Fast trajectory tracking of the steady state of open quantum systems

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We present a control strategy to speed up the steady-state tracking for open quantum systems governed by a time-dependent Markovian master equation. With this fast trajectory tracking strategy, the quantum state strictly follows the instantaneous steady state of open quantum systems and evolves into the target state without any fidelity loss. A compact expression for the counterdiabatic term is derived in terms of a superoperator formalism in Liouville space. To be specific, we utilize a universal method to transform the counterdiabatic superoperator into the ordinary operator form. It can be seen from the operator form that in addition to the coherent control field, an incoherent control is required for the fast trajectory tracking. The control strategy is successfully applied in two examples: One is a two-level quantum system with a time-dependent Hamiltonian coupling to a heat bath, the other is a two-level system following an isothermal process.

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I. INTRODUCTION

One of the main goals of quantum control theory is to characterize the achievable dynamics of a system in relation to a given set of control fields. Such goals have led to the formulation of quantum optimal control theory [1–3]. A quantum trajectory tracking scheme provides a powerful strategy to identify applied fields to control a quantum state into the target state along a desired trajectory in Hilbert space [4–8]. Generally speaking, quantum trajectory tracking control typically involves two steps: (i) specifying a trajectory in Hilbert space according to the goal of the control; (ii) determining control fields to force the quantum state to follow strictly the specified trajectory in time.

Based on the quantum adiabatic theorem, adiabatic trajectory tracking, as a sort of quantum control strategy for tracking state, offers a way to produce target eigenstates [9–12]. The adiabatic theorem of closed quantum systems states that if we prepare a quantum system in an eigenstate of the initial Hamiltonian, it will remain in that instantaneous eigenstate if the evolution time is infinitely long, even though the eigenvalue itself could be time-dependent [13,14]. The performance of adiabatic trajectory tracking is dictated by a long evolution time compared to the inverse of the power of the energy gap [15,16]. However, in a real-world setting, physical systems cannot avoid interacting with their environment, which results in an unwanted transition between instantaneous eigenstates [17–19]. Thus the performance of the adiabatic tracking control is limited by the competition between the adiabatic time scale and the typically short decoherence time scales. Such a competition requires an optimal time scale for adiabatic processes in decohering environments [20,21]. A straightforward method for overcoming this dilemma are shortcuts to adiabaticity (STAs) or the fast eigenstate tracking of the Hamiltonian. The common approach in STAs is to nullify the nonadiabatic coupling by introducing so-called counterdiabatic control [22]. Recently, STAs were used in the ground-state tracking control of open quantum systems, which illustrates that the trajectory tracking control based on the STAs is more robust than the adiabatic tracking method [19,23]. However, the decoherence cannot be eliminated in the whole fast eigenstate tracking process. To completely compress decoherence and design robust, fast processes or devices, the approach of shortcuts to adiabatic decoherence-free subspaces is used, in which quantum states in time-dependent decoherence-free subspaces are protected by the symmetry of the environment [24,25].

On the other hand, adiabatic tracking control can also be used in mixed-state engineering of open quantum systems, which is ensured by the adiabatic theorem of open quantum systems [20]. To accelerate the adiabatic mixed-state tracking, the transitionless quantum driving method for open quantum systems is generalized into open quantum systems [26]. The method states that if one prepares the initial state of open quantum systems in the quasieigenstates (Jordan blocks) of the reference Liouvillian superoperator $\mathcal{L}(t)$, the quantum state will strictly follow the instantaneous quasieigenstates to the final state nonadiabatically by adding the counterdiabatic terms in the reference Liouvillian superoperator. Nevertheless, when this transitionless quantum driving method for open quantum systems is used to accelerate the adiabatic mixed-state tracking, two difficulties have to be faced in formulating a control strategy. First, it is difficult to calculate the superoperator and determine the Jordan decomposition in practice, especially for complicated disturbances. Second, one cannot directly determine concrete control fields using the counterdiabatic term with the superoperator form.

To overcome those difficulties, we focus our attention here on tracking adiabatically the instantaneous steady state, which

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is built on a rigorous generalization of the quantum adiabatic theorem for open systems proposed by Venuti *et al.* [27]. As mentioned in Ref. [27], the adiabaticity of open quantum systems is defined in terms of their closeness to the instantaneous steady state. In analogy to the closed-system case, in which the system follows the instantaneous (pure) eigenstates of the Hamiltonian in the adiabatic limit of arbitrarily large total evolution time, the open system follows the instantaneous steady state of the Liouvillian. This kind of adiabatic tracking control has recently proved to be effective in the study of general time-dependent Liouvillians with higher-dimensional kernels [27–29]. Moreover, adiabatic steady-state tracking has been widely applied in the fields of the quantum-information process [30,31], quantum control [19,32,33], quantum annealing [34,35], and quantum nonequilibrium thermodynamics [36,37].

The remainder of this paper is organized as follows. In Sec. II, we present a strategy for tracking the steady state of an open system. By restricting the evolution of open quantum systems to ideal adiabatic evolution [14], the counterdiabatic superoperator can be expressed as the time derivative of the spectral projection of the instantaneous steady state. Therefore, our strategy only requires information about the steady state of open quantum systems, but the complete spectral decomposition of the Liouvillians is not necessary, which makes our method more implementable in practice. The expression for the counterdiabatic terms in ordinary operator form is also given, which shows that both incoherent control and coherent control are required in the fast steady-state tracking strategy. In Sec. III, we present some applications of the fast trajectory tracking of the steady state. First, we consider an open two-level system with a time-dependent Hamiltonian (Sec. III A) or a time-dependent Lindbladian (Sec. III B). The results show that the incoherent control is indispensable, since the evolution of the spectral projection of the instantaneous steady state is nonunitary. Moreover, the incoherent control terms in two models can be transformed into Lindblad form, which helps us to understand the physical mechanism of incoherent counterdiabatic control [38]. The numeral results illustrate that our method can transfer the quantum state into the final state without any fidelity loss at an arbitrary moment. It also shows that there are parts of counterdiabatic terms with negative decoherence strength, which is unphysical and cannot be achieved in experiment. Therefore, we propose a modulated tracking control strategy in Sec. III C. In this modulated control strategy, we ignore the terms with negative decoherence strength, and we amplify the other terms. The numerical results show that the fidelity is robust against such modification if the amplification parameters are carefully chosen. In Sec. III D, we present an example of the Ξ-type three-level system coupling to a broadband time-dependent squeezing vacuum reservoir. It can be seen that if the steady state is pure, the counterdiabatic control is totally coherent. Finally, we conclude in Sec. IV.

II. THE FAST TRAJECTORY TRACKING OF A STEADY STATE

We assume that the evolution of a finite-dimensional quantum system described by a linear, time-local master equation $\partial_t\rho(t) = \mathcal{L}_0(t)\rho(t)$, in which $\mathcal{L}_0(t)$ is a time-dependent Liouvillian. The final state of an open quantum system can be expressed as $\rho(t_f) = \mathcal{E}_0(t_f, 0)\rho(0)$, where t_f is the total evolution time, and the evolution operator $\mathcal{E}_0(t_f, 0)$ satisfies

$$\partial_t\mathcal{E}_0(t, t_0) = \mathcal{L}_0(t)\mathcal{E}_0(t, t_0), \tag{1}$$

with $\mathcal{E}_0(t, t) = \mathbf{I}$. We further assume that the Liouvillian $\mathcal{L}_0(t)$ can be written in Lindblad form, i.e.,

$$\mathcal{L}_0(t)\rho = -i[H_0(t), \rho] + \sum_\alpha \mathcal{D}[L_\alpha](\rho), \tag{2}$$

where $H_0(t)$ is the time-dependent Hamiltonian of open quantum systems and $\mathcal{D}[L_\alpha]$ is the time-dependent Lindbladian

$$\mathcal{D}[L_\alpha](\rho) = L_\alpha(t)\rho L_\alpha^\dagger(t) - \tfrac{1}{2}\{L_\alpha^\dagger(t)L_\alpha(t), \rho\}, \tag{3}$$

with the Lindblad operator $L_\alpha(t)$. We assume that Eq. (2) describes a Markovian dynamics with a time-dependent generator, and the corresponding evolution operator $\mathcal{E}(t, t_0)$ is a completely positive trace-preserving (CPTP) map for any $t \geqslant t_0$ [39]. We further assume that for all time t, $\mathcal{L}_0(t)$ admits a unique (instantaneous) steady state $\rho_0(t)$, which satisfies

$$\mathcal{L}_0(t)\rho_0(t) = 0. \tag{4}$$

In the following, we are ready to informally state the adiabatic theorem of the adiabatic steady-state process: if a system is initialized at $t = 0$ in the steady state of the initial Liouvillian $\mathcal{L}_0(0)$,the final state at $t = t_f$ will be close to the instantaneous steady state of the final Liouvillian $\mathcal{L}_0(t_f)$, provided the Liouvillian changes sufficiently slowly [27]. This theorem is the theoretical foundation of the adiabatic tracking control strategy of the steady state.

Our focus is on accelerating the adiabatic tracking of steady states with the reference Liouvillian $\mathcal{L}_0(t)$. To transfer the quantum state of open quantum systems along the trajectory of the instantaneous steady state of $\mathcal{L}_0(t)$, we introduce the counterdiabatic superoperator $\mathcal{L}_c(t)$ into the master equation. Therefore, the evolution of open quantum systems is governed by the following master equation:

$$\partial_t\rho(t) = [\mathcal{L}_0(t) + \mathcal{L}_c(t)]\rho(t), \tag{5}$$

and the corresponding evolution operator $\mathcal{E}(t)$ satisfies

$$\partial_t\mathcal{E}(t, t_0) = [\mathcal{L}_0(t) + \mathcal{L}_c(t)]\mathcal{E}(t, t_0), \tag{6}$$

with $\mathcal{E}(t_0, t_0) = \mathbf{I}$. On the other hand, the ideal adiabatic evolution is represented by a superoperator $V(t)$ that satisfies the intertwining property: $V(t)P(0) = P(t)V(t)$, where $P(t)$ denotes the spectral projection of the instantaneous steady state. It has been shown that a possible choice for $V(t)$ is given by the solution of the differential equation [14],

$$\partial_t V(t) = [\partial_t P(t), P(t)]\, V(t), \tag{7}$$

with $V(0) = \mathbf{I}$. (The details can be found in Appendix A.) However, $V(t)$ is not CPTP in general. Instead, it has been proven that $V(t)P(0)$ is a CPTP map since it can be written as a product of projectors [27]. Since the fast trajectory tracking of the quantum steady state requires that the quantum state evolves into the target state along the instantaneous steady state of open quantum systems, the evolution operator $\mathcal{E}(t, t_0)$ must completely cover the ideal adiabatic evolution $V(t)$. In

other words, the following conditions (the STAs condition) have to be guaranteed: (i) $\mathcal{E}(t)P(0)$ must be the same with $V(t)P(0)$ at any moment t,

$$\mathcal{E}(t)P(0) - V(t)P(0) = \mathbf{0},\ \forall\, t; \tag{8}$$

(ii) the equivalency between $\mathcal{E}(t)P(0)$ and $V(t)P(0)$ must be dynamical stable, i.e.

$$\partial_t[\mathcal{E}(t)P(0) - V(t)P(0)] = \mathbf{0}. \tag{9}$$

We are interested in how to design a proper counterdiabatic superoperator $\mathcal{L}_c(t)$ to compel the open quantum system following the instantaneous steady state into the final state. According to the STA condition, we obtain

$$\begin{aligned}\mathbf{0} &= \partial_t[\mathcal{E}(t)P(0) - V(t)P(0)] \\ &= [\mathcal{L}_0(t) + \mathcal{L}_c(t)]\mathcal{E}(t)P(0) \\ &\quad - [\partial_t P(t), P(t)]V(t)P(0),\end{aligned} \tag{10}$$

in which Eqs. (6) and (7) have been used. Further, we use the other STA condition Eq. (8), i.e., $\mathcal{E}(t)P(0) = V(t)P(0)$, to obtain

$$\begin{aligned}&\partial_t[\mathcal{E}(t)P(0) - V(t)P(0)] \\ &\quad = \{\mathcal{L}_0(t) + \mathcal{L}_c(t) - [\partial_t P(t), P(t)]\}P(t)V(t),\end{aligned} \tag{11}$$

where the intertwining property, i.e., $P(t)V(t) = V(t)P(0)$, has been considered. By differentiating $P^2(t)$, $\partial_t P^2(t) = \partial_t P(t)P(t) + P(t)\partial_t P(t)$. After right multiplying by $P(t)$, we obtain $P(t)\partial_t P(t)P(t) = 0$, so that

$$[\partial_t P(t), P(t)]P(t)V(t) = \partial_t P(t)P(t)V(t). \tag{12}$$

On the other hand, since $P(t)$ is the projector on the instantaneous steady state,

$$\mathcal{L}_0(t)P(t) = 0. \tag{13}$$

Considering Eqs. (12) and (13), the STA condition Eq. (11) can be rewritten as

$$\begin{aligned}\mathbf{0} &= \partial_t[\mathcal{E}(t)P(0) - V(t)P(0)] \\ &= [\mathcal{L}_c(t) - \partial_t P(t)]P(t)V(t).\end{aligned} \tag{14}$$

As a result, if the counterdiabatic superoperator is designed as follows:

$$\mathcal{L}_c(t) = \partial_t P(t), \tag{15}$$

the STA condition is satisfied definitely. This is the main result obtained in this paper.

Since the counterdiabatic superoperator $\mathcal{L}_c(t)$ is not convenient to determine the control scheme on the open quantum system, we transform the counterdiabatic superoperator $\mathcal{L}_c(t)$ into the ordinary operators form. Here, we switch to "bra-ket" notation for the superoperator [40]:

$$\begin{aligned}\mathcal{L}_c(t)\rho_0(t) &\leftrightarrow \hat{\mathcal{L}}_c(t)|\rho_0(t)\rangle\rangle, \\ \mathrm{Tr}\{X^\dagger Y\} &\leftrightarrow \langle\langle X|Y\rangle\rangle, \\ X\rho_0(t)Y^\dagger &\leftrightarrow (X\otimes Y^*)|\rho_0(t)\rangle\rangle,\end{aligned}$$

where $*$ denotes complex conjugation. Since any physical operator X can always be written as $X = \sum_{\mu=0}^{N^2-1} x_\mu T_\mu$ with the SU(N) Hermitian generators $\{T_\mu\}|_{\mu=0}^{N^2-1}$, the counterdiabatic superoperator can be expanded by the bases $\{T_\mu \otimes T_\nu^*\}$, i.e.,

$$\hat{\mathcal{L}}_c(t) = \sum_{\mu=0}^{N^2-1}\sum_{\nu=0}^{N^2-1} c_{\mu\nu}(t)T_\mu \otimes T_\nu^*. \tag{16}$$

$c_{\mu\nu}(t)$ is a time-dependent expanding coefficient, which can be determined by

$$c_{\mu\nu}(t) = \mathrm{Tr}(\hat{\mathcal{L}}_c(t)T_\mu \otimes T_\nu^*). \tag{17}$$

Here, we set $T_0 = \mathbf{I}$, where $\mathbf{I}$ is the identity operator. Without loss of generality, we denote $c_{\nu 0}$ as C_ν. Further, the counterdiabatic superoperator can be rewritten in the ordinary operator form,

$$\mathcal{L}_c(t)\rho(t) = \sum_{\mu=0}^{N^2-1}\sum_{\nu=0}^{N^2-1} c_{\mu\nu}(t)T_\mu\rho(t)T_\nu. \tag{18}$$

Because $\mathcal{L}_c(t)$ does not break the hermiticity of the density matrix, one may obtain $\mathcal{L}_c(t)\rho(t) = [\mathcal{L}_c(t)\rho(t)]^\dagger$. Therefore, according to Eq. (18), the coefficient matrix $\mathbf{c} = (c_{\mu\nu})$ is Hermitian, i.e., $c_{\mu\nu} = c_{\nu\mu}^*$. The expansion as Eq. (18) can be rewritten as

$$\begin{aligned}\mathcal{L}_c(t)\rho(t) &= \sum_{\nu=0}^{N^2}[C_\nu(t)T_\nu\rho(t) + C_\nu^*(t)\rho(t)T_\nu] \\ &\quad + \sum_{\mu=1}^{N^2-1}\sum_{\nu=1}^{N^2-1} c_{\mu\nu}(t)T_\mu\rho(t)T_\nu \\ &= i\sum_{\nu=0}^{N^2-1}\mathrm{Im}(C_\nu(t))[T_\nu, \rho(t)] \\ &\quad + \sum_{\nu=0}^{N^2-1}\mathrm{Re}(C_\nu(t))\{T_\nu, \rho(t)\} \\ &\quad + \sum_{\mu=1}^{N^2-1}\sum_{\nu=1}^{N^2-1} c_{\mu\nu}(t)T_\mu\rho(t)T_\nu,\end{aligned} \tag{19}$$

where $\mathrm{Im}(C_\nu(t))$ and $\mathrm{Re}(C_\nu(t))$ are the imaginary and real part of $C_\nu(t)$, respectively. In Eq. (19), the first term denotes the coherent control part of the counterdiabatic term $\mathcal{L}_c(t)$ with the control Hamiltonian

$$H_c(t) = \sum_{\nu=0}^{N^2}\mathrm{Im}(C_\nu(t))T_\nu. \tag{20}$$

The last two terms are the incoherent controls on the open quantum systems, in which the second term is the non-Hermitian part of the control Hamiltonian and the third term describes the trajectory jump caused by $\mathcal{L}_c(t)$.

When the steady state is a pure state, the fast trajectory tracking of the steady state will degenerate into the STAs for the adiabatic decoherence-free subspaces [24]. Assume that the steady state of open quantum systems is a pure state, which can be expanded by the bases of the time-dependent decoherence-free subspaces (TDFSs) $\{|\Phi_i(t)\rangle\}_{i=1}^M$ with $M \leqslant N$ [41]. The unique steady state can be expressed

as $\rho_0(t) = |\phi(t)\rangle\langle\phi(t)|$ with $|\phi(t)\rangle = \sum_{i=1}^{M} a_i(t)|\Phi_i(t)\rangle$. The corresponding superoperator projector on this TDFS reads $\mathcal{P}(t) = P(t) \otimes P^*(t)$, where $P(t) = |\phi(t)\rangle\langle\phi(t)|$ is the ordinary project operator. Placing $\mathcal{P}(t)$ into Eq. (15), we immediately obtain

$$\hat{\mathcal{L}}_c(t) = \partial_t P(t) \otimes P^*(t) + P(t) \otimes \partial_t P^*(t),$$

which can also be expressed in the ordinary form

$$\mathcal{L}_c(t)\rho_0(t) = \partial_t P(t)\rho_0(t)P(t) + P(t)\rho_0(t)\partial_t P(t).$$

Considering $\rho_0(t)P(t) = P(t)\rho_0(t) = \rho_0(t)$ and $\partial_t P(t) = |\partial_t\phi(t)\rangle\langle\phi(t)| + |\phi(t)\rangle\langle\partial_t\phi(t)|$, the counterdiabatic term can be written as

$$\mathcal{L}_c(t)\rho_0(t) = -i[H_c(t), \rho_0(t)]$$

with the counterdiabatic Hamiltonian

$$H_c(t) = i|\partial_t\phi(t)\rangle\langle\phi(t)|,$$

which is the very condition used in the STAs of the adiabatic decoherence-free subspaces [24,42].

III. EXAMPLES

A. Example 1: The two-level system coupling to the heat bath

To illustrate the fast trajectory tracking control of the steady state, we consider a two-level system with Hamiltonian

$$H_0(t) = \frac{\hbar\omega_0}{2}\sigma_z + H_c(t), \tag{21}$$

where ω_0 is the frequency difference, $H_c(t)$ denotes the general control Hamiltonian, and σ_z is the z component of the Pauli matrix. The two-level system couples to a bosonic heat reservoir at finite temperature T. In the interaction picture with respect to $\hbar\omega_0\sigma_z/2$, the dynamics of the two-level system is governed by the following Lindblad master equation:

$$\begin{aligned}\partial_t\rho(t) &= \mathcal{L}_0(t)\rho(t)\\ &= -i\big[H_c^I(t), \rho(t)\big]\\ &\quad + \gamma(N+1)\big(\sigma_-\rho(t)\sigma_+ - \tfrac{1}{2}\{\sigma_+\sigma_-, \rho(t)\}\big)\\ &\quad + \gamma N\big(\sigma_+\rho(t)\sigma_- - \tfrac{1}{2}\{\sigma_-\sigma_+, \rho(t)\}\big),\end{aligned} \tag{22}$$

in which $\gamma = \gamma_0\omega_0$ denotes the decoherence strength, and $N = [\exp(\hbar\omega_0/kT) - 1]^{-1}$ is the mean excitation number. Both of them are associated with the spectral density $J(\omega) \propto \omega_0$ and the temperature T of the environment. Here we assume that the control Hamiltonian takes the form

$$H_c^I(t) = \Omega(t)\sigma_x, \tag{23}$$

with the time-dependent control field $\Omega(t) = \Omega_0\cos(\omega t)$, where ω is the change rate of the control field.

In the following, we use the "bra-ket" notation for the superoperator, and we reshape the density matrix into a 1×4 complex vector. The density matrix can be written as $|\rho(t)\rangle\rangle = (\rho_{00}(t), \rho_{01}(t), \rho_{10}(t), \rho_{11}(t))^{\mathbf{T}}$, where $\rho_{ij}(t) = \langle i|\rho(t)|j\rangle$ with $i = 0, 1$. $|0\rangle$ and $|1\rangle$ are the eigenstates of σ_z. On the other hand, the superoperator $\hat{\mathcal{L}}_0(t)$ can be expressed as a 4×4 matrix,

$$\hat{\mathcal{L}}_0(t) = \gamma\begin{pmatrix} -N-1 & \frac{i\Omega}{2\gamma} & -\frac{i\Omega}{2\gamma} & N \\ \frac{i\Omega}{2\gamma} & -\frac{2N+1}{2} & 0 & -\frac{i\Omega}{2\gamma} \\ -\frac{i\Omega}{2\gamma} & 0 & -\frac{2N+1}{2} & \frac{i\Omega}{2\gamma} \\ N+1 & -\frac{i\Omega}{2\gamma} & \frac{i\Omega}{2\gamma} & -N \end{pmatrix}. \tag{24}$$

The steady state of a two-level system is given by the condition $\hat{\mathcal{L}}_0(t)|\rho_0(t)\rangle\rangle = 0$, which can be written as a 1×4 complex vector,

$$|\rho_0\rangle\rangle = \frac{1}{p}\begin{pmatrix} N(2N+1) + (\Omega/\gamma)^2 \\ -i\Omega/\gamma \\ i\Omega/\gamma \\ (N+1)(2N+1) + (\Omega/\gamma)^2 \end{pmatrix} \tag{25}$$

with the normalization factor $p = (2N+1)^2 + 2(\Omega/\gamma)^2$. Obviously, due to $\langle\langle\rho_0|\rho_0\rangle\rangle = 1$, one can obtain $\langle\langle\rho_0| = \langle\langle I| = (1, 0, 0, 1)$. As mentioned in the adiabatic theorem of the adiabatic steady-state process, when the control field $\Omega(t)$ is varied slowly enough, the quantum state of the two-level system will follow the instantaneous steady state Eq. (25) into the target state [27].

In the following, we determine the counterdiabatic superoperator for the open two-level system. As above, the projector on the instantaneous steady state can be expressed as $P(t) = |\rho_0(t)\rangle\rangle\langle\langle\rho_0(t)|$. By considering the counterdiabatic superoperator $\hat{\mathcal{L}}_c(t)$, the evolution of the quantum state is described by the following master equation:

$$\partial_t\rho(t) = [\mathcal{L}_0(t) + \mathcal{L}_c(t)]\rho(t). \tag{26}$$

Putting Eq. (25) into Eq. (15), the counterdiabatic superoperator can be written as

$$\hat{\mathcal{L}}_c(t) = f(t)\begin{pmatrix} -\Omega_0\sin(2\omega t)z_1(t) & 0 & 0 & -\Omega_0\sin(2\omega t)z_1(t) \\ i\sin(\omega t)z_2(t) & 0 & 0 & i\sin(\omega t)z_2(t) \\ -i\sin(\omega t)z_2(t) & 0 & 0 & -i\sin(\omega t)z_2(t) \\ \Omega_0\sin(2\omega t)z_1(t) & 0 & 0 & \Omega_0\sin(2\omega t)z_1(t) \end{pmatrix} \tag{27}$$

with

$$f(t) = \frac{\omega\gamma\Omega_0}{\left(\gamma^2(2N+1)^2 + 2\Omega_0^2\cos(\omega t)^2\right)^2}, \tag{28}$$

$$z_1(t) = \gamma(2N+1), \tag{29}$$

$$z_2(t) = 2\Omega_0^2\cos(\omega t)^2 - \gamma^2(2N+1)^2. \tag{30}$$

Further, we express the counterdiabatic superoperator $\mathcal{L}_c(t)$ into the ordinary operator form, which can be expanded by the Pauli matrices as Eq. (18),

$$\begin{aligned}\mathcal{L}_c(t)\rho(t) &= c_y(t)\{\sigma_y, \rho(t)\} + c_z(t)\{\sigma_z, \rho(t)\}\\ &\quad + ic_z(t)\sigma_x\rho(t)\sigma_y - ic_z(t)\sigma_y\rho(t)\sigma_x\\ &\quad - ic_y(t)\sigma_x\rho(t)\sigma_z + ic_y(t)\sigma_z\rho(t)\sigma_x,\end{aligned} \tag{31}$$

with

$$c_y(t) = -2f(t)\sin(\omega t)z_2(t),$$
$$c_z(t) = -2\Omega_0 f(t)\sin(2\omega t)z_1(t).$$

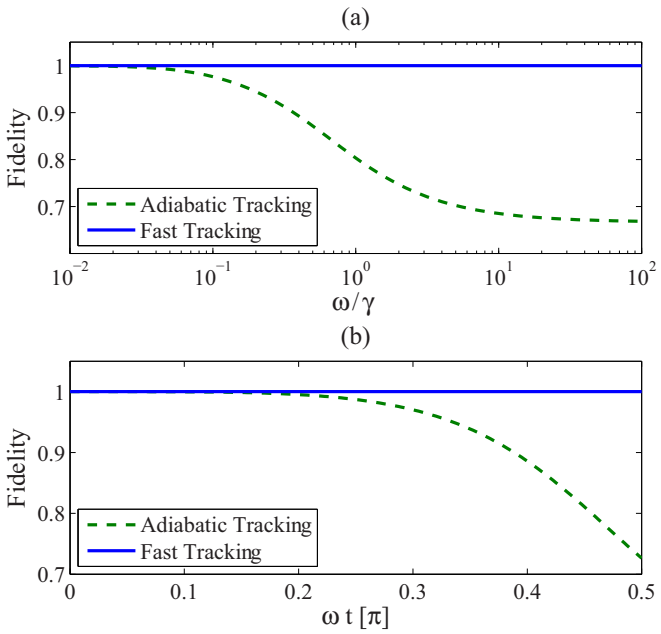

FIG. 1. (a) The final fidelity vs the change rate ω with $\omega t_f = \pi/2$; (b) the evolution of the fidelity with the change rate $\omega = \gamma$ for the adiabatic tracking control (green dash line) and the fast tracking control (blue solid line) at the zero-temperature limit ($T = 0$). The control field strength is set to be $\Omega_0 = 10\gamma$. The above parameters are in units of γ.

Noting that $[\sigma_x, \sigma_y] = 2i\sigma_z$ and $[\sigma_z, \sigma_x] = 2i\sigma_y$, the counterdiabatic term $\mathcal{L}_c(t)$ can be transformed into the following master equation:

$$\mathcal{L}_c(t)\rho(t) = ic_y(t)\left(\tfrac{1}{2}\{\sigma_z\sigma_x, \rho(t)\} - \sigma_x\rho(t)\sigma_z\right) + ic_z(t)\left(\tfrac{1}{2}\{\sigma_y\sigma_x, \rho(t)\} - \sigma_x\rho(t)\sigma_y\right) + \text{H.c.} \quad (32)$$

Since the coefficient matrix

$$\mathbf{c}(t) = \begin{pmatrix} 0 & ic_z(t) & ic_y(t) & 0 \\ -ic_z(t) & 0 & 0 & 0 \\ -ic_y(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (33)$$

is Hermitian, it can be diagonalized with a unitary transformation u, i.e., $u\mathbf{c}u^\dagger = \text{diag}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. Therefore, we can rewrite the counterdiabatic term in diagonal form,

$$\mathcal{L}_c(t)\rho(t) = \sum_{i=1}^{2}\gamma_i(t)\left(\frac{1}{2}\{L_i^\dagger(t)L_i(t), \rho(t)\} - L_i(t)\rho(t)L_i^\dagger(t)\right), \quad (34)$$

in which the "damping rates" are

$$\gamma_1 = -\gamma_2 = \sqrt{c_y^2(t) + c_z^2(t)}, \quad (35)$$

and the Lindblad operators can be written as

$$L_1(t) = L_2^\dagger(t) = \frac{\sqrt{2}}{2}\left(i\sigma_x + \frac{c_z(t)}{\gamma_1}\sigma_y + \frac{c_y(t)}{\gamma_1}\sigma_z\right). \quad (36)$$

In Fig. 1, we plot the fidelity between the instantaneous steady state $\rho_0(t)$ and the quantum state $\rho(t)$ governed by the master equation (26) with (blue solid lines) and without (green dashed lines) the counterdiabatic term at zero bath temperature $T = 0$. When the adiabatic condition is satisfied, i.e., $\omega \ll \gamma$, the quantum state of the two-level system follows the instantaneous steady state with high fidelity, which has been shown in Fig. 1(a). If $\omega \sim \gamma$ or $\omega > \gamma$, the adiabatic condition is broken, so that the fidelity decreases with the increase of ω. This can also be illustrated by Fig. 1(b) (green dashed line), in which we choose $\omega = \gamma$. The quantum state divides from the ideal adiabatic evolution, and the fidelity decreases monotonously with the evolution. On the other hand, when the counterdiabatic term is considered, the quantum state of the open two-level system will strictly track the trajectory of the instantaneous steady state, no matter how large ω/γ is, which has been shown by blue solid lines in Figs. 1(a) and 1(b).

It has been shown in Eq. (34) that the counterdiabatic control is totally incoherent. This is very essential for the fast trajectory tracking of the steady state for this model. According to the steady state Eq. (25), the initial steady state approaches the maximal mixed state at the initial moment with $N = 0$ and $\Omega_0 \gg \gamma$, but the final steady state at $t_f = \pi/2\omega$ is a pure state $|0\rangle$. The purity of the instantaneous steady state is given by

$$p(t) = \frac{2\,\Omega_0^4\cos(\omega t)^4 + 4\,\gamma^2\Omega_0^2\cos(\omega t)^2 + \gamma^4}{\left(\gamma^2 + 2\,\Omega_0^2\cos(\omega t)^2\right)^2}, \quad (37)$$

which illustrates the increase of the purity with the control process. Obviously, the coherent control scheme only induces the unitary evolution of quantum states. Yet the unitary transformation does not change the purity of the quantum state. Therefore, the incoherent control is essential for the fast trajectory tracking control strategy of the steady state.

B. Example 2: The isothermal process of the two-level system

An isothermal process is one in which, as the piston moves, the system remains in equilibrium at all times [43]. Meanwhile, the system is in contact with a heat source so that the temperature T of the gas in the cylinder remains fixed. Here we consider the isothermal process of the two-level system, which is governed by the master equation (22). We assume that the energy splitting $\omega_0(t)$ is changed in time and the coherent control field is absent ($\Omega_0 = 0$), while the temperature of the heat bath remains constant. Since the isothermal process requires that the quantum system must remain in its steady state, the adiabatic evolution of the two-level system is necessary in this process. In this section, we consider how to perform the fast tracking control strategy of the quantum isothermal process, and we present the counterdiabatic term correspondingly.

In the interaction picture, the Hamiltonian $H_I(t)$ vanishes, due to the absence of the coherent control field, i.e., $\Omega_0 = 0$. Instead, the decoherence strength and the mean excitation number depend on the modulated frequency

$$\gamma = \gamma_0\omega_0(t), \qquad N = \{\exp[\hbar\omega_0(t)/kT] - 1\}^{-1}. \quad (38)$$

At this time, the reference Lindbladian superoperator can be written as

$$\hat{\mathcal{L}}_0 = \gamma \begin{pmatrix} -N-1 & 0 & 0 & N \\ 0 & -\frac{2N+1}{2} & 0 & 0 \\ 0 & 0 & -\frac{2N+1}{2} & 0 \\ N+1 & 0 & 0 & -N \end{pmatrix}. \tag{39}$$

The instantaneous steady state of $\hat{\mathcal{L}}_0$ is a Gibbs state,

$$|\rho_0(t)\rangle\rangle = \tfrac{1}{2}[|I\rangle\rangle + z(t)|\sigma_z\rangle\rangle], \tag{40}$$

where $|I\rangle\rangle = (1,0,0,1)^{\mathbf{T}}$, $|\sigma_z\rangle\rangle = (1,0,0,-1)^{\mathbf{T}}$, and

$$z(t) = -\tanh\left(\frac{\hbar\omega_0(t)}{2kT}\right). \tag{41}$$

The projector on the steady state can be obtained immediately, i.e., $P(t) = |\rho_0(t)\rangle\rangle\langle\langle \mathbf{I}|$. Using Eq. (15), the counterdiabatic superoperator can be written as

$$\hat{\mathcal{L}}_c(t) = \frac{\partial_t z(t)}{2}|\sigma_z\rangle\rangle\langle\langle \mathbf{I}|. \tag{42}$$

Following steps of Eqs. (18) and (19), the counterdiabatic term can be rewritten into Lindblad form, i.e.,

$$\mathcal{L}_c(t)\rho = \gamma_c(t)\mathcal{D}[\sigma_+](\rho) - \gamma_c(t)\mathcal{D}[\sigma_-](\rho), \tag{43}$$

where

$$\begin{aligned}\gamma_c(t) &= \frac{\hbar\partial_t\omega_0(t)\exp[\hbar\omega_0(t)/kT]}{kT\{\exp[\hbar\omega_0(t)/kT]+1\}^2} \\ &= -\frac{\partial_t N}{(2N+1)^2}\end{aligned} \tag{44}$$

and

$$\begin{aligned}\mathcal{D}[\sigma_+](\rho) &= \tfrac{1}{2}\{\sigma_-\sigma_+, \rho\} - \sigma_+\rho\sigma_-, \\ \mathcal{D}[\sigma_-](\rho) &= \tfrac{1}{2}\{\sigma_+\sigma_-, \rho\} - \sigma_-\rho\sigma_+.\end{aligned} \tag{45}$$

At this time, the two-level system is governed by the following master equation:

$$\begin{aligned}\partial_t\rho(t) = {} & \gamma\left(N + \frac{\gamma_c}{\gamma} + 1\right)\left(\sigma_-\rho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho(t)\}\right) \\ & + \gamma\left(N - \frac{\gamma_c}{\gamma}\right)\left(\sigma_+\rho(t)\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho(t)\}\right).\end{aligned} \tag{46}$$

In Figs. 2(a) and 2(b), we plot the fidelity between $\rho(t)$ and $\rho_0(t)$ for the adiabatic tracking control (green dashed lines) and the fast trajectory tracking control (blue solid lines), in which the frequency difference depends on time linearly, i.e., $\omega_0(t) = \omega_i + \omega t$ with the initial energy difference $\omega_i = 0.1\gamma$. From Fig. 2(a), the result still illustrates that the fast trajectory tracking control forbids the transition from the steady state $\rho_0(t)$ and the other part of the Hilbert space. On the contrary, when the counterdiabatic term is absent, the isothermal process only occurs in the adiabatic case. On the other hand, the isothermal process with the fast trajectory tracking can be achieved with arbitrary bath temperature T, which is illustrated by Fig. 2(b). It is worth noting that, for the adiabatic tracking control, the fidelity is very close to 1 both in the low-temperature limit and the high-temperature limit.

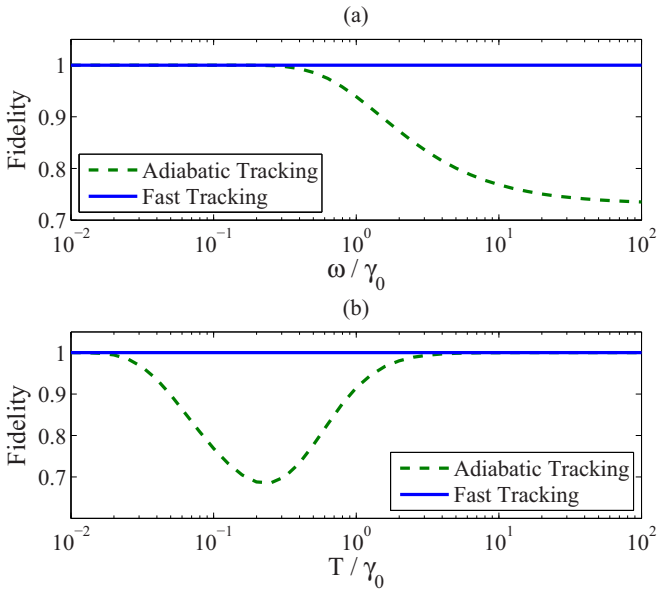

FIG. 2. (a) The final fidelity vs the change rate ω at the zero-temperature limit ($T = 0.1\gamma_0$), and (b) the fidelity vs the temperature of the bath T with the change rate $\omega = 10\gamma_0$ for the adiabatic engineering program (green dashed line) and the superadiabatic engineering program (blue solid line). The total evolution time satisfies $\omega_i + \omega t_f = \pi/2$ with $\omega_i = 0.1$. The above parameters are in units of γ.

This can be understand by Eq. (44): In the high-temperature limit, the excited number $N \to \infty$, so that $\gamma_c \to 0$; in the low-temperature limit, $kT\exp[\hbar\omega_0(t)/kT] \to \infty$, so that

$$\gamma_c(t) \approx \frac{\hbar\partial_t\omega_0(t)}{kT\exp[\hbar\omega_0(t)/kT]} \to 0.$$

C. Modulated tracking control strategy

Generally speaking, it is difficult to realize the counterdiabatic term such as Eq. (43) in experiment. From Eq. (46), we can see that, if $N \geqslant \gamma_c/\gamma$, the decoherence strength is positive. Otherwise, one of two terms in the Lindbladian will be negative. The same trouble also appears in the example of Sec. III A. In the following, we apply the isothermal process of the two-level system as an example to show how to modulate the counterdiabatic term overcoming this difficulty.

The trouble comes from the counterdiabatic term with the negative decoherence strength in Eq. (43), which is impossible in the theory of open quantum systems. However, when we check the counterdiabatic term Eq. (43) in detail, it can be found that the counterdiabatic term $\hat{\mathcal{L}}_c(t)$ provides another way to control the transition between $|0\rangle$ and $|1\rangle$: the superoperator $\hat{\mathcal{D}}[\sigma_+]$ in the first term of Eq. (43) incoherently transfers the population from $|0\rangle$ to $|1\rangle$, and the effect of the superoperator $\hat{\mathcal{D}}[\sigma_-]$ in the second term is inverse. In other words, the effect of the term with the negative decoherence strength [the first term in Eq. (43)] is to tone up the population on $|0\rangle$. The aims of the transition of two terms in Eq. (43) are the same. Therefore, we may drop the term with the negative decoherence strength, and "isomorphically" amplify the term with the positive decoherence strength. In such a case, the

modulated counterdiabatic term can be written as

$$\mathcal{L}_c^m(t)\rho = \begin{cases} A\,\gamma_c(t)\,\mathcal{D}[\sigma_+](\rho) & \text{if } \gamma_c(t) > 0, \\ -A\,\gamma_c(t)\,\mathcal{D}[\sigma_-](\rho) & \text{if } \gamma_c(t) < 0, \end{cases}$$

with the dimensionless amplification parameter A and the Lindbladian $\mathcal{D}[\cdot]$ given by Eq. (3). The numerical result is presented in Fig. 3. Note that the final fidelity with $A = 0$ is the result of the adiabatic tracking control, which has been illustrated by the green dashed line in Fig. 2(b) within the interval $T/\gamma \in [0.1, 10]$. When $A \neq 0$, the counterdiabatic term takes part in the dynamics of open quantum systems. As a whole, the dependence of the final fidelity on A is not monotonous. The final fidelity increases with the amplification parameter A at the beginning, then it decreases gradually. Therefore, we can choose a proper amplification parameter to get the maximum of the final fidelity. As shown in Fig. 3(a), more than 0.999 of the final fidelity F can be achieved, as long as the proper amplification parameter is chosen regardless of the environment temperature T. To illustrate how well the fidelity can be reached, we plot the infidelity (the deviation of the fidelity F from unity) in logarithmic scale $\log_{10}(1-F)$ in Fig. 3(b). It can be seen from the figure that a very high fidelity can be achieved by the modulated tracking control strategy, and the infidelity is far beyond 10^{-8} with a proper amplification parameter.

In fact, the effect of $\hat{\mathcal{L}}_c^m(t)$ is to continuously project the quantum state on $|1\rangle$ or $|0\rangle$, which is known as the quantum Zeno effect. This mechanism can be verified by Fig. 3(c), which presents the evolution of the fidelity with $T = 10^{-0.25}\gamma$ and $A = 5$. At the early time of the evolution, the quantum state driven by the modulated tracking scheme (red dot-dashed line) is pushed out of the trajectory given by the instantaneous steady state, just like the case of the adiabatic steady-state tracking (green dashed line). Due to the quantum Zeno effect induced by the environment, the quantum state is drawn back to the ideal trajectory gradually. Even though the modulated engineering program cannot restrict the quantum state evolving with the instantaneous steady state strictly, it is far better than the adiabatic tracking program.

D. Example 3: The Ξ-type three-level system coupling to a broadband time-dependent squeezing vacuum reservoir

Consider a Ξ-type atom coupled to a time-dependent broadband squeezed vacuum [44,45] in a squeezed vacuum initial state,

$$|\text{vac}(\eta)\rangle = K(\eta)|\text{vac}\rangle,$$

where $K(\eta)$ is a multimode squeezing transformation. In the rotating frame and under the Born-Markovian approximation, the evolution of the atom is governed by the following master equation [44]:

$$\begin{aligned}\partial_t\rho(t) &= \mathcal{L}_0(t)\rho(t) \\ &= -\frac{\gamma}{2}\{R^\dagger(\eta)R(\eta)\rho(t) \\ &\quad + \rho(t)R^\dagger(\eta)R(\eta) - 2R(\eta)\rho(t)R^\dagger(\eta)\}, \end{aligned} \tag{47}$$

where

$$R(\eta) = S\cosh(r) + \exp(i\phi)S^\dagger \sinh(r). \tag{48}$$

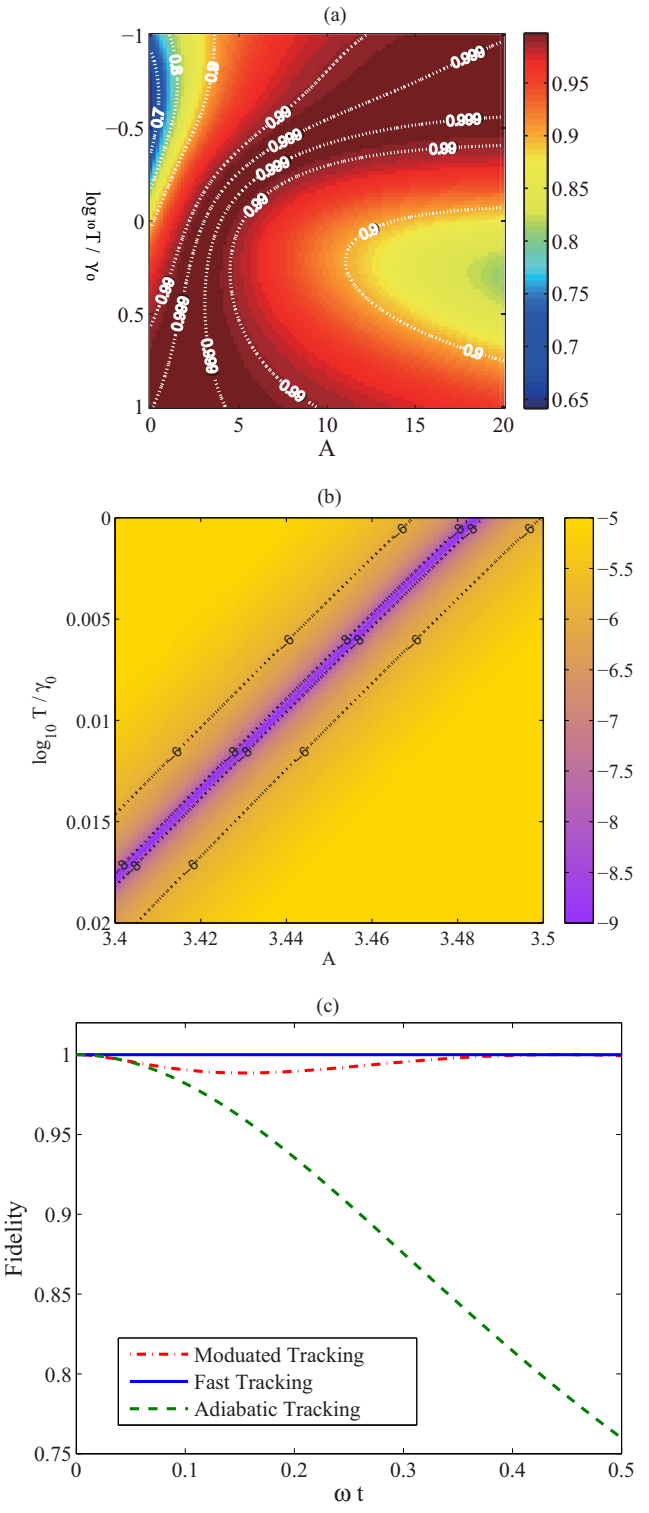

FIG. 3. (a) The final fidelity F vs the dimensionless amplification parameter A and bath temperature T; (b) the final infidelity vs the dimensionless amplification parameter A and bath temperature T in logarithmic scale; (c) the evolution of the fidelity with $A = 5$ and $T = 10^{-0.25}\gamma$. The other parameters are the same as those in Fig. 2.

$S = |-1\rangle\langle 0| + |0\rangle\langle -1|$ is the atomic operator. $\eta(t) = r\exp(i\phi)$ is the squeezing parameter with polar coordinates $\phi \in [0, 2\pi]$ and $r > 0$ called the phase and amplitude of the

squeezing, respectively. Here we assume that the phase of the squeezing parameter changes in time $\phi = \omega_0 t$. The steady state of the three-level system satisfies $\mathcal{L}_0(t)\rho_0(t) = 0$, which leads to the following instantaneous steady state:

$$\rho_0(t) = |\Phi_0(t)\rangle\langle\Phi_0(t)|. \tag{49}$$

The state

$$|\Phi_0(t)\rangle = c(r)|-1\rangle - \exp(i\omega_0 t)s(r)|1\rangle, \tag{50}$$

with $c(r) = \cosh(r)/\sqrt{\cosh(2r)}$ and $s(r) = \sinh(r)/\sqrt{\cosh(2r)}$, satisfies $R(\eta)|\Phi_0(t)\rangle = 0$. It is not difficult to transform the density matrix into

$$|\rho_0(t)\rangle\rangle = |\Phi_0(t)\rangle \otimes [|\Phi_0(t)\rangle]^*. \tag{51}$$

Thus, the projector on the steady state is obtained immediately, i.e., $P(t) = |\rho_0(t)\rangle\rangle\langle\langle \mathbf{I}|$. Taking the projector $P(t)$ into Eq. (15), the counterdiabatic terms can be written as

$$\begin{aligned}\hat{\mathcal{L}}_c(t) = \{&\partial_t|\Phi_0(t)\rangle \otimes [|\Phi_0(t)\rangle]^* \\ &+|\Phi_0(t)\rangle \otimes [\partial_t|\Phi_0(t)\rangle]^*\}\langle\langle \mathbf{I}|.\end{aligned} \tag{52}$$

Next, we expand the counterdiabatic superoperator $\hat{\mathcal{L}}_c(t)$ by SU(3) Hermitian generators $\{T_\mu\}_{\mu=0}^8$ (the details of the generators can be found in Appendix B),

$$\begin{aligned}\hat{\mathcal{L}}_c(t) = &ic_r(\sin(\omega t)T_4 \otimes T_0^* - \cos(\omega t)T_5 \otimes T_0^*) \\ &-ic_r(\sin(\omega t)T_0 \otimes T_4^* - \cos(\omega t)T_0 \otimes T_5^*),\end{aligned} \tag{53}$$

with the constant $c_r = \omega \tanh(2r)$. Therefore, the ordinary operator form of the counterdiabatic terms can be expressed as

$$\mathcal{L}_c(t)\rho(t) = -i[H_c(t), \rho(t)], \tag{54}$$

where the counterdiabatic Hamiltonian is

$$H_c(t) = \begin{pmatrix} 0 & 0 & \omega \tanh(2r)e^{-i\omega t} \\ 0 & 0 & 0 \\ \omega \tanh(2r)e^{i\omega t} & 0 & 0 \end{pmatrix}.$$

When we consider the counterdiabatic terms Eq. (54), the dynamics of the three-level system is described by the following master equation:

$$\begin{aligned}\partial_t\rho(t) = &-i[H_c(t), \rho(t)] - \frac{\gamma}{2}\{R^\dagger(\eta)R(\eta)\rho(t) \\ &+\rho(t)R^\dagger(\eta)R(\eta) - 2R(\eta)\rho(t)R^\dagger(\eta)\}.\end{aligned} \tag{55}$$

In Fig. 4, we plot the fidelity between the instantaneous steady state $\rho_0(t)$ and the quantum state $\rho(t)$ governed by the master equation (55) with (blue solid lines) and without (green dashed lines) the counterdiabatic term. It is not a surprise that both the final fidelity with any change rate ω_0 [Fig. 4(a)] and the fidelity in any moment [Fig. 4(b)] are exactly 1 for the fast tracking control strategy. It should be mentioned that as predicted in Sec. II, if the steady state of an open quantum system is pure, the counterdiabatic control is coherent. The example presented here illustrates this prediction.

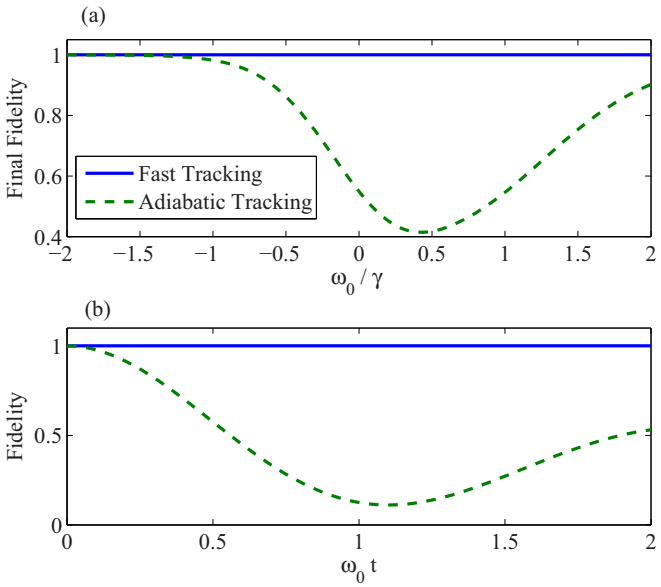

FIG. 4. (a) The final fidelity vs the change rate ω_0/γ and (b) the evolution of the fidelity with the change rate $\omega_0 = 10\gamma$ for the adiabatic tracking control (green dashed line) and the fast tracking control (blue solid line). The squeezed amplitude is set to be $r = 2$. The above variables are in units such that $\gamma = 1$.

IV. DISCUSSION AND CONCLUSION

We have worked out a framework to control open quantum systems tracking the trajectory of the instantaneous steady state by adding counterdiabatic terms to the Liouvillian superoperator. By requiring the equivalence between the evolution of open quantum systems and the ideal adiabatic evolution of the quantum steady state, the counterdiabatic superoperator is derived. We find that the counterdiabatic terms can be simply expressed as the time derivative of the spectral projection of the instantaneous steady state. When the counterdiabatic term is transformed into the ordinary operator form, we can see that both coherent control and incoherent control are needed for achieving fast trajectory tracking of the steady state. To show the effectiveness of the method and illustrate the proposal concretely, we apply the method to an open two-level system with the time-dependent Hamiltonian and the time-dependent Lindbladian. With the method, the counterdiabatic terms are totally incoherent in both examples, which can be written in Lindblad form. Numerical simulation confirms that fast trajectory tracking works well, i.e., the open quantum system approaches the target state along the instantaneous steady state at almost fidelity 1.

As shown in the examples, the "decay rate" of a part of the counterdiabatic terms is negative. By dropping the term with a negative decay rate and amplifying the other terms in the counterdiabatic term, we present a modulated trajectory tracking strategy to overcome this defect. When the amplification parameters are chosen properly, a satisfactory final fidelity can be obtained. Also, we notice that, despite the final fidelity approaches 1 at the end of the fast tracking control by choosing a proper amplification parameter, the quantum state cannot track the trajectory of the instantaneous steady state in the whole evolution. However, we can overcome this dilemma by changing amplification parameters with time. The Krotov-type optimal control technique can be used to determine the amplification parameters, which updates all controls within a

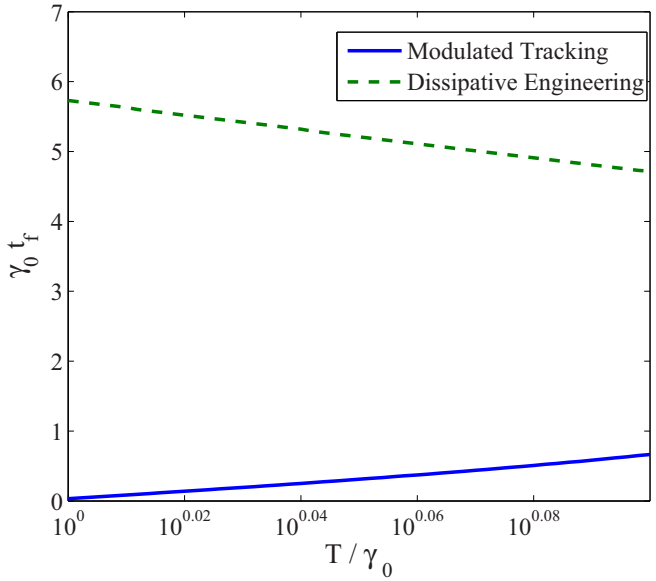

FIG. 5. The control period $\gamma_0 t_f$ vs the reservoir temperature T with the final infidelity 10^{-8} for the dissipative engineering (green dashed line) and the modulated tracking control (blue solid line). The change rate is $\omega = \pi/2t_f$, and the dimensionless amplification parameter is $A = 4$. The decoherence strength and energy splitting used in the dissipative engineering scheme are set to be $\gamma_d = A\gamma_0$ and $\omega_d = (0.1 + \pi/2)\gamma_0$. The above parameters are in units of γ.

single time slice once before proceeding on to the next time slice [46,47]. By applying the optimal control technique, we may expect that the quantum state of the open quantum system follows the trajectory of the instantaneous steady state with extremely high fidelity. On the other hand, even though the final fidelity is satisfactory for the modulated tracking control strategy, it is also a challenge in realizing such modulated incoherent controls in experiment. To realize such complex Lindblad operators, one must tailor the interaction between the open quantum system and its surroundings. The reservoir engineering technology is a candidate for accomplishing this mission. In a recent proposal [48,49], the authors proposed a scheme to entangle two atomic ensembles of cesium at room temperature by engineering a reservoir [50,51]. These techniques, together with measurement [23], can realize the time-dependent incoherent control. For example, the time-dependent Lindblad operators may be achieved by modulating the detuning between the pumping field and the atom or by modifying the Zeeman splitting. (A detailed relation between Lindblad operators and the parameters of the environment can be found in Refs. [48,49].) The parameters in the Lindblad operators can be modulated in experiment [52]. All these together can lead to incoherent control in the counterdiabatic terms.

Finally, we should mention that if the goal of the control is to prepare the quantum state on the steady state of open quantum systems, the dissipative preparing scheme is also a good choice [53,54]. The dissipative preparing scheme states that the engineered coupling between the open quantum system and its surrounding would guide the initial state into the target steady state, no matter what the initial state of the open quantum system is. Although this scheme can be easily performed in the laboratory [55], there are two obvious defects: First, the dissipative preparing scheme is a passive control strategy. Only if the control period approaches infinity does the open quantum system decay into the target state completely. Second, the path of the quantum state evolving in Hilbert space is totally unpredicted. Those disadvantages restrict the application in certain fields that require the control period and the quantum state trajectory in Hilbert space. Compared with the dissipative preparing scheme, the fast trajectory tracking of the steady state not only requires that the quantum state of open quantum systems completely reaches the target steady state within a finite control period, but it also specifies the trajectory of the quantum state evolving in Hilbert space. In fact, even if the final fidelity cannot reach 1, the modulation tracking control strategy is more effective than the dissipative engineering scheme. First, as illustrated in Fig. 3(b), a 10^{-8} infidelity can be reached with the reservoir temperature $\log_{10} T = 0.01$ and the amplification parameter $A = 3.4350$ at the final moment $\gamma_0 t_f = \pi/20$. But for the dissipative engineering scheme, the total control period will take up more time. To illustrate the advantage of the modulated tracking control strategy more clearly, we check the total control period to reach the infidelity 10^{-8} with a fixed amplification parameter $A = 4$. The dynamics of the dissipative engineering scheme can be described by the following time-independent master equation:

$$\begin{aligned}\partial_t \rho(t) &= \mathcal{L}_d \rho(t) \\ &= \gamma_d(N+1)\big(\sigma_-\rho(t)\sigma_+ - \tfrac{1}{2}\{\sigma_+\sigma_-, \rho(t)\}\big) \\ &\quad + \gamma_d N\big(\sigma_+\rho(t)\sigma_- - \tfrac{1}{2}\{\sigma_-\sigma_+, \rho(t)\}\big),\end{aligned}$$

where the mean excitation number is

$$N = [\exp(\hbar\omega_d/kT) - 1]^{-1}, \tag{56}$$

with a fixed energy splitting $\omega_d = \omega t_f = (0.1 + \pi/2)\gamma_0$, and the decoherence strength is $\gamma_d = A\gamma_0$. For the isothermal

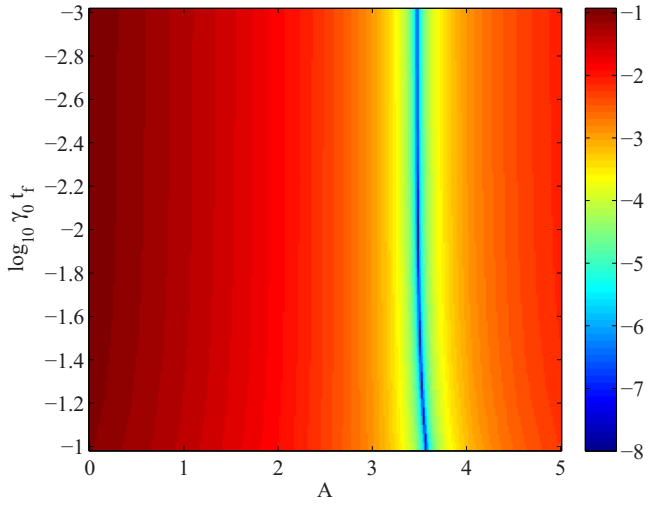

FIG. 6. The final infidelity vs the control period $\gamma_0 t_f$ and the dimensionless amplification parameter A for the modulated tracking control. The change rate is $\omega = \pi/2t_f$, and the reservoir temperature is set to be $T = \gamma_0$. The above parameters are in units of γ.

process, it is easy to verify that the steady state of this time-independent master equation is the very target state for the modulated tracking control strategy in Sec. III C. As shown in Fig. 5, the control period of the modulated tracking strategy is always shorter than the total evolution time of the dissipative engineering scheme. Moreover, for a fixed reservoir temperature T, we can choose a proper amplification parameter to reach a desired fidelity with an arbitrary control period which is illustrated in Fig. 6. Therefore, we can conclude that our method is more effective in the field of high-precision control processes and finite-time thermodynamic processes.

ACKNOWLEDGMENT

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APPENDIX A: THE DERIVATION OF EQ. (7)

Consider the following differential equation:

$$\partial_t X(t) = A(t)X(t), \tag{A1}$$

$$A(t) = [\partial_t P(t), P(t)] = \partial_t P(t)P(t) - P(t)\partial_t P(t). \tag{A2}$$

We assume that $V(t)$ is the solution of Eq. (A1) with the initial condition $V(0) = \mathbf{I}$. It is not difficult to verify that the general solution of Eq. (A1) can be written as $X(t) = V(t)X(0)$. Also, we have

$$\partial_t V(t) = A(t)V(t). \tag{A3}$$

Because $P(t)$ is the projector on the steady state, we have $P^2(t) = P(t)$. Differentiation on both sides gives

$$\partial_t P(t) = \partial_t P(t)P(t) + P(t)\partial_t P(t).$$

Multiplying by $P(t)$ from the left and right sides,

$$P(t)\partial_t P(t)P(t) = 0. \tag{A4}$$

Thus, the following equations can be obtained when we consider Eq. (A2):

$$\begin{aligned} P(t)A(t) &= -P(t)\partial_t P(t), \\ A(t)P(t) &= \partial_t P(t)P(t). \end{aligned} \tag{A5}$$

By considering $\partial_t P(t) = \partial_t P(t)P(t) + P(t)\partial_t P(t)$, we have

$$\partial_t P(t) = A(t)P(t) - P(t)A(t) = [A(t), P(t)]. \tag{A6}$$

Differentiating $P(t)V(t)$,

$$\begin{aligned} \partial_t[P(t)V(t)] &= [\partial_t P(t) + P(t)A(t)]V(t) \\ &= A(t)P(t)V(t), \end{aligned} \tag{A7}$$

where Eq. (A3) is used. Therefore, $P(t)V(t)$ is a solution of Eq. (A1). According to the intertwining property $V(t)P(0) = P(t)V(t)$, $V(t)P(0)$ is also a solution of Eq. (A1), so that we can use $V(t)P(0)$ as a criterion of the ideal adiabatic evolution Eq. (8). On the other hand, from the intertwining property, we obtain

$$P(t) = V(t)P(0)V^{-1}(t), \tag{A8}$$

which means that the superoperator $V(t)$ transforms the initial steady state of $\mathcal{L}_0(0)$ isometrically onto the instantaneous steady state of $\mathcal{L}_0(t)$. Therefore, $V(t)$ is a superoperator about the ideal adiabatic steady-state evolution, which is a solution of Eq. (7).

APPENDIX B: THE GENERATORS OF SU(3): THE GELL-MANN MATRICES

The generators, $\{T_\mu\}_{\mu=1}^8$, of the Lie algebra su(3) of SU(3) in the defining representation are

$$T_\mu = \lambda_\mu/2,$$

where $\{\lambda_\mu\}$, the Gell-Mann matrices, are the SU(3) analog of the Pauli matrices for SU(2):

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

These λ_μ span all traceless Hermitian matrices H of the Lie algebra. Here, we set T_0 as a 3×3 identity matrix:

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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