String Theory Effects on Five-Dimensional Black Hole Physics

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We review recent developments in understanding quantum/string corrections to BPS black holes and strings in five-dimensional supergravity. These objects are solutions to the effective action obtained from M-theory compactified on a Calabi-Yau threefold, including the one-loop corrections determined by anomaly cancellation and supersymmetry. We introduce the off-shell formulation of this theory obtained through the conformal supergravity method and review the methods for investigating supersymmetric solutions. This leads to quantum/string corrected attractor geometries, as well as asymptotically flat black strings and spinning black holes. With these solutions in hand, we compare our results with analogous studies in four-dimensional string-corrected supergravity, emphasizing the distinctions between the four and five dimensional theories.

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1. Introduction

One of the great successes of string theory has been in elucidating the microphysics of black holes. As is expected for any candidate theory of quantum gravity, string theory has provided an accounting of the Bekenstein-Hawking entropy of many black holes in terms of a microscopic counting of states

$$S_{BH} = \frac{A}{4G} = \log \Omega_{string} \ . \tag{1}$$

However, it must be emphasized that the Bekenstein-Hawking entropy is only a leading order result, derived from the classical Einstein-Hilbert action. Any theory of quantum gravity will in general contain higher dimension operators in the low energy effective Lagrangian, *i.e.* terms which contain more than two derivatives of the fundamental fields. The area law for the black hole entropy is therefore only valid in the limit that the black hole is much larger than the Planck and string scales. Analogously, the explicit counting of states is usually done in the limit of large mass and charge, where powerful formulas for the asymptotic degeneracies are available. Thus one expects corrections on both sides of (1), and matching these corrections leads to an even more detailed understanding of string theory and black holes. An ambitious long term goal is to verify (1) *exactly*, as this would surely signal that we have achieved a fundamental understanding of quantum gravity.

Recent years have seen much progress in analyzing the effects of string and quantum gravity corrections to black holes. The Bekenstein-Hawking area law formula has been generalized to the Bekenstein-Hawking-Wald entropy.[1] The Wald formula applies to any diffeomorphism and gauge invariant local effective action, and so can be applied to effective Lagrangians arising in string theory or otherwise. The Wald formula greatly simplifies for an extremal black hole with a near horizon AdS_2 or AdS_3 factor, as is often the case. For AdS_2 the entropy function formalism[2] is appropriate, while for AdS_3 c-extremization[3] is most efficient. Supersymmetric black hole solutions in four dimensional supergravity with R^2 corrections were found in Refs. 4–7, and their entropies successfully matched with a microscopic counting, including subleading corrections.[8] Much recent work has also been stimulated by the connection with topological strings, starting with the OSV conjecture.[9]

An especially interesting application of these ideas is to so-called *small black holes*, those whose Bekenstein-Hawking entropy vanishes at the leading order. In two-derivative supergravity these correspond to solutions with a naked singularity, but higher-order corrections provide a string/Planck scale horizon to cloak the singularity.[10−13] This provides a beautiful

example of how quantum and string effects can smooth out a classical singularity. In some cases, the entropies of these small black holes can be reproduced from the microscopic side, even though they are far from the regime of classical general relativity.

The simplest examples of small black holes are 5D extended string solutions, which can be identified in one duality frame as fundamental heterotic strings. When these carry momentum they have an event horizon and a corresponding entropy that matches that of the heterotic string. We can alternatively consider the solution without any momentum, in which case we obtain a completely smooth and horizon free solution representing an unexcited heterotic string.[14]

1.1. *Why five dimensions?*

In this review we focus on stationary solutions which preserve some fraction of supersymmetry. Four and five dimensional black holes have a privileged status, for it is only these that can be supersymmetric.[a] Five dimensions is especially interesting for a number of reasons. In four dimensions, single-center black hole solutions preserving supersymmetry can carry any number of electric and magnetic charges, but no angular momentum, and the horizon is restricted to be an S^2.[b] The space of solutions is much richer in five dimensions. First of all, there are supersymmetric, electrically charged, rotating (BMPV) black holes.[17−19] Second, there are supersymmetric solutions carrying magnetic charge; such objects are *black strings* extended in one spatial direction.[20,21] These can carry momentum and traveling waves. Finally, there are *black rings*[22,23] which have horizons with $S^1 \times S^2$ topology. These objects carry electric charges and magnetic dipoles, the latter usually referred to as *dipole charges*. Additionally, they carry angular momenta in two independent planes. As a further motivation for focussing on five dimensions, we note that four dimensional black holes can be recovered by the Kaluza-Klein reduction of five dimensional solutions along a circle isometry.

Although the full effective action of string theory is expected to contain an infinite series of higher derivative terms, we confine our attention to the leading order corrections, which are four-derivative terms. On the one hand, this is a product of necessity, since terms with

[a]By a D-dimensional black hole we mean one whose geometry is asymptotic to D-dimensional Minkowski space, possibly times a compact space. In six dimensions there are supersymmetric black strings but no black holes (at least at the two-derivative level.)

[b]That 4D black holes have horizons with S^2 topology was proven by Hawking[15] in the context of general relativity. However it is possible that $S^1 \times S^1$ horizons are allowed in a higher derivative theory; see Ref. 16 for a recent discussion.

more derivatives are not fully known, especially in the off-shell formalism that is needed in order for explicit calculations to be feasible. Fortunately, as we will see, most of the physics that we wish to uncover, such as singularity resolution, is captured already at the four-derivative level. In fact, for black holes containing a near horizon AdS$_3$ factor, a non-renormalization theorem[3,24] can be proven, stating that the entropy gets no corrections from terms with more than four-derivatives. This theorem allows us to systematically compute the entropy of small black holes, even though we would, *a priori*, expect to need to use the entire series of higher derivative terms since there is no small expansion parameter to control the derivative expansion. This argument also applies to non-BPS and near extremal black holes, and explains the fact that we have excellent control over the microscopic and macroscopic entropies of these objects even though they are non-supersymmetric. These powerful consequences of AdS$_3$ underlie all successful entropy matchings in string theory, and provide yet another motivation for working in five dimensions. Conversely, the full symmetries of these solutions are not manifest in the description of these black holes in terms of four-dimensional supergravity.

1.2. *Finding solutions to $D = 5$ R^2-corrected supergravity*

It is useful to think of five-dimensional supergravity as arising from the dimensional reduction of M-theory on a Calabi-Yau threefold. The theory contains some number of vector multiplets, determined by the Hodge numbers of the Calabi-Yau. It then turns out that the action up to four derivatives is completely determined in terms of two additional pieces of topological data, namely the triple intersection numbers and second Chern class of the Calabi-Yau. The task of finding solutions to this theory is greatly simplified by working in a fully off-shell formalism.[25−28] This means that enough auxiliary fields are introduced so that the supersymmetry transformations are independent of the action.[c] This is a great advantage, because the supersymmetry transformation laws are very simple, while the explicit action is quite complicated. In looking for BPS solutions we first exhaust the conditions implied by unbroken supersymmetry; in the off-shell formalism it follows that this part of the analysis proceeds the same whether one considers the two, four, or even higher derivative solutions. Much of the solution thereby can be determined without great effort. Only at the

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[c]A familiar example of this is $N = 1$ supersymmetric field theory in four dimensions, where the superspace construction ensures that the supersymmetry algebra closes without having to use the equations of motion.

very end do we need to consider some of the equations of motion in order to complete the solutions. In general, we find that the full solution can be expressed algebraically in terms of a single function, which obeys a nonlinear ordinary differential equation. This equation is straightforward to solve numerically.

1.3. *Overview of results*

We now give an overview of the results contained in this review. We begin with a review of standard two-derivative $D = 5$ supergravity coupled to an arbitrary number of vector multiplets,[29] and discuss the relevant solutions of this theory: black holes, black strings, and black rings. These are the solutions that we want to generalize to the higher derivative context. We next turn to the formalism of $D = 5$ R^2-corrected supergravity, obtained via the gauge fixing of a superconformal theory. The technical virtue of introducing the physically extraneous superconformal symmetry is that it facilitates the construction of fully off-shell multiplets. Our goal is to highlight the main conceptual steps of the superconformal program, while leaving the technical details to the original literature.[25−28] The four derivative part of the action is to be thought of as the supersymmetric completion of a certain mixed gauge-gravitational Chern-Simons term. This term is related to gauge and gravitational anomalies, and it is this relation that leads to the non-renormalization theorem mentioned above.

All of the solutions that we consider have the property of having a near horizon region with enhanced supersymmetry. This fact implies that the near horizon geometries are much simpler to obtain than the full asymptotically flat solutions, since some of the equations of motion can be traded for the simpler conditions following from enhanced supersymmetry. Therefore, in the interests of pedagogy, we first show how to obtain the near horizon solutions directly, postponing the full solutions until later. The near horizon geometries fall into two classes, depending on whether an AdS_2 or AdS_3 factor is present, and we analyze each in turn. We also exhibit the higher derivative version of the attractor mechanism, which fixes the moduli in terms of the electric and magnetic fluxes in the near horizon region.

Our discussion of the asymptotically flat solutions naturally divides into two parts. Half BPS solutions have a distinguished Killing vector formed out of the Killing spinors, and the analysis hinges on whether this Killing vector is timelike or null.[30] The timelike case gives rise to 5D black holes[17−19] and black rings,[22,23] while the null case corresponds to 5D black strings.[20,21] We show how to systematically construct these solutions, starting by applying

the conditions of unbroken supersymmetry, and then imposing the equations of motion for the Maxwell and auxiliary fields.

After constructing the full solutions and observing that they indeed contain the near horizon regions with enhanced supersymmetry, we turn to evaluating the black hole entropy. This is not completely straightforward, since Wald's formula does not directly apply, as the action contains non-gauge invariant Chern-Simons terms. We apply two different strategies, depending on whether an AdS_3 or AdS_2 factor is present. In the AdS_3 case, finding the black hole entropy can be reduced to finding the generalized Brown-Henneaux central charges[31] of the underlying Virasoro algebras.[d] For a general Lagrangian, an efficient c-extremization formula is available,[3] which reduces the computation of the central charges to solving a set of algebraic equations. In the supersymmetric context the procedure is even simpler, since the central charges can be read off from the coefficients of the Chern-Simons terms.[35,3,24] We carry out both procedures and show that they agree. For AdS_2, there is a similar extremization recipe based on the so-called entropy function.[2] Applying the entropy function here requires a bit of extra work, since the Chern-Simons terms need to be rewritten in a gauge invariant form in order for Wald's formula to apply. We carry this out and obtain the explicit entropy formulas for our black hole solutions. The results turn out to be remarkably simple. In the case of non-rotating 5D black holes the effects of the higher derivative terms manifest themselves simply as a shift in the charges.

We are also able to shed light on the connection between four and five dimensional black holes. One way to interpolate between these solutions is by placing the 5D black object at or near the tip of a Taub-NUT geometry, as in Refs. 36–40. The Taub-NUT contains a circle of freely adjustable size, which is to be thought of as the Kaluza-Klein circle that takes us from five to four dimensions. When the circle is large the solution is effectively five dimensional, while in the other limit we have a four dimensional black hole. In the case of BPS black holes, the entropy is independent of moduli (except possibly for isolated jumps at walls of marginal stability), and so this interpolation lets us relate the entropies of four and five dimensional black holes.[37] At the two-derivative level this works out very simply. In Section 10 we also review how the full solutions can be mapped back and forth, by demonstrating

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[d]To be precise, the central charges take into account the contributions from all *local* terms in the effective action. Additional nonlocal terms are also present due to the fact that the black hole has a different topology than Minkowski space. These contributions come from the worldlines of particles winding around the horizon.[32−34]

the equivalence of the BPS equations in the two cases.[41] On the other hand, higher derivative corrections bring in some new complications. As we shall see explicitly, due to the existence of curvature induced charge densities, the relations between the four and five dimensional charges gets corrected in a nontrivial way. Also, for reasons that we discuss, we find that there is apparently no simple relation between the two sets of BPS equations.

1.4. *Additional references*

This review is mainly based on Refs. 42, 14, and 43. However, we have also taken the opportunity to include some new results, in particular extending the higher derivative BPS equations to the general case.

The literature on higher derivative corrections to supersymmetric black holes is by now quite large. There are a number of reviews available which overlap with some of the topics discussed here, for example Refs. 44–50. Further references on supergravity solutions in the presence of higher derivatives include Refs. 51–61. Some references relating higher derivative corrections to the topological string and the OSV conjecture include Refs. 62–66. Works discussing small black holes in the context of AdS/CFT include Refs. 54 and 67–69. For more on supergravity and the attractor mechanism in five dimensions see Refs. 19, 21 and 70–75. Fundamental string solutions are further discussed in Refs. 76–80.

2. On-shell Formalism for $D = 5$ Supergravity

In this section we briefly review the two-derivative supergravity theory in which we are interested, $N = 2$ supergravity in five dimensions coupled to an arbitrary number of vector multiplets. We review how this theory is embedded in string theory as a dimensional reduction of eleven-dimensional supergravity on a Calabi-Yau threefold.[81,82] We also summarize the panoply of supersymmetric solutions to this theory, including their origin as wrapped M-branes in the higher-dimensional theory. This is all done in the familiar "on-shell" formalism (in contrast to the formalism to be introduced subsequently in this review).

2.1. *M-theory on a Calabi-Yau threefold*

We begin with the action

$$S_{11} = -\frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G} \left(R + \frac{1}{2}|\mathcal{F}_4|^2 \right) + \frac{1}{12\kappa_{11}^2} \int \mathcal{A}_3 \wedge \mathcal{F}_4 \wedge \mathcal{F}_4 , \qquad (2)$$

which is the bosonic part of the low energy eleven-dimensional supergravity theory. The perturbative spectrum contains the graviton G_{MN}, the gravitino Ψ_M, and the three-form potential $\mathcal{A}_3$ with field strength $\mathcal{F}_4 = d\mathcal{A}_3$. This theory is maximally supersymmetric, possessing 32 independent supersymmetries. There are spatially extended half-BPS solitons, the $M2$ and $M5$-branes, which carry the electric and magnetic charges, respectively, of the flux $\mathcal{F}_4$.

The five-dimensional theory of interest is obtained via compactification on a Calabi-Yau threefold CY_3 and depends only on topological data of the compactification manifold. Let J_I be a basis of closed $(1,1)$-forms spanning the Dolbeault cohomology group $H^{(1,1)}(CY_3)$ and let $h^{(1,1)} = dim\left(H^{(1,1)}(CY_3)\right)$. We can then expand the Kähler form J on CY_3 as

$$J = M^I J_I , \qquad I = 1 \ldots h^{(1,1)} . \tag{3}$$

By de Rham's theorem[e] we can choose a basis of two-cycles ω^K for the homology group $H_2(CY_3)$ such that

$$\int_{\omega^K} J_I = \delta_I^K . \tag{4}$$

Thus the real-valued expansion coefficients M^I can be understood as the volumes of the two-cycles ω^I

$$M^I = \int_{\omega^I} J . \tag{5}$$

The M^I are known as *Kähler moduli* and they act as scalar fields in the effective five-dimensional theory. We will often refer to the M^I simply as the *moduli* since the other Calabi-Yau moduli, the *complex structure moduli*, lie in $D = 5$ hypermultiplets and are decoupled for the purposes of investigating stationary solutions.[f] We will therefore largely ignore the hypermultiplets in the following.

The eleven-dimensional three-form potential can be decomposed after compactification as

$$\mathcal{A}_3 = A^I \wedge J_I , \tag{6}$$

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[e]This theorem asserts the duality between the homology group $H_2(\mathcal{M})$ and the *de Rham* cohomology $H^2(\mathcal{M})$ for a manifold $\mathcal{M}$. For a Calabi-Yau threefold there are no $(0,2)$ or $(2,0)$ forms in the cohomology so we have a duality between H_2 and the *Dolbeault* cohomology $H^{(1,1)}$.
[f]By *decoupled*, we mean that they can be set to constant values in a way consistent with the BPS conditions and equations of motion of the theory.

where A^I is a one-form living in $D = 5$. This results in $h^{(1,1)}$ vector fields in the five-dimensional effective theory. Since the J_I are closed, the field strengths are given by

$$\mathcal{F}_4 = F^I \wedge J_I \ , \tag{7}$$

where $F^I = dA^I$. The eleven-dimensional Chern-Simons term reduces to

$$\int_{\mathcal{M}_{11}} \mathcal{A}_3 \wedge \mathcal{F}_4 \wedge \mathcal{F}_4 = \int_{CY_3} J_I \wedge J_J \wedge J_K \int_{\mathcal{M}_5} A^I \wedge F^J \wedge F^K$$
$$= c_{IJK} \int_{\mathcal{M}_5} A^I \wedge F^J \wedge F^K \ , \tag{8}$$

where in the last line we have used the definition of the Calabi-Yau manifold *intersection numbers*

$$c_{IJK} = \int_{CY_3} J_I \wedge J_J \wedge J_K \ . \tag{9}$$

The nomenclature arises since c_{IJK} can be regarded as counting the number of triple intersections of the four-cycles ω_I, ω_J, and ω_K, which are basis elements of the homology group $H_4 (CY_3)$. This basis has been chosen to be dual to the previously introduced basis of $H_2 (CY_3)$, *i.e.* with normalized inner product

$$\left(\omega^I, \omega_J\right) = \delta^I_J \ , \tag{10}$$

where this inner product counts the number of intersections of the cycles ω^I and ω_J.

The above is almost sufficient to write down the $D = 5$ Lagrangian, but there is an important constraint that must be considered separately. To fill up the five-dimensional supersymmetry multiplets one linear combination of the aforementioned vectors must reside in the gravity multiplet. This vector is called the *graviphoton* and is given by

$$A^{grav}_\mu = M_I A^I_\mu \ , \tag{11}$$

where the M_I are the volumes of the basis four-cycles ω_I

$$M_I = \frac{1}{2} \int_{\omega_I} J \wedge J = \frac{1}{2} \int_{CY_3} J \wedge J \wedge J_I = \frac{1}{2} c_{IJK} M^J M^K \ . \tag{12}$$

Since one combination of the vectors arising from compactification does not live in a vector multiplet, the same must be true of the scalars. It turns out that the total Calabi-Yau volume, which we call $\mathcal{N}$, sits in a hypermultiplet. Due to the decoupling of hypermultiplets we can simply fix the value of the volume, and so we arrive at the *very special geometry* constraint

$$\mathcal{N} \equiv \frac{1}{3!} \int_{CY_3} J \wedge J \wedge J = \frac{1}{6} c_{IJK} M^I M^J M^K = 1 \ . \tag{13}$$

Due to the above considerations the index I runs over $1 \dots (n_V + 1)$, where n_V is the number of independent vector multiplets in the effective theory.

Choosing units[g] such that $\kappa_{11}^2 = \kappa_5^2 \mathcal{N} = 2\pi^2$, the action for the theory outlined above is

$$S = \frac{1}{4\pi^2} \int_{\mathcal{M}_5} d^5 x \sqrt{|g|} \mathcal{L} \ , \tag{14}$$

with Lagrangian

$$\mathcal{L} = -R - G_{IJ} \partial_a M^I \partial^a M^J - \frac{1}{2} G_{IJ} F_{ab}^I F^{Jab} + \frac{1}{24} c_{IJK} A_a^I F_{bc}^J F_{de}^K \epsilon^{abcde} \ . \tag{15}$$

The metric on the scalar moduli space is[83]

$$G_{IJ} = \frac{1}{2} \int_{CY_3} J_I \wedge *J_J = -\frac{1}{2} \partial_I \partial_J (\ln \mathcal{N})|_{\mathcal{N}=1} = \frac{1}{2} \left(\mathcal{N}_I \mathcal{N}_J - \mathcal{N}_{IJ} \right) \ , \tag{16}$$

where the $*$ denotes Hodge duality within the Calabi-Yau and $\mathcal{N}_I$ and $\mathcal{N}_{IJ}$ denote derivatives of $\mathcal{N}$ with respect to the moduli

$$\mathcal{N}_I \equiv \partial_I \mathcal{N} = \frac{1}{2} c_{IJK} M^J M^K = M_I \ , \quad \mathcal{N}_{IJ} \equiv \partial_I \partial_J \mathcal{N} = c_{IJK} M^K \ . \tag{17}$$

As previously stated, the eleven-dimensional theory is maximally supersymmetric with 32 independent supersymmetries. A generic Calabi-Yau manifold has $SU(3)$ holonomy, reduced from $SU(4) (\cong SO(6))$ for a generic six-dimensional manifold; thus it preserves 1/4 supersymmetry, or 8 independent supersymmetries. More precisely, this is the number of explicit supersymmetries for general c_{IJK}; for special values of the c_{IJK} there are more supersymmetries which are implicit in our formalism. These values correspond to compactification on a manifold $\mathcal{M}$ of further restricted holonomy; for $\mathcal{M} = T^2 \times K3$ there are 16 supersymmetries, while for $\mathcal{M} = T^6$ there are 32.

2.2. *M-branes and $D = 5$ solutions*

Eleven-dimensional supergravity has asymptotically flat solutions with non-trivial four-form flux. In the full quantum description, these solutions are understood to be sourced by certain solitonic objects, the *M*-branes. Specifically, the *M2-brane* is an extended object with a $(2+1)$-dimensional worldvolume which carries the unit electric charge associated with $\mathcal{A}_3$. Conversely, the *M5-brane* carries the unit magnetic charge of $\mathcal{A}_3$ and has a $(5+1)$-dimensional worldvolume. The worldvolumes of these objects can be wrapped around various cycles in a Calabi-Yau and so lead to sources in the effective five-dimensional theory.

[g]Equivalently, our units are such that the five-dimensional Newton's constant is $G_5 = \frac{\pi}{4}$.

The five-dimensional theory has a wealth of interesting supersymmetric solutions including black holes, black strings and black rings. As indicated above, these each can be embedded into M-theory as a bound state of M-branes. In particular, wrapping an $M2$-brane around one of the basis two-cycles ω^I leads to a five-dimensional solution carrying electric charges

$$q_I \equiv -\int_{S^3} \frac{\delta S}{\delta F^I} = \frac{1}{2\pi^2} \int_{S^3} G_{IJ} \star F^J \;, \tag{18}$$

where the integral is taken over the asymptotic three-sphere surrounding the black hole. Wrapping an $M5$ brane around one of the basis four-cycles ω_I gives an infinitely extended one-dimensional string,[h] carrying the magnetic charges

$$p^I = -\frac{1}{2\pi} \int_{S^2} F^I \;, \tag{19}$$

where one integrates over the asymptotic two-sphere surrounding the string. Further, there are dyonic solutions constructed from both $M2$ and $M5$-branes. These can take the form of either infinite strings with an extended electric charge density along their volume, or a black ring, where the $M5$-branes contribute non-conserved magnetic dipole moments.

In the next section, we briefly describe the various supersymmetric solutions of the $D = 5$ on-shell supergravity.

2.3. *Solutions of the two-derivative theory*

In this section we collect the standard black hole, black string, and black ring solutions in the two-derivative theory. The remainder of this review consists of finding the analogous solutions in the presence of higher derivatives.

When the distinguished Killing vector is timelike, the 5D metric and gauge field strengths take the form[i]

$$ds^2 = e^{4U(x)}(dt + \omega)^2 - e^{-2U(x)}ds_B^2 \;, \tag{20}$$

$$F^I = d[M^I e^{2U}(dt + \omega)] + \Theta^I \;. \tag{21}$$

The 4D base metric ds_B^2, with coordinates x, is required by supersymmetry to be hyperKähler.[j] U, ω, and Θ^I are, respectively, a function, a one-form, and a two-form on the base.

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[h]This is not to be confused with a fundamental string, although special configurations are dual to an infinite heterotic string.

[i]We derive these results is section 7.

[j]In general, the base metric can have indefinite signature,[84−86] but here we restrict to Euclidean signature.

The Bianchi identity implies that Θ^I is closed. The moduli obey the *very special geometry* constraint

$$\frac{1}{6}c_{IJK}M^I M^J M^K = 1 \ . \tag{22}$$

Half BPS solutions are those obeying the following set of equations[30,74]

$$\Theta^I = - \star_4 \Theta^I \ , \tag{23}$$

$$\nabla^2(M_I e^{-2U}) = \frac{1}{2}c_{IJK}\Theta^J \cdot \Theta^K \ , \tag{24}$$

$$d\omega - \star_4 d\omega = -e^{-2U}M_I \Theta^I \ , \tag{25}$$

where the above equations should all be understood as tensor equations on the 4D base space. When higher derivatives are added, the metric and gauge fields will still take the form (20,21), Θ^I will remain closed and anti-self-dual, but the remaining two equations (24,25) will be modified.

2.3.1. *Solutions with Gibbons-Hawking base*

If we take the base space to be a Gibbons-Hawking space[87] then the above equations admit a general solution in terms of locally harmonic functions.[30,88] The Gibbons-Hawking space is

$$ds^2_B = (H^0)^{-1}(dx^5 + \chi)^2 + H^0 dx^m dx^m \ , \tag{26}$$

where H^0 is harmonic on $\mathbb{R}^3$, up to isolated singularities, and the 1-form χ obeys

$$\vec{\nabla} \times \vec{\chi} = \vec{\nabla}H^0 \ , \tag{27}$$

and $\vec{\nabla}$ denotes the gradient on $\mathbb{R}^3$. The compact coordinate x^5 has period 4π.

The closed, anti-self-dual 2-form Θ^I can now be expressed as

$$\Theta^I = \frac{1}{2}(dx^5 + \chi) \wedge \Lambda^I - \frac{1}{4}H^0 \epsilon_{mnp}\Lambda^I_m dx^n \wedge dx^p \ , \tag{28}$$

with

$$\Lambda^I = d\left(\frac{H^I}{H^0}\right) \ , \tag{29}$$

for some set of functions H^I harmonic on $\mathbb{R}^3$. We can then solve (24) as

$$M_I - \frac{1}{8}\frac{c_{IJK}H^J H^K}{H^0} = H_I \ , \tag{30}$$

with H_I harmonic on $\mathbb{R}^3$. Imposing the special geometry constraint (22) allows us to solve for e^{-2U} in terms of M^I and the harmonic functions.

We next decompose ω as

$$\omega = \omega_5(dx^5 + \chi) + \hat{\omega} \ , \tag{31}$$

with

$$\hat{\omega} = \hat{\omega}_m dx^m \ . \tag{32}$$

Equation (25) then becomes

$$\vec{\nabla} \times \vec{\hat{\omega}} = H^0 \vec{\nabla} \omega_5 - \omega_5 \vec{\nabla} H^0 - \frac{1}{2}\Big(H^0 H_I + \frac{1}{8}c_{IJK} H^J H^K\Big)\vec{\nabla}\left(\frac{H^I}{H^0}\right) \ . \tag{33}$$

Taking the gradient we get

$$\nabla^2 \omega_5 = \nabla^2 \Big[\frac{1}{4}\frac{H_I H^I}{H^0} + \frac{1}{48}\frac{c_{IJK} H^I H^J H^K}{(H^0)^2}\Big] \ , \tag{34}$$

and we then solve for ω_5 in terms of another harmonic function H_0,

$$\omega_5 = \frac{1}{4}\frac{H_I H^I}{H^0} + \frac{1}{48}\frac{c_{IJK} H^I H^J H^K}{(H^0)^2} + H_0 \ . \tag{35}$$

Finally, substituting back into (33) gives

$$\vec{\nabla} \times \vec{\hat{\omega}} = H^0 \vec{\nabla} H_0 - H_0 \vec{\nabla} H^0 - \frac{1}{4}[H_I \vec{\nabla} H^I - H^I \vec{\nabla} H_I] \ . \tag{36}$$

To summarize, the full solution is given by choosing harmonic functions $(H^0, H^I; H_0, H_I)$, in terms of which we have

$$M_I e^{-2U} = \frac{1}{8}\frac{c_{IJK} H^J H^K}{H^0} + H_I \ , \tag{37}$$

$$e^{-2U} = \frac{1}{3}\left(H_I M^I + \frac{1}{8}\frac{c_{IJK} M^I H^J H^K}{H^0}\right) \ , \tag{38}$$

$$\Theta^I = \frac{1}{2}(dx^5 + \chi) \wedge d\left(\frac{H^I}{H^0}\right) - \frac{1}{4}H^0 \epsilon_{mnp}\partial_m\left(\frac{H^I}{H^0}\right)dx^n \wedge dx^p \ , \tag{39}$$

$$\omega = \omega_5(dx^5 + \chi) + \hat{\omega} \ , \tag{40}$$

$$\omega_5 = \frac{1}{4}\frac{H_I H^I}{H^0} + \frac{1}{48}\frac{c_{IJK} H^I H^J H^K}{(H^0)^2} + H_0 \ , \tag{41}$$

$$\vec{\nabla} \times \vec{\hat{\omega}} = H^0 \vec{\nabla} H_0 - H_0 \vec{\nabla} H^0 + \frac{1}{4}(H^I \vec{\nabla} H_I - H_I \vec{\nabla} H^I) \ . \tag{42}$$

We usually take the harmonic functions to have isolated singularities where $\vec{\nabla}^2 H \propto \delta^{(3)}(\vec{x}-\vec{x}_i)$, in which case (42) implies a nontrivial integrability constraint on the harmonic functions, obtained by taking the divergence of both sides. We now consider various examples.

2.3.2. *5D static black hole*

This corresponds to the choice,

$$H^0 = \frac{1}{|\vec{x}|} \ , \quad H^I = 0 \ ; \quad H_0 = 0 \ , \quad H_I = H_I^\infty + \frac{q_I}{4|\vec{x}|} \ . \tag{43}$$

In this case $\omega = \Theta^I = 0$, and we have a spherically symmetric 5D black hole carrying the electric charges q_I. Note that the Gibbons-Hawking space becomes simply flat $\mathbb{R}^4$ in nonstandard coordinates. The near horizon geometry is $\mathrm{AdS}_2 \times S^3$ with scale sizes $\ell_A = \frac{1}{2}\ell_S$. The near horizon moduli are $M_I = \frac{q_I}{4\ell_A^2}$, from which ℓ_A can be computed from the special geometry constraint (13). In particular, if we define the dual charges q^I through

$$q_I = \frac{1}{2}c_{IJK}q^J q^K \ , \tag{44}$$

then $\ell_A = \frac{1}{2}Q^{1/2}$ with[k]

$$Q^{3/2} = \frac{1}{6}c_{IJK}q^I q^J q^K \ . \tag{45}$$

The entropy is $S = 2\pi\sqrt{Q^3}$. Finding explicit expressions in terms of the charges q_I requires that the equation (44) can be inverted to find q^I, but this is only possible for special choices of c_{IJK}.

2.3.3. *5D spinning (BMPV) black hole*

To add rotation we take $H_0 = \frac{q_0}{16|\vec{x}|}$ and keep the other harmonic functions as in (43). The integer normalized eigenvalues of the $SU(2)_L \times SU(2)_R$ rotation group are $(J_L = q_0, \ J_R = 0)$. A nontrivial fact is that the functions M^I and U are unaffected by the inclusion of angular momentum.[l] The near horizon geometry is now described by a circle fibered over an $\mathrm{AdS}_2 \times S^2$ base, and the entropy is $S = 2\pi\sqrt{Q^3 - \frac{1}{4}q_0^2}$, with ℓ_A the same as in the static case.

2.3.4. *5D spinning black hole on Taub-NUT = 4D black hole*

We now take the base to be a charge p^0 Taub-NUT by changing H^0 and H_0 to

$$H^0 = 1 + \frac{p^0}{|\vec{x}|} \ , \quad H_0 = H_0^\infty + \frac{q_0}{16|\vec{x}|} \ . \tag{46}$$

The integrability condition fixes $H_0^\infty = \frac{q_0}{16p^0}$. Taub-NUT is asymptotically $\mathbb{R}^3 \times S^1$, and so this gives rise to a 4D black hole after reduction on the S^1, with magnetic charge $(p^0, \ p^I = 0)$

---

[k]Note that Q has the same dimension as the physical electric charges q_I.
[l]This will no longer be true with higher derivatives.

and electric charges (q_0, q_I). For $p^0 \neq 1$ the Taub-NUT has a $\mathbb{Z}_{p^0}$ singularity at the origin; the full 5D metric is smooth, however. The near horizon geometry is again a circle fibered over an $\text{AdS}_2 \times S^2$ base. The near horizon moduli and ℓ_A are unchanged, and the entropy is $S = 2\pi\sqrt{p^0 Q^3 - \frac{1}{4}(p^0 q_0)^2}$.

2.3.5. *5D black ring*

We now choose the harmonic functions

$$H^0 = \frac{1}{|\vec{x}|}, \ H^I = \frac{p^I}{|\vec{x} + R\hat{n}|}, \ H_0 = \frac{q_0}{16}\left(\frac{1}{|\vec{x} + R\hat{n}|} - \frac{1}{R}\right), \ H_I = H_I^\infty + \frac{\overline{q}_I}{4|\vec{x} + R\hat{n}|} \ , \tag{47}$$

where $\hat{n}$ is an arbitrary unit vector in $\mathbb{R}^3$. The parameter R is interpreted as the ring radius, and is fixed by the integrability condition to be $R = \frac{-q_0}{4p^I H_I^\infty}$. The conserved electric charges measured at infinity are $q_I = \overline{q}_I + \frac{1}{2}c_{IJK}p^J p^K$, and there are also non-conserved magnetic dipole charges p^I. The near horizon geometry is $\text{AdS}_3 \times S^2$ with $\ell_A = 2\ell_S = (\frac{1}{6}c_{IJK}p^I p^J p^K)^{\frac{1}{3}}$. The near horizon moduli are $M^I = \frac{p^I}{\ell_A}$. The black ring entropy is given by setting $p^0 = 1$ in formula (49) below.

2.3.6. *5D black ring on Taub-NUT = 4D two-center black hole*

This is the most general case that we'll consider, with harmonic functions

$$H^0 = 1 + \frac{p^0}{|\vec{x}|}, \ H^I = \frac{p^I}{|\vec{x} + R\hat{n}|}, \ H_0 = \frac{q_0}{16}\left(\frac{1}{|\vec{x} + R\hat{n}|} - \frac{1}{R}\right), \ H_I = H_I^\infty + \frac{\overline{q}_I}{4|\vec{x} + R\hat{n}|} \ , \tag{48}$$

and the radius is now determined by $1 + \frac{p^0}{R} = \frac{4p^I H_I^\infty}{-q_0}$. From the 4D perspective this solution is a two-centered geometry, with one center being a magnetic charge p^0 and the other being a 4D dyonic black hole with charges proportional to (p^I, q_0, $\overline{q}_I$). The entropy of the black hole is[89]

$$S = 2\pi\sqrt{p^0 Q^3 - (p^0)^2 J^2} \ , \tag{49}$$

with

$$Q^{\frac{3}{2}} = \frac{1}{6}c_{IJK}y^I y^J y^K \ ,$$
$$c_{IJK}y^J y^K = 2\overline{q}_I + \frac{c_{IJK}p^J p^K}{p^0} \ ,$$
$$J = \frac{q^0}{2} + \frac{\frac{1}{6}c_{IJK}p^I p^J p^K}{(p^0)^2} + \frac{p^I \overline{q}_I}{2p^0} \ . \tag{50}$$

2.3.7. *5D black string*

Starting with the black ring but taking $H^0 = 1$, so that the base becomes $\mathbb{R}^3 \times S^1$, yields a 5D black string carrying electric and magnetic charges.[90] An even simpler black string is obtained by taking the limit $H^0 \to 0$. In this limit the metric on the base degenerates, but the full 5D solution is well behaved:

$$ds^2 = \frac{4}{(\frac{1}{6}c_{IJK}H^I H^J H^K)^{1/3}} \left(dt dx^5 + \hat{H}_0(dx^5)^2 \right) - \frac{1}{4}(\frac{1}{6}c_{IJK}H^I H^J H^K)^{2/3}d\vec{x}^2 \ ,$$

$$F^I = d\left[\frac{\frac{1}{3}H^I H_I}{(\frac{1}{6}c_{IJK}H^I H^J H^K)^{2/3}} \right] \wedge dx^5 - \frac{1}{4}\epsilon_{mnp}\partial_m H^I dx^n \wedge dx^p \ ,$$

$$M^I = \frac{H^I}{(\frac{1}{6}c_{IJK}H^I H^J H^K)^{1/3}} \ , \tag{51}$$

with

$$\hat{H}_0 = H_0 - \frac{1}{2}c^{IJ}H_I H_J \ , \tag{52}$$

and c^{IJ} is the inverse of $c_{IJ} = c_{IJK}H^K$. The near horizon geometry is locally $\mathrm{AdS}_3 \times S^2$ with $\ell_A = 2\ell_S = (\frac{1}{6}c_{IJK}p^I p^J p^K)^{\frac{1}{3}}$, and the near horizon moduli are $M^I = \frac{p^I}{\ell_A}$. Note that these are the same as for the black ring. The entropy is

$$S = 2\pi\sqrt{\frac{1}{6}c_{IJK}p^I p^J p^K \hat{q}_0} \ , \tag{53}$$

with $\hat{q}_0 = q_0 - \frac{1}{32}C^{IJ}\overline{q}_I\overline{q}_J$, and C^{IJ} is the inverse of $C_{IJ} = c_{IJK}p^K$. Here we continued to identify $x^5 \cong x^5 + 4\pi$, but we are to free to drop the identification and obtain an infinite 5D black string.

Note that by taking the $H^0 \to 0$ limit the formerly timelike Killing vector $\frac{\partial}{\partial t}$ has become null. The solution (51) therefore does not appear directly in the classification based on timelike supersymmetry, but rather lies in the domain of null supersymmetry.

There are also more general solutions without translation invariance along the string[30,91] but we will refrain from discussing them in detail here.

3. Conformal Supergravity

The low energy limit of a supersymmetric compactification of string theory is a supergravity theory. While the Lagrangians of these theories can in principle be extracted from string S-matrix computations, in practice a more efficient method is to work directly in field theory, by demanding invariance under local supersymmetry. This approach typically uses the so-called Noether method. In this procedure one starts with an action invariant under global

supersymmetry, and then attempts to incorporate local invariance iteratively. For a two-derivative Lagrangian the possible matter couplings are usually known, and the process of constructing the action and transformation rules for the fields only involves a finite number of steps. The incorporation of higher derivative terms increases enormously the possible terms in the action and transformation rules. For example, one might start by including a specific four-derivative interaction and using the two-derivative transformation rules. This will generate additional four-derivative terms that will necessitate modifications to the supersymmetry transformations. Now, these modified transformations will generate six-derivative terms in the Lagrangian and so forth. In general, it may take many steps, if not an infinite number, for this iterative procedure to terminate, making the construction extremely difficult and tedious.

A more systematic approach to obtaining an invariant action is by constructing off-shell representations of the supersymmetry algebra. The advantage of this formalism is that the construction of invariants is well-defined since the transformation rules are fixed. Now, the theory we are aiming for is an off-shell version of Poincaré supergravity. However, it turns out that the construction of off-shell multiplets is greatly simplified by first considering a theory with a larger gauge invariance, and then at the end gauge fixing down to Poincaré supergravity. In five dimensions, it turns out that extending conformal supergravity to a gauge theory described by the superalgebra $F(4)$ gives an irreducible off-shell realization of the gravity and matter multiplets. The cost of this procedure is the inclusion of additional symmetries and compensating fields, which have no physical degrees of freedom. The construction of a supergravity theory from the gauge theory is first done by imposing constraints that identify the gauge theory as a gravity theory. Then, gauge fixing appropriately the values of certain compensating fields, one reduces the superconformal theory to Poincaré supergravity. This has been extensively studied for $d \leqslant 6$ superconformal theories; for more details we refer the reader to Refs. 25–28, 44, and 92–94.

One of the fruitful applications of this formalism is the construction of higher-derivative Lagrangians. Specifically, we will consider the four-derivative corrections to $N = 2$ supergravity which arise from string theory. In five dimensional theories, there is a special mixed gauge-gravitational Chern-Simons term given by

$$\mathcal{L}_{\text{CS}} = \frac{c_{2I}}{24 \cdot 16} \epsilon_{abcde} A^{Ia} R^{bcfg} R^{de}{}_{fg} \ . \tag{54}$$

The coefficient of this term is precisely determined in string/M-theory by $M5$-brane anomaly

cancellation via anomaly inflow.[95] The constants c_{2I} are understood as the expansion coefficients of the second Chern class of the Calabi-Yau threefold on which the eleven-dimensional M-theory is compactified. In Ref. 28 all terms related by supersymmetry to (54) were derived using the superconformal formalism.

Our present goal is to simply introduce the main concepts of the superconformal formalism and the specific results we will use to study black holes and other such objects in the next sections. In the following subsections we outline this construction by first describing the full superconformal algebra and the necessary identifications to obtain the gravity theory. The field content and transformation rules for the gravity and matter multiplets are given, and we briefly explain how to obtain invariant actions for these multiplets. At the end of this section, after gauge fixing the superconformal theory, we present the R^2 supersymmetric completion of $N = 2$ supergravity.

3.1. *Superconformal formalism*

The five dimensional theory is obtained by first constructing a gauge theory with gauge symmetry given by the supergroup $F(4)$. The generators $\mathbf{X}_A$ and corresponding gauge fields h_μ^A for this theory are

$$\mathbf{X}_A : \ \mathbf{P}_a \ , \ \mathbf{M}_{ab} \ , \ \mathbf{D} \ , \ \mathbf{K}_a \ , \ \mathbf{U}_{ij} \ , \ \mathbf{Q}_i \ , \ \mathbf{S}^i$$
$$h_\mu^A : \ e_\mu^a \ , \ \omega_\mu^{ab} \ , \ b_\mu \ , \ f_\mu^a \ , \ V_\mu^{ij} \ , \ \psi_\mu^i \ , \ \phi_\mu^i \tag{55}$$

where $a, b = 0, \ldots, 4$ are tangent space indices, $\mu, \nu = 0, \ldots, 4$ are (curved) spacetime indices and $i, j = 1, 2$ are $SU(2)$ indices. The generators of the Poincaré algebra are translations $\mathbf{P}_a$ and Lorentz transformations $\mathbf{M}_{ab}$. The special conformal transformations and dilatations are generated by $\mathbf{D}$ and $\mathbf{K}_a$, respectively. $\mathbf{U}_{ij}$ is the generator of $SU(2)$ and the fermionic generators for supersymmetry and conformal supersymmetry are the symplectic Majorana spinors $\mathbf{Q}_i$ and $\mathbf{S}^i$.

The next step is to construct from the superconformal gauge theory a conformal supergravity theory, *i.e.* our symmetries have to be realized as space-time symmetries rather than internal symmetries. In order to make this a theory of diffeomorphisms of spacetime one needs to identify the non-compact translations $\mathbf{P}_a$ with general coordinate transformations generated by $\mathcal{D}_a$. This procedure is well known[96] and it is achieved by applying torsion-less

constraints[m] over the curvatures, which are

$$\hat{R}^a_{\mu\nu}(P) = 0 \ , \quad \gamma^\mu \hat{R}^i_{\mu\nu}(Q) = 0 \ , \quad \hat{R}_\mu{}^a(M) = 0 \ . \tag{56}$$

Here the curvatures are defined as commutators of the conformal supercovariant derivatives, that is

$$[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu] = -\hat{R}^A_{\mu\nu}\mathbf{X}_A \ , \tag{57}$$

with

$$\hat{\mathcal{D}}_\mu = \partial_\mu - h^A_\mu \mathbf{X}_A \ , \tag{58}$$

where we are summing over $\mathbf{X}_A = \{\mathbf{M}_{ab}, \ \mathbf{D}, \ \mathbf{K}_a, \ \mathbf{U}^{ij}, \ \mathbf{Q}_i, \ \mathbf{S}^i\}$. By solving (56), some of the gauge fields will become dependent fields. Assuming that the vielbein e^a_μ is invertible, the first constraint will determine the connection ω^{ab}_μ. The second and third constraints fix ϕ^i_μ and f^a_μ, respectively, making them dependent fields as well.[25,26]

The final step in constructing the off-shell gravity multiplet is adding auxiliary fields. In order to understand this, it is useful to track the number of independent bosonic and fermionic components. Before imposing (56) the gauge fields are composed of 96 bosonic and 64 fermionic gauge fields. Explicitly, the number of independent gauge fields is the total number of components h^A_μ minus the number of generators $\mathbf{X}_A$:

$$\begin{aligned} h^A_\mu : \ &e^a_\mu \ , \ \omega^{ab}_\mu \ , \ b_\mu \ , \ f^a_\mu \ , \ V^{ij}_\mu \ , \ \psi^i_\mu \ , \ \phi^i_\mu \ . \\ \# : \ &20 \ , \ 40 \ , \ 4 \ , \ 20 \ , \ 12 \ , \ 32 \ , \ 32 \ . \end{aligned} \tag{59}$$

The curvature constraints fix the connections ω^{ab}_μ, ϕ^i_μ and f^a_μ, eliminating their degrees of freedom. The new number of degrees of freedom is then the total number of components of the remaining gauge fields minus the total number of the generators $\mathbf{X}_A$. This counting results in 21+24 degrees of freedom. Adding auxiliary fields, which will include extra transformation rules and modifications to the supersymmetry algebra, solves this final mismatch in the number of bosonic and fermionic degrees of freedom. The procedure has been outlined in Ref. 94.

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[m]In the literature, (56) are often called the *conventional constraints*.

3.1.1. *Weyl multiplet*

The construction sketched above gives the irreducible Weyl multiplet (denoted by $\mathbf{W}$) describing $32 + 32$ degrees of freedom. The multiplet contains the following fields,

$$e_\mu^a \ , \ V_\mu^{ij} \ , \ b_\mu \ , \ v_{ab} \ , \ D \ , \ \psi_\mu^i \ , \ \chi^i \ . \tag{60}$$

As we mentioned before, in order to have a closed algebra it is necessary to include compensators, *i.e.* auxiliary fields. For the Weyl multiplet, the non-propagating fields are an antisymmetric two-form tensor v_{ab}, a scalar field D and an $SU(2)$ Majorana spinor χ^i. The $\mathbf{Q}$-$\mathbf{S}$ supersymmetry and $\mathbf{K}$ transformation rules for the bosonic fields in the multiplet are

$$\begin{aligned}
\delta e_\mu^a &= -2i\bar\epsilon\gamma^a\psi_\mu \ , \\
\delta V_\mu^{ij} &= -3i\bar\epsilon^i\phi_\mu^j + 2i\bar\epsilon^i\gamma\cdot v\psi_\mu^j - \frac{i}{8}\bar\epsilon^i\gamma_\mu\chi^j + 3i\bar\eta^i\psi_\mu^j + (i\leftrightarrow j) \ , \\
\delta b_\mu &= -2i\bar\epsilon\phi_\mu - 2i\bar\eta\psi_\mu - 2\xi_{K\mu} \ , \\
\delta v_{ab} &= -\frac{i}{8}\bar\epsilon\gamma_{ab}\chi - \frac{3}{2}i\bar\epsilon\hat{R}_{ab}(Q) \ , \\
\delta D &= -i\bar\epsilon\gamma^a\hat{\mathcal{D}}_a\chi - 8i\bar\epsilon\hat{R}_{ab}(Q)v^{ab} + i\bar\eta\chi \ ,
\end{aligned} \tag{61}$$

and for the fermionic fields we have

$$\begin{aligned}
\delta\psi_\mu^i &= \mathcal{D}_\mu\epsilon^i + \frac{1}{2}v^{ab}\gamma_{\mu ab}\epsilon^i - \gamma_\mu\eta^i \ , \\
\delta\chi^i &= D\epsilon^i - 2\gamma^c\gamma^{ab}\hat{\mathcal{D}}_a v_{bc}\epsilon^i + \gamma\cdot\hat{R}^i{}_j(U)\epsilon^j - 2\gamma^a v^{bc}v^{de}\epsilon_{abcde}\epsilon^i + 4\gamma\cdot v\eta^i \ ,
\end{aligned} \tag{62}$$

where $\delta \equiv \bar\epsilon^i\mathbf{Q}_i + \bar\eta^i\mathbf{S}_i + \xi_K^a\mathbf{K}_a$. The superconformal covariant derivative $\hat{\mathcal{D}}_a$ appearing in (61) and (62) is defined in (58). The un-hatted derivative $\mathcal{D}_a$ is defined similarly but the sum is only over $\mathbf{X}_A = \left\{\mathbf{M}_{ab}, \mathbf{D}, \mathbf{U}_{ij}\right\}$.

3.1.2. *Matter multiplets*

In this section we will describe the properties and transformation rules for matter multiplets coupled to five-dimensional conformal supergravity. The three matter multiplets relevant for our purposes are the vector multiplet, hypermultiplet and linear multiplet.

- **Vector multiplet:** The off-shell components of the vector multiplet $\mathbf{V}$ are,

$$M^I \ , \ A_\mu^I \ , \ Y_{ij}^I \ , \ \Omega_i^I \ .$$

   M^I are scalar fields and A_μ^I gauge fields. The multiplet also contains a $SU(2)$ triplet auxiliary field Y_{ij}^I and the $SU(2)$ Majorana spinor Ω_i^I. The index I labels the gener-

ators of the gauge group G. For brevity, we consider G as $n_V + 1$ copies of $U(1)$; the generalization to non-Abelian gauge groups is discussed in Refs. 25 and 26. The **Q** and **S** transformation rules for the fermion in the vector multiplet is,

$$\delta\Omega^{Ii} = -\frac{1}{4}\gamma \cdot \hat{F}^I \epsilon^i - \frac{1}{2}\gamma^a \hat{\mathcal{D}}_a M^I \epsilon^i + Y^{Ii}_j \epsilon^j - M^I \eta^i \ ,$$

with the field strength given by

$$\hat{F}^I_{\mu\nu} = 2\partial_{[\mu} A^I_{\nu]} + 4i\bar{\psi}_{[\mu}\gamma_{\nu]}\Omega^I - 2i\bar{\psi}_\mu\psi_\nu M^I \ .$$

- **Hypermultiplet:** The components of **H**, the hypermultiplet, are

$$\mathcal{A}^\alpha_i \ , \ \zeta^\alpha \ , \ \mathcal{F}^i_\alpha \ ,$$

where the index $\alpha = 1\cdots 2r$ represents $USp(2r)$. The scalars $\mathcal{A}^\alpha_i$ are anti-hermitian, ζ^α is a Majorana spinor and $\mathcal{F}^i_\alpha$ are auxiliary fields. For our discussion, the relevant supersymmetry transformation is given by

$$\delta\zeta^\alpha = \gamma^a \hat{\mathcal{D}}_a \mathcal{A}^\alpha_j \epsilon^j - \gamma \cdot v \mathcal{A}^\alpha_j \epsilon^j + 3\mathcal{A}^\alpha_j \eta^j \ . \tag{63}$$

As we will discuss shortly, (63) will allow us to consistently preserve the Poincaré gauge by performing a compensating $\mathbf{S}^i$ transformation, *i.e.* fix η^i in terms of ϵ^i. The covariant derivative appearing in (63) is

$$\hat{\mathcal{D}}_\mu \mathcal{A}^\alpha_i = \left(\partial_\mu - \frac{3}{2}b_\mu\right)\mathcal{A}^\alpha_i - V_{\mu i}{}^j \mathcal{A}^\alpha_j - 2i\bar{\psi}_{\mu i}\zeta^\alpha \ .$$

- **Linear multiplet:** The components of the linear multiplet **L** are

$$L^{ij} \ , \ E^a \ , \ N \ , \ \varphi^i \ .$$

The scalar L^{ij} is symmetric in $SU(2)$ indices, E_a is a vector, N is a scalar and φ^i is a $SU(2)$ Majorana spinor. In addition, the algebra will close if the vector satisfies $\hat{\mathcal{D}}^a E_a = 0$. The transformation rule for the scalar L^{ij} reads

$$\delta L^{ij} = i\bar{\epsilon}^i \varphi^j + (i \leftrightarrow j) \ . \tag{64}$$

An interesting property of the linear multiplet is that it can be used as a "kinetic multiplet", *i.e.* the components of a matter (or Weyl) multiplet can be embedded into the linear multiplet. These embedding formulae are constructed by noticing that any symmetric, real bosonic combination of fields which is invariant under **S**-supersymmetry leads to the transformation rule in (64) with the appropriate identification of φ^i. The construction of the remaining components is done by repeated supersymmetry transformations. We denote such an embedding of a multiplet **X** into a linear multiplet as

L[**X**]. This is the key ingredient that allows the construction of invariants, which is discussed in the next section.

3.2. *Constructing invariant actions*

The first step towards constructing Lagrangians is done by identifying a quantity invariant under supersymmetry transformations. As discussed in Ref. 26, the contraction of some given linear and vector multiplets

$$
\begin{aligned}
\mathcal{L}(\mathbf{L} \cdot \mathbf{V}) \equiv \quad & Y^{ij} \cdot L_{ij} + 2i\bar{\Omega} \cdot \varphi + 2i\bar{\psi}_i^a \gamma_a \Omega_j \cdot L^{ij} \\
& - \frac{1}{2} A_a \cdot \left(E^a - 2i\bar{\psi}_b \gamma^{ba} \varphi + 2i\bar{\psi}_b^{(i} \gamma^{abc} \psi_c^{j)} L_{ij} \right) \\
& + \frac{1}{2} M \cdot \left(N - 2i\bar{\psi}_b \gamma^b \varphi - 2i\bar{\psi}_a^{(i} \gamma^{ab} \psi_b^{j)} L_{ij} \right) \ ,
\end{aligned}
\tag{65}
$$

transforms as a total derivative under all gauge transformations in (55). By exploiting the kinetic property of the linear multiplet, *i.e.* embedding of the Weyl or matter multiplets into **L**, the invariant density (65) is the building block for constructing supersymmetric actions. Here we will only present the construction of the invariants relevant for $N = 2$ ungauged supergravity.

First, let us consider the dynamics of the vector multiplets. The action we are pursuing should describe a Yang-Mills system coupled to gravity. In five dimensions the gauge field interactions will have a Chern-Simons term of the form $A \wedge F \wedge F$. By inspecting (65), the Chern-Simons term can be included in $A_a \cdot E^a$ by appropriately embedding **L**[**V**], where the field E_a will take the form

$$
E_a \sim \epsilon_{abcde} F^{bc} F^{de} + \dots \ .
\tag{66}
$$

After carefully performing this embedding, the bosonic terms in (65) gives the following Lagrangian for the vector multiplet,

$$
\begin{aligned}
\mathcal{L}_{B,V}(\mathbf{L} \cdot \mathbf{V}) &= -Y^{ij} \cdot L_{ij}[\mathbf{V}] + \frac{1}{2} A_a \cdot E^a[\mathbf{V}] - \frac{1}{2} M \cdot N[\mathbf{V}] \\
&= \mathcal{N} \left(\frac{1}{2} D - \frac{1}{4} R + 3v^2 \right) + 2\mathcal{N}_I v^{ab} F_{ab}^I + \frac{1}{4} \mathcal{N}_{IJ} F_{ab}^I F^{Jab} \\
&\quad - \mathcal{N}_{IJ} \left(\frac{1}{2} \mathcal{D}^a M^I \mathcal{D}_a M^J + Y_{ij}^I Y^{Jij} \right) + \frac{1}{24e} c_{IJK} A_a^I F_{bc}^J F_{de}^K \epsilon^{abcde} \ .
\end{aligned}
\tag{67}
$$

At this level, the function $\mathcal{N}$ is defined as an arbitrary cubic function of the scalars M^I, a condition that arises such that the embedding of the linear multiplet preserves the symmetry

transformations. Using the same notation as in the two-derivative on-shell theory, we can write

$$\mathcal{N} = \frac{1}{6}c_{IJK}M^I M^J M^K \ , \tag{68}$$

where c_{IJK} are constants, and $\mathcal{N}_I$ and $\mathcal{N}_{IJ}$ are derivatives of $\mathcal{N}$ with respect to M^I

$$\mathcal{N}_I \equiv \frac{\partial\mathcal{N}}{\partial M^I} = \frac{1}{2}c_{IJK}M^J M^K \ , \qquad \mathcal{N}_{IJ} \equiv \frac{\partial^2\mathcal{N}}{\partial M^I \partial M^J} = c_{IJK}M^K \ . \tag{69}$$

Note that in the off-shell theory $\mathcal{N}$ is not fixed, in contrast to (13) in the on-shell theory.

The hypermultiplet can also be embedded in the linear multiplet, **L**[**H**]. The resulting bosonic sector of the Lagrangian contains the kinetic terms for the hyperscalar $\mathcal{A}_\alpha^i$ and its coupling to the Weyl multiplet,

$$\begin{aligned}
\mathcal{L}_{B,H}(\mathbf{L}\cdot\mathbf{V}) &= -Y^{ij}\cdot L_{ij}[\mathbf{H}] + \frac{1}{2}A_a\cdot E^a[\mathbf{H}] - \frac{1}{2}M\cdot N[\mathbf{H}] \\
&= 2\mathcal{D}^a\mathcal{A}_i^\alpha\mathcal{D}_a\mathcal{A}_\alpha^i + \mathcal{A}^2\left(\frac{1}{4}D + \frac{3}{8}R - \frac{1}{2}v^2\right) \ .
\end{aligned} \tag{70}$$

The hypermultiplet can be decoupled from the remaining fields in the theory. In this framework, the decoupling is understood as the gauge fixing that will reduce the superconformal symmetries to the super-Poincaré group. In a suitable gauge, we will find that (67) and (70) lead to a canonical normalization for the Ricci scalar, *i.e.* the conventional Einstein-Hilbert term.

As we mentioned at the beginning of this section, we are interested in studying the supersymmetric Lagrangian containing the mixed gauge-gravitational Chern-Simons term (54). Similar to the construction of the vector Lagrangian, the term $A_a\cdot E^a$ guides the form of the embedding, where now we need

$$E_a \sim \epsilon_{abcde}\text{Tr}(\hat{R}^{bc}\hat{R}^{de}) + \dots \ . \tag{71}$$

This requires an embedding of the Weyl multiplet into the linear multiplet, **L**[**W**²], and the

invariant (65) turns into

$$
\begin{aligned}
\mathcal{L}_{B,W}(\mathbf{L}\cdot\mathbf{V}) &= \frac{c_{2I}}{24}\left(-Y^{Iij}\cdot L_{ij}[\mathbf{W^2}] + \frac{1}{2}A_a^I\cdot E^a[\mathbf{W^2}] - \frac{1}{2}M^I\cdot N[\mathbf{W^2}]\right) \\
&= \frac{c_{2I}}{24}\left[\frac{1}{16}\epsilon_{abcde}A^{Ia}C^{bcfg}C^{de}{}_{fg} - \frac{1}{12}\epsilon_{abcde}A^{Ia}\hat{R}^{bcij}(U)\hat{R}^{de}{}_{ij}(U)\right.\\
&\quad + \frac{1}{8}M^I C^{abcd}C_{abcd} - \frac{1}{3}M^I\hat{R}^{abij}(U)\hat{R}_{abij}(U) - \frac{4}{3}Y_{ij}^I v_{ab}\hat{R}^{abij}(U) \\
&\quad + \frac{1}{3}M^I C_{abcd}v^{ab}v^{cd} + \frac{1}{2}F^{Iab}C_{abcd}v^{cd} + \frac{1}{12}M^I D^2 + \frac{1}{6}F^{Iab}v_{ab}D \\
&\quad + \frac{8}{3}M^I v_{ab}\hat{\mathcal{D}}^b\hat{\mathcal{D}}_c v^{ac} + \frac{4}{3}M^I\hat{\mathcal{D}}^a v^{bc}\hat{\mathcal{D}}_a v_{bc} + \frac{4}{3}M^I\hat{\mathcal{D}}^a v^{bc}\hat{\mathcal{D}}_b v_{ca} \\
&\quad - \frac{2}{3}M^I\epsilon_{abcde}v^{ab}v^{cd}\hat{\mathcal{D}}_f v^{ef} + \frac{2}{3}F^{Iab}\epsilon_{abcde}v^{cf}\hat{\mathcal{D}}_f v^{de} \\
&\quad + F^{Iab}\epsilon_{abcde}v^c{}_f\hat{\mathcal{D}}^d v^{ef} - \frac{4}{3}F^{Iab}v_{ac}v^{cd}v_{db} - \frac{1}{3}F^{Iab}v_{ab}v^2 \\
&\quad \left. + 4M^I v_{ab}v^{bc}v_{cd}v^{da} - M^I(v^{ab}v_{ab})^2\right] ,
\end{aligned}
\tag{72}
$$

where the overall coefficient in $\mathcal{L}_{B,W}$ is fixed by the anomaly cancellation condition. The double covariant derivative of v_{ab} reads

$$
v_{ab}\hat{\mathcal{D}}^b\hat{\mathcal{D}}_c v^{ac} = v_{ab}\mathcal{D}^b\mathcal{D}_c v^{ac} - \frac{2}{3}v^{ac}v_{cb}R^b{}_a - \frac{1}{12}v^2 R ,
\tag{73}
$$

where $\mathcal{D}_a$ is the covariant derivative with respect to $\mathbf{M}_{ab}$, $\mathbf{D}$, $\mathbf{U}^{ij}$. In (72), C_{abcd} is the Weyl tensor

$$
C_{abcd} = R_{abcd} - \frac{2}{3}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{1}{6}g_{a[c}g_{d]b}R ,
\tag{74}
$$

with R_{abcd} the Riemann tensor, R_{ab} and R the Ricci tensor and scalar, respectively. The bosonic components of the curvature tensor associated to the $SU(2)$ symmetry is given by

$$
\hat{R}_{\mu\nu}{}^{ij}(U) = \partial_\mu V_\nu^{ij} - V_{\mu k}{}^i V_\nu^{kj} - (\mu\leftrightarrow\nu) .
\tag{75}
$$

Finally, the bosonic terms of the five dimensional Lagrangian for the superconformal theory are given by

$$
\mathcal{L}_B = \mathcal{L}_{B,V} + \mathcal{L}_{B,H} + \mathcal{L}_{B,W} .
\tag{76}
$$

3.3. *Poincaré supergravity*

Our main interest is five dimensional Poincaré supergravity. Starting from the superconformal theory, it is possible to gauge fix the additional conformal symmetries and consistently obtain an off-shell representation of $N = 2$ supergravity. This requires choosing the *vevs* of certain fields associated with the conformal group and the R-symmetry, which spontaneously breaks

the superconformal symmetry. The procedure does not make use of the equations of motion, and the number of symmetries and degrees of freedom eliminated is balanced. This makes the process reversible and therefore, the conformal theory is gauge equivalent to Poincaré supergravity.[97]

We will start by considering the Weyl multiplet coupled to $n_V + 1$ vector fields and one hypermultiplet.[n] The Lagrangian describing the bosonic sector of the conformal theory is given by (76). The first step towards gauge fixing the theory is to notice that the dilatational field b_μ only appears in the Lagrangians (67), (70), (72) through the covariant derivatives of the matter fields. This allows us to fix special conformal transformations by choosing the gauge $b_\mu = 0$.

In order to have the canonical normalization for the Ricci scalar in (76), our gauge choice for the dilatational group is $\mathcal{A}^2 = -2$. Notice that in the two-derivative theory this gauge choice, combined with the equations of motion of the auxiliary field D, gives the very special geometry constraint $\mathcal{N} = 1$.

The $SU(2)$ symmetry is fixed by identifying the indices in the hypermultiplet scalar, *i.e.* $\mathcal{A}^i_\alpha = \delta^i_\alpha$. Finally, since we restricted the discussion to an Abelian gauge group for the vector multiplet, the auxiliary fields V^{ij}_μ and Y^I_{ij} will only appear quadratically in (67) and (70). Therefore, it is appropriate for the ungauged theory to set both Y^I_{ij} and V^{ij}_μ to zero.

Summarizing, our gauge choice is given by:

$$\mathcal{A}^i_\alpha = \delta^i_\alpha \ , \ \mathcal{A}^2 = -2 \ ,$$
$$b_\mu = 0 \ , \ V^{ij}_\mu = 0 \ , \ Y^I_{ij} = 0 \ . \tag{77}$$

Substituting (77) into (67) and (70) gives rise to the two-derivative Lagrangian

$$\mathcal{L}_0 = -\frac{1}{2}D - \frac{3}{4}R + v^2 + \mathcal{N}\left(\frac{1}{2}D - \frac{1}{4}R + 3v^2\right) + 2\mathcal{N}_I v^{ab} F^I_{ab}$$
$$+ \mathcal{N}_{IJ}\left(\frac{1}{4}F^I_{ab}F^{Jab} + \frac{1}{2}\partial_a M^I \partial^a M^J\right) + \frac{1}{24}c_{IJK}A^I_a F^J_{bc}F^K_{de}\epsilon^{abcde} \ . \tag{78}$$

---

[n]One could include additional hyper multiplets (or other matter fields not discussed here), which would require the inclusion of non-dynamical multiplets in order to consistently eliminate the extra gauge symmetries, obscuring the procedure.

Similarly, the higher-derivative Lagrangian (72) becomes

$$
\begin{aligned}
\mathcal{L}_1 = \frac{c_{2I}}{24} \Big(& \frac{1}{16} \epsilon_{abcde} A^{Ia} R^{bcfg} R^{de}{}_{fg} + \frac{1}{8} M^I C^{abcd} C_{abcd} + \frac{1}{12} M^I D^2 + \frac{1}{6} F^{Iab} v_{ab} D \\
& + \frac{1}{3} M^I C_{abcd} v^{ab} v^{cd} + \frac{1}{2} F^{Iab} C_{abcd} v^{cd} + \frac{8}{3} M^I v_{ab} \mathcal{D}^b \mathcal{D}_c v^{ac} \\
& - \frac{16}{9} M^I v^{ac} v_{cb} R_a{}^b - \frac{2}{9} M^I v^2 R + \frac{4}{3} M^I \mathcal{D}^a v^{bc} \mathcal{D}_a v_{bc} + \frac{4}{3} M^I \mathcal{D}^a v^{bc} \mathcal{D}_b v_{ca} \\
& - \frac{2}{3} M^I \epsilon_{abcde} v^{ab} v^{cd} \mathcal{D}_f v^{ef} + \frac{2}{3} F^{Iab} \epsilon_{abcde} v^{cf} \mathcal{D}_f v^{de} + F^{Iab} \epsilon_{abcde} v^c{}_f \mathcal{D}^d v^{ef} \\
& - \frac{4}{3} F^{Iab} v_{ac} v^{cd} v_{db} - \frac{1}{3} F^{Iab} v_{ab} v^2 + 4 M^I v_{ab} v^{bc} v_{cd} v^{da} - M^I (v^2)^2 \Big) ,
\end{aligned}
\tag{79}
$$

where $\mathcal{N}$, $\mathcal{N}_I$ and $\mathcal{N}_{IJ}$ are defined in (68) and (69). The symbol $\mathcal{D}_a$ now refers to the usual covariant derivative of general relativity and should not be confused with the conformal covariant derivatives of the previous sections. Indeed, the presence of the auxiliary fields D and v_{ab} are the only remnants of the superconformal formalism.

As we mentioned before, the supersymmetry transformations are also affected by the gauge fixing. In particular the parameter η^i associated to **S**-supersymmetry is fixed. The BPS condition for the hypermultiplet fermion follows from (63)

$$
\gamma^a \hat{\mathcal{D}}_a \mathcal{A}_j^\alpha \epsilon^j - \gamma \cdot v \mathcal{A}_j^\alpha \epsilon^j + 3 \mathcal{A}_j^\alpha \eta^j = 0 .
\tag{80}
$$

For the field configuration (77), we can solve (80) for η^i,

$$
\eta^i = \frac{1}{3} \gamma \cdot v \epsilon^i .
\tag{81}
$$

Replacing (81) in the transformation rules for the remaining fermionic fields, we obtain the following the residual supersymmetry transformations[o]

$$
\begin{aligned}
\delta \psi_\mu &= \left(\mathcal{D}_\mu + \frac{1}{2} v^{ab} \gamma_{\mu ab} - \frac{1}{3} \gamma_\mu \gamma \cdot v \right) \epsilon , \\
\delta \Omega^I &= \left(-\frac{1}{4} \gamma \cdot F^I - \frac{1}{2} \gamma^a \partial_a M^I - \frac{1}{3} M^I \gamma \cdot v \right) \epsilon , \\
\delta \chi &= \left(D - 2 \gamma^c \gamma^{ab} \mathcal{D}_a v_{bc} - 2 \gamma^a \epsilon_{abcde} v^{bc} v^{de} + \frac{4}{3} (\gamma \cdot v)^2 \right) \epsilon .
\end{aligned}
\tag{82}
$$

It is the vanishing of these transformations which constitute the BPS conditions in the off-shell Poincaré supergravity.

---

[o] We now leave the i indices implicit since they play very little role in what follows. See Ref. 79 for a discussion of this point.

4. Off-shell Poincaré Supergravity

In the last section we provided the off-shell Lagrangian and supersymmetry transformations for five-dimensional supergravity with R^2 terms. This forms the starting point for our detailed analysis of corrections to black holes and similar objects, so let us briefly summarize the theory and make some general comments before investigating any specific solutions.

The degrees of freedom from the Weyl (gravity) multiplet are

$$e_\mu{}^a \ , \psi_\mu \ , v_{ab} \ , D \ , \chi \ , \tag{83}$$

where $e_\mu{}^a$ is the vielbein, ψ_μ is the gravitino, v_{ab} is an anti-symmetric two-form, D is a scalar, and χ a Majorana fermion. The last three fields are auxiliary fields, representing non-physical degrees of freedom. Coupled to the above fields are a number of $U(1)$ vector multiplets containing

$$A_\mu^I \ , M^I \ , \Omega^I \ , \tag{84}$$

which are the gauge fields, Kähler moduli, and gauginos, respectively. The index $I = 1 \ldots n_V +$ 1 runs over all of the gauge fields in the theory, although only n_V of them are dynamically independent.

The theory is described by the action

$$S = \frac{1}{4\pi^2} \int \ d^5x \sqrt{g} \left(\mathcal{L}_0 + \mathcal{L}_1 \right) \ , \tag{85}$$

where the bosonic part of the leading (two-derivative) Lagrangian is (78) and the bosonic higher derivative corrections are described by (79). The supersymmetry variations of the fermionic fields around bosonic backgrounds are given by (82).

Note that the four-derivative Lagrangian (79) is proportional to the constants c_{2I}, which can be thought of as the effective expansion parameters of the theory. Furthermore, the expansion coefficients c_{2I} make no appearance in the supersymmetry transformations (82) for the supersymmetry algebra is completely off-shell, *i.e.* independent of the action of the theory.

4.1. *Integrating out the auxiliary fields*

We have termed the fields v_{ab}, D, and χ as auxiliary fields. This nomenclature is clear from the viewpoint of the superconformal symmetry of Section 3, where these fields were added to compensate for the mismatch between the number of bosonic and fermionic degrees of

freedom. However, focusing on the bosonic fields, from the point of view of the leading-order action (78) the fields v_{ab} and D are also auxiliary variables in the sense of possessing algebraic equations of motion. It can be easily seen that substituting the equations of motion for v_{ab} and D into (78) leads to the on-shell two-derivative supergravity Lagrangian (15), complete with the very special geometry constraint (13).

When the higher-derivative corrections encapsulated in (79) are taken into account, the two-form v_{ab} no longer has an algebraic equation of motion. It seems fair to now ask in what sense it is still an auxiliary field. To sensibly interpret this, we must recall that the Lagrangian including stringy corrections should be understood as an effective Lagrangian, *i.e.* part of a derivative expansion suppressed by powers of the five-dimensional Planck scale. Thus, it is only sensible to integrate out the auxiliary fields iteratively, in an expansion in inverse powers of the Planck mass or, equivalently, in powers of the constants c_{2I}.

4.2. *Comments on field redefinitions*

In higher-derivative theories of gravity, the precise form of the Lagrangian is ambiguous due to possible field redefinitions. For example, one may consider

$$g_{\mu\nu} \to g_{\mu\nu} + aRg_{\mu\nu} + bR_{\mu\nu} + \dots \, , \qquad (86)$$

for some dimensionful constants a and b, or generalizations involving the matter fields. Field redefinitions leave the leading order Einstein-Hilbert action invariant, but can change the coefficients and form of the R^2 terms. Since they mix terms of different orders in derivatives it is generally ambiguous to label certain terms as "two-derivative" or "higher-derivative".

One of the advantages of the off-shell formalism we employ is that it addresses these ambiguities. The reason is that the off-shell supersymmetry transformations are independent of the action, yet they do not mix different orders in derivatives (if we assign the auxiliary fields v_{ab} and D derivative orders of one and two, respectively). General field redefinitions of the form (86) would modify the supersymmetry algebra and mix orders of derivatives. Thus if we restrict to variables where the supersymmetry transformations take their off-shell forms, *e.g.* in (82), then most of the field redefinition ambiguity is fixed. In our formalism it is therefore meaningful to label terms by their order in derivatives.[28]

4.3. *Modified very special geometry*

In the on-shell theory, there is a constraint imposed by hand

$$\mathcal{N} \equiv \frac{1}{6} c_{IJK} M^I M^J M^K = 1 \ . \tag{87}$$

This is known as the *very special geometry* constraint and indicates that all of the Kähler moduli are not independent fields. Interestingly, the off-shell formalism does not require this to be imposed externally. Rather, the equation of motion for D following from the two-derivative Lagrangian (78) is precisely this condition. This immediately implies that (87) does not hold in the presence of higher-derivative corrections since D also appears in the four-derivative Lagrangian (79). Indeed, the D equation of motion following from the full Lagrangian $\mathcal{L}_0 + \mathcal{L}_1$ is

$$\mathcal{N} = 1 - \frac{c_{2I}}{72} \left[F_{ab}^I v^{ab} + M^I D \right] \ . \tag{88}$$

Very special geometry is an interesting mathematical structure in its own right and the modified very special geometry is also likely to be an interesting structure, but we will have little to say about that here. Indeed, it would be of much interest to explore this topic further. For the present purposes, we use (88) as just another equation in specifying our solutions.

4.4. *Isometries and projections on Killing spinors*

In the following we will be investigating supersymmetric solutions to the theory described above. While we will consider maximally supersymmetric solutions, for which the supersymmetry parameter ϵ in the BPS conditions is understood to be unconstrained, we will also discuss asymptotically flat solutions such as black holes and black strings. These asymptotically flat solutions break some fraction of supersymmetry and so ϵ is expected to satisfy some sort of projective constraint(s). We can derive this projection in the following way (analogous to that of Ref. 30 in the on-shell formalism) which is generally applicable. Assume the existence of some spinor ϵ satisfying the BPS condition from the gravitino variation

$$\left\{ \mathcal{D}_\mu + \frac{1}{6} v^{ab} e_\mu{}^c \left(\gamma_{abc} - 4\eta_{ac}\gamma_b \right) \right\} \epsilon = 0 \ . \tag{89}$$

Now define the vector, $V_\mu = -\bar{\epsilon}\gamma_\mu\epsilon$ and use (89) to compute its covariant derivative

$$\mathcal{D}_\mu V_\nu = -\frac{1}{6} v^{ab} e_\mu{}^c e_\nu{}^d \bar{\epsilon} \left([\gamma_{abc}, \gamma_d] + 4\eta_{ac}\{\gamma_b, \gamma_d\} \right) \epsilon \ ,$$
$$= -\frac{1}{6} v^{ab} e_\mu{}^c e_\nu{}^d \bar{\epsilon} \left(\gamma_{abcd} + 8\eta_{ac}\eta_{bd} \right) \epsilon \ . \tag{90}$$

The right-hand side in the second line is anti-symmetric under exchange of μ and ν, thus V_μ is a Killing vector. One can now use various Fierz identities[30] to derive a projection obeyed by the Killing spinor

$$V^\mu \gamma_\mu \epsilon = -f\epsilon \,, \tag{91}$$

where $f = \sqrt{V^\mu V_\mu}$. Since there is only one condition on ϵ, this argument leads to solutions which preserve half of the supersymmetries.

The details of the supersymmetry analysis are qualitatively different for solutions with a null isometry ($f^2 = 0$) and those with a timelike isometry ($f^2 > 0$). We will study these two cases in turn in subsequent sections. The analysis proceeds as follows. One introduces a metric *ansatz* with an isometry that we identify with $V_\mu = -\bar{\epsilon}\gamma_\mu \epsilon$. This determines a projection obeyed by the Killing spinor via (91). One then uses the BPS conditions to obtain as much information as possible about the undetermined functions of the *ansatz*. In the off-shell formalism, the results of this analysis are completely independent of the action. Equations of motion, which do of course depend on the precise form of the action, are then imposed as needed to completely specify the solution.

5. Attractor Solutions

An important property of extremal black holes is attractor behavior. The literature on the attractor mechanism is extensive but the original works which first explored the phenomenon include Refs. 70, 71 and 98–100. Furthermore, useful reviews which approach the subject from different viewpoints include Refs. 46, 50, 101, and 102.

In general, there exist BPS and non-BPS extremal black holes, and both display attractor behavior. The non-BPS branch is quite interesting, but it will not be discussed here (see Refs. 3, 69, and 103–107 for discussion).

We would like to reconsider BPS attractors within the higher derivative setting developed in this review. First, recall that attractor behavior involves two related aspects:

- **Attractor mechanism:** Within a fixed basin of attraction, the scalar fields flow to constants at the black hole horizon which depend on the black hole charges alone. In particular the endpoint of the attractor flow is independent of the initial conditions, *i.e.* the values of the asymptotic moduli.
- **The attractor solution:** The limiting value of the geometry (and the associated matter fields) near the black hole horizon constitutes a solution in its own right,

independently of the flow. One remarkable feature is that the attractor solution has enhanced, in fact maximal, supersymmetry. This property is highly constraining, and so the solution can be analyzed in much detail.

We will ultimately derive complete, asymptotically flat solutions, from which attractor solutions are extracted by taking appropriate near horizon limits. But since this method obscures the intrinsic simplicity of the attractor solutions, it is instructive to construct the attractor solutions directly. This is what we do in this section.

5.1. *Maximal supersymmetry in the off-shell formalism*

As we have emphasized, the attractor solution has maximal supersymmetry. Thus we consider the vanishing of the supersymmetry variations (82), which we repeat for ease of reference

$$0 = \left(\mathcal{D}_\mu + \frac{1}{2} v^{ab} \gamma_{\mu ab} - \frac{1}{3} \gamma_\mu \gamma \cdot v \right) \epsilon \ ,$$

$$0 = \left(-\frac{1}{4} \gamma \cdot F^I - \frac{1}{2} \gamma^a \partial_a M^I - \frac{1}{3} M^I \gamma \cdot v \right) \epsilon \ ,$$

$$0 = \left(D - 2\gamma^c \gamma^{ab} \mathcal{D}_a v_{bc} - 2\gamma^a \epsilon_{abcde} v^{bc} v^{de} + \frac{4}{3} (\gamma \cdot v)^2 \right) \epsilon \ . \tag{92}$$

The supersymmetry parameter ϵ should be subject to no projection conditions if the solution is to preserve maximal supersymmetry. Therefore, terms with different structures of γ-matrices cannot cancel each other on the attractor solution. The gaugino variation (the middle equation in (92)) therefore demands

$$M^I = \text{constant} \ , \tag{93}$$

and also

$$F^I = -\frac{4}{3} M^I v \ . \tag{94}$$

Constancy of the scalar fields is a familiar feature of attractors in two-derivative gravity. The values of the constants will be determined below. The second result (94) is special to the off-shell formalism in that it identifies the auxiliary two-form v with the graviphoton field strength.

We next extract the information from the third equation in (92). We can write it as

$$\left[\left(D - \frac{8}{3} v^2 \right) - 2\gamma^{abc} \mathcal{D}_a v_{bc} + 2\gamma^a \left(\mathcal{D}^b v_{ba} - \frac{1}{3} \epsilon_{abcde} v^{bc} v^{de} \right) \right] \epsilon = 0 \ , \tag{95}$$

by using the algebraic identities

$$\gamma^{ab} \gamma^{cd} = -(\eta^{ac} \eta^{bd} - \eta^{ad} \eta^{bc}) - (\gamma^{ac} \eta^{bd} - \gamma^{bc} \eta^{ad} + \gamma^{bd} \eta^{ac} - \gamma^{ad} \eta^{bc}) + \gamma^{abcd} \ ,$$

$$\gamma^a \gamma^{bc} = \eta^{ab}\gamma^c - \eta^{ac}\gamma^b + \gamma^{abc} \ ,$$

$$\gamma_{abcde} = \epsilon_{abcde} \ . \tag{96}$$

Again, maximal supersymmetry precludes any cancellation between different tensor structures of the γ-matrices. We therefore determine the value of the D-field as

$$D = \frac{8}{3}v^2 \ , \tag{97}$$

and we find that the auxiliary two-form v must satisfy

$$\epsilon^{abcde}\mathcal{D}_a v_{bc} = 0 \ ,$$

$$\mathcal{D}^b v_{ba} - \frac{1}{3}\epsilon_{abcde}v^{bc}v^{de} = 0 \ . \tag{98}$$

Both equations support the identification of the auxiliary field v with the graviphoton field strength in minimal supergravity. The first equation is analogous to the Bianchi identity and the second is analogous to the two-derivative equation of motion for a gauge field with Chern-Simons coupling. Note that v satisfies the equations of motion of two-derivative minimal supergravity even though, here, we have not assumed an action yet.

The final piece of information from maximal supersymmetry is the vanishing of the gravitino variation, corresponding to the first equation in (92). This equation is identical to the gravitino variation of minimal supergravity, with the auxiliary two-form v taking the role of the graviphoton field strength. The solutions to minimal supergravity have been classified completely.[30] Adapted to our notation, the solutions with maximal supersymmetry are:

- Flat space.
- A certain class of pp-waves.
- Generalized Gödel space-times.
- AdS$_3 \times S^2$ with geometry

$$ds^2 = \ell_A^2 ds_{AdS}^2 - \ell_S^2 d\Omega_2^2 \ , \quad \text{with} \quad \ell_A = 2\ell_S \ . \tag{99}$$

  Note that supersymmetry relates the two radii. Additionally, v is proportional to the volume form on S^2

$$v = \frac{3}{4}\ell_S \epsilon_{S^2} \ . \tag{100}$$

- AdS$_2 \times S^3$ with geometry

$$ds^2 = \ell_A^2 ds_{AdS}^2 - \ell_S^2 d\Omega_3^2 \ , \quad \text{with} \quad \ell_A = \frac{1}{2}\ell_S \ , \tag{101}$$

Again, supersymmetry relates the two radii. In this case v is proportional to the volume form on AdS_2

$$v = \frac{3}{4}\ell_A \epsilon_{AdS_2} \ . \tag{102}$$

- The near horizon BMPV solution or the *rotating attractor.*

The computations leading to the above classification of solutions to minimal supergravity use the equations of motion and the Bianchi identity for the field strength, as well as the on-shell supersymmetry transformations. Presently, we analyze the consequences of maximal supersymmetry in the off-shell formalism, but do not wish to apply the equations of motion yet, because they depend on the action. Fortunately, we found in (98) that supersymmetry imposes the standard two-derivative equation of motion and Bianchi identity for the auxiliary two-tensor v, which in turn can be identified with the graviphoton in minimal supergravity. In our context, the classification therefore gives precisely the conditions for maximal supersymmetry, with no actual equations of motion imposed.

We will not repeat the general classification of Ref. 30 but just explain why the possibilities are so limited and derive the quantitative results given above. First recall that there exists an integrability condition obtained from the commutator of covariant derivatives acting on a spinor

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\,\epsilon = \frac{1}{4}R_{\mu\nu ab}\gamma^{ab}\epsilon \ . \tag{103}$$

We can evaluate the left-hand side of (103) by differentiating and then antisymmetrizing the BPS condition resulting from the gravitino variation, *i.e.* the first equation in (92). The resulting equations are rather unwieldy, but they can be simplified to purely algebraic conditions by using reorderings akin to (96) along with the supersymmetry conditions (98). The terms proportional to the tensor structure γ^{ab} give the Riemann tensor

$$\begin{aligned}
R_{\mu\nu\rho\sigma} = {}& -\tfrac{16}{9}v_{\mu\nu}v_{\rho\sigma} - \tfrac{4}{3}(v_{\mu\rho}v_{\nu\sigma} - v_{\mu\sigma}v_{\nu\rho}) + \tfrac{2}{9}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma})v^2 \\
& -\tfrac{4}{9}\left(g_{\mu\sigma}v_{\rho\tau}v^\tau{}_\nu - g_{\mu\rho}v_{\sigma\tau}v^\tau{}_\nu - g_{\nu\sigma}v_{\rho\tau}v^\tau{}_\mu + g_{\nu\rho}v_{\sigma\tau}v^\tau{}_\mu\right) \ .
\end{aligned} \tag{104}$$

Conceptually, we might want to start with a v that solves equations (98), since then the geometry is completely determined by (104). But the two sets of equations are of course entangled. Also, one must further check that the gravitino variation does in fact vanish, and not just its commutator.

The most basic solutions for the study of black holes and strings are the $\mathrm{AdS}_3 \times S^2$ and $\mathrm{AdS}_2 \times S^3$ geometries. For these, the solutions to the supersymmetry conditions (98) are

given by magnetic and electric fluxes, as in (100) and (102). In each case we can insert in (104) and verify that the geometry is in fact maximally symmetric and that the scales ℓ_A, ℓ_S are related to those of v in the manner indicated.

5.2. *The magnetic attractor solution*

So far we have just analyzed the consequences of supersymmetry. In order to determine the solutions completely we also need information from the equations of motion. We next show how this works in the case of the simplest nontrivial attractor solution, the $AdS_3 \times S^2$ that is interpreted as the near horizon geometry of a magnetic string.

The key ingredient beyond maximal supersymmetry is the modified very special geometry constraint

$$\begin{aligned}
\frac{1}{6}c_{IJK}M^I M^J M^K &= 1 - \frac{c_{2I}}{72}\left(F^I \cdot v + M^I D\right) \ , \\
&= 1 - \frac{c_{2I}}{54}M^I v^2 \ , \\
&= 1 - \frac{c_{2I}}{12}\frac{M^I}{\ell_A^2} \ .
\end{aligned} \tag{105}$$

We first used the D equation of motion (88) and then simplified using (94) and (97). In the last line we used

$$v^2 = \frac{9}{8\ell_S^2} = \frac{9}{2\ell_A^2} \ , \tag{106}$$

from (100) and (99).

In $AdS_3 \times S^2$ the field strengths (94) become

$$F^I = -\frac{4}{3}M^I v = -\frac{1}{2}M^I \ell_A \epsilon_{S^2} \ . \tag{107}$$

In our normalization the magnetic fluxes are fixed as

$$F^I = -\frac{p^I}{2}\epsilon_{S^2} \ , \tag{108}$$

so we determine the scalar fields as

$$M^I = \frac{p^I}{\ell_A} \ . \tag{109}$$

Inserting this into the modified very special geometry constraint (105) we finally determine the precise scale of the geometry

$$\ell_A^3 = \frac{1}{6}c_{IJK}p^I p^J p^K + \frac{1}{12}c_{2I}p^I \equiv p^3 + \frac{1}{12}c_2 \cdot p \ . \tag{110}$$

The preceding three equations specify the attractor solution completely in terms of magnetic charges p^I.

5.3. *The electric attractor solution*

The electric attractor solution is the $\mathrm{AdS}_2 \times S^3$ near horizon geometry of a non-rotating 5D black hole. The scales of the geometry are

$$\ell \equiv \ell_A = \frac{1}{2}\ell_S \ , \tag{111}$$

and the auxiliary two-form v in (102) gives

$$v^2 = -\frac{9}{8\ell^2} \ , \tag{112}$$

so that the modified very special geometry constraint becomes

$$\frac{1}{6}c_{IJK}M^I M^J M^K = 1 - \frac{c_{2I}M^I}{54}v^2 = 1 + \frac{c_{2I}M^I}{48\ell^2} \ . \tag{113}$$

We can write this in a more convenient way by introducing the rescaled moduli

$$\hat{M}^I = 2\ell M^I \ , \tag{114}$$

so that

$$\ell^3 = \frac{1}{8}\left[\frac{1}{6}c_{IJK}\hat{M}^I\hat{M}^J\hat{M}^K - \frac{c_{2I}}{12}\hat{M}^I\right] \ . \tag{115}$$

This equation gives the scale of the geometry in terms of the rescaled moduli.

We would often like to specify the solution in terms its electric charges, rather than the rescaled moduli. Electric charges may be defined as integration constants in Gauss' law, a step that depends on the detailed action of the theory. We will carry out this computation in Section 7.7 and find that

$$q_I = \frac{1}{2}c_{IJK}\hat{M}^J\hat{M}^K - \frac{c_{2I}}{8} \ . \tag{116}$$

If the q_I are given, this relation determines the rescaled moduli $\hat{M}^I$ and so, through (115), the scale ℓ.

5.3.1. *Rotating attractors*

Just as the non-rotating 5D black hole can be generalized to include rotation, the electric attractor just studied is a special case of a more general rotating attractor. Here we just briefly state the result, deferring more discussion until after we have derived the full, corrected, rotating black hole solution. The attractor solution is

$$ds^2 = w^2\left[(1 + (e^0)^2)(\rho^2 d\tau^2 - \frac{d\rho^2}{\rho^2} - d\theta^2 - \sin^2\theta d\phi^2) - (dy + \cos\theta d\phi)^2\right] \ ,$$
$$v = -\frac{3}{4}w(d\tau \wedge d\rho - e^0 \sin\theta d\theta \wedge d\phi) \ . \tag{117}$$

The geometry describes a spatial circle nontrivially fibered over $\mathrm{AdS}_2 \times S^2$. The two-form v is a solution to the supersymmetry conditions (98) written in the convenient form

$$d \star v + \frac{4}{3} v \wedge v = 0 \ . \tag{118}$$

The modified special geometry constraint follows by inserting v in the second line of (105). As we discuss in more detail in Section 7.7, the parameters w and e^0 specify the scale sizes and angular momentum of the solution.

6. Black Strings and Null Supersymmetry

We now begin investigating asymptotically flat solutions which preserve only a fraction of the supersymmetry of the theory. We begin with black string solutions, which were first discussed in Ref. 20. In particular, we will study corrections to the Calabi-Yau black strings studied in Ref. 21. These solutions each have at least one null isometry so we will determine the off-shell supersymmetry conditions for any such spacetime. The conditions from supersymmetry do not completely specify the solution and we will require more conditions on the functions in our *ansatz*, including equations of motion from the full higher-derivative Lagrangian. We will comment some on the general case, then specialize to purely magnetically charged strings which carry no momentum along their length; this is precisely the case studied in Ref. 14. Under these assumptions we will only need to use the equation of motion for D and the Bianchi identity for F^I to completely specify the solution.

6.1. *Metric ansatz*

As argued in Ref. 30 the most general metric, up to diffeomorphisms, with lightlike killing vector $V = \partial_{y^+}$, is

$$ds^2 = e^{2U_1} \left(\mathcal{F}(dy^-)^2 + 2dy^+ dy^- \right) - e^{-4U_2} \delta_{ij} \left(dx^i + a^i dy^- \right) \left(dx^j + a^j dy^- \right) \ , \tag{119}$$

where $i, j = 1, 2, 3$ and the undetermined functions $U_{1,2}$, $\mathcal{F}$, and a^i are independent of y^+. We choose for our vielbeins

$$e^{\hat{+}} = e^{U_1} (dy^+ + \frac{1}{2} \mathcal{F} dy^-) \ , \quad e^{\hat{-}} = e^{U_1} dy^- \ , \quad e^{\hat{i}} = e^{-2U_2} (dx^i + a^i dy^-) \ . \tag{120}$$

The spin connections which follow from this choice of vielbeins are

$$\omega^{\hat{+}}_{\ \hat{+}} = e^{-U_1} \left(a^i \partial_i U_1 - \partial_- U_1 \right) e^{\hat{-}} \ ,$$

$$\omega^{\hat{+}}_{\ \hat{i}} = \frac{1}{2} e^{2U_2} \partial_i \mathcal{F} e^{\hat{-}} + \partial_i U_1 e^{2U_2} e^{\hat{+}} - e^{-U_1} \left(S_{ij} + 2 \left(\partial_- U_2 - a^k \partial_k U_2 \right) \delta_{ij} \right) e^{\hat{j}} \ ,$$

$$\omega^{\hat{-}}_{\ \hat{i}} = \partial_i U_1 e^{2U_2} e^{\hat{-}} \ ,$$
$$\omega^{\hat{i}}_{\ \hat{j}} = -e^{-U_1} A_{ij} e^{\hat{-}} + 2e^{2U_2}\left(\partial_i U_2 \delta_{jk} - \partial_j U_2 \delta_{ik}\right) e^{\hat{k}} \ , \tag{121}$$

where we have defined

$$S_{ij} = \frac{1}{2}\left(\partial_i a^j + \partial_j a^i\right) \ ,$$
$$A_{ij} = \frac{1}{2}\left(\partial_i a^j - \partial_j a^i\right) \ . \tag{122}$$

6.2. *Supersymmetry conditions*

We now substitute our *ansatz* into the supersymmetry conditions (82) to determine the bosonic backgrounds which preserve supersymmetry. Following (91), we look for Killing spinors which satisfy the projection

$$\gamma_{\hat{+}}\epsilon = \gamma^{\hat{-}}\epsilon = 0 \ . \tag{123}$$

Equivalent forms of the projection are

$$\gamma_{\hat{+}\hat{-}}\epsilon = \epsilon \ ,$$
$$\gamma_{\hat{i}\hat{j}\hat{k}}\epsilon = -\varepsilon_{ijk}\epsilon \ ,$$
$$\gamma_{\hat{i}\hat{j}}\epsilon = \varepsilon_{ijk}\gamma_{\hat{k}}\epsilon \ , \tag{124}$$

where we have used the gamma matrix and orientation conventions

$$\gamma_{abcde} = \varepsilon_{abcde} \ ,$$
$$\varepsilon_{\hat{+}\hat{-}\hat{i}\hat{j}\hat{k}} = \varepsilon_{ijk} \ . \tag{125}$$

with $\varepsilon_{123} = 1$. Furthermore, the isometry $V = \partial_{y^+}$ ensures that ϵ has no y^+ dependence.

Around bosonic backgrounds, the condition for supersymmetry from the variation of the gravitino is

$$\delta\psi_\mu = \left[\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} + \frac{1}{6}v^{ab}e_\mu^{\ c}\left(\gamma_{abc} - 4\eta_{ac}\gamma_b\right)\right]\epsilon = 0 \ . \tag{126}$$

The y^+ component is rather simple. After applying the projection (123) we find

$$-\frac{1}{3}e_+^{\hat{+}}v^{\hat{-}\hat{i}}\gamma_{\hat{i}}\epsilon = 0 \ , \tag{127}$$

from which it follows $v^{\hat{-}\hat{i}} = -v_{\hat{+}\hat{i}} = 0$.

The y^- component gives

$$0 = \left[\left(\partial_- + \frac{1}{2}\omega_-^{\hat{+}\hat{-}} - \frac{1}{6}\varepsilon_{ijk}e_-^{\hat{i}}v^{\hat{j}\hat{k}}\right) + \left(\frac{1}{2}\omega_-^{\hat{-}\hat{k}} + \frac{1}{6}v^{\hat{i}\hat{j}}e_-^{\hat{-}}\varepsilon_{ijk}\right)\gamma_{\hat{-}\hat{k}}\right.$$
$$\left. + \left(-\frac{2}{3}v^{\hat{+}\hat{-}}e_-^{\hat{-}}\right)\gamma_{\hat{-}} + \left(\frac{1}{4}\omega_-^{\hat{i}\hat{j}}\varepsilon_{ijk} - v^{\hat{+}\hat{k}}e_-^{\hat{-}} + \frac{1}{3}v^{\hat{+}\hat{-}}e_-^{\hat{k}} - \frac{2}{3}v^{\hat{i}\hat{k}}e_-^{\hat{i}}\right)\gamma_{\hat{k}}\right]\epsilon \ . \tag{128}$$

Vanishing of $\gamma_{\hat{-}}$ terms imply $v^{\hat{+}\hat{-}} = 0$. Vanishing of $\gamma_{\hat{-}\hat{i}}$ terms requires

$$v^{\hat{i}\hat{j}} = \frac{3}{2} e^{2U_2} \varepsilon_{ijk} \partial_k U_1 \ . \tag{129}$$

Cancellation of $\gamma_{\hat{i}}$ terms yields

$$v^{\hat{+}\hat{i}} = -v_{\hat{-}\hat{i}} = \frac{1}{4} e^{-U_1} \varepsilon_{ijk} \partial_j a^k + e^{-U_1} \varepsilon_{ijk} a^j \partial_k \left(U_1 - U_2 \right) \ . \tag{130}$$

Finally, cancellation of terms without γ's yields

$$\left(\partial_- - \frac{1}{2} \partial_- U_1 \right) \epsilon = 0 \ . \tag{131}$$

The x^i components of the gravitino variation yield the additional constraints that $U_1 = U_2 \equiv U$ and

$$\left(\partial_i - \frac{1}{2} \partial_i U \right) \epsilon = 0 \ . \tag{132}$$

Combining the above equation with (131), we can solve for the Killing spinor

$$\epsilon = e^{U/2} \epsilon_0 \ , \tag{133}$$

where ϵ_0 is a constant spinor satisfying the projection (123).

We have now exhausted the conditions following from stationarity of the gravitino under supersymmetry transformations. Requiring the gaugino to be stationary about a bosonic background yields the condition

$$\delta \Omega^I = \left(-\frac{1}{4} \gamma \cdot F^I - \frac{1}{2} \gamma^a \partial_a M^I - \frac{1}{3} M^I \gamma \cdot v \right) \epsilon = 0 \ . \tag{134}$$

For the present *ansatz*, and utilizing the previously found values for v_{ab}, we can write the above as

$$\left[-\frac{1}{4} \left(2F^I_{\hat{-}\hat{+}} - 2F^I_{\hat{+}\hat{i}} \gamma_{\hat{-}\hat{i}} + F^I_{\hat{i}\hat{j}} \varepsilon_{ijk} \gamma_{\hat{k}} \right) + \frac{1}{2} e^{2U} \partial_i M^I \gamma_{\hat{i}} - e^{2U} \partial_k U \gamma_{\hat{k}} \right] \epsilon = 0 \ . \tag{135}$$

Requiring terms with like γ-matrix structure to cancel we find that the only non-trivial field strength components are

$$F^I_{\hat{i}\hat{j}} = \varepsilon_{ijk} e^{4U} \partial_k \left(e^{-2U} M^I \right) \ , \tag{136}$$

and $F^I_{\hat{-}\hat{i}}$, which is unconstrained.

The final supersymmetry condition is obtained by imposing stationarity of the auxiliary fermion

$$\delta \chi = \left[D - 2\gamma^c \gamma^{ab} \mathcal{D}_a v_{bc} - 2\gamma^a \varepsilon_{abcde} v^{bc} v^{de} + \frac{4}{3} (\gamma \cdot v)^2 \right] \epsilon = 0 \ . \tag{137}$$

Using the projection (123) and the known values for v_{ab} yields the single condition

$$D = 6e^{4U}\nabla^2 U \;, \tag{138}$$

where $\nabla^2 = \delta_{ij}\partial_i\partial_j$.

6.2.1. *Summary of supersymmetry conditions*

As a quick summary, we now restate the results of the above supersymmetry analysis. The metric *ansatz* with lightlike isometry $V = \partial_{y^+}$ which is consistent with supersymmetry is

$$ds^2 = e^{2U}\left(\mathcal{F}\left(dy^-\right)^2 + 2dy^+ dy^-\right) - e^{-4U}\delta_{ij}\left(dx^i + a^i dy^-\right)\left(dx^i + a^i dy^-\right) \;, \tag{139}$$

or in a null orthonormal frame

$$e^{\hat{+}} = e^{U}(dy^+ + \frac{1}{2}\mathcal{F}dy^-) \;, \quad e^{\hat{-}} = e^{U}dy^- \;, \quad e^{\hat{i}} = e^{-2U}(dx^i + a^i dy^-) \;. \tag{140}$$

The non-trivial auxiliary fields are

$$v^{\hat{i}\hat{j}} = \frac{3}{2}e^{2U}\varepsilon_{ijk}\partial_k U \;,$$
$$v^{\hat{+}\hat{i}} = -v_{\hat{-}\hat{i}} = \frac{1}{4}e^{-U}\varepsilon_{ijk}a^j\partial_k U \;,$$
$$D = 6e^{4U}\nabla^2 U \;. \tag{141}$$

The gauge field strengths are given by

$$F^I = F^I_{\hat{-}\hat{i}}e^{\hat{-}} \wedge e^{\hat{i}} + \frac{1}{2}\varepsilon_{ijk}e^{4U}\partial_k\left(e^{-2U}M^I\right)e^{\hat{i}} \wedge e^{\hat{j}} \;. \tag{142}$$

All of the undetermined functions in the above (that is U, $\mathcal{F}$, a^i, M^I and $F^I_{\hat{-}\hat{i}}$) are independent of the isometry coordinate y^+, but otherwise unconstrained.

6.3. *Bianchi identity*

We have shown in the previous section how supersymmetry partially determines the gauge field strengths. However, these field strengths (142) do not manifestly follow from exterior differentiation of some one-form potentials. Therefore we must impose the Bianchi identity

$$dF^I = 0 \;, \tag{143}$$

which should result in further non-trivial conditions. Physically, this is because supersymmetry is consistent with any extended distribution of magnetic charges, while here we are considering solutions away from their isolated sources.

The two non-trivial conditions from the Bianchi identity are

$$0 = \partial_{[k} F^I_{ij]} \; ,$$
$$0 = \partial_- F^I_{ij} + \partial_j F^I_{-i} - \partial_i F^I_{-j} \; . \tag{144}$$

In coordinate frame the field strength components can be written as

$$F^I_{ij} = \varepsilon_{ijk} \partial_k \left(e^{-2U} M^I \right) \; ,$$
$$F^I_{-i} = e^{-U} F^I_{-\hat{i}} - \varepsilon_{ijk} a^j \partial_k \left(e^{-2U} M^I \right) \; . \tag{145}$$

The relations (144) are more usefully expressed after contraction with ε_{ijk} and substitution of (145). The first equation becomes

$$\nabla^2 \left(e^{-2U} M^I \right) = 0 \; , \tag{146}$$

allowing us to write the moduli in terms of harmonic functions H^I on $\mathbb{R}^3$

$$M^I = e^{2U} H^I \; . \tag{147}$$

It is important to note that the H^I have arbitrary dependence on the null coordinate y^-. The second equation in (144) then becomes

$$\partial_- \partial_k H^I = \varepsilon_{kij} \partial_i \left(e^{-U} F^I_{-\hat{j}} \right) - \partial_i a^k \partial_i H^I + \partial_i \left(a^i \partial_k H^I \right) \; . \tag{148}$$

Thus $F^I_{-\hat{j}}$ is determined up to integration constants by the Bianchi identities in terms of metric functions and the harmonic functions which describe the moduli.

6.4. *Modified very special geometry*

As determined in (88), the equation of motion for D yields the modified very special geometry constraint

$$\mathcal{N} = 1 - \frac{c_{2I}}{72} \left(F^I_{ab} v^{ab} + M^I D \right) \; . \tag{149}$$

Substituting in the results from supersymmetry and the Bianchi identity, specifically equations (141), (142), and (147), we find

$$e^{-6U} = \frac{1}{6} c_{IJK} H^I H^J H^K + \frac{c_{2I}}{24} \left(\nabla U \cdot \nabla H^I + 2 H^I \nabla^2 U \right) \; , \tag{150}$$

where $\nabla_i = \partial_i$ are derivatives on $\mathbb{R}^3$. This equation thus specifies the metric function $U(y^-, x^i)$ in terms of the harmonic functions H^I.

6.5. *Other equations of motion and the general solution*

So far, we have used only one equation of motion in addition to the BPS conditions and Bianchi identity. Supersymmetry puts constraints on the metric and determines the auxiliary fields in terms of the moduli M^I and metric functions U and a^i as summarized in equations (139)–(141). Supersymmetry also constrains the gauge field strengths F^I to be of the form (142). The Bianchi identity for F^I provides two relations. First, it specifies the moduli in terms of U and some arbitrary harmonic functions H^I through (147). The second relation (148) completes the specification of the gauge field strengths (up to integration constants) in terms of U, a^i and the harmonic functions H^I. Now examining the equation of motion for D specifies U in terms of the harmonic functions H^I through (150).

There remain the functions $\mathcal{F}$ and a^i which are not yet specified. No solutions with higher-derivative corrections have yet been found which have non-zero values for these fields, but we can make a few comments. Having already examined the Bianchi identity, we have fixed completely the magnetic part of the gauge field strength. The electric part, $F^I_{\hat{-}\hat{i}}$, should be further constrained by the Maxwell equation, *i.e.* the equation of motion for the gauge field. Combining this with (148) should be enough to completely specify $F^I_{\hat{-}\hat{i}}$ and a^i.

To specify the final undetermined function $\mathcal{F}$, we turn to the equation of motion from the metric. Since the Ricci tensor of the metric is

$$R_{\hat{-}\hat{-}} = \frac{1}{2}e^{4U}\nabla^2\mathcal{F} + \dots \ , \tag{151}$$

we expect that $\mathcal{F}$ is determined by the $(\hat{-}\hat{-})$ component of the Einstein equation.

As stated already, the most general solutions with null supersymmetry are not yet known. In the next section, we will examine the full solution for a certain special case.

6.6. *The magnetic string solution*

We now simplify our analysis by specializing to string solutions which carry no momentum or electric charge ($M2$-brane charge) and have an additional null isometry $\tilde{V} = \partial_{y^-}$. In terms of our *ansatz* this corresponds to setting

$$\begin{aligned}
\mathcal{F} &= 0 \ , \\
a^i &= 0 \ , \\
F^I_{\hat{-}\hat{i}} &= 0 \ ,
\end{aligned} \tag{152}$$

and assuming that the remaining undetermined functions of the solution are also independent of the other null coordinate y^-. Furthermore, the dependence of undetermined functions on the x^i is assumed to be spherically symmetric, *i.e.* dependent only on the radial variable $r^2 = \delta_{ij} x^i x^j$. These are the solutions discussed in Ref. 14.

It is convenient to now use spherical coordinates so that the metric takes the form

$$ds^2 = 2e^{2U} dy^+ dy^- - e^{-4U} \left(dr^2 + r^2 d\Omega_2^2 \right) \ . \tag{153}$$

In these coordinates, the gauge fields and auxiliary field v_{ab} are given by

$$
\begin{aligned}
F_{\theta\phi}^I &= \partial_r \left(e^{-2U} M^I \right) r^2 \sin\theta \ , \\
v_{\theta\phi} &= \frac{3}{2} e^{-2U} r^2 \sin\theta \partial_r U \ ,
\end{aligned}
\tag{154}
$$

and the auxiliary scalar is

$$D = 6e^{4U} \nabla^2 U \ , \tag{155}$$

where $\nabla^2 = r^{-2} \partial_r \left(r^2 \partial_r \right)$ is the Laplacian on $\mathbb{R}^3$.

6.6.1. *Maxwell equation and Bianchi identity*

Since we have narrowed our search to solutions with no electric charge, we do not expect to have any constraints from the Maxwell equation. Indeed, it can be straightforwardly verified that the equations of motion for the A_μ^I are identically satisfied for the *ansatz* described by equations (153)–(155). Thus we get no new information from these equations of motion.

For magnetic solutions the nontrivial condition arises from the Bianchi identity $dF^I = 0$. As found in (147), this determines the moduli to be

$$M^I = e^{2U} H^I \ , \tag{156}$$

where H^I is some y^- independent function which is harmonic on the three-dimensional base $\mathbb{R}^3$. Here we look for single-center solutions on $\mathbb{R}^3$ so

$$M^I e^{-2U} = H^I = M_\infty^I + \frac{p^I}{2r} \ , \tag{157}$$

with M_∞^I the value of M^I in the asymptotically flat region where $U = 0$ and p^I is some constant. By using (154) we see that the field strengths are given by

$$F^I = -\frac{p^I}{2} \epsilon_{S^2} \ . \tag{158}$$

In our units, this identifies p^I as the integer quantized magnetic flux of F^I

$$p^I = -\frac{1}{2\pi} \int_{S^2} F^I \; . \tag{159}$$

It is worth noting that the magnetic charge does not get modified after including higher derivatives since it is topological, *i.e.* the Bianchi identity is not corrected by higher-order effects. We will find in Section 7 that this does not hold for electric charges which are instead governed by the equations of motion for the gauge fields.

6.6.2. *D equation of motion*

So far, by imposing the conditions for supersymmetry and integrating the Bianchi identity, we have been able to write our solution in terms of one unknown function $U(r)$. To determine this remaining function we use the equation of motion for the auxiliary field D. As stated earlier in (150) this is given by

$$e^{-6U} = \frac{1}{6}c_{IJK}H^I H^J H^K + \frac{c_{2I}}{24}\left(\nabla H^I \cdot \nabla U + 2H^I \nabla^2 U\right) \; . \tag{160}$$

Here H^I are the harmonic functions defined in (157) and we used

$$\mathcal{N} = \frac{1}{6}c_{IJK}H^I H^J H^K e^{6U} \; . \tag{161}$$

The D constraint (160) is now an ordinary differential equation that determines $U(r)$. Its solution specifies the entire geometry and all the matter fields.

6.6.3. *Magnetic attractors*

We can solve (160) exactly in the near horizon region. This case corresponds to vanishing integration constants in (157) so that

$$H^I = \frac{p^I}{2r} \; . \tag{162}$$

Then (160) gives

$$e^{-6U} = \frac{1}{8r^3}\left(p^3 + \frac{1}{12}c_2 \cdot p\right) = \frac{\ell_S^3}{r^3} \; , \tag{163}$$

where $p^3 = \frac{1}{6}c_{IJK}p^I p^J p^K$. The geometry in this case is $\text{AdS}_3 \times S^2$ with the scale ℓ_S in agreement with the magnetic attractor solution developed in Section 5.2.

6.6.4. *Corrected geometry for large black strings*

One way to find solutions to (160) is by perturbation theory. This strategy captures the correct physics when the solution is regular already in the leading order theory, *i.e.* for large black strings. Accordingly, the starting point is the familiar solution

$$e^{-6U_0} = \frac{1}{6}c_{IJK}H^I H^J H^K \; , \tag{164}$$

to the two-derivative theory. This solves (160) with $c_{2I} = 0$.

Although c_{2I} is not small it will be multiplied by terms that are of higher order in the derivative expansion. It is therefore meaningful to expand the full solution to (160) in the form

$$e^{-6U} = e^{-6U_0} + c_{2I}\varepsilon^I + \frac{1}{2}c_{2I}c_{2J}\varepsilon^{IJ} + \dots \; , \tag{165}$$

where $\varepsilon^I(r), \varepsilon^{IJ}(r), \dots$ determine the corrected geometry with increasing precision.

Inserting (165) in (160) and keeping only the terms linear in c_{2I} we find the first order correction[p]

$$\varepsilon^I = \frac{1}{24}(\nabla H^I \cdot \nabla U_0 + 2H^I \nabla^2 U_0) \; . \tag{166}$$

Iterating, we find the second order correction

$$\varepsilon^{IJ} = -\frac{1}{72}\left(\nabla H^I \cdot \nabla(e^{6U_0}\varepsilon^J) + 2H^I \nabla^2(e^{6U_0}\varepsilon^J)\right) \; , \tag{167}$$

where the first order correction ε^I is given by (166). Higher orders can be computed similarly. In summary, we find that starting from a smooth solution to the two-derivative theory we can systematically and explicitly compute the higher order corrections. The series is expected to be uniformly convergent.

In the near horizon limit (162), the full solution (163) is recovered exactly when taking the leading correction (166) into account. As indicated in (163) the effect of the higher derivative corrections is to expand the sphere by a specific amount (which is small for large charges). The perturbative solution gives approximate expressions for the corrections also in the bulk of the solution. Numerical analysis indicates that the corrections remain positive so at any value of the isotropic coordinate r the corresponding sphere is expanded by a specific amount.

[p]It is understood that the correction ε^I is only defined in the combination $c_{2I}\varepsilon^I$.

In this section we have focused on large black strings, that is, those which are non-singular in the leading supergravity description. We will later turn to *small strings*, particularly the important case of *fundamental strings*, in Section 9.

7. Timelike Supersymmetry – Black Holes and Rings

We now turn to the case in which the Killing vector V^μ is timelike over some region of the solution. This class of solutions includes 5D black holes and black rings. The analysis that follows is mainly taken from Refs. 14 and 43, with some further generalizations included.

7.1. *Metric ansatz*

We start with a general metric *ansatz* with Killing vector $\frac{\partial}{\partial t}$,

$$ds^2 = e^{4U_1(x)}(dt + \omega)^2 - e^{-2U_2(x)}ds_B^2 \ . \tag{168}$$

Here ω is a 1-form on the 4D base B with coordinates x^i with $i = 1,\ldots,4$. We choose vielbeins

$$e^{\hat{t}} = e^{2U_1}(dt + \omega) \ , \quad e^{\hat{i}} = e^{-U_2}\tilde{e}^{\hat{i}} \ , \tag{169}$$

where $\tilde{e}^{\hat{i}}$ are vielbeins for ds_B^2. The corresponding spin connection is

$$\omega^{\hat{t}}{}_{\hat{i}} = 2e^{U_2}\tilde{\nabla}_{\hat{i}}U_1 e^{\hat{t}} + \frac{1}{2}e^{2U_1+U_2}d\omega_{\hat{i}\hat{j}}\tilde{e}^{\hat{j}} \ ,$$
$$\omega^{\hat{i}}{}_{\hat{j}} = \tilde{\omega}^{\hat{i}}{}_{\hat{j}} + \frac{1}{2}e^{2U_1+2U_2}d\omega_{\hat{i}\hat{j}}e^{\hat{t}} + e^{U_2}\tilde{\nabla}_{\hat{i}}U_2 e^{\hat{j}} - e^{U_2}\tilde{\nabla}_{\hat{j}}U_2 e^{\hat{i}} \ . \tag{170}$$

We will adopt the following convention for hatted indices. Hatted indices of five dimensional tensors are orthonormal with respect to the full 5D metric, whereas those of tensors defined on the base space are orthonormal with respect to ds_B^2. For example, $d\omega$ is defined to live on the base, and so obeys $d\omega_{ij} = \tilde{e}^{\hat{k}}_i \tilde{e}^{\hat{l}}_j d\omega_{\hat{k}\hat{l}}$. Furthermore, the tilde on $\tilde{\nabla}_{\hat{i}}$ indicates that the $\hat{i}$ index is orthonormal with respect to the base metric. To avoid confusion, we comment below when two different types of hatted indices are used in a single equation.

The Hodge dual on the base space is defined as

$$\star_4 \alpha_{\hat{i}\hat{j}} = \frac{1}{2}\epsilon_{\hat{i}\hat{j}\hat{k}\hat{l}}\alpha^{\hat{k}\hat{l}} \ , \tag{171}$$

with $\epsilon_{\hat{1}\hat{2}\hat{3}\hat{4}} = 1$. A 2-form on the base space can be decomposed into self-dual and anti-self-dual forms,

$$\alpha = \alpha^+ + \alpha^- \ , \tag{172}$$

where $\star_4 \alpha^\pm = \pm\alpha^\pm$.

Equation (91) tells us to look for supersymmetric solutions with a Killing spinor obeying the projection

$$\gamma^{\hat{t}}\epsilon = -\epsilon \ , \tag{173}$$

with a useful alternative form being

$$\alpha^{-\hat{i}\hat{j}}\gamma_{\hat{i}\hat{j}}\epsilon = 0 \ , \tag{174}$$

where $\alpha^{-\hat{i}\hat{j}}$ is any two-form that is anti-self-dual on the 4D base space. The strategy we employ is the same as for the null projection discussed in the previous section: we first exhaust the conditions implied by unbroken supersymmetry, and then impose some of the equations of motion or other constraints.

7.2. *Supersymmetry conditions*

There are three supersymmetry conditions we need to solve. Following the same procedure as in the previous section we first impose a vanishing gravitino variation,

$$\delta\psi_\mu = \left[\mathcal{D}_\mu + \frac{1}{2}v^{ab}\gamma_{\mu ab} - \frac{1}{3}\gamma_\mu\gamma\cdot v\right]\epsilon = 0 \ . \tag{175}$$

Evaluated in our background, the time component of equation (175) reads

$$\left[\partial_t - e^{2U_1+U_2}\partial_i U_1\gamma_{\hat{i}} - \frac{2}{3}e^{2U_1}v^{\hat{t}\hat{i}}\gamma_{\hat{i}} - \frac{1}{4}e^{4U_1+2U_2}d\omega_{ij}\gamma^{\hat{i}\hat{j}} - \frac{1}{6}e^{2U_1}v_{\hat{i}\hat{j}}\gamma^{\hat{i}\hat{j}}\right]\epsilon = 0 \ , \tag{176}$$

where we used the projection (173). The terms proportional to $\gamma_{\hat{i}}$ and $\gamma_{\hat{i}\hat{j}}$ give the conditions

$$v_{\hat{t}\hat{i}} = \frac{3}{2}e^{U_2}\tilde{\nabla}_{\hat{i}}U_1 \ ,$$
$$v_{\hat{i}\hat{j}}^+ = -\frac{3}{4}e^{2U_1}d\omega_{\hat{i}\hat{j}}^+ \ . \tag{177}$$

The spatial component of the gravitino variation (175) simplifies to

$$\left[\tilde{\nabla}_i + \frac{1}{2}\partial_j U_2\gamma_{\hat{i}\hat{j}} + v^{\hat{t}\hat{k}}e_i{}^{\hat{j}}\left(\gamma_{\hat{j}\hat{k}} - \frac{2}{3}\gamma_{\hat{j}}\gamma_{\hat{k}}\right) - e_i{}^{\hat{k}}\left(v_{\hat{k}\hat{j}}^- + \frac{1}{4}e^{2U_1}d\omega_{\hat{k}\hat{j}}^-\right)\gamma^{\hat{j}}\right]\epsilon = 0 \ , \tag{178}$$

where we used the results from (177). The last term in (178) relates the anti-self-dual pieces of v and $d\omega$,

$$v_{\hat{i}\hat{j}}^- = -\frac{1}{4}e^{2U_1}d\omega_{\hat{i}\hat{j}}^- \ . \tag{179}$$

To forestall confusion, we note that in equations (177) and (179) the indices on v are orthonormal with respect to the full 5D metric, while those on $d\omega$ are orthonormal with respect to the base metric.

The remaining components of (178) impose equality of the two metric functions $U_1 = U_2 \equiv U$ and determine the Killing spinor as

$$\epsilon = e^{U(x)}\epsilon_0 \ , \tag{180}$$

with ϵ_0 a covariantly constant spinor on the base, $\tilde{\nabla}_i \epsilon_0 = 0$. This implies that the base space is hyperKähler.[q]

The gaugino variation is given by

$$\delta\Omega^I = \left[-\frac{1}{4}\gamma \cdot F^I - \frac{1}{2}\gamma^a \partial_a M^I - \frac{1}{3}M^I \gamma \cdot v \right]\epsilon = 0 \ . \tag{181}$$

This condition determines the electric and self-dual pieces of F^I_{ab},

$$\begin{aligned} F^{I\hat{t}\hat{i}} &= e^{-U}\tilde{\nabla}_{\hat{i}}(e^{2U}M^I) \ , \\ F^{I+} &= -\frac{4}{3}M^I v^+ \ . \end{aligned} \tag{182}$$

Defining the anti-self-dual form

$$\Theta^I = -e^{2U}M^I d\omega^- + F^{I-} \ , \tag{183}$$

the field strength can be written as

$$F^I = d(M^I e^{\hat{t}}) + \Theta^I \ . \tag{184}$$

The Bianchi identity implies that Θ^I is closed. We emphasize that Θ^I, or more precisely F^{I-}, is undetermined by supersymmetry. These anti-self-dual components are important for black ring geometries but vanish for rotating black holes.

Finally, the variation of the auxiliary fermion is

$$\delta\chi = \left[D - 2\gamma^c\gamma^{ab}\mathcal{D}_a v_{bc} - 2\gamma^a \epsilon_{abcde}v^{bc}v^{de} + \frac{4}{3}(\gamma \cdot v)^2 \right]\epsilon = 0 \ . \tag{185}$$

Using equations (177) and (179), the terms proportional to one or two gamma matrices cancel identically. The terms independent of $\gamma_{\hat{i}}$ give an equation for D, which reads

$$D = 3e^{2U}(\tilde{\nabla}^2 U - 6(\tilde{\nabla}U)^2) + \frac{1}{2}e^{8U}(3d\omega^+_{\hat{i}\hat{j}}d\omega^{+\hat{i}\hat{j}} + d\omega^-_{\hat{i}\hat{j}}d\omega^{-\hat{i}\hat{j}}) \ . \tag{186}$$

---

[q]Recall that there is an implicit $SU(2)$ index on the spinor ϵ. One can then construct three distinct two-forms, $\Phi^{ij}_{ab} = \bar{\epsilon}^i\gamma_{ab}\epsilon^j$, which enjoy an $SU(2)$ algebra. This algebra defines the hyperKähler structure of the base space B; see Ref. 30 for details.

7.3. *Maxwell equations*

The part of the action containing the gauge fields is

$$S^{(A)} = \frac{1}{4\pi^2} \int d^5x \sqrt{g} \left(\mathcal{L}_0^{(A)} + \mathcal{L}_1^{(A)} \right) \ , \tag{187}$$

where the two-derivative terms are

$$\mathcal{L}_0^{(A)} = 2\mathcal{N}_I v^{ab} F_{ab}^I + \frac{1}{4} \mathcal{N}_{IJ} F_{ab}^I F^{Jab} + \frac{1}{24} c_{IJK} A_a^I F_{bc}^J F_{de}^K \epsilon^{abcde} \ , \tag{188}$$

and the four-derivative contributions are

$$\mathcal{L}_1^{(A)} = \frac{c_{2I}}{24} \left(\frac{1}{16} \epsilon^{abcde} A_a^I R_{bc}{}^{fg} R_{defg} + \frac{2}{3} \epsilon_{abcde} F^{Iab} v^{cf} \mathcal{D}_f v^{de} + \epsilon_{abcde} F^{Iab} v^c{}_f \mathcal{D}^d v^{ef} \right.$$
$$\left. + \frac{1}{6} F^{Iab} v_{ab} D + \frac{1}{2} F^{Iab} C_{abcd} v^{cd} - \frac{4}{3} F^{Iab} v_{ac} v^{cd} v_{db} - \frac{1}{3} F^{Iab} v_{ab} v^2 \right) \ . \tag{189}$$

Variation of (187) with respect to A_μ^I gives,

$$\nabla_\mu \left(4\mathcal{N}_I v^{\mu\nu} + \mathcal{N}_{IJ} F^{J\mu\nu} + 2 \frac{\delta \mathcal{L}_1}{\delta F_{\mu\nu}^I} \right) = \frac{1}{8} c_{IJK} F_{\alpha\beta}^J F_{\sigma\rho}^K \epsilon^{\nu\alpha\beta\sigma\rho} + \frac{c_{2I}}{24 \cdot 16} \epsilon^{\nu\alpha\beta\sigma\rho} R_{\alpha\beta\mu\gamma} R_{\sigma\rho}{}^{\mu\gamma} \ , \tag{190}$$

with

$$2 \frac{\delta \mathcal{L}_1}{\delta F^{Iab}} = \frac{c_{2I}}{24} \left(\frac{1}{3} v_{ab} D - \frac{8}{3} v_{ac} v^{cd} v_{db} - \frac{2}{3} v_{ab} v^2 + C_{abcd} v^{cd} \right.$$
$$\left. + \frac{4}{3} \epsilon_{abcde} v^{cf} \mathcal{D}_f v^{de} + 2 \epsilon_{abcde} v^c{}_f \mathcal{D}^d v^{ef} \right) \ , \tag{191}$$

and

$$\frac{\delta \mathcal{L}_1}{\delta F_{\mu\nu}^I} = e_a{}^\mu e_b{}^\nu \frac{\delta \mathcal{L}_1}{\delta F_{ab}^I} \ . \tag{192}$$

A lengthy computation is now required in order to expand and simplify (190). After making heavy use of the conditions derived from supersymmetry, we eventually find that the spatial components of (190) are satisfied identically, while the time component reduces to

$$\tilde{\nabla}^2 \left[M_I e^{-2U} - \frac{c_{2I}}{24} \left(3(\tilde{\nabla}U)^2 - \frac{1}{4} e^{6U} d\omega^{+\hat{i}\hat{j}} d\omega_{\hat{i}\hat{j}}^+ - \frac{1}{12} e^{6U} d\omega^{-\hat{i}\hat{j}} d\omega_{\hat{i}\hat{j}}^- \right) \right]$$
$$= \frac{1}{2} c_{IJK} \Theta^J \cdot \Theta^K + \frac{c_{2I}}{24} \frac{1}{8} \tilde{R}^{\hat{i}\hat{j}\hat{k}\hat{l}} \tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}} \ , \tag{193}$$

where $\Theta^I \cdot \Theta^J = \Theta^{I\hat{i}\hat{j}} \Theta_{\hat{i}\hat{j}}^J$ and $\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$ is the Riemann tensor of the metric on the base. Note also that the indices on $\Theta_{\hat{i}\hat{j}}^I$ are defined to be orthonormal with respect to the metric on the base.

7.4. *D equation*

The equation of motion for the auxiliary field D was given in (88). In the present case it becomes

$$\mathcal{N} = 1 - \frac{c_{2I}}{24}e^{2U}\left[M^I\left(\tilde{\nabla}^2 U - 4(\tilde{\nabla}U)^2 \right) + \tilde{\nabla}_{\hat{i}}M^I\tilde{\nabla}_{\hat{i}}U \right.$$
$$\left. + \frac{1}{4}e^{6U}M^I\left(d\omega^{+\hat{i}\hat{j}}d\omega^+_{\hat{i}\hat{j}} + \frac{1}{3}d\omega^{-\hat{i}\hat{j}}d\omega^-_{\hat{i}\hat{j}} \right) - \frac{1}{12}e^{4U}\Theta^I_{\hat{i}\hat{j}}d\omega^{-\hat{i}\hat{j}} \right] . \quad (194)$$

7.5. *v equation*

The final ingredient needed to completely determine the general solution is the v equation of motion. In fact, for the explicit solutions considered in this review, namely the spinning black holes, this information is not needed. It is however needed to determine the black ring solution, and so we display the result. The full v equation of motion is rather forbidding, and so we simplify by considering just a flat base space. Furthermore, simplifications result upon contracting the v equation with $d\omega$. It turns out that the v equation contracted with $d\omega^+$ is automatically satisfied given our prior results, and so, after a lengthy calculation, we are left with

$$\frac{1}{4}d\omega^{-\hat{i}\hat{j}}d\omega^-_{\hat{i}\hat{j}} + \frac{1}{8}e^{-2U}M_I\Theta^{I\hat{i}\hat{j}}d\omega^-_{\hat{i}\hat{j}} = -\frac{c_{2I}}{16\cdot 24}d\omega^{-\hat{i}\hat{j}}\left[-\frac{1}{6}e^{-6U}\tilde{\nabla}^2(e^{6U}\Theta^I_{\hat{i}\hat{j}}) \right.$$
$$+ 4\tilde{\nabla}_{\hat{j}}\tilde{\nabla}_{\hat{k}}(e^{2U}M^I d\omega^+_{\hat{i}\hat{k}}) + \frac{1}{3}\tilde{\nabla}^2(e^{2U}M^I d\omega^-_{\hat{i}\hat{j}})$$
$$\left. + \frac{1}{6}e^{6U}\Theta^I_{\hat{i}\hat{j}}\left((3d\omega^{+\hat{k}\hat{l}}d\omega^+_{\hat{k}\hat{l}} + d\omega^{-\hat{k}\hat{l}}d\omega^-_{\hat{k}\hat{l}}) \right) \right] . \quad (195)$$

7.6. *Spinning black holes on Gibbons-Hawking space*

We now focus on 5D electrically charged spinning black hole solutions. The main simplification here is that we take[r]

$$d\omega^- = \Theta^I = 0 . \quad (196)$$

Then, to determine the full solution the relevant equations (193) and (194) become

$$\tilde{\nabla}^2\left[M_I e^{-2U} - \frac{c_{2I}}{24}\left(3(\tilde{\nabla}U)^2 - \frac{1}{4}e^{6U}d\omega^{+\hat{i}\hat{j}}d\omega^+_{\hat{i}\hat{j}} \right) \right] = \frac{c_{2I}}{24\cdot 8}\tilde{R}^{\hat{i}\hat{j}\hat{k}\hat{l}}\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}} , \quad (197)$$

$$\mathcal{N} = 1 - \frac{c_{2I}}{24}e^{2U}\left[M^I\left(\tilde{\nabla}^2 U - 4(\tilde{\nabla}U)^2 \right) + \tilde{\nabla}_{\hat{i}}M^I\tilde{\nabla}_{\hat{i}}U + \frac{1}{4}e^{6U}M^I d\omega^{+\hat{i}\hat{j}}d\omega^+_{\hat{i}\hat{j}} \right] . \quad (198)$$

---

[r]The two-derivative BMPV solution enjoys such a property. We include (196) as part of our *ansatz* to investigate higher-derivative corrections to this solution.

The base space is now taken to be a Gibbons-Hawking space with metric

$$ds_B^2 = (H^0)^{-1}(dx^5 + \vec{\chi} \cdot d\vec{x})^2 + H^0 d\vec{x}^2 \ , \tag{199}$$

with H^0 and $\vec{\chi}$ satisfying[s]

$$\vec{\nabla} H^0 = \vec{\nabla} \times \vec{\chi} \ , \tag{200}$$

which in turn implies that H^0 is harmonic on $\mathbb{R}^3$, up to isolated singularities. The x^5 direction is taken to be compact, $x^5 \cong x^5 + 4\pi$, and an isometry direction for the entire solution. We note a few special cases. Setting $H^0 = 1/|\vec{x}|$ yields the flat metric on $\mathbb{R}^4$ in Gibbons-Hawking coordinates. Taking $H^0 = 1$ yields a flat metric on $\mathbb{R}^3 \times S^1$. A more interesting choice is the charge p^0 Taub-NUT space with

$$H^0 = H_\infty^0 + \frac{p^0}{|\vec{x}|} \ . \tag{201}$$

For general p^0 the geometry has a conical singularity but the $p^0 = 1$ case is non-singular.

With this choice of base space we find

$$\tilde{R}^{\hat{i}\hat{j}\hat{k}\hat{l}}\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}} = 2\tilde{\nabla}^2 \left(\frac{(\tilde{\nabla} H^0)^2}{(H^0)^2} \right) + \cdots \ , \tag{202}$$

where the dots represent δ-functions due to possible isolated singularities in H^0, such as in (201). Thus we can write $\tilde{R}^{\hat{i}\hat{j}\hat{k}\hat{l}}\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$ as a total Laplacian

$$\tilde{R}^{\hat{i}\hat{j}\hat{k}\hat{l}}\tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}} = \tilde{\nabla}^2 \Phi \equiv \tilde{\nabla}^2 \left(2\frac{(\tilde{\nabla} H^0)^2}{(H^0)^2} + \sum_i \frac{a_i}{|\vec{x} - \vec{x}_i|} \right) \ , \tag{203}$$

for some coefficients a_i. We can now solve (197) as

$$M_I e^{-2U} - \frac{c_{2I}}{24} \left[3(\tilde{\nabla} U)^2 - \frac{1}{4} e^{6U} d\omega^{+\hat{i}\hat{j}} d\omega_{\hat{i}\hat{j}}^+ \right] - \frac{c_{2I}}{24 \cdot 8} \Phi = H_I \ , \tag{204}$$

where $\tilde{\nabla}^2 H_I = 0$. The choice

$$H_I = 1 + \frac{q_I}{4\rho} \ , \quad \rho = |\vec{x}| \ , \tag{205}$$

identifies the q_I as the conserved 5D electric charges in the case that the base is $\mathbb{R}^4$. In Section 10.3.1 we will closely study the case of a Taub-NUT base space and see that there are modifications to the asymptotics controlled by the coefficients a_i in (203).

---

[s]Take care not confuse $\vec{\nabla}$, the gradient on $\mathbb{R}^3$, with $\tilde{\nabla}$, the covariant derivative on the four-dimensional Gibbons-Hawking space.

The rotation, as encoded in $d\omega^+$, is determined uniquely from closure and self-duality to be

$$d\omega^+ = -\frac{J}{8\rho^2} x^m \left(\tilde{e}^{\hat{5}} \wedge \tilde{e}^{\hat{m}} + \frac{1}{2} \epsilon_{\hat{m}\hat{n}\hat{p}} \tilde{e}^{\hat{n}} \wedge \tilde{e}^{\hat{p}} \right) , \tag{206}$$

where $\tilde{e}^a$ are the obvious vielbeins for the Gibbons-Hawking metric (199) and the orientation is $\epsilon_{\hat{5}\hat{m}\hat{n}\hat{p}} = 1$. With this normalization, J is the angular momentum of the 5D spinning black hole (note that for the supersymmetric black hole the two independent angular momenta in 5D must be equal.)

Now that $d\omega^+$ has been specified the full solution can found as follows. After using (204) to find M_I we determine M^I by solving $M_I = \frac{1}{2} c_{IJK} M^J M^K$ (this can be done explicitly only for special choices of c_{IJK}). We then insert M^I into (198) to obtain a nonlinear, second order, differential equation for $U = U(\rho)$. This last equation typically can be solved only by numerical integration. However, the near horizon limit of the solution can be computed analytically as we do next.

7.7. *The rotating attractor revisited*

We can employ these formulae to make the rotating attractor discussed in Section 5.3.1 more explicit. For this we take the base to be flat $\mathbb{R}^4$, deferring the Taub-NUT case to the next section.

The attractor solution corresponds to dropping the constant in the harmonic functions (201,205) and considering a metric factor of the form

$$e^{2U} = \frac{\rho}{\ell^2} . \tag{207}$$

The resulting geometry takes the form of a circle fibered over an $\text{AdS}_2 \times S^2$, as detailed in (117). Inserting the various functions into the modified very special geometry constraint (198) and the relation expressing flux conservation (204) we verify that this *ansatz* gives an exact solution. Importantly, we also find the relation between the parameters of the attractor solution and the charges measured at infinity.

In more detail, to display the near horizon solution it is useful to define the rescaled quantities

$$\hat{M}^I = 2\ell M^I , \quad \hat{J} = \frac{1}{8\ell^3} J , \tag{208}$$

where ℓ can be identified with the radii of the AdS_2 and S^2 factors in string frame.

We then have the following procedure: given asymptotic charges (J, q_I) we find the rescaled variables $(\hat{J}, \hat{M}^I)$ by solving the equations (198,204) written in the form

$$J = \left(\frac{1}{3!} c_{IJK} \hat{M}^I \hat{M}^J \hat{M}^K - \frac{c_{2I} \hat{M}^I}{12} (1 - 2\hat{J}^2) \right) \hat{J} \,,$$

$$q_I = \frac{1}{2} c_{IJK} \hat{M}^J \hat{M}^K - \frac{c_{2I}}{8} \left(1 - \frac{4}{3} \hat{J}^2 \right) \,. \tag{209}$$

With the solution in hand we compute

$$\ell^3 = \frac{1}{8} \left(\frac{1}{3!} c_{IJK} \hat{M}^I \hat{M}^J \hat{M}^K - \frac{c_{2I} \hat{M}^I}{12} (1 - 2\hat{J}^2) \right) \,,$$

$$M^I = \frac{1}{2\ell} \hat{M}^I \,, \tag{210}$$

to find the values for the physical scale of the solution ℓ and the physical moduli M^I, written as functions of (J, q_I). A novel feature of the higher derivative attractor mechanism is that the fixed values of the moduli depend on the angular momentum as well as the electric charges. From (209) it is clear that the J dependence only appears through the higher derivative terms.

In general it is of course rather difficult to invert (209) explicitly. This is the situation also before higher derivative corrections have been taken into account and/or if angular momentum is neglected. However, in the large charge regime we can make the dependence on the higher derivative corrections manifest in an inverse charge expansion. Let us define the dual charges q^I through

$$q_I = \frac{1}{2} c_{IJK} q^J q^K \,. \tag{211}$$

We also define[t]

$$Q^{3/2} = \frac{1}{3!} c_{IJK} q^I q^J q^K \,, \tag{212}$$

$$C_{IJ} = c_{IJK} q^K \,. \tag{213}$$

Each of these quantities depend on charges and Calabi-Yau data but not on moduli.

With the definitions (211)-(213) we can invert (209) for large charges (*i.e.* expand to first order in c_{2I}) and find

$$\hat{M}^I = q^I + \frac{1}{8} \left(1 - \frac{4}{3} \frac{J^2}{Q^3} \right) C^{IJ} c_{2J} + \dots \,,$$

---

[t] The $\frac{3}{2}$ power is introduced so that Q has the same dimension as the physical charges q_I.

$$\hat{J} = \frac{J}{Q^{3/2}} \left(1 + \frac{c_2 \cdot q}{48 Q^{3/2}} \left[1 - 4 \frac{J^2}{Q^3} \right] \right) + \dots \;, \tag{214}$$

where C^{IJ} is the inverse of the matrix C_{IJ} defined in (213). Then (210) gives the physical scale of the geometry and the physical moduli as

$$\ell = \frac{1}{2} Q^{1/2} \left(1 - \frac{c_2 \cdot q}{144 Q^{3/2}} \left[1 - 4 \frac{J^2}{Q^3} \right] \right) + \dots \;,$$

$$M^I = \frac{q^I}{Q^{1/2}} \left(1 + \frac{c_2 \cdot q}{144 Q^{3/2}} \left[1 - 4 \frac{J^2}{Q^3} \right] \right) + \frac{1}{8 Q^{1/2}} \left(1 - \frac{4}{3} \frac{J^2}{Q^3} \right) C^{IJ} c_{2J} + \dots \;. \tag{215}$$

7.8. *Example: $K3 \times T^2$ compactifications*

We can find more explicit results in the special case of $K3 \times T^2$ compactifications. In this case $c_{1ij} = c_{ij}$, $i,j = 2, \dots 23$ are the only nontrivial intersection numbers and $c_{2,i} = 0$, $c_{2,1} = 24$ are the 2nd Chern-class coefficients. We define c^{ij} to be the inverse of the $K3$ intersection matrix c_{ij}.

We first derive the attractor solution. Our procedure instructs us to first find the hatted variables in terms of conserved charges by inverting (209). In the present case we find

$$\hat{M}^1 = \sqrt{\frac{\frac{1}{2} c^{ij} q_i q_j + \frac{4J^2}{(q_1+1)^2}}{q_1 + 3}} \;, \qquad \hat{M}^i = \sqrt{\frac{q_1 + 3}{\frac{1}{2} c^{ij} q_i q_j + \frac{4J^2}{(q_1+1)^2}}} \, c^{ij} q_j \;, \tag{216}$$

and

$$\hat{J} = \sqrt{\frac{q_1 + 3}{\frac{1}{2} c^{ij} q_i q_j + \frac{4J^2}{(q_1+1)^2}}} \frac{J}{q_1 + 1} \;. \tag{217}$$

All quantities of interest are given in terms of these variables. For completeness, we display the entropy of this solution here, although it will be derived later in Section 8.2 for an arbitrary Calabi-Yau compactification. For a spinning black hole the entropy is given by (244) which, after substitution of (216) and (217) and the intersection numbers and Chern class coefficients for $K3 \times T^2$, becomes

$$S = 2\pi \sqrt{\frac{1}{2} c^{ij} q_i q_j (q_1 + 3) - \frac{(q_1 - 1)(q_1 + 3)}{(q_1 + 1)^2} J^2} \;. \tag{218}$$

In the case of $K3 \times T^2$ the charge q_1 corresponding to M2-branes wrapping T^2 is apparently special; the higher order corrections to the entropy are encoded entirely in the modified functional dependence on q_1.

We now turn to the full asymptotically flat solution in the static case $J = 0$. The full solution can be expressed explicitly in terms of the function U, which obeys a nonlinear ODE

requiring a numerical treatment. We first invert $M_I = \frac{1}{2}c_{IJK}M^JM^K$ as

$$M^1 = \sqrt{\frac{c^{ij}M_iM_j}{2M_1}} \; , \quad M^i = c^{ij}M_j\sqrt{\frac{2M_1}{c^{kl}M_kM_l}} \; . \tag{219}$$

Substituting into (204) gives

$$M^1 = \left(\frac{e^{2U}c^{ij}H_iH_j}{2H_1 + 6U'^2}\right)^{1/2} \; , \quad M^i = \left(\frac{e^{2U}c^{ij}H_iH_j}{2H_1 + 6U'^2}\right)^{-1/2} e^{2U}c^{ij}H_j \; , \tag{220}$$

where $'$ denotes differentiation with respect to $r = 2\sqrt{\rho}$ (in terms of r the base space metric becomes $dr^2 + r^2 d\Omega_3^2$). The special geometry constraint (198) is

$$\frac{1}{2}c_{ij}M^iM^jM^1 - 1 + e^{2U}\left[(U'' + \frac{3}{r}U' - 4U'^2)M^1 + U'M^{1'}\right] = 0 \; . \tag{221}$$

The problem is now to insert (220) into (221) and solve for $U(r)$.

This is straightforward to solve numerically given specific choices of charges. Consider a small black hole, $q_1 = 0$ with $q_2 = q_3 = 1$, $c^{23} = 1$. We also assume $H = H_2 = H_3 = 1 + \frac{1}{r^2}$ are the only harmonic functions not equal to unity. Then (221) becomes

$$HU'' + (1 + 3U'^2)\left[\left(\frac{3}{r} + \frac{1}{r^3}\right)U' + H\right] - e^{-3U}(1 + 3U'^2)^{3/2} = 0 \; . \tag{222}$$

The boundary conditions are fixed by matching to the small r behavior

$$e^{-2U} \sim \frac{\ell_S^2}{r^2} \; , \tag{223}$$

with $\ell_S = 3^{-1/6}$. The result of the numerical solution for $U(r)$ is shown in Fig. 1.
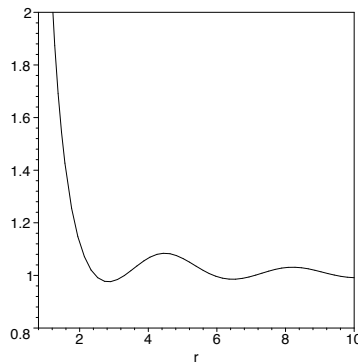

Fig. 1.   Numerical solution of equation (222); the curve represents $e^{-2U(r)}$ for small values of r. The oscillatory behavior is characteristic of higher derivative theories and will be discussed further in Section 9.1.3.

7.9. *Comments on black rings*

Black rings[22,23] incorporate nonzero Θ^I and $d\omega^-$. After choosing the base space, the two-form Θ^I can be determined by the requirements of closure and anti-self duality. In the two-derivative limit $d\omega^-$ can be determined from the v equation of motion according to

$$d\omega^- = \frac{1}{2}e^{-2U}M_I\Theta^I \ . \tag{224}$$

In the higher derivative case there is instead equation (195), which has not yet been solved. The full black ring solution is therefore not available at present. We can, however, find the near horizon geometry of the black ring and an expression for its entropy. This question will be revisited in the Section 8.3.

8. Black Hole Entropy and Extremization Principles

An important application of the solutions we construct is to the study of gravitational thermodynamics. The higher derivative corrections to the supergravity solutions are interesting for this purpose because they are sensitive to details of the microscopic statistical description.

The black hole entropy is famously given by the Bekenstein-Hawking area law

$$S = \frac{1}{4G_D}A_{D-2} \ . \tag{225}$$

This expression applies only when the gravitational action is just the standard Einstein-Hilbert term. In general, one must use instead the Wald entropy formula[u]

$$S = -\frac{1}{8G_D}\int_{\text{hor}}d^{D-2}x\sqrt{h}\frac{\delta\mathcal{L}_D}{\delta R_{\mu\nu\rho\sigma}}\epsilon^{\mu\nu}\epsilon^{\rho\sigma} \ . \tag{226}$$

This reduces to (225) for the two-derivative action, but generally the density one must integrate over the event horizon is more complicated than the canonical volume form. In practice, it is in fact rather cumbersome to evaluate (226) and evaluate the requisite integral but there is a short-cut that applies to black holes with near horizon geometry presented as a fibration over $\text{AdS}_2 \times S^2$. Then the Wald entropy (226) is the Legendre transform of the on-shell action[v] up to an overall numerical factor. This general procedure is known as the

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[u]Theories with gravitational Chern-Simons terms may violate diffeomorphism invariance. Then Wald's formula does not apply and one must use a further generalization due to Tachikawa.[57]
[v]This refers to the usual notion of "on-shell action", *i.e.* the action evaluated on a solution to all of the equations of motion. This is not to be confused with the sense of "on-shell" that we have been using throughout this review, *i.e.* with only the auxiliary field equations of motion imposed.

entropy function formalism.[2] In Section 8.2 we apply the entropy function formalism to our five dimensional black hole solutions with $AdS_2 \times S^3$ near horizon geometry.

Although we analyze a theory in five dimensions, we can discuss four dimensional black holes by adding excitations to black strings with $AdS_3 \times S^2$ near string geometry. For large excitation energy the black hole entropy is given by Cardy's formula

$$S = 2\pi \left[\sqrt{\frac{c_L}{6}\left(h_L - \frac{c_L}{24}\right)} + \sqrt{\frac{c_R}{6}\left(h_R - \frac{c_R}{24}\right)} \right] , \tag{227}$$

where h_L, h_R are eigenvalues of the AdS_3 energy generators $L_0, \bar{L}_0$. Since Cardy's formula can be justified in both the gravitational description[w] and also in the dual CFT, the central charge becomes a proxy for the entropy in the $AdS_3 \times S^2$ setting. It is therefore the central charge that we want to compute for our solutions. The central charge is convenient to compute because it is just the on-shell action, up to an overall numerical factor. This methodology is known as c-extremization.[3] In Section 8.1 we apply c-extremization to our five dimensional black string solutions with $AdS_3 \times S^2$ near horizon geometry.

The entropy function formalism and the c-extremization procedure can be carried out while keeping arbitrary the scales of the AdS and sphere geometries, as well as matter fields consistent with the symmetries. These parameters are then determined by the extremization procedure in a manner independent of supersymmetry. The computations therefore constitute an important consistency check on the explicit Lagrangian and other parts of the framework.

8.1. *Black strings and c-extremization*

The general c-extremization procedure considers a $AdS_3 \times S^{D-3}$ solution to a theory with action of the form

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{g} \mathcal{L} + S_{CS} + S_{\text{bndy}} . \tag{228}$$

The Chern-Simons terms (if any) are collected in the term S_{CS}, and S_{bndy} are the terms regulating the infrared divergences at the boundary of AdS_3. The total central charge

$$c = \frac{1}{2}(c_L + c_R) , \tag{229}$$

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[w]In fact Cardy's formula (227) agrees with the Wald entropy whenever diffeomorphism invariance applies ($c_L = c_R$),[3,51] or with Tachikawa's generalization[57] when $c_L \neq c_R$.

is essentially the trace anomaly of the CFT, which in turn is encoded in the on-shell action of the theory. The precise relation is

$$c = -\frac{3\Omega_{D-3}}{8G_D}\ell_A^3\ell_S^{D-3}\mathcal{L}_{\text{ext}} , \tag{230}$$

with the understanding that the action must be extremized over all parameters, with magnetic charges through S^{D-3} kept fixed.

We want to apply this formalism to the black string attractor solution found in Section 5.2. The isometries of the near horizon region determines the form of the solution as

$$ds^2 = \ell_A^2 ds_{AdS}^2 - \ell_S^2 d\Omega_2^2 ,$$
$$F^I = -\frac{p^I}{2}\epsilon_2 ,$$
$$v = V\epsilon_2 ,$$
$$M^I = mp^I . \tag{231}$$

In Section 5.2 we used maximal supersymmetry and the modified very special geometry constraint to determine the parameters ℓ_A, ℓ_S, V, m and the auxiliary scalar D in terms of the magnetic charges p^I as

$$V = \frac{3}{8}\ell_A , \quad D = \frac{12}{\ell_A^2} , \quad m = \frac{1}{\ell_A} ,$$
$$\ell_S = \frac{1}{2}\ell_A , \quad \ell_A^3 = p^3 + \frac{1}{12}c_2 \cdot p , \tag{232}$$

where

$$p^3 \equiv \frac{1}{6}c_{IJK}p^I p^J p^K . \tag{233}$$

However, it is instructive to use just the *ansatz* (231) for now. Inserting this *ansatz* into the leading order action (78) we find

$$\mathcal{L}_0 = 2\left(\frac{1}{4}(p^3 m^3 - 1)D - \frac{1}{4}(p^3 m^3 + 3)\left(\frac{3}{\ell_A^2} - \frac{1}{\ell_S^2}\right)\right.$$
$$\left. + \frac{1}{\ell_S^4}\left((3p^3 m^3 + 1)V^2 + 3p^3 m^2 V\right) + \frac{3p^3}{\ell_S^4}\frac{m}{8}\right) , \tag{234}$$

and the four derivative action (79) yields

$$\mathcal{L}_1 = \frac{c_2 \cdot p}{24}\left[\frac{m}{4}\left(\frac{1}{\ell_A^2} - \frac{1}{\ell_S^2}\right)^2 + \frac{2}{3}\frac{V^3}{\ell_S^8} + 4m\frac{V^4}{\ell_S^8} + m\frac{D^2}{12} + \frac{D}{6}\frac{V}{\ell_S^4}\right.$$
$$\left. - \frac{2}{3}m\frac{V^2}{\ell_S^4}\left(\frac{3}{\ell_A^2} + \frac{5}{\ell_S^2}\right) + \frac{1}{2}\frac{V}{\ell_S^4}\left(\frac{1}{\ell_A^2} - \frac{1}{\ell_S^2}\right)\right] . \tag{235}$$

According to c-extremization we now need to extremize the c-function

$$c(\ell_A, \ell_S, V, D, m) = -6\ell_A^3\ell_S^2(\mathcal{L}_0 + \mathcal{L}_1) , \tag{236}$$

with respect to all variables. The resulting extremization conditions are quite involved. For example, the variation of (236) with respect to m gives

$$\frac{3p^3 m^2}{4}\left(D - \frac{3}{\ell_A^2} + \frac{1}{\ell_S^2}\right) + \frac{3p^3}{\ell_S^4}\left(3m^2 V^2 + 2mV + \frac{1}{8}\right) +$$
$$+ \frac{c_2 \cdot p}{48}\left[\frac{1}{4}\left(\frac{1}{\ell_A^2} - \frac{1}{\ell_S^2}\right)^2 + 4\frac{V^4}{\ell_S^8} + \frac{D^2}{12} - \frac{2}{3}\frac{V^2}{\ell_S^4}\left(\frac{3}{\ell_A^2} + \frac{5}{\ell_S^2}\right)\right] = 0 . \tag{237}$$

It would be very difficult to solve equations with such complexity without any guidance. Fortunately we already determined the attractor solution (232) and it is straightforward to verify that it does indeed satisfy (237). We can similarly vary the c-function (236) with respect to ℓ_A, ℓ_S, V, D and show that the resulting equations are satisfied by the attractor solution (232). Thus the attractor solution extremizes the c-function (236) as it should.

Since we have proceeded indirectly we have not excluded the possibility that c-extremization could have other solutions with the same charge configuration. Such solutions would not be supersymmetric. This possibility further imposes the point that c-extremization is logically independent from the considerations using maximal supersymmetry that determined the attractor solution in the first place. The success of c-extremization therefore constitutes a valuable consistency check on the entire framework.

At this point we have verified that the c-function is extremized on the attractor solution (232). The central charge is now simply the value of the (236) on that solution. The computation gives

$$c = 6p^3 + \frac{3}{4}c_2 \cdot p . \tag{238}$$

In order to put this result in perspective, let us recall the microscopic interpretation of these black strings.[8] We can interpret $N = 2$ supergravity in five dimensions as the low energy limit of M-theory compactified on some Calabi-Yau threefold CY_3. The black string in five dimensions corresponds to a M5-brane wrapping a 4-cycle in CY_3 that has component p^I along the basis four-cycle ω_I. The central charges of the effective string CFT are known to be[8,35]

$$c_L = 6p^3 + \frac{1}{2}c_2 \cdot p , \quad c_R = 6p^3 + c_2 \cdot p , \tag{239}$$

where c_{IJK} are the triple intersection numbers of the CY_3, and c_{2I} are the expansion coefficients of the second Chern class. Computing the total central charge (229) from (239) we find precise agreement with our result (238) found by c-extremization.

It is worth noting that the simple form of the central charge comes about in a rather nontrivial way in the c-extremization procedure. The radius of curvature ℓ_A from the last line

of (232) introduces powers of $(p^3 + \frac{1}{12}c_2 \cdot p)^{1/3}$ in the denominator of the Lagrangian (234-235). It is only due to intricate cancellations that the final result (238) becomes a polynomial in the charges p^I.

8.2. *Black hole entropy*

We want to compute the entropy of black hole solutions with $\mathrm{AdS}_2 \times S^3$ near horizon geometry. As mentioned in the introduction to this section the most efficient method to find the entropy is by use of the entropy function,[2] which amounts to computing the Legendre transform of the Lagrangian density evaluated on the near horizon solution. Some care is needed because the 5D action contains non-gauge invariant Chern-Simons terms while the entropy function method applies to gauge invariant actions.

We first review the general procedure for determining the entropy from the near horizon solution, mainly following Ref. 108. The general setup is valid for spinning black holes as well as black rings.

The near horizon geometries of interest take the form of a circle fibered over an $\mathrm{AdS}_2 \times S^2$ base:

$$ds^2 = w^{-1}\Big[v_1\Big(\rho^2 d\tau^2 - \frac{d\rho^2}{\rho^2}\Big) - v_2(d\theta^2 + \sin^2\theta d\phi^2)\Big] - w^2\Big(dx^5 + e^0\rho d\tau + p^0\cos\theta d\phi\Big)^2 \; ,$$
$$A^I = e^I \rho d\tau + p^I \cos\theta + a^I\Big(dx^5 + e^0 \rho d\tau + p^0 \cos\theta d\phi\Big) \; ,$$
$$v = -\frac{1}{4\mathcal{N}}M_I F^I \; . \tag{240}$$

The parameters w, $v_{1,2}$, a^I and all scalar fields are assumed to be constant. Kaluza-Klein reduction along x^5 yields a 4D theory on $\mathrm{AdS}_2 \times S^2$. The solution carries the magnetic charges p^I, while e^I denote electric potentials.[x]

Omitting the Chern-Simons terms for the moment, let the action be

$$I = \frac{1}{4\pi^2}\int d^5x \sqrt{g}\mathcal{L} \; . \tag{241}$$

Define

$$f = \frac{1}{4\pi^2}\int d\theta d\phi dx^5 \sqrt{g}\mathcal{L} \; . \tag{242}$$

Then the black hole entropy is

$$S = 2\pi\Big(e^0\frac{\partial f}{\partial e^0} + e^I\frac{\partial f}{\partial e^I} - f\Big) \; . \tag{243}$$

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[x]An important point, discussed at length in Section 10, is that e^I are conjugate to 4D electric charges, which differ from the 5D charges.

Here w, $v_{1,2}$ etc. take their on-shell values. One way to find these values is to extremize f while holding fixed the magnetic charges and electric potentials. The general extremization problem would be quite complicated given the complexity of our four-derivative action. Fortunately, in the cases of interest we already know the values of all fields from the explicit solutions.

The Chern-Simons term is handled by first reducing the action along x^5 and then adding a total derivative to $\mathcal{L}$ to restore gauge invariance in the resulting 4D action (it is of course not possible to restore gauge invariance in 5D).[43]

The result of this computation is the entropy formula

$$S = 2\pi \sqrt{1 - \hat{J}^2} \left(\frac{1}{6} c_{IJK} \hat{M}^I \hat{M}^J \hat{M}^K + \frac{1}{6} \hat{J}^2 c_{2I} \hat{M}^I \right) , \tag{244}$$

where the rescaled moduli are evaluated at their attractor values (209).

We can also express the entropy in terms of the conserved charges. We first use (210) to find an expression in terms of geometrical variables

$$S = 2\pi \sqrt{(2\ell)^6 - J^2} \left(1 + \frac{c_{2I} M^I}{48 \ell^2} \right) , \tag{245}$$

and then expand to first order in c_{2I} using (215) to find

$$S = 2\pi \sqrt{Q^3 - J^2} \left(1 + \frac{c_2 \cdot q}{16} \frac{Q^{3/2}}{(Q^3 - J^2)} + \cdots \right) . \tag{246}$$

From the standpoint of our 5D supergravity action (244) is an exact expression for the entropy. But as a statement about black hole entropy in string theory it is only valid to first order in c_{2I}, since we have only kept terms in the effective action up to four derivatives. The situation here is to be contrasted with that for 5D black strings, where anomaly arguments imply that the entropy is uncorrected by terms beyond four derivatives. The anomaly argument relies on the presence of an AdS$_3$ factor, which is absent for the 5D black holes considered in this section.

8.3. *Black ring entropy*

Although we have not yet determined the complete black ring solution we can compute its entropy by applying the entropy function formalism to the black ring attractor.

For the black ring the near horizon solution is

$$ds^2 = w^{-1} v_3 \left[\left(\rho^2 d\tau^2 - \frac{d\rho^2}{\rho^2} \right) - d\Omega^2 \right] - w^2 \left(dx^5 + e^0 \rho d\tau \right)^2 ,$$

$$A^I = -\frac{1}{2}p^I \cos\theta d\phi - \frac{e^I}{e^0}dx^5 \ . \tag{247}$$

Further details of the solution follow from the fact that the near horizon geometry is a magnetic attractor. The near horizon geometry is a product of a BTZ black hole and an S^2, and there is enhanced supersymmetry. These conditions imply

$$\begin{aligned}
M^I &= \frac{p^I}{2we^0} \ , \\
v_3 &= w^3(e^0)^2 \ , \\
D &= \frac{3}{w^2(e^0)^2} \ , \\
v &= -\frac{3}{4}we^0 \sin\theta d\theta \wedge d\phi \ .
\end{aligned} \tag{248}$$

The computation of the entropy in terms of the entropy function proceeds as in the case of the spinning black hole. The result is

$$S = \frac{2\pi}{e^0}\left(\frac{1}{6}c_{IJK}p^I p^J p^K + \frac{1}{6}c_{2I}p^I\right) \ . \tag{249}$$

The entropy is expressed above in terms of magnetic charges p^I and the potential e^0, but the preferred form of the entropy would be a function of the conserved asymptotic charges. To get a formula purely in terms of the charges (p^I, q_I) and the angular momenta we need to trade away e^0. But for this one needs knowledge of more than just the near horizon geometry, which, as we noted above, is not available at present.

Let us finally note that the entropy can be expressed in geometric variables as

$$S = (2-\mathcal{N})\frac{A}{\pi} = (2-\mathcal{N})\frac{A}{4G_5} \ , \tag{250}$$

where A is the area of the event horizon. In the two-derivative limit we have $\mathcal{N}=1$ and we recover the Bekenstein-Hawking entropy.

9. Small Black Holes and Strings

One of the main motivations for studying higher derivative corrections is their potential to regularize geometries that are singular in the lowest order supergravity approximation.[3,11−13,109] One version of this phenomenon occurs for black holes possessing a nonzero entropy, where the effect of the higher derivative terms is not to remove the black hole singularity, but rather to shield it with an event horizon. The resulting spacetime is then qualitatively similar to that of an ordinary "large" black hole. Examples of this occur for both four and five dimensional black holes in string theory. A second, and in many ways

more striking, example pertains to the case in which the solution has a vanishing entropy. In this case the singularity, instead of being shielded by a finite size event horizon, is smoothed out entirely. Our five dimensional string solutions provide an explicit realization of this.

To realize the latter type of solutions, we consider magnetic string solutions whose charge configurations satisfy $p^3 = \frac{1}{6}c_{IJK}p^Ip^Jp^K = 0$. We refer to these as *small strings*. Recall from Section 5.2 that our string solutions had a near horizon $AdS_3 \times S^2$ geometry with AdS scale size given by

$$\ell_A^3 = p^3 + \frac{1}{12}c_2 \cdot p \; . \tag{251}$$

For small strings the geometry is singular in the two derivative approximation, since $\ell_A^3 = 0$. Conversely, $\ell_A^3 \neq 0$ when the correction proportional to c_{2I} is taken into account. Thus it appears that a spacetime singularity has been resolved. To understand the causal structure we can note that our metric is a particular example of the general class of geometries studied in Ref. 20. The resulting Penrose diagram is like that of the M5-brane in eleven dimensions. In particular, the geometry is completely smooth, and there is no finite entropy event horizon.

We should, however, close one potential loophole. In principle, it could be that the actual near string geometry realized in the full asymptotically flat solution is not the regular solution that is consistent with the charges, but instead a deformed but still singular geometry. In order to exclude this possibility we must construct the complete solution that smoothly interpolates between the regular near horizon geometry and asymptotically flat space. In this section we present such an interpolating solution, thereby confirming that the singularity is indeed smoothed out.

Since the near string geometry after corrections are taken into account has an AdS_3 factor, it is natural to ask whether the AdS/CFT correspondence applies, and to determine what special features the holography might exhibit. This question has attracted significant attention recently and remains an active area of inquiry.[77−80]

A particularly important example of a small string is obtained when the Calabi-Yau is $K3 \times T^2$, and the only magnetic charge that is turned on is that corresponding to an $M5$-brane wrapping the $K3$. The resulting 5D string is then dual, via IIA-heterotic duality, to the fundamental heterotic string.[110,111] We will focus on this particular example in this section.

9.1. *The small string: explicit solution*

Let M^1 be the single modulus on the torus and M^i be the moduli of $K3$ where $i = 2, \ldots, 23$. The charge configuration of interest specifies the harmonic functions as

$$H^1 = M^1_\infty + \frac{p^1}{2r} \ ,$$
$$H^i = M^i_\infty \ , \qquad i = 2, \ldots, 23 \ . \tag{252}$$

The only nonvanishing intersection numbers are $c_{1ij} = c_{ij}$ where c_{ij} is the intersection matrix for $K3$. To simplify, we choose M^i_∞ consistent with $\frac{1}{2} c_{ij} M^i_\infty M^j_\infty = 1$, so that (161) becomes

$$\mathcal{N} e^{-6U} = \frac{1}{6} c_{IJK} H^I H^J H^K = H^1 \ . \tag{253}$$

The master equation (160) now becomes

$$H^1 = e^{-6U} - \left[\partial_r H^1 \partial_r U + 2H^1 \frac{1}{r^2} \partial_r (r^2 \partial_r U) \right] \ , \tag{254}$$

where we used $c_2(K3) = 24$ and $c_{2i} = 0$. We can write this more explicitly as

$$1 + \frac{p^1}{2r} = e^{-6U} - 2(1 + \frac{p^1}{2r}) U'' - \frac{4}{r} \left(1 + \frac{3p^1}{8r} \right) U' \ , \tag{255}$$

where primes denote derivatives with respect to r. Note that we set $M^1_\infty = 1$; a general value can be restored by a rescaling of p^1 and a shift of U.

In our units distance r is measured in units of the 5D Planck length. The parameter p^1 is a pure number counting the fundamental strings. For a given p^1, it is straightforward to integrate (255) numerically. Instead, to gain some analytical insight we will take $p^1 \gg 1$ so as to have an expansion parameter. We will analyze the problem one region at a time.

9.1.1. *The $AdS_3 \times S^2$-region*

This is the leading order behavior close to the string. According to our magnetic attractor solution in the form (163) we expect the precise asymptotics

$$e^{-6U} \to \frac{\ell_S^3}{r^3} \ , \ r \to 0 \ , \tag{256}$$

where the S^2-radius is given by

$$\ell_S = \left(\frac{p^1}{4} \right)^{1/3} \ . \tag{257}$$

For $p^1 \gg 1$ this is much smaller than the scale size of a large string, which from (251) has scale $\sim p$. However, it is nevertheless much larger than the 5D Planck scale. The modulus

describing the volume of the internal T^2 is

$$M^1 = \frac{p^1}{2\ell_S} = 2^{-1/3}(p^1)^{2/3} \ , \tag{258}$$

which also corresponds to the length scale $(p^1)^{1/3}$.

9.1.2. *The near-string region*

We next seek a solution in the entire range $r \ll p^1$ which includes the scale (257) but reaches further out. In fact, it may be taken to be all of space in a scaling limit where $p^1 \to \infty$.

In the near string region (255) reduces to

$$\frac{p^1}{2r} = e^{-6U} - \frac{p^1}{r}U'' - \frac{3p^1}{2r^2}U' \ . \tag{259}$$

We can scale out the string number p^1 by substituting

$$e^{-6U(r)} = \frac{p^1}{4r^3}e^{-6\Delta(r)} \ , \tag{260}$$

which amounts to

$$U(r) = \frac{1}{2}\ln\frac{r}{\ell_S} + \Delta(r) \ . \tag{261}$$

This gives

$$\Delta'' + \frac{3}{2r}\Delta' + \frac{1}{4r^2}(1 - e^{-6\Delta}) + \frac{1}{2} = 0 \ , \tag{262}$$

which describes the geometry in the entire region $r \ll p^1$. The asymptotic behavior at small r is

$$\Delta(r) = -\frac{1}{13}r^2 + \frac{3}{(13)^3}r^4 + \frac{20}{9(13)^4}r^6 + \cdots \ . \tag{263}$$

Since $\Delta(r) \to 0$ smoothly as $r \to 0$ we have an analytical description of the approach to the $\text{AdS}_3 \times S^2$ region.

The asymptotic behavior for large r is also smooth. Expanding in $u = \frac{1}{r}$ we find

$$\Delta(r) = -\frac{1}{6}\ln(2r^2) - \frac{1}{36}\frac{1}{r^2} + \frac{13}{12 \cdot 36}\frac{1}{r^4} + \cdots \ . \tag{264}$$

It is straightforward to solve (262) numerically. Fig. 2 shows the curve that interpolates between the asymptotic forms (263) and (264). The oscillatory behavior in the intermediate region is characteristic of higher derivative theories. We comment in more detail below.

In the original variable $U(r)$ the approximation (264) gives

$$e^{-6U} = \frac{p^1}{2r}\left(1 + \frac{1}{6r^2} - \frac{1}{6r^4} + \cdots\right) \ , \tag{265}$$

for large r. The leading behavior, $e^{-6U} = H^1 \sim \frac{p^1}{2r}$, agrees with the near string behavior in two-derivative supergravity. In the full theory this singular region is replaced by a smooth geometry.

9.1.3. *The approach to asymptotically flat space*

We still need to analyze the region where r is large, meaning $r \sim p^1$ or larger. Here we encounter some subtleties in matching the solution on to the asymptotically flat region.

In the asymptotic region the full equation (254) simplifies to

$$1 + \frac{p^1}{2r} = e^{-6U} - 2(1 + \frac{p^1}{2r})U'' \ . \tag{266}$$

Terms with explicit factors of $1/r$ were neglected, but we kept derivatives with respect to r so as to allow for Planck scale structure, even though $r \sim p^1 \gg 1$. Changing variables as

$$e^{-6U} = (1 + \frac{p^1}{2r})e^{-6W} \ , \tag{267}$$

we find

$$W'' = \frac{1}{2}(e^{-6W} - 1) \simeq -3W \ . \tag{268}$$

The expansion for small W is justified because (265) imposes the boundary condition $W \to 0$ for $r \ll p^1$.

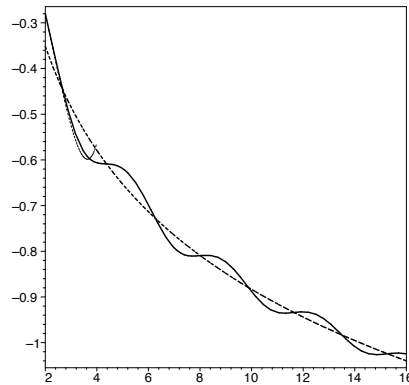The solution $W = 0$ expected from two-derivative supergravity is in fact a solution to

Fig. 2. Analytical and numerical results for $\Delta(r)$ in the near string region. The solid curve describes the numerical solution of (262). The dotted curve represent the analytical solution for small values of r given by (263), and the dashed curved is the approximate solution for large values of r (264).

(268), but there are also more general solutions of the form

$$W = A\sin(\sqrt{3}r + \delta) \ . \tag{269}$$

The amplitude of this solution is undamped, so it is not really an intrinsic feature of the localized string solution we consider. Instead it is a property of fluctuations about flat space, albeit an unphysical one. The existence of such spurious solutions is a well-known feature of theories with higher derivatives, and is related to the possibility of field redefinitions.[11−13,112] In the present context the issue is that the oscillatory solutions can be mapped to zero by a new choice of variables, such as $\tilde{W} = (\nabla^2 - 3)W$.

   To summarize, modulo the one subtlety associated with field redefinitions, we have found a smooth solution interpolating between the near horizon $\mathrm{AdS}_3 \times S^2$ attractor and asymptotically flat space. The solution is completely regular, the causal structure being the same as that of an M5-brane in eleven dimensions. While our result is highly suggestive of the existence of a smooth solution of the full theory with all higher derivative corrections included, we cannot establish with certainty that this is the case. The reason is that for small strings there is no small parameter suppressing even higher derivative terms. Indeed, it is easy to check that in the near horizon region terms in the action with more than four derivatives contribute at the same order as those included in the present analysis. As a result, the precise numerical results for the attractor moduli and scale size are expected to receive corrections of order unity. On the other hand, it seems highly plausible that the solution will remain smooth even after these additional corrections have been taken into account.

9.2. *Holography for the fundamental string*

The small string solutions are not merely regular, they exhibit an $\mathrm{AdS}_3 \times S^2$ near string geometry. This raises the possibility that the AdS/CFT correspondence applies to small strings.[77−80] As noted already, in a particular duality frame these solutions correspond to $n = p^1$ fundamental heterotic strings. The underlying theory in this case is Heterotic string theory compactified on T^5, with the fundamental strings extended along one of the noncompact spatial directions. The near horizon solution in this frame has string coupling $g_s \sim 1/\sqrt{n}$ and curvature of order the string scale. Therefore, for a large number of strings, quantum effects are suppressed while α' corrections are of order unity. This implies that the proper description of these solutions is really in terms of a worldsheet CFT, since this will capture all of the α' corrections rather than just the leading ones used in our supergravity approach.

In trying to establish an AdS/CFT correspondence in this case, the following features are expected:

- The attractor geometry has an $SL(2,\mathbb{R}) \times OSp(4^*|4)$ isometry group[79] (for a recent discussion on the super-isometry group of 5D small black hole attractors see Ref. 113). It is not completely clear whether the full theory also is $SL(2,\mathbb{R}) \times OSp(4^*|4)$ invariant. If it is, then since the general argument of Brown and Henneaux[31] implies that the $SL(2,\mathbb{R})$ symmetries are enhanced to Virasoro algebras, the symmetry algebra of the theory will be some superconformal extension of $OSp(4^*|4)$, with affine $SU(2) \times Sp(4)$ R-currents.[79,80]

- It is expected that the superconformal algebra has $(0,8)$ supersymmetry. In our supergravity solution based on M-theory on $K3 \times T^2$ (or equivalently Heterotic on T^5), only half of the supersymmetry is manifest since we work in an $N = 2$ formalism.

- Based on the worldsheet structure of the heterotic string, the central charges of the theory are expected to be $c_R = 12n$ and $c_L = 24n$, possibly with subleading $1/n$ corrections. For our supergravity solution, we can infer that $c_L - c_R = 12n$, since this combination is determined by a diffeomorphism anomaly.[24] To determine $c_L + c_R$ via c-extremization one needs to use the full set of higher derivative corrections, which are not known. As we saw in Section 8.1, the four-derivative action yields the expected $c_L + c_R = 36n$, but this result is not reliable for small strings. Alternatively, once the precise superconformal algebra has been established, we can use the R-symmetry anomaly to determine c_L. If this algebra contains a $(0,4)$ subalgebra, then the desired $c_L = 24n$ will follow.

The most immediate difficulty with all these expectations is the absence of a conventional superconformal algebra with $(0,8)$ supersymmetry. In particular, one can start with currents corresponding to Virasoro, R-symmetry, and local supersymmetry, and then look for a consistent operator product expansion in which only these currents (plus central terms) appear in the singular parts. One finds that it is impossible to satisfy the Jacobi identities. Faced with this problem, one is led to consider the *nonlinear superconformal algebras*.[114] These are algebras with bilinears of the R-currents appearing in the OPEs of supercurrents.[115,116] One of the appealing features of these algebras is that their central charges are completely determined algebraically in terms of the level of the R-current algebra. In particular, this would give exact expressions for nontrivial quantum corrections to the spacetime central charges.[80]

Unfortunately there are serious difficulties with this optimistic scenario. There are at least two major problems:

- The nonlinear algebras of interest do not permit unitary highest weight representations. The Jacobi identities either imply that the central charge is negative, or else one of the R-currents has negative level. Either way, this would seem unacceptable for the spacetime theory. Firstly, unitarity is of course sacred in quantum mechanical descriptions. Secondly, it would be extremely surprising to find instabilities in a BPS system with so much supersymmetry.
- The nonlinear $(0,8)$ supersymmetry does not have any obvious $(0,4)$ subalgebra. The actual central charge determined from the nonlinear algebra does not agree with the expectations.

There is no logical inconsistency in this state of affairs, since we were careful to emphasize that some our assumptions are optimistic and not backed by explicit computations. Obviously, we must be cautious in extending our usual AdS/CFT expectations to the unfamiliar terrain of small strings.

The simplest consistent modification of the expectations is that the superconformal symmetry of this theory is something other than the nonlinear algebra based on $OSp(4^*|4)$. There might instead be some more exotic W-algebra, which perhaps contains an $(0,4)$ subalgebra. This remains a fascinating direction for future research. To motivate further study, we note that understanding holography for fundamental strings could lead to an example of AdS/CFT in which both the bulk and boundary sides of the duality are analytically tractable.

10. Comparing 4D and 5D Solutions

There is a rich web of interconnections between supergravity theories in diverse dimensions, and it is illuminating to consider the relations between solutions to these different theories. A solution with a spacelike isometry can be converted to a lower dimensional one by Kaluza-Klein reduction along the isometry direction. Conversely, a solution can be uplifted to one higher dimension by interpreting a gauge field as the off-diagonal components of a higher dimensional metric.

Here we will be concerned with the relation between 4D and 5D solutions. The BPS equations governing general 4D supersymmetric solutions are well established, including the

contributions from a class of four-derivative corrections.[7] On the other hand, in this review we have obtained the corresponding 5D BPS equations. What is the relation between the two?

We first address this question at the two-derivative level, and show (following Ref. 41) that the 4D BPS equations can be mapped to a special case of the 5D BPS equations. That is to say, the general 4D BPS solution can be interpreted as the rewriting of a 5D solution. Note, though, that the space of solutions is larger in 5D in the sense that the general 5D solution has no spacelike isometry and hence can't be reduced to 4D.

We then turn to the generalization of this correspondence with four-derivative corrections included, and find that there is apparently no simple relation between the two sets of solutions. We discuss the likely reason for this mismatch.

An interesting application of this circle of ideas is to the so-called 4D/5D connection, which gives a relation between the entropies of black holes in four and five dimensions.[37] This connection involves 5D solutions whose base metric is a Taub-NUT. The Taub-NUT geometry interpolates between $\mathbb{R}^4$ at the origin and $\mathbb{R}^3 \times S^1$ at infinity, and the size of the S^1 is freely adjustable. By placing a black hole at the origin and dialing the S^1 radius we can thereby interpolate between black holes with 4D and 5D asymptotics. Since the attractor mechanism implies that the BPS entropy is independent of moduli, we conclude that the 4D and 5D black hole entropy formulas are closely related. Higher derivative corrections turn out to introduce an interesting twist to this story.[43] The relation between the 4D and 5D black hole charges is not the naive one expected from the lowest order solutions, but is rather modified due to the fact that higher derivative terms induce delocalized charge densities on the Taub-NUT space. We work out the corrected charge dictionary explicitly.

10.1. *Relation between 4D and 5D BPS equations*

We now show how to relate the BPS equations governing 4D and 5D solutions. At the two-derivative level, we find that the two sets of equations are equivalent. On the other hand, we shall see that this is apparently no longer the case once higher derivative terms are included.

10.1.1. *Two-derivative BPS equations*

The field content of 4D $N = 2$ supergravity consists of the metric, gauge fields a^A, and complex moduli Y^A (we neglect the hypermultiplet fields, which decouple in the context of

BPS black holes). The A label runs over the values $(0, I)$, where I denotes the corresponding 5D label. The extra a^0 vector corresponds to the Kaluza-Klein gauge field that arises upon reduction from 5D to 4D. The action is fixed by the prepotential F, which we take to be of the form arising from compactification on CY_3,

$$F = -\frac{1}{6}\frac{C_{IJK}Y^I Y^J Y^K}{Y^0} \ . \tag{270}$$

See, *e.g.*, Ref. 44.

Supersymmetry fixes the metric and field strengths to be of the form

$$ds_{4d}^2 = e^{2g}(dt + \sigma)^2 - e^{-2g}dx^m dx^m \ ,$$
$$f^A = d[e^{2g}(Y^A + \overline{Y}^A)(dt + \sigma)] + \frac{i}{2}\epsilon_{mn}{}^p \nabla_p (Y^A - \overline{Y}^A) dx^m dx^n \ , \tag{271}$$

with $\sigma = \sigma_m dx^m$, and ϵ_{mnp} is the volume form of $dx^m dx^m$.

The moduli are determined in terms of harmonic functions (h_A, h^A) on $\mathbb{R}^3$ by

$$Y^A - \overline{Y}^A = ih^A \ , \tag{272}$$

$$F_A - \overline{F}_A = ih_A \ , \tag{273}$$

with $F_A = \frac{\partial F}{\partial Y^A}$. We also have

$$i[\overline{Y}^A F_A - \overline{F}_A Y^A] - e^{-2g} = 0 \ , \tag{274}$$

$$h^A \vec{\nabla} h_A - h_A \vec{\nabla} h^A - \vec{\nabla} \times \vec{\sigma} = 0 \ . \tag{275}$$

The above system of equations fixes the form of the general BPS solution. We now review the solution of these equations. Equation (272) is trivially solved as

$$Y^A = \text{Re}Y^A + \frac{i}{2}h^A \ . \tag{276}$$

We next solve (273) as

$$Y^I = -\frac{|Y^0|}{\sqrt{h^0}}x^I + \frac{Y^0}{h^0}h^I \ , \tag{277}$$

where

$$\frac{1}{2}C_{IJK}x^J x^K = h_I + \frac{1}{2}\frac{C_{IJK}h^J h^K}{h^0} \ . \tag{278}$$

Similarly, Y^0 can be found from the equation $F_0 - \overline{F}_0 = ih_0$.

The metric function e^{-2g} is now determined from (274) as

$$e^{-2g} = i[\overline{Y}^A F_A - \overline{F}_A Y^A] = \frac{(h^0)^2 h_0 + h^0 h_I h^I + \frac{1}{3}C_{IJK}h^I h^J h^K}{2\text{Re}Y^0} \ , \tag{279}$$

and σ is determined from (275).

10.1.2. *4D-5D dictionary*

The dictionary between 4D and 5D solutions has been studied in Ref. 41. 5D solutions are related to those in 4D by performing Kaluza-Klein reduction along the x^5 circle, which amounts to writing the 5D solution as

$$ds_{5d}^2 = e^{2\phi}\left(e^{2g}(dt + \hat{\omega})^2 - e^{-2g}dx^m dx^m\right) - e^{-4\phi}(dx^5 + a^0)^2 \ , \tag{280}$$

$$A^I = \Phi^I(dx^5 + a^0) - 2^{-\frac{1}{3}}a^I \ . \tag{281}$$

Comparing with the 5D solutions as discussed in Section 2.3.1, we can read off

$$e^{-4\phi} = \frac{e^{-2U}}{H^0} - e^{4U}\omega_5^2 \ , \tag{282}$$

$$e^{2g} = \frac{e^{2U+2\phi}}{H^0} \ , \tag{283}$$

$$a^0 = \chi - e^{4U+4\phi}\omega_5(dt + \hat{\omega}) \ . \tag{284}$$

To relate the 4D and 5D BPS equations we make the further identifications

$$\begin{aligned}
\sigma &= \hat{\omega} \ , \\
Y^0 &= -\frac{1}{2}\left(e^{2U+2\phi}\omega_5 - i\right)H^0 \ , \\
Y^I &= 2^{-\frac{4}{3}}\left[-e^{-2U+2\phi}M^I + \frac{1}{2}(e^{2U+2\phi}\omega_5 - i)H^I\right] \ , \\
h^I &= -2^{-\frac{4}{3}}H^I \ , \\
h_I &= 2^{-\frac{2}{3}}H_I \ .
\end{aligned} \tag{285}$$

It is then straightforward to check that the 4D and 5D equations are mapped to each under this identification. For example, (273) with $A = I$ and $A = 0$ yield, respectively, (30) and (35). Also, (274) maps to the special geometry constraint (22). This establishes the equivalence of 4D and 5D BPS solutions at the two-derivative level.

10.2. *Higher derivative case*

Above, we demonstrated the equivalence of the two-derivative BPS equations in 4D and 5D. Does this equivalence extend to the higher derivative BPS equations?

Four-derivative corrections are included into the 4D BPS equations via the generalized prepotential

$$F = -\frac{1}{6}\frac{c_{IJK}Y^I Y^J Y^K}{Y^0} - \frac{c_{2I}}{24 \cdot 64}\frac{Y^I}{Y^0}\Upsilon \ . \tag{286}$$

Equations (271)-(273) remain valid, while (274)-(275) are modified to[7]

$$i[\overline{Y}^I F_I - \overline{F}_I Y^I] - e^{-2g} = 128ie^g\vec{\nabla} \cdot [(\vec{\nabla}e^{-g})(F_\Upsilon - \overline{F}_{\overline{\Upsilon}})]$$

$$-32ie^{4g}(\vec{\nabla} \times \vec{\sigma})^2(F_\Upsilon - \overline{F}_\Upsilon)$$
$$-64e^{2g}(\vec{\nabla} \times \vec{\sigma}) \cdot \vec{\nabla}(F_\Upsilon + \overline{F}_\Upsilon) \, , \tag{287}$$

$$H^I\nabla_p H_I - H_I\nabla_p H^I - (\vec{\nabla} \times \vec{\sigma})_p = -128i\nabla^q[\nabla_{[p}(e^{2g}(\vec{\nabla} \times \vec{\sigma})_{q]}(F_\Upsilon - \overline{F}_\Upsilon))]$$
$$-128\nabla^q[2\nabla_{[p}g\nabla_{q]}(F_\Upsilon + \overline{F}_\Upsilon)] \, , \tag{288}$$

where $F_\Upsilon = \frac{\partial F}{\partial \Upsilon}$, and after taking the derivative we are instructed to set

$$\Upsilon = -64(\vec{\nabla}g - \frac{1}{2}ie^{2g}\vec{\nabla} \times \vec{\sigma})^2 \, . \tag{289}$$

We can now try to compare with the four-derivative BPS equations in 5D, as presented in Section 7. Without going into the details, it turns out that if we continue to use the same dictionary as in two-derivative case, then we find that the 4D and 5D BPS equations do not agree. In particular, while the 5D equations are expressed covariantly in terms of the 4D base space, this property is not realized upon writing the 4D BPS equations in 5D language.

Before concluding that the equations are indeed physically different we should consider the possibility of including corrections to the dictionary. A little thought shows that this is unlikely to work. Any such correction would have to involve c_{2I}, since we have already demonstrated agreement when $c_{2I} = 0$. But both the 4D and 5D BPS equations are linear in c_{2I}, and this property will be upset upon introducing a c_{2I} dependent change of variables. In the same vein, we note that both the 4D and 5D field strengths are determined just by supersymmetry, without using the explicit form of the action, and furthermore precisely map into each other under the uncorrected dictionary. Again, this feature will be disturbed by including corrections to the dictionary.

We therefore conclude that the two sets of BPS equations are in fact different. This is actually not so surprising for the reason that it is known that the 4D supergravity action usually used does not include the full set of terms relevant for black hole solutions. This follows from the observation in Ref. 117 that this action gives the wrong result for the entropy of extremal non-BPS equations. On the other hand, the 5D action used here does give the right answer,[117] in accord with the general arguments based on anomalies.[3,24] There is therefore no reason why the BPS equations derived from these actions should agree. Indeed, what is surprising is that the 4D action does manage to give the right answer for the entropy of 4D BPS black holes, even if not for the full geometry. An interesting open question is to determine what terms in the 4D action are missing, and to then verify agreement with the 5D BPS equations.

10.3. *Quantum/string corrections to the 4D/5D connection*

We now turn to the relation between the entropies of four and five dimensional black holes. To illustrate the salient issues we consider the simplest case of electrically charged, non-rotating, 5D black holes, and their 4D analogues. At the two-derivative level the following relation holds[37]

$$S_{5D}(q_I) = S_{4D}(q_I, p^0 = 1) \ . \tag{290}$$

This formula is motivated by placing the 5D black hole at the tip of Taub-NUT. Since Taub-NUT is a unit charge Kaluza-Klein monopole, this yields a 4D black hole carrying magnetic charge $p^0 = 1$. On the other hand, suppose that we sit at a fixed distance from the black hole and then expand the size of the Taub-NUT circle to infinity. Since Taub-NUT looks like $\mathbb{R}^4$ near the origin it is clear that this limiting process gives back the original 5D black hole. Finally, the moduli independence of the entropy yields (290).

The preceding argument contains a hidden assumption, namely that the act of placing the black hole in Taub-NUT does not change its electric charge. But why should this be so? In fact it is not, as was first noticed in Ref. 14 and further studied in Ref. 43. The reason is that higher derivative terms induce a delocalized charge density on the Taub-NUT, so that the charge carried by the 4D black hole is actually that of the 5D black hole plus that of the Taub-NUT.

To expand on this point, let us return to the general solutions of Section 7.6, *i.e.* spinning black holes on a Gibbons-Hawking base. The Maxwell equations led to (197) which demonstrates that the curvature on the base space provides a delocalized source for the gauge field. This effect should be expected simply from the fact that we deal with an action with a $\int A^I \wedge R \wedge R$ Chern-Simons term.

We now proceed to make explicit the relation between the charges.

10.3.1. *Relation between 4D and 5D charges*

Consider a general action of gauge fields in the language of forms

$$S = \frac{1}{4\pi^2} \int_{\mathcal{M}_5} \star_5 \mathcal{L}\left(A^I, F^I\right) \ . \tag{291}$$

The Euler-Lagrange equations of motion are

$$d \star_5 \frac{\partial \mathcal{L}}{\partial F^I} = \star_5 \frac{\partial \mathcal{L}}{\partial A^I} \ . \tag{292}$$

Since the left side is exact, we see that this identifies a divergenceless current

$$j_I = \frac{\partial \mathcal{L}}{\partial A^I} \ . \tag{293}$$

The conserved charge is obtained by integrating $\star_5 j_I$ over a spacelike slice Σ, suitably normalized. Through the equations of motion and Stoke's theorem this can be expressed as an integral over the asymptotic boundary of Σ

$$Q_I = -\frac{1}{4\pi^2} \int_{\partial\Sigma} \star_5 \frac{\partial \mathcal{L}}{\partial F^I} \ , \tag{294}$$

which clearly reproduces the conventional $Q \sim \int \star F$ for the Maxwell action.

For the present case, we consider solutions where the gauge fields fall off sufficiently fast that only the two-derivative terms in the Lagrangian lead to non-zero contributions to the surface integral in (294). Our charge formula is then

$$Q_I = -\frac{1}{2\pi^2} \int_{\partial\Sigma} \left(\frac{1}{2} \mathcal{N}_{IJ} \star_5 F^J + 2M_I \star_5 v \right) \ . \tag{295}$$

For our solutions with timelike supersymmetry, we can identify Σ with the hyperKähler base space.

Now let us compare the charge computations for two distinct solutions, one with a flat $\mathbb{R}^4$ base space and another on Taub-NUT, considering just the non-rotating black hole for simplicity. As mentioned previously, both $\mathbb{R}^4$ and Taub-NUT can be written as

$$ds^2 = (H^0)^{-1}(dx^5 + \vec{\chi} \cdot d\vec{x})^2 + H^0 \left(d\rho^2 + \rho^2 d\Omega_2\right) \ , \tag{296}$$

where

$$H^0 = \eta + \frac{1}{\rho} \ , \tag{297}$$

with $\eta = 0$ for $\mathbb{R}^4$ and $\eta = 1$ for Taub-NUT. The coordinate x^5 is compact with period 4π, and we choose the orientation $\epsilon_{\hat\rho\hat\theta\hat\phi\hat5} = 1$. Using the formulas from Section 7 for the gauge fields and auxiliary field, we see that in Gibbons-Hawking coordinates

$$Q_I = \lim_{\rho\to\infty} \left[-4\rho^2 \partial_\rho \left(e^{-2U} M_I \right) \right] \ , \tag{298}$$

which is independent of the Gibbons-Hawking function H^0.

Now recall the result from the higher-derivative Maxwell equation (204)

$$M_I e^{-2U} - \frac{c_{2I}}{8} (\tilde\nabla U)^2 - \frac{c_{2I}}{24 \cdot 8} \Phi = 1 + \frac{q_I}{4\rho} \ , \tag{299}$$

where Φ is defined in (203) as

$$\tilde{R}^2_{\hat{i}\hat{j}\hat{k}\hat{l}} = \tilde\nabla^2 \Phi \equiv \tilde\nabla^2 \left(2 \frac{(\tilde\nabla H^0)^2}{(H^0)^2} + \sum_i \frac{a_i}{|\vec{x} - \vec{x}_i|} \right) \ . \tag{300}$$

Both $\mathbb{R}^4$ and $p^0 = 1$ Taub-NUT are completely smooth geometries and so there are no singularities in the corresponding $\tilde{R}^2$. On the other hand, (300) has manifestly singular terms unless the a_i are chosen to cancel singularities in the H^0 dependent term. Comparing with (297), we see that smoothness is assured when the a_i are chosen such that

$$\tilde{R}^2_{\hat{i}\hat{j}\hat{k}\hat{l}} = \tilde{\nabla}^2 \left(2\frac{(\tilde{\nabla}H^0)^2}{(H^0)^2} - \frac{2}{\rho} \right) \ . \tag{301}$$

The solution (299) is now fully specified as

$$M_I e^{-2U} - \frac{c_{2I}}{8}(\tilde{\nabla}U)^2 - \frac{c_{2I}}{24 \cdot 4}\left(\frac{(\tilde{\nabla}H^0)^2}{(H^0)^2} - \frac{1}{\rho} \right) = 1 + \frac{q_I}{4\rho} \ . \tag{302}$$

In the absence of stringy corrections, *i.e.* for $c_{2I} = 0$, we have $M_I e^{-2U} = 1 + \frac{q_I}{4\rho}$ which gives $Q_I = q_I$ independent of the base space geometry. However, including these corrections we see that aymptotically[y]

$$M_I e^{-2U} = 1 + \frac{1}{4}\left(q_I - \eta\frac{c_{2I}}{24} \right)\rho^{-1} + O(\rho^{-2}) \ , \tag{303}$$

yielding the asymptotic charge

$$Q_I = q_I - \eta\frac{c_{2I}}{24} \ . \tag{304}$$

The preceding computation tells us that formula (290) gets modified to

$$S_{5D}(q_I) = S_{4D}(q_I - \frac{c_{2I}}{24}, p^0 = 1) \ . \tag{305}$$

An analogous, and more complicated, relation holds in the case of rotating black holes; see Ref. 43 for the details.

An interesting open question is whether (305) is further corrected by terms with even more than four derivatives.

11. Discussion

We conclude this review with a discussion of several open problems and future directions.

11.1. *Black rings*

In the two-derivative gravity theory the most general known supersymmetric black solution with a connected horizon is a black ring, from which black holes and black strings can be

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[y]We are ignoring the $\frac{c_{2I}}{8}(\tilde{\nabla}U)^2$ term since one can check that it falls off too rapidly as $\rho \to \infty$ to contribute to Q_I.

obtained as special cases. We therefore would like to find a black ring solution taking into account the higher derivative corrections considered in this review.

While the full black ring solution is not yet known, most ingredients have been identified. First, the supersymmetry analysis in Section 7.2 was carried out for the general black ring. Subsequently, we determined Gauss' law for the ring solution and also found the modified very special geometry constraint. The only missing ingredient is the equation of motion for the auxiliary two form v, needed to relate the magnetic dipole charges in Θ^I with the angular momentum described by $d\omega$. Once all the pieces are in place we must of course integrate the equations of motion, repeating the steps for obtaining the black ring solution in two-derivative gravity. It is not obvious that the higher derivative equations of motion will be integrable, but it is encouraging that we were able to write Gauss' law for the black ring (193) in a form that is manifestly integrable.

11.2. *Some other approaches*

In this review we have collected our results for black hole entropy obtained from an off-shell supersymmetric action in five dimensions. To put things in perspective, it is useful to compare and contrast our results with those obtained using other actions. The comparison is easiest to make in the case of 5D black strings, or equivalently, 4D black holes with $p^0 = 0$, since this is where the most results are available. To keep matters simple, we also set $q_{I \neq 0} = 0$. These black holes have near horizon geometry $AdS_3 \times S^2$ in 5D, and $AdS_2 \times S^2$ in 4D, the latter obtained by Kaluza-Klein reduction along the 5D string.

We first recall our results for this case. The entropy is given by

$$S = 2\pi \sqrt{\frac{c_L}{6}\left(h_L - \frac{c_L}{24}\right)} + 2\pi \sqrt{\frac{c_R}{6}\left(h_R - \frac{c_R}{24}\right)} \ , \tag{306}$$

where the central charges are

$$c_L = c_{IJK}p^I p^J p^K + c_{2I}p^I \ , \quad c_R = c_{IJK}p^I p^J p^K + \frac{1}{2}c_{2I}p^I \ , \tag{307}$$

and

$$q_0 = h_R - h_L - \frac{c_R - c_L}{24} \ . \tag{308}$$

In general, the result (306) corresponds to a near extremal black hole. An extremal BPS black hole corresponds to ($h_L > \frac{c_L}{24}$, $h_R = \frac{c_R}{24}$), while ($h_L = \frac{c_L}{24}$, $h_R > \frac{c_R}{24}$) yields an extremal non-BPS black hole. In all cases, the entropy formula is in precise agreement with the microscopic entropy counting obtained in Ref. 8. This agreement, and in particular the fact

that terms with more than four derivatives do not contribute, is explained by anomaly based reasoning,[3,24] which, we emphasize, only applies to black holes with a near horizon AdS$_3$ factor. The validity of (306) also requires that one of $h_{L,R}$ be much larger than the central charge, a parameter region which may or may not be realizable after using the freedom to perform duality rotations.[118] Another way to understand why terms with more than four derivatives do not contribute is by noting that all such terms either vanish on the solutions or can be removed by a field redefinition.[59]

Now we turn to the results using other actions. One well known approach consists of using a 4D action that is the off-shell supersymmetric completion of the square of the Riemann tensor.[4−7] This action gives agreement with (306), and hence with the microscopic entropy, for extremal BPS black holes, but disagreement otherwise. As discussed in Ref. 55, the likely reason for the failure of this action in the non-BPS case is that it is just the minimally supersymmetric action for the gravity multiplet, and does not take into account the extra terms allowed by the presence of vector multiplets. Given that this action is apparently incomplete, it is somewhat surprising that it nevertheless gives the correct results in the BPS case.

Another action that has attracted significant attention is the six-dimensional action[119,117]

$$
\mathcal{L}_{\text{two−loop}} = 2e^{-2\Phi} \left[R_{KLMN} R^{KLMN} - \frac{1}{2} R_{KLMN} H_P{}^{KL} H^{PMN} \right.
$$
$$
\left. - \frac{1}{8} H_K{}^{MN} H^{KPQ} H_{PQ}{}^L + \frac{1}{24} H_{KLM} H_{PQ}{}^K H_R{}^{LP} H^{RMQ} \right] , \qquad (309)
$$

derived from the condition of conformal invariance for the string theory σ-model at two-loop order.[120] This action corresponds to the bosonic four derivative terms of an on-shell supersymmetric action. That is to say, the action truncated at four derivatives is only supersymmetric up to five derivative terms, and the algebra only closes on-shell. We therefore expect this action to give the right correction to first subleading order in the charges, but not beyond. Indeed, it has been established[117] that (309) agrees with (306) to first subleading order, for both BPS and non-BPS extremal black holes.

To close this discussion, let us note that besides these 5D black strings / 4D black holes, we have also obtained results for the entropies of other black objects, such as 5D spinning black holes. In general, these black holes do not have an AdS$_3$ factor. Consequently, we expect that our entropy formulas are valid to first subleading order in the charges, but we have no argument that they remain valid at higher orders. At present, this issue is difficult

to study, since we lack a microscopic description of such black holes (we only have such a description for $N = 4$ black holes, in which case we can find an AdS$_3$ factor by making a duality rotation.)

11.3. *The Gauss-Bonnet action*

Yet another approach to higher derivative corrections is to consider the Gauss-Bonnet type action

$$\mathcal{L}_{\mathrm{GB}} = \frac{1}{192} c_{2I} M^I \left(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2 \right) . \tag{310}$$

The overall coefficient is determined by the coefficient of the $R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}$ term obtained from the string S-matrix. Our action (79) is the supersymmetric completion of that term, with the coefficient of the protected gauge-gravity Chern-Simons term coming out the same as when it is determined using anomalies. In (310) the *Riemann squared* term has been completed instead by forming the Gauss-Bonnet invariant.

Including only the Gauss-Bonnet term is not compatible with supersymmetry, and so *a priori*, there is no reason why this approach will give the correct black hole entropy. Nevertheless, one can check that the Gauss-Bonnet action (310) has the same on-shell value as the supersymmetric action (79) when evaluated on the magnetic attractor solution (discussed Section 5.2). This agreement extends to the non-rotating electric attractor (discussed in Section 5.3), but not to rotating attractors (Section 5.3.1). Combining with the diffeomorphism anomaly from the gauge-gravity Chern-Simons term one can therefore recover both the left and right central charges of magnetic strings. In four dimensions the corresponding term gives the correct result for extremal BPS black holes,[8] but it fails in the non-BPS case.[117]

The successes of the Gauss-Bonnet combination suggest that it may allow a supersymmetric completion. However, until that has been established clearly one must be cautious when using this term, because there is no clear argument that explains the agreements that have been found, and also the success is not universal.

11.4. *Higher dimensions*

In this review we have focussed on higher derivative corrections in five dimensions. As we have discussed, many of the same issues have already been confronted in four dimensions. In Section 10 we discussed the interrelation between four and five dimensions, including some unresolved puzzles in that regard.

An interesting future direction is to consider dimensions six and higher. Ultimately we would like to understand how higher derivative corrections in ten or eleven dimensions modify various solutions. For example, many of the standard brane solutions are singular in the lowest order approximation, but it could be that they are regular once corrections are taken into account. Criteria for determining in which cases this hypothesis is valid have not yet been established. Since we have found a smooth solution for the heterotic string in five dimensions, it seems natural to look for similar smooth solutions for heterotic or type II strings in ten dimensions. On general grounds, we would expect these to have a near horizon $AdS_3 \times S^7$ geometry.

A useful intermediate step would be to understand higher derivative corrections in six dimensions. At one level, six dimensions is expected to be a relatively straightforward lift of the five dimensional examples with $AdS_3 \times S^2$ and $AdS_2 \times S^3$ near horizon geometries. On the other hand, in six dimensions it is not generally possible to write Lorentz invariant actions, because of self-duality conditions on tensor fields. Related to this, anomalies in six dimensions have a much richer structure. These complications introduce new features which would be interesting to develop.

Acknowledgments

Appendix A. Conventions

We briefly summarize our conventions which are largely that of Refs. 14 and 43. Latin indices $a, b, \ldots$ denote tangent space indices and curved space-time indices are denoted by Greek indices $\mu, \nu, \ldots$. The metric signature is mostly minus, $\eta_{ab} = \mathrm{diag}(+, -, -, -, -)$. Covariant derivatives of spinors are defined as

$$\mathcal{D}_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} \; , \tag{A.1}$$

where ω^{ab} are the spin-connection one forms related to the vielbein through the Cartan equation

$$de^a + \omega^a{}_b \wedge e^b = 0 \; . \tag{A.2}$$

Our convention for the two-form curvature is

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d \ . \tag{A.3}$$

The scalar curvature is then, *e.g.*, $R = \frac{p(p-1)}{\ell_A^2} - \frac{q(q-1)}{\ell_S^2}$ for $\mathrm{AdS}_p \times S^q$. The Weyl tensor is given by

$$C_{abcd} = R_{abcd} - \frac{2}{3}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{1}{6}g_{a[c}g_{d]b}R \ . \tag{A.4}$$

Gamma-matrices satisfy the usual Clifford algebra

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} \ . \tag{A.5}$$

Totally anti-symmetric products of p - gamma-matrices are denoted by

$$\gamma_{a_1 \cdots a_p} \ , \tag{A.6}$$

and is normalized such that $\gamma_{abcde} = \varepsilon_{abcde}$, where $\varepsilon_{01234} = 1$. The operation $\gamma \cdot \alpha$, where $\alpha_{a_1 \cdots a_p}$ is a p-form, is understood as

$$\gamma \cdot \alpha = \gamma^{a_1 \cdots a_p} \alpha_{a_1 \cdots a_p} \ . \tag{A.7}$$

Finally, we take $G_5 = \frac{\pi}{4}$ and measure moduli in units of $2\pi\ell_{11}$. In these units the charges are quantized (for review see Ref. 46).

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