Problem 1 (20%)**.** Prove that, for any vector $v \in \mathbb{R}^n$,

$$\frac{|v|_1}{\sqrt{n}} \;\le\; \|v\|_2 \;\le\; |v|_1,$$

where $|v|_1 := \sum_i |v_i|$ is the L_1-norm and $\|v\|_2 := \left(\sum_i v_i^2\right)^{1/2}$ is the L_2-norm of v.

Hint: Use the Cauchy-Schwarz inequality, i.e., $|u \cdot v| \le \|u\|_2 \|v\|_2$ for any $u, v \in \mathbb{R}^n$.

Problem 2 (20%)**.** Let A be a square symmetric matrix and λ be an eigenvalue of A. Prove that, for any $k \in \mathbb{N}$, λ^k is an eigenvalue of A^k.

Problem 3 (20%)**.** Let G be an n-vertex d-regular bipartite graph and A be the normalized adjacency matrix of G. Prove that, there exists a vector $v \in \mathbb{R}^n$ such that

$$Av = -v.$$

Generalize the construction to non-regular bipartite graphs, i.e., for any bipartite graph G' with column-normalized adjacency matrix A', prove that A' has an eigenvalue -1.

Note: A' is also called the *random-walk* matrix of G'.

Problem 4 (20%)**.** Let $G = (V, E)$ be a d-regular graph and P be a random walk of length t in G. Prove that, for any edge $e \in E$ and any $1 \le i \le t$,

$$\Pr\left[\; e \text{ is the } i^{th}\text{-edge of } P \;\right] \;=\; \frac{1}{|E|}.$$

Hint: Prove by induction on i.

Problem 5 (20%)**.** Let $G = (V, E)$ be an (n, d, λ)-expander and $S \subseteq V$ be a vertex subset. Prove that,

$$\Pr_{(u,v)\in E}\left[\, u, v \in S \,\right] \;\le\; \frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right),$$

i.e., for any $(u, v) \in E$, the probability that both u, v are in S is bounded by $\frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right)$.

Hint: Use the fact that $|E(S, S)| = (d|S| - |E(S, T)|)/2$. Apply the crossing lemma.

Problem 1 (20%). Prove that, for any vector $v \in \mathbb{R}^n$,

$$\frac{|v|_1}{\sqrt{n}} \le \|v\|_2 \le |v|_1,$$

where $|v|_1 := \sum_i |v_i|$ is the L_1-norm and $\|v\|_2 := (\sum_i v_i^2)^{1/2}$ is the L_2-norm of v.

Hint: Use the Cauchy-Schwarz inequality, i.e., $|u \cdot v| \le \|u\|_2\|v\|_2$ for any $u, v \in \mathbb{R}^n$.

we want to proof $\|v\|_2 \le |v|_1$ and $\frac{|v|_1}{\sqrt{n}} \le \|v\|_2$

①

$\|v\|_2 \le |v|_1$

$|v|_1 = \sum_{i=1}^{n} v_i$ & $\|v\|_2 = \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}}$

$|v|_1^2 = \left(\sum_{i=1}^{n} v_i\right)^2 = \sum_{i=1}^{n} v_i^2 + 2 \cdot \sum_{1 \le i < j \le n} v_i v_j \ge \left(\left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}}\right)^2 = \sum_{i=1}^{n} v_i^2 = \left(\|v\|_2\right)^2$

$\Rightarrow \|v\|_2 \le |v|_1$ #

②

$\frac{|v|_1}{\sqrt{n}} \le \|v\|_2$

$v = (v_1, v_2, v_3 \dots v_n)$, $l = (1, 1, \dots 1)$

$v \cdot l = \sum_{i=1}^{n} v_i \le \|v\|_2 \|l\|_2 = \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}} \cdot \sqrt{n} = \|v\|_2 \sqrt{n}$

$\Rightarrow |v|_1 \le \|v_2\| \sqrt{n} \equiv \frac{|v|_1}{\sqrt{n}} \le \|v_2\|$

by ① ② 可推得 $\frac{|v|_1}{\sqrt{n}} \le \|v\|_2 \le |v|_1$ #

Problem 2 (20%). Let A be a square symmetric matrix and λ be an eigenvalue of A. Prove that, for any $k \in \mathbb{N}$, λ^k is an eigenvalue of A^k.

1. 根據題意，$A^k v = \lambda^k v$ ，for any $k \in \mathbb{N}$

2. By induction, $k=1$, we have λ is eigenvalue of A , $Av = \lambda v$

3. $k=2$，左右同乘 A。

$$A \cdot Av = A \cdot \lambda v = \lambda \cdot Av = \lambda \cdot \lambda v = \lambda^2 v$$

$$\Rightarrow A^2 v = \lambda^2 v$$

4. 以此類推，for $k+1$

$$A^{k+1} v = A \cdot A^k v = A \cdot \lambda^k v = \lambda^{k+1} v \Rightarrow A^{k+1} v = \lambda^{k+1} v$$

5. 因得證，λ^k is an eigenvalue of A^k $\forall$ $k \in \mathbb{N}$ #

Problem 3 (20%)**.** Let G be an n-vertex d-regular bipartite graph and A be the normalized adjacency matrix of G. Prove that, there exists a vector $v \in \mathbb{R}^n$ such that

$$Av = -v.$$

Generalize the construction to non-regular bipartite graphs, i.e., for any bipartite graph G' with column-normalized adjacency matrix A', prove that A' has an eigenvalue -1.

Note: A' is also called the *random-walk* matrix of G'.

Let $G = (L \cup R)$, L、R 分別是 bipartite 的兩集合

$v \in \mathbb{R}^n$, $v_i = \begin{cases} +1, & i \in L \\ -1, & i \in R \end{cases}$

$A_{ij} = \begin{cases} \frac{1}{d}, & 若 (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$

$i \in L$，所有 neighbor 在 R，且 $v_j = -1$，$v_i = +1$

$$(Av)_i = \sum_{j:(i,j)\in E} A_{ij} v_j = \sum_{j \in N(i)} \frac{1}{d}(-1) = \frac{-1}{d} \cdot |N(i)| = \frac{-1}{d} \cdot d = -1 = -v_i$$

$i \in R$，所有 neighbor 在 L，且 $v_j = +1$，$v_i = -1$

$$(Av)_i = \sum_{j:(i,j)\in E} \frac{1}{d}(+1) = \frac{1}{d} \cdot d = 1 = -v_i$$

$$\Rightarrow (Av) = -v$$

Problem 4 (20%). Let $G = (V, E)$ be a d-regular graph and P be a random walk of length t in G. Prove that, for any edge $e \in E$ and any $1 \leq i \leq t$,

$$\Pr[\ e \text{ is the } i^{th}\text{-edge of } P\ ] = \frac{1}{|E|}.$$

Hint: Prove by induction on i.

I.

設 X_i 為 random walk 在第 i 步落在的 vertex

1. $\forall i\ (1 \leq i \leq t)$, then $X_i \sim \text{Uniform}(V)$

   令 $i=1$，則 X_1 是起點。

2. 設 $X_i = v$ 為目前的 vertex，$X_{i+1} = w$ 為下一步的 vertex

   By induction，任取 i，if X_i 是 uniform，求走到下一步 w 的 $\Pr[X_{i+1} = w]$。

3. 則 $\Pr[X_{i+1} = w] = \sum_{v \in N(w)} \Pr[X_i = v] \cdot \frac{1}{d}$ ($v \to w$ 的機率)

$$= \sum_{v \in N(w)} \frac{1}{n} \cdot \frac{1}{d} = \frac{|N(w)|}{nd} = \frac{d}{nd} = \frac{1}{n}$$

4. X_{i+1} 仍是 uniform。

II.

1. 設 $E_i = \{X_i, X_{i+1}\}$，$e = \{u, v\}$

2. 則 $\Pr(E_i = e) = \Pr(X_i = u, X_{i+1} = v) + \Pr(X_i = v, X_{i+1} = u)$

$$= \frac{1}{n} \cdot \frac{1}{d} + \frac{1}{n} \cdot \frac{1}{d} = \frac{2}{nd} = \frac{1}{|E|}$$

3. $|E| = nd/2 \quad \forall i$

4. 推得 $e \in E$，$1 \leq i \leq t$，$\Pr[e \text{ is the } i^{th} \text{ edge}] = \frac{1}{|E|}$ #

Problem 5 (20%)**.** Let $G = (V, E)$ be an (n, d, λ)-expander and $S \subseteq V$ be a vertex subset. Prove that,

$$\Pr_{(u,v)\in E}[\, u, v \in S \,] \ \le\ \frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right),$$

i.e., for any $(u, v) \in E$, the probability that both u, v are in S is bounded by $\frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right)$.

Hint: Use the fact that $|E(S, S)| = (d|S| - |E(S, T)|)/2$. Apply the crossing lemma.

1、由題目知 $G = (V, E)$ and (n, d, λ) – expander, then $|E| = \frac{nd}{2}$

2、we want to proof "$\Pr_{(u,v)\in E}[u, v \in S] \le \frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right)$

3、$\chi \in \{0,1\}^n$, $\chi_i = \begin{cases} 1, & i \in S \\ 0, & i \notin S \end{cases}$

4、$\chi = \frac{|S|}{n} \cdot \ell + y$, $\ell = (1, 1, \ldots 1)^T$, $\ell^T y = 0$

5、$\chi^T P \chi = \left(\frac{|S|}{n} \ell + y\right)^T P \left(\frac{|S|}{n} \ell + y\right)$

$= \frac{|S|^2}{n^2} \ell^T P \ell + 2 \frac{|S|}{n} \ell^T P y + y^T P y$

又 $\ell^T P y = (P\ell)^T y = \ell^T y = 0$

6、$y^T P y \le \lambda \|y\|^2 = \lambda\left(|S| - \frac{|S|^2}{n}\right)$

7、推得 $\chi^T P \chi \le \frac{|S|^2}{n} + \lambda\left(|S| - \frac{|S|^2}{n}\right) = |S|\left(\frac{|S|}{n} + \lambda\right) - \lambda \frac{|S|^2}{n} \le |S|\left(\frac{|S|}{n} + \lambda\right)$

8、$\Pr_{(u,v)\in E}[u, v \in S]$

$= \frac{1}{n} \chi^T P \chi \le \frac{1}{n}\left[|S|\left(\frac{|S|}{n} + \lambda\right)\right] = \frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right)$ #