Extended Essay in Mathematics

Methods for solving a recurrence relation:

What are the differences between using iteration, the characteristic root technique, and generating functions as methods for solving the recurrence relation $a_n = ra_{n-1} + nd$, where $r, d \in \mathbb{Q}$, and finding the initial conditions of the recurrence sequence when $r = 6/7, d = -6/7$, and $a_n = 0$?

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1 Introduction

This extended essay aims to examine the differences between solving recurrence relations by using iteration, the characteristic root technique, and generating functions, and to see whether it differs when not knowing the initial conditions. I became interested in this question when I encountered the sixth problem from the 1967 International Mathematics Olympiad (IMO) that featured finding the initial conditions of a specific recurrence sequence that has integer terms and reaches zero.

At first, we are going to define recurrence relations and their possible characteristics using the recurrence relation from the IMO problem as an example. We will find that the recurrence relation in the problem is a non-homogeneous first-order recurrence relation. Then, if possible, we will use iteration, the characteristic root technique, and generating functions to find closed formulas of a general version of this recurrence relation. We do not know if it is even possible because usually, when solving recurrence relations, the initial conditions are given. Then, we will try to extract the initial conditions of our specific sequence, knowing that the last term is 0 and that all of the terms are integers. Finally, we will be able to reflect on the differences and similarities between these three methods as ways of solving non-homogeneous first-order recurrence relations and finding the initial conditions of a specific sequence. Also, we can see whether finding the initial conditions of the sequence in the IMO problem is even possible using all of our methods.

2 Sequences

A sequence is an ordered list of numbers, which we call terms. More formally, a sequence can be defined as a function from natural numbers (indices of terms) to real numbers. If we want to use variables to represent terms in a sequence we can write them out like this:

$$a_0, a_1, a_2, a_3, \ldots$$

where a_n is the n-th term of the sequence, or when referring to the entire sequence, we will write $(a_n)_{n\in\mathbb{N}}$. Sequences are commonly defined in two ways: by a closed formula or a recursive definition.

Definition 1 (Closed formula)**.** A closed formula for a sequence $(a_n)_{n\in\mathbb{N}}$ is a formula for a_n using a fixed finite number of operations on n.

Definition 2 (Recursive definition)**.** A recursive definition for a sequence $(a_n)_{n\in\mathbb{N}}$ consists of a recurrence relation: an equation relating a term of the sequence to previous terms (terms with smaller index) and an initial condition: a list of a few terms of the sequence. (Levin, 2018a)

To better understand these definitions we will use them to define arithmetic sequences. So, an arithmetic sequence with the terms $a_0, a_1, a_2, a_3, \ldots$ and a common difference d, has the recursive relation $a_n = a_{n-1} + d$, because every term has a difference of d with its previous term. This is not enough to define the specific sequence, only to know the relation between every term. For example, while the sequences

$$1, 3, 5, 7, \ldots \quad \text{and} \quad 0, 2, 4, 6, \ldots$$

have the same recursive relation of $a_n = a_{n-1} + 2$, they are definitely not the same sequence. Therefore, to define these sequences we would need to know the first terms which, together with the recursive relation, let us define all the following terms and hence the entire sequences. The number of terms we need to know as initial conditions is equal to the degree of the recurrence relation, which is the number k in a relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$. So, to define the Fibonacci sequence we would need to know two of the initial terms $a_0 = 0$ and $a_1 = 1$ because it is described by the relation $a_n = a_{n-1} + a_{n-2}$, where $k = 2$ (Lucas, 1891).

On the other hand, a closed formula of an arithmetic sequence would be $a_n = a_0 + nd$, which defines every term using n. This definition also demands the initial conditions like the recursive definition but, instead of the previous terms, it includes n which enables us to calculate a term without knowing the previous one. This fact makes the closed formula more useful. However, finding the closed formula is usually more difficult and requires some skill, while the recurrence relation can be more easily determined just by inspection. Hence, finding the closed formula of a sequence using the recursive definition is also called solving a recurrence relation. (Levin, 2018b)

2.1 The problem

We will define the type of recurrence relation from the IMO problem and examine its characteristics.

In a sports contest, there were m medals awarded on n successive days ($n > 1$). On the first day, one medal and $\frac{1}{7}$ of the remaining $m - 1$ medals were awarded. On the second day, two medals and $\frac{1}{7}$ of the now remaining medals were awarded; and so on. On the n-th and last day, the remaining n medals were awarded. How many days did the contest last, and how many medals were awarded altogether? (IMO, 1967)

The relationship in this problem can be written down as the recurrence relation

$$a_n = a_{n-1} - \frac{1}{7} \cdot (a_{n-1} - n) - n, \quad n \in \mathbb{Z}^+,$$

where $(a_n) \in \mathbb{Z}^+ \cup \{0\}$. Then, a_n represents amount of medals left to be awarded on the n-th day of the contest, $a_0 = m$, as it is the amount of medals on the first day, and $a_n = 0$, because all the medals have been awarded after n days.

We can simplify this to

$$a_n = \frac{6}{7} \cdot a_{n-1} - \frac{6}{7} \cdot n, \quad n \in \mathbb{Z}^+, \tag{1}$$

or a more general recurrence relation

$$a_n = r \cdot a_{n-1} + n \cdot d, \quad n \in \mathbb{Z}^+,$$

where $r, d \in \mathbb{Q}$. Both of these are *linear non-homogeneous recurrence relations with constant coefficients*. Recurrence relation (1) is *linear* because all terms on the right-hand side contain no more than one factor of the form a_k. It is *non-homogeneous*, because there is a term $\frac{6}{7}n$, which is not of the form a_k and is therefore called the inhomogeneity. Finally, it has *constant* coefficients due to the coefficient of a_{n-1} not being a function of n. The same is also true for the general recurrence relation.

3 Iteration

Iteration is the most basic way to solve recurrence relations. The main idea of this technique is to repeatedly iterate the next term starting with the initial condition and generalise it until finding a function for a_n.

3.1 Closed formula

We will use iteration to get a closed formula for our general recurrence relation of the form $a_n = r \cdot a_{n-1} + n \cdot d$, where $a_0, n \in \mathbb{Z}^+$. We will write out the terms and substitute in previous ones to see if we find a pattern:

$$a_1 = ra_0 + d,$$

$$a_2 = ra_1 + 2d = r^2 a_0 + rd + 2d = r^2 a_0 + d(r+2),$$

$$a_3 = ra_2 + 3d = r^3 a_0 + r^2 d + 2rd + 3d = r^3 a_0 + d(r^2 + 2r + 3),$$

$$a_4 = ra_3 + 4d = r^4 a_0 + r^3 d + 2r^2 d + 3rd + 4d = r^3 a_0 + d(r^3 + 2r^2 + 3r + 4).$$

Noticing that

$$a_n = r^n a_0 + d \underbrace{\left(r^{n-1} + 2r^{n-2} + 3r^{n-3} + \cdots + (n-2)r^2 + (n-1)r + n\right)}_{\sum_{i=1}^{n} ir^{n-i}},$$

we predict that the sum $\sum_{i=1}^{n} ir^{n-i}$ will form. Later, we are going to indirectly prove that it indeed forms, by proving the closed formula using induction. Furthermore, we see that the sum $\sum_{i=1}^{n} ir^{n-i}$ is actually the sum of the multiple partial geometric series:

$$\begin{aligned}
r^{n-1}+r^{n-2}+r^{n-3}+\cdots+r^2+r&+1,\\
r^{n-2}+r^{n-3}+\cdots+r^2+r&+1,\\
r^{n-3}+\cdots+r^2+r&+1,\\
&\vdots\\
r^2+r&+1,\\
r&+1,\\
&1.
\end{aligned}$$

This means that we can apply the geometric partial sum formula

$$\sum_{i=0}^{n-1} a^i = \frac{1-a^n}{1-a}$$

and add them together to get that

$$\begin{aligned}
\sum_{i=1}^{n} i r^{n-i} &= \frac{1-r^n}{1-r}+\frac{1-r^{n-1}}{1-r}+\frac{1-r^{n-2}}{1-r}+\cdots+\frac{1-r^3}{1-r}+\frac{1-r^2}{1-r}+\frac{1-r}{1-r}\\
&= \frac{1-r^n+1-r^{n-1}+1-r^{n-2}+\cdots+1-r^3+1-r^2+1-r}{1-r}\\
&= \frac{n-r\left(r^{n-1}+r^{n-2}+r^{n-3}+\cdots+r^2+r+1\right)}{1-r}.
\end{aligned}$$

Again we have a geometric series on which we can use the geometric sum formula to find that

$$\sum_{i=1}^{n} i r^{n-i} = \frac{n-r\left(\frac{1-r^n}{1-r}\right)}{1-r} = \frac{n(1-r)-r(1-r^n)}{(1-r)^2}.$$

Therefore we have found the closed formula

$$a_n = r^n a_0 + d\left[\frac{n(1-r) - r(1-r^n)}{(1-r)^2}\right].$$

We will verify that it is true by using mathematical induction.

Theorem 1. *The recurrence relation* $a_n = ra_{n-1} + nd$, *where* $r \neq 1$ *and* $a_0, n \in \mathbb{Z}^+$, *has a closed formula of* $a_n = r^n a_0 + d\left[\frac{n(1-r)-r(1-r^n)}{(1-r)^2}\right]$.

Proof. Let $P(n)$ be the following statement:

$$a_n = r^n a_0 + d\left[\frac{n(1-r) - r(1-r^n)}{(1-r)^2}\right], \qquad \text{for } n \in \mathbb{Z}^+.$$

<u>Base case.</u> For $n = 1$, we have to show that $a_1 = ra_0 + d$:

$$\begin{aligned} a_0 &= r^1 a_0 + d\left[\frac{1(1-r) - r\left(1-r^1\right)}{(1-r)^2}\right] \\ &= ra_0 + d\left[\frac{(1-r)(1-r)}{(1-r)^2}\right] \\ &= ra_0 + d. \end{aligned}$$

Hence $P(1)$ holds.

<u>Inductive step.</u> We show the implication $P(k) \Rightarrow P(k+1)$ for any positive integer k. We assume that $P(k)$ holds for some $k \geq 1$. In other words, we assume that

$$a_k = r^k a_0 + d\left[\frac{k(1-r) - r\left(1-r^k\right)}{(1-n)^2}\right] \tag{2}$$

for some $k \geq 1$. To show that $P(k+1)$ holds, we need to show that

$$a_{k+1} = r^{k+1} a_0 + d\left[\frac{(k+1)(1-r) - r\left(1-n^{k+1}\right)}{(1-r)^2}\right].$$

Using the original recurrence relation and the assumption from equation (2), we get that

$$
\begin{aligned}
a_{k+1} &= ra_k + d(k+1) \\
&= r \cdot r^k a_0 + r \cdot d \left[\frac{k(1-r) - r\left(1 - r^k\right)}{(1-r)^2} \right] + d(k+1) \\
&= r^{k+1} a_0 + d \left[\frac{r\left(k(1-r) - r\left(1-n^k\right)\right) + (k+1)(1-r)^2}{(1-r)^2} \right],
\end{aligned} \tag{3}
$$

and, as the rest of the equation matches equation (3), we only need to focus on the numerator in the brackets and show that it equals to $(k+1)(1-r) - r\left(1 - n^{k+1}\right)$:

$$
\begin{aligned}
&r\left(k(1-r) - r\left(1-r^k\right)\right) + (k+1)(1-r)^2 = \\
=&kr - kr^2 - n + r^{k+1} + k - 2kr + kr^2 + 1 - 2r + r^2 = \\
=&k - kr - r + 1 - r + r^{k+2} = \\
=&(k+1)(1-r) - r\left(1 - n^{k+1}\right).
\end{aligned}
$$

Therefore, $P(k+1)$ holds as well.

Conclusion. $P(1)$ holds, and for every $k \geq 1$, if $P(k)$ holds, then also $P(k+1)$ holds. Hence, by mathematical induction, we have shown that $P(n)$ holds for all $n \in \mathbb{Z}^+$. □

3.2 Finding the initial conditions

Using the values of $r = \frac{6}{7}$ and $d = -\frac{6}{7}$ for our specific recurrence sequence we get the closed formula

$$\begin{aligned} a_n &= \left(\frac{6}{7}\right)^n a_0 - \frac{6}{7}\left[\frac{n\left(1-\frac{6}{7}\right) - \frac{6}{7}\left(1-\left(\frac{6}{7}\right)^n\right)}{\left(1-\frac{6}{7}\right)^2}\right] \\ &= \left(\frac{6}{7}\right)^n a_0 - \frac{6}{7}\left[\frac{\frac{1}{7}\left(n-6\left(1-\left(\frac{6}{7}\right)^n\right)\right)}{\left(\frac{1}{7}\right)^2}\right] \\ &= \left(\frac{6}{7}\right)^n a_0 - 6\left[n-6\left(1-\left(\frac{6}{7}\right)^n\right)\right]. \end{aligned}$$

Now we can use our knowledge that $a_n = 0$ to get an equation for a_0 so that

$$\begin{aligned} \left(\frac{6}{7}\right)^n a_0 - 6\left[n-6\left(1-\left(\frac{6}{7}\right)^n\right)\right] &= 0 \\ \left(\frac{6}{7}\right)^n a_0 &= 6\left[n-6\left(1-\left(\frac{6}{7}\right)^n\right)\right] \\ a_0 &= \frac{7^n\left[n-6\left(1-\left(\frac{6}{7}\right)^n\right)\right]}{6^{n-1}}, \end{aligned}$$

which we can further simplify to get that

$$\begin{aligned} a_0 &= \frac{7^n n - 7^n \cdot 6\left(\frac{7^n-6^n}{7^n}\right)}{6^{n-1}} \\ &= \frac{7^n n - 7^n \cdot 6 + 6^{n+1}}{6^{n-1}} \\ &= \frac{7^n(n-6)}{6^{n-1}} + 36. \end{aligned}$$

We can find possible values for n, because we know that $a_0 \in \mathbb{Z}^+$. The above equation means that $7^n(n-6)$ has to be divisible by 6^{n-1}, because otherwise $a_0 \notin \mathbb{Z}^+$. Hence, as 7^n and 6^{n-1} have no common factors, $(n-6)/6^{n-1}$ has to be an integer. We know that $n > 1$, which means that $|n-6| < 6^{n-1}$, so $|(n-6)/6^{n-1}| < 1$. Therefore, the only possible integer for it to equal is 0 when $n = 6$, which means that $a_0 = 36$.

4 The characteristic root technique

We can notice is that our closed formula consisted of the two parts: $r^n a_0$, which is actually what we would get for the closed formula of the recurrence relation $a_n = ra_{n-1}$, and

$$\left[\frac{n(1-r) - r(1-r^n)}{(1-r)^2}\right],$$

which resulted from the dn which was brought along inside a_{n-1} during iteration. So theoretically, to solve our recurrence relation, we might somehow have at first solved the recurrence relation $a_n = ra_{n-1}$ and then solved the second part which includes nd.

More precisely, we would have to separate the homogeneous part from the non-homogeneous, solve them separately, and then add them together. This is allowed because we can imagine them as two different sequences which we can later add to get our actual sequence. In summary, the characteristic root technique is used to solve non-homogeneous recurrence relations by first solving the homogeneous and then the non-homogeneous (particular) parts. We will prove that this addition is allowed.

Theorem 2. *If a_n^h is a solution to the homogeneous recurrence relation $a_n = ra_{n-1}$, and a_n^p is a solution to the non-homogeneous recurrence relation $a_n = ra_{n-1} + nd$, then $a_n^h + a_n^p$ is also a solution to the non-homogeneous recurrence relation.*

Proof. To prove that $a_n^h + a_n^p$ is a solution to the relation $a_n = ra_{n-1} + nd$, we have to show that

$$a_n^h + a_n^p = r\left[a_{n-1}^h + a_{n-1}^p\right] + nd.$$

If a_n^h is a solution to the homogeneous recurrence relation $a_n = ra_{n-1}$, then $a_n^h = r \cdot a_{n-1}^h$, and if a_n^p is a solution to the non-homogeneous recurrence relation $a_n = ra_{n-1} + nd$, then $a_n^p = r \cdot a_{n-1}^p + nd$. By adding these two equations, we get that

$$\begin{aligned} a_n^h + a_n^p &= r \cdot a_{n-1}^h + r \cdot a_{n-1}^p + nd \\ &= r\left[a_{n-1}^h + a_{n-1}^p\right] + nd \end{aligned}$$

Therefore, $a_n^h + a_n^p$ is also a solution to the relation $a_n = ra_{n-1} + nd$. $\square$

Hence, we will split our solution to the recurrence relation into homogeneous and particular solutions so that $a_n = a_n^h + a_n^p$.

4.1 Homogeneous

We have established that when solving a recurrence relation we are looking for the closed formula, a function of n, which satisfies the recurrence relation. In other words, we can replace the terms in a recurrence relation with their closed formulas (Levin, 2018c).

The homogeneous part a_n^h of the recurrence relation is $a_n = ra_{n-1}$. We assume that x^n $(x \neq 0)$ is a solution of this recurrence relation, so we will insert it into the recurrence relation and get that

$$x^n = rx^{n-1},$$

now we can find the solution for x by bringing x^n in front:

$$\begin{aligned} x^n - rx^{n-1} &= 0 \\ x^n(x-r) &= 0 \\ x &= r. \end{aligned}$$

Therefore, we found that the relation $a_n = ra_{n-1}$ has a solution $a_n = r^n$. Actually, it has the general solution $a_n = \alpha r^n$, for some constant α based on the initial conditions. This is because, when we plug it in, it satisfies the recurrence relation.

We can prove that this is the solution using iteration.

Theorem 3. *The recurrence relation $a_n = ra_{n-1}$, where $r \neq 1$ and $n \in \mathbb{Z}^+$, has a closed formula of $a_n = \alpha r^n$.*

Proof. Let $P(n)$ be the following statement:

$$a_n = \alpha r^n, \qquad \text{for } n \in \mathbb{Z}^+.$$

<u>Base case.</u> For $n = 1$, we have that $a_1 = \alpha r^1$. Then $\alpha = a_0$. Hence $P(1)$ holds.
<u>Inductive step.</u> We show the implication $P(k) \Rightarrow P(k+1)$ for any positive integer k. We assume that $P(k)$ holds for some $k \geq 1$. In other words, we assume that

$$a_k = \alpha r^k \tag{4}$$

for some $k \geq 1$. To show that $P(k+1)$ holds, we need to show that

$$a_{k+1} = \alpha r^{k+1}.$$

Using the original recurrence relation and the assumption from equation (4), we get

$$\begin{aligned}a_{k+1} &= ra_k \\ &= r \cdot \alpha r^k \\ &= \alpha r^{k+1}.\end{aligned}$$

Therefore, $P(k+1)$ holds as well.

<u>Conclusion.</u> $P(1)$ holds, and for every $k \geq 1$, if $P(k)$ holds, then also $P(k+1)$ holds. Hence, by mathematical induction, we have shown that $P(n)$ holds for all $n \in \mathbb{Z}^+$. $\square$

Therefore, the homogeneous solution is of the form

$$a_n^h = \alpha r^n,$$

where the constant coefficient α depends on the specific recurrence sequence.

We can now show that all solutions to our non-homogeneous recurrence relation are in the form of $a_n^h + a_n^p$, instead of only showing that it is possible for some solutions. We will do that by assuming an arbitrary solution to the recurrence relation and then showing that it is equal to $a_n^h + a_n^p$.

Theorem 4. *For the recurrence relation $a_n = ra_{n-1} + nd$, if a_n^h is the general solution to the homogeneous recurrence part and a_n^p is a solution to the non-homogeneous part, then all solutions to the non-homogeneous recurrence relation take the form $a_n^h + a_n^p$.*

Proof. Let f_n be any solution to the recurrence relation $a_n = ra_{n-1} + nd$, so that

$$f_n = rf_{n-1} + nd.$$

We have to show that $f_n = a_n^h + a_n^p$. This is equivalent to showing that

$$f_n - a_n^p = a_n^h,$$

which we can do by showing that $f_n - a_n^p$ satisfies the homogeneous relation $a_n = ra_{n-1}$.

By subtracting a_n^p from f_n, we get that

$$f_n - a_n^p = rf_{n-1} - ra_{n-1}^p + nd - nd,$$

which simplifies to

$$f_n - a_n^p = r\left(f_{n-1} - a_{n-1}^p\right).$$

Therefore, $f_n - a_n^p$ satisfies the homogeneous relation $a_n = ra_{n-1}$ and hence belongs to a_n^h. Consequently, if $f_n - a_n^p = a_n^h$, then $f_n = a_n^h + a_n^p$. $\square$

4.2 Particular

Similarly, we make an educated guess that the particular solution will be of the form $a_n^p = \beta_1 n + \beta_0$, which is a polynomial of the same degree as the inhomogeneity (MacGillivray, 2021). We will prove that this is the solution.

Theorem 5. *Suppose the inhomogeneity of a recurrence relation with the homogeneous part $a_n = ra_{n-1}$ is a polynomial $\gamma_1 n + \gamma_0$ of n in the first degree, where $\gamma_0, \gamma_1 \in \mathbb{Q}$. Then there is a particular solution a_n^p, which is a polynomial in the form of $\beta_1 n + \beta_0$, where $\beta_0, \beta_1 \in \mathbb{Q}$.*

Proof. To prove that $a_n^p = \beta_1 n + \beta_0$ is a solution to the relation $a_n = ra_{n-1} + \gamma_1 n + \gamma_0$, we have to show that

$$\beta_1 n + \beta_0 = r\left[\beta_1(n-1) + \beta_0\right] + \gamma_1 n + \gamma_0 \tag{5}$$

is true for $\beta_0, \beta_1 \in \mathbb{Q}$.

We can bring all variables in equation (5) to the left-hand side and simplify to get that

$$n\left(\beta_1 - \beta_1 r - \gamma_1\right) + \beta_1 r - \beta_0 r + \beta_0 - \gamma_0 = 0.$$

This lets us view the left hand side as a polynomial of n, and for a polynomial to be equal to zero, all of its coefficients have to also be equal to zero. Hence we get a system of equations

$$\begin{cases} \beta_1 - \beta_1 r - \gamma_1 = 0 \\ \beta_1 r - \beta_0 r + \beta_0 - \gamma_0 = 0 \end{cases}$$

from which we find that

$$\beta_1 = \frac{\gamma_1}{1-r},$$

and that

$$\beta_0 = \frac{-\beta_1 r - \gamma_0}{1-r} = \frac{-\gamma_1 r - \gamma_0(1-r)}{(1-r)^2}.$$

Therefore, the particular solution is

$$a_n^p = \frac{\gamma_1 n}{1-r} + \frac{-\gamma_1 r - \gamma_0(1-r)}{(1-r)^2},$$

with β_0, β_1, and $\gamma_0, \gamma_1 \in \mathbb{Q}$. $\square$

Substituting our variables into the particular solution in the proof of theorem (5), so that $\gamma_0 = 0$ and $\gamma_1 = d$, we get the particular solution

$$a_n^p = \frac{nd}{1-r} + \frac{-rd}{(1-r)^2}.$$

4.3 Closed formula

The homogeneous and particular solutions can be combined to give us the closed formula

$$a_n = \underbrace{\alpha r^n}_{a_n^h} + \underbrace{\frac{nd}{1-r} + \frac{-rd}{(1-r)^2}}_{a_n^p}.$$

Now we can set $n = 0$ so that

$$a_0 = \alpha r^0 + \frac{0 \cdot d}{1-r} - \frac{rd}{(1-r)^2}$$

and rearrange to find that

$$\alpha = a_0 + \frac{rd}{(1-r)^2}.$$

Hence, the closed formula is

$$a_n = r^n \left[a_0 + \frac{rd}{(1-r)^2} \right] + \frac{nd}{1-r} + \frac{-rd}{(1-r)^2}$$

or

$$a_n = r^n a_0 + d \left[\frac{n(1-r) - r(1-r^n)}{(1-r)^2} \right],$$

which is the same as with iteration.

5 Generating functions

Another way of describing a sequence is through the sum of a formal power series, whose coefficients are our sequence's terms. This sum is called the generating function. A formal power series is an infinite sum of the form

$$\sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots,$$

where we take a_n to represent the coefficient of the n-th term. The difference between a formal power series and a regular one is that we never consider the numerical value of the variable x and whether the series actually converges. We will just assume that it does and use the function to which it should converge to as a representation of our sequence. Therefore we can just treat this series as an object that records our sequence in its coefficients. (Wilf, 1994)

A power series that we know well is the geometric series with the ratio x so that

$$\sum_{n\geq 0} x^n = 1 + x + x^2 + x^3 + \dots,$$

and we also know that it converges to

$$\sum_{n\geq 0} x^n = \frac{1}{1-x}, \text{ for } |x| < 1.$$

If we now considered the geometric series as a generating function, instead of a real power series, it would describe the sequence $1, 1, 1, 1, \dots$ because every term in the series is multiplied by a coefficient of 1. Therefore we could also represent this repeating sequence with the function $1/(1-x)$, which is the fundamental generating function.

5.1 The generating function

We will solve our recurrence relation with generating functions by defining a function that encodes our unknown sequence and then expressing our recurrence relation in terms of that function. So, we will be looking for the function

$$A(x) = \sum_{n \geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

To find the function $A(x)$ from our relation $a_n = ra_{n-1} + nd$, we will multiply every relation from $a_1 \ldots a_n$ by x_n and sum them from $n = 0$ to ∞. Then we will try to express these sums in terms of the function $A(x)$. (Wilf, 1994)

We will start with the recurrence relations for each value n:

$$\begin{aligned}
a_1 &= ra_0 + d, \quad n = 1, \\
a_2 &= ra_1 + 2d, \quad n = 2, \\
a_3 &= ra_2 + 3d, \quad n = 3, \\
a_4 &= ra_3 + 4d, \quad n = 4, \\
&\vdots \\
a_n &= ra_{n-1} + nd, \quad n \geq 1,
\end{aligned}$$

each of which we will multiply by x^n to get the equations:

$$\begin{aligned}
a_1 x &= r a_0 x + x d, \\
a_2 x^2 &= r a_1 x^2 + 2x^2 d, \\
a_3 x^3 &= r a_2 x^3 + 3x^3 d, \\
a_4 x^4 &= r a_3 x^4 + 4x^4 d, \\
&\vdots \\
a_n x^n &= r a_{n-1} x^n + n x d, \quad n \geq 1.
\end{aligned}$$

We can now sum all of these and relate them to our generating function $A(x)$.

The left hand side of the equation is equal to

$$a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots,$$

which is just our function $A(x)$ without the first term a_0. Hence,

$$\text{LHS} = A(x) - a_0.$$

On the right hand side, the term $r a_n$ forms the sum

$$r a_0 x + r a_1 x^2 + r a_2 x^3 + r a_3 x^4 + \cdots = r x (a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots),$$

which is just $r x \cdot A(x)$. Finally, the sum of the nd terms takes the form of

$$\sum_{n \geq 1} n x^n d = x d + 2x^2 d + 3x^3 d + 4x^4 d + 5x^5 d + \dots$$

We can bring xd forward so that

$$\sum_{n\geq 1} nx^n d = xd\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots\right),$$

and notice that the series in the parenthesis is actually the derivative of the geometric series $x + x^2 + x^3 + x^4 + x^5 + \dots$ Therefore, we can replace it and get that

$$\sum_{n\geq 1} nx^n d = xd\left[f'(x + x^2 + x^3 + x^4 + x^5 + \dots)\right],$$

which we can further substitute with its generating function so that

$$\sum_{n\geq 1} nx^n d = xd\left[f'\left(\frac{x}{1-x}\right)\right],$$

and finally take its derivative, so we get

$$\begin{aligned}\sum_{n\geq 1} nx^n d &= xd\left[\frac{1}{(1-x)^2}\right]\\ &= \frac{xd}{(1-x)^2}.\end{aligned}$$

Now, combining all of these we end up with the equation

$$A(x) - a_0 = rx \cdot A(x) + \frac{xd}{(1-x)^2},$$

from which we can find the generating function of our recurrence sequence

$$\begin{aligned}A(x) &= \frac{a_0}{1-rx} + \frac{xd}{(1-x)^2(1-rx)}\\ &= \frac{xd + a_0(1-x)^2}{(1-x)^2(1-rx)}.\end{aligned}$$

5.2 Closed formula

To find the closed formula, we have to use partial fraction decomposition to split $A(x)$ into rational functions whose coefficients we can find:

$$\begin{aligned} A(x) &= \frac{xd + a_0(1-x)^2}{(1-x)^2(1-rx)} \\ &= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-rx}. \end{aligned}$$

We see that our function $A(x)$ is actually comprised of three different generating functions, each of which represents a different simpler sequence. Therefore we can find the sequences represented by these and add them together to find the closed formula.

The first one, $\frac{A}{1-x}$, is the simplest, as it is just a geometric series with the ratio x multiplied by A. Hence it represents the series $\sum_{n=0}^{\infty} Ax^n = A + Ax + Ax^2 + Ax^3 + \ldots$, which contains the sequence $A, A, A, A, \ldots$ Hence we can represent this part of the closed formula as just A.

Next, we can see that the function $\frac{B}{(1-x)^2}$ is the derivative of geometric series with the ratio x multiplied by the constant B. But what sequence does this represent? We can find that out by performing the same actions on the geometric series. So we start with the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

and then take its derivative with respect to x and end up with

$$\sum_{n=0}^{\infty} nx^{n-1} = 0 + 1 + 2x + 3x^2 + \ldots$$

Now, we will redefine $n \to n+1$, so that we get rid of the first 0 term

$$\sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \dots$$

and finally multiply by B to get

$$B \cdot \sum_{n=0}^{\infty}(n+1)x^n = B + 2Bx + 3Bx^2 + 4Bx^3 + \dots$$

Hence we can represent this part of the closed formula as $(n+1)B$.

Finally we have the function $\frac{C}{1-rx}$ which is the geometric series with the ratio rx multiplied by C,

$$\frac{C}{1-rx} = Crx + Cr^2x^2 + Cr^3x^3 + \dots = C \cdot \sum_{n=0}^{\infty}(rx)^n.$$

Hence this part of the closed formula is Cr^n.

Putting all of them together we get that

$$a_n = A + (n+1)B + Cr^n.$$

To find A, B, and C, we will start with the partial fraction decomposition

$$\frac{xd + a_0(1-x)^2}{(1-x)^2(1-rx)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-rx}$$

which we will multiply through with the denominator on the left-hand side to get the expression

$$xd + a_0(1-x)^2 = A(1-x)(1-rx) + B(1-rx) + C(1-x)^2.$$

Now we can open the parenthesis so

$$xd + a_0 - 2a_0x + a_0x^2 = A - Arx - Ax + Arx^2 + B - Brx + C - 2Cx + Cx^2$$

and factorise to get the coefficients of x on both sides:

$$a_0x^2 + (d - 2a_0)x + a_0 = (Ar + C)x^2 + (-A(r+1) - Br - 2C)x + (A + B + C)$$

We can now use them to create a system of three linear equations,

$$\begin{cases} rA+ & & C = a_0 \\ (1+r)A+ & rB+ & 2C = 2a_0 - d \\ A+ & B+ & C = a_0. \end{cases}$$

Hence, we can find A, B, and C by solving this system with row reduction by performing row operations to reduce this matrix to reduced row echelon form:

$$\left[\begin{array}{rrr|r} r & 0 & 1 & a_0 \\ -(1+r) & -r & -2 & d - 2a_0 \\ 1 & 1 & 1 & a_0 \end{array}\right] \underset{-R_1+R_3 \to R_1}{\xrightarrow{R_2+(R_1+R_3)\to R_2}} \left[\begin{array}{rrr|r} 1-r & 1 & 0 & 0 \\ 0 & 1-r & 0 & d \\ 1 & 1 & 1 & a_0 \end{array}\right] \longrightarrow$$

$$\underset{R_2/(1-r)\to R_2}{\xrightarrow{R_1-R_2/(1-r)\to R_1}} \left[\begin{array}{rrr|r} 1-r & 0 & 0 & -\frac{d}{1-r} \\ 0 & 1 & 0 & \frac{d}{1-r} \\ 1 & 1 & 1 & a_0 \end{array}\right] \underset{R_3-(R_1+R_2)\to R_3}{\xrightarrow{R_1/(1-r)\to R_1}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & -\frac{d}{(1-r)^2} \\ 0 & 1 & 0 & \frac{d}{1-r} \\ 0 & 0 & 1 & a_0 + \frac{rd}{(1-r)^2} \end{array}\right].$$

Therefore,

$$\begin{cases} A = -\dfrac{d}{(1-r)^2} \\ B = \dfrac{d}{1-r} \\ C = a_0 + \dfrac{rd}{(1-r)^2} \end{cases}$$

which, together with $a_n = A + (n+1)B + Cr^n$ means that the closed formula is

$$\begin{aligned} a_n &= -\frac{d}{(1-r)^2} + \frac{d(n+1)}{1-r} + r^n \left[a_0 + \frac{rd}{(1-r)^2}\right] \\ &= r^n a_0 + d \left[\frac{n(1-r) - r(1-r^n)}{(1-r)^2}\right] \end{aligned}$$

which is the same as the previous two methods.

6 Analysis

6.1 Closed formulas

One way to compare our three methods is to look at the differences between the closed formulas. We got the formula

$$a_n = r^n a_0 + d\left[\frac{n}{1-r} + \frac{-r(1-r^n)}{(1-r)^2}\right] \tag{6}$$

by using iteration,

$$a_n = r^n\left[a_0 + \frac{rd}{(1-r)^2}\right] + \frac{dn}{1-r} + \frac{-rd}{(1-r)^2} \tag{7}$$

by using characteristic roots, and

$$a_n = r^n\left[a_0 + \frac{rd}{(1-r)^2}\right] + \frac{d(n+1)}{1-r} - \frac{d}{(1-r)^2} \tag{8}$$

by using generating functions.

The closed formulas (7) and (8) are the most similar. This may indicate similarities in their way of solving recurrence relations. The only difference between them is in the last two fractions which differ due to n being redefined to $n+1$ when deriving the generating function $B/(1-x)^2$.

We can also compare the homogeneous and particular solutions

$$a_n = \underbrace{r^n\left[a_0 + \frac{rd}{(1-r)^2}\right]}_{a_n^h} + \underbrace{\frac{dn}{1-r} + \frac{-rd}{(1-r)^2}}_{a_n^p}$$

to the generating function $A(x)$,

$$A(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-rx}.$$

Here, the generating function $C/(1-rx)$ corresponds to the homogeneous solutions while the rest form the particular solutions. Another connection between $C/(1-rx)$ and the homogeneous solutions is that the denominator $1-rx$ came from the transformation of the recurrence relation $a_n = ra_{n-1}$, which represents the homogeneous part of the recurrence relation.

One the other hand, we can look at the original generating function

$$A(x) = \frac{a_0}{1-rx} + \frac{xd}{(1-x)^2(1-rx)}$$

and see that the first part $a_0/(1-rx)$ generates a sequence with the closed formula $r^n a_0$, which is the first part of the closed formula (6). Therefore the second fraction should generate the sequence which is expressed by the second part of the formula (6). This is different from the decomposed generating function and the closed formula (7) as in those the r^n term included also $rd/(1-r)^2$ while the iteration and original generating function has r^n multiplied only by a_0.

Hence, the characteristic root technique, like the partial fraction decomposition when using generating functions, separates the sum $d \cdot \sum_{i=1}^{n} ir^{n-i}$ from iteration into the homogeneous and the particular parts. Fortunately, we could derive the closed formula of this sum, but with more terms, like with second order recurrence sequences, this may be impossible. This separation of a sequence into more simple sequences therefore lets us solve recurrence relations to which we cannot find the closed formula right away.

6.2 Initial conditions

We also wanted to know whether finding the initial conditions would be any different between our methods. We found that there are no differences because we achieved the same closed formula with all of our methods, and then could use that to find the initial conditions. This shows that the methods we used were still able to solve recurrence relations even without knowing the specific initial conditions. Hence the initial conditions are only needed for defining a specific sequence's closed formula by inserting them into the relation's closed formula.

6.3 Comparison

Although all three methods gave in essence the same closed formula, they differed in their approach.

Iteration was probably the most easily understandable as it just relied on us discovering a pattern when generating terms. However, this also means that if we somehow could not have discerned the pattern, then finding the closed formula would have been impossible. For example, with higher order recurrence relations like the Fibonacci sequence, iteration is much harder, because we would have to keep track of more previous terms than just one. Therefore, while iteration worked for our recurrence relation, it may not be foolproof.

Our second method, the characteristic root technique, was the fastest of our three methods, but it also relied the most on previous knowledge and experience. Hence, compared with iteration, it was difficult to understand why and how it actually worked. One difference of this method was that we had to split our recurrence relation into homogeneous and particular solutions, which we did not have to do with generating functions or iteration. It also relied on us making an educated guess, which is not very intuitive and means that, if we encountered something new, we would not know how to go forward.

Finally, while using generating functions to solve our recurrence relation was not the fastest, it did not require us to have any inherent knowledge. It was based around the geometric series, which we were already familiar with, but needed to use in an unconventional way. Therefore, it was understandable why it worked, if we understood the concept of generating functions. The most difficult part was probably transforming the recurrence relation into generating functions, but compared to the characteristic root technique, we did not have to make any guesses or have pre-existing knowledge. Using generating functions possibly also guards against the problem of higher order relations, as each term can be expressed with the generating function.

6.4 Conclusion

We have shown that solving the linear first-order non-homogeneous recurrence relation $a_n = ra_{n-1} + nd$, where $r, d \in \mathbb{Q}$ is possible using iteration, the characteristic root technique, and generating functions without knowing the initial conditions. We found that these methods produce the same closed formula for this recurrence relation. Additionally, we have shown that it is possible to find the initial conditions of a specific recurrence sequence, when $r = 6/7, d = -6/7$, and $a_n = 0$, by finding them using the common closed formula.

A further investigation could be to find whether these methods still work for higher order recurrence relations. It is probable that iteration would not work for higher order relations, as it would be too hard to track each term. Additionally, it would be interesting to see if there are any recurrence relations that cannot be solved without knowing the initial conditions. As well as finding out what other ways of defining a recurrence sequence there are, because we have shown in this essay that the first terms are not always necessary.

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