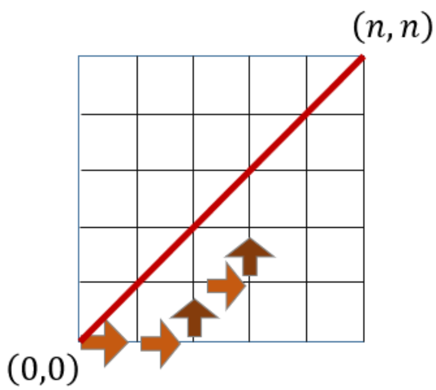
Problem 1 (20%). Let X, Y be discrete random variables. The variance of a random variable X is defined as $\mathrm{Var}[X] := \mathrm{E}\left[(X - \mathrm{E}[X])^2\right]$. Prove that

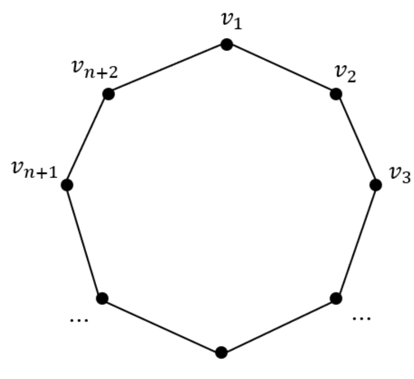
1. $\mathrm{E}[aX + bY] = a \cdot \mathrm{E}[X] + b \cdot \mathrm{E}[Y]$ for any constant a, b.
2. If X and Y are independent, then $\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y]$ and
   $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.
3. $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$. *Hint:* Use the fact that $\mathrm{E}[X \cdot \mathrm{E}[X]] = \mathrm{E}[X]^2$.

Problem 2 (20%). Consider the slides #2. Prove that the graphs H_i defined in the proof of Theorem 3 are bicliques.

Problem 3 (20%). For any integer $n \geq 1$, consider the grid points (r, c) with $1 \leq r, c \leq n$. Let C_n be the number of possible paths from $(0, 0)$ to (n, n) that use only $\rightarrow$ and $\uparrow$ and that never cross the diagonal $r = c$. See also the Figure (a) below. For convenience, define $C_0 := 1$.



(a)



(b)

For any integer $n \geq 2$, consider the convex $(n+2)$-gon with vertices labeled with $v_1, v_2, \ldots, v_{n+2}$. Let P_n denote the number of possible ways to triangulate the polygon. It follows that $P_2 = 2$, $P_3 = 5$, etc. For convenience, also define $P_0 := 1$ and $P_1 := 1$.

1. Prove that for any $n \geq 2$, P_n satisfies the recurrence
$$P_n \;=\; \sum_{0 \leq k < n} P_k \cdot P_{n-k-1}.$$
2. Prove that for any $n \geq 2$, C_n satisfies the same recurrence
$$C_n \;=\; \sum_{0 \leq k < n} C_k \cdot C_{n-k-1}.$$

Note that this proves that P_n also equals the n^{th}-Catalan number.

Problem 4 (20%)**.** Let $\mathcal{F}$ be a family of subsets, where

$$|A| \geq 3 \text{ for any } A \in \mathcal{F} \qquad \text{and} \qquad |A \cap B| = 1 \text{ for any } A, B \in \mathcal{F},\ A \neq B.$$

Suppose that $\mathcal{F}$ is not 2-colorable. Let x, y be any elements that appear in $\mathcal{F}$, i.e., $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$ for some $A, B \in \mathcal{F}$. Prove that:

1. x belongs to at least two members of $\mathcal{F}$.
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Hint: Construct proper coloring to prove the properties. For (1), consider a particular A with $x \in A \in \mathcal{F}$. Color $A \setminus \{x\}$ red and the remaining blue. Show that this leads to the conclusion of (1). For (2), consider particular A, B with $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$. Color $(A \cup B) \setminus \{x, y\}$ red and the remaining blue. Prove that it leads to (2).

Problem 5 (20%)**.** Let $G = (A \cup B, E)$ be a bipartite graph, d be the minimum degree of vertices in A and D the maximum degree of vertices in B. Assume that $|A|d \geq |B|D$.

Show that, for every subset $A_0 \subseteq A$ with the density α defined as $\alpha := |A_0|/|A|$, there exists a subset $B_0 \subseteq B$ such that:

1. $|B_0| \geq \alpha \cdot |B|/2$,
2. every vertex of B_0 has at least $\alpha D/2$ neighbors in A_0, and
3. at least half of the edges leaving A_0 go to B_0.

Hint: Let B_0 consist of all vertices in B that have at least $\alpha D/2$ neighbors in A_0. First prove (3) and then (1).

Problem 1 (20%). Let X, Y be discrete random variables. The variance of a random variable X is defined as $\mathrm{Var}[X] := \mathrm{E}\left[(X - \mathrm{E}[X])^2\right]$. Prove that

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1. $$\begin{aligned} E[aX+bY] &= \sum_{\forall x,y} (aX+bY)\cdot Pr[X=x, Y=y] \\ &= \sum_x \sum_y (aX+bY)\cdot Pr[X=x, Y=y] \\ &= a\sum_x X \cdot Pr[X=x, Y=y] + b\sum_y Y \cdot Pr[X=x, Y=y] \\ &= aE[X] + bE[Y] \ \# \end{aligned}$$

2. $$\begin{aligned} E[X\cdot Y] &= \sum_{\forall x,y} (X\cdot Y)\cdot Pr[X=x, Y=y] \\ &= \sum_x (X)\cdot Pr[X] \cdot \sum_y (Y)\cdot Pr[Y] \\ &= E[X]\cdot E[Y] \ \# \end{aligned}$$

$$\begin{aligned} Var[X+Y] &= E\left[(X+Y-E[X+Y])^2\right] \\ &= E\left[(X-E[X]+Y-E[Y])^2\right] \\ &= E\left[(X-E[X])^2\right] + E\left[(Y-E[Y])^2\right] + \underbrace{2E\left[(X-E[X])(Y-E[Y])\right]}_{=0} \\ &= Var[X] + Var[Y] \ \# \end{aligned}$$

3. $$\begin{aligned} Var[X] &= E\left[(X-E[X])^2\right] \\ &= E\left[X^2 - 2XE[X] + (E[X])^2\right] \\ &= E[X^2] - 2E[X]E[X] + (E[X])^2 \\ &= E[X^2] - 2(E[X])^2 + (E[X])^2 \\ &= E[X^2] - (E[X])^2 \ \# \end{aligned}$$

Problem 2 (20%). Consider the slides #2. Prove that the graphs H_i defined in the proof of Theorem 3 are bicliques.

Slides:

Theorem 3: if $n = 2^m$, then $bc(K_n) = n \cdot \log_2 n$

For any $1 \le i \le m$, define H_i as follows

- $V(H_i) = V(K_n)$
- For any $u, v \in V(K_n)$, $(u,v) \in E(H_i)$ if the i^{th}- coordinates of u and v differ

that H_i is biclique

---

pf:

- ∵ $n = 2^m$ 且 K_n with a coordinate $\{0,1\}^m$

  ∴ 可得知 $V(K_n) = \{0,1\}^m$

- 把 vertices set $V(K_n)$ 分成兩個不相交的 subsets
- we defined:

  S_0 是第 i 個座標為 0 的 vertices set

  $S_0 = \{u \in V(K_n) \mid u_i = 0\}$

  S_1 是第 i 個座標為 1 的 vertices set

  $S_1 = \{v \in V(K_n) \mid v_i = 1\}$

- 根據上述的 H_i 的定義

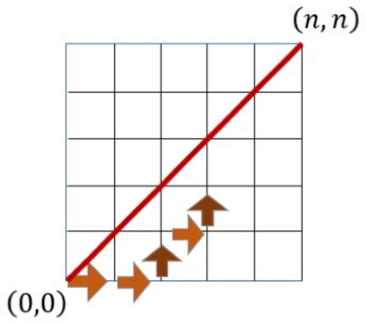
  $u \in S_0$, $v \in S_1$, 則 $(u,v) \in E(H_i)$

  當 u_i, v_i 在 S_0 or S_1, $(u_i, v_i) \notin E(H_i)$

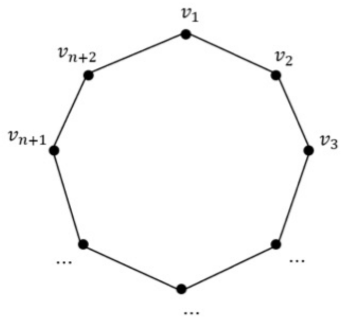
  (∵ i^{th}- coordinates 相同)

- 由上述推導 H_i 為 biclique

Problem 3 (20%). For any integer $n \geq 1$, consider the grid points (r, c) with $1 \leq r, c \leq n$. Let C_n be the number of possible paths from $(0, 0)$ to (n, n) that use only $\rightarrow$ and $\uparrow$ and that never cross the diagonal $r = c$. See also the Figure (a) below. For convenience, define $C_0 := 1$.



(a)



(b)

$$P_n = \sum_{0 \le k < n} P_k \cdot P_{n-k-1}$$

For any integer $n \geq 2$, consider the convex $(n+2)$-gon with vertices labeled with $v_1, v_2, \ldots, v_{n+2}$. Let P_n denote the number of possible ways to triangulate the polygon. It follows that $P_2 = 2$, $P_3 = 5$, etc. For convenience, also define $P_0 := 1$ and $P_1 := 1$.

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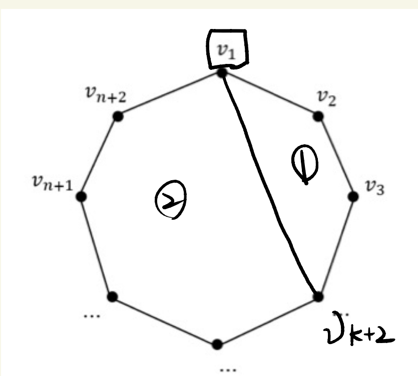
Note that this proves that P_n also equals the n^{th}-Catalan number.

$v_1, v_2 \cdots v_{n+2}$

1. 依題意，$n \geq 2$ 且有一個 凸 $(n+2)$ 邊形，代表有 v_i $(1 \leq i \leq n+2)$ 個 vertices，to calculate the # of possible ways to triangulate the polygon。可以先考慮固定一個 vertix，if we choose v_1，代表 v_{n+2} 和 v_2 不能選，再任選一点 v_i $(3 \leq i \leq n+1)$ 形成一條對角線，會將 pic. (b) 分割成 2 個 小凸多邊形 (① 和 ②)

因此，把 $i = k+2$ 做改寫 且 $1 \leq k \leq n-1$



① 有 $k+2$ 個 vertices $(v_1, v_2 \ldots v_{k+2})$

$\Rightarrow$ ① 的 # of possible ways to triangulate the polygon $= k$ (記為 P_k)

② 有 $(v_1, v_{k+2} \ldots v_{n+2})$ 有 $(n+2)-(k+2)+1 = n-k+1$ 個 vertices

$\Rightarrow$ ② 的 # of possible ways to triangulate the polygon $= n-k-1$ (記為 P_{n-k-1})

$\because$ ① and ② 是獨立，當固定 diagnol 時，可能的 # $= P_k \times P_{n-k-1}$

又 k $(0 \leq k \leq n-1)$ $\Rightarrow P_n = \sum_{0 \leq k < n} P_k P_{n-k-1}$ #

2. $$C_n = \sum_{0 \le k < n} C_k \cdot C_{n-k-1}$$

依題意從 $(0,0)$ 走到 (n,n) $\Rightarrow$ 共走了 $2n$ 步 ($\uparrow \times n$ and $\rightarrow \times n$)

令第一次碰到 diagnol 時的座標 $(k+1, k+1)$，且 $1 \le k+1 \le n$。

ie. $0 \le k \le n-1 \equiv 0 \le k < n$

令以 $(k+1, k+1)$ 為界，分成 2 段路徑，

①: $(0,0) \rightarrow (k+1, k+1)$ 前 and ② $(k+1, k+1) \rightarrow (n,n)$

①: 由 C_n (n^{th} - Catalan #) 定義

$(0,0) \rightarrow (k+1, k+1)$ 之前 可以寫成 C_k

② 由 C_n (n^{th} - Catalan #) 定義

$(k+1, k+1) \rightarrow (n,n)$ 可以寫成 C_{n-k-1}

$\because n-(k+1)$

① and ② 互不影響，可能的路徑數 $C_k \cdot C_{n-k-1}$

又 $0 \le k < n \Rightarrow C_n = \sum_{0 \le k < n} C_k \cdot C_{n-k-1}$ #

Problem 4 (20%). Let $\mathcal{F}$ be a family of subsets, where

$$|A| \geq 3 \text{ for any } A \in \mathcal{F} \qquad \text{and} \qquad |A \cap B| = 1 \text{ for any } A, B \in \mathcal{F},\, A \neq B.$$

Suppose that $\mathcal{F}$ is not 2-colorable. Let x, y be any elements that appear in $\mathcal{F}$, i.e., $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$ for some $A, B \in \mathcal{F}$. Prove that:

1. x belongs to at least two members of $\mathcal{F}$.
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Hint: Construct proper coloring to prove the properties. For (1), consider a particular A with $x \in A \in \mathcal{F}$. Color $A \setminus \{x\}$ red and the remaining blue. Show that this leads to the conclusion of (1). For (2), consider particular A, B with $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$. Color $(A \cup B) \setminus \{x, y\}$ red and the remaining blue. Prove that it leads to (2).

已知

① $A \in F$ 且至少3個元素

②. $|A \cap B| = 1$

③. F not 2-colorable

④. $x \in A \in F$，$y \in B \in F$

若 F no 2-colorable

則 ②

pf:

1. x belongs to at least two members of F

at least 2 → no 2-c

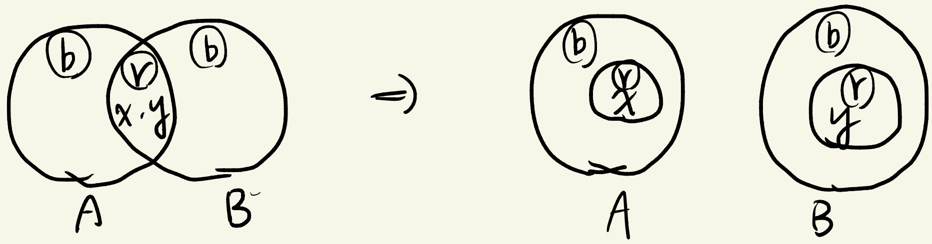
only 1 → 2-c

- if x only belongs to A and $A \in F$
- consider the F is 2-colorable
- 把 x 塗成 blue and A 中的其他 elements 塗成 red
  且 x 只在 A 出現。
- 不會讓其他 sets 因為 A 中的 x，而變成 monochromatic
- ∵ $P \to Q \equiv \neg Q \to \neg P$
- 固得證 x belongs to at least two members of F 且 F is no
  2-colorable

pf.

2. There exists some $C \in F$ such that $\{x, y\} \subseteq C$

- if there no exists $C \in F$ such that $\{x, y\} \subseteq C$
- $x \in A$, $y \in B$, 把 x, y 塗成 red, 而其他 elements 塗成 blue



- 根據上圖, 這些 sets 是 2 - colorable
- then F is 2 - colorable

$\because P \to Q \equiv \neg Q \to \neg P$

$\therefore$ 因得證 There exists some $C \in F$ such that $\{x, y\} \subseteq C$

Problem 5 (20%). Let $G = (A \cup B, E)$ be a bipartite graph, d be the minimum degree of vertices in A and D the maximum degree of vertices in B. Assume that $|A|d \geq |B|D$.

Show that, for every subset $A_0 \subseteq A$ with the density α defined as $\alpha := |A_0|/|A|$, there exists a subset $B_0 \subseteq B$ such that:

1. $|B_0| \geq \alpha \cdot |B|/2$,
2. every vertex of B_0 has at least $\alpha D/2$ neighbors in A_0, and
3. at least half of the edges leaving A_0 go to B_0. $|E(A_0, B_0)| \geq \frac{1}{2}|E(A_0, B)|$

Hint: Let B_0 consist of all vertices in B that have at least $\alpha D/2$ neighbors in A_0. First prove (3) and then (1).

依題意 deg() of $B \leq D$ and deg() of $A \geq d$

所以 D 是 upper bound and d 是 lower bound

$\Rightarrow |A|d \leq |E| \leq |B|D \Rightarrow |A|d \leq |B|D$

由於 assume $|A|d \geq |B|D \Rightarrow |A|d = |B|D$

定義

$B_0 = \{ b \in B \mid \deg_{A_0}(b) \geq \frac{\alpha D}{2} \}$ i.e. 只要有足夠多 neighbor 的 b 就放去 B_0，反之留在 B'

pf:

3. at least half of the edges leaving A_0 go to B_0

- Let $a \in A$, $b \in B$, $B' = B \setminus B_0$
- if $b \in B'$，則 $\deg_{A_0}(b) < \frac{\alpha D}{2}$
- $|E(A_0, B')| = \sum_{b \in B'} \deg_{A_0}(b) < \sum_{b \in B'} \frac{\alpha D}{2} = |B'| \cdot \frac{\alpha D}{2}$

又 $|E(A_0, B)| = \sum_{a \in A_0} \deg_B(a) \geq \alpha \underbrace{|A|}_{\text{vertices}} \underbrace{d}_{\text{degree}}$

- if $|E(A_0, B')| > \frac{1}{2}|E(A_0, B)|$

則 $|E(A_0, B')| > \frac{1}{2}|E(A_0, B)| > \frac{1}{2}\alpha|A|d$

$= \sum_{b \in B'} \deg_{A_0}(b) = |E(A_0, B')| > \frac{1}{2}\alpha|A|d$ $\qquad |B| < \frac{|A|d}{D}$

又 $\because \sum_{b \in B'} \deg_{A_0}(b) < |B'|\frac{\alpha D}{2} \Rightarrow |B'|\frac{\alpha D}{2} > |A|\frac{\alpha d}{2} \Rightarrow |B'| > \frac{|A|d}{D}$

but 題目是 $|A|d \geq |B|D$ 和假設矛盾 ($\because$ 會使 $B' > B$, but $B' \subseteq B$)

$\therefore\ |E(A_0, B')| \leq \frac{1}{2}|E(A_0, B)|$

因此 $|E(A_0, B_0)| \geq \frac{1}{2}|E(A_0, B)|$ #

pf:

1. $|B_0| \geq \alpha \frac{|B|}{2}$

由 (3) 知, $|E(A_0, B_0)| \geq \frac{1}{2}|E(A_0, B)|$, 又 $|E(A_0, B)| \geq \alpha |A| d$

因此 $|E(A_0, B_0)| \geq \frac{1}{2}\alpha |A| d$

- if $|B_0| < \alpha \frac{|B|}{2}$

則 $\frac{1}{|B_0|}\sum_{b \in B_0} deg_{A_0}(b) > \frac{\frac{1}{2}\alpha |A| d}{\frac{\alpha |B|}{2}} = \frac{\alpha |A| d}{\alpha |B|} = \frac{|A| d}{|B|}$

由題目知 $|A|d \geq |B|D \Rightarrow \frac{|A|d}{|B|} \geq D$

$\frac{1}{|B_0|}\sum_{b \in B_0} deg_{A_0}(b) > D \Rightarrow$ 矛盾 ($\because deg_{A_0}(b) \leq D$)

固 $|B_0| \geq \frac{\alpha |B|}{2}$ #

pf

2.

由定義

$B_0 = \{ b \in B \mid deg_{A_0}(b) \geq \frac{\alpha D}{2} \}$, $\forall\, b \in B_0$ 則 $deg_{A_0}(b) \geq \frac{\alpha D}{2}$