Shortcut to Equilibration of an Open Quantum System: supplementary material

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I. CANONICAL INVARIANCE AND REPRESENTATION OF THE SYSTEM IN TERMS OF THE GENERALIZED GIBBS STATE COEFFICIENTS.

Canonical invariance is a property of the density operator, which preserves the generalized Gibbs form throughout the dynamics. This property is inter-linked with a closed Lie algebra of operators, that are employed as a basis for the generalized Gibbs state. Utilizing the structure of the Lie algebra the generalized Gibbs state can be represented in a product form [1, 2]. Specificaly, for the closed set $\mathcal{B} = \{\hat{b}^\dagger\hat{b}, \hat{b}^2, \hat{b}^{\dagger 2}\}$ the state is given by equation (5) of the Main Part (MP). Next, we calculate $d\tilde{\rho}_S(t)/dt\,(\tilde{\rho}_S(t))^{-1}$ explicitly, using the generalized Gibbs state

$$\frac{\partial\tilde{\rho}_S(t)}{\partial t}(\tilde{\rho}_S(t))^{-1} = \dot{\gamma}\hat{b}^2 + \dot{\beta}e^{\gamma\hat{b}^2}\hat{b}^\dagger\hat{b}e^{-\gamma\hat{b}^2} + \dot{\gamma}^* e^{\gamma\hat{b}^2}e^{\beta\hat{b}^\dagger\hat{b}}\hat{b}^{\dagger 2}e^{-\beta\hat{b}^\dagger\hat{b}}e^{-\gamma\hat{b}^2} \quad . \tag{1}$$

Introducing the Baker-Campbell-Hausdorff relation, leads to

$$\frac{\partial\tilde{\rho}_S(t)}{\partial t}(\tilde{\rho}_S(t))^{-1} = \hat{b}^2\left(2\dot{\beta}\gamma + \dot{\gamma} + 4\gamma^2\dot{\gamma}^*e^{2\beta}\right) + \hat{b}^\dagger\hat{b}\left(\dot{\beta} + 4\gamma\dot{\gamma}^*e^{2\beta}\right) + \hat{b}^{\dagger 2}\dot{\gamma}^*e^{2\beta} + \hat{I}2\gamma\dot{\gamma}^*e^{2\beta} \quad . \tag{2}$$

By substituting Eq. (5) MP into the master equation, Eq. (2) MP, we obtain an expansion of $d\tilde{\rho}_S(t)/dt\,(\tilde{\rho}_S(t))^{-1}$ in terms of the operators of $\mathcal{B}$. Such a property is termed canonical invariance, it implies that an initial state that belongs to the class of canonical states, will remain in this class throughout the evolution [3]. The expression reads

$$\frac{\partial\tilde{\rho}_S(t)}{\partial t}(\tilde{\rho}_S(t))^{-1} = \hat{b}^2A + \hat{b}^\dagger\hat{b}B + \hat{b}^{\dagger 2}C + \hat{I}D \quad , \tag{3}$$

TABLE I

coefficient	value	coefficient	value
a_1	$2\gamma e^{\beta}$	c_1	$-2\gamma^* e^{2\beta}$
a_2	$2\gamma\left(1-4\lvert\gamma\rvert^2 e^{2\beta}\right)$	c_2	$-2\gamma^* e^{\beta}$
b_1	e^{β}	d_1	e^{β}
b_2	$1-8\lvert\gamma\rvert^2 e^{2\beta}$	d_2	$-4\lvert\gamma\rvert^2 e^{2\beta}$
b_3	$e^{-\beta}-4\lvert\gamma\rvert^2 e^{\beta}$	d_3	$-2\lvert\gamma\rvert^2 e^{\beta}+1$

where

$$A = k_\downarrow\left(a_1 - \frac{1}{2}a_2\right) - \frac{1}{2}k_\uparrow a_2 \tag{4}$$

$$B = k_\downarrow\left(b_1 - \frac{1}{2}b_2 - \frac{1}{2}\right) + k_\uparrow\left(b_3 - \frac{1}{2}b_2 - \frac{1}{2}\right) \tag{5}$$

$$C = -\frac{1}{2}(k_\downarrow + k_\uparrow)c_1 + k_\uparrow c_2$$

$$D = k_\downarrow\left(d_1 - \frac{1}{2}d_2\right) + k_\uparrow\left(-\frac{1}{2}(d_3+1)\right) \quad .$$

The values of of the coefficients are summarized in Table I. To satisfy both Eq. (2) and (3) the coefficients multiplying each operator must be equal. Comparing terms, leads to four coupled differential equations

$$2\dot{\beta}\gamma + \dot{\gamma} + 4\gamma^2\dot{\gamma}^*e^{2\beta} = \gamma\left(\left(2e^{\beta}-1\right)k_\downarrow + 4e^{2\beta}(k_\downarrow + k_\uparrow)\lvert\gamma\rvert^2 - k_\uparrow\right) \tag{6}$$

$$\dot{\beta} + 4\gamma\dot{\gamma}^*e^{2\beta} = e^{-\beta}\left(\left(e^{\beta}-1\right)\left(e^{\beta}k_\downarrow - k_\uparrow\right) + 4e^{2\beta}\lvert\gamma\rvert^2\left(e^{\beta}(k_\downarrow + k_\uparrow) - k_\uparrow\right)\right) \tag{7}$$

$$\dot{\gamma}^*e^{2\beta} = k_\downarrow e^{2\beta}\gamma^* + k_\uparrow\gamma^*\left(e^{2\beta} - 2e^{\beta}\right) \tag{8}$$

$$2\gamma\dot{\gamma}^*e^{2\beta} = k_\downarrow\left(e^{\beta}\left(2e^{\beta}\lvert\gamma\rvert^2 + 1\right)\right) + k_\uparrow\left(2e^{2\beta}\lvert\gamma\rvert^2 - 1\right) \tag{9}$$

After some algebraic manipulations we obtain the simplified form

$$\dot{\beta} = k_\downarrow\left(e^{\beta}-1\right) + k_\uparrow\left(e^{-\beta} - 1 + 4e^{\beta}\lvert\gamma\rvert^2\right) \tag{10}$$

$$\dot{\gamma} = (k_\downarrow + k_\uparrow)\gamma - 2k_\downarrow\gamma e^{-\beta} \quad ,$$

These coupled differential equations completely determine the system's dynamics.

II. LOWER BOUND OF THE PROTOCOL DURATION

The validity of the *inertial theorem* is quantified by the inertial parameter Υ [4]. When $\Upsilon \ll 1$ the inertial solution is a good approximation of the true dynamics. For the harmonic oscillator, Υ takes the form $\Upsilon \sim \frac{\mu'(\theta)}{(2\kappa)^2}$, $(\theta = \theta(t) = \int_0^t \omega(t')\,dt'$ and $\kappa = \sqrt{4-\mu^2})$, which explicitly becomes

$$\Upsilon = \frac{1}{(2\kappa)^2}\left(\frac{\ddot{\omega}}{\omega^3} - 2\mu^2\right) \quad . \tag{11}$$

Transforming variables to the dimensionless parameter $s = t/t_f$ and constraining the inertial parameter by $\Upsilon < f \ll 1$, introduces a lower bound for the protocol duration:

$$t_f > f \cdot \max_s \left(\frac{1}{\omega}\sqrt{\frac{\omega''(s)}{2\omega(s)}\frac{1}{8-(\mu(s))^2}}\right) \quad . \tag{12}$$

III. QUENCH PROTOCOL

When the parametric quantum harmonic oscillator is in a Gaussian state [2], it is convenient to analyze the system in terms of three time-dependent operators, the Hamiltonian Eq. (1) MP, Lagrangian $\hat{L}(t) = \frac{\hat{P}^2}{2m} - \frac{1}{2}m\omega^2(t)\hat{Q}^2$, and the position-momentum correlation operator $\hat{C}(t) = \frac{\omega(t)}{2}\left(\hat{Q}\hat{P} + \hat{P}\hat{Q}\right)$. The *quench* protocol includes an initial abrupt shift in frequency from $\omega(0) = \omega_i$ to $\omega(t_f) = \omega_f$. The sudden transformation is approximately isolated, as the bath's influence on the system occurs on a much longer timescale. Moreover, for a sudden quench the system state remains unchanged. Hence, time-independent operators, such as $\hat{Q}(0)$ and $\hat{P}(0)$ do not vary. This property allows expressing the operators after the sudden quench $\hat{H}'$, $\hat{L}'$ and $\hat{C}'$ in terms of the operators at initial time (for the sudden quench)

$$\begin{aligned}\hat{H}' &= \tfrac{1}{2}\left[\hat{H}(0)\left(1+\tfrac{\omega^2(t)}{\omega^2(0)}\right) + \hat{L}(0)\left(1-\tfrac{\omega^2(t)}{\omega^2(0)}\right)\right] \\ \hat{L}' &= \tfrac{1}{2}\left[\hat{H}(0)\left(1-\tfrac{\omega^2(t)}{\omega^2(0)}\right) + \hat{L}(0)\left(1+\tfrac{\omega^2(t)}{\omega^2(0)}\right)\right] \\ \hat{C}' &= \tfrac{\omega(t)}{\omega(0)}\hat{C}(0) \quad .\end{aligned} \tag{13}$$

The sudden change in frequency generates coherence, which is manifested by non-vanishing values of $\langle\hat{L}(t)\rangle$ and $\langle\hat{C}(t)\rangle$ [5].

Once the system is quenched, the frequency remains constant and the system relaxes towards equilibrium. Such dynamics were derived in Ref. [5], where the state's evolution is expressed as a matrix vector multiplication $\vec{v}(t) = \mathcal{U}_S(t,0)\vec{v}'$, with $\vec{v}(t) = \{\hat{H}(t), \hat{L}(t), \hat{C}(t), \hat{I}\}$, $\vec{v}' = \{\hat{H}', \hat{L}', \hat{C}', \hat{I}\}$, $\hat{I}$ is the identity operator and

$$\mathcal{U}_S(t,0) = \begin{bmatrix} R & 0 & 0 & \langle\hat{H}\rangle_{eq}(1-R) \\ 0 & Rc & -Rs & 0 \\ 0 & Rs & Rc & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad . \tag{14}$$

Here, $R = e^{-\Gamma t}$ with $\Gamma = k_\downarrow - k_\uparrow$, $c = \cos(2\omega_f t)$, $s = \sin(2\omega_f t)$ and $\langle\hat{H}\rangle_{eq} = \frac{\hbar\omega_f}{2}coth\left(\frac{\hbar\omega_f}{2k_BT}\right)$. Utilizing Eq. (13) and (14), the evolution of the quenched system is completely defined.

A. Asymptotic behaviour of the fidelity of the quench procedure

The fidelity is a measure of the similarity between two quantum states. It was introduced by Uhlmann as the maximal quantum-mechanical transition probability between the two states' purifications in an enlarged Hilbert space [6–8]. For two displaced squeezed thermal states the fidelity obtains the form [9, 10]

$$\mathcal{F}(\hat{\rho}_1, \hat{\rho}_2) = \frac{2}{\sqrt{\Delta+\delta} - \sqrt{\delta}} \exp\left[-\vec{u}^T (A_1 + A_2)^{-1} \vec{u}\right] \quad , \tag{15}$$

where $\Delta = \det(A_1 + A_2)$, $\delta = (\det(A_1) - 1)(\det(A_2) - 1)$, with

$$A_i = 2\begin{bmatrix} \sigma^i_{QQ} & \sigma^i_{PQ}/\hbar \\ \sigma^i_{PQ}/\hbar & \sigma^i_{PP}/\hbar^2 \end{bmatrix} \quad , \tag{16}$$

where σ_{QQ}, σ_{PP} and σ_{PQ} are the variances and covariance of the position and momentum operators. The vector $\vec{u}$ is given by, $\vec{u} = \{\langle\hat{Q}\rangle_2 - \langle\hat{Q}\rangle_1, \langle\hat{P}\rangle_2 - \langle\hat{P}\rangle_1\}^T$. In the equilibration process the system's state remains centered at the origin (for all the considered protocols) and is compared to a thermal state. For such a case the calculation of the fidelity is greatly simplified, namely $\vec{u} = 0$ and the fidelity obtains the form,

$$\mathcal{F}(\hat{\rho}_S(t_f), \hat{\rho}_{th}) = \frac{2}{\sqrt{\Delta+\delta} - \sqrt{\delta}} \quad . \tag{17}$$

For the *quench* procedure, once the sudden transition to the final frequency ω_f the system relaxes to a thermal state, following an exponential decay rate. For a time-independent Hamiltonian Eq. (2) MP, [11], reduces to the standard master equation for the harmonic oscillator [12–14]:

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_S(t) = &-i\omega\left[\hat{a}^\dagger\hat{a}, \hat{\rho}_S(t)\right] \\ &+ k_\downarrow\left(\hat{a}\rho_S(t)\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger\hat{a}, \hat{\rho}_S(t)\}\right) \\ &+ k_\uparrow\left(\hat{a}^\dagger\hat{\rho}_S(t)\hat{a} - \frac{1}{2}\{\hat{a}\hat{a}^\dagger, \hat{\rho}_S(t)\}\right) \quad ,\end{aligned} \tag{18}$$

where the annihilation operator is given by $\hat{a} = \left(\sqrt{\frac{m\omega(0)}{2\hbar}}\hat{Q} + \frac{i}{\sqrt{2\hbar m\omega(0)}}\hat{P}\right)$. The solution of the master equation can be represented in the Heisenberg picture, obtaining the form

$$\hat{a}(t) = \hat{a}(0)\, e^{-i\omega - k/2t} \tag{19}$$

$$\hat{a}^{\dagger}\hat{a}(t) = \hat{a}^{\dagger}\hat{a}(0)\, e^{-kt} + N\left(1 - e^{-kt}\right) \quad . \tag{20}$$

Next, we write the elements of A_i in terms of the creation annihilation operators, and neglect terms that decay with a rate $2k$. This leads to a simplified form for the fidelity

$$\mathcal{F}\left(\hat{\rho}_S(t_f), \hat{\rho}_{th}\right) \approx \frac{2}{\sqrt{c_1 + c_2 e^{-kt}} - \sqrt{c_3 + c_4 e^{-kt}}} \quad , \tag{21}$$

where $k \equiv k_{\downarrow} - k_{\uparrow}$

$$c_1 = \left(4\sigma_{PP}^{th}\sigma_{QQ}^{th}/\hbar^2 + 1\right)^2 \tag{22}$$

$$c_2 = 4\left(4\sigma_{PP}^{th}\sigma_{QQ}^{th}/\hbar^2 + 1\right)\left(c_{PP}\sigma_{QQ}^{th} + c_{QQ}\sigma_{PP}^{th}/\hbar^2\right)$$

$$c_3 = \left(4\sigma_{PP}^{th}\sigma_{QQ}^{th}/\hbar^2 - 1\right)^2$$

$$c_4 = 4\left(4\sigma_{PP}^{th}\sigma_{QQ}^{th}/\hbar^2 - 1\right)\left(\sigma_{PP}^{th}c_{QQ}/\hbar^2 + \sigma_{QQ}^{th}c_{PP}\right) \quad .$$

Here, c_{PP} and c_{QQ} are time-independent parameters, defined by the evolution of the system's variances

$$\sigma_{QQ} = c_{QQ}e^{-kt} + \sigma_{QQ}^{th} \tag{23}$$

$$\sigma_{PP} = c_{PP}e^{-kt} + \sigma_{PP}^{th} \; .$$

In the asymptotic limit, equation (21) can be expanded in orders of e^{-kt}, and the fidelity's asymptotic behaviour reads: $1 - \mathcal{F}(t) \propto e^{-kt}$.

IV. ADIABATIC LIMIT

In the limit of infinite protocol time duration $t_f \to \infty$ the STE result converges to the adiabatic solution. This can be seen by studying the change in $y(t)$. Differentiating Eq. (9) MP leads to

$$\dot{y}(t) = \left(\frac{t}{t_f}\right)^2 \left(3c_3 + 4c_4\left(\frac{t}{t_f}\right) + 5c_5\left(\frac{t}{t_f}\right)^2\right). \tag{24}$$

Hence, in the limit $t_f \to \infty$, $\dot{y}$ vanishes. Moreover, the effective frequency converges to $\alpha(t) \to \omega(t)$ (Eq. (4) MP). Substituting this result into Eq. (10) MP gives the adiabatic solution

$$y = \frac{k_{\uparrow}^{adi}(t)}{k_{\downarrow}^{adi}(t)} \quad , \tag{25}$$

where the excitation and decay rate, in the adiabatic limit, are $k_{\downarrow}^{adi}(t) = k_{\uparrow}^{adi}(t)\, e^{\omega(t)/k_B T} = \frac{\omega(t)|\vec{d}|^2}{8\pi\varepsilon_0\hbar c}\left(1 + N(\omega(t))\right)$. Writing Eq. (25) in terms of β, the system state (Eq. (5)) obtains the form

$$\hat{\rho}_S(t) = Z_{th}(t)\, e^{-\hbar\omega(t)\left(\hat{a}^{\dagger}(t)\hat{a}(t) + \frac{1}{2}\right)}, \tag{26}$$

with $Z_{th}(t) = \mathrm{tr}\left(\exp\left(-\hbar\omega(t)\left(\hat{a}^{\dagger}(t)\hat{a}(t) + \frac{1}{2}\right)\right)\right)$. Thus, in the adiabatic limit the STE solution converges to the adiabatic state.

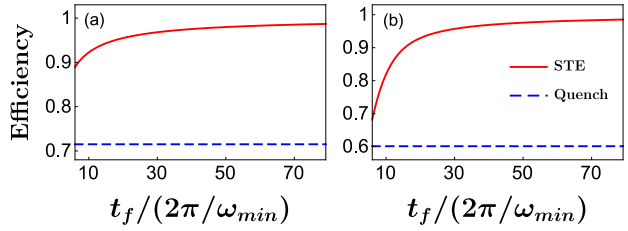

FIG. 1: The efficiency relative to the optimal adiabatic result for the *short-cut to equilibration* (red) and *quench* (blue) protocols, for (a) expansion protocol, (b) compression protocol.

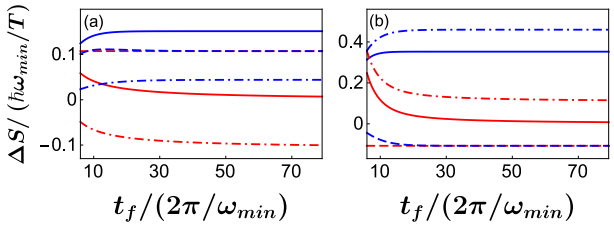

FIG. 2: The change in entropy during the (a) expansion and (b) compression processes for the STE (red) and *quench* (blue) protocols as a function of protocol duration. The change in system entropy ΔS_{sys}, bath entropy ΔS_B and global entropy ΔS_U are indicated by dashed, dashed-dot, and continuous lines respectively.

V. ENTROPY CALCULATION

The *shortcut to equilibration* procedure induces a swift change in the system's entropy ΔS_{sys}. An expansion protocol is accompanied by an increase in the system entropy, while a compression is followed by a decrease in entropy, Figure 2. The STE transforms the system to the target state with high precision, which is almost independent on the protocol duration t_f. As a result, the change in system entropy ΔS_{sys} is independent of t_f.

In contrast, the global entropy generation depends on the trajectory between initial and final states. For large protocol times the STE approaches the adiabatic limit and the global entropy generation ΔS_U vanishes. The heat exchange with the bath increases for shorter protocol duration, which in turn, increases the change in the bath's entropy, $\Delta S_B = -Q_{sys}/T_B$. During the expansion (compression) process the system energy decreases (increases), and heat is transferred from (to) the bath, this is accompanied by a decrease (increase) in the bath entropy see Fig. 2.

The change in entropy associated with the *quench* protocol is composed of a fast isoentropic process, followed

by a natural decay towards equilibrium. During the relaxation stage the coherence decays, leading to a rise in the system entropy. For both expansion and compression quench protocols energy flows from the system to the bath, resulting in an increase in the bath entropy and a global entropy production. For both compression and expansion the global irreversible entropy generation of the *quench* exceeds the STE result.

VI. ESTIMATION OF THE FIDELITY OF THE SHORTCUT TO EQUILIBRATION PROTOCOL

The accuracy of the STE depends on two factors, the accuracy of the reverse engineering protocol and the validity of the inertial solution. We find that the reverse engineering achieves a fidelity with an accuracy $\mathcal{A} = -\log_{10}(1-\mathcal{F}) < 10^{-10}$, for the final state $\tilde{\rho}_S(\beta(t_f),\mu(t_f)) = Z^{-1}e^{\beta(t_f)\hat{b}^\dagger\hat{b}(\mu(t_f))}$. However, the STE is based on the inertial solution for the free propagator. This accuracy can be checked by comparing the fidelity of the inertial solution with respect to an exact numerical solution. The obtained accuracy is $\mathcal{A} < 10^{-5}$, see Fig. 3 MP.

VII. SHORTCUT TO EQUILIBRATION FOR A SU(2) ALGEBRA

We engineer a shortcut to equilibration protocol in the framework of the SU(2) algebra. The procedure follows a similar approach as described in the MP. We expect that a similar result can be obtained for systems described by a closed Lie algebra, once the non-adiabatic master equation [4, 11] is obtained.

We consider a driven system characterized by the SU(2) algebra. The system is described by the Hamiltonian

$$\hat{H}_S(t) = \omega(t)\hat{S}_z + \varepsilon(t)\hat{S}_x \quad , \tag{27}$$

here, $\hat{S}_i$ is the $i = x,y,z$ spin operator. The time-dependent Rabi frequency is $\Omega(t) = \sqrt{\omega^2(t)+\varepsilon^2(t)}$.

Constructing the NAME and solving the open system dynamics requires an explicit solution of the closed dynamics. This is achieved in the framework of the inertial theorem [4]. In this framework, it is natural to employ a time-dependent operator basis: $\{\hat{H},\hat{L},\hat{C},\hat{I}\}$, with $\hat{H}(t) = \omega(t)\hat{S}_z + \varepsilon(t)\hat{S}_x$, $\hat{L}(t) = \varepsilon(t)\hat{S}_z - \omega(t)\hat{S}_x$, $\hat{C}(t) = \Omega(t)\hat{S}_y$. The derivation in [4] Sec. II B leads to the inertial solution for Hamiltonian Eq. (27). This solution describes the system dynamics in Liouville space, given in terms of eigenoperators of the free propagator. The eigenoperators associated with Hamiltonian Eq. (27) read

$$\hat{\chi} = \frac{1}{\Omega(0)\kappa}\left(\hat{H}_S(0) + \mu(t)\hat{C}(0)\right) \tag{28}$$

$$\hat{\sigma}(0) = \frac{1}{2\Omega(0)\kappa^2(0)}\left(-\mu(0)\hat{H}_S(0) - i\kappa\hat{L}(0) + \hat{C}(0)\right) \tag{29}$$

and $\sigma^\dagger(t)$. Here, the adiabatic parameter is $\mu(t) = (\dot{\omega}\varepsilon - \omega\dot{\varepsilon})/\Omega^3$ and $\kappa(t) = \sqrt{1+\mu(t)^2}$.

Following the derivation in [4] Appendix C, assuming an Ohmic boson bath, we obtain the non-adiabatic master equation in the interaction picture for the driven open system

$$\frac{d}{dt}\tilde{\rho}_S(t) = k_\downarrow(t)\left(\hat{\sigma}\tilde{\rho}_S(t)\hat{\sigma}^\dagger - \frac{1}{2}\{\hat{\sigma}^\dagger\hat{\sigma},\tilde{\rho}_S(t)\}\right) + k_\uparrow(t)\left(\hat{\sigma}^\dagger\tilde{\rho}_S(t)\hat{\sigma} - \frac{1}{2}\{\hat{\sigma}\hat{\sigma}^\dagger,\tilde{\rho}_S(t)\}\right) \quad , \tag{30}$$

with $\hat{\sigma} \equiv \sigma(0)$, $\hat{\sigma}^\dagger \equiv \hat{\sigma}^\dagger(0)$ and

$$k_\downarrow(t) = \frac{2\alpha(t)}{\hbar c}(1+N(\alpha(t))) \tag{31}$$

$$k_\uparrow(t) = \frac{2\alpha(t)}{\hbar c}N(\alpha(t)) \quad , \tag{32}$$

where

$$\alpha(t) = \kappa(t)\Omega(t) \quad . \tag{33}$$

The solution of the dynamics (Eq. (30)) is obtained by employing a generalized canonical form:

$$\tilde{\rho}_S(t) = Z^{-1}e^{\gamma\hat{\sigma}}e^{\beta\hat{\chi}}e^{\gamma^*\hat{\sigma}^\dagger} \quad , \tag{34}$$

with the partition function

$$Z = tr\left(e^{\gamma\hat{\sigma}}e^{\beta\hat{\chi}}e^{\gamma^*\hat{\sigma}^\dagger}\right) = \frac{1}{4}e^{-\frac{\beta}{2}}\left(4e^{\beta} + \frac{e^{\beta}|\gamma|^2}{\kappa^2} + 4\right) \quad , \tag{35}$$

where β, γ and $\gamma*$ are the Lagrange multipliers [1].

Representing the density operator in the generalized canonical form (Eq. (34)) the system dynamics are completely determined by the time-dependent parameters β and γ. Next, we substitute Eq. (34) in Eq. (30), multiply from the right by $\tilde{\rho}_S^{-1}(t)$ and comparing terms proportionate to $\hat{\chi}$, $\hat{\sigma}$ and $\hat{\sigma}^\dagger$ (similar derivation as in Sec. I) leads to dynamical equations for the parameters

$$\dot{\beta} = \frac{\gamma e^{\beta}}{2\kappa^2}\dot{\gamma}^* - k_\downarrow\frac{4\kappa^2\left(e^{\beta}+1\right)+|\gamma|^2e^{\beta}}{16\kappa^4} + k_\uparrow\frac{\left(|\gamma|^2+e^{-\beta}4\kappa^2\right)\left(4\left(e^{\beta}+1\right)\kappa^2+e^{\beta}|\gamma|^2\right)}{64\kappa^6} \tag{36}$$

$$\dot{\gamma} = k_\downarrow\frac{e^{-\beta}\gamma}{8\kappa^2} - k_\uparrow\frac{\gamma\left(2\left(1+2e^{-\beta}\right)\kappa^2+|\gamma|^2\right)}{16\kappa^4} \quad ,$$

For an initial stationary thermal state $\tilde{\rho}_S(0) = Z^{-1}e^{-\hat{H}(0)/k_BT}$, implying that $\gamma(0) = 0$ and from Eq. (29) $\mu(0) = 0$. For an initial thermal state Eq. (36) reduces to a single equation, and the open system dynamics is fully characterized by

$$\dot{\beta} = \frac{1}{4\kappa^2(t)}\left[k_\uparrow(t)\left(1+e^{-\beta}\right) - k_\downarrow(t)\left(e^{\beta}+1\right)\right] \quad . \quad (37)$$

Following a similar procedure as described in the MP, we perform a change of variables $y \equiv e^{\beta}$ to obtain an equation of the Riccati form

$$\dot{y}(t) = \frac{1}{4\kappa^2(t)}\left[-y^2(t)\,k_\downarrow(t) + y\left(k_\uparrow(t) - k_\downarrow(t)\right) + k_\downarrow(t)\right] \quad . \quad (38)$$

Next, we can substitute a polynomial in t which satisfies the boundary condition $\beta(0) = -\hbar\Omega(0)/k_BT$, $\beta(t_f) = -\hbar\Omega(t_f)/k_BT$, $\dot{\beta}(0) = \dot{\beta}(t) = \dot{\alpha}(0) = \dot{\alpha}(t_f) = 0$ and solve for $\alpha(t)$ and $\Omega(t)$ from Eq. (38) and Eq. (33) accordingly.

VIII. CALCULATION OF OBSERVABLES

We define a time-dependent basis in Liouville space $\vec{v}(t) = \{\hat{H}_S(t), \hat{L}(t), \hat{C}(t), \hat{I}\}^T$, where $\hat{L}(t) = \frac{\hat{p}^2}{2m} - \frac{1}{2}\omega^2(t)\,\hat{q}^2$, $\hat{C}(t) = \frac{\omega(t)}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$ and $\hat{H}_S(t)$ is the Hamiltonian, Eq. (1) MP. Any operator within the algebra can be represented in terms of the basis operators, hence, the problem reduces to the solution of $\langle\vec{v}(t)\rangle$. In a vector notation we write

$$\begin{bmatrix} \langle\hat{H}_S(t)\rangle \\ \langle\hat{L}(t)\rangle \\ \langle\hat{C}(t)\rangle \\ \hat{I} \end{bmatrix} = \langle\vec{v}(t)\rangle = \mathrm{tr}\left(\vec{v}(t)\,\rho_S(t)\right) \quad . \quad (39)$$

Alternatively, Eq. (39) can be expressed in the interaction representation relative to the free dynamics

$$\langle\vec{v}(t)\rangle = \mathrm{tr}\left(\vec{v}_H(t)\,\tilde{\rho}_S(t)\right) \quad , \quad (40)$$

where the free dynamics are given by

$$\vec{v}_H(t) = \mathcal{U}_S(t)\,\vec{v}(0) \quad , \quad (41)$$

here $\mathcal{U}_S(t)$ is the propagator in Liouville space, obtained from the inertial solution [4].

Next, we express $\vec{v}_H(0)$ in terms of the instantaneous basis $\vec{b}(\mu) = \{\hat{b}^2(\mu), \hat{b}^\dagger\hat{b}(\mu), \hat{b}^{\dagger 2}(\mu), \hat{I}\}^T$, utilizing the transformation

$$\vec{v}(0) = \mathcal{M}(\mu)\,\vec{b}(\mu) \quad . \quad (42)$$

with $\mathcal{M}(\mu) = \mathcal{M}_1(\mathcal{M}_2)^{-1}$

$$\mathcal{M}_1 = \begin{bmatrix} \frac{1}{2}m\omega(0)^2 & 0 & \frac{1}{2m} & 0 \\ -\frac{1}{2}m\omega(0)^2 & 0 & \frac{1}{2m} & 0 \\ 0 & \frac{\omega(0)}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad (43)$$

TABLE II

Coefficient	Value [atomic units]
Oscillator mass	1
Compression $\omega(0)$	5
Compression $\omega(t_f)$	10
Expansion $\omega(0)$	10
Expansion $\omega(t_f)$	5
Bath temperature	2
Coupling prefactor	$\frac{\lvert\vec{d}\rvert^2}{8\pi\varepsilon_0\hbar c} = 0.02$

and

$$\mathcal{M}_2 = \frac{1}{4\hbar\kappa}\begin{bmatrix} m\omega(0)(\kappa+i\mu)^2 & 2(i\kappa-\mu) & -\frac{4}{m\omega(0)} & 0 \\ 4m\omega(0) & 2\mu & \frac{4}{m\omega(0)} & -4\hbar\kappa \\ m\omega(0)(\kappa-i\mu)^2 & -2(i\kappa+\mu) & -\frac{4}{m\omega(0)} & 0 \\ 0 & 0 & 0 & 4\hbar\kappa \end{bmatrix} \quad . \quad (44)$$

Substituting Eq. (41) and (42) into (40) gives

$$\langle\vec{v}(t)\rangle = \mathcal{U}_S(t)\,\mathcal{M}(\mu)\,\langle\vec{b}(\mu)\rangle \quad . \quad (45)$$

For an initial thermal state the density matrix is of the form $\tilde{\rho}_S(t) = Z^{-1}e^{\beta(t)\hat{b}^\dagger\hat{b}(\mu)}$ (Z and $\hat{b}(\mu)$ are defined in the MP). The eigenoperators satisfy the Bosonic annihilation creation commutation relations, $\left[\hat{b}(\mu), \hat{b}^\dagger(\mu)\right] = 1$, utilizing the these commutation relations we obtain

$$\langle\hat{b}^\dagger\hat{b}(\mu)\rangle = \frac{\left(e^{-2\beta}-1\right)}{2\left(e^{-\beta}-1\right)^2} - \frac{1}{2}$$
$$\langle\hat{b}^2(\mu)\rangle = 0 \quad . \quad (46)$$

Inserting these relations into the operator vector leads to

$$\langle\vec{b}(\mu)\rangle = \{0, \langle\hat{b}^\dagger\hat{b}(\mu)\rangle, 0, \hat{I}\}^T \quad , \quad (47)$$

and by substituting Eq. (47) into Eq. (45) we obtain the dynamics for $\langle\vec{v}(t)\rangle$.

To conclude, $\beta(t)$ completely determines the vector $\langle\vec{b}(\mu)\rangle$ (Eq. (47) and (45)), which leads the expectation value of any system observable within the Heisenberg-Weyl algebra.

IX. NUMERICAL DETAILS

The value of $\omega(t)$ is assessed by a numerical solution of equations (10) MP and (4) MP, employing a built in Matlab solver. The solution is used to calculate the system's evolution according to the inertial solution [4]. The validity of the inertial approximation has been verified, with the inertial parameter obtaining maximum values of $\Upsilon \approx 0.1\,(0.04)$ for the compression (expansion) protocols.

The model parameters are summarized in Table II.

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