ELSEVIER

Nuclear Physics B 500 (1997) 75-93

NUCLEAR PHYSICS B

Black holes and critical points in moduli space

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Received 17 March 1997; accepted 15 May 1997

Abstract

We study the stabilization of scalars near a supersymmetric black hole horizon using the equation of motion of a particle moving in a potential and background metric. When the relevant 4-dimensional theory is described by special geometry, the generic properties of the critical points of this potential can be studied. We find that the extremal value of the central charge provides the minimal value of the BPS mass and of the potential under the condition that the moduli space metric is positive at the critical point. This is a property of a regular special geometry. We also study the critical points in all $N \geqslant 2$ supersymmetric theories. We relate these ideas to the Weinhold and Ruppeiner metrics introduced in the geometric approach to thermodynamics and used for the study of critical phenomena. © 1997 Elsevier Science B.V.

PACS: 04.70.-s; 04.70.Dy; 04.65.+e; 05.70.Jk

1. Introduction

In this paper we intend to tie together some recent (and not so recent) work on 4-dimensional black holes in $N = 2$ ungauged supergravity theories [1-7]. These theories have *two types of geometries*: space-time geometry and moduli space geometry, so-called special geometry [8-10] in the space of the scalar fields of the theory. Various properties of space-time singularities including black holes have been studied for a long time. Much less is known about the singularities of the moduli space. In view of the recent understanding that there is a web of connections between different versions of string theories and supergravities, one can view the study of these two types of geometries

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PII S0550-3213(97)00324-6

as a useful tool for clarifying such connections. An interesting example of an interplay between the two types of singularities is provided by the massless black holes [11]. Such solutions have been constructed and studied before [12,13] in the heterotic string theory and have found to have naked singularities [13]. More recently these solutions have been reexamined with account taken of the first loop corrections of the heterotic string [14]. These corrections modify the prepotential of $N = 2$ supergravity model from $STU + aU^3$. The massless black holes with some charges negative and some charges positive have the following time components of the metric

$$g_{tt}^2 = 4\left(h_0 + \frac{q_0}{r}\right)\left(\left(h^1 - \frac{p^1}{r}\right)\left(h^2 - \frac{p^2}{r}\right)\left(h^3 + \frac{p^3}{r}\right) + a\left(h^3 + \frac{p^3}{r}\right)^3\right). \quad (1)$$

The classical solution ($a = 0$) has some naked singularity which makes the black hole repulsive to all matter [13]. In the internal space this singularity corresponds to a vanishing cycle. If one includes the quantum corrections one may remove the naked singularities from the space-time by a proper choice of the parameters, still keeping the mass vanishing. The quantum correction proportional to a seem to act as a regulator which removes some of the singularities of the space-time metric. However, by looking into the scalar metric in the moduli space $g_{k\bar{k}} = \partial_k \partial_{\bar{k}} K(z, \bar{z})$ one can find that some components of the scalar metric given by the second derivative of the Kähler potential become negative, at least for $a = \frac{1}{3}$ which is the actual number coming from the first loop calculation in heterotic string theory. This signals that the moduli space geometry becomes singular as the price for having the space-time geometry singularity free. Note that the change in sign of the scalar metric without a change of the signature of the space-time means that the Lagrangian has the wrong sign for the kinetic term for scalars. Moreover, the fact that the sign of the scalar metric changes from the positive to the negative one means that somewhere in the space-time the metric vanishes and the special geometry is singular.

In general it may be useful to study the properties of these two geometries together. The purpose of this paper is to set up the relevant connections between space-time and special geometry. In particular this will allow us to establish the properties of the critical points of the central charge in supersymmetric theories. We will find out when it is a minimum, when it is a maximum and whether one should expect the uniqueness or non-uniqueness of the critical points. Without the use of the special geometry in the moduli space one can only try to address these issues on a case by case basis. However, special geometry will allow us to have a clear answer to all these problems.

The generalization of this study to the higher supersymmetries $N > 2$ is also possible using the recently developed new formulation of extended supergravities with built-in symplectic structure [15]. The basic difference between $N = 2$ and $N > 2$ is in the structure of the moduli space: for $N = 2$ the scalar manifold in not necessarily a coset manifold whereas for $N > 2$ it is a coset manifold.

Upon identifying the second derivative of the BPS mass and of the black hole potential with the metric on the moduli space of $N = 2$ theory we have realized the deep connection of this construction to the geometric approach to thermodynamics where the Weinhold and Ruppeiner metrics were used for the study of critical phenomena [16].

In Section 2 we mainly review some work on black holes and 1-dimensional geodesic motion which was introduced earlier [1] and adapt this work with the purpose of relating it to the special geometry. In Section 3 we focus on extreme and double-extreme configurations and find that equations of motion of a particle in a potential together with regularity requirement are sufficient to explain the attractor mechanism for the scalars near the horizon which was understood before [3,4] with the help of supersymmetry. Section 4 contains new results: we identify the potential of $N = 2$ theory with the first symplectic invariant of special geometry which is homogeneous and of degree 2 in electric and magnetic charges. The critical points of the central charge are found to be also the critical points of the potential. We find the correlation between the sign of the second derivative of the potential at the critical point and the sign of the scalar metric at a given critical point. This relates the singularities of the moduli space to the critical behavior of the potentials defining the motion in space-time. In Section 5 we study the critical points of the BPS mass and of the black holes potential in arbitrary extended supergravity with or without matter and calculate the second derivatives at the critical points. Section 6 is devoted to the connection to the third geometry used in the geometric approach to thermodynamics.

2. Geodesic action with a constraint

The class of theories we wish to consider has Lagrangian

$$-\frac{R}{2}+\tfrac{1}{2}G_{ab}\partial_\mu\phi^a\partial_\nu\phi^b g^{\mu\nu}-\tfrac{1}{4}\mu_{\Lambda\Sigma}\mathcal{F}^\Lambda_{\mu\nu}\mathcal{F}^\Sigma_{\lambda\rho}g^{\mu\lambda}g^{\nu\rho}-\tfrac{1}{4}\nu_{\Lambda\Sigma}\mathcal{F}^\Lambda_{\mu\nu}{}^*\mathcal{F}^\Sigma_{\lambda\rho}g^{\mu\lambda}g^{\nu\rho}. \tag{2}$$

We restrict our attention to static solutions and make the ansatz for the metric in the form

$$ds^2=e^{2U}dt^2-e^{-2U}\gamma_{mn}dx^m dx^n. \tag{3}$$

The effective 3-dimensional Lagrangian from which the field equations can be derived takes the form

$$\tfrac{1}{2}R[\gamma_{mn}]-\tfrac{1}{2}\gamma^{mn}\partial_m\hat{\phi}^a\partial_n\hat{\phi}^b\hat{G}_{ab}, \tag{4}$$

where the "hatted" scalar fields include in addition to the scalar fields ϕ^a of the 4-dimensional theories also the function U defining the metric as well as the electrostatic ψ^Λ and magnetic static χ_Λ potentials

$$\hat{\phi}^a=(U,\phi^a,\psi^\Lambda,\chi_\Lambda). \tag{5}$$

The metric $\hat{G}$ of the enlarged scalar manifold is independent of ψ^Λ and χ_Λ as required by gauge independence. Now consider spherically symmetric solutions. We specify the ansatz

$$\gamma_{mn}dx^m dx^n = \frac{c^4 d\tau^2}{\sinh^4 c\tau} + \frac{c^2}{\sinh^2 c\tau}(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{6}$$

and discover that the effective 1-dimensional Lagrangian from which the radial equations may be derived is a pure geodesic action:

$$\hat{G}_{ab}\frac{d\hat{\phi}^a}{d\tau}\frac{d\hat{\phi}^b}{d\tau}\,, \tag{7}$$

together with the constraint that

$$\hat{G}_{ab}\frac{d\hat{\phi}^a}{d\tau}\frac{d\hat{\phi}^b}{d\tau} = c^2\,. \tag{8}$$

Because $\hat{G}$ is independent of ψ^Λ and χ_Λ due to gauge invariance, we have constants of motion

$$p^\Lambda = \hat{G}^{\Lambda\Sigma}\frac{d\hat{\chi}_\Sigma}{d\tau}\,, \qquad q_\Lambda = \hat{G}_{\Lambda\Sigma}\frac{d\hat{\psi}^\Sigma}{d\tau}\,. \tag{9}$$

We may now replace the pure geodesic action by the

$$\left(\frac{dU}{d\tau}\right)^2 + G_{ab}\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau} + e^{2U}V(\phi,(p,q)) \tag{10}$$

and the constraint by

$$\left(\frac{dU}{d\tau}\right)^2 + G_{ab}\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau} - e^{2U}V(\phi,(p,q)) = c^2\,. \tag{11}$$

Here $V(\phi,(p,q))$ is a particular potential function constructed from the scalar dependent positive definite couplings $\mu_{\Lambda\Sigma}$ and $\nu_{\Lambda\Sigma}$ of vector fields and $c^2 = 2ST$, where S is the entropy and T is the temperature of the black hole [2]. Specifically

$$V = \tfrac{1}{2}(p,q)\mathcal{M}\begin{pmatrix} p \\ q \end{pmatrix}, \tag{12}$$

where

$$\mathcal{M} = \begin{vmatrix} \mu + \nu\mu^{-1}\nu & \nu\mu^{-1} \\ \mu^{-1}\nu & \mu^{-1} \end{vmatrix}. \tag{13}$$

It is clear that the properties of the black holes in theories of this type are governed entirely by the metric G_{ab} on the scalars and the potential function $V(\phi,(p,q))$.

3. Extreme and double-extreme holes

We begin by considering extreme holes when $c^2 = 2ST = 0$ and $\gamma_{mn}dx^m dx^n$ is given by

$$\gamma_{mn}dx^m dx^n = \frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{14}$$

The geometry becomes

$$ds^2 = e^{2U}dt^2 - e^{-2U}\left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}(d\theta^2 + \sin^2\theta d\varphi^2)\right]\,. \tag{15}$$

Evidently, to obtain a finite area solution we must have that

$$e^{-2U} \longrightarrow \left(\frac{A}{4\pi}\right)\tau^2 \qquad \text{as} \quad \tau \to -\infty\,. \tag{16}$$

We also require that this expression for our solution is not infinite near the horizon,

$$G_{ab}\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau}e^{2U}\tau^4 < \infty\,. \tag{17}$$

This leads to

$$G_{ab}\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau}\left(\frac{4\pi}{A}\right)\tau^2 \to X^2 \qquad \text{as} \quad \tau \to -\infty\,. \tag{18}$$

Now we can substitute this into the constraint and we get

$$\frac{1}{\tau^2} + \left(\frac{X^2 A}{\tau^2 4\pi}\right) - \frac{4\pi}{A}\frac{V(p,q,\phi_h)}{\tau^2} = 0\,. \tag{19}$$

This leads to

$$A \leqslant 4\pi V(p,q,\phi_h)\,. \tag{20}$$

The near horizon geometry becomes equal to

$$ds^2 = \frac{4\pi}{A\tau^2}dt^2 - \left(\frac{A}{4\pi}\right)\left[\frac{d\tau^2}{\tau^2} + (d\theta^2 + \sin^2\theta d\varphi^2)\right]\,. \tag{21}$$

It is useful to change the variables as follows:

$$\rho = -\frac{1}{\tau}\,, \qquad \omega = \log\rho \tag{22}$$

and bring the near horizon geometry to the form of the product space $\mathrm{AdS}_2 \times \mathrm{S}^2$:

$$ds^2 = \frac{4\pi}{A}e^{2\omega}dt^2 - \left(\frac{A}{4\pi}\right)d\omega^2 - \left(\frac{A}{4\pi}\right)(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{23}$$

In these coordinates the value of the derivatives of the moduli as the function of ω enters in the term $G_{ab}\partial_\mu\phi^a\partial_\nu\phi^b g^{\mu\nu}$ as follows:

$$G_{ab}\frac{d\phi^a}{d\omega}\frac{d\phi^b}{d\omega}\left(\frac{4\pi}{A}\right) \to X^2 \qquad \text{as} \quad \omega \to \infty\,. \tag{24}$$

Now it is obvious that only $X^2 = 0$ is consistent with the requirement that the moduli do not blow up near the horizon. Indeed, if

$$\frac{d\phi^a}{d\omega} = \text{const.} \qquad \text{as} \quad \omega \to \infty \tag{25}$$

and the moduli near the horizon have to be linear in ω they would not be finite near the horizon. Thus we have proved that for $c = 0$ extremal black holes from the single requirement that the geometry as well as moduli are regular near the horizon in the suitably chosen coordinates it follows that the area of the horizon equals the value of the potential near horizon,

$$\frac{A}{4\pi} = V(p, q, \phi_h)\,. \tag{26}$$

This property near the horizon applies both to extremal and double-extremal black holes. Double-extremal black holes [5] have the constant moduli, so for them the term $G_{ab}\partial_\mu\phi^a\partial_\nu\phi^b g^{\mu\nu}$ vanishes everywhere and the equality (26) between the area of the horizon and the value of the potential near the horizon follows from the constraint equation (11) immediately. Thus we have also shown that the area of the horizon of extreme black holes coincides with the area of the horizon of the double-extreme black holes with the same values of charges and is given by the value of the potential

$$A_{\text{extr}} = A_{\text{double-extr}} = 4\pi V(p, q, \phi_h)\,. \tag{27}$$

This universality was understood before [3,4] as a consequence of supersymmetry. Here the universal properties of the area of the horizon of the extremal black holes are deduced only from the requirement of the regularity of the configuration.

The equation of motion for ϕ^a is

$$\frac{D\phi^a}{D\tau^2} = \tfrac{1}{2}\frac{\partial V}{\partial\phi^a} e^{2U}\,. \tag{28}$$

Near the horizon, taking into account that $d\phi^a/d\omega = \tau d\phi^a/d\tau = 0$, we get

$$\frac{d^2\phi^a}{d\tau^2} \to \tfrac{1}{2}\frac{\partial V}{\partial\phi^a}\left(\frac{4\pi}{\tau^2 A}\right)\,. \tag{29}$$

The solution of this equation near the horizon is

$$\phi^a = \left(\frac{2\pi}{A}\right)\frac{\partial V}{\partial\phi^a}\log\tau + \phi^a_h\,. \tag{30}$$

Above we have omitted a term linear in τ since it will lead to the singular dilaton at the horizon. Eq. (30) shows that unless the derivative of the scalar potential over the scalar field vanishes near the horizon, one can not have a regular value of the scalars near the horizon. Thus

$$\left(\frac{\partial V}{\partial\phi^a}\right)_h = 0\,. \tag{31}$$

Finally we note that the behavior at infinity, at $\tau \to 0$, is $U \to M\tau$ leads to the following constraint between the black hole mass, scalar charges, scalar metric and the potential, valid regardless of whether the hole is extreme or not:

$$M^2 + G_{ab}\Sigma^a\Sigma^b - V(p,q,\phi^a_\infty) = c^2 , \tag{32}$$

where ϕ^a_∞ are the values of the scalars at spatial infinity and the scalar charges are defined via the expansion of the scalars at infinity. Of course in the extreme limit we set $c = 0$. The double-extreme black holes have a vanishing scalar charge and the constant fixed value of scalars everywhere:

$$M^2 = V(p,q,\phi^a_{\text{fix}}) , \tag{33}$$

$$\frac{A}{4\pi} = V(p,q,\phi^a_{\text{fix}}) \tag{34}$$

and the fixed value of the scalars is defined by the extremization of the potential

$$\left(\frac{\partial V(\phi,p,q)}{\partial \phi^a}\right)_{\text{fix}} = 0 . \tag{35}$$

Note that in this description of the extremal and double-extremal black holes there was no use of supergravity and/or supersymmetry. We have used the bosonic field equations of the theory and the requirement that the extremal configuration is regular near the horizon, including the regularity of scalars. In the next section we will specify this study to the case of $N = 2$ supergravity and special geometry.

We have arrived at the following picture. We may associate with each critical point ϕ^a_{fix} of the potential $V(\phi,p,q)$ on the manifold of scalars $\mathcal{M}_\phi$ a supersymmetric Bertotti–Robinson vacuum state. Extreme black hole solutions correspond to dynamical trajectories in the moduli space $\mathcal{M}_\phi$ starting from the point ϕ^a_∞ at spatial infinity and ending on a critical point ϕ^a_{fix}. Double-extreme holes with frozen moduli correspond to trivial point trajectories. One could also consider trajectories running between two different critical points but these would not correspond to asymptotically flat solutions. Thus the extreme solutions may be said to spatially interpolate between different vacua. Finding explicit trajectories which effect this interpolation and which satisfy the second-order dynamical equations of motion is in general quite difficult, although a number of solutions are known. However, we shall see shortly that using special geometry one is able to reduce this problem to the easier one of finding the solutions of a set of first-order differential describing the steepest descent curves of another potential function whose physical significance is that it determines the central charge.

4. Critical points of the central charge and of the potential in special geometry

Now we consider the special case for which the scalars field manifold $\mathcal{M}_4$ is a Kähler manifold with complex coordinates z^i and Kähler potential K so that

$$G_{ab}d\phi^a d\phi^b = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^n} dz^i d\bar{z}^n \,. \tag{36}$$

The bosonic part of the action of $N = 2$ supergravity interacting with some number of vector multiplets is [4]

$$-\frac{R}{2} + G_{i\bar{j}}\partial_\mu z^i \partial_\nu \bar{z}^{\bar{j}} g^{\mu\nu} + \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{F}^{\Sigma}_{\lambda\rho}g^{\mu\lambda}g^{\nu\rho} + \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}{}^*\mathcal{F}^{\Sigma}_{\lambda\rho}g^{\mu\lambda}g^{\nu\rho} \,, \tag{37}$$

Here the positive definite metric $G_{i\bar{j}}$ on the scalar manifold as well as scalar dependent negative definite vector couplings $\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}$ and $\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}$ can be derived from the prepotential or from the symplectic section which defines a particular $N = 2$ theory. The symplectic invariant I_1 of the special geometry constructed in [10]

$$\begin{aligned} I_1 &= |Z(z,p,q)|^2 + |D_i Z(z,p,q)|^2 \\ &= -\tfrac{1}{2}(p,q) \begin{vmatrix} \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}\mathrm{Im}\mathcal{N}^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}\mathrm{Im}\mathcal{N}^{-1} \\ -\mathrm{Im}\mathcal{N}^{-1}\mathrm{Re}\mathcal{N} & \mathrm{Im}\mathcal{N}^{-1} \end{vmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \end{aligned} \tag{38}$$

can be identified with the potential

$$V(p,q,\phi^a) = I_1 = |Z(z,p,q)|^2 + |D_i Z(z,p,q)|^2 \,. \tag{39}$$

From Eqs. (12), (13) of the present paper and from Eq. (56) of Ref. [10] it follows that the identification requires that

$$\nu + i\mu = -\mathcal{N} = -\mathrm{Re}\mathcal{N} - i\mathrm{Im}\mathcal{N} \,. \tag{40}$$

Here Z is the central charge [9], the charge of the graviphoton in $N = 2$ supergravity and $D_i Z$ is the Kähler covariant derivative of the central charge:

$$Z(z,\bar{z},q,p) = e^{K(z,\bar{z})/2}(X^\Lambda(z)q_\Lambda - F_\Lambda(z)\,p^\Lambda) = (L^\Lambda q_\Lambda - M_\Lambda p^\Lambda) \,. \tag{41}$$

We will use the abbreviation

$$\left|\frac{dz}{d\tau}\right|^2 = G_{ab}\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^n}\frac{dz^i}{d\tau}\frac{d\bar{z}^n}{d\tau} \,. \tag{42}$$

The first symplectic invariant of special geometry I_1 is positive definite. Our 1-dimensional Lagrangian is

$$\begin{aligned} \mathcal{L}\left(U(\tau), z^i(\tau), \bar{z}^i(\tau)\right) = \left(\frac{dU}{d\tau}\right)^2 + \left|\frac{dz}{d\tau}\right|^2 \\ + e^{2U}\left(|Z(z,p,q)|^2 + |D_i Z(z,p,q)|^2\right) . \end{aligned} \tag{43}$$

The constraint becomes

$$\left(\frac{dU}{d\tau}\right)^2 + \left|\frac{dz}{d\tau}\right|^2 - e^{2U}\left(|Z(z,p,q)|^2 + |D_i Z(z,p,q)|^2\right) = c^2 \,. \tag{44}$$

[4] Here we follow notation of [5] where the black holes were studied in the context of the special geometry. In this context, in contrast to Eq. (2), the vector field strength has an additional factor 1/2.

At infinity, at $\tau \to 0$, $U \to M\tau$ we get

$$M^2(z_\infty, \bar{z}_\infty, p, q) - |Z(z_\infty, p, q)|^2 = c^2 . \tag{45}$$

The BPS configuration has the mass equal to the central charge in supersymmetric theories

$$M^2(z_\infty, \bar{z}_\infty, p, q) - |Z(z_\infty, p, q)|^2 , \qquad c = 0 \tag{46}$$

since the second and the fourth terms in the l.h.s. of Eq. (44) cancel at $\tau \to 0$ and $U \to M\tau$.

By using some properties of special geometry and of the central charge and its covariant derivatives one can rewrite the action as

$$\mathcal{L} = \left(\frac{dU}{d\tau} \pm e^U |Z|\right)^2 + \left|\frac{dz^i}{d\tau} \pm e^U G^{i\bar{k}} \bar{D}_{\bar{k}} \bar{Z}\right|^2 \pm 2\frac{d}{d\tau}\left(e^U |Z|\right) . \tag{47}$$

Thus we may solve the second-order equations by making the ansatz that the following first-order equations hold: [5]

$$\frac{dU}{d\tau} = e^U |Z| , \tag{48}$$

$$\frac{dz^i}{d\tau} = e^U G^{i\bar{k}} \bar{D}_{\bar{k}} |Z| . \tag{49}$$

These first-order equations immediately give (by evaluating at infinity)

$$M = |Z|(z_0, p, q) , \tag{50}$$

$$\Sigma^i = G^{i\bar{k}} \bar{D}_{\bar{k}} \bar{Z} . \tag{51}$$

From the behavior at the horizon $\tau \to \infty$ we get

$$\left(\frac{A}{4\pi}\right)^{1/2} = |Z|(z_h) , \tag{52}$$

$$D^i Z(z_h) = 0 , \tag{53}$$

where also

$$\partial_i V = \partial_i \left(|Z(z)|^2 + |D_i Z(z)|^2\right) . \tag{54}$$

Thus we have recovered and specified in this framework of special geometry the equations obtained earlier. To study the critical points of the potential we need a few identities from special geometry [8–10]:

$$D_i D_j Z = i C_{ijk} G^{k\bar{k}} \bar{D}_{\bar{k}} \bar{Z} , \tag{55}$$

$$D_i \bar{D}_{\bar{j}} \bar{Z} = G^{i\bar{j}} \bar{Z} , \tag{56}$$

$$\bar{D}_{\bar{m}} Z = 0 . \tag{57}$$

[5] These first-order equations were derived using supersymmetry in [3,17]. The derivation here is new and does not use supersymmetry.

Here C_{ijk} is a completely symmetric covariantly holomorphic tensor.

The critical points of the potential coincide with the critical point of the central charge. Using the identities above one gets

$$\partial_i V = 2(D_i Z)\bar{Z} + iC_{ijk}G^{j\bar{m}}G^{k\bar{k}}\bar{D}_{\bar{m}}\bar{Z}\bar{D}_{\bar{k}}\bar{Z}. \tag{58}$$

Thus indeed

$$D_i Z = \bar{D}_{\bar{k}}\bar{Z} = 0 \quad \Longrightarrow \quad \partial_i V = \bar{\partial}_{\bar{i}} V = 0. \tag{59}$$

In what follows we will address the following problem: when is the extremum of the central charge and of the potential a minimum and when is it a maximum. Note that the second covariant derivative of the moduli of the central charge at the critical point coincides with the partial (non-covariant) second derivative. We have to calculate

$$D_i D_j |Z| = \partial_i \partial_j |Z|, \qquad \bar{D}_{\bar{i}} D_j |Z| = \bar{\partial}_{\bar{i}} \partial_j |Z| \quad \text{at} \quad D_i|Z| = \partial_i |Z| = 0 \tag{60}$$

and the conjugate to this. We start with the second derivative of the moduli of the central charge and use some identities of special geometry,

$$D_i D_j |Z| = -\tfrac{1}{4}|Z|^{-3}(\bar{Z})^2 D_i Z D_j Z + \tfrac{1}{2}|Z|^{-1} iC_{ijk}G^{k\bar{k}}\bar{D}_{\bar{k}}\bar{Z}. \tag{61}$$

It follows that at the critical point

$$\partial_i \partial_j |Z| = \bar{\partial}_{\bar{i}} \bar{\partial}_{\bar{j}} |Z| = 0. \tag{62}$$

The mixed derivatives are

$$\bar{D}_{\bar{i}} D_j |Z| = \tfrac{1}{4}|Z|^{-1}\bar{D}_{\bar{i}}\bar{Z} D_j Z + \tfrac{1}{2}G_{\bar{i}j}|Z|. \tag{63}$$

It follows that at the critical point we get

$$\left(\bar{\partial}_i \partial_j |Z|\right)_{\text{cr}} = \tfrac{1}{2}G_{\bar{i}j}|Z|_{\text{cr}}. \tag{64}$$

This means that whenever the metric is positive in the moduli space at the critical point, the BPS mass reaches its minimum. However, if the scalar metric is singular and changes the sign the connection established above only signals that the BPS mass at the critical point reaches the maximum outside the range of validity of regular special geometry. In general, when the metric changes the sign we have some sort of a phase transition and there is a breakdown of the effective Lagrangian unless new massless states appear.

Let us now proceed with the evaluation of the second derivative of the potential V at the critical point where the value of the potential is given by the square of the central charge at the critical point of the central charge,

$$\bar{\partial}_{\bar{i}} V = \partial_i V = 0, \qquad V_{\text{cr}} = |Z_{\text{cr}}|^2. \tag{65}$$

Upon some extensive use of the identities of special geometry we conclude that at the critical points of the potential which are simultaneously the critical points of the central charge we get

$$D_i D_j V = 0 \quad \text{at} \quad D_i Z = \partial_i |Z| = 0 \tag{66}$$

and

$$\left(\bar{D}_{\bar{i}} D_j V\right)_{\rm cr} = \left(\bar{\partial}_{\bar{i}} \partial_j V\right)_{\rm cr} = 2G_{\bar{i}j} V_{\rm cr} . \tag{67}$$

Thus again the sign of the second derivative of the potential is defined by the sign of the metric on the moduli space at the critical point where the central charge and the potential have vanishing derivatives.

Now we recall that the extremal condition for the central charge was brought to an equivalent form of the stabilization equation in [4,6] under the condition that the special geometry is not singular. We found that all moduli from the vector multiplets become functions of electric and magnetic charges. The stabilization equation of special geometry is [4,6]

$$\begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \mathrm{Re} \begin{pmatrix} 2i\bar{Z}L^\Lambda \\ 2i\bar{Z}M_\Lambda \end{pmatrix} \equiv \mathrm{Re} \begin{pmatrix} 2iY^I \\ 2iF_I(Y) \end{pmatrix} . \tag{68}$$

The fixed values of the moduli of special geometry in terms of electric and magnetic charges has been found in many examples before [3–6]. They correspond to the fixed values to which scalar fields are attracted near the black hole horizon. All known examples solve both the extremality condition of the central charge $\partial_i|Z| = 0$ and the stabilization equation (68). In fact they have been found mostly by solving the easier equation: the stabilization equation (68):

$$(z^i)_{\rm cr} = z^i(p, q) . \tag{69}$$

For many of these solutions the entropy of $N = 2$ black holes has been understood from the microscopic point of view [7] via the counting of the string or M-theory states.

The issue of the uniqueness of the critical points has not been studied before, we have mainly focused on the goal of finding critical points. Now that we have studied the second derivatives of the central charge and of the potential at the critical points one can address the issue of the uniqueness of the critical points.[6] Under the assumption that we limit ourself exclusively to study only the potentials and their critical points in the range of applicability of special geometry (the scalar metric is strictly positive) we may expect that the minimum of the potential, which is also the minimum of the BPS mass, is unique. Indeed we have proved that the second derivative at the critical point is positive when the scalar metric is positive and the critical point is a minimum. This concerns any critical point in this class and therefore it has to be unique at least for the continuous branch of the potential. However, if we were to relax the condition that the scalar metric is positive we might find some disconnected branches of the potential exhibiting some maxima and various other phenomena. We refer the reader to the 5d case [18] where we present particular examples of such situations. In the 4d context the analogous investigation would amount to studying all possible branches of the potentials

[6] The detailed analysis of the uniqueness issue of the critical points (with particular examples) has been performed in the context of the 5-dimensional very special geometry in [18]. Here we outline how this analysis may be extended to 4-dimensional special geometry.

for various interesting theories and calculating the value of the scalar metric at the critical points where it will become a particular function of charges, according to the stabilization equation (68) (in case of a negative scalar metric the use of the stability equation (68) may be questionable; rather one may rely on the direct solutions of extremality condition $\partial_i|Z| = 0$ for exhibiting the critical point).

5. Critical points of moduli space of extended supergravities

Here we will use the recent geometric formulation of extended supergravities in which the duality symmetries of the theory are manifest [15]. Below we will present the minimal amount of information on this which will allow us to describe the critical points of these theories. The reader is referred for the details of the new construction to [15].

All $d = 4, N > 2$ supergravities have scalar fields whose kinetic Lagrangian is described by sigma models of the form G/H. Here G is a non-compact group acting as an isometry group on the scalar manifold while H is the isotropy group. For $N \leqslant 4$ a supergravity can be coupled to matter. For such cases the isotropy group is given by a direct product $H = H_{\mathrm{Aut}} \otimes H_{\mathrm{matter}}$ of the automorphism group of the supersymmetry algebra H_{Aut} and the isotropy group of the matter multiplets H_{matter}. In pure supergravity, when matter is absent, H is just H_{Aut}. The supergravity theory is completely specified in terms of the geometry of the coset space and in particular in terms of the coset representatives L. For the case of our interest which are $D = 4$ $N > 2$ theories the group G has to be embedded into $Sp(2n, \mathbb{R})$ group. The construction therefore presents symplectic sections of a $Sp(2n_v, \mathbb{R})$ bundle over G/H, given by $f = (f^{\Lambda}_{AB}, h = h_{\Lambda AB})$ and $(f^{\Lambda}_{I}, h_{\Lambda I})$. Here AB are indices[7] in the antisymmetric representation of $H_{\mathrm{Aut}} = SU(N) \times U(1)$ and I is the index of the fundamental representation of H_{matter}. The graviphoton central charges $Z_{AB}, \bar{Z}^{AB}$ and the matter charges $Z_I, \bar{Z}^I$ are defined as linear combination of quantized electric and magnetic charges and moduli as follows:

$$Z_{AB} = f^{\Lambda}_{AB} q_{\Lambda} - h_{\Lambda AB} p^{\Lambda}, \tag{70}$$

$$Z_I = f^{\Lambda}_{I} q_{\Lambda} - h_{\Lambda I} p^{\Lambda}. \tag{71}$$

The crucial observation which will make it possible to establish a complete universality of critical phenomena in extended supergravities is the following. The manifestly symplectic form of supergravity supplies a simple and completely general expression for the black hole potential V presented in Eq. (12) of this paper: it is given in Eq. (3.66) of [15] upon identification between the scalar couplings in (2) and in manifestly symplectic form of extended supergravities in [15],

$$V = \tfrac{1}{2} Z_{AB} \bar{Z}^{AB} + Z^I \bar{Z}_I. \tag{72}$$

[7] Upper $SU(N)$ indices label objects in the complex conjugate representation of $SU(N)$.

The differential relations among charges follow from their definition with the use of a vielbein P of G/H. The embedded vielbein of the coset consists of blocks:

$$\mathcal{P} = \begin{pmatrix} P_{ABCD} & P_{ABJ} \\ P_{JAB} & P_{IJ} \end{pmatrix} . \tag{73}$$

The differential equations which we will use for the study of the critical points are

$$\nabla Z_{AB} = \tfrac{1}{2}\bar{Z}^{CD} P_{ABCD} + \bar{Z}_I P^I_{AB} , \tag{74}$$

$$\nabla Z^I = \tfrac{1}{2}\bar{Z}^{CD} P^I_{CD} + \bar{Z}_J P^{JI} . \tag{75}$$

Now we have sufficient amount of the information on the properties of central charges and moduli space of $N \geqslant 2$ theories to study the critical points. We will first focus on pure supergravity without matter and afterwards will study supergravities coupled to matter.

(1) Critical points in pure $N \geqslant 2$ supergravities with $Z_I \equiv 0$

The potential and the differential relation simplify:

$$V = \tfrac{1}{2} Z_{AB} \bar{Z}^{AB} . \tag{76}$$

$$\nabla Z_{AB} = \tfrac{1}{2} \bar{Z}^{CD} P_{ABCD} . \tag{77}$$

Using Eq. (77) to find the derivatives of the central charge we get

$$\partial_i V = \tfrac{1}{4}\bar{Z}^{CD}\bar{Z}^{AB} P_{ABCD,i} + \tfrac{1}{4} Z_{AB} Z_{CD} P^{ABCD}{}_{,i} . \tag{78}$$

Since P_{ABCD} is completely antisymmetric the solution to this equation exists where only one eigenvalue of the central charge matrix is non-vanishing and other eigenvalues vanish [4]:

$$\partial_i V = 0 \quad \text{at} \quad Z^{12} \neq 0 , \quad Z_{\text{other}} = 0 . \tag{79}$$

The non-vanishing eigenvalue of the central charge matrix is the BPS mass at the critical point. For the second derivative we get

$$D_j \partial_i V \mid_{\partial_i V = 0} = \partial_j \partial_i V = \tfrac{1}{2} Z_{LM} P^{ABLM}{}_{,j} \bar{Z}^{CD} P_{ABCD,i} . \tag{80}$$

At present it is not clear if one can simplify this expression and bring it to the form close to what has been found in $N = 2$ case. However, we expect further developments in this direction.

Matter coupled supergravities $N \geqslant 2$ with $Z_I \neq 0$

The potential (72) and the differential relation (75) now have a contribution from the matter charges. The critical point is defined by

$$\begin{aligned} \partial_i V &= \tfrac{1}{4}\bar{Z}^{CD}\bar{Z}^{AB} P_{ABCD,i} + \tfrac{1}{4} Z_{AB} Z_{CD} P^{ABCD}{}_{,i} + \tfrac{1}{2}\bar{Z}_I P^I_{AB,i}\bar{Z}^{AB} + \tfrac{1}{2} Z^I P^{AB}_{I,i} Z_{AB} \\ &= 0 . \end{aligned} \tag{81}$$

The configurations with

$$Z^{12} \neq 0\,, \qquad Z_{\text{other}} = 0\,, \qquad Z^I = \bar{Z}_I = 0 \tag{82}$$

solves the extremization condition (81) for the potential. The evaluation of the second derivative of the potential at the critical point proceeds with the use of differential relations (75):

$$(\partial_j \partial_i V)_{\text{cr}} = Z_{CD} \bar{Z}^{AB} (P^{CDPQ}{}_{,j} P_{ABPQ,i} + \tfrac{1}{2} P^{CD}_{Ij} P^I_{ABi})\,. \tag{83}$$

Again, it remains to be seen if by using the coset space geometry one can simplify this to a form analogous to a simple result in $N = 2$ theory.

6. Geometry, thermodynamics and critical points

In Section 2 we described a general formalism for constructing 4-dimensional spherically symmetric solutions. In some ways the most symmetrical formulation is the pure geodesics formulation involving the fields $\hat{\phi}$ comprising the scalars ϕ^a, the potentials for the vectors ψ^A, χ_A and the Newtonian potential U all on an equal footing. This naturally arises from dimensional reduction, three dimensions which automatically places U and other Kaluza–Klein scalars and vectors on the same footing. Physically it is more convenient to eliminate the potentials ψ^A and χ_A in favour of their conjugate conserved charges. This gives rise to the potential $V(p, q, \phi,)$ while we have the scalars ϕ^a and the Newtonian potential U remain on the same footing and are described by a simple dynamical model involving motion in the U, ϕ space with a potential. As we have seen, the essential properties of the extreme black holes, such as the area of the event horizon, are given by the values of the potential $V(q, p, \phi)$ thought of as a function on the moduli space of scalars $\mathcal{M}_\phi$ at its critical points ϕ^a_{fix}. Now it becomes especially interesting to ask how the mass M and area A depend on the moduli. In particular about their second covariant derivatives. At this point it is worth recalling some standard geometrical ideas in ordinary thermodynamics, see Ref. [16] for a review.

Some time ago Weinhold suggested using as a metric the Hessian of the energy M, considered as a function of the $n+1$ extensive variables $N^\mu = (S, N^a)$, where S is the entropy and N^a, $a = 1, \dots, n$ are conserved numbers. Note in this formulation of an ordinary gas the volume is included as one of the N^a's. In the case of black holes, the N^a's include conserved charges, angular momenta and also (see Ref. [2]) the values ϕ^∞ of the moduli at infinity. Thus in conventional thermodynamics the Weinhold metric $W_{\mu\nu}$ is given by

$$W_{\mu\nu} = \frac{\partial M}{\partial N^\mu \partial N^\nu}\,. \tag{84}$$

In conventional thermodynamics the Weinhold metric is positive semi-definite because of the fact that the energy is least among equilibrium configurations with given entropy S, and total numbers N^a. Because

$$dM = TdS + \mu_a dN^a, \tag{85}$$

the dual function to M is of course G the Gibbs free energy, which should be thought of as function of the intensive variables $\mu_\mu = (T, \mu_a)$ given by

$$G = M - TS - \mu_a N^a, \tag{86}$$

whence

$$dG = -SdT - N^a d\mu_a. \tag{87}$$

Thus the inverse metric $W^{\mu\nu}$ is given by

$$W^{\mu\nu} = -\frac{\partial G}{\partial\mu_\mu \partial\mu_\nu}. \tag{88}$$

Note that the negative sign arises because of the conventional choice of sign made in the thermodynamics literature when defining the Legendre transformation.

Some time after Weinhold, Ruppeiner focused attention on the entropy S considered as a function of the extensive variables M and N^a. It is convenient to define extensive charges $Q^\mu = (M, N^a)$ and conjugate variables $\beta_\mu = (1/T, -\mu_a/T)$. Ruppeiner observed that fluctuations of the system are governed by the Ruppeiner metric

$$S_{\mu\nu} = -\frac{\partial S}{\partial Q^\mu \partial Q^\nu}. \tag{89}$$

The inverse metric is given by

$$S^{\mu\nu} = \frac{\partial \Gamma}{\partial\beta_\mu \partial\beta_\nu}, \tag{90}$$

where Γ is the Legendre transform of the entropy S, i.e.

$$\Gamma = \frac{G}{T} = -S + \frac{M}{T} - N^a \frac{\mu_a}{T}. \tag{91}$$

More symmetrically

$$S + \Gamma = \beta_\mu Q^\mu. \tag{92}$$

An interesting question to ask is how these two metrics are related. The answer is, perhaps, surprisingly, that they are conformally related and the conformal factor is the temperature; in other words,

$$W_{\mu\nu} dN^\mu dN^\nu = TS_{\mu\nu} dQ^\mu dQ^\nu. \tag{93}$$

To see this note that

$$W_{\mu\nu} dN^\mu dN^\nu = dT \otimes_s dS + d\mu_a \otimes_s dN^a \tag{94}$$

while

$$-S_{\mu\nu} dQ^\mu dQ^\nu = d\frac{1}{T} \otimes_s dM + d\frac{\mu_a}{T} \otimes_s dQ^a. \tag{95}$$

It is interesting to observe that it is the conformal geometry which is physically relevant. Thus ratios of specific heats should be conformal invariants.

We shall now relate these geometric thermodynamic ideas to the work of this paper. The first thing to note is that in general, for non-extreme holes these metrics will not be positive definite because of the fact that non-extreme black holes with have negative specific heats.

The heat capacity $C = (\partial M/\partial T)_{p,q,\phi}$ is related to the second derivative of the mass over the entropy at fixed values of charges. For dilaton non-extreme black holes the change in the sign of the heat capacity has been studied in [19–21]. It has been shown for generic $U(1)^2$ that in the process of the black hole evaporation the temperature increases, reaches the maximum and rapidly drops to zero when the mass of the black hole reaches the value of the central charge. Specific heat blows up when the temperature reaches the maximum and changes the sign, see Figs. 4, 6, 7 of Ref. [21]. The change of the sign of the heat capacity happens at the non-vanishing temperature and means that the corresponding component of the Weinhold metric undergoes the same type of changes.

The second important point is that, as pointed out in [2], when considering black holes with scalars we must augment the usual extensive thermodynamic variables such as the area A and the conserved charges (q,p) with the values of the moduli at infinity ϕ_∞.

This has the consequence that the thermodynamic configuration space is no longer flat $\mathbb{R}^k \equiv (A,q,p)$ (where k is one plus two times the number of electric charges, but becomes its product with the scalar moduli space $\mathbb{R}^k \times \mathcal{M}_\phi$.

It was shown in [2] that the thermodynamic variables conjugate to the moduli ϕ^a_∞, i.e. the analogue of the chemical potentials μ_a are minus the scalar charges, i.e. one has the relation

$$dM = TdS + \psi^A dq_A + \chi_A dp^A - \Sigma_a d\phi^a_\infty \,. \tag{96}$$

The fact that the scalar moduli space $\mathcal{M}_\phi$ is no longer flat complicates, but of course does not invalidate, the usual thermodynamic formalism involving Legendre transformations (technically one should speak of Legendre submanifolds, etc.) and in particular one is allowed to extend the definition of the Weinhold and Ruppeiner metrics to the scalar moduli space $\mathcal{M}_\phi$ provided we replace the ordinary derivative by the covariant derivative with respect to the metric G_{ab} on the moduli space.

In the present paper we have been considering extreme black holes for which the temperature $T = 0$ and it is the Weinhold metric W_{ab} which seems to be the more appropriate one to consider. This may be defined by

$$W_{ab} = \nabla_a \nabla_b M(p,q,\phi) \,. \tag{97}$$

The Ruppeiner metric governs fluctuations and naively diverges (see the relevant equation above) if $T \to 0$. This is in agreement with the arguments presented in [20,21] that near extreme the thermodynamics breaks down. However, one might consider a renormalized definition of the Ruppeiner metric

$$S_{ab} = \tfrac{1}{4}\nabla_a\nabla_b A(p,q,\phi) \,. \tag{98}$$

Note that if the mass M considered as a function of the scalars is at a critical point the first derivative vanishes and the covariant derivative may be replaced by the partial derivative. For a general thermodynamic substance or for a general black hole one expects to be able to say very little about the Weinhold metric. In the case of extreme black holes it is given by

$$W_{ab} = \frac{1}{2\sqrt{V}}\nabla_a\nabla_b V \,. \tag{99}$$

As to the Ruppeiner metric, because the area A of the event horizon depends only on the values of the scalars on the horizon and is independent of their values at infinity it follows that

$$S_{ab} = 0 \,. \tag{100}$$

By contrast for black hole arising form special geometry we are able to make rather more precise statements about the Weinhold metric. We find the remarkable result that the Weinhold metric is proportional to the metric G_{ab} on the moduli space.

7. Conclusion

In conclusion we have found the properties of the critical points of the BPS mass in the range of applicability of the special geometry. Supersymmetric states in the spectrum of $N = 2$ theory have the properties that their mass equals the central charge $M^{\mathrm{BPS}} = |Z|$. The central charge Z is defined in a generic point of moduli space to be a particular function of moduli and electric and magnetic charges and therefore the BPS mass in $N = 2$ theory is given by

$$M^{\mathrm{BPS}} = M(z,\bar{z},p,q) \,. \tag{101}$$

When the derivative of the BPS mass over the moduli at fixed values of electric and magnetic charges vanishes we call this a critical point of the moduli space,

$$\left(\frac{\partial}{\partial z^i} M(z,\bar{z},p,q)\right)_{\mathrm{cr}} = 0 \quad \Longrightarrow \quad z_{\mathrm{cr}} = z(p,q) \,. \tag{102}$$

The critical value of the BPS mass coincides with the value of the entropy of the black hole with the corresponding charges: $\pi M(p,q)_{\mathrm{cr}} = S(p,q)$.

In this paper we have calculated the second derivatives of the BPS mass at the critical point. The result is simple and universal for all possible $N = 2$ supergravities interacting with arbitrary number of vector multiplets. It shows that the second derivative is proportional to the metric in the moduli space and the critical value of the BPS mass,

$$\left(\frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^j} M(z,\bar{z},p,q)\right)_{\mathrm{cr}} = \tfrac{1}{2} G_{i\bar{j}}(z_{\mathrm{cr}},\bar{z}_{\mathrm{cr}}) M(p,q)_{\mathrm{cr}} \,. \tag{103}$$

Thus, as long as the values of the mass $M_{\rm cr}$ is positive and the scalar metric on the moduli space $(G_{i\bar{j}})_{\rm cr}$ is positive definite at the critical point, the BPS mass reaches its unique minimum at the critical point. This fact was already applied to the study of the energy of the bound states of branes [22] and it was pointed out that the extremum of the central charge describes the bound states with minimal energy.

If however, any of these two positivity conditions are violated, the analysis based on regular special geometry on $N = 2$ supersymmetric theories does not apply: one has to include the possibility of vanishing moduli and vanishing BPS mass [11] and of various singularities of special geometry, in particular the change in the sign of the metric of the scalar manifold. This will extend the study performed here to the interesting cases relevant to possible "phase transitions" between different vacua in different theories as suggested in [23]. The examples of such behavior in the context of the 5-dimensional Calabi–Yau type black holes will be presented in [18].

An interesting outcome of our analysis is the relation of the metric on the moduli space G_{ab} with the thermodynamic metric $W_{\mu\nu}$ introduced by Weinhold [16]. For a general thermodynamic system it would seem to be very difficult to say much about the Weinhold metric. In the present case we are dealing, quite literally, with special geometry and in the extreme case we have found that they are proportional. It would be of interest to extend this analysis to the non-extreme case and this we plan to do in the future. One motivation for studying the Weinhold metric is that one might imagine that in a more exact quantum theory of gravity in which space-time geometry may not play the same pre-eminent role that it does in classical and semi-classical general relativity, one will still be able to talk about the thermodynamic properties of "black holes" but at a more abstract level. One needs therefore some principle to determine the thermodynamic surface giving the equation of state of the system. The thermodynamic properties are encoded in the Weinhold metric. In theories based on an underlying geometric structure, such as $N = 2$ theories which are based on special geometry it is not unreasonable to hope that the metric on moduli space and the Weinhold metric continue to be closely related in the full quantum regime.

Acknowledgements

We would like to thank E. Cremmer and R. D'Auria for discussions on the analysis covered in Section 5 and A. Chou, J. Rahmfeld, S-J. Rey, M. Shmakova and W.K. Wong for the discussions of the non-uniquness of critical points and discontinuity of the potentials. The work of S. F. was supported in part by DOE grant DE-FGO3-91ER40662 and by European Commission TMR programme ERBFMRX-CT96-0045 (INFN, Frascati). R.K. was supported by the NSF grant PHY-9219345.

References

[1] G.W. Gibbons, Nucl. Phys. B 207 (1982) 337;
P. Breitenlohner, D. Maison and G. Gibbons, Commun. Math. Phys. 120 (1988) 295.

[2] G. Gibbons, R. Kallosh and B. Kol, Phys. Rev. Lett. 77 (1996) 4992, hep-th/9607108;
D.A. Rasheed, hep-th/9702087.
[3] S. Ferrara, R. Kallosh and A. Strominger, Phys. Rev. D 52 (1995) 5412, hep-th/9508072;
A. Strominger, Phys. Lett. B 383 (1996) 39, hep-th/9602111.
[4] S. Ferrara and R. Kallosh, Phys. Rev. D 54 (1996) 1514, hep-th 9602136; Phys. Rev. D 54 (1996) 1525, hep-th 9603090.
[5] R. Kallosh, M. Shmakova and W.K. Wong, Phys. Rev. D 54 (1996) 6284, hep-th/9607077;
K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W.K. Wong, Phys. Rev. D 54 (1996) 6293 hep-th/9608059;
G.L. Cardoso, D. Lüst and T. Mohaupt, Phys. Lett. B 388 (1996) 266 hep-th/9608099.
[6] K. Behrndt, G. Lopes Cardoso, B. de Wit, R. Kallosh, D. Lüst and T. Mohaupt, Nucl. Phys. B 488 (1997) 236, hep-th/9610105;
Soo-Jong Rey, hep-th/9610157;
M. Shmakova, hep-th/9612076.
[7] D. Kaplan, D. Lowe, J. Maldacena and A. Strominger, Phys. Rev. D 55 (1997) 4898, hep-th/9609204;
K. Behrndt and T. Mohaupt, hep-th/9611140;
J.M. Maldacena, hep-th/9611163.
[8] B. de Wit and A. van Proyen, Phys. Lett. B 293 (1992) 94;
S. Ferrara and A. Strominger, *in* Strings '89, ed. R. Arnowitt, R. Bryan, M.J. Duff, D.V. Nanopoulos and C.N. Pope (World Scientific, Singapore, 1989) p. 245;
A. Strominger, Commun. Math. Phys. 133 (1990) 163;
L.J. Dixon, V.S. Kaplunovsky and J. Louis, Nucl. Phys. B 329 (1990) 27;
P. Candelas and X.C. de la Ossa, Nucl. Phys. B 355 (1991) 455;
L. Castellani, R. D' Auria and S. Ferrara, Phys. Lett. B 241 (1990) 57; Class. Quant. Grav. 7 (1990) 1767;
R. D'Auria, S. Ferrara and P. Fré, Nucl. Phys. B 359 (1991) 705;
B. de Wit and A. Van Proeyen, *in* Quaternionic Structures in Mathematics and Physics, ILAS/FM-6/1996 (Sissa, Trieste) hep-th/9505097; Nucl. Phys. B (Proc. Suppl.) 45B,C (1996) 196.
[9] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, Nucl. Phys. B 444 (1995) 92, hep-th/9502072.
[10] A. Ceresole, R. D'Auria, S. Ferrara, Proceedings of the Trieste workshop on Mirror Symmetry and S-Duality, Trieste 1995, ed. K.S. Narain and E. Gava; hep–th 9509160.
[11] A. Strominger, Nucl. Phys. B 451 (1995) 96, hep-th/9504090
[12] K. Behrndt, Nucl. Phys. B 455 (1995) 188, hep-th/9506106;
M. Cvetič and D. Youm, Phys. Lett. B 359 (1995) 87, hep-th/9507160;
K.L. Chan and M. Cvetič, Phys. Lett. B 375 (1996) 98, hep-th/9512188;
G.W. Gibbons and D.A. Rasheed, Nucl. Phys. B 476 (1996) 515, hep-th/9604177;
T. Ortin, Phys. Rev. Lett. 76 (1996) 3890, hep-th/9602067.
[13] R. Kallosh and A. Linde, Phys. Rev. D 52 (1995) 7137, hep-th/9507022.
[14] K. Behrndt, Phys. Lett. B 396 (1997) 77, hep-th/9610232.
[15] L. Andrianopoli, R. D'Auria and S. Ferrara, hep-th/9612105.
[16] G. Ruppeiner, Rev. Mod. Phys. 67 605 (1995); 68 (1996) 313 (E).
[17] P. Fré, hep-th/9701054.
[18] A. Chou, R. Kallosh, J. Rahmfeld, S-J. Rey, M. Shmakova and W.K. Wong, hep-th/9704142.
[19] G.W. Gibbons and K. Maeda, Nucl. Phys. B 298 (1988) 741.
[20] J. Preskill, P. Schwarz, A. Shapere, S. Trivedi and F. Wilczek, Mod. Phys. Lett. A 6 (1991) 2353.
[21] R. Kallosh, A. Linde, T. Ortín, A. Peet and A. Van Proeyen, Phys. Rev. D 46 (1992) 5278.
[22] R. Kallosh, Phys. Rev. D 55 (1997) 3241, hep-th/9611162.
[23] E. Witten, Nucl. Phys. B 471 (1996) 195, hep-th/9603150.