Problem 1 (6%). Consider the Turán's theorem and its proof, and let n be a multiple of k, where $k \geq 2$ is an integer.

Construct a graph $G = (V, E)$ with n vertices that contains no $(k+1)$-clique such that the number of edges attains the upper-bound given in the Turán's theorem, i.e.,

$$|E| \; = \; \left(1 - \frac{1}{k}\right) \cdot \frac{n^2}{2}.$$

Justify your answer.

Problem 2 (7%). Let $k > 0$ be an integer and let $p(n)$ be a function of n with $p(n) = \Omega\left((6k \ln n)/n\right)$ for large n. Prove that "almost surely" the random graph $G = G(n, p)$ has no independent set of size $n/2k$, i.e., show that

$$\Pr\left[\, \alpha(G) \geq \frac{n}{2k} \,\right] \; = \; o(1).$$

Problem 3 (7%). Let K_n denote the complete graph with n vertices. Show that it is possible to color K_n with at most $3\sqrt{n}$ colors so that there are no monochromatic triangles.

Problem 1 (6%). Consider the Turán's theorem and its proof, and let n be a multiple of k, where $k \geq 2$ is an integer.

Construct a graph $G = (V, E)$ with n vertices that contains no $(k+1)$-clique such that the number of edges attains the upper-bound given in the Turán's theorem, i.e.,

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Justify your answer.

1. Set a graph $G = (V, E)$, with n vertices and contains no $(k+1)$-clique and a maximal # of edges

2. Since above, when we add a new edge to G will create a $(k+1)$-clique

3. G contains at least one k-clique, and we let A be a k-clique in G, and let $B := V \setminus A$; e_A, e_B, e_{AB} denotes edge # in A, in B, and between A and B respectively.

4. Hence, $e_A = \binom{k}{2} = \frac{k \cdot k-1}{2}$

5. By induction hypothesis, $e_B \leq \left(1-\frac{1}{k}\right) \cdot \frac{(n-k)^2}{2}$

6. Each $v \in B$ is adjacent to at most $k-1$ vertices in A.

   Hence, $e_{AB} \leq (k-1) \cdot (n-k)$

7. Hence, $|E| = e_A + e_B + e_{AB} \leq \frac{k^2-k}{2} + \left(1-\frac{1}{k}\right) \cdot \frac{(n-k)^2}{2} + (k-1) \cdot (n-k)$

$$\frac{k^2-k}{2} + \frac{\left(n^2-2nk+k^2-\frac{n^2}{k}+2n-k\right)}{2} + \left(nk-k^2-n+k\right)$$

$$= \frac{\cancel{k^2}-\cancel{k}+n^2-\cancel{2nk}+\cancel{k^2}-\frac{n^2}{k}+\cancel{2n}-\cancel{k}+\cancel{2nk}-\cancel{2k^2}-\cancel{2n}+\cancel{2k}}{2}$$

$$= \left(1-\frac{1}{k}\right) \cdot \frac{n^2}{2} \ \#$$

Problem 2 (7%). Let $k > 0$ be an integer and let $p(n)$ be a function of n with $p(n) = \Omega((6k \ln n)/n)$ for large n. Prove that "almost surely" the random graph $G = G(n,p)$ has no independent set of size $n/2k$, i.e., show that

$$\Pr\left[\alpha(G) \geq \frac{n}{2k}\right] = o(1).$$

1. Let r denotes independent set of size at least $n/2k$

   i.e. $r = \lfloor n/2k \rfloor$

2. the probability of choose any subset S of size r ($S \subseteq V$), that S is independent set is

   $$\Pr[S \text{ is independent set}] = (1-p)^{\binom{r}{2}}$$

   $$\Rightarrow \Pr[S \text{ is independent set}, |S| = r] \leq \binom{n}{r}(1-p)^{\binom{r}{2}}$$

3. 分別對 $\binom{n}{r}$ 和 $(1-p)^{\binom{r}{2}}$ 做估計的 upper bound

   ① $\binom{n}{r}$:

   $$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n \times (n-1) \times \cdots \times (n-r+1)}{r!} \leq \frac{n^r}{r!}, \quad 又\ r! \geq \left(\frac{r}{e}\right)^r$$

   $$\Rightarrow \binom{n}{r} \leq \frac{n^r}{(r/e)^r} = \left(\frac{en}{r}\right)^r = e^{r \ln(en/r)}$$

   ② $(1-p)^{\binom{r}{2}}$:

   由 $(1-p) \leq e^{-p}$, $(1-p)^{\binom{r}{2}} \leq e^{-p \cdot \binom{r}{2}}$

   $$又\ \binom{r}{2} \approx \frac{r^2}{2} = \frac{n^2}{2 \cdot 4k^2} = \frac{n^2}{8k^2}$$

   $$p(n) \geq c \cdot \left(\frac{6k \cdot \ln n}{n}\right) \Rightarrow p \cdot \binom{r}{2} \geq c \cdot \left(\frac{6k \cdot \ln n}{n}\right) \cdot \frac{n^2}{8k^2} = \frac{c}{8k} n \ln n$$

   $$因此\ (1-p)^{\binom{r}{2}} \leq e^{\left(-\frac{c}{8k} \cdot n \ln n\right)}$$

4. $$\binom{n}{r}(1-p)^{\binom{r}{2}} \leq e^{r \cdot \ln\left(\frac{en}{r}\right)} \times e^{\left(-\frac{c}{8k} \cdot n \ln n\right)}$$

   當 n 很大時 $e^{r \cdot \ln\left(\frac{en}{r}\right)} \times e^{\left(-\frac{c}{8k} \cdot n \ln n\right)}$ 趨近於 0

   $$\Rightarrow \Pr\left[\alpha(G) \geq \frac{n}{2k}\right] = o(1) \ \#$$

Problem 3 (7%). Let K_n denote the complete graph with n vertices. Show that it is possible to color K_n with at most $3\sqrt{n}$ colors so that there are no monochromatic triangles.

1. For each edge, choose one of $3\sqrt{n}$ colors uniformly at random to color the edges

2. The probability a given three vertices form a monochromatic triangles is $\left(\frac{1}{3\sqrt{n}}\right)^2 = \frac{1}{9n}$

3. The # of triangles whose three edge colorings are not independent of a given triangle is $3n-8$

4. By LLL, it holds that $e \cdot \frac{1}{9n}(3n-7) \leq 1$

hence, there is a nonzero probability that there are no monochromatic triangles.