Problem 1: The Susskind-Glogower operators

THE SG operators (also called exponential operators) are given by

$$\begin{aligned}\hat{E} &= \frac{1}{(\hat{n}+1)^{1/2}}\hat{a} = \left(1-\frac{1}{2}\hat{n}+\dots\right)\hat{a} \\ \hat{E}^{\dagger} &= \hat{a}^{\dagger}\frac{1}{(\hat{n}+1)^{1/2}} = \hat{a}^{\dagger}\left(1-\frac{1}{2}\hat{n}+\dots\right)\end{aligned} \tag{1}$$

Let us consider some state $|n\rangle$ and write

$$\begin{aligned}\hat{E}|n\rangle &= \left(1-\frac{1}{2}\hat{n}+\dots\right)\hat{a}|n\rangle \\ &= \sqrt{n}\left(1-\frac{1}{2}\hat{n}+\dots\right)|n-1\rangle \\ &= \sqrt{n}\left(1-\frac{n-1}{2}+\dots\right)|n-1\rangle \\ &= \sqrt{n}\frac{1}{\sqrt{1+(n-1)}}|n-1\rangle \\ &= |n-1\rangle\end{aligned} \tag{2}$$

For $n=0$, we have $\hat{E}|n\rangle = 0$. On the other hand,

$$\begin{aligned}\hat{E}^{\dagger}|n\rangle &= \hat{a}^{\dagger}\left(1-\frac{1}{2}\hat{n}+\dots\right)|n\rangle \\ &= \hat{a}^{\dagger}\left(1-\frac{1}{2}n+\dots\right)|n\rangle \\ &= \frac{1}{\sqrt{n+1}}\hat{a}^{\dagger}|n\rangle \\ &= \frac{1}{\sqrt{n+1}}\sqrt{n+1}|n+1\rangle \\ &= |n+1\rangle\end{aligned} \tag{3}$$

Problem 2: An example featuring the phase operator

The exponential eigenstates are given by

$$\left|\phi\right\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle \tag{4}$$

We consider the state vector

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right) \tag{5}$$

and calculate the conjugate square of their inner product by writing

$$\begin{aligned}\left\langle\phi|\psi\right\rangle &= \sum_{n=0}^{\infty}\frac{e^{-in\phi}}{\sqrt{2}}\langle n|\left(|0\rangle + e^{i\theta}|1\rangle\right) \\ &= \sum_{n=0}^{\infty}\frac{e^{-in\phi}}{\sqrt{2}}\left(\delta_{n,0} + e^{i\theta}\delta_{n,1}\right) \\ &= \frac{1}{\sqrt{2}}\left(1 + e^{i(\theta-\phi)}\right)\end{aligned} \tag{6}$$

so that

$$\left|\left\langle\phi|\psi\right\rangle\right|^2 = 1 + \cos(\theta-\phi) \tag{7}$$

This gives the phase distribution of

$$\mathscr{P}\left(\phi\right) = \frac{1+\cos(\theta-\phi)}{2\pi} \tag{8}$$

Problem 3: Commutation relations of the obvious analogs

We start with

$$\begin{aligned}\left[\hat{E},\hat{n}\right] &= \sum_{n=0}^{\infty}|n\rangle\langle n+1|\sum_{m=0}^{\infty}m|m\rangle\langle m| - \sum_{m=0}^{\infty}m|m\rangle\langle m|\sum_{n=0}^{\infty}|n\rangle\langle n+1| \\ &= \sum_{m,n}\delta_{n+1,m}m|n\rangle\langle m| - \sum_{m,n}\delta_{m,n}m|m\rangle\langle n+1| \\ &= \sum_{n}(n+1)|n\rangle\langle n+1| - \sum_{n}n|n\rangle\langle n+1| \\ &= \hat{E}\end{aligned} \tag{9}$$

Similarly,

$$\left[\hat{E}^{\dagger},\hat{n}\right] = -\hat{E}^{\dagger} \tag{10}$$

Now,

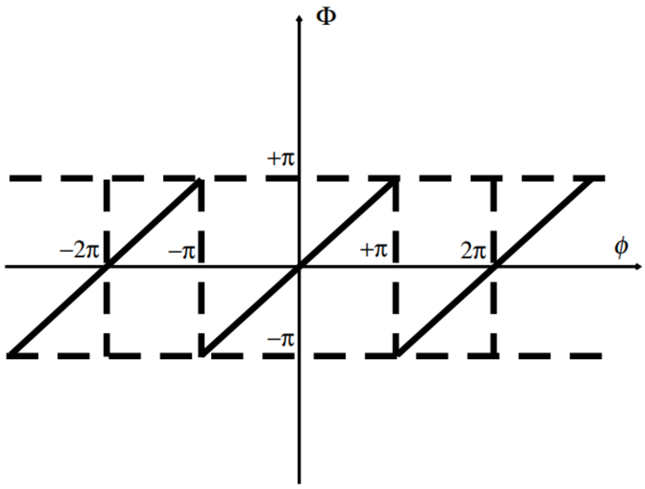
$$\begin{aligned}\hat{C} &= \frac{1}{2}\left(\hat{E}+\hat{E}^{\dagger}\right) \\ \hat{S} &= \frac{1}{2i}\left(\hat{E}-\hat{E}^{\dagger}\right)\end{aligned} \tag{11}$$

It is thus easy to see that

$$\begin{aligned}\left[\hat{C},\hat{n}\right] &= \frac{1}{2}\left(\left[\hat{E},\hat{n}\right]+\left[\hat{E}^{\dagger},\hat{n}\right]\right) = \frac{1}{2}\left(\hat{E}-\hat{E}^{\dagger}\right) = i\hat{S} \\ \left[\hat{S},\hat{n}\right] &= \frac{1}{2i}\left(\left[\hat{E},\hat{n}\right]-\left[\hat{E}^{\dagger},\hat{n}\right]\right) = \frac{1}{2i}\left(\hat{E}+\hat{E}^{\dagger}\right) = -i\hat{C}\end{aligned} \tag{12}$$

Problem 4: The problem with a certain periodic solution to the nonperiodicity of ϕ

One periodic function $\Phi(\phi)$ we can define to fix the nonperiodicity of ϕ is like this:



We would like to calculate $\left[\Phi(\phi),\hat{L}_z\right]$, so we need to get the expression for Φ. We can do this by using the Fourier series. Let us consider the interval $\phi\in[-\pi,\pi]$. We have

$$\Phi\left(\phi\right) = \phi \tag{13}$$

The Fourier coefficients are

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi e^{-in\phi}d\phi = \frac{i^{2n+1}}{n} \tag{14}$$

for $n\neq 0$. For $n=0$, we simply have $c_0=0$. We thus have

$$\Phi(\phi) = \phi \equiv \sum_{n=-\infty}^{-1}\frac{i^{2n+1}}{n}e^{in\phi} + \sum_{n=1}^{\infty}\frac{i^{2n+1}}{n}e^{in\phi} \tag{15}$$

Let us calculate the commutation using one term for a given n of the series. The commutation relation follows immediately. We have

$$\begin{aligned}\left[\frac{i^{2n+1}}{n}e^{in\phi},\hat{L}_z\right]|\psi\rangle &= \left[\frac{i^{2n+1}}{n}e^{in\phi},-i\frac{\partial}{\partial\phi}\right]|\psi\rangle \\ &= \frac{i^{2n+1}}{n}e^{in\phi}(-i)\frac{\partial|\psi\rangle}{\partial\phi}-(-i)\frac{\partial}{\partial\phi}\left(\frac{i^{2n+1}}{n}e^{in\phi}|\psi\rangle\right) \\ &= \frac{i^{2n+2}}{n}\left(-e^{in\phi}\frac{\partial|\psi\rangle}{\partial\phi}+ine^{in\phi}|\psi\rangle+e^{in\phi}\frac{\partial|\psi\rangle}{\partial\phi}\right) \\ &= -i(-1)^n e^{in\phi}|\psi\rangle\end{aligned} \tag{16}$$

We can use

$$-1\equiv e^{-i(2k+1)\pi} \tag{17}$$

where k is an integer, to write

$$\left[\frac{i^{2n+1}}{n}e^{in\phi},\hat{L}_z\right]=-ie^{in[\phi-(2k+1)\pi]} \tag{18}$$

We can then write

$$\left[\Phi,\hat{L}_z\right]=-i\left(\sum_{n=-\infty}^{-1}e^{in[\phi-(2k+1)\pi]}+\sum_{n=1}^{\infty}e^{in[\phi-(2k+1)\pi]}\right) \tag{19}$$

Next, we use the fact that

$$\sum_{n=-\infty}^{\infty}e^{inx}=\sum_{n=-\infty}^{-1}e^{inx}+1+\sum_{n=1}^{\infty}e^{inx} \tag{20}$$

to write

$$\left[\Phi,\hat{L}_z\right]=-i\left(\sum_{n=-\infty}^{\infty}e^{in[\phi-(2k+1)\pi]}-1\right) \tag{21}$$

Lastly, we can use

$$2\pi\delta(x)\equiv\sum_{-\infty}^{\infty}e^{inx} \tag{22}$$

and absorb the negative sign at the front to end up with

$$\left[\Phi,\hat{L}_z\right]=i\left\{1-2\pi\delta\left(\phi-[2k+1]\pi\right)\right\} \tag{23}$$

Problem 5: Trigonometric identity (⋆)

We use (11) to write

$$\hat{C}^2+\hat{S}^2=\frac{1}{2}\left(\hat{E}\hat{E}^\dagger+\hat{E}^\dagger\hat{E}\right) \tag{24}$$

We have

$$\begin{aligned}\hat{E}\hat{E}^\dagger &= 1 \\ \hat{E}^\dagger\hat{E} &= 1-|0\rangle\langle 0|\end{aligned} \tag{25}$$

so

$$\hat{C}^2+\hat{S}^2=\frac{1}{2}(1+1-|0\rangle\langle 0|)=1-\frac{1}{2}|0\rangle\langle 0| \tag{26}$$