric tensors such as $\omega_{\mu\nu} = -\omega_{\nu\mu}$ have a particularly useful structure when considered together. Note that antisymmetry is preserved under coordinate transformations so it is a tensorial property. Using the coordinate differentials $\mathrm{d}x^\mu$, we can construct differential p-forms for $p = 1, 2, \ldots, D$ as

$$\omega^{(1)} = \omega_\mu(x)\mathrm{d}x^\mu\,,$$

$$
\begin{aligned}
\omega^{(2)} &= \frac{1}{2}\omega_{\mu\nu}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \\
&\vdots \\
\omega^{(p)} &= \frac{1}{p!}\omega_{\mu_1\mu_2\cdots\mu_p}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \mathrm{d}x^{\mu_p}\,. \qquad (6.11)
\end{aligned}
$$

The wedge product is defined as antisymmetric, i.e. $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho = -\mathrm{d}x^\rho \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, etc. At each point we have an element of the p-fold antisymmetric tensor product of the tangent space, so the differential form $\omega^{(p)}$ is a smooth assignment is a smooth assignment of an element of this tensor product as the point varies over M. The space of p-forms is denoted by $\Lambda^p(M)$. By convention the scalars are considered to be 0-forms.

There is an exterior algebra and calculus of p-forms, which we will not develop in detail. See [39, 40, 41, 42, 43] for more complete discussions. Rather we will state some key properties without proof and write the specific examples needed later to discuss frames, connections, and curvature. In the exterior algebra, a p-form $\omega^{(p)}$ and a q-form $\omega^{(q)}$ can be multiplied to give a $(p+q)$-form if $p+q \le D$. The product vanishes if $p+q > D$. The product satisfies $\omega^{(p)} \wedge \omega^{(q)} = (-)^{pq}\omega^{(q)} \wedge \omega^{(p)}$ and it is associative.

Some examples are

$$
\begin{aligned}
\omega^{(1)} \wedge \tilde\omega^{(1)} &= \omega_\mu \mathrm{d}x^\mu \wedge \tilde\omega_\nu \mathrm{d}x^\nu \\
&= \tfrac{1}{2}(\omega_\mu\tilde\omega_\nu - \omega_\nu\tilde\omega_\mu)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \\
\omega^{(1)} \wedge \omega^{(2)} &= \omega_\mu \mathrm{d}x^\mu \wedge \tfrac{1}{2}\omega_{\nu\rho}(x)\mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \\
&= \tfrac{1}{6}(\omega_\mu\omega_{\nu\rho} + \omega_\nu\omega_{\rho\mu} + \omega_\rho\omega_{\mu\nu})\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho\,. \qquad (6.12)
\end{aligned}
$$

The explicit anti-symmetrization in the second line of each example is not necessary, since it is implicit in the wedge products of the $\mathrm{d}x^\mu$. But it is convenient to indicate that the covariant tensor field associated with each form is anti-symmetric.

The exterior calculus is based on the exterior derivative, which maps p-forms into $(p+1)$-forms as follows

$$
\mathrm{d}\omega^{(p)} = \frac{1}{p!}\partial_\mu\omega_{\mu_1\mu_2\cdots\mu_p}\mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \mathrm{d}x^{\mu_p}\,. \qquad (6.13)
$$

Ex. 6.5 *Show that the operation* d *is nilpotent, i.e.* $\mathrm{d}(\mathrm{d}\omega^{(p)}) = 0$ *on any* p*-form and that it satisfies the distributive property*

$$
\mathrm{d}(\omega^{(p)} \wedge \omega^{(q)} = \mathrm{d}\omega^{(p)} \wedge \omega^{(q)} + (-)^p\omega^{(p)} \wedge \mathrm{d}\omega^{(q)}\,. \qquad (6.14)
$$

On forms of degree $0, 1, 2$

$$
\begin{aligned}
\mathrm{d}\phi &= \partial_\mu\phi \mathrm{d}x^\mu\,, \\
\mathrm{d}\omega^{(1)} &= \tfrac{1}{2}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \\
\mathrm{d}\omega^{(2)} &= \tfrac{1}{6}(\partial_\mu\omega_{\nu\rho} + \partial_\nu\omega_{\rho\mu} + \partial_\rho\omega_{\mu\nu})\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho\,. \qquad (6.15)
\end{aligned}
$$

A p-form that satisfies $\mathrm{d}\omega^{(p)} = 0$ is called closed. A p-form $\omega^{(p)}$ that can be expressed as $\omega^{(p)} = \mathrm{d}\omega^{(p-1)}$ is called exact. The Poincaré lemma implies that locally any closed p-form can be expressed as $\mathrm{d}\omega^{(p-1)}$, but $\omega^{(p-1)}$ may not be well defined globally on M.

Consider the vectors V_i, $i = 1, \ldots, p$, which are explicitly

$$V_i = V_i{}^\mu \frac{\partial}{\partial x^\mu}\,. \tag{6.16}$$

A p-form may also be defined as a map from p vectors to a real number, which is

$$\omega^{(p)}(V_1, \ldots, V_p) = \omega_{\mu_1\mu_2\cdots\mu_p} V_1{}^{\mu_1} V_2{}^{\mu_2} \ldots V_2{}^{\mu_p}\,. \tag{6.17}$$

We saw that the exterior derivative is a map from p-forms into $(p+1)$-forms. There is also an interior derivative, which maps p forms into $(p-1)$-forms. The latter depends on a vector V and is denoted as i_V. It is defined as follows:

$$\begin{aligned}
(i_V\omega^{(p)})(V_1, \ldots, V_{p-1}) &= \omega^{(p)}(V, V_1, \ldots, V_{p-1})\,,\\
(i_V\omega^{(p)}) &= \frac{1}{(p-1)!} V^\mu \omega_{\mu\mu_1\ldots\mu_{p-1}} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \mathrm{d}x^{\mu_{p-1}}\,,\\
i_V\left(\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \mathrm{d}x^{\mu_p}\right) &= V^{\mu_1}\mathrm{d}x^{\mu_2} \wedge \ldots \mathrm{d}x^{\mu_p} - V^{\mu_2}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_3} \ldots \mathrm{d}x^{\mu_p} +
\end{aligned} \tag{6.18}$$

Ex. 6.6 *Show that these are equivalent definitions.*

Ex. 6.7 *Prove that also the internal derivative is nilpotent, i.e.* $i_V i_V = 0$.

With the internal and external derivative, the Lie derivative, which we introduced in Sec. (6.2), has a simple expression on p-forms:

$$\mathcal{L}_V = \mathrm{d}i_V + i_V\mathrm{d}\,. \tag{6.19}$$

The formulas that we gave in (6.8) for e.g. $\mathcal{L}_V\omega_\mu$ are in this context the values of the components[1] of $\mathcal{L}_V(\omega_\mu \mathrm{d}x^\mu)$, i.e. they are $(\mathcal{L}\omega)_\mu$ and not $\mathcal{L}(\omega_\mu)$.

Ex. 6.8 *Check this expression on 0-forms (where the first term vanishes by definition), on 1-forms and on 2-forms.*

Differential forms have a natural application to the theories of electromagnetism, Yang-Mills theory, and to the antisymmetric tensor gauge theories that appear in higher dimensional supergravity. However, we need to bring in some other ideas in the next section before discussing these physical applications.

[1] Hence we warn the reader that one should not use them in $\mathcal{L}_V(\omega_\mu\mathrm{d}x^\mu) = (\mathcal{L}_V\omega_\mu)\mathrm{d}x^\mu + \omega_\mu\mathcal{L}_V\mathrm{d}x^\mu$ and then use (6.8) for the first term. When one uses the definitions with forms $\omega_\mu(x)$ is a 0-form and $\mathcal{L}_V\omega_\mu = i_V\mathrm{d}\omega_\mu = V^\nu\partial_\nu\omega_\mu$.

6.4 The metric and frame field on a manifold

We now introduce the additional structure of a metric on a manifold M. In general relativity the metric is of primary importance in describing the geometry of spacetime and the dynamics of gravity. In theories such as supergravity where there are fermions coupled to gravity, one must use an auxiliary quantity, the frame field (more commonly called the vierbein or vielbein), which we discuss in detail. The metric tensor is quadratically related to the frame field.

A metric on inner product on a real vector space V is a non-degenerate bilinear map from $V \otimes V \to \mathbb{R}$. The inner product of two vectors $u,\, v \in V$ is a real number denoted by (u, v). The inner product must satisfy the following properties:
i) bilinearity– $(u, c_1 v_1 + c_2 v_2) = c_1(u, v_1) + c_2(u, v_2)$ and
$(c_1 v_1 + c_2 v_2, u) = c_1(v_1, u) + c_2(v_2, u)$.
ii) non-degeneracy– if $(u, v) = 0$ for all $v \in V$, then $u = 0$.
iii) symmetry– $(u, v) = (v, u)$.

The metric on a manifold is a smooth assignment of an inner product map on each $T_p(M) \otimes T_p(M) \to \mathbb{R}$. In local coordinates the metric is specified by a covariant second rank symmetric tensor field $g_{\mu\nu}(x)$, and the inner product of two contravariant vectors $U^\mu(x)$ and $V^\mu(x)$ is $g_{\mu\nu}(x)U^\mu(x)V^\nu(x)$, which is a scalar field. In particular the metric gives a formula for the length s of a curve $x^\mu(\lambda)$ with tangent vector $\mathrm{d}x^\mu/\mathrm{d}\lambda$:

$$s_{12} = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sqrt{g_{\mu\nu}(x(\lambda))(\mathrm{d}x^\mu/\mathrm{d}\lambda)(\mathrm{d}x^\nu/\mathrm{d}\lambda)}\,. \tag{6.20}$$

Thus it is most convenient to summarize the properties of a given metric by the line element

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu\,. \tag{6.21}$$

Non-degeneracy means that $\det g_{\mu\nu} \neq 0$, so the inverse metric $g^{\mu\nu}(x)$ exists as a rank 2 symmetric contravariant tensor, which satisfies

$$g^{\mu\rho} g_{\rho\nu} = g_{\nu\rho} g^{\rho\mu} = \delta^\mu_\nu\,. \tag{6.22}$$

The metric tensor and its inverse may be used to lower and raise indices, e.g. $V_\mu(x) = g_{\mu\nu}V^\nu(x)$ and $\omega^\mu(x) = g^{\mu\nu}(x)\omega_\nu(x)$, thus providing a natural isomorphism between the spaces of contravariant and covariant vectors and tensors.

In a gravity theory in spacetime, the metric has signature $-++\ldots+$. Concretely this means that the the metric tensor $g_{\mu\nu}$ may be diagonalized by an orthogonal transformation, i.e. $(O^{-1})_\mu{}^a = O^a{}_\mu$, and

$$g_{\mu\nu} = O^a{}_\mu D_{ab} O^b{}_\nu\,, \tag{6.23}$$

with positive eigenvalues λ^a in $D_{ab} = \mathrm{diag}(-\lambda^0, \lambda^1, \ldots, \lambda^{D-1})$.

Ex. 6.9 *Show that $\lambda^a(x) > 0$ holds throughout M if the metric is non-degenerate. In another coordinate system the transformed metric $g'_{\rho\sigma} =$*

$(\mathrm{d}x^\mu/\mathrm{d}x'^\rho)(\mathrm{d}x^\nu/\mathrm{d}x'^\sigma)g_{\mu\nu}$ *may be diagonalized giving another set of eigenvalues* λ'^a, *in general different from the* λ^a. *Show that the* $\lambda'^a > 0$. *Thus the signature of a metric is a global invariant.*

The construction above, which involved only matrix linear algebra, allows us to define an important auxiliary quantity in a theory of gravity, namely

$$e^a_\mu(x) \equiv \sqrt{\lambda^a(x)}O^a{}_\mu(x)\,. \tag{6.24}$$

In 4 dimensions this quantity is commonly called the tetrad or vierbein. In general dimension the term vielbein is frequently used, but we prefer the term frame field for reasons that should become clear as we discuss its properties.

Note that

$$g_{\mu\nu}(x) = e^a_\mu(x)\eta_{ab}e^b_\nu(x)\,, \tag{6.25}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$ is the metric of flat D-dimensional Minkowski spacetime. Further, e^a_μ is a non-singular $D \times D$ matrix, with $\det e^a_\mu = \sqrt{-\det g}$. The inverse frame field $e^\mu_a(x)$ satisfies $e^a_\mu e^\mu_b = \delta^a_b$ and $e^\mu_a e^a_\nu = \delta^\mu_\nu$.

Ex. 6.10 *Show that*

$$e^\mu_a = g^{\mu\nu}\eta_{ab}e^b_\nu\,, \qquad e^\mu_a g_{\mu\nu}e^\nu_b = \eta_{ab}\,. \tag{6.26}$$

The last relation shows that the (inverse) frame field can be used to relate a general metric of signature $-++..+$ *to the Minkowski metric.*

Given the metric $g_{\mu\nu}(x)$, the frame field $e^a_\mu(x)$ is not uniquely determined. Any local Lorentz transformation $\Lambda^a{}_b(x)$, which leaves η_{ab} invariant, produces an equally good frame field

$$e'^a_\mu(x) = \Lambda^{-1\,a}{}_b(x)e^b_\mu(x)\,. \tag{6.27}$$

We require that there is no preferred Lorentz frame, which means that the frame field and geometrical quantities derived from it must be used in a way that is covariant with respect to the local Lorentz transformation (6.27). The results (6.25),(6.26) indicate that the frame field e^a_μ transforms as a covariant vector under coordinate transformations, while e^μ_a transforms as a contravariant vector, viz.

$$e'^a_\mu(x') = \frac{\partial x^\rho}{\partial x'^\mu}e^a_\rho(x)\,, \qquad e'^\mu_a(x') = \frac{\partial x'^\mu}{\partial x^\rho}e^\rho_a(x)\,. \tag{6.28}$$

The equations (6.27), (6.28) give the basic transformations associated with the frame field. Note that the position of Lorentz indices $a, b, \ldots$ has no special significance since they can be freely raised and lowered using η^{ab} and η_{ab}. Thus we may use $e_{a\mu}(x) = \eta_{ab}e^b_\mu$ if needed.

The second relation of (6.26) indicates that the e^μ_a form an othonormal[2] set of vectors in the tangent space of M at each point. Since $\det e^\mu_a \neq 0$, we have a basis

[2] A more precise term is 'pseudo-orthonormal' because of the indefinite spacetime metric. We will use the simpler and less awkward "orthonormal".

of each tangent space. Any contravariant vector field has a unique expansion in the new basis, i.e. $V^\mu(x) = V^a(x)e_a^\mu(x)$ with $V^a(x) = V^\mu(x)e_\mu^a(x)$. The $V^a(x)$ are the frame components of the original vector field $V^a(x)$. They transform as a set of D scalar fields under coordinate transformations, and as a vector under Lorentz transformations, i.e. $V'^a(x) = \Lambda^{-1\,a}{}_b(x)V^b(x)$. The same may be done for covariant vectors, i.e. $\omega_\mu(x) = \omega_a(x)e_\mu^a(x)$ with $\omega_a(x) = \omega_\mu(x)e_a^\mu(x)$. These constructions may be extended to tensor fields of any rank in a straightforward way.

Thus we may use e_μ^a and e_a^μ to transform vector and tensor fields back and forth between a coordinate basis with indices μ, $\nu, \ldots$ and a local Lorentz basis with indices $a, b \ldots$ in which the metric is η_{ab}. Invariants such as the inner product may be calculated in either basis.

Ex. 6.11 *Show that*

$$U^\mu(x)V_\mu(x) = g_{\mu\nu}(x)U^\mu(x)V^\nu(x) = \eta_{ab}U^a(x)V^b(x) = U^a(x)V_a(x)\,. \tag{6.29}$$

At the level of differential operators the change of basis in the tangent space is expressed as

$$E_a \equiv e_a^\mu(x)\frac{\partial}{\partial x^\mu}\,. \tag{6.30}$$

This makes it clear that the local Lorentz basis is a non-coordinate basis. If there were local coordinates y^a such that $E_a = \partial/\partial y^a$, these differential operators would commute. However the commutator

$$[E_a, E_b] = \Omega^c_{ab}E_c\,, \tag{6.31}$$

where the $\Omega^c_{ab} = e^c_\mu \mathcal{L}_{e_a} e^\mu_b = -e^c_\mu \mathcal{L}_{e_b} e^\mu_a$ are the frame components of the Lie bracket, which do not vanish in a general manifold.

Ex. 6.12 *Show that* $\Omega^c_{ab} = e^\mu_a e^\nu_b(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu)$.

We can also use the frame field e_μ^a to define a new basis in the spaces $\Lambda^p(M)$ of differential forms. The local Lorentz basis of 1-forms is

$$e^a \equiv e^a_\mu(x)\mathrm{d}x^\mu\,. \tag{6.32}$$

This is the dual basis to (6.30), since the pairing is given by $\langle e^a|E_b\rangle = \delta^a_b$. For 2-forms the basis consists of the wedge products $e^a \wedge e^b$, and so on.

In a field theory containing only bosonic fields, which are always vectors or tensors, the use of local frames is unnecessary, although it is an option that is convenient for some purposes, as the latter part of this chapter will show. **XXX Does it really show this? Need an example of use of Cartan structure eqtns. to calculate connection and curvature.** Local frames are a necessity to treat the coupling of fermion fields to gravity, because spinors are defined by their special transformation properties under Lorentz transformations.

Induced metrics. In many applications of differential geometry one encounters a manifold of dimension D which can be viewed as a surface embedded in flat Minkowski or Euclidean space of dimension $D+1$. We discuss the Euclidean case for $D=2$. Suppose that our surface is described by the equation

$$f(x,y,z)=0\,. \tag{6.33}$$

On the surface the differential vanishes, viz.

$$\mathrm{d}f=\frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y+\frac{\partial f}{\partial z}\mathrm{d}z=0\,. \tag{6.34}$$

The intrinsic geometry of the surface is determined by the Euclidean metric

$$\mathrm{d}s^2=\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\,. \tag{6.35}$$

To find it one can, in principle, solve (6.33) to eliminate one variable and then use (6.34) to find a relation among the coordinate differentials. When this information is inserted in (6.35), one has the induced metric. Voila!

This is often easier said than done, so we confine our discussion to the solvable and instructive example of the unit 2-sphere for the embedding equation (6.33) is

$$x^2+y^2+z^2\;=\;1\,. \tag{6.36}$$

Let's proceed using spherical coordinates

$$z=r\cos\theta\,,\qquad x=r\sin\theta\cos\varphi\,,\qquad y=r\sin\theta\sin\varphi\,. \tag{6.37}$$

The embedding equation becomes simply $r^2=1$, so we can eliminate the coordinate r and write the differentials

$$\begin{aligned}\mathrm{d}z &= -\sin\theta\,\mathrm{d}\theta\,,\\ \mathrm{d}x &= \cos\theta\cos\varphi\,\mathrm{d}\theta-\sin\theta\sin\varphi\,\mathrm{d}\varphi\,,\\ \mathrm{d}y &= \cos\theta\sin\varphi\,\mathrm{d}\theta+\sin\theta\cos\varphi\,\mathrm{d}\varphi\,.\end{aligned} \tag{6.38}$$

Upon substitution in (6.35) one finds the induced metric

$$\mathrm{d}s^2\;=\;\mathrm{d}\theta^2\;+\;\sin^2\theta\,\mathrm{d}\varphi^2\,. \tag{6.39}$$

This is a commonly used and quite useful metric on S_2, but it is evidently singular at the north and south poles where the metric tensor is not invertible. One can do somewhat better using one of the two sets of coordinates defined by the stereographic projection in Sec. 6.1, and this is the subject of the following exercise.

Ex. 6.13 *Reexpress the metric (6.39) in the coordinates* $X=\cos\varphi\tan(\theta/2)$, $Y=\sin\varphi\tan(\theta/2)$. *Show that the new metric is*

$$\mathrm{d}s^2=\frac{\mathrm{d}X^2\;+\;\mathrm{d}Y^2}{4(1\;+\;X^2\;+Y^2)^2}\,. \tag{6.40}$$

6.5 Volume forms and integration

The equations of motion in any field theory are most conveniently packaged in the action integral. In a gravitational theory this requires integration over the curved spacetime manifold. We thus need a procedure of integration that is invariant under coordinate transformations. The volume form is the key to this procedure.

On a D-dimensional manifold, one may choose *any* top degree D-form $\omega^{(D)}$ as a volume form and define the integral

$$\begin{aligned} I &= \int \omega^{(D)} \\ &= \frac{1}{D!}\int \omega_{\mu_1\cdots\mu_D}(x)\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_D} \\ &= \int \omega_{01\cdots D-1}\mathrm{d}x^0\mathrm{d}x^1\ldots\mathrm{d}x^{D-1}\,. \end{aligned} \tag{6.41}$$

The anti-symmetric tensor $\omega_{\mu_1\cdots\mu_D}(x)$ has only one independent component, and we have used this fact in the last line above to write the integral so that it may be performed by the rules of multi-variable calculus. For the same reason any two D-forms $\tilde{\omega}(D)$ and $\omega^{(D)}$ must be related by $\tilde{\omega}(D) = f\,\omega^{(D)}$, where $f(x)$ is a scalar field. Thus the definition (6.41) includes $\int f\,\omega^{(D)}$.

Ex. 6.14 *Show that in a new coordinate system with coordinates $x'^{\mu}(x^{\nu})$ the integral I in (6.41) takes the form*

$$I = \frac{1}{D!}\int \omega'_{\mu_1\cdots\mu_D}(x')\mathrm{d}x'^{\mu_1}\wedge\ldots\mathrm{d}x'^{\mu_D} \tag{6.42}$$

and is thus coordinate invariant.

Although there are many possible volume forms, there are two types that usually appear in the context of physics. The first, which is the more specialized, occurs when the physical theory contains form fields. As an example, on a 3-manifold the wedge product $\omega^{(1)}\wedge\omega^{(2)}$ can be chosen as a volume form. Using (6.12) we see that

$$\begin{aligned} I &= \int \omega^{(1)}\wedge\omega^{(2)} \\ &= \tfrac{1}{6}\int (\omega_\mu\omega_{\nu\rho}+\omega_\nu\omega_{\rho\mu}+\omega_\rho\omega_{\mu\nu})\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho \\ &= \int (\omega_0\omega_{12}+\omega_1\omega_{20}+\omega_2\omega_{01})\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\,. \end{aligned} \tag{6.43}$$

The integral is coordinate invariant, and it does not involve the metric on M. The action integral of the simplest Chern-Simons field theory, in which $\omega^{(2)} = \mathrm{d}\omega^{(1)}$, takes this form.

The second type of volume form is far more common in physics and we call it the canonical volume form. There are several ways to introduce it, and we will use the frame field $e^a_\mu(x)$ and the basis of frame 1-forms e^a for this purpose. As a preliminary we define the Levi-Civita alternating symbol in local frame components

$$\varepsilon_{a_1 a_2 \cdots a_D} = \begin{cases} +1 & a_1 a_2 \cdots a_D \text{ an even permutation of } 01\ldots(D-1) \\ -1 & a_1 a_2 \cdots a_D \text{ an odd permutation of } 01\ldots(D-1) \\ 0 & \text{otherwise.} \end{cases} \tag{6.44}$$

Under (proper) Lorentz transformations, i.e., $\det \Lambda^a{}_b = 1$, this is an invariant tensor that takes the same form in any Lorentz frame. As usual Lorentz indices are raised with η^{ab}. Note that $\varepsilon^{01\cdots(D-1)} = -1$.

Note that the Levi-Civita symbol provides a useful formula for the determinant of any $D \times D$ matrix $A^a{}_b$, namely

$$\det A \, \varepsilon_{b_1 b_2 \cdots b_D} = \varepsilon_{a_1 a_2 \cdots a_D} A^{a_1}{}_{b_1} A^{a_2}{}_{b_2} \ldots A^{a_D}{}_{b_D} \,, \tag{6.45}$$

and that there are systematic identities for the contraction of p of the $D = p + q$ indices, as we saw in (3.8).

We use the frame fields to transform this to the coordinate basis, but insert factors e^{-1} and $e = \det e^a_\mu$, i.e.

$$\begin{aligned} \varepsilon_{\mu_1 \mu_2 \cdots \mu_D} &= e^{-1} \varepsilon_{a_1 a_2 \cdots a_D} e^{a_1}_{\mu_1} e^{a_2}_{\mu_2} \cdots e^{a_D}_{\mu_D} \,, \\ \varepsilon^{\mu_1 \mu_2 \cdots \mu_D} &= e \, \varepsilon^{a_1 a_2 \cdots a_D} e^{\mu_1}_{a_1} e^{\mu_2}_{a_2} \cdots e^{\mu_D}_{a_D} \,. \end{aligned} \tag{6.46}$$

Note that these definitions ensure that $\varepsilon^{\mu_1 \cdots \mu_D}$ and $\varepsilon_{\mu_1 \cdots \mu_D}$ take the constant values on the right side of (6.44), as can be seen using (6.45). The tensor $\varepsilon^{\mu_1 \mu_2 \cdots \mu_D}$ is as such a *tensor density*. However, it is important to recognize that $\varepsilon^{\mu_1 \mu_2 \cdots \mu_D}$ can not be obtained from raising the indices of $\varepsilon_{\mu_1 \mu_2 \cdots \mu_D}$ in the usual way using the inverse of the metric. Therefore expressions like $\varepsilon^{\mu_1 \cdots \mu_p}{}_{\mu_{p+1} \cdots \mu_D}$ are not well defined. There is no such problem for $\varepsilon^{a_1 \cdots a_p}{}_{b_{p+1} \cdots b_D}$.

Ex. 6.15 *Prove, using (6.45) that both $\varepsilon^{\mu_1 \mu_2 \cdots \mu_D}$ and $\varepsilon_{\mu_1 \mu_2 \cdots \mu_D}$ takes values ± 1. This should guarantee that they are invariant under infinitesimal changes of the frame field. Show also directly that $\delta \varepsilon^{\mu_1 \mu_2 \cdots \mu_D} = 0$ for any δe^μ_a using the formula for matrices that*

$$\delta \det M = (\det M) \operatorname{Tr}(M^{-1} \delta M) \,, \tag{6.47}$$

and the Schouten identity, see (3.10).

With these preliminaries, the canonical volume form is defined as

$$\begin{aligned} \mathrm{d}V &\equiv e^0 \wedge e^1 \wedge \ldots \wedge e^{D-1} \\ &= \frac{1}{D!} \varepsilon_{a_1 \cdots a_D} e^{a_1} \wedge \cdots \wedge e^{a_D} \\ &= \frac{1}{D!} e \, \varepsilon_{\mu_1 \cdots \mu_D} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_D} \end{aligned}$$

$$
\begin{aligned}
&= e\,\mathrm{d}x^0 \ldots \mathrm{d}x^{D-1} \\
&= \mathrm{d}^D x\,\sqrt{-\det g}\,. \qquad (6.48)
\end{aligned}
$$

Note that the determinant of the frame field e_μ^a appears in a natural fashion. In the last line we give the abbreviated notation we will use in most applications. For example, given the Lagrangian of a system of fields, such as the kinetic Lagrangian $\mathcal{L} = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ of a scalar field, the action integral is written as

$$
S = \int \mathrm{d}V\,\mathcal{L} = \int \mathrm{d}^D x\sqrt{-\det g}\,\mathcal{L}\,. \qquad (6.49)
$$

6.6 Hodge duality of forms

The dimension of the component spaces of p- and q-forms with $p+q=D$ is the same, so that it is possible to define a $1:1$ map between them. This map is the Hodge duality map from $\Lambda^p(M) \to \Lambda^q(M)$, which is quite useful in the physics of supergravity. The map is denoted by $\Omega^{(q)} =^* \omega^{(p)}$.

Since the map is linear we can define it on a basis of p-forms and then extend to a general form. It is convenient to use the local frame basis initially and define

$$
{}^*e^{a_1} \wedge \ldots e^{a_p} = \frac{1}{q!} e^{b_1} \wedge \ldots e^{b_q} \varepsilon_{b_1\cdots b_q}{}^{a_1\cdots a_p}\,. \qquad (6.50)
$$

A general p-form can be expressed in this basis, and we can proceed to define its dual via

$$
\begin{aligned}
\Omega^{(q)} =^* \omega^{(p)} &= {}^*\left(\frac{1}{p!}\omega_{a_1\cdots a_p} e^{a_1} \wedge \ldots e^{a_p}\right) \\
&= \frac{1}{p!}\omega_{a_1\cdots a_p}\,{}^*e^{a_1} \wedge \ldots e^{a_p}\,. \qquad (6.51)
\end{aligned}
$$

Ex. 6.16 *Show that the frame components of $\Omega^{(q)}$ are given by*

$$
\Omega_{b_1\cdots b_q} = ({}^*\omega)_{b_1\cdots b_q} = \frac{1}{p!}\varepsilon_{b_1\cdots b_q}{}^{a_1\cdots a_p}\omega_{a_1\cdots a_p}\,. \qquad (6.52)
$$

These formulas are far less complicated than they look since there is only one independent term in each sum. For example, for $D=4$ the dual of a 3-form is a 1-form. For basis elements we have ${}^*e^1 \wedge e^2 \wedge e^3 = e^0$ and ${}^*e^0 \wedge e^1 \wedge e^2 = e^3$. For components, $({}^*\omega)_0 = \omega_{123}$ and $({}^*\omega)_3 = \omega_{012}$.

The duality has an important involutive property, which can be inferred from the following sequence of operations on basis elements:

$$
{}^*({}^*e^{a_1} \wedge \ldots e^{a_p}) = \frac{1}{q!}{}^*e^{b_1} \wedge \ldots e^{b_q}\,\varepsilon_{b_1\cdots b_q}{}^{a_1\cdots a_p}
$$

$$
\begin{aligned}
&= \frac{1}{p!q!} e^{c_1} \wedge \ldots e^{c_p} \varepsilon_{c_1 \cdots c_p}{}^{b_1 \cdots b_q} \varepsilon_{b_1 \cdots b_q}{}^{a_1 \cdots a_p} \\
&= -(-)^{pq} e^{c_1} \wedge \ldots e^{c_p} \delta^{a_1 \cdots a_p}_{c_1 \cdots c_p} \\
&= -(-)^{pq} e^{a_1} \wedge \ldots e^{a_p} \,. \qquad (6.53)
\end{aligned}
$$

This leads to the general relation ${}^*({}^*\omega^{(p)}) = -(-)^{pq}\omega^{(p)}$. This is the correct relation for a Lorentzian signature manifold. For Euclidean signature the involution property is ${}^*({}^*\omega^{(p)}) = (-)^{pq}\omega^{(p)}$.

For even dimension $D = 2m$, it is possible to impose the constraint of self-duality (or anti-self-duality) on forms of degree m, i.e. $\Omega^{(m)} = \pm{}^*\Omega^{(m)}$. In a given dimension this condition is consistent only if duality is a strict involution, i.e. $-(-)^{m^2} = -(-)^m = +1$ for Lorentzian signature and $(-)^m = +1$ for Euclidean signature. Thus we have self-dual Yang-Mills instantons in 4 Euclidean dimensions and a self-dual $F^{(5)}$ is possible in $D = 10$ and indeed appears in Type IIB Lorentzian signature supergravity as we will see later.

The duality relations defined above in a frame basis are easily transformed to a coordinate basis using the relations $e^a = e^a_\mu(x)\mathrm{d}x^\mu$ and $\mathrm{d}x^\mu = e^\mu_a(x)e^a$. For coordinate basis elements the duality map is

$$
{}^*(\mathrm{d}x^{\mu_1} \wedge \ldots \mathrm{d}x^{\mu_p}) = \frac{1}{q!} e\, g^{\mu_1\rho_1} \ldots g^{\mu_p\rho_p} \mathrm{d}x^{\nu_1} \wedge \ldots \mathrm{d}x^{\nu_p} \varepsilon_{\nu_1 \cdots \nu_q \rho_1 \cdots \rho_p} \,. \qquad (6.54)
$$

For anti-symmetric tensor components, we have

$$
({}^*\omega)_{\mu_1 \cdots \mu_q} = \frac{1}{p!} e\, \varepsilon_{\mu_1 \cdots \mu_q \rho_1 \cdots \rho_p} g^{\nu_1\rho_1} \ldots g^{\nu_p\rho_p} \omega_{\nu_1 \cdots \nu_p} \,. \qquad (6.55)
$$

Following the discussion in Sec. 6.5, we may take as a volume form on M the wedge product ${}^*\omega^{(p)} \wedge \omega^{(p)}$ of any p form and its Hodge dual. The integral of this volume form is simply the standard invariant norm of the tensor components of $\omega^{(p)}$, i.e.

$$
\int {}^*\omega^{(p)} \wedge \omega^{(p)} = \frac{1}{p!} \int \mathrm{d}^D x \sqrt{-g}\, \omega^{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p} \,. \qquad (6.56)
$$

Ex. 6.17 *Prove (6.56). Use the definitions above and those in Sec. 6.5 and the fact*

$$
e^{a_1} \wedge \ldots \wedge e^{a_q} \wedge e^{b_1} \wedge \ldots \wedge e^{b_p} = -\varepsilon^{a_1 \cdots a_q b_1 \cdots b_p} \mathrm{d}V \,, \qquad (6.57)
$$

where $\mathrm{d}V$ *is the canonical volume element of (6.48).*

Ex. 6.18 *Show that the volume form* $\mathrm{d}V$ *can also be written as* ^{*1}.

Ex. 6.19 *Compare these definitions with Sec. 4.2.1, to obtain*

$$
\tilde{F}_{\mu\nu} = -\mathrm{i}\,({}^*F)_{\mu\nu} \,. \qquad (6.58)
$$

See that the factor i *ensures that the tilde operation squares to the identity. Self-duality is then possible for complex 2-forms.*

6.7 p-form gauge fields

With the terminology of the previous section, we can rewrite the simplest kinetic actions of scalars and gauge vectors in the following way

$$\begin{aligned} S_0 &= -\tfrac{1}{2}\int {}^*F^{(1)}\wedge F^{(1)}\,, \qquad F^{(1)} = \mathrm{d}\phi\,,\\ S_1 &= -\tfrac{1}{2}\int {}^*F^{(2)}\wedge F^{(2)}\,, \qquad F^{(2)} = \mathrm{d}A^{(1)}\,. \end{aligned} \tag{6.59}$$

In each case we have a Bianchi identity: $\mathrm{d}F^{(1)} = 0$ and $\mathrm{d}F^{(2)} = 0$, whose solution is that these field strengths can be written as the differential of a lower form. For the form $A^{(1)}$, which describes the photon, there is a gauge transformation that can be written as $\delta A^{(1)} = \mathrm{d}\Lambda^{(0)}$.

This suggests that there is a generalization. When we call the previous actions the theories for a 0-form and a 1-form, we can describe a p-form in terms of its field strength as

$$S_p = -\tfrac{1}{2}\int {}^*F^{(p+1)}\wedge F^{(p+1)}\,, \qquad F^{(p+1)} = \mathrm{d}A^{(p)}\,. \tag{6.60}$$

Ex. 6.20 *Translate this in components and find*

$$\begin{aligned} S_p &= -\frac{1}{2\,(p+1)!}\int \mathrm{d}^D x\,\sqrt{-g}\,F^{\mu_1\cdots\mu_{p+1}}F_{\mu_1\cdots\mu_{p+1}}\,,\\ F_{\mu_1\cdots\mu_{p+1}} &= (p+1)\partial_{[\mu_1}A_{\mu_2\ldots\mu_{p+1}]}\,. \end{aligned} \tag{6.61}$$

Again there is a gauge transformation $\delta A^{(p)} = \mathrm{d}\Lambda^{(p-1)}$, as such transformations of the p-form gauge field leave $F^{(p+1)}$ invariant. The number of independent components in $\Lambda^{(p-1)}$ is $\binom{D}{p-1}$. However, not all the components of $\Lambda^{(p-1)}$ are independent symmetries, as all transformations where $\Lambda^{(p-1)} = \mathrm{d}\Lambda'^{(p-2)}$ are not effective transformations of the gauge field. But not all components of the latter are to be subtracted from the gauge symmetries. Indeed, when $\Lambda'^{(p-2)} = \mathrm{d}\Lambda''^{(p-3)}$, these are not zero modes. In this way, we can count the remaining gauge-invariant components of a p form as

$$\binom{D}{p} - \binom{D}{p-1} + \binom{D}{p-2} - \ldots = \binom{D-1}{p}\,, \tag{6.62}$$

the number of components of an p-form in $D-1$ dimensions. We thus find that the off-shell degrees of freedom (field variables minus symmetries) form a representation of $\mathrm{SO}(D-1)$, as we saw before for other fields. If furthermore the field equations of massless fields are used, the independent components form representations of $\mathrm{SO}(D-2)$. This is the on-shell counting of degrees of freedom. Thus the number of physical degrees of freedom is $\binom{D-2}{p}$. This suggests that a p-form has the same degrees of freedom as a $(D-2-p)$-form. Indeed, we will now show that not only the numbers match, but that the physical content is the same.

One further indication in this direction is that the field equation of the action (6.60) is

$$\mathrm{d}^* F^{(p+1)} = 0\,, \tag{6.63}$$

while the Bianchi identity that says that $F^{(p+1)}$ is an exact form is

$$\mathrm{d}F^{(p+1)} = 0\,. \tag{6.64}$$

When one defines a dual form $G^{(D-p-1)} = {}^*F^{(p+1)}$, then these two equations have a similar form, and $\mathrm{d}G^{(D-p-1)} = 0$ can then be interpreted as the statement that it is the field strength of a $(D-p-2)$-form.

We can easily prove this duality by the following steps. First, we rewrite the action (6.60) as

$$S_p = -\int \left[\tfrac{1}{2}{}^*F^{(p+1)} \wedge F^{(p+1)} + b^{(D-p-2)} \wedge \mathrm{d}F^{(p+1)}\right]. \tag{6.65}$$

The latter term is a Lagrange multiplier term where the field equation of $b^{(D-p-2)}$ implies that $F^{(p+1)}$ is the field strength of a p-form. If we include that term, we can consider $F^{(p+1)}$ and $b^{(D-p-2)}$ as the independent fields. Therefore, we can first consider the field equation for $F^{(p+1)}$, which states that

$${}^*F^{(p+1)} = (-)^{D-p}\,\mathrm{d}b^{(D-p-2)}\,. \tag{6.66}$$

As this is an algebraic field equation, it can be used in the action, which leads to the action of the type (6.60), where $b^{(D-p-2)}$ takes the role of $A^{(p)}$. This proves the equivalence of a p-form with a $(D-2-p)$-form.

The simplest case are the gauge antisymmetric tensors (2-forms) in 4 dimensions. The statement is thus that the simplest kinetic actions for such fields are equivalent to actions for scalars. However, the arguments above are only true for the simplest actions, i.e. with abelian gauge fields. In non-abelian field theories, antisymmetric tensors can lead to non-equivalent theories.

The duality transformations for gauge vectors in 4-dimensions is a *self-duality.* A 1-form is dual to another 1-form. In fact, it transforms electric in magnetic components, and this is the duality that we discussed in Sec. 4.2.

6.8 Connections and covariant derivatives

A covariant derivative on a manifold is a rule to differentiate a tensor of type (p,q) producing a tensor of type $(p,q+1)$. It is well known that one needs to introduce the affine connection $\Gamma^\rho_{\mu\nu}(x)$ to accomplish this. On vector fields the covariant derivative is defined by

$$\begin{aligned} \nabla_\mu V^\rho &= \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu\,, \\ \nabla_\mu V_\nu &= \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho\,, \end{aligned} \tag{6.67}$$

and it is straightforward to extend this definition to tensors. Most discussions of the affine connection in the physics literature are based on the idea of parallel transport and use a coordinate basis. See [44, 45].

Since supergravity requires the frame field $e^a_\mu(x)$, and we have used the frame 1-forms e^a extensively, it is natural to introduce the affine connection in this framework and then make contact with the more common treatment. In frames the affine connection is specified by the 1-forms $\omega^{ab} = \omega_\mu{}^{ab}(x)\mathrm{d}x^\mu$. Although more general connections can be considered, the connection required for gravitational theories with fermions is anti-symmetric in Lorentz indices, $\omega^{ab} = -\omega^{ba}$, and we impose this condition ab initio. As we will see, anti-symmetry means that ω^{ab} is a connection for the Lorentz group $O(D-1,1)$. Indeed covariance under local Lorentz transformations (6.27) of the frame is the guiding principle of the discussion. See [38] for a roughly similar treatment. The components $\omega_\mu{}^{ab}(x)$ of ω^{ab} are usually called the spin connection because they are essential in the description of spinors on manifolds. We will use the terms spin connection or frame connection for $\omega_\mu{}^{ab}(x)$ or ω^{ab}.

6.8.1 The first structure equation and the spin connection $\omega_{\mu ab}$

Given the frame 1-forms e^a, we examine the 2-forms

$$\mathrm{d}e^a = \tfrac{1}{2}(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,. \tag{6.68}$$

The anti-symmetric components transform as a (0,2) tensor under coordinate transformations, but not as a local Lorentz vector. This is most quickly seen at the 2-form level, where, using (6.27)

$$\mathrm{d}e'^a = \mathrm{d}(\Lambda^{-1\,a}{}_b e^b) = \Lambda^{-1\,a}{}_b \mathrm{d}e^b + \mathrm{d}\Lambda^{-1\,a}{}_b \wedge e^b\,. \tag{6.69}$$

The second term spoils the vector transformation property. To cancel it we add the contribution from a 2-form involving the spin connection and consider

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b \equiv T^a\,. \tag{6.70}$$

If $\omega^a{}_b$ is defined to transform under local Lorentz transformations as

$$\omega'^a{}_b = \Lambda^{-1\,a}{}_c \mathrm{d}\Lambda^c{}_b + \Lambda^{-1\,a}{}_c\,\omega^c{}_d\,\Lambda^d{}_b\,, \tag{6.71}$$

then T^a does indeed transform as a vector, i.e. $T'^a = \Lambda^{-1\,a}{}_b T^b$. The 2-form T^a is called the torsion 2-form of the connection, and (6.70) is called the first Cartan structure equation.

Ex. 6.21 *Confirm that T^a transforms as a Lorentz vector if the connection transforms as in (6.71).*

There is a lot to be said about the connection 1-form ω^{ab}, and we begin with some properties that should be familiar from the study of non-abelian gauge theories. The components ${\omega_\mu}^{ab}(x)$ transform as a covariant vector under coordinate transformations, while (6.71) implies that

$$\omega'_\mu{}^a{}_b = \Lambda^{-1\,a}{}_c\,\partial_\mu\Lambda^c{}_b + \Lambda^{-1\,a}{}_c\,\omega_\mu{}^c{}_d\,\Lambda^d{}_b\,. \tag{6.72}$$

These are exactly the gauge transformation properties of a Yang-Mills potential for the group $O(D-1,1)$. The situation is even more familiar for Euclidean signature manifolds in which frame fields transform under local $SO(D)$ rotations, and an anti-symmetric ${\omega_\mu}^{ab}$ transforming as in (6.72) is a gauge potential for the compact group $SO(D)$.

Thus local Lorentz covariance is implemented like Yang-Mills gauge covariance for the gauge group $O(d-1,1)$.

In Sec. 6.5 we showed that any (vector) or tensor field on M can be transformed from a coordinate basis to a local Lorentz basis, where the tensor components of a type (p,q) tensor take the form $T^{a_1\cdots a_p}_{b_1\cdots b_q}(x)$. Let us consider the simplest cases of vectors V^a, U_a and type $(0,2)$ tensors T_{ab}, which transform as

$$\begin{aligned} V'^a(x) &= \Lambda^{-1\,a}{}_b(x)V^b(x)\,, \\ U'_a(x) &= U_b(x)\Lambda^b{}_a(x)\,, \\ T'_{ab}(x) &= T_{cd}(x)\Lambda^c{}_a(x)\Lambda^d{}_b(x)\,, \end{aligned} \tag{6.73}$$

respectively. The extension of these local Lorentz transformation rules to Lorentz tensors of type (p,q) is straightforward.

We use the spin connection to define local Lorentz covariant derivatives as

$$\begin{aligned} D_\mu V^a &= \partial_\mu V^a + \omega_\mu{}^a{}_b V^b\,, \\ D_\mu U_a &= \partial_\mu U_a - U_b\omega_\mu{}^b{}_a = \partial_\mu U_a + \omega_{\mu a}{}^b U_b\,, \\ D_\mu T_{ab} &= \partial_\mu T_{ab} - T_{cb}\omega_\mu{}^c{}_a - T_{ac}\omega_\mu{}^c{}_b\,. \end{aligned} \tag{6.74}$$

The extension to type (p,q) Lorentz tensors involves p connection terms with $\omega_\mu{}^{a_i}{}_c$ contracted from the left and q terms with $-\omega_\mu{}^c{}_{b_i}$ contracted on the right. Recall ${\omega_\mu}^{ab} = \omega_\mu{}^a{}_c\eta^{cb}$ and $\omega_{\mu a}{}^b = \eta_{ac}\omega_\mu{}^c{}_d\eta^{db}$, etc.

Ex. 6.22 *Use (6.71) to show that $D_\mu V^a$, $D_\mu U_a$ and $D_\mu T_{ab}$ do transform as Lorentz tensors, i.e. as in (6.73). They also transform as covariant vectors under coordinate transformations.*

Ex. 6.23 *Show that the Lorentz covariant derivative obeys the Leibnitz product rule, e.g. $D_\mu(V^aU_b) = (D_\mu V^a)U_b + V^aD_\mu U_b$. Show also that it commutes with index contractions, e.g. $\delta^c_a D_\mu(V^aT_{bc}) = D_\mu(V^cT_{bc})$.*

Let's apply (6.74) to the Lorentz metric tensor η_{ab}. We have

$$D_\mu\eta_{ab} = -\eta_{cb}\omega_\mu{}^c{}_a - \eta_{ac}\omega_\mu{}^c{}_b = -\omega_{\mu ba} - \omega_{\mu ab} = 0\,. \tag{6.75}$$

The metric has vanishing covariant derivative because η_{ab} is an invariant tensor of the Lorentz group. A direct consequence is that scalar products V^aU_a are preserved under parallel transport[3] with the spin connection [42]. Thus our connection is metric preserving.

The Cartan structure equation (6.70) is important, both conceptually and practically. The least familiar element may be the torsion 2-form T^a or its components, the torsion tensor $T_{\mu\nu}{}^a = -T_{\nu\mu}{}^a$. Indeed in most applications of differential geometry to gravity, the torsion vanishes, and one deals with a torsion-free, metric-preserving connection. This is also called the Levi-Civita connection, for which the structure equation reads $\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0$. However non-vanishing torsion can arise from coupling of gravity to certain matter fields, and this does occur in supergravity. The geometrical effect of torsion is seen in the properties of an infinitesimal 'parallelogram' constructed by the parallel transport of two vector fields. The parallelogram closes for a torsion-free connection, but not if there is torsion, see [42].

Let's examine the tensor components of the structure equation (6.70). It is convenient to refer all quantities to a coordinate basis. The goal is to express $\omega_{\mu[\rho\nu]} = \omega_{\mu ab}e^a_\rho e^b_\nu$ in terms of

$$\Omega_{[\mu\nu]\rho} = (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu)\; e_{a\rho}\,, \tag{6.76}$$

and the torsion tensor $T_{[\mu\nu]\rho} = T_{\mu\nu}{}^a e_{a\rho}$. We use [..] to indicate the antisymmetric pair of indices. The structure equation then reads

$$T_{[\mu\nu]\rho} = \Omega_{[\mu\nu]\rho} + \omega_{\mu[\rho\nu]} - \omega_{\nu[\rho\mu]}\,. \tag{6.77}$$

We simply have $\frac{1}{2}D(D-1)$ equations for the $\frac{1}{2}D(D-1)$ independent 'unknowns' $\omega_{\mu[\rho\nu]}$. It is a standard exercise, outlined below, to find the unique solution

$$\begin{aligned}
\omega_{\mu[\nu\rho]} &= \omega_{\mu[\nu\rho]}(e) + K_{\mu[\nu\rho]}\,, &(6.78)\\
\omega_{\mu[\nu\rho]}(e) &= \tfrac{1}{2}(\Omega_{[\mu\nu]\rho} - \Omega_{[\nu\rho]\mu} + \Omega_{[\rho\mu]\nu}) = \omega_{\mu ab}(e)e^a_\nu e^b_\rho\,, &\\
\omega_\mu{}^{ab}(e) &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma{}^c\,, &(6.79)\\
K_{\mu[\nu\rho]} &= -\tfrac{1}{2}(T_{[\mu\nu]\rho} - T_{[\nu\rho]\mu} + T_{[\rho\mu]\nu})\,. &(6.80)
\end{aligned}$$

The unique torsion-free spin connection appears in (6.79), and what is conventionally called the contortion tensor is defined in (6.80).

Ex. 6.24 *Obtain the result in (6.78)-(6.80) from the following combination of (6.77) and two permutations:* $T_{[\mu\nu]\rho} - T_{[\nu\rho]\mu} + T_{[\rho\mu]\nu}$.

We now bring our discussion of Lorentz covariant derivatives of fields in the frame basis to a close by discussing a more general viewpoint. In general one can consider

[3] The infinitesimal parallel transport of a vector from the point with coordinates x^μ to the nearby point $x^\mu + \Delta x^\mu$ is defined by $\tilde{V}^a(x) = V^a(x) + \omega_\mu{}^a{}_b V^b(x)\Delta x^\mu$.

fields that transform in any finite dimensional matrix representation of $\mathrm{O}(D-1,1)$. The matrix representative of a proper Lorentz transformation takes the form

$$\Lambda(x) = \exp\left(\tfrac{1}{2}\lambda^{ab}(x) m_{[ab]}\right) . \tag{6.81}$$

The anti-symmetric array $\lambda^{ab}(x)$ contains the boost and rotation parameters of the transformation. The $\frac{1}{2}D(D-1)$ matrices $m_{[ab]}$ are a representation of the Lie algebra $\mathfrak{so}(D-1,1)$, as we saw e.g. in (1.31), and therefore satisfy the commutation relations (1.34) (in both these equations the $\mu, \nu, \ldots$ indices are now replaced by $a, b, \ldots$ indices). Row and column indices of the representation are suppressed in the equations in this paragraph. If $\Phi(x)$ is a field whose frame components transform in an n-dimensional representation, then it transforms under Lorentz transformations as

$$\Phi'(x) = \Lambda^{-1}(x)\Phi(x) , \tag{6.82}$$

and its covariant derivative is defined as

$$D_\mu \Phi(x) = \left(\partial_\mu + \tfrac{1}{2}\omega_\mu{}^{ab} m_{[ab]}\right)\Phi(x) . \tag{6.83}$$

We are particularly interested in two representations, The first is the defining D-dimensional vector representation in which the generators are given by (1.31).In this case the covariant derivative of a frame vector $V(x)$ computed using (6.83) and (1.31) is the same as in (6.74).

Ex. 6.25 *Show this.*

The second representation is the Dirac (or Majorana) spinor representation, which is very important for supergravity. As discussed in Ch. 3, it has dimension $2^{[D/2]}$, and the generators are the second rank Clifford matrices $\frac{1}{2}\gamma^{ab}$. In a gravitational theory, spinors must be described through their local frame components. The local Lorentz transformation rule and covariant derivative of a spinor field are

$$\begin{aligned} \Psi'(x) &= \exp\left(-\tfrac{1}{4}\lambda^{ab}(x)\gamma_{ab}\right)\Psi(x) , \\ D_\mu \Psi(x) &= \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}(x)\gamma_{ab}\right)\Psi(x) . \end{aligned} \tag{6.84}$$

6.8.2 The affine connection $\Gamma^\rho_{\mu\nu}$

Our next task is to transform Lorentz covariant derivatives of vector and tensor frame fields to the coordinate basis where they become covariant derivatives with respect to general coordinate transformations and take the familiar form in (6.67), but include torsion. It should be emphasized that no new structure on the manifold is required. We simply reexpress the information in the spin connection $\omega_\mu{}^a{}_b$ in a coordinate basis where it is contained in the affine connection $\Gamma^\rho_{\mu\nu}$.

We note that the quantity $\nabla_\mu V^\nu \equiv e_a^\nu D_\mu V^a$ is the transform to coordinate basis of a frame vector field and coordinate covariant vector field. It is necessarily a type $(1,1)$ tensor under coordinate transformations. The following manipulations will bring it to the form in (6.67)

$$\begin{aligned}\nabla_\mu V^\rho &\equiv e_a^\rho D_\mu V^a \\ &= e_a^\rho D_\mu (e_\nu^a V^\nu) \\ &= \partial_\mu V^\rho + e_a^\rho (\partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b) V^\nu \,. \end{aligned} \tag{6.85}$$

Ex. 6.26 *Show by similar manipulation that $\nabla_\mu V_\nu \equiv e_\nu^a D_\mu V_a$ also takes the form in (6.67), namely*

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - e_a^\rho (\partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b) V_\rho \,. \tag{6.86}$$

These results can be extended to Lorentz tensors of type (p,q). The conclusion is that the transformation to coordinate basis given by

$$\nabla_\mu T^{\rho_1 \cdots \rho_p}_{\nu_1 \cdots \nu_q} \equiv e^{\rho_1}_{a_1} \ldots e^{\rho_p}_{a_p} \, e^{b_1}_{\nu_1} \ldots e^{b_q}_{\nu_q} \, D_\mu T^{a_1 \cdots a_p}_{b_1 \cdots b_q} \tag{6.87}$$

defines a tensor of type $(p, q+1)$ with the properties of the conventional covariant derivative, and that the affine connection is related to the spin connection by

$$\Gamma^\rho_{\mu\nu} = e_a^\rho (\partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b) \,. \tag{6.88}$$

We now show that this definition of $\Gamma^\rho_{\mu\nu}$ does satisfy the expected properties, noting first that it can be rewritten as

$$\partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b - \Gamma^\sigma_{\mu\nu} e_\sigma^a = 0 \,. \tag{6.89}$$

This property is called the 'vielbein postulate' in some discussions of the spin connection. The vielbein postulate leads to the more familiar metric postulate or metricity property

$$\nabla_\mu g_{\nu\rho} \equiv \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = 0 \,, \tag{6.90}$$

as the following exercise shows.

Ex. 6.27 *Obtain (6.90) by contracting (6.89) with $e_{a\rho}$ and adding the same expression with ν and ρ interchanged.*

The metric postulate means that the metric tensor is covariantly constant. Hence lengths and scalar products of vectors are preserved under parallel transport, and covariant differentiation commutes with index raising, e.g. $\nabla_\mu V^\rho = g^{\rho\nu} \nabla_\mu V_\nu$.

Although it follows from our discussion that the connection defined by (6.88) transforms correctly under coordinate transformations, it is worthwhile for readers to show this explicitly.

Ex. 6.28 *Consider the definition (6.88) in two different coordinate systems and show that $\Gamma'^{\rho}_{\mu\nu}(x')$ and $\Gamma^{\rho}_{\mu\nu}(x)$ are related by the conventional transformation property*

$$\Gamma'^{\rho}_{\mu\nu}(x') = \frac{\partial x'^{\rho}}{\partial x^{\sigma}}\left(\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} x'^{\nu}} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\Gamma^{\sigma}_{\alpha\beta}\right) . \qquad (6.91)$$

Note that the first term in (6.91) cancels in the difference between any two connections, such as the infinitesimal variation $\delta\Gamma^{\rho}_{\mu\nu}(x)$, which occurs in the variational principles needed for gravitational field theories. Thus $\delta\Gamma^{\rho}_{\mu\nu}(x)$ transforms as a tensor. It follows from (6.72) that the same property holds for the variation $\delta\omega_{\mu ab}$ of the spin connection, which transforms as a type $(0,2)$ local Lorentz tensor.

Last, but hardly least, we substitute the explicit form (6.78) of the spin connection in the definition (6.88). After some calculation we obtain the explicit formula for $\Gamma^{\rho}_{\mu\nu}$

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \Gamma^{\rho}_{\mu\nu}(g) - K_{\mu\nu}{}^{\rho} , && (6.92)\\ \Gamma^{\rho}_{\mu\nu}(g) &= \tfrac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) . && (6.93) \end{aligned}$$

(Remember that K is antisymmetric in the last two indices, i.e. $K_{\mu\nu}{}^{\rho} = -K_{\mu}{}^{\rho}{}_{\nu}$). The first term, written in detail in (6.93), is the torsion-free connection, frequently called the Christoffel symbol and denoted by $\left\{{\rho \atop \mu\nu}\right\}$ instead of our $\Gamma^{\rho}_{\mu\nu}(g)$. It is well known that it is the unique *symmetric* affine connection that satisfies (6.90). When torsion is present the affine connection is not symmetric, rather

$$\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu} = -K_{\mu\nu}{}^{\rho} + K_{\nu\mu}{}^{\rho} = T_{\mu\nu}{}^{\rho} . \qquad (6.94)$$

The Lie derivative operation defined (for some examples) in (6.8) maps any type (p,q) tensor into another type (p,q) tensor, yet does not require a metric or connection. In most physical applications of differential geometry there is a natural metric and connection, and it is convenient to rewrite Lie derivatives in terms of covariant derivatives. *If the connection is symmetric*, and thus given by (6.93), this is easily done since the connection cancels pairwise among the various terms in (6.8), leading to

$$\begin{aligned} \mathcal{L}_V U^{\mu} &= V^{\rho}\nabla_\rho U^{\mu} - (\nabla_\rho V^{\mu})U^{\rho} , \\ \mathcal{L}_V \omega_{\mu} &= V^{\rho}\nabla_\rho \omega_{\mu} + (\nabla_\mu V^{\rho})\omega_{\rho} , \\ \mathcal{L}_V T^{\mu}_{\nu} &= V^{\rho}\nabla_\rho T^{\mu}_{\nu} - (\nabla_\rho V^{\mu})T^{\rho}_{\nu} + (\nabla_\nu V^{\rho})T^{\mu}_{\rho} . && (6.95) \end{aligned}$$

If there is torsion then there are additional terms.

Ex. 6.29 *Derive (6.95) from (6.8). Show that the Lie derivative of the metric tensor is (for vanishing K or T tensors)*

$$\mathcal{L}_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu . \qquad (6.96)$$

For 'mixed' quantities with both coordinate and frame indices, it is useful to distinguish between local Lorentz and coordinate covariant derivatives. Thus, for a vector-spinor field $\Psi_\mu(x)$, we define both

$$\begin{aligned} D_\mu \Psi_\nu &\equiv \left(\partial_\mu + \tfrac{1}{4}\omega_{\mu ab}\gamma^{ab}\right)\Psi_\nu \,, \\ \nabla_\mu \Psi_\nu &= D_\mu \Psi_\nu - \Gamma^\rho_{\mu\nu}\Psi_\rho \,. \end{aligned} \tag{6.97}$$

The first transforms as a local Lorentz spinor, but not a tensor, and the second transforms as a spinor and type $(0,2)$ tensor. After anti-symmetrization in $\mu\nu$, both derivatives yield $(0,2)$ tensors, but they differ by a torsion term, viz.

$$\nabla_\mu \Psi_\nu - \nabla_\nu \Psi_\mu = D_\mu \Psi_\nu - D_\nu \Psi_\mu - T_{\mu\nu}{}^\rho \Psi_\rho \,. \tag{6.98}$$

Similarly, the vielbein transforms as a Lorentz vector and a coordinate vector, such that ∇ contains both connections and the vielbein postulate (6.89) can be written as

$$\nabla_\mu e^a_\nu \;=\; \partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu - \Gamma^\sigma_{\mu\nu} e^a_\sigma \;=\; 0 \,. \tag{6.99}$$

6.8.3 Partial integration

The covariant derivatives without torsion are convenient to make integration by parts. The key relation on the Christoffel connection is

$$\partial_\mu \sqrt{-g} = \sqrt{-g}\,\Gamma^\rho_{\rho\mu}(g) \,. \tag{6.100}$$

It implies, using the definitions (6.67) and (6.92),

$$\int \mathrm{d}^D x \sqrt{-g}\,\nabla_\mu V^\mu = \int \mathrm{d}^D x\, \partial_\mu \left(\sqrt{-g}\, V^\mu\right) - \int \mathrm{d}^D x \sqrt{-g}\, K_{\nu\mu}{}^\nu V^\mu \,. \tag{6.101}$$

Ex. 6.30 *Observe how the metric dependent connections $\Gamma^\rho_{\mu\nu}$ required in the covariant derivative on the left-hand side is 'automatically constructed' from derivatives of the metric on the right-hand side.*

The first term is the total derivative that we often neglect in field theory due to the assumption of vanishing fields at infinity. The second term shows the violation of the manipulations of integration by parts in case of torsion. It is proportional to

$$K_{\nu\mu}{}^\nu = -T_{\nu\mu}{}^\nu \,. \tag{6.102}$$

6.9 The second structure equation and the curvature tensor

In Sec. 6.8.1 we discussed the fact that the spin connection $\omega_{\mu ab}$ transforms as a Yang-Mills gauge potential for the group $O(D-1,1)$, see (6.72). It then follows that the quantity

$$R_{\mu\nu ab} \equiv \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b \tag{6.103}$$

has the properties of a Yang-Mills field strength; transforming as a type $(0,2)$ Lorentz tensor under local Lorentz transformations. See (6.74). Because of anti-symmetry in $\mu\nu$, it is also a (0,2) tensor under general coordinate transformation, called the curvature tensor. Thus we can define the curvature 2-form

$$\rho^{ab} = \frac{1}{2} R_{\mu\nu}{}^{ab}(x) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,. \tag{6.104}$$

Using (6.103) it is easy to see that the curvature 2-form is related to the connection 1-form by

$$\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = \rho^{ab} \,. \tag{6.105}$$

It is this equation that is known as the second Cartan structure equation. Needless to say, it is the metric, curvature , and (if present) the torsion tensor, which carry the basic local information about a space-time manifold that are needed for gravitational physics.

Ex. 6.31 *Consider the effect on $R_{\mu\nu ab}$ of an infinitesimal variation of the spin connection. Show that*

$$\delta R_{\mu\nu ab} = D_\mu \delta\omega_{\nu ab} - D_\nu \delta\omega_{\mu ab} \,. \tag{6.106}$$

This fact is related to the discussion below Ex. 6.28, which argued that $\delta\omega_{\mu ab}$ transforms as a tensor.

One immediate application of the Cartan structure equations (6.70),(6.105) is to derive the Bianchi identities for the curvature tensor.

Ex. 6.32 *Apply the exterior derivative to (6.70)and (6.105) and obtain, respectively the 3-form relations*

$$\begin{aligned} &\rho^{ab} \wedge e_b = \mathrm{d}T^a + \omega^{ab} \wedge T_b \,, \\ &\mathrm{d}\rho^{ab} + \omega^a{}_c \wedge \rho^{cb} - \rho^{ac} \wedge \omega_c{}^b = 0 \,. \end{aligned} \tag{6.107}$$

Show that the components of these 3-form equations read, using $R_{\mu\nu\rho}{}^a = R_{\mu\nu b}{}^a e^b_\rho$,

$$\begin{aligned} &R_{\mu\nu\rho}{}^a + R_{\nu\rho\mu}{}^a + R_{\rho\mu\nu}{}^a = -D_\mu T_{\nu\rho}{}^a - D_\nu T_{\rho\mu}{}^a - D_\rho T_{\mu\nu}{}^a \,, \\ &D_\mu R_{\nu\rho}{}^{ab} + D_\nu R_{\rho\mu}{}^{ab} + D_\rho R_{\mu\nu}{}^{ab} = 0 \,. \end{aligned} \tag{6.108}$$

The derivatives D_μ etc. in these equations are Lorentz covariant derivatives and contain only the spin connection. However, each relation is a type $(0,3)$ coordinate tensor because of anti-symmetry. The first relation is the conventional first Bianchi identity for the curvature tensor, but corrected by torsion terms on the right side. This has no analogue in Yang-Mills theory. The second relation is the usual Bianchi identity of non-abelian gauge theory, and is called the second Bianchi identity for the curvature.

The commutator of Lorentz covariant derivatives leads to important relations, in both gauge theory and gravity, which we call Ricci identities. We write the identity for a field $\Phi(x)$ transforming in a representation of the proper Lorentz group with generators M^{ab} together with special cases of interest, namely frame vector fields $V^a(x)$ and spinor fields $\Psi(x)$,

$$\begin{aligned} [D_\mu, D_\nu]\Phi &= \tfrac{1}{2}R_{\mu\nu ab}M^{ab}\Phi\,, \\ [D_\mu, D_\nu]V^a &= R_{\mu\nu}{}^a{}_b V^b\,, \\ [D_\mu, D_\nu]\Psi &= \tfrac{1}{4}R_{\mu\nu ab}\gamma^{ab}\Psi\,. \end{aligned} \tag{6.109}$$

There are also generalized Ricci identities for the commutator of coordinate covariant derivatives, but they contain torsion terms. For example, for vector fields $V^\rho(x)$, one can derive by direct computation

$$[\nabla_\mu, \nabla_\nu]V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma - T_{\mu\nu}{}^\sigma \nabla_\sigma V^\rho\,, \tag{6.110}$$

in which the curvature tensor appears as

$$R_{\mu\nu}{}^\rho{}_\sigma = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\tau}\Gamma^\tau_{\nu\sigma} - \Gamma^\rho_{\nu\tau}\Gamma^\tau_{\mu\sigma}\,. \tag{6.111}$$

Ex. 6.33 *Verify (6.110). Show that* $R_{\mu\nu}{}^\rho{}_\sigma = R_{\mu\nu ab}e^{a\rho}e^b_\sigma$ *by evaluating* $[\nabla_\mu, \nabla_\nu]e^\rho_a = 0$. *Prove the generalized second Bianchi identity*

$$\nabla_\mu R_{\nu\rho}{}^{\sigma\tau} + \nabla_\nu R_{\rho\mu}{}^{\sigma\tau} + \nabla_\rho R_{\mu\nu}{}^{\sigma\tau} = T_{\mu\nu}{}^\xi R_{\xi\rho}{}^{\sigma\tau} + T_{\nu\rho}{}^\xi R_{\xi\mu}{}^{\sigma\tau} + T_{\rho\mu}{}^\xi R_{\xi\nu}{}^{\sigma\tau}\,. \tag{6.112}$$

Ex. 6.34 *Derive*

$$\delta R_{\mu\nu}{}^\rho{}_\sigma = \nabla_\mu \delta\Gamma^\rho_{\nu\sigma} - \nabla_\nu \delta\Gamma^\rho_{\mu\sigma}\,, \tag{6.113}$$

which is closely related to (6.106)

The Ricci tensor is defined as $R_{\mu\nu} = R_\mu{}^\sigma{}_{\nu\sigma}$ and the curvature scalar is $R = g^{\mu\nu}R_{\mu\nu}$.

Ex. 6.35 *Show that* $R_{\mu\nu} = R_{\nu\mu}$ *if and only if there is no torsion. Show that the same holds for the symmetry equation* $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$.

Below is an exercise designed to show that the Hilbert action for general relativity, $S \sim \int \mathrm{d}^D x\sqrt{-g}R$ can be written as the integral of a volume form involving the frame and curvature forms. It is this form of the action that reveals how the mathematical framework of connections with torsion is realized in the physical setting of gravity coupled to fermions. We will explore this in the next chapter, but the exercise brings together some of the ideas discussed in this chapter.

Ex. 6.36 *Show that*

$$\frac{1}{(D-2)!}\int \varepsilon_{abc_1\ldots c_{(D-2)}} e^{c_1} \wedge \cdots \wedge e^{c_{(D-2)}} \wedge \rho^{ab} = \int \mathrm{d}^D x\sqrt{-g}R\,. \tag{6.114}$$

Hint: express $\rho^{ab} = \frac{1}{2}R_{cd}{}^{ab}e^c \wedge e^d$ *and use (6.57) and (6.48).*

6.10 The nonlinear σ-model

The principal application of differential geometry in this book is to theories of gravity and supergravity in spacetime, viewed as a differentiable manifold. The matter couplings in these theories, including fermions and p-form gauge fields also require the ideas of differential geometry reviewed in this chapter. There is still another physical application of differential geometry that is important in supergravity. This application involves the dynamics of scalar fields in flat spacetime. Physicists usually call it the nonlinear σ-model because it first appeared as a description of the low-energy interactions of the triplet of $\vec{\pi}$ mesons which involve the composite field $\sigma = (f_\pi^2 + \vec{\pi}^2)^{1/2}$. The notation $\vec{\pi}$ indicates that the three pion fields transform in the fundamental representation of an SO(3) internal symmetry. In mathematics the same type of field theory may be called the theory of harmonic maps.

To begin the discussion let us consider an n-dimensional Riemannian manifold M_n. The name Riemannian means that M_n has a smooth Euclidean signature metric. In local coordinates called $\phi^i,\ i = 1, 2 \ldots, n$, the metric tensor is $g_{ij}(\phi)$. In the nonlinear σ-model the coordinates are fields $\phi(x)$ in which the $x^\mu,\ \mu = 1, 2, \ldots D$ are the Cartesian coordinates of a flat spacetime M_D. We assume that M_D is of Minkowski signature, but our discussion requires little or no change for Euclidean signature. Thus we deal with maps from spacetime to the internal space or target space M_n, see Fig. 6.3.

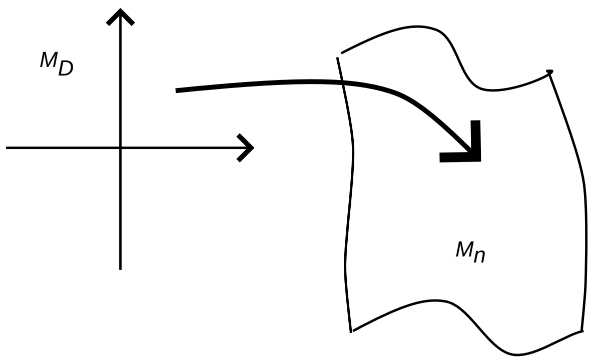

Fig. 6.3. *Scalar fields as maps from spacetime to the target manifold.*

We postulate that the dynamics of these maps is governed by the action

$$S[\phi] = -\frac{1}{2} \int \mathrm{d}^D x \, g_{ij}(\phi) \eta^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \,. \tag{6.115}$$

Spacetime indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$.

Ex. 6.37 *Compute the Euler variation of this action. Observe that the Christoffel connection (6.93) is obtained in the process and express the equation of motion in* *the form:*

$$\Box\phi^i + \Gamma^i_{jk}(\phi)\partial^\mu\phi^i\partial_\mu\phi^j \ = 0\,. \tag{6.116}$$

For Euclidean signature the d'Alembertian $\Box$ *is replaced by the Laplacian* ∇^2*. This is why the solutions are called harmonic maps; they are a nonlinear generalization of harmonic functions. Indeed the conventional Laplace equation is obtained in the special case when the target space is also flat.*

Suppose that we use a different set of coordinates $\phi'^i(\phi)$ on the target space M_n. In these coordinates the metric tensor becomes

$$g'_{ij}(\phi') = \frac{\partial\phi^k}{\partial\phi'^i}g_{k\ell}(\phi)\frac{\partial\phi^\ell}{\partial\phi'^j}\,, \tag{6.117}$$

and the action changes to

$$S'[\phi'] \ = \ -\frac{1}{2}\int \mathrm{d}^D x\, g'_{ij}(\phi')\eta^{\mu\nu}\partial_\mu\phi'^i\partial_\nu\phi'^j\,. \tag{6.118}$$

This means that the equation of motion for $\phi'^i(x)$ has the same structure form as in (6.116), but with metric and connection expressed in the new coordinates. Our viewpoint is that significant physical information may be expressed in either set of coordinates.

It is interesting to consider the case when the 'spacetime' degenerates to the one dimensional line with a time coordinate t. The action (6.115) is still perfectly sensible and the equation of motion (6.116) reduces to a set of n coupled nonlinear differential equations

$$\frac{\mathrm{d}^2\phi^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\frac{\mathrm{d}\phi^j}{\mathrm{d}t}\frac{\mathrm{d}\phi^k}{\mathrm{d}t} = 0\,. \tag{6.119}$$

Readers should recognize this as the equation for geodesic[4] curves on M_n.

Physically, the nonlinear sigma model on the line describes the motion of a particle of mass m in a curved target space. Indeed, any solution $\phi^i(t)$ of (6.119) describes a possible trajectory of a particle. For the special case when the ϕ^i are Cartesian coordinates of a flat target space, the equation (6.119) reduces to the statement that $\mathrm{d}^2\phi^i/\mathrm{d}t^2 = 0$. The particle moves freely with no acceleration. In this case the kinetic energy $E = m\delta_{ij}(\mathrm{d}\phi^i/\mathrm{d}t)(\mathrm{d}\phi^j/\mathrm{d}t)/2$ is conserved. There is also a conserved energy for motion on a general target space.

Ex. 6.38 *Show that* $E \equiv \frac{1}{2}m\, g_{ij}(\phi)(\mathrm{d}\phi^i/\mathrm{d}t)(\mathrm{d}\phi^j/\mathrm{d}t)$ *is conserved, i.e.* $\mathrm{d}E/\mathrm{d}t = 0$*, for any solution of the equation of motion (6.119).*

---

[4] Geodesics are an important subject in differential geometry both for true spacetimes and Euclidean signature manifolds. We do not discuss geodesics since they are peripheral to the main thrust of this book. Readers are referred to a general relativity textbook such as [44, 45].

Many different target spaces appear in specific applications of the nonlinear σ-model. They may be compact, for example, the spheres S^n, or non-compact. As an example that will be relevant later in the book, we discuss the Poincaré plane. This is the non-compact two-dimensional manifold with coordinates X, Y with $Y > 0$. The Riemannian metric is defined by the line element

$$\mathrm{d}s^2 = \frac{\mathrm{d}X^2 + \mathrm{d}Y^2}{Y^2} . \tag{6.120}$$

The action that describes particle motion on the Poincaré plane is

$$S = \frac{1}{2} \int \mathrm{d}t \frac{1}{Y^2} (\dot{X}^2 + \dot{Y}^2) , \tag{6.121}$$

in which we have introduced the notation $\dot{X} = \mathrm{d}X/\mathrm{d}t$.

The purpose of the following exercise is to illustrate a method to compute the connection components of a specific metric by what might be called the 'geodesic' method. It is frequently faster to use this method rather than the general definition (6.93).

Ex. 6.39 *Compute the variation of the action (6.121) quite directly to obtain*

$$\delta S = - \int \mathrm{d}t \Big(\delta X \left[\frac{1}{Y^2} \ddot{X} - \frac{2}{Y^3} \dot{Y} \dot{X} \right] + \delta Y \left[\frac{1}{Y^2} \ddot{Y} - \frac{1}{Y^3} (\dot{X}^2 - \dot{Y}^2) \right] \Big) . \tag{6.122}$$

The equations of motion then read:

$$\ddot{X} - \frac{2}{Y} \dot{Y} \dot{X} = 0 , \qquad \ddot{Y} + \frac{1}{Y} (\dot{X}^2 - \dot{Y}^2) . \tag{6.123}$$

Compare with the general form (6.119) and identify the connection coefficients for the Poincaré plane metric in an obvious notation as:

$$\begin{aligned} \Gamma^X_{XY} &= \Gamma^X_{YX} = -\frac{1}{Y} , \qquad \Gamma^X_{XX} = \Gamma^X_{YY} = 0 , \\ \Gamma^Y_{XX} &= \frac{1}{Y} = -\Gamma^Y_{YY} , \qquad \Gamma^Y_{XY} = \Gamma^Y_{YX} = 0 . \end{aligned} \tag{6.124}$$

Ex. 6.40 *Here is an exercise on geodesics of the Poincaré plane metric. Show that there are geodesics that are the straight vertical lines $X = x_0$, $Y = y_0 \mathrm{e}^{kt}$. Can you show that the general geodesic curve is a semicircle of any radius r_0 whose center is at any point $(X_0, 0)$ on the X-axis of the plane? If not, then wait until the next section.*

6.11 Symmetries and Killing vectors

It is frequently and correctly said that the equations of a gravitational theory are invariant under general coordinate transformations, and the same is true for the equations of the nonlinear σ-model. What this really means, however, is that the equations are constructed from the same elements, vectors, tensors, covariant derivatives, etc. in any coordinate system. This has some content, but it is only a special class of coordinate transformations that lead to symmetries of a system. As discussed in Ch. 1, a symmetry is a transformation that takes one solution of the equations of motion into another. The equations themselves must take the same form in two coordinate transformations related by a symmetry. For each global symmetry there is a conserved Noether current.[5]

Although the situation is similar for both gravitation and the nonlinear σ-model, we prefer to discuss symmetries in the latter setting. The equations of motion (6.116) (and also (6.119)) take the same form in the two sets of coordinates ϕ^i and ϕ'^i if the metric g'_{ij} in (6.117) has the same functional form as the original metric, i.e. if

$$g'_{ij}(\phi) \;=\; g_{ij}(\phi)\,. \tag{6.125}$$

A symmetry transformation of the metric is frequently called an isometry.

We now assume that the coordinate transformation is continuous. This means that the functional relation $\phi'^i(\phi)$ depends continuously on parameters θ^A. In the limit of small θ^A it is assumed that the transformation is close to the identity and thus approximated by $\phi'^i = \phi^i + \theta^A k_A{}^i(\phi)$. The $k_A{}^i(\phi)$ are vector fields, which are the generators of the coordinate transformation. Thus we have an infinitesimal coordinate transformation, whose effect on the various tensor fields on the target space is given by Lie derivatives as defined in (6.10). In particular, the condition (6.125) for a symmetry holds to first order in θ^A if

$$\delta g_{ij} \;\equiv\; g'_{ij}(\phi) - g_{ij}(\phi) \;=\; -\theta^A \mathcal{L}_{k_A}\, g_{ij} \;=\; 0\,. \tag{6.126}$$

This means that each $k_A{}^i$ is a vector field that satisfies

$$\nabla_i k_{jA} + \nabla_j k_{iA} = 0\,, \qquad k_{iA} = g_{ij} k^j_A\,, \qquad \nabla_i k_{jA} = \partial_i k_{jA} - \Gamma^k_{ij}(g) k_{kA}\,, \tag{6.127}$$

where ∂_i is the ordinary derivative with respect to ϕ^i and $\Gamma^k_{ij}(g)$ is the Christoffel connection on M_n (we are not dealing with torsion here). A vector field with this property is called a Killing vector, and each Killing vector is associated with a symmetry. To summarize; for each continuous isometry of the target space metric there is a Killing vector k^i_A.

Symmetries of the nonlinear σ-model are Noether symmetries[6] as discussed in Sec. 1.3. The Killing symmetries are a special case of (1.65) in which the infinitesimal internal symmetry variations of the fields $\phi^i(x)$ take the form

$$\delta(\theta)\phi^i = \theta^A k_A{}^i(\phi)\,. \tag{6.128}$$

[5] We restrict to symmetries that leave the action invariant, see (1.5). Symmetries that transform solutions of the equations of motion to other solutions may not have a Noether current.

[6] Killing vectors are not related to the quantity K^μ_A of Sec. 1.3. That quantity denoted the total derivative that appears for spacetime transformations of a Lagrangian system. Killing symmetries are internal symmetries, so K^μ_A vanishes.

As opposed to the symmetries discussed in Ch. 1, the Killing symmetries are frequently nonlinear in the fields. It is easy to see that this transformation leaves the σ-model action invariant. To first order in θ^A, the transformation of (6.115) is[7]

$$\begin{aligned}
\delta S[\phi] &= S[\phi'] - S[\phi] \\
&= \int \mathrm{d}^D x\, \theta^A \left[\frac{\partial_\mu g_{ij}}{\partial \phi_k} k_A{}^k\, \partial_\mu \phi^i\, \partial^\mu \phi^j \right. \\
&\qquad \left. + g_{ij}(\phi)\, \frac{\partial k_A{}^i}{\partial \phi^k} \partial_\mu \phi^k\, \partial^\mu \phi^j + g_{ij}(\phi)\, \partial_\mu \phi^i\, \frac{\partial k_A{}^j}{\partial \phi^k} \partial_\mu \phi^k \right] \\
&= \int \mathrm{d}^D x\, \theta^A \left[(\nabla_i k_{jA} + \nabla_j k_{iA}) \partial^\mu \phi^i \partial_\mu \phi^j \right] = 0\,. \qquad (6.129)
\end{aligned}$$

Thus the formula (1.69) applies immediately and tells us that

$$J^\mu_A = g_{ij}(\partial^\mu \phi^i) k_A{}^j(\phi)\,, \qquad (6.130)$$

is a conserved Noether current for the symmetry.

Ex. 6.41 *It is an instructive exercise to show explicitly that the current satisfies $\partial_\mu J^\mu_A = 0$, if k^i_A is a Killing vector and $\phi^i(x)$ is any solution of the σ-model equations of motions (6.116).*

As described in Sec.6.2 it is frequently useful to encode a Killing vector in a differential operator

$$k_A \equiv k^j_A \frac{\partial}{\partial \phi^j}\,. \qquad (6.131)$$

One can then define the action of the symmetry transformation of any scalar function $f(\phi)$ on the target space as $\delta f \equiv \theta^A k_A f$. The elementary variation (6.128) is then the special case when $f(\phi) = \phi^i$.

The isometry group of a given metric on M_n is frequently non-abelian. Then there are several linearly independent Killing vectors. As in Sec.6.2, the Lie algebra is determined from the commutators of the differential operators, i.e.

$$[k_A\,, k_B] = f_{AB}{}^C k_C = f_{AB}{}^C k^i_C \frac{\partial}{\partial \phi^i}\,. \qquad (6.132)$$

The vector field k^i_C is also a Killing vector. With the transformation parameters included, one finds that

$$\left[\theta^A_1 k_A\,, \theta^B_2 k_B\right] = \theta^C_3 k_C\,, \qquad \theta^C_3 = \theta^B_2 \theta^A_1 f_{AB}{}^C\,. \qquad (6.133)$$

Thus nonlinear Killing symmetries compose in the same way as the linear internal symmetries in (1.22). Although Killing vectors $k_A{}^i(\phi)$ are generically nonlinear

[7] The symmetry acts on the dynamical variables ϕ^i of the nonlinear σ-model, and the transformation of $g_{ij}(\phi)$ is induced by the variations $\delta\phi^i$.

functions of the ϕ^k, the isometry group of a manifold M_n can include linearly realized matrix symmetries, which act in the same way as the internal symmetries of Sec. 1.2.2. In this case the Killing vectors ${k_A}^i$ are related to the matrix generators t_A by ${k_A}^i(\phi) = -(t_A)^i{}_j\phi^j$ and the differential operators by

$$k_A = -(t_A)^i{}_j\phi^j\frac{\partial}{\partial\phi^i}\,. \tag{6.134}$$

Ex. 6.42 *Check that for products of Killing vectors of the form (6.134) we have*

$$k_Ak_B = (t_Bt_A)^i{}_j\phi^j\frac{\partial}{\partial\phi^i} + \text{ a term symmetric in } A \leftrightarrow B\,. \tag{6.135}$$

This interchange of the matrices is the underlying reason for introducing − signs in the definitions of transformations defined by matrices already started in (1.15).

Finally we discuss how to recognize when a given metric has symmetries and how to obtain the Killing vectors. One simple situation occurs when the metric tensor g_{ij} does not depend on a particular coordinate, say ϕ^k. Then 'translation' $\phi^k \to \phi^k + c$ is an isometry, and $k_k = \partial/\partial\phi^k$ is the differential operator which generates the symmetry. In general, it may not be easy to examine a metric tensor and determine its isometries, Indeed there are target space metrics that have important applications in supergravity and string theory and have no continuous isometries.[8]

The isometry group of the Poincaré plane metric (6.120) is the Lie group $\mathrm{SL}(2,\mathbb{R})$. Its applications in supergravity and string theory are fundamental, so it is worthwhile to study. The group may be defined as the group of 2×2 real matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \qquad ad - bc = 1\,. \tag{6.136}$$

First we will study the action of finite group transformations and then identify three Killing vectors as infinitesimal symmetries. For this purpose it is very convenient to use a complex coordinate $Z = X + \mathrm{i}Y$ on the upper half-plane. The line element (6.120) becomes

$$\mathrm{d}s^2 = \frac{\mathrm{d}Z\mathrm{d}\bar{Z}}{Y^2}\,. \tag{6.137}$$

$\mathrm{SL}(2,\mathbb{R})$ transformations act as nonlinear maps

$$Z \to Z' = \frac{aZ+b}{cZ+d} = X' + \mathrm{i}Y' \tag{6.138}$$

of the upper half-plane onto itself. Up to some trivial redefinitions these are the transformations mentioned in (4.53) for a scalar field that can appear in the kinetic

[8] For interested readers we note that compact Calabi-Yau metrics have no isometries.

terms of a Maxwell gauge field. It is a matter of straightforward algebra to show that

$$
\begin{aligned}
X' &= \frac{ac(X^2+Y^2)+(ad+bc)X+bd}{|cZ+d|^2}\,, \\
Y' &= \frac{Y}{|cZ+d|^2}\,, \\
\mathrm{d}Z' &= \frac{\mathrm{d}Z}{(cZ+d)^2}\,.
\end{aligned} \tag{6.139}
$$

Then, by direct substitution we find that the line element behaves as

$$
\mathrm{d}s^2 = \frac{\mathrm{d}Z\mathrm{d}\bar{Z}}{Y^2} = \frac{\mathrm{d}Z'\mathrm{d}\bar{Z}'}{Y'^2}\,, \tag{6.140}
$$

corresponding to a Lagrangian

$$
\mathcal{L} = \frac{\partial_\mu Z \partial^\mu \bar{Z}}{(Z-\bar{Z})^2}\,. \tag{6.141}
$$

It has the same form in both coordinate systems, so finite $\mathrm{SL}(2,\mathbb{R})$ transformations are indeed isometries.

Ex. 6.43 *The straightforward algebra is highly recommend. Note that Y' is positive whenever Y is positive. This shows that the transformation maps the upper half-plane into itself. It would fail for complex a, b, c, d.*

We can use the finite isometry (6.138) in a simple and elegant way to obtain the general geodesic of the Poincaré metric. In Ex. 6.40 readers found vertical lines with exponential t dependence are particularly simple geodesics. For the present purpose, it is sufficient to consider the special case $Z_0(t) = \mathrm{i}e^t$. Under the isometry (6.138) this curve is mapped to

$$
Z(t) = \frac{a\mathrm{i}e^t + b}{c\mathrm{i}e^t + d}\,. \tag{6.142}
$$

Since a symmetry maps any solution of the equations of motion into another solution, the curves $Z(t) = X(t) + \mathrm{i}Y(t)$ are also solutions of (6.119) for every choice of a, b, c, d. Thus we obtain a large family of geodesics! In the theory of complex variables, the maps are well known conformal transformation which map straight lines into circles.

We could obtain the Killing vectors by expanding (6.138) around the unit transformations. However, as an illustrative example, in the following exercise the student is invited to start again from the metric (i.e. the Lagrangian) and derive the Killing vectors systematically.

Ex. 6.44 *Consider Z and $\bar{Z}$ as the independent fields, rather than X and Y. The metric components are then*

$$g_{ZZ} = g_{\bar{Z}\bar{Z}} = 0\,, \qquad g_{Z\bar{Z}} = g_{\bar{Z}Z} = \frac{1}{(Z-\bar{Z})^2}\,. \tag{6.143}$$

Prove that the non-vanishing components of the Christoffel connection are Γ^Z_{ZZ} and its complex conjugate $\Gamma^{\bar{Z}}_{\bar{Z}\bar{Z}}$. Calculate them and show that there are three Killing vectors (or symmetries):

$$k_1^Z = 1\,, \qquad k_2^Z = Z\,, \qquad k_3^Z = Z^2\,, \tag{6.144}$$

each with conjugate components $k_A^{\bar{Z}}$. Show that their Lie brackets give a Lie algebra whose non-vanishing structure constants are

$$f_{12}{}^1 = 1\,, \qquad f_{13}{}^2 = 2\,, \qquad f_{23}{}^3 = 1\,. \tag{6.145}$$

This is the algebra $\mathfrak{su}(1,1) = \mathfrak{so}(2,1) = \mathfrak{sl}(2)$.

7
The first and second order formulations of general relativity

In this chapter we discuss and compare the first and second order formulations of gravitational dynamics. The most familiar setup is the second order formalism in which the metric tensor or the frame field is the dynamical variable describing gravity. If fermions are present one must use the frame field. The curvature tensor and covariant derivatives are constructed from the torsion-free connections $\Gamma^{\rho}_{\mu\nu}(g)$ and $\omega_{\mu ab}(e)$, see (6.79),(6.92). The name 'second order' refers to the fact that the gravitational field equation is second order in derivatives of $g_{\mu\nu}$ or e^{a}_{μ}.

In the first order (or Palatini) formalism one starts with an action in which e^{a}_{μ} and $\omega_{\mu ab}$ are independent variables, and the Euler-Lagrange equations are first order in derivatives. Without matter, the solution of the $\omega_{\mu ab}$ field equation simply sets $\omega_{\mu ab} = \omega_{\mu ab}(e)$. When this result is substituted in the field equation for e^{a}_{μ}, one finds the conventional Einstein equations, exactly as they emerge in the second order formulation for frame fields. When gravity is coupled to spinor fields, the $\omega_{\mu ab}$ field equation contains terms bilinear in the spinors, and its solution is $\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}$, as in (6.78), with contortion tensor determined as a bilinear expression in the spinor fields. It is in this way that the mathematical formalism of connections with torsion is realized in physics.

When the result $\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}$ is substituted in the other field equations, one finds that the full effects of the torsion could have been obtained in the second order formalism with an added set of quartic fermion terms in the action. One may then ask why bother? Why introduce the complication of torsion when its physical effects can be described more conventionally? The answer is that the proof that a supergravity theory is invariant under local supersymmetry transformations is greatly simplified by the fact that quartic terms in the gravitino field can be organized within the connection $\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}$.

7.1 Second order formalism for gravity and bosonic matter

In this section we review the conventional treatment of gravity coupled to bosonic matter. To be definite we consider the simplest and most common matter fields, a real scalar field $\phi(x)$ and an abelian gauge field with gauge potential $A_\mu(x)$ and field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The action functional of the coupled system is the sum of the Hilbert action for gravity plus the action for matter fields:

$$\begin{aligned} S &= \int \mathrm{d}^D x \sqrt{-\det g}\left(\frac{1}{2\kappa^2} g^{\mu\nu} R_{\mu\nu}(g) + L\right), \\ L &= -\tfrac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\,. \end{aligned} \tag{7.1}$$

The constant $\kappa^2 = 8\pi G_N$ is the gravitational coupling constant[1]. The Ricci tensor $R_{\mu\nu}(g)$ contains the torsion-free connection.

Ex. 7.1 *Prove that κ^2 should have dimension of a length to the power $(D-2)$ in order that the action be dimensionless.*

The matter field Lagrangian was obtained from the Minkowski space kinetic Lagrangians discussed in Chapters 1 and 4 by the minimal coupling prescription. This consists of the rules:
i. replace the Minkowski metric $\eta_{\mu\nu}$ by the spacetime metric tensor $g_{\mu\nu}(x)$.
ii. replace each derivative ∂_μ by the appropriate covariant derivative ∇_μ with connection $\Gamma^\rho_{\mu\nu}(g)$.
iii. Use the canonical volume form (6.48).
These rules incorporate the equivalence principle of general relativity and the principle of general covariance. The Lagrangian transforms as a scalar under coordinate transformations, so the matter action is invariant. The second rule is not really needed for the scalar and vector gauge fields of our matter system. For scalars, no connection is needed since $\partial_\mu\phi = \nabla_\mu\phi$. The same is true for the gauge field, since $F_{\mu\nu}$ can be written as $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The Christoffel connection cancels by symmetry.

The equations of motion are obtained by requiring that the action is stationary with respect to variations of the three independent fields.

Ex. 7.2 *Show that the scalar and gauge field equations are*

$$\begin{aligned} \partial_\mu(\sqrt{-\det g}\, g^{\mu\nu}\partial_\nu\phi) &= \sqrt{-\det g}\, g^{\mu\nu}\,\nabla_\mu\partial_\nu\phi = 0\,, \\ \partial_\mu(\sqrt{-\det g}\, F^{\mu\nu}) &= \sqrt{-\det g}\,\nabla_\mu F^{\mu\nu} = 0\,. \end{aligned} \tag{7.2}$$

Use (6.101) without the torsion term.

[1] Or: (for $D=4$) $\kappa = M_P^{-1}$ with $M_P \equiv M_{\mathrm{Planck}}/\sqrt{8\pi} \sim 2\times 10^{18}$ GeV.

Ex. 7.3 *Obtain the Einstein field equations*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}\,, \tag{7.3}$$

$$T_{\mu\nu} \equiv -2 \frac{1}{\sqrt{-\det g}} \frac{\delta(\sqrt{-\det g}\, L)}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi + F_\mu{}^\rho F_{\nu\rho} + g_{\mu\nu} L\,. \tag{7.4}$$

from

$$\delta S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x [\delta(\sqrt{-\det g} g^{\mu\nu}) R_{\mu\nu} + \sqrt{-\det g} g^{\mu\nu} \delta R_{\mu\nu} + \ldots]\,, \tag{7.5}$$

where ... indicates the metric variation of the matter field terms that give the stress tensor $T_{\mu\nu}$ in (7.4). Note that (6.113) implies that $\delta R_{\mu\nu} = \nabla_\rho \delta \Gamma^\rho_{\mu\nu} - \nabla_\mu \delta \Gamma^\rho_{\nu\rho}$. Thus the term in (7.5) containing $\delta R_{\mu\nu}$ is the integral of a total derivative, which vanishes.

7.2 Gravitational fluctuations of flat spacetime

In the absence of matter, the gravitational field satisfies the equation

$$R_{\mu\nu} = 0\,. \tag{7.6}$$

One solution is flat Minkowski spacetime with metric tensor $g_{\mu\nu} = \eta_{\mu\nu}$. In this section we study weak gravitational perturbations of Minkowski space, perturbations that are described by metrics of the form

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x). \tag{7.7}$$

We will work systematically to first order in the gravitational coupling constant κ. The symmetric tensor fluctuation $h_{\mu\nu}(x)$ is a gauge field. We will count the physical degrees of freedom and determine its propagator, as we have done for the free vector A_μ and vector-spinor ψ_μ fields in Ch. 4 and 5, respectively. The free equation of motion for $h_{\mu\nu}$ is the linearization of the exact Ricci tensor, obtained by substituting the metric ansatz (7.7) and retaining terms of order κ. We ask readers to do this as an exercise.

Ex. 7.4 *Obtain the linearized Christoffel connection*

$$\Gamma^{\rho\,\mathrm{Lin}}_{\mu\nu} = \tfrac{1}{2}\kappa\, \eta^{\rho\sigma} (\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu})\,. \tag{7.8}$$

Indices are raised using $\eta^{\mu\nu}$. Then use the variational formula $\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\mu \delta \Gamma^\rho_{\nu\rho}$ to determine the linearized Ricci tensor

$$R^{\mathrm{Lin}}_{\mu\nu} = -\tfrac{1}{2}\kappa\, [\Box h_{\mu\nu} - \partial^\rho(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho}) + \partial_\mu \partial_\nu h^\rho_\rho]\,. \tag{7.9}$$

The gauge properties of the free field system are obtained by linearizing the exact transformation rules under infinitesimal diffeomorphisms (See Sec. 6.2)

$$\begin{aligned}\delta g_{\mu\nu} &= \kappa\left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\right), \\ \delta R_{\mu\nu} &= \kappa\left(\xi^\rho \nabla_\rho R_{\mu\nu} + \nabla_\mu \xi^\rho R_{\rho\nu} + \nabla_\nu \xi^\rho R_{\mu\rho}\right),\end{aligned} \tag{7.10}$$

in which we have specified that the arbitrary vector $\xi(x)$ is of order κ (so that the $\eta_{\mu\nu}$ term in (7.7) is not changed). Since the right-hand sides of these transformation rules are of order κ or higher, the linearization is simply

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu\,. \tag{7.11}$$

Thus we learn that the gravitational fluctuation field is a gauge field with gauge transformation (7.11), which obeys the gauge-invariant wave equation

$$R^{\mathrm{Lin}}_{\mu\nu} = 0\,. \tag{7.12}$$

Ex. 7.5 *Show explicitly that $R^{\mathrm{Lin}}_{\mu\nu}$ in (7.9) is invariant under the gauge transformation (7.11).*

For most applications of the linearized equation (7.5), it is desirable to fix the gauge. The Lorentz covariant de Donder gauge condition $\partial^\rho h_{\rho\nu} - \partial_\nu h^\rho_\rho/2 = 0$ is commonly used. It leads to the simple wave equation $\Box h_{\mu\nu} = 0$, which shows convincingly that the field describes massless particles. However, the de Donder gauge condition does not fix the gauge completely since $\delta(\partial^\rho h_{\rho\nu} - \partial_\nu h^\rho_\rho/2) = \Box \xi_\nu$. This makes the argument for counting physical degrees of freedom somewhat indirect, see [44].

Instead, we will choose a non-covariant gauge condition, namely the D conditions

$$\partial^i h_{i\mu} = 0\,, \tag{7.13}$$

where the sum includes only the space coordinates $i = 1, \ldots, D-1$. We will show that this fixes the gauge completely and then proceed to count the physical modes of the field.

First note that the variation of the gauge condition is

$$\delta(\partial^i h_{i\mu}) = \nabla^2 \xi_\mu + \partial^i \partial_\mu \xi_i = 0\,. \tag{7.14}$$

Let $\mu \to j$ and contract with ∂^j to learn that $\nabla^2 \partial^i \xi_i = 0$. Thus $\partial^i \xi_i$ does not contain degrees of freedom, see page 76. With this information incorporated, (7.14) tells us that $\xi_\mu(x) \equiv 0$.

We now turn to the wave equation specified by (7.9),(7.12). To count the degrees of freedom, we use the gauge condition (7.13) to write it as

$$\Box h_{\mu\nu} - \partial^0(\partial_\mu h_{0\nu} + \partial_\nu h_{\mu 0}) + \partial_\mu \partial_\nu (h_{ii} - h_{00}) = 0\,. \tag{7.15}$$

We now distinguish the specific components

$$
\begin{aligned}
\mu = \nu = 0 \qquad &: \quad \nabla^2 h_{00} + \partial_0^2 h_{ii} = 0\,, \\
\mu = \nu = i \qquad &: \quad 2\nabla^2 h_{ii} - \partial_0^2 h_{ii} - \nabla^2 h_{00} = 0\,,
\end{aligned}
\tag{7.16}
$$

in which the indices i are summed. The sum of the two equations tells us that $\nabla^2 h_{ii} = 0$, so that $h_{ii} = 0$. Going back to the 00 equation, we learn that $h_{00} \equiv 0$. With the information just learned incorporated, the $\mu = 0,\ \nu = i$ components of (7.15) becomes $\nabla^2 h_{0i} = 0$, so that $h_{0i} \equiv 0$.

Only components h_{ij} remain, and they satisfy the simple wave equation

$$\Box h_{ij} = 0\,. \tag{7.17}$$

There are $D(D-1)/2$ distinct components h_{ij}, but they satisfy the constraints $\partial^i h_{ij} = 0$ from the gauge condition (7.13) and the trace condition $h_{ii} = 0$, which was found above. Thus the number of independent functions that must be supplied as initial data are $D(D-1)/2 - (D-1) - 1 = D(D-3)/2$ together with their time derivatives. A graviton in D spacetime dimensions thus has $D(D-3)/2$ degrees of freedom. As discussed in Ch. 4, the states of a massless particle in D dimensions carry an irreducible representation of the orthogonal group $\mathrm{SO}(D-2)$. For the graviton, this is the traceless symmetric tensor representation, which indeed has dimension $D(D-3)/2$.

7.3 Second order formalism for gravity and fermions

In this section we explore the most commonly used framework for the coupling of spinor fields to gravity. The frame field $e_\mu^a(x)$ and covariant derivatives constructed with the torsion-free connection $\omega_\mu{}^{ab}(e)$ play an essential role. It is instructive to consider the simplest case of a massless Dirac field $\Psi(x)$. The action functional is

$$S = S_2 + S_{1/2} = \int \mathrm{d}^D x\, e\, \left[\frac{1}{2\kappa^2} e_a^\mu e_b^\nu R_{\mu\nu}{}^{ab}(e) - \frac{1}{2}\bar{\Psi}\gamma^\mu \nabla_\mu \Psi + \frac{1}{2}\bar{\Psi}\overleftarrow{\nabla}_\mu \gamma^\mu \Psi \right]\,. \tag{7.18}$$

The first term contains the curvature tensor (6.103)

$$R_{\mu\nu ab} \;=\; \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b\,, \tag{7.19}$$

with $\omega \to \omega(e)$, see (6.79).

Ex. 7.6 *Check that inserting (7.19) in S_2 gives an action $S_2(e,\omega)$ for which the derivative w.r.t. ω vanishes by virtue of the expression $\omega(e)$, i.e.*

$$\left.\frac{\delta S_2}{\delta \omega_{\mu ab}}\right|_{\omega=\omega(e)} \;=\; 0\,. \tag{7.20}$$

The fermion action contains the curved space γ^μ matrix, whose construction and properties we discuss below, and the covariant derivatives

$$\begin{aligned} \nabla_\mu \Psi &= D_\mu \Psi = (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\Psi\,, \\ \bar{\Psi}\overleftarrow{\nabla}_\mu &= \bar{\Psi}\overleftarrow{D}_\mu = \bar{\Psi}(\overleftarrow{\partial}_\mu - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\,. \end{aligned} \tag{7.21}$$

Note that the total covariant derivative ∇_μ and the local Lorentz derivative coincide for a spinor field Ψ, but not for a field such as the gravitino Ψ_μ with additional coordinate indices, see (6.97).

The fermion action is a covariant version of the anti-symmetric derivative form introduced in Sec. 2.7.2. The necessary calculations are simpler using this form. In the present second order formalism it differs by a total derivative from the covariant version of (2.44).

The procedure to covariantize Dirac-Clifford matrices begins with the observation that the Clifford algebra is closely linked to the properties of spinors on the space-time manifold and is therefore defined in local frames. The generators are the *constant* matrices γ^a, which satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, and higher rank elements γ^{ab}, $\gamma^{abc}, \ldots$ are defined as anti-symmetric products of the generators as in Ch. 3.

Frame fields are used to transform frame *vector* indices to a coordinate basis. For example, $\gamma_\mu = e_{a\mu}\gamma^a$ or $\gamma^\mu = e_a^\mu \gamma^a = g^{\mu\nu}\gamma_\nu$. Thus γ_μ transforms as a covariant vector under coordinate transformations. But it also has (suppressed) row and column *spinor* indices and is therefore a Lorentz bi-spinor.[2] The covariant derivative of γ_μ is therefore

$$\nabla_\mu \gamma_\nu = \partial_\mu \gamma_\nu + \tfrac{1}{4}\omega_\mu{}^{ab}[\gamma_{ab}, \gamma_\nu] - \Gamma^\rho_{\mu\nu}\gamma_\rho\,. \tag{7.22}$$

The spin connection appears with the commutator as required for a bi-spinor.

Ex. 7.7 *Derive the very useful result that the covariant derivative of γ_ν vanishes. Specifically show that*

$$\nabla_\mu \gamma_\nu = \gamma^a(\partial_\mu e_{a\nu} + \omega_{\mu ab} e_\nu^b - \Gamma^\rho_{\mu\nu} e_{a\rho}) = 0\,. \tag{7.23}$$

In the last step use (6.89). Note that $\nabla_\mu\gamma_\nu = 0$ holds for any affine connection, with or without torsion, provided that one uses the total covariant derivative.

The result (7.23) implies that covariant derivatives commute with multiplication by gamma matrices. For example, if $\Psi(x)$ is a Dirac spinor field, then $\nabla_\mu(\gamma_\nu\Psi) = \gamma_\nu\nabla_\mu\Psi$.

Ex. 7.8 *Show that $\bar{\Psi}\gamma^\mu\nabla_\mu\Psi$ is invariant under infinitesimal local Lorentz transformations $\delta\Psi = -\frac{1}{4}\lambda_{ab}\gamma^{ab}\Psi$, $\delta\bar{\Psi} = \frac{1}{4}\lambda_{ab}\bar{\Psi}\gamma^{ab}$, $\delta e_a^\mu = -\lambda_a{}^b e_b^\mu$. Show that it transforms*

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[2] The archetypical example of a bi-spinor is the product $\Psi_\alpha\bar{\Psi}_\beta$ of a Dirac spinor and its adjoint, where α, β are spinor indices. Under infinitesimal local Lorentz transformations, this bi-spinor transforms as $\delta(\Psi\,\bar{\Psi}) = -\frac{1}{4}\lambda_{ab}(x)[\gamma^{ab}, \Psi\,\bar{\Psi}]$.

as a scalar under infinitesimal coordinate transformations $\delta\Psi = \xi^\rho\partial_\rho\Psi$, $\delta e_a^\mu = \xi^\rho\partial_\rho e_a^\mu - \partial_\rho\xi^\mu e_a^\rho$, *etc. These transformations are Lie derivatives, as in (6.8), under which local frame indices are inert.*

The fermion equation of motion obtained from (7.18) is the (massless) covariant Dirac equation

$$\gamma^\mu\nabla_\mu\Psi = 0\,. \tag{7.24}$$

We now outline the derivation of the Einstein equation starting from the variation of (7.18) with respect to $e^{a\mu}$. The goal is to bring this equation to the form (7.3) with conserved symmetric fermion stress tensor $T_{\mu\nu}$, which will turn to be the covariant extension of the stress tensor (2.64) of Ch. 2. The variation of the action is

$$\begin{aligned}\delta S \;=\; & \int \mathrm{d}^D x\, e\, \left[\frac{1}{\kappa^2}\left(e^{b\nu}R_{\mu\nu ab} - \frac{1}{2}e_{a\mu}R\right)\delta e^{a\mu}\right.\\ & \left. -\frac{1}{2}\bar\Psi\gamma^a\overleftrightarrow{\nabla}_\mu\Psi\delta e^{a\mu} - \frac{1}{8}\bar\Psi\{\gamma^\mu,\gamma^{ab}\}\Psi\delta\omega_{\mu ab}\right]\,.\end{aligned} \tag{7.25}$$

We used here (7.20), and dropped a term proportional to the fermion Lagrangian L because we assume that $\Psi(x)$ satisfies (7.24). We continue to omit terms that vanish by the fermion equation of motion.

We now study the last term involving the variation $\delta\omega_{\mu ab}$ of the spin connection. When integrated by parts this term will contribute to the stress tensor. The anti-commutator in this term is equal to twice the third rank Clifford matrix $e_c^\mu\gamma^{cab}$. The discussion below Ex. 6.28 and the transformation properties (6.71),(6.72) tell us that the variation of the spin connection $\omega(e)$ is a tensor. The result follows from $\omega_{[\mu\nu]}{}^b = \partial_{[\mu}e_{\nu]}{}^b$ and hence $e^a_{[\nu}\delta\omega_{\mu]ab} = \nabla_{[\mu}\delta e_{\nu]b}$. Hence, we can write this term as

$$-\tfrac{1}{4}\bar\Psi\gamma^{\mu\nu\rho}\Psi e_\nu^a e_\rho^b\delta\omega_{\mu ab} = -\tfrac{1}{4}\bar\Psi\gamma^{\mu\nu\rho}\Psi e_\rho^b\nabla_\mu\delta e_{\nu b}\,. \tag{7.26}$$

The next step is to integrate (7.26) by parts, including the sign change this brings in (7.25). We then use the distributive property of ∇_μ, the Dirac equation (7.24), and γ-matrix manipulation to rewrite (7.26) as

$$\tfrac{1}{4}e_{a\rho}\delta e_\nu^a\bar\Psi(\gamma^\nu\overleftrightarrow{\nabla}^\rho - \gamma^\rho\overleftrightarrow{\nabla}^\nu)\Psi = -\tfrac{1}{4}\bar\Psi(\gamma_a\overleftrightarrow{\nabla}_\mu - \gamma_\mu\overleftrightarrow{\nabla}_\rho e_a^\rho)\Psi\,\delta e^{a\mu}\,. \tag{7.27}$$

To reach the last form we use $e_{a\rho}\delta e_\nu^a = -e_{b\rho}e_\mu^b\delta e_a^\mu e_\nu^a = -g_{\rho\mu}e_\nu^a\delta e_a^\mu$. Finally we insert this result in (7.25) and combine terms to obtain

$$\delta S = \int \mathrm{d}^D x\left[\frac{1}{\kappa^2}\left(e^{b\nu}R_{\mu\nu ab} - \frac{1}{2}e_{a\mu}R\right)\delta e^{a\mu} - \frac{1}{4}\bar\Psi\left[\gamma_a\overleftrightarrow{\nabla}_\mu + \gamma_\mu e_a^\rho\overleftrightarrow{\nabla}_\rho\right]\Psi\delta e^{a\mu}\right]\,. \tag{7.28}$$

The variational condition $e_\nu^a\delta S/\delta e^{a\mu} = 0$ then gives the Einstein equation in the form

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu} \equiv -\kappa^2\tfrac{1}{4}\bar\Psi\left[\gamma_\mu\overleftrightarrow{\nabla}_\nu + \gamma_\nu\overleftrightarrow{\nabla}_\mu\right]\Psi\,. \tag{7.29}$$

The stress tensor is indeed the covariant version of the symmetric $T_{\mu\nu}$ derived in Ch. 2.

7.4 The first order formalism for gravity and fermions

Let us now describe the first order formalism for the coupling of fermions to the gravitational field. As discussed at the beginning of this chapter, the key points are that the frame field e_μ^a and spin connection $\omega_{\mu ab}$ appear as independent variables in the first order action. The field equation for $\omega_{\mu ab}$ may then be solved, giving a connection with torsion, $\omega_{\mu ab} = \omega_{\mu ab}(g) + K_{\mu ab}$. The contortion tensor is bilinear in the fermions. It is interesting to see how connections with torsion arise in a physical setting.

We can still use (7.18) and (7.21) for the first order action and covariant derivatives with the understanding that the connection ω_μ^{ab} is an independent quantity, and that the curvature tensor $R_{\mu\nu}{}^{ab}$ is constructed from this connection. See (7.19). Note that the first order spinor actions with anti-symmetric and right acting D_μ are not equivalent, since a torsion term appears upon partial integration. See Sec. 6.8.3.

Our goal is to solve the field equations for the connection. We will obtain $\delta S/\delta\omega_\mu{}^{ab}$, which is the change in the action due to a small variation $\delta\omega_\mu{}^{ab}$ of the connection. Note that $\delta\omega_\mu{}^{ab}$ is a local Lorentz tensor and covariant coordinate vector.

The variation of the gravitational action, using (6.106), is

$$\delta S_2 = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\; e_a^\mu e_b^\nu \left(D_\mu \delta\omega_\nu{}^{ab} - D_\nu\delta\omega_\mu{}^{ab}\right) . \tag{7.30}$$

Due to the anti-symmetry in $\mu\nu$, we can replace the Lorentz covariant derivatives with fully covariant derivatives up to torsion as in (6.98), and with the anti-symmetry in ab, we have then

$$\delta S_2 = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\; e_a^\mu e_b^\nu \left(2\nabla_\mu \delta\omega_\nu{}^{ab} + T_{\mu\nu}{}^\rho\delta\omega_\rho{}^{ab}\right) . \tag{7.31}$$

Using (6.99), the covariant derivatives commute with the vielbeins, and the first term is a total covariant derivative. Comparing with (6.101), and omitting the boundary terms, we obtain the connection variation

$$\begin{aligned}
\delta S_2 &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\; \left(-2K_{\rho\mu}{}^\rho e_a^\mu e_b^\nu \delta\omega_\nu{}^{ab} + T_{ab}{}^\rho\delta\omega_\rho{}^{ab}\right)\\
&= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\; \left(T_{\rho a}{}^\rho e_b^\nu - T_{\rho b}{}^\rho e_a^\nu + T_{ab}{}^\nu\right)\delta\omega_\nu{}^{ab} ,
\end{aligned} \tag{7.32}$$

where we used (6.102) and made the $[ab]$ anti-symmetry explicit.

Ex. 7.9 *It is instructive to obtain the formula (7.32) from the connection variation of the wedge product form of the gravitational action in (6.114). Do this.*

The connection variation of the spinor action is simpler. Using (7.21) we find

$$\begin{aligned}\delta S_{1/2} &= -\tfrac{1}{8}\int \mathrm{d}^D x\, e\; \bar\Psi\{\gamma^\nu\,,\,\gamma_{ab}\}\Psi\,\delta\omega_\nu^{ab} \\ &= -\tfrac{1}{4}\int \mathrm{d}^D x\, e\; \bar\Psi\,\gamma^\nu{}_{ab}\,\Psi\,\delta\omega_\nu^{ab}\,. \end{aligned}\tag{7.33}$$

The connection field equation can now be identified as the coefficient of $\delta\omega_\nu{}^{ab}$ in $\delta S_2 + \delta S_{1/2} = 0$. We find directly an equation for the torsion tensor

$$T_{ab}{}^\nu - T_{a\rho}{}^\rho\, e_b^\nu + T_{b\rho}{}^\rho\, e_a^\nu \;= \tfrac12 \kappa^2 \bar\Psi\,\gamma_{ab}{}^\nu\,\Psi\,. \tag{7.34}$$

Since the trace of the right side vanishes, the trace of the torsion vanishes too, and the torsion of the spinor field with $\overleftrightarrow{D}_\mu$ kinetic term is simply given by the totally anti-symmetric tensor

$$T_{ab}{}^\nu = \tfrac12\kappa^2\bar\Psi\,\gamma_{ab}{}^\nu\,\Psi = -2K^\nu{}_{ab}\,. \tag{7.35}$$

The physical effects of torsion are rather unexciting, but they are worth discussing for pedagogical reasons. We substitute $\omega = \omega(e)+K$ in the first order action (7.18).

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\; \Big[& R(g) - \kappa^2\bar\Psi\gamma^\mu\overleftrightarrow{\nabla}_\mu\Psi \\ & -2\nabla_\mu K_\nu{}^{\nu\mu} + K_{\mu\nu\rho}K^{\nu\mu\rho} - K_\rho{}^\rho{}_\mu K_\sigma{}^{\sigma\mu} - \tfrac12\bar\Psi\gamma_{\mu\nu\rho}\Psi\,K^{\mu\nu\rho}\Big]\,, \end{aligned}\tag{7.36}$$

The connection and curvature that appear in (7.36) are torsion-free, so the ∇K term is a total derivative, which can be dropped. We substitute the specific form of the torsion tensor from (7.35) and obtain the physically equivalent second order action

$$S = \frac12\int \mathrm{d}^D x\, e\; \left[\frac{1}{\kappa^2}R(g) - \bar\Psi\gamma^\mu\overleftrightarrow{\nabla}_\mu\Psi + \frac{1}{16}\kappa^2(\bar\Psi\gamma_{\mu\nu\rho}\Psi)(\bar\Psi\gamma^{\mu\nu\rho}\Psi)\right]\,. \tag{7.37}$$

Note that the $(K_\rho{}^\rho{}_\mu)^2$ term vanishes in this theory because the torsion is totally anti-symmetric. Physical effects in the fermion theories with and without torsion differ only by the presence of the quartic Ψ^4 term, which is a dimension 6 operator suppressed by the Planck scale. This term generates 4-point contact Feynman diagrams in fermion-fermion scattering amplitudes. The contact diagrams give short range amplitudes and dominate the long range graviton exchange diagrams for large angle scattering. The contribution of these diagrams could thus, in principle, be detected in experiments, so the fermion theories with and without torsion are physically inequivalent.

8

$\mathcal{N} = 1$ Global Supersymmetry in $D = 4$

In global SUSY the scope of symmetries included in quantum field theory is extended from Poincaré and internal symmetry transformations, with charges $M_{[\mu\nu]}$, P_μ, and T_A, to include spinor charges Q^i_α, where α is a spacetime spinor index, and $i = 1, \ldots \mathcal{N}$ is the index of the defining representation of an SO(N) global internal symmetry. . We will assume that the Q^i_α are 4-component Majorana spinors, although an equivalent formulation using 2-component Weyl spinors is also commonly used. In this chapter we will study the simplest case where $\mathcal{N} = 1$ and there is a single spinor charge Q_α. This case is called $\mathcal{N} = 1$ SUSY or simple SUSY. Theories with $\mathcal{N} > 1$ spinor charges, called extended SUSY theories, are discussed in Sec. 12.1.

In $\mathcal{N} = 1$ global SUSY the Poincaré generators and Q_α join in a new algebraic structure, that of a superalgebra. A superalgebra contains two classes of elements, even and odd. From the physics viewpoint, they can be called bosonic (B) and fermionic (F). Their structure relations include both commutators and anticommutators in the pattern $[B, B] = B$, $[B, F] = F$, $\{F, F\} = B$. The bosonic charges span a Lie algebra. In SUSY the subalgebra of the bosonic charges $M_{[\mu\nu]}$ and P_μ is the Lie algebra of the Poincaré group discussed in Ch. 1, while the new structure relations involving Q_α are

$$\begin{aligned}
\left\{Q_\alpha, \bar{Q}^\beta\right\} &= -\tfrac{1}{2}(\gamma_\mu)_\alpha{}^\beta P^\mu \,,\\
\left[M_{[\mu\nu]}, Q_\alpha\right] &= -\tfrac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta \,,\\
[P_\mu, Q_\alpha] &= 0 \,.
\end{aligned} \tag{8.1}$$

Note that these are the classical (anti)commutator relations, which we are going to use in this chapter. You might think about them as Poisson brackets of currents. We will discuss this further in Ch. 11.

Ex. 8.1 *Use (2.42) to translate the anticommutator of supersymmetries to an anticommutator of Q and $Q^\dagger$. Then use the correspondence principle that says that* *quantum (anti)commutators are obtained from the classical relations by multiplying with the imaginary unit* i. *This gives the quantum relation*

$$\left\{Q_\alpha, (Q^\dagger)^\beta\right\}_{\text{qu}} = \tfrac{1}{2}\left(\gamma_\mu\gamma^0\right)_\alpha{}^\beta P^\mu\,. \tag{8.2}$$

Consider this for a particle at rest. The positivity of $QQ^\dagger + Q^\dagger Q$ for all of the four components of the supersymmetry generators imposes that particles in global supersymmetry should have a positive energy $E = P^0$.

Many SUSY theories, but not all, are invariant under a chiral U(1) symmetry called $\mathrm{U}(1)_R$. We denote the generator by T_R. This acts on Q_α via

$$[T_R, Q_\alpha] = -\mathrm{i}(\gamma_*)_\alpha{}^\beta Q_\beta\,. \tag{8.3}$$

but this generator T_R is not required. Other internal symmetries, which commute with Q_α and frequently called outside charges, can also be included.[1]

There are two important theorems that severely limit the type of charges and algebras that can be realized in an interacting relativistic quantum field theory in $D = 4$ (strictly speaking in a theory with a non-trivial S-matrix in flat space). According to the Coleman-Mandula (CM) theorem [46, 47], in the presence of massive particles, bosonic charges are limited to $M_{[\mu\nu]}$ and P_μ plus (optional) scalar internal symmetry charges, and the Lie algebra is the direct sum of the Poincaré algebra and a (finite dimensional) compact Lie algebra for internal symmetry.

If superalgebras are admitted, the situation is governed by the Haag-Lopuszański-Sohnius (LHS) theorem [48, 47], and the algebra of symmetries admits spinor charges Q^i_α. If there is only one Q_α, then the superalgebra must agree with the $\mathcal{N} = 1$ Poincaré SUSY algebra in (8.1). When $\mathcal{N} > 1$, the possibilities are restricted to the extended SUSY algebras discussed in Sec. 12.1. The main thought that we wish to convey is that SUSY theories realize the most general symmetry possible within the framework of the few assumptions made in the hypotheses of the C-M and H-L-S theorems.[2] They also unify bosons and fermions, the two broad classes of particles found in Nature.

The parameters of global SUSY transformations are *constant* Majorana spinors ϵ_α. In supergravity SUSY is gauged, necessarily with the Poincaré generators, since they are joined in the superalgebra (8.1). This means that gravity is included, so the spinor parameters become arbitrary functions functions $\epsilon_\alpha(x)$ on a curved spacetime manifold. It is logically possible to skip ahead to Ch. 9 where $\mathcal{N} = 1$, $D = 4$ supergravity is presented. But much important background will be missed, and we encourage only readers quite familiar with global SUSY to do this. We

[1] The matrices γ^μ, $\gamma_{\mu\nu}$, γ_*, which appear in (8.1),(8.3) are matrices of the Clifford algebra which are discussed in Sec. 3.1. Majorana spinors are discussed in Sec. 3.3.

[2] In theories that contain only massless fields and are scale invariant at the quantum level, there are the additional possibilities of conformal and superconformal symmetries. The superconformal algebras contain the Poincaré SUSY algebras as subalgebras. They will be discussed later.

endeavor to give a succinct, pedagogical treatment of *classical* aspects of SUSY field theories. This material is certainly elegant, and part of the reason that SUSY is so appealing. However, there is much more in the deep results that have been discovered in perturbative and non-perturbative *quantum* supersymmetry that we cannot include.

The purpose of this chapter is to move as quickly as possible to an understanding of the structure of the major *interacting* SUSY field theories. At the classical level an interacting field theory is simply one in which the equations of motion are nonlinear. In Sec. 8.4, we give a short survey the particle representations of extended Poincaré SUSY algebras (at least for the massless case) and discuss some interesting new features that appear.

8.1 Basic SUSY Field Theory

SUSY theories contain both bosons and fermions, which are the basis states of a particle representation of the SUSY algebra (8.1)-(8.3). We give a systematic treatment of these representations in Sec. 8.4, but start with an informal discussion here. The states of particles with momentum $\vec{p}$ and energy $E(\vec{p}) = \sqrt{\vec{p}^2 + m_{B,F}^2}$ are denoted by $|\vec{p}, B\rangle$ and $|\vec{p}, F\rangle$, where the labels B, F include particle helicity. SUSY transformations connect these states, e.g. $Q_\alpha|\vec{p}, B\rangle = |\vec{p}, F\rangle$ and $Q_\alpha|\vec{p}, F\rangle \propto |\vec{p}, B\rangle$. Since $[P^\mu\,,\,Q_\alpha] = 0$, the transformed states have the same momentum and energy, hence the same mass, so $m_B^2 = m_F^2$.

The simplest representations of the algebra that lead to the most basic SUSY field theories are

i) the chiral multiplet, which contains a self-conjugate spin 1/2 fermion described by the Majorana field $\chi(x)$ plus a complex spin 0 boson described by the scalar field $Z(x)$. Alternatively, $\chi(x)$ may be replaced by the Weyl spinor $P_L\chi$ and/or $Z(x)$ by the combination $Z(x) = (A(x) + \mathrm{i}B(x))/\sqrt{2}$ where A and B are a real scalar and pseudoscalar, respectively. A chiral multiplet can be either massless or massive.

ii) the gauge multiplet consisting of a massless spin 1 particle, described by a vector gauge field $A_\mu(x)$, plus its spin 1/2 fermionic partner, the gaugino, described by a Majorana spinor $\lambda(x)$ (or the corresponding Weyl field $P_L\lambda$).

8.1.1 Conserved super-currents

It follows from our discussion of the Noether formalism for symmetries that the spinor charge should be the integral of a conserved vector-spinor current, the supercurrent $\mathcal{J}_\alpha^\mu$, hence

$$Q_\alpha = \int \mathrm{d}^3x\, \mathcal{J}_\alpha^0(\vec{x}, t)\,. \tag{8.4}$$

If the current is conserved for all solutions of the equations of motion of a theory then the theory has a fermionic symmetry. By the H-L-S theorem this symmetry must be supersymmetry!

Therefore we begin the technical discussion of SUSY in quantum field theory by displaying such conserved currents,[3] first for *free* fields and then for one non-trivial *interacting* system. Consider a free scalar field $\phi(x)$ satisfying the Klein-Gordon equation $(\Box - m^2)\phi = 0$ and a spinor field $\Psi(x)$ satisfying the Dirac equation $(\partial\!\!\!/ - m)\Psi = 0$.

Ex. 8.2 *Show that the current $\mathcal{J}^\mu = (\partial\!\!\!/ - m)\phi\gamma^\mu\Psi$ is conserved for all field configurations satisfying the Klein-Gordon and Dirac equations.*

It is no surprise to find unusual conserved currents in a *free* theory. In fact the current $\mathcal{J}^\mu{}_\nu = (\partial\!\!\!/ - m)\phi\gamma^\mu\partial_\nu\Psi$, which gives a charge that violates the H-L-S theorem is conserved. Such currents cannot be extended to include interacting fields. For similar reasons conservation of $\mathcal{J}^\mu$ at the free level holds whether ϕ, Ψ are real or complex. To extend to interactions we will have to take $\phi \to Z$, a complex scalar, and Ψ a Majorana spinor. Note also that the current in Ex. 8.2 is conserved for any spacetime dimension D, this is another property that fails with interactions.

As the second example let's look at the free gauge multiplet with vector potential A_μ and field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ satisfying the Maxwell equation $\partial^\mu F_{\mu\nu} = 0$ and a spinor λ satisfying $\partial\!\!\!/\lambda = 0$. Let's show that the current $\mathcal{J}^\mu = \gamma^{\nu\rho}F_{\nu\rho}\gamma^\mu\lambda$ is conserved. We have

$$\partial_\mu\mathcal{J}^\mu = \partial_\mu F_{\nu\rho}\gamma^{\nu\rho}\gamma^\mu\lambda + \gamma^{\nu\rho}F_{\nu\rho}\partial\!\!\!/\lambda\,. \tag{8.5}$$

The last term vanishes. To treat the first term we manipulate the γ matrices as we learned in Sec. 3.1.4:

$$\gamma^{\nu\rho}\gamma^\mu = \gamma^{\nu\rho\mu} + 2\gamma^{[\nu}\eta^{\rho]\mu} \tag{8.6}$$

When inserted in the first term of (8.5) we see that the first term vanishes by the gauge field Bianchi identity (4.11), and the second one by the Maxwell equation.

8.1.2 SUSY Yang-Mills Theory

With little more work we can now exhibit an important interacting theory, $\mathcal{N} = 1$ SUSY Yang-Mills theory and its conserved supercurrent. The theory contains the gauge boson $A_\mu^A(x)$ and its SUSY partner, the Majorana spinor gaugino $\lambda^A(x)$ in the adjoint representation of a simple, compact, non-abelian gauge group G. The action is[4]

$$S = \int \mathrm{d}^4x\,\left[-\tfrac{1}{4}F^{\mu\nu A}F^A_{\mu\nu} - \tfrac{1}{2}\bar\lambda^A\gamma^\mu D_\mu\lambda^A\right]\,. \tag{8.7}$$

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[3] The spinor index α on the current and on most spinorial quantities will normally be suppressed.

[4] We will mostly assume in this chapter that the group has an invariant metric δ_{AB}, and as such two 'upper' group indices can be contracted.

For details of the notation see sections 3.4.1 and 4.3. Note that the gaugino action vanishes unless $\lambda^A(x)$ is anti-commuting! The Euler-Lagrange equations (and gauge field Bianchi identity) are:

$$\begin{aligned} &D^\mu F^A_{\mu\nu} = -\tfrac{1}{2} g f_{BC}{}^A \bar{\lambda}^B \gamma_\nu \lambda^C \,, \\ &D_\mu F^A_{\nu\rho} + D_\nu F^A_{\rho\mu} + D_\rho F^A_{\mu\nu} = 0 \,, \\ &\gamma^\mu D_\mu \lambda^A = 0 \,. \end{aligned} \tag{8.8}$$

The super-current is

$$\mathcal{J}^\mu \;=\; \gamma^{\nu\rho} F^A_{\nu\rho} \gamma^\mu \lambda^A \,. \tag{8.9}$$

The proof that it is conserved begins as in the free (abelian) case:

$$\begin{aligned} \partial_\mu \mathcal{J}^\mu \;&=\; \partial_\mu F^A_{\nu\rho} \gamma^{\nu\rho} \gamma^\mu \lambda^A + \gamma^{\nu\rho} F^A_{\nu\rho} \gamma^\mu D_\mu \lambda^A \\ &=\; -D^\mu F^A_{\mu\nu} \gamma^\nu \lambda^A \\ &=\; \tfrac{1}{2} g f_{ABC} \gamma^\nu \lambda^A \bar{\lambda}^B \gamma_\nu \lambda^C \,. \end{aligned} \tag{8.10}$$

A new ingredient is now required. We will show that the last expression vanishes by performing a Fierz rearrangement as discussed in Sec. 3.2.3. We rename the index A in that equation to I to avoid confusion with the gauge indices here. We write

$$f_{ABC} \gamma^\nu \lambda^A \bar{\lambda}^B \gamma_\nu \lambda^C = \frac{1}{2^m} f_{ABC} \sum_I v_I \Gamma^I \lambda^A \bar{\lambda}^B \Gamma_I \lambda^C \,, \tag{8.11}$$

In Ex. 3.25 it was shown that $v_I = (-)^{r_I}(D - 2r_I)$ where r_I is the tensor rank of the Clifford basis element $\Gamma_I \to \gamma_{\nu_1\nu_2\ldots\nu_{r_I}}$. We are primarily interested in the case $D = 2m = 4$ but leave the notation general to look into other possibilities below. For $D = 4$, the possible contributing Clifford basis elements are restricted to rank $1 \le r_I \le 4$. However, it was shown in (3.49) that, for anti-commuting Majorana spinors, each bilinear $\bar{\Psi}_1 \Gamma_I \Psi_2$ has a definite symmetry under exchange of $\Psi_1 \leftrightarrow \Psi_2$. Since the Lie algebra indices of $\bar{\lambda}^B \Gamma_I \lambda^C$ are anti-symmetrized, only anti-symmetric Clifford bilinears can contribute in (8.11), which according to table 3.1 for 4 dimensions with the preferred choices are the rank $r_I = 1, 2$ bilinears. Since $v_I = -2, 0$ for $r_I = 1, 2$, (8.11) reduces to[5]

$$f_{ABC} \gamma^\nu \lambda^A \bar{\lambda}^B \gamma_\nu \lambda^C = -\tfrac{1}{2} f_{ABC} \gamma^\nu \lambda^A \bar{\lambda}^B \gamma_\nu \lambda^C, \tag{8.12}$$

so the last line of (8.10) vanishes, and the supercurrent (8.9) is conserved!

Thus we have established the existence of our first interacting SUSY field theory. Notice that basic relations of non-abelian gauge symmetry such as the Bianchi identity and of the relativistic description of spin by the Dirac/Clifford algebra and the anti-commutativity of fermion fields are all blended in the proof. Readers

[5] These manipulations are the equivalent to those with spinor indices in Ex. 3.26.

whose intellectual curiosity is not excited by this are advised to put this book aside permanently and watch television instead of reading it.

The two main approaches to SUSY field theories are the approach of this chapter in which we deal with the separate field components describing each physical particle in the theory and the superspace approach in which the separate fields are grouped in superfields. The latter approach is not used in this book, but shortly discussed in Appendix 14.A of Ch. 14 (see references there). A Fierz relation is always required to establish supersymmetry in the 'components' approach to any interacting SUSY theory. This is one reason why the existence and field content of SUSY field theories depend so markedly on the spacetime dimension. The Fierz relation is also informative of the type of fermion required in the theory.

Suppose we try to construct a super-Yang-Mills (SYM) theory in which the superpartner of the gauge field is a complex fermion Ψ^A in the adjoint representation. Then the gauge current $\bar{\Psi}^A\gamma_\nu\Psi^A$ would appear in the gauge field equation of motion and the cubic quantity $f_{ABC}\gamma^\nu\Psi^A\bar{\Psi}^B\gamma_\nu\Psi^C$ in the last line of (8.10). However, many more terms contribute in a Fierz re-ordering of this quantity, because the Clifford bilinears are no longer restricted by symmetry under exchange. In fact the quantity does not vanish for complex Ψ^A, and there is no SYM theory with the field content A_μ^A, Ψ^A.[6]

Here is an exercise in which readers are asked to show that SYM theories with gauge field A_μ plus a specific type of spinor ψ^A and the supercurrent $\mathcal{J}^\mu = \gamma^{\nu\rho}F^A_{\nu\rho}\gamma^\mu\psi^A$ do exist in certain spacetime dimensions [49].

Ex. 8.3 *Study the appropriate Fierz rearrangement and, using the results of Ex. 3.26, show that the supercurrent is conserved in the following cases:*
i) Majorana spinors in $D = 3$,
ii)Majorana (or Weyl) spinors in $D = 4$, which is the case analyzed above,
iii) symplectic-Weyl spinors in $D = 6$, and
iv) Majorana-Weyl spinors in $D = 10$.

Notice that in every case, the number of on-shell degrees of freedom of the gauge field, namely $D - 2$, matches those of the fermion, which are $2.2^{[(D-2)/2]}/k$ (real) where $k = 1$ for a complex Dirac fermion, $k = 2$ for a Majorana, a Weyl or a symplectic-Weyl fermion, and $k = 4$ for a Majorana-Weyl fermion. Equality of the total number of boson and fermion states is a necessary condition for SUSY. This basic fact will be proved for massless physical states in Sec. 8.4 and in general in Sec. 12.3.1.

Ex. 8.4 *Why is there no SYM theory (containing only A_μ^A, λ^A) when $D = 2$? Count the degrees of freedom.*

[6] An $\mathcal{N} = 2$ SYM theory can be constructed if one adds a complex scalar Φ^A in the adjoint representation. See Ch. 19.

8.1.3 SUSY transformation rules

Although global SUSY can be formulated using conserved supercurrents as the primary vehicle, as was done above, it is usually more convenient to emphasize the idea of SUSY field variations involving spinor parameters ϵ_α under which actions must be invariant. The field variations are also called SUSY transformation rules. One link to the conserved current formalism is provided by the canonical formalism in which the field variations are obtained by computing

$$\delta\Phi(x) = \{\bar{\epsilon}^\alpha Q_\alpha, \Phi(x)\}_{\rm PB} = -{\rm i}[\bar{\epsilon}^\alpha Q_\alpha, \Phi(x)]_{\rm qu}. \tag{8.13}$$

where Φ denotes any field of the system under study. A brief description of Poisson brackets (PB) and commutation relations in the canonical formalism is given in Sec. 1.4 . A link in the opposite direction is provided by the Noether formalism, which produces a conserved super-current given field variations under which the action is invariant. One reason to emphasize the field variations, ab initio, is that this avoids some subtleties in the canonical formalism for gauge theories and for Majorana spinors.

It is for the reason above that the next exercise involves the free scalar-spinor $\phi - \Psi$ system of Ex. 8.2. The spinors Ψ, the supersymmetry parameters ϵ and the supersymmetry generator Q are Majorana spinors. They all anticommute. For the canonical formalism, one can either consider Ψ and $\bar{\Psi}$ as independent variables, or use Dirac brackets to obtain

$$\begin{aligned} \{\phi(x), \partial_0\phi(y)\}_{\rm PB} &= -\{\partial_0\phi(x), \phi(y)\}_{\rm PB} = \delta^3(\vec{x}-\vec{y}), \\ \left\{\Psi_\alpha(x), \bar{\Psi}^\beta(y)\right\}_{\rm PB} &= \left\{\bar{\Psi}^\beta(x), \Psi_\alpha(y)\right\}_{\rm PB} = \left(\gamma^0\right)_\alpha{}^\beta \delta^3(\vec{x}-\vec{y}). \end{aligned} \tag{8.14}$$

Ex. 8.5 *Use* $\bar{Q} = \frac{1}{\sqrt{2}}\int {\rm d}^3\vec{x}\,\bar{\Psi}\gamma^0(\not{\partial}+m)\phi$ *or* $Q = \frac{1}{\sqrt{2}}\int {\rm d}^3\vec{x}\,(\not{\partial}-m)\phi\gamma^0\Psi$ *to obtain (remember the correspondence principle, see Sec. 1.4)*

$$\begin{aligned} \delta\phi(x) = -{\rm i}[\bar{\epsilon}Q, \phi(x)]_{\rm qu} &= \{\bar{\epsilon}Q, \phi(x)\}_{\rm PB} = \frac{1}{\sqrt{2}}\bar{\epsilon}\Psi(x), \\ \delta\Psi(x) = -{\rm i}[\bar{Q}\epsilon, \Psi(x)]_{\rm qu} &= \{\bar{\epsilon}Q, \Psi(x)\}_{\rm PB} = \frac{1}{\sqrt{2}}(\not{\partial}+m)\phi\epsilon, \end{aligned} \tag{8.15}$$

Note that $[\bar{Q}\epsilon, \Psi_\alpha(x)]_{\rm qu} = -\{\bar{Q}^\beta, \Psi_\alpha(x)\}_{\rm qu}\epsilon_\beta$ *(where* $\{.,.\}$ *denotes an anti-commutator).*

8.2 SUSY field theories of the chiral multiplet

The physical fields of the chiral multiplet are a complex scalar $Z(x)$ and the Majorana spinor $\chi(x)$. It simplifies the structure in several ways to bring in a complex scalar auxiliary field $F(x)$. The field equations of F are algebraic, so F can be eliminated from the system at a later stage. The set of fields Z, $P_L\chi$, F constitute a chiral multiplet, and their conjugates $\bar{Z}$, $P_R\chi$, $\bar{F}$ are an anti-chiral multiplet. The treatment is streamlined because we use the chiral projections $P_L\chi$ and $P_R\chi$, but can still regard χ as a Majorana spinor. See Sec. 3.4.2.

Our program is to present the SUSY transformation rules of these multiplets, discuss invariant actions, and then study the SUSY algebra (8.1)-(8.3). The spinor parameter ϵ is a Majorana spinor, whose spinor components anti-commute with each other and with components of χ and $\bar{\chi}$.

The transformation rules of the chiral multiplet are:

$$\begin{aligned} \delta Z &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi\,, \\ \delta P_L\chi &= \frac{1}{\sqrt{2}}P_L(\partial\!\!\!/ Z + F)\epsilon\,, \\ \delta F &= \frac{1}{\sqrt{2}}\bar{\epsilon}\,\partial\!\!\!/ P_L\chi\,. \end{aligned} \tag{8.16}$$

The anti-chiral multiplet transformation rules are:

$$\begin{aligned} \delta \bar{Z} &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_R\chi\,, \\ \delta P_R\chi &= \frac{1}{\sqrt{2}}P_R(\partial\!\!\!/ \bar{Z} + \bar{F})\epsilon\,, \\ \delta \bar{F} &= \frac{1}{\sqrt{2}}\bar{\epsilon}\,\partial\!\!\!/ P_R\chi\,. \end{aligned} \tag{8.17}$$

Note that the form of the transformation rules for the physical components is similar to those of the 'toy model' in Ex. 8.5.

Ex. 8.6 *Show that the variations $\delta\bar{Z}$, $\delta P_R\chi$, $\delta\bar{F}$ are precisely the adjoints of δZ, $\delta P_L\chi$, δF.*

There are two basic actions, which are separately invariant under the transformation rules above. The first is the free kinetic action

$$S_{\rm kin} = \int \mathrm{d}^4x\left[-\partial^\mu\bar{Z}\partial_\mu Z - \bar{\chi}\partial\!\!\!/ P_L\chi + \bar{F}F\right]\,, \tag{8.18}$$

in which we have presented the spinor term in chiral form. The interaction is determined by an arbitrary holomorphic function, the superpotential $W(Z)$. Given this we define the action

$$S_F = \int \mathrm{d}^4x\left[FW'(Z) - \tfrac{1}{2}\bar{\chi}P_L W''(Z)\chi\right]\,. \tag{8.19}$$

(The reason for the apparent extra derivative will be explained shortly.) Note that S_F involves only the fields of the chiral multiplet and no anti-chiral components.

Thus the action S_F is not Hermitian, so we must also consider the conjugate action $S_{\bar{F}} = (S_F)^\dagger$. The complete action of the chiral multiplet is the sum

$$S = S_{\rm kin} + S_F + S_{\bar{F}}\,. \tag{8.20}$$

Ex. 8.7 *Consider the superpotential* $W = \frac{1}{2}mZ^2 + \frac{1}{3}gZ^3$*, which gives the first SUSY theory considered by Wess and Zumino in 1973 [50]. Obtain the equations of motion for all fields, then eliminate* F *and* $\bar{F}$ *and show that the correct equations of motion for the physical fields are obtained if you first eliminate* F *and* $\bar{F}$ *by solving their algebraic equations of motion and substituting the result in the action. Substitute* $Z = (A + {\rm i}B)/\sqrt{2}$ *and show that the action (after elimination of auxiliary fields) takes the form*

$$\begin{aligned} S_{WZ} \;&=\; \int {\rm d}^4x \left[\frac{1}{2}(-(\partial A)^2 - m^2A^2 - (\partial B)^2 - m^2B^2 - \bar{\chi}(\not{\partial} - m)\chi)\right. \\ &\qquad \left. + \frac{g}{\sqrt{2}}\bar{\chi}(A + {\rm i}\gamma_* B)\chi + \frac{mg}{\sqrt{2}}(A^3 + AB^2) + \frac{g^2}{4}(A^2 + B^2)^2\right]\,. \end{aligned} \tag{8.21}$$

From the viewpoint of a particle theorist this is a parity conserving, renormalizable theory with equal mass fields and Yukawa plus quartic interactions.

Let's outline the proof that the actions $S_{\rm kin}$ and S_F are invariant under the SUSY transformation (8.16),(8.17). It is rather intricate, so trusting readers may wish to move ahead. For $S_{\rm kin}$ the work is simplified by an observation that is correct in any representation of the Clifford algebra, but clearest in the Weyl representation (2.19). The projections $P_L\epsilon$ and $\bar{\epsilon}P_L$ involve the same half of the components of the Majorana ϵ, while $P_R\epsilon$ and $\bar{\epsilon}P_R$ involve the conjugate components. We write the total variation $\delta S = \delta_{P_L\epsilon}S + \delta_{P_R\epsilon}S$, temporarily separating the two chiral projections of ϵ in the transformation rules. Since $S_{\rm kin}$ is Hermitian, it is sufficient to compute $\delta_{P_L\epsilon}S_{\rm kin}$; then $\delta_{P_R\epsilon}S$ is its adjoint. In the calculation we temporarily allow $\epsilon(x)$ to be an arbitrary function in Minkowski spacetime since that provides a simple way [51] to obtain the Noether current for SUSY. We also need $\delta\bar{\chi}P_R = -\frac{1}{\sqrt{2}}\bar{\epsilon}\left(\not{\partial}\bar{Z} - \bar{F}\right)P_R$, a result most easily obtained by using (3.54) and (3.52) (remember that $t_0 = -t_1 = 1$ in 4 dimensions).

Now that we have prepared the way, let's calculate

$$\begin{aligned} \delta_{P_L\epsilon}S_{\rm kin} \;&=\; -\frac{1}{\sqrt{2}}\int {\rm d}^4x \left[\partial^\mu\bar{Z}\partial_\mu(\bar{\epsilon}P_L\chi) - \bar{\epsilon}(\not{\partial}\bar{Z})\not{\partial}P_L\chi\right. \\ &\qquad \left. + \bar{\chi}\not{\partial}(P_LF\epsilon) - (\bar{\epsilon}\,\not{\partial}P_R\chi)F\right]\,. \end{aligned} \tag{8.22}$$

We have included all $P_L\epsilon$ and $\bar{\epsilon}P_L$ terms and dropped others. The $\bar{Z}\chi$ and $F\chi$ terms are independent and must vanish separately if we are to have a symmetry (when

ϵ is constant). After a Majorana flip in the last term, we find that the $F\chi$ terms combine to (even for $\epsilon(x)$)

$$-\frac{1}{\sqrt{2}}\int \mathrm{d}^4x\,\partial_\mu(\bar\chi\gamma^\mu P_L F\epsilon)\,, \tag{8.23}$$

which vanishes. The $\bar Z\chi$ terms can then be processed by substituting

$$\begin{aligned}\partial_\mu(\bar\epsilon P_L\chi) &= (\partial_\mu\bar\epsilon)P_L\chi + \bar\epsilon P_L\partial_\mu\chi\,,\\ \bar\epsilon P_L\gamma^\mu\gamma^\nu(\partial_\mu\bar Z)\partial_\nu\chi &= \bar\epsilon P_L[(\partial^\mu\bar Z)\partial_\mu\chi + \gamma^{\mu\nu}\partial_\nu(\partial_\mu\bar Z\chi)]\,,\end{aligned} \tag{8.24}$$

in (8.22). Two of the four terms cancel. After partial integration and use of $\eta^{\mu\nu} - \gamma^{\mu\nu} = \gamma^\nu\gamma^\mu$, we find the net result

$$\delta_{P_L\epsilon}S_{\rm kin} = -\frac{1}{\sqrt{2}}\int \mathrm{d}^4x\partial_\mu\bar\epsilon\; P_L(\partial\!\!\!/\bar Z)\gamma^\mu\chi\,. \tag{8.25}$$

This shows that $\delta S_{\rm kin}$ vanishes for constant ϵ, which is enough to prove SUSY. The remaining term is a contribution to the super-current of the complete theory in (8.20), and we will include it below.

Since SUSY for the *free* action $S_{\rm kin}$ is not worth celebrating, we move on to discuss the interaction term S_F. The variation δS_F under the transformations (8.16) has the structure

$$\begin{aligned}\delta S_F &= \int \mathrm{d}^4x\,\big[\delta F W'(Z) + \delta Z F W''(Z)\\ &\qquad -\delta\bar\chi P_L\chi W''(Z) - \tfrac12\delta Z\bar\chi P_L\chi W'''(Z)\big]\,,\end{aligned} \tag{8.26}$$

where we have taken the derivatives of $W(Z)$ required to include all sources of the δZ variation. After use of (8.16) we combine terms. Two $FP_L\chi$ terms cancel and we are left with the net result

$$\delta S_F = \frac{1}{\sqrt{2}}\int \mathrm{d}^4x\left[\bar\epsilon\partial\!\!\!/\left(W'P_L\chi\right) - \frac{1}{2}W'''\bar\epsilon P_L\chi\bar\chi P_L\chi\right]\,. \tag{8.27}$$

The last term vanishes, since $P_L\chi$ has two independent components and any cubic expression vanishes by anti-commutativity! Thus δS_F vanishes for constant ϵ and is supersymmetric. It is clear that $\delta S_{\bar F}$ is just the conjugate of (8.27). At last SUSY is established at the interacting level!

The remaining $\partial_\mu\bar\epsilon$ terms in (8.25) and (8.27) plus their conjugates combine to give the Noether super-current of the interacting theory. This can be written as

$$\mathcal{J}^\mu = \frac{1}{\sqrt{2}}[P_L(\partial\!\!\!/\bar Z - F) + P_R(\partial\!\!\!/ Z - \bar F)]\gamma^\mu\chi\,, \tag{8.28}$$

in which one must use the auxiliary field equations of motion $F = -\overline{W}'(\bar Z)$ and $\bar F = -W'(Z)$.

Ex. 8.8 *Show that the current is conserved for all solutions of the equations of motion of the theory (8.20).*

Ex. 8.9 *Given the component fields $\bar{Z}$, $P_R\chi$, $\bar{F}$ of an anti-chiral multiplet, show that $\bar{F}$, $P_L\partial\!\!\!/\chi$, $\Box\bar{Z}$ transform in the same way as Z, $P_L\chi$, F components of a chiral multiplet. See (8.16),(8.17).*

8.2.1 The SUSY algebra

In this section we will study the realization of the SUSY algebra on the components of a chiral multiplet. It is convenient and interesting that the $\{Q,\bar{Q}\}$ anti-commutator in (8.1) is realized in classical manipulations as the commutator of two successive variations of the fields with distinct (anti-commuting) parameters ϵ_1, ϵ_2.

The variation of a generic field $\Phi(x)$ is given in (8.13). As we discuss the classical setup here, we will denote the Poisson bracket operation $\{\,.\,,\,.\,\}_{\rm PB}$ now as $[\,.\,,\,.\,]$, distinguishing with the symmetric bracket $\{\,.\,,\,.\,\}$. The commutator of successive variations δ_1, δ_2 of $\Phi(x)$, with parameters ϵ_1, ϵ_2, respectively, is (recall that $\bar{\epsilon}Q=\bar{Q}\epsilon$ for Majorana spinors)

$$\begin{aligned}
[\delta_1,\delta_2]\Phi(x) &= \big[\bar{\epsilon}_1 Q,\big[\bar{Q}\epsilon_2,\Phi(x)\big]\big]-(\epsilon_1\leftrightarrow\epsilon_2)\\
&= \bar{\epsilon}_1^\alpha[\{Q_\alpha,\bar{Q}^\beta\},\Phi(x)]\epsilon_{2\beta}\\
&= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\,\partial_\mu\Phi(x)\,.
\end{aligned}\tag{8.29}$$

The standard Jacobi identity has been used to reach the second line and the first relation in (8.1) to obtain the last line. The key result is that the commutator of two SUSY variations is an infinitesimal spacetime translation with parameter $-\frac{1}{2}\,\bar{\epsilon}_1\gamma^\mu\epsilon_2$.

Let's carry out the computation of $[\delta_1,\delta_2]Z(x)$ on the scalar field of a chiral multiplet. Using (8.16) we write

$$\begin{aligned}
[\delta_1,\delta_2]Z &= \frac{1}{\sqrt{2}}\delta_1(\bar{\epsilon}_2P_L\chi)-[1\leftrightarrow 2]\\
&= \tfrac{1}{2}\bar{\epsilon}_2P_L(\partial\!\!\!/ Z+F)\epsilon_1-[1\leftrightarrow 2]\\
&= -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu Z\,.
\end{aligned}\tag{8.30}$$

The symmetry properties of Majorana spinor bilinears (see (3.49)) have been used to reach the final result, which clearly shows the promised infinitesimal translation.

The analogous computation of $[\delta_1,\delta_2]P_L\chi(x)$ is more complex because a Fierz rearrangement is required. We outline it here:

$$\begin{aligned}
[\delta_1,\delta_2]P_L\chi &= \frac{1}{\sqrt{2}}P_L(\partial\!\!\!/\delta_1 Z+\delta_1 F)\epsilon_2-[1\leftrightarrow 2]\\
&= \tfrac{1}{2}P_L\gamma^\mu\epsilon_2\bar{\epsilon}_1P_L\partial_\mu\chi+\tfrac{1}{2}P_L\epsilon_2\bar{\epsilon}_1\partial\!\!\!/ P_L\chi-[1\leftrightarrow 2]\\
&= -\tfrac{1}{8}(\bar{\epsilon}_1\Gamma_A\epsilon_2)P_L(\gamma^\mu\Gamma^A+\Gamma^A\gamma^\mu)P_L\partial_\mu\chi-[1\leftrightarrow 2]\,.
\end{aligned}\tag{8.31}$$

Each term in the second line was reordered as in Ex. 3.27 (with $\bar{\Psi}_1$ of (3.67) removed). We now find a great deal of simplification. Because of the anti-symmetrization in $\epsilon_1 \leftrightarrow \epsilon_2$ the only non-vanishing bilinears are $\Gamma_A \rightarrow \gamma_\nu$ or $\gamma_{\nu\rho}$. However, only γ^ν survives the chiral projection in the last factor. Thus we find the expected result

$$[\delta_1, \delta_2]P_L\chi = -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2 P_L\partial_\mu\chi \tag{8.32}$$

as the just reward for our labor.

Ex. 8.10 *It is quite simple to demonstrate that*

$$[\delta_1, \delta_2]F \;=\; -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu F\,, \tag{8.33}$$

Zealous readers should do it.

The auxiliary field F can be eliminated from the action by substituting the value $F = \overline{W}'(\bar{Z})$, which is the solution of its equation of motion from (8.20), without affecting the classical (or quantum) dynamics. Here is an exercise to show that SUSY is also maintained after elimination.

Ex. 8.11 *Consider the theory after elimination of F and $\bar{F}$. Show that the action*

$$S = \int \mathrm{d}^4x \left[-\partial^\mu\bar{Z}\partial_\mu Z - \bar{\chi}\not{\partial}P_L\chi - \overline{W}'W' + \tfrac{1}{2}\bar{\chi}(P_L W'' \,+\, P_R\overline{W}'')\chi\right]\,, \tag{8.34}$$

is invariant under the transformations rules (8.16) and their conjugates (8.17). Show that $[\delta_1, \delta_2]Z$ is exactly the same as in (8.30), but $[\delta_1, \delta_2]P_L\chi$ is modified as follows

$$[\delta_1, \delta_2]P_L\chi = -\tfrac{1}{4}\bar{\epsilon}_1\gamma^\mu\epsilon_2 P_L\left[\partial_\mu\chi - \gamma_\mu(\not{\partial} - \overline{W}'')\chi\right]\,. \tag{8.35}$$

We find the spacetime translation plus an extra term that vanishes for any solutions of the EOM's.

Since the commutator of symmetries must give a symmetry of the action[7] and translations are a known symmetry, the remaining transformation, namely

$$\begin{aligned}\delta Z &= 0\,,\\ \delta\chi &= v^\mu\gamma_\mu(\not{\partial} - P_L W'' - P_R\overline{W}'')\chi\,,\end{aligned} \tag{8.36}$$

is itself a symmetry for any constant vector v^μ. However, its Noether charge vanishes when the fermion equation of motion is satisfied, so it has no physical effect. Such a symmetry is sometimes called a ‘zilch symmetry’.

[7] The argument is easy: a symmetry is a transformation such that $S_{,i}\delta(\epsilon)\phi^i = 0$, where $S_{,i}$ is the functional derivative to all fields ϕ^i. Applying a second transformation gives $S_{,ij}\delta(\epsilon_1)\phi^i\delta(\epsilon_2)\phi^j + S_{,i}\delta(\epsilon_2)\delta(\epsilon_1)\phi^i = 0$. Taking the commutator, the first term vanishes, and the second term says that the commutator defines a symmetry.

Although nothing physically essential is changed by eliminating auxiliary fields, we can nevertheless see that they play a useful role:
i) it is only with F, $\bar{F}$ included that the form of the SUSY transformation rules (8.16),(8.17) is independent of the superpotential $W(Z)$.
ii) the SUSY algebra is also universal on all components of the chiral multiplet when F is included. The phrase used in the literature is that the SUSY algebra is 'closed off-shell' when auxiliary fields are included and 'closed only on-shell' when they are eliminated.
iii) auxiliary fields are very useful in determining the terms in a SUSY Lagrangian describing interactions between different multiplets such as the SUSY gauge theories described in the next section.
iv) When one considers local supersymmetry and quantizes the theory, ghosts have to be considered. With auxiliary fields the ghost interactions are quadratic in the ghosts, but without auxiliary fields also quartic ghost interactions are needed.

It is also the case that auxiliary fields are known only for a few extended SUSY theories in 4-dimensional spacetime and also unavailable for dimension $D > 4$. Indeed many of the most interesting SUSY theories have no known auxiliary fields.

Although we hope that some readers enjoy the detailed manipulations needed to study SUSY theories, we suspect that many are fed up with Fierz rearrangement and would like a more systematic approach. To a large extent the superspace formalism does exactly that and has many advantages. Unfortunately, complete superspace methods are also unavailable when auxiliary fields are not known.

8.3 SUSY Gauge Theories

The basic SUSY gauge theory is the $\mathcal{N} = 1$ SYM theory containing the gauge multiplet A_μ^A, λ^A where A is the index of the,adjoint representation of a compact, non-abelian gauge group G. This theory was described in Sec. 8.1.2. The discussion there focused on the conserved supercurrent and will be extended to include field variations, auxiliary fields, and the SUSY algebra.

We assume that the group has an invariant metric that can be chosen as δ_{AB}. This is the case for 'reductive groups', i.e. products of compact semisimple groups and Abelian factors, i.e. $G = G_1 \otimes G_2 \otimes \ldots$, where each factor is a simple group or U(1). The normalization of the generators is fixed in each factor, which can lead to different gauge coupling constants $g_1, g_2, \ldots$ for each of these factors. We have taken here the normalizations of the generators where these coupling constants are not explicitly appearing. One can replace everywhere t_A with $g_i\, t_A$ and $f_{AB}{}^C$ with $g_i\, f_{AB}{}^C$, where g_i can be chosen independently in each simple factor, to re-install these coupling constants. Usually one also redefines then the parameters θ^A to $\frac{1}{g_i}\theta^A$. This leads to the formulae with coupling constant g in Sec. 4.3. Further note that for these algebras, the structure constants can be written as f_{ABC}, which are completely antisymmetric.

8.3.1 SUSY Yang-Mills vector multiplet

Our first objective is to obtain the SUSY variations δA_μ^A and $\delta\lambda^A$ under which the action (8.7) is invariant. This will give 'on-shell' supersymmetry; then we will add the auxiliary field. We will organize the presentation to make use of previous work in Secs. 8.1.1 and 8.1.2, which established that the super-current (8.9) is conserved.

The variation of (8.7) is

$$\delta S = \int \mathrm{d}^4x \left[\delta A_\nu^A D^\mu F_{\mu\nu}^A - \delta\bar\lambda^A\gamma^\mu D_\mu\lambda^A + \tfrac12 f_{ABC}\delta A_\mu^A\bar\lambda^B\gamma^\mu\lambda^C\right] . \tag{8.37}$$

We first note that the forms

$$\delta A_\mu^A = -\tfrac12\bar\epsilon\gamma_\mu\lambda^A\,, \qquad \delta\lambda^A = \tfrac14\gamma^{\rho\sigma}F_{\rho\sigma}^A\epsilon\,, \tag{8.38}$$

are determined, up to constant factors, by Lorentz and parity symmetry and by the dimensions (in units of l^{-1}) of the quantities involved. Denoting the dimension of any quantity x by $[x]$, we have $[\epsilon] = -1/2$, $[A_\mu] = 1$, $[\lambda] = 3/2$. Note that if we use the assumed form for δA_μ, then the last term in (8.37) vanishes by the Fierz rearrangement identity (8.12). We substitute both assumed variations, assuming that $\epsilon(x)$ is a general function, and integrate by parts in the second term of (8.37) to obtain

$$\begin{aligned}\delta S &= -\tfrac12\int \mathrm{d}^4x\, \left[\bar\epsilon\gamma^\nu\lambda^A D^\mu F_{\mu\nu}^A + \tfrac12\bar\epsilon\gamma^{\rho\sigma}\gamma^\mu\lambda^A D_\mu F_{\rho\sigma}^A + \tfrac12\partial_\mu\bar\epsilon\gamma^{\rho\sigma}\gamma^\mu F_{\rho\sigma}^A\lambda^A\right]\\ &= -\tfrac12\int \mathrm{d}^4x\,[\bar\epsilon\gamma^\nu\lambda^A D^\mu F_{\mu\nu}^A - \bar\epsilon\gamma^\nu\lambda^A D^\mu F_{\mu\nu}^A + \tfrac12\partial_\mu\bar\epsilon\gamma^{\rho\sigma}\gamma^\mu F_{\rho\sigma}^A\lambda^A]\,,\end{aligned} \tag{8.39}$$

where (8.6) and the gauge field Bianchi identity were used to reach the final line. Thus δS vanishes for constant ϵ, establishing supersymmetry, while the supercurrent $\mathcal{J}^\mu$ of (8.9) appears in the last term![8]

The auxiliary field required for the gauge multiplet is a real pseudoscalar field D^A in the adjoint representation of G. This fact follows from the superspace formulation. The auxiliary field enters the action and transformation rules in the quite simple fashion

$$S = \int \mathrm{d}^4x\, \left[-\tfrac14F^{\mu\nu A}F_{\mu\nu}^A - \tfrac12\bar\lambda^A\gamma^\mu D_\mu\lambda^A + \tfrac12 D^AD^A\right] , \tag{8.40}$$

$$\begin{aligned}\delta A_\mu^A &= -\tfrac12\bar\epsilon\gamma_\mu\lambda^A\,,\\ \delta\lambda^A &= \left[\tfrac14\gamma^{\mu\nu}F_{\mu\nu}^A + \tfrac12\mathrm{i}\gamma_*D^A\right]\epsilon\,,\\ \delta D^A &= \tfrac12\mathrm{i}\,\bar\epsilon\gamma_*\gamma^\mu D_\mu\lambda^A\,, \qquad D_\mu\lambda^A \equiv \partial_\mu\lambda^A + \lambda^C A_\mu{}^B f_{BC}{}^A\,.\end{aligned} \tag{8.41}$$

[8] Those who want to keep track of detailed conventions, can observe here that our conventions identify $\frac14\mathcal{J}^\mu$ as the Noether current.

Ex. 8.12 *Show that the $\delta S = 0$. Only terms involving D^A need to be examined.*

The fields of the gauge multiplet transform also under an internal gauge symmetry:

$$\begin{aligned}
\delta(\theta)A_\mu^A &= \partial_\mu\theta^A + \theta^C A_\mu{}^B f_{BC}{}^A\,, \\
\delta(\theta)\lambda^A &= \theta^C\lambda^B f_{BC}{}^A\,, \\
\delta(\theta)D^A &= \theta^C D^B f_{BC}{}^A\,.
\end{aligned} \tag{8.42}$$

Let us first remark that the commutator of these internal gauge transformations and supersymmetry vanishes.

Ex. 8.13 *Use the transformation rules above to derive the commutator algebra of the supersymmetries for the gauge multiplet*

$$\begin{aligned}
[\delta_1,\delta_2]\,A_\mu^A &= -\tfrac{1}{2}\bar\epsilon_1\gamma^\nu\epsilon_2 F_{\nu\mu}^A\,, \\
[\delta_1,\delta_2]\,\lambda^A &= -\tfrac{1}{2}\bar\epsilon_1\gamma^\nu\epsilon_2 D_\nu\lambda^A\,, \\
[\delta_1,\delta_2]\,D^A &= -\tfrac{1}{2}\bar\epsilon_1\gamma^\nu\epsilon_2 D_\nu D^A\,.
\end{aligned} \tag{8.43}$$

It is no surprise that the commutator of two gauge-covariant variations from (8.41) is gauge covariant, but at first glance the result seems to disagree with the spacetime translation required by (8.1) and (8.29). Note that in all 3 cases in ex. 8.13 the difference between the covariant result in (8.43) and a translation is a gauge transformation by the field-dependent gauge parameter $\theta^A = \frac{1}{2}\bar\epsilon_1\gamma^\nu\epsilon_2 A_\nu^A$. The covariant forms that occur in (8.43) are called gauge-covariant translations. The conclusion is that on the fields of a gauge theory the SUSY algebra closes on gauge-covariant translations. See [52] and Sec. 4.1.5 for further information on this issue.

8.3.2 Chiral multiplets in SUSY gauge theories

We now present and briefly discuss the class of SUSY theories in which the gauge multiplet A_μ^A, λ^A, D^A is coupled to a chiral matter multiplet $Z^\alpha, P_L\chi^\alpha, F^\alpha$ in an arbitrary finite dimensional representation $\mathbf{R}$ of G with matrix generators $(t_A)^\alpha{}_\beta$. Under an infinitesimal gauge transformation with parameters $\theta^A(x)$ the fermions transformas

$$\begin{aligned}
\delta P_L\chi^\alpha &= -\theta^A(t_A)^\alpha{}_\beta P_L\chi^\beta\,, \\
\delta P_R\chi_\alpha &= -\theta^A(t_A)_\alpha^{*\,\beta} P_R\chi_\beta\,,
\end{aligned} \tag{8.44}$$

with similar rules for the other fields. Representation indices are suppressed in most formulas. Thus we can write covariant derivatives of the various fields as

$$D_\mu\lambda^A = \partial_\mu\lambda^A + f_{BC}{}^A A_\mu^B\lambda^C\,,$$

$$\begin{aligned} D_\mu Z &= \partial_\mu Z + t_A A_\mu^A Z\,, \\ D_\mu P_L\chi &= \partial_\mu P_L\chi + t_A A_\mu^A P_L\chi\,, \\ D_\mu P_R\chi &= \partial_\mu P_R\chi + t_A^* A_\mu^A P_R\chi\,. \end{aligned} \tag{8.45}$$

The system need not contain a superpotential, but superpotentials $W(Z^\alpha)$, which should be both holomorphic and gauge invariant, are optional. It is useful to express the condition of gauge invariance of $W(Z^\alpha)$ as

$$\delta_{\text{gauge}} W = W_\alpha \delta_{\text{gauge}} Z^\alpha = -W_\alpha \theta^A (t_A)^\alpha{}_\beta Z^\beta = 0\,. \tag{8.46}$$

The action of the general theory is the sum of several terms

$$S = S_{\text{gauge}} + S_{\text{matter}} + S_{\text{coupling}} + S_W + S_{\overline{W}}\,. \tag{8.47}$$

The form of some terms agrees with expressions given earlier in this chapter. Since convenience is a virtue and repetition is no sin, we shall write everything here.

$$S_{\text{gauge}} = \int \mathrm{d}^4x \left[-\tfrac14 F^{\mu\nu A}F_{\mu\nu}^A - \tfrac12 \bar\lambda^A\gamma^\mu D_\mu\lambda^A + \tfrac12 D^A D^A\right]\,, \tag{8.48}$$

$$S_{\text{matter}} = \int \mathrm{d}^4x \left[-D^\mu \bar Z D_\mu Z - \bar\chi\gamma^\mu P_L D_\mu\chi + \bar F F\right]\,, \tag{8.49}$$

$$S_{\text{coupling}} = \int \mathrm{d}^4x \left[-\sqrt2(\bar\lambda^A \bar Z t_A P_L\chi - \bar\chi P_R t_A Z\lambda^A) + \mathrm{i}\, D^A \bar Z t_A Z\right]\,, \tag{8.50}$$

$$S_F = \int \mathrm{d}^4x \left[F^\alpha W_\alpha + \tfrac12 \bar\chi^\alpha P_L W_{\alpha\beta}\chi^\beta\right]\,, \tag{8.51}$$

$$S_{\bar F} = \int \mathrm{d}^4x \left[\bar F_\alpha \overline{W}^\alpha + \tfrac12 \bar\chi_\alpha P_R \overline{W}^{\alpha\beta}\chi_\beta\right]\,. \tag{8.52}$$

The full action is invariant under SUSY transformation rules, which are unchanged from (8.41). The modified gauge-covariant transformation rules of the chiral and anti-chiral multiplets are

$$\begin{aligned} \delta Z &= \frac{1}{\sqrt2}\bar\epsilon P_L\chi\,, \\ \delta P_L\chi &= \frac{1}{\sqrt2}P_L(\gamma^\mu D_\mu Z + F)\epsilon\,, \\ \delta F &= \frac{1}{\sqrt2}\bar\epsilon P_R\gamma^\mu D_\mu\chi - \bar\epsilon P_R\lambda^A t_A Z\,, \end{aligned} \tag{8.53}$$

and

$$\begin{aligned} \delta \bar Z &= \frac{1}{\sqrt2}\bar\epsilon P_R\chi\,, \\ \delta P_R\chi &= \frac{1}{\sqrt2}P_R(\gamma^\mu D_\mu \bar Z + \bar F)\epsilon\,, \\ \delta \bar F &= \frac{1}{\sqrt2}\bar\epsilon P_L\gamma^\mu D_\mu\chi - \bar\epsilon P_L\lambda^A (t_A)^* \bar Z\,. \end{aligned} \tag{8.54}$$

Some modifications in the action and transformation rules above, notably the introduction of gauge covariant derivatives, are clearly required in a gauge theory, but other additions such as the form of the action S_{coupling} are surely not obvious. The best explanation is that they are dictated by the superspace formalism. However all features can be explained from the component viewpoint. For example, in the SUSY variation δS_{matter} many terms cancel by the same manipulations required to show that δS_{kin} of (8.18) vanishes by simply replacing ∂_μ by D_μ. But there are extra terms due the variation δA_μ^A,

$$\delta S_{\text{matter}} = \int \mathrm{d}^4x\, \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A \left(\bar{\chi}t_A\gamma^\mu P_L\chi - \bar{Z}t_A D^\mu Z + D^\mu\bar{Z}t_A Z\right) , \tag{8.55}$$

which involves the gauge current of the matter fields, and there is a correction to (8.24) due to the gauge Ricci identity similar to (4.85), and proportional to $F_{\mu\nu}^A\bar{\epsilon}P_L\gamma^{\mu\nu}\bar{Z}t_A\chi$. These terms are cancelled by the variations of Z and χ in $\delta S_{\text{coupling}}$. A complete demonstration that the total action (8.47) is invariant under the variations (8.41),(8.53) requires quite delicate calculations, which we recommend only for sufficiently diligent readers. The reader will also be invited to explain the extra terms in (8.53) from algebra considerations below in exercise 14.1

Ex. 8.14 *Show that the action (8.47) is supersymmetric. Why does one need the variation $\delta F^\alpha W_\alpha(Z)$ induced by the last term in δF?*

8.4 Massless supersymmetry multiplets

We will give a brief introduction to the theory of massless unitary representations of the supersymmetry algebras in 4 dimensions. You can consult e.g. Sec. 3 of the review of M. Sohnius [53] for a more complete treatment.

Wigner [54] defined a particle as a unitary representation of the Poincaré group. To classify these, he made a distinction between massive and massless states. For the massless case, he obtained that a particle consists of 2 helicity states $h = \pm s$, where s is the spin of the particle. Spin 0 particles have only one helicity state.

The theory of representations of the supersymmetry algebra has been investigated by Salam and Strathdee [55]. They construct the physical states of a unitary representation of the $\mathcal{N}$-extended algebra as representations of a Clifford algebra of dimension $2\mathcal{N}$.

The minimal representation has thus dimension $2^{\mathcal{N}}$. In fact, one can divide the relevant parts of the supersymmetry generators in $\mathcal{N}$ helicity-creation operators, say C_+^i where $i = 1, \ldots, \mathcal{N}$ and $\mathcal{N}$ helicity-annihilation operators, say $C_-^{\underline{i}}$. As these are supersymmetry operators, they change the helicity by $\pm 1/2$. The states are then built from a Clifford vacuum that has itself an helicity κ, which we may denote as $|\kappa>$. With the $\mathcal{N}$ creation operators, one obtains in the supersymmetry multiplets states

$$|\kappa>,\ C_+^i|\kappa>,\ C_+^i C_+^j|\kappa>,\ ,\ldots , \tag{8.56}$$

of helicity $(\kappa, \kappa + 1/2, \ldots, \kappa + \mathcal{N}/2)$. If this set of helicity states is not self-CPT conjugate, which means that states appear with helicity h and that there is no corresponding state with helicity $-h$, then one has to introduce also the conjugate states $(-\kappa, -\kappa - 1/2, \ldots, -\kappa - \mathcal{N}/2)$, i.e. the multiplet with $\kappa' = -\kappa - \mathcal{N}/2$. As we thus anyway have to accompany the construction from κ with one from κ', to classify all the possibilities it is sufficient to start from $\kappa = -\mathcal{N}/4$ or the higher half-integer from this number.[9]

For $\mathcal{N} = 1$ we thus find for for $\kappa = 0$ the multiplet $(0, 1/2)$ to be complemented with $(0, -1/2)$ leading to the fields of the chiral multiplet. For $\kappa = 1/2$, we find fields with spin $(1/2, 1)$, i.e. the gauge multiplet. In principle, the next multiplet has spins $(1, 3/2)$. However, there is no interacting field theory known for this multiplet without supergravity. This is due to the fact that we saw that field theories for spin 3/2 fields involve a local supersymmetry. Hence, they should by the supersymmetry algebra also involve local translations, and hence general relativity. Therefore, we find the spin 3/2 particle only in the multiplet $(3/2, 2)$. This is the supergravity theory that we will consider in Ch. 9.

The problem that we mentioned above for field theories of spin 3/2 particles implies that global supersymmetry multiplets should be restricted to the spins ≤ 1, i.e. helicities $-1 \leq h \leq 1$. With the above rules, that means that we are restricted to $\mathcal{N} \leq 4$.

Despite a lot of efforts, it is still difficult if not impossible to construct interacting field theories in Minkowski space with particles of spin $\geq 5/2$. With the same principle as above, this implies that supergravity theories are restricted to $\mathcal{N} \leq 8$, which is a very important fact of life for supergravity.

Ex. 8.15 *With this information, the student should check the tables of number of fields with different spins in multiplets for different $\mathcal{N}$. These tables appear later in the book, see tables 12.1 and 12.2 in Sec. 12.3.2.*

[9] There is still a more delicate hermiticity requirement for $\mathcal{N} = 2$ such that the multiplet $(-1/2, 0, 0, 1/2)$ also has to be doubled.

9

$\mathcal{N} = 1$ pure supergravity in 4 dimensions

In earlier chapters we have reviewed the ideas and implementation of Lorentz invariance, relativistic spin, gauge principles, global supersymmetry, and spacetime geometry. We will now begin to study how these elements combine in supergravity. The key idea is that supersymmetry holds *locally* in a supergravity theory. The action is invariant under SUSY transformations in which the spinor parameters $\epsilon(x)$ are arbitrary functions of the spacetime coordinates. The SUSY algebra (see (8.1) and (8.29)) will then involve local translation parameters $\bar{\epsilon}_1\gamma^\mu\epsilon_2$ which must be viewed as diffeomorphisms. Thus local supersymmetry requires gravity.

A supergravity theory is a nonlinear, and thus interacting, field theory that necessarily contains the gauge or gravity multiplet plus, optionally, other matter multiplets of the underlying global supersymmetry algebra. The gauge multiplet consists of the frame field $e_\mu^a(x)$ describing the graviton, plus a specific number $\mathcal{N}$ of vector-spinor fields $\Psi_\mu^i(x)$, $i = 1, \ldots, \mathcal{N}$, whose quanta are the gravitinos, the supersymmetric partners of the graviton. In the basic case of $\mathcal{N} = 1$ supergravity in $D = 4$ spacetime dimensions, the gauge multiplet consists entirely of the graviton and one Majorana spinor gravitino. In all other cases, both $\mathcal{N} \geq 2$ in $D = 4$ dimensions and $\mathcal{N} \geq 1$ for $D \geq 5$, additional fields are required in the gauge multiplet.

Supergravity theories exist for spacetime dimensions $D \leq 11$. For each dimension D, a specific type of spinor is required, e.g. Majorana or Weyl. For $D = 4$, theories exist for $\mathcal{N} = 1, 2, \ldots, 8$. Beyond these limits local supersymmetry fails and the classical equations of motion are inconsistent.

In this chapter we will concentrate on the basic $\mathcal{N} = 1$, $D = 4$ supergravity theory. We will discuss the form of the action and transformation rules and prove local supersymmetry [2]. The principal terms of the $\mathcal{N} = 1$, $D = 4$ action are an important part of the structure of all supergravity theories, and the initial steps in the proof of local SUSY are universal, that is, applicable in any dimension. We discuss these universal steps in the next section, and then refocus and complete the proof for $\mathcal{N} = 1$, $D = 4$ supergravity.

The approach in this chapter is important to understand the structure of supergravity. . Both approaches are valuable. The approach below is most easily extended to higher dimension, while the superconformal approach is better suited to the derivation and understanding of matter couplings.

9.1 The universal part of supergravity

The part of the supergravity action we will study in this section consists of the sum of the Hilbert action for gravity in second order formalism plus a local Lorentz and diffeomorphism invariant extension of the free gravitino action of (5.2) multiplied by 1/2 because we deal with dimensions in which $t_3 = 1$ and rescaled by the factor $1/\kappa^2$ for convenience.[1]

The action is

$$\begin{aligned} S &= S_2 + S_{3/2}\,, & (9.1)\\ S_2 &= \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega)\,, & (9.2)\\ S_{3/2} &= -\frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\, \bar\psi_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho\,, & (9.3)\end{aligned}$$

where e stands for the determinant of e_μ^a. The gravitino covariant derivative is given by

$$D_\nu \psi_\rho \equiv \partial_\nu \psi_\rho + \tfrac{1}{4}\omega_{\nu ab}\gamma^{ab}\psi_\rho\,. \qquad (9.4)$$

It is the torsion-free spin connection $\omega_{\nu ab}(e)$, given in (6.79), that is used exclusively in this section. We need not include the Christoffel connection term $\Gamma^\sigma_{\nu\rho}(g)\psi_\sigma$ in (9.4) because the connection (6.93) is symmetric, and the term vanishes in the action $S_{3/2}$ since the Lagrangian is antisymmetric in $\nu\rho$.

We also need transformation rules, and we postulate the rules

$$\begin{aligned} \delta e_\mu^a &= \tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,, & (9.5)\\ \delta\psi_\mu &= D_\mu\epsilon(x) \equiv \partial_\mu\epsilon + \tfrac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon\,. & (9.6)\end{aligned}$$

The gravitino is the gauge field of local supersymmetry, so it is natural to postulate (9.6) as the curved space generalization of (5.1). For the frame field, (9.5) is the simplest form consistent with the tensor structure required and the Bose-Fermi character of supersymmetry.[2] In Ch. 11 we will give a group-theoretical argument for the supersymmetry transformations postulated here.

[1] The derivation we present in this section uses transposition properties of gamma matrices ($t_3 = 1$) and reality properties that are strictly valid only for Majorana spinors in $D = 4$, 10, 11, mod 8 dimensions. However the method can be easily modified to apply to any desired type of spinor.

[2] The possible form $\delta e_\mu^a \sim \bar\epsilon\gamma_\mu\psi^a$ may be considered explicitly and shown to be incompatible with local supersymmetry.

Ex. 9.1 *Deduce from (9.5) that*

$$\delta e_a^\mu = -\tfrac{1}{2}\bar{\epsilon}\gamma^\mu\psi_a\,, \qquad \delta e = \tfrac{1}{2}e\,(\bar{\epsilon}\gamma^\rho\psi_\rho). \tag{9.7}$$

The action and transformation rules above are not quite the whole story even for the $\mathcal{N}=1$, $D=4$ theory, but they can be completed by incorporating the first order formalism with torsion. This is done in the next section. For all other theories one must also add terms describing the other fields in the gauge multiplet. The variation of the action (9.1) consists of linear terms in ψ_μ from the frame field variation of S_2 and the gravitino variation of $S_{3/2}$ and cubic terms from the frame field variation of $S_{3/2}$. It are the linear terms that are universal, and we now proceed to calculate them.

The variation of the gravitational action under (9.5) is obtained as in Sec. 7.3, see the first part of (7.28)

$$\delta S_2 = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\left(-\bar{\epsilon}\gamma^\mu\psi^\nu\right)\,. \tag{9.8}$$

We now study the gravitino variation. In the second order formalism, partial integration is valid, so it is sufficient to vary $\delta\bar{\psi}_\mu$ and multiply by 2, obtaining

$$\begin{aligned}\delta S_{3/2} &= -\frac{1}{\kappa^2}\int \mathrm{d}^D x\, e\,\bar{\epsilon}\overleftarrow{D}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho \\ &= \frac{1}{\kappa^2}\int \mathrm{d}^D x\, e\,\bar{\epsilon}\gamma^{\mu\nu\rho}D_\mu D_\nu\psi_\rho = \frac{1}{8\kappa^2}\int \mathrm{d}^D x\, e\,\bar{\epsilon}\gamma^{\mu\nu\rho}R_{\mu\nu ab}\gamma^{ab}\psi_\rho\,.\end{aligned} \tag{9.9}$$

We integrated by parts and used (7.23), to move to the second line[3] and then used the Ricci identity to obtain the last expression.

We now need some Dirac algebra to evaluate the product $\gamma^{\mu\nu\rho}\gamma^{ab}$. This product was already discussed as an example in (3.20). We thus find[4]

$$\begin{aligned}\gamma^{\mu\nu\rho}\gamma^{ab}R_{\mu\nu ab} &= \gamma^{\mu\nu\rho ab}R_{\mu\nu ab} + 6R_{\mu\nu}{}^{[\rho}{}_b\gamma^{\mu\nu]b} + 6\gamma^{[\mu}R_{\mu\nu}{}^{\rho\nu]} \\ &= \gamma^{\mu\nu\rho ab}R_{\mu\nu ab} + 2R_{\mu\nu}{}^{\rho}{}_b\gamma^{\mu\nu b} + 4R_{\mu\nu}{}^{\mu}{}_b\gamma^{\nu\rho b} \\ &\quad +4\gamma^\mu R_{\mu\nu}{}^{\rho\nu} + 2\gamma^\rho R_{\mu\nu}{}^{\nu\mu}\,,\end{aligned} \tag{9.10}$$

where in the second line we split the symmetry of $[\mu\nu\rho]$ according to the position of ρ. Anti-symmetry in $[\mu\nu]$ is incorporated because $R_{\mu\nu ab}$ is present in every term. The first term and second term vanish because of the first Bianchi identity (6.108) (without torsion). The third term vanishes because of the symmetry of the Ricci tensor, $R_{\nu b} = R_{\mu\nu}{}^{\mu}{}_b$ being contracted with the antisymmetric $\gamma^{\nu\rho b}$. We are thus left with

$$\delta S_{3/2} = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, e\,(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)(\bar{\epsilon}\gamma^\mu\psi^\nu)\,. \tag{9.11}$$

[3] We can replace ∇_μ by D_μ in the second line because of the antisymmetry and absence of torsion.

[4] Note that we apply (3.20) with a mixture of tangent space indices and curved indices. These can be freely mixed, as the equations are related by multiplication with the invertible frame fields.

We now observe the exact cancellation between (9.8) and (9.11) which shows that local supersymmetry holds to linear order in ψ_μ for any spacetime dimension D that allows Majorana spinors.[5] Notice that the calculation involves an intimate mix of the key properties of Riemannian geometry and Dirac algebra!

9.2 Supergravity in the First Order Formalism

Beyond linear order it becomes complicated to establish local supersymmetry for any supergravity theory. It is therefore very useful to recognize simplifications which lead to convenient organization of terms. One important simplification [3] is to express the action and transformation rules in the first order formalism. We continue to work with the D-dimensional action (9.1) and transformation rules (9.5), (9.6), but we regard the spin connection $\omega_{\mu ab}$ as an independent variable. There is no Christoffel connection term in $S_{3/2}$. It would be inconsistent with local supersymmetry to include it.[6]

As discussed in Ch. 7, in the first order formalism the equation of motion for the spin connection can be solved to obtain a connection with torsion. This result can then be substituted in the action to obtain the physically equivalent second order form of the theory with torsion-free connection and explicit 4-fermion contact terms. We now carry out this process for the supergravity action (9.2). We will use the connection variation (7.32) of S_2 derived in Ch. 7.

For $S_{3/2}$ we easily obtain

$$\delta S_{3/2} = -\frac{1}{8\kappa^2}\int \mathrm{d}^D x\, e\, (\bar\psi_\mu \gamma^{\mu\nu\rho}\gamma_{ab}\psi_\rho)\delta\omega_\nu{}^{ab}\,. \tag{9.12}$$

The Clifford algebra relation needed to simplify the spinor bilinear in (9.12) is again the one given in (3.20). We restrict here to the dimensions in which we have Majorana spinors, $D = 2, 3, 4, 10, 11$. Then, according to table 3.1 spinor bilinears of rank 3 are symmetric, which clashes with the antisymmetry in the indices μ, ρ on the gravitinos. Therefore, we obtain

$$\bar\psi_\mu \gamma^{\mu\nu\rho}\gamma_{ab}\psi_\rho = \bar\psi_\mu \left(\gamma^{\mu\nu\rho}{}_{ab} + 6\gamma^{[\mu} e^\nu{}_{[b} e^{\rho]}{}_{a]}\right)\psi_\rho\,. \tag{9.13}$$

It is now very easy to solve the connection field equation

$$\delta S_2 + \delta S_{3/2} = 0\,, \tag{9.14}$$

using (7.32). The trace structure in the two terms matches exactly, so the unique solution for the torsion is

$$T_{ab}{}^\nu = \tfrac{1}{2}\bar\psi_a\gamma^\nu\psi_b + \tfrac{1}{4}\bar\psi_\mu\gamma^{\mu\nu\rho}{}_{ab}\psi_\rho\,. \tag{9.15}$$

[5] Restrictions on D appear if local SUSY is imposed beyond linear order.

[6] Note that a Christoffel term is permitted by diffeomorphism invariance since the antisymmetric part of $\Gamma^\sigma_{\nu\rho}$ is proportional to the torsion tensor. See (6.94).

The fifth rank tensor term is one of the complications of supergravity for $D \geq 5$, but it simply vanishes when $D = 4$.

For gravity coupled to a spin 1/2 Dirac field, we showed in (7.36), (7.37) how to obtain the physically equivalent second order form of the theory by substituting the value of the torsion tensor in the first order action. Here is an exercise to do the same for supergravity.

Ex. 9.2 *For $D = 4$ substitute the torsion tensor (9.15) in the action (9.1) to obtain the second order action of supergravity*

$$\begin{aligned} S &= \frac{1}{2\kappa^2} \int \mathrm{d}^4 x \, e \, \left[R(e) - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + \mathcal{L}_{\mathrm{SG,torsion}} \right] , \qquad (9.16) \\ \mathcal{L}_{\mathrm{SG,torsion}} &= -\frac{1}{16} \left[(\bar{\psi}^\rho \gamma^\mu \psi^\nu)(\bar{\psi}_\rho \gamma_\mu \psi_\nu + 2\bar{\psi}_\rho \gamma_\nu \psi_\mu) - 4(\bar{\psi}_\mu \gamma \cdot \psi)(\bar{\psi}^\mu \gamma \cdot \psi) \right] , \end{aligned}$$

in which the curvature $R(e)$ and the gravitino covariant derivative now contains the torsion-free connection

$$D_\nu \psi_\rho \equiv \partial_\nu \psi_\rho + \tfrac{1}{4} \omega_{\nu ab}(e) \gamma^{ab} \psi_\rho \, . \qquad (9.17)$$

The theory expressed by (9.16) contains 4-point contact diagrams for gravitino scattering. Their physical effects are certainly not dramatic, but they are necessary for the consistency of the theory, which does not otherwise obey the requirement of local supersymmetry. Local supersymmetry means that the action (9.16) is invariant under the transformation rules (9.5),(9.6), with $\omega = \omega(e) + K$, see (6.78).

9.3 The 1.5 order formalism

The second order action (9.16) for $\mathcal{N} = 1$, $D = 4$ supergravity is complete, and it possesses complete local supersymmetry under the transformation rules (9.5), (9.6) with the connection

$$\begin{aligned} \delta \psi_\mu &= D_\mu \epsilon \equiv \partial_\mu \epsilon + \tfrac{1}{4} \omega_{\mu ab} \gamma^{ab} \epsilon \, , \\ \omega_{\mu ab} &= \omega_{\mu ab}(e) + K_{\mu ab} \, , \\ K_{\mu\nu\rho} &= -\tfrac{1}{4} (\bar{\psi}_\mu \gamma_\rho \psi_\nu - \bar{\psi}_\nu \gamma_\mu \psi_\rho + \bar{\psi}_\rho \gamma_\nu \psi_\mu) \, , \qquad (9.18) \end{aligned}$$

which includes the gravitino torsion. We will prove this invariance property in the next section.

Let us now think schematically about the structure of the proof. The variation of the action contains terms which are first, third, and fifth order in the gravitino field. They are independent and must cancel separately. In the first construction of the theory [2], which used the second order formalism, the first and third order terms were treated analytically, but it required a computer calculation to show that the complicated order $(\psi_\mu)^5$ term vanishes.[7]

[7] Such terms come from the variation of the order $(\psi_\mu)^4$ contact terms in (9.16) with respect to the frame field and the torsion part of $\delta \psi_\mu$.

In the first order form of the theory [3], the fifth order variation is avoided, but one must specify a new transformation rule $\delta\omega_{\mu ab}$, since the connection is an independent variable. This approach becomes quite complicated when matter multiplets are coupled to supergravity.

For the reasons above, the simplest treatment of supergravity uses a formalism intermediate between the first and second order versions we have discussed. It is therefore called the 1.5 order formalism [56, 57], and it is applicable to supergravity theories in any dimension.

One is really working in the second order formalism since there are only two independent fields, e^a_μ, ψ_μ. However, since the second order Lagrangian is obtained by substituting ω from (9.18) in the first order action, one can simplify the proof of invariance by retaining the original grouping of terms. To see more concretely how this works, let's consider an action which is a functional of the three variables $S[e,\,\omega,\,\psi]$. We use the chain rule to calculate its variation in the second order formalism, viz.

$$\delta S[e,\,\omega(e)+K,\,\psi] = \int \mathrm{d}^D x\,\left[\frac{\delta S}{\delta e}\delta e + \frac{\delta S}{\delta\omega}\delta(\omega(e)+K) + \frac{\delta S}{\delta\psi}\delta\psi\right]. \tag{9.19}$$

In particular, the variation $\delta(\omega(e)+K)$ is calculated using (9.5) and (9.18). However, no calculation is needed since the second term vanishes because the expression for ω in (9.18)is obtained by solving the algebraic field equation $\delta S/\delta\omega = 0$. Thus, in the 1.5 order formalism, we can neglect *all* $\delta\omega$ variations as we proceed to establish local supersymmetry.

Summary of the prescription for δS in the 1.5 order formalism:

1. Use the first order form of the action $S[e,\omega,\psi]$ and the transformation rules δe, $\delta\psi$ with connection ω unspecified.
2. Ignore the connection variation and calculate

$$\delta S = \int \mathrm{d}^D x\,\left[\frac{\delta S}{\delta e}\delta e + \frac{\delta S}{\delta\psi}\delta\psi\right]. \tag{9.20}$$

3. Substitute ω from (9.18) in the result, which must vanish for a consistent supergravity theory.

9.4 Local supersymmetry of $\mathcal{N}=1$, $D=4$ supergravity

In this section we will use the 1.5 order formalism to prove that $\mathcal{N}=1$, $D=4$ supergravity is a consistent gauge theory, invariant under the transformation rules (9.5) and (9.18) with arbitrary $\epsilon(x)$. The proof is specific to $D=4$ spacetime dimensions, so we first simplify the gravitino action by introducing the highest rank Clifford element $\gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$. This is related to the third rank Clifford matrices by

$$\gamma^{abc} = -\mathrm{i}\varepsilon^{abcd}\gamma_*\gamma_d\,, \qquad \gamma^{\mu\nu\rho} = -\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_\sigma\,. \tag{9.21}$$

The first relation holds in local frames and the second, which we need, in the coordinate basis of spacetime. The Levi-Civita tensor density, with $\varepsilon^{0123} = -1$, appears in both relations. Using (9.21) and the fact that $\det(e_\mu^a e_a^\nu) = 1$, we can rewrite the gravitino action (9.3) as

$$S_{3/2} = \frac{\mathrm{i}}{2\kappa^2} \int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_* \gamma_\sigma D_\nu \psi_\rho\,. \tag{9.22}$$

This is the way that supergravity was written before more than 4 dimensions were considered. The advantage of this form is that the frame field variation δe is needed only in γ_σ rather than in four positions in $e\,\gamma^{\mu\nu\rho}$. As we can also ignore the $\delta\omega$ variation of $R_{\mu\nu ab}$ in the 1.5 order formalism, we should consider the following terms

$$\delta S \;=\; \delta S_2 + \delta S_{3/2,e} + \delta S_{3/2,\psi} + \delta S_{3/2,\bar{\psi}}\,, \tag{9.23}$$

where the first term is the variation of the gravity action, the second one is due to the frame variation of $S_{3/2}$, while the third and fourth are the variations of ψ and $\bar{\psi}$, respectively. We must obtain the ψ and $\bar{\psi}$ variations separately because partial integration of the local Lorentz derivative D_μ must be done carefully. Indeed, we will encounter a number of subtleties because of the connection with torsion.

The Ricci tensor with torsion is not symmetric, and we call it $R_{\mu\nu}(\omega)$ as a reminder. But it is easy to check that (9.8) is still valid, and can be rewritten as

$$\delta S_2 = \frac{1}{2\kappa^2} \int \mathrm{d}^4x\, e\, (R_{\mu\nu}(\omega) - \frac{1}{2} g_{\mu\nu} R(\omega))(-\bar{\epsilon}\gamma^\mu \psi^\nu)\,. \tag{9.24}$$

The second term is as mentioned above only due to the variation of γ_σ, i.e.

$$\delta S_{3/2,e} = \frac{\mathrm{i}}{4\kappa^2} \int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma} (\bar{\epsilon}\gamma^a \psi_\sigma)(\bar{\psi}_\mu \gamma_* \gamma_a D_\nu \psi_\rho)\,. \tag{9.25}$$

The ψ variation of $S_{3/2}$ is still simple:

$$\begin{aligned} \delta S_{3/2,\psi} \;&=\; \frac{\mathrm{i}}{2\kappa^2} \int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_* \gamma_\sigma D_\nu D_\rho \epsilon \\ &=\; \frac{\mathrm{i}}{16\kappa^2} \int \mathrm{d}^4x\, \bar{\psi}_\mu \varepsilon^{\mu\nu\rho\sigma} \gamma_* \gamma_\sigma \gamma^{ab} R_{\nu\rho ab}(\omega)\epsilon\,. \end{aligned} \tag{9.26}$$

The curvature tensor appears through the commutator of covariant derivatives.

Next we write the $\bar{\psi}_\mu$ variation and exchange the spinors $D_\mu\epsilon$ and ψ_ρ (see (3.49) with $t_3 = 1$) to obtain

$$\delta S_{3/2,\bar{\psi}} = \frac{\mathrm{i}}{2\kappa^2} \int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\rho \overleftarrow{D}_\nu \gamma_* \gamma_\sigma D_\mu \epsilon\,. \tag{9.27}$$

With some thought one see that the left-acting derivative $\bar{\psi}_\rho \overleftarrow{D}_\nu = \partial_\nu \bar{\psi}_\rho - \frac{1}{4}\bar{\psi}_\rho \omega_{\nu ab}\gamma^{ab}$ can be partially integrated and acts distributively to give

$$\begin{aligned} \delta S_{3/2,\bar{\psi}} &= \frac{-\mathrm{i}}{2\kappa^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\rho\gamma_*[(D_\nu\gamma_\sigma)D_\mu\epsilon + \gamma_\sigma D_\nu D_\mu\epsilon] \\ &= \frac{-\mathrm{i}}{2\kappa^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\rho\gamma_*\left[\frac{1}{2}T_{\nu\sigma}{}^a\gamma_a D_\mu\epsilon - \frac{1}{8}\gamma_\sigma\gamma^{ab}R_{\mu\nu ab}(\omega)\epsilon\right] . \end{aligned} \tag{9.28}$$

As indicated in (7.23) the full covariant $\nabla_\nu\gamma_\sigma = 0$. But supergravity employs only the local Lorentz covariant derivative D_ν. When we add back the Christoffel connection and use antisymmetry in $\nu\sigma$, we obtain with (6.94) the torsion tensor in (9.28).

Note that the $R(\omega)$ terms in (9.26) and (9.28) are equal, so that we obtain

$$\delta S_{3/2,\psi} + \delta S_{3/2,\bar{\psi}} = \frac{-\mathrm{i}}{2\kappa^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\rho\gamma_*\left[\frac{1}{2}T_{\nu\sigma}{}^a\gamma_a D_\mu\epsilon - \frac{1}{4}\gamma_\sigma\gamma_{ab}R_{\mu\nu}{}^{ab}(\omega)\epsilon\right] . \tag{9.29}$$

The last term can be treated using the methods for γ-matrix manipulations that we learned in Sect. 3.1.4:

$$\gamma_\sigma\gamma_{ab} = \gamma_{\sigma ab} + 2e_{\sigma[a}\gamma_{b]} = \mathrm{i}e_\sigma^d\varepsilon_{abcd}\gamma_*\gamma^c + 2e_{\sigma[a}\gamma_{b]} . \tag{9.30}$$

(The last expression is only valid for $D = 4$). We now consider these two terms separately. In the first term we encounter the contraction of two Levi-Civita symbols and Riemann tensor. Using (3.8), we write

$$\begin{aligned} \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd}e_\sigma^d R_{\nu\rho}{}^{ab}(\omega) &= -2e\left[e_a^\mu e_b^\nu e_c^\rho + e_b^\mu e_c^\nu e_a^\rho + e_c^\mu e_a^\nu e_b^\rho\right] R_{\nu\rho}{}^{ab}(\omega) \\ &= 4e\left[R_c{}^\mu(\omega) - \tfrac{1}{2}e_c^\mu R(\omega)\right] . \end{aligned} \tag{9.31}$$

When this relation is inserted in (9.29) and we use $\bar{\psi}_\mu\gamma^c\epsilon = -\bar{\epsilon}\gamma^c\psi_\mu$, the result exactly cancels δS_2 in (9.24). The work so far has brought us to the level of linear local supersymmetry proven in Sec. 9.1, although the present cancellation includes cubic terms from the torsion contribution to (9.24).

The contribution of the last term in (9.30) to the integrand of (9.29) involves the factor

$$\varepsilon^{\mu\nu\rho\sigma}R_{\nu\rho\sigma b}(\omega) = -\varepsilon^{\mu\nu\rho\sigma}D_\nu T_{\rho\sigma b} , \tag{9.32}$$

in which we have used the modified first Bianchi identity, derived in (6.108), where it is shown that the derivative D_ν contains only the spin connection acting on the index b. This leaves us with

$$\delta S_2 + \delta S_{3/2,\psi} + \delta S_{3/2,\bar{\psi}} = \frac{-\mathrm{i}}{4\kappa^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_*\gamma_a\left[T_{\rho\sigma}{}^a D_\nu\epsilon + (D_\nu T_{\rho\sigma}{}^a)\epsilon\right] . \tag{9.33}$$

The other remaining variations are in (9.25), which we will now rewrite using the torsion tensor. Therefore, we first reorder the spinors in the integrand using Fierz rearrangement technique of Sec. 3.2.3. Because of its importance here we repeat the result of Ex. 3.25

$$(\gamma^\mu)_\alpha{}^\beta(\gamma_\mu)_\gamma{}^\delta = \frac{1}{2^m}\sum_A v_A(\Gamma_A)_\alpha{}^\delta(\Gamma^A)_\gamma{}^\beta\,. \tag{9.34}$$

For even spacetime dimension $D = 2m$ the sum extends over the rank r_A independent elements of the Clifford algebra $0 \leq r_A \leq D$, and the coefficients $v_A = (-)^{r_A}(D - 2r_A)$. When applied to the integrand of (9.25), this reordering identity reads:

$$\begin{aligned}(\bar\epsilon\gamma^a\psi_{[\sigma})(\bar\psi_{\mu]}\gamma_a\gamma_*D_\nu\psi_\rho) &= -\tfrac14\sum_A(-)^{r_A}(4-2r_A)(\bar\epsilon\Gamma_A\gamma_*D_\nu\psi_\rho)(\bar\psi_{[\mu}\Gamma^A\psi_{\sigma]})\\ &= \tfrac12(\bar\epsilon\gamma_a\gamma_*D_\nu\psi_\rho)(\bar\psi_\mu\gamma^a\psi_\sigma)\\ &= (\bar\epsilon\gamma_*\gamma_aD_\nu\psi_\rho)T_{\mu\sigma}{}^a\,.\end{aligned} \tag{9.35}$$

In the left-hand side, we indicated the antisymmetrization in $[\mu\sigma]$, due to the multiplication with $\varepsilon^{\mu\nu\rho\sigma}$ in (9.25). This implies a remarkable simplification in the second line. Indeed, from the symmetries explained in Sec. 3.1.8, only $r_A = 1$ and 2 should be considered. The vanishing of v_2 for $D = 4$ implies then that only the rank 1 $\Gamma^A \to \gamma^a$ survives. We have rewritten the last line using the $D = 4$ torsion tensor (9.15). We insert this in (9.25), reorder the $(\bar\epsilon \ldots D\psi)$ bilinear, and exchange the indices $\mu\rho$ to obtain

$$\delta S_{3/2,e} = \frac{-\mathrm{i}}{4\kappa^2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}T_{\rho\sigma}{}^a\bar\psi_\mu\overleftarrow{D}_\nu\gamma_*\gamma_a\epsilon\,. \tag{9.36}$$

We have now reached the final step of the proof in which we combine these with the uncancelled torsion terms in (9.33) to obtain the sum:

$$\delta S = \frac{-\mathrm{i}}{4\kappa^2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\partial_\nu[T_{\rho\sigma}{}^b\bar\psi_\mu\gamma_*\gamma_b\epsilon] \equiv 0\,. \tag{9.37}$$

The integrand is a total derivative because the local Lorentz derivative works distributively. Indeed, the spin connection cancels among the three terms from (9.33) and (9.36). This proves that $\mathcal{N} = 1$, $D = 4$ supergravity is locally supersymmetric and thus consistent as a classical theory of the graviton and gravitino! The basic relations of differential geometry with torsion and the Clifford algebra and spinor anti-commutativity combine in the proof in a very striking way!

9.5 The algebra of local supersymmetry

The commutator of two local SUSY transformations should realize an algebra which is compatible with that of global SUSY. On any component field Φ of a chiral multiplet we found in Ch. 8, see (8.29)

$$[\delta_1, \delta_2]\,\Phi = -\tfrac12\bar\epsilon_1\gamma^\mu\epsilon_2\partial_\mu\Phi\,. \tag{9.38}$$

It is natural to expect that the local extension of the global result (9.38) should be a general coordinate transformation with parameter $\xi^\mu(x) = -\frac{1}{2}\bar{\epsilon}_1(x)\gamma^\mu\epsilon_2(x)$. However, the general formalism for symmetries requires only that the commutator closes on a sum of the gauge symmetries of the theory, in this case a sum of general coordinate, local Lorentz, and local SUSY transformations. Furthermore, the gauge parameters that appear in the commutator can be field dependent, a phenomenon already encountered in SUSY gauge theories in Sec. 8.3.1. Thus it is not immediately clear what we will find in supergravity, and it is well advised to do the computation.

The computation is quite simple for the frame field. Here it is:

$$\begin{aligned}
[\delta_1, \delta_2]\, e_\mu^a &= \tfrac{1}{2}\delta_1\bar{\epsilon}_2\gamma^a\psi_\mu - (1 \leftrightarrow 2) = \tfrac{1}{2}\bar{\epsilon}_2\gamma^a\delta_1\psi_\mu - (1 \leftrightarrow 2) \\
&= \tfrac{1}{2}\bar{\epsilon}_2\gamma^a D_\mu\epsilon_1 - (1 \leftrightarrow 2) \\
&= \tfrac{1}{2}(\bar{\epsilon}_2\gamma^a D_\mu\epsilon_1 + D_\mu\bar{\epsilon}_2\gamma^a\epsilon_1) \\
&= D_\mu\xi^a\,, \qquad \xi^a = \tfrac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1 = -\tfrac{1}{2}\bar{\epsilon}_1\gamma^a\epsilon_2\,.
\end{aligned} \tag{9.39}$$

Under the expected general coordinate transformation $x'^\mu = x^\mu - \xi^\mu(x)$, the frame field transforms as a covariant vector, viz.

$$e_\mu^a(x') = \frac{\partial x^\nu}{\partial x'^\mu} e_\nu^a\,. \tag{9.40}$$

The infinitesimal form is

$$\delta_\xi e_\mu^a = \xi^\rho\partial_\rho e_\mu^a + \partial_\mu\xi^\rho e_\rho^a\,. \tag{9.41}$$

Let us 'covariantize' the derivatives by adding and subtracting the ω and Γ connection terms. This must be done with some care since there is torsion. The symbol ∇_ρ includes all appropriate connections.

$$\begin{aligned}
\delta_\xi e_\mu^a &= \xi^\rho\nabla_\rho e_\mu^a - \xi^\rho\omega_\rho{}^a{}_b e_\mu^b + \xi^\rho\Gamma^\sigma_{\rho\mu}e_\sigma^a + \nabla_\mu\xi^\rho\, e_\rho^a - \Gamma^\rho_{\mu\sigma}\xi^\sigma e_\rho^a \\
&= \nabla_\mu\xi^\rho e_\rho^a - \xi^\rho\omega_\rho{}^a{}_b e_\mu^b + \xi^\rho T_{\rho\mu}{}^a\,.
\end{aligned} \tag{9.42}$$

Since $\nabla_\rho e_\mu^a = 0$ (see (6.89) which is valid with torsion), we have dropped it in moving to the second line. Since $e_\rho^a\nabla_\mu\xi^\rho = D_\mu\xi^a$, the first term in (9.42) matches the supergravity result (9.39). The last two terms seem mysterious, but, in the light of the remarks above, let's try to interpret them as field dependent symmetry transformations. The second term in (9.42) is simply a local Lorentz transformation of the frame field with field dependent parameter $\hat{\lambda}_{ab} = \xi^\rho\omega_{\rho ab}$. To interpret the third term, we use the explicit form (9.15) of the torsion tensor to write

$$\xi^\rho T_{\rho\mu}{}^a = \tfrac{1}{2}(\xi^\rho\bar{\psi}_\rho)\gamma^a\psi_\mu\,. \tag{9.43}$$

This is just a local SUSY transformation of e_μ^a with field dependent $\hat{\epsilon} = \xi^\rho\psi_\rho$. Thus we have derived

$$[\delta_1, \delta_2]\, e_\mu^a \,=\, \left(\delta_\xi - \delta_{\hat{\lambda}} - \delta_{\hat{\epsilon}}\right) e_\mu^a\,. \tag{9.44}$$

The combination of symmetries that is on the right-hand side is reminiscent of the transformations that we discussed in Ex. 4.4. This combination is a '*covariant general coordinate transformation*', which we will discuss more systematically in Sec. 11.3.2.

It is more difficult to calculate the SUSY commutator on the gravitino field largely because Fierz rearrangement is required. The result is

$$[\delta_1, \delta_2]\,\psi_\mu = \xi^\rho(D_\rho\psi_\mu - D_\mu\psi_\rho) + \ldots\,. \tag{9.45}$$

We will discuss the omitted terms ... momentarily. First we note that, as done above, one can manipulate the formula for a general coordinate transformation of a vector-spinor field to bring (9.45) to the same form as (9.44), namely

$$[\delta_1, \delta_2]\,\psi_\mu \;=\; \left(\delta_\xi - \delta_{\hat\lambda} - \delta_{\hat\epsilon}\right)\psi_\mu + \ldots\,. \tag{9.46}$$

Ex. 9.3 *Derive (9.45) including the terms ... and verify (9.46). If you need details, see [58].*

The omitted terms ... vanish when ψ_μ satisfies its equation of motion. Such terms do not affect the commutator algebra on physical states, so they can be dropped for most purposes. Similar equation of motion terms also appear in the commutator algebra of global SUSY, after elimination of auxiliary fields, see Ex. 8.11 and the discussion which follows it. Their presence in (9.46) means that the local SUSY algebra also 'closes only on-shell'. As in global SUSY, the physical fields e^a_μ, ψ_μ are only an 'on-shell multiplet' which can be completed to an 'off-shell multiplet' by adding auxiliary fields (for $\mathcal{N}=1$, $D=4$ only!). The auxiliary fields are important in the formulation of systematic methods for the coupling of chiral and gauge multiplets to supergravity.

Finally we note that the field-dependent gauge transformations of this section can be reinterpreted as modifications of the algebra of local supersymmetry leading to the 'soft algebras' discussed in Ch. 11.

10

$D = 11$ supergravity

The basic $\mathcal{N} = 1$, $D = 4$ supergravity theory discussed in Ch. 9 has been generalized in several ways. In four dimensions, one can couple the gravity multiplet $(e_\mu^a,\, \psi_\mu)$ to gauge $(A_\mu^A,\, \lambda^A)$ and chiral $(z^\alpha,\, P_L\chi^\alpha)$ multiplets and promote the global SUSY theories of Ch. 8 to local SUSY.

Another important generalization of supergravity is to spacetime dimension $5 \leq D \leq 11$. The two different classical $D = 10$ supergravities, called Type IIA and Type IIB respectively, are the low energy limits of the superstring theories of the same name. Type IIB and gauged $D = 5$ supergravities have important applications to the AdS/CFT correspondence. Most higher dimensional supergravities are quite complicated, but the maximum dimension $D = 11$ theory has a relatively simple structure. It is an important theory for at least two reasons. First, many interesting lower dimensional cases, such as the maximal $\mathcal{N} = 8$, $D = 4$ theory, can be obtained through dimensional reduction. Second, $D = 11$ supergravity, together with its $M2$- and $M5$-brane solutions is the basis of the extended object theory called M-theory, which is widely considered to be the master theory that contains the various string theories. Thus we devote this chapter to $D = 11$ supergravity.

10.1 $D \leq 11$ from dimensional reduction

Before we embark on a technical discussion of the theory, let's review the argument why $D = 11$ is the largest spacetime dimension allowed for supergravity. The argument is based on a generalization of the dimensional reduction technique we discussed in Sec. 5.3. There we studied Kaluza-Klein compactifications of $D+1$-dimensional fields on $\text{Minkowski}_D \times S^1$. That discussion did not include symmetric tensor fields, but it is clear that the Fourier modes of a symmetric tensor h_{MN} in $D+1$ dimensions give rise to symmetric tensor fields $h_{\mu\nu k}$, vector fields $h_{\mu D k}$, and scalar fields h_{DDk} in D dimensions. Here k is the Fourier mode number.

More generally we can consider the compactification of a D' dimensional theory on a product spacetime $M_{D'} = M_D \times X_d$, with $D' = D + d$ and X_d a compact

d-dimensional internal space. Fields of the lower dimensional theory arise from harmonic expansion on X_d of the various higher dimensional fields. In a Kaluza-Klein compactification one keeps the entire infinite set of harmonic modes, which describe both massless and massive fields in D dimensions. In the related process called dimensional reduction one keeps only a finite set of modes which must be a *consistent truncation* of the full set. Usually the modes which are kept are the massless or light modes, and the omitted modes are heavy modes. A consistent truncation is one in which the field equations of the omitted heavy modes are not sourced by the light modes which are kept. Thus setting the heavy modes to zero is consistent with the field equations.[1]

To show that $D = 11$ is the maximal dimension, we consider the toroidal compactification of a D' dimensional theory to $M_4 \times T^{D'-4}$. Fourier modes on the torus are generalized Fourier modes labeled by integers $k_1, \dots k_{D'-4}$. Only the lowest modes with $k_i = 0$ are massless in four dimensions, and these are retained in the 4-dimensional theory. We don't yet know the full field content of the putative D'-dimensional supergravity, but we can anticipate that it must contain the metric tensor and at least one gravitino of the simplest spinor type (e.g. Majorana) permitted in dimension D'.

We need to see which 4-dimensional fields arise from the gravitino in the truncation.[2]. Suppose that $D' = 11$, in which the simplest spinor is a 32 component Majorana spinor. The 11 matrices Γ^M which generate the Dirac-Clifford algebra in 11 dimensions can be represented as tensor products of 4×4 γ^μ and 8×8 $\hat{\gamma}^i$ as

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \times \mathbb{1}\,, & \mu &= 0,1,2,3\,, \\ \Gamma^i &= \gamma_* \times \hat{\gamma}^i\,, & i &= 4,5,6,7,8,9,10\,. \end{aligned} \qquad (10.1)$$

Here $\mathbb{1}$ is the 8×8 unit matrix, and $\gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$, while the $\hat{\gamma}^i$ are the generating elements of the Clifford algebra in 7 dimensional Euclidean space. In this basis the 11-dimensional gravitino field is labelled as $\Psi_{M\,\alpha a}$ in which $\alpha = 1,2,3,4$ is a 4-dimensional spinor index and $a = 1, \dots 8$ is the index on which the $\hat{\gamma}^i$ (and their products) act.

From the 4-dimensional standpoint $\Psi_{\mu\,\alpha a}$ transforms under Lorentz transformations as a set of 8 gravitinos, while $\Psi_{i\,\alpha a}$ transforms as a set of $7 \times 8 = 56$ spin 1/2 fields. However, note that 8 Majorana gravitinos plus 56 Majorana is the complete fermion content of the $\kappa = -2$ (maximal spin 2) particle representation of the $\mathcal{N} = 8$ SUSY algebra, see Sec. 8.4, and table 12.1 in Sec. 12.3.2. If we started with a single gravitino in $D' \geq 12$ dimensions or more gravitinos in 11 dimensions, then the dimensionally reduced theory would contain more than 8 gravitinos which can only be accommodated in a representation that involves spins $\geq 5/2$ for which no consistent interactions are known.

[1] The truncation to massless modes on a torus is consistent [59, 60].

[2] In this section and the next, we use upper case $M, N \dots$ to denote a vector index in D' dimensions, $i, j, \dots$ for a direction on the torus, and reserve $\mu\nu, \dots$ for 4-dimensional vector indices.

Ex. 10.1 *Show that the product representation of the 11 Γ^μ matrices defined above does satisfy $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$. Compute commutators of the set of 28 matrices consisting of the 21 independent $\gamma^{ij} = \gamma^{[i}\gamma^{j]}$, plus the 7 matrices $\mathrm{i}\hat{\gamma}^k$ and show that they are a basis for an 8 dimensional representation of the Lie algebra of the* SO(8) *group that acts on the 8 gravitinos of the reduced theory.* SO(8) *has three inequivalent 8-dimensional representations.*

10.2 The field content of $D = 11$ supergravity

The previous argument has taught us quite a bit about the theory. We know that it contains the gravitational field whose quantum excitations (see Sec. 7.3) transform in the traceless symmetric tensor representation of $\mathrm{SO}(D-2)$ of dimension $D(D-3)/2$. In 11 dimensions this contains 44 bosonic states. There is also one Majorana spinor gravitino, whose excitations (see Sec. 5.1) transform in a vector-spinor representation of $\mathrm{SO}(D-2)$ of dimension $(D-3)\,2^{[(D-2)/2]}$. For $D = 11$ this contains 128 real fermion states.

The theory must contain an equal number of boson and fermion states, so we are missing 84 bosons. Where are they? Recall the discussion in Sec. 6.7 that bosons in spacetime dimension D can be described by p-form gauge fields or equivalently anti-symmetric tensor $A_{M_1,M_2,\ldots,M_p}$ potentials of rank $p < D$. The excitations of such a field transform in the rank p anti-symmetric tensor representation of $SO(D-2)$, which contains $\binom{D-2}{p}$ states. For rank 3 in 11 dimensions, this contains exactly 84 quantum degrees of freedom. Thus, Cremmer, Julia, and Scherk, [61] who first formulated supergravity in 11 dimensions, made the elegant hypothesis that the theory should contain the metric tensor g_{MN}, the 3-form potential A_{MNP}, and the Majorana vector-spinor Ψ_M.

We already know that upon dimensional reduction on T^7, the vector-spinor produces the 8 gravitinos and 56 spin 1/2 fermions of $\mathcal{N} = 8$, $D = 4$ supergravity. The metric tensor components $g_{\mu\nu}$ give the 4-dimensional spacetime metric, while $g_{\mu i}$ produces 7 spin one particles, and g_{ij} contains 28 scalars. The $\mathcal{N} = 8$ theory contains 28 vectors, and the missing 21 are supplied by the 3-form components $A_{\mu ij}$. The form components A_{ijk} contain 35 scalars, and the components $A_{\mu\nu i}$ give an additional 7 scalars. In this way the field assignment of [61] accounts for the 35 scalars and 35 pseudoscalars of the dimensionally reduced theory.[3]

10.3 Construction of the action and transformation rules

We now know the field content of the theory, and we need to be more precise and determine the Lagrangian and transformation rules. To find them we start with an initial ansatz for the action compatible with the expected symmetries and use some of the ideas of Ch. 8 and Ch. 9 to finalize the construction. Several additional

[3] The field components $A_{\mu\nu\rho}$ contain no degrees of freedom in $D = 4$.

calculations are needed to demonstrate local SUSY completely. We present some of these and refer readers to the literature [61, 62] for the rest.

To start we note that a theory containing the 3-form potential $A_{\mu\nu\rho}$ must be invariant under a gauge transformation involving a gauge parameter $\theta_{\nu\rho}$ which is a 2-form. The theory thus involves a gauge invariant 4-form field strength $F_{\mu\nu\rho\sigma}$. The basic equations are

$$\begin{aligned} \delta A_{\mu\nu\rho} &= 3\partial_{[\mu}\theta_{\nu\rho]} \equiv \partial_\mu\theta_{\nu\rho} + \partial_\nu\theta_{\rho\mu} + \partial_\rho\theta_{\mu\nu}\,, \\ F_{\mu\nu\rho\sigma} &= 4\partial_{[\mu}A_{\nu\rho\sigma]} \equiv \partial_\mu A_{\nu\rho\sigma} - \partial_\nu A_{\rho\sigma\mu} + \partial_\rho A_{\sigma\mu\nu} - \partial_\sigma A_{\mu\nu\rho}\,, \\ \partial_{[\tau}F_{\mu\nu\rho\sigma]} &\equiv 0\,. \end{aligned} \tag{10.2}$$

The last equation contains 5 terms when written in full. It is the Bianchi identity which follows from the fact that $F = \mathrm{d}A$ when expressed as a differential form.

Ex. 10.2 *Readers should prove this identity!*

Another important ingredient we need is the exchange property of spinor bilinears $\bar\chi\Gamma^A\lambda$ where Γ^A is a general element of rank r of the Clifford algebra. As we saw in (3.49), we have the same properties as in 4 dimensions:

$$\bar\chi\gamma^{\mu_1\mu_2\ldots\mu_r}\lambda = t_r\bar\lambda\gamma^{\mu_1\mu_2\ldots\mu_r}\chi\,, \qquad t_0 = t_3 = 1\,, \quad t_1 = t_2 = -1\,, \quad t_{r+4} = t_r\,. \tag{10.3}$$

We postulate that the action contains the universal graviton and gravitino terms of Sec. 9.1 plus the covariant kinetic action for the 3-form plus additional terms (denoted by ...) which we must find. Thus we write

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, e\, \left[e^{a\mu}e^{b\nu}R_{\mu\nu ab} - \bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} + \ldots\right]. \tag{10.4}$$

Initially we use the second order formalism with torsion-free spin connection $\omega_{\mu ab}(e)$. We also need transformation rules and make the ansatz

$$\begin{aligned} \delta e_\mu^a &= \tfrac12\bar\epsilon\gamma^a\psi_\mu\,, \\ \delta\psi_\mu &= D_\mu\epsilon + \left(a\,\gamma^{\alpha\beta\gamma\delta}{}_\mu + b\,\gamma^{\beta\gamma\delta}\delta_\mu^\alpha\right)F_{\alpha\beta\gamma\delta}\epsilon\,, \\ \delta A_{\mu\nu\rho} &= -c\,\bar\epsilon\gamma_{[\mu\nu}\psi_{\rho]} = -\tfrac13 c\,\bar\epsilon(\gamma_{\mu\nu}\psi_\rho + \gamma_{\nu\rho}\psi_\mu + \gamma_{\rho\mu}\psi_\nu)\,. \end{aligned} \tag{10.5}$$

For $\delta A_{\mu\nu\rho}$ and the new terms of $\delta\psi_\mu$ we have postulated general expressions consistent with coordinate and gauge symmetries which contain the numerical constants a, b, c.

We will determine these constants and other useful information by temporarily treating the fields ψ_μ and $A_{\mu\nu\rho}$ as a *free* system with global SUSY in D-dimensional Minkowski space. The free action is (dropping here an irrelevant factor κ^2)

$$S_0 = \frac12\int \mathrm{d}^{11}x\, \left[-\bar\psi_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}\right]. \tag{10.6}$$

Of course there is no true supermultiplet containing only ψ_μ and $A_{\mu\nu\rho}$, but for a *free* theory, this need not be an obstacle. We now follow the same steps as in the discussion of super-Yang-Mills theory in Sec. 8.3.1. Using (3.52) we have

$$\delta\bar{\psi}_\mu = \bar{\epsilon}\left(-a\gamma^{\alpha\beta\gamma\delta}{}_\mu + b\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)F_{\alpha\beta\gamma\delta}\,, \tag{10.7}$$

we compute the variation

$$\begin{aligned}\delta S_0 &= \int \mathrm{d}^{11}x\,\bar{\epsilon}\left[\left(a\gamma^{\alpha\beta\gamma\delta}{}_\mu - b\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)F_{\alpha\beta\gamma\delta}\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - \tfrac{1}{6}c\gamma_{\nu\rho}\psi_\sigma\partial_\mu F^{\mu\nu\rho\sigma}\right]\\ &= \int \mathrm{d}^{11}x\bar{\epsilon}\left[\left(-a\gamma^{\alpha\beta\gamma\delta}{}_\mu + b\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)\partial_\nu F_{\alpha\beta\gamma\delta}\gamma^{\mu\nu\rho}\psi_\rho - \tfrac{1}{6}c\gamma_{\nu\rho}\psi_\sigma\partial_\mu F^{\mu\nu\rho\sigma}\right].\end{aligned} \tag{10.8}$$

We used partial integration to obtain the last line, assuming that ϵ is constant as in global SUSY.

We need matrix identities to reduce the products of γ matrices in (10.8) to sums over rank 6, rank 4, and rank 2 elements Γ^A of the Clifford algebra. In the spirit of the discussion at the end of section 3.1.4 (the first identity is (3.22)), we write

$$\begin{aligned}\gamma^{\alpha\beta\gamma\delta}{}_\mu\gamma^{\mu\nu\rho}F_{\alpha\beta\gamma\delta} &= (D-6)\gamma^{\alpha\beta\gamma\delta\nu\rho}F_{\alpha\beta\gamma\delta} + 8(D-5)\gamma^{\alpha\beta\gamma[\nu}F^{\rho]}{}_{\alpha\beta\gamma}\\ &\quad -12(D-4)\gamma^{\alpha\beta}F_{\alpha\beta}{}^{\nu\rho}\,,\\ \gamma^{\beta\gamma\delta}\gamma^{\mu\nu\rho}F_{\mu\beta\gamma\delta} &= -\gamma^{\nu\rho\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta} - 6\gamma^{\alpha\beta\gamma[\nu}F^{\rho]}{}_{\alpha\beta\gamma} + 6\gamma^{\alpha\beta}F_{\alpha\beta}{}^{\nu\rho}\,.\end{aligned} \tag{10.9}$$

Ex. 10.3 *Conscientious readers should verify these identities.*

When inserted in the first term of (10.8) both rank 6 terms vanish due to the Bianchi identity. The sum of the two rank 4 contributions must vanish while the rank 2 terms must cancel with the second term of (10.8). These conditions lead to the following two numerical relations among a, b, c:

$$\begin{aligned}8(D-5)a + 6b &= 0\,,\\ 12(D-4)a + 6b &= \tfrac{1}{6}c\,.\end{aligned} \tag{10.10}$$

It is only the case $D = 11$ which is relevant here, and the solution of (10.10) in this case is $a = c/216$, $b = -8a$. Thus for the free theory we have found the transformation rules

$$\begin{aligned}\delta\psi_\mu &= \partial_\mu\epsilon + \frac{c}{216}\left(\gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)F_{\alpha\beta\gamma\delta}\epsilon\,,\\ \delta A_{\mu\nu\rho} &= -c\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]}\,.\end{aligned} \tag{10.11}$$

To fix the remaining parameter c, we examine the commutator of two SUSY transformations and require that this agree with the local supergravity algebra discussed in Sec. 9.5. It is simplest to work with the gauge potential, so we write (for constant ϵ)

$$[\delta_1, \delta_2] A_{\mu\nu\rho} \;=\; -\frac{1}{216} c^2 \bar{\epsilon}_2 \gamma_{[\mu\nu} \left(\gamma^{\alpha\beta\gamma\delta}{}_{\rho]} - 8\gamma^{\beta\gamma\delta} \delta^{\alpha}_{\rho]} \right) \epsilon_1 F_{\alpha\beta\gamma\delta} \;-(1 \leftrightarrow 2)\,. \qquad (10.12)$$

It would be good practice to work out the detailed identities for the products of γ-matrices in (10.12), which involve contributions from Clifford elements of rank $1, 3, 5, 7$, but this task can be simplified by the following observations:
i. since the spinor parameters ϵ_1, ϵ_2 are anti-symmetrized, only the rank 1 and rank 5 terms can contribute.
ii. the first product has no rank 1 part.
iii. except for index changes, the second product is already given in (3.20).

The SUSY algebra must contain a spacetime translation involving the rank 1 bilinear $\bar{\epsilon}_1 \gamma^\sigma \epsilon_2$ from the product $\gamma_{\mu\nu}\gamma^{\beta\gamma\delta}$. Using (3.20) we obtain the rank 1 contribution to the commutator

$$[\delta_1, \delta_2] A_{\mu\nu\rho} \;=\; -\tfrac{4}{9} c^2 \bar{\epsilon}_1 \gamma^\sigma \epsilon_2 F_{\sigma\mu\nu\rho}\,. \qquad (10.13)$$

SUSY requires that the rank 5 contribution actually cancels between the two terms of (10.12), and it does as the following exercise shows.

Ex. 10.4 *Show that the rank 5 terms cancel in the expression* $\gamma_{[\mu\nu} \left(\gamma_{\rho]}{}^{\alpha\beta\gamma\delta} - 8\gamma^{\beta\gamma\delta}\delta^{\alpha}_{\rho]} \right) F_{\alpha\beta\gamma\delta}$.

The interpretation of the result (10.13) is straightforward if we refer to the detailed form of the field strength in (10.2). The first term $\partial_\sigma A_{\mu\nu\rho}$ is the spacetime translation we are looking for, while the remaining terms just add up to a gauge transformation of the 3-form potential with field-dependent gauge parameter proportional to $\theta_{\mu\nu} = -\bar{\epsilon}_1 \gamma^\sigma \epsilon_2 A_{\sigma\mu\nu}$. Such field-dependent gauge transformations were already found in SUSY gauge theories, see Ex. 8.13, and were found also in the local algebra in Sec. 9.5. We must normalize the coefficient of the translation term to agree with these results, which were (and should be) uniform for all fields. Thus we fix the parameter $c^2 = 9/8$, and we choose the positive root $c = 3/2\sqrt{2}$.

Recall that we have been studying the global supersymmetry of the free system of ψ_μ and $A_{\mu\nu\rho}$ in flat spacetime. There is another important piece of information from that study, namely the effective supercurrent of the system, obtained by allowing the spinor parameter ϵ in (10.8) to depend on x^μ. After partial integration we find that δS_0 contains a term proportional to $D_\nu \epsilon$ whose coefficient is the supercurrent

$$\mathcal{J}^\nu = \frac{\sqrt{2}}{96} \left(\gamma^{\alpha\beta\gamma\delta\nu\rho} F_{\alpha\beta\gamma\delta} + 12 \gamma^{\alpha\beta} F_{\alpha\beta}{}^{\nu\rho} \right) \psi_\rho\,. \qquad (10.14)$$

As we will see below this provides a new term in the $D = 11$ supergravity action.

Ex. 10.5 *Show that* $\partial_\nu \mathcal{J}^\nu \;=\; 0$ *if* $F_{\alpha\beta\gamma\delta}$ *and* ψ_ρ *satisfy their free equations of motion (and Bianchi identity).*

To extend results on the free ψ_μ, $A_{\mu\nu\rho}$ system to the interacting supergravity theory, we introduce a general frame field $e^a_\mu(x)$ and consider general $\epsilon(x)$. We then have the transformation rules

$$\begin{aligned} \delta e^a_\mu &= \frac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,, \\ \delta\psi_\mu &= D_\mu\epsilon + \frac{\sqrt{2}}{288}\left(\gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right)F_{\alpha\beta\gamma\delta}\epsilon\,, \\ \delta A_{\mu\nu\rho} &= -\frac{3\sqrt{2}}{4}\bar\epsilon\gamma_{[\mu\nu}\psi_{\rho]}\,, \end{aligned} \tag{10.15}$$

and the action

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, e\, \Big[& e^{a\mu}e^{b\nu}R_{\mu\nu ab} - \bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} \\ & - \frac{\sqrt{2}}{96}\bar\psi_\nu\left(\gamma^{\alpha\beta\gamma\delta\nu\rho}F_{\alpha\beta\gamma\delta} + 12\gamma^{\alpha\beta}F_{\alpha\beta}{}^{\nu\rho}\right)\psi_\rho + \dots\Big]\,. \end{aligned} \tag{10.16}$$

The previous discussion ensures that all terms in δS of the form $\bar\epsilon R_{\mu\nu ab}\psi_\rho$ and $\bar\epsilon F_{\alpha\beta\gamma\delta}\psi_\rho$ cancel. The curvature terms vanish by the universal manipulations of Sec. 9.1. For terms linear in $F_{\alpha\beta\gamma\delta}$, the calculations done above in the free limit are essentially the same in a general background geometry. The only new feature is the term $(D_\nu\bar\epsilon)\mathcal{J}^\nu$ which is canceled by the $\delta\bar\psi_\nu = D_\nu\bar\epsilon$ variation of the last term written in (10.16). We still leave $\dots$ in (10.16) because the action is not yet complete.

For the next step it is simpler to rewrite (10.16) as the integral of a Lagrangian, namely

$$S = \frac{1}{\kappa^2}\int \mathrm{d}^{11}x\, e\, L\,, \tag{10.17}$$

and to study variations of the Lagrangian δL. We consider terms in δL of order $\bar\epsilon F^2\psi$ which come from the frame field variation of the order F^2 term in L and the $\delta\psi \sim F\epsilon$ variation of the order $\bar\psi F\psi$ term. The two contributions are

$$\delta L_{FF} = \frac{1}{48}(4\bar\epsilon\gamma^\mu\psi^\nu - \frac{1}{2}g^{\mu\nu}\bar\epsilon\gamma\cdot\psi)F_\mu{}^{\rho\sigma\tau}F_{\nu\rho\sigma\tau}\,, \tag{10.18}$$

$$\begin{aligned} \delta L_{\bar\psi F\psi} = & \frac{1}{96\cdot 144}\,\bar\epsilon\left(\gamma^{\alpha'\beta'\gamma'\delta'}{}_\nu + 8\gamma^{\beta'\gamma'\delta'}\delta^{\alpha'}_\nu\right)F_{\alpha'\beta'\gamma'\delta'} \\ & \times\left(\gamma^{\alpha\beta\gamma\delta\nu\rho}F_{\alpha\beta\gamma\delta} + 12\gamma^{\alpha\beta}F_{\alpha\beta}{}^{\nu\rho}\right)\psi_\rho\,. \end{aligned} \tag{10.19}$$

The products of γ-matrices in (10.19) contain sums of rank 9,7,5,3, and 1 terms. A detailed treatment requires rather complicated identities, so we will be content here to summarize the results and refer readers who need more information to the literature [61, 62].

It turns out that the rank 1 terms cancel between (10.18) and (10.19), and the several rank 3, 5 and 7 terms cancel within (10.19). However, there are rank 9 terms in (10.19) which can be obtained from the products

$$\begin{aligned}\gamma^{\alpha'\beta'\gamma'\delta'}{}_{\nu}\gamma^{\alpha\beta\gamma\delta\nu\rho} &= (D-9)\gamma^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho}+\ldots = 2\,\gamma^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho}+\ldots\,,\\ \gamma^{\beta'\gamma'\delta'}\,\gamma^{\alpha\beta\gamma\delta\alpha'\rho} &= -\gamma^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho}\,,\end{aligned} \tag{10.20}$$

where ... indicate lower rank contributions which we omit. Thus the rank 9 term

$$\delta L_{FF}+\delta L_{\bar\psi F\psi} = -\frac{1}{16\cdot 144}\bar\epsilon\gamma^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho}\psi_\rho\,F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}\,, \tag{10.21}$$

survives in the sum of (10.18) and (10.19), and we must find a way to cancel it.

To cancel the high-rank gamma matrix, recall the discussion of Sec. 3.1.7 of the Dirac-Clifford algebra for spacetimes of odd dimension $D=2m+1$. For $D=11$, the generating matrices are the 32×32 matrices $\gamma^0,\gamma^1,\ldots\gamma^9,\gamma^{10}=\gamma_*$, with $\gamma_*=\gamma^0\gamma^1\ldots\gamma^9$ (i.e. we take the + sign in (3.39)). The rank 9 Clifford element is related to rank 2 by (3.40):

$$\gamma^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho} = -\frac{1}{2e}\,\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho\mu\nu}\,\gamma_{\nu\mu}\,. \tag{10.22}$$

This allows to rewrite (10.21) as

$$\begin{aligned}e\left(\delta L_{FF}+\delta L_{\bar\psi F\psi}\right) &= \frac{1}{32\cdot 144}\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho\mu\nu}\bar\epsilon\gamma_{\nu\mu}\psi_\rho\,F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}\,, \qquad (10.23)\\ &= \frac{4}{3\sqrt2\cdot 32\cdot 144}\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\rho\mu\nu}(\delta A_{\mu\nu\rho})\,F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}\,.\end{aligned}$$

This suggests that one can add a term in the Lagrangian of the following form to cancel this variation:

$$\begin{aligned}S_{\mathrm{C-S}} &= -\frac{\sqrt2}{(144\kappa)^2}\int \mathrm{d}^{11}x\,\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\mu\nu\rho}F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}A_{\mu\nu\rho}\,,\\ &= -\frac{\sqrt2}{6\kappa^2}\int F^{(4)}\wedge F^{(4)}\wedge A^{(3)}\,,\end{aligned} \tag{10.24}$$

where we used the form notation in the last line, with $F^{(4)}=\mathrm{d}A^{(3)}$, which simplifies the formula. Such a term is called a Chern-Simons term, and has some special properties. First of all, using integration by parts one sees that a variation of $A^{(3)}$ gives 3 similar terms as

$$\begin{aligned}\delta\int F^{(4)}\wedge F^{(4)}\wedge A^{(3)} &= \int\left[2\mathrm{d}\delta A^{(3)}\wedge F^{(4)}\wedge A^{(3)}+F^{(4)}\wedge F^{(4)}\wedge\delta A^{(3)}\right]\\ &= 3\int F^{(4)}\wedge F^{(4)}\wedge\delta A^{(3)}\,,\end{aligned} \tag{10.25}$$

where we used the Bianchi identity $\mathrm{d}F^{(4)} = 0$. This shows how such a variation of (10.24) cancels (10.23). Further, this term does not produce other variations, as there are no frame fields in the expression.

One might also wonder whether such a term is invariant under gauge transformations of the form (10.2) in view of the explicit presence of the gauge field. For that one notes that this transformation can be written as $\delta A^{(3)} = \mathrm{d}\theta^{(2)}$. Plugging this into (10.25), integrating by parts and using the Bianchi identity as above, the gauge invariance is guaranteed. These are typical properties of Chern-Simons actions.

We have now reached the point where the major terms in the Lagrangian and transformation rules have been determined. Although there is more work to be done to complete the theory and establish local supersymmetry, the further modifications are quite simple. We prefer to write the full action and transformation rules and then interpret the changes below.

The full action is

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\, e\, \Big[e^{a\mu}e^{b\nu}R_{\mu\nu ab}(\omega) - \bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu(\tfrac12(\omega+\hat\omega))\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} \\ &\qquad - \frac{\sqrt2}{192}\bar\psi_\nu\left(\gamma^{\alpha\beta\gamma\delta\nu\rho} + 12\gamma^{\alpha\beta}g^{\gamma\nu}g^{\delta\rho}\right)\psi_\rho(F_{\alpha\beta\gamma\delta}+\hat F_{\alpha\beta\gamma\delta}) \qquad (10.26) \\ &\qquad - \frac{2\sqrt2}{(144)^2}\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\mu\nu\rho}F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}A_{\mu\nu\rho}\Big]\,. \qquad (10.27) \end{aligned}$$

The ‘hatted’ connection and field strength that appear above are given by

$$\begin{aligned} \omega_{\mu ab} &= \omega_{\mu ab}(e) + K_{\mu ab}\,, \\ \hat\omega_{\mu ab} &= \omega_{\mu ab}(e) - \tfrac14(\bar\psi_\mu\gamma_b\psi_a - \bar\psi_a\gamma_\mu\psi_b + \bar\psi_b\gamma_a\psi_\mu)\,, \\ K_{\mu ab} &= -\tfrac14(\bar\psi_\mu\gamma_b\psi_a - \bar\psi_a\gamma_\mu\psi_b + \bar\psi_b\gamma_a\psi_\mu) + \tfrac18\bar\psi_\nu\gamma^{\nu\rho}{}_{\mu ab}\psi_\rho\,, \\ \hat F_{\mu\nu\rho\sigma} &= 4\,\partial_{[\mu}A_{\nu\rho\sigma]} + \tfrac32\sqrt2\,\bar\psi_{[\mu}\gamma_{\nu\rho}\psi_{\sigma]}\,. \end{aligned} \qquad (10.28)$$

This action is invariant under the transformation rules

$$\begin{aligned} \delta e_\mu^a &= \frac12\bar\epsilon\gamma^a\psi_\mu\,, \\ \delta\psi_\mu &= D_\mu(\hat\omega)\epsilon + \frac{\sqrt2}{288}\left(\gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\gamma^{\beta\gamma\delta}\delta_\mu^\alpha\right)\hat F_{\alpha\beta\gamma\delta}\epsilon\,, \\ \delta A_{\mu\nu\rho} &= -\frac{3\sqrt2}{4}\bar\epsilon\gamma_{[\mu\nu}\psi_{\rho]}\,. \end{aligned} \qquad (10.29)$$

The connection ω incorporates the torsion tensor (9.15).

The ‘hatted’ connection and field strength in (10.28) are ’supercovariant.’ In this context a supercovariant quantity is one whose local SUSY transformation does not contain the derivative $\partial_\mu\epsilon$ of the SUSY parameter $\epsilon(x)$. The principle which guided the way in which these quantities were introduced in the action is that the equations of motion must transform supercovariantly. Like any symmetry, local supersymmetry transforms the equations of motion of the theory among themselves.

10.4 The algebra of $D = 11$ supergravity

It is again instructive to evaluate the commutator of two supersymmetry transformations. The result, given by

$$[\delta_Q(\epsilon_1),\, \delta_Q(\epsilon_2)] \;=\; \delta_{\rm gct}(\xi^\mu) + \delta_L(\lambda^{ab}) + \delta_Q(\epsilon_3) + \delta_A(\theta_{\mu\nu})\,, \tag{10.30}$$

is the sum of a general coordinate transformation, plus field-dependent local Lorentz, supersymmetry and 3-form gauge transformations (as given in (10.2)). The parameters of these transformations are

$$\begin{aligned}
\xi^\mu &= \tfrac{1}{2}\epsilon_2\gamma^\mu\epsilon_1\,,\\
\lambda^{ab} &= -\xi^\mu\hat\omega_\mu{}^{ab} + \frac{1}{288}\sqrt{2}\bar\epsilon_1\left(\gamma^{ab\mu\nu\rho\sigma}\hat F_{\mu\nu\rho\sigma} + 24\gamma_{\mu\nu}\hat F^{ab\mu\nu}\right)\epsilon_2\,,\\
\epsilon_3 &= -\xi^\mu\psi_\mu\,,\\
\theta_{\mu\nu} &= -\xi^\rho A_{\rho\mu\nu} + \tfrac{1}{4}\sqrt{2}\bar\epsilon_1\gamma_{\mu\nu}\epsilon_2\,.
\end{aligned} \tag{10.31}$$

As was the case in $D = 4$ supergravity, the algebra is realized on the gravitino only if its field equations are satisfied. A similar feature appeared in the global SUSY theories studied in in Ch. 8, but we saw that auxiliary fields could be added to the theory to achieve off-shell closure of the SUSY algebra (which means closure for field configurations that need not satisfy the equations of motion). Much effort has been devoted to a search for auxiliary fields for $D = 11$ supergravity, but no solution has ever been found.

The terms in (10.31), which contain the spinor bilinears $\bar\epsilon_1\Gamma^{(2)}\epsilon_2$ and $\bar\epsilon_1\Gamma^{(6)}\epsilon_2$ have special significance. (The $\Gamma^{(6)}$ and $\Gamma^{(5)}$ bilinears are duals in $D = 11$ dimensions and thus equivalent.) These terms are nonvanishing in the classical BPS $M2$ and $M5$ brane solutions of the theory. The BPS property means that the brane solutions preserve a global supersymmetry algebra in which the non-vanishing $\Gamma^{(2)}$ and $\Gamma^{(5)}$ terms are central charges.

11
General gauge theory

In the previous chapters we studied the construction of the simplest supergravity theories, namely $D = 4$ and $D = 11$, $\mathcal{N} = 1$ supergravity, and we proved invariance under local supersymmetry. But $\mathcal{N} = 1$ supergravity in 4 dimensions contains much more. Chiral multiplets and gauge multiplets of global supersymmetry (see chapter 8) can be coupled to supergravity yielding the rich structure of matter-coupled $\mathcal{N} = 1$ supergravities with many applications.

To investigate matter-coupled theories, we will use more advanced methods, which will be developed in the chapters ahead. To prepare for this, we will sharpen our knives in this chapter and discuss a general formulation of gauge theory of the type needed for supergravity. This consists largely of the refinement and extension of the notations and concepts used for symmetries earlier in this book. We will formalize several manipulations and concepts that we encountered in the previous sections. This will allow us to perform calculations with covariant derivatives more effectively. However, we will also see how the supergravity transformation rules postulated in (9.5),(9.6) are actually determined by the structure constants of the Poincaré supersymmetry algebra. In Sec. 11.3, we will explain how the general rules of gauge theory can be applied in gravity theories. We will see that some formulas must be modified.

11.1 Symmetries

In section 1.2 we defined symmetries as maps of the configuration space such that solutions of the equations of motion are transformed into new solutions. As we did there, we restrict the discussion to continuous symmetries that leave the action invariant. We consider only infinitesimal transformations, and hence are concerned with the Lie algebra rather than the Lie group. However, we need also extensions of these mathematical concepts, e.g. allowing structure constants to depend on fields and thus becoming rather 'structure functions'. We use the same formalism to describe both spacetime and internal symmetries and also supersymmetries.

11.1.1 Global symmetries

The algebra. An infinitesimal symmetry transformation is determined by a parameter, which we denote in general by ϵ^A and an operation $\delta(\epsilon)$, which depends linearly on the parameter, and acts on the fields of the dynamical system under study. For global symmetries, often called rigid symmetries, the parameters do not depend on the spacetime point where the symmetry operation is applied. Since the symmetry operation is linear in ϵ we can write it in general as

$$\delta(\epsilon) = \epsilon^A T_A\,, \tag{11.1}$$

in which T_A is an operator on the space of fields.[1] It describes the symmetry transformation with the parameter stripped. We call T_A the field space generator of the transformation. The notation of (11.1) and the formalism we develop below apply to all the types of symmetries encountered in this book. Internal, spacetime, and supersymmetry can be viewed as special cases. However, to be concrete, we begin by rephrasing the discussion of (linearly realized) internal symmetry of Sec. 1.2.2 in the present notation. (See also Sec. 4.3.1.) Then we will discuss how spacetime and supersymmetries fit the pattern.

Suppose that the matrices $(t_A)^i{}_j$ are the matrix generators of a representation of a Lie algebra with commutator

$$[t_A, t_B] = f_{AB}{}^C t_C\,, \tag{11.2}$$

and suppose that the system contains a set of fields ϕ^i that transforms in this representation. Then the transformation rule (1.19) can be expressed as the action of the generator T_A as follows:

$$T_A\phi^i = -(t_A)^i{}_j\phi^j\,. \tag{11.3}$$

The Lie algebra is determined by the commutator of two such transformations, so it is important to define the product of two symmetry operations carefully. By the product $\delta(\epsilon_1)\delta(\epsilon_2)\phi^i$, we mean that we first make a transformation with parameter ϵ_2 followed by another with parameter ϵ_1. In the notation of (11.1), the product operation reads

$$\delta(\epsilon_1)\delta(\epsilon_2)\phi^i = \epsilon_1^A T_A\left(\epsilon_2^B T_B\phi^i\right)\,. \tag{11.4}$$

As discussed in section 1.2.2, a symmetry transformation acts on the fields, which are the dynamical variables of the system, and not on the matrices which are the result of a prior transformation. Therefore, using (11.3), the explicit form of the product operation becomes

$$\begin{aligned}
\delta(\epsilon_1)\delta(\epsilon_2)\phi^i &= \epsilon_1^A T_A\epsilon_2^B\left[-(t_B)^i{}_j\phi^j\right]\\
&= \epsilon_1^A\epsilon_2^B(-t_B)^i{}_j T_A\phi^j\\
&= \epsilon_1^A\epsilon_2^B(-t_B)^i{}_j(-t_A)^j{}_k\phi^k\,.
\end{aligned} \tag{11.5}$$

[1] When the theory has a Hamiltonian description and Poisson brackets, $T_A\phi$ is the operator $\Delta_A\phi$ as defined by Poisson brackets in (1.81).

It is important to realize that in the second line, the transformation operator acts on the field ϕ^j, and not on the 'numbers' $(t_B)^i{}_j$. This is similar to the way in which we calculate commutators as e.g. in (9.39) where δ_1 is similar to the T_A here and acts on ψ_μ and goes after γ^a. The commutator is then

$$\begin{aligned}
[\delta(\epsilon_1), \delta(\epsilon_2)]\, \phi^i &= \epsilon_2^B \epsilon_1^A\, [T_A, T_B]\, \phi^i \\
&= \epsilon_2^B \epsilon_1^A\, ([t_B, t_A]\, \phi)^i = -\epsilon_2^B \epsilon_1^A f_{AB}{}^C (t_C \phi)^i \\
&= \epsilon_2^B \epsilon_1^A f_{AB}{}^C T_C \phi^i\, .
\end{aligned} \tag{11.6}$$

Therefore the commutator of the generators T_A, namely

$$[T_A, T_B] = f_{AB}{}^C T_C\, , \tag{11.7}$$

conforms to the matrix commutator (11.2). The minus sign in the transformation rule (11.3) is necessary to obtain the same result in (11.7) as in (11.2), and it is a general feature of symmetry transformations defined as left multiplication by a matrix.

Although every case of internal symmetry in which the symmetry transformations act linearly on the fields (linearly realized internal symmetry) can be expressed as matrix transformations, it is sometimes convenient to use the more general notation with T_A. For example the transformation rules for fields Ψ^α in a complex representation of a compact symmetry group, their conjugates $\bar\Psi_\alpha$, and fields ϕ^B in the adjoint representation were defined in (4.78). In each case we obtain the generator T_A by stripping the parameter ϵ^A and write

$$\begin{aligned}
T_A \Psi^\alpha &= -(t_A)^\alpha{}_\beta \Psi^\beta\, , \\
T_A \bar\Psi_\alpha &= \bar\Psi_\beta (t_A)^\beta{}_\alpha\, , \\
T_A \phi^B &= -f_{AC}{}^B \phi^C\, .
\end{aligned} \tag{11.8}$$

Here the matrix generator t_A acts by left multiplication on Ψ^α, but by right multiplication on $\bar\Psi_\alpha$.[2] On adjoint fields we again have left action by the matrix generators $(t_A)^B{}_C = f_{AC}{}^B$.

Ex. 11.1 *Show that (11.7) holds for repeated symmetry transformations of $\bar\Psi_\alpha$ and ϕ^B. Recall that the second transformation always acts only on the fields.*

Our conventions are easily extended to include spacetime symmetries. For Lorentz transformations we extract the field space generator $M_{[\mu\nu]}$ from (1.40),(1.41) for scalar fields and from (2.21),(2.23) for spinor fields. Stripping symmetry parameters, we obtain

$$\begin{aligned}
M_{[\mu\nu]} \phi(x) &= -L_{[\mu\nu]} \phi(x)\, , \\
M_{[\mu\nu]} \Psi(x) &= -(L_{[\mu\nu]} + \tfrac{1}{2}\gamma_{\mu\nu}) \Psi(x)\, .
\end{aligned} \tag{11.9}$$

[2] Because the matrices t_A are anti-Hermitian for a compact group, this is equivalent to left action by $-(t_A)^\dagger$. See Sec. 4.3.1.

The differential operator $L_{[\mu\nu]}$ is defined in (1.38). Its commutators realize the Lie algebra (1.34) of the Lorentz group. The same is true of the matrix generators $\Sigma_{[\mu\nu]} = \frac{1}{2}\gamma_{\mu\nu}$ for the spinor representation. For global translations we extract from (1.54)

$$P_\mu = \partial_\mu = \frac{\partial}{\partial x^\mu}, \tag{11.10}$$

which acts by differentiation on fields belonging to any representation of the Lorentz group. The commutators of the field space generators $M_{[\mu\nu]}$ and P_μ conform to the Lie algebra (1.57)-(1.59) of the Poincaré group (provided they are calculated with the convention that the second transformation acts only on the fields resulting from the first).

Ex. 11.2 *Readers should verify this.*

Let's return to the equations (11.5) and (11.6). The order in which the symmetry parameters are written is irrelevant for internal symmetries since the parameters commute. However, the order written in (11.6) will also be valid for fermionic symmetries for which the parameters anti-commute. Indeed we now proceed to formulate global supersymmetry in the present framework.

Here we need the notation for spinor indices that was explained in Sec. 3.2.2. Remember that ordinary spinors carry a lower spinor index, while Dirac conjugate (barred) spinors carry an upper spinor index. Gamma matrix indices are raised and lowered with the charge conjugation matrix. Symmetries of gamma matrices are determined by (3.61), such that in four dimensions $(\gamma^\mu)_{\alpha\beta}$ is symmetric. Flipping indices up-down gives a sign specified by (3.62). In four dimensions this is a minus sign.

For supersymmetry the generators T_A are replaced by the four-component Majorana spinor Q_α and the parameters ϵ^A by the anti-commuting conjugate (Majorana) spinor $\bar\epsilon^\alpha$. A supersymmetry transformation is an operation on fields denoted by

$$\delta(\epsilon) = \bar\epsilon^\alpha Q_\alpha. \tag{11.11}$$

As an explicit example of supersymmetry we rewrite the transformation rule (8.16) of the scalar field $Z(x)$ of a chiral multiplet:

$$Q_\alpha Z = \frac{1}{\sqrt{2}}(P_L\chi)_\alpha. \tag{11.12}$$

In parallel with (11.4), we write the product of two SUSY transformations as

$$\delta(\epsilon_1)\delta(\epsilon_2) = (\bar\epsilon_1)^\alpha Q_\alpha(\bar\epsilon_2)^\beta Q_\beta = (\bar\epsilon_2)^\beta(\bar\epsilon_1)^\alpha Q_\alpha Q_\beta. \tag{11.13}$$

Note that there is no sign change when $(\bar\epsilon_2)_\beta$ is moved through the bosonic quantity $(\bar\epsilon_1)_\alpha Q^\alpha$. It is then easy to check that the commutator of two transformations is

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = (\bar\epsilon_2)^\beta(\bar\epsilon_1)^\alpha\left(Q_\alpha Q_\beta + Q_\beta Q_\alpha\right). \tag{11.14}$$

When applied to any field of a supersymmetric theory this result has the same structure as the first line of (11.6), except that the anticommutator appears. Of course, this is entirely expected for the fermionic elements of a superalgebra, whose structure relation reads

$$\{Q_\alpha, Q_\beta\} = f_{\alpha\beta}{}^C T_C\,, \tag{11.15}$$

with structure constants $f_{\alpha\beta}{}^C$, which are symmetric in α, β and connect fermionic elements to a sum of bosonic elements T_C.

In Sec. 8.2.1, we computed the commutator of SUSY variations on the fields of a chiral multiplet and found

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = -\tfrac{1}{2}\bar\epsilon_1\gamma^\mu\epsilon_2 P_\mu = \tfrac{1}{2}\bar\epsilon_2\gamma^\mu\epsilon_1 P_\mu \;=\; -\tfrac{1}{2}\epsilon_2^\beta(\gamma^\mu)_{\beta\alpha}\epsilon_1^\alpha P_\mu\,, \tag{11.16}$$

where P_μ acts on the fields of the chiral multiplet as ∂_μ. Thus the structure constants of supersymmetry are given by

$$\{Q_\alpha, Q_\beta\} = -\tfrac{1}{2}(\gamma^\mu)_{\alpha\beta}P_\mu\,, \qquad \to \qquad f_{\alpha\beta}{}^\mu = -\tfrac{1}{2}(\gamma^\mu)_{\alpha\beta} = f_{\beta\alpha}{}^\mu\,. \tag{11.17}$$

Finally we would like to show that the treatments of supersymmetry and internal symmetry can be united in a common notation. Toward this end we insert the definition (11.15) in (11.14) and obtain

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = (\bar\epsilon_2)^\beta(\bar\epsilon_1)^\alpha f_{\alpha\beta}{}^C T_C\,. \tag{11.18}$$

This relation has the same structure as the last line of (11.6). Thus the result

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta(\epsilon_3^C = \epsilon_2^B\epsilon_1^A f_{AB}{}^C)\,, \tag{11.19}$$

holds for both bosonic and fermionic symmetries. The presence of a fermionic symmetry is signalled only by the fact that its parameter (called $\bar\epsilon^\alpha$ for supersymmetry) is anticommuting. The left side of (11.19) then contains the necessary anticommutator of the generators as in (11.14).

A superalgebra also contains structure relations of bosonic and fermionic elements, and they are realized by commutators of the generators. In supersymmetry the commutator of a Lorentz generator and a supercharge component was defined in (8.1). Here is a relevant exercise for the reader.

Ex. 11.3 *Use (11.12) and (11.9) to show that the commutator of the field space generators $M_{[\mu\nu]}$ and Q_α conforms conforms to (8.1) when acting on $Z(x)$, specifically*

$$\left[M_{[\mu\nu]}, Q_\alpha\right] Z = -\tfrac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta Z\,. \tag{11.20}$$

The nonlinear σ-model and Killing symmetries. Internal symmetries do not always act linearly. We now consider symmetries in the non-linear σ-model, which was discussed in Sec. 6.10. It contains scalar fields that transform as in (6.128), where the Killing vectors $k_A^i(\phi)$ should satisfy (6.127). Hence, in this case the symmetry generator is

$$T_A\phi^i = k_A^i(\phi)\,. \tag{11.21}$$

The Lie brackets (6.9) of the set of Killing vector fields determine the Lie algebra of the isometry group of M_n. The Lie bracket, which is the commutator of the differential operators k_A, reads as in (6.132)

$$[k_A, k_B] = f_{AB}{}^C k_C\,, \tag{11.22}$$

or, in components,

$$k_A^j\partial_j k_B^i - k_B^j\partial_j k_A^i = f_{AB}{}^C k_C^i\,. \tag{11.23}$$

It is then clear that the commutator of the field space generators T_A conforms to (11.7). The following exercise confirms this.

Ex. 11.4 *Use (11.21) to obtain*

$$\delta(\epsilon_1)\delta(\epsilon_2)\phi^i = \delta(\epsilon_1)\epsilon_2^A k_A^i = \epsilon_2^A\epsilon_1^B k_B^j\partial_j k_A^i\,. \tag{11.24}$$

Use (11.23) to show that for any smooth scalar function $f(\phi^i)$ on M_n,

$$[T_A, T_B]f(\phi^i) = f_{AB}{}^C T_C f(\phi^i)\,. \tag{11.25}$$

11.1.2 Local symmetries and gauge fields

We now consider gauge symmetries and use the notation of the previous section in which symmetries are indexed by $A, B, C\ldots$. For each symmetry there is a field space generator T_A, but the parameters $\epsilon^A(x)$ are arbitrary functions on spacetime. To realize local symmetry in Lagrangian field theory, one needs a gauge field, which we will generically denote by $B_\mu^A(x)$, for every gauged symmetry. The gauge fields transform as

$$\delta(\epsilon)B_\mu{}^A \equiv \partial_\mu\epsilon^A + \epsilon^C B_\mu{}^B f_{BC}{}^A\,. \tag{11.26}$$

It is easy to check that these transformations satisfy the algebra (11.19). This definition is modeled on the transformation of Yang-Mills fields in (4.82), but the definition is valid for internal and spacetime symmetries and for supersymmetry. The gauge fields for each case transform in the same way, although each case has its own specific notation in which the generic index A is appropriately changed.

Ex. 11.5 *Show that the commutator of two gauge transformations (11.26) conforms to the structure of (11.19). Specifically, assume that the generators T_A form a bosonic Lie algebra and show that*

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\,B_\mu^A = \partial_\mu\epsilon_3^A + \epsilon_3^C B_\mu{}^B f_{BC}{}^A\,, \tag{11.27}$$

where $\epsilon_3^C = \epsilon_2^B\epsilon_1^A f_{AB}{}^C$. The proof requires the Jacobi identity (4.75) for Lie algebras. The result (11.27) is also valid for the more general situation of a superalgebra.

Here you must take the anti-commutativity of parameters into account and use the graded Jacobi identity

$$\epsilon_2^B \epsilon_1^A \epsilon_3^C f_{AB}{}^D f_{CD}{}^E + \textit{cyclic } 1 \to 2 \to 3 = 0\,, \qquad \textit{assuming constant } f_{AB}{}^C\,. \quad (11.28)$$

We will be applying (11.26) in theories of gravity. Therefore, we distinguish between coordinate indices $\mu, \nu, \rho \ldots$ and local frame indices $a, b, c, \ldots$, and we now use $M_{[ab]}$ and P_a to denote the generators of local Lorentz transformations and translations. There are some subtleties for local translations, which are actually general coordinate transformations, which we will discuss in Sec. 11.3, but the other major ingredients of our treatment of gauge symmetries are developed in this section. Table 11.1 indicates both the generic notation and the particular cases we are concerned with.[3]

Table 11.1. *Gauge transformations, parameters and gauge fields*

generic gauge symmetry T_A	parameter ϵ^A	gauge field B_μ^A
local translations P_a	ξ^a	e_μ^a
Lorentz transformations $M_{[ab]}$	λ^{ab}	$\omega_\mu{}^{ab}$
Supersymmetry Q_α	$\bar{\epsilon}^\alpha$	$\bar{\psi}_\mu^\alpha$
Internal symmetry T_A	θ^A	A_μ^A

The structure constants $f_{BC}{}^A$ are the important data in the formula (11.26), and we extract them from the symmetry algebras in each specific case. Table 11.2 gives examples for the most common (anti)commutators (remember that the calculation takes into account factors of 2 in summations over pairs of antisymmetric indices as in Ex. 1.4).

Ex. 11.6 *The spin connection field $\omega_\mu{}^{ab}$ transforms under Lorentz transformations*[4] *as*

$$\delta\omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} + 2\omega_{\mu c}{}^{[a}\lambda^{b]c}\,. \qquad (11.29)$$

[3] The same index A is used both for the general case and for internal symmetry, but the distinction will be clear from the context of each application. For the other symmetries, the index A is replaced by either an index a, or an antisymmetric pair $[ab]$ or a spinor index. E.g. $\xi^a = e_\mu^a \xi^\mu$ is the parameter of translations. Our convention for spinors is that barred spinors carry an upper index. Thus we have written barred spinors in the Table. But we consider only Majorana spinors in supersymmetry, so they are linearly related to ϵ or ψ_μ.

[4] The orbital part of Lorentz transformations is not included here. Why this can be omitted will be explained in section 11.3.1.

Table 11.2. *Useful commutators, structure constants and third parameter in the commutation relation (11.19).*

(anti)commutators	structure constants	third parameter
$\left[M_{[ab]}, M_{[cd]}\right] = 4\eta_{[a[c}M_{[d]b]]}$	$f_{[ab][cd]}{}^{[ef]} = 8\eta_{[c[b}\delta_{a]}^{[e}\delta_{d]}^{f]}$	$\lambda_3^{ab} = -2\lambda_1{}^{[a}{}_c\lambda_2{}^{b]c}$
$\left[P_a, M_{[bc]}\right] = 2\eta_{a[b}P_{c]}$	$f_{a,[bc]}{}^d = 2\eta_{a[b}\delta_{c]}^d$	$\xi_3^a = -\lambda_2^{ab}\xi_{1b} + \lambda_1^{ab}\xi_{2b}$
$[P_a, P_b] = 0$		
$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}(\gamma^a)_{\alpha\beta}P_a$	$f_{\alpha\beta}{}^a = -\frac{1}{2}(\gamma^a)_{\alpha\beta}$	$\xi_3^a = \frac{1}{2}\bar\epsilon_2\gamma^a\epsilon_1$
$\left[M_{[ab]}, Q\right] = -\frac{1}{2}\gamma_{ab}Q$	$f_{[ab],\alpha}{}^\beta = -\frac{1}{2}\left(\gamma_{ab}\right)_\alpha{}^\beta$	$\epsilon_3 = \frac{1}{4}\lambda_1^{ab}\gamma_{ab}\epsilon_2 - \frac{1}{4}\lambda_2^{ab}\gamma_{ab}\epsilon_1$
$[P_a, Q] = 0$		

Check that this equation corresponds to (11.26). The gauge field B_μ^A is then replaced by $\omega_\mu{}^{ab}$. You must replace each of the indices A, B, C by antisymmetric pairs $[ab]$, etc., and insert factors $\frac{1}{2}$ as in Ex. 1.4.

When we started with $\mathcal{N} = 1$ supergravity, we simply *assumed* the transformations of the frame field and the gravitino given in (9.5),(9.6). However, now we can consider this again in the context of the gauged super-Poincaré algebra, and it turns out that this ansatz is actually *determined* by (11.26) as the following exercise shows.

Ex. 11.7 *Use (11.26) to calculate the supersymmetry transform of the vierbein e_μ^a, using information from table 11.2. Since the exercise asks only for the supersymmetry transformation, the index on the parameter can be restricted to a spinor index α. On the other hand, the A index is a for translations. Therefore, the first term of (11.26) does not contribute. From the second term you should obtain (9.5). Consider now in the same way the transformation of the gravitino and re-obtain (9.6).*

Although the results from this exercise are encouraging, there are also some puzzling features. As Ex. 11.5 shows, the transformations (11.26) do automatically satisfy the commutator algebra. But for spacetime symmetries, these commutator transformations are not always the ones that we expect. For example, the supersymmetry commutator $[\delta_Q(\epsilon_1)\,,\,\delta_Q(\epsilon_2)]$ vanishes on the gravitino. This result is correct since Table 11.2 and (11.26) tell us that $\delta_P\psi_{\mu\alpha} = 0$, but it disagrees with the results for the local supersymmetry algebra found in Sec. 9.5. Similarly $\delta_Q\omega_\mu{}^{ab} = 0$. This is consistent because the spin connection is an independent field here, but the result differs from Ch. 9 in which the spin connection is related to frame field and gravitino. Therefore we will need to modify the present setup before we can apply it to construct gravitational theories. The new ingredients will be discussed in Sec. 11.3.

First we will discuss some other issues which will concern us in applications to gravity and supergravity. They are situations in which the commutator of symmetry transformations contains the terms corresponding to the conventional structure constants $f_{AB}{}^{C}$ of the algebra *plus other terms.* Algebras modified in this way frequently occur in supersymmetry and supergravity.

11.1.3 Modified symmetry algebras

An important aspect of supergravity is the fact that the symmetry structure is not that of a Lie algebra in the mathematical sense. Rather it has a modified symmetry structure, which we want to highlight in this section.

Soft algebras. In supersymmetry we often encounter a generalization of standard Lie algebras, which we call 'soft algebras'. This means that the usual structure constants $f_{AB}{}^{C}$ can depend on fields and are thus called 'structure functions'. We already encountered this phenomena when we studied the commutator of SUSY transformations in supersymmetric gauge theories. Consider the result (8.43), which we can write as

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{\mathrm{P}}(\xi) - \delta_{\mathrm{gauge}}(\theta^A = A^A_\mu \xi^\mu)\,, \qquad \xi^\mu = \tfrac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_1\,. \tag{11.30}$$

The first term is a translation, which acts as $\xi^\mu\partial_\mu$ on the fields. The second term was called a field dependent gauge transformation in Ch. 8. However, we now interpret it as a modification of the SUSY algebra in which the commutator of supercharges contains a gauge transformation giving the new 'structure function'

$$f_{\alpha\beta}{}^{A} = -\tfrac{1}{2}A^A_\mu(\gamma^\mu)_{\alpha\beta}\,, \tag{11.31}$$

which depends on the gauge field A_μ.

It is not hard to find the reason for the modified algebra in (11.30). The SUSY transformation rules (8.53) and (8.41) of the chiral and gauge multiplets are covariant under non-abelian gauge transformations, so their commutator is also covariant. However, the action of δ_P on any component field is not covariant. Instead its gauge transform contains the derivative of the gauge parameter $\theta^A(x)$. The second term in (11.30) restores gauge covariance.

The extension to soft algebras does not affect most of the formulas we have developed for gauge symmetries. Jacobi identities are modified due to the variation of the structure functions, but we will not need this explicitly. For the interested reader we point out that this type of generalized gauge theory is quite naturally described in the framework of the Batalin-Vilkovisky or field-antifield formalism. See [63, 64].

When one has a solution of the field equations of the full theory, one may plug in the values of the fields into this soft algebra. Then we have structure constants, and hence the consistency of usual algebra theory says that it should fit in normal algebras. These encode the remaining symmetries for that particular solution. The soft algebra thus allows for the final solution various supersymmetry algebras such as the anti-de Sitter algebra or algebras with central charges.

Zilch symmetries. A zilch symmetry is one whose transformation rules vanish on solutions of the equations of motion. The corresponding Noether current then vanishes. Any action with at least two fields has zilch symmetries. Indeed consider an action $S(\phi)$ containing fields ϕ^i. Consider the transformation

$$\delta\phi^i = \epsilon h^{ij}\frac{\delta S}{\delta\phi^j}\,, \tag{11.32}$$

for any antisymmetric matrix $h^{ij} = -h^{ji}$. One finds that

$$\delta S = \frac{\delta S}{\delta\phi^i}\epsilon h^{ij}\frac{\delta S}{\delta\phi^j} = 0\,. \tag{11.33}$$

We assumed bosonic fields here, but the idea extends to fermions if we change the symmetry properties of h. This is in fact the way that we encountered such a zilch symmetry in (8.36). In that case the h^{ij} is represented by $v^\mu(\gamma_\mu)^{\alpha\beta}$, which is symmetric in the indices α, β that refer to the spinor fields. The transformation (8.36) is a zilch symmetry since it vanishes if the field χ satisfies the equation of motion.

The existence of zilch symmetries implies that symmetries are not uniquely defined. One can add to any symmetry transformation $T_A\phi^i$ a zilch symmetry $h_A^{ij}\frac{\delta S}{\delta\phi^j}$, and it is still a symmetry. We view this as a change of basis in the algebra of symmetries.

Open algebras. As we saw already in Ch. 8, zilch symmetries sometimes occur in the commutators of conventional symmetries such as SUSY. We found this in the algebra of supersymmetry transformations of a chiral multiplet after elimination of the auxiliary field. See Ex. 8.11 and the following discussion.

From first principles it follows that the commutator of two symmetries of the action is also a symmetry. The result of the commutator can thus be expanded in the set of all symmetries, as indicated in (11.19). However, this expansion may also include the zilch symmetries. As such it makes sense to write the algebra

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\,\phi^i = \text{minimal susy algebra} + \eta^{ij}(\epsilon_1, \epsilon_2)\,\frac{\delta S}{\delta\phi^j} \tag{11.34}$$

Thus, in a general gauge theory, a commutator of symmetries of the form (11.19) will only be valid after equations of motion are used. In this case algebra of field transformations is said to be closed only 'on-shell'. If one has a basis such that (11.19) is valid without use of the equations of motion, then the commutator closes for any configuration of fields, whether the equations of motion are satisfied or not. In this case we say that the algebra is closed 'off-shell'. One also uses the terminology '*closed supersymmetry algebra*' when the algebra holds without zilch symmetries. When zilch symmetries enter the algebra, it is called an '*open supersymmetry algebra*'. But remember that in this case the algebra does close when the field equations are satisfied or when the (infinite set of) zilch symmetries are included.

11.2 Covariant quantities

We now want to consider field theories in which the Lagrangian contains both gauge fields B_μ^A and other fields ϕ^i, sometimes called 'matter fields', whose transformation rules

$$\delta(\epsilon)\phi^i(x) = \epsilon^A (T_A\phi^i)(x) \tag{11.35}$$

do not involve derivatives of the gauge parameters. Readers are already familiar with the important case of gauged non-abelian internal symmetry discussed in Sec. 4.3.2. In that section we saw how to define and use covariant derivatives and field strengths, which also transform without derivatives of the ϵ^A, and which are the building blocks of the physical gauge theories.

In this section we want to define such quantities within the more general framework considered in this chapter. We begin with the definition

> *A* ***covariant quantity*** *is a local function that transforms under all local symmetries with no derivatives of a transformation parameter.*

Our considerations apply in a straightforward way to gauged internal symmetry, Lorentz transformations and SUSY, but local translations require special care and are mainly discussed in Sec. 11.3.

11.2.1 Covariant derivatives

One very important covariant quantity is the covariant derivative of a field ϕ^i (whether elementary or composite) whose gauge transformation rule is of the form in (11.35). The ordinary derivative $\partial_\mu\phi^i$ is certainly not a covariant quantity since

$$\delta(\epsilon)\partial_\mu\phi^i = \epsilon^A\partial_\mu(T_A\phi^i) + (\partial_\mu\epsilon^A)T_A\phi^i\,. \tag{11.36}$$

The second term spoils covariance but we can correct for this by adding a term involving the gauge fields and defining

$$\begin{aligned} \mathcal{D}_\mu\phi^i &\equiv (\partial_\mu - \delta(B_\mu))\,\phi^i \\ &= \left(\partial_\mu - B_\mu^A T_A\right)\phi^i\,. \end{aligned} \tag{11.37}$$

The notation $\delta(B_\mu)$ means that the covariant derivative is constructed by the specific prescription to subtract the gauge transform of the field with the gauge field itself as the symmetry parameter. As is clear from the second line of (11.37) there is a sum over all gauge transformations of the theory. As an example of this construction we rewrite the spinor covariant derivative of (4.83) and transformation rule in the present notation[5]

$$\delta(\theta)\Psi = -\theta^A t_A \Psi\,, \qquad \mathcal{D}_\mu \Psi = \left(\partial_\mu + A_\mu^A t_A\right)\Psi\,. \tag{11.38}$$

It is easy to check using (11.26) that the gauge transformation of the covariant derivative does not contain derivatives of the gauge parameters. Hence, the covariant derivative is a covariant quantity. We now prove the stronger result that gauge transformations commute with covariant differentiation on fields ϕ for which the algebra is closed.[6] Thus we can apply (11.19). We first write an auxiliary relation which is (11.19) applied to ϕ as in (11.35), with ϵ_1 replaced by B_μ and ϵ_2 by ϵ:

$$\epsilon^A \delta(B_\mu)(T_A\phi) - B_\mu^A \delta(\epsilon)(T_A\phi) = \epsilon^B B_\mu^A f_{AB}{}^C (T_C\phi)\,. \tag{11.39}$$

Here, we use that the algebra is satisfied without adding equations of motion. We then write, using (11.26) in the second line and (11.39) in the third,

$$\begin{aligned}
\delta(\epsilon)\mathcal{D}_\mu\phi &= \partial_\mu\left(\epsilon^A(T_A\phi)\right) - B_\mu^A\delta(\epsilon)(T_A\phi) - \left(\delta(\epsilon)B_\mu^A\right)(T_A\phi)\\
&= \epsilon^A\partial_\mu(T_A\phi) - B_\mu^A\delta(\epsilon)(T_A\phi) - \epsilon^C B_\mu^B f_{BC}{}^A(T_A\phi)\\
&= \epsilon^A\left[\partial_\mu(T_A\phi) - \delta(B_\mu)(T_A\phi)\right] = \epsilon^A\mathcal{D}_\mu(T_A\phi)\,.
\end{aligned} \tag{11.40}$$

Hence,

Gauge transformations and covariant derivatives commute on fields on which the algebra is satisfied.

In summary we repeat the definition of the covariant derivative in slightly more general form.

The covariant derivative of any covariant quantity is given by the operator

$$\mathcal{D}_\mu = \partial_\mu - \delta(B_\mu)\,, \tag{11.41}$$

acting on that quantity. The instruction $\delta(B_\mu)$ means compute all gauge transformations of the quantity, with the potential B_μ^A as the gauge parameter.

Ex. 11.8 *Symmetries of the nonlinear σ-model are generated by Killing vectors $k_A^i(\phi)$ which in general determine a Lie algebra as defined in (11.22). The elementary fields ϕ^j are local coordinates of the target space. Suppose that the symmetry is gauged. Show that the covariant derivative $\mathcal{D}_\mu\phi^i = \partial_\mu\phi^i - A_\mu^A k_A^i$ transforms as*

$$\delta\mathcal{D}_\mu\phi^i = \theta^A\mathcal{D}_\mu k_A^i = \theta^A\left(\partial_j k_A^i\right)\mathcal{D}_\mu\phi^j\,. \tag{11.42}$$

[5] The gauge coupling of Ch. 4 is now set to $g=1$.

[6] Remember that this is not a trivial statement, see the remarks above on open algebras. Thus, e.g. in the chiral multiplet without auxiliary fields, (8.36) implies that the statements below do apply on Z, but not for χ.

11.2.2 Curvatures

We can use the covariant derivative to define the next important set of quantities in any gauge theory of an algebra. For each generator of the algebra the curvature $R_{\mu\nu}{}^A$ is a second rank antisymmetric tensor which is also a covariant quantity. In Yang-Mills theory the curvature was called a field strength and was defined in (4.85),(4.86) as a commutator of covariant derivatives acting on a covariant field.

We now proceed in the same way in this more general framework. Using (11.40) and (11.39) (with now ϵ replaced by B_ν), we obtain

$$\begin{aligned} [\mathcal{D}_\mu, \mathcal{D}_\nu] &= -\delta(R_{\mu\nu}) \,, \\ R_{\mu\nu}{}^A &= 2\partial_{[\mu} B_{\nu]}{}^A + B_\nu{}^C B_\mu{}^B f_{BC}{}^A \,. \end{aligned} \tag{11.43}$$

Here again $\delta(R_{\mu\nu})$, means that one takes a sum over all gauge symmetries and replaces the parameters ϵ^A with $R_{\mu\nu}{}^A$.

Curvatures are covariant quantities, which transform as

$$\delta(\epsilon) R_{\mu\nu}{}^A = \epsilon^C R_{\mu\nu}{}^B f_{BC}{}^A \,. \tag{11.44}$$

They satisfy Bianchi identities, which are, using the definition (11.41),

$$\mathcal{D}_{[\mu} R_{\nu\rho]}{}^A = 0 \,. \tag{11.45}$$

The curvatures and Bianchi identities discussed in Sec. 6.9 are a special case of (11.43) in which the Lie algebra is the algebra of the Lorentz group $\mathfrak{so}(D-1,1)$. Here is a small exercise to verify this.

Ex. 11.9 *Obtain the curvature tensor (6.103) for Lorentz symmetry using the approach of this chapter. Use the definition (11.43) and the structure constants in table 11.2.*

We mentioned earlier that gauged local translations, generated by P_a, are more subtle and require special treatment. The following exercise will illustrate this.

Ex. 11.10 *Translations occur on the right-hand side of the commutators* $[M_{ab}, P_c]$ *and* $\{Q_\alpha, Q_\beta\}$*, see table 11.2. Use this information to show that the curvature for the translation generator* P^a *is*

$$R_{\mu\nu}(P^a) = 2\partial_{[\mu} e_{\nu]}{}^a + 2\omega_{[\mu}{}^{ab} e_{\nu]b} - \tfrac{1}{2}\bar\psi_\mu \gamma^a \psi_\nu \,. \tag{11.46}$$

The last term is the torsion tensor of $\mathcal{N}=1$, $D=4$, *see (9.15), which, by (6.94), is the antisymmetric part of the connection. Hence, the equation (11.46) is just the antisymmetric part of (6.89) in* $[\mu\nu]$*, which is equivalent to the first Cartan structure equation (6.70). Therefore, the curvature for spacetime translations vanishes:*

$$R_{\mu\nu}(P^a) = 0 \,. \tag{11.47}$$

The last exercise indicates already that something is special when local translations are included. In the next section we will discuss the special features of such theories.

11.3 Gauged spacetime translations

The main issue that we have to address here is the reconciliation of the concept of gauge theories developed earlier in this chapter with general coordinate transformations, which we studied in chapter 6.

11.3.1 Gauge transformations for the Poincaré group

The Lie algebra of the Poincaré group includes both translations P_a and Lorentz transformations $M_{[ab]}$. In theories of gravity both symmetries are gauged by parameters $\xi^a(x)$ and $\lambda^{ab}(x)$, which are arbitrary functions on the spacetime manifold.

Special issues arise for *local translations*. The first issue is quite simple. Under global transformations of the Poincaré group, it was shown in Ch. 1 that a scalar field in Minkowski spacetime transforms as

$$\delta(a,\lambda)\phi(x) = \left[a^\mu\partial_\mu - \tfrac{1}{2}\lambda^{\mu\nu}L_{[\mu\nu]}\right]\phi(x) = \left[a^\mu + \lambda^{\mu\nu}x_\nu\right]\partial_\mu\phi(x)\,. \tag{11.48}$$

In curved spacetime we replace $a^\mu \to \xi^\mu(x)$ which is an arbitrary spacetime dependent function. The effect of the second term, which is sometimes called the 'orbital part' of a Lorentz transformation, can be included in $\xi^\mu(x)$. In this way we perform a change of basis in the set of gauge transformations. Originally the independent ones were parametrized by $a^\mu(x)$ and $\lambda^{ab}(x)$. From now on we use $\xi^\mu(x) = a^\mu(x) + \lambda^{\mu\nu}(x)x_\nu$ and $\lambda^{ab}(x)$ as a basis of independent transformations. Thus (11.48) is replaced by

$$\delta_{\rm gct}(\xi)\phi(x) = \xi^\mu(x)\partial_\mu\phi(x)\,, \tag{11.49}$$

in curved spacetime. In a similar fashion we extend this rule to spinors and frame vectors as

$$\begin{aligned}\delta_{\rm gct}(\xi)\Psi(x) &= \xi^\mu(x)\partial_\mu\Psi(x)\,,\\ \delta_{\rm gct}(\xi)V^a(x) &= \xi^\mu(x)\partial_\mu V^a(x)\,.\end{aligned} \tag{11.50}$$

Fields referred to local frames effectively transform as scalars under diffeomorphisms. Lorentz transformations are still an important part of this framework. They are described by parameters $\lambda^{ab}(x)$ and they act on the frame indices as we discuss below.

For a vector field we have a choice of using either the coordinate basis or local frame. The components are related by $V_\mu = e_\mu^a V_a$. The components V_μ transform under general coordinate transformations by the Lie derivative, as explained in (6.10), namely

$$\delta_{\rm gct}(\xi)V_\mu = \mathcal{L}_\xi V_\mu = \xi^\nu\partial_\nu V_\mu + V_\nu\partial_\mu\xi^\nu\,. \tag{11.51}$$

Notice that according to the rules of Sec. 1.2.3 a vector field transforms under Lorentz transformations as $\delta V_\mu = -\lambda_\mu{}^\nu V_\nu$, but that term is already included in $\partial_\mu\xi^\nu$, such that there are no separate Lorentz transformations anymore for V_μ.

Further, we introduce the frame vector parameter $\xi^a(x)$ related to ξ^μ by $\xi^a = \xi^\mu e_\mu^a$. Then (11.49) can be rewritten as

$$\delta_{\rm gct}\phi(x) \,=\, \xi^a(x)e_a^\mu(x)\partial_\mu\phi(x) \,=\, \xi^a(x)\partial_a\phi(x)\,, \tag{11.52}$$

with $\partial_a\phi = e_a^\mu\partial_\mu\phi$. This defines the local translations, and we have seen that they have absorbed the 'orbital part' of Lorentz transformations.

The 'spin part' of Lorentz transformations is implemented as *local Lorentz transformations* in curved spacetime. The rule for these is simple to state and is implicit in the discussion of Ch. 6.

Rule – Only fields carrying local frame indices transform under local Lorentz transformations. The transformation rule involves the appropriate matrix generator. Thus for scalars ϕ, spinors Ψ, and frame vectors V^a we have

$$\begin{aligned}
\delta(\lambda)\phi &= 0\,,\\
\delta(\lambda)\Psi &= -\tfrac14\lambda^{ab}\gamma_{ab}\Psi\,,\\
\delta(\lambda)V^a &= -\lambda^a{}_bV^b\,,\qquad \delta(\lambda)V_a = V_b\lambda^b{}_a\,.
\end{aligned} \tag{11.53}$$

Thus the main results of this section are to establish ξ^a and λ^{ab} as the 'basis' for gauge parameters of the Poincaré group and to define the associated transformation rules of the fields we will encounter.

Note that (11.51), (11.50) and (11.53) are compatible if the transformation rule of the frame field is

$$\delta e_\mu^a = \xi^\nu\partial_\nu e_\mu^a + e_\nu^a\partial_\mu\xi^\nu - \lambda^{ab}e_{\mu b}\,. \tag{11.54}$$

We have combined Lorentz and coordinate transformations, and we have simply applied the previously stated rules to a field with both coordinate and frame indices. To complete the list of transformation rules we include the rule for the transformation of the spin connection ω_μ^{ab}, namely

$$\delta\omega_\mu{}^{ab} = \xi^\nu\partial_\nu\omega_\mu{}^{ab} + \omega_\nu{}^{ab}\partial_\mu\xi^\nu + \partial_\mu\lambda^{ab} - \lambda^a{}_c\omega_\mu{}^{cb} + \omega_\mu{}^{ac}\lambda_c{}^b\,. \tag{11.55}$$

The Lorentz term is just the infinitesimal limit of (6.71).

11.3.2 Covariant derivatives and general coordinate transformations

In this section, we first show why the definitions of covariant derivatives made in Sec. 11.2 need to be modified when applied to an algebra with spacetime symmetries. Then we will improve the definitions. Though this makes the formalism more involved, a good and clear definition of covariant derivatives is needed for supergravity. When one calculates transformations of covariant quantities, the improved formalism avoids (with the help of lemmas to be introduced in Sec. 11.3.3) heavy calculations of terms that later cancel.

It is easy that the definition (11.41) needs to be changed when the gauge group includes local translations. Otherwise we would write the covariant derivative of a scalar field (with no internal symmetry present) as

$$\mathcal{D}_\mu\phi \;=\; \partial_\mu\phi - e_\mu^a(x)\partial_a\phi(x) = 0\,. \tag{11.56}$$

So the previous definition is rather useless. It is not hard to repair the definition by removing general coordinate transformations from the sum over gauge transformations. Thus, from now on, we define the covariant derivative of any covariant field ϕ as

$$\mathcal{D}_\mu\phi \equiv \partial_\mu\phi - B_\mu{}^A T_A\phi\,, \tag{11.57}$$

with general coordinate transformations omitted in the sum over the gauge group indices A. The same restriction applies to the gauge transformation $\delta(\epsilon)\phi$ of (11.1). We discuss below how this changes the rules established in Sec. 11.2. We use the term 'standard gauge transformations' to refer to the set of gauge transformations that remain in (11.57). These include local Lorentz, local supersymmetry, and local internal symmetry transformations.

A second peculiar feature of translations was encountered in Ex. 11.10. Namely the curvature component for local translations in gravity or supergravity vanishes, see (11.46). *From now on we will always impose (11.47) as a constraint*, which determines the connection in the presence of torsion due to the gravitino field.[7] It was shown in Ex. 6.24 how to solve this constraint, which we found there as the first Cartan structure equation, to express the spin connection in terms of the frame field and the torsion tensor. Thus the spin connection becomes a 'composite gauge field' in the formalism we are now developing.

We now make a further change in the basis of gauge transformations. Namely we replace general coordinate transformations by *gauge covariant* general coordinate transformations. To motivate this we consider a set of scalar fields ϕ^i transforming under an internal symmetry as $\delta(\theta)\phi^i(x) = -\theta^A(x)t_A{}^i{}_j\phi^j$. Using the previous definition their transformation under a coordinate transformation would be

$$\delta(\xi)\phi^i \;=\; \xi^\mu\partial_\mu\phi^i\,. \tag{11.58}$$

This is correct, but it has the undesirable property that it does not transform covariantly under internal symmetry. We fix this by adding a field dependent gauge transformation and thus define

$$\delta_{\rm cgct}(\xi)\phi^i \;\equiv\; \xi^\mu\partial_\mu\phi^i + (\xi^\mu A_\mu{}^A)\,t_A{}^i{}_j\phi^j\,. \tag{11.59}$$

A good indication that this is a quite natural modification already came in Ex. 8.13, where readers showed that such field dependent gauge transformations occur in the commutator of global SUSY transformations in a SUSY gauge theory. This

[7] For application to ordinary gravity, we just set $\psi_\mu = 0$.

led to the concept of soft algebras in (11.30). It also appeared in Sec. 9.5 when we first discussed the algebra of local supersymmetry.

We thus conclude that for any field that transforms under one or more of the standard gauge transformations, we define its '*covariant general coordinate transformation*' [52, 65] by

$$\delta_{\rm cgct}(\xi) = \delta_{\rm gct}(\xi) - \delta(\xi^\mu B_\mu)\,. \tag{11.60}$$

For every standard gauge transformation this contains a term in which the parameter of the transformation is replaced by the scalar product of ξ^μ with the standard gauge field $B_\mu{}^A$.

We now discuss the form of the covariant general coordinate transformations on the several types of fields that we use.

Coordinate scalars. Note that the two terms in the example (11.59) can be grouped into the standard covariant derivative, so we can write

$$\delta_{\rm cgct}(\xi)\phi \;=\; \xi^\mu\mathcal{D}_\mu\phi = \xi^a\mathcal{D}_a\phi\,. \tag{11.61}$$

We thus find that cgct transform the scalars to

$$\mathcal{D}_a\phi = e_a{}^\mu\mathcal{D}_\mu\phi\,,\qquad \mathcal{D}_\mu\phi = \partial_\mu\phi - B_\mu^A(T_A\phi)\,. \tag{11.62}$$

The same situation occurs for the spinor fields of $\mathcal{N}=1$ multiplets, which are also general coordinate scalars when they are coupled to supergravity. Consider the fermion χ of a chiral multiplet and the gaugino λ of an abelian gauge multiplet. For them the gauge covariant translation and subsequent covariant derivative requires a field-dependent SUSY transformation and thus involves their SUSY partners Z, F, $A_\mu(x)$ and D, as well as the gravitino ψ_μ.

$$\begin{aligned}\delta_{\rm cgct}(\xi)P_L\chi &= \xi^\mu P_L(\partial_\mu\chi + \frac14\omega_\mu{}^{ab}\gamma_{ab}\chi - \frac{1}{\sqrt2}\gamma^\nu\mathcal{D}_\nu Z\psi_\mu - F\psi_\mu)\\ &= \xi^\mu P_L\mathcal{D}_\mu\chi\,,\\ \delta_{\rm cgct}(\xi)\lambda &= \xi^\mu\left(\partial_\mu\lambda + \tfrac14\omega_\mu{}^{ab}\gamma_{ab}\lambda + \tfrac14\gamma^{\rho\sigma}F_{\rho\sigma}\psi_\mu - \tfrac12{\rm i}\gamma_*D\right) = \xi^\mu\mathcal{D}_\mu\lambda\,.\end{aligned} \tag{11.63}$$

The frame field. The frame field e_μ^a is the gauge field of local translations. Its standard gauge transformations are defined in (11.26). As described until now, those standard gauge transformations are local Lorentz and local SUSY transformations. However, in later chapters we will include conformal and superconformal transformations. Therefore we now use the general notation in which all such standard gauge fields are denoted by $B_\mu{}^A$. As discussed above general coordinate transformations are excluded in a sum with gauge index A, but they should otherwise be kept. Therefore we define the sum over index

B below to include only standard gauge transformations. Local translations (with gauge parameter ξ^a) are indicated separately. The computation is in fact a formalization of the computation that we did in Sec. 9.5, and leads to a simple result:

$$\begin{aligned}\delta_{\rm cgct}(\xi)e_\mu^a &= \xi^\nu\partial_\nu e_\mu^a + e_\nu^a\partial_\mu\xi^\nu - \xi^\nu B_\nu^C B_\mu{}^B f_{BC}{}^a - \xi^\nu B_\nu^C e_\mu{}^b f_{bC}{}^a \\ &= \partial_\mu\xi^a + \xi^\nu\left(2\partial_{[\nu}e_{\mu]}^a - B_\nu^C B_\mu{}^B f_{BC}{}^a - B_\nu^C e_\mu^b f_{bC}{}^a - e_\nu^c B_\mu^B f_{Bc}{}^a\right) \\ &\quad + \xi^c B_\mu^B f_{Bc}{}^a \\ &= \partial_\mu\xi^a + \xi^c B_\mu{}^B f_{Bc}{}^a - \xi^\nu R_{\mu\nu}{}^a\,. \end{aligned} \tag{11.64}$$

As discussed above, the last term vanishes as a constraint on the spin connection. The result is that the cgct of the frame field is given by the first two terms. This agrees exactly with the general rule (11.26) in which the frame field is considered to be the gauge field of translations. This confirms the identification of the cgct as the appropriate modification of local translations.

Gauge fields. We now compute the cgct for the gauge field $B_\mu{}^A$ of a standard gauge transformation. For simplicity we assume that the structure constant $f_{aB}{}^A = 0$, , where A is the particular symmetry index of interest, the index a indicates local translations, and B is general. This assumption is satisfied in the $\mathcal{N} = 1$ SUSY algebra[8], as shown in Table 11.2 above. The cgct of B_μ^A is as the sum of a Lie derivative plus field dependent standard gauge transformation:

$$\delta_{\rm cgct}(\xi)B_\mu^A = \xi^\nu\partial_\nu B_\mu^A + B_\nu^A\partial_\mu\xi^\nu - \partial_\mu\left(\xi^\nu B_\nu^A\right) - \xi^\nu B_\nu^C B_\mu{}^B f_{BC}{}^A = \xi^\nu R_{\nu\mu}{}^A\,. \tag{11.65}$$

Thus the cgct of a gauge field involves its curvature. An example already appeared in the commutator of global SUSY transformations on the Yang-Mills potential in (8.43).

There are important cases in supergravity in which (11.26) (with indices restricted to standard gauge transformations) is not the complete transformation law. Indeed, the transformation (11.26) involves only gauge fields, and multiplets are often composed of gauge fields together with non-gauge fields. Therefore, often (11.26) has to be completed with additional terms. To allow for this, we write the more general form

$$\delta(\epsilon)B_\mu^A = \partial_\mu\epsilon^A + \epsilon^C B_\mu{}^B f_{BC}{}^A + \epsilon^B\mathcal{M}_{\mu B}{}^A\,. \tag{11.66}$$

One case is where gauge transformations do not satisfy $f_{aB}{}^A = 0$, and the latter term thus contains terms of the form $\epsilon^B e_\mu^a f_{aB}{}^A$ as we will encounter later. Another example is the Yang-Mills gauge field A_μ^A, whose transformation law under

[8] In some situations that occur later in this book, this simplification is not generally valid. See next paragraph.

supersymmetry in (8.41) is not related to its gauge properties. This is due to the presence of the non-gauge field λ^A in the gauge multiplet. In this case the modified gauge transformation law reads

$$\delta(\epsilon)A_\mu^A = \partial_\mu\epsilon^A + \epsilon^C A_\mu{}^B f_{BC}{}^A \; - \; \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A\,. \tag{11.67}$$

In this case the new term involves only the local SUSY parameter $\bar{\epsilon}^\alpha$. Formally, we thus have for the latter case (with spinor index α referring to supersymmetry)

$$\mathcal{M}_{\mu\alpha}{}^A = -\tfrac{1}{2}\left(\gamma_\mu\lambda^A\right)_\alpha\,. \tag{11.68}$$

Other modifications can also occur, and they lead to a change in the definition of the curvature which appears in (11.65) which we discuss in the next section.

In practical calculations it turns out that the $\mathcal{M}_{\mu B}{}^A$ are the important terms, while the other parts of the transformations (11.66) automatically appear in the formalism of covariant derivatives which we discuss next.

11.3.3 Covariant derivatives and curvatures in a gravity theory.

In this section we will find the modifications needed in the treatment of covariant derivatives and curvatures in section 11.2. We maintain the same definition of a covariant quantity used on page 208. In particular, since the general coordinate transformation of a covariant quantity should not involve a derivative of the parameter ξ^μ, a covariant quantity must be a world scalar. Hence, the covariant derivative that we will use is $\mathcal{D}_a\phi$ as defined in (11.62), rather than $\mathcal{D}_\mu\phi$.

The modifications made for covariant derivatives have their counterpart for curvatures. First of all we have to distinguish again translations from standard gauge transformations. Hence, we define now curvatures

$$r_{\mu\nu}{}^A = 2\partial_{[\mu}B_{\nu]}{}^A + B_\nu{}^C B_\mu{}^B f_{BC}{}^A\,, \tag{11.69}$$

where the sums over B and C involve only standard gauge transformations. Secondly, if gauge fields transform with extra terms as in (11.66), one has to 'covariantize also for these. Hence the modified curvature is

$$\widehat{R}_{\mu\nu}^A = r_{\mu\nu}{}^A - 2B_{[\mu}{}^B\mathcal{M}_{\nu]B}{}^A\,. \tag{11.70}$$

Finally, as argued above for the covariant derivative, the covariant curvature should carry Lorentz indices rather than world indices in order to be a world scalar. Thus

$$\widehat{R}_{ab}{}^A = e_a{}^\mu e_b{}^\nu\widehat{R}_{\mu\nu}{}^A\,. \tag{11.71}$$

11.3.4 Calculating transformations of covariant quantities

Covariant quantities are the building blocks of general supergravity theories and the construction of these theories frequently requires the gauge transform of one or more covariant quantities. An *ab initio* computation in every case involves very tedious details. Fortunately, there is a common structure in such calculations, and the purpose of this section is to exploit it to obtain useful shortcuts which will be applied many times in the next few chapters.

The common structure and the shortcuts derived from it require the application of the following principles:

1. The covariant derivative $\mathcal{D}_a$ of a covariant quantity is a covariant quantity, and so is the curvature $\widehat{R}_{ab}$.
2. The gauge transformation of a covariant quantity does not involve a derivative of a parameter.[9]
3. If the algebra closes on the fields, then the transformation of a covariant quantity is a covariant quantity, i.e. gauge fields only appear either included in covariant derivatives or in curvatures.

We postpone the proof of these statements to the Appendix 11.A. Instead we illustrate how they are applied here. The importance of these facts is that they tell us in advance that explicit gauge fields only occur inside covariant derivatives and curvatures. This simplifies our (lives and our) calculations.

Consider first a set of scalar fields ϕ^i in the setting of a gravity theory with a gauged internal symmetry group. Their covariant derivatives are given in (11.62), but with $B_\mu^A = A_\mu^A$, the usual Yang-Mills gauge fields. The scalars transform under gauge transformations as

$$\delta\phi^i = -\theta^A (t_A)^i{}_j \phi^j \,. \tag{11.72}$$

We want to calculate the standard transformation of the covariant derivative $\mathcal{D}_a\phi$. In this example the standard transformations include only local Lorentz and internal symmetry. Distributing the transformations on the various fields gives:

$$\delta\mathcal{D}_a\phi = (\delta e_a{}^\mu)\,\mathcal{D}_\mu\phi + e_a{}^\mu\partial_\mu\delta\phi - e_a{}^\mu(\delta A_\mu^A)(T_A\phi) - e_a{}^\mu A_\mu^A\delta(T_A\phi)\,. \tag{11.73}$$

In the first term we need the local Lorentz part of (11.54), which we can also find using the commutator $[P, M]$ from Table 11.2:

$$\delta e_\mu{}^a = -\lambda^{ab} e_{\mu b}\,, \qquad \Rightarrow \qquad \delta e_a{}^\mu = -\lambda_{ab} e^{b\mu}\,. \tag{11.74}$$

The first term then becomes

$$\delta_{\rm Lor}(\lambda)\mathcal{D}_a\phi^i = -\lambda_a{}^b\mathcal{D}_b\phi^i\,. \tag{11.75}$$

Rather than work out the remaining terms explicitly we wish to show how the principles above eliminate much of the work. We can forget the $\partial_\mu\theta^A$ which appears

[9] This is not a derived result; rather, it is the *definition* of a covariant quantity, but we include it here to have all the important facts together for further reference.

when (11.72) is inserted in the second term of (11.73) because the first principle tells us that $\mathcal{D}_a\phi$ is a covariant quantity and the second principle says that its transformations cannot contain derivatives of a gauge parameter. So the derivative term must cancel at the end and we can simply drop it at the beginning. The second term of (11.73) then reduces to $-e_a^\mu\theta^A(t_A)^i{}_j\partial_\mu\phi^j$. The remaining parts of the third and fourth terms of (11.73) contain the gauge field explicitly, and the third principle tells us that gauge fields can only 'covariantize' the derivative $\partial_\mu\phi^i$. Thus we can write the final answer as

$$\delta\mathcal{D}_a\phi^i \;=\; -\lambda_a{}^b\mathcal{D}_b\phi^i - \theta^A(t_A)^i{}_j\mathcal{D}_a\phi^j\,. \tag{11.76}$$

In this example, it is easy to perform a complete calculation to verify this result. However, for more complicated calculations in supergravity we will be happy that we have these principles that save us a lot of calculational work. An illustration is provided in Appendix 11.A.2.

Ex. 11.11 *One can derive a general formula for the transformation of curvatures, correcting (11.44) for the effects that gauge fields transform with matter-like terms. To apply the methods explained earlier, the decomposition in (11.70) is most useful. Indeed, explicit gauge fields appear in $r_{ab}{}^A$ only quadratically. Show that*

$$\delta(\epsilon)\widehat{R}_{ab}{}^A = \epsilon^B\widehat{R}_{ab}{}^C f_{CB}{}^A + 2\epsilon^B\mathcal{D}_{[a}\mathcal{M}_{b]B}{}^A - 2\epsilon^C\mathcal{M}_{[aC}{}^B\mathcal{M}_{b]B}{}^A\,. \tag{11.77}$$

To do this, you should use the principles of this section, i.e. first deleting all terms that have explicit gauge fields, and at the end replacing ordering derivatives with covariant ones. Similarly, show that the Bianchi identity becomes

$$\mathcal{D}_{[a}\widehat{R}_{bc]}{}^A - 2\widehat{R}_{[ab}{}^B\mathcal{M}_{c]B}{}^A = 0\,. \tag{11.78}$$

Ex. 11.12 *Check that the commutator of the covariant derivatives in the context of theories with gravity still gives the curvature, i.e.*

$$[\mathcal{D}_a,\mathcal{D}_b]\,\phi = -\widehat{R}_{ab}{}^A(T_A\phi)\,. \tag{11.79}$$

You can use the principles mentioned in this section. That means that all terms with undifferentiated gauge fields can be omitted, and $2\partial_{[\mu}B_{\nu]}{}^A$ can be replaced by the curvature. Furthermore, you will need that the curvature of translations vanishes to eliminate (the covariantization of) $\partial_{[\mu}e_{\nu]}{}^c$.

11.A Appendix: manipulating covariant derivatives.

11.A.1 Proof of the main lemma

Lemma on covariant derivatives. *If a covariant quantity ϕ transforms into covariant quantities under standard gauge transformations, its covariant derivative*

$\mathcal{D}_a\phi$ given by (11.62) is a covariant quantity. Moreover, if the algebra closes on the field ϕ then the gauge transformations of $\mathcal{D}_a\phi$ involve only covariant quantities.

The proof closely resembles (11.40), but the separation of cgct from the other gauge transformations and the modification (11.66) change a few steps. There is one extra term due to the last term in (11.66). Furthermore one should take into account that the sum over C in the right-hand side of (11.39) includes translations. There are no other modifications because we consider only the standard gauge transformations. The result is therefore

$$\delta(\epsilon)\mathcal{D}_\mu\phi = \epsilon^A\mathcal{D}_\mu(T_A\phi) - \epsilon^B\mathcal{M}_{\mu B}{}^A(T_A\phi) + \epsilon^B B_\mu^A f_{AB}{}^c\mathcal{D}_c\phi\,. \tag{11.80}$$

The last term can be recognized in the transformation of the vielbein:

$$\delta(\epsilon)e_\mu^c = \epsilon^B B_\mu^A f_{AB}{}^c + \epsilon^B e_\mu^a f_{aB}{}^c\,, \tag{11.81}$$

where we again distinguish standard gauge transformations and translations in the sum over B or b. Therefore, the result can be reexpressed as the transformation of $\mathcal{D}_a\phi$:

$$\delta(\epsilon)\mathcal{D}_a\phi = \epsilon^A\mathcal{D}_a(T_A\phi) - \epsilon^B\mathcal{M}_{aB}{}^A(T_A\phi) - \epsilon^A f_{aA}{}^b\mathcal{D}_b\phi\,. \tag{11.82}$$

This is exactly what we want and like. Indeed, $\mathcal{D}_\mu\phi$ is not a covariant quantity because it is not a world scalar, and thus transforms under gct with a derivative on the parameter ξ. On the other hand, $\mathcal{D}_a\phi$ is again a world scalar. Moreover, we see that the right-hand side of (11.82) does not contain explicit gauge fields (which do occur on the right-hand side of (11.80)). Hence, the transformation of $\mathcal{D}_a\phi$ is a covariant quantity.

This is a very important result that simplifies many calculations in supergravity. We find that standard gauge transformations commute with a $\mathcal{D}_a$ covariant derivative up to two type of terms. These are

- the terms generated by the extra transformations of gauge fields (last term in (11.66)) hidden in the covariant derivative;
- for gauge symmetries whose commutator with translations leads to another translation, there is an action on $\mathcal{D}_a$ similar to the action on P_a in the algebra.

These are the terms without explicit standard gauge fields in the transformation of (11.62). The first term originates from the fact that the transformation of B_μ^A contains the last term in (11.66). The second term appears because the transformation of the (inverse) vierbein $e_a{}^\mu$ in (11.62) gets a contribution from the last term in (11.81), which does not involve a standard gauge field:

$$\begin{aligned} \delta e_a^\mu &= -e_a^\nu\epsilon^B e_\nu^c f_{cB}{}^b e_b^\mu + \dots\,, \\ (\delta e_a^\mu)\mathcal{D}_\mu &= -\epsilon^B f_{aB}{}^b\mathcal{D}_b\,. \end{aligned} \tag{11.83}$$

Hence, the result can be obtained from the rules given in Sec. 11.3.4.

11.A.2 Examples in supergravity

We illustrate here the use of the tricks with covariant derivatives and curvatures in a setting that is a bit advanced for the readers of this chapter because it uses supergravity.

The first exercise shows how useful the lemma is.

Ex. 11.13 *Consider the scalars Z of chiral multiplets. We assume moreover that they transform under a Yang-Mills gauge group as $\delta_{\rm YM}Z = -\theta^A t_A Z$. This means that the full set of gauge transformations contains (covariant) general coordinate transformations and as standard gauge transformations: supersymmetry, Lorentz transformations and Yang-Mills transformations. The definition of the covariant derivative on Z is therefore*

$$\mathcal{D}_a Z = e_a{}^\mu \left(\partial_\mu Z - \frac{1}{\sqrt{2}} \bar{\psi}_\mu P_L \chi + A_\mu^A t_A Z \right) . \qquad (11.84)$$

Prove that (assuming that the gravitino has only the gauge transformations that follow from the algebra) this leads to

$$\begin{array}{rcl} \delta(\epsilon)\mathcal{D}_a Z &=& \frac{1}{\sqrt{2}}\bar{\epsilon}\mathcal{D}_a P_L\chi - \theta^A t_A \mathcal{D}_a Z \\ && +\tfrac{1}{2}\bar{\epsilon}\gamma_a\lambda^A t_A Z \\ && -\lambda_a{}^b \mathcal{D}_b Z \, . \end{array} \qquad (11.85)$$

The first line is easy. It follows from commuting transformations and covariant derivatives. The second line is due to the extra term in the supersymmetry transformation of the gauge vector, and the third line is the one due to transformation of the frame field as written above.

Ex. 11.14 *To illustrate (11.71), show that in a supersymmetric theory the covariant field strength of a vector in an abelian vector multiplet is*

$$\widehat{F}_{ab} = e_a{}^\mu e_b{}^\nu \left(2\partial_{[\mu} A_{\nu]} + \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda \right) . \qquad (11.86)$$

We now demonstrate the calculation of the transformation of a covariant field strength by calculating the supersymmetry transform of $\widehat{F}_{ab}$. The first principle implies that this is a covariant quantity. Though we have not yet derived the coupling of the gauge multiplet to supergravity, we assume that the full supersymmetry rule of the gauge field A_μ is the one given in (8.41). Furthermore let us assume that the gravitino has, together with the transformation determined by the algebra, an extra term denoted by Υ_μ. Its form is irrelevant for this exercise:

$$\delta_{\rm susy}(\epsilon)\psi_\mu = \left(\partial_\mu + \tfrac{1}{4}\gamma^{ab}\omega_{\mu ab} \right) \epsilon + \Upsilon_\mu \epsilon \, . \qquad (11.87)$$

When considering the transformation of A_ν in the first term of (11.86), we can ignore terms in which the derivative ∂_μ acts on the gauge parameter $\bar{\epsilon}^\alpha$. Indeed, principle 2 implies that such terms must cancel. The derivative thus acts only on the gaugino λ. Since the supersymmetry algebra is supposed to be closed on the vector multiplet, the third principle implies that this derivative should become a covariant derivative. This leads to

$$\delta_{\text{susy}}(\epsilon)\widehat{F}_{ab} = \bar{\epsilon}\gamma_{[a}\mathcal{D}_{b]}\lambda + \ldots\,. \qquad (11.88)$$

Now we still have to consider the second term in (11.86). An important simplification is that we do not have to act with δ_{susy} on λ. Indeed, such terms would leave an explicit ψ_μ in the result, and the principle 3 says that such terms do not appear. Thus, we need only consider (11.87). With the same principles, one sees that only the last term is relevant as the other contain a derivative on the parameter or an explicit gauge field $\omega_{\mu ab}$. Hence the final result is

$$\delta_{\text{susy}}(\epsilon)\widehat{F}_{ab} = \bar{\epsilon}\gamma_{[a}\mathcal{D}_{b]}\lambda + \bar{\lambda}\gamma_{[a}\Upsilon_{b]}\epsilon\,. \qquad (11.89)$$

The same principles also determine the modification due to supergravity of the transformation law of the gaugino given in (8.41). The modified transformation law will contain the covariant curvature, i.e.

$$\delta\lambda = \left[\tfrac{1}{4}\gamma^{ab}\widehat{F}_{ab} + \tfrac{1}{2}\mathrm{i}\gamma_* D\right]\epsilon\,. \qquad (11.90)$$

Since we have already calculated the supersymmetry transformation of $\widehat{F}_{ab}$, it is easy to calculate the commutator of two supersymmetries on the gaugino. In fact, it can be compared to the calculation in the global case. If the student still has his/her calculation for exercise 8.13, he/she has only to replace each F by $\widehat{F}$ and D_μ by $\mathcal{D}_\mu$, and nothing changes, except for the contribution of the Υ-term to (11.89). This shows how these tricks are used in practice.

12
Survey of supergravities

We have meanwhile seen a few supergravity theories and have written down some general aspects of gauge theories that may include local supersymmetry. We also studied in Ch. 3 the properties of spinors in various dimensions. All this information is sufficient to get an idea of which supergravity theories can exist and what are the general properties that these possess. This will be the subject of the present chapter.

12.1 Minimal and extended superalgebras

The minimal supersymmetry algebra is the one that we saw in (11.17):

$$\{Q_\alpha, Q_\beta\} = -\tfrac{1}{2}(\gamma^a)_{\alpha\beta}P_a \,. \tag{12.1}$$

The supersymmetries commute with translations and are a spinor of Lorentz transformations. We have seen that in many cases the final algebra is more complicated due to the appearance of structure functions, but let us first restrict ourselves to strict algebras, i.e. the ones that are obtained for 'vanishing fields' or constant fields.

12.1.1 4 dimensional minimal algebras

Let us first consider the algebras in $D = 4$. The supercharges Q_α are Majorana spinors and we can also choose to represent them by the chiral projections P_LQ, and P_RQ is then the charge conjugate of the P_LQ. When one considers the commutator including the parameters as in (11.16), it is clear that the parameters ϵ_1 and ϵ_2 should have opposite chirality in 4 dimensions (use (3.54)). The corresponding calculation with spinor indices uses

$$(P_LQ)_\alpha = (P_L)_\alpha{}^\gamma Q_\gamma = -t_0 Q_\gamma (P_L)^\gamma{}_\alpha \,, \tag{12.2}$$

which is based on (3.61) with $t_0 = t_4$ (and that γ_* involves a $\gamma^{(4)}$). Hence if we multiply (12.1) at the left-hand side with P_L, then the right-hand side is proportional to $P_L\gamma^a = \gamma^a P_R$, which shows that the Q_β should be right-handed. Thus the non-vanishing part of (12.1) involves only the anticommutator between two supercharges of opposite chirality, i.e.

$$\{(P_LQ)_\alpha, (P_RQ)_\beta\} = -\tfrac{1}{2}(P_L\gamma^a)_{\alpha\beta}P_a\,. \tag{12.3}$$

Extended superalgebras means that there are several supercharges, with a label $i = 1, \ldots, \mathcal{N}$. It is in that case convenient to use the position of the index i up or down to indicate at once the chiral projections of the supersymmetry generators, thus

$$Q_i = P_LQ_i\,, \qquad Q^i = P_RQ^i\,. \tag{12.4}$$

The Majorana spinors are thus $Q^i + Q_i$, and Q_i is the charge conjugate of Q^i. The minimal extended algebra is then

$$\left\{Q_{\alpha i}, Q_\beta{}^j\right\} = -\tfrac{1}{2}(P_L\gamma^a)_{\alpha\beta}\delta_i^j P_a\,. \tag{12.5}$$

12.1.2 Central charges in 4 dimensions

For extended supersymmetry the algebra can be modified by '*central charges*'. Central charges appeared first in the classical work of Haag–Łopuszański–Sohnius [48]. The simplest example occurs for $\mathcal{N} = 2$, where (12.5) can be extended with a nonvanishing commutator for two supercharges of the same chirality:

$$\{Q_{\alpha i}, Q_{\beta j}\} = \varepsilon_{ij}P_{L\alpha\beta}Z\,, \qquad \left\{Q_\alpha{}^i, Q_\beta{}^j\right\} = \varepsilon^{ij}P_{R\alpha\beta}\bar{Z}\,. \tag{12.6}$$

The matrices P_L are antisymmetric in the spinor indices, and thus one needs an antisymmetric ε^{ij} for consistency of the superalgebra.

Ex. 12.1 *Check that these anticommutator relations are consistent with symmetry in $(i\alpha) \leftrightarrow (j\beta)$ and with chirality projections.*

The generators Z and its complex conjugate $\bar{Z}$ commute in this algebra with everything else and are thus really 'central' in the mathematical sense. Due to the presence of ε^{ij}, such an algebra is only possible for extended supersymmetry. In case of $\mathcal{N} = 4$ one could have 6 independent complex central charges (an antisymmetric 4×4 matrix).

These central charges are important for supersymmetric solutions. The latter means that Q applied to such a configuration gives zero. Consider the configuration in the rest frame, such that $P^a = E\delta_0^a$. Then, with the minimal algebra (12.5) this can only be consistent for a solution that has zero energy. In the presence of central charges that do not vanish on these solutions, the right-hand side of (12.5) can be cancelled by that of (12.6), and hence supersymmetric solutions of non-zero energy become possible.

12.1.3 Superalgebras in higher dimensions

When we consider the symmetry of (12.1) in $(\alpha\beta)$, we find from (3.61) that this is only consistent if $t_1 = -1$. This is according to (3.84) exactly the condition to allow (pseudo-) Majorana spinors. Furthermore, according to (3.54) the left-right projection in (12.3) is only consistent for 4 or 8 dimensions, but not e.g. for 10 dimensions. Indeed, in 4 or 8 dimensions we have the rule as in (12.2), while for 10 (or 6 dimensions) the γ_* involves 10 (or 6) γ matrices and thus we have to use $t_{10} = t_6 = t_2 = -t_0$, such that

$$(P_L Q)_\alpha = (P_L)_\alpha{}^\beta Q_\beta = -t_0 Q_\beta (P_R)^\beta{}_\alpha \,. \tag{12.7}$$

Moving the P_R past the γ^a it is again P_L, and thus for 10 or 6 dimensions the non-zero anticommutator acts between left and left chiral supersymmetries or between right and right supersymmetries. Hence, in these cases one can have the supersymmetry algebra with only left-handed or only right-handed supersymmetry. This is also consistent with the fact that in 10 dimensions Majorana spinors can be chiral (Majorana-Weyl spinors, see table 3.2). Thus in this case one can have a chiral supersymmetry algebra (called type I). When one has two supersymmetries, one can choose to have them of different chirality (called IIA) or of the same chirality (called IIB). As according to table 3.2 the minimal spinors are 16 dimensional, we cannot have more than 2 supersymmetries for a supergravity field theory in 10 dimensions. Indeed, remember the argument in Sec. 10.1 that dimensional reduction to 4 dimensions should not give more that $\mathcal{N} = 8$, which means that the spinor generators should have at most $4 \times 8 = 32$ real components.

In the cases that $t_1 = 1$, i.e. when there are no Majorana fermions, then (12.1) is not consistent with the symmetry of an anticommutator. This is so for dimensions 5, 6 and 7 when according to table 3.2 we have to use symplectic spinors. Hence in these cases there is an antisymmetric tensor ε^{ij} that appears in the reality condition, see (3.83). We can use this object then also to build an anticommutator in these cases and write a consistent algebra as

$$\left\{ Q^i_\alpha, Q^j_\beta \right\} = (\gamma^a)_{\alpha\beta} \varepsilon^{ij} P_a \,. \tag{12.8}$$

In the case of 6 dimensions, we mentioned already that a chiral projection of the supersymmetries is consistent. Hence in this case one can have algebras of the type (p, q) with p chiral and q antichiral supersymmetries. In order to stay in the limit of 32 supercharges, one should have that $p + q \leq 4$.

12.1.4 'Central charges' in higher dimensions

The name 'central charges' has been generalized to include other generators that can appear in the anticommutator of supersymmetries similar to (12.6), but which may not be Lorentz scalars. E.g. in $D = 11$ the properties of the spinors allow us to extend the anticommutator as [66]

$$\{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu + \gamma^{\mu\nu}_{\alpha\beta} Z_{\mu\nu} + \gamma_{\alpha\beta}^{\mu_1\cdots\mu_5} Z_{\mu_1\cdots\mu_5}\,. \tag{12.9}$$

The allowed structures on the right-hand side are determined by the last entry in table 3.2 (remember that the indications there are modulo 4, which thus allows the 5-index object). The ‘central charges’ Z are no longer Lorentz scalars, and thus do not commute with the Lorentz generators. They are therefore not ‘central’ in the group-theoretical meaning of the word, but play in the physical context the same role as the ones in (12.6). They also allow the construction of supersymmetric solutions by the mechanism mentioned in Sec. 12.1.2, and therefore got also the name ‘central charges’ despite their non-zero commutator with the Lorentz generators. The second and third terms in (12.9) are in this way important for the existence of M2 and M5 branes. In general, branes can be classified by considering the possibilities for central charges.

12.2 The R-symmetry group

R-symmetry is the name that is given to symmetries connecting the different supersymmetry generators and commuting with the Lorentz generators. The Lorentz generators connect the different supersymmetry generators according to

$$\left[M_{[ab]}, Q_\alpha{}^i\right] \;=\; -\tfrac{1}{2}(\gamma_{ab})_\alpha{}^\beta Q_\beta{}^i\,. \tag{12.10}$$

We now investigate the existence of other transformations T_A that mix the supersymmetries. We investigate this first in 4 dimensions. We parametrize the action of the T_A on the supersymmetries by matrices $(U_A)_i{}^j$ and $(U_A)^i{}_j$:

$$[T_A, Q_{\alpha i}] = (U_A)_i{}^j Q_{\alpha j}\,, \qquad \left[T_A, Q^i_\alpha\right] = (U_A)^i{}_j Q^j_\alpha\,. \tag{12.11}$$

The latter equation is related to the the former by charge conjugation, thus $(U_A)^i{}_j$ is the complex conjugate of $(U_A)_i{}^j$. The consistency should be determined by Jacobi identities. The Jacobi identity $[TTQ]$, i.e.[1]

$$[[T_A, T_B], Q] \;=\; [T_A, [T_B, Q]] - [T_B, [T_A, Q]] \qquad \to \qquad f_{AB}{}^C U_C = [U_B, U_A]\,, \tag{12.12}$$

imply that the matrices U_A form a representation of the algebra of the T_A. The generators T_A should commute with P_a as follows already from the Coleman–Mandula theorem, and as follows from the fact that they are external to the Poincaré algebra. This can be used in the $[TQ_iQ^j]$ Jacobi identity:

$$[T_A, \{Q_{\alpha i}, Q_\beta{}^j\}] \;=\; \{[T_A, Q_{\alpha i}], Q_\beta{}^j\} + \{[T_A, Q_\beta{}^j], Q_{\alpha i}\}\,. \tag{12.13}$$

[1] See that super-Jacobi identities are most easily written in this form: the first term on the right-hand side has the same order of the generators as the left hand side, but other arrangement of brackets, then the missing term should satisfy the obvious symmetry property, in this case antisymmetry in T_A versus T_B, in (12.13) symmetry between the two Q-operators.

This leads to

$$0 = (U_A)_i{}^j + (U_A)^j{}_i\,, \qquad \text{i.e.} \quad (U_A)_i{}^j = -(U_A)^j{}_i \equiv -\left((U_A)_j{}^i\right)^* . \tag{12.14}$$

Thus the matrices U are anti-Hermitian matrices. These are the generating matrices of $\mathrm{U}(\mathcal{N})$.

For $\mathcal{N}=1$ this leaves only the possibility that up to normalization $-U^1{}_1 = U_1{}^1 = \mathrm{i}/2$. This gives for the Majorana $Q = Q_L + Q_R = Q_1 + Q^1$

$$[T, Q] = \tfrac{1}{2}\mathrm{i}P_L Q - \tfrac{1}{2}\mathrm{i}P_R Q = \tfrac{1}{2}\mathrm{i}\gamma_* Q\,. \tag{12.15}$$

Hence, this $-2T$ corresponds to the generator T_R in (8.3).

We can make a general statement about the automorphism groups in view of table 3.2. The 4-dimensional example is general for even dimensions that have Majorana spinors, but no Majorana–Weyl spinors, i.e. 4 and 8 dimensions. In fact, in this formulation we have not used the Majorana property, but rather the Weyl-spinor formulation. Indeed, in all even dimensions we can use Weyl spinors, and this leads to spinors of the same dimension as the Majorana ones (they are the same in another notation).

When we have the possibility of Majorana–Weyl spinors (MW), then the reality condition says that the U for the left-chiral and those for the right-chiral spinors should each be real. The non-trivial $\{Q, Q\}$ anticommutator is in this case also between supersymmetries of equal chirality, which leads analogous to (12.14) now to a condition of antisymmetry for each of them separately. Therefore, we obtain $\mathrm{SO}(\mathcal{N}_L)\times\mathrm{SO}(\mathcal{N}_R)$.

For odd dimensions, we cannot use the chiral formulation. If there are Majorana spinors (M), the result is similarly $\mathrm{SO}(\mathcal{N})$. If there are symplectic Majorana (S) conditions, then the same argument of the Jacobi identity $[TQQ]$ leads to the preservation of the symplectic metric, and the automorphism algebra is thus reduced to $\mathrm{USp}(\mathcal{N})$ (with even $\mathcal{N}$). If there are symplectic Weyl spinors (SW), then there are two such factors, for the left and for the right-handed sector. In summary, the automorphism groups are obtained from the entries of table 3.2:

$$\begin{array}{lll}
D=10\ : & MW\ : & \mathrm{SO}(\mathcal{N}_L)\times\mathrm{SO}(\mathcal{N}_R)\,,\\
D=9\ : & M \text{ and } D \text{ odd}\ : & \mathrm{SO}(\mathcal{N})\,,\\
D=8 \text{ and } D=4\ : & M \text{ and } D \text{ even}\ : & \mathrm{U}(\mathcal{N})\,,\\
D=7 \text{ and } D=5\ : & S\ : & \mathrm{USp}(\mathcal{N})\,,\\
D=6\ : & SW\ : & \mathrm{USp}(\mathcal{N}_L)\times\mathrm{USp}(\mathcal{N}_R)\,.
\end{array} \tag{12.16}$$

12.3 Multiplets

12.3.1 On- and off-shell multiplets and degrees of freedom

It may already be clear to the reader from the construction of multiplets in Sec. 8.4 or from the explicit examples, that the multiplets have an equal number of bosonic and fermionic degrees of freedom. More strictly stated, the following theorem holds.[2]

> **Theorem:** *There are an equal number of bosonic and fermionic degrees of freedom in any realization of the supersymmetry algebra (12.1) when translations are an invertible operation.*

To interpret this theorem we remind the reader to the terminology of on-shell and off-shell degrees of freedom that we introduced on page 76. But first, let us explain the origin of this theorem.

Proof: Consider the commutator (12.1) on the space of all bosons. The first Q transforms the space to a space of fermions, while the second one brings us back to bosons, translated by P_μ.

$$\text{bosons} \xrightarrow{Q} \text{fermions} \xrightarrow{Q} \text{bosons translated by } P_\mu$$

The latter is an invertible operator, which proves that the last space is as large as the first one. Therefore, the middle one, the space of the fermions, should have at least the same dimension. Repeating the argument with bosons and fermions interchanged leads to the conclusion that the number of bosonic states must be at least equal to the number of fermionic ones. Combining the two results, proves the well-known fact that there are an equal number of bosonic and of fermionic states. ∎

The theorem is valid when the algebra (12.1) holds. So this equality of bosonic and fermionic states should hold e.g. for on-shell states. I.e. for the chiral multiplet, without the auxiliary field F we can apply this only for on-shell states. Then we count 2 bosonic states for the complex Z, and also the fermions have 2 on-shell degrees of freedom. So we have a $2+2$ on-shell multiplet. When the auxiliary field F is included, the algebra is also satisfied off-shell. Thus we have in this case the 4 off-shell degrees of freedom of χ, while the complex Z and F give together also 4 real off-shell degrees of freedom. In this case, we say that we have a $4+4$ off-shell multiplet. These two ways of counting are called *on-shell counting* and *off-shell counting*.

On the other hand, the equality of boson and fermion states is not a general fact of a superalgebra. One needs the invertibility of the square of fermionic generators.

For on-shell degrees of freedom it is clear that the gauge degrees of freedom are not included. But they should not be included in off-shell counting either. This is the way that we defined the number of off-shell degrees of freedom. Why we do so, should be clear from the algebra. The theorem holds only when the $\{Q,Q\} = P$ algebra holds. When there are gauge symmetries, we saw already in the simplest case (illustrated in (11.30)) that the algebra is modified by gauge transformations.

[2] The algebra (12.1) is part of the extended algebras, so that this theorem holds also for extended supersymmetry, and can be generalized also for the case of symplectic structures as in (12.8).

Thus, in order to have the minimal algebra we will have to eliminate the extra gauge terms. Therefore we have to consider only gauge-invariant states (or identify states that differ by a gauge transformation) in all countings. E.g., a gauge vector in 4 dimensions counts off-shell for 3 degrees of freedom, balancing the 4 off-shell ones of the gaugino and the 1 real degree of freedom of the auxiliary field D.

12.3.2 Multiplets in 4 dimensions

We mentioned already in Sec. 8.4 that in 4 dimensions multiplets of global supersymmetry are restricted to $\mathcal{N} \leq 4$ and those of local supersymmetry to $\mathcal{N} \leq 8$ by the restrictions of field theory. Here are the most important multiplets of on-shell fields.

The supergravity multiplet is the minimal multiplet that contains the graviton. It contains the set of fields that represents the spacetime susy algebra and has gauge fields for the supersymmetries. The number of fields is given in table 12.1. These are on-shell multiplets. The fields of spin $s > 0$ have 2

Table 12.1. *Pure supergravity multiplets in 4 dimensions according to spin* s. $\mathcal{N} = 7$ *gives the same result as* $\mathcal{N} = 8$.

s	$\mathcal{N}=1$	$\mathcal{N}=2$	$\mathcal{N}=3$	$\mathcal{N}=4$	$\mathcal{N}=5$	$\mathcal{N}=6$	$\mathcal{N}=8$
2	1	1	1	1	1	1	1
$\frac{3}{2}$	1	2	3	4	5	6	8
1		1	3	6	10	16	28
$\frac{1}{2}$			1	4	11	26	56
0				2	10	30	70

degrees of freedom (helicity $+s$ and $-s$). Therefore you can understand e.g. the 2 rather than 1 in the entry of spin 0 for $\mathcal{N} = 4$ to obtain an equal number of bosonic as fermionic degrees of freedom.

Vector multiplets are the multiplets with fields up to spin 1. They exist for $\mathcal{N} \leq 4$. The multiplet for $\mathcal{N} = 1$ is the one that we encountered in Sec. 8.1.2, and was called the gauge multiplet. Their content in general is given in table 12.2. The gauge fields of these multiplets can gauge an extra Yang-Mills group that is not contained in the supersymmetry algebra as we saw already in Sec. 8.3. In the final soft algebra, however, some mixing always occurs as we saw in (11.30).

Chiral and hypermultiplets are multiplets with only spins 0 and 1/2. For $\mathcal{N} = 1$ these are the chiral multiplets that we know from Sec. 8.2. For $\mathcal{N} = 2$ a similar multiplet is the 'hypermultiplet'. Table 12.2 contains the field content of these multiplets. They do not exist for $\mathcal{N} > 2$. These multiplets may also transform under the gauge group defined by the vector multiplet, as we have illustrated in Sec. 8.3.2. They form thus a representation of these gauge groups.

Table 12.2. *Matter multiplets in 4 dimensions*

s	$\mathcal{N}=1$	$\mathcal{N}=2$	$\mathcal{N}=3,4$
1	1	1	1
$\frac{1}{2}$	1	2	4
0		2	6

s	$\mathcal{N}=1$	$\mathcal{N}=2$
$\frac{1}{2}$	1	2
0	2	4

The vector and chiral multiplets are called *matter multiplets*. These exist for global supersymmetry, and they can be coupled to supergravity theories (which is then called 'matter-coupled supergravity').

Let us still remark that in global supersymmetry one can only have compact gauge groups if one requires positive kinetic energies, but in supergravity some non-compact gauge groups are possible without spoiling the positivity of the kinetic energies. However, the list of possible non-compact groups is restricted for any $\mathcal{N}$.

12.3.3 Multiplets in more than 4 dimensions

As we go along the exposition here, you might consult Table 12.3, which gives an overview of supersymmetric theories. We restrict ourselves here to dimensions $D \geq 4$, to Minkowski spacetimes and to theories with positive definite kinetic terms. The most relevant source in this respect is the paper of Strathdee [67] that analyses the representations of supersymmetries.

We saw in Sec. 10.2 how the fields of the $\mathcal{N} = 8$ supergravity multiplet are all contained in the simpler multiplet in 11 dimensions. The relation was made by dimensional reduction on the torus. Remember that the number of components of the supersymmetry generators did not change in such a torus dimensional reduction. This lead also to the argument that 11 dimensions is the highest dimension [68] in which supergravity can be constructed.

We started in Sec. 10.2 with the 32-component supersymmetry in 11 dimensions, and ended up with $\mathcal{N} = 8$ in 4 dimensions, which contains again $8 \times 4 = 32$ real components. Hence we may consider the multiplets that we saw above in 4 dimensions, and consider whether they can be obtained in other ways from higher-dimensional multiplets.

The supergravity multiplet for 32 supersymmetries. Let us start with the biggest one, the $\mathcal{N} = 8$. We saw already that this can be obtained from simple supergravity in 11 dimensions, that we studied in Ch. 10. The 11-dimensional theory [61] is the basis of 'M-theory', and is therefore indicated as M in the table. Let us start now from this one, and dimensionally reduce it on tori. In this way we will get supergravity multiplets in any $D < 11$ with 32 real supersymmetries. The three fields, graviton, gravitino and 3-index gauge field are representations of the Lorentz group in 11 dimensions, and split in different representations in the lower dimension, as we did it explicitly in Sec. 10.2 for the reduction to 4 dimensions. Here, we will concentrate how the supersymmetry generators reduce in lower dimensions similar to the reduction of the 11 dimensional $\mathcal{N} = 1$ supersymmetry to $\mathcal{N} = 8$ in 4 dimensions.

When we reduce to 10 dimensions, the supersymmetry of 11 dimensions will split in a left and right chiral one (each of 16 components), and thus the multiplet becomes the supergravity multiplet for type IIA supergravity in 10 dimensions. This is the theory of the massless sector of IIA string theory, and that is why we have indicated it as IIA [69, 70, 71].

The other theory with 32 supercharges in 10 dimensions is the one with two supercharges of the same chirality [72, 73, 74]. It cannot be obtained by dimensional reduction from 11 dimensions, as indicated by its place in Table 12.3. It is the massless sector of IIB superstring theory involves doublets of spinors of the same chirality (thus also with 32 real supersymmetries).

In 9 dimensions, an irreducible spinor is a 16-component Majorana spinor. Thus, the theories above reduce to $\mathcal{N} = 2$ in $D = 9$. In odd dimensions there is no chirality, and the theory is therefore unique. Indeed, it turns out that the IIA and IIB supergravities reduce to the same supergravity in 9 dimensions. The fact that both these supergravities are mapped to the same $D = 9$ theory is a basic ingredient in the understanding of dualities between superstring theories.

Also in 8 dimensions the irreducible spinors are 16-component Majorana spinors, and thus the supergravity theory with 32 supersymmetries reduces to $\mathcal{N} = 2$. For 7 dimensions, the irreducible spinors are symplectic Majorana spinors and we need supersymmetry generators Q^i with $i = 1, \ldots, 4$ to describe the 32 supercharges.

In 6 dimensions, the spinors are symplectic Majorana-Weyl. We use here the notation (p, q) theories, which indicates e.g. for the left-chiral Q_i that $i = 1, \ldots, 2p$ and the right-handed Q^i exist for $i = 1, \ldots, 2q$. Left and right is just a convention, i.e. a (p, q) theory is the same as a (q, p) theory. As each of these chiral generators have 4 components, the supergravities that we describe here should have $p + q = 4$. The dimensional reduction from the above theories gives a $(2, 2)$ theory. The $(3, 1)$ and $(4, 0)$ theories exist in principle [75]. However, the graviton field is not represented by a metric tensor $g_{\mu\nu}(x)$, but by a more complicated tensor field [76]. Thus, these theories are different in the sense that they are not based on a dynamical metric tensor. They have not been constructed beyond the linear level.

Dimensional reduction to 5 dimensions gives a theory with supersymmetries Q^i that are symplectic Majorana and $i = 1, \ldots, 8$, while dimensional reduction to the

$\mathcal{N} = 8$ theory in 4 dimensions was discussed in Sec. 10.2.

When there are 32 supersymmetries, the supergravity multiplet for any theory is unique and this multiplet is only known on-shell, i.e. it is not known (and there are no-go theorems) how to add auxiliary fields to obtain off-shell closure.

The supergravity multiplet for less than 32 supersymmetries. Only in 4 dimensions (and lower, but these we do not discuss here) we can consider a multiple of 4 supersymmetries that is not a multiple of 8. We saw already in Table 12.1, that $\mathcal{N} = 7$, i.e. 28 supersymmetries in 4 dimensions has the same field content as $\mathcal{N} = 8$. In fact, it turns out that if one constructs in 4 dimensions a field theory with $\mathcal{N} = 7$, then it automatically has an eighth local supersymmetry. That is why it is not mentioned in the table.

24 real supercharges are possible for dimensions up to 6. For the 6-dimensional case the possibilities are similar situation as those discussed for 32 supersymmetries. In a $(2, 1)$ theory the graviton is described by a metric tensor, but for a $(3, 0)$ theory one needs another representation of the Lorentz group, and there is no non-linear action known. In 4 and 5 dimensions these are the $\mathcal{N} = 6$ theories.

With 16 real supersymmetries one can go up to 10 dimensions, where this is one chiral Majorana spinor. The supergravity theory reduces to $\mathcal{N} = 4$ in 4 dimensions, and in the dimensions in-between to the theories mentioned in Table 12.3, which the reader can now understand using the arguments mentioned for 32 supersymmetries. In 6 dimensions there is also a $(2, 0)$ supergravity. It contains a self-dual antisymmetric tensor, such that (similar to the IIB theory in $D = 10$) the construction of an action is not straightforward.

With 8 supersymmetries, we can have the minimal theories in 5 and 6 dimensions (with symplectic Majorana or symplectic Majorana-Weyl spinors respectively), and the $\mathcal{N} = 2$ theory in 4 dimensions. Finally with 4 supersymmetries there is the gravitational multiplet that we discussed in Ch. 9.

Vector multiplets With 16 supersymmetries or less, global supersymmetry is possible. The highest dimension is $D = 10$ which allows a vector multiplet containing a vector and one Majorana-Weyl spinor. This multiplet can be reduced along the vertical line for 16 supersymmetries in Table 12.3. It leads finally to the vector multiplet with $\mathcal{N} = 4$ in Table 12.2.

There are also vector multiplets for less than 16 supersymmetries. Similar to the situation with $\mathcal{N} = 7$ in supergravity, if one constructs a global supersymmetric theory with $\mathcal{N} = 3$ in 4 dimensions, it automatically has a fourth supersymmetry. However, in this case, there is the possibility of having only three of the four supersymmetries local. Thus $\mathcal{N} = 3$ is only meaningful in supergravity. This explains the lowest line of the table.

For 8 supersymmetries, the vector multiplet can be constructed for $D = 6, 5$ and 4. In 6 dimensions it has just a vector and a symplectic-chiral spinor. Reducing it to 5 dimensions gives a multiplet that has also a real scalar, and in $\mathcal{N} = 2$ in

4 dimensions it has a complex scalar, remnant of the two components of the 6-dimensional vector in the compactified directions. The vector multiplets for $\mathcal{N}=1$ in 4 dimensions were already treated in Ch. 8.

Tensor multiplets There exist also multiplets with fields that are antisymmetric tensors $T_{\mu\nu}$. We have seen in Sec. 6.7 that in 4 dimensions an antisymmetric tensor is equivalent to a scalar, and thus we can omit them and refer to multiplets with scalars. This is at least true when the antisymmetric tensors have the standard kinetic terms as described in (6.61). We will here restrict to this case, but one should be aware that more general theories do exist when antisymmetric tensor fields cannot be dualized.

In 5 dimensions, an antisymmetric tensor is dual to a vector. Therefore, again for the trivial kinetic terms, we can again omit them. However in 'gauged supergravities', see below, the antisymmetric tensors play a non-trivial role.

In 6 dimensions, however, antisymmetric tensors can have self-dual properties. They are therefore physically different fields than the vectors or scalars. They can be real and self-dual (which means in fact that their field strength, the 3-form, is self-dual). They do appear in some supergravity multiplets as we mentioned above. Matter multiplets (multiplets that do not contain the graviton) with antisymmetric tensors, are called 'antisymmetric tensor multiplets'. They exist e.g. for minimal supersymmetry, (1,0). The non-chiral (1,1) supersymmetry does not allow such multiplets, but they do occur for (2,0) supersymmetry. On the other hand, (2,0) supersymmetry does not allow vector multiplets.

Scalar multiplets Finally, there are the multiplets that have only scalars and spinors, as the chiral and hypermultiplets in 4 dimensions. For more than 4 dimensions, these are of the type of the hypermultiplets of $\mathcal{N}=2$.

12.4 Supergravity theories: towards a catalogue

The analysis of Sections 12.1.3 and 12.3.3 indicate the possible dimensions and extensions for supergravity theories, which are the non-empty entries in Table 12.3. As mentioned before, we consider theories with a Minkowski metric and positive definite kinetic terms. We also restrict the Lagrangians to have not more that one spacetime derivative in fermion terms or two spacetime derivatives for the bosonic kinetic terms.

12.4.1 The basic theories and kinetic terms

The table that we discussed, indicates the basic theories. Their field content is determined by what is indicated, but in that table we do not discriminate different actions with the same field content. For most of the theories, the kinetic terms are determined by the field content. In fact this is true for all theories with more than

8 real supercharges. All the supergravity theories with more than 16 supercharges have unique kinetic terms for every entry in the table. E.g. we had not any choice in the determination of the kinetic terms of the $D = 11$ supergravity in Ch. 10.

For the theories with 16 supercharges, we have the choice of the number of vector multiplets coupled to supergravity (or tensor multiplets when we consider (2,0) in 6 dimensions), but once we give this number, the kinetic terms of the supergravity theory are determined. One can also have global supersymmetric field theories with these multiplets. Also for these the kinetic terms are fixed once the field content is given.

Theories with 8 or 4 supersymmetries are not fixed by the discrete choices of number of multiplets and gauging. In these cases the model depends on some functions that can vary by infinitesimal variations. It is in these models that auxiliary fields are most useful.

Finally remember, that in the table we have neglected the multiplets that involve antisymmetric tensors $A_{\mu\nu}$ for dimensions 5 and 4, where they are dual to vectors and scalars as long as one does not involve 'gaugings' as explained in the next section. E.g. for the chiral multiplet that we saw in Sec. 8.2, there is an alternative where one of the scalars is replaced by an antisymmetric tensor.

12.4.2 Deformations and gauged supergravities

Up to here, the theories that we described are determined by their kinetic terms. They all exist in a version without a potential for the scalar fields, nor a cosmological constant. These we call the undeformed, or basic, theories. These are all classified and well-known. However, there are many deformations. Most of these are known as gauged supergravities, but there are also other deformations. E.g. for $\mathcal{N} = 1$, the addition of a superpotential is a deformation unrelated to gauging. Similarly a massive theory of $D = 10$ supergravity was already found by Romans [77] in 1985, and meanwhile several supergravities have been found that have a version with a cosmological constant.

The most common deformations are the so-called gauged supergravities. In string theory they appear when branes carry fluxes. From the supergravity point of view, they are the theories in which some of the vectors gauge a non-Abelian group. The deformation is then proportional to the gauge coupling constant. The number of generators of the gauge group is equal to the number of vector fields.[3] This counting includes as well vectors in the supergravity multiplet as those in vector multiplets. In general in supergravity the kinetic terms of these vectors are mixed, in the sense that the matrices f_{AB} in (4.62) are extended to the non-Abelian case, and are non-diagonal between both type of vectors. Therefore, we should not make a distinction between the vectors in the supergravity multiplets and those in gauge multiplets.

[3] One could say less or equal, but considering that any vector transforms at least as $\delta A_\mu = \partial_\mu\theta$, even if the θ transformation does not act on anything else, we can say that there is a U(1) factor.

The gauge group is in principle arbitrary, but the requirement of positive kinetic terms gives restrictions on possible non-compact gauge groups for any supergravity.

The operations of this group can act on various fields. The easiest example is the one that we encountered in $\mathcal{N} = 1$, when we considered the chiral multiplet in SUSY gauge theories, see Sec. 8.3.2. We saw then that the transformations of the gauge group can act on the chiral multiplet. In a global supersymmetric theory these gauge groups commute with the supersymmetries. However, in supergravity the gauge groups always mix with the R-symmetry group, which means that they do not commute with supersymmetry.

Such gaugings have various consequences:

1. There are extra supersymmetry transformations of the fermions. In global supersymmetry we can see this because the auxiliary field D of the vector multiplet gets a non-zero value. These terms in the fermion supersymmetry transformations are sometimes called 'fermion shifts'.

2. A scalar potential is generated. As we will see later, this is related to the previous item as the potential is built out of the squares of the fermion shifts.

3. Similarly, there are other new terms in the action like fermion mass terms, see e.g. (8.50), which becomes a mass term if Z gets a non-zero vacuum value.

One can consider the deformation in the following way. The basic theories lead to the kinetic terms for the scalars. The latter have isometries, according to our discussion in Sec. 6.11. These isometries are global symmetries. It turns out that they nearly always extend to global symmetries of the full supergravity action. E.g. for the 4-dimensional theories they have an embedding in the symplectic group of dualities (see Sec. 4.2). There are generalizations of this mechanism for other dimensions. A subgroup of these global symmetries can be promoted to a gauge group[4]

For the basic theories we could restrict ourselves to one field realization of physical particles with a certain spin. E.g. in 5 dimensions, an antisymmetric tensor is dual to a vector, and we can omit it. But this does not hold anymore for the deformed theories, e.g. the gauged supergravities. This can be understood by realizing that spin-1 fields A_μ^A transform always in the adjoint representation of the gauge group. However, the antisymmetric tensors allow other representations. Hence, by considering also these fields, one can get more general gauged theories. One can also understand this by realizing that the duality between antisymmetric tensors and vectors holds only for the massless case. The gauging allows us to generate mass terms.

The last years, various new results have been obtained in this direction [34, 35], which are in the case of $D = 4$ based on an extension of the idea of duality invariance (introducing also magnetic gauge fields). Another approach starts from

[4] There are some extra requirements, e.g. that there are enough gauge fields, but we just give the general ideas in this section.

infinite-dimensional symmetries [78, 79, 80, 81, 82, 83]. A complete catalogue of deformations of the supergravity theories is not yet known. However, we believe that all supersymmetric field theories (with a finite number of fields and field equations that are at most quadratic in derivatives) belong to one of the entries in table 12.3.

12.5 General characteristics of the actions

The full action of a supergravity theory is very complicated. It contains 4-fermion couplings, couplings between fermions and vectors (as dipole moments), ... We show here some general structure of the bosonic terms in 4 dimensions. The basic theories contain (we thus do not consider antisymmetric tensors) the graviton, represented by the vierbein e_μ^a, a number of vectors A_μ^A with field strengths $F_{\mu\nu}^A$, a number of scalars φ^u, $\mathcal{N}$ gravitinos ψ_μ^i, and a number of spin 1/2 fermions. The pure bosonic terms of such an action are[5]

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm bos} &= \tfrac{1}{2}R + \tfrac{1}{4}(\mathrm{Im}\,\mathcal{N}_{AB})F_{\mu\nu}^A F^{\mu\nu B} - \tfrac{1}{8}(\mathrm{Re}\,\mathcal{N}_{AB})e^{-1}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^A F_{\rho\sigma}^B \\ &\quad -\tfrac{1}{2}g_{uv}(\varphi)\,D_\mu\varphi^u\,D^\mu\varphi^v - V(\varphi)\,. \end{aligned} \tag{12.17}$$

The determinant of the frame field is an overall factor, which we moved to the left hand side. The first term gives the pure gravity action. Then there are the kinetic terms for the spin-1 fields,[6] which we discussed already in section 4.2. The scalars have kinetic terms determined by a symmetric tensor $g_{uv}(\varphi)$. This is the main ingredient for the identification of geometries, which we will treat in section 12.6. The scalars couple 'minimally' to the vectors with a covariant derivative for the gauge symmetries as in (8.45). Finally, there is the potential $V(\varphi)$ for the scalars with properties that are determined by the supersymmetry.

12.6 Scalars and geometry

We finally consider the kinetic terms of the scalars in supersymmetric theories. the scalars have usually an action of the form of a nonlinear σ-model as we discussed it in Sec. 6.10. We saw their that the values of the scalars span a Riemannian manifold, M_n, with dimension equal to the number of scalars in the model. The metric on this manifold is the matrix $g_{uv}(\varphi)$ that defines the kinetic energies of the scalars. This metric is part of the definition of the theory, in the sense that it is a given function before we start deriving Euler-Lagrange equations from the action. This is to be contrasted with $g_{\mu\nu}(x)$, the metric of spacetime, which is, together with $\varphi^u(x)$, a dynamical field describing the graviton.

---

[5] We use here the notation $\mathcal{N}_{AB}$ for the metric of the vectors, as it is mostly used in extended supersymmetry. This matrix thus replaces $f_{AB} = \mathrm{i}\mathcal{N}_{AB}$ that we used in section 4.2.

[6] Some theories may have more general terms, called Chern-Simons terms, which have some similarities with those that we saw for $D = 11$ in Sec. 10.3.

The scalar manifold can have isometries, i.e. symmetries of the induced metric $\mathrm{d}s^2 = g_{uv}(\varphi)\,\mathrm{d}\varphi^u\,\mathrm{d}\varphi^v$. We discussed in Sec. 6.11 how they are generated by 'Killing vectors'. In many cases all points in the manifold M_n (any scalar field configuration) can be reached from any other point by a symmetry operation. In that case, the manifold is called a *homogeneous manifold*. In order that locally the symmetries connect all neighbouring points, there should be n independent Killing vectors at any point. The example that we saw in Ex. 6.44 is of this type. The manifold is 2-dimensional. There are 3 Killing vectors, but at any point in the domain of the scalar fields ($\mathrm{Im}\, Z > 0$) only two of them are independent. There is one generator that leaves a point invariant. The subgroup of the isomorphism group G that leaves a point invariant is called the isotropy group H. The manifold in this case can be identified with the coset G/H.

Ex. 12.2 *Take an arbitrary point in the manifold of Ex. 6.44, and find the isotropy generator.*

Ex. 12.3 *Why do the isotropy generators define a group? How do you associate the manifold to the coset space?*

In the example, the structure of the algebras $\mathfrak{g}$ of the group G and $\mathfrak{h}$ of the group H has some special properties. We can define a complementary space $\mathfrak{k}$ to $\mathfrak{h}$ such that any $g \in \mathfrak{g}$ can be written as

$$g = h + k\,, \qquad h \in \mathfrak{h}\,, \qquad k \in \mathfrak{k}\,, \tag{12.18}$$

and for any $h_1, h_2 \in \mathfrak{h}$ and $k_1, k_2 \in \mathfrak{k}$

$$[h_1,\, h_2] \in \mathfrak{h}\,, \qquad [h_1,\, k_1] \in \mathfrak{k}\,, \qquad [k_1,\, k_2] \in \mathfrak{h}\,. \tag{12.19}$$

When we use the Cartan-Killing metric in the algebra $\mathfrak{g}$, the decomposition is orthogonal. Any simply connected homogeneous space for which the isomorphism and isotropy algebra have this structure is called a *symmetric space*. Such spaces are also characterized by the fact that the curvature tensor is covariantly constant.

Ex. 12.4
1. *Check that the example used above is a symmetric space.*
2. *Which of the equations (12.19) are specific for a symmetric space, and how should we write them for a general homogeneous space?*

It turns out that the scalar manifolds of all supergravities with more than 8 supersymmetries are symmetric of the type that G is a non-compact group and H is its maximal compact subgroup. These are shown in table 12.4. Notice that H contains always the R-symmetry group, and has another factor in case that there are matter multiplets.

The theories with 4 supersymmetries ($\mathcal{N} = 1$ in 4 dimensions, but one might also consider lower-dimensional theories) are in general not of this form. The manifold can be an arbitrary Kähler geometry, a geometry with a (closed) complex structure. In supergravity, these Kähler manifolds have an extra property and are then called Kähler-Hodge manifolds.

Beautiful structures emerge for theories with 8 supercharges ($\mathcal{N}=2$ if in $D=4$). These theories all belong to a class that was baptized *special geometries* [84, 85], including some real [86], some Kähler geometries [9] and all the quaternionic geometries [87]. Especially, the scalars that by supersymmetry are directly related to vectors have a geometrically distinct structure, special Kähler geometry [9]. This is a subclass of the Kähler geometries discussed above, with an extra symplectic symmetry structure related to the duality transformations of the vectors shown in section 4.2. Scalars in hypermultiplets exhibit quaternionic structures, with many relations with special Kähler manifolds [88, 89].

Specifically, the manifolds that occur in supergravity actions are

$$\begin{aligned} D=6 &\ :\ \frac{\mathrm{O}(1,n)}{\mathrm{O}(n)} \times \text{quaternionic-Kähler manifold}, \\ D=5 &\ :\ \text{very special real manifold} \times \text{quaternionic-Kähler manifold}, \\ D=4 &\ :\ \text{special Kähler manifold} \times \text{quaternionic-Kähler manifold}. \end{aligned} \tag{12.20}$$

These geometries determine the general couplings of supergravity to matter multiplets in $D=6$ [90, 91], $D=5$ [92] and $D=4$ [93, 94]. There exist also versions of these geometries for global supersymmetry, leading to rigid (the terminology 'rigid' is here more common than the terminology global that we use in this book) Kähler manifolds [95, 96] and hyper-Kähler manifolds.

Table 12.3. *Supersymmetry and supergravity theories in dimensions 4 to 11.* An entry represents the possibility to have supergravity theories in a specific dimension D with the number of (real) supersymmetries indicated in the top row. We first repeat for every dimension the type of spinors that can be used. Every entry allows different possibilities. Theories with more than 16 supersymmetries can have different gaugings. Theories with up to 16 (real) supersymmetry generators allow 'matter' multiplets. The possibility of vector multiplets is indicated with $\heartsuit$. Tensor multiplets in $D=6$ are indicated by $\diamondsuit$. Multiplets with only scalars and spin-$\frac{1}{2}$ fields are indicated with $\clubsuit$. At the bottom is indicated whether these theories exist only in supergravity, or also with just global supersymmetry.

D	susy	32	24	20	16	12	8	4
11	M	M						
10	MW	IIA | IIB			I $\heartsuit$			
9	M	$\mathcal{N}=2$			$\mathcal{N}=1$ $\heartsuit$			
8	M	$\mathcal{N}=2$			$\mathcal{N}=1$ $\heartsuit$			
7	S	$\mathcal{N}=4$			$\mathcal{N}=2$ $\heartsuit$			
6	SW	$(2,2)$ | $(3,1)$ | $(4,0)$	$(2,1)$ | $(3,0)$		$(1,1)$ $\heartsuit$ | $(2,0)$ $\diamondsuit$		$(1,0)$ $\heartsuit, \diamondsuit, \clubsuit$	
5	S	$\mathcal{N}=8$	$\mathcal{N}=6$		$\mathcal{N}=4$ $\heartsuit$		$\mathcal{N}=2$ $\heartsuit, \clubsuit$	
4	M	$\mathcal{N}=8$	$\mathcal{N}=6$	$\mathcal{N}=5$	$\mathcal{N}=4$ $\heartsuit$	$\mathcal{N}=3$ $\heartsuit$	$\mathcal{N}=2$ $\heartsuit, \clubsuit$	$\mathcal{N}=1$ $\heartsuit, \clubsuit$
		SUGRA			SUGRA/SUSY	SUGRA	SUGRA/SUSY	

Table 12.4. *Scalar geometries in theories with more than 8 supersymmetries (and dimension ≥ 4).* The theories are ordered as in table 12.3. Note that the R-symmetry group, mentioned in (12.16), is always a factor in the isotropy group. For more than 16 supersymmetries, there is only a unique supergravity (up to gaugings irrelevant to the geometry), while for 16 and 12 supersymmetries there is a number n indicating the number of vector multiplets that are included.

D	32	24	20	16	12
10	$\mathrm{O}(1,1) \;\Big|\; \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$				
9	$\frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} \otimes \mathrm{O}(1,1)$			$\frac{\mathrm{O}(1,n)}{\mathrm{O}(n)} \otimes \mathrm{O}(1,1)$	
8	$\frac{\mathrm{SL}(3)}{\mathrm{SU}(2)} \otimes \frac{\mathrm{SL}(2)}{\mathrm{U}(1)}$			$\frac{\mathrm{O}(2,n)}{\mathrm{U}(1)\times \mathrm{O}(n)} \otimes \mathrm{O}(1,1)$	
7	$\frac{\mathrm{SL}(5)}{\mathrm{USp}(4)}$			$\frac{\mathrm{O}(3,n)}{\mathrm{USp}(2)\times \mathrm{O}(n)} \otimes \mathrm{O}(1,1)$	
6	$\frac{\mathrm{O}(5,5)}{\mathrm{USp}(4)\times \mathrm{USp}(4)} \;\Big|\; \frac{\mathrm{F}_4}{\mathrm{USp}(6)\times \mathrm{USp}(2)} \;\Big|\; \frac{\mathrm{E}_6}{\mathrm{USp}(8)}$	$\frac{\mathrm{SU}^*(4)}{\mathrm{USp}(4)} \;\Big|\; \frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)}$		$\frac{\mathrm{O}(4,n)}{\mathrm{O}(n)\times SO(4)} \otimes \mathrm{O}(1,1) \;\Big|\; \frac{\mathrm{O}(5,n)}{\mathrm{O}(n)\times \mathrm{USp}(4)}$	
5	$\frac{\mathrm{E}_6}{\mathrm{USp}(8)}$	$\frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)}$		$\frac{\mathrm{O}(5,n)}{\mathrm{USp}(4)\times \mathrm{O}(n)} \otimes \mathrm{O}(1,1)$	
4	$\frac{\mathrm{E}_7}{\mathrm{SU}(8)}$	$\frac{SO^*(12)}{\mathrm{U}(6)}$	$\frac{\mathrm{SU}(1,5)}{\mathrm{U}(5)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,n)}{\mathrm{SU}(4)\times \mathrm{SO}(n)}$	$\frac{\mathrm{SU}(3,n)}{\mathrm{U}(3)\times \mathrm{SU}(n)}$

13
Complex Manifolds

The material of this chapter is largely mathematical, although we will include several physically motivated examples of the geometric constructions.. The major prerequisite is Chapter 6 on the differential geometry of real manifolds. A complex manifold is a real manifold of even dimension $2n$ on which one can choose n complex coordinates z^α in a smooth fashion. More rigorously there is a cover of M_{2n} by open sets U_I. On each U_I there is a 1:1 continuous map $\psi_I(p) = (z^1, z^2, \ldots, z^n)$, where $z^\alpha \in \mathbb{C}$. On intersections the compound maps $\psi_J \circ \psi_I^{-1}$ are analytic. It is not always possible to define such a complex structure on a real M_{2n}. There are important topological restrictions which, in the main, are beyond the scope of this book. (For interested readers we suggest the references [97, 98]). Locally it is always possible to introduce complex coordinates z^α by combining real coordinates ϕ^i, and this is the approach we take in our initial technical discussion below.

Complex manifolds are included in this book for a very simple reason. The scalar fields of supersymmetric theories in four spacetime dimensions are a set of complex fields z^α which can be viewed as coordinates of an important type of complex manifold known as a Kähler manifold. We will discuss Kähler manifolds extensively in this chapter. They appear in a natural way both in global supersymmetry and supergravity.

13.1 The local description of complex and Kähler manifolds

For local considerations,[1] one can view an n-dimensional complex manifold as a $2n$-dimensional real manifold parameterized by n complex coordinates. To obtain complex coordinates one can start with a real coordinate set $\phi^1, \ldots, \phi^n, \phi^{n+1}, \ldots, \phi^{2n}$ and define

$$\begin{aligned} z^\alpha &= \phi^\alpha + \mathrm{i}\phi^{\alpha+n}\,, \qquad \alpha = 1, 2, \ldots, n\,, \\ \bar{z}^{\bar{\alpha}} &= \phi^\alpha - \mathrm{i}\phi^{\alpha+n} = \bar{z}^\alpha\,. \end{aligned} \tag{13.1}$$

[1] We follow the pedagogical treatment of [99].

We then take z^a to be the set of $2n$ complex coordinates where the index a runs first through the n unbarred or 'holomorphic' coordinates and then through the n barred or 'anti-holomorphic' coordinates. We consider the map $\phi^i \to z^a$ defined in (13.1) as a coordinate transformation of the type normally considered in differential geometry, and we use the standard transformation formulas of Chapter 6. Complexity causes no difficulty.

We assume that M possesses a Euclidean signature metric structure, with Christoffel connection, and curvature tensor, all as defined in Chapter 6. For convenience we record the relevant formulas again:

$$\begin{aligned} \mathrm{d}s^2 &= g_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j\,, && (13.2)\\ \Gamma^i_{jk} &= \tfrac{1}{2}g^{im}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk})\,, && (13.3)\\ R_{ij}{}^k{}_l &= \partial_i\Gamma^k_{jl} - \partial_j\Gamma^k_{il} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}\,. && (13.4)\end{aligned}$$

The transformation rules of vector fields under a real coordinate change $\phi^i \to \phi'^i$ are

$$\begin{aligned} V'^i(\phi') &= \frac{\partial\phi'^i}{\partial\phi^j}V^j(\phi)\,,\\ V'_i(\phi') &= \frac{\partial\phi^j}{\partial\phi'^i}V_j(\phi)\,. \end{aligned} \qquad (13.5)$$

Covariant derivatives are defined (see (6.67)) by

$$\begin{aligned} \nabla_j V^i &= \partial_j V^i + \Gamma^i_{jk}V^k\,,\\ \nabla_j V_i &= \partial_j V_i - \Gamma^k_{ji}V_k\,. \end{aligned} \qquad (13.6)$$

The transformation rules and covariant derivatives extend to higher rank tensors as discussed in Chapter 6.

All quantities above can be expressed in complex coordinates by applying the transformation formulas to the map $\phi^i \to z^a$. For example,

$$V^i(\phi) \to \tilde{V}^a(z) = \frac{\partial z^a}{\partial\phi_j}V^j(\phi)\,. \qquad (13.7)$$

We then separate the unbarred and barred components

$$\tilde{V}^\alpha = V^\alpha + \mathrm{i}V^{\alpha+n}\,, \qquad \tilde{V}^{\bar\alpha} = V^\alpha - \mathrm{i}V^{\alpha+n}\,, \qquad \alpha = 1,\ldots n\,. \qquad (13.8)$$

The 'splitting' of an index a into α and $\bar\alpha$ is not preserved by general transformation of complex coordinates $z'^\alpha = f^\alpha(z,\bar z)$, but it is preserved under the special class of 'holomorphic' coordinate transformations $z'^\alpha = f^\alpha(z)$. Under this subgroup of diffeomorphisms the holomorphic indices α of any tensor transform into holomorphic indices α' and anti-holomorphic $\bar\alpha$ into $\bar\alpha'$.

The Riemannian metric is expressed in complex coordinates as

$$\mathrm{d}s^2 \;=\; g_{ab}\mathrm{d}z^a\mathrm{d}z^b \;=\; g_{ij}\frac{\partial\phi^i}{\partial z^a}\frac{\partial\phi^j}{\partial z^b}\mathrm{d}z^a\mathrm{d}z^b\,. \tag{13.9}$$

The complex metric tensor $g_{ab} = g_{ba}$ is obtained from g_{ij} as indicated. The general form of the line element is

$$\mathrm{d}s^2 \;=\; 2g_{\alpha\bar\beta}\mathrm{d}z^\alpha\mathrm{d}\bar z^{\bar\beta} + g_{\alpha\beta}\mathrm{d}z^\alpha\mathrm{d}z^\beta + g_{\bar\alpha\bar\beta}\mathrm{d}\bar z^{\bar\alpha}\mathrm{d}\bar z^{\bar\beta}\,. \tag{13.10}$$

This form is real. This is insured by the transformation in (13.9) which implies that $g_{\alpha\bar\beta} = \bar g_{\beta\bar\alpha}$ and $g_{\alpha\beta} = \bar g_{\bar\alpha\bar\beta}$. Covariant derivatives are written as

$$\begin{aligned} \nabla_b\tilde V^a &= \partial_b\tilde V^a + \Gamma^a_{bc}\tilde V^c\,,\\ \nabla_b\tilde V_a &= \partial_b V_a - \Gamma^c_{ba}\tilde V_c\,. \end{aligned} \tag{13.11}$$

The connection Γ^a_{bc} is just the Christoffel connection (13.3) expressed in terms of g_{ab}.

$$\Gamma^a_{bc} \;=\; \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})\,. \tag{13.12}$$

Note that $g^{ad}g_{bd} = \delta^a_b$.

We now define two conditions on the metric g_{ab} which are preserved by holomorphic coordinate transformations. The metric is said to be Hermitian if there are choices of coordinates in which $g_{\alpha\beta} = g_{\bar\alpha\bar\beta} = 0$. The line element then takes the Hermitian form

$$\mathrm{d}s^2 = 2g_{\alpha\bar\beta}\mathrm{d}z^\alpha\mathrm{d}\bar z^{\bar\beta}\,. \tag{13.13}$$

Coordinate systems in which this form holds are said to be adapted to the Hermitian structure. The Hermitian form is a restriction on the metric. It is not possible to transform the general complex form (13.9) to Hermitian form.

Ex. 13.1 *Formulate the equations for a coordinate transformation $z'^a = f^a(z,\bar z)$ from coordinates in which the line element takes the general form in (13.9) to coordinates in which $g'_{\alpha\beta} = 0$. Show that in general there are too many equations to possess a solution.*

As is customary we write the holomorphic index of $g_{\alpha\bar\beta}$ or its inverse $g^{\alpha\bar\beta}$ on the left. Thus $g^{\alpha\bar\gamma}g_{\beta\bar\gamma} = \delta^\alpha_\beta$.

Given a Hermitian metric, we can define the fundamental 2-form

$$\Omega \;=\; -2\mathrm{i}g_{\alpha\bar\beta}\mathrm{d}z^\alpha\wedge\mathrm{d}\bar z^{\bar\beta}\,. \tag{13.14}$$

Ex. 13.2 *Show that Ω is a real 2-form, i.e. $\Omega = \bar\Omega$.*

A manifold with Hermitian metric is a Kähler manifold if its fundamental form (then called the Kähler form) is closed, i.e. $\mathrm{d}\Omega = 0$. The exterior derivative is given by

$$\mathrm{d}\Omega \;=\; -\mathrm{i}(\partial_\gamma g_{\alpha\bar\beta} - \partial_\alpha g_{\gamma\bar\beta})\mathrm{d}z^\gamma \wedge \mathrm{d}z^\alpha \wedge \mathrm{d}\bar z^{\bar\beta} + \text{c.c.}\,, \tag{13.15}$$

so that the necessary and sufficient condition for a Kähler manifold is

$$\partial_\gamma g_{\alpha\bar\beta} - \partial_\alpha g_{\gamma\bar\beta} \;=\; 0\,. \tag{13.16}$$

This condition implies that locally (i.e. in each coordinate patch) the metric can be represented as

$$g_{\alpha\bar\beta} \;=\; \partial_\alpha \partial_{\bar\beta} K(z,\bar z)\,. \tag{13.17}$$

The real function $K(z,\bar z)$ is called the Kähler potential. It is not uniquely determined since the change

$$K(z,\bar z) \to K'(z,\bar z) \;=\; K(z,\bar z) + f(z) + \bar f(\bar z)\,, \tag{13.18}$$

leaves $g_{\alpha\bar\beta}$ invariant. On a topologically non-trivial Kähler manifold, it is frequently the case that there is no globally defined potential. Instead the potentials $K(z,\bar z)$ and $K'(z',\bar z)$ on overlapping coordinate charts are related by a transformation similar to (13.18). These transformations are called Kähler transformations.

The Hermitian and Kähler conditions on the metric lead to simplifications of the connection Γ^a_{bc}. Hermiticity implies that the connection components $\Gamma^\alpha_{\bar\beta\bar\gamma}$ (and conjugates $\Gamma^{\bar\alpha}_{\beta\gamma}$) vanish. For a Kähler metric, there are the additional conditions $\Gamma^\alpha_{\beta\bar\gamma} = \Gamma^{\bar\alpha}_{\bar\beta\gamma} = 0$, so the only nonvanishing connection components are those of the form

$$\Gamma^\alpha_{\beta\gamma} \;=\; g^{\alpha\bar\delta}\partial_\beta g_{\gamma\bar\delta}\,, \qquad \Gamma^{\bar\alpha}_{\bar\beta\bar\gamma} \;=\; g^{\delta\bar\alpha}\partial_{\bar\beta} g_{\delta\bar\gamma}. \tag{13.19}$$

These formulas are much simpler than the conventional real (13.3) and general complex (13.12) forms.

Ex. 13.3 *Take the trace of the first relation in (13.19) and derive*

$$\Gamma^\gamma_{\alpha\gamma} \;=\; \partial_\alpha \log\det g_{\beta\bar\delta}\,, \tag{13.20}$$

which is very similar to its analogue for Riemannian manifolds.

The curvature tensor on a complex manifold is written as R_{abcd}. It is defined by applying the standard formula (13.4) in complex coordinates z^a with Γ^a_{bc}, and it has the usual index symmetries. On a Kähler manifold, there are important simplifications because many connection components vanish. The non-vanishing curvature components are:

$$\begin{aligned} R_{\gamma\bar\delta}{}^\alpha{}_\beta &= -R_{\bar\delta\gamma}{}^\alpha{}_\beta\,, \qquad R_{\gamma\bar\delta}{}^{\bar\alpha}{}_{\bar\beta} \;=\; -R_{\bar\delta\gamma}{}^{\bar\alpha}{}_{\bar\beta}\,, \\ R_{\alpha\bar\beta\gamma\bar\delta} &= -R_{\bar\beta\alpha\gamma\bar\delta} = R_{\gamma\bar\delta\alpha\bar\beta}\,, \end{aligned} \tag{13.21}$$

and these are given by the simple formulas

$$
\begin{aligned}
R_{\gamma\bar{\delta}}{}^{\alpha}{}_{\beta} &= -\partial_{\bar{\delta}}\Gamma^{\alpha}_{\beta\gamma}\,, \\
R_{\alpha\bar{\beta}\gamma\bar{\delta}} &= \partial_{\gamma}\partial_{\bar{\delta}}g_{\alpha\bar{\beta}} - g^{\eta\bar{\epsilon}}\partial_{\gamma}g_{\alpha\bar{\epsilon}}\partial_{\bar{\delta}}g_{\eta\bar{\beta}}\,.
\end{aligned} \tag{13.22}
$$

Ex. 13.4 *Prove the symmetry properties* $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\gamma\bar{\beta}\alpha\bar{\delta}}$ *and* $R_{\gamma\bar{\delta}}{}^{\alpha}{}_{\beta} = R_{\beta\bar{\delta}}{}^{\alpha}{}_{\gamma}$. *Show that the form* $R_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta}$, *with two pairs of holomorphic indices, is symmetric in* $(\alpha\beta)$ *and in* $(\gamma\delta)$.

There are simplifications in the Bianchi identities (6.108) satisfied by the curvature tensor. The forms given for Riemannian geometry in (6.108) are also valid in complex coordinates. The restrictions on curvature components for a Kähler manifold then imply

$$
\begin{aligned}
R_{\alpha\bar{\beta}\gamma\bar{\delta}} &= R_{\gamma\bar{\beta}\alpha\bar{\delta}}\,, \\
\nabla_{\epsilon}R_{\alpha\bar{\beta}\gamma\bar{\delta}} &= \nabla_{\gamma}R_{\alpha\bar{\beta}\epsilon\bar{\delta}}\,.
\end{aligned} \tag{13.23}
$$

The first relation is a symmetry property checked in Ex. 13.4. Finally, note the Ricci identities (see (6.110) for vanishing torsion):

$$
\begin{aligned}
[\nabla_{\alpha}, \nabla_{\beta}]V^{\gamma} &= 0\,, & [\nabla_{\alpha}, \nabla_{\beta}]V^{\bar{\gamma}} &= 0\,, \\
\left[\nabla_{\alpha}, \overline{\nabla}_{\bar{\beta}}\right]V^{\gamma} &= R_{\alpha\bar{\beta}}{}^{\gamma}{}_{\delta}V^{\delta}\,, & [\nabla_{\alpha}, \overline{\nabla}_{\bar{\beta}}]V^{\bar{\gamma}} &= R_{\alpha\bar{\beta}}{}^{\bar{\gamma}}{}_{\bar{\delta}}V^{\bar{\delta}}\,, \\
\left[\nabla_{\alpha}, \overline{\nabla}_{\bar{\beta}}\right]V_{\gamma} &= R_{\alpha\bar{\beta}\gamma}{}^{\delta}V_{\delta}\,, & [\nabla_{\alpha}, \overline{\nabla}_{\bar{\beta}}]V_{\bar{\gamma}} &= R_{\alpha\bar{\beta}\bar{\gamma}}{}^{\bar{\delta}}V_{\bar{\delta}}\,.
\end{aligned} \tag{13.24}
$$

The Ricci tensor of a Kähler metric is defined in the usual way, namely as $R_{ab} = g^{cd}R_{acbd} = R_{ba}$. It can then be shown that the components $R_{\alpha\beta}$ and $R_{\bar{\alpha}\bar{\beta}}$ vanish and that mixed components can be written in the form

$$
R_{\alpha\bar{\beta}} = g^{\bar{\gamma}\gamma}R_{\alpha\bar{\gamma}\bar{\beta}\gamma} = -R_{\alpha\bar{\beta}\gamma}{}^{\gamma} = -\partial_{\alpha}\partial_{\bar{\beta}}(\log\det g_{\gamma\bar{\delta}})\,. \tag{13.25}
$$

Ex. 13.5 *Derive these properties of the Ricci tensor.*

13.2 Mathematical structure of Kähler manifolds

Most mathematical treatments of complex differential geometry (for example, Ch. XXX of [97]) start with the definition of an almost complex structure. This is a (real-valued) tensor $J_i{}^j(\phi)$ on the tangent space of a real manifold M_d with the property $J_i{}^k\,J_k{}^j = -\delta_i{}^j$. The analogy with $\mathrm{i}\cdot\mathrm{i} = -1$ is obvious.

Ex. 13.6 *Show that this property can be satisfied only if the dimension is even,* $d = 2n$ *and that the eigenvalues of* J *are* $\pm\mathrm{i}$.

The almost complex structure maps the tangent space $T(M_{2n})$ onto itself, acting on basis vectors as

$$J\frac{\partial}{\partial\phi^i} \equiv J_i{}^j\frac{\partial}{\partial\phi^j}\,. \tag{13.26}$$

The almost complex structure allows one to express the conditions that define a Kähler metric directly in terms of real coordinates on M_{2n}. This can be useful in physical applications. We summarize the mathematical treatment in this section using both real and complex coordinates.

One can show [97, 100] that a real manifold M_{2n} equipped with an almost complex structure is a complex manifold (of dimension n) if and only if the Nijenhuis tensor vanishes:

$$N_{ij}{}^k \equiv J_i{}^\ell(\partial_\ell J_j{}^k - \partial_j J_\ell{}^k) - J_j{}^\ell(\partial_\ell J_i{}^k - \partial_i J_\ell{}^k) \,=\, 0\,. \tag{13.27}$$

Ex. 13.7 *Show that $N_{ij}{}^k$ transforms as a tensor under diffeomorphisms despite the absence of covariant derivatives.*

The almost complex structure is then called the complex structure of M_{2n}. There is then a covering of M_{2n} by coordinate charts U_I with complex coordinates z^α, $\bar z^{\bar\alpha} = \overline{z^\alpha}$ as used in Sec. 13.1. In the coordinate basis $\partial/\partial z^\alpha$ $\partial/\partial\bar z^{\bar\alpha}$ the almost complex structure is of the form

$$J = \begin{pmatrix} \mathrm{i}\delta_\alpha{}^\beta & 0 \\ 0 & -\mathrm{i}\delta_{\bar\alpha}{}^{\bar\beta}\,. \end{pmatrix} \tag{13.28}$$

We now suppose that M_{2n} has a torsion-free (i.e. symmetric) connection Γ^k_{ij} and that $J_i{}^j$ is covariantly constant, i.e.

$$\nabla_k J_i{}^j = \partial_k J_i{}^j - \Gamma^\ell_{ki}J_\ell{}^j + \Gamma^j_{k\ell}J_i{}^\ell = 0\,. \tag{13.29}$$

This is sufficient to ensure that $N_{ij}{}^k \,=\, 0$.

Ex. 13.8 *Show that in complex coordinates the condition (13.29) implies that $\Gamma^\beta_{\gamma\bar\alpha} = 0$ and $\Gamma^{\bar\beta}_{\gamma\alpha} = 0$. The connection thus has only pure holomorphic and anti-holomorphic components $\Gamma^\alpha_{\beta\gamma}$ and $\Gamma^{\bar\alpha}_{\bar\beta\bar\gamma}$.*

We now assume that M_{2n} has a Riemannian metric $g_{ij}(\phi)$. This metric is Hermitean if it is invariant under the action of the almost complex structure, namely if

$$J_i{}^k g_{k\ell} J_j{}^\ell = g_{ij}\,, \qquad \text{i.e.} \qquad J\,g\,J^T = g\,. \tag{13.30}$$

The second condition is equivalent to the first, but stated in matrix notation.

Ex. 13.9 *Express this condition in complex coordinates and show that it is equivalent to $g_{\alpha\beta} \,=\, g_{\bar\alpha\bar\beta} \,=\, 0$, as used in Sec. 13.1.*

Finally, we demand that the affine connection in (13.29) is the Levi-Civita connection. We then have two covariantly constant tensors:

$$\begin{aligned}\nabla_k J_i{}^j &= \partial_k J_i{}^j - \Gamma^\ell_{ki} J_\ell{}^j + \Gamma^j_{k\ell} J_i{}^\ell \;=\; 0\,,\\ \nabla_k g_{ij} &= \partial_k g_{ij} - \Gamma^\ell_{ki} g_{\ell j} - \Gamma^\ell_{kj} g_{i\ell} \;=\; 0\,.\end{aligned}\qquad(13.31)$$

We then define a Kähler manifold as a real M_{2n} with almost complex structure J and Hermitian metric g such that J is covariantly constant with respect to the Levi-Civita connection. These conditions imply that a Kähler manifold admits complex coordinate charts and is thus a complex manifold.

In real coordinates the Kähler form is defined as the 2-form

$$\Omega = -J_{ij} \mathrm{d}\phi^i \wedge \mathrm{d}\phi^j\,, \qquad J_{ij} = J_i{}^k g_{kj}\,. \qquad(13.32)$$

Ex. 13.10 *Use (13.31) to show that Ω is a closed 2-form. Conversely, show that* $\mathrm{d}\Omega = 0$ *implies that $J_i{}^j$ is covariantly constant. Show that (13.32) is equivalent to (13.14) in complex coordinates.*

We can now see that the approach to Kähler manifolds in this section has led us to the conditions (13.31) and then to the key result that the Kähler 2-form is closed. This is exactly the condition used to *define* a Kähler manifold in Sec. 13.1. Thus all other results in that section apply. In particular, the properties of the curvature tensor obtained in Sec. 13.1 are valid.

The main role of Kähler manifolds in supersymmetry and supergravity is that they serve as the scalar field target space in supersymmetric versions of the nonlinear σ-model discussed in Sec. 6.10. For global $\mathcal{N} = 1$ SUSY in four spacetime dimensions the σ-model can be defined without any further conditions or structure. The coupling to $\mathcal{N} = 1$ supergravity requires the further condition that the target space is a Kähler-Hodge manifold. This is discussed in Sec. 17.A. For $\mathcal{N} = 2$ global supersymmetry the target space must have three covariantly constant complex structures $J_{Ii}{}^j$, $I = 1, 2, 3$. Manifolds with this structure are called hyper-Kähler manifolds and are described in Sec. 19.4 together with the quaternionic Kähler manifolds required for $\mathcal{N} = 2$ supergravity. A subclass of Kähler manifolds called special Kähler manifolds are needed for supersymmetric theories of the $\mathcal{N} = 2$ vector multiplet. We discuss them in Sec. 19.3 when these theories are explained.

Finally, we point out that Kähler manifolds are special cases of *symplectic manifolds*, which are defined as real manifolds M_{2n} with a closed invertible 2-form similar to Ω is called a *symplectic manifold.* Among other applications, symplectic manifolds play an important role in Hamiltonian mechanics. For example, see [101].

13.3 The Kähler manifolds CP^n

The complex projective spaces called CP^n, $n = 1, 2...$ are an interesting yet simple example of Kähler manifolds. The manifold CP^n is defined as the space of complex lines in flat $\mathbb{C}^{n+1}$. Two points in $\mathbb{C}^{n+1}$, denoted by the complex $(n+1)$-tuples $(X^1, X^2, \ldots, X^{n+1})$ and $(X'^1, X'^2, \ldots, X'^{n+1})$ respectively, are defined to lie on the same complex line if there is a complex number w such that

$$(X^1, X^2, \ldots, X^{n+1}) = (wX'^1, wX'^2, \ldots, wX'^{n+1}) . \tag{13.33}$$

The space of lines thus defined has complex dimension n. If $X^{n+1} \neq 0$, we can define a set of n coordinates $z^\alpha = X^\alpha / X^{n+1}$. These can be viewed as intrinsic coordinates which describe all lines with $X^{n+1} \neq 0$, and one can proceed to cover the space with analogously defined coordinates, $z^\alpha_{\{I\}} = X^\alpha / X^I$ with $I = 1, \ldots n+1$, that describe all lines with $X^I \neq 0$.

We now use the ideas of induced metrics and the nonlinear σ-model, discussed in Sections 6.4 and 6.10, respectively, to obtain a Kähler metric on CP^n. Let $X^I(x)$, $I = 1, 2, \ldots, n+1$, denote a set of complex scalar fields over flat Minkowski spacetime of real dimension D. The scalars X^I define an $(n+1)$-tuple or point in $\mathbb{C}^{n+1}$. We denote the complex conjugates of X^I as $\bar{X}_I$ and impose the constraint

$$X^I \bar{X}_I = 1 , \qquad I = 1, 2, \ldots, n+1 . \tag{13.34}$$

This can be thought of as part of the process of choosing a representative of the complex line associated with the $(n+1)$-tuple $(X^1, X^2, \ldots, X^{n+1})$. Indeed, from the definition (13.33) it is clear that two points that satisfy (13.34) and lie on the same line must be related by (13.33) with $|w| = 1$. We now incorporate the fact that the common phase of the fields $X^I(x)$ is irrelevant by introducing a U(1) gauge potential $\mathcal{A}_\mu(x)$ and covariant derivative in the Lagrangian

$$\mathcal{L} = -(\partial_\mu + \mathrm{i}\mathcal{A}_\mu)\bar{X}_I \, (\partial^\mu - \mathrm{i}\mathcal{A}^\mu) X^I . \tag{13.35}$$

The gauge potential $\mathcal{A}_\mu$ is an auxiliary non-dynamical field in this system. Its Euler-Lagrange equation is algebraic with solution, after use of the constraint (13.34),

$$\mathcal{A}_\mu = -\frac{1}{2}\mathrm{i}\bar{X}_I \overleftrightarrow{\partial}_\mu X^I . \tag{13.36}$$

This result may be inserted in (13.35) to obtain the equivalent Lagrangian (including the constraint (13.34))

$$\mathcal{L} = -\partial_\mu \bar{X}_I \partial^\mu X^I - \frac{1}{4}(\bar{X}_I \overleftrightarrow{\partial}_\mu X^I)(\bar{X}_J \overleftrightarrow{\partial}^\mu X^J) , \qquad \bar{X}_I X^I = 1 . \tag{13.37}$$

Ex. 13.11 *Show that the Lagrangian (13.37) is gauge invariant under the* U(1) *transformation* $X^I(x) \to X'^I(x) = \mathrm{e}^{\mathrm{i}\theta(x)} X^I(x)$ *Show that any solution* $X^I(x)$ *of the new system (13.37) is also a solution of the equations of motion of (13.35) and constraint (13.34).*

We can now simultaneously fix the gauge and solve the constraint by expressing the X^I in terms of n independent complex fields $z^\alpha(x)$ and their complex conjugates $\bar{z}_\alpha$ defined by

$$(X^1,\ldots,X^n,X^{n+1}) \;=\; \frac{1}{\sqrt{1+\bar{z}_\alpha z^\alpha}}(z^1,\ldots,z^n,1)\,. \tag{13.38}$$

Note that $z^\alpha = X^\alpha/X^{n+1}$, as in the coordinate chart defined in the first paragraph of this section. We substitute (13.38) in the Lagrangian of (13.37) and find the Lagrangian

$$\begin{aligned}\mathcal{L} \;&=\; -\frac{1}{1+\bar{z}z}\left(\delta_\alpha^\beta - \frac{\bar{z}_\alpha z^\beta}{1+\bar{z}z}\right)\partial_\mu z^\alpha \partial^\mu \bar{z}_\beta \;=\; -\frac{1}{1+\bar{z}z}\left(\delta_{\alpha\bar{\beta}} - \frac{\bar{z}_\alpha z_{\bar{\beta}}}{1+\bar{z}z}\right)\partial_\mu z^\alpha \partial^\mu \bar{z}^{\bar{\beta}}\,,\\ &\bar{z}z \equiv \bar{z}_\alpha z^\alpha = z^\alpha \delta_{\alpha\bar{\beta}}\bar{z}^{\bar{\beta}}\,.\end{aligned} \tag{13.39}$$

Note that we raise and lower indices here with $\delta_{\alpha\bar{\beta}}$ and not with the full metric $g_{\alpha\bar{\beta}}$.

We can now interpret this result in the light of the discussion of the nonlinear σ model in Sec. 6.10. There we associated Lagrangians of the form

$$\mathcal{L} = -\frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j\,, \tag{13.40}$$

with maps from Minkowski space to a Riemannian target manifold with coordinates ϕ^i and metric

$$\mathrm{d}s^2 = g_{ij}(\phi)\mathrm{d}\phi^i\mathrm{d}\phi^j\,. \tag{13.41}$$

The form of (13.39) suggest that we interpret z^α and $\bar{z}^{\bar{\beta}}$ as local coordinates of a complex manifold with Hermitian metric defined by

$$\mathrm{d}s^2 \;=\; 2\,g_{\alpha\bar{\beta}}\mathrm{d}z^\alpha\mathrm{d}z^{\bar{\beta}} \;=\; 2\,\frac{1}{1+\bar{z}z}\left(\delta_{\alpha\bar{\beta}} - \frac{\bar{z}_\alpha z_{\bar{\beta}}}{1+\bar{z}z}\right)\mathrm{d}z^\alpha\mathrm{d}\bar{z}^{\bar{\beta}}\,. \tag{13.42}$$

Ex. 13.12 *Show that the metric tensor of (13.42) satisfies the Kähler condition (13.16). Show that the metric tensor can be obtained as in (13.17) from the Kähler potential $K = \ln(1+\bar{z}z)$.*

This Kähler metric on CP^n is known as the Fubini-Study metric, named after the Italian and German mathematicians who first found it.

Ex. 13.13 *Use (13.25) to calculate the Ricci tensor of the CP^n metric (13.42) and show that it satisfies*

$$R_{\alpha\bar{\beta}} \;=\; (n+1)g_{\alpha\bar{\beta}}\,. \tag{13.43}$$

A Kähler metric whose Ricci tensor satisfies $R_{\alpha\bar{\beta}} = kg_{\alpha\bar{\beta}}$ is called a Kähler-Einstein metric.

Ex. 13.14 . *The metric tensor of the Poincaré plane was defined in real form in (6.120) and in the complex Hermitian form*

$$\mathrm{d}s^2 = \mathrm{d}Z\mathrm{d}\bar{Z}/(\mathrm{Im}Z)^2\,, \tag{13.44}$$

in (6.137). Show that the metric of this manifold of complex dimension 1 is a Kähler metric. What is the Kähler potential?

13.4 Symmetries of Kähler Metrics

13.4.1 Holomorphic Killing vectors and moment maps

Since a complex manifold is a special case of a real manifold, the formalism of Killing vectors that was introduced in Sec. 6.11 certainly applies. However, Kähler manifolds are defined using two basic structures: the Hermitian metric and the covariantly constant complex structure. We consider symmetries that preserve both structures. For each continuous symmetry there is a Killing vector, which generates an infinitesimal variation of local coordinates. It can be expressed equivalently in real or complex form as

$$\delta\phi^i = \theta k^i(\phi)\,, \qquad \text{or} \qquad \delta z^a = \theta k^a(z,\bar{z})\,. \tag{13.45}$$

We now require that the Lie derivative of the metric tensor *and* the complex structure *both* vanish:

$$\mathcal{L}_k\, g_{ij} \;=\; \nabla_i k_j + \nabla_j k_i \;=\; 0\,, \tag{13.46}$$

$$\mathcal{L}_k\, J_i{}^j \;=\; \nabla_i k^\ell J_\ell{}^j - \nabla_\ell k^j J_i{}^\ell \;=\; 0\,. \tag{13.47}$$

The first condition is the definition of a Killing vector for any manifold, real or complex, and the second is special to Kähler manifolds.

In complex coordinates J takes the form (13.28), so the holomorphic components, i.e. $i \to \alpha$, $j \to \beta$ of the condition (13.47) vanish trivially. However, when we replace i by $\bar{\alpha}$ and j by β, we find the condition $\partial_{\bar{\alpha}} k^\beta = 0$, which says that the components k^β are functions of the z^α and not the $\bar{z}^{\bar{\alpha}}$. For this reason a Killing vector which satisfies (13.47) is called a *holomorphic Killing vector*. In other words, the symmetry map $\delta z^a = \theta k^a$ preserves the split between holomorphic and antiholomorphic coordinates, and we have

$$k^a \;=\; \{k^\alpha(z),\; k^{\bar{\alpha}}(\bar{z})\}\,. \tag{13.48}$$

The Kähler manifolds are symplectic manifolds, as we mentioned at the end of Sect. 13.2, because $\mathrm{d}\Omega = 0$. The standard example in physics of a symplectic manifold is the Hamiltonian phase space with invertible closed 2-form $\mathrm{d}q \wedge \mathrm{d}p$. The symmetries, which preserve this form, are canonical transformations, which are all characterized by a generating function. The same applies to holomorphic symmetries of Kähler metrics and the corresponding functions are called the *moment maps* [102, 97].

The existence of moment maps (which are also called Killing potentials) can be simply derived from the fact that (13.46) and (13.47) imply that the Lie derivative of the two-form Ω vanishes. Indeed this gives, using the vanishing of $\mathrm{d}\Omega$,

$$0 = \mathcal{L}_k\Omega = (i_k\mathrm{d} + \mathrm{d}i_k)\,\Omega = \mathrm{d}i_k\Omega\,. \tag{13.49}$$

The Poincaré lemma then implies that there exists a function $\mathcal{P}$ with the property

$$i_k\Omega = -2\mathrm{d}\mathcal{P}\,. \tag{13.50}$$

The scalar function $\mathcal{P}(z,\bar z)$, which is real-valued since (13.50) is a real equation, is the moment map we have been seeking. Note that $\mathcal{P}$ is not determined uniquely; one is free to add a real constant, $\mathcal{P}(z,\bar z) \to \mathcal{P}(z,\bar z) + \xi$.

The condition (13.50) involves 1-forms, but it can also be expressed as an equality of covariant vectors. In complex coordinates $z^a = \{z^\alpha,\, \bar z^{\bar\alpha}\}$, this equality reads

$$\begin{aligned} k_\alpha = g_{\alpha\bar\beta}\,k^{\bar\beta}(\bar z) &= \mathrm{i}\partial_\alpha\,\mathcal{P}(z,\bar z)\,,\\ k_{\bar\alpha} = g_{\beta\bar\alpha}\,k^{\beta}(z) &= -\mathrm{i}\partial_{\bar\alpha}\,\mathcal{P}(z,\bar z)\,. \end{aligned} \tag{13.51}$$

The Killing equation (13.46) splits in these coordinates into two conditions

$$\nabla_\alpha k_\beta + \nabla_\beta k_\alpha = 0\,, \tag{13.52}$$

$$\nabla_a k_{\bar\beta} + \overline{\nabla}_{\bar\beta} k_\alpha = 0\,. \tag{13.53}$$

The first of these is automatic since $\nabla_\alpha g_{\beta\bar\gamma}V^{\bar\gamma}(\bar z) = g_{\beta\bar\gamma}\partial_\alpha V^{\bar\gamma}(\bar z) = 0$ for a holomorphic vector.[2] The second condition is satisfied whenever we have a moment map $\mathcal{P}(z,\bar z)$ due to (13.51). Hence, the symmetries of a Kähler manifold are characterized by real functions $\mathcal{P}(z,\bar z)$ such that

$$k^\alpha(z) = -\mathrm{i}g^{\alpha\bar\beta}\partial_{\bar\beta}\mathcal{P}(z,\bar z)\,, \tag{13.54}$$

is holomorphic. Note that $\mathcal{P}$ is always a function of both z^α and $z^{\bar\alpha}$ because the lower vector components k_α, $k_{\bar\alpha}$ also have this property.

As a conclusion, all symmetries of a Kähler manifold can be found by searching for the real functions $\mathcal{P}(z,\bar z)$ that satisfy the property (we take for convenience the complex conjugate of (13.54) and drop an invertible factor)

$$\nabla_\alpha\partial_\beta\mathcal{P}(z,\bar z) = 0\,. \tag{13.55}$$

Ex. 13.15 *In the special case of the flat Kähler metric $g_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$, solve the holomorphy condition of (13.54) and determine that the moment map is in general of the form*

$$\mathcal{P} = z^\alpha p_{\alpha\bar\beta}\bar z^{\bar\beta} + z^\alpha q_\alpha + \bar z^{\bar\alpha}\bar q_{\bar\alpha} + \xi\,. \tag{13.56}$$

[2] The vector V^γ is holomorphic but need not be Killing vector. Note that $\partial_\alpha V^{\bar\gamma}$ is covariant since the connection coefficients $\Gamma^\alpha_{\bar\beta\gamma}$ vanish for Kähler metrics.

Determine the reality conditions on the constant coefficients $p_{\alpha\bar\beta}$, q_α and ξ. Check explicitly that the related vectors k^α are indeed symmetries of the Lagrangian $\partial_\mu z^\alpha \delta_{\alpha\bar\beta}\partial^\mu \bar z^{\bar\beta}$. Note that the symmetries are holomorphic, but the moment map depends on both z^α and $z^{\bar\alpha}$.

The Killing vector relations (13.52),(13.53) state the fact that the Kähler metric $g_{\alpha\bar\beta}$ is invariant under the isometry. However, the Kähler potential need not be invariant. It must satisfy only the weaker condition

$$\delta K = \theta\left(k^\alpha\partial_\alpha + k^{\bar\alpha}\partial_{\bar\alpha}\right)K(z,\bar z) \;=\; \theta\left[F(z) + \bar F(\bar z)\right]\,, \tag{13.57}$$

with an arbitrary holomorphic function $F(z)$. If this function does not vanish, it means that $K(z,\bar z)$ changes by a Kähler transformation under the action of the symmetry. This is sufficient for an invariant metric. This fact allows us to find a solution to (13.54) when we know the Killing vector:

$$\mathcal{P}(z,\bar z) \;=\; \mathrm{i}\left[k^\alpha\partial_\alpha K(z,\bar z) \;-\; F(z)\right] = -\mathrm{i}\left[k^{\bar\alpha}\partial_{\bar\alpha}K(z,\bar z) \;-\; \bar F(\bar z)\right]\,. \tag{13.58}$$

This solution is general, since we can absorb the real constant ξ in $F(z)\to F(z)+\mathrm{i}\xi$, which does not influence (13.57).

Ex. 13.16 *Prove (13.58) by applying $\partial_{\bar\beta}$ to deduce (13.54). The second term is unnecessary for this calculation, but it is needed in order to have a real $\mathcal{P}$. Prove this using (13.57).*

13.4.2 Algebra of holomorphic Killing vectors

In many cases the (holomorphic) symmetry group of a Kähler metric is non-abelian. To discuss this important case, we consider a set of holomorphic Killing vectors $k_A^\alpha(z)$. For each Killing vector there is a Killing potential $\mathcal{P}_A(z,\bar z)$. For convenience we repeat the main formulae for the Killing vectors and moment maps indicating the extra index:

$$\begin{aligned}
&k_A{}^\alpha(z) \;=\; -\mathrm{i}g^{\alpha\bar\beta}\partial_{\bar\beta}\mathcal{P}_A(z,\bar z)\,,\\
&(k_A^\alpha(z)\partial_\alpha + k_A^{\bar\alpha}(\bar z)\partial_{\bar\alpha})K(z,\bar z) \;=\; F_A(z) + \bar F_A(\bar z)\,,\\
&\mathcal{P}_A(z,\bar z) \;=\; \mathrm{i}(k_A^\alpha\partial_\alpha K(z,\bar z) \;-\; F_A(z)) \;=\; -\mathrm{i}(k_A^{\bar\alpha}\partial_{\bar\alpha}K(z,\bar z) \;-\; \bar F_A(\bar z))\,.
\end{aligned} \tag{13.59}$$

The holomorphic Killing vector $k_A^\alpha(z)$, $k_A^{\bar\alpha}(\bar z)$ generate a Lie algebra. The Lie algebra structure is obtained by expressing the Lie bracket relation (6.132) in complex coordinates, assuming holomorphy. This gives

$$k_A^\beta\partial_\beta k_B^\alpha - k_B^\beta\partial_\beta k_A^\alpha \;=\; f_{AB}{}^C k_C^\alpha\,, \tag{13.60}$$

together with the complex conjugate. Note that if k_A and k_B are holomorphic vectors, then their Lie bracket is necessarily holomorphic.

By adjustment of the additive constants, the $\mathcal{P}_A$ can be chosen to transform in the adjoint representation,

$$(k_A^\alpha \partial_\alpha + k_A^{\bar\alpha} \partial_{\bar\alpha}) \mathcal{P}_B(z, \bar z) \;=\; f_{AB}{}^C \mathcal{P}_C \,. \tag{13.61}$$

The $\mathcal{P}_A$ are then uniquely fixed for (simple) non-abelian symmetries, but the ambiguity $\mathcal{P} \to \mathcal{P} + \xi$ remains for U(1) factors of the symmetry group.

We have introduced the Killing potentials $\mathcal{P}_A$, the possible constants ξ_A, and the quantities F_A using purely mathematical considerations. It is remarkable that they also occur naturally in supersymmetry, as we will discuss in Chapter 14. We will see later that the moment maps that are used in supersymmetry should satisfy the extra condition (13.61), sometimes denoted as 'equivariance relation', which can, using the last equation of (13.59), also be written as

$$k_A{}^\alpha g_{\alpha\bar\beta} k_B{}^{\bar\beta} - k_B{}^\alpha g_{\alpha\bar\beta} k_A{}^{\bar\beta} \;=\; \mathrm{i} f_{AB}{}^C \mathcal{P}_C \,. \tag{13.62}$$

13.4.3 The Killing vectors of CP^1

It is time for an example, and we will discuss the holomorphic Killing vectors of CP^1 which span the Lie algebra of SU(2). One way to see that this is the expected symmetry group is to observe that the elementary SU(2) transformations of the doublet (X^1, X^2) give a holomorphic global symmetry of the Lagrangian (13.37). In these variables the symmetries are generated by the differential operators

$$k_A \;=\; -\mathrm{i}\frac{1}{2}(\sigma_A)^i{}_j X^j \frac{\partial}{\partial X^i} \,, \tag{13.63}$$

where the σ_A are the Pauli matrices. We use the relation $z = X^1/X^2$ and the chain rule to write

$$\begin{aligned}
k_1 &= -\mathrm{i}\frac{1}{2}(X^1 \frac{\partial z}{\partial X^2} + X^2 \frac{\partial z}{\partial X^1}) \frac{\partial}{\partial z} \;=\; -\mathrm{i}\frac{1}{2}(1 - z^2)\frac{\partial}{\partial z} \,, \\
k_2 &= -\frac{1}{2}(X^1 \frac{\partial z}{\partial X^2} - X^2 \frac{\partial z}{\partial X^1}) \frac{\partial}{\partial z} \;=\; \frac{1}{2}(1 + z^2)\frac{\partial}{\partial z} \,, \\
k_3 &= -\mathrm{i}\frac{1}{2}(X^1 \frac{\partial z}{\partial X^1} - X^2 \frac{\partial z}{\partial X^2}) \frac{\partial}{\partial z} \;=\; -\mathrm{i} z\frac{\partial}{\partial z} \,.
\end{aligned} \tag{13.64}$$

The components[3] $k_A^z(z)$ can be read immediately from the expressions on the right.

Ex. 13.17 *Show that the Lie brackets of the vectors $k_A = k_A{}^z \partial/\partial z$ satisfy the structure relations $[k_A\,,\, k_B] \;=\; \varepsilon_{ABC} k_C$ of* SU(2)*.*

[3] We use the common notation in which vector and tensor components are denoted by the name of the coordinate, in this case z, rather than the number

The next step is to find the Killing potentials. Although not all these 'data' are needed, we record the Kähler potential, metric tensor, and connection:

$$K = \ln(1 + z\bar{z})\,, \qquad g_{z\bar{z}} = (1 + z\bar{z})^{-2}\,, \qquad \Gamma^z_{zz} = -2\bar{z}(1 + z\bar{z})^{-1}\,. \tag{13.65}$$

The Killing potentials can be obtained by solving the three differential equations

$$k_A^z \;=\; -\mathrm{i}g^{z\bar{z}}\partial_{\bar{z}}\mathcal{P}_A \;=\; -\mathrm{i}(1 + z\bar{z})^2\,\partial_{\bar{z}}\mathcal{P}_A\,, \tag{13.66}$$

together with there complex conjugates. The solutions are

$$\mathcal{P}_1 \;=\; \frac{1}{2}\frac{z + \bar{z}}{1 + z\bar{z}}\,, \qquad \mathcal{P}_2 = -\mathrm{i}\frac{1}{2}\frac{z - \bar{z}}{1 + z\bar{z}}\,, \qquad \mathcal{P}_3 \;=\; -\frac{1}{2}\frac{1 - z\bar{z}}{1 + z\bar{z}}\,. \tag{13.67}$$

Ex. 13.18 *Consider $\mathcal{P}'_3 = \mathcal{P}_3 + \xi$, where ξ is a constant. Obviously also $\mathcal{P}'_3$ satisfies (13.66). Show that the requirement that the $\mathcal{P}_A$ satisfy (13.61) with the structure constants $f_{AB}{}^C = \varepsilon_{ABC}$ of $\mathrm{SU}(2)$ fixes $\xi = 0$.*

Ex. 13.19 *Apply (13.59) to obtain the $F_A(z)$*

$$F_1 \;=\; \mathrm{i}\frac{1}{2}z\,, \qquad F_2 \;=\; \frac{1}{2}z\,, \qquad F_3 = -\mathrm{i}\frac{1}{2}\,. \tag{13.68}$$

Note that the Kähler potential is invariant under the third isometry k_3, but still $F_3 \neq 0$. Its value is fixed by the requirement (13.61).

We mentioned in Sec. 12.6 that Kähler manifolds are not in general symmetric manifolds (see (12.19) for the essential property). In fact, a general Kähler manifold does not even possess sufficient Killing vectors to promote it to a homogeneous space. There are Kähler manifolds that are symmetric. They are products of the following lists of irreducible symmetric Kähler spaces

$$\begin{array}{ccc} \dfrac{\mathrm{SU}(p,q)}{\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)}\,, & \dfrac{\mathrm{SO}^*(2n)}{\mathrm{U}(n)}\,, & \dfrac{\mathrm{Sp}(2n)}{\mathrm{U}(n)}\,, \\ \dfrac{\mathrm{SO}(n,2)}{\mathrm{SO}(n)\times\mathrm{SO}(2)}\,, & \dfrac{\mathrm{E}_6}{\mathrm{SO}(10)\times\mathrm{U}(1)}\,, & \dfrac{\mathrm{E}_7}{\mathrm{E}_6\times\mathrm{U}(1)}\,. \end{array} \tag{13.69}$$

14
General actions with $\mathcal{N}=1$ supersymmetry

In this chapter we will introduce the multiplet calculus of global $\mathcal{N}=1$ supersymmetry which provides a procedure to construct more general actions than those discussed in chapter 8. The theories considered there all have standard quadratic kinetic terms. Here we will find kinetic terms involving non-trivial metrics constructed from the scalar fields of chiral multiplets. These include the Kähler metrics discussed in chapter 13. Study of the multiplet calculus at the level of global SUSY will make it easier to understand the superconformal multiplet calculus, which we will use to construct supergravity theories later in the book.[1]

The basic multiplet calculus constructions are derived in Secs. 14.1-14.3. They are combined and extended in Sec. 14.4 to obtain the general $\mathcal{N}=1$ supersymmetric gauge theory in which Killing symmetries of the Kähler metric are gauged. In Sec. 14.5 we discuss some physical properties of SUSY gauge theories, notably the important question of spontaneous breakdown of supersymmetry.

Multiplet calculus constructions are rigorous and complete, but they can be technically complicated. The same constructions are simpler and more natural in the well known superspace formalism of $\mathcal{N}=1$, $D=4$ global supersymmetry. This insightful and elegant method allows an easy construction of supersymmetric actions and has other benefits. We do not use superspace methods in this book because it is very complicated to extend them to supergravity. We do include a short Appendix in which the basic superfields are discussed and related to the multiplet formulas in the main part of the chapter.

14.1 Multiplets

A super-multiplet of fields is a set of boson and fermion fields which transform among themselves under supersymmetry, such that the commutator algebra of two

[1] We consider only chiral and gauge multiplets. There are other multiplets, such as the antisymmetric tensor multiplet [103, 104], which has the same physical content as the chiral multiplet.

SUSY variations is

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu\,. \tag{14.1}$$

The simplest example is the chiral multiplet consisting of two complex scalars Z and F and the left chiral projection of a Majorana spinor χ. The SUSY transformation rules were given in (8.16), and the algebra was studied in Sec. 8.2.1. In the superspace formalism the component fields, such as Z, $P_L\chi$, and F are packaged in a single superfield.

In our approach the SUSY transformation rules of a supermultiplet are built by starting from the lowest dimension field and introducing the additional fields needed to obtain a closed realization of the algebra of (14.1).

14.1.1 Chiral multiplets

To illustrate our approach to the multiplet calculus, let us reconsider the chiral multiplet. A chiral multiplet is now *defined* as one which contains a complex scalar Z whose SUSY transformation involves only the chiral projection $P_L\epsilon$, or equivalently $\bar{\epsilon}P_L$, of the constant Majorana SUSY parameter ϵ. We therefore write the transformation rule

$$\delta Z = \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi\,. \tag{14.2}$$

The field that multiplies $\bar{\epsilon}P_L$ is *defined* as the spinor component $P_L\chi$ of the multiplet. Lorentz invariance requires that it is a spinor rather than, say, a vector-spinor. Since $P_L\chi$ must also transform, we postulate the most general Lorentz covariant form

$$\delta P_L\chi = \frac{1}{\sqrt{2}}P_L\left(F + \gamma^\mu X_\mu + \gamma^{\mu\nu}T_{\mu\nu}\right)\epsilon\,, \tag{14.3}$$

in which F, X_μ and $T_{\mu\nu}$ are to be determined by demanding that (14.1) holds for the field Z. This simple calculation requires the symmetry properties of spinor bilinears, see e.g. Table 3.1 at page 54. The result is i) that $T_{\mu\nu} = 0$, ii) that $X_\mu = \partial_\mu Z$, and iii) that F drops out of the commutator and is thus not constrained. So we add F as a new field of the multiplet. Its transformation rule must take the form

$$\delta F = \bar{\epsilon}\left(P_L\lambda + P_R\psi\right)\,, \tag{14.4}$$

where λ and ψ are to be determined by imposing the supersymmetry algebra on χ. This time we need a Fierz rearrangement identity; those of (3.69) are most appropriate. We find that λ must vanish in order to remove unwanted terms of the form $\bar{\epsilon}_1P_R\gamma^{\mu\nu}\epsilon_2$ in the commutator and that $P_R\psi = \not{\partial}P_L\chi$. Next we must check that the SUSY commutator on F is consistent with (14.1), and it is.

To summarize, we constructed the chiral multiplet transformation rules (8.16) as a closed realization of supersymmetry starting with the assumption that its lowest dimension component is a complex scalar Z whose SUSY transformation involves only $P_L\epsilon$.

Ex. 14.1 *In (8.53) we showed that SUSY variations of chiral multiplets are modified when gauge multiplets are present. In this exercise we outline how to rework the construction of chiral multiplets in this more general situation. Simply repeat the process above with the ordinary derivative ∂_μ in (14.1) replaced by the gauge covariant derivative D_μ defined in (8.45), i.e.*

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = -\tfrac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2 D_\mu \,. \tag{14.5}$$

Check that in the first step, the modification leads to what one should expect: $X_\mu = D_\mu Z$ rather than $\partial_\mu Z$. In the second step, when calculating the supersymmetry commutator on $P_L\chi$, one needs to transform A_μ^A, which appears in the covariant derivative. Its transformation was given in (8.41), and involves λ^A. Show that this leads to the modified transformation of F as shown in (8.53).

Although the fields Z, $P_L\chi$ and F of a chiral multiplet can certainly be elementary fields in a Lagrangian, it is important to realize that a chiral multiplet is a more general object whose components can be composites of elementary fields. For example, two elementary chiral multiplets can be multiplied to form a composite multiplet as the following exercise shows.

Ex. 14.2 *Consider two chiral multiplets with components Z^i, $P_L\chi^i$, F^i, $i = 1, 2$. Show that the quadratic combinations $Z^3 = Z^1Z^2$, $P_L\chi^3 = P_L(Z^1\chi^2 + Z^2\chi^1)$ and $F^3 = F^1Z^2 + F^2Z^1 - \bar{\chi}^1P_L\chi^2$ also transform under (8.16). Thus we have constructed a new chiral multiplet which can be considered to be the product of the first two.*

In Sec. 8.2 we introduced interactions of an elementary chiral multiplet using the superpotential $W(Z)$, an arbitrary holomorphic function. Let's now show that $W(Z)$ is the lowest component of a composite chiral multiplet. Using the chain rule we can obtain its SUSY variation

$$\delta W(Z) \;=\; \frac{1}{\sqrt{2}}W'(z)\bar{\epsilon}P_L\chi \,. \tag{14.6}$$

Comparing with (14.2), we see that $W(Z)$ transforms with the chiral projection $\bar{\epsilon}P_L$, and we can thus identify the χ component of the composite multiplet

$$\chi(W) \;\equiv\; W'(Z)\chi \,. \tag{14.7}$$

Ex. 14.3 *Complete the construction of this multiplet as follows.*
i. Compute $\delta\chi(W)$ using the chain rule and a Fierz rearrangement. Compare with (8.16) to identify the F-component

$$F(W) \;=\; W'F \;-\; \tfrac{1}{2}W''\bar{\chi}P_L\chi \,. \tag{14.8}$$

ii. As the final check show that $\delta F(W) \;=\; \bar{\epsilon}\not{\partial}P_L\chi(W)/\sqrt{2}$.

14.1.2 Real multiplets

Another basic multiplet of $\mathcal{N}=1$ SUSY is the real multiplet, which corresponds to a real superfield. By *definition*, the lowest component is a real scalar, $C(x)$, and one adds the additional components necessary to satisfy the SUSY algebra (12.1) on all fields. The SUSY variation of C is the most general real expression: $\delta C=\bar{\epsilon}\chi$, which is real if χ is a Majorana fermion. To agree with common practice we rename $\chi\to \mathrm{i}\gamma_*\zeta$. We now proceed as in the previous section, writing a general ansatz for $\delta\zeta$, and restricting it by requiring that the SUSY algebra is preserved on C. After several such steps we find the closed *real multiplet* containing the component fields

$$(C,\zeta,\mathcal{H},B_\mu,\lambda,D)\,, \tag{14.9}$$

with the transformation rules

$$\begin{aligned}
\delta C &= \tfrac12\mathrm{i}\bar{\epsilon}\gamma_*\zeta\,,\\
\delta P_L\zeta &= \tfrac12 P_L\left(\mathrm{i}\mathcal{H}-\not{B}-\mathrm{i}\not{\partial}C\right)\epsilon\,,\\
\delta\mathcal{H} &= -\mathrm{i}\bar{\epsilon}P_R\left(\lambda+\not{\partial}\zeta\right)\,,\\
\delta B_\mu &= -\tfrac12\bar{\epsilon}\left(\gamma_\mu\lambda+\partial_\mu\zeta\right)\,,\\
\delta\lambda &= \tfrac12\left[\gamma^{\rho\sigma}\partial_\rho B_\sigma+\mathrm{i}\gamma_*D\right]\epsilon\,,\\
\delta D &= \tfrac12\mathrm{i}\,\bar{\epsilon}\gamma_*\gamma^\mu\partial_\mu\lambda\,.
\end{aligned}\tag{14.10}$$

The fermions ζ and λ are Majorana fields, the bosons C, B_μ and D are real, while $\mathcal{H}$ is complex. Chiral projectors are used in the second and third entries for convenience, and the conjugate relations are easily obtained.

Ex. 14.4 *Show that* $\delta\bar{\mathcal{H}}=\mathrm{i}\bar{\epsilon}P_L(\lambda+\not{\partial}\zeta)$ *and that* $\delta P_R\zeta=\frac12 P_R(-\mathrm{i}\bar{\mathcal{H}}-\not{B}+\mathrm{i}\not{\partial}C)\epsilon$.

One important application of the real multiplet is to the SUSY gauge theories discussed in Ch. 8. To explain this application observe first that the variations of the fields B_μ, λ and D are very similar to the abelian case of (8.41), and they would agree exactly if the term $\partial_\mu\zeta$ in δB_μ were absent. In fact the components C, ζ, $\mathcal{H}$ can be eliminated by a supersymmetric generalization of a gauge transformation. To see how this is done, suppose we are given a chiral multiplet Z, $P_L\chi$, F and its anti-chiral conjugate $\bar{Z}$, $P_R\chi$, $\bar{F}$. Then $\operatorname{Im}Z$ is a real scalar, which becomes the lowest component of a real multiplet determined by the chiral transformation laws (and their conjugates). The fields of this multiplet are viewed as parameters of the 'supergauge' transformation, which is defined as the shift

$$\left(C\to C+\operatorname{Im}Z,\ \zeta\to\zeta-\frac{1}{\sqrt{2}}\chi,\ \mathcal{H}\to\mathcal{H}+\mathrm{i}F,\ B_\mu\to B_\mu+\partial_\mu\operatorname{Re}Z,\ \lambda\to\lambda,\ D\to D\right). \tag{14.11}$$

Clearly we can choose parameters to make the transformed C, ζ, $\mathcal{H}$ components vanish, while $\operatorname{Re}Z$ acts as an abelian gauge parameter for the vector B_μ. Of course, we then rename $B_\mu \to A_\mu$ and $\operatorname{Re} Z \to \theta$, so that A_μ, λ and D are the standard fields of the gauge multiplet with the usual gauge transformation. The 'supergauge' transformation just described is said to take the theory to Wess-Zumino gauge. The corresponding transformation in the superspace formalism is described in Appendix 14.A. The non-abelian extension of this procedure gives the result that the real multiplet in Wess-Zumino gauge includes the fields A_μ^A, λ^A and D^A. Their SUSY transformations are exactly those of (8.41) with the gauge covariant algebra given in (8.43).

14.2 Generalized actions by multiplet calculus

In the superspace formalism, the actions of supersymmetric field theories are expressed as integrals over both the x^μ and θ^α coordinates of superspace. To construct the same actions using components, we need only note in (8.16) and (14.10) that the SUSY variations of the highest dimension fields, F for a chiral multiplet and D for a real multiplet, are total spacetime derivatives. Therefore the integrals

$$S_F = \int \mathrm{d}^4x\, F\,, \qquad S_D = \int \mathrm{d}^4x\, D\,, \tag{14.12}$$

are invariant under SUSY,

$$\begin{aligned} \delta S_F &= \int \mathrm{d}^4x\, \delta F = \frac{1}{\sqrt{2}}\int \mathrm{d}^4x\, \bar\epsilon \partial\!\!\!/ P_L\chi = 0\,,\\ \delta S_D &= \int \mathrm{d}^4x\, \delta D = \frac{1}{2}\mathrm{i}\int \mathrm{d}^4x\, \bar\epsilon\gamma_*\partial\!\!\!/\lambda = 0\,. \end{aligned} \tag{14.13}$$

The actions of a supersymmetric field theories are easily obtained by choosing F and D to be components of composite multiplets constructed from elementary chiral and gauge multiplets. The SUSY field theories discussed in Ch. 8 and some important extensions can be obtained in this way.

14.2.1 The superpotential

We start with the simplest case. Suppose that we have a set of n elementary chiral multiplets with components $(Z^\alpha, \chi^\alpha, F^\alpha)$, $\alpha = 1,\ldots,n$. We now consider the composite chiral multiplet whose lowest component is a holomorphic function $W(Z^\alpha)$ of the scalars. By the same construction in (14.6)–(14.8), we find the χ and F components

$$\begin{aligned} \chi(W) &= W_\alpha\chi^\alpha\,, \qquad W_\alpha \equiv \frac{\partial W}{\partial Z^\alpha}\,,\\ F(W) &= W_\alpha F^\alpha - \tfrac12 W_{\alpha\beta}\bar\chi^\alpha P_L\chi^\beta\,, \qquad W_{\alpha\beta} \equiv \frac{\partial^2 W}{\partial Z^\alpha \partial Z^\beta}\,. \end{aligned} \tag{14.14}$$

The integral of $F(W)$ is a SUSY invariant.

14.2.2 Kinetic terms for chiral multiplets

We now describe a similar construction which involves the D-term of a composite real multiplet whose lowest component is $K(Z^\alpha,\bar Z^{\bar\alpha})$, an arbitrary real function of chiral multiplet scalars Z^α and their anti-chiral complex conjugates which we call $\bar Z^{\bar\alpha}$. It is convenient to include an overall factor of 1/2. We list the components of this multiplet which can be obtained using the chain rule, the variations of (8.16), (8.17), and then comparing with the general transformations of (14.10). Derivatives of K are denoted by subscripts $\alpha,\bar\beta,\ldots$ as in (14.14). We find that the components of this composite real multiplet are

$$\begin{aligned}
C(\tfrac12 K) &= \tfrac12 K\,,\\
\zeta(\tfrac12 K) &= -\frac{1}{\sqrt2}\mathrm{i}P_LK_\alpha\chi^\alpha+\frac{1}{\sqrt2}\mathrm{i}P_RK_{\bar\alpha}\chi^{\bar\alpha}\,,\\
\mathcal{H}(\tfrac12 K) &= -K_\alpha F^\alpha+\frac12 K_{\alpha\beta}\bar\chi^\alpha P_L\chi^\beta\,,\\
B_\mu(\tfrac12 K) &= \frac12\mathrm{i}K_\alpha\partial_\mu Z^\alpha-\frac12\mathrm{i}K_{\bar\alpha}\partial_\mu\bar Z^{\bar\alpha}+\frac12\mathrm{i}K_{\alpha\bar\beta}\bar\chi^\alpha P_L\gamma_\mu\chi^{\bar\beta}\,,\\
P_R\lambda(\tfrac12 K) &= \frac{1}{\sqrt2}\mathrm{i}K_{\alpha\bar\beta}P_R\left[(\partial\!\!\!/\bar Z^{\bar\beta})\chi^\alpha-F^\alpha\chi^{\bar\beta}\right]+\frac{\mathrm{i}}{2\sqrt2}K_{\alpha\beta\bar\gamma}\chi^{\bar\gamma}\bar\chi^\alpha P_L\chi^\beta\,,\\
D(\tfrac12 K) &= K_{\alpha\bar\beta}\left(-\partial_\mu Z^\alpha\partial^\mu\bar Z^{\bar\beta}-\tfrac12\bar\chi^\alpha P_L\partial\!\!\!/\chi^{\bar\beta}-\tfrac12\bar\chi^{\bar\beta}P_R\partial\!\!\!/\chi^\alpha+F^\alpha\bar F^{\bar\beta}\right)\\
&\quad+\tfrac12\left[K_{\alpha\beta\bar\gamma}\left(-\bar\chi^\alpha P_L\chi^\beta\bar F^{\bar\gamma}+\bar\chi^\alpha P_L(\partial\!\!\!/Z^\beta)\chi^{\bar\gamma}\right)+\text{h.c.}\right]\\
&\quad+\tfrac14 K_{\alpha\beta\bar\gamma\bar\delta}\,\bar\chi^\alpha P_L\chi^\beta\bar\chi^{\bar\gamma}P_R\chi^{\bar\delta}\,.
\end{aligned}\tag{14.15}$$

The general result (14.13) guarantees that $S=\int\mathrm{d}^4x\,D(K/2)$ is a SUSY invariant. It is a generalized kinetic term for chiral multiplets. Indeed the first term of $D(K/2)$ suggests that we interpret $K(Z^\alpha,\bar Z^{\bar\alpha})$ as the Kähler potential for the metric $g_{\alpha\bar\beta}$. Then

$$S(K)=-\int\mathrm{d}^4x\,g_{\alpha\bar\beta}\partial_\mu Z^\alpha\partial^\mu\bar Z^{\bar\beta}+\ldots\,,\qquad g_{\alpha\bar\beta}=K_{\alpha\bar\beta}=\partial_\alpha\partial_{\bar\beta}K\,.\tag{14.16}$$

The first term is the kinetic action for the nonlinear σ-model on a Kähler target space, and the remaining terms give the $\mathcal{N}=1$ supersymmetric extension of this σ-model. We will discuss its properties in Sec. 14.3. In the special case $K=\sum_\alpha Z^\alpha Z^{\bar\alpha}$, the Kähler metric $g_{\alpha\bar\beta}=\delta_{\alpha\bar\beta}$ is flat, and $S(K)$ reduces to the conventional chiral multiplet kinetic term in (8.18).

14.2.3 Kinetic terms for gauge multiplets

Our final application gives a *generalized kinetic action for gauge multiplets*, which includes interactions with chiral multiplets. The result is a supersymmetric extension of the theories considered in Sec. 4.2. Suppose that we have a set of *abelian* gauge multiplets with components A_μ^A, λ^A, D^A labelled by upper indices $A, B = 1, \ldots, n$. From the gaugino components λ^A, λ^B of any pair of these multiplets, we construct the complex Lorentz scalar $\bar{\lambda}^A P_L \lambda^B$. From (8.41) one can see that its SUSY transformation involves only the chiral projection $P_L\epsilon$. Hence, there is a composite chiral multiplet whose lowest component is $\bar{\lambda}^A P_L \lambda^B$. We also consider another set of composite chiral multiplets whose lowest components $f_{AB}(Z^\alpha) = f_{BA}(Z^\alpha)$ are holomorphic functions of chiral multiplet scalars Z^α. (The trivial case in which f_{AB} is simply a symmetric matrix of complex numbers is also allowed.) By the method of Ex. 14.2, one can show that the product of these two multiplets (multiplied by the convenient factor $\frac{1}{4}$) has the following components:

$$\begin{aligned}
Z(f) &= -\tfrac{1}{4} f_{AB} \bar{\lambda}^A P_L \lambda^B \,, \\
P_L\chi(f) &= \frac{1}{2\sqrt{2}} f_{AB} \left(\tfrac{1}{2}\gamma^{\mu\nu} F_{\mu\nu}^{-A} - \mathrm{i}D^A\right) P_L\lambda^B - \frac{1}{4} f_{AB\alpha} P_L\chi^\alpha \bar{\lambda}^A P_L \lambda^B \,, \\
F(f) &= \frac{1}{4} f_{AB} \left(-2\bar{\lambda}^A P_L \not{\partial}\lambda^B - F_{\mu\nu}^{-A} F^{\mu\nu\,-B} + D^A D^B\right) \\
&\quad + \frac{1}{2\sqrt{2}} f_{AB\alpha} \bar{\chi}^\alpha \left(-\frac{1}{2}\gamma^{\mu\nu} F_{\mu\nu}^{-A} + \mathrm{i}D^A\right) P_L\lambda^B - \frac{1}{4} f_{AB\alpha} F^\alpha \bar{\lambda}^A P_L \lambda^B \\
&\quad + \frac{1}{8} f_{AB\alpha\beta} \bar{\chi}^\alpha P_L \chi^\beta \bar{\lambda}^A P_L \lambda^B \,.
\end{aligned} \tag{14.17}$$

We use the notation $f_{AB\alpha} = \partial_\alpha f_{AB}$, etc. and the (anti-)self-dual tensor $F_{\mu\nu}^-$ introduced in (4.35).

It follows from (14.13) that $\int \mathrm{d}^4x\, F(f)$ is a supersymmetric action. In the simplest case, namely $f_{AB} = \delta_{AB}$, this action reduces to the kinetic action of a set of n free gauge multiplets, specifically the action (8.7) with gauge coupling $g = 0$. Note also that imaginary constant terms in f_{AB} give total derivatives in $F(f)$, which give vanishing contribution to the action integral. The F-component of a chiral multiplet is complex, so in the general case $f_{AB}(Z^\alpha)$, we must take the sum of $\int \mathrm{d}^4x\, F(f)$ plus its complex conjugate to obtain a Hermitian action.

14.3 Kähler geometry from chiral multiplets

We now discuss the properties of the action $\int \mathrm{d}^4x\, D(K/2)$, obtained from (14.15), in more detail. We interpret $K(Z, \bar{Z})$ as a Kähler potential and $g_{\alpha\bar{\beta}} = K_{\alpha\bar{\beta}}$ as the metric.[2] The action then describes the $\mathcal{N} = 1$ extension of the nonlinear σ-model with a Kähler manifold as target space. It is interesting to see how the

[2] Note that $D(K/2)$ is invariant under the Kähler transformation $K \to K + f(Z) + \bar{f}(\bar{Z})$ discussed in Ch. 13. The Kähler transformation is implemented as the shift of the real multiplet of (14.15) by the real part of a chiral multiplet. This is similar to the transformation of the elementary real multiplet to Wess-Zumino gauge discussed in Sec. 14.1.2.

mathematics of Kähler geometry, discussed in Ch. 13, emerges from the construction of the theory via the multiplet calculus. We will use the formulas (13.19)-(13.22) for the connection and curvature tensor of a Kähler metric $g_{\alpha\bar\beta}$.

We will bring the Lagrangian $D(K/2)$ into a form in which its geometrical content is manifest. The first step is to note the solution of the auxiliary field equation in (14.15)

$$F^\alpha = \tfrac12 g^{\alpha\bar\beta} K_{\gamma\beta\bar\beta}\bar\chi^\gamma P_L\chi^\beta = \tfrac12\Gamma^\alpha_{\gamma\beta}\bar\chi^\gamma P_L\chi^\beta\,. \tag{14.18}$$

After substitution of this result in the action and some rearrangement, one finds the equivalent action

$$\begin{aligned} S(K)|_F \;=\; \int \mathrm{d}^4x\, \Big[& g_{\alpha\bar\beta}\left(-\partial_\mu Z^\alpha\partial^\mu\bar Z^{\bar\beta} - \tfrac12\bar\chi^\alpha P_L\nabla\!\!\!\!/\,\chi^{\bar\beta} - \tfrac12\bar\chi^{\bar\beta}P_R\nabla\!\!\!\!/\,\chi^\alpha\right) \\ & +\tfrac14 R_{\alpha\bar\gamma\beta\bar\delta}\,\bar\chi^\alpha P_L\chi^\beta\bar\chi^{\bar\gamma}P_R\chi^{\bar\delta}\Big]\,. \end{aligned}\tag{14.19}$$

Fermion derivatives are defined by

$$P_L\nabla_\mu\chi^\alpha \;=\; P_L\left(\partial_\mu\chi^\alpha + \Gamma^\alpha_{\beta\gamma}\chi^\gamma\partial_\mu Z^\beta\right)\,,\qquad P_R\nabla_\mu\chi^{\bar\alpha} \;=\; P_R\left(\partial_\mu\chi^{\bar\alpha} + \Gamma^{\bar\alpha}_{\bar\beta\bar\gamma}\chi^\gamma\partial_\mu\bar Z^{\bar\beta}\right)\,. \tag{14.20}$$

As we will see very soon these derivatives are covariant under reparametrizations of the target space.

Let us compare the result (14.19) with (8.18). We see that the Kähler σ-model contains new nonlinear interactions, including quartic fermion terms. These couplings are described by the connection and curvature tensor of the target space, quantities that have geometrical significance. The situation is similar to string theory in which couplings typically have a geometric interpretation.

In fact we now show that each of the four terms in (14.19) is invariant under transformations of the fields Z^α, $P_L\chi^\alpha$, which are natural extensions of reparametrizations of the target space to include the fermions. Bosons transform as $Z'^\alpha = Z'^\alpha(Z)$. The Z'^α will have fermion partners χ'^α determined by the chiral multiplet transformation rules and related to χ^α by the chain rule. We write

$$\begin{aligned} \delta Z'^\alpha &= \frac{1}{\sqrt2}\bar\epsilon P_L\chi'^\alpha \\ &= \frac{\partial Z'^\alpha}{\partial Z^\beta}\delta Z^\beta = \frac{\partial Z'^\alpha}{\partial Z^\beta}\frac{1}{\sqrt2}\bar\epsilon P_L\chi^\beta\,. \end{aligned}\tag{14.21}$$

Thus reparametrization of the Z^α is accompanied by a transformation of the fermions, and we have the formulas

$$Z'^\alpha = Z'^\alpha(Z)\,,\qquad P_L\,\chi'^\alpha = \frac{\partial Z'^\alpha}{\partial Z^\beta}P_L\,\chi^\beta\,. \tag{14.22}$$

We see that the χ^α transform as tangent vectors on the target space.

Ex. 14.5 *Show that the spacetime derivatives $\partial_\mu Z^\alpha$ and $\nabla_\mu \chi^\alpha$ also transform as tangent vectors.*

The transformations of the fields $\bar{Z}^{\bar{\alpha}}$ and $\chi^{\bar{\alpha}}$ are the conjugates of those above. It is then quite obvious that the four terms in (14.19) are each invariant under reparametrization.

Ex. 14.6 *Obtain the transformation of the F^α auxiliary fields under reparametrization from the SUSY transform of $P_L\chi'^\alpha$ as follows:*

$$\delta P_L \chi'^\alpha \equiv \frac{1}{\sqrt{2}} P_L \left(\not{\partial} Z'^\alpha + F'^\alpha \right) = \delta \left(\frac{\partial Z'^\alpha}{\partial Z^\beta} P_L \chi^\beta \right) . \tag{14.23}$$

Work out the SUSY variation of the product in the last term. After a Fierz rearrangement you should find that

$$F'^\alpha = \frac{\partial Z'^\alpha}{\partial Z'^\beta} F^\beta - \frac{1}{2} \frac{\partial^2 Z'^\alpha}{\partial Z^\beta \partial Z^\gamma} (\bar{\chi}^\beta P_L \chi^\gamma) \tag{14.24}$$

The connection-like term in the transformation is compatible with (and required by) the solution (14.18).

The solution (14.18) applies to the case when the complete theory under study is specified by the action integral of the D-term $D(K/2)$ in (14.15). The solution for F^α will change if we modify the theory by adding a superpotential or include interactions with gauge multiplets. To allow for these generalizations it is convenient to define the new auxiliary field

$$\tilde{F}^\alpha = F^\alpha - \tfrac{1}{2} g^{\alpha\bar{\beta}} K_{\gamma\beta\bar{\beta}} \bar{\chi}^\gamma P_L \chi^\beta = F^\alpha - \tfrac{1}{2} \Gamma^\alpha_{\gamma\beta} \bar{\chi}^\gamma P_L \chi^\beta . \tag{14.25}$$

The full kinetic action integral of $D(K/2)$ can then be written as

$$S(K) = S(K)|_F + \int \mathrm{d}^4 x \, g_{\alpha\bar{\beta}} \tilde{F}^\alpha \bar{\tilde{F}}^{\bar{\beta}} , \tag{14.26}$$

where $S(K)|_F$ in the expression in (14.19). This form of the action is manifestly invariant under reparametrization of the target space since the first term $S(K)|_F$, given in (14.19), is constructed from covariant elements and the auxiliary $\tilde{F}^\alpha$ now transforms as a tangent vector.

14.4 General couplings of chiral multiplets and gauge multiplets

In the actions discussed in Sec. 14.2, all gauge multiplets are effectively abelian, i.e. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and possible internal symmetries of chiral multiplets are all global symmetries. A theory of this type is said to possess 'ungauged supersymmetry.' The main purpose of this section is to derive more general supersymmetric theories in which internal symmetries are gauged. In these new theories the fields of gauge and chiral multiplets are coupled in a more intimate way, but the actions are still built from the previous three SUSY invariants, determined by a real function $K(Z,\bar{Z})$, a holomorphic function $W(Z)$ and a holomorphic symmetric tensor $f_{AB}(Z)$.

First, it is useful to combine the previous results for ungauged supersymmetry into the single action:

$$\begin{aligned} S &= S(K)+S(W)+S(f)\,, \\ S(K) &= \int \mathrm{d}^4x\, D(\tfrac{1}{2}K) = \int \mathrm{d}^4x\, \mathcal{L}_{\rm kin,chir}\,, \\ S(W) &= \int \mathrm{d}^4x\, F(W)+\text{h.c.} = \int \mathrm{d}^4x\, \mathcal{L}_{\rm pot,chir}\,, \\ S(f) &= \int \mathrm{d}^4x\, F(f)+\text{h.c.} = \int \mathrm{d}^4x\, \mathcal{L}_{\rm kin,gauge}\,. \end{aligned} \tag{14.27}$$

They describe, respectively, kinetic and potential terms for chiral multiplets, and kinetic terms of the gauge multiplets, which can also depend on chiral multiplets fields. Note that we now include the hermitian conjugates of $S(W)$ and $S(f)$. Using results of the previous sections, we find the following explicit Lagrangians:

$$\begin{aligned} \mathcal{L}_{\rm kin,chir} &= g_{\alpha\bar\beta}\Big[-\partial_\mu Z^\alpha\partial^\mu\bar Z^{\bar\beta} - \tfrac12\bar\chi^\alpha P_L \not{\nabla}\chi^{\bar\beta} - \tfrac12\bar\chi^{\bar\beta}P_R\not{\nabla}\chi^\alpha \\ &\qquad + \Big(F^\alpha - \tfrac12\Gamma^\alpha_{\gamma\beta}\bar\chi^\gamma P_L\chi^\beta\Big)\Big(\bar F^{\bar\beta} - \tfrac12\Gamma^{\bar\beta}_{\bar\gamma\bar\alpha}\bar\chi^{\bar\gamma}P_R\chi^{\bar\alpha}\Big)\Big] \\ &\quad +\tfrac14 R_{\alpha\bar\gamma\beta\bar\delta}\,\bar\chi^\alpha P_L\chi^\beta\bar\chi^{\bar\gamma}P_R\chi^{\bar\delta}\,, \\ \mathcal{L}_{\rm pot,chir} &= W_\alpha F^\alpha - \tfrac12 W_{\alpha\beta}\bar\chi^\alpha P_L\chi^\beta + \text{h.c.}\,, \\ \mathcal{L}_{\rm kin,gauge} &= -\frac14\,\mathrm{Re}\,f_{AB}\left(2\bar\lambda^A\not{D}\lambda^B + F^A_{\mu\nu}F^{\mu\nu\,B} - 2D^AD^B\right) \\ &\quad +\frac18(\mathrm{Im}\,f_{AB})\varepsilon^{\mu\nu\rho\sigma}F^A_{\mu\nu}F^B_{\rho\sigma} + \frac14\mathrm{i}(\partial_\mu\,\mathrm{Im}\,f_{AB})\bar\lambda^A\gamma_*\gamma^\mu\lambda^B \\ &\quad +\left[\frac{1}{2\sqrt2}f_{AB\alpha}\bar\chi^\alpha\left(-\frac12\gamma^{\mu\nu}F^{-A}_{\mu\nu}+\mathrm{i}D^A\right)P_L\lambda^B - \frac14 f_{AB\alpha}F^\alpha\bar\lambda^AP_L\lambda^B\right. \\ &\quad \left. +\frac18 f_{AB\alpha\beta}\bar\chi^\alpha P_L\chi^\beta\bar\lambda^AP_L\lambda^B + \text{h.c.}\right]. \end{aligned} \tag{14.28}$$

The Kähler metric $g_{\alpha\bar\beta}$ of the scalar target space appears together with its connection and curvature tensor; $\nabla_\mu\chi$ is defined in (14.20). In the gauge multiplet Lagrangian we have split the contributions from the real parts and the imaginary parts of f_{AB}, and integrated by parts in one term. This shows that imaginary *constant* terms in f_{AB} do not contribute. It is also makes clear that $\mathrm{Re}\,f_{AB}$ can be interpreted as a metric in the gauge multiplet sector. Thus our theories contain two metrics, $g_{\alpha\bar\beta}$ for chiral multiplets and $\mathrm{Re}\,f_{AB}$ for gauge multiplets. It is a physical requirement that both metrics are positive definite, so that the energy of any classical field configuration is positive.

The introduction of gauged internal symmetries is sometimes called a 'deformation' of the ungauged theory. The process involves several steps, which are discussed in the next two subsections:

1. The target space space Kähler metric must have a Lie group of symmetries generated by holomorphic Killing vectors. A subgroup is chosen as the gauge group.
2. The theory must contain gauge multiplets for this group. These are promoted from Maxwell vector fields to Yang-Mills vectors.
3. Gauge and chiral multiplets are then coupled. This requires the 'reconstruction' of the D-term $D(K/2)$ using the SUSY transformation rules of gauged supersymmetry.

14.4.1 Global symmetries of the SUSY σ-model

We now assume that the target space Kähler metric has a continuous isometry group. As discussed in Ch. 13, this means that there are a set of holomorphic Killing vectors $k_A{}^\alpha(Z)$, which are related to real scalar moment maps $\mathcal{P}_A(Z,\bar{Z})$ by

$$k_A{}^\alpha(Z) = -\mathrm{i}g^{\alpha\bar{\beta}}\partial_{\bar{\beta}}\mathcal{P}_A(Z,\bar{Z})\,. \tag{14.29}$$

Lie brackets of the Killing vectors close as in (13.60) on a Lie algebra with structure constants $f_{AB}{}^C$.

The simplest example of holomorphic Killing vectors occurs for a flat Kähler metric $g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$. In this case the symmetries are linear transformations of the coordinates given by

$$\delta(\theta)Z^\alpha = -\theta^A(t_A)^\alpha{}_\beta Z^\beta\,. \tag{14.30}$$

The $(t_A)^\alpha{}_\beta$ are the matrix generators of a Lie algebra such as SU(n). The Killing vectors and their moment maps are (see exercise 13.15)

$$k_A{}^\alpha = -(t_A)^\alpha{}_\beta Z^\beta\,,\qquad \mathcal{P}_A = -\mathrm{i}\bar{Z}^{\bar{\alpha}}\delta_{\bar{\alpha}\alpha}(t_A)^\alpha{}_\beta Z^\beta\,. \tag{14.31}$$

Coupling constants could be introduced according to our discussion in the introduction to Sec. 8.3, Here we should then also replace $k_A{}^\alpha$ by $g_ik_A{}^\alpha$ and $\mathcal{P}_A$ by $g_i\mathcal{P}_A$ with the g_i for simple or U(1) factor as explained.

A more general Killing symmetry is really a special case of a reparametrization of the target space whose effect on chiral multiplet fields Z^α, χ^α, F^α was derived in Section 14.3. It is the special case of an infinitesimal reparametrization generated by a holomorphic Killing vector. Thus we write $Z'^\alpha(Z) = Z^\alpha + \theta^A k_A{}^\alpha(Z)$ and work to first order in the constant scalar parameters θ^A. Therefore we define the symmetry transformation of the fields Z^α by

$$\delta(\theta)Z^\alpha = \theta^A k_A{}^\alpha(Z)\,, \tag{14.32}$$

The analogous transformations of χ^α and F^α are just the infinitesimal versions of (14.22) and (14.24), namely

$$\delta(\theta)P_L\chi^\alpha = \theta^A\frac{\partial k_A{}^\alpha(Z)}{\partial Z^\beta}P_L\chi^\beta\,, \tag{14.33}$$

$$\delta(\theta)F^\alpha = \theta^A\frac{\partial k_A{}^\alpha(Z)}{\partial Z^\beta}F^\beta - \frac{1}{2}\theta^A\frac{\partial^2 k_A{}^\alpha(Z)}{\partial Z^\beta\partial Z^\gamma}\bar\chi^\beta P_L\chi^\gamma\,. \tag{14.34}$$

These expressions can also be derived from the requirement that internal $\delta(\theta)$ and SUSY transformations $\delta(\epsilon)$ commute.

So far we have been working at the level of global internal symmetry. The action $S(K)$ is invariant under the transformations (14.32), (14.33) and (14.34), while $S(W)$ is invariant if $W(Z)$ is invariant, i.e. if

$$W_\alpha k_A{}^\alpha = 0\,. \tag{14.35}$$

14.4.2 Gauge and SUSY transformations for chiral multiplets

We now want to promote some of these Killing symmetries to gauge symmetries, with arbitrary functions $\theta^A(x)$ as gauge parameters, while maintaining $\mathcal{N}=1$ supersymmetry. Of course, the subset of symmetries that we choose to gauge should form a closed algebra.[3] From now on, the index A refers only to the Killing symmetries which are gauged.

Gauging is an involved process, but the first steps are clear. The theory must contain a gauge multiplet A_μ^A, λ^A, D^A for each Killing vector. In general there are both non-abelian and abelian gauge multiplets whose SUSY and gauge transformation rules and gauge covariant algebra were discussed in Sec. 8.3.1, see (8.41), (8.42) and (8.43) and abelian multiplets.

The next step is to define gauge covariant derivatives for non-linear Killing symmetries:

$$D_\mu Z^\alpha = \partial_\mu Z^\alpha - A_\mu^A k_A{}^\alpha(Z)\,, \tag{14.36}$$

$$D_\mu P_L\chi^\alpha = \partial_\mu P_L\chi^\alpha - A_\mu^A\frac{\partial k_A{}^\alpha(Z)}{\partial Z^\beta}P_L\chi^\beta\,. \tag{14.37}$$

With these ingredients we can obtain the SUSY transformation rules of chiral multiplet components. We extend the method of Ex. 14.1 to Killing symmetries, imposing the gauge covariant SUSY algebra (14.5) in each of the three steps. This procedure leads to transformation rules which include the covariant derivatives (14.36), (14.37) and which generalize (8.53):

$$\delta Z^\alpha = \frac{1}{\sqrt{2}}\bar\epsilon P_L\chi^\alpha\,,$$

[3] We consider only 'electric gaugings', but we refer readers to [35] for an interesting generalization which includes both 'electric' and 'magnetic' gauge potentials. The new formulation embodies the duality symmetry discussed in Sec. 4.2.

$$\begin{aligned}
\delta P_L\chi^\alpha &= \frac{1}{\sqrt{2}}P_L(\not{D}Z^\alpha + F^\alpha)\epsilon\,,\\
\delta F^\alpha &= \frac{1}{\sqrt{2}}\bar\epsilon\,\not{D}P_L\chi^\alpha + \bar\epsilon P_R\lambda^A k_A{}^\alpha(Z)\,.
\end{aligned}\tag{14.38}$$

Ex. 14.7 *Show that both $D_\mu Z^\alpha$ and $D_\mu P_L\chi^\alpha$ transform without derivatives of $\theta^A(x)$ under a gauge transformation. For χ^α you should obtain*

$$\delta(\theta)D_\mu P_L\chi^\alpha = \theta^A\frac{\partial k_A{}^\alpha(Z)}{\partial Z^\beta}D_\mu P_L\chi^\beta\,.\tag{14.39}$$

14.4.3 Actions of chiral multiplets in a gauge theory

We can now generalize our results for supersymmetric actions to the situation in which chiral multiplets transform under gauged Killing symmetries and are therefore coupled to gauge multiplets. The potential terms in $S(W)$ of (14.27) are the easiest to discuss. The superpotential $W(Z)$ must be gauge invariant, i.e. it must satisfy (14.35). The previous construction of the composite chiral multiplet is then unchanged, and the Lagrangian $\mathcal{L}_{\text{pot,chir}}$ is correct in gauged supersymmetry.

Gauged Killing symmetry does require modification of the kinetic action. To find the changes we need to reconstruct the D-term of the composite real multiplet $D(K/2)$ of (14.15) using the new transformation rules (14.38) for chiral multiplets. Since these rules involve gauge multiplet components, we also need their transformation rules (8.41) in the process.

The scalar $K(Z,\bar Z)$ is the Kähler potential. We saw in Sec. 13.4 that this is not necessarily invariant under a Killing symmetry, but can change by a Kähler transformation as shown in (13.57). However, it simplifies the reconstruction of the real multiplet in gauged supersymmetry to assume that K is gauge invariant. This allows us to use the formulas of (14.10) which were derived by imposing the SUSY algebra (12.1) for gauge-invariant quantities. We will show that the final result for the new D-term is valid even for non-invariant K.

We now outline the construction of the modified $D(K/2)$ focusing on the new terms required in gauged supersymmetry. We start with the component $C(K/2) = K(Z,\bar Z)$, compute the variation $\delta C(K/2)$ using δZ^α in (14.38), use (14.10) to identify the component $\zeta(K/2)$, and then repeat this process for higher dimension components of the composite multiplet. The components C, ζ and $\mathcal{H}$ are the same as in (14.15). In $B_\mu(K/2)$ the covariant derivative (14.36) appears rather than ∂_μ. This was to be expected, and similar 'trivial covariantizations' also occur in $\lambda(K/2)$ and $D(K/2)$. The first essential change occurs in the step when we compute $\delta\mathcal{H}(K/2)$ using δF^α in (14.38). This leads to the new expression for the λ-component:

$$P_R\lambda(\tfrac12 K) = \frac{1}{\sqrt{2}}\mathrm{i}K_{\alpha\bar\beta}P_R\left[(\not{D}\bar Z^{\bar\beta})\chi^\alpha - F^\alpha\chi^{\bar\beta}\right] + \frac{\mathrm{i}}{2\sqrt{2}}K_{\alpha\beta\bar\gamma}\chi^{\bar\gamma}\bar\chi^\alpha P_L\chi^\beta - \mathrm{i}P_R\lambda^A k_A{}^\alpha K_\alpha\,.\tag{14.40}$$

The next step is to compute $\delta P_R\lambda(K/2)$ and use (14.10) to identify the new form of the D-component. There is no need to repeat previous work, so we concentrate on new terms due to the gauged Killing symmetry. These terms come from three sources: i) from the $\delta A_\mu{}^A$ variation of the covariant derivative in (14.40), ii) from the gauge multiplet modification of δF^α, and iii) from the variation of the last term in (14.40). Fierz rearrangement identities, given in (3.69), are needed at this stage. The result for the new D-component is

$$D(\tfrac{1}{2}K) \;=\; \ldots - \mathrm{i}D^A k_A{}^\alpha K_\alpha - \sqrt{2}g_{\alpha\bar\beta}\bar\lambda^A\left(P_L\chi^\alpha k_A{}^{\bar\beta} + P_R\chi^{\bar\beta}k_A{}^\alpha\right). \tag{14.41}$$

where the ... stand for the terms already written in (14.15) with derivatives replaced by covariant derivatives.

To complete our discussion we need some properties of moment maps derived in Sec. 13.4.2. We are assuming that the Kähler potential is invariant under the Killing symmetries which are gauged. Then $F(z)$ in (13.57) is restricted to be an imaginary constant, and (13.58) requires that moment maps take the form

$$\mathcal{P}_A = \mathrm{i}k_A^\alpha K_\alpha + p_A\,. \tag{14.42}$$

where the p_A are arbitrary real constants. The p_A are called Fayet-Iliopoulos (F-I) terms [105]. They are related to a simple new type of supersymmetric action available for abelian gauge multiplets. Indeed if $D(x)$ is the auxiliary field of an abelian gauge multiplet and $\lambda(x)$ is the gaugino in this multiplet, then $\delta D = \bar\epsilon\not{\partial}\lambda$ is a total derivative, see (8.41). The integral $\int \mathrm{d}^4x\, D(x)$ is then invariant under SUSY.

To explore the physics of F-I terms in a more general way, we consider the conditions under which we can replace the action integral $D(K/2)$ in (14.41) by the integral of

$$D(\tfrac{1}{2}K) \;=\; \ldots - D^A\mathcal{P}_A - \sqrt{2}\bar\lambda^A\left(P_L\chi^\alpha k_{A\alpha} + P_R\chi^{\bar\beta}k_{A\bar\beta}\right). \tag{14.43}$$

The difference is the integral

$$S_{\rm FI} = -\int \mathrm{d}^4x\, p_A D^A\,, \tag{14.44}$$

and the replacement is valid if the integral S_{FI} is invariant under supersymmetry and gauge transformations. Both conditions require

$$p_A f_{BC}{}^A = 0\,, \tag{14.45}$$

which must hold for all choices of the indices B, C. Thus, the constants p_A vanish for generators that occur at the right-hand side of commutation relations. (In mathematics terminology this is the derived algebra.) This is the case for generators in simple non-abelian factors of the gauge group. Conversely, if A is the index of an abelian factor of the gauge group, we can choose $p_A \neq 0$ as a possible F-I coupling. In general the linear equations (14.45) admit $n_{\rm FI}$ non-trivial linearly independent 'vectors' p_A, and the gauged theory contains $n_{\rm FI}$ independent F-I couplings.

Ex. 14.8 *Consider a simple non-abelian gauge group and suppose that K is gauge invariant. Then the moment maps are given by $\mathcal{P}_A = \mathrm{i}k_A^\alpha \partial_\alpha K$. Show that the equivariance condition (13.61) is satisfied.* Hint*: carefully commute derivatives.*

The discussion above shows that we can use $D(K/2)$ of (14.43) in the action integral of chiral multiplets in gauged supersymmetry if the target space Kähler potential $K(Z,\bar Z)$ is gauge invariant. Fayet-Iliopoulos terms appear as an integral of the form $S_{\rm FI}$ contained within

$$S(K) = \int \mathrm{d}^4x \, D\left(\tfrac12 K\right) \,. \tag{14.46}$$

It is quite interesting that the mathematics of constant shifts in moment maps $\mathcal{P}_A$ for abelian Killing symmetries corresponds to new terms in the general action integral of gauged supersymmetry.

Let's consider the situation when K is not invariant, and the moment map is corrected by the extra term shown in (13.58). We now outline an argument to show that the spacetime integral of the D-term (14.43) is invariant under SUSY, whether or not K is gauge invariant. We apply the component transformation rules of (14.38) and (8.41) to the first term of (14.43), obtaining

$$\begin{aligned} -\delta \int \mathrm{d}^4x \, D^A \mathcal{P}_A \;\; = \;\; & -\int \mathrm{d}^4x \left[\frac12 \mathrm{i}\bar\epsilon\gamma_*\gamma^\mu \left(D_\mu\lambda^A\right)\mathcal{P}_A \right. \\ & \left. + \frac{1}{\sqrt2}\mathrm{i}D^A\,\bar\epsilon\left(P_L\chi^\alpha k_{A\alpha} - P_R\chi^{\bar\beta}k_{A\bar\beta}\right)\right] \end{aligned} \tag{14.47}$$

We will show that the first term cancels with part of the $\delta P_L\chi^\alpha \sim P_L\gamma^\mu D_\mu Z$ plus conjugate $\delta P_R\bar\chi^{\bar\beta}$ variations of the second term in (14.43). These variations are

$$-\int \mathrm{d}^4x \bar\lambda^A\left(P_L D_\mu Z^\alpha k_{A\alpha} + P_R D_\mu \bar Z^{\bar\beta} k_{A\bar\beta}\right)\gamma^\mu\epsilon \,. \tag{14.48}$$

We isolate the axial vector term $\bar\lambda\gamma_*\gamma^\mu\epsilon$ of this expression. After a Majorana flip and use of (13.51), it becomes

$$\begin{aligned} & -\tfrac12 \mathrm{i}\bar\epsilon\gamma_*\gamma^\mu\lambda^A\left(D_\mu Z^\alpha\partial_\alpha + D_\mu\bar Z^{\bar\alpha}\partial_{\bar\alpha}\right)\mathcal{P}_A \\ & = -\tfrac12 \mathrm{i}\bar\epsilon\gamma_*\gamma^\mu\lambda^A\left[\partial_\mu\mathcal{P}_A - A_\mu^B\left(k_B^\alpha\partial_\alpha + k_B^{\bar\alpha}\partial_{\bar\alpha}\right)\mathcal{P}_A\right] \\ & = -\tfrac12 \mathrm{i}\bar\epsilon\gamma_*\gamma^\mu\lambda^A\left[\partial_\mu\mathcal{P}_A - A_\mu^B f_{BA}{}^C\mathcal{P}_C\right] \,. \end{aligned} \tag{14.49}$$

In the last step we used the equivariance condition (13.61) to write the covariant derivative $D_\mu\mathcal{P}_A = \partial_\mu\mathcal{P}_A - A_\mu^B f_{BA}{}^C\mathcal{P}_C$. The $D_\mu\lambda^A$ term in (14.47) then combines with the $D_\mu \mathcal{P}_A$ term in (14.49 into a total derivative which vanishes in the integrated $\delta S(K)$. Thus equivariant moment maps are necessary for the consistent gauging of Killing symmetries. The moment maps $\mathcal{P}_A$ used in (14.43) must be equivariant!

There are several other terms in the variation $\delta S(K)$ that depend on the Killing vector, and they must all cancel. For example, the $\bar{\lambda}\gamma^\mu\epsilon$ terms in (14.48) cancel with terms that originate from the variation of A_μ^A in the kinetic term $D_\mu Z D^\mu \bar{Z}$ of $\mathcal{L}_{\text{kin,chiral}}$ with derivatives replaced by covariant derivatives. Similarly, the second term in (14.47) cancels against the $\delta\bar{\lambda}^A \sim \bar{\epsilon}D^A$ variation of the second term in (14.43). The cancellation of these variations and others which we have not mentioned do not depend on the assumption that $K(Z,\bar{Z})$ is gauge invariant. The conclusion of this argument is that the action $S(K)$ (14.46) is invariant under SUSY transformations if we use the D-term of (14.43) with moment maps that satisfy the equivariance condition (13.61) or, equivalently, (13.62). The addition of F-I constants p_A satisfying (14.45), is consistent with this equivariance equation.

14.4.4 General kinetic action of the gauge multiplet

The general kinetic terms for Abelian gauge multiplets derived in Sec. 14.2.3 include holomorphic functions $f_{AB}(Z)$ of chiral multiplet scalars. The Lagrangian given in (14.28) is also correct for non-abelian gauge multiplets provided that several changes required for non-abelian gauge invariance are made. These changes are
i. Use the non-abelian gauge field strength in $F_{\mu\nu}^{-A}$,
ii. Replace $\partial_\mu\lambda^A$ by the Yang-Mills covariant derivative $D_\mu\lambda^A$,
iii. The quantity $f_{AB}(Z^\alpha)$ is no longer arbitrary but must transform as the direct product of adjoint representations, specifically as[4]

$$\delta f_{AB}(Z) = f_{AB\alpha}\theta^C k_C^\alpha = 2\,\theta^C f_{C(A}{}^D f_{B)D}(Z)\,. \tag{14.50}$$

This last requirement is needed both for gauge invariance of the action and for global SUSY. Indeed, an action cannot be supersymmetric invariant unless it is gauge invariant. The latter statement follows from the fact that the commutator of two supersymmetries includes a gauge symmetry, see e.g. (11.30).

It is quite common to assume that the 'tensor' f_{AB} is proportional to the Cartan-Killing metric on each simple factor of the gauge group. For a compact simple Lie algebra, one can choose a basis in which f_{AB} is proportional to δ_{AB}. In string theory applications, there is often still a gauge-invariant proportionality factor depending on moduli-fields. In the simplest case $f_{AB} = \delta_{AB}$, the Lagrangian $\mathcal{L}_{\text{kin,gauge}}$ in (14.28) reduces to that of (8.7) and is gauge invariant.

[4] There is a more general possibility related to the fact that imaginary constant parts in $f_{AB}(Z)$ do not contribute to the action. The gauge transformation may have extra terms $\mathrm{i}\theta^C C_{AB,C}$, where the $C_{AB,C} = C_{BA,C}$ are real constants. If the completely symmetric component vanishes, $C_{(AB,C)} = 0$, additional Chern-Simons terms are needed to restore supersymmetry. Otherwise, gauge anomalies (which are accompanied by supersymmetry anomalies) play a role. See [106, 107].

As mentioned after (4.62), the real part of f_{AB} should be positive definite in order that the kinetic terms of the vectors are positive definite. This implies that if one uses the Cartan-Killing metric, the gauge group should be compact (and one uses the negative of the Cartan-Killing metric, see Appendix B).

14.4.5 Requirements for an $\mathcal{N} = 1$ SUSY gauge theory

We now summarize the results derived earlier in this section. Actions for both gauged and ungauged supersymmetry are determined by three functions of the chiral multiplet scalars: the Kähler potential $K(Z,\bar{Z})$ (up to Kähler transformations (13.18)), a superpotential $W(Z)$ and a holomorphic symmetric matrix $f_{AB}(Z)$ (up to imaginary constants). The possible ways to gauge the theory are characterized by[5]

1. The choice of a gauge group which must be a subgroup of the holomorphic isometry group of the Kähler target space. Information on the Lie algebra of this group is encoded in the real structure constants $f_{AB}{}^{C} = -f_{BA}{}^{C}$, which should satisfy Jacobi identities as in (4.75).
2. Gauge transformations of the chiral multiplet, determined by real moment maps $\mathcal{P}_A(Z,\bar{Z})$.

We summarize the conditions that these should satisfy. The moment maps should satisfy

$$\nabla_\alpha \partial_\beta \mathcal{P}_A(Z,\bar{Z}) \,=\, 0\,. \tag{14.51}$$

This ensures that the Killing vectors

$$k_A{}^\alpha(Z) \,=\, \mathrm{i} g^{\alpha\bar{\beta}} \partial_{\bar{\beta}} \mathcal{P}_A(Z,\bar{Z})\,, \tag{14.52}$$

are holomorphic. Furthermore, they should satisfy the 'equivariance condition' (13.62)

$$k_A{}^\alpha g_{\alpha\bar{\beta}} k_B^{\bar{\beta}} - k_B^\alpha g_{\alpha\bar{\beta}} k_A{}^{\bar{\beta}} \,=\, \mathrm{i} f_{AB}{}^C \mathcal{P}_C\,. \tag{14.53}$$

This condition restricts the constant contributions to the moment maps, and ensures the closure of the algebra of Killing vectors.

A superpotential may be included if it is gauge invariant. Thus we require (14.35)

$$k_A{}^\alpha \partial_\alpha W(Z) \,=\, 0\,. \tag{14.54}$$

Finally, we require gauge invariant kinetic terms for vector multiplets, so the condition

$$k_C^\alpha(Z) f_{AB\alpha}(Z) \,=\, 2 f_{C(A}{}^D f_{B)D}(Z) + \mathrm{i} C_{AB,C}\,, \tag{14.55}$$

[5] These gaugings are sometimes called 'deformations', see Sec. 12.4.2. Technically, the superpotential is also already a deformation in the sense that it determines different actions with the same kinetic terms.

must be satisfied. See footnote 4 for a brief discussion of the constants $C_{AB,C}$.

In physical applications of global supersymmetry, kinetic terms must be positive, so the matrices $g_{\alpha\bar\beta}$ and $\mathrm{Re}\, f_{AB}$ must be positive definite.

When all these requirements are satisfied, we have an $\mathcal{N}=1$ supersymmetric gauge theory with Lagrangian

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{kin,chir}} + \mathcal{L}_{\text{kin,gauge}} + \mathcal{L}_{\text{pot,chir}} \\
\mathcal{L}_{\text{kin,chir}} &= g_{\alpha\bar\beta}\Big[-D_\mu Z^\alpha D^\mu \bar Z^{\bar\beta} - \tfrac12 \bar\chi^\alpha P_L \hat{\not{D}}\chi^{\bar\beta} - \tfrac12 \bar\chi^{\bar\beta} P_R \hat{\not{D}}\chi^\alpha \\
&\qquad + \left(F^\alpha - \tfrac12 \Gamma^\alpha_{\gamma\beta}\bar\chi^\gamma P_L\chi^\beta\right)\left(\bar F^{\bar\beta} - \tfrac12 \Gamma^{\bar\beta}_{\bar\gamma\bar\alpha}\bar\chi^{\bar\gamma} P_R \chi^{\bar\alpha}\right)\Big] \\
&\quad + \tfrac14 R_{\alpha\beta\bar\gamma\bar\delta}\,\bar\chi^\alpha P_L\chi^\beta \bar\chi^{\bar\gamma} P_R \chi^{\bar\delta} \\
&\quad - D^A\mathcal{P}_A - \sqrt2 \bar\lambda^A\left(P_L\chi^\alpha k_{A\alpha} + P_R\chi^{\bar\beta} k_{A\bar\beta}\right) , \\
\mathcal{L}_{\text{pot,chir}} &= W_\alpha F^\alpha - \tfrac12 W_{\alpha\beta}\bar\chi^\alpha P_L \chi^\beta + \text{h.c.}, \\
\mathcal{L}_{\text{kin,gauge}} &= -\frac14 \mathrm{Re}\, f_{AB}\left(2\bar\lambda^A \not{D}\lambda^B + F^A_{\mu\nu}F^{\mu\nu\, B} - 2D^A D^B\right) \\
&\quad + \frac18 (\mathrm{Im}\, f_{AB})\varepsilon^{\mu\nu\rho\sigma}F^A_{\mu\nu}F^B_{\rho\sigma} + \frac14 \mathrm{i}(\mathrm{Im}\, D_\mu f_{AB})\bar\lambda^A\gamma_*\gamma^\mu\lambda^B \\
&\quad + \left[\frac{1}{2\sqrt2} f_{AB\alpha}\bar\chi^\alpha\left(-\frac12\gamma^{\mu\nu}F^{-\,A}_{\mu\nu} + \mathrm{i}D^A\right)P_L\lambda^B - \frac14 f_{AB\alpha}F^\alpha\bar\lambda^A P_L\lambda^B \right. \\
&\quad \left. + \frac18 f_{AB\alpha\beta}\bar\chi^\alpha P_L\chi^\beta \bar\lambda^A P_L \lambda^B + \text{h.c.}\right] ,
\end{aligned} \tag{14.56}$$

where

$$\begin{aligned}
D_\mu Z^\alpha &= \partial_\mu Z^\alpha - A_\mu{}^A k_A{}^\alpha , \\
P_L \hat D_\mu \chi^\alpha &= \partial_\mu P_L\chi^\alpha - A^A_\mu \frac{\partial k_A{}^\alpha(Z)}{\partial Z^\beta} P_L \chi^\beta + \Gamma^\alpha_{\beta\gamma} P_L\chi^\gamma \partial_\mu Z^\beta , \\
D_\mu \lambda^A &= \partial_\mu \lambda^A + A_\mu{}^B f_{BC}{}^A \lambda^C , \\
F^A_{\mu\nu} &= \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + f_{BC}{}^A A^B_\mu A^C_\nu , \\
D_\mu f_{AB} &= \partial_\mu f_{AB} - 2A^C_\mu f_{C(A}{}^D f_{B)D} .
\end{aligned} \tag{14.57}$$

Ex. 14.9 *Write the full action for one chiral multiplet and one gauge multiplet, with Kähler metric $K = Z\bar Z$, kinetic matrix $f_{11} = 1$, and moment map $\mathcal{P}_1 = -gZ\bar Z + \xi$. Show that the superpotential can only be a constant.*

14.5 The physical theory

14.5.1 Elimination of auxiliary fields.

The Lagrangian of (14.56) contains the complex auxiliary fields F^α (and their conjugates $\bar F^{\bar\beta}$) of chiral multiplets, and the real auxiliaries D^A of gauge multiplets. The

Euler-Lagrange equations of these fields do not contain spacetime derivatives. They are purely algebraic and can be easily solved to express auxiliary fields in terms of the physical components. The dynamical content of the theory is preserved if we substitute the solutions for F^α and D^A back in the Lagrangian and transformation rules. The SUSY algebra changes in this process, but the physical content is unchanged. We saw examples in Ch. 8. We now discuss the elimination of auxiliary fields in the general Lagrangian (14.56) of gauged supersymmetry.

We first discuss the elimination of the auxiliary fields F^α (and $\bar{F}^{\bar{\beta}}$). The relevant terms in the Lagrangian of (14.56) are

$$\mathcal{L}_{\text{aux},F} \;=\; g_{\alpha\bar{\beta}}F^\alpha\bar{F}^{\bar{\beta}} - F^\alpha f_\alpha - \bar{F}^{\bar{\beta}}\bar{f}_{\bar{\beta}}\,, \tag{14.58}$$

with

$$f_\alpha = -W_\alpha + \tfrac{1}{2}g_{\alpha\bar{\beta}}\Gamma^{\bar{\beta}}_{\bar{\gamma}\bar{\alpha}}\bar{\chi}^{\bar{\gamma}}P_R\chi^{\bar{\alpha}} + \frac{1}{4}f_{AB\alpha}\bar{\lambda}^A P_L\lambda^B\,. \tag{14.59}$$

The equations of motion

$$g_{\alpha\bar{\beta}}\bar{F}^{\bar{\beta}} \;=\; f_\alpha\,, \qquad g_{\alpha\bar{\beta}}F^\alpha \;=\; \bar{f}_{\bar{\beta}}\,, \tag{14.60}$$

can be immediately solved using the inverse Kähler metric $g^{\alpha\bar{\beta}}$,

$$F^{\bar{\beta}} \;=\; g^{\alpha\bar{\beta}}f_\alpha\,, \qquad F^\alpha \;=\; g^{\alpha\bar{\beta}}\bar{f}_{\bar{\beta}}\,. \tag{14.61}$$

When these results are substituted in (14.58), one obtains

$$\mathcal{L}_{\text{aux},F} \;=\; -g^{\alpha\bar{\beta}}f_\alpha\bar{f}_{\bar{\beta}}\,. \tag{14.62}$$

The negative sign and the appearance of an inverse metric are general features of the elimination process. We will study the physical content of (14.62), but we first consider the elimination of the D^A auxiliaries.

The Lagrangian (14.56) contains the following terms involving the D^A:

$$\begin{aligned}\mathcal{L}_{\text{aux},D} \;&=\; \frac{1}{2}(\operatorname{Re} f_{AB})D^AD^B \\ &\quad + D^A\left[-\mathcal{P}_A + \frac{1}{2\sqrt{2}}\mathrm{i}f_{AB\alpha}\bar{\chi}^\alpha P_L\lambda^B - \frac{1}{2\sqrt{2}}\mathrm{i}\bar{f}_{AB\bar{\alpha}}\bar{\chi}^{\bar{\alpha}}P_R\lambda^B\right]\,.\end{aligned} \tag{14.63}$$

The solution of the field equation for D^A thus gives

$$\operatorname{Re} f_{AB}D^B = \mathcal{P}_A - \frac{1}{2\sqrt{2}}\mathrm{i}f_{AB\alpha}\bar{\chi}^\alpha P_L\lambda^B + \frac{1}{2\sqrt{2}}\mathrm{i}\bar{f}_{AB\bar{\alpha}}\bar{\chi}^{\bar{\alpha}}P_R\lambda^B\,. \tag{14.64}$$

Substitution in (14.63) gives the physically equivalent action

$$\begin{aligned}\mathcal{L}_{\text{aux},D} \;&=\; -\frac{1}{2}\,(\operatorname{Re} f)^{-1\,AB}\left(\mathcal{P}_A - \frac{1}{2\sqrt{2}}\mathrm{i}f_{AC\alpha}\,\bar{\chi}^\alpha P_L\lambda^C + \frac{1}{2\sqrt{2}}\mathrm{i}\bar{f}_{AC\bar{\alpha}}\,\bar{\chi}^{\bar{\alpha}}P_R\lambda^C\right)\times \\ &\quad \left(\mathcal{P}_B - \frac{1}{2\sqrt{2}}\mathrm{i}f_{BD\beta}\,\bar{\chi}^\beta P_L\lambda^D + \frac{1}{2\sqrt{2}}\mathrm{i}\bar{f}_{BD\bar{\beta}}\,\bar{\chi}^{\bar{\beta}}P_R\lambda^D\right)\,.\end{aligned} \tag{14.65}$$

Again a negative sign appears together with the inverse metric $(\operatorname{Re} f)^{-1\,AB}$.

14.5.2 The scalar potential

The scalar potential in a classical field theory is defined as the sum of all terms in the Lagrangian that contain only scalar fields and no spacetime derivatives. The sum is multiplied by -1 because the basic structure of a Lagrangian in classical mechanics is $\mathcal{L} = T - V$, kinetic $-$ potential. The Hamiltonian is $T + V$. The lowest energy state of the theory is found by minimizing the Hamiltonian. It is usually assumed that this vacuum state is Lorentz invariant and translation invariant. Lorentz invariance requires that gauge fields and fermions vanish in the vacuum, and translation invariance means that scalar kinetic terms, which contain spacetime derivatives, can be ignored. Thus the classical approximation to the vacuum state is obtained by minimizing the scalar potential. The values of scalar fields at the minimum are usually called 'vacuum expectation values.'

In $\mathcal{N} = 1$, $D = 4$ supersymmetric field theories, the scalar potential arises entirely from the elimination of auxiliary fields. We retain only the scalar terms in (14.62) and (14.65) and write the potential

$$V = g^{\alpha\bar{\beta}} W_\alpha \overline{W}_{\bar{\beta}} + \tfrac{1}{2} \left(\mathrm{Re}\, f\right)^{-1\,AB} \mathcal{P}_A \mathcal{P}_B \,. \tag{14.66}$$

The first term is called the F-term and the second term the D-term in accordance with their origin from the auxiliary fields. Minimization of this potential determines the vacuum state of the supersymmetric theory.

It is useful to recognize that this form arises from the auxiliary scalar contributions to fermion transformations. The general structure for any supersymmetric theory is[6]

$$V = (\delta_{\mathrm{s}}\text{fermion})\,(\text{metric})\,(\delta_{\mathrm{s}}\text{fermion}) \ , \tag{14.67}$$

where δ_{s}fermion refers to the scalar parts of the supersymmetry transformations of all fermions. These are often called *fermion shifts*. Specifically, in gauged global supersymmetry

$$\begin{aligned} \delta_{\mathrm{s}} P_L \chi^\alpha &\equiv \frac{1}{\sqrt{2}} F^\alpha = -\frac{1}{\sqrt{2}} g^{\alpha\bar{\beta}} \overline{W}_{\bar{\beta}} \,, \\ \delta_{\mathrm{s}} P_L \lambda^A &\equiv \frac{1}{2} \mathrm{i} D^A = \frac{1}{2} \mathrm{i} \left(\mathrm{Re}\, f\right)^{-1\,AB} \mathcal{P}_B \,. \end{aligned} \tag{14.68}$$

Only the scalar parts of the on-shell values of the auxiliary fields are included. Using (14.68) and the complex conjugate expressions for the P_R projections of the fermions, (14.66) can be rewritten as

$$V = 2 \left(\delta_{\mathrm{s}} P_L \chi^\alpha\right) g_{\alpha\bar{\beta}} \left(\delta_{\mathrm{s}} P_R \chi^{\bar{\beta}}\right) + 2 \left(\delta_{\mathrm{s}} P_L \lambda^A\right) \mathrm{Re}\, f_{AB} \left(\delta_{\mathrm{s}} P_R \lambda^B\right) \,. \tag{14.69}$$

The structure (14.67) is very important in supersymmetry, and we will see later that it applies also in supergravity. Thus it is generally valid.

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[6] The general rule is that the metric in all cases is determined by the fermion kinetic terms

Notice that the fermion variations in (14.66) and (14.69) are contracted with the appropriate metric in the chiral and gauge sectors of the theory. These metrics are positive definite. It then follows that the potential is positive semi-definite. Positivity of the potential is universal in global supersymmetry.[7] This is a consequence of the SUSY algebra. Indeed the trace of (8.2) is

$$\left\{Q_\alpha\,,\,(Q^\dagger)^\alpha\right\}_{\rm qu} = 2H\,. \tag{14.70}$$

so the expectation value of the Hamiltonian in any state of the theory is non-negative!

14.5.3 The vacuum state and SUSY breaking

As discussed in the previous section, the vacuum state is determined in the classical approximation by minimizing the scalar potential $V(Z,\bar{Z})$ given in (14.66) with respect to the scalar fields Z^α of the chiral multiplets in the theory. There may be multiple configurations of the Z^α's which realize the *same* minimum value of the potential. In this case the vacuum configuration is not unique, and any one of these configurations determines a possible vacuum state.

One important physical question of interest is whether symmetries of the action are broken in the vacuum. If so we say that the symmetry is *spontaneously* broken. We assume that readers are familiar with the consequences of spontaneously broken *internal* symmetry. If the symmetry in question is not gauged, i.e. it is a global symmetry, then the Goldstone theorem tells us that the theory contains a massless scalar particle. If the broken symmetry is a gauge symmetry, then, according to the Higgs mechanism, the Goldstone boson disappears but there is a massive spin 1 particle. In the classical approximation this information about the particle mass spectrum is contained in the quadratic terms in the power series expansion of the Lagrangian about the vacuum configuration. There is a more formal way, see [108], to characterize the possibility of breaking of a global internal symmetry. The symmetry is unbroken if the charge operator T_A annihilates the vacuum state, i.e. if $T_A|0\rangle = 0$, and the symmetry is broken if $T_A|0\rangle \neq 0$.

In this section we are primarily interested in whether supersymmetry is spontaneously broken in the vacuum state. If the elementary particles and interactions observed in Nature come from a supersymmetric theory then SUSY breaking is vital. The reason is that unbroken supersymmetry requires that for every bosonic particle, elementary or composite, there is a fermion of the same mass and conversely. This is certainly not the situation we observe, so if supersymmetry is relevant to experiment, it must appear as a broken symmetry.

At the quantum level the question of SUSY breaking is the question whether the supercharge components Q_α annihilate the vacuum state or not, i.e. whether

[7] We will see later that there are negative contributions in supergravity because gravitinos are gauge fields.

$Q_\alpha|0\rangle = 0$ or $Q_\alpha|0\rangle \neq 0$. The key to this question is contained in (14.70). The left side is the sum of four non-negative terms. Applying the anti-commutator to the vacuum state we can see the following:

1. If all four supercharges annihilate the vacuum state, then so does the Hamiltonian. Thus *unbroken supersymmetry requires that the vacuum energy vanishes.*

2. If at least one component of the supercharge does not annihilate the vacuum, then the Hamiltonian does not annihilate the vacuum. Thus *broken supersymmetry requires positive vacuum energy.*

In the classical approximation this leads to an astonishingly simple criterion for spontaneous SUSY breaking. *If the minimum value of the scalar potential $V(Z,\bar{Z})$ is zero, then supersymmetry is unbroken. If the minimum value is positive, then supersymmetry is broken.* Since the potential $V(Z,\bar{Z})$ in (14.66) is a non-negative quadratic form in the quantities $W_\alpha(Z)$ and $\mathcal{P}_A(Z,\bar{Z})$, we can restate the criterion in the following simple way. The vacuum is supersymmetric if and only if there is a configuration of the scalar fields Z^α, $\bar{Z}^{\bar{\alpha}}$ such that the algebraic equations

$$W_\alpha(Z) = 0 \quad \text{and} \quad \mathcal{P}_A(Z,\bar{Z}) = 0\,, \tag{14.71}$$

have a common solution. As we can see from (14.61) and (14.64), it is equivalent to say that supersymmetry is preserved if all the auxiliary fields F^α and D^A vanish in the vacuum and is otherwise broken. (Recall that fermion fields vanish in a Lorentz invariant vacuum.)

Readers may question whether the classical approximation provides the correct answer to the properties of the vacuum of the theory. Fortunately there is a remarkable non-renormalization theorem [109] which ensures that there are no perturbative quantum corrections to the classical superpotential $W(Z)$. The equations (14.71) are then valid to all orders in quantum perturbation theory. In some SUSY gauge theories there are non-perturbative additions to the superpotential which change the situation [110, 111].

Much is known about the structure of the equations of (14.71) and whether or not there is a solution that produces exact supersymmetry.[8] It is worthwhile to mention one general theorem, see Sec. 27.4 of [47], which applies to gauged supersymmetry. In that framework, a common solution of the $W_\alpha = 0$ equations is part of an 'orbit' of solutions related by complexified gauge transformations. There is always a point on the orbit where the moment maps $\mathcal{P}(Z,\bar{Z})_A$ also vanish, unless the moment maps contain Fayet-Iliopoulos terms. Thus the question of SUSY breaking is controlled by the F-term conditions unless there are are Fayet-Iliopoulos couplings.

We study the equations first in the simplest context of ungauged supersymmetry with canonical Kähler metric $g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$. Suppose that the theory contains n chiral

[8] A useful recent reference is [111].

multiplets. The vacuum is supersymmetric if there is a common root of the n equations $\partial_\alpha W(Z) = 0$. These are n *complex* equations in n *complex* variables, so for a generic superpotential there is a common solution. The superpotential must be special in some way to obtain broken supersymmetry.

Here is an example from [47] and [111] of the O'Raifeartaigh mechanism [112], which shows how this can happen. Suppose that there are three complex scalars which we call X_1, X_2 and Z and that the superpotential is

$$W = X_1 g_1(Z) + X_2 g_2(Z) . \tag{14.72}$$

The vacuum would be supersymmetric if the three equations

$$\partial_{X_1} W = g_1(Z) = 0 , \qquad \partial_{X_2} W = g_2(Z) = 0 , \qquad \partial_Z W = X_1 g_1'(Z) + X_2 g_2'(Z) = 0 , \tag{14.73}$$

have a common root. However, the first two equations cannot both be satisfied unless the holomorphic functions $g_1(Z)$ and $g_2(Z)$ are specially related. Thus the situation is that once the form (14.72) is chosen, supersymmetry breaking is generic.

Once it is established that there is no solution of the $\partial_\alpha W(Z) = 0$ conditions, one proceeds to find the SUSY breaking vacuum by minimizing the potential. In this example the potential is

$$V \ = \ |g_1(Z)|^2 \ + \ |g_2(Z)|^2 + |X_1 g_1(Z) + X_2 g_2(Z)|^2 . \tag{14.74}$$

Generically, the stationary conditions $\partial V/\partial X_i = 0$ are solved by fixing the ratio X_1/X_2 so that the third term of V vanishes. The vacuum expectation value of Z is then fixed at the minimum of the first two terms. Only the ratio of X_1 and X_2 has been fixed, so their common complex scale is not determined. Thus there is a degenerate subspace of SUSY breaking vacua in this model at least in the classical approximation.[9]

14.5.4 Supersymmetry breaking and the Goldstone fermion

We now discuss an important feature of spontaneous breakdown of SUSY, namely the fact that the theory necessarily contains a massless fermion commonly called the Goldstino. The Goldstino field is a linear combination of the elementary fermion fields in the Lagrangian, which we will identify in the classical approximation. We work first in the simple context of ungauged supersymmetry with flat Kähler metric.

To be definite we take a model which contains n chiral multiplets Z^α, $P_L\chi^\alpha$, F^α with holomorphic superpotential $W(Z)$. Since we are interested in SUSY breaking, we assume that the n equations $W_\alpha(Z) = 0$ have no common solution. Instead we proceed to minimize the scalar potential

$$V(Z, \bar{Z}) = W_\alpha(Z) \overline{W}^\alpha(\bar{Z}) , \tag{14.75}$$

[9] Classical degeneracies of SUSY breaking vacua are expected to be 'lifted' by perturbative radiative corrections.

by finding the solution $Z^\alpha = Z_0^\alpha$ of the stationary condition

$$\partial_\alpha V \;=\; W_{\alpha\beta}(Z_0)\overline{W}^\beta(\bar{Z}_0) = 0\,, \tag{14.76}$$

which gives the global minimum with $V(Z_0, \bar{Z}_0) > 0$. The condition (14.76) shows that, at minimum, the complex symmetric matrix $W_{\alpha\beta}$ has a zero mode with eigenvector $\overline{W}^\beta$. This vector is non-vanishing because $V > 0$. The matrix $W_{\alpha\beta}$, evaluated at minimum, is the mass matrix of the fermions of the model, so there is a massless fermion in the spectrum associated with the null eigenvector. This massless fermion is the Goldstino.

To identify it precisely, and to prepare for the more complicated case of gauged SUSY, let us write the Lagrangian of a 'toy model' containing a set of n free chiral fermions $P_L\psi^i$ (and their charge conjugates $P_R\psi_i$), namely

$$\mathcal{L} = -\bar{\psi}_i \not{\partial} P_L \psi^i + \tfrac{1}{2} M_{ij}\bar{\psi}^i P_L \psi^j + \tfrac{1}{2}\bar{M}^{ij}\bar{\psi}_i P_R \psi_j\,, \tag{14.77}$$

with $\bar{M}^{ij}$ being the complex conjugate of M_{ij}. The equations of motion are

$$\not{\partial} P_L \psi^i = \bar{M}^{ij} P_R \psi_j \qquad\qquad \not{\partial} P_R \psi_i = M_{ij} P_L \psi^j\,. \tag{14.78}$$

Suppose now that the mass matrix has a zero mode eigenvector $\bar{v}^i$. Thus $M_{ij}\bar{v}^j = 0$. The linear combination

$$P_L \upsilon = v_i P_L \psi^i\,, \tag{14.79}$$

is the corresponding massless fermion. To see this we simply compute

$$\not{\partial} P_L \upsilon \;=\; v_i \not{\partial} P_L \psi^i = v_i \overline{M}^{ij} P_R \psi_j \;=\; 0\,. \tag{14.80}$$

In the last step we used the complex conjugate of the zero mode equation.

The toy model result can be applied immediately. Using (the conjugate of) the null eigenvector $\overline{W}^\alpha$, we identify the Goldstino field

$$P_L \upsilon = -\frac{1}{\sqrt{2}}\, W_\alpha P_L \chi^\alpha = P_L \chi^\alpha\, \delta_s P_R \chi_\alpha\,. \tag{14.81}$$

We used the fermion shifts (14.68) (and $P_R\chi_\alpha = g_{\alpha\bar{\beta}} P_R \chi^{\bar{\beta}}$) to write the last form which turns out to generalize to gauged supersymmetry. The Goldstone fermion transforms non-trivially under supersymmetry

$$\delta_s P_L \upsilon = \delta_s P_L \chi^\alpha g_{\alpha\bar{\beta}} \delta_s P_R \chi^{\bar{\beta}}\,. \tag{14.82}$$

We will now discuss the Goldstino in the general gauged supersymmetric theory with Lagrangian given in (14.56). After elimination of auxiliary fields one obtains the scalar potential (14.66). We assume that SUSY is broken in the vacuum, so there is no common solution to the equations of (14.71). The vacuum then corresponds to the minimum positive value of V. The stationary condition $\partial_\alpha V = 0$ is very relevant and we will write it explicitly below.

The major complication in the general theory is that the fermions $P_L\chi^\alpha$ and $P_L\lambda^A$ mix. This is seen in the fermion mass term which is obtained from (14.56), (14.59), (14.62) and (14.65),

$$\begin{aligned}
\mathcal{L}_{m\,\mathrm{fermions}} &= -\tfrac{1}{2}m_{\alpha\beta}\bar\chi^\alpha P_L\chi^\beta - m_{\alpha A}\bar\chi^\alpha P_L\lambda^A - \tfrac{1}{2}m_{AB}\bar\lambda^A P_L\lambda^B + \text{h.c.}\,,\\
m_{\alpha\beta} &= \nabla_\alpha\partial_\beta W \equiv \partial_\alpha\partial_\beta W - \Gamma^\gamma_{\alpha\beta}\partial_\gamma W\,,\\
m_{\alpha A} &= \mathrm{i}\sqrt{2}\left[\partial_\alpha\mathcal{P}_A - \tfrac{1}{4}f_{AB\,\alpha}\,(\mathrm{Re}\,f)^{-1\,BC}\,\mathcal{P}_C\right] = m_{A\alpha}\,,\\
m_{AB} &= -\tfrac{1}{2}f_{AB\,\alpha}g^{\alpha\bar\beta}\overline{W}_{\bar\beta}\,.
\end{aligned} \tag{14.83}$$

The quantities $m_{\alpha\beta}$, $m_{\alpha A}$, m_{AB} are evaluated in the vacuum configuration $Z^\alpha = Z_0^\alpha$. Thus we find the fermion mass matrix of the system in the block form

$$\begin{pmatrix} m_{\alpha\beta} & m_{\alpha B}\\ m_{A\beta} & m_{AB}\end{pmatrix}. \tag{14.84}$$

We will show that the vector

$$-\sqrt{2}\begin{pmatrix}\delta_s P_L\chi^\beta\\ \delta_s P_L\lambda^A\end{pmatrix} = \begin{pmatrix}\overline{W}^\beta\\ -\mathrm{i}\mathcal{P}^B/\sqrt{2}\end{pmatrix} = \begin{pmatrix} g^{\beta\bar\gamma}\overline{W}_{\bar\gamma}\\ -\mathrm{i}(\mathrm{Re}\,f)^{-1\,BC}\mathcal{P}_C/\sqrt{2}\end{pmatrix} \tag{14.85}$$

is a zero eigenvector of the mass matrix, non-vanishing because $V > 0$ in the SUSY breaking vacuum. To show this we need two facts. First we note that the stationary condition $\partial_\alpha V = 0$ can be written neatly in terms of the elements of the mass matrix, namely as

$$\partial_\alpha V = m_{\alpha\beta}\overline{W}^\beta + m_{\alpha A}(-\mathrm{i}\mathcal{P}^A/\sqrt{2}) = 0\,. \tag{14.86}$$

This is just the statement that the 'top row' of the zero mode equation is satisfied. The 'bottom row' condition is also satisfied, since

$$m_{\beta A}\overline{W}^\beta + m_{AB}(-\mathrm{i}\mathcal{P}^B/\sqrt{2}) = \sqrt{2}k_A^{\bar\alpha}\overline{W}_{\bar\alpha} = 0 \tag{14.87}$$

Here we have used the relation (13.51) between Killing vectors and moment maps. The last expression vanishes simply because the superpotential must be gauge invariant!

The toy model then tells us that the Goldstino field is obtained by contraction of χ^α and λ^A with the appropriate components of the conjugate of the null eigenvector

$$P_L v = -\frac{1}{\sqrt{2}}P_L\left[W_\alpha\chi^\alpha + \frac{1}{\sqrt{2}}\mathrm{i}\mathcal{P}_A\lambda^A\right]. \tag{14.88}$$

It is useful to incorporate the results in (14.68) so that we recognize the general structure

$$P_L v = P_L\left[\chi^\alpha g_{\alpha\bar\beta}\delta_{\mathrm{s}}P_R\chi^{\bar\beta} + \lambda^A(\mathrm{Re}\,f)_{AB}\delta_{\mathrm{s}}P_R\lambda^B\right]. \tag{14.89}$$

The Goldstone fermion is always the linear combination in which all elementary fermion fields are multiplied by the fermion shifts and summed with the appropriate metric. Note that (14.86) and (14.87) can also be reexpressed as contractions with the fermion shifts.

14.A Appendix: Superspace

In this Appendix we briefly discuss the superspace formalism and its relation to the basic multiplets treated in the main part of the chapter. The superspace method was first formulated in [113, 103] and it is presented in detail in many textbooks (for example, see [114, 115, 116, 117, 47, 118, 119]). In the superspace approach Minkowski spacetime is extended to an 8-dimensional 'supermanifold' whose coordinates are the usual commuting x^μ plus 4 anti-commuting coordinates θ_α, which transform as a Majorana spinor under Lorentz transformations. It is frequently convenient to use the chiral projections $P_L\theta$ and $P_R\theta$, which transform as 2-component Weyl spinors. They are denoted by θ_α and $\bar\theta_{\dot\alpha}$ in many presentations of 4-dimensional superspace. Supersymmetry transformations are 'motions' in the superspace under which coordinates are shifted, viz. $x^\mu \to x'^\mu = x^\mu + \frac14\bar\epsilon\gamma^\mu\theta$, $\theta \to \theta' = \theta - \epsilon$.

The multiplets discussed earlier in this chapter are described by functions on superspace, denoted by $\Phi(x,\theta)$, which are called superfields. A superfield can be expanded in a power series in θ_α, and such expansions terminate at order $(\theta)^4$ because the θ_α anti-commute. The various components of a multiplet are the coefficients in this series expansion. To illustrate this, we consider a real superfield, which satisfies $\Phi(x,\theta) = \bar\Phi(x,\theta)$. Its series expansion is

$$\begin{aligned}\Phi(x,\theta) &= C + \tfrac12 \mathrm{i}\bar\theta\gamma_*\zeta - \tfrac18\bar\theta P_L\theta\mathcal{H} - \tfrac18\bar\theta P_R\theta\bar{\mathcal{H}} - \tfrac18\mathrm{i}\bar\theta\gamma_*\gamma^\mu\theta B_\mu \\ &\quad -\tfrac18\mathrm{i}\bar\theta\theta\bar\theta\gamma_*\left(\lambda + \tfrac12\partial\!\!\!/\zeta\right) + \tfrac1{32}\bar\theta P_L\theta\bar\theta P_R\theta\left(D + \tfrac12\Box C\right). \qquad (14.90)\end{aligned}$$

The value at $\theta = 0$ defines a real scalar field, i.e. $\Phi(x,0) = C(x)$, which we identify with the lowest component of the real multiplet (14.9). Higher components of (14.9) appear in the θ-dependent terms, and the entire structure encodes the component transformation rules (14.10) in the precise way we will describe below.

Ex. 14.10 *Check that $\bar\theta\theta\bar\theta\theta = 2\bar\theta P_L\theta\bar\theta P_R\theta$. Hint: Since there are only two components in $P_L\theta$, the bilinear $\bar\theta P_L\theta$ exhausts the left-chiral components, hence e.g. $\theta\bar\theta P_L\theta = P_R\theta\bar\theta P_L\theta$.*

Supersymmetry transformations, defined above as shifts in superspace, are implemented by the differential operators[10]

$$\begin{aligned}\mathbb{Q}_\alpha &= \frac{\overrightarrow{\partial}}{\partial\bar\theta^\alpha} - \frac14(\gamma^\mu\theta)_\alpha\frac{\partial}{\partial x^\mu}, \\ \overline{\mathbb{Q}}^\alpha &\equiv \mathcal{C}^{\alpha\beta}\mathbb{Q}_\beta = -\frac{\overrightarrow{\partial}}{\partial\theta_\alpha} + \frac14(\bar\theta\gamma^\mu)^\alpha\frac{\partial}{\partial x^\mu}. \qquad (14.91)\end{aligned}$$

The anti-commutator of the operators $\mathbb{Q}$ is

$$\{\mathbb{Q}_\alpha, \overline{\mathbb{Q}}^\beta\} = \tfrac12(\gamma^\mu)_\alpha{}^\beta\partial_\mu. \qquad (14.92)$$

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[10] The sign of the derivative of a bosonic quantity with respect to θ, a fermionic quantity, depends whether one derives from the left or right. Therefore we write $\overrightarrow{\partial}$ to indicate that the derivative acts from the left.

The variation of a superfield is defined as

$$\delta\Phi \equiv \bar{\epsilon}\mathbb{Q}\,\Phi \;=\; \overline{\mathbb{Q}}\epsilon\,\Phi\,. \tag{14.93}$$

When one commutes the supersymmetries operators, one should apply the operators on the fields, as we have seen in Ch. 1 for iterated symmetry transformations, see (1.61). This leads to

$$\begin{aligned}\delta(\epsilon_1)\delta(\epsilon_2)\Phi &= \bar{\epsilon}_2^\alpha\mathbb{Q}_\alpha\overline{\mathbb{Q}}^\beta\epsilon_{1\beta}\,\Phi\,,\\ [\delta(\epsilon_1),\delta(\epsilon_2)]\,\Phi &= \bar{\epsilon}_2\{\mathbb{Q}_\alpha,\overline{\mathbb{Q}}^\beta\}\epsilon_{1\beta}\,\Phi = \tfrac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu\Phi\,.\end{aligned} \tag{14.94}$$

Note the difference with the calculation in (11.13)-(11.14), which is the origin of the difference of sign between (14.92) and (11.17). The result agrees with the SUSY algebra studied in Sec. 8.2.1 and also with (11.16).

To identify the SUSY variations of components from the superfield variation, we write the θ expansion of $\delta\Phi$:

$$\delta\Phi \equiv \bar{\epsilon}\mathbb{Q}\,\Phi \;=\; \delta C + \tfrac{1}{2}\mathrm{i}\bar{\theta}\gamma_*\delta\zeta + \ldots\,. \tag{14.95}$$

By computing $\mathbb{Q}\Phi$ explicitly and comparing with the expansion (14.90), it is easy to identify $\delta C = \frac{1}{2}\mathrm{i}\bar{\epsilon}\gamma_*\zeta$ and $\delta\zeta = -\frac{1}{2}\mathrm{i}\gamma_*\partial\!\!\!/ C\epsilon + \ldots$, where the $\ldots$ indicate contributions from the $\mathcal{H}$ and B_μ components of Φ. These results agree with (14.10. The complete set of component transformations can be obtained in this way, but the process is tedious. It is easier for the chiral superfield. See Ex. 14.11 below.

The 'covariant derivative'

$$\mathbb{D} = \frac{\vec{\partial}}{\partial\bar{\theta}} + \frac{1}{4}\gamma^\mu\theta\frac{\partial}{\partial x^\mu}\,, \tag{14.96}$$

has many applications in the superspace formalism. This operator anticommutes with the supersymmetry generator $\mathbb{Q}$. Therefore it can be used to impose constraints on a superfield which are compatible with supersymmetry. The most common and most useful constraint defines a chiral superfield. It is defined as a superfield that satisfies

$$P_R\mathbb{D}\Phi = 0\,. \tag{14.97}$$

At $\theta = 0$ this constraint implies that the linear term in the θ expansion of $\Phi(x,\theta)$ involves only $P_L\theta$. Therefore, the SUSY transform of the lowest component involves only $P_L\epsilon$. This is exactly how we defined the chiral multiplet in components! The remainder of the constraint is a covariantization, i.e. making it consistent with the supersymmetry algebra, as we did in Sec. 14.1.1.

To solve the constraint (14.97), it is useful to redefine the superspace coordinates. Specifically we introduce the shifted bosonic coordinate $x_+^\mu = x^\mu + \frac{1}{8}\bar{\theta}\gamma_*\gamma^\mu\theta$. The chain rule then gives

$$\left.\frac{\vec{\partial}}{\partial\bar{\theta}}\right|_{x^+} = \left.\frac{\vec{\partial}}{\partial\bar{\theta}}\right|_{x} - \tfrac{1}{4}\gamma_*\gamma^\mu\theta\frac{\partial}{\partial x^\mu}\,. \tag{14.98}$$

Therefore, when x_+ is used as independent variable, the supersymmetries and covariant derivatives become

$$
\begin{aligned}
P_L\mathbb{Q} = P_L\frac{\overrightarrow{\partial}}{\partial\bar{\theta}}\,, \qquad & P_R\mathbb{Q} = P_R\left(\frac{\overrightarrow{\partial}}{\partial\bar{\theta}} - \frac{1}{2}\gamma^\mu\theta\frac{\partial}{\partial x_+^\mu}\right)\,,\\
P_L\mathbb{D} = P_L\left(\frac{\overrightarrow{\partial}}{\partial\bar{\theta}} + \frac{1}{2}\gamma^\mu\theta\frac{\partial}{\partial x_+^\mu}\right)\,, \qquad & P_R\mathbb{D} = P_R\frac{\overrightarrow{\partial}}{\partial\bar{\theta}}\,. \qquad (14.99)
\end{aligned}
$$

This simplifies the constraint (14.97) and implies that a chiral superfield has an expansion of the form

$$\Phi(x_+,\theta) = Z(x_+) + \frac{1}{\sqrt{2}}\bar{\theta}P_L\chi(x_+) + \frac{1}{4}\bar{\theta}P_L\theta F(x_+)\,. \qquad (14.100)$$

Numerical factors were chosen so that the superspace operator $\bar{\epsilon}P_L\mathbb{Q} + \bar{\epsilon}P_R\mathbb{Q}$ gives the component transformation rules (8.16).

Ex. 14.11 *Verify these transformation rules. Use (3.69) to rearrange products of θ's and the fact that $\bar{\theta}\gamma_{\mu\nu}\theta = 0$ due to the symmetry properties.*

Gauge multiplets are defined as real superfields Φ on which a 'supergauge' transformation acts via

$$\Phi \;\to\; \Phi + \mathrm{i}\left(\Lambda - \bar{\Lambda}\right)\,. \qquad (14.101)$$

The gauge parameters are the components of the chiral superfield Λ. To obtain the component transformations of the gauge multiplet in the Wess-Zumino gauge , one chooses the parameters so that

$$C = 0\,, \qquad \zeta = 0\,, \qquad \mathcal{H} = 0\,. \qquad (14.102)$$

This fixes all components of the superfield Λ except for one real field, which remains as the conventional gauge parameter of an abelian gauge potential.

One nice feature of the superspace formalism is that the product of two superfields is a superfield. Multiplication of superfields is equivalent to multiplication in the multiplet calculus but usually simpler. The following exercise contains an example.

Ex. 14.12 *If Φ is a chiral superfield given by the expansion (14.100), show that the expansion of Φ^2 is*

$$
\begin{aligned}
\Phi(x_+,\theta)^2 \;=\;& Z(x_+)^2 + \sqrt{2}\bar{\theta}P_L Z(x_+)\chi(x_+)\\
& + \tfrac{1}{4}\bar{\theta}P_L\theta\left(2z(x_+)F(x_+) - \bar{\chi}(x_+)P_L\chi(x_+)\right)\,. \qquad (14.103)
\end{aligned}
$$

Check that this result is compatible with the result of Ex. 14.2

With similar manipulations one also obtains one of the main formulas of this chapter: (14.15).

Action integrals in the superspace formalism include integration over the θ variables. One form involves the integral $\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \Phi(x,\theta)$ where Φ is a real superfield. By definition integration over Grassmann variables is equivalent to differentiation. Using the component expansion (14.90), one can see that the θ integral 'selects' the θ^4 component, and we get

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \Phi(x,\theta) = \tfrac{1}{8}\int \mathrm{d}^4x\, D(x)\,. \tag{14.104}$$

This agrees, except for a numerical factor, with the D-term action in (14.12), which we used in the component approach. If Φ is a chiral superfield, then one can form the action integral $\int \mathrm{d}^4x\, \mathrm{d}^2P_L\theta\, \Phi(x,\theta)$. Using (14.100), one can see that this isolates the F-component. The result is proportional to the F-term action of (14.12).

In $\mathcal{N}=1$, $D=4$ global supersymmetry, superfields give natural and elegant analogues of the multiplet calculus constructions presented in the main part of this chapter. There is a well developed superspace approach to $\mathcal{N}=1$, $D=4$ supergravity, see [120, 121, 122, 123, 124] (or the books [114, 115, 116, 117]), in which the geometry of superspace is described by frame and torsion superfields. However, these fields satisfy many constraints which make it complicated to derive physical results. This is why we chose the multiplet calculus as our principle method. Multiplets are defined by the properties of their lowest components and then completed by enforcing the supersymmetry algebra.

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Index

Batalin-Vilkovisky formalism, 206
Bianchi identity, 210

central charges, 223
chiral multiplets, 228
closed algebra, 208
consistent truncation, 189
covariant derivative, 77, 209
covariant general coordinate transformation, 187, 214
covariant quantity, 208
curvature, 210
curvatures, 210
cyclic identity, 58

D-term, 273
deformations, 233
degrees of freedom, 76
dual tensor, 85
duality transformations, 86, 92

embedding tensor formalism, 92
energy-momentum tensor, 26
equivariance condition, 270

F-term, 273
Fayet-Iliopoulos term, 267
fermion shifts, 273
field strength, 78
field-antifield formalism, 206

gauge covariant translation, 85
gauge field, 77
gauge fields, 203
graded Jacobi identity, 204

Hermitian metric, 242, 245
homogeneous manifold, 236
hypermultiplets, 228

Killing symmetries, 202
Killing vector, 145
Killing vector, holomorphic, 249
Kähler manifold, 246
Kähler transformations, 243

Lie Algebra, 15, 19
local Lorentz covariant derivatives, 134

Majorana spinors, 61
Majorana-Weyl representations, 74
Majorana-Weyl spinors, 63
matter fields, 77
minimal coupling, 77
moment map, 249

Noether identity, 78

off-shell closure, 171
off-shell degrees of freedom, 78
on-shell and off-shell closure, 208
on-shell and off-shell counting, 227
on-shell closure, 171
on-shell degrees of freedom, 75
open algebra, 208
open algebras, 207

proper Lorentz group, 18
pseudo-Majorana spinors, 62

R-symmetry, 225
R-symmetry group, 225

self-dual, 85
soft algebra, 206
special Kähler manifolds, 246
spurionic quantity, 92
standard gauge transformations, 213
stress tensor, 26
structure functions, 206
supergauge transformation, 257, 282
supergravity multiplet, 228
symmetric space, 236
symplectic Majorana spinors, 63
symplectic Majorana-Weyl spinors, 63
symplectic manifold, 246
symplectic transformations, 92

vector multiplets, 228

Wess-Zumino gauge, 258, 282
Weyl representation, 35
Weyl spinors, 63

zilch symmetries, 170, 207