Assignment 4

學號: 111550129 姓名: 林彥亨

1. (20pts) The following ordinary difference table is for $f(x) = x + \frac{sin(x)}{3}$. Use it to find

   (a) $f'(0.72)$ from a cubic polynomial.

   (b) $f'(1.33)$ from a quadratic.

   (c) $f'(0.50)$ from a fourth-degree polynomial.

   In each part, choose the best starting i-value.

i	x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	0.30	0.3985	0.2613	-0.0064	-0.0022	0.0003
1	0.50	0.6598	0.2549	-0.0086	-0.0018	0.0004
2	0.70	0.9147	0.2464	-0.0104	-0.0014	0.0005
3	0.90	1.1611	0.2360	-0.0118	-0.0010	
4	1.10	1.3971	0.2241	-0.0128		
5	1.30	1.6212	0.2113			
6	1.50	1.8325				

(a)

$f'(0.72) \Rightarrow$ cubic polynomial $\quad x = 0.72$

$x - x_1 = 0.22, \quad x - x_2 = 0.02, \quad x - x_3 = -0.18$

$$\frac{0.2549}{(1! \cdot (0.2))} - \frac{0.0086(0.22+0.02)}{(2! \cdot (0.2)^2)} - \frac{0.0018(0.22 \cdot 0.02 - 0.22 \cdot 0.18 - 0.02 \cdot 0.18)}{(3! \cdot (0.2)^3)}$$

$= 1.2745 - 0.0258 + 0.0375 \cdot 0.0388 \approx 1.25$ #

(b)

$f'(1.33) \Rightarrow$ quadratic $\quad 0.2113 \quad -0.0128$

$$\frac{0.2113}{0.2} + \frac{2(1.33) - (2(1.3) + 0.2)}{2(0.2)^2} \cdot (-0.0128)$$

$$= 1.0565 + \frac{\overset{-0.14}{\cancel{2.66 - (2.8)}}}{0.08} \times (-0.0128)$$

$= 1.0565 + 0.0224 \approx 1.0789 \approx 1.08$ #

(c)

$x = 0.5, \; \Delta f_1 = 0.2549 \quad \Delta^2 f_1 = -0.0086 \quad \Delta^3 f_1 = -0.0018 \quad \Delta^4 f_1 = 0.0004$

$$f'(x) \approx \frac{\Delta f_i}{h} + \frac{2x - (2x_i + h)}{2h^2} \Delta^2 f_i + \frac{3x^2 - 6xx_i + 3x^2 + 6xh - h^2}{6h^3} \Delta^3 f_i$$

$$+ \frac{4x^3 - 12x^3 x_i + 12x^2 x_i^2 - 4x_i^3 + 12x^2 h - 6xh^2 + h^3}{24h^4} \Delta^4 f_i$$

$$f'(0.5) \approx \frac{0.2549}{0.2} + \frac{2(0.5) - (2(0.5)+0.2)}{2(0.2)^2}(-0.0086) +$$

$$\frac{3(0.5)^2 - 6(0.5)(0.5) + 3(0.5)^2 + 6(0.5)(0.2) - (0.2)^2}{6(0.2)^3}(-0.0018)$$

$$+ \frac{4(0.5)^3 - 12(0.5)^2(0.5) + 12(0.5)(0.5)^2 - 4(0.5)^3 + 12(0.5)^2(0.2) - 6(0.5)(0.2)^2 + (0.2)^3}{24(0.2)^4} \cdot (0.0004)$$

$$\approx 1.29 \ \#$$

2. (20pts) Use the method of undetermined coefficients to obtain the formulas for $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$ at x_0 using five evenly spaced points from x_2 to x_{-2}, together with their error terms.

$$f''(x) = C_{-2}f_{-2} + C_{-1}f_{-1} + C_0 f_0 + C_1 f_1 + C_2 f_2$$

$$p(u) = 1, u, u^2, u^3, u^4 \qquad p''(u) = 0, 0, 2, 6u, 12u^2$$

$$0 \quad 1 \quad 2u \quad 3u^2 \quad 4u^3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & -h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^2 \\ 16h^4 & -h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$f''(x_0) = \frac{(-f_{-2} + 16f_{-1} - 30f_0 + 16f_1 - f_2)}{12h^2} \ \#$$

$$f'''(x_0) = \frac{(-f_{-2} + 2f_{-1} - 2f_1 + f_2)}{2h^3} \ \#$$

$$f^4(x_0) = \frac{f_{-2} - 4f_{-1} + 6f_0 - 4f_1 + f_2}{h^4} \ \#$$

3. (20pts) Simpson's $\frac{1}{3}$ rule, although based on passing a quadratic through three evenly spaced points, actually gives the exact answer if $f(x)$ is a cubic. The implication is that the area under any cubic between $x = a$ and $x = b$ is identical to the area of a parabola that matches the cubic at $x = a$, $x = b$, and $x = \dfrac{a+b}{2}$. Prove this.

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \qquad h = \frac{b-a}{2}$$

$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$f(x) = px^3 + qx^2 + rx + s$$

$$\int_a^b (px^3 + qx^2 + rx + s)\,dx$$

$$= \left[\frac{p}{4}x^4 + \frac{q}{3}x^3 + \frac{r}{2}x^2 + sx + C\right]_a^b$$

$$= \left(\frac{p}{4}b^4 + \frac{q}{3}b^3 + \frac{r}{2}b^2 + sb + \cancel{C}\right) - \left(\frac{p}{4}a^4 + \frac{q}{3}a^3 + \frac{r}{2}a^2 + sa + \cancel{C}\right)$$

$$f(a) = pa^3 + qa^2 + ra + s$$

$$f(b) = pb^3 + qb^2 + rb + s$$

$$f\left(\frac{a+b}{2}\right) = p\left(\frac{a+b}{2}\right)^3 + q\left(\frac{a+b}{2}\right)^2 + r\left(\frac{a+b}{2}\right) + s$$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\Bigg[\left(pa^3 + qa^2 + ra + s\right) + 4\cdot\left(p\left(\frac{a+b}{2}\right)^3 + q\left(\frac{a+b}{2}\right)^2 + r\left(\frac{a+b}{2}\right) + s\right) + \left(pb^3 + qb^2 + rb + s\right)\Bigg]$$

the weefficient of corresponding terms equate.

4. (20pts) Compute the integral of $f(x) = \frac{sin(x)}{x}$ between $x = 0$ and $x = 1$ using Simpson's $\frac{1}{3}$ rule with $h = 0.5$ and then with $h = 0.25$. (Remember that the limit of $\frac{sin(x)}{x}$ at $x = 0$ is 1.) From these two results, extrapolate to get a better result. What is the order of the error after the extrapolation? Compare your answer with the true answer.

$$\int_0^1 f(x)\,dx \approx \frac{h}{3}\left[f(0) + 4f(0.5) + f(1)\right]$$

$h = 0.5$, $f(x) = \frac{\sin(x)}{x}$

$f(0) = \frac{\sin(0)}{0} \Rightarrow \phi \quad \because \lim_{x\to 0} \frac{\sin x}{x} = 1$. we take $f(0) = 1$

$f(0.5) = \frac{\sin(\frac{1}{2})}{\frac{1}{2}} \approx 2 \times 0.4794 = 0.9588$

$f(1) = \frac{\sin(1)}{1} \approx 0.8415$

$\Rightarrow \int_0^1 f(x)\,dx \approx \frac{1}{6}\left[1 + 4\sin(\frac{1}{2}) + \sin(1)\right] \approx \frac{1}{6} \times 5.6769$

$h = 0.25$ $\qquad \approx 0.946116$

$f(0) = 1$

$f(0.25) = \frac{\sin(0.25)}{0.25} \approx 4 \times 0.2474 = 0.9896$

$f(0.5) = \frac{\sin(0.5)}{0.5} \approx 2 \times 0.4794 = 0.9588$

$f(0.75) = \frac{\sin(0.75)}{0.75} \approx \frac{4}{3} \times 0.6816 = 0.9088$

$f(1) = \frac{\sin(1)}{1} \approx 0.8415$

$$\Rightarrow \int_0^1 f(x)\,dx \approx \frac{1}{12}\left[1 + 4\cdot\frac{\sin(0.25)}{0.25} + 2\cdot\frac{\sin(0.5)}{0.25} + 4\,\frac{\sin(0.75)}{0.25} + \sin(1)\right]$$

≈ 0.9460583

the approximate of $\int_0^1 \frac{\sin(x)}{x}\,dx = 0.94608307$

5. (20pts) Evaluate the following integral, and compare your answers to the analytical solution. Use $h = 0.1$ in both directions in parts (a) and (b),

   (a) Using the trapezoidal rule in both directions.

   (b) Using Simpson's $\frac{1}{3}$ rule in both directions.

   (c) Using Gaussian quadrature, three-term formulas in both directions.

$$\int_{-0.2}^{1.4} \int_{0.4}^{2.6} e^x sin(2y)\, dydx$$

```
PS C:\Users\user\Desktop\Assignment4> & C:/ProgramData/anaconda3/python.exe c:/Users/user/Desktop/Assignment4/q5_ab.py
Trapezoidal integral:  0.3683399550766346
Simpson's integral:  0.3692685194703221
```

The result of quesiton 5 part a and b.

```
>> q5_c
Estimated integral using Gaussian quadrature:
 0.37238
```

The result of question 5 part c.

6. (20pts) Please use Monte Carlo Integration to compute the double integral of $f(x, y) = (x-1)^2 + \frac{y^2}{16}$ where $R = [-2, 3] \times [-1, 2]$.

$$\iint_R f(x, y)\, dy\, dx$$

```
>> q6
Estimated integral using Monte Carlo integration: 35.959313
```

The result of the question 6.