Problem 1 (20%)**.** How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 = 21$ with

1. $x_i \geq 0$. 2. $x_i > 0$. 3. $0 \leq x_i \leq 12$.

Problem 2 (20%)**.** Prove the following identities **using path-walking argument**.

1. For any $n, r \in \mathbb{Z}^{\geq 0}$,

$$\sum_{0 \leq k \leq r} \binom{n+k}{k} \;=\; \binom{n+r+1}{r}.$$

2. For any $m, n, r \in \mathbb{Z}^{\geq 0}$ with $0 \leq r \leq m+n$,

$$\sum_{0 \leq k \leq r} \binom{m}{k}\binom{n}{r-k} \;=\; \binom{m+n}{r}.$$

Problem 3 (20%)**.** Let $\mathcal{F}$ be a set family on the ground set X and $d(x)$ be the degree of any $x \in X$, i.e., the number of sets in $\mathcal{F}$ that contains x. Use the double counting principle to prove the following two identities.

$$\sum_{x \in Y} d(x) \;=\; \sum_{A \in \mathcal{F}} |Y \cap A| \;\text{ for any } Y \subseteq X.$$

$$\sum_{x \in X} d(x)^2 \;=\; \sum_{A \in \mathcal{F}} \sum_{x \in A} d(x) \;=\; \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} |A \cap B|.$$

Problem 4 (20%)**.** Let H be a 2α-dense 0-1 matrix. Prove that at least an $\alpha/(1-\alpha)$ fraction of its rows must be α-dense.

Problem 5 (20%)**.** Let $\mathcal{F}$ be a family of subsets defined on an n-element ground set X. Suppose that $\mathcal{F}$ satisfies the following two properties:

1. $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$.

2. For any $A \subsetneq X$, $A \notin \mathcal{F}$, there always exists $B \in \mathcal{F}$ such that $A \cap B = \emptyset$.

Prove that

$$2^{n-1} - 1 \;\leq\; |\mathcal{F}| \;\leq\; 2^{n-1}.$$

Hint: Consider any set $A \subseteq X$ and its complement $\overline{A}$. Apply the conditions given above and prove the two inequalities "$\leq$" and "$\geq$" separately.

Problem 1 (20%). How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 = 21$ with

1. $x_i \geq 0$.
2. $x_i > 0$.
3. $0 \leq x_i \leq 12$.

$H^n_m = C^{n+m-1}_m$

1. $x_i \geq 0$, $\Rightarrow \binom{21+4-1}{4-1} = \binom{24}{3} = \frac{24!}{21!\,3!} = \frac{\cancel{24}^{4} \times 23 \times 22}{\cancel{6}} = 2024$ #

$H^4_{21} = C^{24}_{21} = C^{24}_3$

2. $x_i > 0$, $\Rightarrow (x_1+1) + (x_2+1) + (x_3+1) + (x_4+1) = 21$

$\Rightarrow x_1 + x_2 + x_3 + x_4 = 17$ $\quad H^4_{17}$

$\Rightarrow \binom{17+4-1}{4-1} = \binom{20}{3} = \frac{20!}{17!\,3!} = \frac{20 \times 19 \times \cancel{18}^{3}}{\cancel{6}} = 1140$ #

3. $0 \leq x_i \leq 12$

$\Rightarrow$ set$(x_i \geq 0)$ − set$(x_i > 12)$ (from inclusion-exclusion principle)

set$(x_i > 12)$: $(x_1 + 13) + x_2 + x_3 + x_4 = 21$ ($\because$ 至多只有一個 $x_i > 12$)

因此

$x_1 + x_2 + x_3 + x_4 = 8 \quad \binom{8+4-1}{4-1} = \binom{11}{3} = \frac{11!}{8!\,3!} = \frac{11 \times 10 \times 9}{6} = 165$

$165 \times 4 = 660$ (x_1, x_2, x_3, x_4 共 4 個 choice)

$2024 - 660 = 1364$ #

$H^4_8 = C^{11}_8$

$2024 - C^{11}_8 \times 4$

Problem 2 (20%). Prove the following identities **using path-walking argument**.

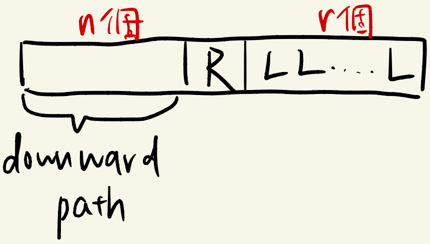
1. For any $n, r \in \mathbb{Z}^{\geq 0}$,
$$\sum_{0 \leq k \leq r} \binom{n+k}{k} = \binom{n+r+1}{r}.$$

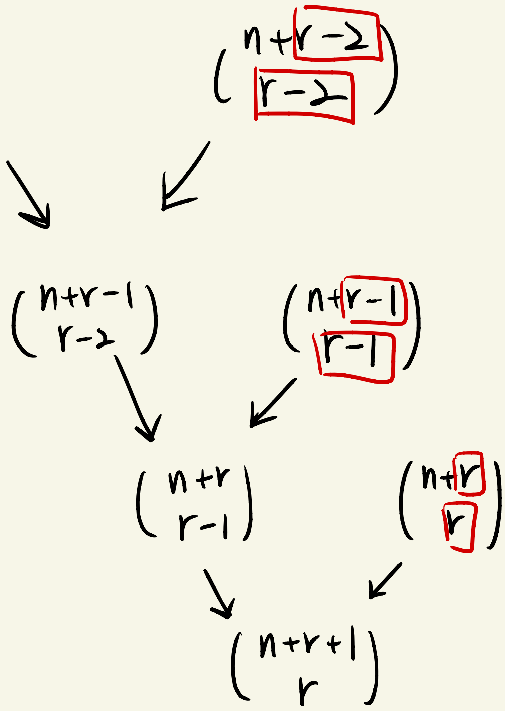
2. For any $m, n, r \in \mathbb{Z}^{\geq 0}$ with $0 \leq r \leq m+n$,
$$\sum_{0 \leq k \leq r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

1. consider a downward path through $\binom{n+k}{k}$ to $\binom{n+r+1}{r}$

RHS is one "R" and some "L"

ie.





hence, we can identify such paths by the last "R" and there are $\sum\limits_{0 \leq k \leq r} \binom{n+k}{k}$ such paths.

2. $\sum\limits_{0 \leq k \leq r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$ $\quad 0 \leq r \leq m+n$

consider the downward path to $\binom{m+n}{r}$

- identify any of such paths by the cell it reaches at m^{th} row
- suppose it is $\binom{m}{k}$

by above argument, there are $\binom{m}{k}\binom{n}{r-k}$ such paths

- taking summation over the cells at m^{th} row

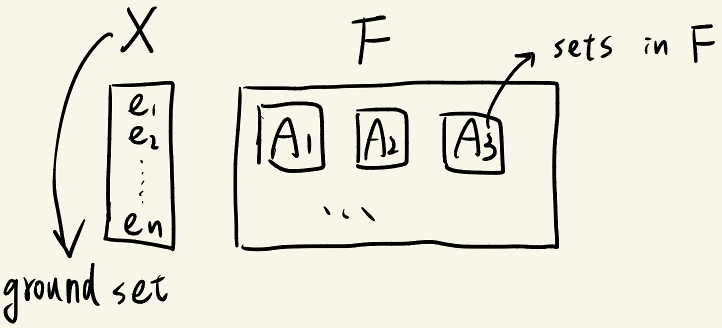
there are $\sum\limits_{0 \leq k \leq r} \binom{m}{k}\binom{n}{r-k}$ such paths

$\Rightarrow$ by double-counting principle, they are equal

Problem 3 (20%). Let $\mathcal{F}$ be a set family on the ground set X and $d(x)$ be the degree of any $x \in X$, i.e., the number of sets in $\mathcal{F}$ that contains x. Use the double counting principle to prove the following two identities.

$$\sum_{x\in Y} d(x) = \sum_{A\in\mathcal{F}} |Y \cap A| \quad \text{for any } Y \subseteq X.$$

$$\sum_{x\in X} d(x)^2 = \sum_{A\in\mathcal{F}} \sum_{x\in A} d(x) = \sum_{A\in\mathcal{F}} \sum_{B\in\mathcal{F}} |A \cap B|.$$

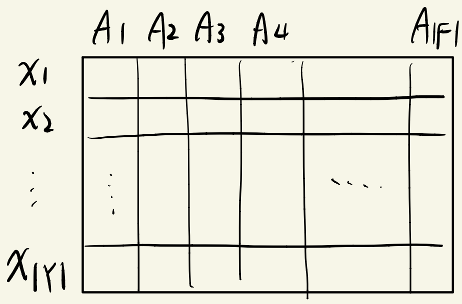


I. $\sum_{x\in Y} d(x) = \sum_{A\in F} |Y \cap A|$ for any $Y \subseteq X$

1. 假設一個由 $|Y| \times |F|$ 組成的 incidence matrix $M = (m_{x,A})$ $x \in Y, A \in F$

where

$$m_{x,a} = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$$



2. $d(x)$ 可以看成是右邊形成的矩陣 M 中 x^{th} row 中的 "1"

3. $|Y \cap A|$ 可以看成是右邊形成的矩陣 M 中 A^{th} col 中的 "1"

4. 因此，LHS 代表將 x 列 "1" 的個數加總

   RHS 代表將 A 列 "1" 的個數加總

5. LHS 和 RHS 都是在算同一個 matrix M 裡面 "1" 的總和

6. 由 double-counting principle，他們是相等

$\text{II}.\ \sum_{x\in X} d(x)^2 = \sum_{A\in F}\sum_{x\in A} d(x) = \sum_{A\in F}\sum_{B\in F} |A\cap B|$

$\sum_{A\in F}\sum_{x\in A} d(x)$：可以看成在由 $|X|\times|F|$ 的 matrix 中 $M=(M_{x_j,A})$

$\sum_{x\in A} d(x)$ 為內層迴圈，$\sum_{A\in F}$ 為外層迴圈，因此，

當內層 x_j 會有 $d(x_j)$ 個，又 $\because A\in F$，必定再走 $d(x_j)$ 次

$d(x_j)\times d(x_j) = d(x_j)^2 \ \{j \mid 1\le j\le |X|\}$

又由 $\sum_{x\in Y} d(x) = \sum_{A\in F} |Y\cap A|$ for any $Y\subseteq X$ 得知

$\Rightarrow \sum_{x\in X} d(x)^2 = \sum_{A\in F}\sum_{x\in A} d(x) = \sum_{A\in F}\sum_{B\in F} |A\cap B|$

Problem 4 (20%). Let H be a 2α-dense 0-1 matrix. Prove that at least an $\alpha/(1-\alpha)$ fraction of its rows must be α-dense.

P $\qquad Q$

$\geq \frac{\alpha}{(1-\alpha)}$

反證

"若P則Q" ≡ "若非Q則非P"

令 $H_{m\times n}$ matrix

1. if 至多 $\frac{\alpha}{(1-\alpha)} \times m$ rows are α-dense (這些 α-dense rows 的 "1" # 是 $\frac{\alpha}{(1-\alpha)} \times m \times n$)

2. 剩下的 rows $= m - \frac{\alpha}{1-\alpha} \times m = m\left(\frac{1-2\alpha}{1-\alpha}\right)$ are non-α-dense

最多的 "1" $< m\left(\frac{1-2\alpha}{1-\alpha}\right) \times \alpha n$

3. total "1"

$$< \left[\frac{\alpha}{(1-\alpha)} m \times n\right] + \left[m\left(\frac{1-2\alpha}{1-\alpha}\right) \times \alpha n\right] = \alpha mn \cdot \left(\frac{1}{1-\alpha} + \frac{1-2\alpha}{1-\alpha}\right)$$

$$= \alpha mn \cdot \left(\frac{2-2\alpha}{1-\alpha}\right) = 2\alpha \cdot mn$$

4. $H_{m\times n}$ 一定不是 2α-dense if 至多 $\frac{\alpha}{1-\alpha}$ 是 α-dense

$\Leftrightarrow$ it must at least an $\frac{\alpha}{1-\alpha}$ fraction of its rows are α-dense

Problem 5 (20%). Let $\mathcal{F}$ be a family of subsets defined on an n-element ground set X. Suppose that $\mathcal{F}$ satisfies the following two properties:

1. $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$.
2. For any $A \subsetneq X$, $A \notin \mathcal{F}$, there always exists $B \in \mathcal{F}$ such that $A \cap B = \emptyset$.

Prove that

$$2^{n-1} - 1 \leq |\mathcal{F}| \leq 2^{n-1}.$$

$2^{n-1} - 1 \leq |F| \leq 2^{n-1}$

①

From F's upper bound:

if F 同時包含 A and $\bar{A}$，則違反①"$A \cap B \neq \phi$"，因為

$A \cap \bar{A} = \phi$ ∴ F 只會有 A or $\bar{A}$ 其中之一

又 X 內有 n 個 elements，F 的 subsets 為 2^n

∴ 從 2^n 個 subsets 中分成 2^{n-1} 對的 A、$\bar{A}$，且擇一放進F

推得 $|F| \leq 2^{n-1}$

②

From F's lower bound:

if F 同時排除 A and $\bar{A}$. 以 A 為例，則可以找

到一個 $B \in F$ 且 $B \cap A = \phi$，因此 $B \subseteq \bar{A}$；

同理以 $\bar{A}$ 為例 $B \cap \bar{A} = \phi$，因此 $B \subseteq A$。矛盾

不可能同時排除2者，必有其一 subset $\in F$

∴ $2^{n-1} - 1 \leq |F|$ (減去 ϕ)

⇒ 結合①、② $2^{n-1} - 1 \leq |F| \leq 2^{n-1}$