Appendix P: Mathematics

Contents

Appendix P: Engineering Mathematics for CNC Design

P.1 Fundamentals: Units and Dimensional Analysis

P.1.1 SI Base Units

Quantity	Unit	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric Current	ampere	A
Temperature	kelvin	K
Amount of Substance	mole	mol

Quantity	Unit	Symbol
Luminous Intensity	candela	cd

P.1.2 Common Derived Units

Quantity	Unit	Symbol	Formula	SI Base
Force	newton	N	F = ma	kg·m/s^2
Pressure/Stress	pascal	Pa	P = F/A	kg/(m·s^2) or N/m^2
Energy/Work	joule	J	W = Fd	kg·m$^{2/s}$2 or N·m
Power	watt	W	P = E/t	kg·m$^{2/s}$3 or J/s
Frequency	hertz	Hz	f = 1/T	s^{-}1
Torque	newton-meter	N·m	τ = F×r	kg·m$^{2/s}$2
Angle	radian	rad	theta = s/r	dimensionless

P.1.3 Dimensional Analysis

Rule: All terms in equation must have same dimensions.

Example: Beam deflection equation

$$\delta = \frac{FL^3}{3EI}$$

Dimensional check: - Left side: $[\delta] = \text{m}$ (length) - Right side: $\frac{[\text{N}]\cdot[\text{m}]^3}{[\text{Pa}]\cdot[\text{m}]^4} = \frac{\text{kg·m/s}^2\cdot\text{m}^3}{(\text{kg/(m·s}^2))\cdot\text{m}^4} = \frac{\text{m}^4}{\text{m}^3} = \text{m}$ [check]

P.2 Algebra and Equation Manipulation

P.2.1 Linear Equations

Standard form: $ax + b = 0$

Solution: $x = -\frac{b}{a}$

Example: Motor sizing requires torque $T = J\alpha$ where $J = 0.05$ kg·m^2, $\alpha = 100$ rad/s^2. Find T:

$$T = 0.05 \times 100 = 5 \text{ N·m}$$

P.2.2 Quadratic Equations

Standard form: $ax^2 + bx + c = 0$

Quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Critical speed of ball screw: $N_{\text{crit}}^2 - 200N_{\text{crit}} - 10000 = 0$

$$N = \frac{200 \pm \sqrt{200^2 - 4(1)(-10000)}}{2(1)} = \frac{200 \pm \sqrt{80000}}{2} = \frac{200 \pm 283}{2}$$

Positive solution: $N = 241.5$ RPM

P.2.3 Simultaneous Linear Equations

Matrix form: Ax $=$ b

Example: Force balance on gantry

$$\begin{cases} F_1 + F_2 = 1000 \text{ N} \\ 2F_1 + F_2 = 1500 \text{ N} \end{cases}$$

Substitution method: - From equation 1: $F_2 = 1000 - F_1$ - Substitute into equation 2: $2F_1 + (1000 - F_1) = 1500$ - Solve: $F_1 = 500$ N, $F_2 = 500$ N

P.2.4 Exponential and Logarithmic Equations

Exponential decay: $y = y_0 e^{-kt}$

Example: Vibration amplitude decay (damping):

$$A(t) = A_0 e^{-\zeta \omega_n t}$$

where ζ = damping ratio, ω_n = natural frequency (rad/s)

Logarithm rules: - $\log(ab) = \log a + \log b$ - $\log(a/b) = \log a - \log b$ - $\log(a^n) = n \log a$ - $\log_b(b^x) = x$

Example: Bearing life calculation involves cube root:

$$L_{10} = \left(\frac{C}{P}\right)^3 \text{ (million revolutions)}$$

Taking log: $\log L_{10} = 3 \log(C/P)$

P.3 Trigonometry

P.3.1 Basic Trigonometric Functions

Right triangle definitions:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1$$

Angle sum formulas:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

P.3.2 Inverse Trigonometric Functions

$$\theta = \arcsin(x), \quad \theta = \arccos(x), \quad \theta = \arctan(x)$$

Example: Gantry beam angle from deflection

$$\theta = \arctan\left(\frac{\delta}{L}\right) = \arctan\left(\frac{0.5\text{ mm}}{1000\text{ mm}}\right) = 0.0005\text{ rad} = 0.029°$$

P.3.3 Angle Conversions

$$\text{radians} = \text{degrees} \times \frac{\pi}{180}$$
$$\text{degrees} = \text{radians} \times \frac{180}{\pi}$$

Example: Stepper motor 1.8° step

$$1.8° = 1.8 \times \frac{\pi}{180} = 0.0314\text{ rad}$$

Steps per revolution: $360°/1.8° = 200$ steps

P.3.4 Practical CNC Applications

Arc interpolation (G02/G03):

Given start point (x_1, y_1), end point (x_2, y_2), radius R:

Center offset from start:

$$I = \pm\sqrt{R^2 - d^2/4}, \quad J = \pm\sqrt{R^2 - d^2/4}$$

where $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

P.4 Vectors and Vector Operations

P.4.1 Vector Notation

2D vector: $\mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ or $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$

3D vector: $\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$ or $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$

P.4.2 Vector Magnitude

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Example: Gantry velocity components $v_x = 10$ m/min, $v_y = 15$ m/min

Resultant velocity:

$$|\mathbf{v}| = \sqrt{10^2 + 15^2} = \sqrt{325} = 18.03 \text{ m/min}$$

P.4.3 Dot Product (Scalar Product)

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = |\mathbf{a}||\mathbf{b}| \cos\theta$$

Applications: - Work: $W = \mathbf{F} \cdot \mathbf{d}$ - Power: $P = \mathbf{F} \cdot \mathbf{v}$ - Angle between vectors: $\theta = \arccos\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$

Example: Cutting force $\mathbf{F} = (100, 50, 30)$ N, tool displacement $\mathbf{d} = (0, 0, -5)$ mm

Work done:

$$W = \mathbf{F} \cdot \mathbf{d} = 100(0) + 50(0) + 30(-5) = -150 \text{ N·mm} = -0.15 \text{ J}$$

P.4.4 Cross Product (Vector Product)

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\mathbf{i} - (a_x b_z - a_z b_x)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

Magnitude: $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$

Applications: - Torque: $\tau = \mathbf{r} \times \mathbf{F}$ - Angular momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

Example: Force $\mathbf{F} = (0, 100, 0)$ N at position $\mathbf{r} = (0.5, 0, 0)$ m

Torque about origin:

$$\tau = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.5 & 0 & 0 \\ 0 & 100 & 0 \end{vmatrix} = (0, 0, 50) \text{ N·m}$$

Torque magnitude: 50 N·m about Z-axis

P.5 Calculus: Differentiation

P.5.1 Basic Derivatives

Function $f(x)$	Derivative $f'(x)$
c (constant)	0
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$1/x$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

P.5.2 Differentiation Rules

Sum/Difference: $(f \pm g)' = f' \pm g'$

Product rule: $(fg)' = f'g + fg'$

Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

P.5.3 Applications in CNC

Velocity from position:

$$v(t) = \frac{dx}{dt}$$

Acceleration from velocity:

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Example: Position profile during trapezoidal move:

Acceleration phase: $x(t) = \frac{1}{2}at^2$

Velocity: $v(t) = \frac{dx}{dt} = at$

Acceleration: $a(t) = \frac{dv}{dt} = a$ (constant)

Instantaneous power:

$$P = \frac{dE}{dt}$$

For motor: $P = \tau\omega$ where $\omega = \frac{d\theta}{dt}$ (angular velocity)

P.5.4 Optimization (Finding Maxima/Minima)

Set $f'(x) = 0$ and solve for x.

Second derivative test: - $f''(x) > 0$: minimum - $f''(x) < 0$: maximum

Example: Minimize deflection by optimizing beam shape

For rectangular beam: $I = \frac{bh^3}{12}$

Given constant area $A = bh$, find h maximizing I:

$b = A/h$, so $I = \frac{Ah^2}{12}$

$\frac{dI}{dh} = \frac{Ah}{6} = 0$ □ No finite maximum (increase h, decrease b)

Practical constraint: buckling limits h/b ratio

P.6 Calculus: Integration

P.6.1 Basic Integrals

Function $f(x)$	Integral $\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} + C$ (n != -1)
$1/x$	$\ln\|x\| + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$

P.6.2 Definite Integrals

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is antiderivative of $f(x)$

P.6.3 Applications in CNC

Distance from velocity:

$$s = \int_0^t v(t)dt$$

Velocity from acceleration:

$$v = \int_0^t a(t)dt$$

Example: Trapezoidal velocity profile

Constant acceleration $a = 2$ m/s^2 for $t = 3$ s:

$$v(t) = \int_0^3 a\,dt = at\Big|_0^3 = 2(3) = 6 \text{ m/s}$$

Distance during acceleration:

$$s = \int_0^3 v(t)dt = \int_0^3 2t\,dt = t^2\Big|_0^3 = 9 \text{ m}$$

Work from force:

$$W = \int_0^d F(x)dx$$

Average value:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example: Average motor torque over acceleration

$$\tau_{\text{avg}} = \frac{1}{T}\int_0^T \tau(t)dt$$

P.7 Differential Equations for Dynamic Systems

P.7.1 First-Order Linear ODE

Standard form: $\frac{dy}{dt} + p(t)y = q(t)$

Solution method: Integrating factor $\mu(t) = e^{\int p(t)dt}$

Example: RC circuit (similar to motor thermal model)

$$\frac{dT}{dt} + \frac{T}{\tau} = \frac{P}{\tau}$$

where T = temperature rise, τ = thermal time constant, P = power dissipation

Solution: $T(t) = P\left(1 - e^{-t/\tau}\right)$ (heating from ambient)

P.7.2 Second-Order Linear ODE (Vibration)

Standard form: $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$

Natural frequency: $\omega_n = \sqrt{\frac{k}{m}}$ rad/s

Damping ratio: $\zeta = \frac{c}{2\sqrt{km}}$

Solutions:

1. **Underdamped ($\zeta < 1$):** Oscillatory

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t + \phi)$$

   where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)

2. **Critically damped ($\zeta = 1$):** Fastest return without overshoot

$$x(t) = (A + Bt)e^{-\omega_n t}$$

3. **Overdamped ($\zeta > 1$):** Slow return, no oscillation

$$x(t) = Ae^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t} + Be^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t}$$

Example: Gantry vibration after step input

Given: $m = 50$ kg, $k = 100,000$ N/m, $c = 500$ N·s/m

$$\omega_n = \sqrt{\frac{100000}{50}} = 44.7 \text{ rad/s} = 7.1 \text{ Hz}$$

$$\zeta = \frac{500}{2\sqrt{50 \times 100000}} = 0.35$$

System is underdamped □ will oscillate at:

$$f_d = \frac{\omega_d}{2\pi} = \frac{44.7\sqrt{1-0.35^2}}{2\pi} = 6.6 \text{ Hz}$$

P.8 Matrix Algebra

P.8.1 Matrix Operations

Addition: $\mathbf{C} = \mathbf{A} + \mathbf{B}$ where $c_{ij} = a_{ij} + b_{ij}$

Multiplication: $\mathbf{C} = \mathbf{AB}$ where $c_{ij} = \sum_k a_{ik}b_{kj}$

Transpose: $\mathbf{A}^T$ where $(A^T)_{ij} = A_{ji}$

Inverse: $\mathbf{A}^{-1}$ where $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

2×2 inverse:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

where $\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21}$

P.8.2 Rotation Matrices

2D rotation by angle θ:

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3D rotation about Z-axis:

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: Rotating coordinate system for angled tool approach

Point $(x, y) = (10, 0)$ mm rotated by $\theta = 45°$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 7.07 \\ 7.07 \end{bmatrix} \text{ mm}$$

P.8.3 Transformation Matrices (Homogeneous Coordinates)

Translation + Rotation in 2D:

$$\mathbf{T} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Point transformation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Application: G-code coordinate transforms (G54-G59 work offsets, G68 rotation)

P.9 Statics and Force Analysis

P.9.1 Newton's Laws

First Law: Object at rest stays at rest unless acted upon by force (inertia)

Second Law: $\mathbf{F} = m\mathbf{a}$

Third Law: Action-reaction pairs (equal magnitude, opposite direction)

P.9.2 Equilibrium Conditions

Static equilibrium:

$$\sum \mathbf{F} = 0 \quad \text{(no net force)}$$
$$\sum \tau = 0 \quad \text{(no net torque)}$$

2D equilibrium (3 equations):

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_z = 0$$

3D equilibrium (6 equations):

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0$$
$$\sum M_x = 0, \quad \sum M_y = 0, \quad \sum M_z = 0$$

P.9.3 Free Body Diagrams (FBD)

Procedure: 1. Isolate body/component 2. Show all external forces (gravity, reactions, applied loads) 3. Choose coordinate system 4. Apply equilibrium equations

Example: Simply supported beam with center load

Given: Beam length $L = 1$ m, load $F = 1000$ N at center

FBD: Reactions R_A and R_B at ends

Equilibrium: - $\sum F_y = R_A + R_B - F = 0$ - $\sum M_A = R_B \cdot L - F \cdot (L/2) = 0$

Solve: $R_B = F/2 = 500$ N, $R_A = 500$ N (symmetry)

P.9.4 Truss Analysis (Method of Joints)

For each joint in equilibrium:

$$\sum F_x = 0, \quad \sum F_y = 0$$

Sign convention: Tension (+), Compression (-)

Example: Simple triangular truss supporting gantry beam

P.10 Dynamics and Kinematics

P.10.1 Linear Motion Equations

Constant acceleration:

$$v = v_0 + at$$

$$s = v_0 t + \frac{1}{2}at^2$$

$$v^2 = v_0^2 + 2as$$

Example: CNC rapid move (G00)

Acceleration phase: $a = 2$ m/s^2, time $t_1 = 1$ s

Final velocity: $v = 0 + 2(1) = 2$ m/s

Distance: $s = 0 + \frac{1}{2}(2)(1)^2 = 1$ m

P.10.2 Rotational Motion Equations

Angular displacement: θ (rad)

Angular velocity: $\omega = \frac{d\theta}{dt}$ (rad/s)

Angular acceleration: $\alpha = \frac{d\omega}{dt}$ (rad/s^2)

Constant angular acceleration:

$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Relationship to linear motion:

$$v = r\omega, \quad a_{\text{tangential}} = r\alpha, \quad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2$$

Example: Spindle acceleration

From rest to $N = 3000$ RPM in $t = 0.5$ s:

$$\omega = \frac{2\pi N}{60} = \frac{2\pi(3000)}{60} = 314 \text{ rad/s}$$

$$\alpha = \frac{\omega - 0}{t} = \frac{314}{0.5} = 628 \text{ rad/s}^2$$

P.10.3 Newton's Second Law for Rotation

$$\tau = I\alpha$$

where I = moment of inertia (kg·m^2), α = angular acceleration (rad/s^2)

Moment of inertia (common shapes):

Shape	Axis	Moment of Inertia
Solid cylinder	Central axis	$I = \frac{1}{2}mR^2$
Hollow cylinder	Central axis	$I = \frac{1}{2}m(R_o^2 + R_i^2)$
Solid sphere	Diameter	$I = \frac{2}{5}mR^2$
Thin rod	Center, perpendicular	$I = \frac{1}{12}mL^2$
Point mass	Distance r	$I = mr^2$

Example: Motor torque for spindle acceleration

Spindle: $m = 5$ kg, $R = 0.05$ m (solid cylinder approximation)

$$I = \frac{1}{2}(5)(0.05)^2 = 0.00625 \text{ kg·m}^2$$

From previous example: $\alpha = 628$ rad/s^2

Required torque: $\tau = I\alpha = 0.00625 \times 628 = 3.93$ N·m

P.10.4 Work-Energy Theorem

$$W = \Delta KE = \frac{1}{2}m(v_f^2 - v_i^2)$$

Rotational kinetic energy:

$$KE_{\text{rot}} = \frac{1}{2}I\omega^2$$

Example: Energy to accelerate servo motor + ball screw

Motor rotor: $I_m = 0.001$ kg·m^2, final speed $\omega_m = 3000$ RPM = 314 rad/s

Ball screw: $I_s = 0.005$ kg·m^2, coupled 1:1

Total inertia (reflected to motor): $I_{\text{total}} = 0.001 + 0.005 = 0.006$ kg·m^2

Energy required:

$$E = \frac{1}{2}(0.006)(314)^2 = 296 \text{ J}$$

If accelerated in 0.5 s: $P_{\text{avg}} = 296/0.5 = 592$ W

P.11 Mechanics of Materials

P.11.1 Stress and Strain

Normal stress: $\sigma = \frac{F}{A}$ (Pa or N/m^2)

Shear stress: $\tau = \frac{F}{A}$ (Pa or N/m^2)

Normal strain: $\epsilon = \frac{\Delta L}{L_0}$ (dimensionless or %)

Shear strain: $\gamma = \frac{\Delta x}{h}$ (rad, dimensionless)

Hooke's Law (elastic region):

$$\sigma = E\epsilon$$

where E = Young's modulus (Pa)

Example: Steel rod under tension

$F = 10,000$ N, $A = 100$ mm^2, $L_0 = 500$ mm, $E = 200$ GPa

$$\sigma = \frac{10000}{100 \times 10^{-6}} = 100 \times 10^6 \text{ Pa} = 100 \text{ MPa}$$

$$\epsilon = \frac{\sigma}{E} = \frac{100 \times 10^6}{200 \times 10^9} = 0.0005 = 0.05\%$$

$$\Delta L = \epsilon L_0 = 0.0005 \times 500 = 0.25 \text{ mm}$$

P.11.2 Bending Stress

Flexure formula:

$$\sigma = \frac{My}{I}$$

where: - M = bending moment (N·m) - y = distance from neutral axis (m) - I = second moment of area (m^4)

Maximum stress (at outer fiber):

$$\sigma_{\text{max}} = \frac{Mc}{I} = \frac{M}{S}$$

where $S = I/c$ = section modulus (m^3)

Example: Gantry beam (rectangular cross-section)

$M = 500$ N·m, $b = 100$ mm, $h = 200$ mm

$$I = \frac{bh^3}{12} = \frac{0.1 \times 0.2^3}{12} = 6.67 \times 10^{-5} \text{ m}^4$$

$$c = h/2 = 0.1 \text{ m}$$

$$\sigma_{\text{max}} = \frac{500 \times 0.1}{6.67 \times 10^{-5}} = 75 \times 10^6 \text{ Pa} = 75 \text{ MPa}$$

P.11.3 Beam Deflection

Cantilever beam (end load):

$$\delta = \frac{FL^3}{3EI}$$

Simply supported beam (center load):

$$\delta = \frac{FL^3}{48EI}$$

Example: Z-axis spindle head deflection (cantilever)

$F = 1000$ N, $L = 0.5$ m, $E = 200$ GPa, $I = 6.67 \times 10^{-5}$ m^4

$$\delta = \frac{1000 \times 0.5^3}{3 \times 200 \times 10^9 \times 6.67 \times 10^{-5}} = 0.000313 \text{ m} = 0.313 \text{ mm}$$

P.11.4 Torsional Stress and Deflection

Shear stress:

$$\tau = \frac{Tr}{J}$$

where: - T = torque (N·m) - r = radius (m) - J = polar moment of inertia (m^4)

Solid circular shaft: $J = \frac{\pi d^4}{32}$

Hollow circular shaft: $J = \frac{\pi(d_o^4 - d_i^4)}{32}$

Angle of twist:

$$\phi = \frac{TL}{GJ}$$

where G = shear modulus (Pa)

Example: Ball screw torsional stiffness

$d = 40$ mm, $L = 1$ m, $G = 80$ GPa (steel)

$$J = \frac{\pi(0.04)^4}{32} = 2.51 \times 10^{-7} \text{ m}^4$$

Torque $T = 50$ N·m:

$$\phi = \frac{50 \times 1}{80 \times 10^9 \times 2.51 \times 10^{-7}} = 0.00249 \text{ rad} = 0.143°$$

P.12 Heat Transfer

P.12.1 Conduction (Fourier's Law)

$$Q = -kA\frac{dT}{dx}$$

Steady-state through wall:

$$Q = \frac{kA\Delta T}{L}$$

where: - Q = heat transfer rate (W) - k = thermal conductivity (W/(m·K)) - A = area (m^2) - ΔT = temperature difference (K) - L = thickness (m)

Thermal resistance: $R_{\text{thermal}} = \frac{L}{kA}$ (K/W)

Example: Motor cooling through aluminum housing

$k = 205$ W/(m·K), $A = 0.01$ m^2, $L = 5$ mm, $\Delta T = 50$ K

$$Q = \frac{205 \times 0.01 \times 50}{0.005} = 20,500 \text{ W}$$

P.12.2 Convection (Newton's Law of Cooling)

$$Q = hA(T_s - T_\infty)$$

where: - h = convection coefficient (W/(m^2·K)) - T_s = surface temperature (K) - T_∞ = fluid temperature (K)

Typical values: - Natural air: $h = 5 - 25$ W/(m^2·K) - Forced air: $h = 10 - 200$ W/(m^2·K) - Water: $h = 500 - 10,000$ W/(m^2·K)

P.12.3 Radiation (Stefan-Boltzmann Law)

$$Q = \epsilon\sigma A(T_s^4 - T_{\text{surr}}^4)$$

where: - ϵ = emissivity (0-1) - $\sigma = 5.67 \times 10^{-8}$ W/(m$^{2\cdot\text{K}}$4) (Stefan-Boltzmann constant) - T in Kelvin (absolute temperature)

P.13 Control Systems Mathematics

P.13.1 Transfer Functions (Laplace Domain)

Laplace transform:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Common transforms:

Time Domain	Laplace Domain
$\delta(t)$ (impulse)	1
$u(t)$ (step)	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

Differentiation: $\mathcal{L}\{\frac{df}{dt}\} = sF(s) - f(0)$

Integration: $\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$

P.13.2 PID Controller

Time domain:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)d\tau + K_d \frac{de}{dt}$$

Laplace domain:

$$U(s) = \left(K_p + \frac{K_i}{s} + K_d s\right) E(s)$$

Transfer function:

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s}$$

Gains: - K_p: Proportional gain (immediate response to error) - K_i: Integral gain (eliminates steady-state error) - K_d: Derivative gain (damping, anticipates error)

P.13.3 First-Order System

Transfer function:

$$G(s) = \frac{K}{\tau s + 1}$$

Step response:

$$y(t) = K(1 - e^{-t/\tau})$$

where τ = time constant (63.2% of final value at $t = \tau$)

P.13.4 Second-Order System

Transfer function:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Step response characteristics: - Rise time: $t_r \approx \frac{1.8}{\omega_n}$ (underdamped) - Settling time: $t_s \approx \frac{4}{\zeta\omega_n}$ (2% criterion) - Peak overshoot: $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ (underdamped)

P.14 Statistics and Uncertainty

P.14.1 Mean and Standard Deviation

Mean (average):

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Standard deviation:

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Example: Measuring part dimension 10 times: Data: 50.01, 50.03, 49.98, 50.02, 50.00, 50.01, 49.99, 50.02, 50.00, 50.01 mm

$$\bar{x} = \frac{500.07}{10} = 50.007 \text{ mm}$$

$$\sigma = 0.015 \text{ mm}$$

P.14.2 Error Propagation

Addition/Subtraction:

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2$$

Multiplication/Division:

$$\left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$$

General function $z = f(x, y)$:

$$\sigma_z^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2$$

P.15 Fourier Analysis (Frequency Domain)

P.15.1 Fourier Series

Periodic signal $f(t)$ with period T:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right]$$

Coefficients:

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos\left(\frac{2\pi nt}{T}\right) dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2\pi nt}{T}\right) dt$$

P.15.2 Frequency Analysis Applications

Vibration analysis: Decompose vibration signal into frequency components

Modal testing: Identify natural frequencies and mode shapes

Chatter detection: Monitor cutting forces in frequency domain

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End of Engineering Mathematics Appendix