Problem 1 (20%). Prove that every set of $n+1$ distinct integers chosen from $\{1, 2, \ldots, 2n\}$ contains a pair of consecutive numbers and a pair whose sum is $2n+1$.

For each n, exhibit two sets of size n to show that the above results are the best possible, i.e., sets of size $n+1$ are necessary.

Hint: Use pigeonholes $(2i, 2i-1)$ and $(i, 2n-i+1)$ for $1 \leq i \leq n$.

Problem 2 (20%). Let $G = (V, E)$ be a graph. Denote by $\chi(G)$ the minimum number of colors needed to color the vertices in V such that, no adjacent vertices are colored the same. Prove that, $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices.

Hint: Order the vertices $v_1, v_2, \ldots, v_n$ and use greedy coloring. Show that it is possible to color the graph using $\Delta(G) + 1$ colors.

Problem 3 (20%). Let $\alpha(G)$ be the *independence number* of a graph G, i.e., the maximum size of any independent set of G. Prove the following dual version of Turán's theorem:

> If G is a graph with n vertices and $nk/2$ edges, where $k \geq 1$, then we have
> $$\alpha(G) \geq n/(k+1).$$

Problem 4 (20%). Consider the following two problems regarding Markov's and Chebyshev's inequalities.

- For any positive integer k, describe a non-negative random variable X such that
$$\Pr[\, X \geq k \cdot \mathrm{E}[X] \,] \;=\; \frac{1}{k}.$$
Note that, this shows that Markov's inequality is as tight as it could possibly be.
- Can you provide an example that shows that Chebyshev's inequality is tight? If not, explain why not.

Problem 5 (20%). Suppose that we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these random bits in any order. Let Y_i be the exclusive-or (XOR) of the i^{th} pair of bits, and let $Y := \sum_{1 \leq i \leq m} Y_i$.

- Show that $Y_i = 0$ and $Y_i = 1$ with probability $1/2$ each.
- Show that $\mathrm{E}[\, Y_i \cdot Y_j \,] = \mathrm{E}[Y_i] \cdot \mathrm{E}[Y_j]$ for any $1 \leq i, j \leq m$ and derive $\mathrm{Var}[Y]$.
- Use Chebyshev's inequality to derive a bound on $\Pr[\, |Y - \mathrm{E}[Y]| \geq n \,]$.

Problem 1 (20%). Prove that every set of $n+1$ distinct integers chosen from $\{1, 2, \ldots, 2n\}$ contains a pair of consecutive numbers and a pair whose sum is $2n+1$.

For each n, exhibit two sets of size n to show that the above results are the best possible, i.e., sets of size $n+1$ are necessary.

Hint: Use pigeonholes $(2i, 2i-1)$ and $(i, 2n-i+1)$ for $1 \le i \le n$.

i. a pair of consecutive numbers:

- 把 $\{1,2,3,\ldots,2n\}$ 分成 n 個不連續的連續對 $(2i, 2i-1)$ for $1 \le i \le n$。 i.e. $(2,1), (4,3), \ldots, (2n, 2n-1)$
- 則我們從這 n 對中各取一個數，否則會有連續數
- 依題意要取 n+1 個 distinct integer，根據 pigeonholes，總共 n 個選 n+1 個 $\Rightarrow$ 必有一對選了 2 次
  $\Rightarrow$ there is a pair of consecutive numbers #

ii. a pair whose sum is $2n+1$

- 把 $\{1,2,3,\ldots,2n\}$ 分成 n 組和為 $2n+1$ 的組合且彼此不重疊 $(i, 2n-i+1)$ for $1 \le i \le n$。 i.e. $(1, 2n), (2, 2n-1), \ldots$
- 則我們從這 n 對中各取一個數，否則會有 $2n+1$
- 依題意要取 n+1 個數，According to Pigeonholes，總共 n 個選 n+1 個 $\Rightarrow$ 必有一對選了 2 次
  $\Rightarrow$ There's a pair whose sum is $2n+1$ #

Problem 2 (20%). Let $G = (V, E)$ be a graph. Denote by $\chi(G)$ the minimum number of colors needed to color the vertices in V such that, no adjacent vertices are colored the same. Prove that, $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices.

Hint: Order the vertices $v_1, v_2, \ldots, v_n$ and use greedy coloring. Show that it is possible to color the graph using $\Delta(G) + 1$ colors.

- 先把 V 中的 vertices 進行排序，令 vertices 為 $v_1, v_2 \cdots v_n$
- 當要對 v_i 進行 coloring，先看 v_i 的相鄰已被著色的頂点，再填入一個不同的顏色。
- ∵ v_i 最多有 $\Delta(G)$ 個 neighbors，i.e. 當填 v_i 顏色時，最多 $\Delta(G)$ 種顏色不能用。
- ∴ 用 $\Delta(G)+1$ 種顏色，可以找到一個可用的顏色給 v_i

$\Rightarrow \chi(G) \leq \Delta(G)+1$ #

Problem 3 (20%)**.** Let $\alpha(G)$ be the *independence number* of a graph G, i.e., the maximum size of any independent set of G. Prove the following dual version of Turán's theorem:

If G is a graph with n vertices and $nk/2$ edges, where $k \geq 1$, then we have

$$\alpha(G) \geq n/(k+1).$$

$\alpha(G) \geq \frac{n}{k+1}$

- since in $G(V, E)$, there are $nk/2$ edges, so the total degree number

$$\sum_{v \in V} d(v) = 2 \cdot |E| = 2 \cdot \frac{nk}{2} = nk$$

- 因此，vertices 的平均 degree 是 $nk/n = k$
- 用 greedy 選出 independent set。按照 degree 大小排序為 $v_1, v_2 \ldots v_n$，從最小 v_i 開始，加入 independent set，並且刪除 G 中和 v_i 有連接的 vertices
- 每次最多刪掉 $\deg(v)+1$ 個 vertices，且 $\deg(v)+1 \leq k+1$ (i.e. $\because \bar{d} = k$ $\therefore$ 存在 $d(v) \leq k$ 的頂點) 並在 independent set 加一個點
- 根據上述，每次刪 $\leq k+1$ 個点，代表最少可以刪 $\geq n/k+1$ 次
- $\alpha(G) \geq n/k+1$ #

Problem 4 (20%). Consider the following two problems regarding Markov's and Chebyshev's inequalities.

- For any positive integer k, describe a non-negative random variable X such that

$$\Pr[\, X \geq k \cdot \mathrm{E}[X] \,] \;=\; \frac{1}{k}.$$

  Note that, this shows that Markov's inequality is as tight as it could possibly be.

- Can you provide an example that shows that Chebyshev's inequality is tight? If not, explain why not.

I. Markov

— 令 X 為 兩点分佈的 隨機 變數

$$\begin{cases} X=0 \;, & p=1-\frac{1}{k} \\ X=a>0 \;, & p=\frac{1}{k} \end{cases}$$

— $E[X] = 0\cdot(1-\frac{1}{k}) + a\cdot\frac{1}{k} = \frac{a}{k}$

— 依題意，要使 $\Pr[X \geq k\cdot E[X]] = \frac{1}{k}$

則 $k\cdot E[X] = k\cdot\frac{a}{k} = a$ ，且 $\Pr[X \geq k\cdot E[X]] = \Pr[X=a] = \frac{1}{k}$ #

II. Chebyshev

$$\Pr[\,|X-E[X]| \geq t\,] \leq \frac{Var[X]}{t^2}$$

— 令 X 為 兩点分佈的 隨機 變數

$$\begin{cases} X=a \;, & p=\frac{1}{2} \\ X=-a \;, & p=\frac{1}{2} \end{cases}$$

— 則 $\begin{cases} E[X] = a\cdot\frac{1}{2} + (-a)\cdot\frac{1}{2} = 0 \\ Var[X] = E[X^2]-(E[X])^2 = a^2-0 = a^2 \end{cases}$

— 使得 $\Pr[\,|X-E[X]| \geq a\,] = \Pr[\,|X| \geq a\,] = \frac{a^2}{a^2} = 1$ #

Problem 5 (20%). Suppose that we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these random bits in any order. Let Y_i be the exclusive-or (XOR) of the i^{th} pair of bits, and let $Y := \sum_{1 \le i \le m} Y_i$.

- Show that $Y_i = 0$ and $Y_i = 1$ with probability 1/2 each.
- Show that $\mathrm{E}[\, Y_i \cdot Y_j \,] = \mathrm{E}[Y_i] \cdot \mathrm{E}[Y_j]$ for any $1 \le i, j \le m$ and derive $\mathrm{Var}[Y]$.
- Use Chebyshev's inequality to derive a bound on $\Pr[\, |Y - \mathrm{E}[Y]| \ge n \,]$.

I.

－令任意對結果 (x, y)

－依題意，$x \neq y = 1$，$x = y = 0$

則 $\begin{cases} \Pr[x \neq y] = \Pr[(0,1)] + \Pr[(1,0)] = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \\ \Pr[x = y] = \Pr[(0,0)] + \Pr[(1,1)] = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \end{cases}$

－因此，$\Pr[Y_i = 1] = \frac{1}{2}$，$\Pr[Y_i = 0] = \frac{1}{2}$ #

II.

－由上題知 $E[Y_i] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

－令 $Y_i \cdot Y_j = X$

當 $\begin{cases} X = 0，則 Y_i \vee Y_j = 0 \\ X = 1，則 Y_i \wedge Y_j = 1 \end{cases} \Rightarrow \begin{cases} \Pr[X = 0] = \frac{3}{4} \\ \Pr[X = 1] = \frac{1}{4} \end{cases}$

－ $E[X] = \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 1 = \frac{1}{4}$

$$\mathrm{Cov}(Y_i, Y_j) := E[\, Y_i \cdot Y_j \,] - E[Y_i] \cdot E[Y_j].$$

$$Y = \sum_{i=1}^{m} Y_i \Rightarrow \mathrm{Var}(Y) = \sum_{i=1}^{m} \mathrm{Var}(Y_i) + \sum_{\substack{i \neq j \\ 1 \le i, j \le m}} \mathrm{Cov}(Y_i, Y_j)$$

then, $\mathrm{Var}[Y_i] = E[Y_i^2] - (E[Y_i])^2 = [1^2 \cdot \frac{1}{2} + 0^2 \cdot \frac{1}{2}] - (\frac{1}{2})^2 = 1 \cdot \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{4}$

又 $\mathrm{Cov}(Y_i, Y_j) = E[Y_i \cdot Y_j] - E[Y_i] \cdot E[Y_j] = 0$ (由上小題知 $E[Y_i \cdot Y_j]$
$= E[Y_i] \cdot E[Y_j]$
$=$

因此，$\mathrm{Var}(Y) = m \cdot \frac{1}{4} = \binom{n}{2} \cdot \frac{1}{4} = \frac{n(n-1)}{8}$ #

III

Chebyshev : $\Pr[|X-E[X]| \geq t] \leq \frac{Var[X]}{t^2}$

所求 : $\Pr[|Y-E[Y]| \geq n] \leq \frac{Var[Y]}{n^2}$

代入 $Var[Y] = \frac{n(n-1)}{8}$

推得 $\Pr[|Y-E[Y]| \geq n] \leq \frac{\frac{n(n-1)}{8}}{n^2} = \frac{n-1}{8n}$ #