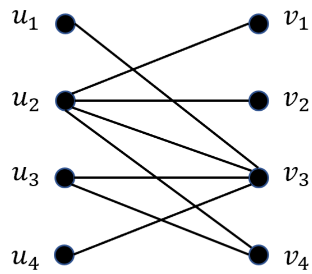
Problem 1 (20%)**.** Consider the following graph. Identify a maximum-size matching and a minimum-size vertex cover for it.



Problem 2 (20%)**.** Let G be a bipartite graph with partite sets A and B, and M, M' be two matchings. Suppose that, M matches the vertices in $S \subseteq A$ and M' matches the vertices in $T \subseteq B$. Prove that there is a matching that matches all the vertices in $S \cup T$.

Hint: Consider $M \cup M'$.

Problem 3 (20%)**.** Let G be a bipartite graph with partite sets X and Y. Prove that G has a matching of size t if and only if for all $A \subseteq X$,

$$|N(A)| \;\geq\; |A| + t - |X| \;=\; t - |X - A|.$$

Hint: Add $|X| - t$ new vertices to Y and connect these vertices to every vertex in X.

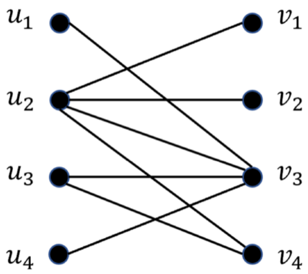
Problem 4 (20%)**.** Let G be a bipartite graph with partite sets X and Y. Define

$$\delta(G) := \max_{A \subseteq X} \left(|A| - |N(A)|\right),$$

i.e., $\delta(G)$ measures the worst violation of the Hall's matching condition. Note that, $\delta(G) \geq 0$ since $A = \emptyset$ is considered as a subset of X. Use the statement in Problem 3 to prove that, G has a maximum matching of size $|X| - \delta(G)$.

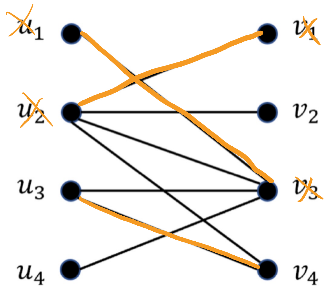
Problem 5 (20%)**.** Let G be a bipartite graph with partite sets X and Y. Assume the same notation $\delta(G)$ as Problem 4. Show that, the largest independent set of G has size $|Y| + \delta(G)$.

Problem 1 (20%). Consider the following graph. Identify a maximum-size matching and a minimum-size vertex cover for it.



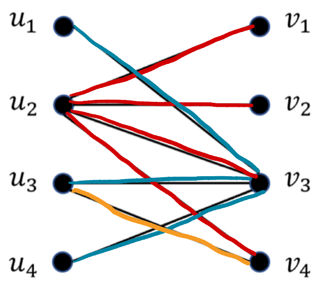
maximum-size matching :

$\{(u_1, v_3), (u_2, v_1), (u_3, v_4)\}$



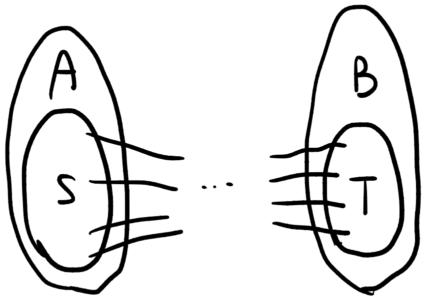
minimum-size vertex cover:

$\{u_2, v_3, u_3\}$



Problem 2 (20%). Let G be a bipartite graph with partite sets A and B, and M, M' be two matchings. Suppose that, M matches the vertices in $S \subseteq A$ and M' matches the vertices in $T \subseteq B$. Prove that there is a matching that matches all the vertices in $S \cup T$.

Hint: Consider $M \cup M'$.



$$M \cup M' = (M \cap M') \cup (M \triangle M')$$

1、Consider $M \cap M'$

① 由 $M \cap M'$ 不會包含全部的 vertices

② 因此令 $S' = S \cap V(M \cap M')$：在 S 裡也被 $M \cap M'$ matching 的点

$T' = T \cap V(M \cap M')$：在 T 裡也被 $M \cap M'$ matching 的点

③ $S' \cup T'$ 的點已經被 match（$\because$ 在 $M \cap M'$ 中）

而 $S \cup T \setminus S' \cup T'$ 的点要透過 $M \triangle M'$ 討論

2、consider $M \triangle M'$

① By #7. Lemma 2. "every component in $M \triangle M'$ is a either path or cycle with even length"

② 因此不管 cycle or path 都交錯的 M 和 M'，只要挑其中之一即可。
這樣即可包含 $S \cup T \setminus S' \cup T'$

3、Final answer：$(M \cap M') \cup \left(\bigcup_{\text{cycle}} \text{either } M \text{ or } M'\right) \cup \left(\bigcup_{\text{path}} \max(M, M')\right)$ #

Problem 3 (20%). Let G be a bipartite graph with partite sets X and Y. Prove that G has a matching of size t if and only if for all $A \subseteq X$,

$$|N(A)| \geq |A| + t - |X| = t - |X - A|.$$

Hall ' theorem

Hint: Add $|X| - t$ new vertices to Y and connect these vertices to every vertex in X.

G has a matching of size t $\Leftrightarrow$ $|N(A)| \geq |A| + t - |X| = t - |X - A|$

I. prove from "$\Rightarrow$" direction

1. Add $|X| - t$ new vertices to Y and connect these vertices to every vertex in X.

2、變成一個 $G' = (X \cup Y', E_G')$ where $E_G' = E_G \cup E_{dummy}$

3、if G 有大小是 t 的 matching，剩下 $|X| - t$ 個 vertices 沒有 match matching，將這 $|X| - t$ 和新加入的 dummy vertices 做 matching。變成 G' 中的 perfect matching 且大小為 $|X|$

4. By Hall's theorem $\Rightarrow$ $|N_{G'}(A)| \geq |A|$ —— ①

又 $\because$ dummy vertices 有 $|X| - t$ 個點，且與 X 相連

$\Rightarrow |N_{G'}(A)| = |N_G(A)| + (|X| - t)$

5、代回 ①，$|N_G(A)| + (|X| - t) \geq |A| \equiv |N_G(A)| \geq |A| + t - |X|$ #

II、prove from "$\Leftarrow$" direction

1. if for all $A \subseteq X$ 都有 $|N_G(A)| \geq |A| + t - |X|$

2、同上再做一個 G'

3、在 G' 中，$|N_{G'}(A)| = |N_G(A)| + |X| - t \geq |A| + t - |X| + |X| - t = |A|$

4. 滿足 Hall's theorem，因此 G' 中存在 matching 可以 match 所有 X

5. 而 G' 的 matching 中，有 $|X| - t$ 條 edge 是 dummy edge，則剩下 t 條邊來自 G

6、可推得 G 有個 size $= t$ 的 matching

Problem 4 (20%)**.** Let G be a bipartite graph with partite sets X and Y. Define

$$\delta(G) := \max_{A \subseteq X} (|A| - |N(A)|),$$

i.e., $\delta(G)$ measures the worst violation of the Hall's matching condition. Note that, $\delta(G) \geq 0$ since $A = \emptyset$ is considered as a subset of X. Use the statement in Problem 3 to prove that, G has a maximum matching of size $|X| - \delta(G)$.

$$\boxed{G \text{ has a matching of size } t \iff |N(A)| \geq |A| + t - |X| = t - |X-A| \quad \text{— ①}}$$

1. By $\delta(G) := \max_{A \subseteq X} (|A| - |N(A)|)$, then $\delta(G) \geq |A| - |N(A)|$

$\Rightarrow |N(A)| \geq |A| - \delta(G)$

2. By ①, $|N(A)| \geq |A| - \delta(G) = |A| + t - |X|$

3. 又因為 $|A| - \delta(G) = |A| + t - |X|$

因此 $\delta(G) = |X| - t \Rightarrow t = |X| - \delta(G)$ #

Problem 5 (20%)**.** Let G be a bipartite graph with partite sets X and Y. Assume the same notation $\delta(G)$ as Problem 4. Show that, the largest independent set of G has size $|Y|+\delta(G)$.

1. By König's theorem, for bipartite graph G,

最大 matching 的大小 $\mu(G)$ + 最大 independent set 的大小 $\alpha(G)$

$= |V|$

$\Rightarrow \alpha(G) = |X| + |Y| - \mu(G)$ — ①

2. By problem 4., $\mu(G) = |X| - \delta(G)$ — ②

3. 將 ② 代入 ①，則 $\alpha(G) = |X| + |Y| - (|X| - \delta(G))$

$= |Y| + \delta(G)$ #