Journal of the Physical Society of Japan **89**, 054002 (2020)
https://doi.org/10.7566/JPSJ.89.054002

Scaling Laws in Synchronization of Metronomic Oscillatory Systems

Jakob Grzesik[1,2], Muhammad Shoufie Ukhtary[1]*, and Riichiro Saito[1]

[1]*Department of Physics, Tohoku University, Sendai 980-8578, Japan*
[2]*Department of Electrical and Computer Engineering, Rice University, Houston, TX 77005-1892, U.S.A.*

(Received August 2, 2019; accepted March 9, 2020; published online April 14, 2020)

We analyze the occurrence of the synchronization phenomenon, in which two or more objects in a system move in unison. We use a purely classical system consisting of N small oscillators exhibiting metronomic behavior coupled to each other through a larger substrate oscillator. We determine the equations of motion of the system and numerically solve them for the position as a function of time to see whether we are able to get the synchronization for N small oscillators. We found that the synchronization occurs after some necessary time intervals, which are referred to as stabilizing time σ and synchronization time τ. By application of scaling laws, we found that τ depends only on the small oscillator's and substrate's masses and damping constants. In the analysis, we investigate the conditions which are required to get a nontrivial synchronization for this classical system.

1. Introduction

Among various phenomena in physics, synchronization is one that has intrigued physicists for centuries not only for its harmony but also for the various situations in which it arises. In a general sense, synchronization is a phenomenon in which two or more objects in a system move and act in unison.[1)] Interest in synchronization historically began with the observations of Dutch physicist Christiaan Huygens, when he noted in 1665 that after a sufficient amount of time, two clock pendulums would come to oscillate in unison.[2)] Since then, scientists across multiple fields have observed and studied synchronizations in many situations, from the applause of an audience,[3)] to fireflies emitting light,[4)] to the human heartbeat[5)] and neurons in the brain.[6,7)]

Another field in which synchronization provides a framework of interest is solid-state physics. For example, a system of carbon nanotubes, when excited by a laser under certain conditions, produces synchronous radial breathing modes. This specific example is known as coherent phonons in carbon nanotubes.[8,9)] Another example of interest is the plasmon, a quantization of the harmonic oscillation of electron density first proposed by David Pines and David Bohm in 1952.[10)]

Synchronization phenomena occur in nonlinear systems, in which the system can go to stable limit-cycle oscillations.[1)] One example of a mathematical approach to understanding synchronization is the Kuramoto model, which describes necessary conditions for coupling forces between oscillators in order to synchronize the oscillators. Kapitaniak et al. discuss N metronomes and show in their plot, the synchronized system in the phase space.[11,12)] However, the synchronization process from the random initial states is not fully discussed. Thus, we feel that the definition of synchronization time is needed for given sets of parameters. Here, synchronization time is defined to be the time needed for a system to synchronize after the system has achieved stable limit-cycle oscillations. Cerdenas et al. discuss the synchronization time of 2 metronomes on a substrate by changing parameters of a feedback control system on the substrate, and they show some reduction of synchronization time.[13)] Ataka and Ohta show another mechanism of synchronization, where they calculate the synchronization time for pulse-coupled oscillators.[14)] In their work, the coupling between oscillators is achieved by transmitting an impulse from one oscillator to all the other oscillators after the activity of the original oscillator exceeds a certain threshold, thereby affecting the activity level of the other oscillators. Due to the transmitted impulse, synchronization can be achieved and they found anomalous behavior of synchronization time, where the synchronization time became infinitely large for certain values of the coupling strength.[14)] In both studies, however, the mathematical rule to determine the synchronization time is not discussed, as they only discuss the parameters reducing the synchronization time. In this paper, we scale the synchronization time as a function of masses of the metronomes, force constants, and damping constants to mathematically determine a formula for synchronization time.

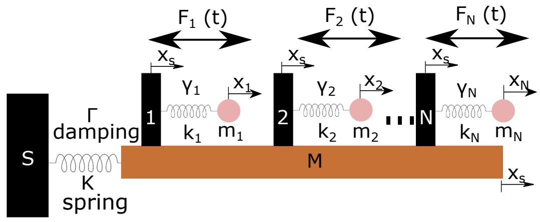

Fig. 1. (Color online) Schematic diagram of the oscillatory system. The system consists of N small oscillators on top of a substrate oscillator. Each with it is own mass m_i, M, damping factor γ_i, Γ, and spring constant k_i, K.

Our goal in this paper is to present a model of synchronization that not only shows occurrences of the synchronization phenomena, but also allows us to determine the synchronization time, and to determine the factors that affect the synchronization time. We use a purely classical model, based on the schematic presented in Fig. 1, in which we have N small oscillators coupled together by a larger substrate oscillator. Each oscillator's interaction is modeled by a Hookean spring, and we introduce external forces to each of the small oscillators. Then, we obtain the equations of motion and solve for the position of each of the small oscillators. After passing stabilization time, the amplitude of each small oscillator becomes steady because the energy loss by damping is balanced by the energy flow from the substrate, then the system progresses to synchronization. We then use these results to observe instances of synchronization and discover what parameters are important in affecting the synchronization time, from which we can determine the scaling law of synchronization time.

We found that in the case of small oscillators of uniform mass, not only is the synchronization inversely proportional to the oscillator damping, it is also nonlinearly dependent on the masses of both the substrate and the small oscillators. Furthermore, due to the energy lost through damping, we establish a simple set of conditions that the system must satisfy in order to observe nontrivial instances of synchronization.

Our paper is organized as follows. In Sect. 2, we will describe our method to model the system of N arbitrary small oscillators on top of a substrate oscillator. We then define synchronization and computational methods to determine synchronization and use them to calculate the time it takes for the system of small oscillators to synchronize fully. In Sect. 3, we will show our results and explain the significant parameters affecting the synchronization of the oscillator particles. We will give our conclusion in Sect. 4.

2. Method

In Fig. 1, we show a schematic diagram of the system that we are modeling, which consists of an oscillating substrate body of mass M, spring constant K, and damping factor Γ and N smaller oscillators which are coupled to each other by their placement on top of the oscillating substrate as depicted. Each of the small oscillators, labeled by a natural number $i = 1, 2, \ldots, N$ has a mass m_i, a spring constant k_i, and a damping factor γ_i. Furthermore, each small oscillator has an applied time-dependent force $F_i(t)$ for mimicking a metronome. Based on our choices for describing $F_i(t)$, the system may be modeled as linear or nonlinear. To describe the position of each oscillator, we consider the displacement x_i from the spring equilibrium position of an oscillator, where $i = s$ for the substrate body and $i = 1, 2, \ldots, N$ for the small oscillators when $x_s = 0$.

The Lagrangian of the system in Fig. 1 is given by

$$L = \frac{1}{2}\left(M\dot{x}_s{}^2 - Kx_s^2 + \sum_{i=1}^{N}[m_i(\dot{x}_i + \dot{x}_s)^2 - k_i(x_i - x_s)^2]\right). \tag{1}$$

Then, we obtain the equations of motion of each particle by using the Euler–Lagrange equations,[15] given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} + \frac{\partial R}{\partial \dot{x}_i} = F_i(t), \quad (i = 1, \ldots, N, \text{ and } s), \tag{2}$$

where $F_i(t)$ is the generalized external forces acting upon the i-th small oscillator. Note that for $i = s$, which corresponds to the substrate oscillator, $F_s = 0$. R represents the Rayleigh dissipation forces, which describe non-conservative forces like friction or damping. In this system, the Rayleigh dissipation forces are given by $\frac{1}{2}\gamma_i(\dot{x}_i + \dot{x}_s)^2$ for the i-th small oscillating particles and $\Gamma\dot{x}_s{}^2$ for the substrate oscillator. By using these considerations and Eqs. (1) and (2), we obtain the equations of motion as follows,

$$m_i(\ddot{x}_i + \ddot{x}_s) = -k_i(x_i - x_s) - \gamma_i(\dot{x}_i + \dot{x}_s) + F_i(t), \quad (i = 1, \ldots, N) \tag{3}$$

and for substrate,

$$\left(M + \sum_{i=1}^{N} m_i\right)\ddot{x}_s + \sum_{i=1}^{N}(m_i\ddot{x}_i) = -\left(K + \sum_{i=1}^{N} k_i\right)x_s - \Gamma\dot{x}_s + \left(\sum_{i=1}^{N} k_i x_i\right). \tag{4}$$

Due to the complexity of the system, it is impossible to analytically solve the equations of motion of all the oscillators, leaving us with only the option to use numerical calculations to observe the system's progression over time and obtain the necessary parameters for synchronization.

In this work, we use the Runge–Kutta method[16] to solve the equations of motion in order to determine the position of all the oscillators in the model as a function of time and investigate the synchronization phenomenon. To implement the Runge–Kutta method, we redefine variables given by Eq. (5) below, in order to make every equation a first-order differential equation.

$$\dot{x}_i = p_i, \quad (i = 1, 2, \ldots, N, \text{ and } s). \tag{5}$$

Applying this change of variables to Eqs. (3) and (4) gives the following first-order differential equations

$$m_i(\dot{p}_i + \dot{p}_s) = -k_i(x_i - x_s) - \gamma(p_i + p_s) + F_i(t), \quad (i = 1, 2, \ldots, N), \tag{6}$$

$$\left(M + \sum_{i=1}^{N} m_i\right)\dot{p}_s + \sum_{i=1}^{N}(m_i p_i) = -\left(K + \sum_{i=1}^{N} k_i\right)x_s - \Gamma p_s + \left(\sum_{i=1}^{N} k_i x_i\right). \tag{7}$$

Equations (5)–(7) give $2(N+1)$ coupled first-order differential equations for x_i and p_i ($i = 1, \ldots, N$ and s) from which the position of each of the oscillators as a function of time can be determined by the Runge–Kutta method. Furthermore, it is also helpful to express these equations in the form of matrix as shown below,

$$B\dot{X} = AX + F, \tag{8}$$

where X, A, B, and F are, respectively, expressed by

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_s \\ p_1 \\ \vdots \\ p_n \\ p_s \end{bmatrix}, \tag{9}$$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 1 \\ -k_1 & 0 & \cdots & k_1 & -\gamma_1 & 0 & \cdots & -\gamma_1 \\ 0 & -k_2 & \ddots & k_2 & 0 & -\gamma_2 & \ddots & -\gamma_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ k_1 & \cdots & k_N & -P & 0 & \cdots & 0 & -\Gamma \end{bmatrix}, \tag{10}$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & m_1 & 0 & \cdots & m_1 \\ 0 & 0 & \ddots & 0 & 0 & m_2 & \ddots & m_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & m_1 & \cdots & m_N & Q \end{bmatrix}, \tag{11}$$

$$F = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ F_1(t) \\ F_2(t) \\ \vdots \\ F_N(t) \\ 0 \end{bmatrix}, \tag{12}$$

with $P = K + \sum_{i=1}^{N} k_i$ and $Q = M + \sum_{i=1}^{N} m_i$. We note that as long as the mass of the substrate M is nonzero, which is the assumption for all cases we look at, the matrix B will have an inverse matrix B^{-1}. Thus, we may obtain expressions for each of the slope functions given by $\dot{X}$ where,

$$\dot{X} = B^{-1}AX + B^{-1}F. \tag{13}$$

By using Eq. (13), we obtain the following slope equations for each of the small oscillators' and substrate's positions and velocity derivatives, which will be used in the Runge–Kutta algorithm,

$$\dot{x}_i = p_i, \quad (i = 1, \ldots, N, s) \tag{14}$$

$$\begin{aligned} \dot{p}_i = &\frac{(-k_i x_i + k_i x_s - \gamma_i p_i - \gamma_i p_s)}{m_i} \\ &- \frac{2\sum_{l=1}^{N} k_l x_l - \left(K + 2\sum_{l=1}^{N} k_l\right) x_s}{M} \\ &+ \frac{\sum_{l=1}^{N} \gamma_l p_l + p_s\left[\left(\sum_{l=1}^{N} \gamma_l\right) - \Gamma\right]}{M} + a_i^{\text{ext}}(t), \\ &\qquad (i = 1, \ldots, N), \end{aligned} \tag{15}$$

$$\begin{aligned} \dot{p}_s = &\frac{2\sum_{l=1}^{N} k_l x_l - \left(K + 2\sum_{l=1}^{N} k_l\right) x_s}{M} \\ &+ \frac{\sum_{l=1}^{N} \gamma_l p_l + \left[\left(\sum_{l=1}^{N} \gamma_l\right) - \Gamma\right]}{M} + a_s^{\text{ext}}(t), \end{aligned} \tag{16}$$

where $a_i^{\text{ext}}(t)$ represents the overall acceleration resulting from the external forces acting upon the particles. If we consider each of the small oscillators having external applied forces of $F_1(t), \ldots, F_N(t)$, then for Eq. (15) we have

$$a_i^{\text{ext}}(t) = \frac{F_i(t)}{m_i} + \frac{F_i(t) - \sum_{j \neq i}^{N} F_j(t)}{M}, \tag{17}$$

in which the second term of Eq. (17) corresponds to the external force from the substrate. Similarly, we find for the substrate oscillator, given by Eq. (16) that,

$$a_s^{\text{ext}}(t) = -\frac{\sum_{j=1}^{N} F_j(t)}{M}. \tag{18}$$

With the slope functions given by Eqs. (14)–(16), we are ready to solve for the position of each particle as a function of time [$x_i(t)$] by applying the Runge–Kutta method. With these numerical solutions, we may now begin analyzing and discussing the emergence of the synchronization phenomenon in this system.

3. Results and Discussion

3.1 Full synchronization

Here, we will focus on two types of synchronization: full and subgroup synchronization. Full synchronization refers to a state in which all of the small oscillators in Fig. 1 are moving in unison. Mathematically, this means that for a fully synchronized system with N small oscillators,

$$x_1(t) = x_2(t) = \cdots = x_N(t), \tag{19}$$

and

$$\dot{x}_1(t) = \dot{x}_2(t) = \cdots = \dot{x}_N(t), \tag{20}$$

for $t \geq \tau$, where τ will be defined as the synchronization time of the system. For the simulation, we define τ as the earliest time that satisfies $|x_i(\tau) - x_j(\tau)| \leq A_s \epsilon$ for all combinations of x_i and x_j ($i \neq j$), where A_s is the amplitude reached when the energy of the system stabilizes and $\epsilon = 0.001$.

For subgroup synchronization, we need to define "subgroups" of a system. Subgroups refer to a collection of small oscillators that have the same position and velocity but do not have the same position and velocity as other oscillators outside the collection. In other words, creating subgroups is a way to partition the small oscillators so that each partition is composed of oscillators whose displacement can be described by a single equation, say $x_q(t)$. Subgroup synchronization is essential to study because it describes how the system fully synchronizes step by step. We will discuss the subgroup synchronization in the Sect. 3.2, where we will show that these subgroups can be treated as single oscillators.

In this section, we focus on full synchronization. We assume that each of the small oscillators has the same mass m, same spring constant k, and same spring damping factor γ. This assumption refers to the "uniform" case. With these simplifying assumptions, we can now replace Eqs. (15) and (16) with the following equations for the accelerations of each oscillator, with Eqs. (21) and (22) representing the small and substrate oscillators respectively.

$$\begin{aligned} \dot{p}_i = &\frac{-kx_i - \gamma p_i + kx_s + N(F(t) - \gamma p_s)}{m} \\ &+ \frac{-2k\left(\sum_{j=1}^{N} x_i\right) + 2kNx_s + Kx_s}{N} \end{aligned}$$

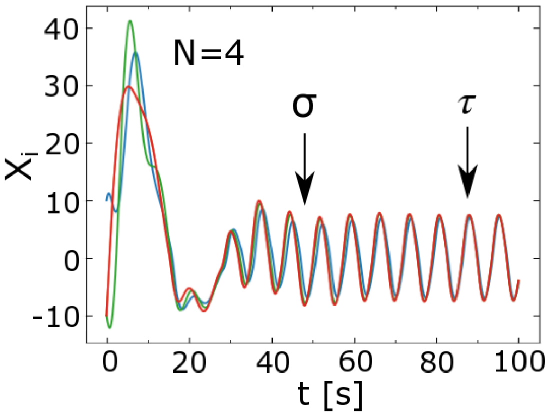

Fig. 2. (Color online) Graphical depiction of synchronization occurring with 4 small oscillators on top of a substrate oscillator. The parameters for this system are as follows: $m = 2\,\mathrm{kg}$, $k = 1.25\,\mathrm{N/m}$, $\gamma = 0.625\,\mathrm{Ns/m}$, $f = 10\,\mathrm{N}$, $A = 0$, $L = 2\,\mathrm{m}$, $M = 160\,\mathrm{kg}$, $\Gamma = 25\,\mathrm{Ns/m}$, $K = 10\,\mathrm{N/m}$. Each of the lines represent a single small oscillator. The system fully synchronizes at around $t = 90\,\mathrm{s}$ (τ), while the amplitudes are stabilized around $t = 50\,\mathrm{s}$ (σ). The initial conditions for each oscillator are $(x_i, \dot{x}_i) : (10, 5), (-10, 15), (-10, -5), (-10, 15)$. It is noted that initial conditions of oscillator 2 and 4 are the same, then they are already synchronized from beginning. That is why we only see three lines.

$$+\frac{N(F(t) - \gamma p_s) - \gamma\left(\sum_{j=1}^{N} p_j\right) + \Gamma p_s}{M}, \qquad (i = 1, \ldots, N), \tag{21}$$

$$\dot{p}_s = \frac{2k\left(-Nx_s + \sum_{j=1}^{N} x_j\right) - Kx_s + \gamma\left(\sum_{j=1}^{N} p_j\right)}{M} + \frac{N(\gamma p_s - F(t)) - \Gamma p_s}{M}. \tag{22}$$

In Fig. 2, we show the numerical solutions of the positions of each oscillator as a function of time in the uniform case, which depicts the positions of 4 small oscillators on top of the substrate oscillator. Here we set random initial conditions for x_i ($i = 1, \ldots, 4$) at $t = 0$ and we apply to each small oscillator the external force that best models the driving force of a metronome F_{met} as $F_i(t)$, which depends on the $x_i(t)$ and $\dot{x}_i(t)$ given below by

$$F_{\mathrm{met}}(t) = \begin{cases} f & A - L < x_i < A + L, \text{ for } \dot{x}_i > 0 \\ -f & -A - L < x_i < -A + L, \text{ for } \dot{x}_i < 0 \\ 0 & \text{else,} \end{cases} \tag{23}$$

where f denotes the magnitude of the force and $2L$ ($A > L$) determines the ranges of x_i around $x_i = A$ for which the force can be applied to the small oscillator.[17] The utilization of this force model of the metronome accentuates the differences between the metronomic oscillators due to randomized initial conditions. One question that arises is whether or not synchronization can even occur when chaos is added to the system with this choice of force. Analysis of the difference between two oscillators' positions shows that this applied force can help maintain the synchronized state of two synchronized oscillators, or either cancel out or intensify the applied forces acting on asynchronous oscillators.

One can see in Fig. 2 that the synchronization of all the oscillators occurs near $t = 90\,\mathrm{s}$, which can be expected as we have defined F_{met} to be nonlinear, creating a system of nonlinear metronomic oscillators that enable interesting nonlinear phenomena like synchronization to occur. For $t < 90\,\mathrm{s}$, the amplitude is gradually damped with frequency $\omega = \sqrt{K/M} = 0.5\,\mathrm{Hz}$ until reaching a stable amplitude of oscillation for each small oscillator at around $t = 50\,\mathrm{s}$ which is kept nonzero by the applied force F_{met}. This stable amplitude occurs once the energy provided by all the F_{met} cancels out the energy lost by the damping factor. We refer to the stable amplitude of each small oscillator as "A_s" and the time it takes to reach steady state as stabilizing time σ. For the simulation, we define σ as the earliest time that satisfies $|A_i(\sigma) - A_j(\sigma)| \leq A_s\epsilon$ for all combination of A_i and A_j ($i \neq j$), where A_i is the maximum amplitude that i-th oscillator attains within a period of time t. In other words, σ is the time by which all of metronomic oscillators attain similar maximum potential energy during oscillation, so that the distinction between each oscillator is its phase and not its amplitude. For synchronization, we are interested in more long-term behavior in which two oscillators' behaviors are seemingly "locked" together.

It is also important to note here a distinction between a trivial synchronization and a nontrivial synchronization. By trivial synchronization, we refer to cases in which the displacement amplitudes of all the small oscillators become 0 due to the damping factor. As this is bound to happen eventually with nonzero positive damping factors, it should not be considered as a result of the coupling forces of the system. Thus, for nontrivial synchronizations, we observe that a system is not merely dying to zero amplitude, but rather, we find that the system acts as a single object with finite nonzero amplitude and velocity for a sufficiently long time. As seen in Fig. 2, the oscillators all have nonzero amplitudes in position when the system reaches full synchronization, which means that the synchronization we observe in the system of Fig. 2 is indeed nontrivial.

While Fig. 2 allows one to easily see synchronization in a small system, it is deficient as a way to accurately record synchronization for a large number of oscillators. In Fig. 3, we show the positions of 10 oscillators as a function of time, where the synchronization is not easy to detect visually. Thus, to better discuss synchronization of more complex systems, it is necessary to construct a mathematical definition of synchronization in order to determine when a system is in the fully synchronized state.

Synchronization can also be understood using phase space of each small oscillator. Let us consider the phase space (x_i, p_i) of each oscillator. The phase angle θ_i of each oscillator at time t is defined by $\tan\theta_i(t) = p_i(t)/x_i(t)$. We will refer the phase angle simply as "phase". By plotting the momentum of the particle with respect to the position of the particle, we can look at the progression of each particle's oscillation, which means that in the case of synchronization, all of the small oscillators' phases will converge into one uniform phase. In Fig. 4, we plot θ_i of $N = 10$ (Fig. 3) for $t = 0, 50, 100\,\mathrm{s}$. As seen in Fig. 4, the phases begin at different initial points and eventually, once synchronization is achieved in Fig. 4(c), all of the oscillators' phases match to a single phase and continue oscillating in phase space in unison at $t = 100\,\mathrm{s}$. It is noted that we normalize the phase space as $(x_i, p_i)/\sqrt{x_i^2 + p_i^2}$ so that we have unit circle in Fig. 4,

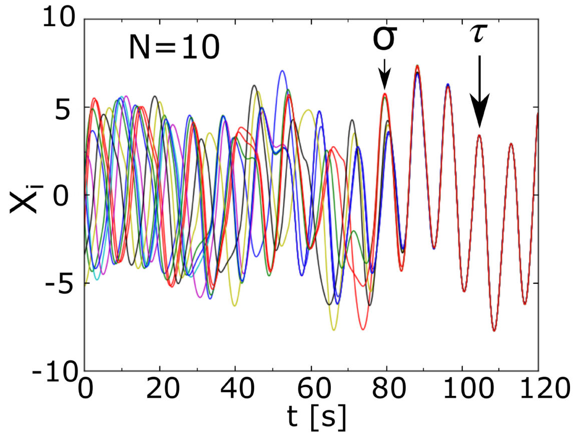

Fig. 3. (Color online) An example of analyzing a system of 10 oscillators, using the following parameters: $m = 5.0\,\mathrm{kg}$, $k = 1.25\,\mathrm{N/m}$, $\gamma = 2.50\,\mathrm{Ns/m}$, $f = 10\,\mathrm{N}$, $A = 0.0\,\mathrm{m}$, $L = 2,0\,\mathrm{m}$, $\Gamma = 1.0\,\mathrm{Ns/m}$, $M = 160.0\,\mathrm{kg}$, $K = 10$ N/m. Due to large number of oscillators, it is hard to analyze the synchronization visually.

otherwise the radius of each circle of Fig. 4 will be different due to different damping coefficients.

In order to avoid trivial synchronization, we keep the amplitudes of the small oscillators nonzero by applying $F_{\mathrm{met}}(t)$ to each of the small oscillators. However, this force also can affect τ. From Eq. (23), we know that $F_{\mathrm{met}}(t)$ for all oscillators is the same when synchronization occurs. Thus, the purpose of applying F_{met} is to keep the amplitude nonzero when two oscillators are synchronized. Before the synchronization, where $F_{\mathrm{met}(i)}(t) \neq F_{\mathrm{met}(j)}(t)$, the applied force plays a role in synchronization to accentuate the randomized differences between the initial conditions of each small oscillators and adds nonlinearity to the system. The randomization of the initial conditions in each set of small oscillators affects τ.

To ensure that the observed synchronizations are nontrivial, we require $F_{\mathrm{met}}(t)$ that provides enough energy to the small oscillators during each period in order to counteract the damping force applied to each oscillator. To do this, we consider the energy lost and gained per cycle of a small oscillator at a steady state with displacement given by Eq. (25). Supposing that the damping force of a single oscillator is given by $F_{\mathrm{damp}} = -\gamma\dot{x}_i$, the energy lost per cycle is given by $E_{\mathrm{damp}} = \pi\gamma\omega A_s^2$, where $\omega = \sqrt{\frac{k}{m}}$. In order for $F_{\mathrm{met}}(t)$ to give the maximum energy to the system, $E_{\max} = 4Lf$, we require that $A_s \geq A + L$ in Eq. (23). Additionally, to ensure that the amplitudes of the small oscillators do not decay to 0, we need $4Lf \geq E_{\mathrm{damp}}$. Thus, for getting nontrivial synchronizations, these two constraints require that γ satisfy the following condition,

$$\gamma \leq \frac{4Lf}{\pi\omega(A+L)^2} := \gamma_\delta. \tag{24}$$

In a system of N small oscillators on a substrate, as we have defined previously, the "stabilizing time" σ to be the time it takes for all N small oscillators to reach stable amplitude A_s. Hence, for all $t > \sigma$, we obtain steady states that are defined by,

$$x_i(t) = A_s e^{i(\omega t + \phi_i)} \tag{25}$$

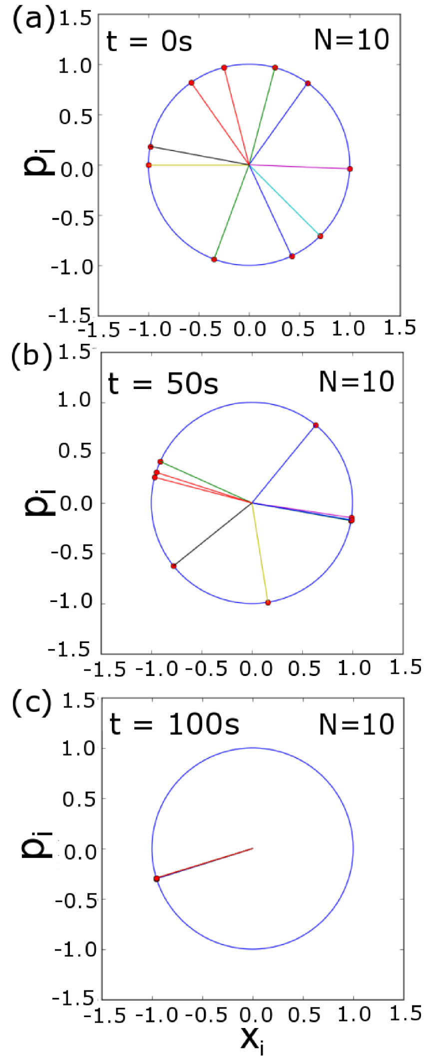

Fig. 4. (Color online) Plotting the phase space (x_i, p_i) for $N = 10$ of each oscillator at (a) $t = 0\,\mathrm{s}$, (b) $t = 50\,\mathrm{s}$, and (c) $t = 100\,\mathrm{s}$, eventually observing the phase synchronization when the phases converge to a single phase. The parameters and initial conditions for this system are the same as those used in Fig. 3. Each of lines represents a metronome.

for $i = 1, \ldots, N$. This state is defined as amplitude relaxation state. The full synchronization of the system occurs after the system stabilized or $\tau > \sigma$. Thus, to reach the synchronization faster, σ should be reduced. To reduce σ, we approximate the initial energy to be close to the energy of the system at a stable state. In that way, there is minimal excess energy that needs to be damped out, leading to a significantly reduced σ. To do this, we plot the total energy of a system consisting of small oscillators on a substrate as a function of time, as shown in Fig. 5. In Fig. 5, we see that after the energy has sufficiently been lost to damping at $t = 180\,\mathrm{s}$, the total energy oscillates around an average central value, which we denote as the stability energy E_s. The oscillations occur due to energy exchanged in and out of the system through the applied forces and damping, respectively. Fourier analysis of these oscillations under different sets of metronome and substrate parameters suggests that the frequency at which the energy oscillates, ω_E is related to the system by $\omega_E \propto \sqrt{\frac{k}{m}}$, which matches well with the discussion used to determine γ_δ.

Once the energy of the system begins oscillating around E_s, the oscillations of metronome have reached a stable

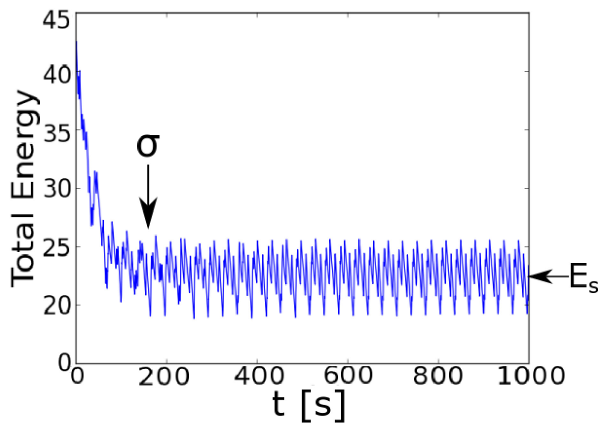

Fig. 5. (Color online) Total energy of a system of small oscillators on a substrate as a function of time, with $N = 8$, $m = 112.7\,\mathrm{kg}$, $k = 1.25\,\mathrm{N/m}$, $\gamma = 2.9673\,\mathrm{Ns/m}$, $M = 2254\,\mathrm{kg}$, $K = 10\,\mathrm{N/m}$, $\Gamma = 37.533\,\mathrm{Ns/m}$, $f = 10\,\mathrm{N}$, $A = 1\,\mathrm{m}$, $L = 0.1\,\mathrm{m}$. In this example, the total energy oscillates around $E_s \approx 23$, from $t > 180\,\mathrm{s}$, making $\sigma \approx 180\,\mathrm{s}$.

amplitude of A_s. A_s can be approximated by balancing energy loss of oscillator by damping E_{damp} with maximum energy from $F_{\mathrm{met}}(t)$ as discussed previously. Thus, $A_s = \sqrt{4Lf/\pi\gamma\omega}$. Assuming that the oscillation in steady-state is nontrivial, we can derive a set of initial conditions that much reduce σ.

First, suppose that the total energy of the system oscillates around the stable energy, E_s, which is obtained by calculating the total energy numerically using random initial conditions. If we set the initial conditions of the N small oscillators so that

$$E_s = \sum_{i=1}^{N}\left(\frac{1}{2}m\dot{x}_i(0)^2 + \frac{1}{2}kx_i(0)^2\right), \tag{26}$$

the stable state can be reached faster. By starting the substrate at rest, $\dot{x}_s(0) = 0$, we choose initial conditions that lead to the small oscillators exchanging energy into the substrate, which then couples the small oscillators together to begin the process of synchronization. Thus, we have that damping's role in synchronization is not overly prevalent, making the pertinent parameters in determining synchronization time more salient.

When synchronization occurs, all the small oscillators move with the same $x_i(t)$, which means that ϕ_i in Eq. (25) is the same for all small oscillators. Then, we can approximate the E_s by assuming that the substrate also oscillates with a steady amplitude of B_s. The substrate reaches a steady nontrivial amplitude when the energy loss from the substrate's damping Γ ($\pi\Gamma\omega B_s^2$) is balanced with the energy coming from all the small oscillators into substrate ($Nm\omega^2 A_s B_s/2$). Therefore, the E_s can be approximated by,

$$E_s = \frac{1}{2}(NkA_s^2 + KB_s^2), \tag{27}$$

where $B_s = Nm\omega^2 A_s/(\pi\Gamma\omega)$.

Next, we have that if the energy E_s is equally distributed between all small oscillators, we get the initial energy of each small oscillator to be

$$\frac{E_s}{N} = \frac{1}{2}m\dot{x}_i(0)^2 + \frac{1}{2}kx_i(0)^2. \tag{28}$$

By defining $A_1 = \sqrt{\frac{2E_s}{Nm}}$, and randomly choosing $0 \leq \theta_i \leq 2\pi$ we get the following initial conditions that satisfy Eq. (26),

$$x_i(0) = \sqrt{\frac{m}{k}}A_1 \sin(\theta_i), \tag{29}$$

$$\dot{x}_i(0) = A_1 \cos(\theta_i). \tag{30}$$

Therefore, we can establish sets of randomized initial conditions that significantly reduce σ and avoid trivial synchronizations of the system.

3.2 *Subgroup synchronization*

In order to obtain full synchronization, which we have previously described as all the small oscillators moving in unison, it is necessary to look at the system's progression through subgroup synchronization. As mentioned in Sect. 3.1, we define a subgroup as a set of oscillators that are already synchronized within that set only. For example, in a totally asynchronous, randomized collection of N small oscillators, all with different positions or velocities, there are N subgroups. Whereas, in a fully synchronized system of small oscillators, there is only one subgroup. In essence, subgroup synchronization is a way of looking at full synchronization step by step, as full synchronization occurs when the last two subgroups synchronize.

In fact, it is possible to treat a subgroup of oscillators as one effective oscillator, with mass, spring constant, and damping factor all determined by the oscillators making up the subgroup. To see why, one needs only to look at the equations of motion of the entire system in Eq. (31) as follows,

$$m_i(\ddot{x}_i + \ddot{x}_s) = -k_i(x_i - x_s) - \gamma_i(\dot{x}_i + \dot{x}_s) + F_i(t), \quad (i = 1, 2, \ldots, N). \tag{31}$$

Suppose we have a system of N small oscillators, with a subgroup consisting of q oscillators. Without loss of generality, due to the symmetry of the system, we may relabel the small oscillators so that the first q oscillators are synchronized. In other words

$$x_i(t) = x_r(t), \quad (i = 1, \ldots, q). \tag{32}$$

Here $x_r(t)$ is the common displacement of the q oscillators in the subgroup. Thus we may rewrite our system of equations as follows

$$\begin{aligned} m_1(\ddot{x}_r + \ddot{x}_s) &= -k_1(x_r - x_s) - \gamma_1(\dot{x}_r + \dot{x}_s) + F_1, \\ &\vdots \\ m_q(\ddot{x}_r + \ddot{x}_s) &= -k_q(x_r - x_s) - \gamma_q(\dot{x}_r + \dot{x}_s) + F_q, \\ &\vdots \\ m_N(\ddot{x}_N + \ddot{x}_s) &= -k_N(x_N - x_s) - \gamma_i N(\dot{x_N} + \dot{x}_s) + F_N. \end{aligned} \tag{33}$$

Next, we apply row operations to each of the first q rows by adding in the rows of each of the other oscillators of the subgroup exactly once. This leaves us with q rows of the form

$$\sum_{i=1}^{q} m_i(\ddot{x}_r + \ddot{x}_s) = -\sum_{i=1}^{q} k_i(x_r - x_s) - \sum_{i=1}^{q}\gamma_i(\dot{x}_r + \dot{x}_s) + \sum_{i=1}^{q} F_i. \tag{34}$$

Appropriately relabeling $m_r = \sum_{i=1}^{q} m_i$, $k_r = \sum_{i=1}^{q} k_i$, $\gamma_r = \sum_{i=1}^{q}\gamma_i$, and $F_r = \sum_{i=1}^{q} F_i$, the original system of equations,

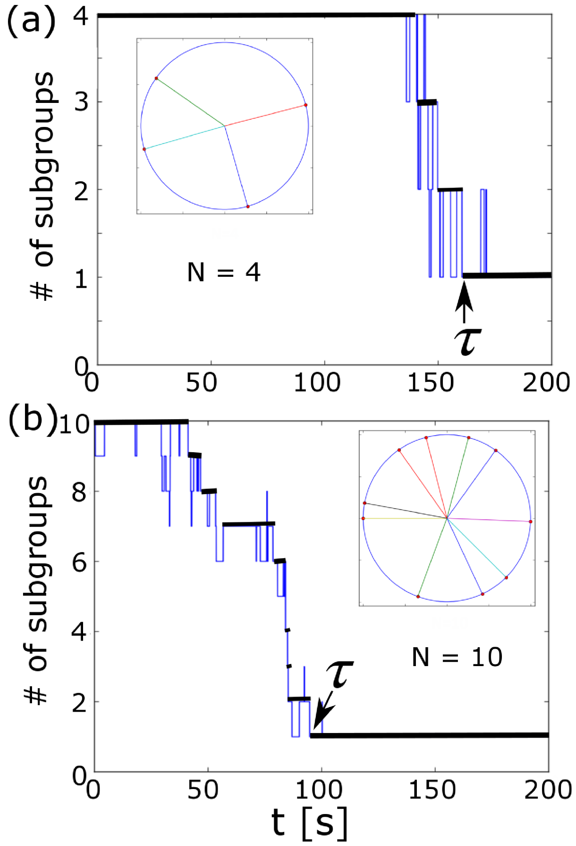

Fig. 6. (Color online) Subgroup progression of two systems, (a) one with 4 small oscillators and (b) one of 10 small oscillators with $m = 0.5$, $k = 1.25$, $\gamma = 10$, $A = 2.5$, $\omega = 2$, $\Gamma = 5$, $M = 10$, $K = 10$. The thick lines represent the effective number of subgroups. Insets: the phase space of initial conditions for (a) and (b).

given in Eq. (31), now effectively becomes a system of $N - q + 1$ small oscillators. The subgroup of q oscillators have now been replaced by a single oscillator, with parameters given by m_r, k_r, γ_r, and F_r. The number of subgroups should satisfy $N = \sum_{i=1}^{N_{\text{sub}}} q_i$.

The concept of a subgroup oscillator suggests that a system of oscillators is best thought as a system of subgroups and that once a subgroup stably forms, it can be replaced and modeled by a single oscillator. For the purpose of determining synchronized subgroups computationally, we define a few mathematical criterion in order to determine whether or not two small oscillators belong in the same subgroup. In order to categorize two oscillators into the same subgroup, we recall our definition of synchronization in position space and phase space. We consider two metronomes to be synchronized if $|x_i(t) - x_j(t)| \leq A_s\epsilon$. In the phase space, two small oscillators are synchronized if $|\theta_i(t) - \theta_j(t)| \leq \pi\epsilon$. Overall, the main roles of A_s and π in our definitions of amplitude and phase synchronization is to help normalize the data relative to the overall magnitude of amplitude oscillations of the fully synchronized system and to the phase differences projected onto the unit circle. Essentially the two synchronization factors scale ϵ to the maximum possible differences between two oscillators' displacements and phases.

Therefore, we approximate any two oscillators that satisfy both synchronization conditions to be in the same subgroup. This, in turn, allows us to track the subgroup progression of a system of small oscillators on a substrate. In Figs. 6(a) and 6(b), we show the number of subgroups as a function of t for $N = 4$ and 10 and we have chosen $\epsilon = 0.001$. Furthermore, we see in both figures that the number of subgroups eventually becomes 1, which indicates synchronization. Therefore, we define τ to be the time it takes for a system of small oscillators on a substrate to enter and stably remain in a single subgroup, which is to say that all small oscillators are phase and amplitude synchronized. In Fig. 6, we consider the effective number of subgroup (N_{sub}) as the number of long lasting subgroups as shown by the thick lines. The number of subgroup becomes more fluctuating close to the transition to lower N_{sub}.

In Fig. 6(a), we have 4 different subgroups at $t = 0$, and then we have only one subgroup when the system is synchronized at $\tau = 170$ s. Therefore, it is important to look at the step by step subgroup synchronizations and to replace a subgroup with an effective oscillator. The system now consists of new effective oscillators replacing the subgroups. With respect to the previously mentioned studies by Kapitaniak et al. the concept of subgroup synchronization can be considered to be the synchronization of what they refer to as "clusters".[11]

3.3 The scaling laws of synchronization time

With all the proper definitions in place, we can begin characterizing the effect of the system parameters $m, k, \gamma, M, K, \Gamma$ on τ, the time it takes for the system of fully synchronize. In order to do so, we consider the potential relationship

$$\tau = m^\alpha k^\beta \gamma^\xi M^\eta K^\lambda \Gamma^\mu. \tag{35}$$

Considering a system that satisfies Eq. (24), we simulated a system of 8 small oscillators on a substrate as a function of γ and plotting τ. Thus, by scaling each of the other parameters, we can evaluate how τ shifts and determine the effect of the scaling. Determining this scaling, in turn, allows us to infer τ's dependence on the other parameters. As an illustration, Fig. 7(a) demonstrates the τ–γ relationship for a base set parameters, and for multiple sets of parameters that differ from this base set. Then, by scaling the γ and the τ axes in an appropriate manner, we can make it so that the τ–γ relationship for the different sets of parameters matches that of the base set of parameters. For example, considering the simulations in which $m = 4m_0$, if we shift the γ axis by a factor of 2, and τ axis by a factor of $4^{0.3}$, we obtain the τ–γ relationship of the base set of parameters. Repeating this to gain an overall trend for various other scaling factors, we settle into a final scaling relationship for each parameter.

Thus, by applying scaling for all sets of parameters, we can infer the following scaling laws on the system parameters and τ.

$$\begin{aligned}&\tau(\gamma, Cm, Dk, EM, FK, G\Gamma)\\&\quad = EGC^{0.3}\tau\left(\frac{\gamma}{EG\sqrt{C}}, m, k, M, K, \Gamma\right)\end{aligned} \tag{36}$$

The scaling laws given in Eq. (36) suggests that τ is independent of the spring constant factors k, K, and the following relations between the powers of each of the remaining parameters in Eq. (35).

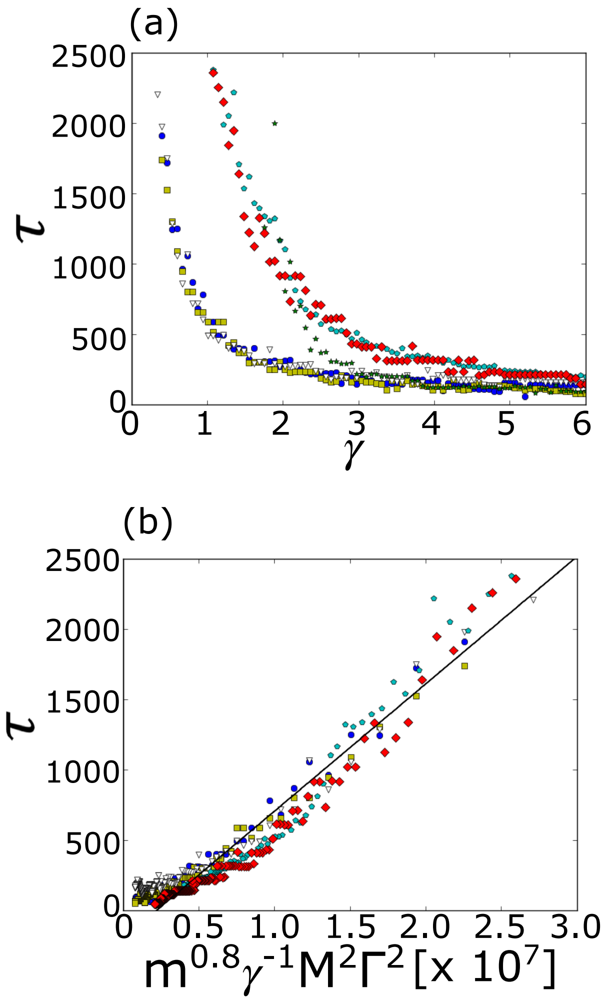

Fig. 7. (Color online) The synchronization time τ as a function of γ used in determining the relationship between τ and the small oscillator and substrate parameters: $m, k, \gamma, M, K, \Gamma$, (a) different m, k, M, K, Γ. In the original case, $m = m_0 = 50$, $k = k_0 = 10$, $M = M_0 = 400$, $K = K_0 = 10$, $\Gamma = \Gamma_0 = 1.5811$. Circle: Original; Star: $\Gamma = 2\Gamma_0$; Diamond: $M = 2M_0$; Triangle: $k = \frac{k_0}{4}$; Square: $K = 2K_0$; Pentagon: $m = 4m_0$. (b) The τ as a function of $m^{0.8}\gamma^{-1}M^2\Gamma^2$ showing linear relationship.

$$\alpha = \frac{-5\zeta + 3}{10}, \tag{37}$$

$$\eta = -\zeta + 1, \tag{38}$$

$$\mu = -\zeta + 1. \tag{39}$$

Then, by applying curve-fitting to all of the τ–γ relations used to investigate the scaling laws, we get that $\zeta = -1$. By using Eqs. (37)–(39), we get the synchronization law produced below.

$$\tau(\gamma, m, k, M, K, \Gamma) \propto m^{0.8}\gamma^{-1}M^2\Gamma^2 \tag{40}$$

Performing simulations within the range of conditions described by Eq. (24), we find that the result given above agrees with the τ dependence on each of the other parameters individually scaled. Defining $\Omega = m^{0.8}\gamma^{-1}M^2\Gamma^2$, we produce Fig. 7(b) by plotting τ as a function of Ω for each set of parameters used in Fig. 7(a). The persistent linear relationship between Ω and τ for each set of parameters further supports the scaling law determined in Eq. (40), with the slope of the τ–Ω relationship being determined by the strictness of ϵ used to establish synchronization. We obtain a linear fitting as $\tau = (8.227 \pm 0.086) \times 10^{-5}\Omega + (-53.45 \pm 5.46)$, with value of $R^2 = 0.9062$.

One thing to note is the overall relationship between the substrate and small oscillators on both the conditions needed for synchronization to occur and τ. In particular, the argument in determining Eq. (24) demonstrates the importance of the energy loss due to damping in the small oscillators in order for obtaining nontrivial amplitude synchronization to occur. Within the determination of τ, we find the substrate's contributions primarily from M and Γ. Intuitively, increasing Γ dissipates the coupling interactions between the small oscillators on the substrate, hence increasing Γ makes it more difficult for synchronization to occur, leading to larger τ. Similarly, increasing M increases the inertia of the substrate, making it more difficult for the small oscillators to affect the substrate and effectively couple to each other. Likewise, having larger metronome mass makes it more difficult to accelerate and hence couple the oscillators. τ decreases with increasing the damping factor γ as the γ damps out differences due to randomized initial conditions, and only the synchronous motion remains. Overall, these equations and conditions identify that the substrate parameters pose constraints on the time it takes for synchronization to occur, while those of the small oscillators also contribute to making synchronization a likely scenario.

4. Conclusion

In conclusion, we are able to look at a classical model of synchronization and are able to arrive at a general formula for the time it takes for the system to fully synchronize in the uniform mass case. After analyzing the system parameters for a uniform system of small oscillators, we found the conditions that led to synchronization in nontrivial cases with oscillators that do not decay to zero amplitude. We found that this corresponds to the case of underdamped oscillators, as overdamped systems cannot be kept at nontrivial amplitudes by the applied metronome force. Overall, we found that the time it takes for synchronization to occur in systems satisfying this nontrivial synchronization condition depends only on the mass and damping factors of the small oscillators and substrate, with the substrate having the most significant effect on the synchronization time. While the parameters of small oscillators have less of an effect on the synchronizations time as the substrate, these parameters determine whether or not the system will synchronize nontrivially.

Overall, we hope that this simple classical model can be extended and applied using quantum mechanics in the field of solid-state physics in order to better explain and understand examples of the synchronization phenomenon that may occur, such as the previously mentioned coherent phonon phenomenon, plasmon and supperradiance.[18,19] Furthermore, this phenomenon has been investigated in optomechanical systems, which have potential uses for quantum communication schemes.[20] Additionally, it has been shown that the notion of classical synchronization naturally extends to superposition quantum mechanically.[21] Thus, it follows that studies in synchronization of classical systems can be used to understand entanglement in their quantum mechanical equivalents further. Thus, we hope to extend this work over a classical case of synchronization by investigating its relationship to the quantum mechanical cases presented in phonons, superradiance, and plasmons, among other exciting instances of synchronization throughout physics.

Acknowledgments J.G. thanks J. Kono, C. J. Stanton, S. Phillips, K. Ogawa, and U. Endo for making the Nakatani RIES program possible. M.S.U. and R.S. acknowledge JSPS KAKENHI Grant Nos. JP18J10199 and JP18H01810, respectively.

*shoufie@flex.phys.tohoku.ac.jp

1) L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
2) A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, U.K., 2003) Vol. 12.
3) Z. Néda, E. Ravasz, Y. Brechet, T. Vicsek, and A.-L. Barabási, Nature **403**, 849 (2000).
4) R. E. Mirollo and S. H. Strogatz, SIAM J. Appl. Math. **50**, 1645 (1990).
5) C. Schäfer, M. G. Rosenblum, J. Kurths, and H.-H. Abel, Nature **392**, 239 (1998).
6) S. Cobb, E. Buhl, K. Halasy, O. Paulsen, and P. Somogyi, Nature **378**, 75 (1995).
7) F. Varela, J.-P. Lachaux, E. Rodriguez, and J. Martinerie, Nat. Rev. Neurosci. **2**, 229 (2001).
8) G. Sanders, A. Nugraha, K. Sato, J. Kim, J. Kono, R. Saito, and C. Stanton, J. Phys.: Condens. Matter **25**, 144201 (2013).
9) R. Saito, G. Dresselhaus, and M. S. Dresselhaus, *Physical Properties of Carbon Nanotubes* (World Scientific, Singapore, 1998).
10) D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).
11) M. Kapitaniak, K. Czolczynski, P. Perlikowski, A. Stefanski, and T. Kapitaniak, Phys. Rep. **517**, 1 (2012).
12) A. Stefański and T. Kapitaniak, Int. J. Solids Struct. **40**, 5175 (2003).
13) I. Lugo-Cérdenas, M. L. Ramirez, G. Maldonado, A. Soria, and J. C. Martínez-García, 9th Int. Conf. Electrical Engineering, Computing Science and Automatic Control (CCE), 2012, p. 1.
14) M. Ataka and T. Ohta, Prog. Theor. Phys. **113**, 55 (2005).
15) H. Goldstein, C. P. Poole, and J. L. Safko, *Classical Mechanics: Pearson New International Edition* (Pearson, Harlow, U.K., 2014).
16) J. C. Butcher, *The Numerical Analysis of Ordinary Differential Equations: Runge–Kutta and General Linear Methods* (Wiley-Interscience, New York, 1987).
17) Y. Kubo, Master's Thesis, Ryukoku University (2010).
18) A. V. Andreev, V. I. Emel'yanov, and Y. A. Il'inskii, Sov. Phys. Usp. **23**, 493 (1980).
19) K. Cong, Q. Zhang, Y. Wang, G. T. Noe, A. Belyanin, and J. Kono, J. Opt. Soc. Am. B **33**, C80 (2016).
20) G.-j. Qiao, H.-x. Gao, H.-d. Liu, and X. X. Yi, Sci. Rep. **8**, 15614 (2018).
21) D. Witthaut, S. Wimberger, R. Burioni, and M. Timme, Nat. Commun. **8**, 14829 (2017).

