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Dark energy as a kinematic effect

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Resumo

Observações realizadas nas últimas três décadas confirmaram que o universo se encontra em um estado de expansão acelerada. Essa aceleração é atribuída à presença da chamada energia escura, cuja origem permanece desconhecida. A maneira mais simples de se modelar a energia escura consiste em introduzir uma constante cosmológica positiva nas equações de Einstein, cuja solução no vácuo é então dada pelo espaço de de Sitter. Isso, por sua vez, indica que a cinemática subjacente ao espaço-tempo deve ser aproximadamente governada pelo grupo de de Sitter $SO(1,4)$, e não pelo grupo de Poincaré $ISO(1,3)$.

Nesta tese, adotamos tal argumento como base para a conjectura de que o grupo que governa a cinemática local é o grupo de de Sitter, com o desvio em relação ao grupo de Poincaré dependendo ponto-a-ponto do valor de um termo cosmológico variável. Com o propósito de desenvolver tal formalismo, estudamos a geometria de Cartan na qual o espaço modelo de Klein é, em cada ponto, um espaço de de Sitter com o conjunto de pseudo-raios definindo uma função não-constante do espaço-tempo. Encontramos que o tensor de torção nessa geometria adquire uma contribuição que não está presente no caso de uma constante cosmológica. Fazendo uso da teoria das realizações não-lineares, estendemos a classe de simetrias do grupo de Lorentz $SO(1,3)$ para o grupo de de Sitter. Em seguida, verificamos que a estrutura da gravitação teleparalela— uma teoria gravitacional equivalente à relatividade geral— é uma geometria de Riemann-Cartan não linear.

Inspirados nesse resultado, construímos uma generalização da gravitação teleparalela sobre uma geometria de de Sitter–Cartan com um termo cosmológico dado por uma função do espaço-tempo, a qual é consistente com uma cinemática localmente governada pelo grupo de de Sitter. A função cosmológica possui sua própria dinâmica e emerge naturalmente acoplada não-minimalmente ao campo gravitacional, analogamente ao que ocorre nos modelos telaparalelos de energia escura ou em teorias de gravitação escalares-tensoriais. Característica peculiar do modelo aqui desenvolvido, a função cosmológica fornece uma contribuição para o desvio geodésico de partículas adjacentes em queda livre. Embora tendo sua própria dinâmica, a energia escura manifesta-se como um efeito da cinemática local do espaço-tempo.

Abstract

Observations during the last three decades have confirmed that the universe momentarily expands at an accelerated rate, which is assumed to be driven by dark energy whose origin remains unknown. The minimal manner of modelling dark energy is to include a positive cosmological constant in Einstein's equations, whose solution in vacuum is de Sitter space. This indicates that the large-scale kinematics of spacetime is approximated by the de Sitter group $SO(1,4)$ rather than the Poincaré group $ISO(1,3)$.

In this thesis we take this consideration to heart and conjecture that the group governing the local kinematics of physics is the de Sitter group, so that the amount to which it is a deformation of the Poincaré group depends pointwise on the value of a nonconstant cosmological function. With the objective of constructing such a framework we study the Cartan geometry in which the model Klein space is at each point a de Sitter space for which the combined set of pseudoradii forms a nonconstant function on spacetime. We find that the torsion receives a contribution that is not present for a cosmological constant. Invoking the theory of nonlinear realizations we extend the class of symmetries from the Lorentz group $SO(1,3)$ to the enclosing de Sitter group. Subsequently, we find that the geometric structure of teleparallel gravity— a description for the gravitational interaction physically equivalent to general relativity— is a nonlinear Riemann–Cartan geometry.

This finally inspires us to build on top of a de Sitter–Cartan geometry with a cosmological function a generalization of teleparallel gravity that is consistent with a kinematics locally regulated by the de Sitter group. The cosmological function is given its own dynamics and naturally emerges nonminimally coupled to the gravitational field in a manner akin to teleparallel dark energy models or scalar-tensor theories in general relativity. New in the theory here presented, the cosmological function gives rise to a kinematic contribution in the deviation equation for the world lines of adjacent free-falling particles. While having its own dynamics, dark energy manifests itself in the local kinematics of spacetime.

Keywords: dark energy, de Sitter special relativity, teleparallel gravity, cosmological function, Cartan geometry.

Subjects: gravitation, cosmology, special relativity.

Contents

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And different forms of ... make laws in different ways. Some operate democratically; in others the aristocrats rule; and in still others a single tyrant makes the laws. It all depends on their various interests. They all claim that what is advantageous to themselves is justice for the people they rule. Anyone who violates this principle they punish as a lawbreaker, and they brand that person as unjust. That is what I mean, sir, when I say that there exists in all states the same principle of justice, and that is the interest of the established .... In all cases the ... has the power, so the only reasonable conclusion is that everywhere there is but one principle of justice: the interest of the stronger.

Thrasymachos in Republic I, 338d-339a, Plato
(Translation by B. Jowett)

1 | Prolegomena

The gravitational force was the first of the four fundamental interactions in physics to have been given a quantitative description when Sir Isaac Newton stated his law of universal gravitation in 1687. Well over three centuries later the ubiquitous gravitational interaction still lacks a solid understanding both at the smallest of unobservable and the largest of observable length scales, considering that Einstein's general relativity remains in full control in between. At very small distances and high energies, i.e., around the Planck energy density, a quantization of the gravitational interaction is expected indispensable [Pad87, Kie07]. The problem of constructing quantum gravity is thus of direct relevance to understand the physics of the interior of black holes and the stages of the universe closely following the big bang, but trying to solve it head-on might result too ambitious.

It is not impossible that general relativity comes short in describing the evolution of the large-scale structure of the universe, so that it may not be the complete story at the classical level already. Until the nineties of the last century the gravitational force had been observed to be attractive only, such that the expansion of the universe was believed to be decelerating. In 1998, however, comparison of the apparent luminosity and redshift for a set of Type Ia supernovae showed for the first time that the expansion of the universe fairly recently started to accelerate [P^+99, R^+98], i.e., gravity has a repulsive component. According to Einstein's equations such a component can only be the result of some sort of energy whose pressure to density ratio is less than $-1/3$, coined *dark energy* due to this exotic equation of state. The simplest candidate for dark energy is the cosmological constant Λ, which has a pressure to density ratio equal to -1 if interpreted as a perfect fluid. Observations indicate it accounts for about 70% of the total energy density, whereas the remaining 30% is almost completely made up of pressureless dust [A+15]. It is part of the standard model of cosmology— the so-called Λ cold dark matter (ΛCDM) model— that dark energy is the cosmological constant, being the simplest model consistent with the present state of observational data. Nevertheless, the absolute constancy of the cosmological constant due to its inability to interact with other forms of energy, as well as the lack of a physical explanation for its observed value which is close to $10^{-52}\,\mathrm{m}^{-2}$, give it a somewhat artificial appearance unless one is willing to interpret it as another fundamental constant of nature. Alternative models for the cosmological constant to explain the observed dark energy come in a number of varieties. Some of these models minimally couple exotic scalar fields to general relativity, while others modify gravity directly [CST06]. Generally they account for some form of dark energy with a dynamical equation of state that mimics a present-day cosmological constant driving the accelerated expansion of the universe.

The existence of dark energy suggests that the large-scale geometry of spacetime must be considered a deformation of de Sitter instead of Minkowski space, for it is the former that solves Einstein's equations in the absence of further contributions to the energy-momentum density. The kinematics of physics at this scale is consequently expected to be governed by the de Sitter group $SO(1,4)$ in place of the Poincaré group $ISO(1,3)$, for what reason its characterization by special relativity is to be replaced with the more general de Sitter special relativity [ABAP07]. The degree to which the kinematic group is deformed from the Poincaré to the de Sitter group depends on the value of Λ, which in the case of dynamical dark energy must be allowed to become time dependent. In addition, there seems no reason to suppose *a priori* that dark energy should remain homogeneous over scales where the cosmological principle breaks down, such that Λ and the corresponding kinematic group become spacetime dependent in principle. According to this scheme, different regions in spacetime are approximated by de Sitter spaces with different cosmological constants whose values are given by the cosmological function Λ. Because the cosmological function modifies the local kinematics of physics one anticipates it to form an integral part of the geometry of spacetime.

The working objective of this thesis consists in finding the precise mathematical structure to implement this scheme and to apply it in order to construct a generalization of the gravitational interaction coupled to the cosmological function. The set of mathematical tools that are necessary to describe the spacetimes we are after is contained in Cartan geometry, which is a nonhomogeneous version of Klein geometry [Sha97]. Klein geometries describe homogeneous spaces in terms of their Lie symmetry groups, which connect any two of the spaces' points. When these symmetries are retained only between points that are separated by an infinitesimal element, Cartan geometry introduces inhomogeneities that are quantified by nonzero values for the curvature or torsion two-forms or both. Accordingly, the space characterized by the Cartan geometry is at each point approximated by the corresponding homogeneous Klein space. It becomes intuitively clear that we need to consider a Cartan geometry modeled on de Sitter space in order to achieve our purpose, but with the peculiarity that the set of cosmological constants, i.e., the cosmological function, is nonconstant on spacetime.

It appears sensible to have the cosmological function implemented in this manner as the obtained geometry reduces to a Riemann–Cartan geometry when Λ vanishes everywhere. This is nothing but a Cartan geometry modeled on Minkowski space, from which it is well known that its zero-torsion variant is the framework that underlies general relativity. When it is curvature instead of torsion that is turned off, the resulting Riemann–Cartan geometry is used to describe teleparallel gravity. This theory is physically equivalent to general relativity in its description of the gravitational interaction, although conceptually it is rather unlike it [AP12]. A key difference between the two alternatives is that the spin connection of teleparallel gravity accounts for inertial effects only, such that it does not bear any gravitational degrees of freedom. Inertial and gravitational effects are therefore logically separated, which makes the description robust against a hypothetical breakdown of the weak equivalence principle. When aiming for a generalization of the gravitational interaction in the presence of the cosmological function, we shall take as our starting point the description of teleparallel gravity rather than the one of general relativity. This way gravitational degrees of freedom are encoded in the torsion of the geometry, while the spin connection represents inertial effects due to the noninertiality of the frame and kinematic effects due to the cosmological function.

To conclude this introductory chapter we give a brief outline of the thesis.

- Basic tools regarding differentiable manifolds, Lie groups and principal fibre bundles are recalled in §2, which are preliminary to a modern treatment of geometry.
- Geometry is the central theme of §3. The abstract geometries of Ehresmann connections are introduced first, after which the Lie theoretic descriptions of homogeneous Klein geometries and nonhomogeneous Cartan geometries are looked at in detail. We additionally revise the relation between Ehresmann and Cartan connections.
- §4, based on Ref. [Jen14], constructs de Sitter–Cartan geometry, which is a Cartan geometry modeled on de Sitter space so that the cosmological function is nonconstant. Afterwards, a nonlinear realization of the Cartan geometry is considered to render the construction $SO(1,4)$ invariant.
- In §5 we review teleparallel gravity and clarify how its historical interpretation as a gauge theory for the Poincaré translations can be understood in terms of a nonlinear Riemann–Cartan geometry.

- §6, based on Ref. [JP16], introduces de Sitter teleparallel gravity, which generalizes teleparallel gravity for the cosmological function using de Sitter–Cartan geometry. It is shown what the phenomenology is of the kinematic effects due to a nonvanishing cosmological function. Finally, we postulate the dynamics of the gravitational field coupled to the cosmological function.
- The thesis is concluded in §7.

2 | Preliminaries

In this first of three mathematically inclined chapters we provide a selective survey of differentiable manifolds and Lie groups preliminarily to the study of Cartan geometry in the following chapter. This review is mainly based on the first chapter of the beautiful exposition [KN96a] and the book [Nak90].

2.1 Differentiable manifolds

2.1.1 Topological spaces and differentiable atlases

Definition. Let $\mathcal{S}$ be a set and let $\mathcal{P}(\mathcal{S}) = \{S \mid S \subseteq \mathcal{S}\}$ be the power set of $\mathcal{S}$, i.e., the collection of subsets of $\mathcal{S}$. A subset $\tau_{\mathcal{S}}$ of $\mathcal{P}(\mathcal{S})$ is a *topology* on $\mathcal{S}$ if the following is true, namely,

(i) The empty set $\varnothing$ and $\mathcal{S}$ itself are in $\tau_{\mathcal{S}}$;
(ii) The union of any number of elements of $\tau_{\mathcal{S}}$ is an element of $\tau_{\mathcal{S}}$;
(iii) The intersection of any finite number of elements of $\tau_{\mathcal{S}}$ is an element of $\tau_{\mathcal{S}}$.

The pair $(\mathcal{S}, \tau_{\mathcal{S}})$ is called a topological space and the subsets of $\mathcal{S}$ that are in the topology are referred to as *open sets*. The elements of $\mathcal{S}$ are called *points*. We shall generally abbreviate notation and denote the topological space simply by $\mathcal{S}$.

This at first sight rather abstract construction gives us a qualitative notion of closeness between points of a set. This notion is obtained when the topology is used to define convergence of sequences in a topological space.

Definition. A sequence of points $x_1, \dots x_n, \dots = \{x_n\}_{n=1}^{\infty}$ *converges* to the point x if for any open set U that contains x there is a natural number n_U such that the tail $\{x_n\}_{n>n_U}$ is in U. The convergence of the sequence is denoted by $x_n \xrightarrow{\tau_{\mathcal{S}}} x$.

The points $\{x_n\}_{n>n_U}$ are said to be closer to x than x_{n_U}. If V is an open subset of U, then $\{x_n\}_{n>n_V}$ is a subtail of $\{x_n\}_{n>n_U}$ and $n_V \geqslant n_U$, so that the points in V are closer to x than the points in U/V. By considering a countable series of open subsets that contain x, the points of the corresponding subtails get arbitrarily close to x. To assure oneself that this abstraction is natural, it is useful to concretize it for the set $\mathbb{R}^d$ of real d-tuples $x = (x^1, \dots x^\mu, \dots x^d)$.

Example. The *usual topology* on $\mathbb{R}^d$ is constructed as follows. For any point x and any positive real number r, the *open ball* with center x and radius r is defined by

$$B(x, r) = \{y \in \mathbb{R}^d \mid \|x - y\| < r\}.$$

The open sets U that constitute the usual topology are those that for each $x \in U$ encompass an open ball with center x. Consider a convergent sequence $x_n \to x$ with respect to the usual topology. Let a collection of open sets that contain x be given by $B(x, r)$ for any strictly positive r. Since the sequence $\{x_n\}$ converges to x, there is for any radius a number n_r such that $B(x, r)$ contains the tail $\{x_n\}_{n>n_r}$, that is, $\|x_n - x\| < r$ for any $n > n_r$. When r gets arbitrary small, the elements of the sequence contained in $B(x, r)$ come arbitrarily close to x, generally denoted by $\lim_{n\to\infty} x_n = x$.

A subset N of $\mathcal{S}$ is a *neighborhood* of a point x if N encompasses an open set to which x belongs. If N is open it is called an *open neighborhood*. Evidently, the elements of the usual topology on $\mathbb{R}^d$ are open neighborhoods of any of its points.

A topological space $(\mathcal{S}, \tau_{\mathcal{S}})$ is a *Hausdorff space* if for any two points there exist disjoint neighborhoods. The set $\mathbb{R}^d$ with the usual topology is an example of a Hausdorff space. Indeed, for any two points x_1 and x_2 there exist $B(x_i, r)$ with $r \leqslant \|x_1 - x_2\|/2$.

Let $\mathcal{S}$ and $\mathcal{T}$ be topological spaces. A function $\varphi : \mathcal{S} \to \mathcal{T}$ is *continuous* if the preimage of any open subset of $\mathcal{T}$ is open in $\mathcal{S}$, i.e., $\varphi^{-1}(V) \in \tau_{\mathcal{S}}$ whenever $V \in \tau_{\mathcal{T}}$. The function φ is *homeomorphic* if it is a bijection and both φ and φ^{-1} are continuous. This implies that open sets of $\mathcal{S}$ are mapped to open sets of $\mathcal{T}$. When there exists a homeomorphism between two topological spaces they are said to be homeomorphic. In particular, if a mapping $\varphi : \mathcal{S} \to \mathbb{R}^d$ is a homeomorphism, $\mathcal{S}$ is homeomorphic to $\mathbb{R}^d$.

Naturally, not every topological space is homeomorphic to $\mathbb{R}^d$. We are nonetheless interested in topological spaces whose open sets are homeomorphic to the open sets in $\mathbb{R}^d$. To provide a topological space with such a structure, we first define it.

Definition. A *differentiable atlas* of dimension d for a topological space $\mathcal{S}$ is a collection of pairs $\{(U_i, \varphi_i)\}$ for which the following is true:

(i) Every U_i is an open set of $\mathcal{S}$ and $\cup_i U_i = \mathcal{S}$;
(ii) Each φ_i is a homeomorphism from U_i onto an open subset of $\mathbb{R}^d$;
(iii) Whenever $U_i \cap U_j$ is nonempty the homeomorphism $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is differentiable, i.e., of class C^∞.

The pairs (U_i, φ_i) are called *coordinate charts* or just charts. The second item makes sense because the sets U_i are topological subspaces of $\mathcal{S}$ with topologies

$$\tau_{U_i} = \{U_i \cap U \mid U \in \tau_{\mathcal{S}}\}.$$

When the last condition is dropped, the atlas is said to be topological instead of differentiable.

Definition. A *differentiable manifold* $\mathcal{M}$ of dimension d is a Hausdorff space on which is defined a d-dimensional smooth atlas.

The above defined manifold is for obvious reasons also called a *real* differentiable manifold. A complex (analytic) d-dimensional manifold is readily constructed by demanding that the homeomorphisms φ_i of the coordinate charts map onto open subsets of $\mathbb{C}^d$. From now on, by a manifold we shall mean a real differentiable manifold.

The smoothness of the transition functions $\varphi_j \circ \varphi_i^{-1}$ can be written as follows in terms of coordinate charts. Consider a nonempty intersection $U_i \cap U_j$ on which there are defined two coordinate systems $x^\mu = \varphi_i$ and $y^\mu = \varphi_j$ taking values in some open set of $\mathbb{R}^d$. The transition function on this open set takes the form $y^\mu(x^\nu) = \varphi_j \circ \varphi_i^{-1}(x^\nu)$. Differentiability of such a mapping means that the functions $y^\mu(x^\nu)$ are smooth, so that $\partial y^\mu / \partial x^\nu$ and higher order derivatives exist.

A mapping f from a d-dimensional manifold $\mathcal{M}$ to a d'-dimensional manifold $\mathcal{M}'$ is differentiable if for every chart (U_i, φ_i) of $\mathcal{M}$ and every chart (V_j, ψ_j) of $\mathcal{M}'$ such that $f(U_i) \subset V_j$, the $\mathbb{R}^{d'}$-valued function $\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \psi_j(V_j)$ is differentiable. We shall denote the latter equally by f or $f^{\mu'}$, where $\mu' = 1, \dots d'$. If we denote $x^\mu = \varphi_i$ and $y^{\mu'} = \psi_j$, the mapping may be expressed as $y^{\mu'} = f^{\mu'}(x^\mu)$. If f is a bijection and its inverse is differentiable, it is called a *diffeomorphism.* One may conclude that $d = d'$ because all the matrices $\psi_j \circ f \circ \varphi_i^{-1}$ are invertible as well. If the diffeomorphism goes from $\mathcal{M}$ to itself, it is also called a *transformation.*

At this point it is appropriate to dedicate a few lines to the relationship between diffeomorphisms and coordinate transformations on a manifold $\mathcal{M}$, on which there are defined two atlases $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$. For any two overlapping charts (U_i, φ_i) and (V_j, ψ_j) a *coordinate transformation* is defined by

$$\psi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap V_j) \to \psi_j(U_i \cap V_j) : x^\mu \mapsto y^\mu(x^\nu) = \psi_j \circ \varphi_i^{-1}(x^\nu).$$

From the definition of a differentiable atlas it follows that a coordinate transformation is a smooth invertible transformation on $\mathbb{R}^d$. Concretely, the points of $\mathcal{M}$ with coordinates x^μ are given new coordinates y^μ, but the points themselves have not changed. On the other hand, given a diffeomorphism $f : \mathcal{M} \to \mathcal{M}$, we may consider its representation on $\mathbb{R}^d$, namely,

$$\varphi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i \cap f^{-1}(U_j)) \to \varphi_j(f(U_i) \cap U_j) : x^\mu \mapsto y^\mu(x^\nu) = \varphi_j \circ f \circ \varphi_i^{-1}(x^\nu).$$

The point with coordinates x^μ is mapped to another point with coordinates y^μ, whereas the coordinate atlas itself has been left untouched. Diffeomorphisms and coordinate transformations can be identified with each other on subdomains of $\mathbb{R}^d$ by choosing $\psi_j = \varphi_j \circ f$ or $f = \varphi_j^{-1} \circ \psi_j$. Whether one prefers to think about the transformation $x^\mu \mapsto y^\mu(x^\nu)$ as a coordinate transformation (the *passive* point of view) or a diffeomorphism (the *active* point of view) is not relevant mathematically.

2.1.2 Tensor fields

There are two subclasses of mappings on a given manifold $\mathcal{M}$ that are worthwhile to consider explicitly. The first consists of the *curves* x_t, which map open intervals of $\mathbb{R}$ into $\mathcal{M}$, i.e., $t \in (a,b) \mapsto x_t \in \mathcal{M}$. We shall assume that a curve does not intersect with itself, although they may form loops. A second important class of mappings are made up by the functions $f : \mathcal{M} \to \mathbb{R}$, which assign a real number to each point of the manifold. The set of differentiable functions on $\mathcal{M}$ will be given the name $\mathscr{F}(\mathcal{M})$. With the help of a coordinate chart (U_i, φ_i) one readily obtains a curve $\varphi_i \circ x_t$ in $\varphi_i(U_i) \subset \mathbb{R}^d$ and a function $f \circ \varphi_i^{-1}$ on $\varphi_i(U_i)$. We generally shall write them as $x_t^\mu = \varphi_i(x_t)$ and $f(x^\mu) = f \circ \varphi_i^{-1}(x^\mu)$, respectively.

The *tangent vector* X to a curve x_t at the point $x = x_{t_0}$ is the mapping $\mathscr{F}(x_{t_0}) \to \mathbb{R}$ defined by

$$X_x f = \frac{d}{dt}(f \circ x_t)|_{t_0}, \quad \text{for any } f \in \mathscr{F}(x_{t_0}), \tag{2.1.1}$$

In terms of a local coordinate system (U_i, φ_i), this is expressed as $\dot{x}_t^\mu \partial_\mu|_{t_0} f(x^\nu)$, where the dot means differentiation with respect to t. The tangent vector to x_t at any point along the curve is therefore given by

$$X^\mu \frac{\partial}{\partial x^\mu} = \frac{dx_t^\mu}{dt}\frac{\partial}{\partial x^\mu}, \tag{2.1.2}$$

which is the directional derivative along the curve. If (V_j, ψ_j) is a second coordinate chart such that $x = x_{t_0} \in V_j$, one also may write (2.1.1) as $\dot{y}_t^\mu \partial_\mu|_{t_0} f(y^\nu)$. It follows directly that the components of a tangent vector at x transform according to

$$X^\mu \mapsto X^\nu \left.\frac{\partial y^\mu}{\partial x^\nu}\right|_x$$

under the coordinate transformation $x^\mu \mapsto y^\mu(x^\nu) = \psi_j \circ \varphi_i^{-1}(x^\nu)$.

Obviously, at any point x the tangent vector is a linear operator on the space $\mathscr{F}(x)$, i.e., for any two functions f and g it is true that $X(f+g) = X(f) + X(g)$. Furthermore, it satisfies Leibniz's rule, i.e., $X(fg) = X(f)g + fX(g)$. Under the natural addition and scalar multiplication, the set of tangent vectors forms a vector space at any x, which is called the *tangent space* $T_x\mathcal{M}$. From (2.1.2) it is evident that the collection $\{\partial/\partial x^\mu\}$ forms a basis for the tangent space, so that $\dim T_x\mathcal{M} = \dim \mathcal{M}$. For obvious reasons it is called a *coordinate basis*.

A *vector field* X on $\mathcal{M}$ is obtained by smoothly assigning a tangent vector at any point in $\mathcal{M}$. More precisely, the function Xf, being defined pointwise by $(Xf)(x) = X_x f$, is differentiable whenever $f \in \mathscr{F}(\mathcal{M})$. Vector fields are thus mappings of $\mathscr{F}(\mathcal{M})$ onto itself, the set of which we shall denote by $\mathscr{X}(\mathcal{M})$. They constitute an infinite-dimensional vector space under the natural addition and scalar multiplication. Moreover, it is a Lie algebra[1] with the bracket given by $[X,Y]f = X(Yf) - Y(Xf)$, which on a local coordinate patch is expressed as $[X,Y]^\mu = X^\nu\partial_\nu Y^\mu - Y^\nu\partial_\nu X^\mu$. The bracket is a commutator, i.e., $[X,Y] = -[Y,X]$ and therefore satisfies the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

If a collection $\{e_a\}$, with $a = 1, \dots \dim \mathcal{M}$, of vector fields on $\mathcal{M}$ is at any point on which they are defined a basis for the tangent space, it is said to be a *frame*. The commutator of two elements of the frame can at any point be written as a linear combination of the frame fields, i.e.,

$$[e_a, e_b] = c_{ab}{}^c(p) e_c, \tag{2.1.3}$$

where the nonconstant $c_{ab}{}^c$ are called the *structure functions* of the frame. Note that the structure functions of a coordinate basis vanish. When it is possible to define a frame on the whole of $\mathcal{M}$, the manifold is said to be *parallelizable*.

The *total differential* df of a function f is the $\mathscr{F}(\mathcal{M})$-linear functional on the set of vector fields $\mathscr{X}(\mathcal{M})$ that is defined pointwise by

$$df_x(X) = X_x(f), \quad \text{for any } X \in \mathscr{X}(\mathcal{M}). \tag{2.1.4}$$

In a local coordinate system (U_i, φ_i) such that $x^\mu = \varphi_i$, (2.1.4) implies $dx^\mu(\partial_\nu) = \delta^\mu_\nu$.

[1] See §2.2 for the definition for a Lie algebra.

Hence, the set $\{dx^\mu\}$ forms a dual set to the coordinate basis and spans the *cotangent space* $T_p^*\mathcal{M}$, which is equal in dimension to $T_p\mathcal{M}$. Because $df(X) = X^\nu df(\partial_\nu) = X^\nu \partial_\nu f = X^\nu \partial_\mu f dx^\mu(\partial_\nu)$, it is obvious that $df = \partial_\mu f dx^\mu$. Of course, not any element $\omega_\mu dx^\mu$ of the cotangent space can be written as a total differential of a function. A *one-form* $\omega : \mathscr{X}(\mathcal{M}) \to \mathscr{F}(\mathcal{M})$ is a smooth assignment of an element of the cotangent space at any point along $\mathcal{M}$ whereby it is meant that $\omega(X)$ is differentiable whenever X is differentiable. From now on the set of vector fields and the set of one-forms are denoted by $T\mathcal{M} = \mathscr{X}(\mathcal{M})$ and $T^*\mathcal{M}$, respectively. We remark that in a local coordinate system one finds that $\omega(X) = \omega_\mu X^\mu$, from which it is readily concluded that a coordinate transformation $x^\mu \mapsto y^\mu(x^\nu)$ induces at any point a concomitant transformation on the components of the one-form according to $\omega_\mu \mapsto \omega_\nu \partial x^\nu / \partial y^\mu$.

It is then possible to interpret a vector field as a mapping of one-forms into functions by identifying $X(\omega) = \omega(X)$, which in a local coordinate system gives meaning to the equality $\partial_\mu dx^\nu = \delta_\mu^\nu$. Such an interpretation paves the way for the introduction of differentiable multilinear mappings

$$T : \underbrace{T^*\mathcal{M} \times \ldots T^*\mathcal{M}}_{r \text{ times}} \times \underbrace{T\mathcal{M} \times \ldots T\mathcal{M}}_{s \text{ times}} \to \mathscr{F}(\mathcal{M}),$$

which are named (r,s)-type tensor fields and whose collection we shall label by $T_s^r\mathcal{M}$. Elements of $T_0^r\mathcal{M}$ are also called *contravariant* tensor fields, while those of $T_s^0\mathcal{M}$ go by the name of *covariant* tensors. The *tensor product* $\otimes$ of two tensor fields $T \in T_{s_1}^{r_1}\mathcal{M}$ and $S \in T_{s_2}^{r_2}\mathcal{M}$ is the $(r_1 + r_2, s_1 + s_2)$-type tensor field defined through

$$\begin{aligned}(T \otimes S)(\omega_1, \ldots \omega_{r_1}, \eta_1, \ldots \eta_{r_2}, X_1, \ldots X_{s_1}, Y_1, \ldots Y_{s_2})& \\ = T(\omega_1, \ldots \omega_{r_1}, X_1, \ldots X_{s_1}) S(\eta_1, \ldots \eta_{r_2}, Y_1, \ldots Y_{s_2})&,\end{aligned}$$

for any set of one-forms $(\{\omega_i\}, \{\eta_j\})$ and vector fields $(\{X_i\}, \{Y_j\})$. A tensor field T is expanded in some local coordinate system according to $T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} \partial_{\mu_1} \otimes \cdots \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \cdots dx^{\nu_s}$.

A mapping $f : \mathcal{M} \to \mathcal{N}$ between two manifolds induces a mapping $f^* : \mathscr{F}(\mathcal{N}) \to \mathscr{F}(\mathcal{M})$ between their sets of functions, defined by

$$(f^*g)(x) = g(f(x)), \quad \text{for any } x \in \mathcal{M}.$$

Since the function g on $\mathcal{N}$ is pulled back to a function f^*g on $\mathcal{M}$, f^*g is called the *pullback* of g by the mapping f. The *pushforward* of a vector $X \in T_x\mathcal{M}$ is the vector f_*X tangent to $\mathcal{N}$ at $f(x)$ such that

$$(f_*X)_{f(x)}(g) = X_x(f^*g), \quad \text{for any } g \in \mathscr{F}(\mathcal{N}).$$

Consequently, the pullback $f^* : T^*\mathcal{N} \to T^*\mathcal{M}$ of a one-form is defined pointwise as

$$(f^*\omega)_x(X) = \omega_{f(x)}(f_*X), \quad \text{for any } X \in T_x\mathcal{M}.$$

The pushforward and pullback are generalized in a straightforward way to act on contravariant and covariant tensor fields. Indeed, for $T \in T_0^r\mathcal{M}$ and $S \in T_s^0\mathcal{N}$ we have

$$(f_*T)_{f(x)}(\omega_1, \dots \omega_r) = T_x(f^*\omega_1, \dots f^*\omega_r), \quad \text{for any } \omega_i \in T^*_{f(x)}\mathcal{N}$$

and

$$(f^*S)_x(X_1, \dots X_s) = S_{f(x)}(f_*X_1, \dots f_*X_s), \quad \text{for any } X_j \in T_x\mathcal{M}.$$

In general it is not possible to transform (r, s)-type tensors, because the pushforward and pullback transform in opposite directions between $\mathcal{M}$ and $\mathcal{N}$. More precisely, the pushforward follows the same direction as f, whereas the pullback goes the other way around. However, if the mapping is a diffeomorphism $\mathcal{M} \to \mathcal{M}$ one may define the pushforward of an arbitrary mixed-type tensor field $T \in T_s^r\mathcal{M}$ by

$$(f_*T)_{f(x)}(\omega_1, \dots \omega_r, X_1, \dots X_s) = T_x(f^*\omega_1, \dots f^*\omega_r, f_*^{-1}X_1, \dots f_*^{-1}X_s)$$

and

$$(f^*T)_x(\omega_1, \dots \omega_r, X_1, \dots X_s) = T_{f(x)}(f^{-1*}\omega_1, \dots f^{-1*}\omega_r, f_*X_1, \dots f_*X_s),$$

for any $\omega_i \in T^*\mathcal{M}$ and $X_j \in T\mathcal{M}$. Obviously, the pullback of a tensor field by f is equal to its pushforward by f^{-1}, i.e., $f^* = f_*^{-1}$, and vice versa.

The product manifold $\mathcal{M} \times \mathcal{N}$ is the set of elements (x, y), where x and y are points in $\mathcal{M}$ and $\mathcal{N}$, respectively. For every point (x, y), the tangent space $T_{(x,y)}(\mathcal{M} \times \mathcal{N})$ can be identified with the direct product $T_x\mathcal{M} \times T_y\mathcal{N}$ as follows. Let $z_t = (x_t, y_t)$ be a curve in $\mathcal{M} \times \mathcal{N}$ and let Z be the tangent to z_t at $t = 0$, i.e.,

$$Zg = \left.\frac{d}{dt}\right|_0 g(z_t) = \left.\frac{d}{dt}\right|_0 g(x_t, y_0) + \left.\frac{d}{dt}\right|_0 g(x_0, y_t) = \bar{X}g + \bar{Y}g,$$

where $\bar{X}$ is the vector tangent to the curve (x_t, y_0) and $\bar{Y}$ is the vector tangent to the curve (x_0, y_t). We then identify Z with (X, Y), where X and Y are the vectors tangent to, respectively, x_t and y_t at $t = 0$. This notation allows us to state the *Leibniz rule* in the following proposition.

Proposition 2.1.1. *Let f be a mapping $\mathcal{M} \times \mathcal{N} \to \mathcal{O}$. If $Z \in T_{(x_0,y_0)}(\mathcal{M} \times \mathcal{N})$ corresponds to $(X, Y) \in T_{x_0}\mathcal{M} + T_{y_0}\mathcal{N}$, then*

$$f_*Z = f_{1*}X + f_{2*}Y,$$

where $f_1 : x \in \mathcal{M} \to f(x, y_0) \in \mathcal{O}$ *and* $f_2 : y \in \mathcal{N} \to f(x_0, y) \in \mathcal{O}$.

Proof. Let Z be the tangent vector to (x_t, y_t) at $t = 0$. In the manner described in the paragraph preceding the proposition we have that $Z = \bar{X} + \bar{Y}$, and hence $f_* Z = f_* \bar{X} + f_* \bar{Y}$. It is easy to see that $f_* \bar{X} = f_{1*} X$ and $f_* \bar{Y} = f_{2*} Y$, which proves what needed to be demonstrated. □

2.1.3 Differential forms

There is a subclass of covariant tensor fields widely used in physics, so that it is appropriate to have it reviewed in a separate discussion. These are the so-called differential forms, now defined.

Definition. A *differential form* ω of degree p, or simply a p-form, is a covariant tensor field of rank p that is totally antisymmetric, i.e., for any set of vector fields $\{X_i\}$

$$\omega(X_{\pi(1)}, \dots X_{\pi(p)}) = \varepsilon(\pi)\, \omega(X_1, \dots X_p),$$

where π is an element of the group of permutations of $(1, \dots p)$ and $\varepsilon(\pi)$ the corresponding sign.

Being a covariant tensor field, a differential form can be interpreted as a multilinear antisymmetric mapping over $\mathscr{F}(\mathcal{M})$ of $\mathscr{X}(\mathcal{M}) \times \dots \mathscr{X}(\mathcal{M})$ into $\mathscr{F}(\mathcal{M})$. We shall denote the module of differential p-forms on $\mathcal{M}$ by $\Omega^p(\mathcal{M})$. It is a short exercise to show that a covariant tensor field is a differential form if and only if it is an eigenstate with eigenvalue 1 of the antisymmetrizer $\mathcal{A}$, whose action is defined by

$$(\mathcal{A}T)(X_1, \dots X_p) = \frac{1}{p!} \sum_{\pi} \varepsilon(\pi)\, T(X_{\pi(1)}, \dots X_{\pi(p)}),$$

on any $T \in T^0_p \mathcal{M}$. Because $\mathcal{A}^2 = \mathcal{A}$, the antisymmetrizer is a projector from the covariant tensor fields of rank p onto the module of differential p-forms. Therefore, and since $\mathcal{A}$ is a linear operator, the elements $dx^{\mu_1} \wedge \dots dx^{\mu_p} = p! \mathcal{A}(dx^{\mu_1} \otimes \dots dx^{\mu_p})$ form a coordinate basis for elements of $\Omega^p(\mathcal{M})$, so that a generic p-form ω may be expanded according to

$$\omega = \frac{1}{r!} \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \dots dx^{\mu_p}.$$

The *exterior* or *wedge product* of a p-form ω with a r-form η is the $(p+r)$-form $\omega \wedge \eta$, defined by

$$\omega \wedge \eta = \frac{(p+r)!}{p! r!} \mathcal{A}(\omega \otimes \eta),$$

which in local coordinates is expressed as

$$\omega \wedge \eta = \frac{1}{p!r!}\omega_{\mu_1\cdots\mu_p}\eta_{\mu_{p+1}\cdots\mu_{p+r}}dx^{\mu_1} \wedge \ldots dx^{\mu_{p+r}}.$$

It is easily seen that the dimension of $\Omega^p(\mathcal{M})$ is given by $\binom{d}{p}$ and that there exist no p-forms for $p > d$. The *exterior algebra* $\Omega(\mathcal{M}) = \Omega^0(\mathcal{M}) \oplus \Omega^1(\mathcal{M}) \oplus \cdots \Omega^d(\mathcal{M})$, where $\Omega^0(\mathcal{M}) = \mathscr{F}(\mathcal{M})$ denotes the functions on $\mathcal{M}$, is therefore closed under the exterior product.

Next, a derivative operator is defined on the exterior algebra.

Definition. *Exterior differentiation* is the linear mapping $d : \Omega^p(\mathcal{M}) \to \Omega^{p+1}(\mathcal{M})$ that is characterized by the following properties:

(i) For any function $f \in \mathscr{F}(\mathcal{M})$, df is the total differential;
(ii) $d \circ d \equiv 0$;
(iii) For any $\omega \in \Omega^p(\mathcal{M})$ and $\eta \in \Omega^r(\mathcal{M})$,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.$$

In terms of a local coordinate system, the exterior derivative of a p-form ω is given by

$$d\omega = \frac{1}{p!}\partial_\mu \omega_{\mu_1\cdots\mu_r} dx^\mu \wedge dx^{\mu_1} \wedge \ldots dx^{\mu_p}.$$

There is an expression concerning differential forms that is useful to remember for future reference. For any vector-valued one-form ω and vector fields X and Y, the following identity holds:

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]). \tag{2.1.5}$$

This equality is a special case from a larger class of identities that are formulated for vector-valued differential forms of any degree and proved in, e.g., [KN96a], Prop. 3.11. Equation (2.1.5) is easily verified by comparing the left-hand side with the right-hand side in a local coordinate chart.

Definition. The *interior product* of a differential form with respect to a vector field X is the linear mapping $i_X : \Omega^p(\mathcal{M}) \to \Omega^{p-1}(\mathcal{M})$, such that:

(i) For each function $f \in \mathscr{F}(\mathcal{M})$, $i_X(f) = 0$;
(ii) For any $\omega \in \Omega^p(\mathcal{M})$,

$$(i_X\omega)(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1}).$$

We shall also make use of the notation $X \rfloor \omega = i_X\omega$. In a local coordinate system the interior product of a p-form may be expressed as

$$i_X\omega = \frac{1}{(p-1)!}X^\mu \omega_{\mu\mu_1\cdot\mu_{p-1}} dx^{\mu_1} \wedge \ldots dx^{\mu_{p-1}}.$$

2.1.4 One-parameter groups of transformations

An *integral curve* x_t of a vector field X on a manifold $\mathcal{M}$ is a curve to which X is tangent at x_t for any value of the parameter t. Given a local coordinate chart, this means that

$$\frac{dx_t^\mu}{dt} = X^\mu(x_t). \tag{2.1.6}$$

For any set of initial conditions x_0^μ the system of ordinary differential equations (2.1.6) has a unique solution [LS90]. Concretely, this expresses that given a point x_0 a unique integral curve of X through the point exists.

Definition. A *one-parameter group of transformations* of $\mathcal{M}$ is a set $\{\phi_t\}_{t\in\mathbb{R}}$ of mappings of $\mathcal{M}$, satisfying the following conditions:

(i) For each $t \in \mathbb{R}$, ϕ_t is a diffeomorphism of $\mathcal{M}$;
(ii) For all $t, s \in \mathbb{R}$ and $x \in \mathcal{M}$, $\phi_t \circ \phi_s(x) = \phi_{t+s}(x)$.

By identifying ϕ_0 with the unit element and ϕ_{-t} with the inverse element of ϕ_t, it is easily understood that a one-parameter group of transformations is an abelian group. Each one-parameter group of transformations ϕ_t naturally induces a vector field on $\mathcal{M}$ as follows. For any point $x \in \mathcal{M}$, a curve through x is traced out by $\phi_t(x)$. The vector field X tangent to the thus obtained curves solves the system

$$\frac{d}{dt}f(\phi_t(x)) = Xf(\phi_t(x)) \quad \text{for any } x \in \mathcal{M}. \tag{2.1.7}$$

Conversely, let X be a vector field on $\mathcal{M}$ and let $\phi_t(x)$ be an integral curve of X that passes through x, such that $\phi_0(x) = x$. This curve solves the set of differential equations (2.1.7) with the initial condition $\phi_0(x) = x$. By the uniqueness of solutions it is easily verified that $\phi_t \circ \phi_s = \phi_{t+s}$, so that ϕ_t is a one-parameter group of transformations that induces X. In this context, one says that the vector field X *generates* the one-parameter group ϕ_t.

Let X be a vector field and let ϕ_t be a one-parameter group of transformations generated by X. The *Lie derivative* of a generic mixed-type tensor field T with respect to the vector field X is defined by

$$\mathscr{L}_X T = \lim_{t\to 0} \frac{1}{t}(T - \phi_{t*}T),$$

which is a tensor field of the same type as T. For a differential form ω, its Lie derivative can be rewritten as

$$\mathscr{L}_X \omega = \lim_{t\to 0} \frac{1}{t}(\phi_t^* \omega - \omega),$$

and it may be verified that there is a useful relation between the Lie derivative, the exterior derivative and the interior product of a differential form [KN96a]:

$$\mathscr{L}_X \omega = d \circ i_X \omega + i_X \circ d\omega. \tag{2.1.8}$$

In terms of a local coordinate chart, it follows from the definition that the Lie derivatives of a vector field Y and one-form ω take the forms

$$(\mathscr{L}_X Y)^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu \quad \text{and} \quad (\mathscr{L}_X \omega)_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu.$$

This furthermore shows that the Lie derivative with respect to X of a vector field Y is equal to the Lie bracket $[X, Y]$.

Proposition 2.1.2. *Let ϕ be a transformation of a manifold $\mathcal{M}$. If X generates the one-parameter group of transformations ϕ_t, the vector field $\phi_* X$ generates the one-parameter group $\phi \circ \phi_t \circ \phi^{-1}$.*

This proposition, a proof for which can be found in [KN96a], pg. 14, implies that X is invariant under the action of ϕ, that is, $\phi_* X = X$, if and only if ϕ commutes with ϕ_t, namely, $[\phi, \phi_t] = 0$ for all t.

Given a transformation ϕ of a manifold $\mathcal{M}$, we are able to conclude that for any two vector fields X and Y on $\mathcal{M}$, it is true that $\phi_*[X, Y] = [\phi_* X, \phi_* Y]$, because

$$\phi_*[X, Y] = \lim_{t\to 0} \frac{1}{t}(\phi_* Y - (\phi \circ \phi_t \circ \phi^{-1})_* \phi_* Y) = [\phi_* X, \phi_* Y], \tag{2.1.9}$$

where X generates ϕ_t.

2.2 Lie groups and Lie algebras

It is possible to endow manifolds with a group structure that is consistent with its differentiable character. The resulting objects, which go by the name of Lie groups, find vast applications in many areas of physics, where they generally represent real-world or abstract groups of continuous transformations. In physical applications they generally take the form of the so-called matrix groups, which will be reviewed in §2.2.2. As they form a subclass of Lie groups, we begin with a discussion on the nature of generic Lie group manifolds.

2.2.1 Generalities

Definition. A *Lie group* is a group which at the same time is a differentiable manifold, such that the group composition and the inverse operation are differentiable mappings. The dimension of the group is the dimension of the manifold.

We will denote by $L_a : g \mapsto ag$ and $R_a : g \mapsto ga$, the left translation, respectively, right translation of a Lie group G by an element $a \in G$. These diffeomorphisms on Lie groups allow us to consider a canonical subclass of vector fields on G. A vector field A is *left invariant* if it is invariant under the action of left translations, i.e., $L_{a*}A = A$ for any $a \in G$. Each element $A_e \in T_eG$ defines a unique left-invariant vector field $A_a = L_{a*}A_e$ on G, while the inverse translation identifies a left-invariant vector field with a unique element of the tangent space at the identity. This bijection is an isomorphism between the tangent space T_eG and the vector space of left-invariant vector fields on G, the latter of which is denoted by $\mathfrak{g}$, so that $\dim \mathfrak{g} = \dim G$. Since $\mathfrak{g}$ is a subset of $\mathscr{X}(G)$, the former inherits from the latter the definition for the Lie bracket. It follows from (2.1.9) that $\mathfrak{g}$ is closed under the Lie bracket, i.e., $[A, B]$ is left invariant whenever A and B are elements of $\mathfrak{g}$.

Definition. The *Lie algebra* $\mathfrak{g}$ of a Lie group G is the set of all left-invariant vector fields on G together with the usual vector addition, scalar multiplication and Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Every $A \in \mathfrak{g}$ generates a one-parameter group of transformations ϕ_t of G, which allows us to define the curve $a_t = \phi_t(e)$ that goes through the origin of G. The left-invariance of A implies that a_t is a one-parameter subgroup of G. We therefore call a_t the one-parameter subgroup generated by A, whose curve is a solution to the set of differential equations $\dot{f}(a_t) = Af(a_t)$ for any $f \in \mathscr{F}(G)$. This equation is generally given the shorthand notation $a_t^{-1}\dot{a}_t f = A_e f$. The correspondence between one-parameter subgroups of a Lie group and elements of its Lie algebra becomes manifest if we introduce the *exponential map* $\exp : \mathfrak{g} \to G$ as follows. For a one-parameter group a_t generated by A, define $\exp A = a_1$. We now show that $\exp tA = a_t$ for any $t \in \mathbb{R}$. If c is a real number, we have that $df(a_{ct})/dt = c\, df(a_{ct})/d(ct) = cAf(a_{ct})$, which proves that a_{ct} is the one-parameter subgroup generated by cA. The left-invariant vector field $B = cA$ also generates b_t, which satisfies $df(b_t)/dt = cAf(b_t)$. From the uniqueness of the solution, we conclude that $a_{ct} = b_t$ and $\exp cA = b_1 = a_c$. This proves the assertion.

Each automorphism ϕ of a Lie group G induces an automorphism ϕ_* of its Lie algebra.[2] To verify this, note first that $\phi_* A$ is left-invariant whenever A is an element of $\mathfrak{g}$, because $\phi \circ L_a = L_{\phi(a)} \circ \phi$. Furthermore, ϕ_* commutes with the Lie bracket, as was shown in (2.1.9).

[2] An automorphism of a Lie group is a diffeomorphism that respects the group structure, whereas an automorphism of a Lie algebra is an invertible mapping that preserves the Lie bracket.

In particular, the conjugation $\mathrm{conj}_a : g \in G \mapsto aga^{-1} \in G$ induces the automorphism

$$\mathrm{Ad}_a : A \in \mathfrak{g} \mapsto \mathrm{Ad}_a(A) = R_{a^{-1}*}A \in \mathfrak{g}.$$

The mapping $\mathrm{Ad} : g \mapsto \mathrm{Ad}_g = \mathrm{Ad}(g)$ is the *adjoint representation* of G on $\mathfrak{g}$. Let ϕ_t be the one-parameter group of transformations of G generated by A, corresponding to which $a_t = \phi_t(e) = \exp tA$ is the one-parameter subgroup of G. For any $g \in G$, $\phi_t(g) = \phi_t \circ L_g(e) = R_{a_t}g$, as Prop. 2.1.2 implies that $[L_g, \phi_t] = 0$. The *adjoint representation* $\mathrm{ad} : A \mapsto \mathrm{ad}_A = \mathrm{ad}(A)$ of $\mathfrak{g}$ on itself is defined as

$$\begin{aligned} \mathrm{ad}_A(B) = \lim_{t\to 0} \frac{1}{t}(B - \mathrm{Ad}_{a_t^{-1}}(B)) &= \lim_{t\to 0} \frac{1}{t}(B - R_{a_t*}B) \\ &= \lim_{t\to 0} \frac{1}{t}(B - \phi_{t*}B) = [A, B]. \end{aligned}$$

Although it is true that the Lie algebra $\mathfrak{g}$ denotes the set of left-invariant vector fields on G, while T_eG is the tangent space at the identity, the canonical isomorphism of left translation allows us to refer to both of them as the Lie algebra of G. We shall make use of this natural identification and denote T_eG by $\mathfrak{g}$ as well.

Note also that a frame e_a of left-invariant vector fields on G forms a basis for the Lie algebra. The structure functions in its commutation relations (2.1.3) are constants, because both sides of $[e_a, e_b] = c_{ab}{}^c e_c$ are left-invariant. They are called, quite unsurprisingly, the *structure constants* of $\mathfrak{g}$.

Definition. The *Maurer-Cartan form* on G is the $\mathfrak{g}$-valued one-form $\omega_G : TG \to \mathfrak{g}$, defined by $\omega_G(X) = L_{g^{-1}*}X$ for $X \in T_gG$.

The Maurer-Cartan form is left invariant, i.e., $L_a^*\omega_G = \omega_G$ for any $a \in G$, since for $X \in T_gG$ we have that

$$(L_a^*\omega_G)(X) = \omega_G(L_{a*}X) = L_{(ag)^{-1}*}L_{a*}X = \omega_G(X).$$

Furthermore, for each left-invariant A, $\omega_G(A)$ is a constant function. Therefore, it follows from (2.1.5) that $d\omega_G(A, B) = -\omega_G([A, B])$ for any two $A, B \in \mathfrak{g}$. Because $\omega_G([A, B]) = [\omega_G(A), \omega_G(B)] = \frac{1}{2}[\omega_G, \omega_G](A, B)$,[3] we obtain the Maurer-Cartan *structural equation*, namely

$$d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0. \tag{2.2.1}$$

Although the structural equation has been derived for left-invariant vector fields, it is true for generic vector fields X and Y on G. The reason is that it is an equation of two-forms

[3] We refer the reader to §2.A for a summary on the notation that is used for Lie algebra-valued differential forms.

which act linearly on vector fields. As any vector field can pointwise be expanded with respect to a basis of $\mathfrak{g}$, Eq. (2.2.1) retains its validity for all elements of $\mathscr{X}(G)$. If the Maurer-Cartan form is expanded as $\omega_G^a e_a$, the structural equation (2.2.1) can equally be decomposed according to

$$d\omega_G^a + \frac{1}{2} c_{bc}{}^a \omega_G^b \wedge \omega_G^c = 0,$$

where we made use of the equality $[\omega_G, \omega_G] = \omega_G^b \wedge \omega_G^c [e_b, e_c]$.

Lie groups are often characterized by their action on manifolds. The following definition generalizes the one-parameter groups of transformations on a manifold that were introduced in §2.1.4 to Lie transformation groups.

Definition. The *action* of a Lie group G on a manifold $\mathcal{M}$ is a differentiable mapping $(a, x) \in G \times \mathcal{M} \mapsto \phi_a(x) \in \mathcal{M}$ that satisfies the conditions:

(i) For each $a \in G$, ϕ_a is a diffeomorphism of $\mathcal{M}$;
(ii) For all $a, b \in G$ and $x \in \mathcal{M}$, $\phi_b \circ \phi_a(x) = \phi_{ab}(x)$, or alternatively, $\phi_b \circ \phi_a(x) = \phi_{ba}(x)$.

Since ϕ_a is a diffeomorphism, so that there is a unique $y = \phi_a(x)$ for each x, and because $\phi_e \circ \phi_a(x) = \phi_a(x)$, ϕ_e is the identity transformation of $\mathcal{M}$. Similarly, one may conclude that $\phi_{a^{-1}} = \phi_a^{-1}$. The action is said to be *transitive* if for any two points x and y, there exists an element a of G such that $y = \phi_a(x)$. If the existence of a point x, for which $\phi_a(x) = x$, implies $a = e$, the action is called *free*. In other words, there are no fixed points for nontrivial elements. The action is *effective* if $\phi_a(x) = x$ for all points x implies $a = e$, that is, if the only element that leaves $\mathcal{M}$ unchanged is the identity of G. Note that a free action is effective but that the converse does not hold in general.

If the action of a Lie group G is given on a manifold $\mathcal{M}$, each left invariant vector field A of G induces a vector field $A^\star$ on $\mathcal{M}$ as follows. Namely, the action ϕ_{a_t} of the one-parameter subgroup $a_t = \exp tA$ on $\mathcal{M}$ induces a vector field $A^\star \in \mathscr{X}(\mathcal{M})$, which satisfies

$$\frac{d}{dt} f(\phi_{a_t}(x)) = A^\star f(\phi_{a_t}(x)) \quad \text{for any } x \in \mathcal{M}.$$

Well-known examples of Lie transformation groups are the left and right action of a Lie group on a manifold, which are defined by, respectively, $L_a(x) = ax$ and $R_a(x) = xa$ for any point x and group element a. In particular the right action is of great importance for its role played in the theory of principal fibre bundles, which are to be reviewed in §2.3. We therefore recall some properties that are proper to the right action of a Lie group on a manifold.

The vector field $A^\star$ that is induced by the right action R_{a_t} is called the *fundamental vector field* generated by A, where as usual $a_t = \exp tA$. We first verify that the mapping $\rho : A \in \mathfrak{g} \mapsto A^\star \in \mathscr{X}(\mathcal{M})$ is a Lie algebra homomorphism, i.e., it is linear and $[A^\star, B^\star] = [A, B]^\star$. Observe that the mapping ρ can be constructed explicitly as follows. For any

$x \in \mathcal{M}$, let $\rho_x : g \in G \mapsto xg \in \mathcal{M}$. One may understand that $\rho_{x*}A_e = A^\star_x = \rho(A)_x$, from which it trivially follows that ρ is a linear operator. To show that it is a Lie algebra homomorphism, we note that for every x

$$\begin{aligned}[A^\star, B^\star]_x &= \lim_{t\to 0}\frac{1}{t}(B^\star_x - (R_{a_t*}B^\star)_x)\\ &= \lim_{t\to 0}\frac{1}{t}\rho_{x*}(B_e - \mathrm{Ad}_{a_t^{-1}}(B_e)) = \rho_{x*}[A,B]_e = [A,B]^\star_x,\end{aligned}$$

where the second equality follows from (let $f \in \mathscr{F}(\mathcal{M})$)

$$\begin{aligned}(R_{a_t*}B^\star_{xa_t^{-1}})f &= (R_{a_t*}\rho_{xa_t^{-1}*}B_e)f\\ &= B_e(f\circ R_{a_t}\circ \rho_{xa_t^{-1}}) = B_e(f\circ\rho_x\circ \mathrm{conj}_{a_t^{-1}}) = (\rho_{x*}\,\mathrm{Ad}_{a_t^{-1}}(B_e))f,\end{aligned}$$

As a corollary, these equalities lead to the interesting observation that

$$R_{g*}A^\star = (\mathrm{Ad}_{g^{-1}}(A))^\star \quad \text{for any } A \in \mathfrak{g}. \tag{2.2.2}$$

Finally, if the right action of a Lie group on a manifold is free, the fundamental vector field corresponding to a nonzero element of the Lie algebra is everywhere nonvanishing. Indeed, if $A^\star$ were vanishing at some point x, $xa_t = x$ for all t. Since the action is free, this implies that a_t is the identity element of G, so that A must be the zero element of $\mathfrak{g}$. At any point x, the Lie algebra spanned by the fundamental vector fields is isomorphic to $\mathfrak{g}$.

2.2.2 Matrix groups

The vector space of linear mappings $\mathbb{R}^n \to \mathbb{R}^n$ consists of the real $n \times n$ matrices, which we denote by $M(n, \mathbb{R})$. Because not every matrix is invertible, $M(n, \mathbb{R})$ is not a group with respect to the matrix multiplication rule. By limiting our attention to those that are invertible, we obtain the largest subspace of $M(n, \mathbb{R})$ that is a group, which is called the general linear group of $\mathbb{R}$ and denoted by

$$Gl(n, \mathbb{R}) = \{g \in M(n, \mathbb{R}) \mid \det g \neq 0\}.$$

Since we shall be concerned only with $\mathbb{R}$-valued matrices, we drop the reference to the set of real numbers and simply write $Gl(n)$. It can be shown that the general linear group is a smooth manifold and that the matrix multiplication and inverse operations are differentiable [Sha97], from which it follows that $Gl(n)$ is a Lie group. There is a natural coordinate system that is given by the matrix entries ${g^i}_j$, from which it is clear that the general linear group is n^2-dimensional.

The determinant of the $n \times n$ matrices forms a continuous mapping from $Gl(n)$ onto $\mathbb{R}/\{0\}$, because it is a polynomial in the matrix entries. Since its image has two disconnected pieces, its preimage also is disconnected in the same way. The component of $Gl(n)$ that has a positive determinant is connected to the identity and is easily seen to be a Lie group as well. From now on we shall only consider the identity component and agree that $Gl(n)$ refers to this group.

A tangent vector X at g can be expanded as $X = X^i{}_j \partial_i{}^j|_g$, where the n^2 real numbers $X^i{}_j$ are arbitrary. Thereby we conclude that $T_g Gl(n) \simeq M(n)$. The left and right translation of $Gl(n)$ on itself are given by the left and right matrix multiplication, respectively. The pushforward of the left translation on $X \in T_g Gl(n)$ is then determined by

$$(L_{a*}X)f = X(f \circ L_a) = X^k{}_l \frac{\partial [ag]^i{}_j}{\partial g_k{}^l} \partial_i{}^j|_{ag} f = a^i{}_k X^k{}_j \partial_i{}^j|_{ag} f.$$

This shows that if $X^i{}_j \in M(n)$ corresponds to $X \in T_g Gl(n)$, the matrix $[aX]^i{}_j$ corresponds to $L_{a*}X \in T_{ag}Gl(n)$. In a similar way, it can be demonstrated that the right translation of X induces the right multiplication on $X^i{}_j$, i.e., $[Xa]^i{}_j$ corresponds to $R_{a*}X \in T_{ga}Gl(n)$. It follows directly that the matrix $[aXa^{-1}]^i{}_j$ is in correspondence with $\mathrm{Ad}_a(X) \in T_{aga^{-1}}Gl(n)$.

As a consequence, if $A \in \mathfrak{gl}(n)$ is a left-invariant vector field on $Gl(n)$ and $A^i{}_j$ is the matrix corresponding to A at the identity, $[gA]^i{}_j$ is the matrix that corresponds to A at g. Consider a second element $B \in \mathfrak{gl}(n)$ for which $B^i{}_j$ corresponds to $B_e \in T_e Gl(n)$. It is a short exercise to show that

$$[A, B]_g f = g^i{}_k([AB]^k{}_j - [BA]^k{}_j)\partial_i{}^j|_e f = [g[A, B]]^i{}_j \partial_i{}^j|_e f,$$

where the commutator on the right-hand side of the second equation is the ordinary matrix commutator. One concludes that the left-invariant vector field $[A, B] \in \mathfrak{gl}(n)$ at the identity corresponds to $[A, B]^i{}_j \in M(n)$. As a result of that is there a Lie algebra isomorphism between the set of left-invariant vector fields on $Gl(n)$ and the set of matrices $M(n)$, equipped with the matrix commutator. Since it is far more practical to use matrices in applications, we shall identify them with the set of left-invariant vector fields, so that in the following $\mathfrak{gl}(n)$ refers to the set of real $n \times n$ matrices, where the composition rule is the matrix commutation operation.

Finally, it is useful to find out how the Maurer–Cartan form looks like on $Gl(n)$. Assume $X^i{}_j$ is the matrix that corresponds to $X \in T_g Gl(n)$. In general, the Maurer–Cartan form sends a tangent vector into the Lie algebra by left translating it to the identity, where it is identified with the corresponding element of the Lie algebra. Since for the general linear group, the Lie algebra $\mathfrak{gl}(n)$ consists of the matrices making up the components of the left-invariant vector fields at the identity, it follows that the Maurer–Cartan form should be a mapping

$$X^i{}_j \partial_i{}^j|_g \mapsto [g^{-1}X]^i{}_j,$$

where $[g^{-1}X]^i{}_j$ is the matrix corresponding to $L_{g^{-1}*}X \in T_eGl(n)$. Therefore, at any point $g \in Gl(n)$, the Maurer–Cartan form is given by the $\mathfrak{gl}(n)$-valued one-form

$$\omega_{Gl(n)} = [g^{-1}]^i{}_k dg^k{}_j : T_gGl(n) \to \mathfrak{gl}(n). \tag{2.2.3}$$

This is the reason why the Maurer–Cartan form is sometimes written formally as $g^{-1}dg$, albeit it only makes sense for matrix Lie groups.

2.3 Principal fibre bundles

In this last section, we review the notion of principal and their associated fibre bundles. The usefulness of fibre bundles for the description of physical phenomena can hardly be understated, as it forms the background on which both the standard model of particle physics and general relativity are formulated. Nevertheless, to implement a description for the interactions these theories attempt to model, the bundle structure is not sufficient and a geometry has to be defined on top through the introduction of connections. A discussion on these connections and what type of geometries they create will be treated in the following chapter. To start with the beginning, let us now define principal fibre bundles.

Definition. A *principal fibre bundle* $P(\mathcal{M}, G)$ over a manifold $\mathcal{M}$ with Lie group G consists of a manifold P and an action of G on P, for which the following conditions are satisfied:

(i) G acts freely on P on the right;

(ii) $\mathcal{M} = P/G$ is the quotient space of P by the equivalence relation induced by the right action of G, and the canonical projection $\pi : P \to \mathcal{M}$ is differentiable;

(iii) P is locally trivial, in the sense that every $x \in \mathcal{M}$ has a neighborhood U_i such that $\pi^{-1}(U_i)$ is isomorphic with $U_i \times G$, i.e., there is a diffeomorphism

$$\psi_i : p \in \pi^{-1}(U_i) \mapsto (\pi(p), \gamma_i(p)) \in U_i \times G,$$

called a *local trivialization*, where γ_i is a mapping from $\pi^{-1}(U_i)$ to G, such that $\gamma_i(pa) = \gamma_i(p)a$ for any $a \in G$.

We shall call P the bundle manifold, $\mathcal{M}$ the base manifold, $\pi^{-1}(x) \simeq G$ the fibre over x, and G the structure group or typical fibre of P. For every two overlapping charts U_i and U_j, we define the *transition functions* $t_{ij}(\pi(p)) = \gamma_i(p)\gamma_j(p)^{-1}$ for each $p \in \pi^{-1}(U_i \cap U_j)$.

The transition functions are constant on the fibres of P, so that they are mappings

$$t_{ij} : U_i \cap U_j \to G.$$

It is directly verified that $t_{ii}(x) = e$ for any $x \in U_i$, that $t_{ji}(x) = t_{ij}(x)^{-1}$ for any $x \in U_i \cap U_j$, and that $t_{ij}(x)t_{jk}(x) = t_{ik}(x)$ for any $x \in U_i \cap U_j \cap U_k$. The transition functions owe their name to the fact that they relate different local trivializations on overlapping charts, namely $\psi_j^{-1}(\pi(p), \gamma_j(p)) = \psi_i^{-1}(\pi(p), \gamma_i(p)) = \psi_i^{-1}(\pi(p), t_{ij}(\pi(p))\gamma_j(p))$ for any $p \in \pi^{-1}(U_i \cap U_j)$.

When there is a manifold F on which G acts on the left, the *associated fibre bundle* with the principal bundle $P(\mathcal{M}, G)$ is constructed as follows. First, an action of G on the product manifold $P \times F$ is defined through

$$(p, f) \in P \times F \mapsto (pa, a^{-1}f), \quad \text{with } a \in G.$$

Consider then the quotient space $E = (P \times F)/G$ of equivalence classes $[p, f] = [pa, a^{-1}f]$, induced by the action of G on $P \times F$. We shall denote this quotient space by $E = P \times_G F = P[F]$. The projection $\pi_E : E \to \mathcal{M}$ is specified by

$$\pi_E([p, f]) = \pi(p),$$

a mapping that is well defined, for $\pi(pa) = \pi(p)$. Because for any $x \in \mathcal{M}$, there is a neighborhood U_i for which $\pi^{-1}(U_i) \simeq U_i \times G$, it is also true that

$$\pi_E^{-1}(U_i) \simeq \frac{U_i \times G \times F}{G} = U_i \times F \quad \text{and} \quad \pi_E^{-1}(x) \simeq F,$$

the latter of which are called the fibres of E. The local trivializations on overlapping charts $U_i \cap U_j$ are given by

$$\begin{aligned}
\tilde{\psi}_j([p, f]) = [\psi_j(p), f] = [\pi(p), \gamma_j(p), f] &= [\pi(p), e, \gamma_j(p)f] \\
&\equiv (\pi_E([p, f]), \varphi_j([p, f])), \\
\tilde{\psi}_i([p, f]) = [\psi_i(p), f] = [\pi(p), \gamma_i(p), f] &= [\pi(p), e, t_{ij}(\pi(p))\gamma_j(p)f] \\
&= (\pi_E([p, f]), t_{ij}([p, f])\varphi_j([p, f])),
\end{aligned}$$

so that $\tilde{\psi}_j^{-1}(\pi_E([p, f]), \varphi_j([p, f])) = \tilde{\psi}_i^{-1}(\pi_E([p, f]), t_{ij}([p, f])\varphi_j([p, f]))$, from which we conclude that the transition functions are the same on a principal bundle P and some associated fibre bundle E.

A *cross section* of a fibre bundle P over a region $U_i \subset \mathcal{M}$ is a mapping $\sigma : U_i \to P$ such that $\pi \circ \sigma$ is the identity mapping on U_i. Similarly, one may consider cross sections of associated fibre bundles.

Proposition 2.3.1. *Let $Q(\mathcal{M}, G)$ be a principal bundle and let F be a manifold on which G acts on the left. There is a one-to-one correspondence between sections of $Q[F] = Q \times_G F$ and mappings $\varphi : Q \to F$ that are G equivariant, i.e., $R_g^* \phi = g^{-1}\phi$.*

Proof. When $\varphi : Q \to F$ is a map that satisfies $\varphi(qg) = g^{-1}\varphi(q)$ for any $g \in G$, a section of $Q[F]$ is given by $\sigma(\pi(q)) \equiv [q, \varphi(q)]$ for each $q \in Q$. This is indeed a section, for it is constant on the fibres:

$$\sigma(\pi(qg)) = [qg, \varphi(qg)] = [q, \varphi(q)] = \sigma(\pi(q)).$$

Conversely, let σ be section of $Q[F]$. Then there must be a map $Q \to F$ so that

$$\sigma(\pi(q)) = [q, \varphi(q)] = [qg, g^{-1}\varphi(q)].$$

Because a section is constant on the fibres, $\sigma(\pi(q)) = \sigma(\pi(qg))$ and $\varphi(qg) = g^{-1}\varphi(q)$, that is, φ is G equivariant. □

Appendix 2.A Lie algebra-valued differential forms

In this dissertation, we use to some extent the index-free language of Lie algebra-valued differential forms. Once familiar, they are very powerful in keeping equations readable and focused on their algebraic structure. For convenience of the reader who is not at home with the notation, we gather the facts necessary to understand its use.

Consider a manifold $\mathcal{M}$ and a Lie algebra $\mathfrak{g}$. The space of $\mathfrak{g}$-valued differential p-forms on $\mathcal{M}$, i.e., $\mathfrak{g} \otimes \Omega^p(\mathcal{M})$, will be denoted by $\Omega^p(\mathcal{M}, \mathfrak{g})$. If $\{E_a\}_{a=1\ldots \dim \mathfrak{g}}$ is a basis for $\mathfrak{g}$, an element η of $\Omega^p(\mathcal{M}, \mathfrak{g})$ can be expanded as $\eta^a \otimes E_a$, which will also be written as $\eta = \eta^a E_a$, and where every $\eta^a \in \Omega^p(\mathcal{M})$.

For any two forms $\eta \in \Omega^p(\mathcal{M}, \mathfrak{g})$ and $\theta \in \Omega^q(\mathcal{M}, \mathfrak{g})$ such that $p + q \leqslant \dim \mathcal{M}$, a bracket operation is defined by

$$(\eta, \theta) \in \Omega^p(\mathcal{M}, \mathfrak{g}) \times \Omega^q(\mathcal{M}, \mathfrak{g}) \mapsto [\eta, \theta] = \eta^a \wedge \theta^b [E_a, E_b] \in \Omega^{p+q}(\mathcal{M}, \mathfrak{g}),$$

where the last bracket is of course just the ordinary Lie bracket of $\mathfrak{g}$. This operator is a graded commutator, namely,

$$[\eta, \theta] = (-1)^{pq+1} [\theta, \eta],$$

and satisfies the graded Jacobi identity (let $\omega \in \Omega^r(\mathcal{M}, \mathfrak{g})$):

$$(-1)^{rp} [[\eta, \theta], \omega] + (-1)^{pq} [[\theta, \omega], \eta] + (-1)^{qr} [[\omega, \eta], \theta] \equiv 0,$$

which can be rewritten in the alternative form

$$[\eta, [\theta, \omega]] = [[\eta, \theta], \omega] + (-1)^{pq} [\theta, [\eta, \omega]].$$

From this identity it follows that

$$[\eta, [\theta, \theta]] = \begin{cases} 2[[\eta, \theta], \theta] & \text{for odd } q, \\ 0 & \text{for even } q. \end{cases}$$

The exterior derivative of differential forms can naturally be extended to act on Lie algebra-valued differential forms, by restricting its action on the form parts, that is,

$$\eta \in \Omega^p(\mathcal{M}, \mathfrak{g}) \mapsto d\eta = d\eta^a E_a \in \Omega^{p+1}(\mathcal{M}, \mathfrak{g}).$$

It obeys a graded Leibniz rule, namely,

$$d[\eta, \theta] = [d\eta, \theta] + (-1)^p [\eta, d\theta].$$

Lastly, we observe that if η and θ are $\mathfrak{g}$-valued one-forms, then for any two vector fields X and Y one has

$$[\eta,\theta](X,Y) = [\eta(X),\theta(Y)] - [\eta(Y),\theta(X)],$$
$$[\eta,\eta](X,Y) = 2[\eta(X),\eta(Y)].$$

In these expressions, we used the notation $\eta(X) = \eta^a(X)E_a$, so that the brackets on the right-hand side reduce to the ordinary bracket of $\mathfrak{g}$.

3 | Cartan geometry

Having reviewed principal fibre bundles in the preceding chapter, we next discuss how geometry is brought into play. Through the course of history geometry has undergone a remarkable evolution— from Euclid's axioms through Riemannian manifolds to its present-day description in terms of connections on principal bundles [Che90]. To the broad public, geometry remains a theory of Euclidean lengths and angles, or at best the analytical equations as developed by Descartes. Within the standard models of physics Ehresmann connections and Riemannian geometry are familiar concepts because of their applications in particle physics and general relativity, respectively. The objective of this chapter is to review the less-known structure of Cartan geometry, which generalizes the local geometry of Riemannian spaces, i.e., Euclidean geometry, to generic Klein geometries, which describe homogeneous spaces defined in terms of symmetry Lie groups. Such generalizations will give us the mathematical tools to tackle the problem of formulating theories of gravity whose local geometry is given by de Sitter space, instead of Minkowski space.

We begin by reviewing the more general Ehresmann connections, which define the abstract geometries underlying Yang–Mills theories. Afterwards, the Klein geometry of homogeneous spaces is introduced, followed by their nonhomogeneous generalizations in Cartan geometry. To conclude, the relation between Ehresmann and Cartan connections is considered.

3.1 Ehresmann connections

Let $Q(\mathcal{M}, G)$ be a principal G bundle over a manifold $\mathcal{M}$. Because G acts freely on the right on Q, the fundamental vector fields on the bundle space vanish nowhere. At any point q they span the *vertical subspace* $V_q \subset T_qQ$, whose dimension is equal to $\mathfrak{g}$. Vertical vectors are tangent to the fibres of Q and form the kernel of the bundle projection. Furthermore, the distribution $q \to V_q$ is right invariant, i.e., $R_{a*}V_q = V_{qa}$ for any $a \in G$, because the fundamental vector fields satisfy (2.2.2). Note, however, that a generic vertical vector field must not be right invariant.

The vertical distribution is canonical on a principal bundle. On the other hand, at any point q we are left with an infinite set of linear complements $H_q = T_qQ - V_q$, each of dimension $\dim \mathcal{M} = \dim Q - \dim G$. The geometric idea behind an *Ehresmann connection* is to single out a smooth distribution $q \to H_q$ that is right invariant, which is to say, a distribution such that [Ehr51]

(i) $T_qQ = H_q + V_q$,
(ii) $R_{g*}H_q = H_{qg}$, and
(iii) H_q depends differentiably on q, so that a smooth vector field X is separated into smooth vector fields X^{h} and X^{v}, where $X_q^{\mathrm{h}} \in H_q$ and $X_q^{\mathrm{v}} \in V_q$.

The subspaces H_q are said to be *horizontal*. Consequently, a connection singles out a unique horizontal complement to every vertical subspace along the principal bundle, which gives rise to a $\mathfrak{g}$-valued one-form as follows. Let ω be the differential form that maps a vector field X on Q into the Lie algebra of G according to

$$\omega_q(X) = A, \quad \text{where } X_q^{\mathrm{v}} = A_q^\star.$$

If an Ehresmann connection is given on Q, then ω annihilates horizontal vector fields, for they have no vertical components by construction. Furthermore, $R_g^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega$, an equality that follows from the right invariance of the horizontal distribution together with (2.2.2). Therefore, it is possible to define an Ehresmann connection on a principal bundle equivalently through the following connection one-form.

Definition. An Ehresmann connection ω on a principal bundle $Q(\mathcal{M}, G)$ is a $\mathfrak{g}$-valued one-form that satisfies:

(i) $\omega(A^\star) = A$, for each $A \in \mathfrak{g}$;
(ii) $R_g^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega$.

The horizontal subspace of T_qQ is for every q defined by the kernel of ω, i.e.,

$$H_q = \{X \in T_qQ \mid \omega(X) = 0\}.$$

Note that the second item of this definition rightly implies that the horizontal distribution is right invariant. Furthermore, the condition $\omega(X) = 0$ is a system of $\dim G$ ordinary differential equations, thus leaving undetermined the $\dim \mathcal{M}$ directions of the horizontal subspaces. The advantage of the definition in terms of a one-form is that it lends itself better to applications than the geometrically more intuitive definition given above. In view of these applications it actually turns out to be necessary to pull back the connection on the bundle space to a family of local connection forms on the base manifold $\mathcal{M}$, since it is in the latter that observational physics takes place.

Let $\{U_i\}$ be an open covering of $\mathcal{M}$, for which ψ_i is a family of local trivializations and t_{ij} a set of corresponding transition functions. For every chart U_i, we denote by σ_i the cross section that is defined by $\sigma_i(x) = \psi_i^{-1}(x, e)$. The *local connection form* is constructed subsequently as

$$\omega_i = \sigma_i^* \omega : TU_i \to \mathfrak{g}, \tag{3.1.1}$$

which is a $\mathfrak{g}$-valued one form on $U_i \subset \mathcal{M}$. The following proposition explains how the local connection forms defined by different local trivialisations are related to each other.

Proposition 3.1.1. *The local connection forms on overlapping charts U_i and U_j are related by*

$$\omega_j = \mathrm{Ad}(t_{ij}^{-1})\omega_i + t_{ij}^* \omega_G. \tag{3.1.2}$$

Proof. On two overlapping charts U_i and U_j, we observe that $\sigma_j(x) = \sigma_i(x) t_{ij}(x)$. If $X \in T_x\mathcal{M}$, it follows from Prop. 2.1.1 that

$$\sigma_{j*}X = R_{t_{ij}*}\sigma_{i*}X + \rho_{\sigma_i *} t_{ij*}X.$$

To understand the right-hand side of this equation, it is useful to consider the following diagrams, where the second represents the pushforward of the first, namely,

$$\begin{tikzcd} x \arrow[r] \arrow[rr, bend right, "\sigma_j"'] & (\sigma_i(x), t_{ij}(x)) \arrow[r] & \sigma_i(x) t_{ij}(x) \end{tikzcd}$$

and (by Prop. 2.1.1)

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "\sigma_{j*}"'] & (\sigma_{i*}X, t_{ij*}X) \arrow[r] & R_{t_{ij}*}\sigma_{i*}X + \rho_{\sigma_i *} t_{ij*}X \end{tikzcd}.$$

Next let $t_{ij*}X$ be equal to $A \in \mathfrak{g}$ at $t_{ij}(x)$. Because $\rho_{q*}A_g = A^\star_{qg}$ for any $q \in Q$ and $g \in G$, it follows that $\omega(\rho_{\sigma_i *} t_{ij*}X) = A = \omega_G(t_{ij*}X)$. We thus find that

$$\omega(\sigma_{j*}X) = (R^*_{t_{ij}}\omega)(\sigma_{i*}X) + \omega_G(t_{ij*}X),$$

which proves the assertion. □

A *horizontal lift* of a curve x_t in $\mathcal{M}$ is a curve $\tilde{x}_t$ in Q such that $\pi(\tilde{x}_t) = x_t$ and the tangent vector to $\tilde{x}_t$ is horizontal for any t. The curve $\tilde{x}_t$ thus satisfies $\omega(\dot{\tilde{x}}_t) = 0$. These conditions add up to a system of $\dim Q$ ordinary differential equations, so that there is a unique horizontal lift of x_t for each initial condition $\tilde{x}_0 = q \in \pi^{-1}(x_0)$.

Since it will be of use when defining the covariant derivative on associated vector bundles, we construct a horizontal lift of a curve x_t that lies within a chart U_i. If we choose a section σ_i on U_i such that $\sigma_i(x_0) = \tilde{x}_0$, a curve g_t exists in G for which $\tilde{x}_t = \sigma_i(x_t)g_t$ and $g_0 = e$. Invoking Prop. 2.1.1, one finds that

$$df(\sigma_i(x_t)g_t)/dt = R_{g_t *}df(\sigma_i(x_t))/dt + \rho_{\sigma_i(x_t)*}df(g_t)/dt.$$

The lift is horizontal, so that ω annihilates the left-hand side, which implies that $\omega(R_{g_t *}\sigma_{i*}\dot{x}_t) = -\omega(\rho_{\sigma_i(x_t)*}\dot{g}_t)$, where $\dot{x}_t$ and $\dot{g}_t$ denote the tangent vectors to x_t and g_t, respectively. To simplify the right-hand side, note that if A is the element of $\mathfrak{g}$ that coincides with $\dot{g}_t$ at a given t, then $\rho_{\sigma_i(x_t)*}\dot{g}_t = A^\star_{\sigma_i(x_t)g_t}$, and thus $\omega(\rho_{\sigma_i(x_t)*}\dot{g}_t) = A_e = g_t^{-1}\dot{g}_t$. Therefore, the lift $\tilde{x}_t = \sigma_i(x_t)g_t$ of x_t is horizontal if and only if the curve g_t solves the system of ordinary differential equations

$$\mathrm{Ad}_{g_t^{-1}}\,\omega_i(\dot{x}_t) = -g_t^{-1}\dot{g}_t, \quad \text{with } g_0 = e. \tag{3.1.3}$$

The right G action by constant elements maps the horizontal lift $\tilde{x}_t$ trough $\tilde{x}_0$ into a horizontal lift $\tilde{x}_t a$ trough $\tilde{x}_0 a$. Indeed, $\pi(\tilde{x}_t a) = x_t$ and $\omega(R_{a*}\dot{\tilde{x}}_t) = (R^*_{a_t}\omega)(\dot{\tilde{x}}_t) + a_t^{-1}\dot{a}_t$, which is equal to zero if and only if a_t is constant along the curve.

Because the curve x_t has a unique horizontal lift through any point in the fibre above x_0, which in turn singles out a unique point in the fibre above a second point x_1, a corresponding map is induced. This mapping

$$x_t : \pi^{-1}(x_0) \to \pi^{-1}(x_1)$$

is called the *parallel displacement along the curve* x_t of the fibre at x_0 to the fibre at x_1. Since horizontal lifts of x_t are related by the right action of constant elements of G, parallel transport commutes with the constant right action.

Finally, the *curvature* of an Ehresmann connection ω is the $\mathfrak{g}$-valued two-form defined as

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] \quad \Longrightarrow \quad \Omega_i = d\omega_i + \frac{1}{2}[\omega_i, \omega_i], \tag{3.1.4}$$

where $\Omega_i = \sigma_i^*\Omega$ is the local curvature two-form, pulled back to the domain U_i of σ_i. The curvature satisfies a Bianchi identity, namely,

$$d\Omega + [\omega, \Omega] \equiv 0 \quad \Longrightarrow \quad d\Omega_i + [\omega_i, \Omega_i] \equiv 0. \tag{3.1.5}$$

On overlapping charts U_i and U_j, the curvature forms Ω_i and Ω_j are related by[1] [Sha97]

$$\Omega_j = \mathrm{Ad}(t_{ij}^{-1})\Omega_i. \tag{3.1.6}$$

From now on, we shall not refer explicitly to the chart U_i on which the connection and its curvature are pulled back, so that the forms on Q and their local versions on $\mathcal{M}$ are denoted by the same symbols. This is not a problem, for their domains will make clear which we are referring to. The section σ chosen to pull back the connection and curvature is referred to as the *gauge*. Obviously, any other section σ' is a valid gauge as well. On their common domain, two gauges are related by the corresponding transition function g^{-1}, i.e., $\sigma' = \sigma g^{-1}$ which is referred to as the *gauge transformation*. The behavior under gauge transformations of an Ehresmann connection and its curvature on the base manifold is determined by Eqs. (3.1.2) and (3.1.6), for $t_{ij} = g^{-1}$.

To conclude this section on Ehresmann connections on principal bundles, we introduce parallel transport and covariant differentiation on associated vector bundles, which is a derivative operator that is consistent with the geometry of the bundle. Let $E = Q[V]$ be the associated bundle with $Q(\mathcal{M}, G)$, such that V is a linear representation space of G. The fibres of E are isomorphic to V as vector spaces, whereby vector addition and scalar multiplication is given by

$$c_1[q, v] + c_2[q, w] = [q, c_1 v + c_2 w].$$

If the set $\{E_a\}_{a=1\ldots\dim V}$ forms a basis for V, then the collection $\{[q, E_a]\}$ is a basis for the fibre over $x = \pi(q)$, because any element $\pi_E^{-1}(x)$ can be written as $[q, v^a E_a] = v^a[q, E_a]$, where $q \in \pi^{-1}(x)$.

Given a curve x_t and a horizontal lift $\tilde{x}_t$, a section $s = [\tilde{x}_t, v(x_t)]$ of the vector bundle E is parallel transported along a curve x_t on $\mathcal{M}$ if the V-valued function $v(x_t)$ is constant. Note that parallel transport is independent of the horizontal lift chosen. Whether a section is parallel transported along x_t therefore only depends on the Ehresmann connection ω, and is quantified by the *covariant derivative* $D_{\dot{x}_t} s = [\tilde{x}_t, \dot{v}(x_t)]$. It follows directly that a section is parallel transported if and only if its covariant derivative vanishes everywhere along x_t.

In order to express the covariant derivative on a local chart $U \subset \mathcal{M}$, we assume that

[1] This relation may also be verified by reconsidering the proof for Prop 3.1.1 if we substitute Ω for ω. More precisely, when we take into account that Ω returns zero if at least one of its arguments is vertical, which is proven in §3.3, one obtains (3.1.6).

x_t lies within U. Let $\sigma : U \to Q$ be a section such that $\sigma(x_t)g_t = \tilde{x}_t$ with $g_0 = e$, and let $e_a(x) = [\sigma(x), E_a]$ be a frame field on U. The covariant derivative of a section $s(x_t) = v^a(x_t)e_a(x_t) = [\tilde{x}_t g_t^{-1}, v(x_t)]$ is given by (note that the G action on V is represented by matrices)

$$D_{\dot{x}_t} s(x_t) = [\tilde{x}_t, d(g_t^{-1})/dt\, v(x_t) + g_t^{-1} dv(x_t)/dt] = [\sigma(x_t), \omega(\dot{x}_t)v(x_t) + \dot{v}_t],$$

where we used (3.1.3). This can be further worked out to give

$$D_{\dot{x}_t}(v^a e_a) = \dot{x}_t^\mu (\partial_\mu v^a + \omega^a{}_{b\mu} v^b) e_a,$$

which leads to the well known expression of the covariant derivative of a field v in component form, that is,

$$(D_\mu v)^a = \partial_\mu v^a + \omega^a{}_{b\mu} v^b. \tag{3.1.7}$$

Under a gauge transformation $\sigma \to \sigma' = \sigma g^{-1}$, we have

$$v^a \mapsto v'^a = g^a{}_b v^b, \quad \omega^a{}_b \mapsto \omega'^a{}_b = g^a{}_c \omega^c{}_d [g^{-1}]^d{}_b + g^a{}_c d[g^{-1}]^c{}_b.$$

3.2 Klein geometry

At the time Felix Klein initiated its *Erlanger Programm* in 1872, of which an English translation was published about twenty years later [Kle93], a collection of homogeneous geometries had come to light, such as, for example, projective and hyperbolic geometry. These geometries differed to a more or less extent from the Euclidean geometry, which for two millennia had been the unique geometry that was described by a consistent set of theorems. Any of the newly conceived geometries had its own theory, which, although showing similarities in its axioms, were above all conspicuous by the degree to which they contrasted.

It was Klein who, instead of giving too much importance to their dissimilarities, appreciated the common structure that existed among these geometries. He observed that each geometry is the study of a set of properties of configurations that belong to some manifold S, and that these geometric properties are invariant under a Lie group G of transformations— the principal group or *Hauptgruppe*— acting on the left on S. An obvious example is given by Euclidean geometry, in which the geometric properties are angles and lengths in three-dimensional space, and the group of transformations is the so-called group of motions, which consists of the rotations $SO(3)$ and the vector group of translations $\mathbb{R}^3$.

Conversely, the geometric properties are characterized by their invariance under the transformations of the principal Lie group. Given a Lie group G and a manifold S of any dimension on which there is defined a left G action, a corresponding geometry is defined, in which the real geometric properties are those that remain invariant under G transformations. Thereby, Klein shifted emphasis of the geometric properties to the principal group, which led to a unified description for "a series of almost distinct theories, which are advancing in comparative independence of each other" [Kle93]. In what follows we take a closer look at Klein's point of view on homogeneous geometries.

Definition. A *homogeneous space* consists of a manifold S and a Lie group G that acts on the left of S in a transitive and effective way. The elements of G are called the symmetries of S, while G is said to be the symmetry group of the homogeneous space.

Given a point $\xi \in S$, H_ξ denotes the subgroup of G that leaves ξ fixed, i.e., $H_\xi(\xi) = \xi$. Because S is transitive under the action of G, it can be identified with the right cosets G/H_ξ, namely,

$$g(\xi) \in S \rightleftharpoons [g] \in G/H_\xi,$$

where $[g]$ is the equivalence class induced by the relation $g \sim gh$ for any $h \in H_\xi$. At first sight, the description of the homogeneous space in terms of its symmetry group apparently forces one to break the symmetry by singling out a point preferred over the others. However, one must take into account that there is nothing special about our choice for ξ to be the *origin* of S, as any other point ξ' will serve as well. The broken symmetries are then understood to be merely hidden. To be sure, if $\xi' = a(\xi)$ were taken the origin, S is identified with $G/H_{\xi'}$, where the isotropy group of ξ' is related with the isotropy group of ξ by the adjoint action, namely, $H_{\xi'} = aH_\xi a^{-1}$.

Instead of describing a geometry with an origin ξ as the pair (S, ξ) and principal group G, we may as well describe it as a pair (G, H_ξ). The change of viewpoint necessitates a choice of origin in a trivial way, since the combined set of isotropy groups of different choices are mutually isomorphic. Nonetheless, it has a physically nontrivial meaning that the group inclusion

$$i : H \to G,$$

is not canonically given, if $i(H) = H_\xi$ denotes the isotropy group of ξ. For now, we shall assume this subtlety understood and denote each one them by the same letter H.

Definition. A *Klein geometry* (G, H) consists of a Lie group G and a closed subgroup $H \subset G$, such that the space of cosets G/H is connected. The manifold $S \simeq G/H$ is called the homogeneous space of the Klein geometry.

The Klein geometry is reductive if there is a direct sum decomposition as vector spaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ that is $\mathrm{Ad}(H)$ invariant, which is equivalent to the conditions

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \quad \text{and} \quad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}.$$

Note that the action of G is effective if and only if the largest subgroup of H normal in G is trivial. The Klein geometry (G, H) is therefore said to be effective in case the identity elements constitute the largest subgroup of H that is normal in G.

The natural projection of G onto its right cosets gives rise to a principal H bundle $G(G/H, H)$, with the right action of H on the bundle space canonically given. The fundamental vector fields generated by elements of $\mathfrak{h}$ are identical to the latter. Indeed, let $A \in \mathfrak{h}$ so that $a_t = \exp(tA)$ is a one-parameter subgroup of H. For any function f on G, one derives

$$A^{\star}_{ga_t} f = \frac{d}{dt} f(ga_t) = L_{g*} A_{a_t} f = A_{ga_t} f.$$

Summarizing, the following structure is associated with an effective Klein geometry (G, H):

(i) The homogeneous space $S \simeq G/H$ on which G acts transitively and effectively;
(ii) A principal fibre bundle $G(G/H, H)$;
(iii) There is a $\mathfrak{g}$-valued one-form on the bundle space, given by the Maurer-Cartan form ω_G, which satisfies
   (a) The linear map $\omega_G : T_g G \to \mathfrak{g}$ is an isomorphism, at any $g \in G$;
   (b) $\omega_G(A^{\star}) = A$, for each fundamental vector field corresponding to $A \in \mathfrak{h}$;
   (c) $R_h^* \omega_G = \mathrm{Ad}(h^{-1})\omega_G$, for every $h \in H$.

After having read the items just summed up, one should pause a moment to assure himself of the power of the Kleinian point of view on homogeneous spaces. The shift from the concrete geometric properties to the abstract Lie theoretic viewpoint allowed one to recognize a fairly rich amount of structure associated with the geometry. It is this Lie theoretic language that allows for a beautiful generalization to nonhomogeneous spaces, and where the homogeneity is retained at the infinitesimal level, which forms the subject of Cartan geometry, to be reviewed in the following section.

When the homogeneous space describes spacetime, the symmetry group is also called the *kinematic group*. Homogeneous spacetimes underly a corresponding theory of special relativity and exhibit metrical properties, which *a priori* are not determined by the symmetry group. For instance, in §4.1 we shall consider de Sitter spaces with different cosmological constants. These have the same symmetry groups but differ in their metrical properties, depending only but crucially on the value of the cosmological constant. The extra information required to introduce a metric on the Klein space G/H can nevertheless be blended without effort into the Lie theoretic structure of Klein geometries, which we sketch as follows. First note that $\mathrm{pr}_{\mathfrak{h}} \circ \omega_G$ is an Ehresmann connection on the bundle $G(G/H, H)$, which we do not verify explicitly as it also follows as a special case from Prop. 3.3.1. Furthermore, the space $H_g = [\mathrm{pr}_{\mathfrak{p}} \circ \omega_G|_g]^{-1}(\mathfrak{p}) \subset T_g G$ is horizontal with respect to the connection $\mathrm{pr}_{\mathfrak{h}} \circ \omega_G$ at any $g \in G$, and is equal in dimension to the tangent space $T_{[g]}G/H$. Therefore, the tangent spaces to the Klein space can be identified with the vector space $\mathfrak{p} = \mathfrak{g}/\mathfrak{h}$, up to the adjoint H action,[2] in accordance with the diagram

$$\begin{tikzcd} T_gG \arrow[dr, "\mathrm{pr}_{\mathfrak{h}} \circ \omega_G"] \arrow[dd, "\pi"] & \\ & \mathfrak{p} \arrow[dl] \\ T_{[g]}G/H & \end{tikzcd}$$

If an $\mathrm{Ad}(H)$ invariant metric is defined on $\mathfrak{p}$, a corresponding metric is induced on the tangent bundle of $S \simeq G/H$. This metric on $\mathfrak{p}$ can be given canonically, such as the Cartan–Killing metric, or defined by hand, and included in the definition of the Klein geometry. The resulting structure is called a *metric Klein geometry*.

3.3 Cartan geometry

The spaces of Klein geometries are perfectly homogeneous, so that they look identical at all points and at every scale. They are adequate to describe physical objects at some scales but generally become quite useless at other scales. For example, when viewed at over a large enough scale our universe looks homogeneous, but such a description clearly becomes insufficient on smaller scales, at which the distribution of matter is nonhomogeneous [WLR99].

Before Klein developed its theory of homogeneous geometry in terms of the principal group, a first notion of nonhomogeneity of manifolds was introduced in the form of Riemannian spaces [Rie54]. More precisely, in Riemannian manifolds the homogeneity of Euclidean spaces disappears, for the distance between two infinitesimally separated points x^μ and $x^\mu + dx^\mu$ ceases to be constant along space, while it is given by the square root of the quadratic form $g_{\mu\nu}(x)dx^\mu dx^\nu$, where the two-form g is called the *metric*. Such a generalization, however, did also break the Euclidean nature at the infinitesimal level of space, since there is no canonical way to relate the geometric properties of infinitesimal elements at different points.[3] This changed when Levi-Civita developed parallel displacement of tangent spaces along curves in space, an operation in which the length of elements and the angles between elements are ensured to be preserved [LC16], or in other words, an operation with respect to which the metric is covariantly constant. In this way, it has meaning to say that the geometry is locally of a Euclidean nature, since

[2] This follows from $H_{gh} = R_{h*}H_g$, see also the discussion following Prop. 3.3.1.

[3] It is true that the tangent space can be shown to have the Euclidean metric, but this is not sufficient to recover an infinitesimal Euclidean geometry, because one should also be able to translate the tangent space at one point to the tangent space at a neighboring point, without modifying the geometric properties, which in this case are the Euclidean length and angles.

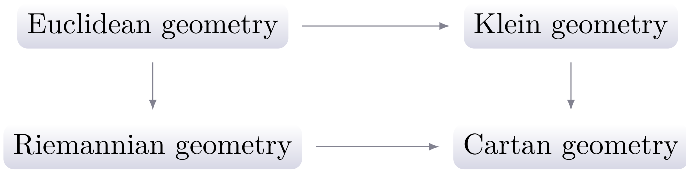

Figure 3.1: Generalization scheme of geometries, where generalization runs along the direction of the arrows. The first row includes homogeneous manifolds in which the symmetry group of Euclidean geometry is generalized to generic Lie groups. The second row are the corresponding nonhomogeneous geometries, which retain the homogeneity of their model geometries at the local level, but which are more general in that they have nonvanishing curvature and torsion. Adapted from [Sha97].

its geometric properties in the tangent space are invariant under rotations and parallel transport. The degree to which nonhomogeneity of the Riemannian space differs from the Euclidean geometry, namely, the degree to which an around an infinitesimal loop parallel transported tangent Euclidean space differs from its preimage, is quantified by the presence of curvature and torsion.

Therefore, Riemannian spaces generalize the flat Euclidean space, for which both curvature and torsion vanish, while their local geometry remains Euclidean. On the other hand, we discussed in the previous section how Klein geometry generalizes Euclidean geometry to generic homogeneous spaces by defining the geometry in terms of a symmetry Lie group. The French mathematician Élie Cartan recognized that both directions of the generalization of Euclidean spaces could be unified in one consistent theory. Just as Riemannian geometry is a nonhomogeneous version of its model Euclidean geometry, any Klein geometry gives way to a nonhomogeneous Cartan geometry [Car23a, Car26, Car35], which retains the geometric properties of the homogeneous space at the infinitesimal level. This scheme of generalizations is summarized in Fig. 3.1.

Our review on Cartan geometry principally follows [Sha97], a must-read for those who want an extended analysis of the subject and some of its applications. Further mathematically rigorous discussions on Cartan geometry are found in [Kob57, AM95], while the articles [Wis09, Wis10, WZ12] are helpful for understanding the geometric intuition that underlies the abstract notions central to the mathematics. In the following, we shall assume that G and H are Lie groups, such that H is a subgroup of G, and denote their Lie algebras by $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

Definition. A *Cartan geometry* (P, A) modeled on $(\mathfrak{g}, H)$ consists of a principal H bundle $P(\mathcal{M}, H)$, on which there exists a $\mathfrak{g}$-valued one-form A— the *Cartan connection*— satisfying the following properties:

(i) The map $A : T_pP \to \mathfrak{g}$ is a linear isomorphism, at any $p \in P$;

(ii) $A(B^\star) = B$, for each fundamental vector field corresponding to $B \in \mathfrak{h}$;
(iii) $R_h^* A = \mathrm{Ad}_{h^{-1}} A$, for every $h \in H$.

A Cartan geometry is said to be reductive if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{g}/\mathfrak{h}$, is a reductive splitting.

The Cartan connection differs from an Ehresmann connection in the first of its defining properties. Whereas the latter connection is valued in the Lie algebra of its structure group, a Cartan connection is valued in a larger algebra that must be of the same dimension as P. This is an important ingredient of Cartan geometries, because it implies that the dimension of the base manifold is equal to the dimension of the homogeneous space of the Klein geometry (G, H), i.e.

$$\dim \mathcal{M} = \dim G/H.$$

The isomorphism A_p can be inverted pointwise to give an injection of $\mathfrak{g}$ into $\mathscr{X}(P)$, namely, $A^{-1} : B \in \mathfrak{g} \mapsto A_p^{-1}(B) \in T_p P$. The H equivariance property of the Cartan connection implies that $A_{ph}^{-1} = R_{h*} \circ A_p^{-1} \circ \mathrm{Ad}_h$, as can be verified from the following commuting diagram:

$$\begin{tikzcd} T_pP \arrow[r, "R_{h*}"] \arrow[d, "A_p"] & T_{ph}P \arrow[d, "A_{ph}"] \\ \mathfrak{g} \arrow[r, "\mathrm{Ad}_{h^{-1}}"] & \mathfrak{g} \end{tikzcd} .$$

Because $A(B^\star) = B$ for each $B \in \mathfrak{h}$ and since A is a linear isomorphism, one concludes that $A^{-1}(B)$ is the fundamental vector field that corresponds to $B \in \mathfrak{h}$. At an arbitrary point p, the restriction of the Cartan connection on the fibres yields

$$A_p(B^\star) = B = \omega_H(B^\star),$$

where on the most right-hand side $B^\star$ denotes the fundamental vector field on the Lie group H. It is in this sense that the Cartan connection is said to restrict to the Maurer-Cartan form on the fibres of P, i.e., $A|_H = \omega_H$.

The *Cartan curvature* is the exterior covariant derivative of the Cartan connection, which is given by

$$F = dA + \frac{1}{2}[A, A], \tag{3.3.1}$$

and satisfies the Bianchi identity

$$dF + [A, F] \equiv 0. \tag{3.3.2}$$

The $\mathfrak{g}$-valued two-form F is strictly horizontal in the sense that it vanishes if at least one of its arguments is a vertical vector field. This can be understood by observing that A restricts to ω_H on the fibres, whose exterior covariant derivative is always zero, see (2.2.1). Explicitly, it is verified as follows. Since F is a two-form, it is enough to check that $F(X^{\mathrm{v}}, Y)$ vanishes, where X^{v} is a vertical and Y a generic vector field on P. At any $p \in P$, there is a $B \in \mathfrak{h}$ such that $X^{\mathrm{v}}_p = B^\star_p = A_p^{-1}(B)$. Thence, $F_p(X^{\mathrm{v}}, Y) = dA_p(A_p^{-1}(B), Y) + [B, A_p(Y)]$, and

$$\begin{aligned} dA_p(A_p^{-1}(B), Y) &= \big(i_{A_p^{-1}(B)} \circ dA\big)_p(Y) \\ &= -\big(d \circ i_{A_p^{-1}(B)} A\big)_p(Y) + \big(\mathscr{L}_{A_p^{-1}(B)} A\big)_p(Y), \end{aligned}$$

where we used (2.1.8). Since $i_{B^\star} A = B$ is a constant function, the first term vanishes. If we denote $b_t = \exp(tB)$, the second term can be written as $\lim_{t\to 0} \frac{1}{t}[(R^*_{b_t} A)_p(Y) - A_p(Y)] = -\operatorname{ad}_B(A_p(Y))$. We thus find that what was looked for:

$$F_p(X^{\mathrm{v}}, Y) = -\operatorname{ad}_B(A_p(Y)) + \operatorname{ad}_B(A_p(Y)) \equiv 0.$$

A Cartan geometry is said to be *flat* if the Cartan curvature vanishes. An example of a flat Cartan geometry modeled on $(\mathfrak{g}, H)$ is the Klein geometry (G, H). More precisely, the pair (G, ω_G) is a flat Cartan geometry, which can easily be verified by comparing the definition for the latter with the structure associated with a Klein geometry discussed in §3.2. Conversely, the space $\mathcal{M}$ of a flat Cartan geometry modeled on $(\mathfrak{g}, H)$ is in the neighborhood of any point of $\mathcal{M}$ isomorphic as a Cartan geometry to an open subset of the Klein space $S \simeq G/H$ [Sha97]. The Cartan curvature is thus a measure that quantifies the nonhomogeneity of $\mathcal{M}$, in comparison with the perfect homogeneity of the Klein space S.

For the following, we restrict our attention to reductive Cartan geometries for which, moreover, the Lie algebra $\mathfrak{g}$ is symmetric. Concretely, there is a reductive splitting as vector spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \tag{3.3.3}$$

such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$. It is the last condition that is necessarily fulfilled in order that a reductive algebra be symmetric. When the Cartan geometry is reductive, it is sensible to consider the corresponding decompositions of the connection and its curvature, thus defined by

$$\begin{aligned} A_{\mathfrak{h}} &= \mathrm{pr}_{\mathfrak{h}} \circ A \\ A_{\mathfrak{p}} &= \mathrm{pr}_{\mathfrak{p}} \circ A \end{aligned} \qquad \text{and} \qquad \begin{aligned} F_{\mathfrak{h}} &= \mathrm{pr}_{\mathfrak{h}} \circ F \\ F_{\mathfrak{p}} &= \mathrm{pr}_{\mathfrak{p}} \circ F, \end{aligned}$$

because they transform reducibly under the right H action.

Proposition 3.3.1. *The one-form $A_{\mathfrak{h}} \in \Omega(P, \mathfrak{h})$ is an Ehresmann connection on P, and the one-form $A_{\mathfrak{p}} \in \Omega(P, \mathfrak{p})$ is a displacement form, namely, H equivariant, i.e.,*

$R_h^* A_{\mathfrak{p}} = \mathrm{Ad}(h^{-1}) A_{\mathfrak{p}}$ *for each* $h \in H$*, and strictly horizontal.*

Proof. Because A is H equivariant and since $\mathfrak{g}$ is reductive, it follows that $R_h^* A_{\mathfrak{h}} + R_h^* A_{\mathfrak{p}} = \mathrm{Ad}(h^{-1}) A_{\mathfrak{h}} + \mathrm{Ad}(h^{-1}) A_{\mathfrak{p}}$. By rearranging this equation so that

$$R_h^* A_{\mathfrak{h}} - \mathrm{Ad}(h^{-1}) A_{\mathfrak{h}} = \mathrm{Ad}(h^{-1}) A_{\mathfrak{p}} - R_h^* A_{\mathfrak{p}},$$

we obtain an equality between an $\mathfrak{h}$-valued left-hand side and a $\mathfrak{p}$-valued right-hand side. This only makes sense if both sides vanish, which proves the H equivariance of $A_{\mathfrak{h}}$ and $A_{\mathfrak{p}}$.

Next let $B^\star$ be the fundamental vector field that corresponds to $B \in \mathfrak{h}$. We have that $B = A(B^\star) = A_{\mathfrak{h}}(B^\star) + A_{\mathfrak{p}}(B^\star)$, or

$$B - A_{\mathfrak{h}}(B^\star) = A_{\mathfrak{p}}(B^\star).$$

This can only be true if both sides vanish, hence $A_{\mathfrak{h}}(B^\star) = B$ and $A_{\mathfrak{p}}(B^\star) = 0$, which completes the proof. □

Note that for any $B \in \mathfrak{p}$, $A_{\mathfrak{h}}(A^{-1}(B)) = 0$ as A is an isomorphism. Consequently, at any $p \in P$ the subspace $A_p^{-1}(\mathfrak{p})$ is horizontal with respect to the Ehresmann connection $A_{\mathfrak{h}}$. Conversely, any element of H_p must be in $A_p^{-1}(\mathfrak{p})$, so that at $A_p^{-1}(\mathfrak{p}) = H_p$. If $\mathfrak{g}$ is reductive, the distribution $p \to A_p^{-1}(\mathfrak{p})$ is right invariant and is therefore equal to the horizontal distribution that corresponds to $A_{\mathfrak{h}}$:

$$H_{ph} = A_{ph}^{-1}(\mathfrak{p}) = R_{h*} A_p^{-1}(\mathrm{Ad}_h(\mathfrak{p})) = R_{h*} H_p,$$

This also implies that the tangent spaces of $\mathcal{M}$ can be identified with $\mathfrak{p}$, up to adjoint H transformations, because $\pi_* : H_p \to T_x\mathcal{M}$ is right invariant. Moreover, if there is defined an $\mathrm{Ad}(H)$ invariant metric on $\mathfrak{p}$, it can be pulled back to give a metric structure on the base manifold $\mathcal{M}$.

The two-forms $F_{\mathfrak{h}}$ and $F_{\mathfrak{p}}$ are H equivariant and horizontal, because the same is true for F, as has been verified above. For a symmetric Lie algebra, the definition for the Cartan curvature (3.3.1) allows us to express them in terms of $A_{\mathfrak{h}}$ and $A_{\mathfrak{p}}$, namely,

$$F_{\mathfrak{h}} = dA_{\mathfrak{h}} + \frac{1}{2}[A_{\mathfrak{h}}, A_{\mathfrak{h}}] + \frac{1}{2}[A_{\mathfrak{p}}, A_{\mathfrak{p}}], \tag{3.3.4a}$$

and

$$F_{\mathfrak{p}} = dA_{\mathfrak{p}} + [A_{\mathfrak{h}}, A_{\mathfrak{p}}]. \tag{3.3.4b}$$

The $\mathfrak{h}$-valued $F_{\mathfrak{h}}$ is called the (corrected) curvature of the geometry. In general, this is not the same as the exterior covariant derivative of the Ehresmann connection $A_{\mathfrak{h}}$, which is given by $dA_{\mathfrak{h}} + \frac{1}{2}[A_{\mathfrak{h}}, A_{\mathfrak{h}}]$ only. The $\mathfrak{p}$-component $F_{\mathfrak{p}}$ of the Cartan curvature is called the *torsion* of the geometry. The Klein geometry (G, H) is therefore a Cartan geometry modeled on $(\mathfrak{g}, H)$ with vanishing curvature and torsion.

We also decompose the Bianchi identity (3.3.2) in an $\mathfrak{h}$-, respectively, $\mathfrak{p}$-valued component, rendering the identities

$$d_{A_{\mathfrak{h}}} \circ d_{A_{\mathfrak{h}}} A_{\mathfrak{h}} \equiv 0, \tag{3.3.5a}$$

and

$$d_{A_{\mathfrak{h}}} \circ d_{A_{\mathfrak{h}}} A_{\mathfrak{p}} + [A_{\mathfrak{p}}, d_{A_{\mathfrak{h}}} A_{\mathfrak{h}}] \equiv 0, \tag{3.3.5b}$$

where $d_{A_{\mathfrak{h}}}$ is the exterior covariant derivative with respect to the Ehresmann connection $A_{\mathfrak{h}}$.

The structures associated with a Cartan connection were heretofore all considered on the bundle space P only. As we already mentioned in §3.1, it is necessary to pull back the geometric objects to the base manifold, for it is here that physical theories are implemented in a concrete manner. Just as an Ehresmann connection and its curvature are pulled back by a family of gauges σ, a Cartan connection and its curvature give rise to a family of local connection and curvature forms, defined on subdomains of $\mathcal{M}$. These local forms will be denoted by the same symbols as the corresponding forms on P. Furthermore, from the transformation behavior of the Cartan connection and its curvature under gauge transformations $\sigma \mapsto \sigma' = \sigma h^{-1}$, one concludes that the local connection A and curvature F transform as

$$A \mapsto \mathrm{Ad}(h)A + h^{-1*}\omega_H \quad \text{and} \quad F \mapsto \mathrm{Ad}(h)F, \tag{3.3.6}$$

where $h^{-1*}\omega_H = hdh^{-1}$ if H is a matrix group. We conclude in mentioning that the $\mathfrak{p}$-valued part of the local connection, i.e., $A_{\mathfrak{p}} : T_x\mathcal{M} \to \mathfrak{p}$ is an isomorphism. This follows from the first condition in the definition of a Cartan connection. Naturally, such an isomorphism only makes sense for reductive Cartan geometries.

3.4 Relation between Ehresmann and Cartan connections

In this final section on Cartan geometry we discuss the relationship between Ehresmann connections and Cartan connections. To be precise, we shall review how certain Ehresmann connections on a principal G bundle restrict to Cartan connections modeled on $(\mathfrak{g}, H)$ on a reduced H bundle.

The reduction of a principal G bundle to a principal bundle with structure group H, H being a subgroup of G, comes down to the restriction of the structure group G to the smaller group H, as will be reviewed in §3.4.1. At various places in physics, such a mechanism generally models a situation in which a larger symmetry group is broken down to an enclosed group describing less symmetry. Such a reduction of symmetries may be forced by hand or may happen by chance, which is the case in spontaneous symmetry breaking.

We already saw that the notion of symmetry breaking is closely related to the Lie theoretic description of homogeneous spaces in Klein geometry, where the isomorphism $S \simeq G/i(H)$ depends on the inclusion $i : H \to G$, $i(H) = H_\xi$ being the isotropy group of an arbitrary point ξ in S. In a similar way, a principal G bundle reduces to a principal H bundle by singling out a section ξ of the associated G bundle with typical fibre S. Conceptually, the section ξ locally singles out a point $\xi(x)$ of the fibre S_x over x, which breaks the symmetry group G of S pointwise to $i_x(H) = H_{\xi(x)}$. Concomitantly, some Ehresmann connections on the G bundle restrict to Cartan connections on the reduced H bundle. A rather rigorous discussion of this reduction process is indispensable to understand how the accompanying breakdown of symmetry can be recovered by observing that the choice of section ξ is completely arbitrary, an observation that will be of use in §4.3.

3.4.1 Reduction of principal fibre bundles

Our short synthesis on the reduction of principal bundles is based on the work [KN96a, Hus66, HJJS08], but the notation used here is slightly adapted so that it will be easier to make connection with its role played later on in this dissertation.

Definition. Let $P(\mathcal{M}, H)$ be a principal H bundle and $Q(\mathcal{M}, G)$ a principal G bundle over the same base manifold $\mathcal{M}$, such that $i_x : H \to G$ is an inclusion, i.e., an injective homomorphism, for each $x \in \mathcal{M}$. Let $\imath : P \to \imath(P) \subset Q$ be a homeomorphism such that

$$\imath(ph) = \imath(p) i_x(h) \quad \text{and} \quad \pi_Q \circ \imath = \pi_P, \quad \text{where } x = \pi_P(p).$$

We then say that P is a *reduction* of Q and that the structure group G is reduced to the group $H \simeq i_x(H)$, while Q is called an *extension* of P.

Note that the inclusion i_x varies along $\mathcal{M}$, but that each $i_x(H)$ is isomorphic with any other, so that the definition makes sense. Because $i_x(H)$ is a subgroup of G we can construct the space $Q/i_x(H)$ of equivalence classes $[q] = [q\, i_x(h)]$, where $\pi_Q(q) = x$. We denote the canonical projection of Q onto the space of equivalence classes by

$$\mu_x : q \in Q \mapsto [q] \in \frac{Q}{i_x(H)}.$$

Just as we could identify a space S that is symmetric under G with the space of cosets G/H_ξ, where H_ξ is the group leaving ξ fixed, it is possible to identify the associated bundle $Q \times_G S$ with the space of equivalence classes $Q/i_x(H)$, where $i_x(H) = H_{\xi(x)}$ for any $x = \pi_Q(q)$, namely,

$$[q, a\xi] \in Q \times_G S \rightleftharpoons [qa] \in \frac{Q}{i_x(H)}.$$

This is a well-defined identification, for $[qa] = [qa\, i_x(h)] \rightleftharpoons [qa, i_x(h)\xi] = [qa, \xi]$.

Given a principal H bundle $P(\mathcal{M}, H)$, we always can extend it to a principal G bundle as follows. Consider the associated fibre bundle $Q = P \times_{i_x(H)} G$, where $x = \pi_P(p)$, which consists of the equivalence classes $[p, g] = [ph, i_x(h)g]$. The manifold Q is turned into a principal fibre bundle over $\mathcal{M}$ with structure group G by including the right G action, naturally defined by $[p, g]g' = [p, gg']$. The right action on the space of equivalence classes $Q = P[G]$ is well defined, because it commutes with the left action, i.e.,

$$[p, g]g' = [ph, i_x(h)g]g' = [ph, i_x(h)gg'] = [p, gg'].$$

The extension is then constructed by

$$\imath : p \in P \mapsto [p, e] \in Q = P \times_{i_x(H)} G.$$

On the other hand, it is generally not possible to reduce a principal bundle $Q(\mathcal{M}, G)$ to a principal H bundle. This is the subject of study in the following proposition.

Proposition 3.4.1. *A principal bundle Q with structure group G is reducible to a principal bundle P with structure group H, if and only if the associated bundle $Q[S]$, where $S \simeq G/H$, admits a globally defined section.*

Proof. First assume that $\imath : P \to Q$ is a reduction. The composed mapping $\tilde{\sigma} = \mu_x \circ \imath : P \to Q/i_x(H) \simeq Q[S]$ is constant on the fibres of P, since for every $h \in H$

$$\mu_x(\imath(ph)) = \mu_x(\imath(p)i_x(h)) = \mu_x(\imath(p)).$$

Hence, the mapping $\sigma = \tilde{\sigma} \circ \pi_P^{-1} : x \in \mathcal{M} \mapsto \tilde{\sigma}(p) \in Q[S]$ is a section, because

$$\pi_{Q[S]}(\sigma(x)) = \pi_{Q[S]}(\tilde{\sigma}(p)) = \pi_{Q[S]}([\imath(p), \xi]) = \pi_Q(\imath(p)) = x,$$

so that $\pi_{Q[S]} \circ \sigma = \mathrm{id}_{\mathcal{M}}$.

Conversely, let $[q, \xi]$ be a section of $Q[S]$. According to Prop. 2.3.1, there is a corresponding G equivariant mapping $\varphi : Q \to S$ such that $\xi = \varphi(q)$. Consider the subspace of Q, given by

$$\imath(P) = \{q \in Q \mid q = \varphi^{-1}(\xi) \text{ if } \pi_Q(q) = \pi_{Q[S]}([q, \xi])\}.$$

Denote by π_P the restriction of π_Q on $\imath(P)$ and let q_1 and q_2 be elements of $\pi_P^{-1}(x)$ for some $x \in \mathcal{M}$. It follows that $\xi = \varphi(q_1) = \varphi(q_2)$ and that there exists an element $g \in G$ such that

$$\begin{tikzcd}
 & & S \simeq G/H \\
P(\mathcal{M},H) \arrow[r, "\imath"] \arrow[ddr] & Q(\mathcal{M},G) \arrow[ur, "\varphi"] \arrow[dr, "\mu_x"] \arrow[dd] & \\
 & & Q \times_G S \arrow[dl] \\
 & \mathcal{M} \arrow[ur, dashed, bend right, "\sigma"'] &
\end{tikzcd}$$

Figure 3.2: The diagram summarizes which manifolds take part in the reduction process and how they are related by the different mappings discussed in the text. The reductions $\imath$ are in one-onto-one correspondence with the sections σ.

$q_1 = q_2 g$. Because φ is G equivariant, we have that $\xi = g\xi$, so that $g \in i_x(H)$, where $i_x(H)$ is the fixed group of ξ. Therefore, the structure group of $\imath(P)$ is everywhere isomorphic with H and $\imath : P \to Q$ is a reduction from the structure group G to $i_x(H) \simeq H$. □

To conclude, we summarize the reduction process diagrammatically in Fig. 3.2.

3.4.2 Restricting Ehresmann to Cartan connections

Consider a reduction $\imath : P \to Q$ from a principal G bundle to a principal H bundle such that the dimension of P equals G. The following proposition specifies which Ehresmann connections on Q are pulled back by the reduction to Cartan connection on P.

Proposition 3.4.2. *Let $\omega \in \Omega(Q, \mathfrak{g})$ be an Ehresmann connection on Q. If*

$$\ker \omega \cap \imath_*(TP) = 0, \tag{3.4.1}$$

then the one-form $A \in \Omega(P, \mathfrak{g})$ defined by

$$A = \imath^* \omega$$

is a Cartan connection on P.

Proof. Because $\ker \omega_p \cap \imath_*(T_p P)$ is zero for every p, $\imath^*\omega = \omega \circ \imath_*$ is a $\mathfrak{g}$-valued one-form on P that has no kernel.

(i) Because $\imath$ is an injection and the dimension of P and G are equal, $\imath^*\omega$ is an isomorphism at every p;

(ii) Let $B^\star$ be the fundamental vector field on P that corresponds to $B \in \mathfrak{h}$, namely, for each function f we have $B^\star f = df(pb_t)/dt$, where $b_t = \exp(tB)$. It follows that

$$(\imath_* B^\star) f = \frac{d}{dt} f(\imath(p) i_x(b_t)), \quad \text{where } x = \pi_P(p),$$

which is fundamental vector field on Q corresponding to $B \in \mathfrak{h}$. It follows that $\imath^*\omega(B^\star) = B$ for each $B \in \mathfrak{h}$.

(iii) For any p it is true that $\imath \circ R_h(p) = \imath(p)i_x(h) = R_{i_x(h)} \circ \imath(p)$, where $x = \pi_P(p)$. Therefore,

$$(R_h^*\imath^*\omega)_p = \left(\imath^* R_{i_x(h)}^*\omega\right)_p = \mathrm{Ad}(i_x(h))(\imath^*\omega)_p.$$

By identifying $i_x(H) \simeq H$ at any $x \in \mathcal{M}$, this proves that $A = \imath^*\omega$ is a Cartan connection on $P(\mathcal{M}, H)$. ▫

We have seen above that there is a one-onto-one correspondence between reduction $\imath : P \to Q$ and section from the associated bundle $Q[S] = Q \times_G S$. Therefore, a Cartan connection on P can be thought as an Ehresmann connection on Q together with a section of $Q[S]$ for which the corresponding reduction satisfies (3.4.1).

4 | Cartan geometry of spacetimes with a nonconstant cosmological function

The means of Cartan geometry now at our disposal, we next present the geometry of spacetimes that are tangentially approximated by de Sitter spaces whose cosmological constants vary over spacetime [Jen14]. For this purpose we consider a Cartan geometry for which the local Klein space is at each point a de Sitter space, but for which the combined set of pseudoradii forms a nonconstant function on spacetime. We begin with a study of de Sitter space as a metric Klein geometry, after which de Sitter–Cartan spacetimes with a cosmological function are introduced. We show that the torsion of such geometries receives a contribution that is not present for a cosmological constant. The structure group of the obtained de Sitter–Cartan geometry is by construction the Lorentz group $SO(1,3)$. Invoking the theory of nonlinear realizations, we extend the class of symmetries to the enclosing de Sitter group $SO(1,4)$, and compute the corresponding spin connection, vierbein, curvature, and torsion.

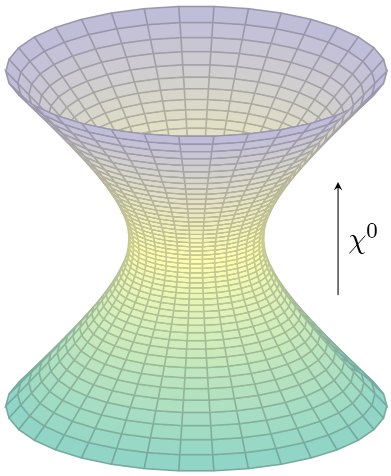

Figure 4.1: Two-dimensional hyperboloid illustrating a de Sitter space with two spacelike dimensions suppressed. The constant time slices represent three-spheres S^3 with time-dependent radii.

4.1 Klein geometry of de Sitter space

Four-dimensional de Sitter space dS is a spacetime that is visualized easiest by embedding it as the hyperboloid[1] [HE73, SSV02]

$$\eta_{AB}\chi^A\chi^B = -\frac{3}{\Lambda} \quad \text{with } \Lambda > 0$$

in the five-dimensional vector space $\mathbb{R}^{1,4}$ parametrized by Cartesian coordinates, see Fig 4.1. The number Λ is the cosmological constant of Einstein's equations [Wal84], for which there is a different de Sitter space for every positive value of the cosmological constant. In what follows we shall discuss de Sitter space explicitly from the Lie theoretic point of view offered by metric Klein geometry, see also [KN96b, Wis10] and §3.2. Such a treatment allows us to relate the cosmological constant of dS to a length scale defined in the translational part of the Lie algebra $\mathfrak{so}(1,4)$. This relation will be important when we define spacetimes with a nonconstant cosmological function in §§4.2–4.3.

The group $SO(1,4)$ acts transitively and effectively on de Sitter spaces with arbitrary cosmological constants, a standard result proven in [KN96a], for example. The Lorentz group in five dimensions is therefore also called the *de Sitter group*. Hence, the de Sitter group is the symmetry group of de Sitter space and the pairs $(dS, SO(1,4))$ form a set of homogeneous spaces, namely, one for each Λ. According to §3.2, in order to describe dS in terms of its symmetry group is it necessary to single out an origin o, which we choose to

[1] We use the convention $\eta_{AB} = \mathrm{diag}(+1,-1,-1,-1,-1)$; see also §4.A.

be the south pole, i.e., the point with coordinates $o^A = (0, 0, 0, 0, -\sqrt{3/\Lambda})$. The isotropy group of the south pole consists of the elements of the de Sitter group that are of the form,

$$\Lambda^A{}_B = \begin{pmatrix} \Lambda^a{}_b & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where } \Lambda^a{}_b \in SO(1,3). \tag{4.1.1}$$

The fixed group of the origin is manifestly isomorphic to the Lorentz group in four dimensions, whereas the matrix above clearly illustrates the inclusion $SO(1,3)_o = i(SO(1,3)) \subset SO(1,4)$. Because the action of the de Sitter group on dS is transitive, coset elements of $SO(1,4)/SO(1,3)_o$ are in one-onto-one correspondence with points of de Sitter space, namely, there is the isomorphism

$$\lambda_o : [\Lambda^A{}_B] \in \frac{SO(1,4)}{SO(1,3)_o} \mapsto \chi^A = \Lambda^A{}_B o^B \in dS. \tag{4.1.2}$$

We shall denote the inclusion $SO(1,3)_o$ by $SO(1,3)$ as well, so that it has been verified that $(SO(1,4), SO(1,3))$ is an effective Klein geometry, wherein the Klein space is isomorphic to the de Sitter space, i.e., $SO(1,4)/SO(1,3) \simeq dS$. Note however that the left-hand side of this isomorphism does not make reference to the cosmological constant of the right-hand side. Therefore, the present description offered by Klein geometry is not sufficient if we want to discriminate between de Sitter spaces with different values for Λ. In concordance with §3.2 we extend the structure to a metric Klein geometry in order to introduce metrical properties on the Klein space. To do so, it is necessary to shift focus to the infinitesimal structure of the principal group, namely, to its Lie algebra, as follows.

To begin with, it should be observed that the Klein geometry $(SO(1,4), SO(1,3))$ is a symmetric space. To be precise, in concordance with the definition of symmetric spaces [KN96b, Loo69], there is an involutive automorphism σ on $SO(1,4)$ induced by the linear transformation $S^A{}_B : (\chi^a, \chi^4) \mapsto (-\chi^a, \chi^4)$, which in the fundamental representation acts according to

$$\sigma(\Lambda)^A{}_B = S^A{}_C \Lambda^C{}_D S^D{}_B, \quad S^A{}_B = \begin{pmatrix} -\mathbb{1}_4 & 0 \\ 0 & 1 \end{pmatrix}.$$

From (4.1.1) one sees that the Lorentz subgroup is invariant under the action of σ. This automorphism on $SO(1,4)$ induces a corresponding automorphism on the algebra $\mathfrak{so}(1,4)$, which obviously is given by the adjoint action as well. Because σ is involutive, the vector space $\mathfrak{so}(1,4)$ can be separated in an eigenspace $\mathfrak{h}$ with eigenvalue 1 and an eigenspace $\mathfrak{p}$ with eigenvalue -1, from which it furthermore follows that

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad \text{and } [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h},$$

for σ commutes with the Lie bracket. Since the elements of the Lorentz subgroup remain invariant under σ, $\mathfrak{so}(1,3) \subset \mathfrak{h}$. In fact, in its fundamental representation elements of the de Sitter algebra are of the form

$$\begin{aligned} \omega^A{}_B &= \begin{pmatrix} \omega^a{}_b & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0_4 & \omega^a/l \\ \omega_a/l & 0 \end{pmatrix}, \\ &= \frac{i}{2}\omega^{ab}[M_{cd}]^a{}_b + i\omega^b[P_b]^a{}_4, \end{aligned} \tag{4.1.3}$$

with $\omega^a = -l\omega^a{}_4$ and $\omega_a = \eta_{ab}\omega^b$, and where the M_{ab} span the Lorentz algebra while the $P_a = M_{a4}/l$ the linear complement $\mathfrak{p} = \mathfrak{so}(1,4)/\mathfrak{so}(1,3)$, which are called the infinitesimal *de Sitter translations* or *transvections*. It is easy to check that $\mathfrak{p}$ is the eigenspace of σ with eigenvalue -1, so that $\mathfrak{h} = \mathfrak{so}(1,3)$ and

$$\mathfrak{so}(1,4) = \mathfrak{so}(1,3) \oplus \mathfrak{p} \tag{4.1.4}$$

is the corresponding Cartan decomposition [KN96b].

The symmetric nature of the de Sitter algebra just verified in the fundamental representation can be observed in a way that does not depend on the representation, when we decompose the commutation relations (4.A.1) for $\mathfrak{so}(1,4)$ according to the reductive splitting (4.1.4):

$$-i[M_{ab}, M_{cd}] = \eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad}, \tag{4.1.5a}$$

$$-i[M_{ab}, P_c] = \eta_{ac}P_b - \eta_{bc}P_a, \tag{4.1.5b}$$

$$-i[P_a, P_b] = -l^{-2}M_{ab}. \tag{4.1.5c}$$

Similarly to the decomposition (4.1.3) in the fundamental representation, an element ω of $\mathfrak{so}(1,4)$ may be decomposed according to the reductive splitting (4.1.4) as $\frac{i}{2}\omega^{ab}M_{cd} + i\omega^a P_a$, which naturally is invariant under the action of the Lorentz group. To obtain an expression for the Lorentz action on $\mathfrak{so}(1,4)$ that is independent of the representation employed, it is useful to compute it first for the fundamental representation. More precisely, if $\omega = \frac{i}{2}\omega^{AB}M_{AB}$ is an element of the de Sitter algebra in a generic representation, we would like to find an explicit expression for the components $\frac{i}{2}[\mathrm{Ad}(\Lambda)(\omega)]^{AB}$ in

$$\begin{aligned} \mathrm{Ad}(\Lambda)(\omega) &= \frac{i}{2}[\mathrm{Ad}(\Lambda)(\omega)]^{AB}M_{AB} \\ &= \frac{i}{2}[\mathrm{Ad}(\Lambda)(\omega)]^{ab}M_{ab} + i[\mathrm{Ad}(\Lambda)(\omega)]^a P_a. \end{aligned}$$

These components are clearly independent of the representation, so that they can be found through computing the adjoint action in the fundamental representation. Doing so, one obtains that

$$[\mathrm{Ad}(\Lambda)(\omega)]^{ab} = \Lambda^a{}_c \omega^{cd} \Lambda^b{}_d \quad \text{and} \quad [\mathrm{Ad}(\Lambda)(\omega)]^a = \Lambda^a{}_b \omega^a. \tag{4.1.6}$$

This way, we have an expression for the adjoint action on the de Sitter algebra for any representation, whereas the part that depends on the representation is contained in the form of the generators M_{ab} and P_a. Note how ω^{ab} and ω^a do not mix under the adjoint action, which reflects the reductive nature of the de Sitter algebra.

We are now in the position to implement the isomorphism between the infinitesimal de Sitter translations $\mathfrak{p}$ and the tangent space to dS at the origin. This isomorphism is canonically given as the pushforward of the isomorphism between $SO(1,4)/SO(1,3)$ and de Sitter space, explicitly written out in (4.1.2). With every $i\xi^a P_a \in \mathfrak{p}$ we identify an element of $T_o dS$ given by (let $f \in \mathscr{F}(dS)$)

$$\begin{aligned} \lambda_{o*} \left.\frac{d}{dt}\right|_0 f(\exp(it\xi^a P_a)) &= \left.\frac{d}{dt}\right|_0 f(\lambda_o(\exp(it\xi^a P_a))) \\ &= \left.\frac{d}{dt}\right|_0 [\lambda_o(\exp(it\xi^a P_a))]^A \left.\frac{\partial}{\partial \chi^A}\right|_o f \end{aligned}$$

where

$$\lambda_o(\exp(it\xi^a P_a)) = \exp(it\xi^a [P_a]^A{}_B) o^B = -\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{3/\Lambda} \end{pmatrix} + \frac{1}{l}\sqrt{\frac{3}{\Lambda}} \begin{pmatrix} \xi^0 \\ \xi^1 \\ \xi^2 \\ \xi^3 \\ 0 \end{pmatrix} t + \mathcal{O}(t^2).$$

Consequently, the isomorphism takes the explicit form given by

$$i\xi^a P_a \in \mathfrak{p} \mapsto \frac{1}{l}\sqrt{\frac{3}{\Lambda}} \xi^a \left.\frac{\partial}{\partial \chi^a}\right|_o \in T_o dS. \tag{4.1.7}$$

Observe how the precise relation between corresponding elements depends upon the length scale l, introduced above in the algebra of de Sitter translations. We will leave the value of l arbitrary for the moment, and first consider its role in the metrical properties of the Klein geometry.

To provide the Klein space with a metric structure, according to §3.2 an $\mathrm{Ad}(SO(1,3))$ invariant metric must be introduced on $\mathfrak{p}$. Such a metric is given naturally when restricting the Killing form of $\mathfrak{so}(1,4)$ to $\mathfrak{p}$. For any two elements X and Y in $\mathfrak{so}(1,4)$, the Killing form is given by $B(X,Y) = 4\,\mathrm{tr}(XY)$ [Hel78]. Therefore, upon restricting this form to $\mathfrak{p}$

we find

$$B(i\xi \cdot P, i\vartheta \cdot P) = 4[i\xi \cdot P]^A{}_B[i\vartheta \cdot P]^B{}_A = \frac{8}{l^2}\eta_{ab}\xi^a\vartheta^b,$$

which is a dimensionless number. Because we would like the inner product on $\mathfrak{p}$ to result in a quantity with a dimension of length squared, we make use of the length scale l to define the symmetric bilinear form $\eta = (l^2/8)B : \mathfrak{p} \times \mathfrak{p} \to \mathbb{R}$, so that

$$\eta(i\xi \cdot P, i\vartheta \cdot P) = \eta_{ab}\xi^a\vartheta^b.$$

The isomorphism (4.1.7) induces a corresponding metric on the tangent space at the origin of dS. More precisely, let $g = \lambda_o^{-1*}\eta : T_o dS \times T_o dS \to \mathbb{R}$ so that

$$g(\xi \cdot \partial, \vartheta \cdot \partial) = \frac{l^2\Lambda}{3}\eta(i\xi \cdot P, i\vartheta \cdot P) = \frac{l^2\Lambda}{3}\eta_{ab}\xi^a\vartheta^b.$$

This line element should be compared with the metric on dS that is induced by the geometry of the embedding vector space $\mathbb{R}^{1,4}$. Since the latter is given by $\eta_{ab}\xi^a\vartheta^b$, it is manifest that the metrical properties of the Klein space $SO(1,4)/SO(1,3)$ coincide with the geometry of a corresponding de Sitter space with cosmological constant Λ if and only if we set the length scale introduced in the Lie algebra of de Sitter translations $\mathfrak{p}$ according to [Wis10]

$$l = \sqrt{\frac{3}{\Lambda}}. \tag{4.1.8}$$

4.2 de Sitter–Cartan geometry

In §3 we saw how Cartan geometries generalize homogeneous model Klein spaces to nonhomogeneous spaces with arbitrary curvature and torsion, while preserving the homogeneity of the model space at the infinitesimal scale of the manifold, when the latter is equipped with an adequate Cartan connection.

In this section we construct the Cartan geometry (P, A) that is modeled on $(\mathfrak{so}(1,4), SO(1,3))$, where P is a principal Lorentz bundle over spacetime $\mathcal{M}$ and A is an $\mathfrak{so}(1,4)$-valued Cartan connection. The homogeneous model space for this geometry is de Sitter space, whereas the corresponding Klein geometry described in §4.1 is an example of the Cartan geometry with vanishing curvature and torsion. The spacetime thus obtained describes a four-dimensional nonhomogeneous universe whose geometry at the infinitesimal scale reduces to the geometry of de Sitter space, which in this sense are called *tangent* de Sitter spaces. For such a spacetime local kinematics is governed by the de Sitter group, something that will be made concrete in §6.

The cosmological constants of the de Sitter spaces tangent to spacetime are determined by a length scale in the de Sitter algebra wherein the Cartan connection is valued. By letting this length scale be spacetime-dependent in an arbitrary way, with the only restriction of forming a smooth function, we shall obtain a geometry in which the cosmological constants of the local de Sitter spaces vary correspondingly. Doing so, there will be defined a nonconstant *cosmological function* Λ on spacetime from the outset. We shall call the Cartan geometry obtained in the manner just outlined a *de Sitter–Cartan geometry* [Jen14].

In the present section we focus in a rather abstract manner on the basic ingredients of a de Sitter–Cartan geometry with a nonconstant cosmological function. More precisely, the structure group $SO(1,3)$ is considered abstractly a subgroup of $SO(1,4)$ without explicitly making reference to the point of dS of which it is the isotropy group, whereas the de Sitter–Cartan connection will be decomposed according to the corresponding reductive splitting. Such a treatment is mathematically correct but not explicit enough if we would like to interpret the Cartan connection on the Lorentz bundle P as an Ehresmann connection on an extended de Sitter bundle Q together with a section in the associated bundle $Q[dS]$ of de Sitter spaces. In such an interpretation the structure group $SO(1,3) \simeq \pi_P^{-1}(x)$ above each $x \in \mathcal{M}$ is given by the isotropy group of the point in $dS \simeq \pi_E^{-1}(x)$ singled out by the given section. For the moment we shall not enter into further details on this issue, and come back to it in the following section. Let us nonetheless emphasize that the geometric structure here outlined is complete, and that the spin connection, vierbein, curvature and torsion of this section can also be seen as gauge-fixed versions of the corresponding objects calculated in the section hereafter, as will become clear there.

A de Sitter–Cartan geometry is thus constructed over spacetime $\mathcal{M}$ when we introduce a $\mathfrak{so}(1,4)$-valued Cartan connection A, which may be decomposed with respect to the reductive splitting (4.1.4) as

$$A = A_{\mathfrak{so}(1,3)} + A_{\mathfrak{p}} = \frac{i}{2} A^{ab} M_{ab} + i A^a P_a. \tag{4.2.1}$$

Under the local Lorentz transformation Λ the Cartan connection transforms according to (see Eq. (3.3.6))

$$A \mapsto \mathrm{Ad}(\Lambda) A + \Lambda d \Lambda^{-1}.$$

Proposition 3.3.1 implies that the $\mathfrak{so}(1,3)$-valued part of A is an Ehresmann connection for the Lorentz bundle P, i.e., a spin connection. This can also be seen directly from the transformation behavior of its components:

$$A^a{}_b \mapsto \Lambda^a{}_c A^c{}_d \Lambda_b{}^d + \Lambda^a{}_c d \Lambda_b{}^c.$$

From now on we shall refer to $A^a{}_b$ as the spin connection of the geometry. While this is correct for the fundamental representation, it must be remembered that the spin connection in an arbitrary representation is really given by $\frac{i}{2} A^{ab} [M_{ab}]^\alpha{}_\beta$.

The $\mathfrak{p}$-valued one-form $A_{\mathfrak{p}}$ is called the *coframe field* and constitutes a pointwise mapping between tangent vectors and infinitesimal de Sitter translations. This mapping induces another map that will be denoted by the same symbol as well as given the same name of coframe field,[2] namely,

$$A_{\mathfrak{p}} : V^{\mu}\partial_{\mu} \in T\mathcal{M} \mapsto iV^{a}p_{a} = iA^{a}{}_{\mu}V^{\mu}p_{a} \in E = P \times_{SO(1,3)} \mathfrak{p}, \tag{4.2.2}$$

where $p_a(x) = [\sigma(x), P_a]$ forms a basis for $\pi_E^{-1}(x)$ at any $x \in U \subset \mathcal{M}$ and $\sigma : U \to P$ is a section. For simplicity's sake, we shall denote the coframe field by its algebraic components $A^a = A^a{}_{\mu}dx^{\mu}$— once again, such a notation is complete only when the fundamental representation is considered— and which rotate as a vector under local Lorentz transformations, i.e.,

$$A^a \mapsto \Lambda^a{}_b A^b.$$

Using this shorthand notation, the mapping (4.2.2) is reformulated as $V^a = A^a(V) = A^a{}_{\mu}V^{\mu}$.

Since at any given point A^a is an isomorphism $T_x\mathcal{M} \to \pi_E^{-1}(x) \simeq \mathfrak{p}$, its inverse exists.[3] Such a bundle map takes elements $V^a \in E$ as its input and results in vector fields over $\mathcal{M}$, hence is of the form $A_a = A_a{}^{\mu}\partial_{\mu}$. Since $V^aA_a = V$, we have the orthogonality condition

$$A^a{}_{\nu}A_a{}^{\mu} = \delta^{\mu}_{\nu}. \tag{4.2.3}$$

The vector fields $A_a{}^{\mu}\partial_{\mu}$ constitute the so-called *vierbein*. We choose the vierbein to form a set of vector fields that are dual with the coframe, i.e., $A^a(A_b) = \delta^a_b$, or

$$A^a{}_{\mu}A_b{}^{\mu} = \delta^a_b. \tag{4.2.4}$$

[2] A comment should be made here to clarify some mathematical subtleties regarding what is meant by the coframe field [Wis10]. The form we denoted hitherto by $A_{\mathfrak{p}}$ is a pulled-back version of the $\mathfrak{p}$-valued part of the Cartan connection $A : TP \to \mathfrak{so}(1,4)$ from §3.3. On the other hand, in the physical literature the coframe field is generally a bundle map $\tilde{A}_{\mathfrak{p}}$ from the tangent bundle to the associated bundle $E = P \times_{SO(1,3)} \mathfrak{p}$, i.e., the diagram

$$\begin{tikzcd} TM \arrow[rr, "\tilde{A}_{\mathfrak{p}}"] \arrow[dr, "\pi"'] & & E \arrow[dl, "\pi_E"] \\ & \mathcal{M} & \end{tikzcd}$$

commutes. If we denote the horizontal lift of a tangent vector X by $\tilde{X}$, the coframe field $\tilde{A}_{\mathfrak{p}}$ is defined through $A_{\mathfrak{p}}$ as

$$\tilde{A}_{\mathfrak{p}} : X \in T_x\mathcal{M} \mapsto [p, A_{\mathfrak{p}}(\tilde{X})] \in \pi_E^{-1}(x) \simeq \mathfrak{p}, \quad \text{where } p \in \pi^{-1}(x).$$

It is shown in [Wis10] that the inverse relation identifies an $A_{\mathfrak{p}}$ on P with each coframe field $\tilde{A}_{\mathfrak{p}}$ on $\mathcal{M}$, so that both forms may be denoted by the same name and symbol.

[3] As was mentioned at the end of §3.3, the mapping (4.2.2) is an isomorphism if the first condition in the definition for a Cartan connection holds. If this condition is relaxed, such that the dimensions of P and

Although denoting the coframe A^a and vierbein A_a by the same letter might seem somewhat confusing, there is good reason to do so. To see this we first note that the mapping (4.2.2) induces a metric structure on $\mathcal{M}$. Indeed, the Killing form η on $\mathfrak{p}$ can be pulled back to give a symmetric bilinear form g on $\mathcal{M}$, i.e., for every two tangent vectors V and W we define $g(V,W) = \eta_{ab}A^a(V)A^b(W)$,[4] so that

$$g_{\mu\nu} = \eta_{ab}{A^a}_\mu {A^b}_\nu.$$

This metric is automatically nonsingular and of Lorentzian signature. As usual, its inverse is denoted by $g^{\mu\nu}$. Spacetime indices may then be raised or lower by g, while the same can be done for algebraic indices with η. From the orthogonality conditions (4.2.3) and (4.2.4), we then see that the coframe and vierbein really are the same object with its indices raised or lowered. Therefore, we shall use the name coframe and vierbein interchangeably.

The vierbein identifies with every vector tangent to spacetime a unique infinitesimal de Sitter translation. Intuitively speaking this signifies that displacements in spacetime are generated by de Sitter translations: a tangent vector singles out a direction in which we are to move in spacetime, while the corresponding infinitesimal de Sitter translation is the actual physical displacement. These generators satisfy the commutation relations (4.1.5c), so that the commutator of two infinitesimal translations in a de Sitter–Cartan spacetime is proportional to a Lorentz rotation. The constant of proportionality is essentially the cosmological constant of the tangent de Sitter spaces. To understand this one should remember from §4.1 that the subspace of de Sitter translations $\mathfrak{p} \subset \mathfrak{so}(1,4)$ with length scale l can be identified with the tangent spaces of a de Sitter space with cosmological constant Λ, according to (4.1.7). Moreover, it was shown that the metrical properties of dS implied by the Killing form on $\mathfrak{p}$ under the identification (4.1.7) coincide with those implied by the embedding pseudo-Euclidean space, if and only if $l = \sqrt{3/\Lambda}$, see (4.1.8). Accordingly, at any point the vierbein constitutes a mapping of the space tangent to spacetime onto a tangent space of the de Sitter space with cosmological constant

$$\Lambda = \frac{3}{l^2}. \tag{4.2.5}$$

Because the Cartan connection is at any point valued in a copy of the de Sitter algebra, the corresponding length scales defined pointwise in $\mathfrak{p}$ can be chosen to form an arbitrary smooth function l on spacetime. In particular, its first order derivatives may be nonvanishing. Consequently, we have obtained a de Sitter–Cartan geometry that at any point x is approximated by a de Sitter space whose cosmological constant $\Lambda(x)$ is spacetime-dependent

$\mathfrak{g}$ remain equal, *without* $A_p : T_pP \to \mathfrak{g}$ being an isomorphism, however, one obtains the definition for a *generalized Cartan geometry* [Wis10]. Then (4.2.2) cannot be assumed to be an isomorphism and the corresponding metric structure on $\mathcal{M}$ will be degenerate.

[4] See §3.2.

in a nontrivial way. The combined set of these constants will be called the *cosmological function* Λ, which in general is nonconstant:

$$d\Lambda \neq 0.$$

With the objects at hand it is possible to define local Lorentz and spacetime covariant derivatives. The covariant derivative of sections of E with respect to the spin connection is given by

$$D_\mu V^a = \partial_\mu V^a + A^a{}_{b\mu} V^b. \tag{4.2.6}$$

Correspondingly, an affine covariant derivative $\nabla = d + \Gamma$ is defined by

$$\Gamma^\rho{}_{\nu\mu} = A_a{}^\rho D_\mu A^a{}_\nu.$$

The vierbein is then covariantly constant with respect to the total covariant derivative, i.e., $D_\mu A^a{}_\nu - \Gamma^\rho{}_{\nu\mu} A^a{}_\rho = 0$. It follows directly that the metric is covariantly constant,[5] namely,

$$\nabla_\rho g_{\mu\nu} = 0.$$

The Cartan curvature of a de Sitter–Cartan geometry is decomposed as

$$F = F_{\mathfrak{so}(1,3)} + F_{\mathfrak{p}} = \frac{i}{2} F^{ab} M_{ab} + i F^a P_a \tag{4.2.7}$$

with respect to the splitting (4.1.4). The $\mathfrak{so}(1,4)$-valued two-form F transforms covariantly under local Lorentz transformations, i.e., $F \mapsto \mathrm{Ad}(\Lambda) F$. This implies that the curvature F^{ab} and torsion F^a rotate as vectors under the action of the Lorentz group:

$$F^a_{\ b} \mapsto \Lambda^a{}_c F^c_{\ d} \Lambda_b{}^d \quad \text{and} \quad F^a \mapsto \Lambda^a{}_b F^b.$$

The symmetric nature of the de Sitter algebra allows us to write the curvature and

[5] A different conclusion can be drawn if one defines the algebraic covariant derivative as

$$\mathcal{D}_\mu V^a = \partial_\mu V^a + A^a{}_{b\mu} V^b - \partial_\mu \ln l \, V^a = D_\mu V^a - \partial_\mu \ln l \, V^a.$$

There is an extra term compared with the standard expression 4.2.6, which explicitly takes into account that the generators $P_a = M_{a4}/l$ change along spacetime. Correspondingly, an affine covariant derivative $\tilde{\nabla} = d + \tilde{\Gamma}$ is defined by

$$\tilde{\Gamma}^\rho{}_{\nu\mu} = A_a{}^\rho \mathcal{D}_\mu A^a{}_\nu = A_a{}^\rho D_\mu A^a{}_\nu - \partial_\mu \ln l \, \delta^\rho_\nu.$$

The vierbein is then covariantly constant with respect to these connections, i.e., $\mathcal{D}_\mu A^a{}_\nu - \tilde{\Gamma}^\rho{}_{\nu\mu} A^a{}_\rho = 0$. On the other hand, one may verify that the metric is *not* covariantly constant, namely,

$$\tilde{\nabla}_\rho g_{\mu\nu} = \partial_\rho \ln l^2 \, g_{\mu\nu}.$$

This conclusion is in concordance with the results of [WZ14]. Note that this equation is completely equivalent with $\nabla_\rho g_{\mu\nu} = 0$, when it is considered that $\tilde{\Gamma} = \Gamma - d \ln l$.

torsion as functions of the spin connection and vierbein. To that end we must make use of (3.3.4) and (4.1.5), after which we find that

$$\begin{aligned} F^a{}_b &= dA^a{}_b + A^a{}_c \wedge A^c{}_b + \frac{1}{l^2} A^a \wedge A_b \\ &= d_A A^a{}_b + \frac{1}{l^2} A^a \wedge A_b, \end{aligned} \tag{4.2.8a}$$

and

$$\begin{aligned} F^a &= dA^a + A^a{}_b \wedge A^b - \frac{1}{l} dl \wedge A^a \\ &= d_A A^a - \frac{1}{l} dl \wedge A^a, \end{aligned} \tag{4.2.8b}$$

where d_A denotes the exterior covariant derivative with respect to the spin connection A^{ab}.

The expressions (4.2.8) for the curvature and torsion in a de Sitter–Cartan geometry differ at two places from the corresponding two-forms in a Riemann–Cartan geometry, which are given by $d_A A^a{}_b$ and $d_A A^a$, respectively [Tra06]. The curvature has an extra term that accounts for the curvature of the local de Sitter spaces. Due to this term, a flat de Sitter–Cartan geometry describes a de Sitter space. In order to see this let us rewrite (4.2.8a) into an expression for the spin curvature

$$d_A A^a{}_b = F^a{}_b - \frac{1}{l^2} A^a \wedge A_b.$$

In a homogeneous de Sitter–Cartan geometry F^{ab} vanishes and $\mathcal{M}$ is identical to the model de Sitter space. This is indeed confirmed by the equation for the spin curvature, since for a homogeneous geometry its Ricci scalar is given by

$$\mathcal{R}[d_A A^a{}_b] = A_b \rfloor A_a \rfloor d_A A^{ab} = -\frac{12}{l^2} = -4\Lambda.$$

In addition, there is a term in the expression (4.2.8b) for the torsion which is new compared with the torsion $d_A A^a$ of a Riemann–Cartan spacetime. The presence of this contribution has its origin in the spacetime-dependence of the length scale l in the algebra of de Sitter transvection $\mathfrak{p}$, and comes about as follows. The torsion is the $\mathfrak{p}$-valued two-form $F_\mathfrak{p} = dA_\mathfrak{p} + [A_{\mathfrak{so}(1,3)}, A_\mathfrak{p}]$.[6] The first term in this expression is expanded as

$$dA_\mathfrak{p} = d(iA^a P_a) = i\, dA^a P_a - i\left(\frac{dl}{l} \wedge A^a\right) P_a,$$

since $P_a = M_{a4}/l$. By use of the relation (4.1.8) between l and the cosmological function

[6] See Eq. (3.3.4b).

Λ, the last term of the torsion can be rewritten as

$$-d\ln l \wedge A^a = \tfrac{1}{2} d\ln\Lambda \wedge A^a,$$

which shows that this contribution depends on the relative infinitesimal change of the cosmological function along spacetime rather than on its absolute change.

The Bianchi identities (3.3.5) for the given de Sitter–Cartan geometry reduce to

$$d_A \circ d_A A^a{}_b \equiv 0, \tag{4.2.9a}$$

and

$$d_A \circ d_A A^a + A^b \wedge d_A A_b{}^a \equiv 0, \tag{4.2.9b}$$

which are identical to the corresponding identities for a Riemann–Cartan geometry [Tra06].

The transformations that are consistent with the given geometry are local Lorentz transformations and spacetime diffeomorphisms, the latter being unphysical as they merely relabel spacetime coordinates [EZ06]. In contrast, we see from (4.2.1) and (4.2.7) that the spin connection and vierbein, and the torsion and curvature form irreducible multiplets with respect to elements of $SO(1,4)$. For example, a local infinitesimal *pure* de Sitter translation $1 + i\epsilon(x)\cdot P$ leads to the following variations of the spin connection and vierbein,

$$\delta_\epsilon A^a{}_b = \frac{1}{l^2}(\epsilon^a A_b - \epsilon_b A^a) \quad \text{and} \quad \delta_\epsilon A^a = -d\epsilon^a - A^a{}_b\epsilon^b + \frac{dl}{l}\epsilon^a, \tag{4.2.10}$$

while for the curvature and torsion it is found that

$$\delta_\epsilon F^a{}_b = \frac{1}{l^2}(\epsilon^a F_b - \epsilon_b F^a) \quad \text{and} \quad \delta_\epsilon F^a = -\epsilon^b F^a{}_b.$$

Due to the reductive nature of $\mathfrak{so}(1,4)$, these geometric objects are well defined up to local Lorentz transformations only. Since local translational symmetry may play an important role in theories of gravity, there is the need to extend the structure group to $SO(1,4)$, while preserving the presence of these different objects, necessary to construct geometric theories of gravity. This will be discussed for the given de Sitter-Cartan geometry in the following section.

4.3 $SO(1,4)$ invariant de Sitter–Cartan geometry

In order to extend the structure group from $SO(1,3)$ to $SO(1,4)$, such that the geometric objects obtained by the decomposition of a Cartan connection and curvature according to the reductive splitting (4.1.4) are well defined, we nonlinearly realize the de Sitter–Cartan connection of §4.2. The formalism of nonlinear realizations was originally developed to systematically study spontaneous symmetry breaking in phenomenological field theory [CWZ69, CCWZ69, Vol73], see also [SS69], in which linearly transforming irreducible multiplets become nonlinear but reducible realizations, when the symmetry group is realized nonlinearly by one of its subgroups. Nonlinear realizations have been applied to gravity in [ISS71, BO75], whereas Stelle and West [SW79, SW80] made use of the formalism to realize connections on spacetime in a nonlinear way. Further discussion on the role of nonlinear realizations in gravitational theories may be found in, for example, [Wis12, TT04, Tre08, HB13].

There is compelling reason why one may expect that nonlinear realizations have their importance in theories of gravity. As we explained in §3.4, a Cartan connection on a principal Lorentz bundle P may be thought of as an Ehresmann connection on a principal $SO(1,4)$ bundle Q over $\mathcal{M}$ that is reduced to P. In essence this is a symmetry breaking process [Wis12], for the reason that it corresponds to singling out a section ξ of the associated bundle $Q[dS] = Q \times_{SO(1,4)} dS$ of tangent de Sitter spaces, thereby reducing the structure group $SO(1,4)$ pointwise to $SO(1,3)_\xi$, the isotropy group of the point $\xi(x)$ in the local de Sitter space $\pi^{-1}_{Q[dS]}(x) \simeq dS$, see also [GG09]. Most importantly, the reduction is not canonical, i.e., the section ξ can be chosen arbitrarily, and the broken symmetries are nonmanifestly restored by realizing them nonlinearly through elements of the Lorentz group. Consequently, decomposing a nonlinear de Sitter-Cartan connection according to the reductive splitting of $\mathfrak{so}(1,4)$ gives way to true geometric objects, well defined with respect to all elements of $SO(1,4)$. These objects, which include a spin connection and vierbein, are indispensable for constructing metric theories of gravity. In short, starting with an $SO(1,4)$-connection which encodes the kinematical group of spacetime at the infinitesimal level, its nonlinear $SO(1,3)$ realization results in the geometric fields with which gravitational theories can be built.

In order to present this section in a self-dependent manner we first review the formalism of nonlinear realizations of the de Sitter group, after which it is applied to Cartan connections to obtain the nonlinear de Sitter-Cartan geometry with a cosmological function.

4.3.1 Nonlinear realizations of the de Sitter group

Apart from the literature cited above we would like to refer the reader to [Zum77] and the Appendices of [WB92], on which the present review is based.

Within some neighborhood of the identity, an element g of $SO(1,4)$ can uniquely be represented in the form

$$g = \exp(i\xi \cdot P)\tilde{h}, \quad \text{with } \tilde{h} \in SO(1,3).$$

Hence, the coordinates ξ^a parametrize a region of the coset space $SO(1,4)/SO(1,3)$ so that they constitute a set of coordinates for that region of the de Sitter space. Note that the elements $\tilde{h}$ by definition constitute the fixed group of the origin $\xi^a = 0$. The parametrization allows us to define the action of $SO(1,4)$ on de Sitter space by

$$g_0 \exp(i\xi \cdot P) = \exp(i\xi' \cdot P)h' \quad \text{with } h' = \tilde{h}'\tilde{h}^{-1}, \tag{4.3.1}$$

and where $\xi' = \xi'(g_0, \xi)$ and $h' = h'(g_0, \xi)$ are in general nonlinear functions of the indicated variables.

To verify that the elements $h'(g_0, \xi)$ form a nonlinear realization of $SO(1,4)$ we compute

$$\begin{aligned} g_1 \exp(i\xi' \cdot P) &= \exp(\xi'' \cdot P)h''(g_1, \xi'), \\ g_1 g_0 \exp(i\xi \cdot P) &= \exp(\xi''' \cdot P)h'''(g_1 g_0, \xi) \\ &= \exp(\xi'' \cdot P)h''(g_1, \xi')h'(g_0, \xi). \end{aligned}$$

It then follows that

$$h'''(g_1 g_0, \xi) = h''(g_1, \xi')h'(g_0, \xi); \quad \xi \xmapsto{g_0} \xi' \xmapsto{g_1} \xi'',$$

which manifestly proves how the group $SO(1,4)$ is realized by its Lorentz subgroup. Remark that one has to keep track of the transformation of the coset parameters under the group composition. This is the trade-off for realizing the de Sitter transformations by elements of the smaller Lorentz group: the latter have become nonlinear functions of the ξ^a.

If σ is a linear representation of $SO(1,4)$ on some vector space V, a corresponding nonlinear realization is constructed as follows. Let $Q[V]$ be the associated vector bundle of $Q(\mathcal{M}, SO(1,4))$ with typical fibre V and denote by ψ some arbitrary section of it. On a local chart, the field ψ transforms according to $\psi(x) \mapsto \psi'(x) = \sigma(g(x))\psi(x)$. Given a section ξ of the associated bundle of de Sitter spaces $Q[dS]$, the nonlinear realization of ψ is pointwise defined as

$$\bar{\psi}(x) = \sigma(\exp(-i\xi(x) \cdot P))\psi(x). \tag{4.3.2}$$

Under a local $SO(1,4)$ transformation $\bar{\psi}$ rotates according to

$$\bar{\psi}' = \sigma(h'(\xi, g_0))\bar{\psi}, \tag{4.3.3}$$

that is, only with respect to its Lorentz indices. The field ψ belonging to a linear irreducible representation of $SO(1,4)$ thus gives way to a nonlinear but reducible realization. The price paid for getting irreducible $SO(1,3)$ representations is their complex nonlinear transformation behavior.

For elements of the Lorentz subgroup, i.e., if $g_0 = h_0 \in SO(1,3)$, the transformations (4.3.1) and (4.3.3) are linear. To see this, we first rewrite (4.3.1) trivially as $h_0 \exp(i\xi \cdot P)h_0^{-1}h_0 = \exp(i\xi' \cdot P)h'$. Because $h_0 \exp(i\xi \cdot P)h_0^{-1} = \exp(i\xi \cdot \mathrm{Ad}(h_0)(P))$ and since the de Sitter algebra is reductive, it follows that

$$\exp(i\xi' \cdot P) = h_0 \exp(i\xi \cdot P)h_0^{-1} \quad \text{and} \quad h' = h_0 \ ,$$

whereas ξ transform linearly as a Lorentz vector, see §4.1.

In principle, this concludes the review on the mathematical framework underlying nonlinear realizations. The remaining part of this subsection will be devoted to computing $\xi'(g_0, \xi)$ and $h'(g_0, \xi)$ explicitly when g_0 is an infinitesimal pure the Sitter translation, i.e., $g_0 = \exp(i\epsilon \cdot P)$, with $\mathcal{O}(\epsilon^2) = 0$, so that $g_0 = \mathbb{1} + i\epsilon \cdot P$.

If we denote $\delta g_0 = i\epsilon \cdot P$, the Taylor series expansion around the identity element of the transformed coset parameters ξ' is given by

$$\xi'(g_0) = \xi'(\mathbb{1}) + \partial_g \xi'(g)|_{\mathbb{1}} \delta g_0 + \mathcal{O}(\delta g_0^2) = \xi + \delta\xi,$$

which induces the following variation on the coset elements:

$$\begin{aligned} \exp(i\xi' \cdot P) &= \exp(i\xi' \cdot P)|_{\mathbb{1}} + \partial_g \exp(i\xi' \cdot P)|_{\mathbb{1}} \delta g_0 + \mathcal{O}(\delta g_0^2) \\ &= \exp(i\xi \cdot P) + \delta\exp(i\xi \cdot P). \end{aligned}$$

Note that the variation in the coset elements depends solely on the variation of the coset coordinates under local de Sitter transformations. Furthermore, up to first order in δg_0 we have that $h' = (\tilde{h} + \delta\tilde{h})\tilde{h}^{-1} = \mathbb{1} + \delta h$. Equation (4.3.1), which determines the variations of ξ and h due to $g_0 = \mathbb{1} + i\epsilon \cdot P$, can then be rewritten in its infinitesimal form as

$$\exp(-i\xi \cdot P)\, i\epsilon \cdot P \exp(i\xi \cdot P) - \exp(-i\xi \cdot P)\, \delta\exp(i\xi \cdot P) = \delta h. \tag{4.3.4}$$

In order to solve (4.3.4) for $\delta\xi$ and δh one first uses the information that $\mathfrak{so}(1,4)$ is symmetric. We explained in §4.1 that the symmetric nature of the de Sitter algebra meant there exists an automorphism such that $\mathfrak{so}(1,3)$ and $\mathfrak{p}$ are eigenspaces with eigenvalues 1, respectively, -1. Applying this automorphism to (4.3.4) and eliminating δh leads to

$$\begin{aligned} &\exp(-i\xi \cdot P)\, \delta\exp(i\xi \cdot P) - \exp(i\xi \cdot P)\, \delta\exp(-i\xi \cdot P) \\ &\qquad = \exp(-i\xi \cdot P)\, i\epsilon \cdot P \exp(i\xi \cdot P) + \exp(i\xi \cdot P)\, i\epsilon \cdot P \exp(-i\xi \cdot P). \end{aligned}$$

When we use the identities (4.B.1) and (4.B.2) this equation takes on the form

$$\frac{1 - \exp(-i\xi \cdot P)}{i\xi \cdot P} \wedge i\delta\xi \cdot P - \frac{1 - \exp(i\xi \cdot P)}{i\xi \cdot P} \wedge i\delta\xi \cdot P$$

$$= \exp(-i\xi \cdot P) \wedge i\epsilon \cdot P + \exp(i\xi \cdot P) \wedge i\epsilon \cdot P,$$

which can be solved for $\delta\xi$, that is,

$$i\delta\xi \cdot P = \frac{i\xi \cdot P \cosh(i\xi \cdot P)}{\sinh(i\xi \cdot P)} \wedge i\epsilon \cdot P. \tag{4.3.5}$$

This gives us the transformed coset parameters $\xi'(\epsilon) = \xi + \delta\xi(\epsilon)$ due to an infinitesimal pure de Sitter translation as prescribed by (4.3.1).

Consequently, one finds $h'(\xi, \epsilon) = \mathbb{1} + \delta h(\xi, \epsilon)$ upon substituting (4.3.5) for (4.3.4) and solving for δh, i.e.,

$$\frac{i}{2}\delta h \cdot M = \frac{1 - \cosh(i\xi \cdot P)}{\sinh(i\xi \cdot P)} \wedge i\epsilon \cdot P. \tag{4.3.6}$$

The right-hand sides of the expressions (4.3.5) and (4.3.6) must be interpreted as power series in the adjoint action of the de Sitter algebra, cf. §4.B. In order to get explicit solutions for these variations we must compute these infinite series of nested commutators, which are given by (4.1.5). This is the final task that hence remains to be done.

The power series for the relevant hyperbolic functions are given by[7] [AS68]

$$\cosh(i\xi \cdot P) = \sum_{n=0}^{\infty} \frac{(i\xi \cdot P)^{2n}}{(2n)!}, \tag{4.3.7a}$$

$$\sinh(i\xi \cdot P) = \sum_{n=0}^{\infty} \frac{(i\xi \cdot P)^{2n+1}}{(2n+1)!}, \tag{4.3.7b}$$

and

$$\operatorname{csch}(i\xi \cdot P) = (i\xi \cdot P)^{-1} + \sum_{n=1}^{\infty} \frac{c_{2n}}{(2n)!}(i\xi \cdot P)^{2n-1}. \tag{4.3.7c}$$

Invoking the identity (4.B.3a) we compute the right-hand side of (4.3.5):

$$\begin{aligned}
&i\xi \cdot P \wedge \operatorname{csch}(i\xi \cdot P) \wedge \cosh(i\xi \cdot P) \wedge i\epsilon \cdot P \\
&= \left(\mathbb{1} + \sum_{n=1}^{\infty} \frac{c_{2n}}{(2n)!}(i\xi \cdot P)^{2n}\right) \wedge \left[\cosh z \left(i\epsilon \cdot P - \frac{\xi \cdot \epsilon\, i\xi \cdot P}{\xi^2}\right) + \frac{\xi \cdot \epsilon\, i\xi \cdot P}{\xi^2}\right] \\
&= \cosh z \left(1 + \sum_{n=1}^{\infty} \frac{c_{2n}}{(2n)!} z^{2n}\right)\left(i\epsilon \cdot P - \frac{\xi \cdot \epsilon\, i\xi \cdot P}{\xi^2}\right) + \frac{\xi \cdot \epsilon\, i\xi \cdot P}{\xi^2} \\
&= \cosh z\, z \operatorname{csch} z \left(i\epsilon \cdot P - \frac{\xi \cdot \epsilon\, i\xi \cdot P}{\xi^2}\right) + \frac{\xi \cdot \epsilon\, i\xi \cdot P}{\xi^2}
\end{aligned}$$

[7] The coefficients in the power series for the hyperbolic cosecant are $c_{2n} = 2(1 - 2^{2n-1})B_{2n}$ with B_i the i-th Bernoulli number, see [AS68].

$$= i\epsilon \cdot P + \Big(\frac{z\cosh z}{\sinh z} - 1\Big)\Big(i\epsilon \cdot P - \frac{\xi \cdot \epsilon\, i\xi \cdot P}{\xi^2}\Big), \quad \text{where } z = l^{-1}(\xi^a \xi_a)^{1/2}.$$

Hence, the variation of the coset coordinates due to an infinitesimal pure de Sitter translation with transformation parameters ϵ^a is given by[8]

$$\delta\xi^a = \epsilon^a + \Big(\frac{z\cosh z}{\sinh z} - 1\Big)\Big(\epsilon^a - \frac{\xi^a \epsilon_b \xi^b}{\xi^2}\Big). \tag{4.3.8}$$

The right-hand side of (4.3.6) is computed in a similar way resulting in the variations

$$\delta h^{ab} = \frac{1}{l^2}\frac{\cosh z - 1}{z \sinh z}(\epsilon^a \xi^b - \epsilon^b \xi^a). \tag{4.3.9}$$

The infinitesimal Lorentz transformation $\mathbb{1} + \frac{i}{2}\delta h^{ab} M_{ab}$ is the nonlinear realization of the infinitesimal de Sitter translation $\mathbb{1} + i\epsilon^a P_a : \xi \mapsto \xi + \delta\xi$.

4.3.2 Nonlinear de Sitter–Cartan geometry

Now that it is clear how irreducible representations of the de Sitter group can be turned into fields transforming nonlinearly only with respect to their Lorentz indices, we shall use this framework to nonlinearly realize $SO(1,4)$ Ehresmann connections. We thus consider an $SO(1,4)$ bundle Q over spacetime $\mathcal{M}$ and a corresponding Ehresmann connection A on $\mathcal{M}$. Under local $SO(1,4)$ transformations, A transforms as

$$A \mapsto g_0 A g_0^{-1} + g_0 d g_0^{-1} = \mathrm{Ad}(g_0)(A + d), \tag{4.3.10}$$

while its curvature $F = dA + \frac{1}{2}[A, A]$ transforms covariantly, i.e.,

$$F \mapsto g_0 F g_0^{-1} = \mathrm{Ad}(g_0) F. \tag{4.3.11}$$

In order to realize A and F nonlinearly we explained in §4.3.1 that it is necessary to single out a section ξ of the associated bundle $Q[dS]$ of de Sitter spaces. Then, in

[8]The computation of (4.3.8) made use of the power series expansion of the hyperbolic cosecant. For a real argument the series only converges on the domain $(-\pi, \pi)$. How can we trust the solution (4.3.8)? Note that (4.3.5) can be rewritten as

$$(i\xi \cdot P)^{-1} \sinh(i\xi \cdot P) \wedge \delta\xi \cdot P = \cosh(i\xi \cdot P) \wedge \epsilon \cdot P,$$

which when worked out gives us

$$z^{-1} \sinh z \Big(\delta\xi \cdot P - \frac{\xi \cdot \delta\xi \xi \cdot P}{\xi^2}\Big) + \frac{\xi \cdot \delta\xi \xi \cdot P}{\xi^2} = \cosh z \Big(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2}\Big) + \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2}\ .$$

This result relies on the power series expansion of the hyperbolic sine, which converges for every value of its argument. It is readily checked that (4.3.8) satisfies the above equation, which confirms its validity.

concordance with the prescription (4.3.2) to construct nonlinear realizations, it follows from (4.3.10) that the nonlinear connection must be defined as [SW80]

$$\bar{A} = \mathrm{Ad}(\exp(-i\xi \cdot P))(A + d). \tag{4.3.12}$$

Under local de Sitter transformations, the $\mathfrak{so}(1,4)$-valued one-form $\bar{A}$ transforms according to

$$\bar{A} \mapsto \mathrm{Ad}(h'(\xi, g_0))(\bar{A} + d).$$

Because elements of $SO(1,4)$ are nonlinearly realized as elements of $SO(1,3)$, $\bar{A}$ is a Cartan connection on a reduced Lorentz bundle, and the reductive decomposition $\bar{A}_{\mathfrak{so}(1,3)} + \bar{A}_{\mathfrak{p}}$ is invariant under local de Sitter transformations. It is then sensible to define the spin connection and vierbein through these projections, namely, as $\omega = \bar{A}_{\mathfrak{so}(1,3)}$ and $e = \bar{A}_{\mathfrak{p}}$, respectively.

The spin connection ω and vierbein e can be expressed in terms of the section ξ and the projections $A_{\mathfrak{so}(1,3)}$ and $A_{\mathfrak{p}}$ of the linear $SO(1,4)$ connection. These relations follow from (4.3.12), in which the different objects appear according to

$$\frac{i}{2}\omega^{ab}M_{ab} + ie^a P_a = \mathrm{Ad}(\exp(-i\xi \cdot P))\Big(\frac{i}{2}A^{ab}M_{ab} + iA^a P_a + d\Big).$$

To carry out the computation of the right-hand side, we first rewrite it with the help of the identities (4.B.1) and (4.B.2) in the form

$$\exp(-i\xi \cdot P) \wedge \big(\frac{i}{2}A^{ab}M_{ab} + iA^a P_a\big) + \frac{1 - \exp(-i\xi \cdot P)}{i\xi \cdot P} \wedge d(i\xi \cdot P).$$

This expression has to be worked out and terms must be collected in two parts— one valued in the Lorentz algebra $\mathfrak{so}(1,3)$ and a second taking values in the subspace of transvections $\mathfrak{p}$. That such a decomposition can be done explicitly follows from the symmetric nature of the de Sitter algebra: for any two elements X and Y in $\mathfrak{h}$ or $\mathfrak{p}$, the element $X \wedge Y$ is in $\mathfrak{h}$ or $\mathfrak{p}$. By using the identities (4.B.3), it is found successively that

$$\begin{aligned}\exp(-i\xi \cdot P) \wedge \frac{i}{2}A^{ab}M_{ab} = \frac{i}{2}\big(A^{ab} + \frac{\cosh z - 1}{l^2 z^2}\xi_c(\xi^b A^{ac} - \xi^a A^{bc})\big)M_{ab} \\ + i\big(z^{-1}\sinh z A^a{}_b \xi^b\big)P_a,\end{aligned}$$

$$\begin{aligned}\exp(-i\xi \cdot P) \wedge iA^a P_a = \frac{i}{2}\left(\frac{\sinh z}{l^2 z}(A^a\xi^b - A^b\xi^a)\right)M_{ab} \\ + i\left(A^a + (\cosh z - 1)\left(A^a - \frac{\xi^b A_b \xi^a}{\xi^2}\right)\right)P_a,\end{aligned}$$

and

$$\frac{1-\exp(-i\xi\cdot P)}{i\xi\cdot P}\wedge d(i\xi\cdot P)=\frac{i}{2}\bigg(\frac{\cosh z-1}{l^2z^2}(d\xi^a\xi^b-d\xi^b\xi^a)\bigg)M_{ab}$$
$$+\,i\bigg(\frac{\sinh z}{z}\bigg(d\xi^a-\frac{\xi^b d\xi_b\xi^a}{\xi^2}\bigg)+\frac{\xi^b d\xi_b\xi^a}{\xi^2}-\frac{dl}{l}\xi^a\bigg)P_a.$$

Collecting these different contributions and separating terms according to whether they are valued in $\mathfrak{so}(1,3)$, respectively $\mathfrak{p}$, one gets the expressions for the spin connection $\omega^a{}_b$ and vierbein e^a, namely,

$$\omega^a{}_b=A^a{}_b-\frac{\cosh z-1}{l^2z^2}\big(\xi^a(d\xi_b-A^c{}_b\xi_c)-\xi_b(d\xi^a+A^a{}_c\xi^c)\big)$$
$$-\frac{\sinh z}{l^2z}(\xi^aA_b-\xi_bA^a)\qquad(4.3.13a)$$

and

$$e^a=A^a+\frac{\sinh z}{z}(d\xi^a+A^a{}_b\xi^b)-\frac{dl}{l}\xi^a$$
$$+\,(\cosh z-1)\bigg(A^a-\frac{\xi^bA_b\xi^a}{\xi^2}\bigg)-\bigg(\frac{\sinh z}{z}-1\bigg)\frac{\xi^b d\xi_b\xi^a}{\xi^2}.\qquad(4.3.13b)$$

These expressions are almost identical to the corresponding objects found by Stelle and West [SW80]. The difference to note is that we have a new term in the expression (4.3.13b) for the vierbein, namely, $-d\ln l\,\xi^a$. This term is present because it is possible that the internal de Sitter spaces are characterized by cosmological constants that are not necessarily equal along spacetime. More precisely, one has to take into account the possibility that the in $\mathfrak{p}$ defined length scale is a nonconstant function, see Sec. 4.2. On the other hand, the results of [SW80] specialize for the case that the local de Sitter spaces have the same pseudo-radius at any point in spacetime. When l is a constant function one naturally recovers the results of [SW80].

Under the action of local de Sitter transformations, the linear curvature F rotates in the adjoint representation, see (4.3.11). Because the adjoint action commutes with exterior differentiation, see (3.1.6), one deduces that the nonlinear Cartan curvature $\bar{F}$ is equal to the exterior covariant derivative of the nonlinear connection, i.e.,

$$\bar{F}=\mathrm{Ad}(\exp(-i\xi\cdot P))(F)=d\bar{A}+\frac{1}{2}[\bar{A},\bar{A}],\qquad(4.3.14)$$

which complies with the structure of a Cartan geometry. The nonlinear Cartan curvature is an $\mathfrak{so}(1,4)$-valued two-form on spacetime, which we decompose once again according to $\bar{F} = \bar{F}_{\mathfrak{so}(1,3)} + \bar{F}_{\mathfrak{p}}$. Since $\bar{F}$ transforms— in general nonlinearly— with elements of $SO(1,3)$, the reductive splitting is invariant under local de Sitter transformations. This suggests that $\bar{F}_{\mathfrak{so}(1,3)}$ and $\bar{F}_{\mathfrak{p}}$ must be considered the genuine curvature and torsion of the Cartan geometry, which are denoted by R, respectively T.

Similar to the computation of the spin connection and vierbein is it possible to write the curvature R and torsion T as a function of ξ, $F_{\mathfrak{so}(1,3)}$ and $F_{\mathfrak{p}}$. Indeed, the definition (4.3.14) implies that

$$\tfrac{i}{2} R^{ab} M_{ab} + i T^a P_a = \mathrm{Ad}(\exp(-i\xi \cdot P))\Big(\frac{i}{2} F^{ab} M_{ab} + i F^a P_a\Big),$$

after which the right-hand side must be worked out and written as sum of an $\mathfrak{so}(1,3)$-valued and a $\mathfrak{p}$-valued part. This computation is to a large extent identical to the derivation of (4.3.13a) and (4.3.13b) and results in

$$R^a{}_b = F^a{}_b - \frac{\cosh z - 1}{l^2 z^2}\xi^c(\xi^a F_{bc} - \xi_b F^a{}_c) - \frac{\sinh z}{l^2 z}(\xi^a F_b - \xi_b F^a), \tag{4.3.15a}$$

and

$$T^a = \frac{\sinh z}{z}\xi^b F^a{}_b + \cosh z\, F^a + (1 - \cosh z)\frac{\xi_b F^b \xi^a}{\xi^2}. \tag{4.3.15b}$$

Moreover, from (4.3.14) it follows that

$$R^a{}_b = d_\omega \omega^a{}_b + \frac{1}{l^2} e^a \wedge e_b \quad \text{and} \quad T^a = d_\omega e^a - \frac{1}{l} dl \wedge e^a.$$

These equations, which express the curvature and torsion in terms of the spin connection and vierbein, are the ones to be expected for a Cartan geometry. Because the exterior covariant derivative of $\bar{F}$ is always zero, there are two Bianchi identities that are formally the same as those given by (4.2.9), i.e.,

$$d_\omega \circ d_\omega \omega^a{}_b \equiv 0 \quad \text{and} \quad d_\omega \circ d_\omega e^a + e^b \wedge d_\omega \omega_b{}^a \equiv 0.$$

When the section ξ is gauge-fixed along spacetime, and for convenience at any point be chosen the origin of the tangent de Sitter spaces, i.e., $\xi^a(x) = 0$, all the expressions reduce to those of §4.2. This is to be expected, because the broken symmetries are not considered, and the geometry is described simply by a $SO(1,4)$ Ehresmann connection for which only $SO(1,3)$-transformations— the isotropy group of $\xi^a = 0$— are taken into account. This has precisely been the way in which the de Sitter–Cartan geometry of §4.2 was set up. In this sense, the linear geometry is a gauge-fixed case of the nonlinear geometry discussed in this section. On the other hand, the de Sitter–Cartan geometry of §4.2 can also be seen as one which describes the formal structure, whereas the different geometric objects of this section are explicit examples of quantities making up such a structure.

Therefore, the different secondary objects introduced in §4.2, such as covariant differentiation, the coframe fields and a metric tensor, among others, and the relations that exists between these objects, are also valid for the nonlinear de Sitter–Cartan geometry of this section.

Appendix 4.A The Lorentz group $SO(1, d-1)$

The vector space $\mathbb{R}^{1,d-1}$ is defined as the vector space $\mathbb{R}^d$ together with the nondegenerate symmetric bilinear form

$$\eta : (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \eta_{ab} x^a y^b \in \mathbb{R}, \quad \text{where } \eta_{ab} = \mathrm{diag}(+1, -1, \dots -1).$$

The Lorentz group $SO(1, d-1)$ in d dimension consists of the elements Λ of $Gl(d)$ that fix η, so that

$$\Lambda^a{}_b \in SO(1, d-1) \iff \Lambda^c{}_a \Lambda^d{}_b \eta_{cd} = \eta_{ab},$$

which have a determinant equal to 1, if we only consider elements of $Gl(d)$ that are connected to the identity. Note that $[\Lambda^{-1}]^a{}_b = \Lambda_b{}^a$. For infinitesimal Lorentz transformations $\Lambda^a{}_b = \delta^a_b + \omega^a{}_b$, the matrices $\omega^a{}_b = \ln \Lambda^a{}_b$ are elements of the Lie algebra $\mathfrak{so}(1, d-1)$, which satisfy $\omega_{ab} = -\omega_{ba}$. This implies that the dimension of the Lorentz algebra is equal to $d(d-1)/2$.

The generators of the algebra $\mathfrak{so}(1, d-1)$ may therefore be denoted as $M_{ab} = -M_{ba}$, where the subscript now enumerates the linearly independent elements, and should not be confused with the representation indices of the fundamental representation. A generic linear representation is generated by the corresponding matrices $[M_{ab}]^\alpha{}_\beta$. The Lorentz transformations are given by the matrix exponential of linear combinations of the generators, i.e.,[9]

$$\Lambda(\omega) = \exp(\tfrac{i}{2}\omega^{ab} M_{ab}), \quad \text{with } M_{ab} = -2i \frac{\partial}{\partial \omega^{ab}} \ln \Lambda(\omega).$$

For example, in the fundamental representation one finds that $[M_{cd}]^a{}_b = -i(\eta_{db}\delta^a_c - \eta_{cb}\delta^a_d)$, which is consistent with $\frac{i}{2}\omega^{cd}[M_{cd}]^a{}_b = \omega^a{}_b$. The generators in the fundamental representation allow us to calculate the commutation relations for $\mathfrak{so}(1, d-1)$, which by definition are independent of the representation. They are given by

$$-i[M_{ab}, M_{cd}] = \eta_{ac} M_{bd} - \eta_{ad} M_{bc} + \eta_{bd} M_{ac} - \eta_{bc} M_{ad}\ . \tag{4.A.1}$$

Appendix 4.B Nonlinear realizations

This appendix gathers background and a couple of intermediate results that were omitted in discussing nonlinear realizations for the de Sitter group in §4.3.1.

To make notation less cluttered, we shall denote the adjoint action of a Lie algebra on itself by

$$X \wedge Y = \mathrm{ad}_X(Y) = [X, Y] \in \mathfrak{g}.$$

[9] A conventional factor of 2 is introduced to account for double counting in the sum.

Note that the symbol $\wedge$ for the adjoint action of a Lie algebra is exclusive to this section and to §4.3.1, and should not be confused with the wedge product of differential forms, which is exclusive to the other sections throughout the dissertation.

Repeated use of the adjoint action will be written as

$$X^k \wedge Y = \mathrm{ad}_X^k(Y) = [X, [X, \ldots [X, Y] \ldots]],$$

so that for a power series $f(X) = \sum_k c_k X^k$

$$f(X) \wedge Y = \sum_k c_k X^k \wedge Y$$

denotes a corresponding series of adjoint actions. Given a second function $g(X) = \sum_l d_l X^l$, one obtains

$$\begin{aligned} g(X) \wedge f(X) \wedge X &= \sum_{kl} c_k d_l \mathrm{ad}_X^l(\mathrm{ad}_X^k(Y)) \\ &= \sum_{kl} c_k d_l X^{k+l} \wedge Y = g(X) f(X) \wedge Y, \end{aligned}$$

because of the linearity of the adjoint action. From this result it follows that the equation $f(X) \wedge Y = Z$ can be solved for $Y = f(X)^{-1} \wedge Z$. Note that the inverse function is supposed to be expressed as a power series. Furthermore, one uses the convention that $\mathbb{1} \wedge X = X \neq [\mathbb{1}, X]$.

The following two identities are useful in carrying out the calculations of the present section and §4.3.1. The first is Hadamard's formula, namely,

$$\exp(X) Y \exp(-X) = \exp(X) \wedge Y. \tag{4.B.1}$$

The other is the Campbell-Poincaré fundamental identity, which is given by

$$\exp(-X)\, \delta\exp(X) = \frac{1 - \exp(-X)}{X} \wedge \delta X. \tag{4.B.2}$$

We now verify the identities ($z = l^{-1}\xi$ and $\xi = (\eta_{ab}\xi^a\xi^b)^{1/2}$)

$$(i\xi \cdot P)^{2n} \wedge \epsilon \cdot P = z^{2n}\left(\epsilon \cdot P - \frac{\xi \cdot \epsilon\, \xi \cdot P}{\xi^2}\right); \quad n \geqslant 1, \tag{4.B.3a}$$

$$(i\xi \cdot P)^{2n+1} \wedge \epsilon \cdot P = \frac{1}{2} l^{-2} z^{2n} (\xi^a \epsilon^b - \xi^b \epsilon^a) M_{ab}; \quad n \geqslant 0, \tag{4.B.3b}$$

$$(i\xi \cdot P)^{2n} \wedge \delta h \cdot M = \delta h^{ab} l^{-2} z^{2n-2} \xi^c (\xi_b M_{ac} - \xi_a M_{bc}); \quad n \geqslant 1, \tag{4.B.3c}$$

and

$$(i\xi \cdot P)^{2n+1} \wedge \delta h \cdot M = \delta h^{ab} z^{2n} (\xi_a P_b - \xi_b P_a); \quad n \geqslant 0, \tag{4.B.3d}$$

which were used in some of the computations carried out in §4.3.1. Equation (4.B.3a) follows when we consider

$$\begin{aligned}
i\xi \cdot P \wedge \epsilon \cdot P &= i\xi^a \epsilon^b [P_a, P_b] = l^{-2} \xi^a \epsilon^b M_{ab}; \\
(i\xi \cdot P)^2 \wedge \epsilon \cdot P &= i\xi^c P_c \wedge l^{-2} \xi^a \epsilon^b M_{ab} \\
&= l^{-2} \xi^a \epsilon^b \xi^c (\eta_{ac} P_b - \eta_{bc} P_a) \\
&= l^{-2} \xi^2 \left(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2} \right); \\
(i\xi \cdot P)^4 \wedge \epsilon \cdot P &= l^{-2} \xi^2 (i\xi \cdot P)^2 \wedge \left(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2} \right) \\
&= l^{-2} \xi^2 (i\xi \cdot P)^2 \wedge \epsilon \cdot P \\
&= (l^{-2} \xi^2)^2 \left(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2} \right); \\
&\vdots \\
(i\xi \cdot P)^{2n} \wedge \epsilon \cdot P &= (l^{-2} \xi^2)^n \left(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2} \right).
\end{aligned}$$

This gives (4.B.3a) for $z = l^{-1}\xi$, while (4.B.3b) readily follows as well:

$$\begin{aligned}
(i\xi \cdot P)^{2n+1} \wedge \epsilon \cdot P &= (i\xi \cdot P) \wedge z^{2n} \left(\epsilon \cdot P - \frac{\xi \cdot \epsilon \xi \cdot P}{\xi^2} \right) \\
&= l^{-2} z^{2n} \xi^a \epsilon^b M_{ab} = \frac{1}{2} l^{-2} z^{2n} (\xi^a \epsilon^b - \xi^b \epsilon^a) M_{ab}.
\end{aligned}$$

In a similar manner Eqs. (4.B.3c) and (4.B.3d) are found by[10]

$$\begin{aligned}
(i\xi \cdot P) \wedge \delta h \cdot M &= \delta h^{ab} \xi^c (-i) [M_{ab}, P_c] = \delta h^{ab} (\xi_a P_b - \xi_b P_a); \\
(i\xi \cdot P)^2 \wedge \delta h \cdot M &= 2\delta h^{ab} i\xi \cdot P \wedge \xi_a P_b \\
&= 2\delta h^{ab} \xi_a \xi^c (-i) [P_b, P_c] \\
&= 2\delta h^{ab} l^{-2} \xi_a \xi^c M_{cb} = \delta h^{ab} l^{-2} \xi^c (\xi_b M_{ac} - \xi_a M_{bc}); \\
(i\xi \cdot P)^4 \wedge \delta h \cdot M &= 2\delta h^{ab} l^{-2} \xi_b \xi^c (i\xi \cdot P)^2 \wedge M_{ac} \\
&= 2\delta h^{ab} l^{-2} \xi_b \xi^c l^{-2} \xi^d (\xi_c M_{ad} - \xi_a M_{cd}) \\
&= 2\delta h^{ab} l^{-2} z^2 \xi^d \xi_b M_{ad}
\end{aligned}$$

[10] We use the notation $\delta h \cdot M = \delta h^{ab} M_{ab}$.

$$
\begin{aligned}
&= \delta h^{ab} l^{-2} z^2 \xi^c (\xi_b M_{ac} - \xi_a M_{bc}); \\
&\vdots \\
(i\xi \cdot P)^{2n} \wedge \delta h \cdot M &= \delta h^{ab} l^{-2} z^{2n-2} \xi^c (\xi_b M_{ac} - \xi_a M_{bc}); \\
(i\xi \cdot P)^{2n+1} \wedge \delta h \cdot M &= 2\delta h^{ab} l^{-2} z^{2n-2} \xi^d \xi^c \xi_b (-i) [M_{ac}, P_d] \\
&= 2\delta h^{ab} l^{-2} z^{2n-2} (\xi_b \xi_a \xi \cdot P - \xi^2 \xi_b P_a) \\
&= \delta h^{ab} z^{2n} (\xi_a P_b - \xi_b P_a).
\end{aligned}
$$

5 | Teleparallel gravity

This chapter marks the beginning of the second part of the thesis, in which we utilize de Sitter–Cartan geometry to generalize the gravitational interaction for a nonvanishing cosmological function. In the present chapter we review teleparallel gravity, a theoretical representation different from general relativity for exactly the same physics. We shall see that the mathematical structure of teleparallel gravity is given by a nonlinear Riemann–Cartan geometry without curvature [JP16]. This will inspire us to build in §6 the generalization for a cosmological function on top of de Sitter–Cartan geometry.

5.1 Introduction

Physically equivalent to general relativity in its description of the gravitational interaction, teleparallel gravity is mathematically and conceptually rather different from Einstein's *opus magnum*. Although the precise implementation of general relativity differs from the one of teleparallel gravity, their geometric structures are related by switching between certain subclasses of Riemann–Cartan spacetimes.

General relativity being the standard model for classical gravity, it is naturally well known that the fundamental field is the ten-component metric, which is accompanied by the zero-torsion Levi-Civita connection. In the absence of gravity the connection only represents inertial effects in a certain reference frame, which is characterized by its curvature equaling zero. Subsequently, Einstein naturally incorporated gravity by allowing for connections that have curvature. This way inertial effects and gravity are locally completely equivalent, while mathematically inseparable, albeit the equivalence cannot be maintained over any finite region of spacetime.

This metrical formulation makes no distinction between local Lorentz and coordinate transformations. Conceptually, however, they arguably are very different. Whereas the former represent physical transformations between observers moving at different velocities, the latter just redefine the way one labels spacetime [EZ06]. The distinction is preserved in principle if the metric and spacetime connection are replaced by a corresponding vierbein and spin connection. This way one obtains the vierbein formalism of general relativity, a formulation even mandatory when fermions are to be coupled to gravity [Dir58, BW57]. The geometry that underlies this description is that of a Riemann–Cartan spacetime without torsion, see §5.2.

There is another possibility to generalize the geometry of Minkowski space to incorporate the dynamics of the gravitational field. In special relativity the vierbein represents a class of idealized observers, whereas the spin connection quantifies the inertial effects due to their reference frames. This representation retains its validity in the presence of gravity, when it is torsion instead of curvature that is turned on by gravitating sources. Since the curvature remains zero, the spin connection keeps its role of representing inertial effects only, which can be transformed away globally by singling out a class of inertial observers. If the vierbein gives rise to a metric equivalent to the one of general relativity, the *teleparallel equivalent theory of general relativity*, or simply teleparallel gravity, is obtained. The fact that inertial and gravitational effects are logically separated in teleparallel gravity historically has been of importance, for it allows for a clear-cut definition of a true energy-momentum tensor for the gravitational field [Møl61, dAGP00a]. This will be further clarified in §§5.3–5.5 when we review the basics of teleparallel gravity.

It is interesting to note that teleparallel gravity originally was devised to form a gauge group for the Poincaré translations [HN67, Cho76]. Although the gauge structure of teleparallel gravity unmistakably bears similarities with the one of the Yang–Mills type theories that describe the electromagnetic, the weak, and the strong interaction, there are also important discrepancies. The geometries that encode the interactions of the standard model are abstract, in the sense that they are unrelated to the tangent structure of spacetime, so that the gauge fields are represented by certain Ehresmann connections, see §3.1. Because gravity changes the geometry of spacetime itself, it is not surprising that the mathematical structure is to be different. In §5.3 we shall argue that the geometric structure underlying teleparallel gravity is that of a nonlinear Riemann–Cartan geometry, and that its interpretation as a translational gauge theory can be understood from the particular properties of such a geometry.

This understanding will have its importance when we aim at generalizing teleparallel gravity for the presence of a cosmological function in §6, where we deform the group governing the local spacetime kinematics from the Poincaré to the de Sitter group. Indeed, since the gauge paradigm does not appear to allow for such a generalization— a point to which we shall come back later— it will be necessary to leave the gauge picture for what it is when generalizing teleparallel gravity for de Sitter kinematics.

5.2 Riemann–Cartan geometry

The geometric structure that underlies teleparallel gravity is a Weitzenböck geometry [Wei23], i.e., a Riemann-Cartan spacetime with vanishing curvature but nonzero torsion. In order to establish a notation and reference that will be used throughout this chapter, we here gather some properties regarding such a geometry.

A Riemann-Cartan spacetime is a manifold $\mathcal{M}$ that comes with a Cartan geometry modeled on $(\mathfrak{iso}(1,3), SO(1,3))$, namely, it is the base manifold of a principal Lorentz bundle and is provided with an $\mathfrak{iso}(1,3)$-valued Cartan connection. Consequently, it is a nonhomogeneous spacetime with arbitrary curvature and torsion, and where the degree of nonhomogeneity is compared to the homogeneous affine Minkowski space M. Such a geometry can be obtained as a special case from the de Sitter–Cartan geometry discussed in §4.2. More precisely, upon considering the limit of a diverging length scale $l \to \infty$ in the commutation relations (4.1.5), one recovers the characterizing relations for the Poincaré algebra $\mathfrak{iso}(1,3)$ through an Inönü–Wigner contraction [IW53, Gil02], namely,

$$-i[M_{ab}, M_{cd}] = \eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad}, \tag{5.2.1a}$$

$$-i[M_{ab}, P_c] = \eta_{ac}P_b - \eta_{bc}P_a, \tag{5.2.1b}$$

$$-i[P_a, P_b] = 0. \tag{5.2.1c}$$

The contraction limit corresponds to the vanishing of the cosmological constant[1] of the homogeneous Klein space dS, which becomes Minkowski space. When we perform the contraction at all points of a de Sitter–Cartan spacetime simultaneously, each of the tangent de Sitter spaces reduces to the affine Minkowski space, and one obtains a Riemann–Cartan spacetime. Correspondingly, the geometric quantities that characterize a Riemann–Cartan spacetime are recovered by evaluating this limit in the respective objects for a de Sitter–Cartan spacetime.

Hence, in concordance with the results of §4.2, there is the following structure associated with a Riemann–Cartan geometry. The basic ingredients are the spin connection ${\omega^a}_{b\mu}$ and the vierbein ${e^a}_\mu$, the latter of which is now valued in the abelian algebra of Poincaré

[1] See (4.1.8).

translations $\mathfrak{t} = \operatorname{span}\{P_a\}$. Their curvature and torsion are denoted by

$$B^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{5.2.2a}$$

and

$$G^a = de^a + \omega^a{}_b \wedge e^b. \tag{5.2.2b}$$

The expression (5.2.2b) for the torsion can be solved for the spin connection, which gives the Ricci theorem

$$\omega^a{}_{b\mu} = \mathring{\omega}^a{}_{b\mu} + K^a{}_{b\mu}, \tag{5.2.3}$$

where $\mathring{\omega}^a{}_{b\mu} = \frac{1}{2}e^c{}_\mu(\Omega_{bc}{}^a + \Omega_b{}^a{}_c + \Omega_c{}^a{}_b)$ is the Levi-Civita spin connection, with

$$\Omega_{abc} = e_b \rfloor e_a \rfloor de_c = e_a{}^\mu e_b{}^\nu(\partial_\mu e_{c\nu} - \partial_\nu e_{c\mu})$$

the coefficients of anholonomy, and where

$$K^a{}_{b\mu} = \frac{1}{2}(G^a{}_{\mu b} + G_\mu{}^a{}_b + G_b{}^a{}_\mu), \quad \text{or} \quad G^a{}_{\mu\nu} = K^a{}_{b\mu}e^b{}_\nu - K^a{}_{b\nu}e^b{}_\mu. \tag{5.2.4}$$

The object denoted by K is called the contortion of $\omega^a{}_{b\mu}$. Note that the contortion is an $\mathfrak{so}(1,3)$-valued one-form that transforms covariantly under local Lorentz transformations, which can be deduced from (5.2.3) and (5.2.4). From (5.2.3) it also follows that the Levi-Civita spin connection is the unique spin connection that has no torsion, while the expression for the coefficients of anholonomy proves that $\mathring{\omega}$ is determined completely by the vierbein and its first order derivatives.

Subsequently, it is straightforward to define algebraic and spacetime covariant differentiation by $D = d + \omega$ and $\nabla = d + \Gamma$, respectively, which are interrelated by

$$\Gamma^\rho{}_{\nu\mu} = e_a{}^\rho D_\mu e^a{}_\nu \quad \text{and} \quad \omega^a{}_{b\mu} = e^a{}_\rho \nabla_\mu e_b{}^\rho.$$

A metric structure is readily constructed as well by contracting the vierbeins, i.e., $g_{\mu\nu} = e^a{}_\mu e_{a\nu}$, a symmetric tensor that is covariantly constant, namely, $\nabla_\rho g_{\mu\nu} = 0$.

The Bianchi identities for a generic Riemann–Cartan geometry are given by

$$dB^a{}_b + \omega^a{}_c \wedge B^c{}_b - \omega^c{}_b \wedge B^a{}_c \equiv 0$$

and

$$dG^a + \omega^a{}_b \wedge G^b + e^b \wedge B_b{}^a \equiv 0.$$

It will turn out to be useful to define the $\mathfrak{so}(1,3)$-valued two-form

$$Q^a{}_b = dK^a{}_b + \omega^a{}_c \wedge K^c{}_b - \omega^c{}_b \wedge K^a{}_c - K^a{}_c \wedge K^c{}_b. \tag{5.2.5}$$

This quantity measures the difference between the curvature of the spin connection ω and the curvature of the Levi Civita spin connection, i.e., $\mathring{B} = d_{\mathring{\omega}}\mathring{\omega}$, namely,

$$B^a{}_b = \mathring{B}^a{}_b + Q^a{}_b, \tag{5.2.6}$$

which is verified directly from (5.2.3). Furthermore, the Bianchi identities can be reformulated in terms of Q as

$$\begin{aligned} dQ^a{}_b + \omega^a{}_c \wedge Q^c{}_b - \omega^c{}_b \wedge Q^a{}_c & \\ - K^a{}_c \wedge Q^c{}_b + K^c{}_b \wedge Q^a{}_c + K^a{}_c \wedge B^c{}_b - K^c{}_b \wedge B^a{}_c &\equiv 0 \end{aligned} \tag{5.2.7a}$$

and

$$e^b \wedge Q_b{}^a - e^b \wedge B_b{}^a \equiv 0. \tag{5.2.7b}$$

Riemann–Cartan spacetimes are the fundamental mathematical structure that underly various theories for the gravitational interaction at the classical level. In its most general form it allows for both curvature and torsion, a setting that is considered in the Einstein–Cartan–Sciama–Kibble (ECSK) theory [Car24, Car23b, Kib61, Sci64]; for an extensive account on the inclusion of torsion in general relativity, see [HVDHKN76].

If one demands the spin connection to have no torsion, the resulting geometry is a Riemann spacetime, see Fig. 5.1. This structure is the one considered in (the vierbein formalism of) Einstein's general relativity. It is physically nonequivalent to ECSK theory, because the geometric objects $\mathring{\omega}$ and e have a combined set of ten off-shell gravitational degrees of freedom, whereas the torsion introduces extra components in the ECSK model.

When it is not the torsion that is required to vanish, but the curvature is set equal to zero, the Weitzenböck spacetime is obtained. The spin connection is *pure gauge*, i.e.,

$$\omega^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu \Lambda_b{}^c, \quad \text{where } \Lambda^a{}_b \in SO(1,3). \tag{5.2.8}$$

It is interesting to note that the combined set of degrees of freedom of the spin connection and vierbein in a Weitzenböck spacetime is the same as in a Riemann geometry. Not only is it interesting; it is fundamental for what follows as it opens up the possibility for a theory of gravity that is equivalent to general relativity, but where the spin connection does not depend on any gravitational degrees of freedom. This theory goes by the name of teleparallel gravity, which indeed is equivalent to general relativity in its description of gravitational phenomena, a statement to be clarified in §5.5.

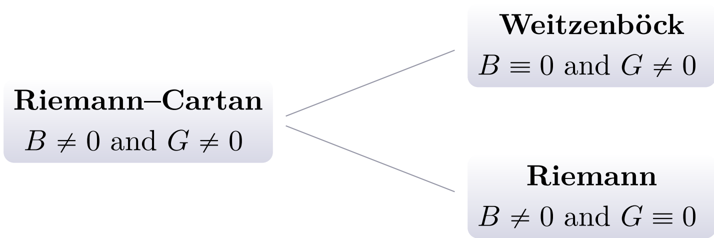

Figure 5.1: This diagram summarizes how a Riemann–Cartan geometry with arbitrary curvature and torsion reduces to a Weitzenböck, respectively, Riemann geometry, when the curvature, respectively, torsion is turned off.

5.3 Fundamentals of teleparallel gravity

In the following paragraphs we briefly review the fundamentals of teleparallel gravity. For an extensive account on the subject we would like to refer the reader to [AP12, dAGP00b, AP04]. Our discussion is based mainly on these references, although we adapt our point of view in order to make the relation with a nonlinear Riemann–Cartan geometry manifest.

In concordance with observational facts we as usual model spacetime as a four-dimensional manifold allowing for events to be specified by one time and three space coordinates. If we furthermore take into account that in the absence of gravity inertial observers at any spacetime point are related by Lorentz transformations, while observers at different points can be linked by Poincaré translations, we would like to maintain such notions at regions in spacetime sufficiently small when gravity is present.[2] This kind of local kinematics is implemented when we provide spacetime with an $\mathfrak{iso}(1,3)$-valued Ehresmann connection A, which we decompose in a Lorentz and translational part, see (4.2.1).[3] We already observed recurrently[4] that such a splitting in practice requires us to single out a point of the Klein space, which for the case here discussed is Minkowski space. As for the de Sitter connection, the splitting withstands local Lorentz transformations but not local Poincaré translations $1 + i\epsilon(x) \cdot P$, for (see also (4.2.10))

$$\delta_\epsilon A^a{}_b = 0 \quad \text{and} \quad \delta_\epsilon A^a = -d\epsilon^a - A^a{}_b \epsilon^b. \tag{5.3.1}$$

Note that, as opposed to A^a, the form $A^a{}_b$ is invariant under translations, so that it is a

[2]Promoting the global symmetries of special relativity to such local ones is really an assumption, because it is possible that turning on gravity alters the kinematical group that governs physics locally. It is nonetheless the most natural assumption, although we will relax it in de Sitter teleparallel gravity in view of the dark energy problem, see §6.

[3]For the moment we set the curvature of A equal to zero, until we specify how to introduce the gravitational degrees of freedom.

[4]See, e.g., §4.1

true spin connection. We therefore define

$$\omega^a{}_b = A^a{}_b \tag{5.3.2}$$

as the spin connection of teleparallel gravity.

Next consider the associated bundle of tangent Minkowski spaces $Q[M] = Q \times_{ISO(1,3)} M$. Because A is a flat connection, i.e., $dA + \frac{1}{2}[A, A] = 0$, spacetime $\mathcal{M}$ is geometrically identical to the model Minkowski space, such that one may identify $\mathcal{M}$ and $M_x = \pi^{-1}_{Q[M]}(x)$ for all $x \in \mathcal{M}$ simultaneously. We do so in an explicit manner as follows. Let ξ^a be a Cartesian coordinate system for M, after which we set $x^\mu = \delta^\mu_a \xi^a$, while for every M_x the point of tangency $\xi^a(x)$ is chosen so that it corresponds in value to $x^\mu \in \mathcal{M}$. This way $x^\mu = \delta^\mu_a \xi^a$ is a (Cartesian) coordinate system for spacetime, while $\xi^a(x)$ forms a section of $Q[M]$. With some imagination it may be seen that such a choice reflects a single identification of (Minkowski) spacetime and all the tangent Minkowski spaces. The tangent space $T_x\mathcal{M}$ of spacetime is consequently identical with the tangent space $T_\xi M_x$, where the corresponding map is the vierbein $e^a{}_\mu = \delta^a_\mu$, since $\partial_\mu = \delta^a_\mu \partial_a$. We are of course free to change coordinate system and under the arbitrary transformation $\delta^\mu_a \xi^a \mapsto x^\mu(\xi)$, the coordinate basis changes according to $\partial_\mu \mapsto \partial_\mu \xi^a \partial_a$, so that the vierbein assumes the form $e^a{}_\mu = \partial_\mu \xi^a$.

There still remains some arbitrariness in the way we identify the tangent Minkowski spaces M_x with spacetime. Firstly, each one of them can undergo a Lorentz rotation, i.e., $\xi^a(x) \mapsto \Lambda^a{}_b(x)\xi^b(x)$. In order for the vierbein to rotate covariantly under local Lorentz transformations, we introduce a connection term, i.e., $e^a{}_\mu = \partial_\mu \xi^a + A^a{}_{b\mu}\xi^b$. This connection term does *not* introduce gravitational degrees of freedom, as it only accounts for inertial effects. To be precise, it is of the form $A^a{}_b = \Lambda^a{}_c d\Lambda_b{}^c$. Lastly, the tangent Minkowski spaces can be acted upon by local Poincaré translations. An infinitesimal translation $1 + i\epsilon(x) \cdot P$ changes the section $\xi^a(x) \mapsto \xi^a(x) + \epsilon^a(x)$. These transformations are of fundamental importance in teleparallel gravity, for they are the gauge transformations on which its construction as a gauge theory of gravity for the translation group $\mathbb{R}^{1,3}$ is based. Because a Lorentz vector does not change components under a translation, the vierbein should remain invariant. Therefore, we must include A^a in its definition:

$$e^a = d\xi^a + A^a{}_b \xi^b + A^a. \tag{5.3.3}$$

From (5.3.1) it follows that e^a is invariant under local Poincaré translations.

The curvature and torsion then take the form

$$B^a{}_b = F^a{}_b \tag{5.3.4a}$$

and

$$G^a = F^a{}_b \xi^b + F^a, \tag{5.3.4b}$$

where the two-forms ${F^a}_b$ and F^a are the exterior covariant derivatives of ${A^a}_b$ and A^a, respectively, with respect to ${A^a}_b$. Because the spin connection is assumed to represent inertial effects only, its curvature (5.3.4a) vanishes, hence the torsion of the geometry is determined entirely by the quantity $F^a = d_A A^a$. This object itself is nonzero only if $A^a \neq d\epsilon^a + {A^a}_b \epsilon^b$ for every choice of ϵ^a, namely, it cannot be set to zero everywhere by a local Poincaré translation. The form A^a is therefore generally given the role of gravitational gauge potential, whereas the torsion is said to be its field strength, in some analogy with Yang-Mills type gauge theories. The analogy is not complete since the torsion is not given by the exterior covariant derivative of the gauge field with respect to itself, which here would be dA^a.

Then what is the mathematical structure underlying teleparallel gravity? We argue it is a nonlinear Riemann–Cartan geometry for which the curvature ${B^a}_b$ of the spin connection is set to zero [JP16]. This can be concluded by considering the limit of an everywhere diverging length scale l for the geometric objects of a de Sitter–Cartan geometry. To be explicit, the spin connection (5.3.2) and vierbein (5.3.3) are clearly recovered from the corresponding objects (4.3.13), while the curvature and torsion (5.3.4) follow from (4.3.15). To specialize for the geometry of teleparallel gravity we just set ${B^a}_b = {F^a}_b = 0$, a condition manifestly consistent with local $ISO(1,3)$ transformations. From this point of view the field A^a is the $\mathfrak{t}$-valued part of a linear $ISO(1,3)$ connection, while the vierbein is the $\mathfrak{t}$-valued projection of the nonlinear Riemann–Cartan connection $\bar{A}$. Furthermore, the invariance of the vierbein e^a and the torsion T^a under local infinitesimal Poincaré translations follows from the fact that they transform nonlinearly with the Lorentz rotation generated by (see (4.3.9))

$$\lim_{l\to\infty} \delta h^{ab} = \frac{1}{l^2}\frac{\cosh z - 1}{z \sinh z}(\epsilon^a \xi^b - \epsilon^b \xi^a) = 0,$$

that is, with the identity transformation. The implied invariance of the vierbein and torsion under these local translations is a crucial ingredient to allow for the interpretation of teleparallel gravity as a gauge theory. Indeed, playing the role of covariant derivative and field strength, respectively, they must transform with the adjoint representation of the gauge group, which is the trivial representation due to abelian nature of $\mathfrak{iso}(1,3)$.

Because the set of de Sitter translations do not constitute a group, it does not appear feasible to construct teleparallel gravity with local kinematics regulated by $SO(1,4)$ through the gauge paradigm. By observing that the structure underlying teleparallel gravity is a nonlinear Riemann–Cartan geometry, it is natural to incorporate local de Sitter kinematics by generalizing for a nonlinear de Sitter–Cartan geometry with a cosmological function. This strategy will be deployed in §6.

5.4 Equations of motion

Now that the geometric structure that underlies teleparallel gravity has been laid out, let us review the equations of motions for a classical test particle moving in a gravitational field, after which we discuss the equations of motion for the gravitational field itself.

5.4.1 Particle mechanics

The action that determines the motion of a point particle with nonzero rest mass m in a gravitational field is defined by $(c = 1)$ [dAP97]

$$\mathcal{S} = -m \int d\tau, \tag{5.4.1}$$

where integration runs along a world line $x^\mu(\tau)$ traced out by the particle between given start and end events, and where $d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$, i.e., τ is the proper time of the particle. By the principal of least action the physical path taken by the particle is the one that extremizes (5.4.1). The equations of motion are thus given by $\delta\mathcal{S} = 0$, where the variation is due to an arbitrary infinitesimal deformation of the world line $\delta x^\mu(\tau)$ that nonetheless vanishes at the start and end events.

Because the four-velocity $u^\mu = dx^\mu/d\tau$ satisfies $d\tau^2 = u^\mu u_\mu d\tau^2$, we have that $u^\mu u_\mu = 1$ and $d\tau = u_\mu dx^\mu = u_a e^a$, where the vierbein is given by (5.3.3). It is readily verified that $\delta d\tau = u_a \delta e^a$ so that

$$\delta\mathcal{S} = -m \int u_a \delta(A^a + d\xi^a + A^a{}_b \xi^b).$$

The variations of the different terms are actually given by the Lie derivatives in the direction δx^μ, i.e., for a zero- and one-form we have, respectively,

$$\delta\xi^a = \partial_\mu \xi^a \delta x^\mu \quad \text{and} \quad \delta A^a{}_b = A^a{}_{b\mu} d\delta x^\mu + \partial_\rho A^a{}_{b\mu} \delta x^\rho dx^\mu.$$

Furthermore, if we integrate by parts terms that contain $d\delta x^\mu$, then the resulting surface terms vanish, as the world line variations there are equal to zero. A straightforward computation leads to

$$\begin{aligned}\delta\mathcal{S} = \int d\tau \delta x^\mu \Bigg[&\left(\frac{du_a}{d\tau} - A^b{}_{a\rho} u_b u^\rho \right) \left(\partial_\mu \xi^a + A^a{}_{b\mu} \xi^b + A^a{}_\mu \right) \\ &- u_a u^\rho \xi^b \left(\partial_\mu A^a{}_{b\rho} - \partial_\rho A^a{}_{b\mu} + A^a{}_{c\mu} A^a{}_{c\rho} - A^a{}_{c\rho} A^a{}_{c\mu} \right) \\ &- u_a u^\rho \left(\partial_\mu A^a{}_\rho - \partial_\rho A^a{}_\mu + A^a{}_{b\mu} A^b{}_\rho - A^a{}_{b\rho} A^b{}_\mu \right) \Bigg].\end{aligned}$$

The last two lines of the integrand can be rewritten as $-u_a u^\rho (\xi^b F^a{}_{b\mu\rho} - F^a{}_{\mu\rho}) = -u_a u^\rho G^a{}_{\mu\rho}$.

When the variation of the action is demanded to vanish we obtain the equations of motion that determine the particle's world line, i.e.,

$$u^\rho D_\rho u^a = u_b u^\rho G^b{}_{\mu\rho} e^{a\mu} = K^a{}_{b\rho} u^b u^\rho. \tag{5.4.2}$$

In teleparallel gravity the spin connection does not have curvature, i.e., $B^a{}_b = 0$, so that $F^a{}_b$ vanishes as well, while G^a is equal to F^a. We remark as a side note that the calculation above remains valid for a generic Riemann–Cartan geometry. Therefore, the equations of motion (5.4.2) also retain their form, as long as one keeps in mind that G^a is the nonlinear torsion (5.3.4b). The equality $G^a = F^a$ in a Weitzenböck spacetime is important for considering teleparallel gravity from the gauge theoretic point of view, because this way the equations of motion take the form of a force equation wherein the right-hand side of (5.4.2) consists of a field strength F^a that corresponds to the gauge field A^a, in analogy with the Lorentz force equation in classical electrodynamics.

Observe that a particle in a gravitational field does not follow geodesics of the Weitzenböck connection, since geodesics are solutions of (5.4.2) with the right-hand side set to zero. The contortion gives rise to a gravitational force that accelerates the particle away from the path it would follow in the absence of gravitating sources. As will be discussed in §5.5, the physical path taken by the particle dictated by (5.4.2) is identical to the one prescribed by general relativity. In particular, the world line does not depend on the particle's mass, for the weak equivalence principle is equally assumed in postulating the action (5.4.1). However, it has been shown in [APV04b] that generalizations of teleparallel gravity can be formulated in which the weak equivalence principle breaks down. On the other hand, it is well known that such a generalization is not reconcilable with the Riemannian spacetime of general relativity.[5]

5.4.2 Gravitational field equations

Although we have found a set of equations that determines the motion of a test particle in the presence of a gravitational field, it remains to be specified how the latter is produced by gravitating sources. In other words, we need to formulate equations of motion for the gravitational field itself, that is, for the vierbein. As has been mentioned before, the presence or not of gravity is encoded in the torsion of the geometry. Therefore, the action for the gravitational field should be the spacetime integral of a scalar made up of at least the torsion. Furthermore, the field equations that follow from the action must be equivalent to those of general relativity, so that they both predict equivalent physical

[5] At the present state of experimental affairs this is not really an argument of much value in favoring teleparallel gravity over general relativity, for there seem to be no indications whatsoever that the weak equivalence principle be false [WSGA12].

behavior. This means that the actions of teleparallel gravity and general relativity are the same— or that the corresponding Lagrangians are equal up to the covariant divergence of a four-vector.

The action for the gravitational field in teleparallel gravity is given by $(c = \hbar = 1)$

$$\mathcal{S}_{\text{tg}} = \frac{1}{2\kappa} \int d^4x \, e \, \mathcal{L}_{\text{tg}}, \tag{5.4.3}$$

where the Lagrangian is defined as

$$\begin{aligned} \mathcal{L}_{\text{tg}} &= \frac{1}{4} G^a{}_{\mu\nu} G_a{}^{\mu\nu} + \frac{1}{2} G^a{}_{\mu\nu} G^{b\mu}{}_\lambda e_a{}^\lambda e_b{}^\nu - G^a{}_{\mu\nu} G^{b\mu}{}_\lambda e_a{}^\nu e_b{}^\lambda \\ &= K^{\mu\rho}{}_\nu K^\nu{}_{\rho\mu} - K^\mu{}_{\rho\mu} K^{\nu\rho}{}_\nu . \end{aligned} \tag{5.4.4}$$

We introduced the number $\kappa = 8\pi G$ with G the gravitational constant and denoted $e = \det e^a{}_\mu$. In §5.5 we shall verify that this action is completely equivalent to the one of general relativity.

The physical vierbein is the one that extremizes (5.4.3), i.e., the one that solves $\delta\mathcal{S} = 0$, where the action is varied with respect to an infinitesimal shift in the components of the vierbein. The variation goes as

$$\delta\mathcal{S} = \frac{1}{2\kappa} \int d^4x \big(\delta e \, \mathcal{L}_{\text{tg}} + e \, \delta\mathcal{L}_{\text{tg}} \big).$$

To further work out the first term in the integrand we use the formula that expresses the variation of the determinant of a matrix in terms of the variations of its components, i.e.,

$$\delta e = e \, e_a{}^\mu \delta e^a{}_\mu .$$

In order to simplify the second contribution in the integral one assumes that the variation of the vierbein goes to zero at infinity, such that surface integrals vanish. One obtains

$$\int d^4x \, e \, \delta\mathcal{L}_{\text{tg}} = \int d^4x \, e \left[\frac{\partial \mathcal{L}_{\text{tg}}}{\partial e^a{}_\mu} - \frac{1}{e} \partial_\rho \left(e \frac{\partial \mathcal{L}_{\text{tg}}}{\partial \partial_\rho e^a{}_\mu} \right) \right] \delta e^a{}_\mu .$$

Combining these results and demanding $\delta\mathcal{S}$ to be zero for any variation in the vierbein gives us the field equations for teleparallel gravity, namely,

$$\partial_\rho \left(e \frac{\partial \mathcal{L}_{\text{tg}}}{\partial \partial_\rho e^a{}_\mu} \right) - e \frac{\partial \mathcal{L}_{\text{tg}}}{\partial e^a{}_\mu} - e \, e_a{}^\mu \mathcal{L}_{\text{tg}} = 0. \tag{5.4.5}$$

When one substitutes the Lagrangian (5.4.4) for $\mathcal{L}_{\text{tg}}$ in the field equations, their explicit form is found:

$$\partial_\rho(e\,W_a{}^{\rho\mu}) - e\,\omega^b{}_{a\rho}W_b{}^{\rho\mu} + e\,G^b{}_{\rho a}W_b{}^{\rho\mu} - e\,e_a{}^\mu\mathcal{L}_{\rm tg} = 0, \tag{5.4.6}$$

where we introduced the *superpotential*

$$W_a{}^{\mu\nu} \equiv G_a{}^{\mu\nu} + G^{\nu\mu}{}_a - G^{\mu\nu}{}_a - 2e_a{}^\nu G^{\lambda\mu}{}_\lambda + 2e_a{}^\mu G^{\lambda\nu}{}_\lambda, \tag{5.4.7}$$

which is a tensor that is antisymmetric in its two spacetime indices. The calculation that leads to these results is summarized in §5.A.

The field equations may also be written as

$$D_\rho(e\,W_a{}^{\rho\mu}) + e\,t_a{}^\mu = 0,$$

where we defined the tensor

$$t_a{}^\mu = G^b{}_{\rho a}W_b{}^{\rho\mu} - e_a{}^\mu\mathcal{L}_{\rm tg}. \tag{5.4.8}$$

Because of the antisymmetry in the last two indices of the superpotential, and since the curvature of ω vanishes, we have that $D_\mu D_\rho(e\,W_a{}^{\rho\mu}) = \frac{1}{2}[D_\rho, D_\mu](e\,W_a{}^{\rho\mu}) = 0$, which in turn implies that

$$D_\mu(e\,t_a{}^\mu) = 0. \tag{5.4.9}$$

The tensor $t_a{}^\mu$ is thus covariantly conserved and can be interpreted as the energy-momentum tensor of the gravitational field [dAGP00a]. Furthermore, in a class of inertial frames the spin connection vanishes everywhere and the covariant derivative reduces to the ordinary derivative. For the corresponding inertial observers the quantities

$$q_a = \int d^3x\, e\, t_a{}^0 \tag{5.4.10}$$

are charges conserved in time, because[6]

$$\partial_0 q_a = -\int d^3x\,\partial_i(e\,t_a{}^i) = 0,$$

as the fields are assumed to vanish at spatial infinity.

From the point of view of noninertial observers the quantities (5.4.10) are *not* conserved in time. As they are accelerating, the observed gravitational field acquires energy and momentum, so that the nonconservedness of q_a is of course very natural— and completely equivalent to noninertial electrodynamics, for example. The charges conserved for such

[6]The superscript i indexes spacelike coordinates.

observers are just the ones derived from ${t_a}^{\mu} - {\omega^b}_{a\rho} {W_b}^{\rho\mu}$, which can be verified directly from the field equations (5.4.6). The second term can be interpreted as the energy-momentum density due to the coupling of gravity to the inertial effects resulting from the noninertial reference frame.

5.5 Equivalence with general relativity

Although teleparallel gravity and general relativity employ different geometric objects to describe the gravitational interaction, the predictions they yield are equivalent on the domain where general relativity applies.

The equations of motion for a test particle moving under the influence of a gravitational field, given in (5.4.2), can be rewritten trivially as

$$u^\rho \partial_\rho u^a + ({\omega^a}_{b\rho} - {K^a}_{b\rho}) u^b u^\rho = 0.$$

We explained in §5.2 that the Levi-Civita spin connection is equal to the difference $\mathring{\omega} = \omega - K$. Therefore, if the flat Weitzenböck geometry is replaced with a Riemannian spacetime, the equations of motion take the form

$$u^\rho \partial_\rho u^\mu + {\mathring{\Gamma}^\mu}_{\rho\lambda} u^\lambda u^\rho = 0,$$

where we used the vierbein to eliminate the Lorentz index in favor of a spacetime index. This is of course the geodesic equation for a Riemannian geometry, and it is well known to determine the world line for a particle moving in a gravitational field according to general relativity [Wal84].

Although both teleparallel gravity and general relativity predict the same world line followed by a particle that interacts with a gravitational field ${e^a}_\mu$, the difference in form of the respective equations of motion indicates that they conceptually speaking have a distinct point of view on the interpretation of the gravitational interaction. General relativity geometrizes the gravitational interaction in the sense that the Levi-Civita connection is determined by the gravitational field. This way, inertial and gravitational effects get mixed up in the same mathematical object, although they are distinguishable physically, when going beyond the infinitesimal structure of spacetime. In teleparallel gravity, on the other hand, the equations of motion (5.4.2) take the form of a force equation with the contortion playing the role of gravitational force, while the connection encodes inertial effects only.

It is no coincidence that both sets of equations of motion lead to the same solution for the particle's worldline. This is so because the generating action of teleparallel gravity was chosen to be equivalent to the one of general relativity, i.e., the particles elapsed proper time. *A priori*, this equivalence is only formally true, since the metric of general relativity and the metric derived from the vierbein of teleparallel gravity may not be the same. In order to show that both theories give rise to the same metrical structure one must verify whether their field equations— and therefore also their solutions— are equivalent. We do so in the remaining paragraphs of this section by proving the total equivalence between the actions for the gravitational field in general relativity and teleparallel gravity.

The Lagrangian of general relativity is the Ricci scalar of the Levi-Civita connection, i.e.,

$$\mathcal{L}_{\text{gr}} = -\mathring{\mathcal{B}} = -\mathring{B}^{ab}{}_{\mu\nu} e_a{}^{\mu} e_b{}^{\nu},$$

where $\mathring{B}$ is the exterior covariant derivative of the Levi-Civita spin connection $\mathring{\omega}$ with respect to itself. Let ω be the Weitzenböck spin connection and K its contortion. The Ricci theorem (5.2.3) implies that $\mathring{B} = B - Q = -Q$, where we observed that $B = d_\omega \omega = 0$ and the two-form Q is defined as in (5.2.5). One thus finds that $\mathcal{L}_{\text{gr}} = \mathcal{Q} = Q^{ab}{}_{\mu\nu} e_a{}^{\mu} e_b{}^{\nu}$. We therefore calculate

$$\begin{aligned}
\mathcal{Q} &= \partial_\mu K^{\mu\nu}{}_{\nu} - D_\mu e_a{}^{\mu} K^{a\nu}{}_{\nu} - D_\mu e_b{}^{\nu} K^{\mu b}{}_{\nu} - \partial_\nu K^{\mu\nu}{}_{\mu} + D_\nu e_a{}^{\mu} K^{a\nu}{}_{\mu} \\
&\quad + D_\nu e_b{}^{\nu} K^{\mu b}{}_{\mu} - K^{\mu}{}_{\rho\mu} K^{\rho\nu}{}_{\nu} + K^{\mu}{}_{\rho\nu} K^{\rho\nu}{}_{\mu} \\
&= 2\partial_\mu K^{\mu\nu}{}_{\nu} + \Gamma^{\mu}{}_{\rho\mu} e_a{}^{\rho} K^{a\nu}{}_{\nu} + \Gamma^{\nu}{}_{\rho\mu} e_b{}^{\rho} K^{\mu b}{}_{\nu} - \Gamma^{\mu}{}_{\rho\nu} e_a{}^{\rho} K^{a\nu}{}_{\mu} \\
&\quad - \Gamma^{\nu}{}_{\rho\nu} e_b{}^{\rho} K^{\mu b}{}_{\mu} - K^{\mu}{}_{\rho\mu} K^{\rho\nu}{}_{\nu} + K^{\mu}{}_{\rho\nu} K^{\rho\nu}{}_{\mu} \\
&= 2\partial_\mu K^{\mu\nu}{}_{\nu} + 2\mathring{\Gamma}^{\mu}{}_{\rho\mu} K^{\rho\nu}{}_{\nu} + 2K^{\mu}{}_{\rho\mu} K^{\rho\nu}{}_{\nu} + 2\mathring{\Gamma}^{\nu}{}_{\rho\mu} K^{\mu\rho}{}_{\nu} + 2K^{\nu}{}_{\rho\mu} K^{\mu\rho}{}_{\nu} \\
&\quad - K^{\mu}{}_{\rho\mu} K^{\rho\nu}{}_{\nu} + K^{\mu}{}_{\rho\nu} K^{\rho\nu}{}_{\mu} \\
&= \frac{2}{e}\partial_\mu (e K^{\mu\nu}{}_{\nu}) + K^{\nu}{}_{\rho\mu} K^{\mu\rho}{}_{\nu} - K^{\mu}{}_{\rho\mu} K^{\nu\rho}{}_{\nu}.
\end{aligned}$$

It follows from (5.4.4) that

$$\mathcal{L}_{\text{tg}} = \mathcal{L}_{\text{gr}} - \frac{2}{e}\partial_\mu (e K^{\mu\nu}{}_{\nu}). \tag{5.5.1}$$

We thus conclude that the actions of teleparallel gravity and general relativity are equivalent up to a surface integral at spacetime infinity. Such a surface term is usually ignored by assuming that the fields at infinity go to zero sufficiently quick. Since the actions are equivalent, the same is true for the corresponding gravitational field equations. As a result, the metric structure determined by teleparallel gravity is the same as the one determined by general relativity for a given current of energy-momentum.

Appendix 5.A Calculation of gravitational field equations

In this section we derive the field equations (5.4.6) for the vierbein in teleparallel gravity. These equations follow when computing (5.4.5) for the Lagrangian (5.4.4). We therefore calculate in succession the partial derivatives of $\mathcal{L}_{\mathrm{tg}}$ with respect to the vierbein and its first order spacetime derivatives, respectively.

We thus begin with

$$\begin{aligned}\frac{\partial \mathcal{L}_{\mathrm{tg}}}{\partial e^c{}_\sigma} = &\frac{1}{4}\frac{\partial G^a{}_{\mu\nu}}{\partial e^c{}_\sigma} G_a{}^{\mu\nu} + \frac{1}{4} G^a{}_{\mu\nu}\frac{\partial G_a{}^{\mu\nu}}{\partial e^c{}_\sigma} + \frac{1}{2}\frac{\partial G^a{}_{\mu\nu}}{\partial e^c{}_\sigma} G^{b\mu}{}_\lambda e_a{}^\lambda e_b{}^\nu \\ &+ \frac{1}{2} G^a{}_{\mu\nu}\frac{\partial G^{b\mu}{}_\lambda}{\partial e^c{}_\sigma} e_a{}^\lambda e_b{}^\nu + \frac{1}{2} G^a{}_{\mu\nu} G^{b\mu}{}_\lambda \frac{\partial (e_a{}^\lambda e_b{}^\nu)}{\partial e^c{}_\sigma} \\ &- \frac{\partial G^a{}_{\mu\nu}}{\partial e^c{}_\sigma} G^{b\mu}{}_\lambda e_a{}^\nu e_b{}^\lambda - G^a{}_{\mu\nu}\frac{\partial G^{b\mu}{}_\lambda}{\partial e^c{}_\sigma} e_a{}^\nu e_b{}^\lambda - G^a{}_{\mu\nu} G^{b\mu}{}_\lambda \frac{\partial (e_a{}^\nu e_b{}^\lambda)}{\partial e^c{}_\sigma}.\end{aligned}$$

In furtherance of the calculation we first compute:

$$\begin{aligned}\frac{\partial G^a{}_{\mu\nu}}{\partial e^c{}_\sigma} &= \omega^a{}_{c\mu}\delta^\sigma_\nu - \omega^a{}_{c\nu}\delta^\sigma_\mu, \\ \frac{\partial g_{\rho\lambda}}{\partial e^c{}_\sigma} = \frac{\partial (e^a{}_\rho e_{a\lambda})}{\partial e^c{}_\sigma} &= e_{c\lambda}\delta^\sigma_\rho + e_{c\rho}\delta^\sigma_\lambda, \\ \frac{\partial g^{\rho\lambda}}{\partial e^c{}_\sigma} &= -g^{\sigma\rho} e_c{}^\lambda - g^{\sigma\lambda} e_c{}^\rho \ .\end{aligned}$$

Subsequently, one also needs the equalities

$$\begin{aligned}\frac{\partial e_a{}^\lambda}{\partial e^c{}_\sigma} &= -e_a{}^\sigma e_c{}^\lambda, \\ \frac{\partial G_a{}^{\mu\nu}}{\partial e^c{}_\sigma} &= \left[\eta_{ab}\omega^b{}_{c\alpha} g^{\alpha\mu} g^{\sigma\nu} + G_a{}^{\sigma\mu} e_c{}^\nu + G_{a\lambda}{}^\mu e_c{}^\lambda g^{\sigma\nu}\right] - \left[\mu \leftrightarrow \nu\right], \\ \frac{\partial G^{b\mu}{}_\lambda}{\partial e^c{}_\sigma} &= g^{\rho\mu}\omega^b{}_{c\rho}\delta^\sigma_\lambda - g^{\sigma\mu}\omega^b{}_{c\lambda} - G^b{}_{\rho\lambda} e_c{}^\mu g^{\sigma\rho} - G^b{}_{\rho\lambda} e_c{}^\rho g^{\sigma\mu}.\end{aligned}$$

We then make use of these intermediate results to work out $\partial \mathcal{L}_{\mathrm{tg}}/\partial e^c{}_\sigma$, which after some algebra is found to be given by

$$\frac{\partial \mathcal{L}_{\mathrm{tg}}}{\partial e^c{}_\sigma} = \omega^a{}_{c\mu} W_a{}^{\mu\sigma} + G^a{}_{\mu c} W_a{}^{\sigma\mu}, \tag{5.A.1}$$

where we used the notation W as defined in (5.4.7).

It is a simpler exercise to find the derivative of the Lagrangian with respect to the first order derivatives of the vierbein. One only needs the expression

$$\frac{\partial G^a{}_{\mu\nu}}{\partial\partial_\rho e^c{}_\sigma} = \delta^\rho_\mu \delta^\sigma_\nu \delta^a_c - \delta^\rho_\nu \delta^\sigma_\mu \delta^a_c.$$

This is sufficient since the derivative operator annihilates the metric $g_{\mu\nu} = e^a{}_\mu e_{a\nu}$ and we can freely raise and lower spacetime indices. Using this information, it is readily found that

$$\frac{\partial \mathcal{L}_{\rm tg}}{\partial\partial_\rho e^c{}_\sigma} = W_c{}^{\rho\sigma}. \tag{5.A.2}$$

Finally, when the expressions (5.A.1) and (5.A.2) are substituted for (5.4.5), the field equations for the vierbein, i.e., Eq. (5.4.6), are recovered.

6 | de Sitter teleparallel gravity

We formulate a theory of gravity in which the local kinematics of physics is regulated by the de Sitter group [JP16]. More precisely, we generalize teleparallel gravity for the cosmological function being built on top of a de Sitter–Cartan geometry. The cosmological function is given its own dynamics and naturally emerges nonminimally coupled to the gravitational field in a manner akin to teleparallel dark energy models or scalar-tensor theories in general relativity. New in the theory here presented, the cosmological function gives rise to a kinematic contribution in the deviation equation for the world lines of adjacent free-falling particles. While having its own dynamics, dark energy manifests itself in the local kinematics of spacetime.

6.1 Introduction

In the preceding chapter we saw how teleparallel gravity models the gravitational interaction in a manner that is in complete equivalence with general relativity from a physical point of view. As is well known, these theories are consistent with the strong equivalence principle [Wei72, AP12], which states that

> all test fundamental physics (including gravitational physics) is not affected, locally, by the presence of a gravitational field [DCLS15].

This means that if we set up an experiment taking place in a region of spacetime sufficiently small and record the results, it is always possible to perform the same experiment in a region absent of gravity and change reference frame such that the same results are obtained. In theory, sufficiently small might be synonymous to infinitesimal, namely, the equivalence is exact only if the experiment performed in the presence of gravity takes place in the tangent structure of spacetime. Alternatively, the equivalence principle implies that gravitational effects at a given event can be turned off by considering a certain local Lorentz transformation.

Furthermore, both teleparallel gravity and general relativity incorporate the assumption that the local kinematics of spacetime is regulated by the Poincaré group. These considerations are the rationale behind Riemann-Cartan geometry underlying both theories of gravity, a fact we verified in §5, while their only difference being the way they accommodate the gravitational degrees of freedom among the geometric objects available.

Be that as it may, there is significant experimental evidence that our universe momentarily undergoes accelerated expansion [PR03, Wei08]. The substance that drives such a stretching of spacetime is generally conjectured to be *dark energy*, which is a term used for the cosmological constant, or a component that acts like one [PR03]. From a conceptual point of view, there are at least two objections that may be posed against the dark energy picture in teleparallel gravity or general relativity. Firstly, the cosmological constant must be a spacetime constant, which follows directly from the gravitational field equations, and it is thus excluded that this form of dark energy may explain possibly different rates of expansion in space and time. This problem is sometimes circumvented by introducing a scalar field with a self-interaction potential, which mimics a cosmological constant when the potential is assumed to vary relatively slowly in spacetime. This brings us to a second point of scepticism, as the new exotic matter is brought on the scene in a manner rather reminiscent of *ad hoc* hypotheses, without good reason why one form over another should be considered.

The presence of dark energy indicates that the large-scale kinematics of spacetime is approximated better by the de Sitter group $SO(1,4)$ [ABAP07]. We shall take this evidence to heart and conjecture that local kinematics is governed by the de Sitter group. Looked at from a mathematical standpoint, this amounts to have the Riemann–Cartan geometry replaced by a Cartan geometry modeled on de Sitter space, see §4. The corresponding spacetime is everywhere approximated by de Sitter spaces, whose combined set of cosmological constants in general varies from event to event, hence resulting in the cosmological function.

In the present chapter we propose an extended theory of gravity as we generalize teleparallel gravity for such a de Sitter–Cartan geometry. Quite similar to the cosmological constant in teleparallel gravity or general relativity, we model the dark energy driving the accelerated expansion by a cosmological function Λ of dimension one over length squared. Fundamentally different, however, the cosmological function alters the kinematics governing physics around any point, such that spacetime is approximated locally by a de Sitter space of cosmological constant Λ. To be exact, a congruence of particles freely falling in an external gravitational field exhibits a relative acceleration, not only due to the nonhomogeneity of the gravitational field, but also because of the local kinematic properties of spacetime that are determined by the cosmological function.

It is opportune to observe that the strong equivalence principle remains formally the same, by which we mean that an observer always can change reference frame so that inertial effects balance gravitational forces at a given event. This is not in contradiction with our assertion that kinematics is locally governed by the de Sitter group. The kinematics reveals itself in the relative acceleration between a congruence of particles, which can be observed only in experiments taking place over regions including different points of spacetime. The tangent space at any spacetime event is identified with the tangent space of the local de Sitter space, which thus continues to be a vector space of signature two.

6.2 Fundamentals of de Sitter teleparallel gravity

Following the arguments exposed in the introduction, we construct de Sitter teleparallel gravity on top of a nonlinear de Sitter–Cartan geometry. The mathematical structure of this geometry was discussed in §4, to which extensive reference will be made during the remainder of this chapter. For the sake of clarity, there will be some repetition of content in the following sections, although details or derivations of formulae are usually omitted, as the reader is invited to consult them in §§4.2–4.3.

To begin with we mention that spacetime continues to be modeled by a four-dimensional manifold, for the necessity to label events by four coordinates is not a property we wish to alter when introducing the cosmological function. An essential and new ingredient with respect to teleparallel gravity, on the other hand, is that the presence of a cosmological function Λ is supposed to change the kinematics that governs physics over a sufficiently small region of spacetime. More precisely, the value of the cosmological function at a given event determines the cosmological constant of the approximating de Sitter space, which therefore can be thought of as being *tangent* to spacetime.[1]

The corresponding change in the kinematical group from the Poincaré group to the de Sitter group lies at the heart for replacing the $\mathfrak{iso}(1,3)$-valued connection A of teleparallel gravity in §5.3 with a similar connection that is valued in the de Sitter algebra $\mathfrak{so}(1,4)$. In order to decompose A in two pieces, of which one takes values in $\mathfrak{so}(1,3)$ and another is valued in the infinitesimal de Sitter translations $\mathfrak{p}$, we must fix a point in de Sitter space dS— or, more precisely, we must fix a section ξ of the associated bundle of de Sitter

[1] The problem of finding the precise value for Λ along spacetime in de Sitter teleparallel gravity will be addressed when we discuss the gravitational field equations in §6.4.

spaces $Q[dS] = Q \times_{SO(1,4)} dS$, because the splitting may be considered pointwise along spacetime. We already verified in (4.2.10) that such a splitting is not preserved under local de Sitter translations $1 + i\epsilon(x) \cdot P$:

$$\delta_\epsilon A^a{}_b = \frac{1}{l^2}(\epsilon^a A_b - \epsilon_b A^a) \quad \text{and} \quad \delta_\epsilon A^a = -d\epsilon^a - A^a{}_b \epsilon^b + \frac{dl}{l}\epsilon^a.$$

In contrast with the transformations (5.3.1) for a Riemann–Cartan geometry in teleparallel gravity, the $\mathfrak{so}(1,3)$-valued part $A^a{}_b$ is not invariant under translations and cannot be taken as the spin connection.

In order to obtain a true spin connection ω and vierbein e, i.e., objects that retain their usual meaning, independent of the section ξ chosen to decompose the de Sitter algebra, we nonlinearly realize A. This is completely equivalent to the manner in which we constructed ω and e for teleparallel gravity in §5.3, the only distinction being that $\xi(x)$ is at any x a point in the local de Sitter space $dS_x = \pi^{-1}_{Q[dS](x)}$, instead of a Minkowski space, so that the vierbein is a mapping from the tangent space at x to the tangent space at $\xi(x)$, i.e., $e^a : V^\mu \in T_x\mathcal{M} \mapsto e^a{}_\mu V^\mu \in T_\xi dS_x$. The spin connection and vierbein were computed in §4.3, i.e., they are given by (4.3.13). Their curvature and torsion are then expressed in (4.3.15).

Furthermore, we verified in §4.3 that the curvature R and torsion T are related to the spin connection and vierbein through

$$R^a{}_b = B^a{}_b + \frac{1}{l^2} e^a \wedge e_b \tag{6.2.1a}$$

and

$$T^a = G^a - d\ln l \wedge e^a, \tag{6.2.1b}$$

were we denoted the exterior covariant derivatives of $\omega^a{}_b$ and e^a, respectively, by

$$B^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{6.2.2a}$$

and

$$G^a = de^a + \omega^a{}_b \wedge e^b, \tag{6.2.2b}$$

which is consistent with the notation chosen in (5.2.2). It is then manifest that the Ricci theorem (5.2.3) holds, i.e., $\omega^a{}_{b\mu} = \mathring{\omega}^a{}_{b\mu} + K^a{}_{b\mu}$, where $\mathring{\omega}^a{}_{b\mu}$ is the Levi-Civita spin connection and $K^a{}_{b\mu}$ is given by (5.2.4), namely,

$$K^a{}_{b\mu} = \frac{1}{2}(G^a{}_{\mu b} + G_\mu{}^a{}_b + G_b{}^a{}_\mu), \quad \text{or} \quad G^a{}_{\mu\nu} = K^a{}_{b\mu} e^b{}_\nu - K^a{}_{b\nu} e^b{}_\mu.$$

We shall continue to call K the contortion, although one must keep in mind that G is not the torsion for a de Sitter–Cartan geometry whenever the cosmological function is nonconstant, as can be seen from (6.2.1b). Furthermore, the Levi-Civita connection is the connection for which the two-form G vanishes, such that the torsion of $\mathring{\omega}$ is nonzero in a de Sitter–Cartan geometry, namely, $\mathring{T}^a = -d\ln l \wedge e^a$.

In §4.2 we already observed that the algebraic covariant derivative of $\mathfrak{p}$-valued forms on spacetime may be defined as

$$D_\mu V^a = \partial_\mu V^a + \omega^a{}_{b\mu} V^b. \tag{6.2.3}$$

Subsequently, spacetime covariant differentiation is introduced as $\nabla = d + \Gamma$, such that $\nabla_\mu V^\rho = e_a{}^\rho D_\mu V^a$. It thus follows that

$$\Gamma^\rho{}_{\nu\mu} = e_a{}^\rho D_\mu e^a{}_\nu,$$

and that the vierbein is covariantly constant, i.e., $D_\mu e^a{}_\nu - \Gamma^\rho{}_{\nu\mu} e^a{}_\rho = 0$. Additionally, one may provide spacetime with a metric structure by constructing the symmetric tensor $g_{\mu\nu} = e^a{}_\mu e_{a\nu}$, which is covariantly constant:

$$\nabla_\rho g_{\mu\nu} = 0. \tag{6.2.4}$$

Finally, we remark that the Bianchi identities are the same as those for a Riemann–Cartan geometry, which were written down in (5.2.7).

Whilst this outline concludes our review of de Sitter–Cartan geometry as the mathematical framework we shall employ to model teleparallel gravity with a cosmological function, there remains to be specified how precisely it intends to accommodate the kinematics due to Λ and the dynamical degrees of freedom of the gravitational field and the cosmological function. These issues are addressed in the remainder of this section.

To begin with, it is postulated that a gravitational field is present if and only if the exterior covariant derivative (6.2.2b) of the vierbein has a value not equal to zero. This characterization to indicate whether or not there are gravitational degrees of freedom is formally the same as in teleparallel gravity, which may be argued to be natural. *A priori*, we would like to generalize for a different kinematics only and not alter the geometrical representation of the dynamics of the gravitational field. Nevertheless, this does not mean that the dynamics itself remains unaltered. On the contrary, we shall see in §6.4 how the presence of a cosmological function modifies the gravitational field equations.

In further similarity with teleparallel gravity, the spin connection does not bear any gravitational degrees of freedom. Being a connection for local Lorentz transformations, it naturally continues to represent fictitious forces existing in a certain class of frames. The final and most important issue that must be settled in specifying for the geometry is then—*How are the local kinematics, whose defining group in the presence of the cosmological function is $SO(1,4)$, accounted for?*

The question is given an answer by postulating that the curvature (6.2.1a) vanishes at every spacetime event, i.e.,

$$B^a{}_{b\mu\nu} = -\frac{\Lambda}{3}(e^a{}_\mu e_{b\nu} - e^a{}_\nu e_{b\mu}). \tag{6.2.5}$$

The curvature of the spin connection hence equals the curvature of the Levi-Civita connection on a de Sitter space with cosmological constant given by Λ, which varies from point to point. If the cosmological function goes to zero over the whole of spacetime, the spin connection becomes the Levi-Civita connection for Minkowski space, or in the presence of torsion, the Weitzenböck connection of teleparallel gravity. The prescription (6.2.5) is therefore consistent with teleparallel gravity, which is recovered in the contraction limit $\Lambda \to 0$. The prescription (6.2.5) to implement the kinematics in the geometric framework is of great importance, for the kinematic effects will be observable as fictitious forces between adjacent free-falling particles, something which will be clarified in the following section.

6.3 Particle mechanics and kinematic effects

The geometric framework and fundamentals now well understood, we are equipped to implement de Sitter teleparallel gravity. In this section we establish the motion of a test particle moving in the presence of a given gravitational field and cosmological function. Thereafter, we determine in §6.4 how the gravitational field and cosmological function themselves are sourced by a certain distribution of matter energy-momentum.

The motion of a particle of nonzero rest mass m in the presence of a gravitational field and cosmological function is determined by the action ($c = 1$)

$$\mathcal{S} = -m \int u_a e^a, \tag{6.3.1}$$

where $u^a = e^a{}_\mu dx^\mu/d\tau$ is the four-velocity of the particle. Hence, as usual (6.3.1) is proportional to the particle's proper time τ. Note that integration goes along a curve that connects two given events, and that the physical world line traced out by the particle is the curve that extremizes (6.3.1).

To be explicit, one looks for the world line $x^\mu(\tau)$ such that the Lie derivative in arbitrary directions $\delta x^\mu(\tau)$ of (6.3.1) vanishes. The variations are set to zero at the end events, since the latter are fixed by construction. Because $\delta u_a e^a = 0$, we only need the expression for the Lie derivative of the vierbein, i.e.,

$$\delta e^a = e^a{}_\mu d\delta x^\mu + \partial_\rho e^a{}_\mu \delta x^\rho dx^\mu. \tag{6.3.2}$$

The variation of the action can then be worked out without much effort, which consequently leads to the result

$$\delta\mathcal{S} = m\int d\tau\delta x^\mu\bigg[\left(\frac{du_a}{d\tau} - \omega^a{}_{b\rho}u_b u^\rho\right)e^a{}_\mu \\ - u_a u^\rho(\partial_\mu e^a{}_\rho - \partial_\rho e^a{}_\mu + \omega^a{}_{b\mu}e^b{}_\rho - \omega^a{}_{b\rho}e^b{}_\mu)\bigg].$$

When this variation is required to vanish for arbitrary δx^μ, one obtains the particle's equations of motion, thus given by

$$u^\rho D_\rho u^a = K^a{}_{b\rho}u^b u^\rho. \tag{6.3.3}$$

Before we discuss this result in further detail, we dedicate a couple of lines on its derivation. Observe that the variation of the vierbein has been substituted directly for the Lie derivative (6.3.2), instead of first replacing the vierbein with the expression $e^a(A^a, A^a{}_b, \xi^a, l)$ given in (4.3.13b), and subsequently taking the Lie derivative. The latter procedure was followed when the particle's equations of motion for teleparallel gravity were derived in §5.4. It does not matter which procedure one chooses, because they are equivalent to each other. This is so, since the expression $e^a(A^a, A^a{}_b, \xi^a, l)$ constitutes a one-form, hence must satisfy (6.3.2). It is a significantly simpler computation to take the Lie derivative directly of e^a, compared to first opening up the expression and then applying the variation. For the sake of completeness, we have included the alternative and more complex computation in §6.A, which confirms (6.3.3). In §5.4 as well, one could have calculated the particle's equations of motion by taking the Lie derivative from the vierbein, instead of first using (5.3.3), although the gain in efficiency would have been marginal.[2]

The equations of motion (6.3.3) are identical in form to the ones (5.4.2) governing particle mechanics in teleparallel gravity, i.e., when the cosmological function vanishes. In particular, (6.3.3) complies with the weak equivalence principle, yet a breakdown of the latter most likely could be coped with along the lines it is done in teleparallel gravity [APV04a]. Despite the fact it is not immediately obvious from (6.3.3), a nonzero cosmological function has an impact on the motion of particles. The first change is rather

[2]When teleparallel gravity is formulated as a gauge theory for the Poincaré translations, first using (5.3.3) and then taking the variations may be preferred from a conceptual point of view, since the field A^a is considered the fundamental gauge potential.

indirect and stems from a modification in the gravitational field equations, thus altering the value of the contortion for a given distribution of energy-momentum that sources gravity.

The second change reflects the alteration in kinematics, now regulated by the de Sitter group. In order to clarify this, we consider a one-parameter family $x_\sigma(\tau)$ of solutions of (6.3.3), parametrized by σ. Hence, these curves are the world lines of a string of neighboring test particles, moving in an external gravitational field and in the presence of a cosmological function. The solutions constitute a two-dimensional surface, to which the vector fields

$$u = \frac{d}{d\tau} = \frac{dx^\mu}{d\tau}\partial_\mu \quad \text{and} \quad v = \frac{d}{d\sigma} = \frac{dx^\mu}{d\sigma}\partial_\mu$$

are tangent. For every value of σ, the vector field u is the four-velocity of the particle with world line $x_\sigma(\tau)$, hence satisfies (6.3.3). The field v is tangent to constant τ slices, i.e., $v(\tau) = \lim_{h\to 0}\frac{1}{h}[x_{\sigma+h}(\tau) - x_\sigma(\tau)]$, and thus connects two infinitesimally separated test particles during their motion in a gravitational field. The vector field [Car04]

$$\mathfrak{a}^a = u^\mu D_\mu(u^\nu D_\nu v^a)$$

is therefore the relative acceleration between the world lines, measured by a free-falling observer.

In order to get a useful expression for the relative acceleration $\mathfrak{a}$, we first note that the definitions of u and v trivially imply that the commutator $[u, v]$ vanishes. Further, one has that

$$u^\mu D_\mu v^a - v^\mu D_\mu u^a = u^\mu v^\nu G^a{}_{\mu\nu},$$

while it follows that

$$\begin{aligned} u^\mu D_\mu(u^\nu D_\nu v^a) &= u^\mu D_\mu(v^\nu D_\nu u^a + u^\lambda v^\nu G^a{}_{\lambda\nu}) \\ &= u^\mu \partial_\mu v^\nu D_\nu u^a + u^\mu v^\nu D_\mu D_\nu u^a + u^\mu D_\mu(u^\lambda v^\nu G^a{}_{\lambda\nu}) \\ &= v^\mu \partial_\mu u^\nu D_\nu u^a + u^\mu v^\nu u^b B^a{}_{b\mu\nu} + u^\mu v^\nu D_\nu D_\mu u^a \\ &\quad + u^\mu D_\mu(u^\lambda v^\nu G^a{}_{\lambda\nu}), \end{aligned}$$

where the last equality depends on the identity $[D_\mu, D_\nu]u^a = u^b B^a{}_{b\mu\nu}$. The third term in the last line can be written as

$$u^\mu v^\nu D_\nu D_\mu u^a = v^\nu D_\nu(u^\mu D_\mu u^a) - v^\nu \partial_\nu u^\mu D_\mu u^a.$$

When we substitute for this identity and the particle's field equations (6.3.3) are invoked as well, we find a final expression for the relative acceleration, namely,

$$\mathfrak{a}^a = u^\mu v^\nu u^b B^a{}_{b\mu\nu} + v^\mu D_\mu (K^a{}_{b\nu} u^b u^\nu) + u^\mu D_\mu (u^\lambda v^\nu G^a{}_{\lambda\nu}), \tag{6.3.4}$$

where $B^a{}_{b\mu\nu}$ is the curvature (6.2.5) of the local de Sitter spaces. The first term is therefore present only when the cosmological function is nonzero.

Equation (6.3.4) is a chief result of de Sitter teleparallel gravity, for it describes what the phenomenology is of the local de Sitter kinematics. The last two terms are dynamical in nature and come from a nonhomogeneous gravitational field. The first term originates in the cosmological function Λ, as can be seen from (6.2.5), and is caused by the kinematics. This contribution manifests itself in that two particles separated by the infinitesimal v^a deviate as if they were moving in a de Sitter space with cosmological constant Λ. Hence, two neighboring free-falling particles have world lines that deviate, not only because they move in a nonhomogeneous gravitational field, but also because of the kinematics that is determined by the cosmological function. According to this approach, dark energy has its origins in the cosmological function and reveals itself as a kinematic effect.

6.4 Dynamics of the gravitational field and the cosmological function

Having specified the particle mechanics caused by a gravitational field and cosmological function in the foregoing section, we now prescribe the dynamics of the latter two themselves.

The gravitational action we shall consider is a reasonable generalization of the action for the gravitational field in teleparallel gravity. Remember that the Lagrangian (5.4.4) for teleparallel gravity is the function

$$\mathcal{L}_{\text{tg}} = \frac{1}{4} G^a{}_{\mu\nu} G_a{}^{\mu\nu} + \frac{1}{2} G^a{}_{\mu\nu} G^{b\mu}{}_\lambda e_a{}^\lambda e_b{}^\nu - G^a{}_{\mu\nu} G^{b\mu}{}_\lambda e_a{}^\nu e_b{}^\lambda,$$

where G^a is the exterior covariant derivative of the vierbein. The geometry underlying teleparallel gravity is a Riemann–Cartan geometry, in which G^a is the torsion of the structure. Because the torsion for a de Sitter–Cartan geometry in the presence of a cosmological function is given by $T^a = G^a - d \ln l \wedge e^a$, it appears a natural proposition to define the Lagrangian $\mathcal{L}_{\text{dStg}}$ for de Sitter teleparallel gravity to be the same function of T^a as $\mathcal{L}_{\text{tg}}$ is a function of G^a. We thus postulate the Lagrangian

$$\mathcal{L}_{\text{dStg}} = \frac{1}{4} T^a{}_{\mu\nu} T_a{}^{\mu\nu} + \frac{1}{2} T^a{}_{\mu\nu} T^{b\mu}{}_\lambda \, e_a{}^\lambda e_b{}^\nu - T^a{}_{\mu\nu} T^{b\mu}{}_\lambda \, e_a{}^\nu e_b{}^\lambda.$$

In order to restate this function as a Lagrangian for the gravitational field and the cosmological function, we substitute the torsion for $G^a + \frac{1}{2} d \ln \Lambda \wedge e^a$, after which a straightforward calculation shows that

$$\mathcal{L}_{\rm dStg} = \mathcal{L}_{\rm tg} - \frac{3}{2} \partial_\mu \ln \Lambda \, \partial^\mu \ln \Lambda - 2G^{\mu\nu}{}_\mu \partial_\nu \ln \Lambda. \tag{6.4.1}$$

The action for de Sitter teleparallel gravity is thus given by ($c = \hbar = 1$)

$$\mathcal{S}_{\rm dStg} = \frac{1}{2\kappa} \int d^4x \, e \left(\mathcal{L}_{\rm tg} - \frac{3}{2} \partial_\mu \ln \Lambda \, \partial^\mu \ln \Lambda - 2G^{\rho\mu}{}_\rho \partial_\mu \ln \Lambda \right), \tag{6.4.2}$$

where $\kappa = 8\pi G_N$ and $e = \det e^a{}_\mu$.

The action (6.4.2) reminds, on the one hand, of the scheme in which scalar-tensor theories modify gravity in the framework of general relativity [BD61, Dic62, Ber68, SF10, Tsu10], or, on the other hand, of teleparallel dark energy, where a scalar field is coupled nonminimally to teleparallel gravity [GLSW11, GLS12, XSL12]. To be precise, it specifies for a gravitational sector modeled by teleparallel gravity— for a spin connection with curvature (6.2.5)— that interacts with the cosmological function due to a nonminimal coupling between the trace of the exterior covariant derivative of the vierbein and the logarithmic derivative of Λ. A theory quite similar in structure was discussed in [Ota16]. Despite the similarity, there is a crucial discrepancy it has in common with any of the other modifications of general relativity or teleparallel gravity that introduce nonminimal couplings to scalar fields. Usually, these fields are added to the theory in a manner rather reminiscent of *ad hoc* hypotheses, and are not an essential feature of the spacetime geometry. In de Sitter teleparallel gravity by contrast, the scalar field is the cosmological function, which forms an integral part of the geometric structure and, moreover, quantifies the kinematics locally governed by the de Sitter group in the sense of §6.3.

Note that the cosmological function appears in the action only through its logarithmic derivative, which is a direct consequence of (6.2.1b). Factors of $\partial_\mu \ln \Lambda$ are naturally dimensionless, which renders a correct overall dimension for the action. Because spacetime coordinates are numbers and the dimension of the vierbein components is that of length, i.e., $[e^a{}_\mu] = L$, while

$$[\kappa] = L^2, \quad [e] = L^4, \quad [g^{\mu\nu}] = L^{-2}, \text{ and } \quad [G^{\rho\mu}{}_\rho] = L^{-2},$$

one confirms that $\mathcal{S}_{\rm dStg}$ indeed has dimension of $\hbar = 1$.

The field equations for the vierbein are obtained by demanding that the functional (6.4.2) attains an extremal value as function of the vierbein, which is the case if

$$\partial_\rho \left(e \frac{\partial \mathcal{L}_{\rm dStg}}{\partial \partial_\rho e^a{}_\mu} \right) - e \frac{\partial \mathcal{L}_{\rm dStg}}{\partial e^a{}_\mu} - e \, e_a{}^\mu \mathcal{L}_{\rm dStg} = 0. \tag{6.4.3}$$

Substituting for the Lagrangian (6.4.1), these equations take the form

$$
\begin{aligned}
0 = {} & \partial_\rho\bigg(e\frac{\partial\mathcal{L}_{\text{tg}}}{\partial\partial_\rho e^a{}_\mu}\bigg) - e\,\frac{\partial\mathcal{L}_{\text{tg}}}{\partial e^a{}_\mu} - e\,e_a{}^\mu\mathcal{L}_{\text{tg}} \\
& - 2\partial_\rho\bigg(e\frac{\partial G^{\lambda\sigma}{}_\lambda}{\partial\partial_\rho e^a{}_\mu}\partial_\sigma\ln\Lambda\bigg) + \frac{3}{2}e\frac{\partial(\partial_\rho\ln\Lambda\,\partial^\rho\ln\Lambda)}{\partial e^a{}_\mu} \\
& + 2e\frac{G^{\lambda\sigma}{}_\lambda}{\partial e^a{}_\mu}\partial_\sigma\ln\Lambda + \frac{3}{2}e\,e_a{}^\mu\partial_\rho\ln\Lambda\,\partial^\rho\ln\Lambda + 2e\,e_a{}^\mu G^{\lambda\sigma}{}_\lambda\partial_\sigma\ln\Lambda. \qquad (6.4.4)
\end{aligned}
$$

The first three terms of the right-hand side make up the field equations (5.4.5) for teleparallel gravity. In §5.A we computed that they reduce to the left-hand side of (5.4.6), a result that can be recycled here. Us thus rests the task to compute the remaining terms of the expression above, an exercise that is synthesized in §6.B. The gravitational field equations are hence found to be given by

$$
\begin{aligned}
& D_\rho(e\,W_a{}^{\rho\mu}) + e\,t_a{}^\mu - 2e\,G^{\rho\mu}{}_\rho e_a{}^\nu\partial_\nu\ln\Lambda - 2e\,e_a{}^\mu\Box\ln\Lambda \\
& \qquad + 2e\,e_a{}^\rho\nabla_\rho\partial^\mu\ln\Lambda - 3e\,e_a{}^\rho\partial_\rho\ln\Lambda\,\partial^\mu\ln\Lambda + \frac{3}{2}e\,e_a{}^\mu\partial_\rho\ln\Lambda\,\partial^\rho\ln\Lambda = 0, \quad (6.4.5)
\end{aligned}
$$

where $\Box$ is the d'Alembertian operator $g^{\mu\nu}\nabla_\mu\partial_\nu = \nabla_\mu\partial^\mu$, while $W_a{}^{\rho\mu}$ and $t_a{}^\mu$ are the superpotential (5.4.7), respectively, the gravitational energy-momentum current (5.4.8).

The equations (6.4.5) thus determine the components of the vierbein, but they do not fix a value for the cosmological function. We thus need a differential equation that needs to be solved by Λ. In de Sitter teleparallel gravity the cosmological function is given its own dynamics, i.e., it is attributed its own field equation $\delta_\Lambda\mathcal{S}_{\text{dStg}} = 0$. We compute

$$
\delta\mathcal{S}_{\text{dStg}} = -\frac{1}{2\kappa}\int d^4x\,e\big(3g^{\mu\rho}\partial_\rho\ln\Lambda\,\delta\partial_\mu\ln\Lambda + 2G^{\rho\mu}{}_\rho\,\delta\partial_\mu\ln\Lambda\big).
$$

Because $\delta\partial_\mu\ln\Lambda = \partial_\mu\delta\ln\Lambda$, it follows that

$$
\begin{aligned}
\delta\mathcal{S}_{\text{dStg}} = \frac{1}{2\kappa}\int d^4x\,\Big\{ & \delta\ln\Lambda\Big[3\partial_\mu\big(e\,\partial^\mu\ln\Lambda\big) + 2\partial_\mu\big(e\,G^{\rho\mu}{}_\rho\big)\Big] \\
& - \partial_\mu\Big[\big(3e\,\partial^\mu\ln\Lambda + 2e\,G^{\rho\mu}{}_\rho\big)\delta\ln\Lambda\Big]\Big\}.
\end{aligned}
$$

The second line can be integrated into a surface term that vanishes, since the variation $\delta\ln\Lambda$ is assumed to be zero at the boundary.[3] If we demand that the variation be equal to

[3] It may not be directly clear that $\delta\ln\Lambda$ vanishes at the boundary. What if the cosmological function goes to zero as well? Note, however, that $\delta\ln\Lambda = -\frac{1}{2}\delta\ln l = -\frac{1}{2}l^{-1}\delta l$. As long as l does not go to zero at the boundary— which would correspond to an infinite cosmological function— a vanishing variation δl at infinity results in the well-defined condition that $\delta\ln\Lambda|_\infty = 0$.

zero, the field equation for Λ is found, namely,

$$\partial_\mu(e\,\partial^\mu \ln\Lambda) + \frac{2}{3}\partial_\mu(e\,G^{\rho\mu}{}_{\rho}) = 0,$$

which we rewrite as

$$\Box \ln\Lambda + G^{\mu\rho}{}_{\mu}\partial_\rho \ln\Lambda = -\frac{2}{3}(\nabla_\mu G^{\rho\mu}{}_{\rho} + G^{\mu}{}_{\rho\mu}G^{\nu\rho}{}_{\nu}). \tag{6.4.6}$$

The coupling of matter fields to the gravitational sector is carried out by taking the sum of the matter action

$$\mathcal{S}_{\mathrm{m}} = \int d^4x\, e\,\mathcal{L}_{\mathrm{m}}$$

and the action (6.4.2) for the gravitational field and cosmological function. The energy-momentum current $\delta(e\mathcal{L}_{\mathrm{m}})/\delta e^a{}_\mu$ of matter is a source for the gravitational field equations (6.4.5), but does not appear in the equation of motion (6.4.6) for the cosmological function. According to this scheme, energy-momentum generates gravity, which in turn sources the cosmological function.

6.5 Concluding remarks

We formulated a theory of gravity consistent with local spacetime kinematics regulated by the de Sitter group, namely, de Sitter teleparallel gravity. It was made plain in §5.3 that teleparallel gravity, a theory physically equivalent to general relativity, has the mathematical structure of a nonlinear Riemann–Cartan geometry. This inspired us to generalize for de Sitter kinematics by considering de Sitter–Cartan geometry in the presence of a nonconstant cosmological function Λ.

The theory has the structure of a gravitational sector described by teleparallel gravity that couples nonminimally to the cosmological function. Dynamical degrees of freedom of the gravitational field are present if and only if the exterior covariant derivative of the vierbein does not vanish. Further, the cosmological function has its own dynamics, sourced by the trace of the exterior covariant derivative of the vierbein, but not directly by the matter energy-momentum current. It is thence similar in form to teleparallel dark energy, or scalar-tensor theories in the framework of general relativity.

A crucial difference between these models and the theory here proposed is that the cosmological function modifies the local kinematics of spacetime. Indeed, at every spacetime point we put forward that the curvature of the spin connection is equal to the curvature of the Levi-Civita connection of a de Sitter space with cosmological constant given by the value of the cosmological function. We saw that such a choice gives rise to a kinematic contribution in the deviation equation for the world lines of adjacent free-falling particles, that is, they undergo a relative acceleration that is kinematic in origin. This result is arguably the one of most importance of this chapter, for it specifies in exactly what manner the kinematics due to the cosmological function are to be observed. Hence, dark energy may be interpreted as a kinematic effect or, alternatively, as the cosmological function causing this effect.

It is interesting to note that there exists a link between the dynamics and kinematics of the theory, in the sense that the value of the cosmological function is determined dynamically by its interaction with the gravitational field, while the resulting value determines the local spacetime kinematics, which in its turn affects the motion of matter. The theory thus gives a precise model that prescribes how the kinematics of high energy physics may be modified locally and becomes spacetime-dependent [Man02]. Although there is a connection between them, dynamics and kinematics remain logically separated in the geometric representation of de Sitter teleparallel gravity. Nontrivial dynamics gives way to the torsion of the de Sitter–Cartan geometry being nonzero, whereas the value of the curvature of the spin connection encodes the inertial effects of a given frame and the local de Sitter kinematics. This is a natural generalization of the geometric representation of teleparallel gravity, which is recovered when the cosmological function vanishes.

To conclude we comment that if the specific dynamical model for de Sitter teleparallel gravity in §6.4 were to be falsified by observational data, one naturally would be forced to find alternative Lagrangians for the gravitational field and cosmological function, but the paradigm of representing dark energy in the kinematics of spacetime according to §6.2 would remain a valid track to be considered further.

Appendix 6.A Explicit calculation of a particle's equations of motion

In §6.3 the equations of motion (6.3.3) were derived for a test particle of mass m moving in an external gravitational field. They were found after a short calculation, since we applied the Lie derivative directly on the vierbein, rather then first substituting it for the expression (4.3.13b) for $e^a(A^a, {A^a}_b, \xi^a, l)$. Here, we shall work out the second equivalent approach as it generalizes the calculation of the particle's equations of motion (5.4.2) in teleparallel gravity.

We thus need to compute $\delta\mathcal{S} = \int u_a \delta e^a$, where the vierbein is replaced by (4.3.13b), which we rewrite here as

$$e^a = \cosh z\, A^a - (\cosh z - 1)\frac{\xi_b A^b \xi^a}{\xi^2} + \frac{\sinh z}{z}(d\xi^a + {A^a}_b\xi^b) - \frac{dl}{l}\xi^a - \left(\frac{\sinh z}{z} - 1\right)\frac{\xi_b d\xi^b \xi^a}{\xi^2}.$$

The variation is a Lie derivative, from which it follows that $\delta\mathcal{S}$ equals

$$\begin{aligned}\int u_a \bigg\{ &\delta\cosh z\, A^a + \cosh z\, \delta {A^a}_\rho dx^\rho + \cosh z\, {A^a}_\rho d\delta x^\rho - \delta\cosh z \frac{\xi_b A^b \xi^a}{\xi^2} \\ &+ (\cosh z - 1)\left[\frac{2\delta\xi}{\xi}\frac{\xi_b A^b\xi^a}{\xi^2} - \frac{\delta\xi_b A^b \xi^a}{\xi^2} - \frac{\xi_b \delta {A^b}_\rho \xi^a}{\xi^2}dx^\rho - \frac{\xi_b A^b \delta\xi^a}{\xi^2}\right. \\ &\left. - \frac{\xi_b {A^b}_\rho \xi^a}{\xi^2} d\delta x^\rho\right] + \delta\Big(\frac{\sinh z}{z}\Big)(d\xi^a + {A^a}_b\xi^b) + \frac{\sinh z}{z}(d\delta\xi^a + \delta {A^a}_{b\rho}\xi^b dx^\rho \\ &+ {A^a}_b\delta\xi^b + {A^a}_{b\rho}\xi^b d\delta x^\rho) + \frac{\delta l dl}{l^2}\xi^a - \frac{d\delta l}{l}\xi^a - \frac{dl}{l}\delta\xi^a - \delta\Big(\frac{\sinh z}{z}\Big)\frac{\xi_b d\xi^b \xi^a}{\xi^2} \\ &+ \Big(\frac{\sinh z}{z} - 1\Big)\left[\frac{2\delta\xi}{\xi}\frac{\xi_b d\xi^b \xi^a}{\xi^2} - \frac{\delta\xi_b d\xi^b \xi^a}{\xi^2} - \frac{\xi_b d\delta\xi^b \xi^a}{\xi^2} - \frac{\xi_b d\xi^b \delta\xi^a}{\xi^2}\right]\bigg\},\end{aligned}$$

where, furthermore,

$$\delta {A^a}_\rho = \partial_\mu {A^a}_\rho \delta x^\mu, \quad \delta {A^a}_{b\rho} = \partial_\mu {A^a}_{b\rho}\delta x^\mu,$$

and

$$\delta z = \delta(l^{-1}\xi) = -l^{-2}\delta l\,\xi + l^{-1}\delta\xi, \quad \text{with } \delta l = \partial_\mu l\,\delta x^\mu, \quad \delta\xi^a = \partial_\mu \xi^a \delta x^\mu.$$

We also used that for an exact one form $d\xi^a$, the Lie derivative commutes with the exterior derivative, in the sense that $\delta d\xi^a = d(\delta x^\mu \partial_\mu \xi^a) = d\delta\xi^a$, which can be verified explicitly.

Subsequently, boundary terms resulting from the integration vanish, because the world line is held fixed at the end events. We then obtain for the variation of the action:

$$
\begin{aligned}
\int \Big[& - du_a \Big\{ \cosh z\, A^a{}_\mu - (\cosh z - 1)\frac{\xi_b A^b{}_\mu \xi^a}{\xi^2} + \frac{\sinh z}{z}(\partial_\mu \xi^a + A^a{}_{b\mu}\xi^b) \\
& - \frac{\partial_\mu l}{l}\xi^a - \Big(\frac{\sinh z}{z} - 1\Big)\frac{\xi_b \partial_\mu \xi^b \xi^a}{\xi^2}\Big\}\delta x^\mu + u_a \delta x^\mu dx^\rho \Big\{ \Big[\frac{\sinh z}{z} A^a{}_{b\rho}\Big(\partial_\mu \xi^b \\
& - \xi^b \frac{\xi_c \partial_\mu \xi^c}{\xi^2}\Big) + \cosh z\,(\partial_\rho \xi^a + A^a{}_{b\rho}\xi^b)\frac{\xi_c \partial_\mu \xi^c}{\xi^2} - \partial_\rho \xi^a \frac{\xi_b \partial_\mu \xi^b}{\xi^2} - \cosh z \Big(\partial_\rho \xi^a \\
& + A^a{}_{b\rho}\xi^b - \xi^a \frac{\xi_b \partial_\rho \xi^b}{\xi^2}\Big)\frac{\partial_\mu l}{l} + \partial_\rho \xi^a \frac{\partial_\mu l}{l} - 2(\cosh z - 1)\xi^a \frac{\xi_b \partial_\rho \xi^b}{\xi^2}\frac{\xi_c A^c{}_\mu}{\xi^2} \\
& + (\cosh z - 1)\partial_\rho \xi^a \frac{\xi_b A^b{}_\mu}{\xi^2} + (\cosh z - 1)\xi^a \frac{\partial_\rho \xi_b A^b{}_\mu}{\xi^2} + z \sinh z \frac{\xi_c \partial_\mu \xi^c}{\xi^2}\Big(A^a{}_\rho \\
& - \xi^a \frac{\xi_b A^b{}_\rho}{\xi^2}\Big) - z \sinh z \Big(A^a{}_\rho - \xi^a \frac{\xi_b A^b{}_\rho}{\xi^2}\Big)\frac{\partial_\mu l}{l}\Big] - \Big[\rho \leftrightarrow \mu\Big]\Big\} + u_a \delta x^\mu dx^\rho \Big\{ \\
& - \Big[\frac{\partial_\rho l}{l}\frac{\sinh z}{z}\Big(\partial_\mu \xi^a + A^a{}_{b\mu}\xi^b - \xi^a \frac{\xi_b \partial_\mu \xi^b}{\xi^2}\Big) + \frac{\partial_\rho l}{l}\frac{\partial_\mu l}{l}\xi^a + \cosh z\, \partial_\rho A^a{}_\mu \\
& - (\cosh z - 1)\xi^a \frac{\xi_b \partial_\rho A^b{}_\mu}{\xi^2} + \frac{\sinh z}{z}\partial_\rho A^a{}_{b\rho}\xi^b\Big] + \Big[\rho \leftrightarrow \mu\Big]\Big\}\Big].
\end{aligned}
$$

The terms between the first pair of curly brackets add up to give the vierbein, while it may be verified that the sum of terms between the second pair of curly brackets is equivalent to

$$
\begin{aligned}
\Big[\omega^a{}_{b\rho} e^b{}_\mu - \frac{\sinh z}{z} A^a{}_{b\rho} A^b{}_{c\mu}\xi^c - \frac{\partial_\rho l}{l}\xi^a \frac{\xi_b \partial_\mu \xi^b}{\xi^2} - \cosh z\, A^a{}_{b\rho} A^b{}_\mu \\
- (\cosh z - 1)\xi^a \frac{A_{bc\rho}\xi^c A^b{}_\mu}{\xi^2} - z \sinh z\, A^a{}_\rho \frac{\xi_c A^c{}_\mu}{\xi^2}\Big] - \Big[\rho \leftrightarrow \mu\Big].
\end{aligned}
$$

This allows us to further work out $\delta \mathcal{S}$, namely,

$$
\begin{aligned}
\int \Big[& - du_a e^a{}_\mu \delta x^\mu + u_a \delta x^\mu dx^\rho \Big\{ \Big[\omega^a{}_{b\rho} e^b{}_\mu - \frac{\sinh z}{z}\xi^c \Big(\partial_\rho A^a{}_{c\rho} + A^a{}_{b\rho} A^b{}_{c\mu} \\
& + \frac{1}{l^2} A^a{}_\rho A_{c\mu}\Big) - \cosh z \Big(\partial_\rho A^a{}_\mu + A^a{}_{b\rho} A^b{}_\mu - \frac{\partial_\rho l}{l} A^a{}_\mu\Big) - (1 - \cosh z)\frac{\xi^a \xi_b}{\xi^2}\Big(\\
& \partial_\rho A^b{}_\mu + A^b{}_{c\rho} A^c{}_\mu - \frac{\partial_\rho l}{l} A^b{}_\mu\Big) - \frac{\partial_\rho l}{l}\Big(\cosh z\, A^a{}_\mu - (\cosh z - 1)\frac{\xi_b A^b{}_\mu \xi^a}{\xi^2} \\
& + \frac{\sinh z}{z}(\partial_\mu \xi^a + A^a{}_{b\mu}\xi^b) - \frac{\partial_\mu l}{l}\xi^a - \Big(\frac{\sinh z}{z} - 1\Big)\frac{\xi_b \partial_\mu \xi^b \xi^a}{\xi^2}\Big)\Big] \\
& - \Big[\rho \leftrightarrow \mu\Big]\Big\}\Big]
\end{aligned}
$$

$$= \int \delta x^\mu \bigg[-du_a e^a{}_\mu + u_a \omega^a{}_{b\rho} e^b{}_\mu dx^\rho - u_a \omega^a{}_{b\mu} e^b{}_\rho dx^\rho - u_a dx^\rho \bigg(\frac{\sinh z}{z} \xi^c F^a{}_{c\rho\mu}$$
$$+ \cosh z \, F^a{}_{\rho\mu} + (1 - \cosh z) \frac{\xi^a \xi_b F^b{}_{\rho\mu}}{\xi^2} \bigg) - \frac{\partial_\rho l}{l} e^a{}_\mu u_a dx^\rho + \frac{\partial_\mu l}{l} e^a{}_\rho u_a dx^\rho \bigg].$$

In these last two lines one may recognize the expression (4.3.15b) for the torsion, i.e.,

$$T^a = \frac{\sinh z}{z} \xi^b F^a{}_b + \cosh z \, F^a + (1 - \cosh z) \frac{\xi_b F^b \xi^a}{\xi^2}.$$

It follows that

$$\delta \mathcal{S} = \int d\tau \delta x^\mu \bigg\{ - e^a{}_\mu \bigg(\frac{du_a}{d\tau} - \omega^b{}_{a\rho} u_b u^\rho + u^\rho \frac{\partial_\rho l}{l} u_a \bigg) + T^a{}_{\mu\rho} \, u_a u^\rho + \frac{\partial_\mu l}{l} \bigg\}.$$

Setting $\delta \mathcal{S} = 0$ and using (6.2.1b), (6.3.3) is recovered.

Appendix 6.B Verification of gravitational field equations

In the following paragraphs it is to be confirmed that the field equations (6.4.5) for the vierbein are correct, in the sense that they follow from the equations (6.4.4).

The first three terms of (6.4.4) were already calculated in 5.A, and shown to be equal to the left-hand side of (6.4.3), i.e.,

$$\partial_\rho \bigg(e \frac{\partial \mathcal{L}_{\text{tg}}}{\partial \partial_\rho e^a{}_\mu} \bigg) - e \frac{\partial \mathcal{L}_{\text{tg}}}{\partial e^a{}_\mu} - e \, e_a{}^\mu \mathcal{L}_{\text{tg}}$$
$$= D_\rho (e \, W_a{}^{\rho\mu}) + e \, G^b{}_{\rho a} W_b{}^{\rho\mu} - e \, e_a{}^\mu \mathcal{L}_{\text{tg}}. \qquad (6.B.1)$$

To calculate the five remaining terms of (6.4.4) we make use of the auxiliary results

$$\frac{\partial G^{\lambda\sigma}{}_\lambda}{\partial \partial_\rho e^a{}_\mu} = e_a{}^\mu g^{\sigma\rho} - e_a{}^\rho g^{\sigma\mu},$$
$$\frac{\partial g^{\lambda\sigma}}{\partial e^a{}_\mu} = -g^{\mu\lambda} e_a{}^\sigma - g^{\mu\sigma} e_a{}^\lambda,$$

and

$$\frac{\partial G^{\lambda\sigma}{}_\lambda}{\partial e^a{}_\mu} = -G^{\mu\sigma}{}_a + e_b{}^\mu g^{\lambda\sigma} \omega^b{}_{a\lambda} - e_b{}^\lambda g^{\mu\sigma} \omega^b{}_{a\lambda} - G^{\lambda\mu}{}_\lambda e_a{}^\sigma - G^\lambda{}_{a\lambda} g^{\mu\sigma}.$$

Substituting for these expressions shows that the last five terms of (6.4.4) equal

$$-2\partial_\rho (e \, e_a{}^\mu g^{\sigma\rho} \partial_\sigma \ln \Lambda - e \, e_a{}^\rho g^{\sigma\mu} \partial_\sigma \ln \Lambda) + 2e \, g^{\rho\sigma} e_b{}^\mu \omega^b{}_{a\rho} \partial_\sigma \ln \Lambda$$

$$
\begin{aligned}
&- 2e\, g^{\mu\sigma} e_b{}^\rho \omega^b{}_{a\rho} \partial_\sigma \ln\Lambda - 2e\, G^{\rho\mu}{}_\rho e_a{}^\sigma \partial_\sigma \ln\Lambda - 2e\, G^\rho{}_{a\rho} g^{\mu\sigma} \partial_\sigma \ln\Lambda \\
&- 2e\, G^{\mu\sigma}{}_a \partial_\sigma \ln\Lambda - 3e\, g^{\mu\rho} e_a{}^\sigma \partial_\rho \ln\Lambda\, \partial_\sigma \ln\Lambda + 2e\, e_a{}^\mu G^{\rho\sigma}{}_\rho \partial_\sigma \ln\Lambda \\
&- \frac{3}{2} e\, e_a{}^\mu \partial_\rho \ln\Lambda\, \partial^\rho \ln\Lambda. \qquad (6.B.2)
\end{aligned}
$$

The first four terms equal $-2D_\rho(e\, e_a{}^\mu g^{\sigma\rho}\partial_\sigma \ln\Lambda) + 2D_\rho(e\, e_a{}^\rho g^{\sigma\mu}\partial_\sigma \ln\Lambda)$, which can be worked out further by noting that $\partial_\mu e = e\, \mathring{\Gamma}^\rho{}_{\rho\mu}$ and that the vierbein is covariantly constant:

$$
\begin{aligned}
&- 2e\, \mathring{\Gamma}^\rho{}_{\rho\nu} e_a{}^\mu g^{\sigma\nu} \partial_\sigma \ln\Lambda + 2e\, \Gamma^\mu{}_{\rho\nu} e_a{}^\rho g^{\sigma\nu} \partial_\sigma \ln\Lambda + 2e\, e_a{}^\mu \Gamma^\sigma{}_{\rho\nu} g^{\rho\nu} \partial_\sigma \ln\Lambda \\
&\quad + 2e\, e_a{}^\mu \Gamma^\nu{}_{\rho\nu} g^{\rho\sigma} \partial_\sigma \ln\Lambda - 2e\, e_a{}^\mu g^{\sigma\nu} \partial_\nu \partial_\sigma \ln\Lambda + 2e\, \mathring{\Gamma}^\rho{}_{\rho\nu} e_a{}^\nu g^{\sigma\mu} \partial_\sigma \ln\Lambda \\
&\quad - 2e\, \Gamma^\nu{}_{\rho\nu} e_a{}^\rho g^{\sigma\mu} \partial_\sigma \ln\Lambda - 2e\, e_a{}^\nu \Gamma^\sigma{}_{\rho\nu} g^{\rho\mu} \partial_\sigma \ln\Lambda - 2e\, e_a{}^\nu \Gamma^\mu{}_{\rho\nu} g^{\rho\sigma} \partial_\sigma \ln\Lambda \\
&\quad + 2e\, e_a{}^\nu g^{\sigma\mu} \partial_\nu \partial_\sigma \ln\Lambda \\
&\quad = 2e\, K^\mu{}_{\rho\nu} e_a{}^\rho g^{\sigma\nu} \partial_\sigma \ln\Lambda + 2e\, e_a{}^\mu \Gamma^\sigma{}_{\rho\nu} g^{\rho\nu} \partial_\sigma \ln\Lambda - 2e\, e_a{}^\mu K^\sigma{}_{\rho\nu} g^{\rho\nu} \partial_\sigma \ln\Lambda \\
&\qquad - 2e\, e_a{}^\mu g^{\sigma\nu} \partial_\nu \partial_\sigma \ln\Lambda - 2e\, K^\nu{}_{\rho\nu} e_a{}^\rho g^{\sigma\mu} \partial_\sigma \ln\Lambda - 2e\, e_a{}^\nu \Gamma^\sigma{}_{\rho\nu} g^{\rho\mu} \partial_\sigma \ln\Lambda \\
&\qquad + 2e\, e_a{}^\nu K^\sigma{}_{\rho\nu} g^{\rho\mu} \partial_\sigma \ln\Lambda + 2e\, e_a{}^\nu g^{\sigma\mu} \partial_\nu \partial_\sigma \ln\Lambda,
\end{aligned}
$$

where we also used the Ricci theorem (5.2.3). With the help of this result, the sum of the right-hand side of (6.B.1) and (6.B.2) becomes

$$
\begin{aligned}
&D_\rho(e\, W_a{}^{\rho\mu}) + e\, G^b{}_{\rho a} W_b{}^{\rho\mu} - e\, e_a{}^\mu \mathcal{L}_{\mathrm{tg}} + 2e\, (K^\mu{}_a{}^\sigma + K^{\sigma\mu}{}_a) \partial_\sigma \ln\Lambda \\
&\quad - 2e\, G^{\mu\sigma}{}_a \partial_\sigma \ln\Lambda - 2e\, G^\rho{}_{\mu\rho} e_a{}^\sigma \partial_\sigma \ln\Lambda - 2e\, e_a{}^\mu g^{\sigma\nu} (\partial_\nu \partial_\sigma \ln\Lambda - \Gamma^\rho{}_{\sigma\nu} \partial_\rho \ln\Lambda) \\
&\quad + 2e\, e_a{}^\nu g^{\sigma\mu} (\partial_\nu \partial_\sigma \ln\Lambda - \Gamma^\rho{}_{\sigma\nu} \partial_\rho \ln\Lambda) - 3e\, e_a{}^\rho \partial_\rho \ln\Lambda\, \partial^\mu \ln\Lambda \\
&\quad + \frac{3}{2} e\, e_a{}^\mu \partial_\rho \ln\Lambda\, \partial^\rho \ln\Lambda.
\end{aligned}
$$

Because $K^\mu{}_a{}^\sigma + K^{\sigma\mu}{}_a = G^{\mu\sigma}{}_a$, we finally rewrite the above expression as

$$
\begin{aligned}
&D_\rho(e\, W_a{}^{\rho\mu}) + e\, G^b{}_{\rho a} W_b{}^{\rho\mu} - e\, e_a{}^\mu \mathcal{L}_{\mathrm{tg}} - 2e\, G^{\rho\mu}{}_\rho e_a{}^\nu \partial_\nu \ln\Lambda - 2e\, e_a{}^\mu \Box \ln\Lambda \\
&\qquad + 2e\, e_a{}^\rho \nabla_\rho \partial^\mu \ln\Lambda - 3e\, e_a{}^\rho \partial_\rho \ln\Lambda\, \partial^\mu \ln\Lambda + \frac{3}{2} e\, e_a{}^\mu \partial_\rho \ln\Lambda\, \partial^\rho \ln\Lambda,
\end{aligned}
$$

which is equal to the right-hand side of (6.4.5).

7 | Conclusions and outlook

In this thesis we investigated the possibility to generalize classical gravity in a manner that deforms the group governing the local kinematics of physics from the Poincaré group $ISO(1,3)$ to the de Sitter group $SO(1,4)$, so that the local de Sitter spaces are characterized by cosmological constants that form a nonconstant cosmological function on spacetime. This generalization is motivated by noting that the accelerated expansion of the universe implies that the large scale structure of spacetime is approximated by a de Sitter space rather than a Minkowski space. From this point of view the observed present-day cosmological constant is the cosmological function for a homogeneous and isotropic universe.

We first recognized that the mathematical framework appropriate to implement such deformed kinematics is contained in Cartan geometry. The precise geometry consists of a spacetime that is equipped with a Cartan connection valued in the de Sitter Lie algebra, namely, a Cartan geometry which is modeled on de Sitter space. The cosmological constants of the local de Sitter spaces were shown to be related to a length scale defined in the translational part of the de Sitter algebra, i.e., in the subspace of de Sitter transvections. Because the Cartan connection is valued pointwise in copies of the de Sitter algebra, we managed to account for a nonconstant cosmological function when the set of length scales is allowed to become an arbitrary differentiable function on spacetime. This led to a new contribution proportional to the logarithmic derivative of the cosmological function in the torsion of the de Sitter–Cartan geometry. This geometry is by definition covariant with respect to local Lorentz transformations, but the structure group was enlarged to the encompassing de Sitter group by constructing a nonlinear realization of the Cartan connection.

With the right mathematical framework at our disposal we set out to formulate a generalization of teleparallel gravity in the presence of a cosmological function. The result was referred to as de Sitter teleparallel gravity. Equivalently to teleparallel gravity with a vanishing cosmological function, gravitational degrees of freedom continued to be represented by the exterior covariant derivative of the vierbein. We saw how the cosmological function gave origin to a kinematic contribution in the deviation equation for the world lines of adjacent free-falling particles, thereby shedding light on the phenomenology of the local de Sitter kinematics. The action we postulated for the dynamics of the theory is a natural generalization of the action for teleparallel gravity and resulted in a coupled system of equations of motion for the gravitational field and the cosmological function. What sets de Sitter teleparallel gravity apart from theories with similar dynamical structures is that the cosmological function determines to what degree the kinematics of physics is ruled by the de Sitter group. In this sense the cosmological function is a true generalization of the cosmological constant in Einstein's equations: dark energy manifests itself as a kinematic effect.

The new paradigm here presented to replace the Poincaré group with the de Sitter group as the set of transformations that govern local kinematics is further motivated by prospective problems to conciliate special relativity with the existence of an invariant length parameter at the Planck scale. Namely, the concomitant replacement of special relativity with an $SO(1,4)$ invariant special relativity allows for the existence of an invariant length parameter proportional to Λ^{-2}, while preserving the constancy of the speed of light [ABAP07]. Because the cosmological function is not restricted to be constant, its value can evolve with cosmological time, so that this model may be suitable to describe the evolution of the universe, which requires different values of the cosmological term at different epochs. For example, a huge cosmological term could drive inflation at the primordial universe. Afterwards, the cosmological term should decay to a small value in order to allow the formation of the structures we see today. Then, to account for the late–time acceleration in the universe expansion rate, the value of the cosmological term should somehow increase [AJP$^+$15].

It goes without saying that the ideas implemented in this thesis constitute just a new scenario for studying cosmology, and further research needs to be conducted. For example, the field equations for the gravitational field and the cosmological function must be solved for a homogeneous and isotropic universe in order to find the time evolution of the scale factor and the cosmological function. Whether the dynamical model here presented is consistent with observed dark energy behavior may be verified in a model-independent way, i.e., cosmographically [Vis05], by comparing the calculated values for low order derivatives of the present-day scale factor with their observed values. Such cosmographic requirements previously have been applied to put constraints on the functional form of modified gravity Lagrangians, see, e.g., [CLS15]. Furthermore, if the Newtonian limit of the field equations were to be obtained, one could verify their predictions for the galactic rotation curves' dependence on the galactocentric radius. Because the cosmological function couples to the gravitational field, it might contribute to the effective mass inside the galactic disk and hence form a component of the dark matter supposedly responsible for the observed flat rotation curves of galaxies. Another interesting question to be answered is how the kinematic contribution due to the cosmological function in the deviation equation for free-falling particles will leave its trace in the Raychaudhuri equation and what will be the implications thereof for the motion of particles in a gravitational field.

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