Problem 1 (20%). Show that, for any positive integer n, there is a multiple of n that contains only the digits 7 or 0.

Hint: Consider all the numbers a_i of the form $77\ldots7$, with i sevens, for $i = 1, 2, \ldots n+1$, and the value a_i modulo n.

Problem 2 (20%). Prove that for any two sets I, J with $I \subseteq J$, we have

$$\sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \begin{cases} 1, & \text{if } I = J, \\ 0, & \text{if } I \neq J. \end{cases}$$

Hint: Rewrite the summation and apply the binomial theorem (in slides # 1a).

Problem 3 (20%). Let $\mathcal{F}$ be a k-uniform k-regular family, i.e., each set has k elements and each element belongs to k sets. Let $k \geq 10$. Show that there exists at least one valid 2-coloring of the elements.

Hint: Define proper events for the sets and apply the symmetric version of the local lemma.

Problem 4 (20%). We proved the asymmetric version of the local lemma in lecture #4. Assume that the statement of this lemma holds. Furthermore, assume that

1. $\Pr[A_i] \leq p$ for all i, and
2. $ep(d+1) \leq 1$.

Prove that $\Pr\left[\bigcap_i \overline{A_i}\right] > 0$, i.e., use Theorem 19.2 to prove the statement of Theorem 19.1.

Hint: Let $x(A_i) = \frac{1}{d+1}$ for all $1 \leq i \leq n$. Use the inequality $\frac{1}{e} \leq \left(1 - \frac{1}{d+1}\right)^d$ obtained by the limit formula of $1/e$ and the fact that it converges from the above.

Problem 5 (20%). Let X be a finite set and $A_1, A_2, \ldots, A_m$ be a partition of X into mutually disjoint blocks. Given a subset $Y \subseteq X$, consider the partition $Y = B_1 \cup B_2 \cup \cdots \cup B_m$ with the blocks B_i defined as $B_i := A_i \cap Y$. For any $1 \leq i \leq m$, we say that the block B_i is λ-*large* if

$$\frac{|B_i|}{|A_i|} \geq \lambda \cdot \frac{|Y|}{|X|}.$$

Show that, for every $\lambda > 0$, at least $(1-\lambda) \cdot |Y|$ elements of Y belong to λ-large blocks.

Problem 1 (20%). Show that, for any positive integer n, there is a multiple of n that contains only the digits 7 or 0.

Hint: Consider all the numbers a_i of the form $77\ldots7$, with i sevens, for $i = 1, 2, \ldots n+1$, and the value a_i modulo n.

根據 hint,

1. Consider all the numbers a_i of the form "$77\ldots7$" with i "7"

i.e. $\underbrace{777\ldots7}_{i個}$

for $i = 1, 2, \ldots, n+1$

2. 對 $a_1, a_2 \ldots a_{n+1}$ 取 mod n，則餘數的範圍 $0, 1, 2 \ldots (n-1)$ 共 n 種可能

3. 由於有 $n+1$ 個數取 mod n，根據鴿籠原理，至少會有 2 個 a_i 有相同的餘數 $\Rightarrow a_i \equiv a_j \pmod{n}$ that $i > j$

4. $a_i - a_j = \underbrace{777\ldots7}_{i-j個7}\underbrace{000\ldots0}_{j個0}$

5. $(a_i - a_j) \bmod n \equiv (a_i \bmod n - a_j \bmod n + n) \bmod n = 0$

6. 因此可推得 for any positive int. n there is a multiple of n that contains only the digits 7 or 0 #

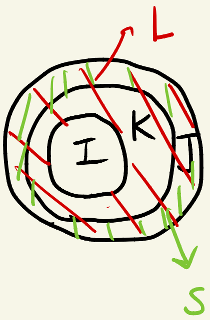
Problem 2 (20%)**.** Prove that for any two sets I, J with $I \subseteq J$, we have

$$\sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \begin{cases} 1, & \text{if } I = J, \\ 0, & \text{if } I \neq J. \end{cases}$$

Hint: Rewrite the summation and apply the binomial theorem (in slides # 1a).

1. If $I = J$

- 唯一集合是 $I = K = J$
- 因此 $K \setminus I = I \setminus I = \phi$，又 $|K \setminus I| = 0$
- $\sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \sum_{I \subseteq K \subseteq J} (-1)^0 = 1$ #



2. If $I \neq J$

- 設 $L = J \setminus I$，由於 $I \subseteq J$ 且 $I \neq J$
  則 L non-empty
- 因此 K 可表示為 $K = I \cup S$ 且 $S \subseteq L$
- $S = K \setminus I \Rightarrow (-1)^{|K \setminus I|} = (-1)^{|S|}$
- 改寫原式成 $\sum_{S \subseteq L} (-1)^{|S|}$
- 令 $s = |S|$，$\sum_{S \subseteq L} (-1)^{|S|}$ 可表示成 $\sum_{0}^{s} (-1)^i \binom{s}{i} = (1-1)^s = 0$ (by 二項式定理) #

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Problem 3 (20%). Let $\mathcal{F}$ be a k-uniform k-regular family, i.e., each set has k elements and each element belongs to k sets. Let $k \geq 10$. Show that there exists at least one valid 2-coloring of the elements.

Hint: Define proper events for the sets and apply the symmetric version of the local lemma.

設 $F = \{S_1, S_2 \dots S_k\}$ be a k-uniform, k-regular family

- By LLL, we define the bad event A_i denote the S_i is not 2-colorable (i.e. S_i 的 elements 都是單色)
- And to prove $\Pr[\cap_i \overline{A_i}] > 0$
- $\Pr[A_i] = \underset{\substack{\downarrow \\ \text{2種color}}}{2} \times (\frac{1}{2})^{\underset{\substack{\downarrow \\ \text{k個}}}{k}} = 2^{1-k} = p$

$e = 2.718$

- 假設當 A_1 和 A_2 有元素重疊，則 A_1、A_2 連一條 edge
- ∵ k-regular ∴ 每個 elements 恰屬於 k 個集合
- if 固定一個 S_i，element 可再屬於 $k-1$ 個集合
  因此最多有 $k \times (k-1)$，最少有一個 element 共享
- $d = k(k-1)$
- $ep(d+1) = e \cdot (2^{1-k}) \cdot (k^2-k+1) \cdots ①$
  when $k=10$ 代入 ①，$e \cdot (2^{-9})(100-10+1) = e \cdot \frac{91}{2^9} = e \cdot \frac{91}{512} \leq 1$
  同理 $k > 10$ 的乘積 < 1
- 由 LLL 得知，有 positive probability 去避免 A_i 的發生
  (i.e. 沒有任何集合是單色)。必存在一種 valid 2-coloring #

Problem 4 (20%). We proved the asymmetric version of the local lemma in lecture #4. Assume that the statement of this lemma holds. Furthermore, assume that

1. $\Pr[A_i] \leq p$ for all i, and
2. $ep(d+1) \leq 1$.

Prove that $\Pr\left[\bigcap_i \overline{A_i}\right] > 0$, i.e., use Theorem 19.2 to prove the statement of Theorem 19.1.

Hint: Let $x(A_i) = \frac{1}{d+1}$ for all $1 \leq i \leq n$. Use the inequality $\frac{1}{e} \leq \left(1 - \frac{1}{d+1}\right)^d$ obtained by the limit formula of $1/e$ and the fact that it converges from the above.

Let $x(A_i) = \frac{1}{d+1}$ $\forall\ 1 \leq i \leq n$

By asymmetric LLL:

$$\Pr[A_i] \leq x_i \cdot \prod_{j:(i,j)\in E} (1-x_j)$$

現在 ∵ $x_i = x = \frac{1}{d+1}$ 且最多有 d 個 vertices 和 i 相鄰

$$\Rightarrow \prod_{j:(i,j)\in E} (1-x_j) \geq (1-x)^d$$

We want to proof:

$$\Pr[A_i] \leq x \cdot (1-x)^d \Leftarrow p \leq \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^d$$

- 已知 $ep(1+d) \leq 1 \equiv p \leq \frac{1}{e(1+d)}$
- ∵ $\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$，∴ finite "$d$" 亦同 $\Rightarrow \left(1 - \frac{1}{d+1}\right)^d \geq \frac{1}{e}$
- 則 $\left(1 - \frac{1}{d+1}\right)^d \cdot \frac{1}{d+1} \geq \frac{1}{d+1} \cdot \frac{1}{e}$ 固得證 #

Problem 5 (20%). Let X be a finite set and $A_1, A_2, \dots, A_m$ be a partition of X into mutually disjoint blocks. Given a subset $Y \subseteq X$, consider the partition $Y = B_1 \cup B_2 \cup \cdots \cup B_m$ with the blocks B_i defined as $B_i := A_i \cap Y$. For any $1 \le i \le m$, we say that the block B_i is λ-*large* if

$$\frac{|B_i|}{|A_i|} \ge \lambda \cdot \frac{|Y|}{|X|}.$$

Show that, for every $\lambda > 0$, at least $(1-\lambda) \cdot |Y|$ elements of Y belong to λ-large blocks.

- 設一個 Block B_i' 不是 λ-large

$$\frac{|B_i'|}{|A_i|} < \lambda \cdot \frac{|Y|}{|X|} \equiv |B_i'| < \lambda \cdot \frac{|Y|}{|X|} \cdot |A_i|$$

- 算"非 λ-large" 的 elements $\Rightarrow$ $I = \{ i \mid B_i \notin \lambda\text{-large} \}$

因此 $\sum_{i \in I} |B_i'| < \sum_{i \in I} \left(\lambda \frac{|Y|}{|X|} \cdot |A_i| \right) = \lambda \frac{|Y|}{|X|} \cdot \sum_{i \in I} |A_i|$

- ∵ A_i $(1 \le i \le m)$ 是 X 的 partition，∴ A_i mutual independent

$\Rightarrow \sum_{i=1}^{m} |A_i| = |X|$

- $\sum_{i \in I} |B_i'| < \lambda \cdot \frac{|Y|}{|X|} \sum_{i \in I} |A_i| \le \lambda \cdot \frac{|Y|}{|X|} \cdot |X| = \lambda \cdot |Y|$ (非 λ-large)

- 因此在 λ-large 內的 $|Y| - \lambda |Y| = (1-\lambda)|Y|$ #