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On the Partial Difference Equations
of Mathematical Physics

Introduction

Problems involving the classical linear partial differential equations of mathematical physics can be reduced to algebraic ones of a very much simpler structure by replacing the differentials by difference quotients on some (say rectilinear) mesh. This paper will undertake an elementary discussion of these algebraic problems, in particular of the behavior of the solution as the mesh width tends to zero. For present purposes we limit ourselves mainly to simple but typical cases, and treat them in such a way that the applicability of the method to more general difference equations and to those with arbitrarily many independent variables is made clear.

Corresponding to the correctly posed problems for partial differential equations we will treat boundary value and eigenvalue problems for elliptic difference equations, and initial value problems for the hyperbolic or parabolic cases. We will show by typical examples that the passage to the limit is indeed possible, i.e., that the solution of the difference equation converges to the solution of the corresponding differential equation; in fact we will find that for elliptic equations in general a difference quotient of arbitrarily high order tends to the corresponding derivative. Nowhere do we assume the existence of the solution to the differential equation problem—on the contrary, we obtain a simple existence proof by using the limiting process.[1] For the case of elliptic equations convergence is

obtained independently of the choice of mesh, but we will find that for the case of the initial value problem for hyperbolic equations, convergence is obtained only if the ratio of the mesh widths in different directions satisfies certain inequalities which in turn depend on the position of the characteristics relative to the mesh.

We take as a typical case the boundary value problem of potential theory. Its solution and its relation to the solution of the corresponding difference equation has been extensively treated during the past few years.[2] However in contrast to the present paper, the previous work has involved the use of quite special characteristics of the potential equation so that the applicability of the method used there to other problems has not been immediately evident.

In addition to the main part of the paper, we append an elementary algebraic discussion of the connection of the boundary value problem of elliptic equations with the random walk problem arising in statistics.

[1] Our method of proof may be extended without difficulty to cover boundary value and eigenvalue problems for arbitrary linear elliptic differential equations and initial value problems for arbitrary linear hyperbolic differential equations.

[2] J. le Roux, "Sur le problem de Dirichlet", *Journ. de mathém. pur. et appl.* (6) 10, 189 (1914). R. G. D. Richardson, "A new method in boundary problems for differential equations", *Trans. of the Am. Math. Soc.* 18, p. 489 ff, (1917). H. B. Philips and N. Wiener, *Nets and the Dirichlet Problem*, Publ. of M.I.T. (1925). Unfortunately these papers were not known by the first of the three authors when he prepared his note "On the theory of partial difference equations," *Gött. Nachr.* 23, X, 1925, on which the present work depends. See also L. Lusternik, "On an application of the direct method in variation calculus," Recueil de la Société Mathém. de Moscou, 1926. G. Bouligand, "Sur le probléme de Dirichlet," *Ann. de la soc. polon. de mathém.* 4, Cracow (1926). On the meaning of the difference expressions and on further applications of them, see R. Courant, "Über direkte Methoden in der Variationsrechnung," *Math. Ann.* 97, p. 711, and the references given therein.

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I. The elliptic case

Section 1. Preliminary remarks

• 1. *Definitions*

Consider a rectangular array of points in the (x, y)-plane, such that for mesh width $h > 0$ the points of the lattice are given by

$$\left. \begin{array}{l} x = nh \\ y = mh \end{array} \right\} \quad m, n = 0, \pm 1, \pm 2, \cdots .$$

Let G be a region of the plane bounded by a continuous closed curve which has no double points. Then the corresponding mesh region, G_h—which is uniquely determined for sufficiently small mesh width—consists of all those mesh points lying in G which can be connected to any other given point in G by a connected chain of mesh points. By a connected chain of mesh points we mean a sequence of points such that each point follows in the sequence one of its four neighboring points. We denote as a *boundary point* of G_h a point whose four neighboring points do not all belong to G_h. All other points of G_h we call *interior points*.

We shall consider functions $u, v, \cdots$ of position on the grid, i.e., functions which are defined only for grid points, but we shall denote them as $u(x, y), v(x, y), \cdots$. For their forward and backward difference quotients we employ the following notation,

$$u_x = \frac{1}{h} [u(x + h, y) - u(x, y)],$$

$$u_y = \frac{1}{h} [u(x, y + h) - u(x, y)].$$

$$u_{\bar{x}} = \frac{1}{h} [u(x, y) - u(x - h, y)],$$

$$u_{\bar{y}} = \frac{1}{h} [u(x, y) - u(x, y - h)].$$

In the same way the difference quotients of higher order are formed, e.g.,

$$(u_x)_{\bar{x}} = u_{x\bar{x}} = u_{\bar{x}x}$$

$$= \frac{1}{h^2} [u(x + h, y) - 2u(x, y) + u(x - h, y)], \text{ etc.}$$

• 2. *Difference expressions and Green's function*

In order to study linear difference expressions of second order, we form (using as a model the theory of partial differential equations), a bilinear expression from the forward difference quotients of two functions, u and v,

$$B(u, v) = au_x v_x + bu_x v_y + cu_y v_x + du_y v_y + \alpha u_x v$$

$$+ \beta u_y v + \gamma uv_x + \delta uv_y + guv,$$

where $a = a(x, y), \cdots, \alpha = \alpha(x, y), \cdots, g = g(x, y)$ are functions on the mesh.

From the bilinear expression of first order we derive a difference expression of second order in the following way: we form the sum

$$h^2 \sum_{G_h} \sum B(u, v)$$

over all points of a region G_h in the mesh, where in $B(u, v)$ the difference quotients between boundary points and points not belonging to the mesh are to be set equal to zero. We now transform the sum through partial summation, i.e., we arrange the sum according to v, and split it up into a sum over the set of interior points, G'_h and a sum over the set of boundary points, Γ_h. Thus we obtain:

$$h^2 \sum_{G_h} \sum B(u, v)$$

$$= -h^2 \sum_{G_h'} \sum vL(u) - h \sum_{\Gamma_h} v\Re(u). \quad (1)$$

$L(u)$ is a linear difference expression of second order defined for all interior points of G_h:

$$L(u) = (au_x)_{\bar{x}} + (bu_x)_{\bar{y}} + (cu_y)_{\bar{x}} + (du_y)_{\bar{y}}$$

$$- \alpha u_x - \beta u_y + (\gamma u)_{\bar{x}} + (\delta u)_{\bar{y}} - gu.$$

$\Re(u)$ is, for every boundary point, a linear difference expression whose exact form will not be given here.

If we arrange $\sum_{G_h} \sum B(u, v)$ according to u, we find

$$h^2 \sum_{G_h} \sum B(u, v)$$

$$= -h^2 \sum_{G_h'} \sum uM(v) - h \sum_{\Gamma_h'} uS(v). \quad (2)$$

$M(v)$ is called the adjoint difference expression of $L(u)$ and has the form

$$M(v) = (av_x)_{\bar{x}} + (bv_y)_{\bar{x}} + (cv_x)_{\bar{y}} + (dv_y)_{\bar{y}}$$

$$+ (\alpha v)_{\bar{x}} + (\beta v)_{\bar{y}} - \gamma v_x - \delta v_y - gv,$$

while $S(v)$ is a difference expression corresponding to $\Re(u)$ for the boundary.

Formulas (1) and (2) give

$$h \sum_{G_h'} \sum [vL(u) - uM(v)]$$

$$+ h \sum_{\Gamma_h'} [v\Re(u) - uS(v)] = 0. \quad (3)$$

Formulas (1), (2), and (3) are called *Green's formula*.

The simplest and most important case results if the bilinear form is symmetric, i.e., if the relations $b = c$, $\alpha = \gamma$, $\beta = \delta$ hold. In this case $L(u)$ is identical with $M(u)$—the self-adjoint case—and it can be derived from the quadratic expression

$$B(u, u) = au_x^2 + 2bu_x u_y + du_y^2$$

$$+ 2\alpha u_x u + 2\beta u_y u + gu^2.$$

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In the following we shall limit ourselves mainly to expressions $L(u)$ which are self-adjoint. The character of the difference expression $L(u)$ depends principally on the nature of those terms in the quadratic form $B(u, u)$ which are quadratic in the first difference quotients. We call this part of $B(u, u)$ the *characteristic form*:

$$P(u, u) = au_x^2 + 2bu_x u_y + du_y^2.$$

We call the corresponding difference expression $L(u)$ elliptic or hyperbolic, depending on whether the function $P(u, u)$ of the difference quotients is (positive) definite or indefinite.

The difference expression $\Delta u = u_{x\bar{x}} + u_{y\bar{y}}$ with which we shall concern ourselves in the following paragraph is elliptic, i.e., it comes from the quadratic expression

$$B(u, u) = u_x^2 + u_y^2 \quad \text{or} \quad u_{\bar{x}}^2 + u_{\bar{y}}^2.$$

The corresponding Green's formulas are

$$h^2 \sum_{G_h} \sum (u_x^2 + u_y^2) = -h^2 \sum_{G_h'} \sum u\Delta u$$
$$- h \sum_{\Gamma_h} u\Re(u) \qquad [\text{Note 3}] \qquad (4)$$

$$h^2 \sum_{G_h'} \sum (v\Delta u - u\Delta v)$$
$$+ h \sum_{\Gamma_h} [v\Re(u) - u\Re(v)] = 0. \qquad (5)$$

The difference expression, $\Delta u = u_{x\bar{x}} + u_{y\bar{y}}$, is obviously the analogue of the differential expression $(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2)$ for a function $u(x, y)$ of the continuous variables x and y. Written out explicitly the difference expression is

$$\Delta u = \frac{1}{h^2} [u(x + h, y) + u(x, y + h)$$
$$+ u(x - h, y) + u(x, y - h) - 4u(x, y)].$$

Therefore $(h^2/4)\Delta u$ is the excess of the arithmetic mean of the functional values at the four neighborhood points over the functional value at the point in question.

Completely similar considerations lead to linear difference expressions of the fourth order and corresponding Green's formula, provided one starts from bilinear difference expressions which are formed from the difference quotients of second order. Consider for example the difference expression

$$\Delta\Delta u = u_{xx\bar{x}\bar{x}} + 2u_{x\bar{x}y\bar{y}} + u_{yy\bar{y}\bar{y}}.$$

This corresponds to the quadratic expression

$$B(u, u) = (u_{x\bar{x}} + u_{y\bar{y}})^2 = (\Delta u)^2,$$

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[3] The boundary expression $\Re(u)$ may be written as follows: Let $u_0, u_1, \ldots, u_\nu$ be values of the function at a boundary point and at its ν neighboring points $(\nu \leq 3)$, then

$$\Re(u) = \frac{1}{h}(u_1 + \cdots + u_\nu - \nu u_0).$$

provided one orders the sum

$$h^2 \sum_{G_h'} \sum \Delta u \Delta v$$

according to v, or equivalently replaces u by Δu in Eq. (5). One must notice however that in the expression $\Delta\Delta u$, the functional value at a point is connected with the values at its neighboring points and at their neighboring points, and accordingly is defined only for such points of the region G_h as are also interior points of the region G_h' (See Section 5). The entirety of such points we designate as G_h''. We obtain then Green's formula

$$h^2 \sum_{G_h'} \sum \Delta u \cdot \Delta v$$
$$= h^2 \sum_{G_h''} \sum v \cdot \Delta\Delta u + h \sum_{\Gamma_h + \Gamma_h'} v \cdot \Re(u), \qquad (6)$$

where $\Re(u)$ is a definable linear difference expression for each point of the boundary strip $\Gamma_h + \Gamma_h'$. Γ_h' indicates the set of boundary points of G_h'.

Section 2. Boundary value and eigenvalue problems

● 1. *The theory of boundary value problems*

The boundary value problem for linear elliptic homogeneous difference equations of second order, which corresponds to the classical boundary value problem for partial differential equations, can be formulated in the following way.

Let there be given a self-adjoint elliptic linear difference expression of second order, $L(u)$, in a mesh region, G_h. $L(u)$ results from a quadratic expression $B(u, u)$ which is positive definite in the sense that it cannot vanish if u_x and u_y do not themselves vanish.

A function, u, is to be determined satisfying in G_h the difference equation $L(u) = 0$, and assuming prescribed values at the boundary points.

Under these requirements there will be exactly as many linear equations as there are interior points of the mesh at which the function u is to be determined.[4] Some of these equations which involve only mesh points whose neighbors also lie in the interior of the region are homogeneous; others which involve boundary points of the mesh region are inhomogeneous. If the right-hand side of the inhomogeneous equations is set equal to zero, that is if $u = 0$ on the boundary, then it follows from Green's formula (1), by setting $u = v$ that $B(u, u)$ vanishes, and further, from the definiteness of $B(u, u)$, that u_x and u_y vanish, and hence that u itself vanishes. Thus the difference equation for zero boundary values has the solution $u = 0$, or in other words the solution is uniquely determined since the difference of two solutions with the

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[4] If the matrix of the linear system of equations corresponding to an arbitrary difference equation of second order, $L(u) = 0$, is transposed, then the transposed set of equations corresponds to the adjoint difference equation $M(v) = 0$. Thus the above self-adjoint system gives rise to a set of linear equations with symmetric coefficients.

same boundary value must vanish. Further, if a linear system of equations with as many unknowns as equations is such that for vanishing right-hand side the unknowns must vanish, then the fundamental theorem of the theory of equations asserts that for an arbitrary right-hand side exactly one solution must exist. In our case there follows at once the existence of a solution of the boundary value problem.

Therefore we see that for elliptic difference equations the uniqueness and existence of the solution of the boundary value problem are related to each other through the fundamental theorem of the theory of linear equations, whereas for partial differential equations both facts must be proved by quite different methods. The basis for this difficulty in the latter case is to be found in the fact that the differential equations are no longer equivalent to a finite number of equations, and so one can no longer depend on the equality of the number of unknowns and equations.

Since the difference expression $\Delta u = 0$ can be derived from the positive definite quadratic expression

$$h^2 \sum_{G_h} \sum (u_x^2 + u_y^2),$$

the boundary value problem for the difference expression is uniquely solvable.

The theory for difference equations of higher order is developed in exactly the same way as that for difference equations of second order; for example one can treat the fourth order difference equation $\Delta\Delta u = 0$. In this case the values of the function must be prescribed on the boundary strip $\Gamma_h + \Gamma_h'$. Evidently here also the difference equation yields just as many linear equations as there are unknown functional values at the points of G_h''. In order to demonstrate the existence and uniqueness of the solution one needs only to show that a solution which has the value zero in the boundary strip $\Gamma_h + \Gamma_h'$ necessarily vanishes identically. To this end we note that the sum over the corresponding quadratic expression

$$h^2 \sum_{G_h''} \sum (\Delta u)^2 \qquad (7)$$

for such a function vanishes, as can be seen by transforming the sum according to Green's formula (6). The vanishing of the sum (7) however implies that Δu vanishes at all points of G_h', and according to the above proof this can only happen for vanishing boundary values if the function u assumes the value zero throughout the region. Thus our assertion is proved, and both the existence and uniqueness of the solution to the boundary value problem for the difference equations are guaranteed.[5]

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[5] For the actual process of carrying through the solution of the boundary value problem by an iterative method, see among others the treatment: "Über Randwertaufgaben bei partiellen Differenzengleichungen" by R. Courant, *Zeitschr. f. angew. Mathematik u. Mechanik* **6**, 322–325 (1925). Also there is a report by H. Henky, in *Zeitschr. f. angew. Math. u. Mech.* **2**, 58 ff (1922).

● 2. *Relation to the minimum problem*

The above boundary value problem is related to the following minimum problem: among all functions $\varphi(x, y)$ defined in the mesh region G_h and assuming given values at the boundary points, that function $\varphi = u(x, y)$ is to be found for which the sum

$$h^2 \sum_{G_h} \sum B(\varphi, \varphi)$$

over the mesh region assumes the least possible value. We assume that the quadratic difference expression of first order, $B(u, u)$ is positive definite in the sense described in Section 1, Part 2. One can show that the difference equation $L(\varphi) = 0$ results from this minimum requirement on the solution $\varphi = u(x, y)$, where $L(\varphi)$ is the difference expression of second order derived previously from $B(\varphi, \varphi)$. In fact this can be seen either by applying the rules of differential calculus to the sums $h^2 \sum_{G_h} \sum B(\varphi, \varphi)$ as a function of a finite number of values of φ at the grid points, or by employing the usual methods from the calculus of variations.

By way of example, solving the boundary value problem of finding the solution to the equation $\Delta\varphi = 0$ which assumes given boundary values, is equivalent to minimizing the sum $h^2 \sum_{G_h} \sum (\varphi_x^2 + \varphi_y^2)$ over all functions which take on the boundary values.

There is a similar correspondence for the fourth-order difference equations, where we limit ourselves to the example $\Delta\Delta\varphi = 0$. The boundary value problem corresponding to this difference equation is equivalent to the problem of minimizing the sum

$$h^2 \sum_{G_h'} \sum (\Delta\varphi)^2$$

for functions that take on given values on the boundary strip Γ_h'. Besides this sum there are yet other quadratic expressions in the second derivatives which give rise to the equation $\Delta\Delta u = 0$ under the process of being minimized. For example this is true in G_h for the sum

$$h^2 \sum_{G_h'} \sum (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2).$$

That the minimum problem posed above always has a solution follows from the theorem that a continuous function of a finite number of variables (the functional values of φ at the grid points) always has a minimum if it is bounded from below and if it tends to infinity as soon as any of the independent variables goes to infinity.[6]

● 3. *Green's function*

It is possible to treat the boundary value problem for the inhomogeneous equation, $L(u) = -f$, in much the same way as the homogeneous case, $L(u) = 0$. For the inhomogeneous case it is sufficient to consider only the case of

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[6] It can easily be verified that the hypotheses for the application of this theorem are satisfied.

zero boundary conditions, since different boundary conditions can be taken care of by adding a suitable solution of the homogeneous equation. To solve the linear system of equations representing the boundary value problem, $L(u) = -f$, we first choose as the function $f(x, y)$ a function which assumes the value $-1/h^2$ at the point $x = \xi$, $y = \eta$ of the mesh. If $K(x, y; \xi, \eta)$ is the solution (vanishing on the boundary) of this difference equation which depends on the parametric point (ξ, η), then the solution for an arbitrary boundary condition can be represented by the sum

$$u(x, y) = h^2 \sum_{(\xi, \eta) \text{ in } G_h} K(x, y; \xi, \eta) f(\xi, \eta).$$

The function $K(x, y; \xi, \eta)$ which depends on the points (x, y) and (ξ, η) is called the Green's function of the differential expressions $L(u)$. If we call the Green's function for the adjoint expression $M(v)$, $\bar{K}(x, y; \xi, \eta)$, then the equivalence

$$K(\bar{\xi}, \bar{\eta}; \xi, \eta) = \bar{K}(\xi, \eta; \bar{\xi}, \bar{\eta}) \quad \text{holds,}$$

as can be seen to follow immediately from Green's formula (3) when $u = K(x, y; \xi, \eta)$, and $v = \bar{K}(x, y; \bar{\xi}, \bar{\eta})$. For a self-adjoint difference expression the above relation gives the symmetric expression

$$K(\bar{\xi}, \bar{\eta}; \xi, \eta) = K(\xi, \eta; \bar{\xi}, \bar{\eta}).$$

● 4. *Eigenvalue problems*

Self-adjoint difference expressions, $L(u)$, give rise to eigenvalue problems of the following type: find the value of a parameter λ, the eigenvalue, such that in G_h, a solution, the eigenfunction can be found for the difference equation $L(u) + \lambda u = 0$, where u is to be zero on the boundary, Γ_h.

The eigenvalue problem is equivalent to finding the principle axes of the quadratic form $B(u, u)$. Exactly as many eigenvalues and corresponding eigenfunctions exist as there are interior mesh points of the region G_h. The system of eigenfunctions and eigenvalues (and a proof of their existence) is given by the minimum problem:

Among all functions, $\varphi(x, y)$, vanishing on the boundary, and satisfying the orthogonality relation

$$h^2 \sum_{G_h} \sum \varphi u^{(\nu)} = 0, \qquad (\nu = 1, \cdots, m - 1)$$

and normalized such that

$$h^2 \sum_{G_h} \sum \varphi^2 = 1,$$

the function, $\varphi = u$, is to be found for which the sum

$$h^2 \sum_{G_h} \sum B(\varphi, \varphi)$$

assumes its minimum value. The value of this minimum

is the mth eigenvalue, and the function for which it is assumed is the mth eigenfunction.[7]

Section 3.[8] *Connections with the problem of the random walk*
The theme of the following is related to a question from the theory of probability, namely the problem of the random walk in a bounded region.[9] We consider the lines of a mesh region G_h as paths along which a particle can move from one grid point to a neighboring one. In this net of streets the particle can wander aimlessly, and it can choose at random one of the four directions leading from each intersection of paths of the net—all four directions being equally probable. The walk ends as soon as a boundary point of G_h is reached because here the particle must be absorbed.
We ask:

1) What is the probability $w(P; R)$ that a random walk starting from a point P reaches a particle point R of the boundary?
2) What is the mathematical expectation $v(P; Q)$ that a random walk starting from P reaches a point Q of G_h without touching the boundary?

This probability or mathematical expectation, respectively, will be defined more precisely by the following process. Assume that at the point P there is a unit concentration of matter. Let this matter diffuse into the mesh with constant velocity, traveling say a mesh width in unit time. At each meshpoint let exactly one-fourth of the matter at the point diffuse outwards in each of the four possible directions. The matter which reaches a boundary point is to remain at that point. If the point of origin P is itself a boundary point, then the matter never leaves that point.
We define the probability $w(P; R)$ that a random walk starting from P reaches the boundary point R (without

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[7] From the orthogonality condition on the eigenfunctions,

$$h^2 \sum_{G_h} \sum u^{(\nu)} u^{(\mu)} = 0, \qquad (\nu \neq \mu)$$

it follows that each function, $g(x, y)$, which vanishes on the boundary of the mesh can be expanded in terms of the eigenfunctions in the form

$$y = \sum_{\nu=1}^{N} c^{(\nu)} u^{(\nu)},$$

where the coefficients are determined from the equations

$$c^{(\nu)} = h^2 \sum_{G_h} \sum g u^{(\nu)}.$$

In this way in particular the following representation for the Green's functions may be derived,

$$K(x, y; \xi, \eta) = -\sum_{\nu=1}^{N} \frac{u^{(\nu)}(x, y) \cdot u^{(\nu)}(\xi, \eta)}{\lambda^{(\nu)}}.$$

[8] Section 3 is not prerequisite to Section 4.
[9] The present treatment is essentially different from the familiar treatments which can be carried through, say for example in the case of Brownian motion for molecules. The difference lies precisely in the way in which the boundary of the region enters.

previously attaining the boundary), as the amount of matter which accumulates at this boundary point over an infinite amount of time.

We define the probability $E_n(P; Q)$ that the walk starting from the point P reaches the point Q in exactly n steps without touching the boundary by the amount of matter which accumulate in n units of time provided P and Q are both interior points. If either P or Q are boundary points then E_n is set equal to zero.

The value $E_n(P; Q)$ is exactly equal to $1/4^n$ times the number of paths of n steps leading from P to Q without touching the boundary. Thus $E_n(P; Q) = E_n(Q; P)$.

We define the mathematical expectation $v(P; Q)$ that one of the paths considered above leads from P to the point Q by the infinite sum of all of these possibilities,

$$v(P; Q) = \sum_{\nu=0}^{\infty} E_\nu(P; Q) \qquad \text{(Note 10)},$$

i.e., for the interior points P and Q, the sum of all the concentrations which have passed through the point Q at different times. This will be assigned the expected value 1 for a concentration originating at Q.

If the amount arriving at the boundary point R in exactly n steps is designated as $F_n(P; R)$, then the probability $w(P; R)$ is given by the series

$$w(P; R) = \sum_{\nu=0}^{\infty} F_\nu(P; R).$$

All the terms of this series are positive and the partial sum is bounded by one (since the concentration reaching the boundary can be made up of only part of the initial concentration), and therefore the convergence of the series is assured.

Now it is easy to see that the probability $E_n(P; Q)$, that is, the concentration reaching the point Q in exactly n steps tends to zero as n increases. For if at any point Q, from which the boundary point R can be reached in m steps, we have $E_n(P; Q) > \alpha > 0$, then at least $\alpha/4^m$ of the concentration at Q will reach the point R after m steps. However, since the sum $\sum_{\nu=0}^{\infty} F_\nu(P; R)$ converges, the concentration reaching R goes to zero with increasing time, where the value of $E_n(P; Q)$ must itself vanish as time increases; that is, the probability of an infinitely long walk remaining in the interior of the region is zero.

From this it follows that the entire concentration eventually reaches the boundary; or in other words that the sum $\sum_R w(P; R)$ over all the boundary points R is equal to one.

The convergence of the infinite series for the mathematical expectation

$$v(P; Q) = \sum_{\nu=0}^{\infty} E_\nu(P; Q)$$

remains to be shown.

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10 The convergence will be shown below.

To this end we remark that the quantity $E_n(P; Q)$ satisfies the following relations

$$E_{n+1}(P; Q) = \tfrac{1}{4}\{E_n(P; Q_1) + E_n(P; Q_2)$$
$$+ E_n(P; Q_3) + E_n(P; Q_4)\}, \qquad [n \geq 1],$$

where Q_1 through Q_4 are the four neighboring points of the interior point Q. That is, the concentration at the point Q at the $n + 1^{\text{st}}$ step consists of $1/4$ times the sum of the concentrations at the four neighboring points at the nth step. If one of the neighbors of Q, for example $Q_1 = R$, is a boundary point then it follows that no concentration flows from this boundary point to Q since the expression $E_n(P; R)$ is zero in this case. Furthermore, for an interior point, $E_0(P; P) = 1$ and of course $E_1(P; Q) = 0$.

From these relationships we find for the partial sum

$$v_n(P; Q) = \sum_{\nu=0}^{n} E_\nu(P; Q)$$

the equation

$$v_{n+1}(P; Q) = \tfrac{1}{4}\{v_n(P; Q_1) + v_n(P; Q_2)$$
$$+ v_n(P; Q_3) + v_n(P; Q_4)\},$$

for $P \neq Q$. For the case of $P = Q$,

$$v_{n+1}(P; P) = 1 + \tfrac{1}{4}\{v_n(P; P_1) + v_n(P; P_2)$$
$$+ v_n(P; P_3) + v_n(P; P_4)\},$$

that is, the expectation that a point goes back into itself consists of the expectation that a nonvanishing path leads from P back again to itself—namely,

$$\tfrac{1}{4}\{v_n(P; P_1) + v_n(P; P_2) + v_n(P; P_3) + v_n(P; P_4)\},$$

together with the expectation unity that expresses the initial position of the concentration itself at this point.

The quantity $v_n(P; Q)$ thus satisfies the following difference equation[11]

$$\Delta v_n(P; Q) = \frac{4}{h^2} E_n(P; Q), \qquad \text{for} \quad P \neq Q,$$

$$\Delta v_n(P; Q) = \frac{4}{h^2}\{E_n(P; Q) - 1\}, \quad \text{for} \quad P = Q.$$

$v_n(P; Q)$ is equal to zero when Q is a boundary point.

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11 This defines the Δ-operation for the variable point Q. This equation can be interpreted as an equation of the heat conduction type. That is, if the function $v_n(P; Q)$ is considered, not as a function of the index n as in our presentation above, but rather as a function of time, t, which is proportional to n, so that $t = n\tau$ and $v_n(P; Q) = v(P; Q; t) = v(t)$, then the above equations can be written in the following form:

$$\Delta v(t) = \frac{4\tau}{h^2} \cdot \frac{v(t + \tau) - v(t)}{\tau} \qquad \text{for} \quad P \neq Q,$$

$$\Delta v(t) = \frac{4\tau}{h^2} \cdot \left(\frac{v(t + \tau) - v(t)}{\tau} - \frac{1}{\tau}\right) \quad \text{for} \quad P = Q.$$

For a similar difference equation which has a parabolic differential equation as its limiting form, see Section 6 of the second half of the paper.

The solution of this boundary value problem for arbitrary right-hand side is uniquely determined as we have explained earlier (Section 2, Part 1), and depends continuously on the right-hand side. Since the variables $E_n(P; Q)$ tend to zero, the solution $v_n(P; Q)$ converges to the solution $v(P; Q)$ of the difference equation

$$\Delta v(P; Q) = 0 \qquad \text{for} \quad P \neq Q$$

$$\Delta v(P; Q) = -\frac{4}{h^2} \quad \text{for} \quad P = Q,$$

with boundary values $v(P; R) = 0$.

Thus we see that the mathematical expectation $v(P; Q)$ exists and is none other than the Green's function for the difference equation $\Delta u = 0$, except for a factor of 4. The symmetry of the Green's function is an immediate consequence of the symmetry of the quantity $E_n(P; Q)$ which was used to define it.

The probability $w(P; R)$ satisfies, with respect to P, the relation

$$w(P; R) = \tfrac{1}{4}\{w(P_1; R) + w(P_2; R)$$

$$+ w(P_3; R) + w(P_4; R)\},$$

and thus the difference equation $\Delta w = 0$. That is, if P_1, P_2, P_3, P_4, are the four neighboring points of the interior point P, then each path from P to R must pass through one of these four directions, and each of the four is equally likely. Furthermore, the probability of going from one boundary point R to another R' is $w(R; R') = 0$ unless the two points R and R' coincide, in which case $w(R; R) = 1$. Thus $w(P; R)$ is that solution of the boundary value problem $\Delta w = 0$ which assumes the value 1 at the boundary point R and the value 0 at all other points of the boundary. Therefore the solution of the boundary value problem for an arbitrary boundary value $u(R)$ has the simple form $u(P) = \sum_R w(P; R)u(R)$, where the sum is to be extended over all the boundary points.[12] If the function $u \equiv 1$ is substituted for u in this expression, then we check the relation $1 = \sum_R w(P; R)$.

The interpretation given above for Green's function as an expectation yields immediately further properties. We mention only the fact that the Green's function decreases if one goes from the region G to a subregion $\bar{G}$ lying within G; that is, the number of possible paths for steps on the mesh leading from one point P to another Q (without touching the boundary), decreases as the region decreases.

Of course, corresponding relationships hold for more than two independent variables. We note only that other elliptic difference equations admit a similar probability interpretation.

If the limit for vanishing mesh width is considered by methods given in the following section, then the Green's function on the mesh goes over to the Green's function of the potential equation except for a numerical factor; a similar relationship holds between the expression $w(P; R)/h$ and the normal derivative of the Green's function at the boundary of the region. In this way, for example, the Green's function for the potential equation could be interpreted as the specific mathematical expectation of going from one point to another,[13] without reaching the boundary.

In going over to the limit of a continuum from the mesh, the influence of the direction in the mesh prescribed for the random walk vanishes. This fact suggests that it would be of some interest to consider limiting cases of more general random walks for which the limitations on the direction of the walk are relaxed. This lies outside of the scope of this presentation, however, and we can only hope to renew the question at some other opportunity.

Section 4. The solution of the differential equation as a limiting form of the solution of the difference equation

● 1. *The boundary value problem of potential theory*

In letting the solution of the difference equation tend to the solution of the corresponding differential equation we shall relinquish the greatest possible degree of generality with regard to the boundary and boundary values in order to demonstrate more clearly the character of our method.[14] Therefore we assume that we are given a simply connected region G with a boundary formed of a finite number of arcs with continuously turning tangents. Let $f(x, y)$ be a given function which is continuous and has continuous partial derivatives of first and second order in a region containing G. If G_h is the mesh region with mesh width h belonging to the region G, then let the boundary value problem for the difference equation $\Delta u = 0$ be solved for the same boundary values which the function $f(x, y)$ assumes on the boundary; let $u_h(x, y)$ be the solution. We shall prove that as $h \to 0$ the function $u_h(x, y)$ converges to a function $u(x, y)$ which satisfies in G the partial differential equation $(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = 0$ and takes the value of $f(x, y)$ at each of the points of the boundary. We shall show further that for any region lying entirely within G the difference quotients of u_h of arbitrary order tend uniformly towards the corresponding partial derivatives of $u(x, y)$.

In the convergence proof it is convenient to replace the boundary condition $u = f$ by the weaker requirement that

$$\frac{1}{r} \iint_{S_r} (u - f)^2 \, dx\,dy \to 0 \quad \text{as} \quad r \to 0,$$

[12] Moreover it is easy to show that the probability $w(P; R)$ of reaching the boundary is the boundary expression $\mathfrak{R}(K(P, Q))$, constructed from the Green's function $K(P; Q)$ in terms of Q, where $u(x, y)$ is to be identified with $w(P, Q)$, and $v(x, y)$ with $v(P, Q)$ in Green's formula (5).

[13] Here the *a priori* expectation of reaching a certain area element is understood to be equal to the area of the element.

[14] We mention however that carrying through our method for more general boundaries and boundary values in no way causes any fundamental difficulty.
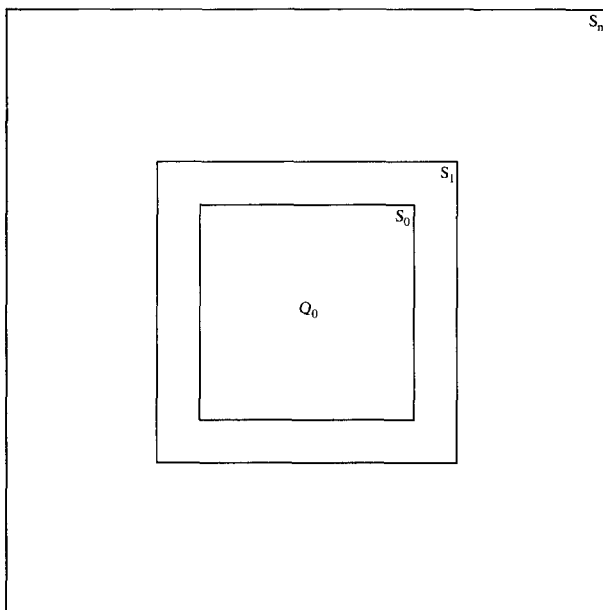
Figure 1

where S_r is that strip of G whose points are at a distance from the boundary smaller than r.[15] The convergence proof depends on the fact that for any subregion G^* lying entirely within G, the function $u_h(x, y)$ and each of its difference quotients is bounded and uniformly continuous as $h \to 0$ in the following sense: For each of these functions, say $w_h(x, y)$, there exists a $\delta(\varepsilon)$ depending only on the subregion and not on h such that

$$|w_h(P) - w_h(P_1)| < \varepsilon$$

provided the mesh points P and P_1 lie in the same subregion of G_h and are separated from each other by a distance less than $\delta(\varepsilon)$.

Once uniform continuity in this sense (equicontinuity) has been established we can in the usual way select from the functions u_h a subsequence of functions which tend uniformly in any subregion G^* to a limit function $u(x, y)$, while the difference quotients of u_h tend uniformly towards

[15] The weaker boundary value requirement does in fact provide the unique characterization of the solution, as can be seen from the easily proved theorem: If the boundary condition above is satisfied for $f(x, y) = 0$ for a function satisfying the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the interior of G and if

$$\iint_G \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy$$

exists, then $u(x, y)$ is identically zero. (See Courant, "Über die Lösungen der Differentialgleichungen der Physik," *Math Ann.* 85, 296 ff.)

In the case of two independent variables the boundary values are actually attained, as follows from the weaker requirement; but in the case of more variables the corresponding result cannot in general be expected since there can exist exceptional points on the boundary at which the boundary value is not taken on even though a solution exists under the weaker requirement.

the partial derivatives of u. The limit function then possesses derivatives of arbitrarily high order in any proper subregion G^* of G and satisfies $\nabla^2 u = 0$ in this region. If we can show also that u satisfies the boundary condition we can regard it as the solution of our boundary value problem for the region G. Since this solution is uniquely determined, it is clear that not only a partial sequence of the functions u_h, but this sequence of functions itself possesses the required convergence properties.

The uniform continuity (equicontinuity) of our quantities may be established by proving the following lemmas.

1) As $h \to 0$ the sums over the mesh region $h^2 \sum_{G_h} \sum u^2$ and $h^2 \sum_{G_h} \sum (u_x^2 + u_y^2)$ remain bounded.[16]

2) If $w = w_h$ satisfies the difference equation $\Delta w = 0$ at a mesh point of G_h, and if, as $h \to 0$ the sum

$$h^2 \sum_{G_h*} \sum w^2,$$

extended over a mesh region G_h^* associated with a subregion G^* of G remains bounded, then for any fixed subregion G^{**} lying entirely within G^* the sum

$$h^2 \sum_{G_h**} \sum (w_x^2 + w_y^2)$$

over the mesh region G_h^{**} associated with G^{**} likewise remains bounded as $h \to 0$. Using 1) there follows from this, since all of the difference quotients w of the function u_h again satisfy the difference equation $\Delta w = 0$, that each of the sums $h^2 \sum_{G_h*} \sum w^2$ is bounded.

3) From the boundedness of these sums there follows finally the boundedness and uniform continuity of all the difference quotients themselves.

• 2. *Proof of the lemmas*

The proof of 1) follows from the fact that the functional values u_h are themselves bounded. For the greatest (or least) value of the function is assumed on the boundary[17] and so is bounded by a prescribed finite value. The boundedness of the sum $h^2 \sum_{G_h} \sum (u_x^2 + u_y^2)$ is an immediate consequence of the minimum property of our mesh function formulated in Part 2 of Section 2 which gives in particular

$$h^2 \sum_{G_h} \sum (u_x^2 + u_y^2) \leq h^2 \sum_{G_h} \sum (f_x^2 + f_y^2),$$

but as $h \to 0$ the sum on the right tends to the integral

$$\iint_G \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 dx dy \right]$$

which, by hypothesis, exists.

[16] Here and in the following we drop the index h from the grid functions.
[17] We note, however, with a view to carrying over the method to other differential equations, that we can relax these conditions. To this end we need only to bring into play the inequality (15) or to use the reasoning of the alternative (see Part 4, Section 4).

To prove 2) we consider the quadratic sum

$$h^2 \sum_{int.\,Q_1} \sum (w_x^2 + w_{\bar{x}}^2 + w_y^2 + w_{\bar{y}}^2),$$

where the summation extends over all the interior points of a square Q_1, (see Fig. 1). We denote the values of the function on the boundary S_1 of the square Q_1, by w_1, and those on the boundary S_0, of Q_0, by w_0. Then Green's formula gives

$$h^2 \sum_{int.\,Q_1} \sum (w_x^2 + w_{\bar{x}}^2 + w_y^2 + w_{\bar{y}}^2) \tag{8}$$

$$= \sum_{S_1} w^2 - \sum_{S_0} w^2 - \sum_{C_1} w^2$$

where S_1 and S_0 are respectively the boundary of Q_1 and the square boundary of the lattice points Q_0 lying within S_1, while C_1 consists of the four corner points of the boundary of Q_1.

We now consider a sequence of concentric squares $Q_0, Q_1, \cdots, Q_n$ with boundaries $S_0, S_1, \cdots, S_n$, where each boundary is separated from the next by a mesh width. Applying the formula to each of these squares and observing that we have always

$$2h^2 \sum_{Q_0} \sum (w_x^2 + w_y^2)$$

$$\leq h^2 \sum_{Q_k} \sum (w_x^2 + w_{\bar{x}}^2 + w_y^2 + w_{\bar{y}}^2), \qquad (k \geq 1)$$

we obtain

$$2h^2 \sum_{Q_0} \sum (w_x^2 + w_y^2)$$

$$\leq \sum_{S_k} w^2 - \sum_{S_{k-1}} w^2 - \sum_{C_k} w^2 \qquad (1 \leq k < n),$$

where C_k consists of the four corner points of the boundary Q_k.

We strengthen the inequality by neglecting the last non-positive term on the right and we then add the n inequalities to obtain

$$2nh^2 \sum_{Q_0} \sum (w_x^2 + w_y^2) \leq \sum_{S_n} w^2 - \sum_{S_0} w^2 \leq \sum_{S_n} w^2.$$

Summing this inequality from $n = 1$ to $n = N$ we get

$$N^2 h^2 \sum_{Q_0} \sum (w_x^2 + w_y^2) \leq \sum_{Q_N} \sum w^2.$$

Diminishing the mesh width h we can make the squares Q_0 and Q_n converge towards two fixed squares lying within G and having corresponding sides separated by a distance a. In this process $Nh \to a$ and we have, independent of the mesh width

$$h^2 \sum_{Q_0} \sum (w_x^2 + w_y^2) \leq \frac{h^2}{a^2} \sum_{Q_N} \sum w^2. \tag{9}$$

For sufficiently small h this inequality holds of course not only for two squares Q_0 and Q_N but with a change in the constant, a, for any two subregions of G such that one
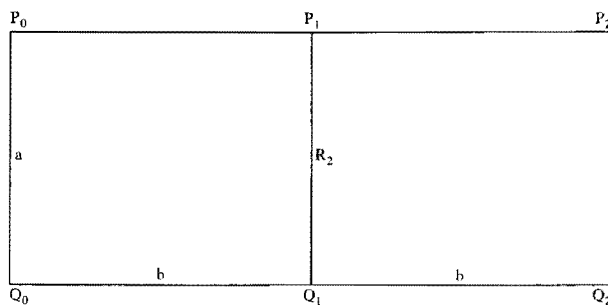

Figure 2

is contained entirely within the other. Thus lemma (2) is proved.[18]

In order to prove the third result, that u_h and all its partial difference quotients w_h remain bounded and uniformly continuous as $h \to 0$, we consider a rectangle R (Fig. 2) with corners P_0, Q_0, P, Q and with sides P_0Q_0 and PQ of length a parallel to the x-axis.

We start with the relation

$$w(Q_0) - w(P_0) = h \sum_{PQ} w_x - h^2 \sum_R \sum w_{xy},$$

and the inequality

$$|w(Q_0) - w(P_0)| \leq h \sum_{PQ} |w_x| + h^2 \sum_R \sum |w_{xy}|, \tag{11}$$

which is a consequence of it.

We then let the side PQ of the rectangle vary between an initial line P_1Q_1, a distance b from P_0Q_0 and a final line P_2Q_2 a distance $2b$ from P_0Q_0, and we sum the corresponding $(b/h) + 1$ inequalities (11). We obtain the estimate

$$|w(P_0) - w(Q_0)| \leq \frac{1}{b+h} h^2 \sum_{R_2} \sum |w_x|$$

$$+ h^2 \sum_{R_2} \sum |w_{xy}|$$

where the summations extend over the entire rectangle, $R_2 = P_0Q_0P_2Q_2$. From Schwarz's inequality it then follows that,

$$|w(P_0) - w(Q_0)| \leq \frac{1}{b} \sqrt{2ab} \sqrt{h^2 \sum_{R_2} \sum w_x^2} \tag{12}$$

$$+ \sqrt{2ab} \sqrt{h^2 \sum_{R_2} \sum w_{xy}^2}.$$

[18] If we do not assume that $\Delta w = 0$, then in place of the inequality (9) we find:

$$h^2 \sum_{G^{**}} \sum (w_x^2 + w_y^2)$$

$$\leq c_1 h^2 \sum_{G^*} \sum w^2 + c_2 h^2 \sum_{G^*} \sum (\Delta w)^2 \tag{10}$$

for suitable constants c_1 and c_2 independent of h, where G^{**} lies entirely within G^*, and G^* in turn is contained in the interior of G.
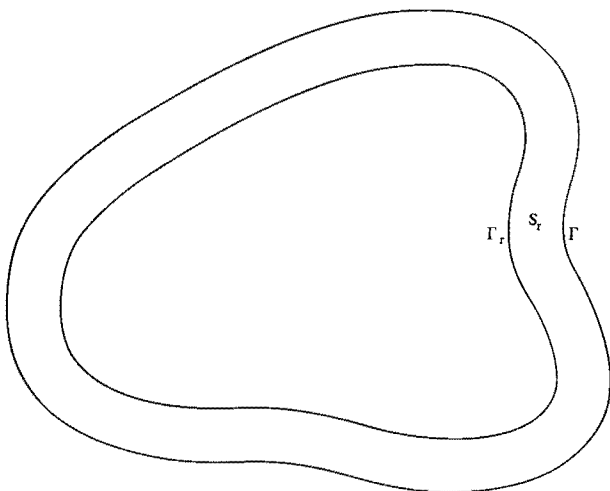
Figure 3

Since, by hypothesis, the sums which occur here multiplied by h^2 remain bounded, it follows that as $a \to 0$ the difference $|w(P_0) - w(Q_0)| \to 0$ independently of the mesh-width, since for each subregion G^* of G the quantity b can be held fixed. Consequently the uniform continuity (equicontinuity) of $w = w_h$ is proved for the x-direction. Similarly it holds for the y-direction and so also for any subregion G^* of G. The boundedness of the function w_h in G^* finally follows from its uniform continuity (equicontinuity) and the boundedness of $h^2 \sum_G \cdot \sum w_h^2$.

By this proof we establish the existence of a subsequence of functions u_h which converge towards a limit function $u(x, y)$ and which do so uniformly together with all their difference quotients, in the sense discussed above for every interior subregion of G. This limit function $u(x, y)$ has throughout G continuous partial derivatives of arbitrary order, and satisfies there the potential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

● 3. *The boundary condition*

In order to prove that the solution satisfies the boundary condition formulated above we shall first of all establish the inequality

$$h^2 \sum_{S_{r,h}} \sum v^2 \leq Ar^2 h^2 \sum_{S_{r,h}} \sum (v_x^2 + v_y^2)$$
$$+ Brh \sum_{\Gamma_h} v^2 \qquad (13)$$

where $S_{r,h}$ is that part of the mesh region G_h which lies inside a boundary strip S_r. This boundary strip S_r consists of all points of G whose distance from the boundary is less than r; it is bounded by Γ and another curve Γ_r (Fig. 3). The constants A and B depend only on the region and not on the function v nor the mesh width h.

In order to prove the above inequality, we divide the boundary, Γ, of G into a finite number of pieces for which the angle of the tangent with one of the x- or y-axes is greater than some positive value (say 30°). Let γ, for instance, be a piece of Γ which is sufficiently steep (in the above sense) relative to the x-axis (see Fig. 4). Lines parallel to the x-axis intersecting γ will cut a section γ_r from the neighboring curve Γ_r, and will define together with γ and γ_r a piece s_r of the boundary strip S_r. We use the symbol $s_{r,h}$ to denote the portion of G_h contained in s_r and denote the associated portion of the boundary Γ_h by γ_h.

We now imagine a parallel to the x-axis to be drawn through a mesh point P_h of $s_{r,h}$. Let it meet the boundary γ_h in a point $\bar{P}_h$. The portion of this line which lies in $s_{r,h}$ we call $p_{r,h}$. Its length is certainly smaller than cr, where the constant c depends only on the smallest angle of inclination of a tangent γ to the x-axis.

Between the values of v at P_h and $\bar{P}_h$ we have the relation

$$v(P_h) = r(\bar{P}_h) \pm h \sum_{P_h \bar{P}_h} v_x.$$

Squaring both sides and applying Schwarz's inequality, we obtain

$$v(P_h)^2 \leq 2v(\bar{P}_h)^2 + 2cr \cdot h \sum_{p_{r,h}} v_x^2.$$

Summing with respect to P_h in the x-direction, we get

$$h \sum_{p_r} v^2 \leq 2crv(\bar{P}_h)^2 + 2c^2 r^2 h \sum_{p_r} v_x^2.$$

Summing again in the y-direction we obtain the relation

$$h \sum_{S_{r,h}} \sum v^2 \leq 2cr \sum_{\Gamma_h} v(\bar{P}_h)^2 + 2c^2 r^2 h \sum_{S_{r,h}} \sum v_x^2. \qquad (14)$$

Writing down the inequalities associated with the other portions of Γ and adding all the inequalities together we obtain the desired inequality (13).[19]

We next set $v_h = u_h - f_h$ so that $v_h = 0$ on Γ_h.

Then since $h^2 \sum_{G_h} \sum (v_x^2 + v_y^2)$ remains bounded as $h \to 0$, we obtain from (13)

$$\frac{h^2}{r} \sum_{S_{r,h}} \sum v^2 \leq \kappa r, \qquad (16)$$

where κ is a constant which does not depend on the function v or the mesh size. Extending the sum on the left to the difference $S_{r,h} - S_{\rho,h}$ of two boundary strips, the inequality (16) still holds with the same constant κ and we can pass to the limit $h \to 0$.

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[19] By similar reasoning we can also establish the inequality

$$h^2 \sum_{G_h} \sum v^2 \leq c_1 h \sum_{\Gamma_h} v^2 + c_2 h^2 \sum_{G_h} \sum (v_x^2 + v_y^2) \qquad (15)$$

in which the constants c_1 and c_2 depend only on the region G and not on the mesh division.
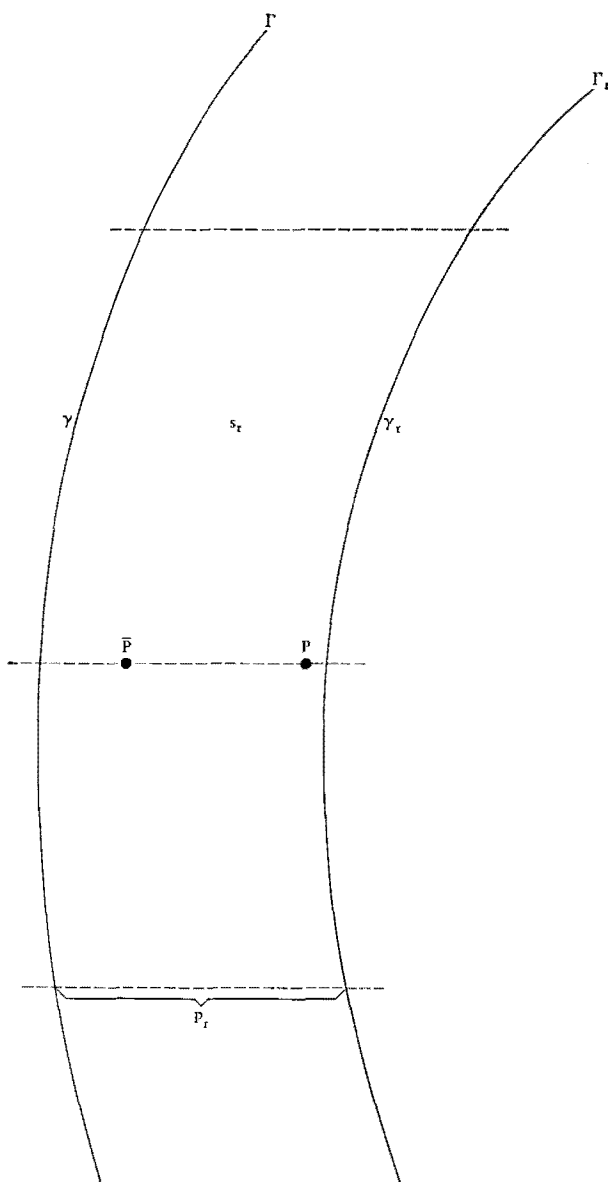
Figure 4

From the inequality (16) we then get

$$\frac{1}{r} \iint_{S_r - S_\rho} v^2 \, dxdy \leq \kappa r, \qquad v = u - f.$$

Now letting the narrower boundary strip S_ρ approach the boundary we obtain the inequality

$$\frac{1}{r} \iint_{S_r} v^2 \, dxdy = \frac{1}{r} \iint_{S_r} (u - f)^2 \, dxdy \leq \kappa r$$

which states that the limit function u satisfies the prescribed boundary condition.

• 4. *Applicability of the method to other problems*

Our method is based essentially on the inequalities arising from the lemmas stated previously since the principal points of the proofs follow from the two last lemmas in Part 1 of Section 4.[20] No use is made of special fundamental solutions or special properties of the difference expression, and therefore the method can be carried over directly to the case of arbitrarily many independent variables as well as to the eigenvalue problem,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0.$$

The same sort of convergence relations will obtain in this case as above.[21] Also the method applies to linear partial differential equations of other types, in particular its application to equations with variable coefficients requires only some minor modifications. The essential difference in this case lies only in proving the boundedness of $h^2 \sum \sum u_h^2$ which of course does not always hold for an arbitrary linear problem. However in case this sum is not bounded it can be shown that the general boundary value problem for the differential equation in question also possesses effectively no solutions, but that in this case there exist nonvanishing solutions of the corresponding homogeneous problem, i.e., eigenfunctions.[22]

• 5. *The boundary value problem* $\Delta\Delta u = 0$

In order to show that the method can be carried over to the case of differential equations of higher order, we will treat briefly the boundary value problem of the differential equation:

$$\frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0.$$

We seek, in G, a solution of this equation for which the values of u and its first derivative are given on the boundary, being specified there by some function $f(x, y)$.

To this end we assume as previously that $f(x, y)$ together with its first and second derivatives is continuous in that region of the plane containing the region G.

We replace our differential equation problem by the new problem of solving the difference equation $\Delta u = 0$ in the mesh region G subject to the condition that at the points of the boundary strip $\Gamma_h + \Gamma_h'$ the function u takes on the values $f(x, y)$. From Section 2 we know that this boundary value problem has a unique solution. We will show that as the mesh size decreases, this solution, in each

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[20] For an application of corresponding integral inequalities see K. Friedrichs, "Die Rand- und Eigenwertprobleme aus der Theorie der elastischen Platten", *Math. Ann.* 98, 222 (1926).
[21] Similarly one proves at the same time that every solution of such a differential equation problem has derivatives of every order.
[22] See Courant-Hilbert, *Methoden der Mathematischen Physik* 1, Ch. III, Section 3, where the theory of integral equations is handled with the help of the corresponding principle of the alternative. See also the Dissertation (Göttingen) of W. v. Koppenfels, which will appear soon.

interior subregion of G, converges to the solution of the differential equation, and that all of its difference quotients converge to the corresponding partial derivatives.

We note first that for the solution $u = u_h$, the sum

$$h^2 \sum_{G_h'} \sum (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)$$

remains bounded as $h \to 0$. That is, by applying the minimum requirements on the solution (Part 2, Section 2) one finds that this sum is not larger than the corresponding sum

$$h^2 \sum_{G_h'} \sum (f_{xx}^2 + 2f_{xy}^2 + f_{yy}^2),$$

and this converges as $h \to 0$ to

$$\iint_G \left(\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \right) dx dy$$

which exists, by hypothesis.

From the boundedness of the sum

$$h^2 \sum_{G_h'} \sum (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)$$

follows immediately the boundedness of $h^2 \sum_{G_h'} \sum (\Delta u)^2$ and also that of

$$h^2 \sum_{G_h'} \sum (u_x^2 + u_y^2) \quad \text{and} \quad h^2 \sum_{G_h} \sum u^2.$$

That is, for arbitrary w the following inequality holds (see Footnote 19),

$$h^2 \sum_{G_h} \sum w^2 \leq ch^2 \sum_{G_h} \sum (w_x^2 + w_y^2) + ch \sum_{\Gamma_h} w^2. \tag{15}$$

Then if one substitutes the first difference quotients of w for w itself in this inequality and applies the expression over the subregion of G_h for which they are defined, one finds the further inequality,

$$h^2 \sum_{G_h} \sum (w_x^2 + w_y^2)$$
$$\leq ch^2 \sum_{G_h'} \sum (w_{xx}^2 + 2w_{xy}^2 + w_{yy}^2)$$
$$+ ch \sum_{\Gamma_h + \Gamma_h'} (w_x^2 + w_y^2),$$

where again the constant c is independent of the function and of the mesh size. We apply this inequality to $w = u_h$ and thus find the boundedness of the sum over $\Gamma_h + \Gamma_h'$ on the right-hand side; by definition these boundary sums converge to the corresponding integral containing $f(x, y)$. Thus from the boundedness of

$$h^2 \sum_{G_h'} \sum (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)$$

follows the boundedness of $h^2 \sum_{G_h} \sum (u_x^2 + u_y^2)$ and thence that of $h^2 \sum_{G_h} \sum u^2$.

For the third step we substitute one after the other the expressions $\Delta u, \Delta u_x, \Delta u_y, \Delta u_{xx}, \cdots$, for w in the inequality

$$h^2 \sum_{G_{**}} \sum (w_x^2 + w_y^2) \leq ch^2 \sum_{G_*} \sum w^2$$
$$+ ch^2 \sum_{G_*} \sum (\Delta w)^2$$

(see Part 2, Section 4) where G^* is a subregion of G containing G^{**} in its interior. The expressions introduced all satisfy the equation $\Delta w = 0$. It follows then that for each expression in turn and for all subregions G^* of G that the sums, $h^2 \sum_{G*} \sum (w_x^2 + w_y^2)$, that is, $h^2 \sum_{G*} \sum (\Delta u_x^2 + \Delta u_y^2)$, $h^2 \sum_{G*} \sum (\Delta u_{xx}^2 + \Delta u_{xy}^2)$, $\cdots$ are bounded together with the sums:

$$h^2 \sum_{G_h} \sum u^2, \qquad h^2 \sum_{G_h} \sum (u_x^2 + u_y^2),$$

$$\text{and} \quad h^2 \sum_{G_h} \sum (\Delta u)^2,$$

which are already known to be bounded.

Finally we substitute into the inequality (10), in place of w, the sequence of functions $u_{xx}, u_{xy}, u_{yy}, u_{xxx}, \cdots$, for which

$$h^2 \sum_{G_h*} \sum (\Delta w)^2, \quad \text{i.e.} \quad h^2 \sum_{G_h*} \sum (\Delta u_{xx})^2, \cdots$$

are bounded as shown above. We then find that for all subregions the sums

$$h^2 \sum_{G_h*} \sum (u_{xxx}^2 + u_{xxy}^2), \quad h^2 \sum_{G_h*} \sum (u_{xyz}^2 + u_{xyy}^2), \cdots$$

remain bounded.

From these facts we can now conclude as previously that from our sequence of mesh functions a subsequence can be chosen which in each interior subregion of G converges (together with all its difference quotients) uniformly to a limit function (or respectively its derivatives) which is continuous in the interior of G.

We have yet to show that this limit function which obviously satisfies the differential equation $\Delta \Delta u = 0$ also takes on the prescribed boundary conditions. For this purpose we say here only that, analogous to the foregoing, the expressions

$$\iint_{S_r} (u - f)^2 \, dx dy \leq cr^2,$$

$$\iint_{S_r} \left[\left(\frac{\partial u}{\partial x} - \frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} - \frac{\partial f}{\partial y} \right)^2 \right] dx dy \leq cr^2$$

hold.[23] That the limit function fulfills these conditions may be seen by carrying over the treatment in Part 3, Section 4 to the function u and its first difference quotient.

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[23] That the boundary values for the function and its derivatives actually are assumed is not difficult to prove. See for instance the corresponding treatment of K. Friedrichs, *loc. cit.*
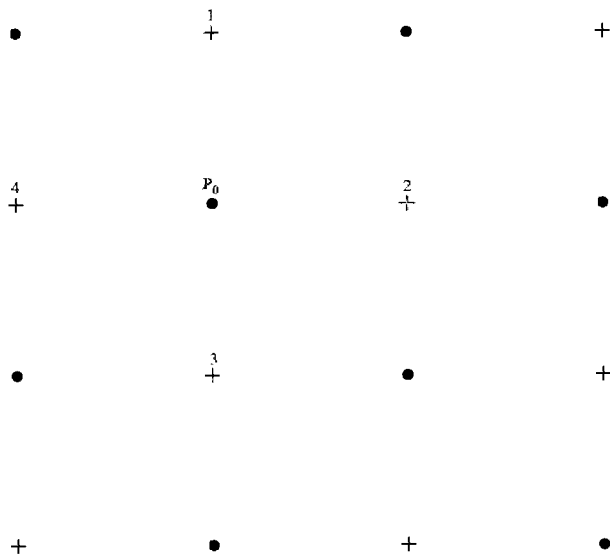
Figure 5

From the uniqueness of our boundary value problem we see furthermore that not only a selected subsequence, but also the original sequence of functions u possesses the asserted convergence properties.

II. The hyperbolic case

Section 1. The equation of the vibrating string

In the second part of this paper we shall consider the initial value problem for linear hyperbolic partial differential equations. We shall prove that under certain hypotheses the solutions of the difference equations converge to the solutions of the differential equations as the mesh size decreases.

We can discuss the situation most easily by considering the example of the approximation to the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

We limit ourselves to the particular initial value problem where the value of the solution u, and its derivatives are given on the line $t = 0$.

In order to find the corresponding difference equation, we construct in the (x, t)-plane a square grid with lines parallel to the axes and with mesh width h. Following the notation of the first part of the paper we replace the differential equation (1) by the difference equation $u_{t\,t} - u_{x\,x} = 0$. If we select a grid point, P_0, then the corresponding difference equation relates the value of the function at this point to the values at four neighboring points. If we characterize the four neighboring values by the four indices 1, 2, 3, 4 (cf. Fig. 5), then the difference

equation assumes the simple form

$$u_1 + u_3 - u_2 - u_4 = 0. \tag{2}$$

Note that the value of the function u at the point P_0 does not appear itself in the equation.

We consider the grid split up into two subgrids, indicated in Fig. 5 by dots and crosses respectively. The difference equation connects the values of the function over each of the subgrids separately, and so we shall consider only one of the two grids. As initial condition the values of the function are prescribed on the two rows of the grid, $t = 0$ and $t = h$. We can give the solution of this initial value problem explicitly; that is, we express the value of the solution at any point S in terms of the values given along the two initial rows. One can see at once that the value at a point of the row $t = 2h$ is uniquely determined by only the three values at the points close to it in the two first rows. The value at a point of the fourth row is uniquely determined by the values of the solution at three particular points in the second and third rows, and through them it is related to certain values in the first two rows. In general to a point S there will correspond a certain region of dependence in the first two rows; it may be found by drawing the lines $x + t =$ const. and $x - t =$ const., through the point S and extending them until they meet the second row at the points α and β respectively (cf. Fig. 6). The triangle $S\alpha\beta$ is called the *triangle of determination* because all the values of u in it remain unchanged provided the values on the first two rows of it are held fixed. The sides of the triangle are called *lines of determination*.

If one denotes the differences of u in the direction of the lines of determination by u' and $u^\backslash$, that is,

$$u_1' = u_1 - u_4, \qquad u_1^\backslash = u_1 - u_2$$

$$u_2' = u_2 - u_3, \qquad u_4^\backslash = u_4 - u_3,$$
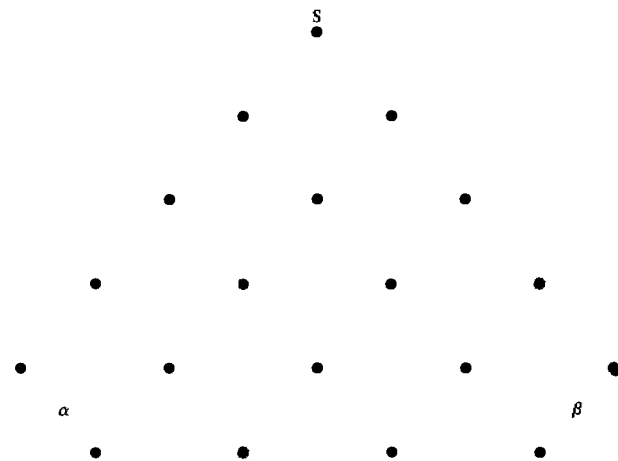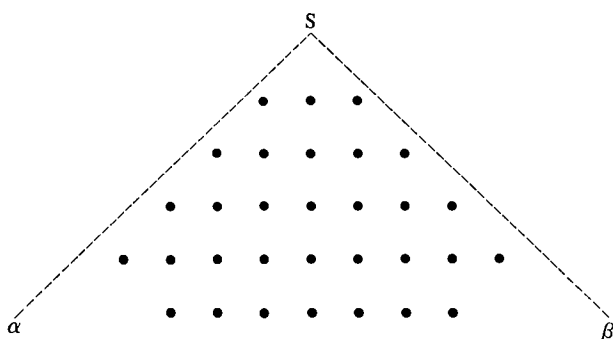
Figure 6

Figure 7

Figure 8

then the difference equation assumes the form

$$u_1' = u_2',$$

i.e., along a line of determination the differences with respect to the other direction of determination are constant, and thus are equal to one of the given differences between the value at two points on the first two rows. Moreover the difference $u_S - u_\alpha$ is a sum of differences u' along the determining line αS, so that using the remark above, we can obtain the final result (in obvious notation):

$$u_S = u_\alpha + \sum_{\alpha_1}^{\beta_1} u'. \tag{3}$$

We now let h go to zero, and let the prescribed values on the second and first rows converge uniformly to a twice continuously differentiable function, $f(x)$, and the difference quotients $u'/h\sqrt{2}$ there converge uniformly to a continuously differentiable function $g(x)$. Evidently the right-hand side of (3) goes over uniformly to the expression

$$f(x - t) + \frac{1}{\sqrt{2}} \int_{x-t}^{x+t} g(\xi)\, d\xi \tag{4}$$

if S converges to the point (x, t). This is the well-known expression for the solution of the wave equation (1) with initial values $u(x, 0) = f(x)$ and $\partial u(x, 0)/\partial t = f'(x) + \sqrt{2}g(x)$. Thus it is shown that as $h \to 0$ the solution of the difference equation converges to the solution of the differential equation provided the initial values converge appropriately (as above).

Section 2. On the influence of the choice of mesh. The domains of dependence of the difference and differential equations

The relationships developed in Section 1 suggest the following considerations.

In the same way that the solution of a linear hyperbolic equation at a point S depends only on a certain part of the initial line—namely the "domain of dependence"
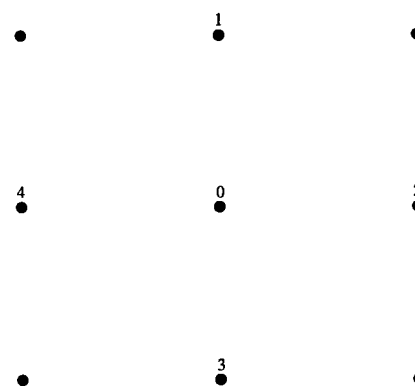
lying between the two characteristics drawn through S, the solution of the difference equation has also at the point S a certain domain of dependence defined by the lines of determination drawn through S. In Section 1 the directions for the lines of determination of the difference equation coincided with the characteristic directions for the differential equation so that the domains of dependence coincided in the limit. This result, however, was based essentially on the orientation of the mesh in the (x,t)-plane, and depended furthermore on the fact that a square mesh had been chosen. We shall now consider a more general rectangular mesh with mesh size equal to h (time interval) in the t-direction and equal to kh (space interval) in the x-direction, where k is a constant. The domain of dependence for the difference equation, $u_{tt} - u_{x\bar{x}} = 0$ for this mesh will now either lie entirely within the domain of dependence of the differential equation, $\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 = 0$, or on the other hand will contain this latter region inside its own domain according as $k < 1$ or $k > 1$ respectively.

From this follows a remarkable fact: if for the case $k < 1$ one lets $h \to 0$, then the solution to the difference equation in general *cannot* converge to the solution of the differential equation. In this case a change in the initial values of the solution of the differential equation in the neighborhood of the endpoints α and β of the domain of dependence (see Fig. 7) causes, according to formula (4), a change in the solution itself at the point (x, t). For the solution of the difference equation at the point S, however, the changes at the points α and β are not relevant since these points lie outside of the domain of dependence of the difference equations. That convergence does occur for the case $k > 1$ will be proved in Section 3. See for example Fig. 9.

If we consider the differential equation

$$2\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \tag{5}$$

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in two space variables, x and y, and time, t, and if we replace it by the corresponding difference equation on a rectilinear grid, then in contrast to the case of only two independent variables it is impossible to choose the mesh division so that the domain of dependence of the difference and differential equations coincide, since the domain of dependence of the difference equation is a quadrilateral while that of the differential equation is a circle. Later (cf. Section 4) we shall choose the mesh division so that the domain of determination of the difference equation contains that of the differential equation in its interior, and shall show that once again convergence occurs.

On the whole an essential result of this section will be that in the case of each linear homogeneous hyperbolic equation of second order one can choose the mesh so that the solution of the difference equation converges to the solution of the differential equation as $h \to 0$, (see for instance Sections 3, 4, 7, 8).

Section 3. Limiting values for arbitrary rectangular grids

Now we consider further the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{1}$$

but impose it now on a rectangular grid with time interval h and space interval kh. The corresponding difference equation is

$$L(u) = \frac{1}{h^2}(u_1 - 2u_0 + u_3)$$
$$- \frac{1}{k^2 h^2}(u_2 - 2u_0 + u_4) = 0, \tag{6}$$

where the indices represent a "fundamental rhombus" with midpoint P_0 and corners P_1, P_2, P_3, P_4 (see Fig. 8). According to the equation $L(u) = 0$ the value of the function u at a point S can be represented by its values on that

Figure 9
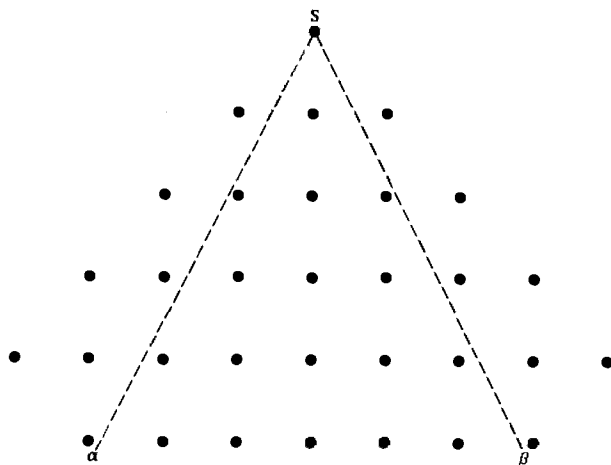
section of the initial rows $t = 0$ and $t = h$ cut out by lines of determination through S parallel to the sides of an elementary rhombus. We assume that the initial values are prescribed in such a way that as $t \to 0$ for fixed k the first difference quotients formed from them converge uniformly to given continuous functions on the line $t = 0$. The initial values can be used to form an explicit representation of the solution of the difference equation (corresponding to (3) in Section 1); however it is too complicated to yield a limiting value easily as $h \to 0$. Thus we will try another approach which will also make it possible for us to treat the general problem.[24]

We multiply the difference expression $L(u)$ by $(u_1 - u_3)$ and form the product using the following identities:

$$(u_1 - u_3)(u_1 - 2u_0 + u_3)$$
$$= (u_1 - u_0)^2 - (u_0 - u_3)^2, \tag{7}$$

$$(u_1 - u_3)(u_2 - 2u_0 + u_4)$$
$$= (u_1 - u_0)^2 - (u_0 - u_3)^2 - \tfrac{1}{2}[(u_1 - u_2)^2$$
$$+ (u_1 - u_4)^2 - (u_2 - u_3)^2 - (u_4 - u_3)^2]. \tag{8}$$

Then we obtain

$$2(u_1 - u_3)L(u) = \frac{2}{h^2}\left(1 - \frac{1}{k^2}\right)[(u_1 - u_0)^2$$
$$- (u_0 - u_3)^2] + \frac{1}{h^2 k^2}[(u_1 - u_2)^2$$
$$+ (u_1 - u_4)^2 - (u_2 - u_3)^2 - (u_4 - u_3)^2]. \tag{9}$$

The product (9) is now summed over all elementary rhombuses of the domain of determination, $S\alpha\beta$. The quadratic difference terms on the right-hand side always appear with alternate signs in two neighboring rhombuses so that they cancel out for any two rhombuses belonging to the triangle $S\alpha\beta$. Only the sums of squared differences over the "boundary" of the triangle remain, and we obtain the relation:

$$h^2 \sum_{S} \sum_{\alpha\beta} 2\frac{u_1 - u_3}{h} L(u)$$

$$= h \sum_{S\alpha} \left[2\left(1 - \frac{1}{k^2}\right)\left(\frac{\dot u}{h}\right)^2 + \frac{1}{k^2}\left(\frac{u'}{h}\right)^2 \right]$$

$$+ h \sum_{S\beta} \left[2\left(1 - \frac{1}{k^2}\right)\left(\frac{\dot u}{h}\right)^2 + \frac{1}{k^2}\left(\frac{u'}{h}\right)^2 \right]$$

$$- h \sum_{I\,II} \left[2\left(1 - \frac{1}{k^2}\right)\left(\frac{\dot u}{h}\right)^2 \right.$$

$$+ \frac{1}{k^2}\left(\frac{u'}{h}\right)^2 + \frac{1}{k^2}\left(\frac{u^\backslash}{h}\right)^2 \right]. \tag{10}$$

[24] For the following see K. Friedrichs and H. Lewy, "Über die Eindeutig-keit...etc.," *Math. Ann.* 98, 192 ff. (1928), where a similar transformation is used for integrals.

Here u' and $u^{\backslash}$ denote differences in the direction of deter-
mination defined in Section 1, while $\dot{u}$ designates the dif-
ference of the functional values at two neighboring points
on a mesh line parallel to the t axis. The range in $\sum_{S\alpha}$
over which $(u')^2$ is taken is the outermost boundary line
$S\alpha$ and its nearest parallel neighbor found by shifting the
points of $S\alpha$ downward by the amount h. There is a similar
range for $(u^{\backslash})^2$ in $\sum_{S\beta}$, and so all of the differences, u', $u^{\backslash}$,
and $\dot{u}$ appear once and only once.

For the solution to the problem $L(u) = 0$ the right-hand
side of (10) vanishes. The sum over the initial rows I and II
which occurs there remains bounded as $h \to 0$ (for fixed k);
specifically it goes over into an integral of the prescribed
function along the initial line. Consequently the sums over
$S\alpha$ and $S\beta$ in (10) also remain bounded. If now $k \geq 1$
as we must require (see previous discussion), then $1 - 1/k^2$
is non-negative, and we find that the individual sums

$$h \sum_{S\alpha} \left(\frac{u'}{h}\right)^2, \qquad h \sum_{S\beta} \left(\frac{u^{\backslash}}{h}\right)^2,$$

extended over any line of determination whatever, remain
bounded.

From this we can derive the "uniform continuity"
(equicontinuity) (cf. Section 4 of the first part of the paper)
of the sequence of grid functions in all directions in the
plane;[25] since the values of u on the initial line are bounded,
there must exist a subset which converges uniformly to a
limit function $u(x, t)$.

Both the first and second difference quotients of the func-
tion u also satisfy the difference equation $L(u) = 0$. Their
initial values can be expressed through the equation
$L(u) = 0$ in terms of the first, second and third difference
quotients of u involving initial values at points on the two
initial lines I and II only. We require that they tend to
continuous limit functions, that is, that the given initial
values $u(x, 0)$, $u_t(x, 0)$ be three times or respectively
twice continuously differentiable with respect to x.

From this we can apply the convergence considerations
set forth above to the first and second difference quotients
of u, as well as to u itself, and we can choose a subsequence
such that these difference quotients converge uniformly
to certain functions, which must be the first or respectively
second derivatives of the limit function $u(x, t)$. Hence the
limit function satisfies the differential equation $\partial^2 u/\partial t^2 -$
$\partial^2 u/\partial x^2 = 0$ which results as the limit of the difference
equation $L(u) = 0$; it represents indeed the solution of the
initial value problem. Since such a solution is uniquely

[25] If S_1 and S_2 are two points at a distance δ, then one connects them by a
path of two segments, S_1S and SS_2, where the former is parallel to one line of
determination and the latter to the other. Then one finds the appraisal,

$$|u_{S_1} - u_S| \leq |u_{S_1} - u_S| + |u_S - u_{S_2}|$$

$$\leq \sqrt{\delta}\,\sqrt{h \sum_{S_1 S} \left(\frac{u'}{h}\right)^2} + \sqrt{\delta}\,\sqrt{h \sum_{SS_2} \left(\frac{u^{\backslash}}{\delta}\right)^2}.$$

determined, every subsequence of mesh functions con-
verges, and therefore the sequence itself converges to the
limit function.

Section 4. The wave equation in three variables

We treat next the wave equation,

$$2\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \tag{11}$$

and consider its relation to the observations on the domain
of dependence discussed in Section 2. The domain of de-
pendence of the differential equation (11) is a circular cone
with axis parallel to the t-direction and with apex angle α,
where $\tan \alpha = 1/\sqrt{2}$. In any rectilinear grid parallel to the
axes we introduce the corresponding difference equation

$$2u_{t\bar{t}} - u_{x\bar{x}} - u_{y\bar{y}} = 0. \tag{12}$$

This equation relates the functional values of u at points
of an elementary tetrahedron. In fact it allows the value of
the function u at a point S to be expressed uniquely in
terms of the values of the function at certain points of the
two initial planes $t = 0$ and $t = h$. At each point S we
obtain a pyramid of determination which cuts out from
the two base planes two rhombuses as domains of de-
pendence.

If we let the mesh widths tend to zero, keeping their
ratios fixed, then we can expect convergence of the se-
quence of mesh functions to the solution of the differential
equation only provided the pyramid of determination
contains the cone of determination of the differential
equation in its interior. The simplest grid with this property
will be one constructed in such a way that the pyramid of
determination is tangent to the exterior of the cone of
determination. Our differential equation is chosen so that
this occurs for a grid of cubes parallel to the axes.

The difference equation (12), in the notation of Fig. 10,
assumes for such a grid the form:

$$L(u) = \frac{2}{h^2}(u'_\alpha - 2u_0 + u_\alpha) - \frac{1}{h^2}(u_1 - 2u_0 + u_3)$$

$$- \frac{1}{h^2}(u_2 - 2u_0 + u_4), \tag{13}$$

in which the functional value, u_0, at the midpoint actually
cancels out. The values of the solution on the two initial
planes must be the values of a function having four con-
tinuous derivatives with respect to $x_1 y_2 t$.

For the convergence proof we again use the method de-
veloped in Section 3. We construct the triple sum

$$h^2 \sum \sum \sum 2\frac{u'_\alpha - u_\alpha}{h} L(u) = 0$$

for the solution to the difference equation, where the
summation is to be extended over all elementary octahe-
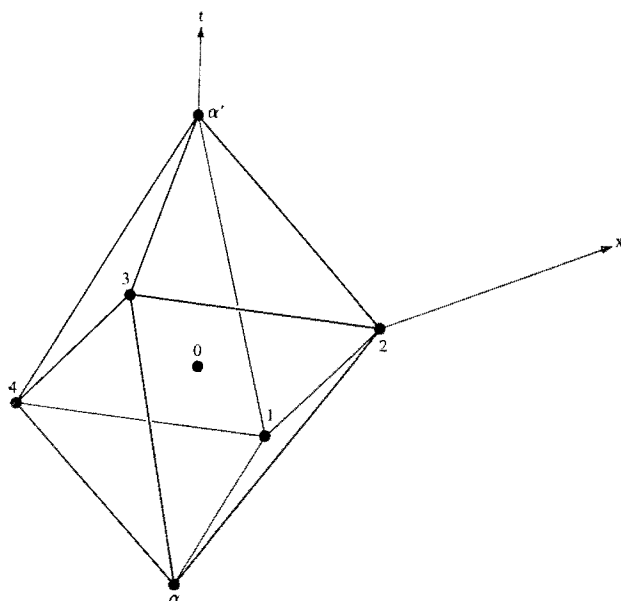drons of the pyramid of determination emanating from

Figure 10

the point S. Then almost exactly as before we find that the values of the function u at the interior points of the pyramid of determination cancel out in the summation and that only the values on the two pyramids called F, and on the two base surfaces I and II remain.

If we denote by u' the difference of the values of the function at two points connected by a line of an elementary octahedron, then we can write the result as

$$\sum_F \sum (u')^2 - \sum_I \sum_{II} (u')^2 = 0, \tag{14}$$

where the sum is extended over all surfaces containing differences u'; each such difference is to appear only once.[26] The double sum over the two initial surfaces stays bounded since it goes over into an integral of the initial values. Therefore the sum over the "surface of determination" F remains bounded.

We now apply these results not to u itself, but to its first, second and third difference quotients, which themselves satisfy the difference equation (13). Their initial values can be expressed using only values on the first two initial planes by means of (13) using first through fourth difference quotients. If $w = w_h$ is one of the difference quotients of any order up to third order, then we know that the sum $h^2 \sum \sum (w'/h)^2$ extended over a surface of determination remains bounded. From this it follows, through exactly the methods used in Section 4 of the first part of the paper, that the function u and its first and second difference quotients are uniformly continuous (equicontinuous). Thus there exists a sequence of mesh widths decreasing to zero such that these quantities, which

[26] The grid ratio has been chosen in such a way that the differences between values of u appearing on the two neighboring surfaces in F do not occur, (as they did in the general case in one dimension treated in Section 3).

are bounded initially, converge to continuous limit functions and, in fact, converge to the solution of the differential equation and to the first and second derivatives of this solution, all exactly as we found earlier (Section 3).

Appendix. Supplements and generalizations

Section 5. Example of a differential equation of first order

We have seen in Section 2 that in the case when the region of dependence of the differential equation covers only a part of the region of dependence of the difference equation, the influence of the rest of the region is not included in the limit. We can demonstrate this phenomenon explicitly by the example of the differential equation of first order, $\partial u/\partial t = 0$ if we replace it by the difference equation

$$2u_t - u_x + u_{\bar{x}} = 0. \tag{15}$$

In the notation of Fig. 5 this becomes

$$u_1 = \frac{u_2 + u_4}{2}. \tag{16}$$

As before, the difference equation connects only the points of a submesh with one another. The initial value problem consists of assigning as initial values for u at points $x = 2ih$ on the row $t = 0$ the values, f_{2i}, assumed there by a continuous function $f(x)$.

We consider the point S at a distance $2nh$ up along the t-axis. It is easy to verify that the solution u at S is represented as

$$u_S = \sum_{i=-n}^{n} \frac{1}{2^{2n}} \begin{bmatrix} 2n \\ n+i \end{bmatrix} f_{2i}. \tag{17}$$

As the mesh size decreases, that is as $n \to \infty$, the sum on the right-hand side tends simply to the value f_0. This can be demonstrated from the continuity of $f(x)$ and from the behavior of the binomial coefficients as n increases (see the following paragraph).

Section 6. The equation of heat conduction

The difference equation (16) of Section 5 can also be interpreted as the analogue of an entirely different differential equation, namely the equation of heat conduction,

$$2 \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{18}$$

In any rectangular mesh with mesh spacing l and h in the time and space directions, respectively, the corresponding difference equation becomes

$$2 \left(\frac{u_1 - u_0}{l} \right) = \left(\frac{u_2 + u_4 - 2u_0}{h^2} \right). \tag{19}$$

In the limit as the mesh size goes to zero the difference equation preserves its form only if l and h^2 are decreased

proportionately. In particular if we set $l = h^2$, then the value u_0 drops out of the equation and the difference equation becomes

$$u_1 = \frac{u_2 + u_4}{2}.$$ (16)

The solution to (16) is given by formula

$$u(0, t) = \sum_{i=-n}^{n} \frac{1}{2^{2n}} \begin{bmatrix} 2n \\ n + i \end{bmatrix} f_{2i}.$$

As the mesh width decreases, a point ξ on the x-axis is always characterized by the index

$$2i = \xi/h.$$ (20)

The mesh width h is related to the ordinate t of a particular point by

$$2nh^2 = t.$$ (21)

We shall investigate what happens to formula (17) as $h \to 0$, that is $n \to \infty$. Using (21) we write (17) in the form

$$u(0, t) = \sum_{i=-n}^{n} \frac{\sqrt{2n}}{2 \cdot 2^{2n} \sqrt{t}} \begin{bmatrix} 2n \\ n + i \end{bmatrix} f_{2i} \cdot 2h.$$ (22)

For the coefficient of $2h f_{2i} = 2h f(\xi)$ we use the abbreviation

$$\frac{1}{2\sqrt{t}} g_{2n}(\xi) = \frac{\sqrt{2n}}{2 \cdot 2^{2n} \sqrt{t}} \begin{bmatrix} 2n \\ n + \frac{\xi}{\sqrt{2t}} \sqrt{n} \end{bmatrix}.$$

The limiting value of the coefficient, which is usually determined by using Stirling's formula, we will calculate here by considering the function $g_{2n}(\xi)$ as the solution of an ordinary difference equation which approaches a differential equation as $h \to 0$. As the difference equation one finds

$$\frac{1}{2h} [g_h(\xi + 2h) - g_h(\xi)] = -\frac{1}{2h} g_h(\xi) \frac{2i + 1}{n + i + 1}$$

(in which we have written $g_h(\xi)$ instead of $g_{2n}(\xi)$). Or

$$\frac{1}{2h} [g_h(\xi + 2h) - g_h(\xi)] = -g_h(\xi) \frac{\xi + h}{t + h\xi + 2h^2}.$$

$g_h(\xi)$ satisfies the normalization condition

$$\sum_{i=-n}^{n} g_h(\xi) \cdot 2h = 2\sqrt{t}.$$

This sum is over the region of dependence of the difference equation, and as $h \to 0$ this covers the entire x-axis.

It can be shown easily that $g_h(\xi)$ converges uniformly to the solution $g(x)$ of the differential equation

$$g'(x) = -g(x)x/t$$

with the auxiliary condition

$$\int_{-\infty}^{\infty} g(x) \, dx = 2\sqrt{t}.$$

From formula (22) after passing to the limit we find

$$u(0, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\xi^2/2t} f(\xi) \, d\xi,$$

which is the known solution of the heat conduction equation.

The results of this section can be carried over directly to the case of the differential equation,

$$4 \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

and so on for even more independent variables.

Section 7. The general homogeneous linear equation of second order in the plane

We consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} - k^2 \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} + \gamma u = 0.$$ (23)

The coefficients are assumed to be twice continuously differentiable with respect to x and t, while the initial values on the line $t = 0$ are assumed three times continuously differentiable with respect to x. We replace the differential equation by the difference equation

$$L(u) = u_{tt}(x, t) - k^2 u_{xx}(x, t)$$
$$+ \alpha u_t + \beta u_x + \gamma u = 0$$ (24)

in a grid with time mesh width h and space mesh width ch so that in a neighborhood of the appropriate part of the initial value line the inequality $1 - k^2/c^2 > \varepsilon > 0$ holds for the constant, c. The initial values are to be chosen as in Section 3.

For the proof of convergence we again transform the sum,

$$h^2 \sum_{S} \sum_{\alpha\beta} 2 \frac{u_1 - u_3}{h} L(u)$$

by using identities (7) and (8). In addition to a sum (see for example (10)) over the doubled boundary of the triangle $S\alpha\beta$ (Fig. 6) one obtains a sum over the entire triangle $S\alpha\beta$ because of the variability of the coefficient k and the presence of lower order derivatives in the differential equation. By using the differentiability of k and the Schwarz inequality one can show that this latter sum is bounded from above by

$$Ch^2 \sum_{S} \sum_{\alpha\beta} \left[\left(\frac{u'}{h}\right)^2 + \left(\frac{u'}{h}\right)^2 + \left(\frac{u}{h}\right)^2 + u^2 \right]$$

where the constant C is independent of the function u,

the mesh width h, and, in a certain neighborhood of the initial line, also independent of the point S.

Again we can estimate an upper bound for $h^2 \sum_{S\alpha\beta} \sum u^2$ by[27]

$$C_1 h^2 \sum_{S\alpha\beta} \sum \left(\frac{\dot{u}}{h}\right)^2 + C_2 h \sum_{III} u^2,$$

where the same properties hold for C_1 and C_2 as are stated above for C.

Thus we obtain an inequality of the form

$$h \sum_{S\alpha} \left[2\left(1 - \frac{k^2}{c^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{c^2}\left(\frac{u'}{h}\right)^2 \right]$$

$$+ h \sum_{S\beta} \left[2\left(1 - \frac{k^2}{c^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{c^2}\left(\frac{u^{\backslash}}{h}\right)^2 \right]$$

$$\leq C_3 h^2 \sum_{S\alpha\beta} \sum \left[\left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{u'}{h}\right)^2 + \left(\frac{u^{\backslash}}{h}\right)^2 \right] + D, \quad (25)$$

where D, for all points S and mesh widths h, is a fixed bound for the sums over the initial line.

As vertices of our triangles we choose a sequence of points $S_0, S_1, \cdots, S_n = S$ lying on a line parallel to the t-axis. By summing the corresponding sequence of inequalities (25) after multiplying by h we obtain the following inequality

$$h^2 \sum_{S} \sum_{S_0\alpha} \left[2\left(1 - \frac{k^2}{c^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{c^2}\left(\frac{u'}{h}\right)^2 \right]$$

$$+ h^2 \sum_{S} \sum_{S_0\beta} \left[2\left(1 - \frac{k^2}{c^2}\right)\left(\frac{\dot{u}}{h}\right)^2 + \frac{k^2}{c^2}\left(\frac{u^{\backslash}}{h}\right)^2 \right]$$

$$\leq nh^3 C_3 \sum_{S\alpha\beta} \sum \left[\left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{u'}{h}\right)^2 + \left(\frac{u^{\backslash}}{h}\right)^2 \right] + nh D.$$

$$(26)$$

Now if we notice that one can express a difference u' or $u^{\backslash}$ in terms of two differences $\dot{u}$ and a difference $u^{\backslash}$ or respectively u', then we see that the left-hand side of (26) is larger than the simpler form

$$C_4 h^2 \sum_{S\alpha\beta} \sum \left[\left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{u'}{h}\right)^2 + \left(\frac{u^{\backslash}}{h}\right)^2 \right],$$

with a suitable constant C_4.

Then by confining the discussion to a sufficiently small neighborhood, $0 \leq t \leq nh = \delta$ of the initial line where δ is small enough so that

$$C_4 - nhC_3 = C_5 > 0,$$

we find from (26),

$$C_5 h^2 \sum_{S\alpha\beta} \sum \left[\left(\frac{\dot{u}}{h}\right)^2 + \left(\frac{u'}{h}\right)^2 + \left(\frac{u^{\backslash}}{h}\right)^2 \right] \leq \frac{C_4}{C_3} D. \quad (27)$$

The bound given by (27) when combined with (25) gives a bound on

$$h \sum_{S\alpha} \left(\frac{u'}{h}\right)^2 + h \sum_{S\beta} \left(\frac{u^{\backslash}}{h}\right)^2,$$

from which, as in Section 3, one can prove the uniform continuity of u.

We apply the inequality (25) now, instead of to the function u itself, to its first and second difference quotients, w, which also satisfy difference equations whose second-order terms are the same as those of (24). In the rest of the terms there will appear lower order differences of u which cannot be expressed in terms w, but they will appear in the above argument in the form of quadratic double sums multiplied by h^2. This is enough to let us reach the same conclusions for the difference equation for w as we found above for u. So we can conclude from this the uniform continuity (equicontinuity) and boundedness of the function u and its first and second derivatives. Consequently a subsequence exists which converges uniformly to the solution of the initial value problem for the differential equation. Again from the uniqueness of the solution we find that the sequence of functions itself converges.

In all of this the assumption must be made that the difference quotients up to third order involving values on the two initial lines converge to continuous limit functions.[28]

Section 8. The initial value problem for an arbitrary linear hyperbolic differential equation of second order

We shall now show that the methods developed so far are adequate for solving the initial value problem for an arbitrary homogeneous linear hyperbolic differential equation of second order. It suffices to limit the discussion to the case of three variables. The development can be extended immediately to the case of more variables. It is easy to see that a transformation of variables can reduce the most general problem of this type to the following: a function $u(x, y, t)$ is to be found which satisfies the differential equation

$$u_{tt} - (au_{xx} + 2bu_{xy} + cu_{yy})$$

$$+ \alpha u_t + \beta u_x + \gamma u_y + \delta u = 0, \quad (28)$$

and which, together with its first derivative, assumes prescribed values on the surface $t = 0$. The coefficients in Eq. (28) are functions of the variables x, y, and t and satisfy the condition

$$a > 0, \quad c > 0, \quad ac - b^2 > 0.$$

We assume that the coefficients are three times differentiable with respect to x, y, and t, and that the initial

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[27] For proof one makes use of the inequality used in Footnote 25.

[28] This assumption and also the assumptions on the differentiability of the coefficients of the differential equation, and further on the restriction to a sufficiently small region of the initial line can be weakened in special cases.
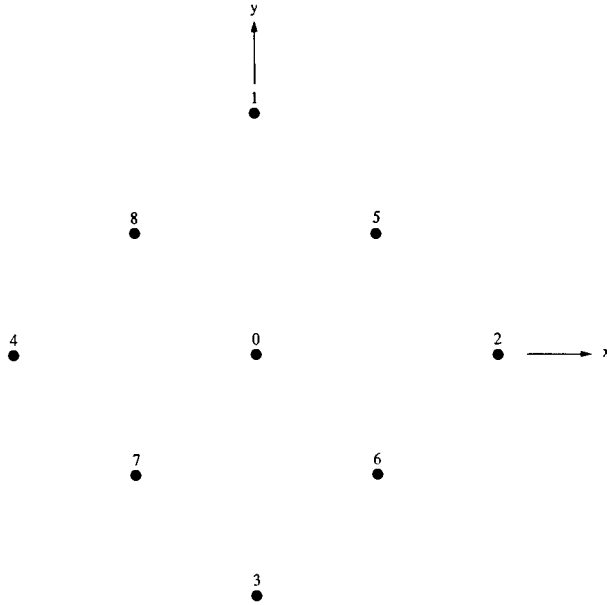
y

8　　　　　5

4　　　　0　　　　2 ——→ x

7　　　　6

3

Figure 11

values u and u_t are respectively four and three times continuously differentiable with respect to x and y.

The coordinates x and y can be drawn from a given point on the initial plane in such a way that $b = 0$ there. Then of course in a certain neighborhood of this point the conditions

$$a - |b| > 0, \qquad c - |b| > 0$$

hold. We restrict our investigation to this neighborhood. We can choose a three times continuously differentiable function $d > 0$ so that

$$\left. \begin{array}{l} a - d \\ c - d \\ d - |b| \end{array} \right\} > \varepsilon > 0 \qquad (29)$$

for some constant ε. Then we put the differential equation into the form

$$\begin{aligned} u_{tt} &- (a - d)u_{xx} - (c - d)u_{yy} \\ &- \tfrac{1}{2}(d + b)(u_{xx} + 2u_{xy} + u_{yy}) \\ &- \tfrac{1}{2}(d - b)(u_{xx} - 2u_{xy} + u_{yy}) \\ &+ \alpha u_t + \beta u_x + \gamma u_y + \delta u = 0. \end{aligned} \qquad (30)$$

We now construct in the space a grid of points, $t = lh$, $x + y = mkh$, $x - y = nkl$ ($l, m, n = \cdots -1, 0, 1, 2 \cdots$) and we replace Eq. (30) by a difference equation $L(u) = 0$ over this mesh. We do this by assigning to each point P_0 the following neighborhood: The point $P_{\alpha'}$ or the point P_α which lies at a distance h or $-h$ respectively along the t-axis from P_0; also the points $P_1, \cdots, P_8$ which lie in the same (x, y)-plane with P_0 (see Fig. 11). These points con-

stitute an "elementary octahedron" with vertices $P_{\alpha'}$, P_α, P_1, P_2, P_3, P_4. For each grid point lying in the interior of G we replace the derivatives appearing in Eq. (30) by difference quotients over the elementary octahedron about P_0.

We replace

$$u_{tt} \quad \text{by} \quad \frac{1}{h^2}(u_{\alpha'} - 2u_0 + u_\alpha)$$

$$u_{xx} \quad \text{by} \quad \frac{1}{k^2 h^2}(u_2 - 2u_0 + u_4)$$

$$u_{yy} \quad \text{by} \quad \frac{1}{k^2 h^2}(u_1 - 2u_0 + u_3)$$

$$u_{xx} + 2u_{xy} + u_{yy} \quad \text{by} \quad \frac{4}{k^2 h^2}(u_6 - 2u_0 + u_8)$$

$$u_{xx} - 2u_{xy} + u_{yy} \quad \text{by} \quad \frac{4}{k^2 h^2}(u_5 - 2u_0 + u_7).$$

The first derivatives in (30) are replaced by the corresponding difference quotients in the elementary octahedron. The coefficients of the difference equation are given the values assumed by the coefficients of the differential equation at the point P_0.

On the initial planes $t = 0$ and $t = h$ we assume that the values of the function are assigned in such a way that as the mesh size approaches zero for fixed k, the function approaches the prescribed initial value, and the difference quotients over the two planes up through differences of fourth order converge uniformly to the prescribed derivatives.

The solution of the difference equation $L(u) = 0$ at a point is uniquely determined by the values on the two base surfaces of the pyramid of determination passing through the point.

To prove convergence we construct a sum

$$h^3 \sum \sum \sum 2 \frac{u_{\alpha'} - u_\alpha}{h} L(u)$$

over all the elementary octahedrons of the pyramid of determination, and we transform it by using identities (7) and (8). This gives one space summation multiplied by h^3 and quadratic in the first difference quotients, and also over a double surface a sum which is multiplied by h^2 and contains the squares of all the difference quotients of the type $u_\alpha - u_0, u_\alpha - u_1, \cdots, u_\alpha - u_8$. In this expression according to (29) the coefficients are larger than some fixed positive constant in all those cases where the ratio $1/k$ between the time and space mesh sizes is taken sufficiently small.

From here on one can proceed exactly as before (Sections 7, 4) and prove that the solution of the difference equation converges to the solution of the differential equation.

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