ITP-UU-02/56
SPIN-02/35
September 2002

SUPERGRAVITY

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Abstract

These notes are based on lectures presented at the 2001 Les Houches Summerschool "Unity from Duality: Gravity, Gauge Theory and Strings".

Contents

1 Introduction

Supergravity plays a prominent role in our ideas about the unification of fundamental forces beyond the standard model, in our understanding of many central features of superstring theory, and in recent developments of the conceptual basis of quantum field theory and quantum gravity. The advances made have found their place in many reviews and textbooks (see, *e.g.*, [1]), but the subject has grown so much and has so many different facets that no comprehensive treatment is available as of today. Also in these lectures, which will cover a number of basic aspects of supergravity, many topics will be left untouched.

During its historical development the perspective of supergravity has changed. Originally it was envisaged as an elementary field theory which should be free of ultraviolet divergencies and thus bring about the long awaited unification of gravity with the other fundamental forces in nature. But nowadays supergravity is primarily viewed as an effective field theory describing the low-mass degrees of freedom of a more fundamental underlying theory. The only candidate for such a theory is superstring theory (for some reviews and textbooks, see, *e.g.*, [2]), or rather, yet another, somewhat hypothetical, theory, called M-theory. Although we know a lot about M-theory, its underlying principles have only partly been established. String theory and supergravity in their modern incarnations now represent some of the many faces of M-theory. String theory is no longer a theory exclusively of strings but includes other extended objects that emerge in the supergravity context as solitonic objects. Looking backwards it becomes clear that there are many reasons why neither superstrings nor supergravity could account for all the relevant degrees of freedom and we have learned to appreciate that M-theory has many different realizations.

Because supersymmetry is such a powerful symmetry it plays a central role in almost all these developments. It controls the dynamics and, because of nonrenormalization theorems, precise predictions can be made in many instances, often relating strong- to weak-coupling regimes. To appreciate the implications of supersymmetry, chapter 2 starts with a detailed discussion of supersymmetry and its representations. Subsequently supergravity theories are introduced in chapter 3, mostly concentrating on the maximally supersymmetric cases. In chapter 4 gauged nonlinear sigma models with homogeneous target spaces are introduced, paving the way for the construction of gauged supergravity. This construction is explained in chapter 5, where the emphasis is on gauged supergravity with 32 supercharges in 4 and 5 spacetime dimensions. These theories can describe anti-de Sitter ground states which are fully supersymmetric. This is one of the motivations for considering anti-de Sitter supersymmetry and the representations of the anti-de Sitter group in chapter 6. Chapter 7 contains a short introduction to superconformal transformations and superconformally invariant theories. This chapter is self-contained, but it is of course related to the discussion on anti-de Sitter representations in chapter 6, as well as to the adS/CFT correspondence.

This school offers a large number of lectures dealing with gravity, gauge theories and string theory from various perspectives. We intend to stay within the supergravity perspective and to try and indicate what the possible implications of supersymmetry and supergravity are for these subjects. Our hope is that the material presented below will offer a helpful introduction to and will blend in naturally with the material presented in other lectures.

2 Supersymmetry in various dimensions

An enormous amount of information about supersymmetric theories is contained in the structure of the underlying representations of the supersymmetry algebra (for some references, see [1, 3, 4, 5, 6]. Here we should make a distinction between a supermultiplet of fields which transform irreducibly under the supersymmetry transformations, and a supermultiplet of states described by a supersymmetric theory. In this chapter[1] we concentrate on supermultiplets of states, primarily restricting ourselves to flat Minkowski spacetimes of dimension D. The relevant symmetries in this case form an extension of the Poincaré transformations, which consist of translations and Lorentz transformations. However, many of the concepts that we introduce will also play a role in the discussion of other superalgebras, such as the anti-de Sitter (or conformal) superalgebras. For a recent practical introduction to superalgebras, see [7].

2.1 The Poincaré supersymmetry algebra

The generators of the super-Poincaré algebra comprise the supercharges, transforming as spinors under the Lorentz group, the energy and momentum operators, the generators of the Lorentz group, and possibly additional generators that commute with the supercharges. For the moment we ignore

[1] The material presented in this and the following chapter is an extension of the second chapter of [6].

these additional charges, often called *central* charges.[2] There are other relevant superalgebras, such as the supersymmetric extensions of the anti-de Sitter (or the conformal) algebras. These will be encountered in due course.

The most important anti-commutation relation of the super-Poincaré algebra is the one of two supercharges,

$$\{Q_\alpha, \bar{Q}_\beta\} = -2iP_\mu \,(\Gamma^\mu)_{\alpha\beta}\,, \tag{2.1}$$

where we suppressed the central charges. Here Γ^μ are the gamma matrices that generate the Clifford algebra $\mathcal{C}(D-1,1)$ with Minkowskian metric $\eta_{\mu\nu} = \mathrm{diag}(-,+,\cdots,+)$.

The size of a supermultiplet depends exponentially on the number of independent supercharge components Q. The first step is therefore to determine Q for any given number of spacetime dimensions D. The result is summarized in table 1. As shown, there exist five different sequences of spinors, corresponding to spacetimes of particular dimensions. When this dimension is odd, it is possible in certain cases to have Majorana spinors. These cases constitute the first sequence. The second one corresponds to those odd dimensions where Majorana spinors do not exist. The spinors are then Dirac spinors. In even dimension one may distinguish three sequences. In the first one, where the number of dimensions is a multiple of 4, charge conjugation relates positive- with negative-chirality spinors. All spinors in this sequence can be restricted to Majorana spinors. For the remaining two sequences, charge conjugation preserves the chirality of the spinor. Now there are again two possibilities, depending on whether Majorana spinors can exist or not. The cases where we cannot have Majorana spinors, corresponding to $D = 6 \bmod 8$, comprise the fourth sequence. For the last sequence with $D = 2 \bmod 8$, Majorana spinors exist and the charges can be restricted to so-called Majorana-Weyl spinors.

One can consider *extended supersymmetry*, where the spinor charges transform reducibly under the Lorentz group and comprise N irreducible spinors. For Weyl charges, one can consider combinations of N_+ positive- and N_- negative-chirality spinors. In all these cases there exists a group $\mathrm{H_R}$ of rotations of the spinors which commute with the Lorentz group and leave the supersymmetry algebra invariant. This group, often referred to as the 'R-symmetry' group, is thus defined as the largest subgroup of the

[2] The terminology adopted in the literature is not always very precise. Usually, all charges that commute with the supercharges, but not necessarily with all the generators of the Poincaré algebra, are called 'central charges'. We adhere to this nomenclature. Observe that the issue of central charges is different when not in flat space, as can be seen, for example, in the context of the anti-de Sitter superalgebra (discussed in chapter 6).

D	$Q_{\rm irr}$	$\rm H_R$	type
3, 9, 11, mod 8	$2^{(D-1)/2}$	$\mathrm{SO}(N)$	M
5, 7, mod 8	$2^{(D+1)/2}$	$\mathrm{USp}(2N)$	D
4, 8, mod 8	$2^{D/2}$	$\mathrm{U}(N)$	M
6, mod 8	$2^{D/2}$	$\mathrm{USp}(2N_+)\times \mathrm{USp}(2N_-)$	W
2, 10, mod 8	$2^{D/2-1}$	$\mathrm{SO}(N_+)\times \mathrm{SO}(N_-)$	MW

Table 1: The supercharges in flat Minkowski spacetimes of dimension D. In the second column, $Q_{\rm irr}$ specifies the real dimension of an irreducible spinor in a D-dimensional Minkowski spacetime. The third column specifies the group $\rm H_R$ for N-extended supersymmetry, defined in the text, acting on N-fold reducible spinor charges. The fourth column denotes the type of spinors: Majorana (M), Dirac (D), Weyl (W) and Majorana-Weyl (MW).

automorphism group of the supersymmetry algebra that commutes with the Lorentz group. It is often realized as a manifest invariance group of a supersymmetric field theory, but this is by no means necessary. There are other versions of the R-symmetry group $\rm H_R$ which play a role, for instance, in the context of the Euclidean rest-frame superalgebra for massive representations or for the anti-de Sitter superalgebra. Those will be discussed later in this chapter. In table 1 we have listed the corresponding $\rm H_R$ groups for N irreducible spinor charges. Here we have assumed that $\rm H_R$ is compact so that it preserves a positive-definite metric. In the latter two sequences of spinor charges shown in table 1, we allow $N_\pm$ charges of opposite chirality, so that $\rm H_R$ decomposes into the product of two such groups, one for each chiral sector.

Another way to present some of the results above, is shown in table 2. Here we list the real dimension of an irreducible spinor charge and the corresponding spacetime dimension. In addition we include the number of states of the shortest[3] supermultiplet of massless states, written as a sum of bosonic and fermionic states. We return to a more general discussion of the R-symmetry groups and their consequences in section 2.5.

2.2 Massless supermultiplets

Because the momentum operators P_μ commute with the supercharges, we may consider the states at arbitrary but fixed momentum P_μ, which, for

[3]By the *shortest* multiplet, we mean the multiplet with the helicities of the states as low as possible. This is usually (one of) the smallest possible supermultiplet(s).

$Q_{\rm irr}$	D	shortest supermultiplet
32	$D=11$	$128+128$
16	$D=10,9,8,7$	$8+8$
8	$D=6,5$	$4+4$
4	$D=4$	$2+2$
2	$D=3$	$1+1$

Table 2: Simple supersymmetry in various dimensions. We present the dimension of the irreducible spinor charge with $2\leq Q_{\rm irr}\leq 32$ and the corresponding spacetime dimensions D. The third column represents the number of bosonic + fermionic *massless* states for the shortest supermultiplet.

massless representations, satisfies $P^2=0$. The matrix $P_\mu\Gamma^\mu$ on the right-hand side of (2.1) has therefore zero eigenvalues. In a positive-definite Hilbert space some (linear combinations) of the supercharges must therefore vanish. To exhibit this more explicitly, let us rewrite (2.1) as (using $\bar Q = iQ^\dagger\Gamma^0$),

$$\{Q_\alpha, Q^\dagger_\beta\} = 2\,(\not{P}\Gamma^0)_{\alpha\beta}\,. \tag{2.2}$$

For light-like $P^\mu=(P^0,\vec P)$ the right-hand side is proportional to a projection operator $(\mathbf{1}+\Gamma_\parallel\Gamma^0)/2$. Here $\Gamma_\parallel$ is the gamma matrix along the spatial momentum $\vec P$ of the states. The supersymmetry anti-commutator can then be written as

$$\{Q_\alpha, Q^\dagger_\beta\} = 2\,P^0\Big(\mathbf{1}+\tilde\Gamma_D\tilde\Gamma_\perp\Big)_{\alpha\beta}\,. \tag{2.3}$$

Here $\tilde\Gamma_D$ consists of the product of all D independent gamma matrices, and $\tilde\Gamma_\perp$ of the product of all $D-2$ gamma matrices in the transverse directions (i.e., perpendicular to $\vec P$), with phase factors such that

$$(\tilde\Gamma_D)^2 = (\tilde\Gamma_\perp)^2 = \mathbf{1}\,,\qquad [\tilde\Gamma_D,\tilde\Gamma_\perp]=0\,. \tag{2.4}$$

This shows that the right-hand side of (2.3) is proportional to a projection operator, which projects out half of the spinor space. Consequently, half the spinors must vanish on physical states, whereas the other ones generate a Clifford algebra.

Denoting the real dimension of the supercharges by Q, the representation space of the charges decomposes into the two chiral spinor representations of SO$(Q/2)$. When confronting these results with the last column in table 2, it turns out that the dimension of the shortest supermultiplet is not just equal to $2^{Q_{\rm irr}/4}$, as one might naively expect. For $D=6$, this is so because the representation is complex. For $D = 3, 4$ the representation is twice as big because it must also accommodate fermion number (or, alternatively, because it must be CPT self-conjugate). The derivation for $D = 4$ is presented in many places (see, for instance, [1, 4, 5]). For $D = 3$ we refer to [8].

The two chiral spinor subspaces correspond to the bosonic and fermionic states, respectively. For the massless multiplets, the dimensions are shown in table 2. Bigger supermultiplets can be obtained by combining irreducible multiplets by requiring them to transform nontrivially under the Lorentz group. We shall demonstrate this below in three relevant cases, corresponding to $D = 11, 10$ and 6 spacetime dimensions. Depending on the number of spacetime dimensions, many supergravity theories exist. Pure supergravity theories with spacetime dimension $4 \leq D \leq 11$ can exist with $Q = 32, 24, 20, 16, 12, 8$ and 4 supersymmetries.[4] Some of these theories will be discussed later in more detail (in particular supergravity in $D = 11$ and 10 spacetime dimensions).

2.2.1 $D = 11$ Supermultipets

In 11 dimensions we are dealing with 32 independent real supercharges. In odd-dimensional spacetimes irreducible spinors are subject to the eigenvalue condition $\tilde{\Gamma}_D = \pm\mathbf{1}$. Therefore (2.3) simplifies and shows that the 16 nonvanishing spinor charges transform according to a single spinor representation of the helicity group SO(9).

On the other hand, when regarding the 16 spinor charges as gamma matrices, it follows that the representation space constitutes the spinor representation of SO(16), which decomposes into two chiral subspaces, one corresponding to the bosons and the other one to the fermions. To determine the helicity content of the bosonic and fermionic states, one considers the embedding of the SO(9) spinor representation in the SO(16) vector representation. It then turns out that one of the **128** representations branches into helicity representations according to $\mathbf{128} \rightarrow \mathbf{44} + \mathbf{84}$, while the second one transforms irreducibly according to the **128** representation of the helicity group.

The above states comprise precisely the massless states corresponding to $D = 11$ supergravity [11]. The graviton states transform in the **44**,

[4] In $D = 4$ there exist theories with $Q = 12, 20$ and 24; in $D = 5$ there exists a theory with $Q = 24$ [9]. In $D = 6$ there are three theories with $Q = 32$ and one with $Q = 24$. So far these supergravities have played no role in string theory. For a more recent discussion, see [10].

the antisymmetric tensor states in the **84** and the gravitini states in the **128** representation of SO(9). Bigger supermultiplets consist of multiples of 256 states. For instance, without central charges, the smallest massive supermultiplet comprises $32768 + 32768$ states. These multiplets will not be considered here.

2.2.2 $D = 10$ Supermultiplets

In 10 dimensions the supercharges are both Majorana and Weyl spinors. The latter means that they are eigenspinors of $\tilde{\Gamma}_D$. According to (2.3), when we have simple (i.e., nonextended) supersymmetry with 16 charges, the nonvanishing charges transform in a chiral spinor representation of the SO(8) helicity group. With 8 nonvanishing supercharges we are dealing with an 8-dimensional Clifford algebra, whose irreducible representation space corresponds to the bosonic and fermionic states, each transforming according to a chiral spinor representation. Hence we are dealing with three 8-dimensional representations of SO(8), which are inequivalent. One is the representation to which we assign the supercharges, which we will denote by $\mathbf{8}_s$; to the other two, denoted as the $\mathbf{8}_v$ and $\mathbf{8}_c$ representations, we assign the bosonic and fermionic states, respectively. The fact that SO(8) representations appear in a three-fold variety is known as *triality*, which is a characteristic property of the group SO(8). With the exception of certain representations, such as the adjoint and the singlet representation, the three types of representation are inequivalent. They are traditionally distinguished by labels s, v and c (see, for instance, [12]).[5]

The smallest massless supermultiplet has now been constructed with 8 bosonic and 8 fermionic states and corresponds to the vector multiplet of supersymmetric Yang-Mills theory in 10 dimensions [13]. Before constructing the supermultiplets that are relevant for $D = 10$ supergravity, let us first discuss some other properties of SO(8) representations. One way to distinguish the inequivalent representations, is to investigate how they decompose into representations of an SO(7) subgroup. Each of the 8-dimensional representations leaves a different SO(7) subgroup of SO(8) invariant. Therefore there is an SO(7) subgroup under which the $\mathbf{8}_v$ representation branches into

$$\mathbf{8}_v \longrightarrow \mathbf{7} + \mathbf{1}.$$

[5] The representations can be characterized according to the four different conjugacy classes of the SO(8) weight vectors, denoted by 0, v, s and c. In this context one uses the notation $\mathbf{1}_0$, $\mathbf{28}_0$, and $\mathbf{35}_0$, $\mathbf{35}_0'$, $\mathbf{35}_0''$ for $\mathbf{35}_v$, $\mathbf{35}_s$, $\mathbf{35}_c$, respectively.

supermultiplet	bosons	fermions
vector multiplet	$\mathbf{8}_v$	$\mathbf{8}_c$
graviton multiplet	$\mathbf{1}+\mathbf{28}+\mathbf{35}_v$	$\mathbf{8}_s+\mathbf{56}_s$
gravitino multiplet	$\mathbf{1}+\mathbf{28}+\mathbf{35}_c$	$\mathbf{8}_s+\mathbf{56}_s$
gravitino multiplet	$\mathbf{8}_v+\mathbf{56}_v$	$\mathbf{8}_c+\mathbf{56}_c$

Table 3: Massless $N=1$ supermultiplets in $D=10$ spacetime dimensions containing $8+8$ or $64+64$ bosonic and fermionic degrees of freedom.

Under this SO(7) the other two 8-dimensional representations branch into

$$\mathbf{8}_s \longrightarrow \mathbf{8}\,, \qquad \mathbf{8}_c \longrightarrow \mathbf{8}\,,$$

where $\mathbf{8}$ is the spinor representation of SO(7). Corresponding branching rules for the 28-, 35- and 56-dimensional representations are

$$\begin{array}{rcl@{\qquad\qquad}rcl} \mathbf{28} & \longrightarrow & \mathbf{7}+\mathbf{21}\,, & & & \\ \mathbf{35}_v & \longrightarrow & \mathbf{1}+\mathbf{7}+\mathbf{27}\,, & \mathbf{56}_v & \longrightarrow & \mathbf{21}+\mathbf{35}\,, \\ \mathbf{35}_{c,s} & \longrightarrow & \mathbf{35}\,, & \mathbf{56}_{c,s} & \longrightarrow & \mathbf{8}+\mathbf{48}\,. \end{array} \tag{2.5}$$

In order to obtain the supersymmetry representations relevant for supergravity we consider tensor products of the smallest supermultiplet consisting of $\mathbf{8}_v+\mathbf{8}_c$, with one of the 8-dimensional representations. There are thus three different possibilities, each leading to a 128-dimensional supermultiplet. Using the multiplication rules for SO(8) representations,

$$\begin{array}{rcl@{\qquad\qquad}rcl} \mathbf{8}_v\times\mathbf{8}_v & = & \mathbf{1}+\mathbf{28}+\mathbf{35}_v\,, & \mathbf{8}_v\times\mathbf{8}_s & = & \mathbf{8}_c+\mathbf{56}_c\,, \\ \mathbf{8}_s\times\mathbf{8}_s & = & \mathbf{1}+\mathbf{28}+\mathbf{35}_s\,, & \mathbf{8}_s\times\mathbf{8}_c & = & \mathbf{8}_v+\mathbf{56}_v\,, \\ \mathbf{8}_c\times\mathbf{8}_c & = & \mathbf{1}+\mathbf{28}+\mathbf{35}_c\,, & \mathbf{8}_c\times\mathbf{8}_v & = & \mathbf{8}_s+\mathbf{56}_s\,, \end{array} \tag{2.6}$$

it is straightforward to obtain these new multiplets. Multiplying $\mathbf{8}_v$ with $\mathbf{8}_v+\mathbf{8}_c$ yields $\mathbf{8}_v\times\mathbf{8}_v$ bosonic and $\mathbf{8}_v\times\mathbf{8}_c$ fermionic states, and leads to the second supermultiplet shown in table 3. This supermultiplet contains the representation $\mathbf{35}_v$, which can be associated with the states of the graviton in $D=10$ dimensions (the field-theoretic identification of the various states has been clarified in many places; see *e.g.* the appendix in [6]). Therefore this supermultiplet will be called the *graviton multiplet.* Multiplication with $\mathbf{8}_c$ or $\mathbf{8}_s$ goes in the same fashion, except that we will associate the $\mathbf{8}_c$ and $\mathbf{8}_s$ representations with fermionic quantities (note that these are the representations to which the fermion states of the Yang-Mills multiplet and the supersymmetry charges are assigned). Consequently, we interchange the boson and fermion assignments in these products. Multiplication with $\mathbf{8}_c$ then leads to $\mathbf{8}_c \times \mathbf{8}_c$ bosonic and $\mathbf{8}_c \times \mathbf{8}_v$ fermionic states, whereas multiplication with $\mathbf{8}_s$ gives $\mathbf{8}_s \times \mathbf{8}_c$ bosonic and $\mathbf{8}_s \times \mathbf{8}_v$ fermionic states. These supermultiplets contain fermions transforming according to the $\mathbf{56}_s$ and $\mathbf{56}_c$ representations, respectively, which can be associated with gravitino states, but no graviton states as those transform in the $\mathbf{35}_v$ representation. Therefore these two supermultiplets are called *gravitino multiplets*. We have thus established the existence of two inequivalent gravitino multiplets. The explicit SO(8) decompositions of the vector, graviton and gravitino supermultiplets are shown in table 3.

By combining a graviton and a gravitino multiplet it is possible to construct an $N = 2$ supermultiplet of 128 + 128 bosonic and fermionic states. However, since there are two inequivalent gravitino multiplets, there will also be two inequivalent $N = 2$ supermultiplets containing the states corresponding to a graviton and two gravitini. According to the construction presented above, one $N = 2$ supermultiplet may be be viewed as the tensor product of two identical supermultiplets (namely $\mathbf{8}_v + \mathbf{8}_c$). Such a multiplet follows if one starts from a supersymmetry algebra based on *two* Majorana-Weyl spinor charges Q with the *same* chirality. The states of this multiplet decompose as follows:

$$\begin{array}{l} \textit{Chiral } N=2 \textit{ supermultiplet } \text{(IIB)} \\ \qquad\qquad (\mathbf{8}_v + \mathbf{8}_c) \times (\mathbf{8}_v + \mathbf{8}_c) \quad \Longrightarrow \end{array} \left\{ \begin{array}{l} \textbf{bosons}: \\ \mathbf{1} + \mathbf{1} + \mathbf{28} + \mathbf{28} + \mathbf{35}_v + \mathbf{35}_c \\ \\ \textbf{fermions}: \\ \mathbf{8}_s + \mathbf{8}_s + \mathbf{56}_s + \mathbf{56}_s \end{array} \right. \tag{2.7}$$

This is the multiplet corresponding to IIB supergravity [14]. Because the supercharges have the same chirality, one can perform rotations between these spinor charges which leave the supersymmetry algebra unaffected. Hence the automorphism group $H_{\rm R}$ is equal to SO(2). This feature reflects itself in the multiplet decomposition, where the $\mathbf{1}$, $\mathbf{8}_s$, $\mathbf{28}$ and $\mathbf{56}_s$ representations are degenerate and constitute doublets under this SO(2) group.

A second supermultiplet may be viewed as the tensor product of a $(\mathbf{8}_v + \mathbf{8}_s)$ supermultiplet with a second supermultiplet $(\mathbf{8}_v + \mathbf{8}_c)$. In this case the supercharges constitute two Majorana-Weyl spinors of opposite chirality. Now the supermultiplet decomposes as follows:

Nonchiral $N=2$ supermultiplet (IIA)

$$(\mathbf{8}_v+\mathbf{8}_s)\times(\mathbf{8}_v+\mathbf{8}_c) \quad\Longrightarrow\quad \left\{\begin{array}{l}\textbf{bosons}:\\ \mathbf{1}+\mathbf{8}_v+\mathbf{28}+\mathbf{35}_v+\mathbf{56}_v\\ \\ \textbf{fermions}:\\ \mathbf{8}_s+\mathbf{8}_c+\mathbf{56}_s+\mathbf{56}_c\end{array}\right. \tag{2.8}$$

This is the multiplet corresponding to IIA supergravity [15]. It can be obtained by a straightforward reduction of $D=11$ supergravity. The latter follows from the fact that two $D=10$ Majorana-Weyl spinors with opposite chirality can be combined into a single $D=11$ Majorana spinor. The formula below summarizes the massless states of IIA supergravity from an 11-dimensional perspective. The massless states of 11-dimensional supergravity transform according to the **44**, **84** and **128** representation of the helicity group SO(9). They correspond to the degrees of freedom described by the metric, a 3-rank antisymmetric gauge field and the gravitino field, respectively. We also show how the 10-dimensional states can subsequently be branched into 9-dimensional states, characterized in terms of representations of the helicity group SO(7):

$$\begin{array}{rcl}\mathbf{44} & \Longrightarrow & \left\{\begin{array}{lcl}\mathbf{1} & \longrightarrow & \mathbf{1}\\ \mathbf{8}_v & \longrightarrow & \mathbf{1}+\mathbf{7}\\ \mathbf{35}_v & \longrightarrow & \mathbf{1}+\mathbf{7}+\mathbf{27}\end{array}\right.\\ \mathbf{84} & \Longrightarrow & \left\{\begin{array}{lcl}\mathbf{28} & \longrightarrow & \mathbf{7}+\mathbf{21}\\ \mathbf{56}_v & \longrightarrow & \mathbf{21}+\mathbf{35}\end{array}\right.\\ \mathbf{128} & \Longrightarrow & \left\{\begin{array}{lcl}\mathbf{8}_s & \longrightarrow & \mathbf{8}\\ \mathbf{8}_c & \longrightarrow & \mathbf{8}\\ \mathbf{56}_s & \longrightarrow & \mathbf{8}+\mathbf{48}\\ \mathbf{56}_c & \longrightarrow & \mathbf{8}+\mathbf{48}\end{array}\right.\end{array} \tag{2.9}$$

Clearly, in $D=9$ we have a degeneracy of states, associated with the group $\mathrm{H_R}=\mathrm{SO}(2)$. We note the presence of graviton and gravitino states, transforming in the **27** and **48** representations of the SO(7) helicity group.

One could also take the states of the IIB supergravity and decompose them into $D=9$ massless states. This leads to precisely the same supermultiplet as the reduction of the states of IIA supergravity. Indeed, the reductions of IIA and IIB supergravity to 9 dimensions, yield the same theory [16, 17, 18]. However, the massive states are still characterized in terms of the group SO(8), which in $D=9$ dimensions comprises the rest-frame rotations. Therefore the Kaluza-Klein states that one obtains when compactifying the ten-dimensional theory on a circle remain *inequivalent* for the

$\mathrm{SU}_+(2)$	$N_+=1$	$N_+=2$	$N_+=3$	$N_+=4$
5				1
4			1	8
3		1	6	27
2	1	4	14	48
1	2	5	14	42
	$(2+2)_{\mathbf{C}}$	$(8+8)_{\mathbf{R}}$	$(32+32)_{\mathbf{C}}$	$(128+128)_{\mathbf{R}}$

Table 4: Shortest massless supermultiplets of $D=6$ N_+-extended chiral supersymmetry. The states transform both in the $\mathrm{SU}_+(2)$ helicity group and under a $\mathrm{USp}(2N_+)$ group. For odd values of N_+ the representations are complex, for even N_+ they can be chosen real. Of course, an identical table can be given for negative-chirality spinors.

IIA and IIB theories (see [19] for a discussion of this phenomenon and its consequences). It turns out that the $Q=32$ supergravity multiplets are unique in all spacetime dimensions $D>2$, except for $D=10$. Maximal supergravity will be introduced in chapter 3. The field content of the maximal $Q=32$ supergravity theories for dimensions $3\leq D\leq 11$ will be presented in two tables (*c.f.* table 10 and 11).

2.2.3 $D=6$ Supermultiplets

In 6 dimensions we have chiral spinors, which are not Majorana. Because the charge conjugated spinor has the same chirality, the chiral rotations of the spinors can be extended to the group $\mathrm{USp}(2N_+)$, for N_+ chiral spinors. Likewise N_- negative-chirality spinors transform under $\mathrm{USp}(2N_-)$. This feature is already incorporated in table 1. In principle we have N_+ positive- and N_- negative-chirality charges, but almost all information follows from first considering the purely chiral case. In table 4 we present the decomposition of the various helicity representations of the smallest supermultiplets based on $N_+=1,2,3$ or 4 supercharges. In $D=6$ dimensions the helicity group SO(4) decomposes into the product of two SU(2) groups: $\mathrm{SO}(4)\cong(\mathrm{SU}_+(2)\times\mathrm{SU}_-(2))/\mathbf{Z}_2$. When we have supercharges of only one chirality, the smallest supermultiplet will only transform under one SU(2) factor of the helicity group, as is shown in table 4.[6]

[6]The content of this table also specifies the shortest *massive* supermultiplets in four dimensions as well as with the shortest *massless* multiplets in five dimensions. The SU(2) group is then associated with spin or with helicity, respectively.

Let us now turn to specific supermultiplets. Let us recall that the helicity assignments of the states describing gravitons, gravitini, vector and (anti)selfdual tensor gauge fields, and spinor fields are $(3,3)$, $(2,3)$ or $(3,2)$, $(2,2)$, $(3,1)$ or $(1,3)$, and $(2,1)$ or $(1,2)$. Here (m,n) denotes that the dimensionality of the reducible representations of the two SU(2) factors of the helicity group are of dimension m and n. For the derivation of these assignments, see for instance one of the appendices in [6].

In the following we will first restrict ourselves to helicities that correspond to at most the three-dimensional representation of either one of the SU(2) factors. Hence we have only $(3,3)$, $(3,2)$, $(2,3)$, $(3,1)$ or $(1,3)$ representations, as well as the lower-dimensional ones. When a supermultiplet contains $(3,2)$ or $(2,3)$ representations, we insist that it will also contain a single $(3,3)$ representation, because gravitini without a graviton are not expected to give rise to a consistent interacting field theory. The multiplets of this type are shown in table 5. There are no such multiplets for more than $Q = 32$ supercharges.

There are supermultiplets with higher SU(2) helicity representations, which contain neither gravitons nor gravitini. Some of these multiplets are shown in table 6 and we will discuss them in due course.

We now elucidate the construction of the supermultiplets listed in table 5. The simplest case is $(N_+, N_-) = (1,0)$, where the smallest supermultiplet is the (1,0) *hypermultiplet*, consisting of a complex doublet of spinless states and a chiral spinor. Taking the tensor product of the smallest supermultiplet with the $(2,1)$ helicity representation gives the (1,0) *tensor multiplet*, with a selfdual tensor, a spinless state and a doublet of chiral spinors. The tensor product with the $(1,2)$ helicity representation yields the (1,0) *vector multiplet*, with a vector state, a doublet of chiral spinors and a scalar. Multiplying the hypermultiplet with the (2,3) helicity representation, one obtains the states of (1,0) *supergravity*. Observe that the selfdual tensor fields in the tensor and supergravity supermultiplet are of opposite selfduality phase.

Next consider $(N_+, N_-) = (2,0)$ supersymmetry. The smallest multiplet, shown in table 4, then corresponds to the (2,0) *tensor multiplet*, with the bosonic states decomposing into a selfdual tensor and a five-plet of spinless states, and a four-plet of chiral fermions. Multiplication with the (1,3) helicity representation yields the (2,0) supergravity multiplet, consisting of the graviton, four chiral gravitini and five selfdual tensors [20]. Again, the selfdual tensors of the tensor and of the supergravity supermultiplet are of opposite selfduality phase.

Of course, there exists also a nonchiral version with 16 supercharges, namely the one corresponding to $(N_+, N_-) = (1,1)$. The smallest multiplet

multiplet	#	bosons	fermions
(1,0) hyper	$4+4$	$(1,1;2,1)+\mathrm{h.c.}$	$(2,1;1,1)$
(1,0) tensor	$4+4$	$(3,1;1,1)+(1,1;1,1)$	$(2,1;2,1)$
(1,0) vector	$4+4$	$(2,2;1,1)$	$(1,2;2,1)$
(1,0) graviton	$12+12$	$(3,3;1,1)+(1,3;1,1)$	$(2,3;2,1)$
(2,0) tensor	$8+8$	$(3,1;1,1)+(1,1;5,1)$	$(2,1;4,1)$
(2,0) graviton	$24+24$	$(3,3;1,1)+(1,3;5,1)$	$(2,3;4,1)$
(1,1) vector	$8+8$	$(2,2;1,1)+(1,1;2,2)$	$(2,1;1,2)$ $+(1,2;2,1)$
(1,1) graviton	$32+32$	$(3,3;1,1)$ $+(1,3;1,1)+(3,1;1,1)$ $+(1,1;1,1)+(2,2;2,2)$	$(3,2;1,2)$ $+(2,3;2,1)$ $+(1,2;1,2)$ $+(2,1;2,1)$
(2,1) graviton	$64+64$	$(3,3;1,1)$ $+(1,3;5,1)+(3,1;1,1)$ $+(2,2;4,2)+(1,1;5,1)$	$(3,2;1,2)$ $+(2,3;4,1)$ $+(1,2;5,2)$ $+(2,1;4,1)$
(2,2) graviton	$128+128$	$(3,3;1,1)$ $+(3,1;1,5)+(1,3;5,1)$ $+(2,2;4,4)+(1,1;5,5)$	$(3,2;4,1)$ $+(2,3;1,4)$ $+(2,1;4,5)$ $+(1,2;5,4)$

Table 5: Some relevant $D=6$ supermultiplets with (N_+,N_-) supersymmetry. The states $(m,n;\tilde{m},\tilde{n})$ are assigned to (m,n) representations of the helicity group $\mathrm{SU}_+(2)\times\mathrm{SU}_-(2)$ and $(\tilde{m},\tilde{n})$ representations of $\mathrm{USp}(2N_+)\times\mathrm{USp}(2N_-)$. The second column lists the number of bosonic + fermionic states for each multiplet.

supersymmetry	#	bosons	fermions
(1,0)	$8+8$	$(5;1)+(3;1)$	$(4;2)$
(2,0)	$24+24$	$(5;1)+(1,1)$ $+(3;1)+(3;5)$	$(4;4)+(2;4)$
(3,0)	$64+64$	$(5;1)+(1;14)$ $+(3;1)+(3;14)$	$(4;6)$ $+(2;6)+(2;14)$
(4,0)	$128+128$	$(5;1)+(3;27)$ $+(1;42)$	$(4;8)+(2;48)$

Table 6: $D=6$ supermultiplets without gravitons and gravitini with $(N,0)$ supersymmetry, a single $(5;1)$ highest-helicity state and at most 32 supercharges. The theories based on these multiplets have only rigid supersymmetry. The multiplets are identical to those that underly the five-dimensional N-extended supergravities. They are all chiral, so that the helicity group in six dimensions is restricted to $\mathrm{SU}(2)\times\mathbf{1}$ and the states are characterized as representations of $\mathrm{USp}(2N)$. The states $(n;\tilde{n})$ are assigned to the n-dimensional representation of SU(2) and the $\tilde{n}$-dimensional representation of $\mathrm{USp}(2N)$. The second column lists the number of bosonic + fermionic states for each multiplet.

is now given by the tensor product of the supermultiplets with (1,0) and (0,1) supersymmetry. This yields the vector multiplet, with the vector state and four scalars, the latter transforming with respect to the (2,2) representation of $\mathrm{USp}(2)\times\mathrm{USp}(2)$. There are two doublets of chiral fermions with opposite chirality, each transforming as a doublet under the corresponding USp(2) group. Taking the tensor product of the vector multiplet with the (2,2) representation of the helicity group yields the states of the (1,1) *supergravity* multiplet. It consists of 32 bosonic states, corresponding to a graviton, a tensor, a scalar and four vector states, where the latter transform under the (2,2) representation of $\mathrm{USp}(2)\times\mathrm{USp}(2)$. The 32 fermionic states comprise two doublets of chiral gravitini and two chiral spinor doublets, transforming as doublets under the appropriate USp(2) group.

Subsequently we discuss the case $(N_+,N_-)=(2,1)$. Here a supergravity multiplet exists [21] and can be obtained from the product of the states of the $(2,0)$ tensor multiplet with the $(0,1)$ tensor multiplet. There is in fact a smaller supermultiplet, which we discard because it contains gravitini but no graviton states.

Finally, we turn to the case of $(N_+,N_-)=(2,2)$. The smallest supermultiplet is given by the tensor product of the smallest (2,0) and (0,2) supermultiplets. This yields the 128 + 128 states of the (2,2) *supergravity* multiplet.

These states transform according to representations of $\mathrm{USp}(4) \times \mathrm{USp}(4)$.

In principle, one can continue and classify representations for other values of (N_+, N_-). As is obvious from the construction that we have presented, this will inevitably lead to states transforming in higher-helicity representations. Some of these multiplets will suffer from the fact that they have more than one graviton state, so that we expect them to be inconsistent at the nonlinear level. However, there are the chiral theories which contain neither graviton nor gravitino states. Restricting ourselves to 32 supercharges and requiring the highest helicity to be a five-dimensional representation of one of the SU(2) factors, there are just four theories, summarized in table 6. For a recent discussion of one of these theories, see [10].

2.3 Massive supermultiplets

Generically massive supermultiplets are bigger than massless ones because the number of supercharges that generate the multiplet is not reduced, unlike for massless supermultiplets where one-half of the supercharges vanishes. However, in the presence of mass parameters the superalgebra may also contain central charges, which could give rise to a shortening of the representation in a way similar to what happens for the massless supermultiplets. This only happens for special values of these charges. The shortened supermultiplets are known as BPS multiplets. Central charges and multiplet shortening are discussed in subsection 2.4. In this section we assume that the central charges are absent.

The analysis of massive supermultiplets takes place in the restframe. The states then organize themselves into representations of the rest-frame rotation group, $\mathrm{SO}(D-1)$, associated with spin. The supercharges transform as spinors under this group, so that one obtains a Euclidean supersymmetry algebra,

$$\{Q_\alpha, Q^\dagger_\beta\} = 2M\, \delta_{\alpha\beta}\,. \tag{2.10}$$

Just as before, the spinor charges transform under the automorphism group of the supersymmetry algebra that commutes with the spin rotation group. This group will also be denoted by $\mathrm{H_R}$; it is the nonrelativistic variant of the R-symmetry group that was introduced previously. Obviously the nonrelativistic group can be bigger than its relativistic counterpart, as it is required to commute with a smaller group. For instance, in $D = 4$ spacetime dimensions, the relativistic R-symmetry group is equal to $\mathrm{U}(N)$, while the nonrelativistic one is the group $\mathrm{USp}(2N)$, which contains $\mathrm{U}(N)$ as a subgroup according to $\mathbf{2N} = \mathbf{N} + \overline{\mathbf{N}}$. Table 7 shows the smallest massive representations for $N \leq 4$ in $D = 4$ dimensions as an illustration. Clearly the states

spin	$N=1$	$N=2$
1		$1=1$
1/2	$1=1$	$4=2+\bar{2}$
0	$2=1+\bar{1}$	$5=3+1+\bar{1}$
	$N=3$	$N=4$
2		$1=1$
3/2	$1=1$	$8=4+\bar{4}$
1	$6=3+\bar{3}$	$27=15+6+\bar{6}$
1/2	$14=8+3+\bar{3}$	$48=20+\overline{20}+4+\bar{4}$
0	$14=6+\bar{6}+1+\bar{1}$	$42=20'+10+\overline{10}+1+\bar{1}$

Table 7: Minimal $D=4$ massive supermultiplets without central charges for $N\leq 4$. The states are listed as USp($2N$) representations which are subsequently decomposed into representations of U(N).

of given spin can be assigned to representations of the nonrelativistic group $\mathrm{H_R}=\mathrm{USp}(2N)$ and decomposed in terms of irreducible representations of the relativistic R-symmetry group U(N). More explicit derivations can be found in [4].

Knowledge of the relevant groups $\mathrm{H_R}$ is important and convenient in writing down the supermultiplets. It can also reveal certain relations between supermultiplets, even between supermultiplets living in spacetimes of different dimension. Obviously, supermultiplets living in higher dimensions can always be decomposed into supermultiplets living in lower dimensions, and massive supermultiplets can be decomposed in terms of massless ones, but sometimes there exists a relationship that is less trivial. For instance, the $D=4$ *massive* multiplets shown in table 7 coincide with the *massless* supermultiplets of chirally extended supersymmetry in $D=6$ dimensions shown in table 4. In particular the $N=4$ supermultiplet of table 7 appears in many places and coincides with the massless $N=8$ supermultiplet in $D=5$ dimensions, which is shown in tables 10 and 11. The reasons for this are clear. The $D=5$ and the chiral $D=6$ massless supermultiplets are subject to the same helicity group SU(2), which in turn coincides with the spin rotation group for $D=4$. Not surprisingly, also the relevant automorphism groups $\mathrm{H_R}$ coincide, as the reader can easily verify. Since the number of *effective* supercharges is equal in these cases and given by $Q_{\mathrm{eff}}=16$ (remember that only half of the charges play a role in building up massless supermultiplets), the multiplets must indeed be identical.

Here we also want to briefly draw the attention to the relation between off-shell multiplets and massive representations. So far we discussed supermultiplets consisting of states on which the supercharges act. These states can be described by a field theory in which the supercharges generate corresponding supersymmetry variations on the fields. Very often the transformations on the fields do *not* close according to the supersymmetry algebra unless one imposes the equations of motion for the fields. Such representations are called *on-shell* representations. The lack of closure has many consequences, for instance, when determining quantum corrections. In certain cases one can improve the situation by introducing extra fields which do not directly correspond to physical fields. These fields are known as *auxiliary fields*. By employing such fields one may be able to define an *off-shell* representation, where the transformations close upon (anti)commutation without the need for imposing field equations. Unfortunately, many theories do not possess (finite-dimensional) off-shell representations. Notorious examples are gauge theories and supergravity theories with 16 or more supercharges. This fact makes is much more difficult to construct an extended variety of actions for these theories, because the transformation rules are implicitly dependent on the action. There is an off-shell counting argument. according to which the field degrees of freedom should comprise a *massive* supermultiplet (while the states that are described could be massless). For instance, the off-shell description of the $N = 2$ vector multiplet in $D = 4$ dimensions can be formulated in terms of a gauge field (with three degrees of freedom), a fermion doublet (with eight degrees of freedom) and a triplet of auxiliary scalar fields (with three degrees of freedom), precisely in accord with the $N = 2$ entry in table 7. In fact, this multiplet coincides with the multiplet of the currents that couple to an $N = 2$ supersymmetric gauge theory.

The $N = 4$ multiplet in table 7 corresponds to the gravitational supermultiplet of currents [22]. These are the currents that couple to the fields of $N = 4$ conformal supergravity. Extending the number of supercharges beyond 16 will increase the minimal spin of a massive multiplet beyond spin-2. Since higher-spin fields can usually not be coupled, one may conclude that conformal supergravity does not exist for more than 16 charges. For that reason there can be no off-shell formulations for supergravity with more than 16 charges. Conformal supergravity will be discussed in chapter 7.

In section 2.5 we will present a table listing the various groups $\mathrm{H_R}$ for spinors associated with certain Clifford algebras $\mathcal{C}(p, q)$ with corresponding rotation groups $\mathrm{SO}(p, q)$. Subsequently we then discuss some further implications of these results.

D	$\mathrm{H_R}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
11	1			1 [55]			1 [462]
10A	1	1 [1]	1 [10]	1 [45]		1 [210]	$1+1$ [126]
10B	SO(2)		2 [10]		1 [120]		$1+2$ [126]
9	SO(2)	$1+2$ [1]	2 [9]	1 [36]	1 [84]	$1+2$ [126]	
8	U(2)	$3+\bar{3}$ [1]	3 [8]	$1+\bar{1}$ [28]	$1+3$ [56]	$3+\bar{3}$ [35]	
7	USp(4)	10 [1]	5 [7]	$1+5$ [21]	10 [35]		
6	USp(4) $\times$USp(4)	$(4,4)$ [1]	$(1,1)$ $+(5,1)$ $+(1,5)$ [6]	(4,4) [15]	$(10,1)$ $+(1,10)$ [10]		
5	USp(8)	$1+27$ [1]	27 [5]	36 [10]			
4	U(8)	$28+\overline{28}$ [1]	63 [4]	$36+\overline{36}$ [3]			
3	SO(16)	120 [1]	135 [3]				

Table 8: Decomposition of the central extension in the supersymmetry algebra with $Q=32$ supercharge components in terms of p-rank Lorentz tensors. The second row specifies the number of independent components for each p-rank tensor charge. The total number of central charges is equal to $528-D$, because we have not listed the D independent momentum operators

2.4 Central charges and multiplet shortening

The supersymmetry algebra of the maximal supergravities comprises general coordinate transformations, local supersymmetry transformations and the gauge transformations associated with the antisymmetric gauge fields.[7] These gauge transformations usually appear in the anticommutator of two supercharges, and may be regarded as central charges. In perturbation theory, the theory does not contain charged fields, so these central charges simply vanish on physical states. However, at the nonperturbative level, there may be solitonic or other states that carry charges. An example are magnetic monopoles, dyons, or black holes. At the M-theory level, these charges are associated with certain brane configurations. On such states, some of the central charges may take finite values. Without further knowledge about the kind of states that may emerge at the nonperturbative level, we can generally classify the possible central charges, by considering a decomposition of the anticommutator. This anticommutator carries at least two spinor indices and two indices associated with the group $\mathrm{H_R}$. Hence we may write

$$\{Q_\alpha, Q_\beta\} \propto \sum_p \,(\Gamma^{\mu_1\cdots\mu_p}C)_{\alpha\beta}\, Z_{\mu_1\cdots\mu_p}\,, \tag{2.11}$$

where $\Gamma^{\mu_1\cdots\mu_p}$ is the antisymmetrized product of p gamma matrices, C is the charge-conjugation matrix and $Z_{\mu_1\cdots\mu_p}$ is the central charge, which transforms as an antisymmetric p-rank Lorentz tensor and depends on possible additional $\mathrm{H_R}$ indices attached to the supercharges. The central charge must be symmetric or antisymmetric in these indices, depending on whether the $(\Gamma^{\mu_1\cdots\mu_p}C)_{\alpha\beta}$ is antisymmetric or symmetric in α,β, so that the product with $Z_{\mu_1\cdots\mu_p}$ is always symmetric in the combined indices of the supercharges. For given spacetime dimension all possible central charges can be classified.[8] For the maximal supergravities in spacetime dimensions $3\leq D\leq 11$ this classification is given in table 8, where we list all possible charges and their $\mathrm{H_R}$ representation assignments. Because we have 32 supercharge components, the sum of the independent momentum operators and the central charges must be equal to $(32\times 33)/2 = 528$. The results of the table are in direct correspondence with the eleven-dimensional superalgebra with the

[7] There may be additional gauge transformations that are of interest to us. As we discuss in due course, it is possible to have (part of) the automorphism group $\mathrm{H_R}$ realized as a local invariance. However, the corresponding gauge fields are composite and do not give rise to physical states (at least, not in perturbation theory).

[8] For related discussions see, for example, [23, 25, 24] and references therein.

most general central charges,

$$\{Q_\alpha, \bar{Q}_\beta\} = -2iP_M\Gamma^M_{\alpha\beta} + Z_{MN}\Gamma^{MN}_{\alpha\beta} + Z_{MNPQR}\Gamma^{MNPQR}_{\alpha\beta}\,. \qquad (2.12)$$

The two central charges, Z_{MN} and Z_{MNPQR} can be associated with the winding numbers of two- and five-branes.

In order to realize the supersymmetry algebra in a positive-definite Hilbert space, the right-hand side of the anticommutator is subject to a positivity condition, which generically implies that the mass of the multiplet is larger than or equal to the central charges. Especially in higher dimensions, the bound may can take a complicated form. This positivity bound is known as the Bogomol'nyi bound. When the bound is saturated one speaks of BPS states. For BPS multiplets some of the supercharges must vanish on the states, in the same way as half of the charges vanish for the massless supermultiplets. This vanishing of some of the supercharges leads to a shortening of the multiplet. Qualitatively, this phenomenon of multiplet shortening is the same as for massless supermultiplets, but here the fraction of the charges that vanishes is not necessarily equal to 1/2. Hence one speaks of 1/2-BPS, 1/4-BPS supermultiplets, etcetera, to indicate which fraction of the supercharges vanishes on the states. The fact that the BPS supermultiplets have a completely different field content than the generic massive supermultiplets makes that they exhibit a remarkable stability under 'adiabatic' deformations. This means that perturbative results based on BPS supermultiplets can often be extrapolated to a nonperturbative regime.

For higher extended supersymmetry the difference in size of BPS supermultiplets and massive supermultiplets can be enormous in view of the fact that the number of states depend exponentionally on the number of nonvanishing central charges. For lower supersymmetry the multiplets can be comparable in size, but nevertheless they are quite different. For instance, consider $N = 2$ *massive* vector supermultiplets in four spacetime dimensions. Without central charges, such a multiplet comprises 8+8 states, corresponding to the three states of spin-1, the eight states of four irreducible spin-$\frac{1}{2}$ representations, and five states with spin 0. On the other hand there is another massive vector supermultiplet, which is BPS and comprises the three states of spin-1, the four states of two spin-$\frac{1}{2}$ representations and two states of spin-0. These states are subject to a nonvanishing central charge which requires that the states are all doubly degenerate, so that we have again $8 + 8$ states, but with a completely different spin content. When decomposing these multiplets into massless $N = 2$ supermultiplets, the first multiplet decomposes into a massless vector multiplet and a hypermultiplet. Hence this is the multiplet one has in the Higgs phase, where the hypermultiplet provides the scalar degree of freedom that allows the conversion of the massless to massive spin-1 states. This multiplet carries no central charge. The second supermultiplet, which is BPS, appears as a massive charged vector multiplet when breaking a nonabelian supersymmetric gauge theory to an abelian subgroup. This realization is known as the Coulomb phase.

In view of the very large variety of BPS supermultiplets, we do not continue this general discussion of supermultiplets with central charges. In later chapters we will discuss specific BPS supermultiplets as well as other mechanisms of multiplet shortening in anti-de Sitter space.

2.5 On spinors and the R-symmetry group H_R

In this section we return once more to the spinor representations and the corresponding automorphism group H_R, also known as the R-symmetry group. Table 9 summarizes information for spinors up to (real) dimension 32 associated with the groups $\mathrm{SO}(p,q)$, where we restrict $q \leq 2$. From this table we can gain certain insights into the properties of spinors living in Euclidean, Minkowski and (anti-)de Sitter spaces as well as the supersymmetry algebras based on these spinors. Let us first elucidate the information presented in the table. Subsequently we shall discuss some correspondences between the various spinors in different dimensions.

We consider the Clifford algebras $\mathcal{C}(p,q)$ based on $p+q$ generators, denoted by $e_1, e_2, \ldots, e_{p+q}$, with a nondegenerate metric of signature (p,q). This means that p generators square to the identity and q to minus the identity. We list the real dimension of the irreducible Clifford algebra representation, denoted by $d_\mathcal{C}$, and the values r (equal to $0,\ldots,3$), where r is defined by $r \equiv p-q \bmod 4$. The value for r determines the square of the matrix built from forming the product $e_1 \cdot e_2 \cdots e_{p+q}$ of all the Clifford algebra generators. For $r = 0,1$ this square equals the identity, while for $r = 2,3$ the square equals minus the identity. Therefore, for $r = 0$, the subalgebra $\mathcal{C}_+(p,q)$ generated by products of *even* numbers of generators is not simple and breaks into two simple ideals, while for $r = 1$, the *full* Clifford algebra $\mathcal{C}(p,q)$ decomposes into two simple ideals.

We also present the centralizer of the irreducible representations of the Clifford algebra, which, according to Schur's lemma, must form a division algebra and is thus isomorphic to the real numbers (**R**), the complex numbers (**C**), or the quaternions (**H**). This means that the irreducible representation commutes with the identity and none, one or three complex structures, respectively, which generate the corresponding division algebra. Table 9 re-

$d_{\mathcal{C}}$	$\mathcal{C}(p,q)$	r	centralizer	$d_{\mathrm{SO}(p,q)}$	$\mathrm{H_R}$
1	$\mathcal{C}(1,0)$	1	$\mathbf{R}$	1	$\mathrm{SO}(N)$
2	$\mathcal{C}(0,1)$	3	$\mathbf{C}$	$1+1$	$\mathrm{SO}(N)$
2	$\mathcal{C}(1,1)$	0	$\mathbf{R}$	$1+1$	$\mathrm{SO}(N)$
2	$\mathcal{C}(2,0)$	2	$\mathbf{R}$	2	$\mathrm{U}(N)$
2	$\mathcal{C}(2,1)$	1	$\mathbf{R}$	2	$\mathrm{SO}(N)$
4	$\mathcal{C}(0,2)$	2	$\mathbf{H}$	$2+2$	$\mathrm{U}(N)$
4	$\mathcal{C}(1,2)$	3	$\mathbf{C}$	$2+2$	$\mathrm{SO}(N)$
4	$\mathcal{C}(2,2)$	0	$\mathbf{R}$	$2+2$	$\mathrm{SO}(N)$
4	$\mathcal{C}(3,0)$	3	$\mathbf{H}$	4	$\mathrm{USp}(2N)$
4	$\mathcal{C}(3,1)$	2	$\mathbf{C}$	4	$\mathrm{U}(N)$
4	$\mathcal{C}(3,2)$	1	$\mathbf{R}$	4	$\mathrm{SO}(N)$
8	$\mathcal{C}(4,0)$	0	$\mathbf{H}$	$4+4$	$\mathrm{USp}(2N)$
8	$\mathcal{C}(4,1)$	3	$\mathbf{H}$	8	$\mathrm{USp}(2N)$
8	$\mathcal{C}(4,2)$	2	$\mathbf{C}$	8	$\mathrm{U}(N)$
8	$\mathcal{C}(5,0)$	1	$\mathbf{H}$	8	$\mathrm{USp}(2N)$
16	$\mathcal{C}(5,1)$	0	$\mathbf{H}$	$8+8$	$\mathrm{USp}(2N)$
16	$\mathcal{C}(5,2)$	3	$\mathbf{C}$	16	$\mathrm{USp}(2N)$
16	$\mathcal{C}(6,0)$	2	$\mathbf{H}$	$8+8$	$\mathrm{U}(N)$
16	$\mathcal{C}(6,1)$	1	$\mathbf{H}$	16	$\mathrm{USp}(2N)$
16	$\mathcal{C}(7,0)$	3	$\mathbf{C}$	$8+8$	$\mathrm{SO}(N)$
16	$\mathcal{C}(8,0)$	0	$\mathbf{R}$	$8+8$	$\mathrm{SO}(N)$
16	$\mathcal{C}(9,0)$	1	$\mathbf{R}$	16	$\mathrm{SO}(N)$
32	$\mathcal{C}(6,2)$	0	$\mathbf{H}$	$16+16$	$\mathrm{USp}(2N)$
32	$\mathcal{C}(7,1)$	2	$\mathbf{C}$	$16+16$	$\mathrm{U}(N)$
32	$\mathcal{C}(7,2)$	3	$\mathbf{H}$	32	$\mathrm{USp}(2N)$
32	$\mathcal{C}(8,1)$	3	$\mathbf{C}$	$16+16$	$\mathrm{SO}(N)$
32	$\mathcal{C}(9,1)$	0	$\mathbf{R}$	$16+16$	$\mathrm{SO}(N)$
32	$\mathcal{C}(10,0)$	2	$\mathbf{R}$	32	$\mathrm{U}(N)$
32	$\mathcal{C}(10,1)$	1	$\mathbf{R}$	32	$\mathrm{SO}(N)$
64	$\mathcal{C}(8,2)$	2	$\mathbf{H}$	$32+32$	$\mathrm{U}(N)$
64	$\mathcal{C}(9,2)$	3	$\mathbf{C}$	$32+32$	$\mathrm{SO}(N)$
64	$\mathcal{C}(10,2)$	0	$\mathbf{R}$	$32+32$	$\mathrm{SO}(N)$

Table 9: Representations of the Clifford algebras $\mathcal{C}(p,q)$ with $q \leq 2$ and their centralizers, and the $\mathrm{SO}(p,q)$ spinors of maximal real dimension 32 and their R-symmetry group. We also list the dimensions of the Clifford algebra and spinor representation, as well as $r = p - q \bmod 4$.

flects also the so-called periodicity theorem [26], according to which there exists an isomorphism between the Clifford algebras $\mathcal{C}(p+8,q)$ (or $\mathcal{C}(p,q+8)$) and $\mathcal{C}(p,q)$ times the 16×16 real matrices. Therefore, the dimension of the representations of $\mathcal{C}(p+8,q)$ (or $\mathcal{C}(p,q+8)$) and $\mathcal{C}(p,q)$ differs by a factor 16.

Finally the table lists the branching of the Clifford algebra representation into $\mathrm{SO}(p,q)$ spinor representations. When $r=0$ the Clifford algebra representation decomposes into two chiral spinors. Observe that for $r=2$ we can also have chiral spinors, but they are complex so that their real dimension remains unaltered. For $r=2,3$ there are no chiral spinors, but nevertheless in certain cases the Clifford algebra representation can still decompose into two irreducible spinor representations. The last column gives the compact group $\mathrm{H_R}$, consisting of the linear transformations that commute with the group $\mathrm{SO}(p,q)$ and act on N irreducible spinors, leaving a positive-definite metric invariant. For $r=0$, a group $\mathrm{H_R}$ should be assigned to each of the chiral sectors separately. Again, according to Schur's lemma the centralizer of $\mathrm{SO}(p,q)$ must form a division algebra for irreducible spinor representations. Correspondingly, the group $\mathrm{SO}(p,q)$ commutes with the identity and none, one or three complex structures, which leads to $\mathrm{H_R}=\mathrm{SO}(N)$, $\mathrm{U}(N)$, or $\mathrm{USp}(2N)$, respectively. We note that the results of table 9 are in accord with the results presented earlier in tables 1 and 2.

We now discuss and clarify a number of correspondences between spinors living in different dimensions. The first correspondence is between spinors of $\mathrm{SO}(p,1)$ and $\mathrm{SO}(p-1,0)$. According to the table, for any $p>1$, the dimensions of the corresponding spinors differ by a factor two, while their respective groups $\mathrm{H_R}$ always coincide. From a physical perspective, this correspondence can be understood from the fact that $\mathrm{SO}(p-1,0)$ is the *helicity* group of massless spinor states in flat Minkowski space of dimension $D=p+1$. In a field-theoretic context the reduction of the spinor degrees of freedom is effected by the massless Dirac equation and the automorphism groups $\mathrm{H_R}$ that commute with the Lorentz transformations and the transverse helicity rotations, respectively, simply coincide. The two algebras (2.1) and (2.3) thus share the same automorphism group. From a mathematical viewpoint, this correspondence is related to the isomorphism

$$\mathcal{C}(p,q)\cong\mathcal{C}(p-1,q-1)\otimes\mathcal{C}(1,1)\,, \tag{2.13}$$

where we note that $\mathcal{C}(1,1)$ is isomorphic with the real 2×2 matrices.

Inspired by the first correspondence one may investigate a second one between spinors of $\mathrm{SO}(p,1)$ and $\mathrm{SO}(p,0)$ with $p>1$. Physically this correspondence is relevant when considering relativistic massive spinors in flat

Minkowski spacetime of dimension $D = p + 1$, which transform in the rest-frame under p-dimensional spin rotations. As the table shows, this correspondence is less systematic and, indeed, an underlying isomorphism for the corresponding Clifford algebras is lacking. The results of the table should therefore be applied with care. In a number of cases the relativistic spinor transforms irreducibly under the nonrelativistic rotation group. In that case the dimension of the automorphism group $\mathrm{H_R}$ can increase, as it does for $p = 3$ and 10, but not for $p = 5$ and 9. For $p = 3$ (or, equivalently, $D = 4$) the implications of the fact that the nonrelativistic automorphism group $\mathrm{USp}(2N)$ is bigger than the relativistic one, have already been discussed in section 2.3. In the remaining cases, $p = 4, 6, 7, 8$ (always modulo 8), the relativistic spinor decomposes into two nonrelativistic spinors. Because the number of irreducible spinors is then doubled, the nonrelativistic automorphism group has a tendency to increase, but one should consult the table for specific cases.

The third correspondence relates spinors of $\mathrm{SO}(p, 2)$ and $\mathrm{SO}(p, 1)$ with $p > 1$. Again the situation depends sensitively on the value for p. In a number of cases (*i.e.*, $p = 2, 3, 4, 6$) the spinor dimension is the same for both groups. This can be understood from the fact that the Clifford algebra representations are irreducible with respect to $\mathrm{SO}(p, 1)$, so that one can always extend the generators of $\mathrm{SO}(p, 1)$, which are proportional to $\Gamma^{[a}\Gamma^{b]}$, to those of $\mathrm{SO}(p, 2)$ by including the gamma matrices Γ^a. However, the R-symmetry group is not necessarily the same. For $p = 4$ the $\mathrm{SO}(4, 2)$ spinors allow the R-symmetry group $\mathrm{U}(N)$, while for $\mathrm{SO}(4, 1)$ the R-symmetry group is larger and equal to $\mathrm{USp}(2N)$. Therefore theories formulated in flat Minkowski spacetime of dimension $D = 3, 4, 5, 6$ can in principle be elevated to anti-de Sitter space. For $D = 5$ the R-symmetry reduces to $\mathrm{U}(N)$, while for $D = 3, 4, 6$ the R-symmetry remains the same. In the remaining dimensions, $D = 2, 7, 8, 9, 10$, a single Minkowski spinor can not be elevated to anti-de Sitter space, and one must at least start from an even number of flat Minkowski spinors (so that N is even). For these cases, it is hard to make general statements about the fate of the R-symmetry when moving to anti-de Sitter space and one has to consult table 9.

The fourth correspondence is again more systematic, as it is based on the isomorphism (2.13). The correspondence relates spinors of $\mathrm{SO}(p, 2)$ and of $\mathrm{SO}(p-1, 1)$. For all $p > 1$ the spinor dimension differs by one-half while the R-symmetry group remains the same. Observe that $\mathrm{SO}(p, 2)$ can be regarded as the group of conformal symmetries in a Minkowski space of p dimensions, or as the isometry group of an anti-de Sitter space of dimension $p + 1$. This correspondence extends this statement to the level of spinors. It implies that the extension of the Poincaré superalgebra in $D = p$ spacetime dimensions to a superconformal algebra requires a doubling of the number of supercharges. This feature is well known [27] and the two supersymmetries are called Q- and S-supersymmetry. The anticommutator of two S-supersymmetry charges yields the conformal boosts. Both set of charges transform under the R-symmetry group of the Poincaré algebra, which plays a more basic role in the superconformal algebra as its generators appear in the anticommutator of a Q-supersymmetry and an S-supersymmetry charge. In the anti-de Sitter context, the spinor charge is irreducible but has simply twice as many components. We return to the superconformal invariance and related aspects in chapter 7.

It is illuminating to exploit some of the previous correspondences and the relations between various supersymmetry representations in the context of the so-called adS/CFT correspondence [28]. We close this chapter by exhibiting a chain of relationships between various supermultiplets. We start with an $N = 4$ supersymmetric gauge theory in $D = 4$ spacetime dimensions, whose massless states are characterized as representations of the SO(2) helicity group and the R-symmetry group SU(4).[9] Hence we have a field theory with $Q = 16$ supersymmetries, of which only 8 are realized on the massless supermultiplet. This supermultiplet decomposes as follows,

$$(\pm 1, \mathbf{1})_{\mathbf{p}} + (0, \mathbf{6})_{\mathbf{p}} + (\tfrac{1}{2}, \mathbf{4})_{\mathbf{p}} + (-\tfrac{1}{2}, \overline{\mathbf{4}})_{\mathbf{p}} \,, \tag{2.14}$$

where $\mathbf{p}$ indicates the three-momentum, $|\mathbf{p}|$ the energy, and the entries in the parentheses denote the helicity and the SU(4) representation of the states.

Multiplying this multiplet with a similar one, but now with opposite three-momentum $-\mathbf{p}$, yields a multiplet with zero momentum and with mass $M = 2|\mathbf{p}|$. As it turns out the helicity states can now be assembled into states that transform under the 3-dimensional rotation group, so that they can be characterized by their spin. The resulting supermultiplet consists of 128+128 degrees of freedom. While the states of the original multiplet (2.14) were only subject to the helicity group and 8 supersymmetries, the composite multiplet is now a full supermultiplet subject to 16 supersymmetries and the rotation (rather than the helicity) group. Indeed, inspection shows that this composite multiplet is precisely the $N = 4$ massive multiplet shown in table 7. In this form the relevant R-symmetry group is extended to USp(8).

It is possible to cast the above product of states into a product of fields of the 4-dimensional gauge theory. One then finds that the spin-2 operators correspond to the energy-momentum tensor, which is conserved (*i.e.*

[9] Because this multiplet is CPT self-conjugate, the U(1) subgroup of U(4) coincides with the helicity group and plays no independent role here.

divergence-free) and traceless, so that it has precisely the 5 independent components appropriate for spin-2. The spin-1 operators decompose into 15 conserved vectors, associated with the currents of SU(4), and 6 selfdual antisymmetric tensors. The spin-0 operators are scalar composite operators. Furthermore there are 4 chiral and 4 antichiral vector-spinor operators, which are conserved and traceless (with respect to a contraction with gamma matrices) such that each of them correspond precisely to the 4 components appropriate for spin-$\frac{3}{2}$. These are the supersymmetry currents. Finally there are 20 and 4 chiral and antichiral spin-$\frac{1}{2}$ operators. This is precisely the supermultiplet of currents [22], which couples to the fields of conformal supergravity. Because neither the currents nor the conformal supergravity fields are subject to any field equations (unlike the supersymmetric gauge multiplet from which we started, which constitutes only an on-shell supermultiplet), it forms the basis for a proper off-shell theory of $N = 4$ conformal supergravity [22]. The presence of the traceless and conserved energy-momentum tensor and supersymmetry currents, and of the SU(4) conserved currents, is a consequence of the superconformal invariance of the underlying 4-dimensional gauge theory. The $N = 4$ conformal supergravity theory couples consistently to the $N = 4$ supersymmetric gauge theory. Chapter 7 will further explain the general setting of superconformal theories that is relevant in this context.

The off-shell $N = 4$ conformal supergravity multiplet in 4 dimensions can also be interpreted as an on-shell *massless* supermultiplet in 5 dimensions with 32 supersymmetries. Because of the masslessness, the states are annihilated by half the supercharges and are still classified according to SO(3), which now acts as the helicity group; the R-symmetry group coincides with the USp(8) R-symmetry of the relativistic 5-dimensional supersymmetry algebra. Hence this is the same multiplet that describes $D = 5$ maximal supergravity. This theory has a nonlinearly realized $E_{6(6)}$ invariance whose linearly realized subgroup (which is relevant for the spectrum) equals USp(8).

The latter theory can be gauged (we refer to chapter 5 for a discussion of this) in which case it can possess an anti-de Sitter ground state. According to table 9, a fully supersymmetric ground state leads to a U(4) R-symmetry group. As we will discuss in chapter 6, anti-de Sitter space leads to 'remarkable representations'. These are the singletons, which do not have a smooth Poincaré limit because they are associated with possible degrees of freedom living on a 4-dimensional boundary. Because the 5-dimensional anti-de Sitter superalgebra coincides with the 4-dimensional superconformal algebra, the 4-dimensional boundary theory must be consistent with superconformal invariance. Hence it does not come as a surprise that these singleton representations coincide with the supermultiplet of 4-dimensional $N = 4$ gauge theory. This set-up requires the gauge group of 5-dimensional supergravity to be chosen such as to preserve the relevant automorphism group. Therefore the gauge group must be equal to $\mathrm{SO}(6) \cong \mathrm{SU}(4)/\mathbf{Z}_2$. Indeed, this gauging allows for an anti-de Sitter maximally supersymmetric ground state [29]. Thus, the circle closes.

We stress that the above excursion, linking the various supermultiplets in different dimensions by a series of arguments, is purely based on symmetries. It does not capture the dynamical aspects of the adS/CFT correspondence and has no bearing on the nature of the gauge group in 4 dimensions. At this stage we thus have to content ourselves with the existence of this remarkable chain of correspondences. Many aspects of these correspondences will reappear in later chapters.

3 Supergravity

In this chapter we discuss field theories that are invariant under local supersymmetry. Because of the underlying supersymmetry algebra, the invariance under local supersymmetry implies the invariance under spacetime diffeomorphisms. Therefore these theories are necessarily theories of gravity. We exhibit the initial steps in the construction of a supergravity theory, with and without a cosmological term. Then we concentrate on maximal supergravity theories in various dimensions, their symmetries, and dimensional compactifications on tori. At the end we briefly discuss some of the nonmaximal theories

3.1 Simple supergravity

The first steps in the construction of any supergravity theory are usually based on the observation that local supersymmetry implies the invariance under general coordinate transformation. Therefore one must introduce the fields needed to describe general relativity, namely a vielbein field $e_\mu^{\;a}$ and a spin-connection field $\omega_\mu^{\;ab}$. The vielbein field is nonsingular and its inverse is denoted by $e_a^{\;\mu}$. The vielbein defines a local set of tangent frames of the spacetime manifold, while the spin-connection field is associated with (local) Lorentz transformations of these frames. The world indices, $\mu, \nu, \ldots$, and the tangent space indices, $a, b, \ldots$, both run from 0 to $D-1$. For an introduction to the vielbein formalism we refer to [30]. Furthermore one needs one or several gravitino fields, which carry both a world index and a spinor index and which act as the gauge fields associated with local supersymmetry. For simplicity we only consider a single Majorana gravitino field, denoted by ψ_μ, but this restriction is not essential.[10] Hence any supergravity Lagrangian is expected to contain the Einstein-Hilbert Lagrangian of general relativity and the Rarita-Schwinger Lagrangian for the gravitino field,

$$\kappa^2\,\mathcal{L} = -\tfrac{1}{2} e\, R(\omega) - \tfrac{1}{2} e\bar{\psi}_\mu\, \Gamma^{\mu\nu\rho} D_\nu(\omega)\psi_\rho + \cdots\,, \tag{3.1}$$

where the covariant derivative on a spinor ψ reads

$$D_\mu(\omega)\psi = \Big(\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{ab}\,\Gamma_{ab}\Big)\,\psi\,, \tag{3.2}$$

and $\omega_\mu{}^{ab}$ is the spin-connection field whose definition will be discussed in a sequel. The matrices $\frac{1}{2}\Gamma_{ab} = \frac{1}{4}[\Gamma_a, \Gamma_b]$ are the generators of the Lorentz transformations in spinor space, κ^2 is related to Newton's constant and $e = \det(e_\mu{}^a)$. Observe that the spinor covariant derivative on ψ_μ contains no affine connection, as it should not [30].

We note the existence of two covariant tensors, namely the curvature associated with the spin connection $R_{\mu\nu}^{ab}(\omega)$ and the torsion tensor $R_{\mu\nu}^{a}(P)$, which carries this name because it is proportional to the antisymmetric part of the affine connection, $\Gamma^\rho_{[\mu\nu]}$, upon using the vielbein postulate,

$$\begin{aligned} R_{\mu\nu}^{ab}(\omega) &= \partial_\mu\omega_\nu^{ab} - \partial_\nu\omega_\mu^{ab} + \omega_\mu^{ac}\,\omega_{\nu c}{}^{b} - \omega_\nu^{ac}\,\omega_{\mu c}{}^{b}\,, \\ R_{\mu\nu}^{a}(P) &= D_\mu(\omega)e_\nu^{\,a} - D_\nu(\omega)e_\mu^{\,a}\,. \end{aligned} \tag{3.3}$$

We note that these tensors satisfy the Bianchi identities,

$$D_{[\mu}(\omega)R_{\nu\rho]}^{ab}(\omega) = 0\,, \qquad D_{[\mu}(\omega)R_{\nu\rho]}^{a}(P) + R_{[\mu\nu}^{ab}(\omega)\,e_{\rho]\,b} = 0\,. \tag{3.4}$$

It is suggestive to regard $e_\mu^{\,a}$ and $\omega_\mu^{\,ab}$ as the gauge fields of the Poincaré group. In that context $R(\omega)$ is written as $R(M)$, so that P and M denote the translation and the Lorentz generators of the Poincaré algebra. We will use this notation in later chapters when discussing the anti-de Sitter and the conformal algebras. Here we will just use the notation $R(\omega)$ and define its contractions with the inverse vielbeine (related to the Ricci tensor and Ricci scalar) by

$$R_\mu^a(e,\omega) = e_b^{\,\nu}\, R_{\mu\nu}^{ab}(\omega)\,, \qquad R(e,\omega) = e_a^{\,\mu} e_b^{\,\nu}\, R_{\mu\nu}^{ab}(\omega)\,. \tag{3.5}$$

[10] For definiteness we consider a generic supergravity theory with one Majorana gravitino with an antisymmetric charge-conjugation matrix C and gamma matrices Γ_a satisfying $C\Gamma_a C^{-1} = -\Gamma_a^{\rm T}$. Furthermore $\Gamma_\mu = e_\mu{}^a\Gamma_a$. This is the case for $D = 3, 4, 10, 11$ mod 8. For $D = 8, 9$ mod 8, the charge-conjugation matrix is symmetric and $C\Gamma_a C^{-1} = \Gamma_a^{\rm T}$. For $D = 5, 6, 7$ mod 8, Majorana spinors do not exist.

The spin connection can be treated as an independent field (first-order formalism), which is then solved in terms of its field equations, or it can be fixed from the beginning (second-order formalism), for instance, by imposing the constraint,

$$R^a_{\mu\nu}(P) = 0\,. \tag{3.6}$$

Such a constraint is called 'conventional' because it expresses one field in terms of other fields in an algebraic fashion. For pure gravity the first- and the second-order formalism lead to the same result. The constraint (3.6) can be solved algebraically and leads to,

$$\omega_\mu^{ab}(e) = \tfrac{1}{2} e_\mu^{\;c}(\Omega^{ab}{}_c - \Omega^b{}_c{}^a - \Omega_c{}^{ab})\,, \tag{3.7}$$

where the $\Omega_{ab}{}^c$ are the *objects of anholonomity*,

$$\Omega_{ab}{}^c = e^\mu_a e^\nu_b\,(\partial_\mu e^c_\nu - \partial_\mu e^c_\nu)\,. \tag{3.8}$$

From the spin connection one defines the affine connection by $\Gamma_{\mu\nu}{}^\rho = e^\rho_a\, D_\mu(\omega) e_\nu{}^a$, which ensures the validity of the vielbein postulate. With the zero-torsion value (3.7) for the spin connection, the affine connection becomes equal to the Christoffel symbols and $R_{\mu\nu\rho}{}^\sigma = R^{ab}_{\mu\nu}(\omega)\, e_{\rho\, a}\, e_b^{\sigma}$ coincides with the standard Riemann tensor.

The action corresponding to the above Lagrangian is locally supersymmetric up to terms cubic in the gravitino field. The supersymmetry transformations contain the terms,

$$\delta e_\mu{}^a = \tfrac{1}{2}\bar\epsilon\,\Gamma^a\psi_\mu\,, \qquad \delta\psi_\mu = D_\mu(\omega)\epsilon\,, \tag{3.9}$$

where the gravitino variation is the extension to curved spacetime of the spinor gauge invariance of a Rarita-Schwinger field. Extending this Lagrangian to a fully supersymmetric one is not always possible. Usually it requires additional fields of lower spin, whose existence can be inferred from the knowledge of the possible underlying (massless) supermultiplets of states. When the spacetime dimension exceeds eleven, conventional supergravity no longer exists, as we shall discuss in the next section.

Let us now include a cosmological term into the above Lagrangian as well as a suitably chosen masslike term for the gravitino field,

$$\begin{aligned}\mathcal{L} \;\;=\;\; & -\tfrac{1}{2} e\, R(e,\omega) - \tfrac{1}{2} e\bar\psi_\mu\,\Gamma^{\mu\nu\rho} D_\nu(\omega)\psi_\rho \\ & + \tfrac{1}{4} g(D-2) e\,\bar\psi_\mu \Gamma^{\mu\nu}\psi_\nu + \tfrac{1}{2} g^2 (D-1)(D-2)\, e + \cdots\,. \end{aligned} \tag{3.10}$$

As it turns out the corresponding action is still locally supersymmetric, up to terms that are cubic in the gravitino field, provided that we introduce an extra term to the transformation rules,

$$\delta e_\mu{}^a = \tfrac{1}{2}\bar\epsilon\,\Gamma^a\psi_\mu\,, \qquad \delta\psi_\mu = \Big(D_\mu(\omega) + \tfrac{1}{2}g\Gamma_\mu\Big)\,\epsilon\,. \tag{3.11}$$

The Lagrangian (3.10) was first written down in [31] in four space-time dimensions and the correct interpretation of the masslike term was given in [32]. Observe that the variation for ψ_μ may be regarded as a generalized covariant derivative, where ω_μ^{ab} and $e_\mu^{\;a}$ act as gauge fields,[11]

Consistency requires that $g\Gamma_\mu\epsilon$ satisfies the same Majorana constraint as ψ_μ and ϵ. With the conventions that we have adopted this implies that g is real. The reality of g has important consequences, as it implies that the cosmological term is of definite sign. Hence supersymmetry does not a priori forbid a cosmological term, but it must be of definite sign (at least, if the ground state is to preserve supersymmetry). This example does not cover all cases, as one does not always have a single Majorana spinor with the specified charge conjugation properties. Nevertheless the conclusion that the cosmological term must have this particular sign remains, unless one accepts 'ghosts': fields whose kinetic terms are of the wrong sign. For an early discussion, see [33, 34] and references therein. We should point out that there are situations where a cosmological term is not consistent with supersymmetry. Assuming that the theory has an anti-de Sitter or de Sitter ground state, one may verify whether the Minkowski spinors have the right dimension to enable them to live in these spaces. For instance, a Majorana-Weyl spinor in $D = 10$ spacetime dimensions has only half the number of components as a spinor in (anti-)de Sitter space of the same dimension. Therefore, simple supergravity in $D = 10$ dimensions cannot possibly have (anti-)de Sitter ground states. Such a counting argument does not exclude anti-de Sitter ground states in $D = 11$ spacetime dimensions, because $D = 11$ Lorentz spinors can exist in anti-de Sitter space. Here the argument may be invoked that no relevant supersymmetric extension of the anti-de Sitter algebra exists beyond $D = 7$ dimensions [3], but there are also explicit studies ruling out supersymmetric cosmological terms in 11 dimensions [35].

[11] The masslike term in (3.10) is consistent with that interpretation as it can be generated from the Rarita-Schwinger Lagrangian by the same change of the covariant derivative, *i.e.*,

$$-\tfrac{1}{2}e\bar\psi_\mu\Gamma^{\mu\nu\rho}(D_\nu(\omega) + \tfrac{1}{2}g\Gamma_\nu)\psi_\rho\,.$$

The Einstein equation corresponding to (3.10) reads (suppressing the gravitino field),

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\, R + \tfrac{1}{2} g^2 (D-1)(D-2)\, g_{\mu\nu} = 0\,, \tag{3.12}$$

which implies,

$$R_{\mu\nu} = g^2 (D-1)\, g_{\mu\nu}\,, \qquad R = g^2 D(D-1)\,. \tag{3.13}$$

Hence we are dealing with a D-dimensional Einstein space. The maximally symmetric solution of this equation is an anti-de Sitter space, whose Riemann curvature equals

$$R_{\mu\nu}{}^{ab} = 2g^2\, e_\mu{}^{[a}\, e_\nu{}^{b]}\,. \tag{3.14}$$

This solution leaves all supersymmetries intact just as flat Minkowski space does. One can verify this directly by considering the supersymmetry variation of the gravitino field and by requiring that it vanishes in the bosonic background. This happens for spinors $\epsilon(x)$ satisfying

$$\Big(D_\mu(\omega) + \tfrac{1}{2} g \Gamma_\mu \Big)\, \epsilon = 0\,. \tag{3.15}$$

Spinors satisfying this equation are called Killing spinors. Since (3.15) is a first-order differential equation, one expects that it can be solved provided some integrability condition is satisfied. To see this one notes that also $(D_\mu(\omega) + \frac{1}{2} g\Gamma_\mu)(D_\nu(\omega) + \frac{1}{2} g\Gamma_\nu)\epsilon$ must vanish. Antisymmetrizing this expression in μ and ν then yields the (algebraic) integrability condition,

$$\Big(-\tfrac{1}{4} R_{\mu\nu}{}^{ab}\, \Gamma_{ab} + \tfrac{1}{2} g^2\, \Gamma_{\mu\nu} \Big) \epsilon = 0\,. \tag{3.16}$$

Multiplication with Γ^ν yields,

$$\Big(R_{\mu\nu} - g^2 (D-1)\, g_{\mu\nu} \Big) \Gamma^\nu \epsilon = 0\,, \tag{3.17}$$

from which one derives that the Riemann tensor satisfies (3.13). Therefore supersymmetry requires an Einstein space. Requiring full supersymmetry, so that (3.15) holds for any spinor ϵ, implies (3.14) so that the spinor ϵ must live in anti-de Sitter space.

Hence we have seen that supersymmetry can be realized in anti-de Sitter space. We will return to this issue later in chapter 6, where we discuss the (super)multiplet structure in anti-de Sitter space. We stress once more that, in this section, we have restricted ourselves to the graviton-gravitino sector. To construct the full theory usually requires more fields and important restrictions arise on the dimensionality of spacetime. For instance, while minimal supergravity in $D = 4$ dimensions does not require additional fields, in $D = 11$ dimensions an additional antisymmetric gauge field is necessary. The need for certain extra fields can be readily deduced from the underlying massless supermultiplets, which were extensively discussed in the previous chapter.

3.2 Maximal supersymmetry and supergravity

In chapter 2 we restricted ourselves to supermultiplets based on $Q \leq 32$ supercharge components. From the general analysis it is clear that increasing the number of supercharges leads to higher and higher helicity representations. For instance, the maximal helicity, $|\lambda_{\rm max}|$, of a massless supermultiplet in $D = 4$ spacetime dimensions is larger than or equal to $\frac{1}{16}Q$. Therefore, when $Q > 8$ we have $|\lambda_{\rm max}| \geq 1$, so that theories for these multiplets must include vector gauge fields. When $Q > 16$ we have $|\lambda_{\rm max}| \geq \frac{3}{2}$, so that the theory should contain Rarita-Schwinger fields. In view of the supersymmetry algebra an interacting supersymmetric theory of this type should contain gravity, so that in this case we must include $\lambda = 2$ states for the graviton. Beyond $Q = 32$ one is dealing with states of helicity $\lambda > 2$. Those are described by gauge fields that are *symmetric* Lorentz tensors. Symmetric tensor gauge fields for arbitrary helicity states can be constructed (in $D = 4$ dimensions, see, for instance, [36]). However, it turns out that symmetric gauge fields cannot consistently couple, neither to themselves nor to other fields. An exception is the graviton field, which can interact with itself as well as with low-spin matter, but not with other fields of the same spin [37]. By consistent, we mean that the respective gauge invariances of the higher-spin fields (or appropriate deformations thereof) cannot be preserved at the interacting level. Most of the search for interacting higher-spin fields was performed in 4 spacetime dimensions [38], but in higher dimensional spacetimes one expects to arrive at the same conclusions, because otherwise, upon dimensional reduction, these theories would give rise to theories that are consistent in $D = 4$. There is also direct evidence in $D = 3$, where graviton and gravitini fields do not describe dynamic degrees of freedom. Hence, one can write down supergravity theories based on a graviton field and an arbitrary number of gravitino fields, which are topological. However, when coupling matter to this theory in the form of scalars and spinors, the theory does not support more than 32 supercharges. Beyond $Q = 16$ there are four unique theories with $Q = 18, 20, 24$ and 32 [8].

D	$\mathrm{H_R}$	graviton	$p=-1$	$p=0$	$p=1$	$p=2$	$p=3$
11	1	1	0	0	0	1	0
10A	1	1	1	1	1	1	0
10B	SO(2)	1	2	0	2	0	1^*
9	SO(2)	1	$2+1$	$2+1$	2	1	
8	U(2)	1	$5+1+\bar{1}$	$3+\bar{3}$	3	[1]	
7	USp(4)	1	14	10	5		
6	USp(4) $\times$USp(4)	1	(5,5)	(4,4)	$(5,1)$ $+(1,5)$		
5	USp(8)	1	42	27			
4	U(8)	1	$35+\overline{35}$	[28]			
3	SO(16)	1	128				

Table 10: Bosonic field content for maximal supergravities. The $p=3$ gauge field in $D=10$B has a self-dual field strength. The representations [1] and [28] (in $D=8,4$, respectively) are extended to U(1) and SU(8) representations through duality transformations on the field strengths. These transformations can not be represented on the vector potentials. In $D=3$ dimensions, the graviton does not describe propagating degrees of freedom. For $p>0$ the fields can be assigned to representations of a bigger group than $\mathrm{H_R}$. This will be discussed in due course.

Hence the conclusion is that there is a restriction on the number Q of independent supersymmetries, as for $Q>32$ no interacting field theories seem to exist. There have been many efforts to circumvent this bound of $Q=32$ supersymmetries. It seems clear that one needs a combination of the following ingredients in order to do this (for a review, see *e.g.* [39]): (i) an infinite tower of higher-spin gauge fields; (ii) interactions that are inversely proportional to the cosmological constant; (iii) extensions of the super-Poincaré or the super-de Sitter algebra with additional fermionic and bosonic charges. Indeed explicit theories have been constructed which demonstrate this. However, conventional supergravity theories are *not* of this kind. This is the reason why we avoided (*i.e.* in table 5) to list supermultiplets with states transforming in higher-helicity representations. The fact that an infinite number of fields can cure certain inconsistencies is by itself not new. While a massive spin-2 field cannot be coupled to gravity, the coupling of an infinite number of them can be consistent, as can be seen in Kaluza-Klein theory.

In this chapter we review the maximal supergravities in various dimensions. These theories have $Q = 32$ supersymmetries and we restrict our discussion to $3 \leq D \leq 11$. The highest dimension $D = 11$ is motivated by the fact that spinors have more that 32 components in flat Minkowski space for spacetime dimensions $D > 11$. Observe, however, that this argument assumes D-dimensional Lorentz invariance. As was stressed in [40, 41], there are scenarios based on spacetime dimensions higher than $D = 11$, where the extra dimensions can not uniformly decompactify so that the no-go theorem is avoided. The fact that no uniform decompactification is possible is closely related to the T-duality between winding and momentum states that one knows from string theory.

The bosonic fields always comprise the metric tensor for the graviton and a number of $(p+1)$-rank antisymmetric gauge fields. For the antisymmetric gauge fields, it is a priori unclear whether to choose a $(p+1)$-rank gauge field or its dual $(D-3-p)$-rank partner, but it turns out that the interactions often prefer the rank of the gauge field to be as small as possible. Therefore, in table 10, we restrict ourselves to $p \leq 3$, as in $D = 11$ dimensions, $p = 3$ and $p = 4$ are each other's dual conjugates. This table presents all the field configurations for maximal supergravity in various dimensions. Obviously, the problematic higher-spin fields are avoided, because the only symmetric gauge field is the one describing the graviton. In table 11 we also present the fermionic fields, always consisting of gravitini and simple spinors. All these fields are classified as representations of the R-symmetry group $\mathrm{H_R}$. Note that the simplest versions of supergravity (which depend on no other coupling constant than Newton's constant) are manifestly invariant under $\mathrm{H_R}$. Actually, as we will explain in a sequel, the maximal supergravity theories have symmetry groups that are much larger than $\mathrm{H_R}$.

3.3 $D = 11$ Supergravity

Supergravity in 11 spacetime dimensions is based on an "elfbein" field $E_M^{\;A}$, a Majorana gravitino field Ψ_M and a 3-rank antisymmetric gauge field C_{MNP}. With chiral (2,0) supergravity in 6 dimensions, it is the only $Q \geq 16$ supergravity theory without a scalar field. Its Lagrangian can be written as follows [11],

$$\begin{aligned}\mathcal{L}_{11} = \frac{1}{\kappa_{11}^2}\Big[&- \tfrac{1}{2} E\, R(E,\Omega) - \tfrac{1}{2} E \bar\Psi_M \Gamma^{MNP} D_N(\Omega)\Psi_P - \tfrac{1}{48} E (F_{MNPQ})^2 \\ &- \tfrac{1}{3456}\sqrt{2}\,\varepsilon^{MNPQRSTUVWX}\, F_{MNPQ}\, F_{RSTU}\, C_{VWX} \\ &- \tfrac{1}{192}\sqrt{2} E\Big(\bar\Psi_R \Gamma^{MNPQRS}\Psi_S + 12\,\bar\Psi^M \Gamma^{NP}\Psi^Q\Big) F_{MNPQ} + \cdots \Big]\,,\end{aligned} \tag{3.18}$$

D	$\mathrm{H_R}$	gravitini	spinors
11	1	1	0
10A	1	1+1	1+1
10B	SO(2)	2	2
9	SO(2)	2	$2+2$
8	U(2)	$2+\bar{2}$	$2+\bar{2}+4+\bar{4}$
7	USp(4)	4	16
6	USp(4)×USp(4)	$(4,1)+(1,4)$	$(4,5)+(5,4)$
5	USp(8)	8	48
4	U(8)	$8+\bar{8}$	$56+\overline{56}$
3	SO(16)	16	128

Table 11: Fermionic field content for maximal supergravities. For $D=5,6,7$ the fermion fields are counted as symplectic Majorana spinors. For $D=4,8$ we include both chiral and antichiral spinor components, which transform in conjugate representations of $H_{\rm R}$. In $D=3$ dimensions the gravitino does not describe propagating degrees of freedom.

where the ellipses denote terms of order Ψ^4, $E=\det E_M^A$ and $\Omega_M{}^{AB}$ denotes the spin connection. The supersymmetry transformations are

$$\begin{aligned} \delta E_M^A &= \tfrac{1}{2}\,\bar\epsilon\,\Gamma^A\Psi_M\,, \\ \delta C_{MNP} &= -\tfrac{1}{8}\sqrt{2}\,\bar\epsilon\,\Gamma_{[MN}\Psi_{P]}\,, \\ \delta\Psi_M &= D_M(\hat\Omega)\,\epsilon+\tfrac{1}{288}\sqrt{2}\left(\Gamma_M{}^{NPQR}-8\,\delta_M^N\,\Gamma^{PQR}\right)\epsilon\,\hat F_{NPQR}\,. \end{aligned} \tag{3.19}$$

Here the derivative D_M is covariant with respect to local Lorentz transformations,

$$D_M(\Omega)\,\epsilon=\left(\partial_M-\tfrac{1}{4}\Omega_M{}^{AB}\Gamma_{AB}\right)\epsilon\,, \tag{3.20}$$

and $\hat F_{MNPQ}$ is the supercovariant field strength

$$\hat F_{MNPQ}=24\,\partial_{[M}C_{NPQ]}+\tfrac{3}{2}\sqrt{2}\,\bar\Psi_{[M}\Gamma_{NP}\Psi_{Q]}\,. \tag{3.21}$$

The supercovariant spin connection is the solution of the following equation,

$$D_{[M}(\hat\Omega)\,E_{N]}^A-\tfrac{1}{4}\bar\Psi_M\Gamma^A\Psi_N=0\,. \tag{3.22}$$

The left-hand side is the supercovariant torsion tensor.

Note the presence of a Chern-Simons-like term $F \wedge F \wedge C$ in the Lagrangian, so that the action is only invariant under tensor gauge transformations up to surface terms. We also wish to point out that the quartic-Ψ terms can be included into the Lagrangian (3.18) by replacing the spin-connection field Ω by $(\Omega + \hat{\Omega})/2$ in the covariant derivative of the gravitino kinetic term and by replacing F_{MNPQ} in the last line by $(\hat{F}_{MNPQ} + F_{MNPQ})/2$. These substitutions ensure that the field equations corresponding to (3.18) are supercovariant. The Lagrangian is derived in the context of the so-called "1.5-order" formalism, in which the spin connection is defined as a dependent field determined by its (algebraic) equation of motion, whereas its supersymmetry variation in the action is treated as if it were an independent field [42].

We have the following bosonic field equations and Bianchi identities,

$$
\begin{aligned}
R_{MN} &= \tfrac{1}{72} g_{MN}\, F_{PQRS} F^{PQRS} - \tfrac{1}{6} F_{MPQR}\, F_N{}^{PQR} \,, \\
\partial_M \Big(E\, F^{MNPQ} \Big) &= \tfrac{1}{1152} \sqrt{2}\, \varepsilon^{NPQRSTUVWXY} F_{RSTU}\, F_{VWXY} \,, \\
\partial_{[M} F_{NPQR]} &= 0 \,,
\end{aligned}
\tag{3.23}
$$

which no longer depend explicitly on the antisymmetric gauge field. An alternative form of the second equation is [43]

$$
\partial_{[M} H_{NPQRSTU]} = 0 \,, \tag{3.24}
$$

where $H_{MNPQRST}$ is the dual field strength,

$$
H_{MNPQRST} = \frac{1}{7!} E\, \varepsilon_{MNPQRSTUVWX} F^{UVWX} - \tfrac{1}{2} \sqrt{2}\, F_{[MNPQ}\, C_{RST]} \,. \tag{3.25}
$$

One could imagine that the third equation of (3.23) and (3.24) receive contributions from charges that would give rise to source terms on the right-hand side of the equations. These charges are associated with the 'flux'-integral of $H_{MNPQRST}$ and F_{MNPQ} over the boundary of an 8- and a 5-dimensional spatial volume, respectively. In analogy with the Maxwell theory, the integral $\oint H$ may be associated with electric flux and the integral $\oint F$ with magnetic flux. The spatial volumes are orthogonal to a $p = 2$ and a $p = 5$ brane configuration, respectively, and the corresponding charges are 2- and 5-rank Lorentz tensors. These are just the charges that can appear as central charges in the supersymmetry algebra (2.12). Solutions of 11-dimensional supergravity that contribute to these charges were considered in [44, 45, 46].

Finally, the constant $1/\kappa_{11}^2$ in front of the Lagrangian (3.18), which carries dimension $[\text{length}]^{-9} \sim [\text{mass}]^9$, is undetermined and depends on fixing some length scale. To see this consider a continuous rescaling of the fields,

$$E_M^{\ A} \to \mathrm{e}^{-\alpha} E_M^{\ A}\,, \qquad \Psi_M \to \mathrm{e}^{-\alpha/2}\Psi_M\,, \qquad C_{MNP} \to \mathrm{e}^{-3\alpha} C_{MNP}\,. \quad (3.26)$$

Under this rescaling the Lagrangian changes according to

$$\mathcal{L}_{11} \to \mathrm{e}^{-9\alpha}\mathcal{L}_{11}\,. \quad (3.27)$$

This change can then be absorbed into a redefinition of κ_{11},[12]

$$\kappa_{11}^2 \to \mathrm{e}^{-9\alpha}\kappa_{11}^2\,. \quad (3.28)$$

This simply means that the Lagrangian depends on only one dimensional coupling constant, namely κ_{11}. The same situation is present in many other supergravity theories. Concentrating on the Einstein-Hilbert action in D spacetime dimensions, the corresponding scaling property is

$$g_{\mu\nu}^D \to \mathrm{e}^{-2\alpha} g_{\mu\nu}^D\,, \qquad \mathcal{L}_D \to \mathrm{e}^{(2-D)\alpha}\mathcal{L}_D\,, \qquad \kappa_D^2 \to \mathrm{e}^{(2-D)\alpha}\kappa_D^2\,. \quad (3.29)$$

Of course, this implies that the physical value of Newton's constant, does not necessarily coincide with the parameter κ_D^2 in the Lagrangian but it also depends on the precise value adopted for the (flat) metric in the ground state of the theory.

3.4 Dimensional reduction and hidden symmetries

The maximal supergravities in various dimensions are related by dimensional reduction. In this reduction some of the spatial dimensions are compactified on a hypertorus and one retains only the fields that do not depend on the torus coordinates. This corresponds to the theory one obtains when the size of the torus is shrunk to zero. A subset of the gauge symmetries associated with the compactified dimensions survive as internal symmetries. The aim of the present discussion here is to elucidate a number of features related to these symmetries, mainly in the context of the reduction of $D = 11$ supergravity to $D = 10$ dimensions.

We denote the compactified coordinate by x^{10} which now parameterizes a circle of length L.[13] The fields are thus decomposed in a Fourier series as periodic functions in x^{10} on the interval $0 \leq x^{10} \leq L$. This results in a spectrum of massless modes and an infinite tower of massive modes with masses

[12] Note that the rescalings also leave the supersymmetry transformation rules unchanged, provided the supersymmetry parameter ϵ is changed accordingly.

[13] Throughout these lectures we enumerate spacetime coordinates by $0, 1, \ldots, D-1$.

inversely proportional to the circle length L. The massless modes form the basis of the lower-dimensional supergravity theory. Because a toroidal background does not break supersymmetry, the resulting supergravity has the same number of supersymmetries as the original one. For compactifications on less trivial spaces than the hypertorus, this is usually not the case and the number of independent supersymmetries will be reduced. Fully supersymmetric compactifications are rare. For instance, 11-dimensional supergravity can be compactified to a 4-dimensional maximally symmetric spacetime in only two ways such that all supersymmetries remain unaffected [47]. One is the compactification on a torus T^7, the other one the compactification on a sphere S^7. In the latter case the resulting 4-dimensional supergravity theory acquires a cosmological term.

In the formulation of the compactified theory, it is important to decompose the higher-dimensional fields in such a way that they transform covariantly under the lower-dimensional gauge symmetries and under diffeomorphisms of the lower-dimensional spacetime. This ensures that various complicated mixtures of massless modes with the tower of massive modes will be avoided. It is a key element in ensuring that solutions of the lower-dimensional theory remain solutions of the original higher-dimensional one, which is an obvious requirement for having consistent truncations to the massless states. Another point of interest concerns the nature of the massive supermultiplets. Because these originate from supermultiplets that are massless in higher dimensions, they are 1/2-BPS multiplets which are shortened by the presence of central charges corresponding to the momenta in the compactified dimension. Implications of these BPS supermultiplets will be discussed in more detail in section 3.6.

The emergence of new internal symmetries in theories that originate from a higher-dimensional setting, is a standard feature of Kaluza-Klein theories [48]. Following the discussion in [49] we distinguish between symmetries that have a direct explanation in terms of the higher-dimensional symmetries, and symmetries whose origin is obscure from a higher-dimensional viewpoint. Let us start with the symmetries associated with the metric tensor. The 11-dimensional metric can be decomposed according to

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{e}^{4\phi/3}(\mathrm{d}x^{10} + V_\mu\mathrm{d}x^\mu)(\mathrm{d}x^{10} + V_\nu\mathrm{d}x^\nu)\,, \tag{3.30}$$

where the indices μ, ν label the 10-dimensional coordinates and the factor multiplying ϕ is for convenience later on. The massless modes correspond to the x^{10}-independent parts of the 10-dimensional metric $g_{\mu\nu}$, the vector field V_μ and the scalar ϕ. Here the x^{10}-independent component of V_μ acts as a gauge field associated with reparametrizations of the circle coordinate x^{10} with an arbitrary function $\xi(x)$ of the 10 remaining spacetime coordinates x^μ. Specifically, we have $x^{10} \to x^{10} - \xi(x)$ and $x^\mu \to x^\mu$, leading to

$$V_\mu(x) \to V_\mu(x) + \partial_\mu \xi(x)\,. \tag{3.31}$$

The massive modes, which correspond to the nontrivial Fourier modes in x^{10}, couple to this gauge field with a charge that is a multiple of

$$e_{\mathrm{KK}} = \frac{2\pi}{L}\,. \tag{3.32}$$

Another symmetry of the lower-dimensional theory is more subtle to identify.[14] In the previous subsection we noted the existence of certain scale transformations of the $D = 11$ fields, which did not leave the theory invariant but could be used to adjust the coupling constant κ_{10}. In the compactified situation we can also involve the compactification length into the dimensional scaling. The integration over x^{11} introduces an overall factor L in the action (we do not incorporate any L-dependent normalizations in the Fourier sums, so that the 10-dimensional and the 11-dimensional fields are directly proportional). Therefore, the coupling constant that emerges in the 10-dimensional theory equals

$$\frac{1}{\kappa_{10}^2} = \frac{L}{\kappa_{11}^2}\,, \tag{3.33}$$

and is of dimension $[\mathrm{mass}]^8$. However, because of the invariance under diffeomorphisms, L itself has no intrinsic meaning. It simply expresses the length of the x^{10}-periodicity interval, which depends on the coordinatization. Stated differently, we can reparameterize x^{10} by some diffeomorphism, as long as we change L accordingly. In particular, we may rescale L according to

$$L \to \mathrm{e}^{-9\alpha} L\,, \tag{3.34}$$

corresponding to a reparametrization of the 11-th coordinate,

$$x^{10} \to \mathrm{e}^{-9\alpha} x^{10}\,, \tag{3.35}$$

so that κ_{10} remains invariant. Consequently we are then dealing with a *symmetry* of the Lagrangian.

[14] There are various discussions of this symmetry in the literature. Its existence in 10-dimensional supergravity was noted long ago (see, e.g. [9, 50]) and an extensive discussion can be found in [18]. Our derivation here was alluded to in [49], which deals with isometries in $N = 2$ supersymmetric Maxwell-Einstein theories in $D = 5, 4$ and 3 dimensions.

In the effective 10-dimensional theory, the scale transformations (3.26) are thus suitably combined with the diffeomorphism (3.35) to yield an invariance of the Lagrangian. For the fields corresponding to the 11-dimensional metric, these combined transformations are given by[15]

$$e_\mu^a \to \mathrm{e}^{-\alpha} e_\mu^a\,, \qquad \phi \to \phi + 12\alpha\,, \qquad V_\mu \to \mathrm{e}^{-9\alpha} V_\mu\,. \tag{3.36}$$

The tensor gauge field C_{MNP} decomposes into a 3- and a 2-rank tensor in 10 dimensions, which transform according to

$$C_{\mu\nu\rho} \to \mathrm{e}^{-3\alpha} C_{\mu\nu\rho}\,, \qquad C_{11\mu\nu} \to \mathrm{e}^{6\alpha} C_{11\mu\nu}\,. \tag{3.37}$$

The presence of the above scale symmetry is confirmed by the resulting 10-dimensional Lagrangian for the massless (*i.e.*, x^{10}-independent) modes. Its purely bosonic terms read

$$\begin{aligned}\mathcal{L}_{10} \;=\; \frac{1}{\kappa_{10}^2}\Big[& -\tfrac{1}{2} e\,\mathrm{e}^{2\phi/3} R(e,\omega) - \tfrac{1}{8} e\,\mathrm{e}^{2\phi}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 \\ & -\tfrac{1}{48} e\,\mathrm{e}^{2\phi/3}(F_{\mu\nu\rho\sigma})^2 - \tfrac{3}{4} e\,\mathrm{e}^{-2\phi/3}(H_{\mu\nu\rho})^2 \\ & +\tfrac{1}{1152}\sqrt{2}\,\varepsilon^{\mu_1-\mu_{10}}\, C_{11\mu_1\mu_2}\, F_{\mu_3\mu_4\mu_5\mu_6}\, F_{\mu_7\mu_8\mu_9\mu_{10}}\Big]\,,\end{aligned} \tag{3.38}$$

where $H_{\mu\nu\rho} = 6\,\partial_{[\mu} C_{\nu\rho]11}$ is the field strength tensor belonging to the 2-rank tensor gauge field.

The above example exhibits many of the characteristic features of dimensional reduction and of the symmetries that emerge as a result. When reducing to lower dimension one can follow the same procedure a number of times, consecutively reducing the dimension by unit steps, or one can reduce at once to lower dimensions. Before continuing our general discussion, let us briefly discuss an example of the latter based on gravity coupled to an antisymmetric tensor gauge field in $D+n$ spacetime dimensions,

$$\mathcal{L} \propto -\tfrac{1}{2} E\, R - \tfrac{9}{4} E\,(\partial_{[M} B_{NP]})^2\,. \tag{3.39}$$

After compactification on a torus T^n, the fields that are independent of the torus coordinates remain massless fields in D dimensions: the graviton, one tensor gauge field, $2n$ abelian vector gauge fields, and n^2 scalar fields. The

[15] Note that these transformations apply uniformly to all Fourier modes, as those depend on x^{10}/L which is insensitive to the scale transformation. This does not imply that the Lagrangian remains invariant when retaining the higher Fourier modes, because the Kaluza-Klein charges (3.32) depend explicitly on L. This issue will be relevant in section 3.6.

scalar fields originate from the metric and the antisymmetric gauge field with both indices taking values in T^n, so that they are parametrized by a symmetric tensor g_{ij} and an antisymmetric tensor B_{ij}. The diffeomorphisms acting on the torus coordinates x^i which are linear in x^i, *i.e.*, $x^i \to \mathsf{O}^i{}_j\, x^j$, act on g_{ij} and B_{ij} according to $g \to \mathsf{O}^{\mathrm{T}} g\, \mathsf{O}$ and $B \to \mathsf{O}^{\mathrm{T}} B\, \mathsf{O}$. The matrices O generate the group $\mathrm{GL}(n)$, which can be regarded as a generalization of the scale transformations (3.36) and (3.37). The group $\mathrm{GL}(n)$ contains the rotation group $\mathrm{SO}(n)$; its remaining part depends on $\frac{1}{2}n(n+1)$ parameters, exactly equal to the number of independent fields g_{ij}. Special tensor gauge transformations with parameters proportional to $\Lambda_{ij}\, x^j$ induce a shift of the massless scalars B_{ij} proportional to the constants $\Lambda_{[ij]}$. There are thus $\frac{1}{2}n(n-1)$ independent shift transformations, so that, in total, we have now identified $\frac{1}{2}n(3n-1)$ isometries, which act transitively on the manifold (*i.e.* they leave no point on the manifold invariant). Therefore the manifold is homogeneous (for a discussion of such manifolds, see section 4.1). However, it turns out that there exist $\frac{1}{2}n(n-1)$ additional isometries, whose origin is *not* directly related to the higher-dimensional context, and which combine with the previous ones to generate the group $\mathrm{SO}(n,n)$. The homogeneous space can then be identified as the coset space $\mathrm{SO}(n,n)/(\mathrm{SO}(n)\times\mathrm{SO}(n))$.

According to the above, 11-dimensional supergravity reduced on a hypertorus thus leads to a Lagrangian for the massless sector in lower dimensions (the massive sector is discussed in section 3.6), which exhibits a number of invariances that find their origin in the diffeomorphisms and gauge transformations related to the torus coordinates. As already explained, one must properly account for the periodicity intervals of the torus coordinates x^i, but the action for the massless fields remains invariant under continuous $\mathrm{GL}(n)$ transformations. Furthermore, all the scalars that emerge from dimensional reduction of gauge fields are subject to constant shift transformations. These scalars and the scalars originating from the metric transform transitively under the isometry group. Since 11-dimensional supergravity has itself no scalar fields, the rank of the resulting symmetry group in lower dimensions is equal to the rank of $\mathrm{GL}(n)$, and thus to the number of compactified dimensions, *i.e.*, $\mathbf{r} = 11 - D$, where D is the spacetime dimension to which we reduce. In general these extra symmetries are not necessarily symmetries of the full action. In even dimensions, the symmetries may not leave the Lagrangian, but only the field equations, invariant. The reason for this is that the isometries may act by means of duality transformations on field strengths associated with antisymmetric tensor gauge fields of rank $\frac{1}{2}D - 1$ which cannot be implemented on the gauge fields themselves. In 4 dimensions this phenomenon is known as electric-magnetic duality (for a recent review, see [51]); for $D = 6$ we refer to [52]. For supergravity, it is easy to see that the scalar manifold (as well as the rest of the theory) possesses additional symmetries beyond the ones that follow from higher dimensions, because the latter do not yet incorporate the full R-symmetry group of the underlying supermultiplet. We expect that $\mathrm{H_R}$ is also realized as a symmetry, because the maximal supergravity theory that one obtains from compactification on a hypertorus has no additional coupling constants (beyond Newton's constant) which could induce R-symmetry breaking. Therefore we expect that the target space for the scalar fields is an homogeneous space, with an isometry group whose generators belong to a solvable subalgebra associated with the shift transformations, to the subalgebra of $\mathrm{GL}(11-D)$ scale transformations and/or to the subalgebra associated with $\mathrm{H_R}$. Of course, these subalgebras will partly overlap. Usually a counting argument (of the type first used in [53]) then readily indicates what the structure is of the corresponding homogeneous space that is parametrized by the scalar fields. In table 12 we list the isometry group G and the isotropy group $\mathrm{H_R}$ of these scalar manifolds for maximal supergravity in dimensions $3 \leq D \leq 11$. Earlier versions of such tables can, for instance, be found in [9, 50].

A more recent discussion of these isometry groups from the perspectives of string theory and M-theory can be found in, for example, [54, 24]. Here we merely stress a number of characteristic features of the group G. One of them is that $\mathrm{H_R}$ is always the maximal compact subgroup of G. As we mentioned already, another (noncompact) subgroup is the group $\mathrm{GL}(11-D)$, associated with the reduction on an $(11-D)$-dimensional torus. Yet another subgroup is $\mathrm{SL}(2)\times\mathrm{SO}(n,n)$, where $n = 10-D$. This group, which emerges for $D < 10$ can be understood within the string perspective; $\mathrm{SL}(2)$ is the S-duality group and $\mathrm{SO}(n,n)$ is the T-duality group. It also follows from the toroidal compactifications of IIB supergravity, which has a manifest $\mathrm{SL}(2)$ in $D = 10$ dimensions. The group $\mathrm{SO}(n,n)$ is associated with the invariance of toroidal compactifications that involve the metric and an antisymmetric tensor field (*c.f.* (3.39)).

Here we should add that it is generally possible to realize the group $\mathrm{H_R}$ as a *local* symmetry of the Lagrangian. The corresponding connections are then composite connections, governed by the Cartan-Maurer equations. In such a formulation most fields (in particular, the fermions) do not transform under the group G, but only under the local $\mathrm{H_R}$ group. The scalars transform linearly under both the rigid duality group as well as under the local $\mathrm{H_R}$ group; the gauge fields cannot transform under the local group $\mathrm{H_R}$, as this would be in conflict with their own gauge invariance. After fixing a

D	G	H	$\dim[\mathrm{G}] - \dim[\mathrm{H}]$
11	1	1	$0-0=0$
10A	$\mathrm{SO}(1,1)/\mathbf{Z}_2$	1	$1-0=1$
10B	SL(2)	SO(2)	$3-1=2$
9	GL(2)	SO(2)	$4-1=3$
8	$\mathrm{E}_{3(+3)} \sim \mathrm{SL}(3)\times\mathrm{SL}(2)$	U(2)	$11-4=7$
7	$\mathrm{E}_{4(+4)} \sim \mathrm{SL}(5)$	USp(4)	$24-10=14$
6	$\mathrm{E}_{5(+5)} \sim \mathrm{SO}(5,5)$	$\mathrm{USp}(4)\times\mathrm{USp}(4)$	$45-20=25$
5	$\mathrm{E}_{6(+6)}$	USp(8)	$78-36=42$
4	$\mathrm{E}_{7(+7)}$	SU(8)	$133-63=70$
3	$\mathrm{E}_{8(+8)}$	SO(16)	$248-120=128$

Table 12: Homogeneous scalar manifolds G/H for maximal supergravities in various dimensions. The type-IIB theory cannot be obtained from reduction of 11-dimensional supergravity and is included for completeness. The difference of the dimensions of G and H equals the number of scalar fields, listed in table 10.

gauge, the G-transformations become realized nonlinearly (we discuss such nonlinear realizations in detail in chapters 4 and 5). The fields which initially transform only under the local $\mathrm{H_R}$ group, will now transform under the duality group G through field-dependent $\mathrm{H_R}$ transformations. This phenomenon is also realized for the central charges, which transform under the group $\mathrm{H_R}$ as we have shown in table 8. We discuss some of the consequences for the central charges and the BPS states in section 3.6.

3.5 Frames and field redefinitions

The Lagrangian (3.38) does not contain the standard Einstein-Hilbert term for gravity, while a standard kinetic term for the scalar field ϕ is lacking. This does not pose a serious problem. In this form the gravitational field and the scalar field are entangled and one has to deal with the scalar-graviton system as a whole. To separate the scalar and gravitational degrees of freedom, one applies a so-called Weyl rescaling of the metric $g_{\mu\nu}$ by an appropriate function of ϕ. In the case that we include the massive modes, this rescaling may depend on the extra coordinate x^{10}. In the context of Kaluza-Klein theory this factor is known as the 'warp factor'. For these lectures two different Weyl rescalings are particularly relevant, which lead to the so-called Einstein and to the string frame, respectively. They are defined by

$$e_\mu^a = \mathrm{e}^{-\phi/12}\,[e_\mu^a]^{\rm Einstein}\,, \qquad e_\mu^a = \mathrm{e}^{-\phi/3}\,[e_\mu^a]^{\rm string}\,. \tag{3.40}$$

We already stressed that the the compactification length L is just a parameter length with no intrinsic meaning as a result of the fact that one can always apply general coordinate transformations which involve x^{10}. Of course, one may also consider the *geodesic length*, which in the metric specified by (3.30) is equal to $L\,\exp[2\langle\phi\rangle/3]$. In the Einstein frame, the geodesic length of the 11-th dimension is invariant under the $\mathrm{SO}(1,1)$ transformations.

After applying the first rescaling (3.40) to the Lagrangian (3.38) one obtains the Lagrangian in the Einstein frame. This frame is characterized by a standard Einstein-Hilbert term and by a graviton field that is invariant under the scale transformations (3.36, 3.37). The corresponding Lagrangian reads[16]

$$\begin{aligned}\mathcal{L}_{10}^{\rm Einstein} \;=\; \frac{1}{\kappa_{10}^2}\Big[& e\Big[-\tfrac{1}{2}R(e,\omega)-\tfrac{1}{4}(\partial_\mu\phi)^2\Big]-\tfrac{1}{8}e\,\mathrm{e}^{3\phi/2}(\partial_\mu V_\nu-\partial_\nu V_\mu)^2\\ & -\tfrac{3}{4}e\,\mathrm{e}^{-\phi}(H_{\mu\nu\rho})^2-\tfrac{1}{48}e\,\mathrm{e}^{\phi/2}(F_{\mu\nu\rho\sigma})^2\\ & +\tfrac{1}{1152}\sqrt{2}\,\varepsilon^{\mu_1-\mu_{10}}\,C_{11\mu_1\mu_2}\,F_{\mu_3\mu_4\mu_5\mu_6}\,F_{\mu_7\mu_8\mu_9\mu_{10}}\Big]\,.\end{aligned} \tag{3.41}$$

Supergravity theories are usually formulated in this frame, where the isometries of the scalar fields do not act on the graviton.

The second rescaling (3.40) leads to the Lagrangian in the string frame,

$$\begin{aligned}\mathcal{L}_{10}^{\rm string} \;=\; \frac{1}{\kappa_{10}^2}\Big[& e\,\mathrm{e}^{-2\phi}\Big[-\tfrac{1}{2}R(e,\omega)+2(\partial_\mu\phi)^2-\tfrac{3}{4}(H_{\mu\nu\rho})^2\Big]\\ & -\tfrac{1}{8}e\,(\partial_\mu V_\nu-\partial_\nu V_\mu)^2-\tfrac{1}{48}e\,(F_{\mu\nu\rho\sigma})^2\\ & +\tfrac{1}{1152}\sqrt{2}\,\varepsilon^{\mu_1-\mu_{10}}\,C_{11\mu_1\mu_2}\,F_{\mu_3\mu_4\mu_5\mu_6}\,F_{\mu_7\mu_8\mu_9\mu_{10}}\Big]\,.\end{aligned} \tag{3.42}$$

This frame is characterized by the fact that R and $(H_{\mu\nu\rho})^2$ have the same coupling to the scalar ϕ, or, equivalently, that $g_{\mu\nu}$ and $C_{11\mu\nu}$ transform with equal weights under the scale transformations (3.36, 3.37). In string theory ϕ

[16] Note that under a local scale transformation $e_\mu^a \to \mathrm{e}^{\Lambda} e_\mu^a$, the Ricci scalar in D dimensions changes according to

$$R \to \mathrm{e}^{-2\Lambda}\Big[R + 2(D-1)D^\mu\partial_\mu\Lambda + (D-1)(D-2)g^{\mu\nu}\,\partial_\mu\Lambda\,\partial_\nu\Lambda\Big]\,.$$

Observe that gauge fields cannot be redefined by these local scale transformations because this would interfere with their own gauge invariance.

coincides with the dilaton field that couples to the topology of the worldsheet and whose vacuum-expectation value defines the string coupling constant according to $g_{\rm s} = \exp(\langle\phi\rangle)$. The significance of the dilaton factors in the Lagrangian above is well known. The metric $g_{\mu\nu}$, the antisymmetric tensor $C_{\mu\nu 11}$ and the dilaton ϕ always arise in the Neveu-Schwarz sector and couple universally to $\mathrm{e}^{-2\phi}$. On the other hand the vector V_μ and the 3-form $C_{\mu\nu\rho}$ describe Ramond-Ramond (R-R) states and the specific form of their vertex operators forbids any tree-level coupling to the dilaton [55, 18]. In particular the Kaluza-Klein gauge field V_μ corresponds in the string context to the R-R gauge field of type-II string theory. The infinite tower of massive Kaluza-Klein states carry a charge quantized in units of $e_{\rm KK}$, defined in (3.32). In the context of 10-dimensional supergravity, states with R-R charge are solitonic. In string theory, R-R charges are carried by the D-brane states.

For later purposes let us note that the above discussion can be generalized to arbitrary spacetime dimensions. The Einstein frame in any dimension is defined by a gravitational action that is just the Einstein-Hilbert action, whereas in the string frame the Ricci scalar is multiplied by a dilaton term $\exp(-2\phi_D)$, as in (3.41) and (3.42), respectively. The Weyl rescaling which connects the two frames is given by,

$$[e_\mu^a]^{\rm string} = \mathrm{e}^{2\phi_D/(D-2)}\,[e_\mu^a]^{\rm Einstein}\,. \tag{3.43}$$

Let us now return to 11-dimensional supergravity with the 11-th coordinate compactified to a circle so that $0 \leq x^{10} \leq L$. As we stressed already, L itself has no intrinsic meaning and it is better to consider the geodesic radius of the 11-th dimension, which reads

$$R_{10} = \frac{L}{2\pi}\,\mathrm{e}^{2\langle\phi\rangle/3}\,. \tag{3.44}$$

This result applies to the frame specified by the 11-dimensional theory[17]. In the string frame, the above result reads

$$(R_{10})^{\rm string} = \frac{L}{2\pi}\,\mathrm{e}^{\langle\phi\rangle}\,. \tag{3.45}$$

It shows that a small 11-th dimension corresponds to small values of $\exp\langle\phi\rangle$ which in turn corresponds to a weakly coupled string theory. Observe that L is fixed in terms of κ_{10} and κ_{11} (*c.f.* (3.33)).

[17]This is the frame specified by the metric given in (3.30), which leads to the Lagrangian (3.38).

From the 11-dimensional expressions,

$$E_a^{\,M}\partial_M = e_a^{\,\mu}(\partial_\mu - V_\mu\,\partial_{10})\,, \qquad E_{10}^{\,M}\partial_M = \mathrm{e}^{-2\phi/3}\,\partial_{10}\,, \tag{3.46}$$

where a and μ refer to the 10-dimensional Lorentz and world indices, we infer that, in the frame specified by the 11-dimensional theory, the Kaluza-Klein masses are multiples of

$$M^{\mathrm{KK}} = \frac{1}{R_{10}}\,. \tag{3.47}$$

Hence Kaluza-Klein states have a mass and a Kaluza-Klein charge (cf. (3.32)) related by

$$M^{\mathrm{KK}} = |e_{\mathrm{KK}}|\,\mathrm{e}^{-2\langle\phi\rangle/3}\,. \tag{3.48}$$

In the string frame, this result becomes

$$(M^{\mathrm{KK}})^{\mathrm{string}} = |e_{\mathrm{KK}}|\,\mathrm{e}^{-\langle\phi\rangle}\,. \tag{3.49}$$

Massive Kaluza-Klein states are always BPS states, meaning that they are contained in supermultiplets that are 'shorter' than the generic massive supermultiplets because of nontrivial central charges. The central charge here is just the 10-th component of the momentum, which is proportional to the Kaluza-Klein charge.

The surprising insight that emerged, is that the Kaluza-Klein features of 11-dimensional supergravity have a precise counterpart in string theory [54, 56, 55]. There one has nonperturbative (in the string coupling constant) states which carry R-R charges. On the supergravity side these states often appear as solitons.

3.6 Kaluza-Klein states and BPS-extended supergravity

In most of this chapter we restrict ourselves to pure supergravity. However, when compactifying dimensions one also encounters massive Kaluza-Klein states, which couple to the supergravity theory as massive matter supermultiplets. The presence of these BPS states introduces a number of qualitative changes to the theory which we discuss in this section. The most conspicuous change is that the continuous nonlinearly realized symmetry group G is broken to an arithmetic subgroup, known as the U-duality group. This U-duality group has been conjectured to be the exact symmetry group of (toroidally compactified) M-theory [54]. The BPS states (which are contained in M-theory) should therefore be assigned to representations of the U-duality group. Here one naively assumes that the U-duality group acts on the central charges of the BPS states and it is simply defined as the arithmetic subgroup of G that leaves the central-charge lattice invariant. However, central charges are in principle assigned to representations of the group $\mathrm{H_R}$ and not of the group G (although the central charges will eventually, upon gauge fixing, transform nonlinearly under G, via field-dependent $\mathrm{H_R}$ transformations). In most cases, these $\mathrm{H_R}$ representations can be elevated to representations of G, by multiplying with the representatives of the coset space $\mathrm{G/H_R}$ (representatives of coset spaces will be discussed in chapters 4 and 5). In this way, the pointlike (field-dependent) central charges can be assigned to representations of G for spacetime dimensions $D \geq 4$. Similar observations exist for stringlike and membranelike central charges except that in these cases the dimension must be restricted even further [41].

Another aspect of the coupling of the BPS states to supergravity is that the central charges should be related to *local* symmetries, in view of the fact that they appear in the anticommutator of two supercharges and supersymmetry is realized locally. Therefore nonzero central charges must couple to appropriate gauge fields in the supergravity theory. These gauge fields transform (with minor exceptions) linearly with constant matrices under the group G. Inspection of the tables that we have presented earlier, shows that the gauge fields usually appear in the G-representation required for gauging the corresponding central charge. Provided that the central charges can be assigned to the appropriate representations of the U-duality group and that the appropriate gauge fields are available, one may thus envisage a (possibly local field) theory of BPS states coupled to supergravity that is U-duality invariant. This theory would exhibit many of the features of M-theory and describe many of the relevant degrees of freedom.

The Kaluza-Klein states that we encounter in toroidal compactifications of supergravity are a subset of the 1/2-BPS states in M-theory. They carry pointlike central charges and they couple to the Kaluza-Klein photon fields, *i.e.*, the vector gauge fields that emerge from the higher-dimensional metric upon the toroidal compactification. However, they do *not* constitute representations of the U-duality group, because the central charges that they carry are too restricted. This is the reason why retaining the Kaluza-Klein states in the dimensional compactification will lead to a breaking of the U-duality group. From the eleven-dimensional perspective it is easy to see why the central charges associated with the Kaluza-Klein states are too restricted, because under U-duality the central charges related to the momentum operator in the compactified dimensions combine with the two- and five-brane charges (*c.f.* (2.12)) in order to define representations of the U-duality group. However, conventional dimensional compactification does not involve any brane charges. Nevertheless, in certain cases one may still be able to extend the Kaluza-Klein states with other BPS states, so that a U-duality invariant theory is obtained. Such extended theories are called BPS-extended supergravity [40, 41].

The fact that some of the central charges are associated with extra spacetime dimensions (*i.e.* the charges carried by the Kaluza-Klein states) implies that the newly introduced states (associated with wrapped branes) may also have an interpretation in terms of extra dimensions. In this way, the number of spacetime dimensions could exceed eleven, although the theory would presumably not be able to decompactify uniformly to a flat spacetime of more than eleven dimensions. The aim of this section is to elucidate some of these ideas in the relatively simple context of $N=2$ supergravity in $D=9$ spacetime dimensions.

We start by considering the BPS multiplets that are relevant in 9 spacetime dimensions from the perspective of supergravity, string theory and (super)membranes. In 9 dimensions the R-symmetry group and the duality group are equal to $\mathrm{H_R} = \mathrm{SO}(2)$ and $\mathrm{G} = \mathrm{SO}(1,1)\times\mathrm{SL}(2;\mathbf{R})$, respectively. It is well known that the massive supermultiplets of IIA and IIB string theory coincide, whereas the massless states comprise inequivalent supermultiplets for the simple reason that they transform according to different representations of the SO(8) helicity group. When compactifying the theory on a circle, IIA and IIB states that are massless in 9 spacetime dimensions, transform according to identical representations of the SO(7) helicity group and constitute equivalent supermultiplets. The corresponding interacting field theory is the unique $N=2$ supergravity theory in 9 spacetime dimensions. However, the BPS supermultiplets which carry momentum along the circle, remain inequivalent as they remain assigned to the inequivalent representations of the group SO(8) which is now associated with the restframe (spin) rotations of the massive states. Henceforth the momentum states of the IIA and the IIB theories will be denoted as KKA and KKB states, respectively. The fact that they constitute inequivalent supermultiplets, has implications for the winding states in order that T-duality remains valid [19].

In 9 spacetime dimensions with $N=2$ supersymmetry the Lorentz-invariant central charges are encoded in a two-by-two real symmetric matrix Z^{ij}, which can be decomposed as

$$Z^{ij} = b\,\delta^{ij} + a\,(\cos\theta\,\sigma_3 + \sin\theta\,\sigma_1)^{ij}\,. \tag{3.50}$$

Here $\sigma_{1,3}$ are the real symmetric Pauli matrices. We note that the central charge associated with the parameter a transforms as a doublet under the SO(2) R-symmetry group that rotates the two supercharge spinors, while the central charge proportional to the parameter b is SO(2) invariant. Subsequently one shows that BPS states that carry these charges must satisfy the mass formula,

$$M_{\rm BPS} = |a| + |b| \,. \tag{3.51}$$

Here one can distinguish three types of BPS supermultiplets. One type has central charges $b = 0$ and $a \neq 0$. These are 1/2-BPS muliplets, because they are annihilated by half of the supercharges. The KKA supermultiplets that comprise Kaluza-Klein states of IIA supergravity are of this type. Another type of 1/2-BPS mulitplets has central charges $a = 0$ and $b \neq 0$. The KKB supermultiplets that comprise the Kaluza-Klein states of IIB supergravity are of this type. Finally there are 1/4-BPS multiplets (annihilated by one fourth of the supercharges) characterized by the fact that neither a nor b vanishes.

For type-II string theory one obtains these central charges in terms of the left- and right-moving momenta, $p_{\rm L}$, $p_{\rm R}$, that characterize winding and momentum along S^1. However, the result takes a different form for the IIA and the IIB theory as the following formula shows,

$$Z^{ij} = \begin{cases} \tfrac12(p_{\rm L}+p_{\rm R})\delta^{ij} + \tfrac12(p_{\rm L}-p_{\rm R})\sigma_3^{ij}\,, & \text{(for IIB)} \\ \tfrac12(p_{\rm L}-p_{\rm R})\delta^{ij} + \tfrac12(p_{\rm L}+p_{\rm R})\sigma_3^{ij}\,. & \text{(for IIA)} \end{cases} \tag{3.52}$$

The corresponding BPS mass formula is thus equal to

$$M_{\rm BPS} = \tfrac12|p_{\rm L}+p_{\rm R}| + \tfrac12|p_{\rm L}-p_{\rm R}| \,. \tag{3.53}$$

For $p_{\rm L} = p_{\rm R}$ we confirm the original identification of the momentum states, namely that IIA momentum states constitute KKA supermultiplets, while IIB momentum states constitute KKB supermultiplets. For the winding states, where $p_{\rm L} = -p_{\rm R}$, one obtains the opposite result: IIA winding states constitute KKB supermultiplets, while IIB winding states constitute KKA supermultiplets. The 1/4-BPS multiplets arise for string states that have either right- or left-moving oscillator states, so that either $M_{\rm BPS} = |p_{\rm L}|$ or $|p_{\rm R}|$ with $p_{\rm L}^2 \neq p_{\rm R}^2$. All of this is entirely consistent with T-duality[16, 17], according to which there exists a IIA and a IIB perspective, with decompactification radii are that inversely proportional and with an interchange of winding and momentum states. Observe that the 1/4-BPS states will never become massless, so that they don't play a role in what follows.

It is also possible to view the central charges from the perspective of the 11-dimensional (super)membrane [57]. Assuming that the two-brane charge takes values in the compact coordinates labelled by 9 and 10, which can be generated by wrapping the membrane over the corresponding T^2, one readily finds the expression,

$$Z^{ij} = Z_{9\,10}\,\delta^{ij} - (P_9\,\sigma_3^{ij} - P_{10}\,\sigma_1^{ij})\,. \tag{3.54}$$

When compactifying on a torus with modular parameter τ and area A, the BPS mass formula takes the form

$$\begin{aligned} M_{\rm BPS} &= \sqrt{P_9^2 + P_{10}^2} + |Z_{9\,10}| \\ &= \frac{1}{\sqrt{A\,\tau_2}}|q_1 + \tau\,q_2| + T_{\rm m} A\,|p|\,. \end{aligned} \tag{3.55}$$

Here $q_{1,2}$ denote the momentum numbers on the torus and p is the number of times the membrane is wrapped over the torus; $T_{\rm m}$ denotes the supermembrane tension. Clearly the KKA states correspond to the momentum modes on T^2 while the KKB states are associated with the wrapped membranes on the torus. Therefore there is a rather natural way to describe the IIA and IIB momentum and winding states starting from a (super)membrane in eleven spacetime dimensions. This point was first emphasized in [58].

This suggests to consider $N = 2$ supergravity in 9 spacetime dimensions and couple it to the simplest BPS supermultiplets corresponding to KKA and KKB states. As shown in tables 8 and 10 there are three central charges and 9-dimensional supergravity possesses precisely three gauge fields that couple to these charges. From the perspective of 11-dimensional supergravity compactified on T^2, the Kaluza-Klein states transform as KKA multiplets. Their charges transform obviously with respect to an SO(2) associated with rotations of the coordinates labelled by 9 and 10. Hence we have a "double" tower of these charges with corresponding KKA supermultiplets. On the other hand, from the perspective of IIB compactified on S^1, the Kaluza-Klein states constitute KKB multiplets and their charge is SO(2) invariant. Here we have a "single" tower of KKB supermultiplets. However, from the perspective of 9-dimensional supergravity one is led to couple both towers of KKA and KKB supermultiplets simultaneously. In that case one obtains some dichotomic theory[19], which we refer to as BPS-extended supergravity. In the case at hand this new theory describes the ten-dimensional IIA and IIB theories in certain decompactification limits, as well as eleven-dimensional supergravity. But the theory is in some sense truly 12-dimensional with three compact coordinates, although there is no 12-dimensional Lorentz invariance, not even in a uniform decompactification limit, as the fields never depend on all the 12 coordinates! Whether this kind of BPS-extended supergravity offers a viable scheme in a more general context than the one we discuss here, is not known. In the case at hand we know a lot about these couplings from our knowledge of the T^2 compactification of $D = 11$ supergravity and the S^1 compactification of IIB supergravity.

The fields of 9-dimensional $N = 2$ supergravity are listed in table 13, where we also indicate their relation with the fields of 11-dimensional and 10-dimensional IIA/B supergravity upon dimensional reduction. It is not necessary to work out all the nonlinear field redefinitions here, as the corresponding fields can be uniquely identified by their scaling weights under SO(1,1), a symmetry of the massless theory that emerges upon dimensional reduction and is associated with scalings of the internal vielbeine. The scalar field σ is related to G_{99}, the IIB metric component in the compactified dimension, by $G_{99} = \exp(\sigma)$; likewise it is related to the determinant of the 11-dimensional metric in the compactified dimensions, which is equal to $\exp(-\frac{4}{3}\sigma)$. The precise relationship follows from comparing the SO(1,1) weights through the dimensional reduction of IIB and eleven-dimensional supergravity. In 9 dimensions supergravity has two more scalars, which are described by a nonlinear sigma model based on $\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2)$. The coset is described by the complex doublet of fields ϕ^α, which satisfy a constraint $\phi^\alpha\phi_\alpha = 1$ and are subject to a local SO(2) invariance, so that they describe precisely two scalar degrees of freedom ($\alpha = 1, 2$). We expect that the local SO(2) invariance can be incorporated in the full BPS-extended supergravity theory and can be exploited in the construction of the couplings of the various BPS supermultiplets to supergravity.

We already mentioned the three abelian vector gauge fields which couple to the central charges. There are two vector fields A^α_μ, which are the Kaluza-Klein photons from the T^2 reduction of eleven-dimensional supergravity and which couple therefore to the KKA states. From the IIA perspective these correspond to the Kaluza-Klein states on S^1 and the D0 states. From the IIB side they originate from the tensor fields, which confirms that they couple to the IIB (elementary and D1) winding states. These two fields transform under SL(2), which can be understood from the perspective of the modular transformation on T^2 as well as from the S-duality transformations that rotate the elementary strings with the D1 strings. The third gauge field, denoted by B_μ, is a singlet under SL(2) and is the Kaluza-Klein photon on the IIB side, so that it couples to the KKB states. On the IIA side it originates from the IIA tensor field, which is consistent with the fact that the IIA winding states constitute KKB supermultiplets.

From the perspective of the supermembrane, the KKA states are the momentum states on T^2, while the KKB states correspond to the membranes

$D=11$	IIA	$D=9$	IIB	SO(1,1)
$\hat{G}_{\mu\nu}$	$G_{\mu\nu}$	$g_{\mu\nu}$	$G_{\mu\nu}$	0
$\hat{A}_{\mu\,9\,10}$	$C_{\mu\,9}$	B_{μ}	$G_{\mu\,9}$	-4
$\hat{G}_{\mu\,9}$, $\hat{G}_{\mu\,10}$	$G_{\mu\,9}$, C_{μ}	A_{μ}^{α}	$A_{\mu\,9}^{\alpha}$	3
$\hat{A}_{\mu\nu\,9}$, $\hat{A}_{\mu\nu\,10}$	$C_{\mu\nu\,9}, C_{\mu\nu}$	$A_{\mu\nu}^{\alpha}$	$A_{\mu\nu}^{\alpha}$	-1
$\hat{A}_{\mu\nu\rho}$	$C_{\mu\nu\rho}$	$A_{\mu\nu\rho}$	$A_{\mu\nu\rho\sigma}$	2
$\hat{G}_{9\,10}$, $\hat{G}_{9\,9}$, $\hat{G}_{10\,10}$	ϕ, $G_{9\,9}$, C_9	$\left\{ \phi^{\alpha} \right.$	ϕ^{α}	0
		$\left\{ \exp(\sigma) \right.$	$G_{9\,9}$	7

Table 13: The bosonic fields of the eleven dimensional, type-IIA, nine-dimensional $N=2$ and type-IIB supergravity theories. The 11-dimensional and 10-dimensional indices, respectively, are split as $\hat{M} = (\mu, 9, 10)$ and $M = (\mu, 9)$, where $\mu = 0, 1, \ldots 8$. The last column lists the SO(1,1) scaling weights of the fields.

wrapped around the torus. While it is gratifying to see how all these correspondences work out, we stress that, from the perspective of 9-dimensional $N=2$ supergravity, the results follow entirely from supersymmetry.

The resulting BPS-extended theory incorporates 11-dimensional supergravity and the two type-II supergravities in special decompactification limits. But, as we stressed above, we are dealing with a 12-dimensional theory here, although no field can depend nontrivially on all of these coordinates. The theory has obviously two mass scales associated with the KKA and KKB states. We return to them in a moment. Both S- and T-duality are manifest, although the latter has become trivial as the theory is not based on a specific IIA or IIB perspective. One simply has the freedom to view the theory from a IIA or a IIB perspective and interpret it accordingly.

We should discuss the fate of the group $\mathrm{G} = \mathrm{SO}(1,1) \times \mathrm{SL}(2,\mathbf{R})$ of pure supergravity after coupling the theory to the BPS multiplets. The central charges of the BPS states form a discrete lattice, which is affected by this group. Hence, after coupling to the BPS states, we only have a discrete subgroup that leaves the charge lattice invariant. This is the group $\mathrm{SL}(2,\mathbf{Z})$.

The KKA and KKB states and their interactions with the massless theory can be understood from the perspective of compactified 11-dimensional and IIB supergravity. In this way we are able to deduce the following BPS

D	$\mathrm{H_R}$	A_μ	ϕ	χ
10	1	1	0	1
9	1	1	1	1
8	U(1)	1	$1+\bar{1}$	$1+\bar{1}$
7	USp(2)	1	3	2
6	USp(2)$\times$USp(2)	1	$(2,2)$	$(2,1)+(1,2)$
5	USp(4)	1	5	4
4	U(4)	1	6^*	$4+\bar{4}$
3	SO(8)		8	8

Table 14: Field content for maximal super-Maxwell theories in various dimensions. All supermultiplets contain a gauge field A_μ, scalars ϕ and spinors χ and comprises $8+8$ degrees of freedom. In $D=3$ dimensions the vector field is dual to a scalar. The 6^* representation of SU(4) is a selfdual rank-2 tensor.

mass formula,

$$M_{\mathrm{BPS}}(q_1,q_2,p) = m_{\mathrm{KKA}}\, \mathrm{e}^{3\sigma/7}\, |q_\alpha \phi^\alpha| + m_{\mathrm{KKB}}\, \mathrm{e}^{-4\sigma/7}\, |p|\,, \qquad (3.56)$$

where q_α and p refer to the integer-valued KKA and KKB charges, respectively, and m_{KKA} and m_{KKB} are two independent mass scales. This formula can be compared to the membrane BPS formula (3.55) in the 11-dimensional frame. One then finds that

$$m^2_{\mathrm{KKA}}\, m_{\mathrm{KKB}} \propto T_{\mathrm{m}}\,, \qquad (3.57)$$

with a numerical proportionality constant. However, the most important conclusion to draw from (3.56) is that there is no limit in which the masses of both KKA and KKB states will tend to zero. In other words, there is no uniform decompactification limit. Therefore, in spite of the fact that we have more than 11 dimensions, there exists no theory with $Q=32$ supercharges in flat Minkowski spacetime of dimensions $D>11$.

3.7 Nonmaximal supersymmetry: $Q=16$

For completeness we also summarize a number of results on nonmaximal supersymmetric theories with $Q=16$ supercharges, which are now restricted to dimensions $D\leq 10$. Table 14 shows the field representations for the vector multiplet in dimension $3\leq D\leq 10$. This multiplet comprises $8+8$ physical degrees of freedom. We also consider the $Q=16$ supergravity theories. The Lagrangian can be obtained by truncation of (3.38). However, unlike in the case of maximal supergravity, we now have the option of introducing additional matter fields. For $Q = 16$ the matter will be in the form of vector supermultiplets, possibly associated with some nonabelian gauge group. Table 15 summarizes $Q = 16$ supergravity for dimensions $3 \leq D \leq 10$. In $D = 10$ dimensions the bosonic terms of the supergravity Lagrangian take the form [59],

$$\begin{aligned}\mathcal{L}_{10} &= \frac{1}{\kappa_{10}^2}\Big[-\tfrac{1}{2}e\,\mathrm{e}^{2\phi/3}R(e,\omega) \\ &\qquad -\tfrac{3}{4}e\,\mathrm{e}^{-2\phi/3}(H_{\mu\nu\rho})^2 - \tfrac{1}{4}e(\partial_\mu A_\nu - \partial_\nu A_\mu)^2\Big]\,,\end{aligned} \tag{3.58}$$

where, for convenience, we have included a single vector gauge field A_μ, belonging to an abelian vector supermultiplet. A feature that deserves to be mentioned, is that the field strength $H_{\mu\nu\rho}$ associated with the 2-rank gauge field contains a Chern-Simons term $A_{[\mu}\partial_\nu A_{\rho]}$. Chern-Simons terms play an important role in the anomaly cancellations of this theory. Note also that the kinetic term for the Kaluza-Klein vector field in (3.38), depends on ϕ, unlike the kinetic term for the matter vector field in the Lagrangian above. This reflects itself in the extension of the symmetry transformations noted in (3.36, 3.37),

$$\begin{aligned}&e_\mu^a \to \mathrm{e}^{-\alpha}e_\mu^a\,, \qquad \phi \to \phi + 12\alpha\,, \\ &C_{11\mu\nu} \to \mathrm{e}^{6\alpha}C_{11\mu\nu}\,, \qquad A_\mu \to \mathrm{e}^{3\alpha}A_\mu\,.\end{aligned} \tag{3.59}$$

where A_μ transforms differently from the Kaluza-Klein vector field V_μ.

In this case there are three different Weyl rescalings that are relevant, namely

$$\begin{aligned}e_\mu^a &= \mathrm{e}^{-\phi/12}\,[e_\mu^a]^{\text{Einstein}}\,, \\ e_\mu^a &= \mathrm{e}^{-\phi/3}\,[e_\mu^a]^{\text{string}}\,, \\ e_\mu^a &= \mathrm{e}^{\phi/6}\,[e_\mu^a]^{\text{string}'}\,.\end{aligned} \tag{3.60}$$

It is straightforward to obtain the corresponding Lagrangians. In the Einstein frame, the graviton is again invariant under the isometries of the scalar field. The bosonic terms read

$$\begin{aligned}\mathcal{L}_{10}^{\text{Einstein}} &= \frac{1}{\kappa_{10}^2}\Big[-\tfrac{1}{2}e\,R(e,\omega) - \tfrac{1}{4}e(\partial_\mu\phi)^2 \\ &\qquad -\tfrac{3}{4}e\,\mathrm{e}^{-\phi}(H_{\mu\nu\rho})^2 - \tfrac{1}{4}e\,\mathrm{e}^{-\phi/2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2\Big]\,.\end{aligned} \tag{3.61}$$

D	$\mathrm{H_R}$	#	graviton	$p=-1$	$p=0$	$p=1$
10	1	64	1	1		1
9	1	56	1	1	1	1
8	U(1)	48	1	1	$1+\bar{1}$	1
7	USp(2)	40	1	1	3	1
6A	USp(2)×USp(2)	32	1	1	(2,2)	(1,1)
6B	USp(4)	24	1			5^*
5	USp(4)	24	1	1	5+1	
4	U(4)	16	1	$1+\bar{1}$	[6]	
3	SO(8)	$8k$	1	$8k$		

Table 15: Bosonic fields of nonmaximal supergravity with $Q = 16$. In 6 dimensions type-A and type-B correspond to (1,1) and (2,0) supergravity. In the third column, # denotes the number of bosonic degrees of freedom. Note that, with the exception of the 6B and the 4-dimensional theory, all these theories contain precisely one scalar field. The tensor field in the 6B theory is selfdual. In $D = 4$ dimensions, the SU(4) transformations cannot be implemented on the vector potentials, but act on the (abelian) field strengths by duality transformations. In $D = 3$ dimensions supergravity is a topological theory and can be coupled to scalars and spinors. The scalars parametrize the coset space $\mathrm{SO}(8,k)/(\mathrm{SO}(8)\times\mathrm{SO}(k))$, where k is an arbitrary integer.

The second Weyl rescaling leads to the following Lagrangian,

$$\begin{aligned}\mathcal{L}_{10}^{\text{string}} &= \frac{1}{\kappa_{10}^2}\mathrm{e}^{-2\phi}\Big[-\tfrac{1}{2}e\,R(e,\omega)+2e(\partial_\mu\phi)^2 \\ &\qquad -\tfrac{3}{4}e\,(H_{\mu\nu\rho})^2-\tfrac{1}{4}e(\partial_\mu A_\nu-\partial_\nu A_\mu)^2\Big]\,,\end{aligned} \tag{3.62}$$

which shows a uniform coupling to the dilaton. This is the low-energy effective Lagrangian relevant for the heterotic string. Eventually the matter gauge field has to be part of an nonabelian gauge theory based on the groups SO(32) or $\mathrm{E}_8\times\mathrm{E}_8$ in order to be anomaly-free.

Finally, the third Weyl rescaling yields

$$\begin{aligned}\mathcal{L}_{10}^{\text{string}'} &= \frac{1}{\kappa_{10}^2}\Big[e\,\mathrm{e}^{2\phi}\Big[-\tfrac{1}{2}R(e,\omega)+2(\partial_\mu\phi)^2\Big] \\ &\qquad -\tfrac{3}{4}e\,(H_{\mu\nu\rho})^2-\tfrac{1}{4}e\,\mathrm{e}^{\phi}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2\Big]\,.\end{aligned} \tag{3.63}$$

Now the dilaton seems to appear with the wrong sign. As it turns out, this is the low-energy effective action of the type-I string, where the type-I dilaton must be associated with $-\phi$. This is related to the fact that the SO(32) heterotic string theory is S-dual to type-I string theory [60].

4 Homogeneous spaces and nonlinear sigma models

This chapter offers an introduction to coset spaces and nonlinear sigma models based on such target spaces with their possible gaugings. The aim of this introduction is to facilitate the discussion in the next chapter, where we explain the gauging of maximal supergravity, concentrating on the maximal supergravities in $D=4,5$ spacetime dimensions. As discussed earlier, these theories have a nonlinearly realized symmetry group, equal to $\mathrm{E}_{7(7)}$ and $\mathrm{E}_{6(6)}$, respectively. It is possible to elevate the abelian gauge group associated with the vector gauge fields to a nonabelian group, which is a subgroup of these exceptional groups. The construction of these gaugings makes an essential use of the concepts and techniques discussed here.

We start by introducing the concept of a coset space G/H, where H is a subgroup of a group G. Most of this material is standard and can be found in textbooks, such as [61, 62]. Then we discuss the corresponding nonlinear sigma models, based on homogeneous target spaces, and present their description in a form that emphasizes a local gauge invariance associated with the group H. Finally we introduce the so-called *gauging* of this class of nonlinear sigma models. In this introduction we try to be as general as possible but in the examples we restrict ourselves to pseudo-orthogonal groups: SO(n) or the noncompact versions SO(p,q). The latter enable us to include some material on de Sitter and anti-de Sitter spacetimes, which we make use of in later chapters.

4.1 Nonlinearly realized symmetries

As an example consider the n-dimensional sphere S^n of unit radius which we may embed in an $(n+1)$-dimensional real vector space $\mathbf{R}^{n+1}$. The sphere is obviously invariant under SO($n+1$), the group of $(n+1)$-dimensional rotations. Such invariances are called isometries and $\mathrm{G} = \mathrm{SO}(n+1)$ is therefore known as the *isometry group*. A *homogeneous* space is a space where every two points can be connected by an isometry transformation. Clearly the sphere is such a homogeneous space as every two points on S^n can be related by an SO($n+1$) isometry. However, the rotation connecting these two points is not unique as every point on S^n is invariant under an SO(n) subgroup. This group is called the *isotropy group* (or stability subgroup), denoted by H. Obviously, for a homogeneous manifold, the isotropy groups for two arbitrary points are isomorphic (but not identical as one has to rotate between these points). It is convenient to choose a certain point on the sphere (let us call it the north pole) with coordinates in $\mathbf{R}^{n+1}$ given by $(0,\ldots,0,1)$. From the north pole, we can reach each point by a suitable rotation. However, the north pole itself is invariant under the SO(n) isotropy group, consisting of the following orthogonal matrices embedded into SO($n+$ 1),

$$h = \begin{pmatrix} * & \cdots & * & 0 \\ \vdots & * & \vdots & \vdots \\ * & \cdots & * & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}\,, \qquad h \in \mathrm{H}\,. \tag{4.1}$$

Therefore, if a rotation $g_1 \in \mathrm{G}$, with $\mathrm{G} = \mathrm{SO}(n+1)$ maps the north pole onto a certain point on the sphere, then the transformation $g_2 = g_1 \cdot h$ will do the same. Therefore points on the sphere can be associated with the class of group elements $g \in G$ that are equivalent up to multiplication by elements $h \in \mathrm{H}$ from the *right*. Such equivalence classes are called cosets, and therefore the space S^n is a coset space G/H with $\mathrm{G} = \mathrm{SO}(n+1)$ and $\mathrm{H} = \mathrm{SO}(n)$. The sphere is obviously just one particular example of a homogeneous space. Every such space can be described in terms of appropriate G/H cosets based on an isometry group G and a isotropy subgroup H.

A parametrization of the cosets of $\mathrm{SO}(n+1)/\mathrm{SO}(n)$, which assigns a single $SO(n+1)$ element to every coset, is thus equivalent to giving a parametrization of the sphere. It is not difficult to find such a parametrization. One first observes that every element of $\mathrm{SO}(n+1)$ can be decomposed as the product of

$$g(\alpha) = \exp\begin{pmatrix} 0 & \alpha^i \\ -\alpha_j & 0 \end{pmatrix}, \tag{4.2}$$

with some element of H. Here $i, j = 1, \ldots, n$. In view of various applications we extend our notation to noncompact versions of the orthogonal group, so that we can also deal with noncompact spaces. The noncompact groups leave an indefinite metric invariant. For $\mathrm{SO}(p,q)$ we have a diagonal metric with p eigenvalues $+1$ and q eigenvalues -1 (or vice versa as the overall sign is not relevant). Using the same decomposition as in (4.1), we choose this metric of the form diag $(\eta, 1)$, where η is again a diagonal metric with p (or q) eigenvalues equal to -1 and $q-1$ (or $p-1$) eigenvalues equal to $+1$. Elements of $\mathrm{SO}(p,q)$ thus satisfy

$$g^{-1} = \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix} g^{\mathrm{T}} \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.3}$$

so that the metric η is obviously H-invariant. The coset representative (4.2) satisfies the condition (4.3), provided that

$$\alpha_i = \eta_{ij}\, \alpha^j\,. \tag{4.4}$$

When η equals the unit matrix, we are dealing with a compact space. When η has negative eigenvalues the matrix (4.2) is no longer orthogonal and the space will be noncompact.[18] Let us mention some examples. For $\eta = -\mathbf{1}$ we have the hyperbolic space[19] $\mathrm{SO}(n,1)/\mathrm{SO}(n)$, for $\eta = (-,+,\cdots,+)$ we have the de Sitter space $\mathrm{SO}(n,1)/\mathrm{SO}(n-1,1)$, and for $\eta = (-,\cdots,-,+)$, we have the anti-de Sitter space $\mathrm{SO}(n-1,2)/\mathrm{SO}(n-1,1)$. In this way we can thus treat a variety of spaces at the same time. However, note that η,

[18]Noncompact groups are not fully covered by exponentiation, such as in the coset representatives (4.2), because there are disconnected components. We also note that, in the noncompact case, there are other decompositions than (4.1), which offer distinct advantages. We will not discuss these issues here.

[19]We are a little cavalier here with our terminology. Also the de Sitter and anti-de Sitter spaces are hyperbolic, as we shall see later (cf. (4.10)), but they are pseudo-Riemannian. We reserve the term hyperbolic for the Riemannian hyperbolic space. Unlike the de Sitter and anti-de Sitter spaces the former spaces are double-sheeded. In general, coset spaces where H is the maximal compact subgroup of a noncompact group G, have a positive or negative definite metric and are thus Riemannian spaces.

which eventually will play the role of the tangent-space metric, is 'mostly plus' for de Sitter, and 'mostly minus' for the hyperbolic and the anti-de Sitter space. This aspect will be important later on when comparing the curvature for these spaces.

The elements $g(\alpha) \in \mathrm{G}$ define a *representative* of the G/H coset space, and therefore a parametrization of the corresponding space. Of course, the coset representative is not unique. We have decomposed the generators of G into generators h of H and generators k belonging to its complement; the latter have been used to generate the coset representative. In our example, the latter are associated with the last row and column of (4.2), so that they satisfy (schematically),

$$
\begin{aligned}
[\mathsf{h},\mathsf{h}] &= \mathsf{h}\,,\\
[\mathsf{h},\mathsf{k}] &= \mathsf{k}\,,\\
[\mathsf{k},\mathsf{k}] &= \mathsf{h}\,.
\end{aligned}
\tag{4.5}
$$

The first commutation relation states that the h form a subalgebra. The second one implies that the generators k form a representation of H, which ensures that in an infinitesimal neighbourhood of a point invariant under H, the coordinates α^i rotate under H according to that representation. In the more general case the third commutator may also yield the generators k. When they do not, the homogeneous space is *symmetric*. Obviously the above relations involve a choice of basis; the generators k are defined up to additive terms belonging to elements of the algebra associated with H.

Let us now proceed and evaluate (4.2),

$$
g(\alpha) = \begin{pmatrix} \delta^i_j + \alpha^i\alpha_j \dfrac{\cos\alpha - 1}{\alpha^2} & \dfrac{\sin\alpha}{\alpha}\alpha^i \\ & \\ -\dfrac{\sin\alpha}{\alpha}\alpha_j & \cos\alpha \end{pmatrix}\,, \tag{4.6}
$$

where $\alpha^2 = \eta_{ij}\,\alpha^i\alpha^j$. Obviously the space is compact when η is positive, because in that case the parameter space can be restricted to $0 \leq \alpha < \pi$. This corresponds to the sphere S^n. In all other cases the parameter space is obviously noncompact and the sine and cosine may change to the hyperbolic sine and cosine in those parts of the space where α^2 is negative. Observe that the appearance of η in the above formulae is the result of the fact that the generators k are normalized according to $\mathrm{tr}(\mathsf{k}_i\,\mathsf{k}_j) = -2\,\eta_{ij}$.

One may choose a different parametrization of the cosets by making a different decomposition than in (4.2). Different parametrizations are generally related through (coordinate-dependent) H transformations acting from the right. We may also chose different coordinates , such as, for instance,[20]

$$y^i = \alpha^i \frac{\sin\alpha}{\alpha} \;, \tag{4.7}$$

so that the coset representative reads

$$g(y) = \begin{pmatrix} \delta^i_j + y^i y_j \dfrac{\pm\sqrt{1-y^2}-1}{y^2} & y^i \\ & \\ -y_j & \pm\sqrt{1-y^2} \end{pmatrix} . \tag{4.8}$$

where, depending on the sign choice, we parametrize different parts of the space (for the sphere S^n, the upper or the lower hemisphere). For the sphere the range of the coordinates is restricted by $y^2 = \Sigma_i (y^i)^2 \leq 1$. Note that the $n \times n$ submatrix in (4.8) equals the square root of the matrix $\delta^i_j - y^i y_j$.

One may use the coset representative to sweep out the coset space from one point (*i.e.* the 'north pole') in the $(n+1)$-dimensional embedding space. Acting with (4.8) on the point $(0,\ldots,0,1)$ yields the following coordinates in the embedding space,

$$Y^A = (y^i, \pm\sqrt{1-y^2}) \;. \tag{4.9}$$

Using (4.3) one then shows that the coset space is embedded in $n+1$ dimensions according to

$$\eta_{ij}\, Y^i Y^j + (Y^{n+1})^2 = 1 \,. \tag{4.10}$$

Since the $g(y)$ are contained in G one may examine the effect of G transformations acting on $g(y)$, which will induce corresponding transformations in the coset space. To see this, we multiply $g(y)$ by a constant element $o_{\rm G} \in {\rm G}$ from the left. After this multiplication the result is in general no longer compatible with the coset representative $g(y)$, but by applying a suitable y-dependent H transformation, $o_{\rm H}(y)$, from the right, we can again bring $g(y)$ in the desired form. In other words, one has

$$g(y) \longrightarrow o_{\rm G}\, g(y) = g(y')\, o_{\rm H}(y) \,. \tag{4.11}$$

Hence the effect is a change of coordinates $y \to y'$ in a way that satisfies the group multiplication laws. The infinitesimal transformation $y^i \to {y'}^i =$

[20]The coordinates $(y^i, \pm\sqrt{1-y^2})$ are sometimes called *homogeneous* coordinates, because the G-transformations act linearly on these coordinates. Inhomogeneous coordinates are the ratios $y^i/\sqrt{1-y^2}$.

$y^i + \xi^i(y)$ defines the so-called Killing vectors $\xi^i(y)$. Writing $o_{\rm G} \approx \mathbf{1} + \hat{\mathsf{g}}$ and $o_{\rm h} \approx \mathbf{1} + \hat{\mathsf{h}}(y)$, we find the relation,

$$\xi^i(y)\,\partial_i g(y) = \hat{\mathsf{g}}\,g(y) - g(y)\,\hat{\mathsf{h}}(y)\,. \tag{4.12}$$

We return to this result in the next subsection.

Applying this construction to the case at hand, one finds that there are two types of isometries. One corresponding to the group H, which changes the coordinates y^i by constant rotations. The other corresponds to n coordinate dependent shifts,

$$\delta y^i = \epsilon^i\,\sqrt{1-y^2}\,, \tag{4.13}$$

where the ϵ^i are n constant parameters. As the reader can easily verify, both types of transformations take the form of a constant G transformation on the embedding coordinates Y^A which leaves the embedding condition (4.10) invariant.

Coset representatives can be defined in different representations of the group G. The most interesting one is the spinor representation. Assume that we have a representation of the Clifford algebra $\mathcal{C}(p,q)$. The representation transforms (not necessarily irreducibly) under $\mathrm{SO}(p,q)$ generated by the matrices $\frac{1}{2}\Gamma_{ij}$, but in fact it transforms also as a spinor under $\mathrm{SO}(p,q+1)$, as one can verify by including extra generators equal to the matrices $\frac{1}{2}\Gamma_i$. Consequently we can define a representative of $\mathrm{SO}(p,q+1)/\mathrm{SO}(p,q)$ in the spinor representation,

$$g(\alpha) = \exp[\tfrac{1}{2}\Gamma_i\,\alpha^i] = \cos(\alpha/2)\,\mathbf{1} + i\,\frac{\sin(\alpha/2)}{\alpha}\,\alpha^i\Gamma_i\,, \tag{4.14}$$

with α defined as before, $\alpha^2 = \sqrt{\alpha_i\alpha^i}$, and $\{\Gamma_i,\Gamma_j\} = -2\eta_{ij}\,\mathbf{1}$. This construction can applied as well to cosets of other (pseudo-)orthogonal groups. In terms of the coordinates y^i the representative reads

$$g(y) = \tfrac{1}{2}\Big[\sqrt{1+y} + \sqrt{1-y}\,\Big]\,\mathbf{1} + \frac{y^i\Gamma_i}{\sqrt{1+y}+\sqrt{1-y}}\,. \tag{4.15}$$

One can act with this representative on a constant spinor, specified at the north pole, *i.e.* we define $\psi(y) = g^{-1}(y)\,\psi(0)$. The resulting y-dependent spinor $\psi(y)$ is a so-called Killing spinor of the coset space. We shall exhibit this below. Obviously, similar results can be obtained in other representations of G.

4.2 Geometrical quantities

Geometrical quantities of the homogeneous space are defined from the left-invariant one-forms $g^{-1}\mathrm{d}g$, where $g(y) \in \mathrm{G}$, so that the one-forms take their value in the Lie algebra associated with G. It is convenient to use the language of differential forms,but by no means essential. The exterior derivative $\mathrm{d}g(y)$ describes the change of g induced by an infinitesimal variation of the coset-space coordinates y^i. The one-forms $g^{-1}\mathrm{d}g$ are called left-invariant, because they are invariant under left multiplication of g with constant elements of G. The significance of this fact will be clear in a sequel. Because the g's themselves are elements of G, the one-form $g^{-1}\mathrm{d}g$ takes its values in the Lie algebra associated with G. Therefore the one-forms can be decomposed into the generators h and k, introduced earlier, *i.e.*,

$$g^{-1}\mathrm{d}g = \omega + e\,, \tag{4.16}$$

where ω is decomposable into the generators h and e into the generators k. Hence e defines a square matrix, with indices i that label the coordinates and indices a that label the generators k. These one-forms e are thus related to the vielbeine of the coset space, which define a tangent frame at each point of the space. The one-forms ω define the spin connection[21], associated with tangent-space rotations that belong to the group H. Eq. (4.16) is of central importance for the geometry of the coset spaces. As a first consequence we note that the spinor $\psi(y)$, defined with the help of the representative (4.15) at the end of the previous subsection, satisfies the equation,

$$\Big(\mathrm{d} + \omega + e\Big)\psi(y) = 0\,. \tag{4.17}$$

Upon writing this out in terms of the gamma matrices, one recovers precisely the so-called Killing spinor equation (*c.f.* (3.15)).

Let us now proceed and investigate the properties of the one-forms ω and e. In general it is not necessary to specify the coset representative, as different representatives are related by y-dependent H transformation acting from the right on g, *i.e.*,

$$g(y) \longrightarrow g(y)\,h(y)\,, \qquad h(y) \in \mathrm{H}\,. \tag{4.18}$$

This leads to a different parametrization of the coset space. It is straightforward to see how ω and e transform under (4.18),

$$(\omega + e) \longrightarrow h^{-1}(\omega + e)h + h^{-1}\mathrm{d}h\,. \tag{4.19}$$

[21]Observe that in supergravity we have defined the spin connection field with opposite sign.

This equation can again be decomposed (using the first two relations (4.5) in terms of the generators h and k, which yields

$$\begin{aligned} \omega_i &\longrightarrow h^{-1}\omega_i h + h^{-1}\partial_i h\,, \\ e_i &\longrightarrow h^{-1} e_i h\,. \end{aligned} \tag{4.20}$$

Obviously, ω acts as a gauge connection for the local H transformations. Furthermore it follows from (4.27) that ω_i and e_i transform as covariant vectors under coordinate transformations, i.e.

$$\begin{aligned} y^i &\longrightarrow y^i + \xi^i\,, \\ \omega_i &\longrightarrow \omega_i - \partial_i\xi^j\omega_j - \xi^j\partial_j\omega_i\,, \\ e_i &\longrightarrow e_i - \partial_i\xi^j e_j - \xi^j\partial_j e_i\,. \end{aligned} \tag{4.21}$$

We can also define Lie-algebra valued curvatures associated with ω_i and e_i,

$$\begin{aligned} R_{ij}(\mathrm{H}) &= \partial_i\omega_j - \partial_j\omega_i + [\omega_i,\omega_j]\,, \\ R_{ij}(\mathrm{G/H}) &= \partial_i e_j - \partial_j e_i + [\omega_i, e_j] - [\omega_j, e_i]\,. \end{aligned} \tag{4.22}$$

Introducing H-covariant derivatives, we note the relations

$$\begin{aligned} [D_i, D_j] &= -R_{ij}(\mathrm{H})\,, \\ R_{ij}(\mathrm{G/H}) &= D_i e_j - D_j e_i\,. \end{aligned} \tag{4.23}$$

The values of these curvatures follow from the Cartan-Maurer equations. To derive these equations we take the exterior derivative of the defining relation (4.16),

$$\mathrm{d}(g^{-1}\mathrm{d}g) = -(g^{-1}\mathrm{d}g)\wedge(g^{-1}\mathrm{d}g)\,, \tag{4.24}$$

or, in terms of ω and e,

$$\mathrm{d}(\omega + e) = -(\omega + e)\wedge(\omega + e)\,. \tag{4.25}$$

Decomposing this equation in terms of the Lie algebra generators, using the relations (4.5), we find

$$R_{ij}(\mathrm{H}) = -[e_i, e_j]\,, \qquad R_{ij}(\mathrm{G/H}) = 0\,. \tag{4.26}$$

Note that the vanishing of $R_{ij}(G/H)$ is a consequence of the fact that we assumed that the coset space was *symmetric* (see the text below (4.5)).

As we already alluded to earlier, the fields e_i can be decomposed into the generators k and thus define a set of vielbeine e_i^a that specify a tangent frame at each point in the coset space. In the context of differential geometry the indices i are called world indices, because they refer to the coordinates of a manifold, whereas the indices $a, b, \ldots$ that label the generators k are called tangent-space indices (or local Lorentz indices in the context of general relativity). Because the generators k form a representation of H, this group rotates the tangent frames. Usually the group H can be embedded into SO(n) (or a noncompact version thereof) and leaves some target-space metric invariant (we will see the importance of this fact shortly). The quantity ω_i thus acts as the connection associated with rotations of the tangent frames, and therefore we call it the spin connection.

These aspects are easily recognized in the examples we are discussing, because the group H was precisely the (pseudo)orthogonal group. Hence, using the same matrix decomposition as before, we find explicit expressions for the vielbein and the spin connection,

$$g^{-1}\mathrm{d}g = \begin{pmatrix} \omega_i(y)\,\mathrm{d}y^i & e_i(y)\,\mathrm{d}y^i \\ & \\ -e_i(y)\,\mathrm{d}y^i & 0 \end{pmatrix} . \tag{4.27}$$

From (4.8) one readily obtains

$$\begin{aligned} \omega_i^{ab} &= \Big(y^a\,\delta_i^b - y^b\,\delta_i^a\Big)\frac{1\mp\sqrt{1-y^2}}{y^2}\,, \\ e_i^{\,a} &= \delta_i^a + \frac{y_i\,y^a}{y^2}\Big(\pm\frac{1}{\sqrt{1-y^2}}-1\Big)\,, \end{aligned} \tag{4.28}$$

where, as before, indices are raised and lowered with η. Note that ω_i^{ab} is antisymmetric in a, b, which follows from the (pseudo)orthogonality of H. The inverse vielbein reads

$$e_a^{\,i} = \delta_a^i + \frac{y_a\,y^i}{y^2}\Big(\pm\sqrt{1-y^2}-1\Big)\,, \tag{4.29}$$

Furthermore, the curvatures introduced before, are readily identified with the curvature of the spin connection and with the torsion tensor. From the Cartan-Maurer equations, explained above, we thus find in components,

$$R_{ij}^{ab}(\omega) = 2\,e_{[i}^{\,a}\,e_{j]}^{\,b}\,, \qquad D_i\,e_j^{\,a} - D_j e_i^{\,a} = 0\,. \tag{4.30}$$

To define a metric g_{ij} one contracts an H-invariant symmetric rank-2 tensor with the vielbeine. The obvious invariant tensor is η_{ab}, so that

$$g_{ij} = \eta_{ab}\,e_i^{\,a} e_j^{\,b}\,. \tag{4.31}$$

When there are several H-invariant tensors, there is a more extended class of metrics that one may consider, but in the case at hand the metric is unique up to a proportionality factor. In the parametrization (4.28) one obtains for the metric and its inverse,

$$g_{ij} = \eta_{ij} + \frac{y_i\, y_j}{1-y^2}\,, \qquad g^{ij} = \eta^{ij} - y^i\, y^j\,. \tag{4.32}$$

Given the fact that we have already made a choice for η previously, a 'mostly plus' metric requires to include a minus sign in the definition (4.31) for the hyperbolic and anti-de Sitter spaces. This sign is important when comparing to spheres or de Sitter spaces.

From the vielbein postulate, we know that the affine connection is equal to $\Gamma_{ij}{}^k = e_a{}^k\, D_i e_j{}^a$, from which we can define the Riemann curvature. For the examples at hand, this leads to

$$\Gamma_{ij}{}^k = y^k g_{ij}. \tag{4.33}$$

Because the torsion is zero, the connection coincides with Christoffel symbol. Because ω_i differs in sign as compared to the spin connection used in section 3.1, the Riemann tensor is equal to minus the curvature $R_{ij}^{ab}(\omega)$, upon contraction with $\eta_{ac}\, e_k^c\, e_b^l$, and we find the following result for the Riemann curvature tensor,

$$R_{ijk}{}^l = -g_{ki}\,\delta_j^l + g_{kj}\,\delta_i^l\,, \tag{4.34}$$

where g_{ij} is the metric tensor defined by (4.31). Thus the curvature is of definite sign, but we stress that this is related to the signature choice that we made for the metric, as we have discussed above. The Riemann curvature (4.34) is proportional to the metric, which indicates that we are dealing with a maximally symmetric space. This means that the maximal number of isometries (equal to $\frac{1}{2}n(n+1)$) is realized for this space.

All coset spaces have isometries corresponding to the group G. The diffeomorphisms associated with these isometries are generated by Killing vectors $\xi^i(y)$, which we introduced earlier. Combining (4.12) with (4.16), we obtain,

$$\xi^i(y)\,(\omega_i(y) + e_i(y)) = g^{-1}(y)\hat{\mathsf{g}}\, g(y) - \hat{\mathsf{h}}(y)\,. \tag{4.35}$$

Decomposing this equation according to the Lie algebra, we find

$$\begin{aligned} \xi^i(y)\, e_i(y) &= \tilde{\mathsf{g}}(y)\,, \\ \hat{\mathsf{h}}(y) &= -\xi^i(y)\,\omega_i(y) + \tilde{\mathsf{h}}(y)\,, \end{aligned} \tag{4.36}$$

where

$$\begin{aligned}\tilde{\mathsf{h}}(y) &= \Big[g^{-1}(y)\,\hat{\mathsf{g}}\,g(y)\Big]_{\mathrm{H}}\,,\\ \tilde{\mathsf{g}}(y) &= \Big[g^{-1}(y)\,\hat{\mathsf{g}}\,g(y)\Big]_{\mathrm{G/H}}\,.\end{aligned}\tag{4.37}$$

The contribution $\hat{\mathsf{h}}(y)$ is only relevant for those quantities that live in the tangent space.

Now we return to the observation that the left-invariant forms, from which e and ω were constructed, are invariant under the group G. Therefore, it follows that e and ω are both invariant as well. Moreover, we established (*c.f.* (4.11)) that the G-transformation acting on the left can be decomposed into a diffeomorphism combined with a coordinate-dependent H-transformation. Therefore, the vielbein e and the spin connection ω are invariant under these combined transformations. Since the metric is H-invariant by construction, it thus follows that the metric is invariant under the diffeomorphism associated with G. Hence,

$$\delta g_{ij} = D_i\xi_j(y) + D_j\xi_i(y) = 0\,,\tag{4.38}$$

where $\xi_i = g_{ij}\xi^j$ and ξ^i is the so-called Killing vector defined by (4.36). For the vielbein and spin connection, which transform under H, we find

$$\begin{aligned}\partial_i\xi^j\,e_j + \xi^j\,\partial_j e_i + [e_i, \hat{\mathsf{h}}(y)] &= 0\,,\\ \partial_i\xi^j\,\omega_j + \xi^j\,\partial_j\omega_i + \partial_i\hat{\mathsf{h}}(y) + [\omega_i, \hat{\mathsf{h}}(y)] &= 0\,.\end{aligned}\tag{4.39}$$

In terms of $\tilde{\mathsf{h}}(y)$ these results take a more covariant form,

$$\begin{aligned}\partial_i\xi^j\,e_j + \xi^j\,D_j e_i + [e_i, \tilde{\mathsf{h}}(y)] &= 0\,,\\ D_i\tilde{\mathsf{h}}(y) &= R_{ij}(H)\,\xi^j\,.\end{aligned}\tag{4.40}$$

Combining the first equation with the first equation(4.36) yields

$$R_{ij}(G/H)\,\xi^j + [e_i, \tilde{\mathsf{g}}(y)]_{G/H} = 0\,.\tag{4.41}$$

Observe that both terms vanish separately for a symmetric space.

The diffeomorphisms generated by the Killing vector fields will give rise to the group G. This follows from (4.12). Let us label the generators of the group G by indices $\alpha, \beta, \ldots$, and introduce structure constants by

$$[\hat{\mathsf{g}}_\alpha, \hat{\mathsf{g}}_\beta] = f_{\alpha\beta}{}^{\gamma}\,\hat{\mathsf{g}}_\gamma\,.\tag{4.42}$$

The Killing vectors and the corresponding H-transformations then satisfy corresponding group multiplication properties,

$$\begin{aligned} \xi_\beta^j\,\partial_j\xi_\alpha^i - \xi_\alpha^j\,\partial_j\xi_\beta^i &= f_{\alpha\beta}{}^\gamma\,\xi_\gamma^i\,, \\ [\tilde{\mathsf{h}}_\alpha, \tilde{\mathsf{h}}_\beta] &= f_{\alpha\beta}{}^\gamma\,\tilde{\mathsf{h}}_\gamma + \xi_\alpha^i\,\xi_\beta^j\,R_{ij}(H)\,. \end{aligned} \tag{4.43}$$

One can consider fields on the coset space, which are functions of the coset space coordinates assigned to a representation of the group H. On such fields the isometries are generated by the operators,

$$-\xi_\alpha^i D_i + \tilde{\mathsf{h}}_\alpha\,. \tag{4.44}$$

On the basis of the results above one can show that these operators satisfy the commutation relations of the Lie algebra associated with the isometry group G. To show this we note the identity

$$\begin{aligned} (-\xi_\alpha^i D_i + \tilde{\mathsf{h}}_\alpha)(-\xi_\beta^i D_i + \tilde{\mathsf{h}}_\beta) &= \xi_\alpha^i\xi_\beta^j(D_iD_j - R_{ij}(H)) + \tilde{\mathsf{h}}_\alpha\,\tilde{\mathsf{h}}_\beta \\ &\quad + [\xi_\alpha^i(D_i\xi_\beta^j) + \tilde{\mathsf{h}}_\alpha\,\xi_\beta^j + \tilde{\mathsf{h}}_\beta\,\xi_\alpha^j]D_j\,. \end{aligned} \tag{4.45}$$

4.3 Nonlinear sigma models with homogeneous target space

It is now rather straightforward to describe a nonlinear sigma model based on a homogeneous target space by making use of the above framework. One starts from scalar fields which take their values in the homogeneous space, so that the fields $\phi^i(x)$ define a map from the spacetime to the coset space. Hence we may follow the same procedure as before and define a coset representative $\mathcal{V}(\phi^i(x)) \in \mathrm{G}$, which now depends on n fields. Subsequently, one uses the analogue of (4.16), to define Lie-algebra valued quantities $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$,

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} = \mathcal{Q}_\mu + \mathcal{P}_\mu\,, \tag{4.46}$$

where $\mathcal{Q}_\mu$ is decomposable into the generators h and $\mathcal{P}_\mu$ into the generators k. Obviously one has the relations

$$\mathcal{Q}_\mu(\phi) = \omega_i(\phi)\,\partial_\mu\phi^i\,, \qquad \mathcal{P}_\mu(\phi) = e_i(\phi)\,\partial_\mu\phi^i\,. \tag{4.47}$$

The above expressions show that $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ are just the pull backs of the target space connection and vielbein to the spacetime.

The local H transformations depend on the fields $\phi(x)$ and thus indirectly on the spacetime coordinates. Therefore one may elevate these transformations to transformations that depend arbitrarily on x^μ. Under such transformations we have

$$\mathcal{V}(\phi) \to \mathcal{V}(\phi)\,h(x)\,. \tag{4.48}$$

By allowing ourselves to perform such local gauge transformations, we introduced new degrees of freedom into $\mathcal{V}$ associated with the group H. Eventually we will fix this gauge freedom, but until that point $\mathcal{V}$ will just be an unrestricted spacetime dependent element of the group G. After imposing the gauge condition on $\mathcal{V}(x)$ one obtains the coset representative $\mathcal{V}(\phi(x))$. From (4.48) we derive the following local H-transformations,

$$\begin{aligned} \mathcal{Q}_\mu(x) &\longrightarrow h^{-1}(x)\,\mathcal{Q}_\mu(x)\,h(x) + h^{-1}(x)\,\partial_\mu h(x)\,, \\ \mathcal{P}_\mu(x) &\longrightarrow h^{-1}(x)\,\mathcal{P}_\mu(x)\,h(x)\,. \end{aligned} \tag{4.49}$$

Hence $\mathcal{Q}_\mu$ acts as a gauge field associated with the local H transformations. Furthermore both $\mathcal{P}_\mu$ and $\mathcal{Q}_\mu$ are invariant under rigid G-transformations. It is convenient to introduce a corresponding H-covariant derivative,

$$D_\mu\mathcal{V} = \partial_\mu\mathcal{V} - \mathcal{V}\mathcal{Q}_\mu\,, \tag{4.50}$$

so that (4.47) reads

$$\mathcal{V}^{-1}D_\mu\mathcal{V} = \mathcal{P}_\mu\,. \tag{4.51}$$

Just as before, one derives the Cartan-Maurer equations (4.26),

$$\begin{aligned} F_{\mu\nu}(\mathcal{Q}) &= \partial_\mu\mathcal{Q}_\nu - \partial_\nu\mathcal{Q}_\mu + [\mathcal{Q}_\mu,\mathcal{Q}_\nu] = -[\mathcal{P}_\mu,\mathcal{P}_\nu]\,, \\ D_{[\mu}\mathcal{P}_{\nu]} &= \partial_{[\mu}\mathcal{P}_{\nu]} + [\mathcal{Q}_{[\mu},\mathcal{P}_{\nu]}] = 0\,. \end{aligned} \tag{4.52}$$

Here we made use of the commutation relations (4.5)

There are several ways to write down the Lagrangian of the corresponding nonlinear sigma model. Obviously the Lagrangian must be invariant under both the rigid G transformations and the local H transformations. Hence we write

$$\mathcal{L} = \tfrac{1}{2}\mathrm{tr}\left[D_\mu\mathcal{V}^{-1}\,D^\mu\mathcal{V}\right]. \tag{4.53}$$

One can interpret this result in a first- and in a second-order form. In the first one regards the gauge field $\mathcal{Q}_\mu$ as an independent field, whose field equations are algebraic and are solved by (4.51). After substituting the result one obtains the second-order form, which presupposes (4.51) from the beginning. The result can be written as,

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{tr}\left[\mathcal{P}_\mu\,\mathcal{P}^\mu\right]. \tag{4.54}$$

Clearly this Lagrangian is invariant under the group G. At this stage one still has the full gauge invariance with respect to local H transformations and one can impose a gauge restricting $\mathcal{V}$ to a coset representative. When this is not done, the theory is invariant under $\mathrm{G}_{\mathrm{rigid}} \times \mathrm{H}_{\mathrm{local}}$ with both groups acting *linearly*. However, as soon as one imposes a gauge and restricts $\mathcal{V}$ to a coset representative parametrized by certain fields ϕ^i, the residual subgroup is such that the H transformations are linked to the G transformations and depend on the fields ϕ^i. This combined subgroup still generates a representation of the group G, but it is now realized in a nonlinear fashion. In order to deal with complicated supergravity theories that involve homogeneous spaces, the strategy is to postpone this gauge choice till the end, so that one is always dealing with a manifest linearly realized symmetry group $\mathrm{G}_{\mathrm{rigid}} \times \mathrm{H}_{\mathrm{local}}$. As we intend to demonstrate, this strategy allows for a systematic approach, whereas the gauge-fixed approach leads to unsurmountable difficulties (at least, for the spaces of interest). It is straightforward to demonstrate that (4.54) leads to the standard form of the nonlinear sigma model,

$$\mathcal{L} = -\tfrac{1}{2} g_{ij}(\phi)\, \partial_\mu \phi^i\, \partial^\mu \phi^j \,, \tag{4.55}$$

where the target space metric is given by (4.31). In this form the local H invariance is absent, but the invariance under G is still there and realized as target space isometries generated by corresponding Killing vectors.

It is easy to see how to couple matter fields to the sigma model in a way that the invariance under the isometries remains unaffected. Matter fields are assigned to a representation of the local H group, so that they couple to the sigma model fields through the connection $\mathcal{Q}_\mu$ that appears in the covariant derivatives. Usually the fields will remain invariant under the group G as long as one does not fix the gauge and choose a specific coset representative. Also here we can proceed in first- or second-order formalism. In first-order form the equation (4.47) will acquire some extra terms that depend on the matter fields. Upon choosing a gauge, the matter fields transform nonlinearly under the group G with transformations that take the form of ϕ-dependent H-transformations, determined by (4.36). However, gauge fields cannot couple in this way as their gauge invariance would be in conflict with the local invariance under the group H. Therefore, gauge fields have to transform under the rigid group G.

We emphasize that the presentation that we followed so far was rather general; the maximally symmetric spaces that we considered served only as an example. In supergravity we are often dealing with sigma models based on homogeneous, symmetric target spaces. These target spaces are usually noncompact and Riemannian, so that H is the maximally compact subgroup of G. Later in this chapter we will be dealing with the $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$

and $E_{6(6)}/USp(8)$ coset spaces. The exceptional groups are noncompact and are divided by their maximal compact subgroups. The corresponding spaces have dimension 70 and 42, respectively.

4.4 Gauged nonlinear sigma models

Given a nonlinear sigma model with certain isometries, one can gauge some or all of these isometries in the usual way: one elevates the parameters of the isometry group (of a subgroup thereof) to arbitrary functions of the spacetime coordinates and introduces the necessary gauge fields (with their standard gauge-invariant Lagrangian containing a kinetic term) and corresponding covariant derivatives. As explained above, for sigma models based on homogeneous target spaces one can proceed in a way in which all transformations remain linearly realized. To adopt this approach is extremely important for the construction of gauged supergravity theories, as we will discuss in the next section. We will always use the second-order formalism so that the H-connection $\mathcal{Q}_\mu$ will not be an independent field.

Since these new gauge transformations involve the isometry group they must act on the group element $\mathcal{V}$ as a subgroup of G. Hence the covariant derivative of $\mathcal{V}$ is now changed by the addition of the corresponding (dynamical) gauge fields A_μ which take their values in the corresponding Lie algebra (which is a subalgebra of the Lie algebra associated with G). Hence,

$$D_\mu \mathcal{V}(x) = \partial_\mu \mathcal{V}(x) - \mathcal{V}(x)\, \mathcal{Q}_\mu(x) - g\, A_\mu(x)\, \mathcal{V}(x)\,, \tag{4.56}$$

where we have introduced a coupling constant g to keep track of the new terms introduced by the gauging. With this change, the expressions for $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ will change. They remain expressed by (4.51), but the derivative is now covariantized and modified by the terms depending on the new gauge fields A_μ. The consistency of this procedure is obvious as (4.51) is fully covariant. Of course, the original rigid invariance under G transformations from the left is now broken by the embedding of the new gauge group into G.

The modifications caused by the new minimal couplings are minor and the effects can be concisely summarized by the Cartan-Maurer equations,

$$\begin{aligned}
\mathcal{F}_{\mu\nu}(\mathcal{Q}) &= \partial_\mu \mathcal{Q}_\nu - \partial_\nu \mathcal{Q}_\mu + [\mathcal{Q}_\mu, \mathcal{Q}_\nu]\,, \\
&= [\mathcal{P}_\mu, \mathcal{P}_\nu] - g\Big[\mathcal{V}^{-1} F_{\mu\nu}(A)\mathcal{V}\Big]_{\mathrm{H}}\,, \\
D_{[\mu}\mathcal{P}_{\nu]} &= -\tfrac{1}{2} g\Big[\mathcal{V}^{-1} F_{\mu\nu}(A)\mathcal{V}\Big]_{\mathrm{G/H}}\,.
\end{aligned} \tag{4.57}$$

Because $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ now depend on the gauge connections A_μ, according to

$$\mathcal{Q}_\mu = \mathcal{Q}_\mu^{(0)} - g[\mathcal{V}^{-1}A_\mu\mathcal{V}]_{\rm H}\,, \qquad \mathcal{P}_\mu = \mathcal{P}_\mu^{(0)} - g[\mathcal{V}^{-1}A_\mu\mathcal{V}]_{\rm G/H}\,. \tag{4.58}$$

When imposing a gauge condition, the last result for $\mathcal{P}_\mu$ exhibits precisely the Killing vectors (4.36) (in the gauge where $\mathcal{V}$ equals the coset representative). When gauging isometries in a generic nonlinear sigma model (*c.f.* (4.55)), one replaces the derivatives according to $\partial_\mu\phi \to \partial_\mu\phi^i - A_\mu\xi^i(\phi)$, where for simplicity we assumed a single isometry. The modifications in the matter sector arise through the order g contributions to $\mathcal{Q}_\mu$. Note that $\mathcal{P}_\mu$ and $\mathcal{Q}_\mu$ are invariant under the new gauge group (but transform under local H-transformations, as before). In the next subsection we will discuss the application of this formulation to gauged supergravity.

5 Gauged maximal supergravity in 4 and 5 dimensions

The maximally extended supergravity theories introduced in chapter 3 were obtained by dimensional reduction from 11-dimensional supergravity on a hypertorus. In these theories the scalar fields parametrize a G/H coset space (*c.f.* table 12) and the group G is also realized as a symmetry of the full theory. Generically the fields transform as follows. The graviton is invariant, the (abelian) gauge fields transform linearly under G and the fermions transform linearly under the group H. However, in some dimensions the G-invariance is not realized at the level of the action, but at the level of the combined field equations and Bianchi identities. For example, in 4 dimensions the 28 abelian vector fields do not constitute a representation of the group $\mathrm{E}_{7(7)}$. In this case the group G is realized by electric-magnetic duality and acts on the field strengths, rather than on the vector fields. We return to electric-magnetic duality in section 5.3.

It is an obvious question whether these theories allow an extension in which the abelian gauge fields are promoted to nonabelian ones. This turns out to be possible and the corresponding theories are known as gauged supergravities. They contain an extra parameter g, which is the gauge coupling constant. Supersymmetry requires the presence of extra terms of order g and g^2 in the Lagrangian. Apart from the gauge field interactions there are fermionic masslike terms of order g and a scalar potential of order g^2. The latter may give rise to groundstates with nonzero cosmological constant. To explain the construction of gauged supergravity theories, we concentrate on the maximal gauged supergravities in $D = 4$ and 5 spacetime

	$e_\mu^{\ a}$	ψ_μ^i	$\mathcal{F}_{\mu\nu};\mathcal{G}_{\mu\nu};\mathcal{H}_{\mu\nu}$	χ^{ijk}	$u_{ij}{}^{IJ};v^{ijIJ}$
SU(8)	**1**	**8**	**1**	**56**	$\mathbf{28}+\overline{\mathbf{28}}$
$\mathrm{E}_{7(7)}$	**1**	**1**	**56**	**1**	**56**
USp(8)	**1**	**8**	**1**	**48**	$\mathbf{27}+\overline{\mathbf{27}}$
$\mathrm{E}_{6(6)}$	**1**	**1**	**27**	**1**	$\mathbf{27}+\overline{\mathbf{27}}$

Table 16: Representation assignments for the various supergravity fields with respect to the groups G and H. In $D=4$ dimensions these groups are $\mathrm{E}_{7(7)}$ and SU(8), respectively. In $D=5$ dimensions they are $\mathrm{E}_{6(6)}$ and USp(8). Note that the tensors $\mathcal{F}_{\mu\nu}$, $\mathcal{G}_{\mu\nu}$ and/or $\mathcal{H}_{\mu\nu}$ denote the field strengths of the vector fields and/or (for $D=5$) possible tensor fields.

dimensions. An obvious gauging in $D=4$ dimensions is based on the group SO(8), as the Lagrangian has a manifest SO(8) invariance and there are precisely 28 vector fields [63]. This gauging has an obvious Kaluza-Klein origin, and arises when compactifying seven coordinates of $D=11$ supergravity on the sphere S^7, which has an SO(8) isometry group. The group emerges as the gauge group of the compactified theory formulated in 4 dimensions. In this theory the $\mathrm{E}_{7(7)}$ invariance group is broken to a local SO(8) group so that the resulting theory is invariant under $\mathrm{SU}(8)_{\mathrm{local}}\times\mathrm{SO}(8)_{\mathrm{local}}$. In this compactification the four-index field strength acquires a nonzero values when all its indices are in the four-dimensional spacetime. However, it turns out that many other subgroups of $\mathrm{E}_{7(7)}$ can be gauged.

In $D=5$ dimensions the possible gaugings are not immediately clear, as there is no obvious 27-dimensional gauge group. Again the Kaluza-Klein scenario can serve as a guide. While $D=11$ supergravity has no obvious compactification to five dimensions, type-IIB supergravity has a compactification on the sphere S^5. In this solution the five-index (self-dual) field strength acquires a nonzero value whenever the five indices take all values in either S^5 or in the five-dimensional spacetime. Type-IIB supergravity has a manifest SL(2) invariance and the isometry group of S^5 is SO(6), so that the symmetry group of the Lagrangian equals the $\mathrm{SL}(2)\times\mathrm{SO}(6)$ subgroup of $\mathrm{E}_{6(6)}$, where SO(6) is realized as a local gauge group. This implies that 15 of the 27 gauge fields become associated with the nonabelian group SO(6), which leaves 12 abelian gauge fields which are charged with respect to the same group. This poses an obvious problem, as the abelian gauge transformations of these 12 fields will be in conflict with their transformations under the SO(6) gauge group. The solutions is to convert these 12 gauge fields to antisymmetric tensor fields. The Lagrangian can thus be written in a form that is invariant under $\mathrm{USp}(8)_{\mathrm{local}} \times \mathrm{SO}(6)_{\mathrm{local}} \times \mathrm{SL}(2)$ [29]. Also in 5 dimensions other gauge groups are possible. We will briefly comment on this issue at the end of the chapter.

Before continuing with supergravity we first discuss some basic features of the two coset spaces $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ and $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$. Both these exceptional Lie groups can be introduced in terms of 56-dimensional matrices.[22]

5.1 On $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ and $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ cosets

We discuss the $\mathrm{E}_{7(7)}$ and $\mathrm{E}_{6(6)}$ on a par for reasons that will become obvious. To define the groups we consider the fundamental representation, acting on a pseudoreal vector (z_{IJ}, z^{KL}) with $z^{IJ} = (z_{IJ})^*$, where the indices are antisymmetrized index pairs $[IJ]$ and $[KL]$ and $I, J, K, L = 1, \ldots, 8$. Hence the (z_{IJ}, z^{KL}) span a 56-dimensional vector space. Consider now infinitesimal transformations of the form,

$$\begin{aligned} \delta z_{IJ} &= \Lambda_{IJ}{}^{KL}\, z_{KL} + \Sigma_{IJKL}\, z^{KL}\,, \\ \delta z^{IJ} &= \Lambda^{IJ}{}_{KL}\, z^{KL} + \Sigma^{IJKL}\, z_{KL}\,. \end{aligned} \tag{5.1}$$

where $\Lambda_{IJ}{}^{KL}$ and Σ_{IJKL} are subject to the conditions

$$(\Lambda_{IJ}{}^{KL})^* = \Lambda^{IJ}{}_{KL} = -\Lambda_{KL}{}^{IJ}\,, \qquad (\Sigma_{IJKL})^* = \Sigma^{KLIJ}\,. \tag{5.2}$$

The corresponding group elements constitute the group $\mathrm{Sp}(56;\mathbf{R})$ in a pseudoreal basis. This group is the group of electric-magnetic dualities of maximal supergravity in $D = 4$ dimensions. The matrices $\Lambda_{IJ}{}^{KL}$ are associated with its maximal compact subgroup, which is equal to U(28). The defining properties of elements E of $\mathrm{Sp}(56;\mathbf{R})$ are

$$E^* = \omega\, E\, \omega\,, \qquad E^{-1} = \Omega\, E^\dagger\, \Omega\,, \tag{5.3}$$

where ω and Ω are given by

$$\omega = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\,, \qquad \Omega = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}\,. \tag{5.4}$$

The above properties ensure that the sequilinear form,

$$(z_1, z_2) = z_1^{IJ}\, z_{2IJ} - z_{1IJ}\, z_2^{IJ}\,, \tag{5.5}$$

[22]Strictly speaking the isotropy groups are $\mathrm{SU}(8)/\mathbf{Z}_2$ and $\mathrm{USp}(8)/\mathbf{Z}_2$.

is invariant. In passing we note that the real subgroup (in this pseudoreal representation) is equal to the group GL(28).

Let us now consider the $E_{7(7)}$ subgroup, for which the Σ^{IJKL} is fully antisymmetric and the generators are further restricted according to

$$\begin{aligned} &\Lambda_{IJ}{}^{KL} = \delta_{[I}^{[K}\,\Lambda_{J]}{}^{L]}\,, \qquad \Lambda_I{}^J = -\Lambda^J{}_I\,, \\ &\Lambda_I{}^I = 0\,, \qquad \Sigma_{IJKL} = \tfrac{1}{24}\varepsilon_{IJKLMNPQ}\,\Sigma^{MNPQ}\,. \end{aligned} \tag{5.6}$$

Obviously the matrices $\Lambda_I{}^J$ generate the group SU(8), which has dimension 63; since Σ_{IJKL} comprise 70 real parameters, the dimension of $E_{7(7)}$ equals $63{+}70 = 133$. Because SU(8) is the maximal compact subgroup, the number of the noncompact generators minus the number of compact ones is equal to $70-63=7$. It is straightforward to show that these matrices close under commutation and generate the group $E_{7(7)}$. To show this one needs a variety of identities for selfdual tensors [64]; one of them is that the contraction $\Sigma_{IKLM}\,\Sigma^{JKLM}$ is traceless.

However, $E_{7(7)}$ has another maximal 63-dimensional subgroup, which is not compact. This is the group SL(8). It is possible to choose conventions in which the $E_{7(7)}$ matrices have a different block decomposition than (5.1) and where the diagonal blocks correspond to the group SL(8), rather than to SU(8). We note that the subgroup generated by (5.6) with $\Lambda_I{}^J$ and Σ^{IJKL} real, defines the group SL(8; $\mathbf{R}$).

The group $E_{7(7)}$ has a quartic invariant,

$$\begin{aligned} J_4(z) = {}& z_{IJ}z^{JK}z_{KL}z^{LI} - \tfrac{1}{4}(z_{IJ}z^{IJ})^2 \\ &+\tfrac{1}{96}\Big[\varepsilon_{IJKLMNPQ}\,z^{IJ}z^{KL}z^{MN}z^{PQ} + \varepsilon^{IJKLMNPQ}\,z_{IJ}z_{KL}z_{MN}z_{PQ}\Big]\,, \end{aligned} \tag{5.7}$$

which, however, plays no role in the following. For further information the reader is encouraged to read the appendices of [53].

Another subgroup is the group $E_{6(6)}$, for which the restrictions are rather similar. Here one introduces a skew-symmetric tensor Ω_{IJ}, satisfying

$$\Omega_{IJ} = -\Omega_{JI}\,, \quad (\Omega_{IJ})^* = \Omega^{IJ}\,, \quad \Omega_{IK}\Omega^{KJ} = -\delta_I^J\,. \tag{5.8}$$

Now we restrict ourselves to the subgroup of U(8) that leaves Ω_{IJ} invariant. This is the group USp(8). The other restrictions on the generators concern Σ_{IJKL}. Altogether we have the conditions,

$$\begin{aligned} &\Lambda_{IJ}{}^{KL} = \delta_{[I}^{[K}\,\Lambda_{J]}{}^{L]}\,, \qquad \Lambda_I{}^J = -\Lambda^J{}_I\,, \\ &\Lambda_{[I}{}^K\Omega_{J]K} = 0\,, \qquad \Omega_{IJ}\Sigma^{IJKL} = 0\,, \\ &\Sigma_{IJKL} = \Omega_{IM}\,\Omega_{JN}\,\Omega_{KP}\,\Omega_{LQ}\,\Sigma^{MNPQ}\,. \end{aligned} \tag{5.9}$$

The maximal compact subgroup USp(8) thus has dimension $64 - 28 = 36$, while there are $70 - 28 = 42$ generators associated with Σ_{IJKL}. Altogether we thus have $36 + 42 = 78$ generators, while the difference between the numbers of noncompact and compact generators equals $42 - 36 = 6$. These numbers confirm that we are indeed dealing with $\mathrm{E}_{6(6)}$ and its maximal compact subgroup USp(8). Because of the constraints (5.9) the 56-dimensional representation defined by (5.1) is reducible and decomposes into two singlets and a **27** and a $\overline{\mathbf{27}}$ representation. To see this we observe that the following restrictions are preserved by the group,

$$\Omega_{IJ} z^{IJ} = 0\,, \qquad z_{IJ} = \pm \Omega_{IK} \Omega_{JL} z^{KL}\,. \tag{5.10}$$

The first one suppresses the singlet representation and the second one projects out the **27** or the $\overline{\mathbf{27}}$ representation.

The group $\mathrm{E}_{6(6)}$ has a cubic invariant, defined by

$$J_3(z) = z^{IJ} z^{KL} z^{MN}\, \Omega_{JK} \Omega_{LM} \Omega_{NI}\,, \tag{5.11}$$

which plays a role in the $\mathrm{E}_{6(6)}$ invariant Chern-Simons term in the supergravity Lagrangian.

There is another maximal subgroup of $\mathrm{E}_{6(6)}$, which is noncompact, that will be relevant in the following. This is the group $\mathrm{SL}(6) \times \mathrm{SL}(2)$, which has dimension $35 + 3 = 38$, and which plays a role in many of the known gaugings, where the gauge group is embedded into the group SL(6), so that SL(2) remains as a rigid invariance group of the Lagrangian.

5.2 On ungauged maximal supergravity Lagrangians

An important feature of pure extended supergravity theories is that the spinless fields take their values in a homogeneous target space (*c.f.* table 12, where we have listed these spaces). Because the spinless fields always appear in nonpolynomial form, it is vital to exploit the coset structure explained in the previous section in the construction of the supersymmetric action and transformation rules, as well as in the gauging. We will not be complete here but sketch a number of features of the maximal supergravity theories in $D = 4, 5$ where the coset structure plays an important role. We will be rather cavalier about numerical factors, spinor conventions, etcetera. In this way we will, hopefully, be able to bring out the main features of the G/H structure, without getting entangled in issues that depend on the spacetime dimension. For those and other details we refer to the original literature [63, 29].

One starts by introducing a so-called 56-bein $\mathcal{V}(x)$, which is a 56×56 matrix that belongs to the group $\mathrm{E}_{7(7)}$ or $\mathrm{E}_{6(6)}$, depending on whether we are in $D = 4$, or 5 dimensions. A coset representative is obtained by exponentiation of the generators defined in (5.1). Schematically,

$$\mathcal{V}(x) = \exp\begin{pmatrix} 0 & \overline{\Sigma}(x) \\ \Sigma(x) & 0 \end{pmatrix}, \tag{5.12}$$

where the rank-4 antisymmetric tensor Σ satisfies the algebraic restrictions appropriate for the exceptional group. As explained in the previous section, the 56-bein is reducible for $\mathrm{E}_{6(6)}$, but we will use the reducible version in order to discuss the two theories on a par. Our notation will be based on a description in terms of right cosets, just as in the previous sections, which may differ from the notations used in the original references where one sometimes uses left cosets. Hence, we assume that the 56-bein transforms under the exceptional group from the left and under the local SU(8) (or USp(8)) from the right. The 56-bein can be decomposed in block form according to

$$\mathcal{V}(x) = \begin{pmatrix} u^{ij}{}_{IJ}(x) & -v_{klIJ}(x) \\ -v^{ijKL}(x) & u_{kl}{}^{KL}(x) \end{pmatrix}, \tag{5.13}$$

with the usual conventions $u^{ij}{}_{IJ} = (u_{ij}{}^{IJ})^*$ and $v_{ijIJ} = (v^{ijIJ})^*$. Observe that the indices of the matrix are antisymmetrized index pairs $[IJ]$ and $[ij]$. In the above the row indices are $([IJ],[KL])$, and the column indices are $([ij],[kl])$. The latter are the indices associated with the local SU(8) or USp(8). The notation of the submatrices is chosen such as to make contact with [63], where left cosets were chosen, upon interchanging $\mathcal{V}$ and $\mathcal{V}^{-1}$. Observe also that (5.12) is a coset representative, *i.e.* we have fixed the gauge with respect to local SU(8) or USp(8), whereas in (5.13) gauge fixing is not assumed. According to (5.3) the inverse $\mathcal{V}^{-1}$ can be expressed in terms of the complex conjugates of the submatrices of $\mathcal{V}$,

$$\mathcal{V}^{-1}(x) = \begin{pmatrix} u_{ij}{}^{IJ}(x) & v_{ijKL}(x) \\ v^{klIJ}(x) & u^{kl}{}_{KL}(x) \end{pmatrix}. \tag{5.14}$$

Consequently we derive the identities, for $\mathrm{E}_{7(7)}$,

$$\begin{aligned} u^{ij}{}_{IJ}\, u_{kl}{}^{IJ} - v^{ijIJ}\, v_{klIJ} &= \delta^{ij}_{kl}\,, \\ u^{ij}{}_{IJ}\, v^{klIJ} - v^{ijIJ}\, u^{kl}{}_{IJ} &= 0\,, \end{aligned} \tag{5.15}$$

or, conversely,

$$
\begin{aligned}
u^{ij}{}_{IJ}\, u_{ij}{}^{KL} - v_{ijIJ}\, v^{ijKL} &= \delta^{IJ}_{KL}\,, \\
u^{ij}{}_{IJ} v_{ijKL} - v_{ijIJ} u^{ij}{}_{KL} &= 0\,.
\end{aligned} \tag{5.16}
$$

The corresponding equations for $\mathrm{E}_{6(6)}$ are identical, except that the antisymmetrized Kronecker symbols on the right-hand sides are replaced according to

$$
\delta^{ij}_{kl} \rightarrow \delta^{ij}_{kl} - \tfrac{1}{8}\Omega_{kl}\,\Omega^{ij}\,, \qquad \delta^{IJ}_{KL} \rightarrow \delta^{IJ}_{KL} - \tfrac{1}{8}\Omega_{KL}\,\Omega^{IJ}\,. \tag{5.17}
$$

Furthermore the matrices u and v vanish when contracted with the invariant tensor Ω and they are pseudoreal, *e.g.*,

$$
u_{ij}{}^{IJ}\Omega_{IJ} = 0\,, \qquad u_{ij}{}^{KL}\Omega_{IK}\Omega_{JL} = \Omega_{ik}\Omega_{jl}u^{kl}{}_{IJ}\,, \tag{5.18}
$$

with similar identities for the v^{ijIJ}. In this case the (pseudoreal) matrices $u^{ij}{}_{IJ} \pm \Omega_{IK}\,\Omega_{JL}\,v^{ijKL}$ and their complex conjugates define (irreducible) elements of $\mathrm{E}_{6(6)}$ corresponding to the $\mathbf{27}$ and $\overline{\mathbf{27}}$ representations. We note the identity

$$
\Big(u^{ij}{}_{IJ} + \Omega_{IK}\,\Omega_{JL}\,v^{ijKL}\Big)\Big(u_{kl}{}^{IJ} - \Omega^{IM}\,\Omega^{JN}\,v_{klMN}\Big) = \delta^{ij}_{kl} - \tfrac{1}{8}\Omega_{kl}\,\Omega^{ij}\,. \tag{5.19}
$$

In this case we can thus decompose the 56-bein in terms of a 27-bein and a $\overline{27}$-bein.

Subsequently we evaluate the quantities $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$,

$$
\mathcal{V}^{-1}\partial_\mu\mathcal{V} = \begin{pmatrix} \mathcal{Q}_{\mu\, ij}{}^{mn} & \mathcal{P}_{\mu\, ijpq} \\ \\ \mathcal{P}_\mu^{klmn} & \mathcal{Q}_\mu{}^{kl}{}_{pq} \end{pmatrix}\,, \tag{5.20}
$$

which leads to the expressions,

$$
\begin{aligned}
\mathcal{Q}_{\mu\, ij}{}^{kl} &= u_{ij}{}^{IJ}\,\partial_\mu u^{kl}{}_{IJ} - v_{ijIJ}\,\partial_\mu v^{klIJ}\,, \\
\mathcal{P}_\mu^{ijkl} &= v^{ijIJ}\,\partial_\mu u^{kl}{}_{IJ} - u^{ij}{}_{IJ}\,\partial_\mu v^{klIJ}\,.
\end{aligned} \tag{5.21}
$$

The important observation is that $\mathcal{Q}_{\mu ij}{}^{kl}$ and $\mathcal{P}_\mu^{ijkl}$ are subject to the same constraints as the generators of the exceptional group listed in the previous section. Hence, $\mathcal{P}_\mu^{ijkl}$ is fully antisymmetric and subject to a reality constraint. Therefore it transforms according to the 70-dimensional representation of SU(8), with the reality condition,

$$
\mathcal{P}_\mu^{ijkl} = \tfrac{1}{24}\,\varepsilon^{ijklmnpq}\,\mathcal{P}_{\mu\, mnpq}\,, \tag{5.22}
$$

or, to the 42-dimensional representation of USp(8), with the reality condition,

$$\mathcal{P}_{\mu}^{ijkl} = \Omega^{im}\Omega^{jn}\Omega^{kp}\Omega^{lq}\,\mathcal{P}_{\mu\,mnpq}\,, \tag{5.23}$$

Likewise $\mathcal{Q}_\mu$ transforms as a connection associated with SU(8) or USp(8), respectively. Hence $\mathcal{Q}_{\mu\,ij}{}^{kl}$ must satisfy the decomposition,

$$\mathcal{Q}_{\mu\,ij}{}^{kl} = \delta^{[k}_{[i}\,\mathcal{Q}_{\mu\,j]}{}^{l]}\,, \tag{5.24}$$

so that $\mathcal{Q}_{\mu\,i}{}^{j}$ equals

$$\mathcal{Q}_{\mu\,i}{}^{j} = \tfrac{2}{3}\Big[u_{ik}{}^{IJ}\,\partial_\mu u^{jk}{}_{IJ} - v_{ikIJ}\,\partial_\mu v^{jkIJ}\Big]\,. \tag{5.25}$$

Because of the underlying Lie algebra the connections $\mathcal{Q}_{\mu\,i}{}^{j}$ satisfy $\mathcal{Q}_{\mu}{}^{i}_{j} = -\mathcal{Q}_{\mu j}{}^{i}$ and $\mathcal{Q}_{\mu i}{}^{i} = 0$, as well as an extra symmetry condition in the case of USp(8) (cf. (5.9)).

Furthermore we can evaluate the Maurer-Cartan equations (4.52),

$$\begin{aligned} F_{\mu\nu}(\mathcal{Q})_i{}^j &= \partial_\mu\mathcal{Q}_{\nu\,i}{}^j - \partial_\nu\mathcal{Q}_{\mu\,i}{}^j + \mathcal{Q}_{[\mu\,i}{}^k\,\mathcal{Q}_{\nu]k}{}^j = -\tfrac{4}{3}\,\mathcal{P}_{[\mu}{}^{jklm}\,\mathcal{P}_{\nu]iklm}\,,\\ D_{[\mu}\mathcal{P}_{\nu]}^{ijkl} &= \partial_{[\mu}\mathcal{P}_{\nu]}^{ijkl} + 2\mathcal{Q}_{[\mu\,m}{}^{[i}\,\mathcal{P}_{\nu]}^{jkl]m} = 0\,. \end{aligned} \tag{5.26}$$

Observe that the use of the Lie algebra decomposition for G/H is crucial in deriving these equations. Such decompositions are an important tool for dealing with the spinless fields in this nonlinear setting. Before fixing a gauge, we can avoid the nonlinearities completely and carry out the calculations in a transparent way. Fixing the gauge prematurely and converting to a specific coset representative for G/H would lead to unsurmountable difficulties.

Continuing along similar lines we turn to a number of other features that are of interest for the Lagrangian and transformation rules. The first one is the observation that *any* variation of the 56-bein can be written, up to a local H-transformation, as

$$\delta\mathcal{V} = \mathcal{V}\begin{pmatrix} 0 & \overline{\Sigma} \\ \Sigma & 0 \end{pmatrix}\,, \tag{5.27}$$

or, in terms of submatrices,

$$\delta u_{ij}{}^{IJ} = -\Sigma_{ijkl}\,v^{klIJ}\,, \qquad \delta v_{ijIJ} = -\Sigma_{ijkl}\,u^{kl}{}_{IJ}\,. \tag{5.28}$$

where Σ^{ijkl} is the rank-four antisymmetric tensor corresponding to the generators associated with G/H (*i.e.*, the generators denoted by k in the previous chapter). Because Σ takes the form of an H-covariant tensor, the variation (5.28) is consistent with both groups G and H. Under this variation the quantities $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ transform systematically,

$$\begin{aligned} \delta Q_{\mu\,i}{}^{j} &= \tfrac{2}{3}\Big(\Sigma^{jklm}\,\mathcal{P}_{\mu\,iklm} - \Sigma_{iklm}\,\mathcal{P}_\mu^{jklm}\Big)\,, \\ \delta\mathcal{P}_\mu^{ijkl} &= D_\mu\Sigma^{ijkl} = \partial_\mu\Sigma^{ijkl} + 2\mathcal{Q}_{\mu\,m}{}^{[i}\Sigma^{jkl]m}\,. \end{aligned} \tag{5.29}$$

Observe that this establishes that $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ can be assigned to the adjoint representation of the group G, as is already obvious from the decomposition (5.20).

As was stressed above, any variation of $\mathcal{V}$ can be decomposed into (5.27), up to a local H-transformation. In particular this applies to supersymmetry transformations. The supersymmetry variation can be written in the form (5.27), where Σ is an H-covariant expression proportional to the supersymmetry parameter ϵ^i and the fermion fields χ^{ijk}. Hence it must be of the form $\Sigma^{ijkl}\propto\bar\epsilon^{[i}\chi^{jkl]}$, up to complex conjugation and possible contractions with H-covariant tensors, Furthermore Σ must satisfy the restrictions associated with the exceptional group, *i.e.* (5.6) or (5.9).

The supersymmetry variation of the spinor χ^{ijk} contains the quantity $\mathcal{P}_\mu^{ijkl}$, which incorporates the spacetime derivatives of the spinless fields, so that up to proportionality constants we must have a variation,

$$\delta\chi^{ijk}\propto\mathcal{P}_\mu^{ijkl}\,\gamma^\mu\epsilon_l\,. \tag{5.30}$$

The verification of the supersymmetry algebra on $\mathcal{V}$ is rather easy. Under two consecutive (field-dependent) variations (5.28) applied in different orders on the 56-bein, we have

$$[\delta_1,\delta_2]\,\mathcal{V} = \mathcal{V}\begin{pmatrix} 0 & 2\,\delta_{[1}\overline{\Sigma}_{2]} \\ 2\,\delta_{[1}\Sigma_{2]} & 0\end{pmatrix} + \mathcal{V}\left[\begin{pmatrix} 0 & \overline{\Sigma}_1 \\ \Sigma_1 & 0\end{pmatrix},\begin{pmatrix} 0 & \overline{\Sigma}_2 \\ \Sigma_2 & 0\end{pmatrix}\right]\,. \tag{5.31}$$

The last term is just an infinitesimal H-transformations. For the first term we note that $\delta_1\Sigma_2$ leads to a term proportional to $\mathcal{P}_\mu^{ijkl}$, combined with two supersymmetry parameters, ϵ_1 and ϵ_2, of the form $(\bar\epsilon_1^{[i}\gamma^\mu\epsilon_{2\,m})\,\mathcal{P}_\mu^{jkl]m}$. Taking into account the various H-covariant combinations in the actual expressions implied by (5.6) or (5.9), respectively, this contribution can be written in the form

$$[\delta_1,\delta_2]\,\mathcal{V}\propto(\bar\epsilon_1^i\gamma^\mu\epsilon_{2\,i} - \bar\epsilon_2^i\gamma^\mu\epsilon_{1\,i})\;\mathcal{V}\begin{pmatrix} 0 & \overline{\mathcal{P}}_\mu \\ \mathcal{P}_\mu & 0\end{pmatrix}\,. \tag{5.32}$$

This is precisely a spacetime diffeomorphism, up to a local H-transformation proportional to $\mathcal{Q}_\mu$, as follows from (5.20). Hence up to a number of field-dependent H-transformations, the supersymmetry commutator closes on $\mathcal{V}$

into a spacetime diffeomorphism (up to terms of higher-order in the spinors that we suppressed).

Let us now turn to the action. Apart from higher-order spinor terms, the terms in the Lagrangian pertaining to the graviton, gravitini, spinors and scalars take the following form,

$$\begin{aligned} e^{-1}\mathcal{L}_1 &= -\tfrac{1}{2}R(e,\omega) - \tfrac{1}{2}\bar{\psi}^i_\mu\gamma^{\mu\nu\rho}\Big[(\partial_\nu - \tfrac{1}{2}\omega_\nu^{ab}\,\gamma_{ab})\delta_i^j + \tfrac{1}{2}\mathcal{Q}_{\nu\,i}{}^j\Big]\psi_{\rho j} \\ &\quad -\tfrac{1}{12}\bar{\chi}^{ijk}\gamma^\mu\Big[(\partial_\mu - \tfrac{1}{2}\omega_\mu^{ab}\,\gamma_{ab})\delta_k^l + \tfrac{3}{2}\mathcal{Q}_{\mu\,k}{}^l\Big]\chi_{ijl} - \tfrac{1}{12}\mathcal{P}_\mu^{ijkl}\,\mathcal{P}^\mu_{ijkl} \\ &\quad -\tfrac{1}{6}\sqrt{2}\bar{\chi}_{ijk}\gamma^\nu\gamma^\mu\psi_{\nu l}\,\mathcal{P}_\mu^{ijkl}\,. \end{aligned} \tag{5.33}$$

This Lagrangian is manifestly invariant with respect to $\mathrm{E}_{7(7)}$ or $\mathrm{E}_{6(6)}$. Here we distinguish the Einstein-Hilbert term for gravity, the Rarita-Schwinger Lagrangian for the gravitini, the Dirac Lagrangian and the nonlinear sigma model associated with the G/H target space. The last term represents the Noether coupling term for the spin-0/spin-$\frac{1}{2}$ system. For $D=4$ the fermion fields are chiral spinors and we have to add the contributions from the spinors of opposite chirality; for $D=5$ we are dealing with so-called symplectic Majorana spinors. Here we disregard such details and concentrate on the symmetry issues.

The vector fields bring in new features, which are different for spacetime dimensions $D=4$ and 5. In $D=5$ dimensions the vector fields B_μ^{IJ} transform as the $\mathbf{27}$ representation of $\mathrm{E}_{6(6)}$, so that they satisfy the reality constraint $B_{\mu\,IJ} = \Omega_{IK}\Omega_{JL}\,B_\mu^{KL}$, and the Lagrangian is manifestly invariant under the corresponding transformations. It is impossible to construct an invariant action just for the vector fields and one has to make use of the scalars, which can be written in terms of the $\overline{27}$-beine, $u^{ij}{}_{IJ} + v^{ijKL}\Omega_{IK}\Omega_{JL}$, and which can be used to convert $\mathrm{E}_{6(6)}$ to USp(8) indices. Hence we define a USp(8) covariant field strength for the vector fields, equal to

$$\mathsf{F}^{ij}_{\mu\nu} = (u^{ij}{}_{IJ} - v^{ijKL}\Omega_{IK}\Omega_{JL})(\partial_\mu B_\nu^{IJ} - \partial_\nu B_\mu^{IJ})\,. \tag{5.34}$$

The invariant Lagrangian of the vector fields then reads,

$$\begin{aligned} \mathcal{L}_2 &= -\tfrac{1}{16}e\,\mathsf{F}^{ij}_{\mu\nu}\,\mathsf{F}^{\mu\nu kl}\,\Omega_{ik}\Omega_{jl} \\ &\quad -\tfrac{1}{12}\varepsilon^{\mu\nu\rho\sigma\lambda}\,B_\mu^{IJ}\,\partial_\nu B_\rho^{KL}\,\partial_\sigma B_\lambda^{MN}\,\Omega_{JK}\Omega_{LM}\Omega_{NI} \\ &\quad +\tfrac{1}{4}e\,\mathsf{F}^{ij}_{\mu\nu}\,\mathcal{O}^{\mu\nu}_{ij}\,, \end{aligned} \tag{5.35}$$

where we distinguish the kinetic term, a Chern-Simons interaction associated with the $\mathrm{E}_{6(6)}$ cubic invariant (5.11) and a moment coupling with the fermions. Here $\mathcal{O}_{ij}^{\mu\nu}$ denotes a covariant tensor antisymmetric in both spacetime and USp(8) indices and quadratic in the fermion fields, ψ_μ^i and χ^{ijk}. Observe that the dependence on the spinless fields is completely implicit. Any additional dependence would affect the invariance under $\mathrm{E}_{6(6)}$. The result obtained by combining the Lagrangians (5.33) and (5.35) gives the full supergravity Lagrangian invariant under rigid $\mathrm{E}_{6(6)}$ and local USp(8) transformations, up to terms quartic in the fermion fields. We continue the discussion of the $D=4$ theory in the next section, as this requires to first introduce the concept of electric-magnetic duality.

5.3 Electric-magnetic duality and $\mathrm{E}_{7(7)}$

For $D=4$ the Lagrangian is not invariant under $\mathrm{E}_{7(7)}$ but under a smaller group, which acts on the vector fields (but not necessarily on the 56-bein) according to a 28-dimensional subgroup of GL(28). However, the combined equations of motion and the Bianchi identities are invariant under the group $\mathrm{E}_{7(7)}$. This situation is typical for $D=4$ theories with abelian vector fields, where the symmetry group of field equations and Bianchi identities can be bigger than that of the Lagrangian, and where different Lagrangians not related by local field redefinitions, can lead to an equivalent set of field equations and Bianchi identities. However, the phenomenon is not restricted to 4 dimensions and can occur for antisymmetric tensor gauge fields in any even number of spacetime dimensions (see, *e.g.*, [52]). The 4-dimensional version has been known for a long time and is commonly referred to as electric-magnetic duality (for a recent review of this duality in supergravity, see, *e.g.*, [51]). Its simplest form arises in Maxwell theory in four-dimensional (flat or curved) Minkowski space, where one can perform (Hodge) duality rotations, which commute with the Lorentz group and rotate the electric and magnetic fields and inductions according to $\mathbf{E}\leftrightarrow\mathbf{H}$ and $\mathbf{B}\leftrightarrow\mathbf{D}$.

This duality can be generalized to any $D=4$ dimensional field theory with abelian vector fields and no charged fields, so that the gauge fields enter the Lagrangian only through their (abelian) field strengths. These field strengths (in the case at hand we have 28 of them, labelled by antisymmetric index pairs $[IJ]$, but for the moment we will remain more general and label the field strengths by $\alpha,\beta,\ldots$) are decomposed into selfdual and anti-selfdual components $F_{\mu\nu}^{\pm\alpha}$ (which, in Minkowski space, are related by complex conjugation) and so are the field strengths $G^{\pm}_{\mu\nu\,\alpha}$ that appear in the field equations, which are defined by

$$G^{\pm}_{\mu\nu\,\alpha} = \pm\frac{4i}{e}\,\frac{\partial\mathcal{L}}{\partial F^{\pm\,\alpha\mu\nu}}\ . \tag{5.36}$$

Together $F^{\pm\alpha}_{\mu\nu}$ and $G^{\pm}_{\mu\nu\,\alpha}$ comprise the electric and magnetic fields and inductions. The Bianchi identities and equations of motion for the abelian gauge fields take the form

$$\partial^\mu (F^+ - F^-)^\alpha_{\mu\nu} = \partial^\mu (G^+ - G^-)_{\mu\nu\,\alpha} = 0\,, \tag{5.37}$$

which are obviously invariant under *real*, *constant*, rotations of the field strengths $F^\pm$ and $G^\pm$,

$$\begin{pmatrix} F^{\pm\alpha}_{\mu\nu} \\ G^{\pm}_{\mu\nu\,\beta} \end{pmatrix} \longrightarrow \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} F^{\pm\alpha}_{\mu\nu} \\ G^{\pm}_{\mu\nu\,\beta} \end{pmatrix}, \tag{5.38}$$

where U^α_β, $V_\alpha{}^\beta$, $W_{\alpha\beta}$ and $Z^{\alpha\beta}$ are constant, real, $n \times n$ submatrices and n denotes the number of independent gauge potentials. In $N = 8$ supergravity we have 56 such field strengths of each duality, so that the rotation is associated with a 56×56 matrix. The relevant question is whether the rotated equations (5.37) can again follow from a Lagrangian. More precisely, does there exist a new Lagrangian depending on the new, rotated, field strengths, such that the new tensors $G_{\mu\nu}$ follow from this Lagrangian as in (5.36). This condition amounts to an integrability condition, which can only have a solution (for generic Lagrangians) provided that the matrix is an element of the group $\mathrm{Sp}(2n;\mathbf{R})$.[23] This implies that the submatrices satisfy the constraint

$$\begin{aligned} &U^{\mathrm{T}} V - W^{\mathrm{T}} Z = V U^{\mathrm{T}} - W Z^{\mathrm{T}} = \mathbf{1}\,, \\ &U^{\mathrm{T}} W = W^{\mathrm{T}} U\,, \qquad Z^{\mathrm{T}} V = V^{\mathrm{T}} Z\,. \end{aligned} \tag{5.39}$$

We distinguish two subgroups of $\mathrm{Sp}(2n;\mathbf{R})$. One is the invariance group of the combined field equations and Bianchi identities, which usually requires the other fields in the Lagrangian to transform as well. Of course, a generic theory does not have such an invariance group, but maximal supergravity is known to have an $\mathrm{E}_{7(7)} \subset \mathrm{Sp}(56;\mathbf{R})$ invariance. However, this invariance

[23] Without any further assumptions, one can show that in Minkowski spaces of dimensions $D = 4k$, the rotations of the field equations and Bianchi identities associated with n rank-$(k-1)$ antisymmetric gauge fields that are described by a Lagrangian, constitute the group $\mathrm{Sp}(2n;\mathbf{R})$. For rank-$k$ antisymmetric gauge fields in $D = 2k+2$ dimensions, this group equals $\mathrm{SO}(n,n;\mathbf{R})$. Observe that these groups do not constitute an invariance of the theory, but merely characterize an equivalence class of Lagrangians. The fact that the symplectic redefinitions of the field strengths constitute the group $\mathrm{Sp}(2n;\mathbf{R})$ was first derived in [65], but in the context of a duality *invariance* rather than of a *reparametrization.* In this respect our presentation is more in the spirit of a later treatment in [66] for $N = 2$ vector multiplets coupled to supergravity (duality invariances for these theories were introduced in [67]).

group is not necessarily realized as a symmetry of the Lagrangian. The subgroup that is a symmetry of the Lagrangian, is usually smaller and restricted by $Z=0$ and $U^{-1}=V^{\mathrm{T}}$; the subgroup associated with the matrices U equals $\mathrm{GL}(n)$. Furthermore the Lagrangian is not uniquely defined (it can always be reparametrized via an electric-magnetic duality transformation) and neither is its invariance group. More precisely, there exist different Lagrangians with different symmetry groups, whose Bianchi identities and equations of motion are the same (modulo a linear transformation) and are invariant under the same group (which contains the symmetry groups of the various Lagrangians as subgroups). These issues are extremely important when gauging a subgroup of the invariance group, as this requires the gauge group to be contained in the invariance group of the Lagrangian.

Given the fact that we can rotate the field strengths by electric-magnetic duality transformations, we assign different indices to the field strengths and the underlying gauge groups than to the 56-bein $\mathcal{V}$. Namely, we label the fields strengths by independent index pairs $[AB]$, which are related to the index pairs $[IJ]$ of the 56-bein (*c.f.* (5.13)) in a way that we will discuss below. Furthermore, to remain in the context of the pseudoreal basis used previously, we form the linear combinations,

$$\mathsf{F}^{+}_{1\mu\nu\,AB}=\tfrac{1}{2}(i\,G^{+}_{\mu\nu\,AB}+F^{+AB}_{\mu\nu})\,,\quad \mathsf{F}^{+AB}_{2\mu\nu}=\tfrac{1}{2}(i\,G^{+}_{\mu\nu\,AB}-F^{+AB}_{\mu\nu})\,. \tag{5.40}$$

Anti-selfdual field strengths $(\mathsf{F}^{-AB}_{1\mu\nu},\mathsf{F}^{-}_{2\mu\nu\,AB})$ follow from complex conjugation. On this basis the field strengths rotate under $\mathrm{Sp}(56;\mathbf{R})$ according to the matrices E specified in (5.3); the real GL(28) subgroup is induced by corresponding linear transformations of the vector fields.

To exhibit how one can deal with a continuous variety of Lagrangians, which are manifestly invariant under different subgroups of $\mathrm{E}_{7(7)}$, let us remember that the tensors $F^{AB}_{\mu\nu}$ and $G_{\mu\nu\,AB}$ are related by (5.36) and this relationship must be consistent with $\mathrm{E}_{7(7)}$. In order to establish this consistency, the 56-bein plays a crucial role. The relation involves a constant $\mathrm{Sp}(56;\mathbf{R})$ matrix E (so that it satisfies the conditions (5.3)),

$$\mathsf{E}=\begin{pmatrix}\mathsf{U}_{IJ}{}^{AB} & \mathsf{V}_{IJCD}\\ \mathsf{V}^{KLAB} & \mathsf{U}^{KL}{}_{CD}\end{pmatrix}\,. \tag{5.41}$$

On the basis of $\mathrm{E}_{7(7)}$ and SU(8) covariance, the relation among the field strengths must have the form,

$$\mathcal{V}^{-1}\,\mathsf{E}\begin{pmatrix}\mathsf{F}^{+}_{1\mu\nu\,AB}\\ \mathsf{F}^{+AB}_{2\mu\nu}\end{pmatrix}=\begin{pmatrix}\mathsf{F}^{+}_{\mu\nu\,ij}\\ \mathcal{O}^{+kl}_{\mu\nu}\end{pmatrix}\,, \tag{5.42}$$

where $\mathcal{O}^+_{\mu\nu}$ is an SU(8) covariant tensor quadratic in the fermion fields and independent of the scalar fields, which appears in the moment couplings in the Lagrangian. Without going into the details we mention that the chirality and duality of $\mathcal{O}^+_{\mu\nu}$ is severely restricted so that the structure of (5.42) is unique (*c.f.* [63]). The tensor $\mathsf{F}^+_{\mu\nu\,ij}$ is an SU(8) covariant field strength that appears in the supersymmetry transformation rules of the spinors, which is simply defined by the above condition.

Hence the matrix E allows the field strengths and the 56-bein to transform under $\mathrm{E}_{7(7)}$ in an equivalent but nonidentical way. One could consider absorbing this matrix into the definition of the field strengths $(\mathsf{F}_1, \mathsf{F}_2)$, but such a redefinition cannot be carried out at the level of the Lagrangian, unless it belongs to a GL(28) subgroup which can act on the gauge fields themselves. In the basis (5.3) the generators of GL(28) have a block decomposition with SO(28) generators in both diagonal blocks and identical real, symmetric, 28×28 matrices in the off-diagonal blocks. On the other hand, when $\mathsf{E} \in \mathrm{E}_{7(7)}$, it can be absorbed into the 56-bein $\mathcal{V}$. The various Lagrangians are thus encoded in $\mathrm{Sp}(56;\mathbf{R})$ matrices E, up to multiplication by GL(28) from the right and multiplication by $\mathrm{E}_{7(7)}$ from the left, *i.e.* in elements of $\mathrm{E}_{7(7)}\backslash\mathrm{Sp}(56;\mathbf{R})/\mathrm{GL}(28)$.

From (5.42) one can straightforwardly determine the relevant terms in the Lagrangian. For convenience, we redefine the 56-bein by absorbing the matrix E,

$$\hat{\mathcal{V}}(x) = \mathsf{E}^{-1}\,\mathcal{V}(x) = \begin{pmatrix} u^{ij}{}_{AB}(x) & -v^{kl\,AB}(x) \\ & \\ -v^{ij\,CD}(x) & u_{klCD}(x) \end{pmatrix}, \tag{5.43}$$

where we have to remember that $\hat{\mathcal{V}}$ is now no longer a group element of $\mathrm{E}_{7(7)}$. Note, however, that the $\mathrm{E}_{7(7)}$ tensors $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ are not affected by the matrix E and have identical expressions in terms of $\mathcal{V}$ and $\hat{\mathcal{V}}$. This is not the case for the terms in the Lagrangian that contain the abelian field strengths,

$$F^{AB}_{\mu\nu} = \partial_\mu A^{AB}_\nu - \partial_\nu A^{AB}_\mu\,, \tag{5.44}$$

and which take the form,

$$\begin{aligned} \mathcal{L}_3 \;=\;& -\tfrac{1}{8} e\, F^{+AB}_{\mu\nu}\, F^{+CD\,\mu\nu}\,[(u+v)^{-1}]^{AB}{}_{ij}\,(u^{ij}{}_{CD} - v^{ijCD}) \\ & -\tfrac{1}{2} e\, F^{+AB}_{\mu\nu}\,[(u+v)^{-1}]^{AB}{}_{ij}\,\mathcal{O}^{+\mu\nu\,ij} \\ & + \text{h.c.}\,, \end{aligned} \tag{5.45}$$

where the 28×28 matrices satisfy $[(u+v)^{-1}]^{AB}{}_{ij}\,(u^{ij}{}_{CD} + v^{ijCD}) = \delta^{AB}_{CD}$. The SU(8) covariant field strength $\mathsf{F}^{+}_{\mu\nu ij}$ will appear in the supersymmetry transformation rules for the fermions, and is equal to

$$F^{+AB}_{\mu\nu} = (u^{ij}{}_{AB} + v^{ijAB})\,\mathsf{F}^{+}_{\mu\nu ij} - (u_{ij}{}^{AB} + v_{ijAB})\,\mathcal{O}^{+ij}_{\mu\nu}\,. \tag{5.46}$$

Clearly the Lagrangian depends on the matrix E. Because the matrix $\mathsf{E}^{-1}\mathcal{V}$ is an element of $\mathrm{Sp}(56;\mathbf{R})$, the matrix multiplying the two field strengths in (5.45) is symmetric under the interchange of $[AB] \leftrightarrow [CD]$.[24]

In order that the Lagrangian be invariant under a certain subgroup of $\mathrm{E}_{7(7)}$, one has to make a certain choice for the matrix E. According to the analysis leading to (5.38) and (5.39), this subgroup is generated on $\hat{\mathcal{V}}$ by matrices Λ and Σ, just as in (5.1), but with indices $A, B, \ldots$, rather than with $I, J, \ldots$, satisfying

$$\mathrm{Im}\left(\Sigma_{ABCD} + \Lambda_{AB}{}^{CD}\right) = 0\,. \tag{5.47}$$

In order to be a subgroup of $\mathrm{E}_{7(7)}$ as well, they must also satisfy (5.6), but only after a proper conversion of the $I, J, \ldots$ to $A, B, \ldots$ indices. The gauge fields transform under the real subgroup (*i.e.*, the imaginary parts of the generators act exclusively on the 56-bein). A large variety of symmetry groups exists, as one can deduce from the symmetry groups that are realized in maximal supergravity in higher dimensions. One such group whose existence can be directly inferred in this way, is $\mathrm{E}_{6(6)} \times \mathrm{SO}(1,1)$.

5.4 Gauging maximal supergravity; the T-tensor

The gauging of supergravity is effected by switching on the gauge coupling constant, after assigning the various fields to representations of the gauge group embedded in $\mathrm{E}_{7(7)}$ or $\mathrm{E}_{6(6)}$. Only the gauge fields themselves and the spinless fields can transform under this gauge group. Hence the abelian field strengths are changed to nonabelian ones and derivatives of the scalars are covariantized according to

$$\partial_\mu \mathcal{V} \rightarrow \partial_\mu \mathcal{V} - g A^{AB}_\mu\, T_{AB} \mathcal{V}\,, \tag{5.48}$$

where the gauge group generators T_{AB} are 56×56 matrices which span a subalgebra of dimension equal to at most the number of vector fields,

[24]Such symmetry properties follow from the symmetry under interchanging index pairs in the products $(u^{ij}{}_{AB} - v^{ijAB})\,(u^{kl}{}_{AB} + v^{klAB})$ and $(u^{ij}{}_{AB} + v^{ijAB})\,(u_{ij}{}^{CD} + v_{ijCD})$.

embedded in the Lie algebra of $E_{7(7)}$ or $E_{6(6)}$. The structure constants of the gauge group are given by

$$[T_{AB}, T_{CD}] = f_{AB,CD}{}^{EF}\, T_{EF}\,. \tag{5.49}$$

It turns out that the viability for a gauging depends sensitively on the choice of the gauge group and its corresponding embedding. This aspect is most nontrivial for the $D = 4$ theory, in view of electric-magnetic duality. Therefore, we will mainly concentrate on this theory. In $D = 4$ dimensions, one must start from a Lagrangian that is symmetric under the desired gauge group, which requires one to make a suitable choice of the matrix E. In $D = 5$ dimensions, the Lagrangian is manifestly symmetric under $E_{6(6)}$, so this subtlety does not arise. When effecting the gauging the vector fields may decompose into those associated with the nonabelian gauge group and a number of remaining gauge fields. When the latter are charged under the gauge group, then there is a potential obstruction to the gauging as the gauge invariance of these gauge fields cannot coexist with the nonabelian gauge transformations. However, in $D = 5$ this obstruction can be avoided, because (charged) vector fields can alternatively be described as antisymmetric rank-2 tensor fields. For instance, the gauging of $\mathrm{SO}(p, 6-p)$ involves 15 nonabelian gauge fields and 12 antisymmetric tensor fields. The latter can transform under the gauge group, because they are not realized as tensor *gauge* fields. Typically this conversion of vector into tensor fields leads to terms that are inversely proportional to the gauge coupling [68]. However, to write down a corresponding Lagrangian requires an even number of tensor fields.

Introducing the gauging leads directly to a loss of supersymmetry, because the new terms in the Lagrangian lead to new variations. For convenience we now restrict ourselves to $D = 4$ dimensions. The leading variations are induced by the modification (5.48) of the Cartan-Maurer equations. This modification was already noted in (4.57) and takes the form

$$\begin{aligned} F_{\mu\nu}(\mathcal{Q})_i{}^j &= -\tfrac{4}{3}\,\mathcal{P}_{[\mu}{}^{jklm}\,\mathcal{P}_{\nu]iklm} - g\,F_{\mu\nu}^{AB}\,\mathcal{Q}_{AB\,i}{}^j\,,\\ D_{[\mu}\mathcal{P}_{\nu]}^{ijkl} &= -\tfrac{1}{2}g\,F_{\mu\nu}^{AB}\,\mathcal{P}_{AB}^{ijkl}\,, \end{aligned} \tag{5.50}$$

where

$$\mathcal{V}^{-1}T_{AB}\mathcal{V} = \begin{pmatrix} \mathcal{Q}_{AB\,ij}{}^{mn} & \mathcal{P}_{AB\,ijpq} \\ & \\ \mathcal{P}_{AB}^{klmn} & \mathcal{Q}_{AB}{}^{kl}{}_{pq} \end{pmatrix}. \tag{5.51}$$

These modifications are the result of the implicit dependence of $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ on the vector potentials A_μ^{AB}. The fact that the matrices T_{AB} generate a subalgebra of the algebra associated with $\mathrm{E}_{7(7)}$, in the basis appropriate for $\mathcal{V}$, implies that the quantities $\mathcal{Q}_{AB}$ and $\mathcal{P}_{AB}$ satisfy the constraints,

$$\begin{aligned} \mathcal{P}^{ijkl}_{AB} &= \tfrac{1}{24}\,\varepsilon^{ijklmnpq}\,\mathcal{P}_{AB\,mnpq}\,, \\ \mathcal{Q}_{AB\,ij}{}^{kl} &= \delta^{[k}_{[i}\,\mathcal{Q}_{AB\,j]}{}^{l]}\,, \end{aligned} \tag{5.52}$$

while $\mathcal{Q}_{AB i}{}^{j}$ is antihermitean and traceless. It is straightforward to write down the explicit expressions for $\mathcal{Q}_{AB}$ and $\mathcal{P}_{AB}$,

$$\begin{aligned} \mathcal{Q}_{AB\,i}{}^{j} &= \tfrac{2}{3}\Big[u_{ik}{}^{IJ}\,(\Delta_{AB}u^{jk}{}_{IJ}) - v_{ikIJ}\,(\Delta_{AB}v^{jkIJ})\Big]\,, \\ \mathcal{P}^{ijkl}_{AB} &= v^{ijIJ}\,(\Delta_{AB}u^{kl}{}_{IJ}) - u^{ij}{}_{IJ}(\Delta_{AB}v^{klIJ})\,. \end{aligned} \tag{5.53}$$

where $\Delta_{AB}u$ and $\Delta_{AB}v$ indicate the change of submatrices in $\mathcal{V}$ induced by multiplication with the generator T_{AB}. Note that we could have expressed the above quantities in terms of the modified 56-bein $\hat{\mathcal{V}}$, on which the $\mathrm{E}_{7(7)}$ transformations act in the basis that is appropriate for the field strengths, provided we change the generators T_{AB} into

$$\hat{T}_{AB} = \mathsf{E}^{-1}T_{AB}\,\mathsf{E}\,. \tag{5.54}$$

This is done below.

When establishing supersymmetry of the action one needs the Cartan-Maurer equations at an early stage to cancel variations from the gravitino kinetic terms and the Noether term (the term in the Lagrangian proportional to $\bar{\chi}\psi_\mu\mathcal{P}_\nu$). The order-$g$ terms in the Maurer-Cartan equation yield the leading variations of the Lagrangian. They are linearly proportional to the fermion fields and read,

$$\begin{aligned} \delta\mathcal{L} \;=\;& \tfrac{1}{4}g(\bar{\epsilon}_j\gamma^\rho\gamma^{\mu\nu}\psi^i_\rho - \bar{\epsilon}^i\gamma^\rho\gamma^{\mu\nu}\psi_{\rho j})\,\mathcal{Q}_{AB\,i}{}^{j}\,(u^{kl}{}_{AB} + v^{klAB})\,\mathsf{F}^+_{\mu\nu kl} \\ & +\tfrac{1}{288}\varepsilon^{ijklmnpq}\,\bar{\chi}_{ijk}\gamma^{\mu\nu}\epsilon_l\,\mathcal{P}_{AB\,mnpq}\,(u^{rs}{}_{AB} + v^{rsAB})\,\mathsf{F}^+_{\mu\nu rs} \\ & + \text{h.c.} \end{aligned} \tag{5.55}$$

The first variation is proportional to an SU(8) tensor T_i^{jkl}, which is known as the T-tensor,

$$\begin{aligned} T_i^{jkl} &= \tfrac{3}{4}\mathcal{Q}_{AB\,i}{}^{j}\,(u^{kl}{}_{AB} + v^{klAB}) \qquad (5.56)\\ &= \tfrac{1}{2}\Big[u_{im}{}^{CD}\,(\hat{\Delta}_{AB}u^{jm}{}_{CD}) - v_{imCD}\,(\hat{\Delta}_{AB}v^{jmCD})\Big](u^{kl}{}_{AB} + v^{klAB})\,, \end{aligned}$$

where $\hat{\Delta}_{AB}u$ and $\hat{\Delta}_{AB}v$ are the submatrices of $\hat{T}_{AB}\hat{\mathcal{V}}$. Another component of the T-tensor appears in the second variation and is equal to

$$\begin{aligned} T^{mn}_{ijkl} &= \tfrac{1}{2}\mathcal{P}_{AB\,ijkl}\,(u^{mn}{}_{AB} + v^{mnAB}) \qquad (5.57)\\ &= \tfrac{1}{2}\Big[v_{ijCD}\,(\hat{\Delta}_{AB}u_{kl}{}^{CD}) - u_{ij}{}^{CD}(\hat{\Delta}_{AB}v_{klCD})\Big]\,(u^{mn}{}_{AB} + v^{mnAB})\,. \end{aligned}$$

The T-tensor is thus a cubic product of the 56-bein $\hat{\mathcal{V}}$ which depends in a nontrivial way on the embedding of the gauge group into $\mathrm{E}_{7(7)}$. It satisfies a number of important properties. Some of them are rather obvious (such as $T_i^{ijk} = 0$), and follow rather straightforwardly from the definition. We will concentrate on properties which are perhaps less obvious. Apart from the distinction between $\mathcal{V}$ and $\hat{\mathcal{V}}$, which is a special feature of $D = 4$ dimensions, these properties are generic.

First we observe that SU(8) covariantized variations of the T-tensor are again proportional to the T-tensor. These variations are induced by (5.27) and (5.28). Along the same lines as before we can show that the SU(8) tensors $\mathcal{Q}_{AB}$ and $\mathcal{P}_{AB}$ transform in the adjoint representation of $\mathrm{E}_{7(7)}$, which allows one to derive,

$$\begin{aligned} \delta T_i^{jkl} &= \Sigma^{jmnp}\, T^{kl}_{imnp} - \tfrac{1}{24}\varepsilon^{jmnpqrst}\,\Sigma_{imnp}\, T^{kl}_{qrst} + \Sigma^{klmn}\, T^{j}_{imn}\,, \\ \delta T^{mn}_{ijkl} &= \tfrac{4}{3}\Sigma_{p[ijk}\, T^{pmn}_{l]} - \tfrac{1}{24}\varepsilon_{ijklpqrs}\,\Sigma^{mntu}\, T^{pqrs}_{tu}\,. \end{aligned} \tag{5.58}$$

This shows that the SU(8) covariant T-tensors can be assigned to a representation of $\mathrm{E}_{7(7)}$. This property will play an important role below.

Before completing the analysis leading to a consistent gauging we stress that all variations are from now on expressed in terms of the T-tensor, as its variations yield again the same tensor. This includes the SU(8) covariant derivative of the T-tensor, which follows directly from (5.58) upon the substitutions $\delta \to D_\mu$ and $\Sigma \to \mathcal{P}_\mu$. A viable gauging requires that the T-tensor satisfies a number of rather nontrivial identities, as we will discuss shortly, but the new terms in the Lagrangian and transformation rules have a universal form, irrespective of the gauge group. Let us first describe these new terms. First of all, to cancel the variations (5.55) we need masslike terms in the Lagrangian,

$$\begin{aligned} \mathcal{L}_{\rm masslike} &= g\, e\Big\{ \tfrac{1}{2}\sqrt{2}\, A_{1ij}\bar{\psi}^{\,i}_{\mu}\gamma^{\mu\nu}\psi^j_\nu + \tfrac{1}{6}A_{2i}^{jkl}\,\bar{\psi}^i_\mu\gamma^\mu\chi_{jkl} \\ &\qquad + A_3^{ijk,lmn}\,\bar{\chi}_{ijk}\chi_{lmn} + \text{h.c.}\Big\}\,, \end{aligned} \tag{5.59}$$

whose presence necessitates corresponding modifications of the supersymmetry transformations of the fermion fields,

$$\begin{aligned} \delta_g\bar{\psi}^i_\mu &= -\sqrt{2}g\, A_1^{ij}\,\bar{\epsilon}_j\gamma_\mu\,, \\ \delta_g\chi^{ijk} &= -2g\, A_{2l}{}^{ijk}\,\bar{\epsilon}^l\,. \end{aligned} \tag{5.60}$$

Finally at order g^2 one needs a potential for the spinless fields,

$$P(\mathcal{V}) = g^2\Big\{\tfrac{1}{24}|A_{2i}{}^{jkl}|^2 - \tfrac{1}{3}|A_1^{ij}|^2\Big\}\,. \tag{5.61}$$

These last three formulae will always apply, irrespective of the gauge group. Note that the tensors A_1^{ij}, ${A_2}_i{}^{jkl}$ and $A_3^{ijk,lmn}$ have certain symmetry properties dictated by the way they appear in the Lagrangian (5.59). To be specific, A_1 is symmetric in (ij), A_2 is fully antisymmetric in $[jkl]$ and A_3 is antisymmetric in $[ijk]$ as well as in $[lmn]$ and symmetric under the interchange $[ijk] \leftrightarrow [lmn]$. This implies that these tensors transform under SU(8) according to the representations

$$\begin{aligned} A_1 &: \mathbf{36}\,, \\ A_2 &: \mathbf{28} + \mathbf{420}\,, \\ A_3 &: \overline{\mathbf{28}} + \overline{\mathbf{420}} + \overline{\mathbf{1176}} + \overline{\mathbf{1512}}\,. \end{aligned}$$

The three SU(8) covariant tensors, A_1, A_2 and A_3, which depend only on the spinless fields, must be linearly related to the T-tensor, because they were introduced for the purpose of cancelling the variations (5.55). To see how this can be the case, let us analyze the SU(8) content of the T-tensor. As we mentioned already, the T-tensor is cubic in the 56-bein, and as such is constitutes a certain tensor that transforms under $\mathrm{E}_{7(7)}$. The transformation properties were given in (5.58), where we made use of the fact that the T-tensor consists of a product of the fundamental times the adjoint representation of $\mathrm{E}_{7(7)}$. Hence the T-tensor comprises the representations,

$$\mathbf{56} \times \mathbf{133} = \mathbf{56} + \mathbf{912} + \mathbf{6480}\,. \tag{5.62}$$

The representations on the right-hand side can be decomposed under the action of SU(8), with the result

$$\begin{aligned} \mathbf{56} &= \mathbf{28} + \overline{\mathbf{28}}\,, \\ \mathbf{912} &= \mathbf{36} + \overline{\mathbf{36}} + \mathbf{420} + \overline{\mathbf{420}}\,, \\ \mathbf{6480} &= \mathbf{28} + \overline{\mathbf{28}} + \mathbf{420} + \overline{\mathbf{420}} + \mathbf{1280} + \overline{\mathbf{1280}} + \mathbf{1512} + \overline{\mathbf{1512}}\,. \end{aligned} \tag{5.63}$$

Comparing these representations to the SU(8) representations to which the tensors $A_1 - A_3$ (and their complex conjugates) belong, we note that there is a mismatch between (5.63) and (5.62). In view of (5.58) the T-tensor must be restricted by suppressing complete representations of $\mathrm{E}_{7(7)}$ in order that its variations and derivatives remain consistent. This proves that the T-tensor cannot contain the entire $\mathbf{6480}$ representation of $\mathrm{E}_{7(7)}$, so that it must consist of the $\mathbf{28} + \mathbf{36} + \mathbf{420}$ representation of SU(8) (and its complex conjugate). This implies that the T-tensor is decomposable into A_1 and A_2, whereas A_3 is not an independent tensor and can be expressed in terms of

A_2. Indeed this was found by explicit calculation, which gave rise to

$$\begin{aligned} T_i^{jkl} &= -\tfrac{3}{4} A_{2i}{}^{jkl} + \tfrac{3}{2} A_1^{j[k}\,\delta_i^{l]}\,, \\ T_{ijkl}^{mn} &= -\tfrac{4}{3}\delta_{[i}^{[m} T_{jkl]}^{n]}\,, \\ A_3^{ijk,lmn} &= -\tfrac{1}{108}\sqrt{2}\,\varepsilon^{ijkpqr[lm} T_{pqr}^{n]}\,. \end{aligned} \tag{5.64}$$

Note that these conditions are necessary, but not sufficient as one also needs nontrivial identities quadratic in the T-tensors in order to deal with the variations of the Lagrangian of order g^2. One then finds that there is yet another constraint, which suppresses the **28** representation of the T-tensor,

$$T_i^{[jk]i} = 0\,. \tag{5.65}$$

Observe that a contraction with the first upper index is also equal to zero, as follows from the definition (5.56). Hence the T-tensor transforms under $\mathrm{E}_{7(7)}$ according to the **912** representation which decomposes into the **36** and **420** representations of SU(8) and their complex conjugates residing in the tensors A_1 and A_2, respectively,

$$A_1^{ij} = \tfrac{4}{21} T_k^{ikj}\,, \qquad A_{2i}^{jkl} = -\tfrac{4}{3} T_i^{[jkl]}\,. \tag{5.66}$$

Although we concentrated on the $D = 4$ theory, we should stress once more that many of the above features are generic and apply in other dimensions. For instance, the unrestricted T-tensors in $D = 5$ and 3 dimensions belong to the following representations of $\mathrm{E}_{6(6)}$ and $\mathrm{E}_{8(8)}$, respectively[25]

$$\begin{aligned} D = 5 \quad &: \quad \mathbf{27}\times\mathbf{78} = \mathbf{27}+\mathbf{351}+\mathbf{1728}\,, \\ D = 3 \quad &: \quad \mathbf{248}\times\mathbf{248} = \mathbf{1}+\mathbf{248}+\mathbf{3875}+\mathbf{27000}+\mathbf{30380}\,. \end{aligned} \tag{5.67}$$

In these cases a successful gauging requires the T-tensor to be restricted to the **351** and the $\mathbf{1}+\mathbf{3875}$ representations, respectively, which decompose as follows under the action of USp(8) and SO(16),

$$\begin{aligned} \mathbf{351} &= \mathbf{36}+\mathbf{315}\,, \\ \mathbf{1}+\mathbf{3875} &= \mathbf{1}+\mathbf{135}+\mathbf{1820}+\mathbf{1920}\,. \end{aligned} \tag{5.68}$$

These representations correspond to the tensors A_1 and A_2; for $D = 5$ A_3 is again dependent while for $D = 3$ there is an independent tensor A_3 associated with the **1820** representation of SO(16).

[25] The $D = 3$ theory has initially no vector fields, but those can be included by adding Chern-Simons terms. These terms lose their topological nature when gauging some of the $\mathrm{E}_{8(8)}$ isometries [69].

We close with a few comments regarding the various gauge groups that have been considered. As we mentioned at the beginning of this chapter, the first gaugings were to some extent motivated by corresponding Kaluza-Klein compactifications. The S^7 and the S^4 [70] compactifications of 11-dimensional supergravity and the S^5 compactification of IIB supergravity, gave rise to the gauge groups SO(8), SO(5) and SO(6), respectively. Non-compact gauge groups were initiated in [71] for the 4-dimensional theory; for the 5-dimensional theory they were also realized in [29] and in [72]. In $D = 3$ dimensions there is no guidance from Kaluza-Klein compactifications and one has to rely on the group-theoretical analysis described above. In that case there exists a large variety of gauge groups of rather high dimension [69]. Gaugings can also be constructed via a so-called Scherk-Schwarz reduction from higher dimensions [73]. To give a really exhaustive classification remains cumbersome. For explorations based on the group-theoretical analysis explained above, see [74, 75].

6 Supersymmetry in anti-de Sitter space

In section 3.1 we presented the first steps in the construction of a generic supergravity theory, starting with the Einstein-Hilbert Lagrangian for gravity and the Rarita-Schwinger Lagrangians for the gravitino fields. We established the existence of two supersymmetric gravitational backgrounds, namely flat Minkowski space and anti-de Sitter space with a cosmological constant proportional to g^2, where g was some real coupling constant proportional to the the inverse anti-de Sitter radius. The two cases are clearly related and flat space is obtained in the limit $g \to 0$, as can for instance be seen from the expression of the Riemann curvature (*c.f.* (3.14)),

$$R_{\mu\nu\rho}{}^{\sigma} = g^2(g_{\mu\rho}\,\delta^{\sigma}_{\nu} - g_{\nu\rho}\,\delta^{\sigma}_{\mu})\,. \tag{6.1}$$

Because both flat Minkowski space and anti-de Sitter space are maximally symmetric, they have $\frac{1}{2}D(D+1)$ independent isometries which comprise the Poincaré group or the group $\mathrm{SO}(D-1,2)$, respectively. The algebra of the combined bosonic and fermionic symmetries is called the anti-de Sitter superalgebra. Note again that the derivation in section 3.1 was incomplete and in general one will need to introduce additional fields.

In this chapter we will mainly be dealing with simple anti-de Sitter supersymmetry and we will always assume that $3 < D \leq 7$. In that case the bosonic subalgebra coincides with the anti-de Sitter algebra. In $D = 3$ spacetime dimensions the anti-de Sitter group $\mathrm{SO}(2,2)$ is not simple. There exist N-extended versions where one introduces N supercharges, each transforming as a spinor under the anti-de Sitter group. These N supercharges transform under a compact R-symmetry group, whose generators will appear in the $\{Q,\bar{Q}\}$ anticommutator. As we discussed in sect. 2.5, the R-symmetry group is in general not the same as in Minkowski space; according to Table 9, we have $\mathrm{H_R} = \mathrm{SO}(N)$ for $D=4$, $\mathrm{H_R} = \mathrm{U}(N)$ for $D=5$, and $\mathrm{H_R} = \mathrm{USp}(2N)$ for $D=6,7$. For $D>7$ the superalgebra is no longer simple [3]; its bosonic subalgebra can no longer be restricted to the sum of the anti-de Sitter algebra and the R-symmetry algebra, but one needs extra bosonic generators that transform as high-rank antisymmetric tensors under the Lorentz group (see also, [76]). In contrast to this, there exists an (N-extended) super-Poincaré algebra associated with flat Minkowski space of any dimension, whose bosonic generators correspond to the Poincaré group, possibly augmented with the R-symmetry generators associated with rotations of the supercharges.

Anti-de Sitter space is isomorphic to $\mathrm{SO}(D-1,2)/\mathrm{SO}(D-1,1)$ and thus belongs to the coset spaces that were discussed extensively in chapter 4. According to (4.10) it is possible to describe anti-de Sitter space as a hypersurface in a $(D+1)$-dimensional embedding space. Denoting the extra coordinate of the embedding space by Y^-, so that we have coordinates Y^A with $A=-,0,1,2,\ldots,D-1$, this hypersurface is defined by

$$-(Y^-)^2-(Y^0)^2+\vec{Y}^{\,2} = \eta_{AB}\,Y^AY^B = -g^{-2}\,. \qquad (6.2)$$

Obviously, the hypersurface is invariant under linear transformations that leave the metric $\eta_{AB} = \mathrm{diag}\,(-,-,+,+,\ldots,+)$ invariant. These transformations constitute the group $\mathrm{SO}(D-1,2)$. The $\frac{1}{2}D(D+1)$ generators denoted by M_{AB} act on the embedding coordinates by

$$M_{AB} = Y_A\frac{\partial}{\partial Y^B} - Y_B\frac{\partial}{\partial Y^A}\,, \qquad (6.3)$$

where we lower and raise indices by contracting with η_{AB} and its inverse η^{AB}. It is now easy to evaluate the commutation relations for the M_{AB},

$$[M_{AB},M_{CD}] = \eta_{BC}\,M_{AD} - \eta_{AC}\,M_{BD} - \eta_{BD}\,M_{AC} + \eta_{AD}\,M_{BC}\,. \qquad (6.4)$$

Anti-de Sitter space has the topology of $S^1[\mathrm{time}]\times\mathbf{R}^{D-1}[\mathrm{space}]$ and has closed timelike curves. These curves can be avoided by unwrapping S^1, so that one finds the universal covering space denoted by CadS, which has the topology of $\mathbf{R}^D$. There exist no Cauchy surfaces in this space. Any attempt to determine the outcome of some evolution or wave equation from a spacelike surface requires fresh information coming from a timelike infinity which takes a finite amount of time to arrive [77, 78]. Spatial infinity is a timelike surface which cannot be reached by timelike geodesics. There are many ways to coordinatize anti-de Sitter space, but we will avoid using explicit coordinates.

For later use we record the (simple) anti-de Sitter superalgebra, which in addition to (6.4) contains the (anti-)commutation relations,

$$\begin{aligned}
\{Q_\alpha, \bar{Q}_\beta\} &= -\tfrac{1}{2}(\Gamma_{AB})_{\alpha\beta}\, M^{AB}\,, \\
[M_{AB}, \bar{Q}_\alpha] &= \tfrac{1}{2}(\bar{Q}\,\Gamma_{AB})_\alpha\,.
\end{aligned} \tag{6.5}$$

Here the matrices Γ_{AB}, which will be defined later, are the generators of $\mathrm{SO}(D-1,2)$ group in the spinor representation. As we alluded to earlier, this algebra changes its form when considering N supercharges which rotate under R-symmetry, because the R-symmetry generators will appear on the right-hand side of the $\{Q, \bar{Q}\}$ anticommutator.

The relation with the Minkowski case proceeds by means of a so-called Wigner-Inönü contraction. Here one rescales the generators according to $M_{-A} \to g^{-1} P_A$, $Q \to g^{-1/2} Q$, keeping the remaining generators M_{AB} corresponding to the Lorentz subalgebra unchanged. In the limit $g \to 0$, the generators P_A will form a commuting subalgebra and the full algebra contracts to the super-Poincaré algebra.

On spinors, the anti-de Sitter algebra can be realized by the following combination of gamma matrices Γ_a in D-dimensional Minkowski space,

$$M_{AB} = \tfrac{1}{2}\Gamma_{AB} = \begin{cases} \tfrac{1}{2}\Gamma_{ab} & \text{for} \quad A, B = a, b\,, \\[1ex] \tfrac{1}{2}\Gamma_a & \text{for} \quad A = -\,, B = a \end{cases} \tag{6.6}$$

with $a, b = 0, 1, \ldots, D-1$. Our gamma matrices satisfy the Clifford property $\{\Gamma^a\,, \Gamma^b\} = 2\,\eta^{ab}\,\mathbf{1}$, where $\eta^{ab} = \mathrm{diag}\,(-,+,\ldots,+)$ is the D-dimensional Lorentz-invariant metric.[26] Concerning the R-symmetry group in anti-de Sitter space, the reader is advised to consult section 2.5.

Of central importance is the quadratic Casimir operator of the isometry group $\mathrm{SO}(D-1,2)$, defined by

$$\mathcal{C}_2 = -\tfrac{1}{2}\, M^{AB}\, M_{AB}\,. \tag{6.7}$$

The group $\mathrm{SO}(D-1,2)$ has more Casimir operators when $D > 3$, but these are of higher order in the generators and will not play a role in the following.

[26] Note that when the gravitino is a Majorana spinor, the quantities $\Gamma_{AB}\epsilon$ should satisfy the same Majorana constraint.

To make contact between the masslike terms in the wave equations and the properties of the irreducible representations of the anti-de Sitter group, which we will discuss in section 6.1, it is important that we establish the relation between the wave operator for fields that live in anti-de Sitter space, which involves the appropriately covariantized D'Alembertian $\Box_{\rm adS}$, and the quadratic Casimir operator $\mathcal{C}_2$. We remind the reader that fields in anti-de Sitter space are multi-component functions of the anti-de Sitter coordinates that rotate irreducibly under the action of the Lorentz group $\mathrm{SO}(D-1,1)$. The appropriate formulae were given at the end of section 4.2 (*c.f.* (4.44) and (4.45)) and from them one can derive,

$$\mathcal{C}_2 = \Box_{\rm adS}\Big|_{g=1} + \mathcal{C}_2^{\rm Lorentz}\,, \tag{6.8}$$

where $\mathcal{C}_2^{\rm Lorentz}$ is the quadratic Casimir operator for the representation of the Lorentz group to which the fields have been assigned. This result can be proven for any symmetric, homogeneous, space (see, for example, [79]). For scalar fields, the second term in (6.8) vanishes and the proof is elementary (see, *e.g.*, [80]).

Let us now briefly return to the supersymmetry algebra as it is realized on the vielbein field. Using the transformation rules (3.11) The commutator of two supersymmetry transformations yields an infinitesimal general-coordinate transformation and a tangent-space Lorentz transformation. For example, we obtain for the vielbein,

$$\begin{aligned}
[\delta_1,\delta_2]\, e_\mu{}^a &= \tfrac{1}{2}\bar\epsilon_2\,\Gamma^a\,\delta_1\psi_\mu - \tfrac{1}{2}\bar\epsilon_1\,\Gamma^a\,\delta_2\psi_\mu \\
&= D_\mu(\tfrac{1}{2}\bar\epsilon_2\,\Gamma^a\epsilon_1) + \tfrac{1}{2}g\,(\bar\epsilon_2\,\Gamma^{ab}\epsilon_1)\,e_{\mu b}\,.
\end{aligned} \tag{6.9}$$

The first term corresponds to a spacetime diffeomorphism and the second one to a tangent space (local Lorentz) transformation. Here we consider only the gravitational sector of the theory; for a complete theory there are additional contributions, but nevertheless the above terms remain and (6.9) should be realized uniformly on all the fields. In the anti-de Sitter background, were the gravitino field vanishes, the parameters of the supersymmetry transformations are Killing spinors satisfying (3.15) so that the gravitino field remains zero under supersymmetry. Therefore both the gravitino and the vielbein are left invariant under supersymmetry, so that the combination of symmetries on the right-hand side of (6.9) should vanish when ϵ_1 and ϵ_2 are Killing spinors. Indeed, the diffeomorphism with parameter $\xi^\mu = \tfrac{1}{2}\bar\epsilon_2\,\Gamma^\mu\epsilon_1$, is an anti-de Sitter Killing vector (*i.e.*, it satisfies (4.38)), because $D_\mu(g\,\bar\epsilon_2\Gamma_{\nu\rho}\epsilon_1) = -g^2\,g_{\mu[\rho}\xi_{\nu]}$ is antisymmetric in μ and ν. As for all

Killing vectors, higher derivatives can be decomposed into the Killing vector and its first derivative. Indeed, we find $D_\mu(g\,\bar\epsilon_2\Gamma_{\nu\rho}\epsilon_1) = -g^2\, g_{\mu[\rho}\xi_{\nu]}$ in the case at hand. The Killing vector can be decomposed into the $\frac{1}{2}D(D+1)$ Killing vectors of the anti-de Sitter space. The last term in (6.9) is a compensating target space transformation of the type we have been discussing extensively in section 4.2 for generic coset spaces.

6.1 Anti-de Sitter supersymmetry and masslike terms

In flat Minkowski space all fields belonging to a supermultiplet are subject to field equations with the same mass, because the momentum operators commute with the supersymmetry charges, so that P^2 is a Casimir operator. For supermultiplets in anti-de Sitter space this is no longer the case, so that masslike terms will not necessarily be the same for different fields belonging to the same multiplet. We have already discussed the interpretation of masslike terms for the gravitino, following (3.10). This phenomenon will be now illustrated below in a specific example, namely a scalar chiral supermultiplet in $D = 4$ spacetime dimensions. Further clarification from an algebraic viewpoint will be given later in section 6.3.

A scalar chiral supermultiplet in 4 spacetime dimensions consists of a scalar field A, a pseudoscalar field B and a Majorana spinor field ψ. In anti-de Sitter space the supersymmetry transformations of the fields are proportional to a spinor parameter $\epsilon(x)$, which is a Killing spinor in the anti-de Sitter space, *i.e.*, $\epsilon(x)$ must satisfy the Killing spinor equation (3.15). In the notation of this section, this equation reads,

$$\Big(\partial_\mu - \tfrac{1}{4}\omega^{ab}\gamma_{ab} + \tfrac{1}{2}g\, e_\mu{}^a\gamma_a\Big)\epsilon = 0\,, \tag{6.1}$$

where we made the anti-de Sitter vierbein and spin connection explicit. We allow for two constants a and b in the supersymmetry transformations, which we parametrize as follows,

$$\begin{aligned}\delta A &= \tfrac{1}{4}\bar\epsilon\psi\,, \qquad \delta B = \tfrac{1}{4}i\bar\epsilon\gamma_5\psi\,,\\ \delta\psi &= \not{\partial}(A + i\gamma_5 B)\epsilon - (a\,A + ib\,\gamma_5 B)\epsilon\,.\end{aligned} \tag{6.2}$$

In this expression the anti-de Sitter vierbein field has been used to contract the gamma matrix with the derivative. The coefficient of the first term in $\delta\psi$ has been chosen such as to ensure that $[\delta_1, \delta_2]$ yields the correct coordinate transformation $\xi^\mu D_\mu$ on the fields A and B. To determine the constants a and b and the field equations of the chiral multiplet, we consider the closure of the supersymmetry algebra on the spinor field. After some Fierz reordering we obtain the result,

$$[\delta_1, \delta_2]\psi = \xi^\mu D_\mu\psi + \tfrac{1}{16}(a-b)\,\bar\epsilon_2\gamma^{ab}\epsilon_1\,\gamma_{ab}\psi - \tfrac{1}{2}\xi^\rho\gamma_\rho[\not{D}\psi + \tfrac{1}{2}(a+b)\psi]\,. \quad (6.3)$$

We point out that derivatives acting on $\epsilon(x)$ occur in this calculation at an intermediate stage and should not be suppressed in view of (6.1). However, they produce terms proportional to g which turn out to cancel in the above commutator. Now we note that the right-hand side should constitute a coordinate transformation and a Lorentz transformation, possibly up to a field equation. Obviously, the coordinate transformation coincides with (6.9) but the correct Lorentz transformation is only reproduced provided that $a-b=2g$. If we now define $m=\frac{1}{2}(a+b)$, so that the last term is just the Dirac equation with mass m, we find

$$a = m+g\,, \qquad b = m-g\,. \quad (6.4)$$

Consequently, the supersymmetry transformation of ψ equals

$$\delta\psi = \not{D}(A+i\gamma_5 B)\epsilon - m(A+i\gamma_5 B)\epsilon - g(A-i\gamma_5 B)\,\epsilon\,, \quad (6.5)$$

and the fermionic field equation equals $(\not{D}+m)\psi=0$. The second term in (6.5), which is proportional to m, can be accounted for by introducing an auxiliary field to the supermultiplet. The third term, which is proportional to g, can be understood as a compensating S-supersymmetry transformation associated with auxiliary fields in the supergravity sector (see, *e.g.*, [81]). In order to construct the corresponding field equations for A and B, we consider the variation of the fermionic field equation. Again we have to take into account that derivatives on the supersymmetry parameter are not equal to zero. This yields the following second-order differential equations,

$$\begin{aligned}[] [\Box_{\rm adS} + 2g^2 - m(m-g)]\,A &= 0\,,\\ [\Box_{\rm adS} + 2g^2 - m(m+g)]\,B &= 0\,,\\ [\Box_{\rm adS} + 3g^2 - m^2]\,\psi &= 0\,. \end{aligned} \quad (6.6)$$

The last equation follows from the Dirac equation. Namely, one evaluates $(\not{D}-m)(\not{D}+m)\psi$, which gives rise to the wave operator $\Box_{\rm adS}+\frac{1}{2}[\not{D},\not{D}]-m^2$. The commutator yields the Riemann curvature of the anti-de Sitter space. In an anti-de Sitter space of arbitrary dimension D this equation then reads,

$$[\Box_{\rm adS} + \tfrac{1}{4}D(D-1)g^2 - m^2]\psi = 0\,, \quad (6.7)$$

which, for $D=4$ agrees with the last equation of (6.6). A striking feature of the above result is that the field equations (6.6) all have different mass terms, in spite of the fact that they belong to the same supermultiplet [82]. Consequently, the role of mass is quite different in anti-de Sitter space as compared to flat Minkowski space. This will be elucidated later in section 6.2.

For future applications we also evaluate the Proca equation for a massive vector field,

$$D^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) - m^2\, A_\nu = 0\,. \tag{6.8}$$

This leads to[27] $D^\mu A_\mu = 0$, so that (6.8) reads $D^2 A_\nu - [D^\mu, D_\nu]A_\mu - m^2\, A_\nu = 0$ or, in anti-de Sitter space,

$$[\Box_{\mathrm{adS}} + (D-1)g^2 - m^2]\, A_\mu = 0\,. \tag{6.9}$$

This can be generalized to an antisymmetric tensor of rank n, whose field equation reads (antisymmetrizing over indices $\nu_1, \ldots, \nu_n$),

$$(n+1)\, D^\mu \partial_{[\mu} C_{\nu_1\cdots\nu_n]} - m^2\, C_{\nu_1\cdots\nu_n} = 0\,. \tag{6.10}$$

In the same way as before, this leads to

$$[\Box_{\mathrm{adS}} + n(D-n)g^2 - m^2]\, C_{\nu_1\cdots\nu_n} = 0\,. \tag{6.11}$$

The g^2 term in the field equations for the scalar fields can be understood from the observation that the scalar D'Alembertian (in an arbitrary gravitational background) can be extended to a conformally invariant operator (see, *e.g.*, [81]),

$$\Box + \frac{1}{4}\frac{D-2}{D-1}\, R = \Box + \tfrac{1}{4}D(D-2)\, g^2\,, \tag{6.12}$$

which seems the obvious candidate for a massless wave operator for scalar fields. Indeed, for $D=4$, we do reproduce the g^2 dependence in the first two equations (6.6). Observe that the Dirac operator $\not{D}$ is also conformally invariant and so is the wave equation associated with the Maxwell field.

Using (6.8) we can now determine the values for the quadratic Casimir operator for the representations described by scalar, spinor, vector and tensor fields. The quadratic Casimir operator of the Lorentz group takes the values 0, $\frac{1}{8}D(D-1)$, $D-1$ and $n(D-n)$ for scalar, spinor, vector and

[27]When $m \neq 0$, otherwise one can impose this equation as a gauge condition.

tensor fields respectively. Combining this result with (6.12), (6.7), (6.9) and (6.10), (6.8) yields the following values for the quadratic Casimir operators,

$$
\begin{aligned}
\mathcal{C}_2^{\text{scalar}} &= -\tfrac{1}{4}D(D-2) + m^2\,, \\
\mathcal{C}_2^{\text{spinor}} &= -\tfrac{1}{8}D(D-1) + m^2\,, \\
\mathcal{C}_2^{\text{vector}} &= m^2\,, \\
\mathcal{C}_2^{\text{tensor}} &= m^2\,.
\end{aligned}
\tag{6.13}
$$

For spinless fields, m^2 is *not* the coefficient in the mass term of the Klein-Gordon equation, as that is given by the value for $\mathcal{C}_2^{\text{scalar}}$, while, for spinor and vector fields, m and m^2 do correspond to the mass terms in the Dirac and Proca equations, respectively. We thus derive specific values for $\mathcal{C}_2$ for massless scalar, spinor, vector and tensor fields upon putting $m = 0$. The fact that the value of $\mathcal{C}_2$ does not depend on the rank of the tensor field is in accord with the fact that, for $m^2 = 0$, a rank-n and a rank-$(D-n-2)$ tensor gauge field are equivalent on shell (also in curved space).

Hence, we see that the interpretation of the mass parameter is not straightforward in the context of anti-de Sitter space. In the next section we will derive a rather general lower bound on the value of $\mathcal{C}_2$ for the lowest-weight representations of the anti-de Sitter algebra (*c.f.* (6.26)), which implies that the masslike terms for scalar fields can have a negative coefficient μ^2 subject to the inequality,

$$
\mu^2 \geq -\tfrac{1}{4}(D-1)^2\,. \tag{6.14}
$$

This result is known as the the Breitenlohner-Freedman bound [82], which ensures the stability of an anti-de Sitter background against small fluctuations of the scalar fields. For spin-$\frac{1}{2}$ the bound on $\mathcal{C}_2$ implies that $m^2 \geq 0$, whereas for spin-0 we find that $m^2 \geq -\frac{1}{4}$, with m^2 as defined in (6.13).

In the next section we study unitary representations of the anti-de Sitter algebra and this study will confirm some of the results found above. For massless representations of higher spin there is a decoupling of degrees of freedom, which uniquely identifies the massless representations and their values of $\mathcal{C}_2$. In a number of cases this decoupling is more extreme and one obtains a so-called singleton representation which does not have a smooth Poincaré limit. In those cases there is no decoupling of a representation that could be identified as massless and therefore there remains a certain ambiguity in the definition of 'massless' representations. This can also be seen from the observation that (massless) antisymmetric tensor gauge fields of rank $n = D-2$ are on-shell equivalent to massless scalar fields. While we concluded above that these tensor fields lead to $\mathcal{C}_2 = 0$, massless scalar fields have $\mathcal{C}_2 = -\frac{1}{4}D(D-2)$ according to (6.13). The difference may not be entirely surprising in view of the fact that the antisymmetric tensor Lagrangian is not conformally invariant for arbitrary values of D, contrary to the scalar field Lagrangian. At any rate, we have established the existence of two different field representations that describe massless, spinless states which correspond to different values for the anti-de Sitter Casimir operator $\mathcal{C}_2$. At the end of the next section, where we discuss unitary representations of the anti-de Sitter algebra, we briefly return to the issue of massless representations. The connection between the local field theory description and the anti-de Sitter representations tends to be subtle.

6.2 Unitary representations of the anti-de Sitter algebra

In this section we discuss unitary representations of the anti-de Sitter algebra. We refer to [83] for some of the original work, and to [84, 85, 80] where part of this work was reviewed. In order to underline the general features we will stay as much as possible in general spacetime dimensions $D > 3$.[28] The anti-de Sitter isometry group, $\mathrm{SO}(D-1,2)$, is noncompact, which implies that unitary representations will be infinitely dimensional. For these representations the generators are anti-hermitean,

$$M^\dagger_{AB} = -M_{AB}\,. \tag{6.15}$$

Here we note that the cover group of $\mathrm{SO}(D-1,2)$ has the generators $\frac{1}{2}\Gamma_{\mu\nu}$ and $\frac{1}{2}\Gamma_\mu$, acting on spinors which are finite-dimensional objects. These generators, however, have different hermiticity properties.

The compact subgroup of the anti-de Sitter group is $\mathrm{SO}(2)\times\mathrm{SO}(D-1)$ corresponding to rotations of the compact anti-de Sitter time and spatial rotations. It is convenient to decompose the $\frac{1}{2}D(D+1)$ generators as follows. First, the generator M_{-0} is related to the energy operator when the radius of the anti-de Sitter space is taken to infinity. The eigenvalues of this generator, associated with motions along the circle, are quantized in integer units in order to have single-valued functions, unless one passes to the covering space CadS. The energy operator H will thus be defined as

$$H = -iM_{-0}\,. \tag{6.16}$$

Obviously the generators of the spatial rotations are the operators M_{ab} with $a, b = 1, \ldots, D-1$. Note that we have changed notation: here and henceforth

[28]The case of $D = 3$ is special because $\mathrm{SO}(2,2) \cong (\mathrm{SL}(2,\mathrm{R})\times\mathrm{SL}(2,\mathrm{R}))/\mathbf{Z}_2$.

in this chapter the indices $a, b, \ldots$ refer to space indices. The remaining generators M_{-a} and M_{0a} are combined into $D-1$ pairs of mutually conjugate operators,

$$M_a^{\pm} = -iM_{0a} \pm M_{-a}\,, \tag{6.17}$$

satisfying $(M_a^+)^\dagger = M_a^-$. The anti-de Sitter commutation relations then read,

$$\begin{aligned}
[H, M_a^{\pm}] &= \pm M_a^{\pm}\,, \\
[M_a^{\pm}, M_b^{\pm}] &= 0\,, \\
[M_a^+, M_b^-] &= -2(H\,\delta_{ab} + M_{ab})\,.
\end{aligned} \tag{6.18}$$

Clearly, the $M_a^{\pm}$ play the role of raising and lowering operators: when applied to an eigenstate of H with eigenvalue E, application of $M_a^{\pm}$ yields a state with eigenvalue $E \pm 1$. We also give the Casimir operator in this basis,

$$\begin{aligned}
\mathcal{C}_2 &= -\tfrac{1}{2}M^{AB}M_{AB} \\
&= H^2 - \tfrac{1}{2}\{M_a^+, M_a^-\} - \tfrac{1}{2}(M_{ab})^2 \\
&= H(H - D + 1) + J^2 - M_a^+ M_a^-\,,
\end{aligned} \tag{6.19}$$

where J^2 is the total spin operator: the quadratic Casimir operator of the rotation group $\mathrm{SO}(D-1)$, defined by

$$J^2 = -\tfrac{1}{2}(M_{ab})^2\,. \tag{6.20}$$

In simple cases, its value is well known. For $D = 4$ it is expressed in terms of the 'spin' s which is an integer for bosons and a half-integer for fermions and the spin-s representation has dimension $2s + 1$ and $J^2 = s(s + 1)$. For $D = 5$, the corresponding rotation group SO(4) is the product of two SU(2) groups, so that irreducible representations are characterized by two spin values, (s_+, s_-). Their dimension is equal to $(2s_+ + 1)(2s_- + 1)$ and $J^2 = 2(J_+^2 + J_-^2)$ with $J_\pm^2 = s_\pm(s_\pm + 1)$. Summarizing:

$$J^2 = \begin{cases} s(s+1) & \text{for} \quad D = 4\,, \\ 2s_+(s_+ + 1) + 2s_-(s_- + 1) & \text{for} \quad D = 5\,. \end{cases} \tag{6.21}$$

The $\mathrm{SO}(D-1)$ representations for $D > 5$ are specified by giving the eigenvalues of additional (higher-order) $\mathrm{SO}(D-1)$ Casimir operators. A restricted class of representations will be discussed in a sequel; a more general discussion of all possible representations requires a more technical set-up and is outside the scope of these lectures.

In this section we restrict ourselves to the bosonic case, but in passing, let us already briefly indicate how some of the other (anti-)commutators of the simple anti-de Sitter superalgebra decompose (c.f. (6.5)),

$$\begin{aligned}
\{Q_\alpha\,,Q^\dagger_\beta\} &= H\,\delta_{\alpha\beta} - \tfrac{1}{2}iM_{ab}\,(\Gamma^a\Gamma^b\Gamma^0)_{\alpha\beta} \\
&\quad +\tfrac{1}{2}(M^+_a\,\Gamma^a\,(1+i\Gamma^0) + M^-_a\,\Gamma^a\,(1-i\Gamma^0))_{\alpha\beta}\,, \\
[H\,,Q_\alpha] &= -\tfrac{1}{2}i(\Gamma^0\,Q)_\alpha\,, \\
[M^\pm_a\,,Q_\alpha] &= \mp\tfrac{1}{2}(\Gamma_a(1\mp i\Gamma^0)\,Q)_\alpha\,.
\end{aligned} \tag{6.22}$$

For the anti-de Sitter superalgebra, all the bosonic operators can be expressed as bilinears of the supercharges, so that in principle one could restrict oneself to fermionic operators only and employ the projections $(1\pm i\Gamma^0)Q$ as the basic lowering and raising operators. This will be discussed later in section 6.3.

We now turn to irreducible representations of the anti-de Sitter algebra (6.18). We start with the observation that the energy operator can be diagonalized so that we can label the states according to their eigenvalue E. Because application of $M^\pm_a$ leads to the states with higher and lower eigenvalues E, we expect the representation to cover an infinite range of eigenvalues, all separated by integers. For a unitary representation the $M^+_aM^-_a$ term in (6.19) is positive, which implies that the Casimir operator is bounded by

$$\mathcal{C}_2 \leq -\tfrac{1}{4}(D-1)^2 + \Big[J^2 + \Big(E-\tfrac{1}{2}(D-1)\Big)^2\Big]_{\rm minimal}\,, \tag{6.23}$$

where the subscript indicates that one must choose the minimal value that $J^2+(E-\frac{1}{2}(D-1))^2$ takes in the representation. Among other things, this number will depend on whether the eigenvalues E take integer or half-integer values.

Continuous representations cover the whole range of eigenvalues E extending from $-\infty$ to ∞. However, when there is a state with some eigenvalue E_0 that is annihilated by all the M^-_a, then only states with eigenvalues $E>E_0$ will appear in the representation. This is therefore not a continuous representation but a so-called lowest-weight representation. The ground state of this representation (which itself transforms as an irreducible representation of the rotation group and may thus be degenerate) is denoted by $|E_0,J\rangle$ and satisfies

$$M^-_a\,|E_0,J\rangle = 0\,. \tag{6.24}$$

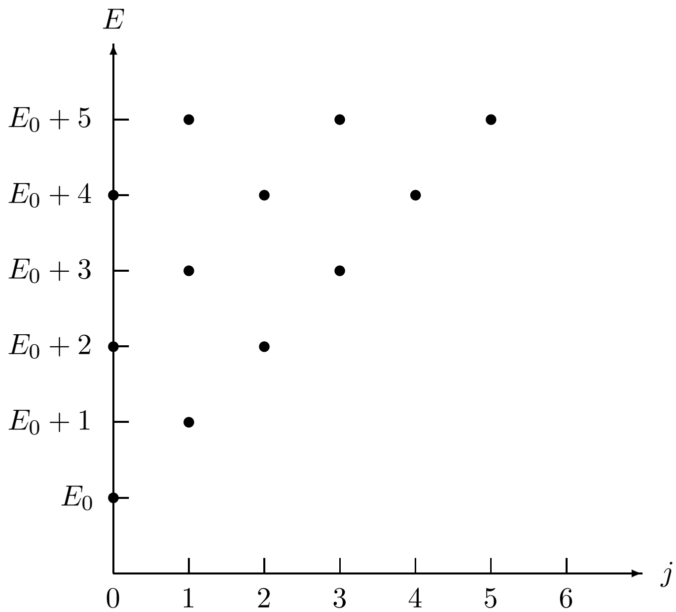

Figure 1: States of the spinless representation in terms of the energy eigenvalues E and the angular momentum j. Each point corresponds to the spherical harmonics of S^{D-1}: traceless, symmetric tensors $Y^{a_1 \cdots a_l}$ of rank $l = j$.

The unitarity upper bound (6.23) on $\mathcal{C}_2$ is primarily useful for continuous representations. For unitary lowest-weight representations one can derive various lower bounds, as we shall see below. Substituting the condition (6.24) in the expression (6.18) applied to the ground state $|E_0, J\rangle$, we derive at once the eigenvalue of the quadratic Casimir operator associated with this representation in terms of E_0 and J^2,

$$\mathcal{C}_2 = E_0(E_0 - D + 1) + J^2 \,. \tag{6.25}$$

Since $\mathcal{C}_2$ is a Casimir operator, this value holds for any state belonging to the corresponding irreducible representation. For real values of E_0 the Casimir operator is bounded by

$$\mathcal{C}_2 \geq J^2 - \tfrac{1}{4}(D-1)^2 \,. \tag{6.26}$$

As we already discussed at the end of the previous section, for scalar fields ($J^2 = 0$) this is just the Breitenlohner-Freedman bound [82]. Additional restrictions based on unitarity will be derived shortly. They generally lead to a lower bound for E_0 and thus to a corresponding lower bound for $\mathcal{C}_2$.

representation	$Y^{A_1\cdots A_l}$	$Y^{B;A_1\cdots A_l}$
D	$l(l+D-3)$	$(l+D-4)(l+1)$
$D=4$	$s=l$	$s=l$
$D=5$	$s_\pm=\frac{1}{2}l$	$s_\pm=s_\mp+1=\frac{1}{2}(l+1)$

Table 17: Two generic SO($D-1$) representations. One is the symmetric traceless tensor representation (corresponding to the spherical harmonics on S^{D-2}) denoted by l-rank tensors $Y^{A_1\cdots A_l}$, and the representation spanned by mixed tensors $Y^{B;A_1\cdots A_l}$ of rank $l+1$ (which is not independent for $D=4$). We list the corresponding eigenvalues of the quadratic Casimir operator J^2, for general D. For $D=4$ these representations are characterized by an integer spin s. For $D=5$ there are two such numbers, $s_\pm$, as we explained in the text.

Unless this bound supersedes (6.26) there can exist a degeneracy in the sense that there are two possible, permissible values for E_0 with the same value for $\mathcal{C}_2$. These two values correspond to two different solutions of the field equations subject to different boundary conditions at spatial infinity.

In what follows we restrict ourselves to lowest-weight representations, because those have a natural interpretation in the limit of large anti-de Sitter radius in terms of Poincaré representations. Alternatively we can construct highest-weight representations, but those will be similar and need not to be discussed separately.

The full lowest-weight representation can now be constructed by acting with the raising operators on the ground state $|E_0,J\rangle$. To be precise, all states of energy $E=E_0+n$ are constructed by an n-fold product of creation operators M_a^+. In this way one obtains states of higher eigenvalues E with higher spin. The simplest case is the one where the vacuum has no spin ($J=0$). For given eigenvalue E, the states decompose into the state of the highest spin generated by the traceless symmetric product of $E-E_0$ operators M_a^+ and a number of lower-spin descendants. These states are all shown in fig. 1.

In the following we consider a number of representations of SO($D-1$) that exist for any dimension. For the bosons we consider the spherical harmonics, spanned by l-rank traceless, symmetric tensors $Y^{a_1\cdots a_l}$. Multiplying such tensors with the vector representation gives rise to two of these representations with rank $l\pm1$, and a 'mixed' representation, spanned by mixed tensors $Y^{b;a_1\cdots a_l}$. Table 17 lists the value of J^2 for these representations, for general D and for the specific cases of $D=4,5$. In a similar table 18 we

representation	$Y^{\alpha;a_1\cdots a_l}$
J^2	$l(l+D-2)+\frac{1}{8}(D-1)(D-2)$
$D=4$	$s=l+\frac{1}{2}$
$D=5$	$s_\pm = s_\mp - \frac{1}{2} = \frac{1}{2}l$

Table 18: The eigenvalues of the quadratic SO$(D-1)$ Casimir operator J^2 for the symmetric tensor-spinor representation spanned by tensors $Y^{\alpha;a_1\cdots a_l}$ for general dimension D and for the specific cases $D=4,5$.

list the value of J^2 for the irreducible symmetric tensor-spinors, denoted by $Y^{\alpha;a_1\cdots a_l}$. They are symmetric l-rank tensor spinors that vanish upon contraction by a gamma matrix and appear when taking products of spherical harmonics with a simple spinor.

Armed with this information it is straightforward to find the decompositions of the spinor representation of the anti-de Sitter algebra. One simply takes the direct product of the spinless representation with a spin-$\frac{1}{2}$ state. That implies that every point with spin j in fig. 1 generates two points with spin $j\pm\frac{1}{2}$, with the exception of points associated with $j=0$, which will simply move to $j=\frac{1}{2}$. The result of this is shown in fig. 2.

However, the spinless and the spinor representations that we have constructed so far are not necessarily irreducible. To see this consider the excited state that has the same spin content as the ground state, but with an energy equal to E_0+2 or E_0+1, for the scalar and spinor representations, respectively, and compare their value for the Casimir operator with that of the corresponding ground state. In this way we find for the scalar

$$E_0(E_0-D+1)=(E_0+2)(E_0-D+3)+\Big|M_a^-|E_0+2,\text{spinless}\rangle\Big|^2 . \quad (6.27)$$

This leads to

$$2E_0+3-D=\tfrac{1}{2}\Big|M_a^-|E_0+2,\text{spinless}\rangle\Big|^2 , \quad (6.28)$$

so that unitary of the representation requires the inequality,

$$E_0 \geq \tfrac{1}{2}(D-3)\,. \quad (6.29)$$

For $E_0=\frac{1}{2}(D-3)$ we have the so-called *singleton* representation[29], where

[29] The singleton representation was first found by Dirac [86] in 4-dimensional anti-de Sitter space and was known as a 'remarkable representation'. In the context of the oscillator method, which we will refer to later, singletons in anti-de Sitter spaces of dimension $D\neq 4$ are called 'doubletons' [87]. In these lectures we will only use the name singleton to denote these remarkable representations.

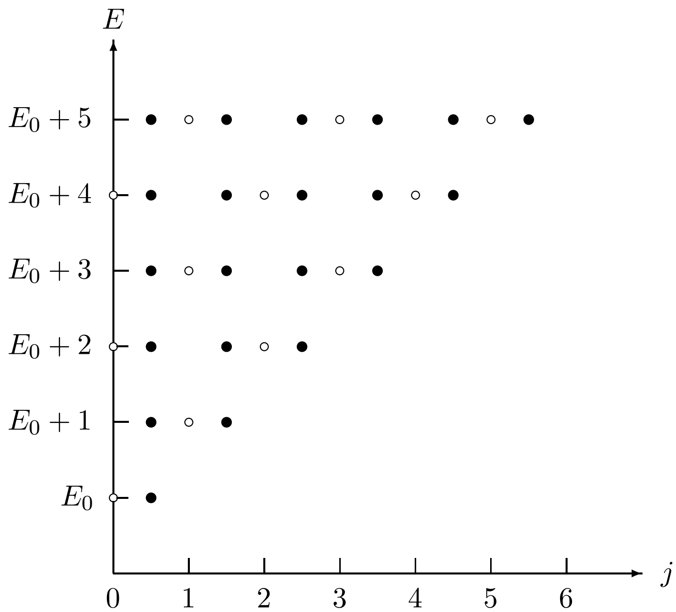

Figure 2: States of the spinor representation in terms of the energy eigenvalues E and the angular momentum; the half-integer values for $j = l + \frac{1}{2}$ denote that we are dealing with a symmetric tensor-spinor of rank l. The small circles denote the original spinless multiplet from which the spinor multiplet has been constructed by a direct product with a spinor.

we have only one state for each given spherical harmonic. The Casimir eigenvalue for this representation equals

$$\mathcal{C}_2(\text{spinless singleton}) = -\tfrac{1}{4}(D+1)(D-3)\,. \tag{6.30}$$

The excited state then constitutes the ground state for a separate irreducible spinless representation, but now with $E_0 = \frac{1}{2}(D+1)$, which, not surprisingly, has the same value for $\mathcal{C}_2$.

For the spinor representation one finds a similar result,

$$\begin{aligned} E_0(E_0 - D + 1) + J^2 \;\; &= \;\; (E_0+1)(E_0 - D + 2) + J^2 \\ &\quad + \Big| M_a^- |E_0 + 1, \text{spinor}\rangle \Big|^2 . \end{aligned} \tag{6.31}$$

As the value for J^2 are the same for the ground state and the excited state one readily derives

$$2E_0 - D + 3 = \Big| M_a^- |E_0 + 1, \text{spinor}\rangle \Big|^2 , \tag{6.32}$$

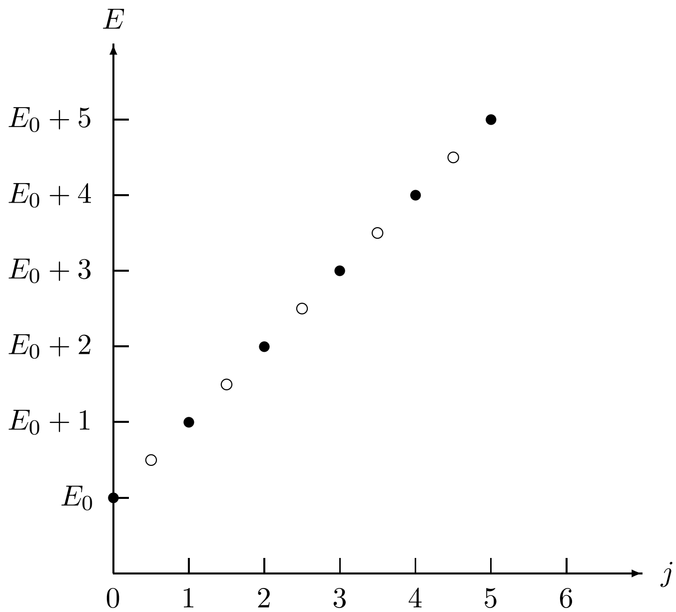

Figure 3: The spin-0 and spin-$\frac{1}{2}$ singleton representations. The solid dots indicate the states of the spin-0 singleton, the circles the states of the spin-$\frac{1}{2}$ singleton. It is obvious that singletons contain much less degrees of freedom than a generic local field. The value of E_0, which denotes the spin-0 ground state energy, is equal to $E_0 = \frac{1}{2}(D-3)$. The spin-$\frac{1}{2}$ singleton ground state has an energy which is one half unit higher, as is explained in the text.

so that one obtains the unitarity bound

$$E_0 \geq \tfrac{1}{2}(D-2)\,. \tag{6.33}$$

For $E_0 = \frac{1}{2}(D-2)$ we have the spinor singleton representation, which again consists of just one state for every value of the total spin. For the spinor representation the value of the Casimir operator equals

$$\mathcal{C}_2(\text{spinor singleton}) = -\tfrac{1}{8}(D+1)(D-2)\,. \tag{6.34}$$

Note that in $D=4$, both singleton representations have the same eigenvalue of the Casimir operator.

The existence of the singletons was first noted by Dirac [86]. These representations are characterized by the fact that they do not exist in the Poincaré limit. To see this, note that Poincaré representations correspond to plane waves which are decomposable into an infinite number of spherical harmonics, irrespective of the size of the spatial momentum (related to the

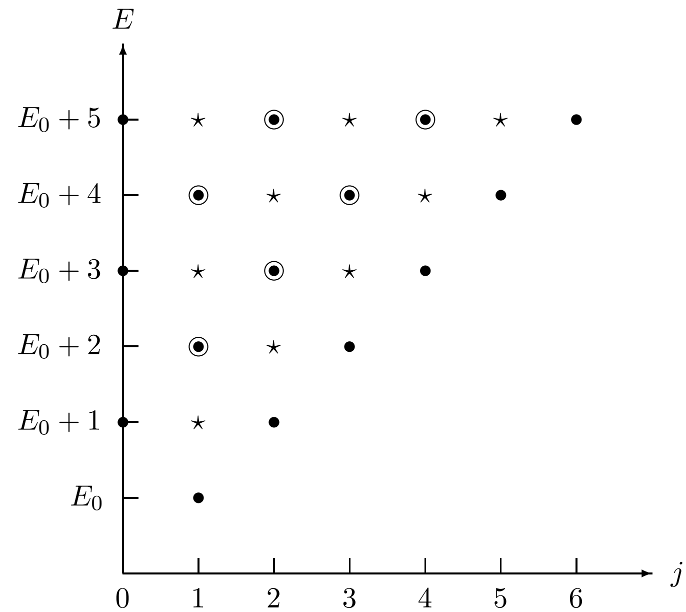

Figure 4: States of the spin-1 representation in terms of the energy eigenvalues E and the angular momentum j. Observe that there are now points with double occupancy, indicated by the circle superimposed on the dots and states transforming as mixed tensors (with $l = j$) denoted by a $\star$. The double-occupancy points exhibit the structure of a spin-0 multiplet with ground state energy $E_0 + 1$. This multiplet becomes reducible and can be dropped when $E_0 = D - 2$, as is explained in the text. The remaining points then constitute a massless spin-1 multiplet, shown in fig. 5.

energy eigenvalue). That means that, for given spin, one is dealing with an infinite, continuous tower of modes, which is just what one obtains in the limit of vanishing energy increments for the generic spectrum shown in, *e.g.*, fig. 1. In contradistinction, the singleton spectrum is different as the states have a single energy eigenvalue for any given value of the spin, as is obvious in fig. 3. Consequently, wave functions that constitute singleton representations do not depend on the radius of the anti-de Sitter spacetime and can be regarded as living on the boundary.

To obtain the spin-1 representation one can take the direct product of the spinless multiplet with a spin-1 state. Now the situation is more complicated, however, as the resulting multiplet contains states of spin lower than that of the ground state. In principle, each point in fig. 1 now generates three points, associated with two spherical harmonics, associated with rank-$j^{\pm}1$

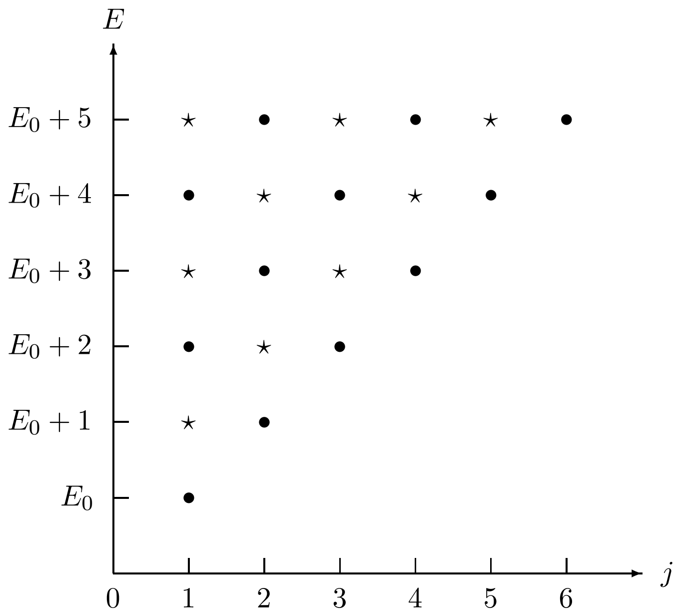

Figure 5: States of the massless $s = 1$ representation in terms of the energy eigenvalues E and the angular momentum j. Now E_0 is no longer arbitrary but it is fixed to $E_0 = D - 2$.

tensors as well as mixed tensors of rank $j + 1$ (so that $l = j$). An exception are the spinless points, which simply move to $j = 1$. The result of taking the product is depicted in fig. 4. This procedure can be extended directly to ground states that transform as a spherical harmonic $Y^{a_1 \cdots a_l}$.

Along the same lines as before, we investigate whether this representation can become reducible for special values of the ground state energy. We compare the value of the Casimir operator for the first excited states with minimal spin to the value for the ground state specified in (6.25). Hence we consider the states with $E = E_0 + 1$ and $j = l - 1$, assuming that the ground state has $l \geq 1$. In that case we find

$$\begin{aligned} \mathcal{C}_2 &= (E_0 + 1)(E_0 - D + 2) + (l - 1)(l + D - 4) - \Big| M_a^- |E_0 + 1, l - 1\rangle \Big|^2 \\ &= E_0(E_0 - D + 1) + l(l + D - 3)\,, \end{aligned} \tag{6.35}$$

so that

$$E_0 - l - D + 3 = \tfrac{1}{2} \Big| M_a^- |E_0 + 1, l - 1\rangle \Big|^2 . \tag{6.36}$$

Therefore we establish the unitarity bound

$$E_0 \geq l + D - 3\,, \qquad (l \geq 1) \tag{6.37}$$

When $E_0 = l + D - 3$, however, the state $|E_0 + 1, l - 1\rangle$ is itself the ground state of an irreducible multiplet, which decouples from the original multiplet together with its corresponding excited states. This can be interpreted as the result of a gauge symmetry. Because these representations have a smooth Poincaré limit they are not singletons and can therefore be regarded as *massless* representations. Hence massless representations with spin $l \geq 1$ are characterized by

$$E_0 = l + D - 3 \,. \qquad (l \geq 1) \tag{6.38}$$

For these particular values the quadratic Casimir operator acquires a minimal value equal to

$$\mathcal{C}_2(\text{massless}) = 2(l-1)(l+D-3) \,. \qquad (l \geq 1) \tag{6.39}$$

We recall that this result is only derived for $l \geq 1$. For certain other cases, the identification of massless representation is somewhat ambiguous, as we already discussed. We return to this issue at the end of this section.

The above arguments can be easily extended to other grounds states, but this requires further knowledge of the various representations of the rotation group, at least for general dimension. This is outside the scope of these lectures. However, in $D = 4, 5$ dimensions this information is readily available. For a spin-s ground state in 4 spacetime dimensions we immediately derive the unitarity bound (for $s > \frac{1}{2}$),

$$E_0 \geq s + 1 \,, \tag{6.40}$$

by following the same procedure as leading to (6.37). When the bound is saturated we obtain a massless representation. The equation corresponding to (6.39) becomes

$$\mathcal{C}_2 = 2(s^2 - 1) \,, \tag{6.41}$$

It turns out that this result applies to all spin-s representations, even to $s = 0, \frac{1}{2}$ conformal fields, for which we cannot use this derivation. This is a special property for $D = 4$ dimensions.

The case of $D = 5$ requires extra attention, because here the rotation group factorizes into two SU(2) groups. We briefly summarize some results. First let us assume the the groundstate has spin (s_+, s_-) with $s_\pm \geq \frac{1}{2}$. In that case we find that the ground state energy satisfies the unitarity bound,

$$E_0 \geq s_+ + s_- + 2 \,. \qquad (s_\pm \geq \tfrac{1}{2}) \tag{6.42}$$

This bound is saturated for massless states, for which the $E = E_0 + 1$ states with spin $(s_+ - \frac{1}{2}, s_- - \frac{1}{2})$ decouples. The corresponding value for the Casimir operator is equal to

$$\mathcal{C}_2 = (s_+ + s_-)^2 + 2s_+(s_+ + 1) + 2s_-(s_- + 1) - 4\,. \tag{6.43}$$

For $s_\pm = \frac{1}{2}l$ these values are in agreement with earlier result.

What remains to be considered are the ground states with spin $(0, s)$. Here we find

$$E_0 \geq 1 + s\,. \tag{6.44}$$

When the bound is saturated we have again a singleton representation. The corresponding values for the Casimir operator are

$$\mathcal{C}_2(\text{singleton}) = 3(s^2 - 1)\,. \tag{6.45}$$

The singleton representations for $s = 0, \frac{1}{2}$ were already found earlier. Note that for $D = 5$ there are thus infinitely many singleton representations, unlike in 4 dimensions, with a large variety of spin values. This is generically the case for arbitrary dimensions $D \neq 4$ and is thus related to the fact that the rotation group is of higher rank.

From the above it is clear that we are dealing with the phenomenon of multiplet shortening for specific values of the energy and spin of the representation, just as in the earlier discussions on BPS multiplets in previous chapters. This phenomenon can be understood from the fact that the $[M_a^+, M_b^-]$ commutator acquires zero or negative eigenvalues for certain values of E_0 and J^2. When viewed in this way, the shortening of the representation is qualitatively similar to the shortening of BPS multiplets based on the anticommutator of the supercharges. Our discussion of the shortening of anti-de Sitter supermultiplets in section 6.3 will support this point of view. The same phenomenon of multiplet shortening is well known and relevant in conformal field theory in $1 + 1$ dimensions.

The purpose of this section was to elucidate the various principles that underlie the anti-de Sitter representations and their relation with the field theory description. Here we are not striving for completeness. There are in fact alternative and often more systematic techniques for constructing the lowest-weight representations. A powerful method to construct the unitary irreducible representations of the anti-de Sitter algebra, is known as the oscillator method [88], which is applicable in any number of spacetime dimensions and which can also be used for supersymmetric extensions of the anti-de Sitter algebra. There is an extensive literature on this; for a recent elementary introduction to this method we refer to [80].

We close this section with a number of comments regarding 'massless' representations and their field-theoretic description. As we demonstrated above, certain representations can, for a specific value of E_0, decouple into different irreducible representations. This phenomenon takes place when some unitarity bound is saturated. In that case one has representations that contain fewer degrees of freedom. When these 'shortened' representations have a smooth Poincaré limit, they are called massless; when they do not, they are called singletons. For the case of spin-0 or spin-$\frac{1}{2}$ representations, for example, the spectrum of states is qualitatively independent of the value for E_0, as long as E_0 does not saturate the unitarity bound and a singleton representation decouples. Therefore the concept of mass remains ambiguous. We have already discussed this in section 6.1, where we emphasized that the absence of mass terms in the field equations is also not a relevant criterion for masslessness. In table 19 we have collected a number of examples of spin-0 and spin-$\frac{1}{2}$ representations with the criteria according to which they can be regarded as massless. One of them is tied to the fact that the corresponding field equation is conformally invariant, as we discussed at the end of section 6.1. Another one follows from the fact that we are dealing with a gauge field. Here the example is an antisymmetric rank-$(D-2)$ gauge field, which is on-shell equivalent to a scalar.

We also invoke a criterion introduced by Günaydin (see [89] and the discussion in [90]), according to which every representation should be regarded as massless that appears in the tensor product of two singleton representations. For instance, it is easy to verify that the product of two spinless singletons leads to an infinite series of higher spin representations that are all massless according to (6.38). However, it also contains the $l=0$ representation with $E_0 = D-3$, to which (6.38) does not apply so that the interpretation as a massless representation is less obvious. It is interesting to consider this criterion for masslessness in $D=5$ dimensions. The tensor product of the singleton representations with spin $(0,s_-)$ and $(s_+,0)$ leads to a ground state with spin (s_+,s_-) and $E_0 = 2+s_++s_-$, which are obviously massless in view of (6.44) and (6.42). Taking the product of two singleton representations, one with spin $(s_1,0)$ and another one with spin $(s_2,0)$ leads to ground states with spin $(s,0)$ and energy $E_0 = 2+s+n$, where n is an arbitrary positive integer. Hence these representations should be regarded as massless.

This interpretation can be tested as follows. In maximal gauged supergravity in 5 dimensions with gauge group SO(6), one of these representations appears as part of the 'massless' supergravity multiplet. This anti-de Sitter representation is described by a (complex) tensor field, whose field equation

spin	E_0	$\mathcal{C}_2$	type
0	$\frac{1}{2}D-1$	$-\frac{1}{4}D(D-2)$	conformal scalar
0	$\frac{1}{2}D$	$-\frac{1}{4}D(D-2)$	conformal scalar
0	$D-3$	$-2(D-3)$	$\in$ singleton $\times$ singleton
0	$D-1$	0	$(D-2)$-rank gauge field
$\frac{1}{2}$	$\frac{1}{2}D-\frac{1}{2}$	$-\frac{1}{8}D(D-1)$	conformal spinor
$\frac{1}{2}$	$D-\frac{5}{2}$	$\frac{1}{8}(D^2-15D+32)$	$\in$ singleton $\times$ singleton

Table 19: Some unitary anti-de Sitter representations of spin 0 and $\frac{1}{2}$ which are massless according to various criteria, and the corresponding values for E_0 and $\mathcal{C}_2$.

takes the form,

$$e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda}\, D_\rho B_{\sigma\lambda} + 2im\, B^{\mu\nu} = 0\,, \tag{6.46}$$

where $m = \pm g$. From this equation one can show that $B_{\mu\nu}$ satisfies (6.10) so that $\mathcal{C}_2 = 1$ (*c.f.* (6.13)). On shell the equation (6.46) projects out the degrees of freedom corresponding to spin $(1,0)$ or $(0,1)$, depending on the sign of m. From this one derives that $E_0 = 3$ (a second solution with $E_0 = 1$ violates the unitarity bound (6.44)).

6.3 The superalgebras $\mathrm{OSp}(N|4)$

In this section we return to the anti-de Sitter superalgebras. We start from the (anti-)commutation relations already presented in (6.18) and (6.22). For definiteness we discuss the case of 4 spacetime dimensions with a Majorana supercharge Q. This allows us to make contact with the material discussed in section 6.1. These anti-de Sitter supermultiplets were first discussed in [82, 91, 84, 85]. In most of the section we discuss simple supersymmetry (*i.e.*, $N=1$), but at the end we turn to more general N. We choose conventions where the 4×4 gamma matrices are given by

$$\Gamma^0 = \begin{pmatrix} -i\mathbf{1} & 0 \\ 0 & i\mathbf{1} \end{pmatrix}\,, \qquad \Gamma^a = \begin{pmatrix} 0 & -i\sigma^a \\ i\sigma^a & 0 \end{pmatrix}\,, \quad a = 1,2,3\,, \tag{6.47}$$

and write the Majorana spinor Q in the form

$$Q = \begin{pmatrix} q_\alpha \\ \varepsilon_{\alpha\beta}\, q^\beta \end{pmatrix}\,, \tag{6.48}$$

where $q^\alpha \equiv q_\alpha^\dagger$, the indices $\alpha, \beta, \dots$ are two-component spinor indices and the σ^a are the Pauli spin matrices. We substitute these definitions into (6.22) and obtain

$$
\begin{aligned}
[H\,, q_\alpha] &= -\tfrac{1}{2} q_\alpha\,, \\
[H\,, q^\alpha] &= \tfrac{1}{2} q^\alpha\,, \\
\{q_\alpha\,, q^\beta\} &= (H\,\mathbf{1} + \vec{J}\cdot\vec{\sigma})_\alpha{}^\beta\,, \\
\{q_\alpha\,, q_\beta\} &= M_a^-\,(\sigma^a\sigma^2)_{\alpha\beta}\,, \\
\{q^\alpha\,, q^\beta\} &= M_a^+\,(\sigma^2\sigma^a)^{\alpha\beta}\,,
\end{aligned}
\tag{6.49}
$$

where we have defined the (hermitean) angular momentum operators $J_a = -\frac{1}{2}\, i\,\varepsilon_{abc} M^{bc}$. We see that the operators q_α and q^α are lowering and raising operators, respectively. They change the energy of a state by half a unit. Observe that the relative sign between H and $\vec{J}\cdot\sigma$ in the third (anti)commutator is not arbitrary but fixed by the closure of the algebra.

In analogy to the bosonic case, we study unitary irreducible representations of the OSp(1|4) superalgebra. We assume that there exists a lowest-weight state $|E_0, s\rangle$, characterized by the fact that it is annihilated by the lowering operators q_α,

$$
q_\alpha |E_0, s\rangle = 0\,. \tag{6.50}
$$

In principle we can now choose a ground state and build the whole representation upon it by applying products of raising operators q^α. However, we only have to study the *antisymmetrized* products of the q^α, because the symmetric ones just yield products of the operators M_a^+ by virtue of (6.49). Products of the M_a^+ simply lead to the higher-energy states in the anti-de Sitter representations of given spin that we considered in section 6.2. By restricting ourselves to the antisymmetrized products of the q^α we thus restrict ourselves to the ground states upon which the full anti-de Sitter representations are build. These ground states are $|E_0, s\rangle$, $q^\alpha|E_0, s\rangle$ and $q^{[\alpha}q^{\beta]}|E_0, s\rangle$. Let us briefly discuss these representations for different s.

The $s = 0$ case is special since it contains less anti-de Sitter representations than the generic case. It includes the spinless states $|E_0, 0\rangle$ and $q^{[\alpha}q^{\beta]}|E_0, 0\rangle$ with ground-state energies E_0 and $E_0 + 1$, respectively. There is one spin-$\frac{1}{2}$ pair of ground states $q^\alpha|E_0, 0\rangle$, with energy $E_0 + \frac{1}{2}$. As we will see below, these states can be described by the scalar field A, the pseudoscalar field B and the spinor field ψ of the scalar chiral supermultiplet, that we studied in section 6.1. Obviously, the bounds for E_0 that we derived in the previous sections should be respected, so that $E_0 > \frac{1}{2}$. For $E_0 = \frac{1}{2}$ the multiplet degenerates and decomposes into a super-singleton, consisting of a spin-0 and a spin-$\frac{1}{2}$ singleton, and another spinless supermultiplet with $E_0 = \frac{3}{2}$.

For $s \geq \frac{1}{2}$ we are in the generic situation. We obtain the ground states $|E_0, s\rangle$ and $q^{[\alpha}q^{\beta]}|E_0, s\rangle$ which have both spin s and which have energies E_0 and $E_0 + 1$, respectively. There are two more (degenerate) ground states, $q^\alpha|E_0, s\rangle$, both with energy $E_0 + \frac{1}{2}$, which decompose into the ground states with spin $j = s - \frac{1}{2}$ and $j = s + \frac{1}{2}$.

As in the purely bosonic case of section 6.2, there can be situations in which states decouple so that we are dealing with multiplet shortening associated with gauge invariance in the corresponding field theory. The corresponding multiplets are then again called massless. We now discuss this in a general way analogous to the way in which one discusses BPS multiplets in flat space. Namely, we consider the matrix elements of the operator $q_\alpha\, q^\beta$ between the $(2s + 1)$-degenerate ground states $|E_0, s\rangle$,

$$\begin{aligned} \langle E_0, s|\, q_\alpha q^\beta \,|E_0, s\rangle &= \langle E_0, s|\{q_\alpha\,, q^\beta\}|E_0, s\rangle \\ &= \langle E_0, s|(E_0\, \mathbf{1} + \vec{J}\cdot\vec{\sigma})_\alpha{}^\beta|E_0, s\rangle\,. \end{aligned} \qquad (6.51)$$

This expression constitutes an hermitean matrix in both the quantum numbers of the degenerate groundstate and in the indices α and β, so that it is $(4s + 2)$-by-$(4s + 2)$. Because we assume that the representation is unitary, this matrix must be positive definite, as one can verify by inserting a complete set of intermediate states between the operators q_α and q^β in the matrix element on the left-hand side. Obviously, the right-hand side is manifestly hermitean as well, but in order to be positive definite the eigenvalue E_0 of H must be big enough to compensate for possible negative eigenvalues of $\vec{J}\cdot\vec{\sigma}$, where the latter is again regarded as a $(4s+2)$-by-$(4s+2)$ matrix. To determine its eigenvalues, we note that $\vec{J}\cdot\vec{\sigma}$ satisfies the following identity,

$$(\vec{J}\cdot\vec{\sigma})^2 + (\vec{J}\cdot\vec{\sigma}) = s(s + 1)\mathbf{1}\,, \qquad (6.52)$$

as follows by straightforward calculation. This shows that $\vec{J}\cdot\vec{\sigma}$ has only two (degenerate) eigenvalues (assuming $s \neq 0$, so that the above equation is not trivially satisfied), namely s and $-(s + 1)$. Hence in order for (6.51) to be positive definite, E_0 must satisfy the inequality

$$E_0 \geq s + 1\,, \quad \text{for } s \geq \tfrac{1}{2}\,. \qquad (6.53)$$

If the bound is saturated, i.e. if $E_0 = s + 1$, the expression on the right-hand side of (6.51) has zero eigenvalues so that there are zero-norm states in the multiplet which decouple. In that case we must be dealing with a massless multiplet. This is the bound (6.40), whose applicability is extended to spin $\frac{1}{2}$. The ground state with $s = \frac{1}{2}$ and $E_0 = \frac{3}{2}$ leads to the massless vector supermultiplet in 4 spacetime dimensions.

As we already mentioned one can also use the oscillator method to construct the irreducible representations. There is an extended literature on this. The reader may consult, for instance, [87, 92].

Armed with these results we return to the masslike terms of section 6.1 for the chiral supermultiplet. The ground-state energy for anti-de Sitter multiplets corresponding to the scalar field A, the pseudo-scalar field B and the Majorana spinor field ψ, are equal to E_0, E_0+1 and $E_0+\frac{1}{2}$, respectively. The Casimir operator therefore takes the values

$$\begin{aligned}
\mathcal{C}_2(A) &= E_0(E_0-3)\,, \\
\mathcal{C}_2(B) &= (E_0+1)(E_0-2)\,, \\
\mathcal{C}_2(\psi) &= (E_0+\tfrac{1}{2})(E_0-\tfrac{5}{2})+\tfrac{3}{4}\,.
\end{aligned} \tag{6.54}$$

For massless anti-de Sitter multiplets, we know that the quadratic Casimir operator is given by (6.41), so we present the value for $\mathcal{C}_2 - 2(s^2-1)$ for the three multiplets, i.e

$$\begin{aligned}
\mathcal{C}_2(A)+2 &= (E_0-1)(E_0-2)\,, \\
\mathcal{C}_2(B)+2 &= E_0(E_0-1)\,, \\
\mathcal{C}_2(\psi)+\tfrac{3}{2} &= (E_0-1)^2\,.
\end{aligned} \tag{6.55}$$

The terms on the right-hand side are not present for massless fields and we should therefore identify them somehow with the common mass parameter m of the supermultiplet. Comparison with the field equations (6.6) shows (for $g=1$) that we obtain the correct contributions provided we make the identification $E_0 = m+1$. Observe that we could have made a slightly different identification here; the above result remains the same under the interchange of A and B combined with a change of sign in m (the latter is accompanied by a chiral redefinition of ψ).

When $E_0 = 2$ there exists, in principle, an alternative field representation for describing this supermultiplet. The spinless representation with $E_0 = 2$ can be described by a scalar field, the spin-$\frac{1}{2}$ representation with $E_0 = \frac{5}{2}$ by a spinor field, and the second spinless representation with $E_0 = 3$ by a rank-2 *tensor* field. The Lagrangian for the tensor supermultiplet is not conformally invariant in 4 dimensions, and this could account for the unusual ground state energy for the spinor representation. We have not constructed this supermultiplet in anti-de Sitter space; in view of the fact that it contains a tensor gauge field, it should be regarded as massless.

From Kaluza-Klein compactifications of supergravity one can deduce that there should also exist shortened *massive* supermultiplets. The reason is that the underlying supergravity multiplet in higher dimensions is shortened because it is massless. When compactifying to an anti-de Sitter ground state with supersymmetry the massless supermultiplets remain shortened by the same mechanism, but also the infinite tower of massive Kaluza-Klein states should comprise shortened supermultiplets. For toroidal compactifications the massive Kaluza-Klein states belong to BPS multiplets whose central charges are the momenta associated with the compactified dimensions. For nontrivial compactifications that correspond to supersymmetric anti-de Sitter ground states, the massive Kaluza-Klein states must be shortened according to the mechanism exhibited in this section. The singleton multiplets decouple from the Kaluza-Klein spectrum. Therefore it follows that there must exist shortened massive representations of the extended supersymmetric anti-de Sitter algebra.

To exhibit this we generalize the previous analysis to the N-extended superalgebra, denoted by $\mathrm{OSp}(N,4)$. As it turns out, the analysis is rather similar. The supercharges now carry an extra $\mathrm{SO}(N)$ index and are denoted by $q_{\alpha i}$ and $q^{\alpha i}$, with $q^{\alpha i} = (q_{\alpha i})^\dagger$ with $i = 1, \ldots, N$. The most relevant change to the (anti)commutators (6.49) is in the third one, which reads

$$\{q_{\alpha i}\,, q^{\beta j}\} = \delta_i^j\,\delta_\alpha^\beta\,H + \delta_i^j\,\vec{J}\cdot\vec{\sigma}_\alpha{}^\beta + \delta_\alpha^\beta\,\vec{T}\cdot\vec{\Sigma}_i{}^j\,, \tag{6.56}$$

where $\vec{T}$ are the hermitean $\frac{1}{2}N(N-1)$ generators of $\mathrm{SO}(N)$ which act on the supercharges in the fundamental representation, generated by the hermitean matrices $\vec{\Sigma}$. The last two anticommutators are given by

$$\begin{aligned}\{q_{\alpha i}\,, q_{\beta j}\} &= M_a^-\,(\sigma^a\sigma^2)_{\alpha\beta}\,\delta_{ij}\,,\\ \{q^{\alpha i}\,, q^{\beta j}\} &= M_a^+\,(\sigma^2\sigma^a)^{\alpha\beta}\,\delta^{ij}\,.\end{aligned} \tag{6.57}$$

The construction of lowest-weight representations proceeds in the same way as before. One starts with a ground state of energy E_0 which has a certain spin and transforms according to a representation of $\mathrm{SO}(N)$ which is annihilated by the $q_{\alpha i}$. Denoting the $\mathrm{SO}(N)$ representation by t (which can be expressed in terms of the eigenvalues of the Casimir operators or Dynkin labels), we have

$$q_{\alpha i}|E_0, s, t\rangle = 0\,. \tag{6.58}$$

Excited states are generated by application of the $q^{\alpha i}$, which are mutually anticommuting, with exception of the combination that leads to the operators M_a^+ which will generate the full anti-de Sitter representations. Hence the generic N-extended representations decompose into ordinary anti-de Sitter representation whose ground states have energy $E_0 + \frac{1}{2}n$ and which can be written as

$$q^{[\alpha_1 i_1} \cdots q^{\alpha_n i_n]} \, |E_0, s, t\rangle \, . \tag{6.59}$$

Here the antisymmetrization applies to the combined (αi) labels. As before the unitarity limits follow from the separate limits on the anti-de Sitter representations and from the right-hand side of the anticommutator (6.56), which decomposes into three terms, namely the Hamiltonian, the rotation generators and the R-symmetry generators, taken in the space of ground state configurations (*c.f.* (6.51)). We have already determined the possible eigenvalues of $\vec{J}\cdot\vec{\sigma}$ which are equal to s or $-(s+1)$. In a similar way one can determine the eigenvalues for $\vec{T}\cdot\vec{\Sigma}$ by noting that it satisfies a polynomial matrix equation such as (6.52) with coefficients determined by the Casimir operators. For instance, for $N=3$ we derive,

$$-(\vec{T}\cdot\vec{\Sigma})^3 + 2(\vec{T}\cdot\vec{\Sigma})^2 + (t^2+t-1)(\vec{T}\cdot\vec{\Sigma}) = t(t+1)\,\mathbf{1}\, , \tag{6.60}$$

where $\vec{T}^2 = t(t+1)\,\mathbf{1}$. This equation shows that the eigenvalues of $\vec{T}\cdot\vec{\Sigma}$ take the values $-t$, 1 or $t+1$, unless $t=0$. Combining these results we find that the right-hand side of (6.56) in the space of degenerate ground state configurations has the following six eigenvalues: E_0+s-t, E_0+s+1, $E_0+s+t+1$, $E_0-s-t-1$, E_0-s or E_0+t+1. All these eigenvalues must be positive, so that in the generic case where s and t are nonvanishing, we derive the unitarity bound, $E_0 \geq 1+s+t$. Incorporating also the possibility that s or t vanishes, the combined result takes the following form,

$$\begin{array}{lll} E_0 \geq 1+s+t & \text{for} & s\geq \frac{1}{2}\,,\ t\geq\frac{1}{2}\,, \\ E_0 \geq 1+s & \text{for} & s\geq\frac{1}{2}\,,\ t=0\,, \\ E_0\geq t & \text{for} & s=0\,,\ t\geq\frac{1}{2}\,. \end{array} \tag{6.61}$$

Whenever one of these bounds is saturated, certain anti-de Sitter representations must decouple. The ground states with $s=0$ and $E_0=t$ define massive shortened representations of the type that appear in Kaluza-Klein compactifications [84]. In the Poincaré limit these representations become all massless.

Obviously these techniques can be extended to other cases, either by changing the number of supersymmetries or by changing the spacetime dimension. There is an extended literature to which we refer the reader for applications and further details.

Before closing the chapter we want to return to the remarkable singleton representations. Long before the formulation of the AdS/CFT correspondence it was realized that supersingleton representations could be described by conformal supersymmetric field theories on a boundary. Two prominent examples were noted (see, *e.g.*, [92, 87]), namely the singleton representations in $D = 5$ and 7 anti-de Sitter space, which correspond to $N = 4, D = 4$ supersymmetric gauge theories and the chiral $(2, 0)$ tensor multiplet in $D = 6$ dimensions. The singletons decouple from the Kaluza-Klein spectrum, precisely because they are related to boundary degrees of freedom. Group-theoretically they are of interest because their products lead to the massless and massive representations that one encountered in the Kaluza-Klein context. Another theme addresses the connection between singletons and higher-spin theories. Here the issue is whether the singletons play only a group-theoretic role or whether they have also a more dynamical significance. We refrain from speculating about these questions and just refer to some recent papers [93, 94, 95]. In [94] the reader may also find a summary of some useful results about singletons as well as an extensive list of references.

In the next chapter we will move to a discussion of superconformal symmetries, which are based on the same anti-de Sitter algebra. We draw the attention of the reader to the fact that in chapter 7, D will always denote the spacetime dimension of the superconformal theory. The corresponding superalgebra is then the anti-de Sitter superalgebra, but in spacetime dimension $D + 1$.

7 Superconformal symmetry

Invariances of the metric are known as isometries. Continuous isometries are generated by so-called Killing vectors, satisfying

$$D_\mu \xi_\nu + D_\nu \xi_\mu = 0\,. \tag{7.1}$$

The maximal number of linearly independent Killing vectors is equal to $\frac{1}{2}D(D + 1)$. A space that has the maximal number of isometries is called maximally symmetric. A weaker condition than (7.1) is,

$$D_\mu \xi_\nu + D_\nu \xi_\mu = \frac{2}{D}\, g_{\mu\nu} D_\rho \xi^\rho\,. \tag{7.2}$$

Solutions to this equation are called *conformal* Killing vectors. Note that the above equation is the traceless part of (7.1). The conformal Killing vectors that are not isometries are thus characterized by a nonvanishing $\xi = D_\mu \xi^\mu$. For general dimension $D > 2$ there are at most $\frac{1}{2}(D+1)(D+2)$ conformal Killing vectors. For $D = 2$ there can be infinitely many conformal Killing vectors. These result can be derived as follows. First one shows that

$$D_\mu D_\nu \xi_\rho = R_{\nu\rho\mu}{}^\sigma\, \xi_\sigma - \frac{1}{D}\Big[g_{\mu\nu}\, D_\rho \xi - g_{\rho\mu}\, D_\nu \xi - g_{\rho\nu}\, D_\mu \xi\Big]\,. \tag{7.3}$$

For Killing vectors (which satisfy $\xi = 0$) this result implies that the second derivatives of Killing vectors are determined by the vector and its first derivatives. When expanding about a certain point on the manifold, the Killing vector is thus fully determined by its value at that point and the values of its first derivatives (which are antisymmetric in view of (7.1)). Altogether there are thus $\frac{1}{2}D(D+1)$ initial conditions to be fixed and they parametrize the number of independent Killing vectors. For conformal Killing vectors, where $\xi \neq 0$ one then proves that $(D-2)D_\mu D_\nu \xi$ and $D^\mu D_\mu \xi$ are determined in terms of lower derivatives. This suffices to derive the maximal number of conformal Killing vectors quoted above for $D > 2$. Both ordinary and conformal Killing vectors generate a group.

In what follows we choose a Minkowski signature for the D-dimensional space, a restriction that is mainly relevant when considering supersymmetry. Flat Minkowski spacetime has the maximal number of conformal Killing vectors, which decompose as follows,

$$\xi^\mu \propto \begin{cases} \xi_{\rm P}^\mu & \text{spacetime translations } (P) \\ \epsilon^\mu{}_\nu\, x^\nu & \text{Lorentz transformations } (M) \\ \Lambda_{\rm D}\, x^\mu & \text{scale transformations } (D) \\ (2\, x^\mu x^\nu - x^2\, \eta^{\mu\nu})\Lambda_{{\rm K}\nu} & \text{conformal boosts } (K) \end{cases} \tag{7.4}$$

Here $\xi_{\rm P}^\mu$, $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$, $\Lambda_{\rm D}$ and $\Lambda_{\rm K}^\mu$ are constant parameters. Obviously $\xi = D(\Lambda_{\rm D} + x_\mu \Lambda_{\rm K}^\mu)$. The above conformal Killing vectors generate the group $\mathrm{SO}(D,2)$. This is the same group as the anti-de Sitter group in $D+1$ dimensions. The case of $D = 2$ is special because in that case the above transformations generate a semisimple group, $\mathrm{SO}(2,2) \cong (\mathrm{SL}(2,\mathrm{R}) \times \mathrm{SL}(2,\mathrm{R}))/Z_2$. This follows directly by writing out the infinitesimal transformations (7.4) for the linear combinations $x \pm t$,

$$\delta(x \pm t) = (\xi_{\rm P}^x \pm \xi_{\rm P}^t) + (\Lambda_{\rm D} \mp \epsilon^{xt})(x \pm t) + \tfrac{1}{2}(\Lambda_{\rm K}^x \mp \Lambda_{\rm K}^t)(x \pm t)^2\,. \tag{7.5}$$

However, for $D = 2$ there are infinitely many conformal Killing vectors, corresponding to two copies of the Virasoro algebra. The corresponding diffeomorphisms can be characterized in terms of two independent functions $f_\pm$ and take the form,

$$\delta x = f_+(x+t) + f_-(x-t)\,, \qquad \delta t = f_+(x+t) - f_-(x-t)\,. \tag{7.6}$$

The fact that, for $D \geq 3$ the anti-de Sitter and the conformal group coincide for dimensions $D+1$ and D, respectively, can be clarified by extending the D-dimensional spacetime parametrized by coordinates x^μ with an extra (noncompact) coordinate y, assuming the line element,

$$\mathrm{d}s^2 = \frac{g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2}{y^2}\,, \tag{7.7}$$

so that the right-hand side of (7.2), which is responsible for the lack of invariance of the line element of the original D-dimensional space, can be cancelled by a scale transformation of extra coordinate y. It is straightforward to derive the nonvanishing Christoffel symbols for this extended space,

$$\{{}_\mu{}^y{}_\nu\} = y^{-1}\, g_{\mu\nu}\,, \quad \{{}_\mu{}^\nu{}_y\} = -y^{-1}\,\delta_\mu^\nu\,, \quad \{{}_y{}^y{}_y\} = -y^{-1}\,, \tag{7.8}$$

where $\{{}_\mu{}^\rho{}_\nu\}$ remains the same for both spaces and all other components vanish. The corresponding expressions for the curvature components are

$$\begin{aligned} R_{\mu\nu\rho}{}^\sigma &= R^D_{\mu\nu\rho}{}^\sigma + 2y^{-2}\, g_{\rho[\mu}\delta^\sigma_{\nu]}\,, \\ R_{\mu y\rho}{}^y &= y^{-2}\, g_{\mu\rho}\,, \\ R_{y\nu y}{}^\sigma &= y^{-2}\delta^\sigma_\nu\,. \end{aligned} \tag{7.9}$$

With these results one easily verifies that the curvature tensor of the $(D+1)$-dimensional extension of a flat D-dimensional Minkowski space is that of an anti-de Sitter spacetime with unit anti-de Sitter radius (i.e. $g = 1$ in (3.14)). This was the reason why we adopted a positive signature in the line element (7.7) for the coordinate y.

Subsequently one can show that the D-dimensional conformal Killing vectors satisfying $D_\mu D_\nu \xi = 0$ can be extended to Killing vectors of the $(D+1)$-dimensional space,

$$\xi^\mu(x,y) = \xi^\mu(x) - \frac{y^2}{2D}\,\partial^\mu\xi(x)\,, \qquad \xi^y(x,y) = \frac{y}{D}\,\xi(x)\,. \tag{7.10}$$

The condition $D_\mu D_\nu \xi = 0$ holds for the conformal Killing vectors (7.4). For $D = 2$ these vectors generate a finite subgroup of the infinite-dimensional conformal group, and only this group can be extended to isometries of the

$(D+1)$-dimensional space. Nevertheless, near the boundary [77] of the space ($y \approx 0$), the conformal Killing vectors generate asymptotic symmetries. Such a phenomenon was first analyzed in [96].

This setting is relevant for the adS/CFT correspondence and there exists an extensive literature on this (see, *e.g.*, [28, 97, 98, 99, 101, 100], and also the lectures presented at this school). Also the relation between the D'Alembertians of the extended and of the original D-dimensional spacetime is relevant in this context. Straightforward calculation yields,

$$\Box_{D+1} = y^2\,\Box^D + (y\,\partial_y)^2 - D\,y\,\partial_y\,. \tag{7.11}$$

Near the boundary where y is small, the fields can be approximated by $y^\Delta\,\phi(x)$. We may compare this to solutions of the Klein-Gordon equation in the anti-de Sitter space, for which we know that the D'Alembertian equals the quadratic Casimir operator $\mathcal{C}_2$. In terms of the ground state energy E_0 of the anti-de Sitter representation, we have $\mathcal{C}_2 = E_0(E_0 - D)$ (observe that we must replace D by $D+1$ in (6.25)), which shows that we have the identification $\Delta = E_0$ or $\Delta = D - E_0$. This identification is somewhat remarkable in view of the fact that E_0 is the energy eigenvalue associated with the SO(2) generator of the anti-de Sitter algebra and not with the noncompact scale transformation of y, which associated with the $\mathrm{SO}(1,1)$ eigenvalue. The identification of the generators is discussed in more detail in the next section.

7.1 The superconformal algebra

From the relation between the conformal and the anti-de Sitter algebra one can determine the superextension of the conformal algebra generated by the above conformal Killing vectors. In comparison to the anti-de Sitter algebra and superalgebra (*c.f.* (6.4) and (6.5)) we make a different decomposition than the one that led to (6.18) and (6.22). We start from a D-dimensional spacetime of coordinates carrying indices $a = 0, 1, \ldots, D-1$, which we extend with *two* extra index values, so that $A = -, 0, 1, \ldots, D-1, D$. For the bosonic generators which generate the group $\mathrm{SO}(D,2)$ we have

$$\begin{aligned} M_{D-} &\longrightarrow D\,,\\ M_{ab} &\longrightarrow M_{ab}\,,\\ M_{Da} &\longrightarrow \tfrac{1}{2}(P_a - K_a)\,,\\ M_{-a} &\longrightarrow \tfrac{1}{2}(P_a + K_a)\,, \end{aligned} \tag{7.12}$$

Here we distinguish the generator D of the dilatations, $\frac{1}{2}D(D-1)$ generators M_{ab} of the Lorentz transformations, D generators P_a of the translations, and D generators K_a of the conformal boosts.

The algebra associated with $\mathrm{SO}(D,2)$ was given in (6.4) and corresponds to the following commutation relations,

$$\begin{aligned} [D,P_a] &= -P_a\,, & [D,K_a] &= K_a\,, \\ [M_{ab},P_c] &= -2\,\eta_{c[a}\,P_{b]}\,, & [M_{ab},K_c] &= -2\,\eta_{c[a}\,K_{b]}\,, \\ [M_{ab},M_{cd}] &= 4\,\eta_{[a[c}\,M_{d]b]}\,, & [P_a,P_b] &= [K_a,K_b]=0\,, \\ [D,M_{ab}] &= 0\,, & [K_a,P_b] &= 2(M_{ab}+\eta_{ab}\,D)\,. \end{aligned} \tag{7.13}$$

To obtain the superextension (for $D\leq 6$) one must first extend the spinor representation associated with the D-dimensional spacetime to incorporate two extra gamma matrices Γ_D and Γ_-. According to the discussion in section 2.5 (see, in particular, table 9) this requires a doubling of the spinor charges,

$$Q\to\mathcal{Q}=\begin{pmatrix} S_\alpha \\ Q_\alpha \end{pmatrix}\,,\qquad \bar{Q}\to\bar{\mathcal{Q}}=(\bar{Q}_\alpha,\bar{S}_\alpha)\,, \tag{7.14}$$

and we define an extended set of gamma matrices Γ_A by,

$$\Gamma_a=\begin{pmatrix}\Gamma^a & 0\\ 0 & -\Gamma^a\end{pmatrix}\qquad \Gamma_D=\begin{pmatrix}0 & \mathbf{1}\\ \mathbf{1} & 0\end{pmatrix}\qquad \Gamma_-=\begin{pmatrix}0 & \mathbf{1}\\ -\mathbf{1} & 0\end{pmatrix}\,. \tag{7.15}$$

The new charges S_α generate so-called *special* supersymmetry transformations [27]. The decomposition of the conjugate spinor is somewhat subtle, to make contact with the Majorana condition employed for the anti-de Sitter algebra.

The anticommutation relation for the spinor charges follows from (6.5) and can be written as

$$\begin{aligned} \{\mathcal{Q},\bar{\mathcal{Q}}\} &= \begin{pmatrix}\{S,\bar{Q}\} & \{S,\bar{S}\}\\ \{Q,\bar{Q}\} & \{Q,\bar{S}\}\end{pmatrix} \\ &= \begin{pmatrix}-\frac{1}{2}\Gamma^{ab}M_{ab}-D & -\Gamma^aK_a\\ -\Gamma^aP_a & -\frac{1}{2}\Gamma_{ab}M_{ab}+D\end{pmatrix}\,, \end{aligned} \tag{7.16}$$

or,

$$\begin{aligned} \{Q_\alpha,\bar{Q}_\beta\} &= -\Gamma^a_{\alpha\beta}\,P_a\,, \\ \{S_\alpha,\bar{S}_\beta\} &= -\Gamma^a_{\alpha\beta}\,K_a\,, \\ \{Q_\alpha,\bar{S}_\beta\} &= -\tfrac{1}{2}\Gamma^{ab}_{\alpha\beta}\,M_{ab}+\eta_{\alpha\beta}\,D\,. \end{aligned} \tag{7.17}$$

The nonvanishing commutators of the spinor charges with the bosonic generators read,

$$\begin{aligned} [M_{ab},\bar{Q}_\alpha] &= \tfrac{1}{2}(\bar{Q}\Gamma_{ab})_\alpha\,, & [M_{ab},\bar{Q}_\alpha] &= \tfrac{1}{2}(\bar{Q}\Gamma_{ab})_\alpha\,, \\ [D,\bar{Q}_\alpha] &= -\tfrac{1}{2}\bar{Q}_a\,, & [D,\bar{S}_\alpha] &= \tfrac{1}{2}\bar{S}_\alpha\,, \\ [K_a,\bar{Q}_\alpha] &= -(\bar{S}\Gamma_a)_\alpha\,, & [P_a,\bar{S}_\alpha] &= -(\bar{Q}\Gamma_a)_\alpha\,. \end{aligned} \tag{7.18}$$

Here we are assuming the same gamma matrix conventions as in the beginning of chapter 3. From the results quoted in the previous chapter, we know that, up to $D = 6$, the bosonic subalgebra will be the sum of the conformal algebra and the R-symmetry algebra. The R-symmetry can be identified from table 9 and the corresponding generators will appear on the righ-hand side of the $\{Q,S\}$ anticommutator; the other (anti)commutation relations listed above remain unchanged. In addition, commutators with the R-symmetry generators must be specified, but those follow from the R-symmetry assignments of the supercharges. The above (anti)commutators satisfy the Jacobi identities that are at most quadratic in the fermionic generators. The validity of the remaining Jacobi identities, which are cubic in the fermionic generators, requires in general the presence of the R-symmetry charges. The results given so far suffice to discuss the most salient features of the superconformal algebra and henceforth we will be ignoring the contributions of the R-symmetry generators. Note also that the numbers of bosonic and fermion generators do not match; this mismatch will in general remain when including the R-symmetry generators.

As before, the matrix on the right-hand side of (7.16) may have zero eigenvalues, leading to shortened supermultiplets. Those multiplets are in one-to-one correspondence with the anti-de Sitter supermultiplets. Its eigenvalues are subject to certain positivity requirements in order that the algebra is realized in a positive-definite Hilbert space.

The abstract algebra can be connected to the spacetime transformations (7.4) in flat spacetime introduced at the beginning of this chapter. To see this we derive how the conformal transformations act on generic fields. In principle, this is an application of the theory of homogeneous spaces discussed in chapter 4 and we will demonstrate this for the bosonic transformations [102]; a supersymmetric extension can be given in superspace. Let us assume that the action of these spacetime transformations denoted by g takes the following form on a generic multicomponent field ϕ,

$$\phi(x)\longrightarrow\phi_g(x) = \mathrm{S}(g,x)\,\phi(g^{-1}x)\,, \tag{7.19}$$

where S is some matrix acting on the components of ϕ. Observe that there exists a subgroup of the conformal group that leaves a point in spacetime invariant and choose, by a suitable translation, this point equal to $x^a = 0$. From (7.4) it then follows that the corresponding stability group of this point is generated by the generators M of the Lorentz group, the generator D of the scale transformations and the generators K of the conformal boosts. Hence we conclude that the matrices $\mathrm{S}(g,0)$ must form a representation of this subgroup, whose generators are denoted by the matrices $\hat{M}_{ab}$, $\hat{D}$ and $\hat{K}_a$. Generic fields are thus assigned to representations of this subgroup.

On the other hand, we want the translation operators to act exclusively on the coordinates x^a, so that according to (7.19) (the generators are anti-hermitean),

$$P_a\,\phi(x) = -\frac{\partial}{\partial x^a}\,\phi(x)\,. \tag{7.20}$$

Hence we may write $\phi(x) = \exp(-x^a P_a)\,\phi(0)$. Subsequently we define the infinitesimal variation $\phi_g(x) \approx \phi(x) + \delta\phi(x)$, where $\delta\phi(x)$ is generated by

$$\delta\phi(x) = \Big[\xi_{\mathrm{P}}^a P_a + \tfrac{1}{2}\epsilon^{ab}M_{ab} + \Lambda_{\mathrm{D}}D + \Lambda_{\mathrm{K}}^a K_a\Big]\phi(x)\,. \tag{7.21}$$

This variation can be converted to the basis defined by the fields at the origin, by sandwiching between $\exp(x^a P_a)$ and $\exp(-x^a P_a)$. The result is then related to the infinitesimal variation of $\mathrm{S}(g,0) \approx \mathbf{1} + \frac{1}{2}\hat{\epsilon}^{ab}\hat{M}_{ab} + \hat{\Lambda}_{\mathrm{D}}\hat{D} + \hat{\Lambda}_{\mathrm{K}}^a\hat{K}$ and terms proportional to P_a. Using the commutation relations (7.13) and using (7.19) and (7.20), it follows that

$$\begin{aligned}\delta\phi(x) \;=\; & -\Big[\xi_{\mathrm{P}}^a - \epsilon^{ab}x_b + \Lambda_{\mathrm{D}}\,x^a - 2x^a\,x_b\Lambda_{\mathrm{K}}^b + x^2\,\Lambda_{\mathrm{K}}^a\Big]\partial_a\phi(x) \\ & +\Big[\Big(\tfrac{1}{2}\epsilon^{ab} - 2\Lambda_{\mathrm{K}}^{[a}x^{b]}\Big)\hat{M}_{ab} + \Big(\Lambda_{\mathrm{D}} - 2\Lambda_{\mathrm{K}}^a x_a\Big)\hat{D} + \Lambda_{\mathrm{K}}^a\hat{K}_a\Big]\phi(x)\,,\end{aligned} \tag{7.22}$$

where the first term represents the conformal Killing vectors parametrized in (7.4). Note that the combination of sign factors is dictated by the algebra (7.13).

The procedure applied above is just a simple example of the construction of induced representations on a G/H coset manifold. Indeed, we are describing flat space as a coset manifold, where the conformal group plays the role of the isometry group G and the stability group plays the role of the isotropy group H. The coset representative equals $\exp(-x^a P_a)$, from which it follows (*c.f.* (4.16)) that the vielbein is constant and diagonal and the connections associated with the stability group are zero. Hence the metric is invariant under the conformal transformations, as established earlier, while the vielbein is invariant after including the compensating transformations represented by the second line of (7.22). Explicit evaluation then shows that the invariance of the flat vielbein requires the compensating tangent-space transformations,

$$\delta e_\mu{}^a \propto \epsilon^{ab}\, e_{\mu,b} - \Lambda_{\rm D}\, e_\mu{}^a \,, \tag{7.23}$$

with parameters specified by (7.22). Note that the special conformal boosts do not act on the tangent space index of the vielbein.

In the next two sections we will discuss how one can deviate from flat space in the context of the conformal group. There are two approaches here which lead to related results. One is to start from a gauge theory of the conformal group. This conformal group has a priori nothing to do with spacetime transformations and the resulting theory is described in some unspecified spacetime. Then one imposes constraints on certain curvatures. This is similar to what we described in section 3, where we imposed a constraint on the torsion tensor (*c.f.* (3.6)), so that the spin connection becomes a dependent field and the Riemann tensor becomes proportional to the curvature of the spin connection field. This approach amounts to imposing the maximal number of conventional constraints. The second approach starts from the coupling to superconformal matter and the corresponding superconformal currents.

7.2 Superconformal gauge theory and supergravity

In principle it is straightforward to set up a gauge theory associated with the superconformal algebra. We start by associating a gauge field to every generator,

$$\begin{array}{lcccccc} \text{generators:} & P & M & D & K & Q & S \\ \text{gauge fields:} & e_\mu^{\ a} & \omega_\mu^{ab} & b_\mu & f_\mu^{\ a} & \psi_\mu & \phi_\mu \\ \text{parameters:} & \xi_{\rm P}^a & \epsilon^{ab} & \Lambda_{\rm D} & \Lambda_{\rm K}^a & \epsilon & \eta \end{array} \tag{7.24}$$

Up to normalization factors, the transformation rules for the gauge fields, which we specify below, follow directly from the structure constants of the superconformal algebra,

$$\begin{aligned}
\delta e_\mu{}^a &= \mathcal{D}_\mu \xi_{\rm P}^a - \Lambda_{\rm D} e_\mu{}^a + \tfrac{1}{2}\bar\epsilon\Gamma^a\psi_\mu \,, \\
\delta\omega_\mu^{ab} &= \mathcal{D}_\mu\epsilon^{ab} + \Lambda_{\rm K}^{[a} e_\mu{}^{b]} - \xi_{\rm P}^{[a} f_\mu{}^{b]} - \tfrac{1}{4}\bar\epsilon\Gamma^{ab}\phi_\mu + \tfrac{1}{4}\bar\psi_\mu\Gamma^{ab}\eta \,, \\
\delta b_\mu &= \mathcal{D}_\mu\Lambda_{\rm D} + \tfrac{1}{2}\Lambda_{{\rm K}a} e_\mu{}^a - \tfrac{1}{2}\xi_{{\rm P}a} f_\mu{}^a + \tfrac{1}{4}\bar\epsilon\phi_\mu - \tfrac{1}{4}\bar\psi_\mu\eta \,, \\
\delta f_\mu{}^a &= \mathcal{D}_\mu\Lambda_{\rm K}^a + \Lambda_{\rm D} e_\mu{}^a + \tfrac{1}{2}\bar\eta\Gamma^a\phi_\mu \,, \\
\delta\psi_\mu &= \mathcal{D}_\mu\epsilon - \tfrac{1}{2}\Lambda_{\rm D}\psi_\mu - \tfrac{1}{2}e_\mu{}^a\Gamma_a\eta + \tfrac{1}{2}\xi_{\rm P}^a\Gamma_a\phi_\mu \,, \\
\delta\phi_\mu &= \mathcal{D}_\mu\eta + \tfrac{1}{2}\Lambda_{\rm D}\phi_\mu - \tfrac{1}{2}f_\mu{}^a\Gamma_a\epsilon + \tfrac{1}{2}\Lambda_{\rm K}^a\Gamma_a\psi_\mu \,.
\end{aligned} \tag{7.25}$$

Here we use derivatives that are covariantized with respect to dilatations and Lorentz transformations, *i.e.*,

$$
\begin{aligned}
\mathcal{D}_\mu \xi_{\rm P}^a &= \partial_\mu \xi_{\rm P}^a + b_\mu \xi_{\rm P}^a - \omega_\mu{}^{ab}\xi_{{\rm P}b}\,,\\
\mathcal{D}_\mu \Lambda_{\rm K}^a &= \partial_\mu \Lambda_{\rm K}^a - b_\mu \Lambda_{\rm K}^a - \omega_\mu{}^{ab}\Lambda_{{\rm K}b}\,,\\
\mathcal{D}_\mu \Lambda_{\rm D} &= \partial_\mu \Lambda_{\rm D}\,,\\
\mathcal{D}_\mu \epsilon &= (\partial_\mu + \tfrac{1}{2} b_\mu - \tfrac{1}{4}\omega_\mu^{ab}\Gamma_{ab})\epsilon\,,\\
\mathcal{D}_\mu \eta &= (\partial_\mu - \tfrac{1}{2} b_\mu - \tfrac{1}{4}\omega_\mu^{ab}\Gamma_{ab})\eta\,.
\end{aligned}
\tag{7.26}
$$

Again we suppressed the gauge fields for the R-symmetry generators.

The above transformation rules close under commutation, up to the commutators of two supersymmetry transformations acting on the fermionic gauge fields. In that case, one needs Fierz reorderings to establish the closure of the algebra, which depend sensitively on the dimension and on the presence of additional generators (for $D = 4$, see, for example, [27]). As an example we list some of the commutation relations that can be obtained from (7.25),

$$
\begin{aligned}
{}[\delta_{\rm P}(\xi_{\rm P}), \delta_{\rm K}(\Lambda_{\rm K})] &= \delta_{\rm D}(\tfrac{1}{2}\Lambda_{\rm K}^a\xi_{\rm P}^b\,\eta_{ab}) + \delta_{\rm M}(\Lambda_{\rm K}^{[a}\xi_{\rm P}^{b]})\,,\\
\{\delta_{\rm Q}(\epsilon_1), \delta_{\rm Q}(\epsilon_2)\} &= \delta_{\rm P}(\tfrac{1}{2}\bar\epsilon_2\Gamma^a\epsilon_1)\,,\\
\{\delta_{\rm S}(\eta_1), \delta_{\rm S}(\eta_2)\} &= \delta_{\rm K}(\tfrac{1}{2}\bar\eta_2\Gamma^a\eta_1)\,,\\
\{\delta_{\rm Q}(\epsilon), \delta_{\rm S}(\eta)\} &= \delta_{\rm M}(\tfrac{1}{4}\bar\epsilon\Gamma^{ab}\eta) + \delta_{\rm D}(-\tfrac{1}{4}\bar\epsilon\eta)\,,\\
{}[\delta_{\rm Q}(\epsilon), \delta_{\rm K}(\Lambda_{\rm K})] &= \delta_{\rm S}(\tfrac{1}{2}\Lambda_{\rm K}^a\Gamma_a\epsilon)\,,\\
{}[\delta_{\rm S}(\eta), \delta_{\rm P}(\Lambda_{\rm P})] &= \delta_{\rm Q}(\tfrac{1}{2}\xi_{\rm P}^a\Gamma_a\eta)\,.
\end{aligned}
\tag{7.27}
$$

For completeness we also present the corresponding curvature tensors of the superconformal gauge theory,

$$
\begin{aligned}
R_{\mu\nu}^a(P) &= 2\,\mathcal{D}_{[\mu}e_{\nu]}^a - \tfrac{1}{2}\bar\psi_{[\mu}\Gamma^a\psi_{\nu]}\,,\\
R_{\mu\nu}^{ab}(M) &= 2\,\partial_{[\mu}\,\omega_{\nu]}^{ab} - 2\,\omega_{[\mu}^{ac}\,\omega_{\nu]\,c}{}^b - 2\,f_{[\mu}^{[a}\,e_{\nu]}^{b]} + \tfrac{1}{2}\bar\psi_{[\mu}\Gamma^{ab}\phi_{\nu]}\,,\\
R_{\mu\nu}(D) &= 2\,\mathcal{D}_{[\mu}b_{\nu]} - f_{[\mu}^a\,e_{\nu]\,a} - \tfrac{1}{2}\bar\psi_{[\mu}\phi_{\nu]}\,,\\
R_{\mu\nu}^a(K) &= 2\,\mathcal{D}_{[\mu}f_{\nu]}^a - \tfrac{1}{2}\bar\phi_{[\mu}\Gamma^a\phi_{\nu]}\,,\\
R_{\mu\nu}(Q) &= 2\,\mathcal{D}_{[\mu}\psi_{\nu]} - e_{[\mu}^{\;a}\,\Gamma_a\phi_{\nu]}\,,\\
R_{\mu\nu}(S) &= 2\,\mathcal{D}_{[\mu}\phi_{\nu]} - f_{[\mu}^{\;a}\,\Gamma_a\psi_{\nu]}\,.
\end{aligned}
\tag{7.28}
$$

These curvature tensors transform covariantly and their transformation rules follow from the structure constants of the superconformal algebra. They also satisfy a number of Bianchi identities which are straightforward to write down. As an example and for future reference we list the first three identities,

$$\begin{aligned}
\mathcal{D}_{[\mu} R^a_{\rho]}(P) + R^{ab}_{[\mu\nu}(M)\, e_{\rho]\,b} - R_{[\mu\nu}(D)\, e_{\rho]}{}^a - \tfrac{1}{2}\bar\psi_{[\rho}\Gamma^a R_{\mu\nu]}(Q) \;&=\; 0\,,\\
\mathcal{D}_{[\mu} R^{ab}_{\nu\rho]}(M) + R^{[a}_{[\mu\nu}(K)\, e_{\rho]}{}^{b]} + R^{[a}_{[\mu\nu}(P)\, f_{\rho]}{}^{b]}&\\
+\tfrac{1}{4}\bar\phi_{[\rho}\Gamma^{ab} R_{\mu\nu]}(Q) + \tfrac{1}{4}\bar\psi_{[\rho}\Gamma^{ab} R_{\mu\nu]}(S) \;&=\; 0\,,\\
\mathcal{D}_{[\mu} R_{\nu\rho]}(D) + \tfrac{1}{2} R^a_{[\mu\nu}(K)\, e_{\rho]\,a} - R^a_{[\mu\nu}(P)\, f_{\rho]\,a}&\\
+\tfrac{1}{4}\bar\phi_{[\rho} R_{\mu\nu]}(Q) - \tfrac{1}{4}\bar\psi_{[\rho} R_{\mu\nu]}(S) \;&=\; 0\,. \qquad (7.29)
\end{aligned}$$

At this stage, the superconformal algebra is not related to symmetries of spacetime. Of course, the gauge fields independently transform as vectors under general coordinate transformations but these transformations have no intrinsic relation with the gauge transformations. This is the reason why, at this stage, there is no need for the bosonic and fermionic degrees of freedom to match, as one would expect for a conventional supersymmetric theory.

There is a procedure to introducing a nontrivial entangling between the spacetime diffeomorphisms and the (internal) symmetries associated with the superconformal gauge algebra, based on curvature constraints. Here one regards the P gauge field $e_\mu{}^a$ as a nonsingular vielbein field, whose inverse will be denoted by $e_a{}^\mu$. This interpretation is in line with the interpretation presented in the previous section, where flat space was viewed as a coset space. In that case, the curvature $R(P)$ has the interpretation of a torsion tensor, and one can impose a constraint $R(P) = 0$, so that the M gauge field ω_μ^{ab} becomes a dependent field, just as in (3.6). The effect of this constraint is also that the P gauge transformations are effectively replaced by general-coordinate transformations. To see this, let us rewrite a P-transformation on $e_\mu{}^a$, making use of the fact that there exists an inverse vielbein $e_a{}^\mu$,

$$\delta e_\mu{}^a = \mathcal{D}_\mu \xi^a_{\rm P} = \partial_\mu \xi^\nu\, e_\nu{}^a - \xi^\nu\, D_\nu e_\mu{}^a + \xi^\nu R^a_{\mu\nu}(P)\,, \qquad (7.30)$$

where $\xi^\mu = \xi^a_{\rm P}\, e_a{}^\mu$. Hence, when imposing the torsion constraint $R(P) = 0$, a P-transformation takes the form of a (covariant) general coordinate transformation. This is completely in line with the field transformations (7.22), where the P-transformations were also exclusively represented by coordinate changes, except that we are now dealing with arbitrary diffeomorphisms.

A constraint such as $R(P) = 0$ is called a *conventional* constraint, because it algebraically expresses some of the gauge fields in terms of the others. Of course, by doing so, the transformation rules of the dependent fields are determined and they may acquire extra terms beyond the original ones presented in (7.25). Because $R(P) = 0$ is consistent with spacetime diffeomorphisms, and the bosonic conformal transformations, the field the field $\omega_\mu{}^{ab}$ will still transform under these symmetries according to (7.25). This is also the case for S-supersymmetry, but not for Q-supersymmetry, because the constraint $R(P) = 0$ is inconsistent with Q-supersymmetry. Indeed, under Q-supersymmetry, the field $\omega_\mu{}^{ab}$ acquires an extra term beyond what was presented in (7.25), which is proportional to $R(Q)$. We will not elaborate on the systematics of this procedure but concentrate on a number of noteworthy features. One of them is that there are potentially more conventional constraints. Inspection of (7.28) shows that constraints on $R(M)$, $R(D)$ and $R(Q)$ can be conventional and may lead to additional dependent gauge fields $f_\mu{}^a$ and ϕ_μ associated with special conformal boosts and special supersymmetry transformations. A maximal set of conventional constraints that achieves just that, takes the form

$$
\begin{aligned}
R^a_{\mu\nu}(P) &= 0\,,\\
e_b{}^b R^{ab}_{\mu\nu}(M) &= 0\,,\\
\Gamma^\mu R_{\mu\nu}(Q) &= 0\,,
\end{aligned}
\tag{7.31}
$$

where, for reasons of covariance, one should include possible modifications of the curvatures due to the changes in the transformation laws of the dependent fields. Other than that, the precise form of the constraints is not so important, because constraints that differ by the addition of other covariant terms result in the addition of covariant terms to the dependent gauge fields, which can easily be eliminated by a field redefinition. Note that $R_{\mu\nu}(D)$ is not independent as a result of the first Bianchi identity on $R^a_{\mu\nu}(P)$ given in (7.29) and should not be constrained.

At this point we are left with the vielbein field $e_\mu{}^a$, the gauge field b_μ associated with the scale transformations, and the gravitino field ψ_μ associated with Q-supersymmetry. All other gauge fields have become dependent. The gauge transformations remain with the exception of the P transformations; we have diffeomorphisms, local Lorentz transformations (M), local scale transformations (D), local conformal boosts (K), Q-supersymmetry and S-supersymmetry. Note that b_μ is the only field that transforms nontrivially on special conformal boosts and therefore acts as a compensator which induces all the K-transformations for the dependent fields. Because the constraints are consistent with all the bosonic transformations, those will not change and will still describe a closed algebra. The superalgebra will, however, not close, as one can verify by comparing the numbers of bosonic and fermionic degrees of freedom. In order to have a consistent superconformal theory one must add additional fields (for a review, see [103]). A practical way to do this makes use of the superconformal multiplet of currents [22], which we will discuss in the next section. This construction is limited to theories with $Q = 16$ supercharges and leads to consistent conformal supergravity theories [104, 105, 22].

We close this section with a comment regarding the number of degrees of freedom described by the above gauge fields. The independent bosonic fields, $e_\mu{}^a$ and b_μ, comprise $D^2 + D$ degrees of freedom, which are subject to the $\frac{1}{2}D^2 - \frac{3}{2}D - 1$ independent, bosonic, gauge invariances of the conformal group. This leaves us with $\frac{1}{2}D(D-1) - 1$ degrees of freedom, corresponding to the independent components of a symmetric, traceless, rank-2 tensor in $D-1$ dimensions, which constitutes an irreducible representation of the Poincaré algebra. This representation is the minimal representation that is required for an off-shell description of gravitons in D spacetime dimensions. A similar off-shell counting argument applies to the fermions, which comprise $(D-2)_{\rm s}$ degrees of freedom after subtracting the gauge degrees of freedom associated with Q- and S-supersymmetry. Here $n_{\rm s}$ denotes the spinor dimension. Hence, the conformal framework is set up to reduce the field representation to the smallest possible one that describes the leading spin without putting the fields on shell. The fact that the fields can exist off the mass shell, implies that they must constitute massive representations of the Lorentz group. Similarly, the supermultiplet of fields on which conformal supergravity is based, comprise the smallest *massive* supermultiplet whose highest spin coincides with the graviton spin.

7.3 Matter fields and currents

In the previous section we described how to set up a consistent gauge theory for conformal supergravity. This theory has an obvious rigid limit, where all the gauge fields are equal to zero, with the exception of the vielbein which is equal to the flat vielbein, $e_\mu{}^a = \delta_\mu^a$. This is the background we considered in section 7.1. In this background we may have (matter) theories that are superconformally invariant under rigid transformations, described by (7.22). Suppose that we couple such a rigidly superconformal matter theory in first order to the gauge fields of conformal supergravity. Hence we write,

$$\begin{aligned}\mathcal{L} \;=&\; \mathcal{L}_{\rm matter} \\ &+ h_\mu^a\,\theta_a{}^\mu + \tfrac{1}{2}\omega_\mu^{ab}\,S^\mu_{ab} + b_\mu\,T^\mu + f_\mu{}^a\,U_a{}^\mu + \bar\psi_\mu J^\mu + \bar\phi_\mu J^\mu_{\rm S}\,, \qquad (7.32)\end{aligned}$$

where h_μ^a denotes the deviation of the vielbein from its flat space value, *i.e.*, $e_\mu{}^a \approx \delta_\mu^a + h_\mu{}^a$. The first term denotes the matter Lagrangian in flat space. The current $\theta_a{}^\mu$ is the energy-momentum tensor. In linearized approximation the above Lagrangian is invariant under *local* superconformal transformations. To examine the consequences of this we need the leading (inhomogeneous) terms in the transformations of the gauge fields (*c.f.* (7.25)),

$$
\begin{array}{rlll}
\text{translations:} & \delta h_\mu^{\ a} = \partial_\mu \xi_{\rm P}^a \,, & & \\
\text{Lorentz:} & \delta\omega_\mu^{ab} = \partial_\mu \epsilon^{ab} \,, & \delta h_\mu^{\ a} = \epsilon^{ab}\,\delta_{\mu\, b} \,, & \\
\text{dilatations:} & \delta b_\mu = \partial_\mu \Lambda_{\rm D} \,, & \delta h_\mu^{\ a} = -\delta_\mu^a\,\Lambda_{\rm D} \,, & \\
\text{conformal boosts:} & \delta f_\mu^{\ a} = \partial_\mu \Lambda_{\rm K}^a \,, & \delta\omega_\mu^{ab} = \Lambda_{\rm K}^{[a}\,\delta_\mu^{b]} \,, & \delta b_\mu = \tfrac12 \Lambda_{{\rm K}\mu} \,, \\
Q\text{-supersymmetry:} & \delta\psi_\mu = \partial_\mu \epsilon \,, & & \\
S\text{-supersymmetry:} & \delta\phi_\mu = \partial_\mu \eta \,, & \delta\psi_\mu = -\tfrac12 \Gamma_\mu \eta \,, &
\end{array}
\tag{7.33}
$$

The variations of the action corresponding to (7.32) under the superconformal transformations, ignoring variations that are proportional to the superconformal gauge fields and assuming that the matter fields satisfy their equations of motion, must vanish. One can verify that this leads to a number of conservation equations for the currents,

$$
\begin{array}{rclrcl}
\partial_\mu \theta_a{}^\mu &=& 0\,, & \partial_\mu U_a{}^\mu - \tfrac12 S^\mu_{a\mu} - \tfrac12 T_a &=& 0\,, \\
\partial_\mu S^\mu_{ab} - 2\,\theta_{[ab]} &=& 0\,, & \partial_\mu J^\mu &=& 0\,, \\
\partial_\mu T^\mu + \theta_\mu{}^\mu &=& 0\,, & \partial_\mu J^\mu_{\rm S} + \tfrac12 \Gamma_\mu J^\mu &=& 0\,,
\end{array}
\tag{7.34}
$$

where we used the flat vielbein to convert world into tangent space indices and vice versa; for instance, we employed the notation $\theta_{ab} = \theta_a{}^\mu\, e_{\mu\, b}$ and $\theta_\mu{}^\mu = \theta_a{}^\mu\, e_\mu{}^a$. Obviously, not all currents are conserved, but we can define a set of conserved currents by allowing an explicit dependence on the coordinates,

$$
\begin{array}{rcl}
\partial_\mu \theta_a{}^\mu &=& 0\,, \\
\partial_\mu \left(S^\mu_{ab} - 2\,\theta_{[a}{}^\mu x_{b]} \right) &=& 0\,, \\
\partial_\mu \left(T^\mu + \theta_a{}^\mu x^a \right) &=& 0\,, \\
\partial_\mu \left(U_a{}^\mu - \tfrac12 S^\mu_{ab} x^b - \tfrac12 T^\mu x^a - \tfrac12 \theta_b{}^\mu \left(x_a x^b - \tfrac12 x^2 \delta_{a)}^b \right) \right) &=& 0\,, \\
\partial_\mu J^\mu &=& 0\,, \\
\partial_\mu \left(J^\mu_{\rm S} + \tfrac12 \Gamma_\nu J^\mu x^\nu \right) &=& 0\,.
\end{array}
\tag{7.35}
$$

In this result one recognizes the various components in (7.22) and in (7.4). For S-supersymmetry one can understand the expression for the current by noting that the following combination of a constant S transformation with a spacetime dependent Q-transformation with $\epsilon = \frac{1}{2}x^\mu\Gamma_\mu\eta$ leaves the gravitino field ψ_μ invariant. Observe that the terms involving the energy-momentum tensor take the form $\theta_a{}^\mu\,\xi^a$, where ξ^a are the conformal Killing vectors defined in (7.4).

So far we have assumed that the gauge fields in (7.32) are independent. However, we have argued in the previous section that it is possible to choose the gauge fields associated with the generators M, K and S, to depend on the other fields. At the linearized level, the fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and ϕ_μ can then be written as linear combinations of curls of the independent gauge fields. After a partial integration, the currents $\theta_a{}^\mu$, T^μ and J^μ are modified by improvement terms: terms of the form $\partial_\nu\,A^{[\nu\mu]}$, which can be included into the currents without affecting their divergence. Hence, the currents S^μ_{ab}, $U_a{}^\mu$ and $J^\mu_{\rm S}$ no longer appear explicitly but are absorbed in the remaining currents as improvement terms. We don't have to work out their explicit form, because we can simply repeat the analysis leading to (7.34), suppressing S^μ_{ab}, $U_a{}^\mu$ and $J^\mu_{\rm S}$. We then obtain the following conditions for the *improved* currents,

$$
\begin{aligned}
\partial^\mu\theta^{\rm imp}_{\mu\nu} = \theta^{\rm imp}_{[\mu\nu]} = \theta^{\rm imp}{}_\mu{}^\mu = 0\,,\\
\partial^\mu J^{\rm imp}_\mu = \Gamma^\mu J^{\rm imp}_\mu = 0\,.
\end{aligned}
\tag{7.36}
$$

Observe that these equations reduce the currents to irreducible representations of the Poincaré group, in accord with the earlier counting arguments given for the gauge fields.

To illustrate the construction of the currents, let us consider a nonlinear sigma model in flat spacetime with Lagrangian,

$$
\mathcal{L} = \tfrac{1}{2}g_{AB}\,\partial_\mu\phi^A\,\partial^\mu\phi^B\,. \tag{7.37}
$$

Its energy-momentum operator can be derived by standard methods and is equal to

$$
\theta_{\mu\nu} = \tfrac{1}{2}g_{AB}\left(\partial_\mu\phi^A\,\partial_\nu\phi^B - \tfrac{1}{2}\eta_{\mu\nu}\,\partial_\rho\phi^A\partial^\rho\phi^B\right)\,. \tag{7.38}
$$

It is conserved by virtue of the field equations; moreover it is symmetric, but not traceless. It is, however, possible to introduce an improvement term,

$$
\begin{aligned}
\theta^{\rm imp}_{\mu\nu} \;=\;& \tfrac{1}{2}g_{AB}(\partial_\mu\phi^A\,\partial_\nu\phi^B - \tfrac{1}{2}\eta_{\mu\nu}\,\partial_\rho\phi^A\partial^\rho\phi^B)\\
&+\frac{D-2}{4(D-1)}\left(\eta_{\mu\nu}\,\partial^2 - \partial_\mu\partial_\nu\right)\chi(\phi)\,.
\end{aligned}
\tag{7.39}
$$

When $\chi(\phi)$ satisfies

$$D_A \partial_B \chi(\phi) = g_{AB} \,, \tag{7.40}$$

the improved energy-momentum tensor is conserved, symmetric and traceless (again, upon using the field equations). This implies that, $\chi_A = \partial_A \chi$ is a homothetic vector.[30] From this result it follows that locally in the target space, χ can be written as

$$\chi = \tfrac{1}{2} g^{AB} \, \chi_A \, \chi_B \,, \tag{7.41}$$

up to an integration constant. Spaces that have such a homothety are cones. To see this, we decompose the target-space coordinates ϕ^A into ϕ and remaining coordinates φ^a, where ϕ is defined by

$$\chi^A \frac{\partial}{\partial \phi^A} = \frac{\partial}{\partial \phi} \,. \tag{7.42}$$

It then follows that $\chi(\phi, \varphi) = \exp[2\phi] \, \hat{\chi}(\varphi)$, where $\hat{\chi}$ is an undetermined function of the coordinates φ^a. In terms of these new coordinates we have $\chi^A = (1, 0, \cdots, 0)$ and $g_{A\phi} = \chi_A = (2\chi, g_{a\phi})$. From this result one proves directly that the metric takes the form,

$$(\mathrm{d}s)^2 = \frac{(\mathrm{d}\chi)^2}{2\chi} + \chi h_{ab}(\varphi) \, \mathrm{d}\varphi^a \mathrm{d}\varphi^b \,, \tag{7.43}$$

where the ϕ-independence of h_{ab} can be deduced directly from (7.40). This result shows that the target space is a cone over a base manifold $\mathcal{M}_{\mathrm{B}}$ parametrized in terms of the coordinates φ^a with metric h_{ab} [106]. In the supersymmetic context it is important to note that, when the cone is a Kähler or hyperkähler space, it must be invariant under U(1) or SU(2). These features play an important role when extending to the supersymmetric case. In that case U(1) or SU(2) must be associated with the R-symmetry of the superconformal algebra.

Coupling the improved energy-momentum tensor (7.39) to gravity must lead to a conformally invariant theory of the nonlinear sigma model and gravity. The relevant Lagrangian reads,

$$e^{-1} \mathcal{L} = \tfrac{1}{2} g_{AB} \, \partial_\mu \phi^A \, \partial^\mu \phi^B - \frac{D-2}{4(D-1)} \, \chi(\phi) \, R \,. \tag{7.44}$$

[30] A homothetic vector satisfies $D_A \chi_B + D_A \chi_B = 2 g_{AB}$ Here we are dealing with an *exact* homothety, for which $D_A \chi_B = D_B \chi_A$, and which can be solved by a potential χ.

Indeed, this Lagrangian is invariant under local scale transformations characterized by the functions $\Lambda_{\rm D}(x)$,

$$\delta_{\rm D}\phi^A = w\Lambda_{\rm D}\chi^A\,, \qquad \delta_{\rm D}g_{\mu\nu} = -2\Lambda_{\rm D}\,g_{\mu\nu}\,. \tag{7.45}$$

where w is the Weyl weight of the scalar fields which is equal to $w=\frac{1}{2}(D-2)$. The transformation of $g_{\mu\nu}$ is in accord with the vielbein scale transformation written down in section 7.2. We should also point out that the coupling with the Ricci scalar can be understood in the context of the results of the previous section. Using the gauge fields of the conformal group, the Lagrangian reads,

$$e^{-1}\,\mathcal{L} = \tfrac{1}{2}g_{AB}\,g^{\mu\nu}(\partial_\mu\phi^A - w\,b_\mu\,\chi^A)\,(\partial_\nu\phi^B - w\,b_\nu\,\chi^B) - \tfrac{1}{2}w\,f_\mu{}^\mu\,\chi\,. \tag{7.46}$$

As one can easily verify from the transformation rules (7.25), this Lagrangian is invariant under local dilatations, conformal boosts and spacetime diffeomorphisms. Upon using the second constraint (7.31) for the gauge field $f_\mu{}^a$ associated with the conformal boosts and setting $b_\mu=0$ as a gauge condition for the conformal boosts, the Lagrangian becomes equal to (7.44), which is still invariant under local dilatations. This example thus demonstrates the relation between improvement terms in the currents and constraints on the gauge fields.

It is possible to also employ a gauge condition for the dilatations. An obvious one amounts to putting χ equal to a constant χ_0, with the dimension of $[\mathrm{mass}]^{D-2}$,

$$\chi = \chi_0\,. \tag{7.47}$$

Substituting the metric (7.43) the Lagrangian then acquires the form,

$$e^{-1}\,\mathcal{L} \propto \tfrac{1}{2}h_{ab}\,\partial_\mu\varphi^a\,\partial^\mu\varphi^b - \frac{D-2}{4(D-1)}\,R\,. \tag{7.48}$$

This Lagrangian describes a nonlinear sigma model with the base manifold $\mathcal{M}_{\rm B}$ of the cone as a target space, coupled to (nonconformal) gravity. The constant χ_0 appears as an overall constant and is inversely proportional to Newton's constant in D spacetime dimensions. Observe that in order to obtain positive kinetic terms, the metric h_{ab} should be negative definite and χ_0 must be positive.

The above example forms an important ingredient in the so-called superconformal multiplet calculus that has been used extensively in the construction of nonmaximal supergravity couplings. There is an extensive literature on this. For an introduction to the 4-dimensional $N=1$ multiplet calculus, see, *e.g.*, [81], for 4-dimensional $N = 2$ vector multiplets and hypermultiplets, we refer to [67, 107].

Acknowledgements
I am grateful to Sergio Ferrara, Murat Günaydin, N.D. Hari Dass, Jan Louis, Francisco Morales, Hermann Nicolai, Peter van Nieuwenhuizen, Soo-Jong Rey, Henning Samtleben, Ergin Sezgin, Kostas Skenderis, Mario Trigiante, Stefan Vandoren and Toine Van Proeyen for many helpful and stimulating discussions.

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