1 Funo et. al. [1]: Quantum Speed Limit for Open Quantum Systems

The quantum speed limit (QSL) for open quantum systems described by Lindblad master equations is given by the following inequality [1]:

$$\tau \geq \frac{d_{\mathrm{tr}}\left(\rho_i, \rho_f\right)}{\hbar^{-1}\left\langle\Delta E\right\rangle_\tau + \hbar^{-1}\left\langle\Delta E_{\mathcal{D}}\right\rangle_\tau + \sqrt{\frac{1}{2}\left\langle\dot{\sigma}\right\rangle_\tau\left\langle A\right\rangle_\tau}} \tag{1.1}$$

Here

$$d_{\mathrm{tr}}\left(\rho_i, \rho_f\right) = \frac{1}{2}\mathrm{Tr}\left\{\sqrt{\left(\rho_i - \rho_f\right)^\dagger\left(\rho_i - \rho_f\right)}\right\}$$

is the trace distance between the initial and final density matrix. Meanwhile,

$$\langle q\rangle_\tau = \frac{1}{\tau}\int_0^\tau q(t)dt$$

is the time-average of some quantity q. The time average appears in the denominator of Eq. (1.1) for the following quantities:

- $\Delta E = \sqrt{\mathrm{Tr}\left\{\hat{H}^2\rho\right\} - \mathrm{Tr}\left\{\hat{H}\rho\right\}^2}$ is the uncertainty in energy.
- $\Delta E_{\mathcal{D}} = \sqrt{\mathrm{Tr}\left\{\hat{H}_{\mathcal{D}}^2\rho\right\}}$. Here

  $$\hat{H}_{\mathcal{D}}(t) = i\hbar\sum_{\substack{m,n \\ p_m \neq p_n}} \frac{\langle m|\mathcal{D}\left[\rho(t)\right]|n\rangle}{p_n - p_m}|m\rangle\langle n|$$

  where $\{|n\rangle\} = \{|n(t)\rangle\}$ is the set of basis states which diagonalizes $\rho(t)$, and p_n is the probability associated with $|n\rangle$. Meanwhile,

  $$\mathcal{D}\left[\rho(t)\right] = \sum_{\epsilon_t,\alpha}\gamma_\alpha\left(\epsilon_t\right)\left[\hat{L}_{\epsilon_t,\alpha}(t)\rho(t)\hat{L}^\dagger_{\epsilon_t,\alpha}(t) - \frac{1}{2}\left\{\hat{L}^\dagger_{\epsilon_t,\alpha}(t)\hat{L}_{\epsilon_t,\alpha}(t), \rho(t)\right\}\right]$$

  where γ_α is the parameter corresponding to the αth nonunitary interaction, and

  $$\hat{L}_{\epsilon_t,\alpha}(t) = \sum_{\epsilon_t = \epsilon_m(t) - \epsilon_n(t)}|\epsilon_n(t)\rangle\left\langle\epsilon_n(t)\middle|\hat{L}_\alpha\middle|\epsilon_m(t)\right\rangle\langle\epsilon_m(t)|$$

  where $|\epsilon_n(t)\rangle$ is the energy eigenstate of the Hamiltonian $\hat{H}$ corresponding to the eigenvalue $\epsilon_n(t)$, is the dissipator representing the nonunitary dynamics. Note that $\mathrm{Tr}\left\{\hat{H}_{\mathcal{D}}\rho\right\} = 0$. ΔE may be interpreted as the velocity for isolated systems, while $\Delta E_{\mathcal{D}}$ may be interpreted as the velocity of the bath-induced unitary dynamics.
- $\dot{\sigma}$ is the quantum entropy production rate, defined by

  $$\dot{\sigma} = \underbrace{\sum_{\omega,\alpha,n,m}}_{\substack{\text{except indices satisfying} \\ (m=n)\wedge(\omega=0)}} W^{\omega,\alpha}_{mn}p_n\ln\left[\frac{W^{\omega,\alpha}_{mn}p_n}{W^{-\omega,\alpha}_{nm}p_m}\right]$$

  where

  $$W^{\omega,\alpha}_{mn} = \gamma_\alpha(\omega)\left|\left\langle m\middle|\hat{L}_{\omega,\alpha}\middle|n\right\rangle\right|^2$$

  and

  $$\hat{L}_{-\omega,\alpha} = \hat{L}^\dagger_{\omega,\alpha}$$

To tell whether $\omega = 0$, the following property may be used:

$$\left[\hat{L}_{\omega,\alpha}, \hat{H}\right] = \omega \hat{L}_{\omega,\alpha}$$

- A is the quantum dynamical activity, which is anologous to the classical dynamical activity. It is defined by

$$A = \frac{1}{2} \underbrace{\sum_{\omega,\alpha,n,m}}_{\substack{\text{except indices satisfying} \\ (m=n)\wedge(\omega=0)}} \left(p_n W_{mn}^{\omega,\alpha} + p_m W_{nm}^{-\omega,\alpha}\right)$$

The third term in the denominator of Eq. (1.1) characterizes the speed of the population transfer via the bath.

Let us do a little bit of math to simplify the equations for computational purposes. First, we set $\hbar = 1$. Notice that we can rewrite Eq. (1.1) as

$$\int_0^\tau \left[\Delta E(t) + \Delta E_{\mathcal{D}}(t)\right] dt + \sqrt{\frac{1}{2}\int_0^\tau \dot{\sigma}(t) dt \int_0^\tau A(t) dt} - d_{\mathrm{tr}}\left(\rho_i, \rho_f\right) \geq 0 \tag{1.2}$$

The quantum speed limit τ_{QSL} can thus be calculated by solving for τ which satisfies the equality. Next, we move on to simplify $\Delta E_{\mathcal{D}}$ by writing

$$\begin{aligned} \hat{H}_{\mathcal{D}}^2 \rho &= -\sum_{\substack{m,n \\ p_m \neq p_n}} \frac{\langle m|\mathcal{D}|n\rangle}{p_n - p_m} |m\rangle\langle n| \sum_{\substack{a,b \\ p_a \neq p_b}} \frac{\langle a|\mathcal{D}|b\rangle}{p_b - p_a} |a\rangle\langle b| \sum_k p_k |k\rangle\langle k| \\ &= -\sum_{\substack{m,n \\ p_m \neq p_n}} \sum_{\substack{n,k \\ p_n \neq p_k}} p_k \frac{\langle m|\mathcal{D}|n\rangle}{p_n - p_m} \frac{\langle n|\mathcal{D}|k\rangle}{p_k - p_n} |m\rangle\langle k| \end{aligned} \tag{1.3}$$

so that

$$\begin{aligned} \mathrm{Tr}\left\{\hat{H}_{\mathcal{D}}^2 \rho\right\} &= -\sum_{\substack{m,n \\ p_m \neq p_n}} p_m \frac{\langle m|\mathcal{D}|n\rangle}{p_n - p_m} \frac{\langle n|\mathcal{D}|m\rangle}{p_m - p_n} \\ &= \sum_{\substack{m,n \\ p_m \neq p_n}} \frac{p_m \langle m|\mathcal{D}|n\rangle \langle n|\mathcal{D}|m\rangle}{\left(p_m - p_n\right)^2} \end{aligned} \tag{1.4}$$

Note that $\mathcal{D}$ is not hermitian.

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2 `del Campo et. al.` [2]: A Less Tight Quantum Speed Limit for Open Quantum Systems

The Lindblad master equation can be concisely written as

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \mathcal{L}\rho$$

By writing

$$\mathcal{L}^\dagger \rho = \frac{i}{\hbar}\left[\hat{H}, \rho\right] + \sum_\alpha \gamma_\alpha \left[\hat{L}_\alpha^\dagger \rho \hat{L}_\alpha - \frac{1}{2}\left\{\hat{L}_\alpha^\dagger \hat{L}_\alpha, \rho\right\}\right]$$

i.e. the hermitian conjugate of $\mathcal{L}\rho$, the quantum speed limit is given by

$$\tau \geq \frac{\left|\mathrm{Tr}\left\{\rho_0 \rho_{\mathrm{target}}\right\} - \mathrm{Tr}\left\{\rho_0^2\right\}\right|}{\overline{\sqrt{\mathrm{Tr}\left\{\left(\mathcal{L}^\dagger \rho_0\right)^2\right\}}}} \tag{2.1}$$

where

$$\overline{X} = \frac{1}{\tau}\int_0^\tau X \mathrm{d}t$$

for a time-dependent $\mathcal{L} = \mathcal{L}(t)$.

We can see that the lower bound (i.e. the QSL) imposed by Eq. (2.1) is not tight. This means that there are other expressions that gives a lower bound which is greater than what we have here, e.g. the one in the previous section. This statement can be justified by noting that the unified Mandelstam-Tamm/Margolus-Levitin bound is tight [3], and Eq. (2.1) does not reduce to any of the two in the nondissipative regime.

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3 `Impens & Guéry-Odelin` [4]: Shortcut to Synchronization

The Hamiltonian in the frame rotating at the frequency of the drive is given by

$$\hat{H} = \Delta \hat{a}^\dagger \hat{a} + \frac{\epsilon_1(t)}{2}\left(\hat{a} + \hat{a}^\dagger\right) + i\frac{\epsilon_2(t)}{2}\left(\hat{a} - \hat{a}^\dagger\right) \tag{3.1}$$

The quantities used by `Impens2023`:

$$\begin{aligned}
\Delta &= 2\pi \times 0.05 \\
\kappa_1 &= \begin{cases} 1 & \text{(weakly nonlinear case)} \\ 0.05 & \text{(strongly nonlinear case)} \end{cases} \\
\kappa_2 &= \begin{cases} 0.05 & \text{(weakly nonlinear case)} \\ 1 & \text{(strongly nonlinear case)} \end{cases} \\
\rho_0 &= \left|\beta_0\right\rangle\left\langle\beta_0\right| \quad \text{(a coherent state)}
\end{aligned} \tag{3.2}$$

To help match the notations to those of `Walter2014` [5], we write

$$\begin{aligned}
\hat{a} &\longleftrightarrow \hat{b} & \langle\alpha\rangle_t &\longleftrightarrow \beta(t) \\
\Delta &\longleftrightarrow -\Delta & \frac{\epsilon_1}{2} &\longleftrightarrow \text{(none)} \\
\frac{\epsilon_2}{2} &\longleftrightarrow \Omega & 2\kappa_i &\longleftrightarrow \gamma_i
\end{aligned}$$

Under these relations, we may rewrite Eq. (3.2) as

$$\hat{H} = -\Delta \hat{b}^\dagger \hat{b} + \Omega_1(t)\left(\hat{b} + \hat{b}^\dagger\right) + i\Omega_2(t)\left(\hat{b} - \hat{b}^\dagger\right) \tag{3.3}$$

where we have defined $\Omega_1 \equiv \Omega_1/2$ and $\Omega_1 \equiv \Omega_2/2$. The evolution of the density matrix ρ is given by

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i\left[\hat{H}, \rho\right] + \gamma_1 \mathcal{D}\left[\hat{b}^\dagger\right]\rho + \gamma_2 \mathcal{D}\left[\hat{b}^2\right]\rho \tag{3.4}$$

where $\mathcal{D}\left[\hat{O}\right] \equiv \hat{O}\rho\hat{O}^\dagger - \frac{1}{2}\left\{\hat{O}^\dagger\hat{O}, \rho\right\}$. This new notation is going to be used in this section and in `pyqosc`.

The objective is **to reach the steady state (i.e. the synchronized state) in a shorter time period by devising an optimal driving $\Omega_{1,2}^{(\text{short})}(t)$**. More specifically, the goal is to reach the steady state of the system driven by the Ω_2 term alone. If the driving with whatever constant value of Ω_2 (it must be constant so that the steady state can exist) achieves the steady state within some time period t_{ss}, then we target the steady state to be achieved within a shorter time period $\tau < t_{\text{ss}}$ with the optimal driving.

In principle, it is ρ which determines whether steady-state is reached. For harmonically driven quantum van der Pol oscillator, however, it is simpler to work with the quantity $\langle\beta\rangle_t = \mathrm{Tr}\left\{\rho\hat{b}\right\}$ (we write $\langle\beta\rangle_t$ instead of $\beta(t)$) thanks to the comparably simple equation of motion

$$\frac{\mathrm{d}\langle\beta\rangle_t}{\mathrm{d}t} = i\Delta\langle\beta\rangle_t + \frac{\gamma_1}{2}\langle\beta\rangle_t - \gamma_2\left\langle|\beta|^2\beta\right\rangle_t - (\Omega_2 + i\Omega_1) \tag{3.5}$$

(here $\left\langle|\beta|^2\beta\right\rangle_t = \mathrm{Tr}\left\{\rho(t)\hat{b}^\dagger\hat{b}\hat{b}\right\}$) and the fact that $\langle\beta\rangle_t$ is simply a point in the complex plane. We can characterize the steady state by **$\langle\beta\rangle_t$ getting fixed to a certain value in the complex phase plane rotating at the frequency of the drive**.

At $t = 0$, $\langle\beta\rangle_t$ starts out at some value $\langle\beta\rangle_0$ and moves toward the steady-state value of $\langle\beta\rangle_\infty$. The trajectory depends on the dynamics of the oscillator, meaning that if we change $\Omega_{1,2}$ then the trajectory changes as well. What is the shortest trajectory between two points in a plane? The answer is a straight line. The goal is thus **to find $\Omega_{1,2}$ which drives the system to evolve along the straight line connecting $\langle\beta\rangle_0$ and $\langle\beta\rangle_\infty$.**

I am not so sure about this. Speaking of a trajectory of the shortest time, I think of the brachistochrone problem. Although the shortest line connecting the initial and final point is a straight line, the trajectory between those points that takes the least time to track is not a straight line. For example, this can be caused by the nature of gravity which tends to pull the object downwards. I suspect that this problem has a similar twist. For example, forcing a straight line trajectory might only be useful if there is no detuning (that is, the phase point is stationary in the rotating phase space) and might take more effort with detuning present (where the phase point travels in a spiral-like trajectory and there are regions where the motion of the phase point opposes the target trajectory). **I thus find it interesting to try out different trajectories and see the difference in the average amplitude of the resulting driving.**

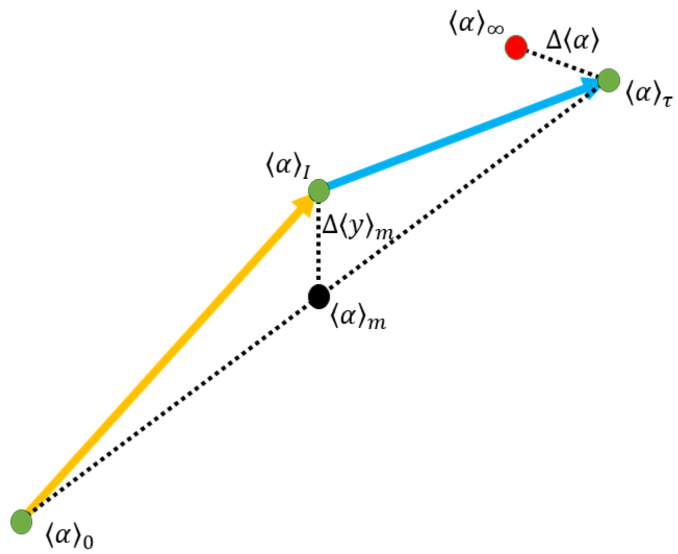

Figure 1: Cartoon of the shortcut trajectory.

The time evolution of $\langle\beta\rangle_t$ given by Eq. (3.5) depends on the moment $\left\langle|\beta|^2\beta\right\rangle_t$, so we need to make sure that **$\left\langle|\beta|^2\beta\right\rangle_t$ also reaches its steady state $\left\langle|\beta|^2\beta\right\rangle_\infty$.** Note that technically any observable of the system must match its steady state, but in this problem we only concern ourselves with $\langle\beta\rangle_t$ and $\left\langle|\beta|^2\beta\right\rangle_t$. Better results should be obtained if more observables are matched to their steady-state values. To measure how close $\left\langle|\beta|^2\beta\right\rangle_{tau}$ is to $\left\langle|\beta|^2\beta\right\rangle_\infty$, `Impens2023` devised the distance

$$\Delta_3\left(\Delta\langle y\rangle_m, \tau\right) \equiv \sqrt{\sum_{j=x,y}\left[\left\langle|\beta|^2 j\right\rangle_\tau - \left\langle|\beta|^2 j\right\rangle_\infty\right]^2} \tag{3.6}$$

First, we **set a target time τ at which $\langle\beta\rangle_t$ gets sufficiently close to $\langle\beta\rangle_\infty$ with $\left\langle|\beta|^2\beta\right\rangle_t$ sufficiently close to $\left\langle|\beta|^2\beta\right\rangle_\infty$.** The steady state values $\langle\beta\rangle_\infty \equiv \langle x\rangle_\infty + i\langle y\rangle_\infty$ and $\left\langle|\beta|^2\beta\right\rangle_\infty \equiv \left\langle|\beta|^2 x\right\rangle_\infty + i\left\langle|\beta|^2 y\right\rangle_\infty$ are given since we can calculate the steady-state density matrix ρ_∞, e.g. by using the `qutip.steadystate` function of QuTiP. The resulting shortcut driving $\Omega_{1,2}^{\text{short}}(t)$ shall be given to the system for $t \in [0,\tau]$. Afterwards, the system is driven with $\Omega_1(t) = 0$ and $\Omega_2(t) = \Omega_2$ where it continues towards synchronization. As can be testified, the driving brings the system close, but not exactly to synchronization. It is thus more appropriate to say that **we want to design a driving that can take the system close to the synchronized state of the original driving with less time, after which the original driving will do the rest of the work.**

With τ, $\langle\beta\rangle_0$, and $\langle\beta\rangle_\infty$ given, we may now write the value of $\langle\beta\rangle_t$ for $t \in (0,\tau)$. With a linear trajectory, we simply have

$$\langle\beta\rangle_t = \langle\beta\rangle_0 + \frac{t}{\tau}\left[\langle\beta\rangle_\infty - \langle\beta\rangle_0\right] \tag{3.7}$$

But note that a straight trajectory might give us a result for which the condition for $\left\langle |\beta|^2\beta \right\rangle_t$ is not satisfied. To circumvent this, **we can let the trajectory "relax" a little by defining a tunable intermediate point $\langle\beta\rangle_I$ reached at $\tau/2$.** This intermediate point splits the trajectory into two straight lines. Impens2023 defined

$$\langle\beta\rangle_I = \langle x\rangle_m + i\left[\langle y\rangle_m + \Delta\langle y\rangle_m\right] \tag{3.8}$$

where $\Delta\langle y\rangle_m$ is chosen such that the condition for $\left\langle |\beta|^2\beta \right\rangle_t$ is satisfied. Here $\langle x\rangle_m + i\langle y\rangle_m \equiv \langle\beta\rangle_m$ is the midpoint of the original straight trajectory. A longer trajectory does not mean a longer time needed; the magnitude of the drive can be increased so that the target state is reached under the same duration. The equation for $\langle\beta\rangle_t$ now reads

$$\langle\beta\rangle_t = \begin{cases} \langle\beta\rangle_0 + \frac{2t}{\tau}\left[\langle\beta\rangle_I - \langle\beta\rangle_0\right], & t \in \left[0, \frac{\tau}{2}\right] \\ \langle\beta\rangle_I + \frac{2t}{\tau}\left[\langle\beta\rangle_\infty - \langle\beta\rangle_I\right], & t \in \left[\frac{\tau}{2}, \tau\right] \end{cases} \tag{3.9}$$

With this equation, the next step is to **feed the values of $\langle\beta\rangle_t$ to Eq. (3.5)**. However, we do not know the value of $\left\langle |\beta|^2\beta \right\rangle_t$. To get away with this, we can **invoke the semiclassical approximation**:

$$\left\langle |\beta|^2\beta \right\rangle_t \approx \left|\langle\beta\rangle_t\right|^2 \langle\beta\rangle_t \tag{3.10}$$

Note that this approximation is less accurate in the strongly nonlinear case. We thus have

$$\Omega_2(t) + i\Omega_1(t) = -\frac{\mathrm{d}\langle\beta\rangle_t}{\mathrm{d}t} + i\Delta\langle\beta\rangle_t + \frac{\gamma_1}{2}\langle\beta\rangle_t - \gamma_2\left|\langle\beta\rangle_t\right|^2\langle\beta\rangle_t \tag{3.11}$$

We are not done. If the values of $\Omega_{1,2}$ we obtain from Eq. (3.11) (with the suitable choice of τ and $\Delta\langle y\rangle_m$) are used to evolve the system, we might find that $\langle\beta\rangle_\tau$ is not close enough to $\langle\beta\rangle_\infty$, i.e.

$$\langle\beta\rangle_\tau = \langle\beta\rangle_\infty + \Delta\langle\beta\rangle \tag{3.12}$$

In the case that the offset $\Delta\langle\beta\rangle$ is not small enough to our liking, we may iterate the procedure with a modified target:

$$\langle\beta\rangle_{\text{target}}^{(k+1)} = \langle\beta\rangle_{\text{target}}^{(k)} - \Delta\langle\beta\rangle^{(k)} \tag{3.13}$$

where $k = 1, 2, \ldots$ denotes the iteration number and $\langle\beta\rangle_{\text{target}}^{(1)} \equiv \langle\beta\rangle_\infty$. Impens2023 probably confused this, but

$$\Delta\langle\beta\rangle^{(k)} = \langle\beta\rangle_\tau^{(k)} - \langle\beta\rangle_\infty \tag{3.14}$$

i.e. the offset is always calculated with respect to the steady-state value, as it is what we want to achieve. Each iteration should reduce the value of the error $\Delta\langle\beta\rangle^{(k)}$ obtained. **When the error gets small enough, we can stop and finally get the optimal $\Omega_{1,2}$.**

4 Formulating the Research Problem

Does shortcut-to-synchronization violate the quantum speed limit? While I can not analytically prove this, it is suggestive that StS does NOT violate the QSL. The QSL is inversely proportional to energy fluctuations, which is proportional to the Hamiltonian. This means that the larger the driving amplitude is, the smaller the QSL gets. A shortcut driving modulates the driving amplitude in time, thus changing the energy fluctuations and hence the QSL. To drive the system close to the steady state within a shorter time period, larger driving amplitude will be needed. Consequently, the QSL will drop accordingly.

My idea for this research is **to compare the performance of the StS algorithm devised by Ref. [4] for various target trajectories, initial states, and target time.** I am interested in how these trajectories perform for variations in the system's parameters, particularly the detuning Δ. This stems from the fact that a nonzero Δ means that the phase point of the system is rotating in the rotating frame. This means that a driving may either promote or oppose the motion of the phase point depending on the direction of the drive.

To compare the performance, it may be a good idea to come up with an expression of the figure of merit (FoM). Keep in mind that the point of the StS algorithm is to come up with a shortcut driving that takes us to the steady state corresponding to whatever constant-amplitude driving we have for the system. In the previous section, the constant amplitude is set to be $\Omega_1 = 0, \Omega_2 = 1$, but let us be more general here and write $\Omega_1 = 0, \Omega_2 = \Omega_2$. Let ρ_{ss} be the steady state corresponding to this Hamiltonian. The shortcut driving is followed by the $\Omega_1 = 0, \Omega_2 = \Omega_2$ constant driving at $t = \tau$ to reach the steady state and keep the system there. Here are some factors we can consider:

- The **average total amplitude** $\left\langle\left|\Omega^{(\text{short})}\right|\right\rangle_\tau$ of the shortcut driving, defined by

$$\left\langle\left|\Omega^{(\text{short})}\right|\right\rangle_\tau \equiv \frac{1}{\tau}\int_0^\tau \left\{\left|\Omega_1^{\text{short}}(t)\right| + \left|\Omega_2^{\text{short}}(t)\right|\right\} dt \tag{4.1}$$

  in the continuous case. Smaller average total amplitude means less work, which is better. To take into account different values of Ω_2, we can define what I call the **amplitude ratio**

$$\alpha = \frac{\left\langle\left|\Omega^{(\text{short})}\right|\right\rangle_\tau}{\Omega_2} \tag{4.2}$$

  The smaller this value, the more energy efficient the shortcut driving is. I am pretty sure that α is no less than unity, since driving the system to synchronization in a smaller time period must require more work.

- The trace distance $d_{\text{tr}}\left(\rho_\tau^{(\text{short})}, \rho_{\text{ss}}\right)$ between the shortcut-driven system at time τ and the target steady-state. Ideally, we target the steady state to be reached at τ, i.e. that $d_{\text{tr}}\left(\rho_\tau^{(\text{short})}, \rho_{\text{ss}}\right) = 0$. However, this is generally not the case. Larger value means worse performance. Additionally, the trace distance $d_{\text{tr}}\left(\rho_0, \rho_{\text{ss}}\right)$ between the initial density matrix and the steady state density matrix depends on the initial state. It may be the case that $d_{\text{tr}}\left(\rho_\tau^{(\text{short})}, \rho_{\text{ss}}\right)$ is large since $d_{\text{tr}}\left(\rho_0, \rho_{\text{ss}}\right)$ is large to begin with. To this end, I would like to introduce the **fractional trace distance reduction**, defined as

$$\chi \equiv \frac{d_{\text{tr}}\left(\rho_0, \rho_{\text{ss}}\right) - d_{\text{tr}}\left(\rho_\tau^{(\text{short})}, \rho_{\text{ss}}\right)}{d_{\text{tr}}\left(\rho_0, \rho_{\text{ss}}\right)} \tag{4.3}$$

  This quantity tells us how much the trace distance is reduced due to the shortcut driving. Ideally, we want $d_{\text{tr}}\left(\rho_\tau^{(\text{short})}, \rho_{\text{ss}}\right) = 0$ so that $\chi = 1$. Generally, χ is somewhere between 0 and 1. The larger χ is, the better the shortcut driving.

- Two shortcut drivings may end up with the same $d_{\mathrm{tr}}\left(\rho_\tau^{(\mathrm{short})},\rho_{\mathrm{ss}}\right)$, but the time it takes for the original driving to drive the system to synchronization might differ. We may additionally define what I would like to call the **speed-up ratio**:

$$\sigma \equiv \frac{T_{\mathrm{ss}}}{T_{\mathrm{ss}}^{(\mathrm{short})}} \tag{4.4}$$

This quantity tells us how much faster the steady state is attained with the use of the shortcut driving compared to the constant driving. Here T_{ss} is the time it takes for the system to reach the steady state under the constant $\Omega_1 = 0, \Omega_2 = \Omega_2$ driving, while $T_{\mathrm{ss}}^{(\mathrm{short})}$ is the time it takes to reach the steady state under the shortcut driving plus the constant driving for the rest of the way. The larger σ is, the better.

Putting these quantities together, the FoM can be expressed as

$$\mathrm{FoM} = \frac{\sigma\chi}{\alpha} \tag{4.5}$$

Larger FoM means better performance of the shortcut driving.

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