Extrema of two-type reducible branching Brownian motion

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The model

In a two-type reducible branching Brownian motion (BBM):

- Type 1 particles move as Brownian motion with diffusion coefficient σ^2. They split at rate β into two children of type 1; and give brith to type 2 particles at rate α.
- Type 2 particles move as standard Brownian motion and branch at rate 1 into two type 2 children, but can NOT produce children of type 1.

Initially we have a type 1 particle starting from the origin.

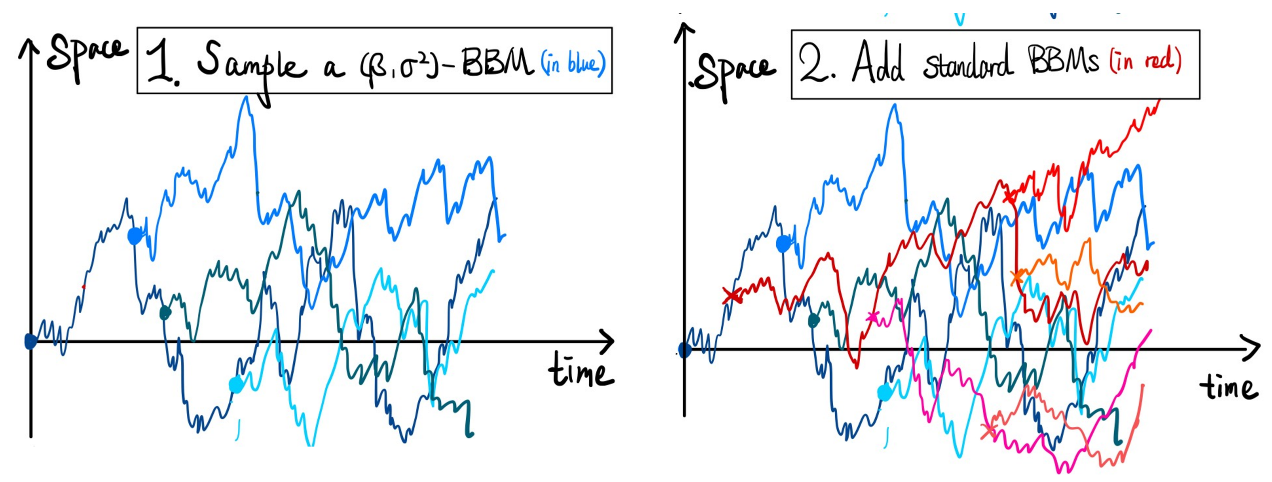


Figure 1. How to sample a two-type reducible BBM

Questions

Denote the process by $\{X_i(t)\}_{1\leq i\leq n(t)}$, where $X_i(t)$ is the position of the i-th particle at time t. Let M_t be its maximum at time t, i.e.,

$$M_t := \max_{1\leq i\leq n(t)} X_i(t)$$

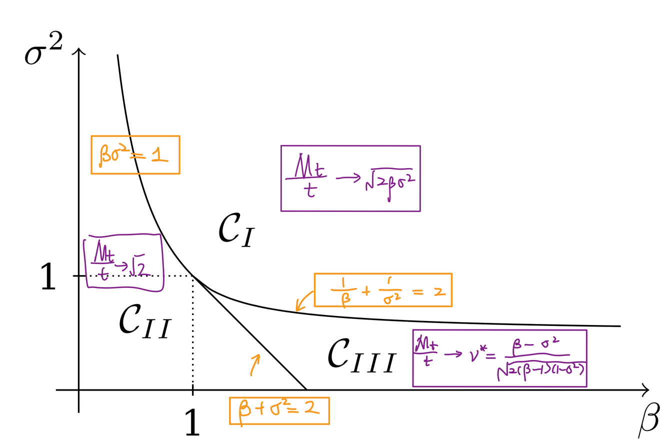
Motivated by the study of extreme values of BBM, branching random walk (BRW), and 2-dim discrete Gaussian free field, we are interested in the following questions.

- **Asymptotic behavior of M_t.** One should except that there are constants C_1, C_2 depending on β, σ^2 such that
$$M_t = C_1 t - C_2 \log t + O_{\mathbb{P}}(1).$$
- **Asymptotic behavior of extremal particles.** One should excepe that for $m(t) = C_1 t + C_2 \log t$, the limit of the *extremal process*
$$\sum_{i=1}^{n(t)} \delta_{X_i(t)-m(t)}$$
converges in law to certain decorated poisson point process (DPPP).

Leading order of the maximum

Biggins'12 [3] obtained the spreading speed $\lim_{t\to\infty} \frac{M_t}{t}$ (in a more general setting.)

- If $(\beta,\sigma^2) \in \mathscr{C}_I$ (resp. $\mathscr{C}_I$), type 1 (resp. type 2) particles are dominating: $M_t/t \to \sqrt{2\beta\sigma^2}$ (resp. $\sqrt{2}$) = speed of BBM with single type 1 (resp. type 2).
- If$(\beta,\sigma^2) \in \mathscr{C}_{III}$,
$$M_t/t \to v^* = \frac{\beta-\sigma^2}{\sqrt{2(1-\sigma^2)(\beta-1)}} > \max\{\sqrt{2\beta\sigma^2}, \sqrt{2}\}.$$
This was called **anomalous spreading**, as the speed of the two-type process is strictly larger than the speed of both single type particle systems.



Subleading order of the maximum: Doule jump

Belloum-Mallein'21 [2] investigated the subleading order of the maximum M_t and the limiting extremal processes, when the parameter (β,σ^2) are interior points of regions $\mathscr{C}_I, \mathscr{C}_{II}, \mathscr{C}_{III}$. **Belloum'22+** [1] considered a special critical case $\beta = \sigma^2 = 1$. **M.-Ren'23+** [4] investigated the case that (β,σ^2) lies on the *boundries* between $\mathscr{C}_I, \mathscr{C}_{II}, \mathscr{C}_{III}$. Set $\mathscr{B}_{I,II} = \partial\mathscr{C}_I \cap \partial\mathscr{C}_{II} \backslash \{(1,1)\}$.

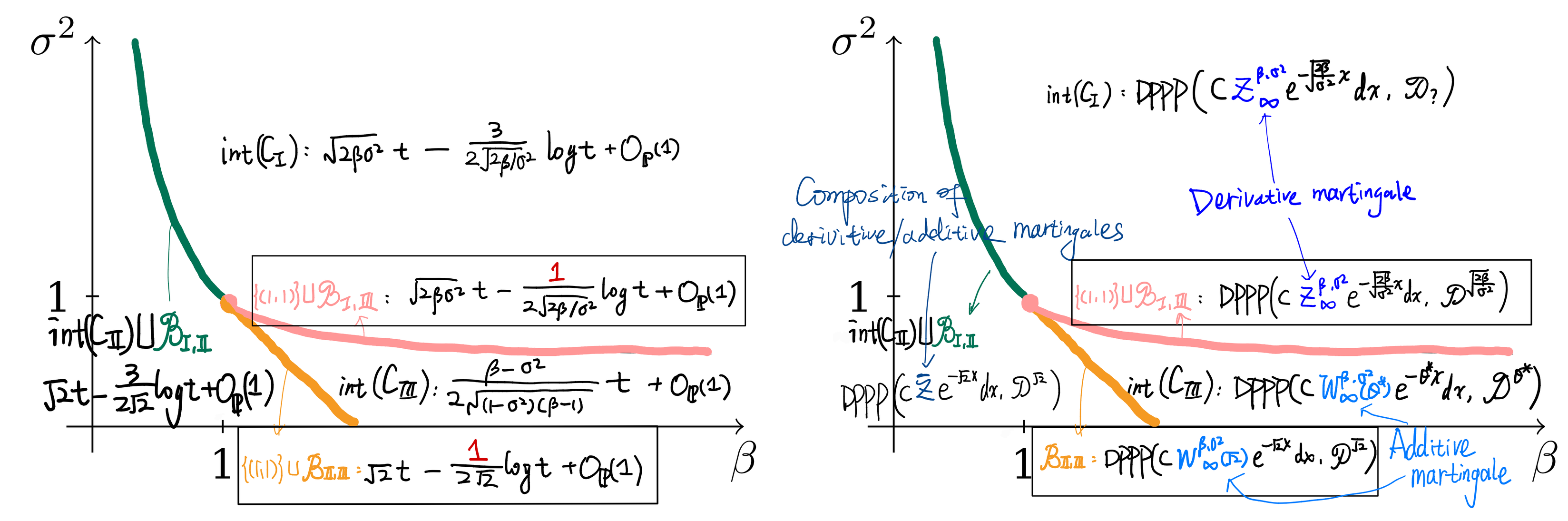


Figure 2. Phase diagram of two-type BBM. Left hand side is the order of maximum M_t. Right hand side is the limitting extremal process. DPPP stands for decorated Poisson point process. The decorations $\mathcal{D}^\rho, \rho \geq \sqrt{2}$ are the limit of $\sum_{i=1}^{n(t)} \delta_{Y_i(t)-M_t^Y}$ conditioned on $M_t > \rho t$, where Y is standard BBM. Decoration $\mathcal{D}_?$ was obtained implicitly.

Note that a **double jump** occurs when parameters (β,σ^2) cross the boundary of the anomalous spreading region $\mathscr{C}_{III}$, and only a single jump occurs when (β,σ^2) cross $\mathscr{B}_{I,II}$. See the following figures for a visual description.

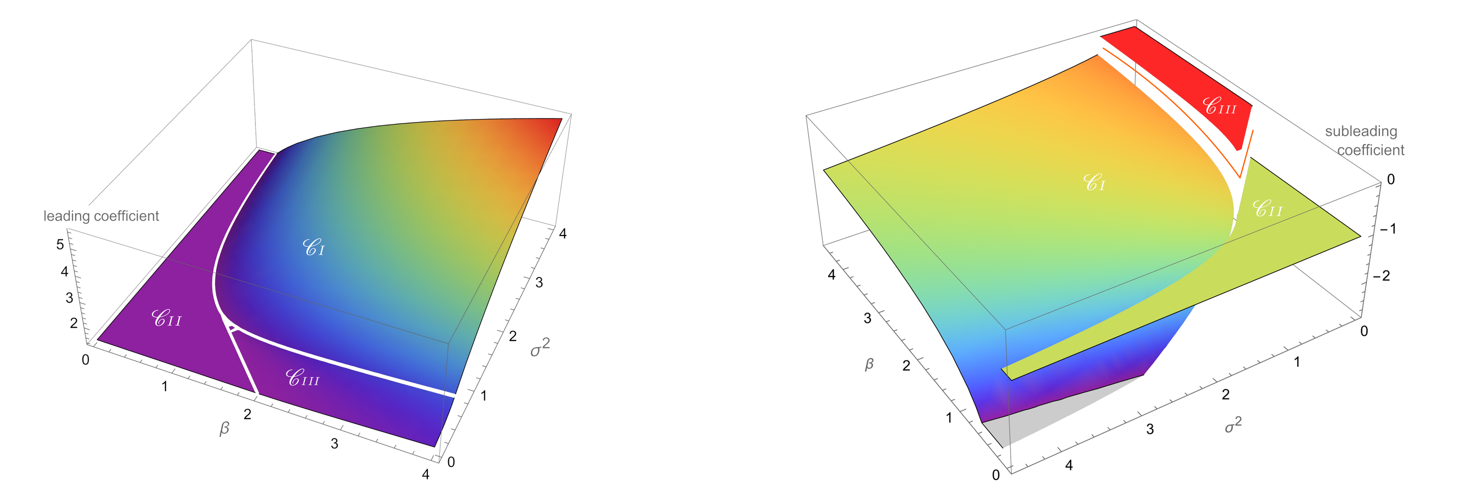
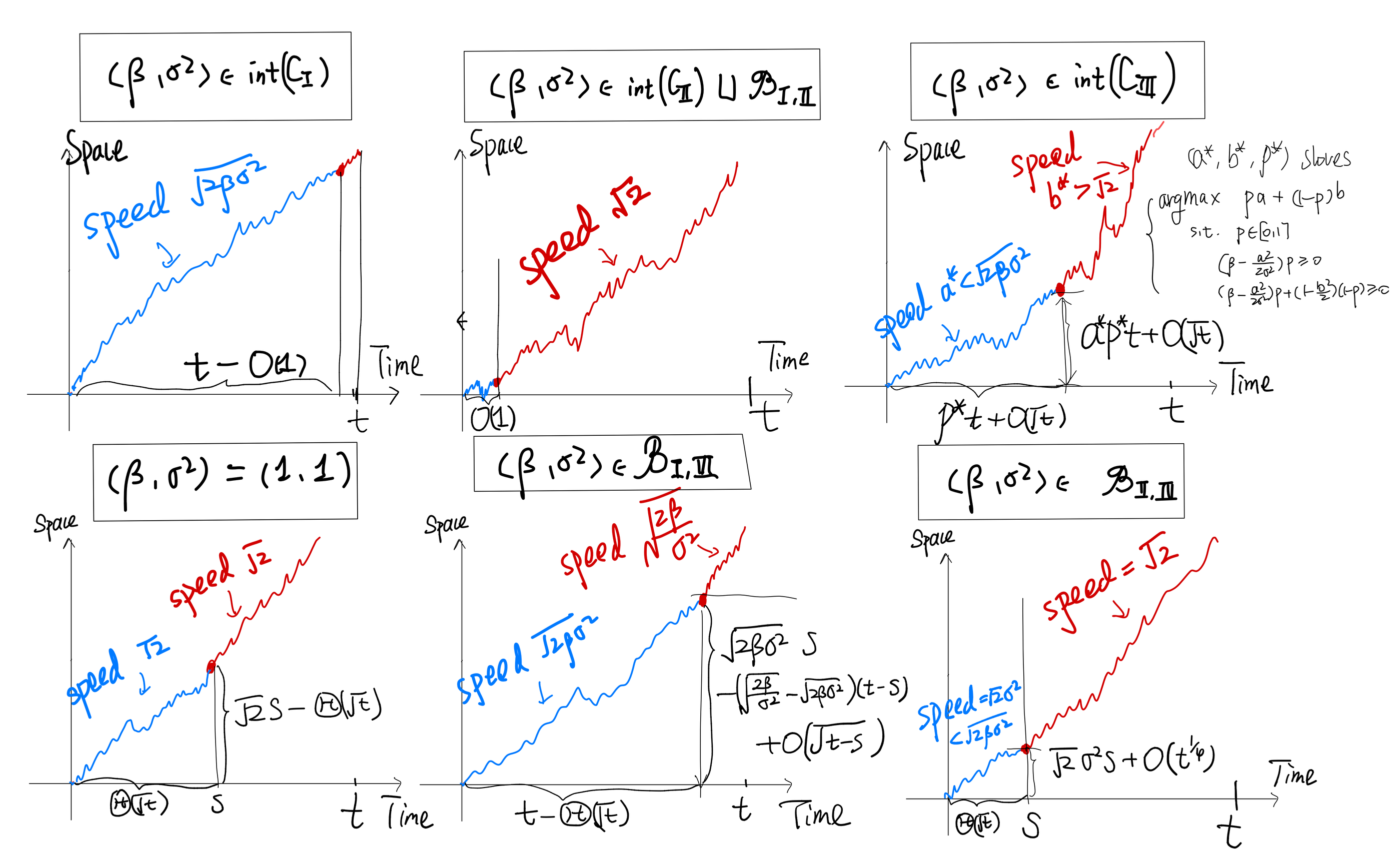


Figure 3. Images for coefficient of term t(as a function of (β,σ^2)) and coefficient of term $\log t$ in the asymptotic fomula of M_t. The first funcation is continuous, but the second one has jumps.

Localization of extremal particles

One of the key observation that will be needed in the proof is a localization result on the position of the ancestors of extremal particles.



Intermediate phases: From 0 to 3

To get a continuous phase transation, in the following we shall modify the model. We assume that parameters (β,σ^2) **depends on the** ***time horizon*** t and are close to the boundaries $\mathscr{B}_{I,III}$, $\mathscr{B}_{II,III}$. The corresponding distribution of the two-type process is denote by $\mathbb{P}_{(\beta_t,\sigma_t^2)}$. We set

$$\frac{1}{\beta_t} + \frac{1}{\sigma_t^2} = 2 \pm \frac{1}{t^h} \quad (\beta_t,\sigma_t^2) \to (\beta,\sigma^2) \in \mathscr{B}_{I,III} \text{ or} \tag{H1}$$

$$\beta_t + \sigma_t^2 = 2 \pm \frac{1}{t^h} \quad (\beta_t,\sigma_t^2) \to (\beta,\sigma^2) \in \mathscr{B}_{II,III} \tag{H2}$$

Theorem[M.-Ren, in preparation]: Let $v_t^* = \frac{\beta_t-\sigma_t^2}{2\sqrt{(1-\sigma_t^2)(\beta_t-1)}}$. Define

$$m_{h,+}^{1,3}(t) = v_t^* t - \frac{\min\{h,1/2\}}{\sqrt{2\beta/\sigma^2}} \log t \,;\; m_{h,-}^{1,3}(t) = \sqrt{2\beta_t\sigma_t^2}\, t - \frac{3-4\min\{h,1/2\}}{2\sqrt{2\beta/\sigma^2}} \log t$$

$$m_{h,+}^{2,3}(t) = v_t^* t - \frac{\min\{h,1/2\}}{\sqrt{2}} \log t \,;\; m_{h,-}^{2,3}(t) = \sqrt{2}t - \frac{3-4\min\{h,1/2\}}{2\sqrt{2}} \log t.$$

Under assumption (Hi), $\{M_t - m_{h,\pm}^{i,3}(t), \mathbb{P}_{(\beta_t,\sigma_t^2)}\}$, $i = 1,2$ converges in law. The limiting distribution is the same (up to a shift depends on h) as the limiting distribution of centered M_t under $\mathbb{P}_{(\beta,\sigma^2)}$. Same result holds for the extremal processes.

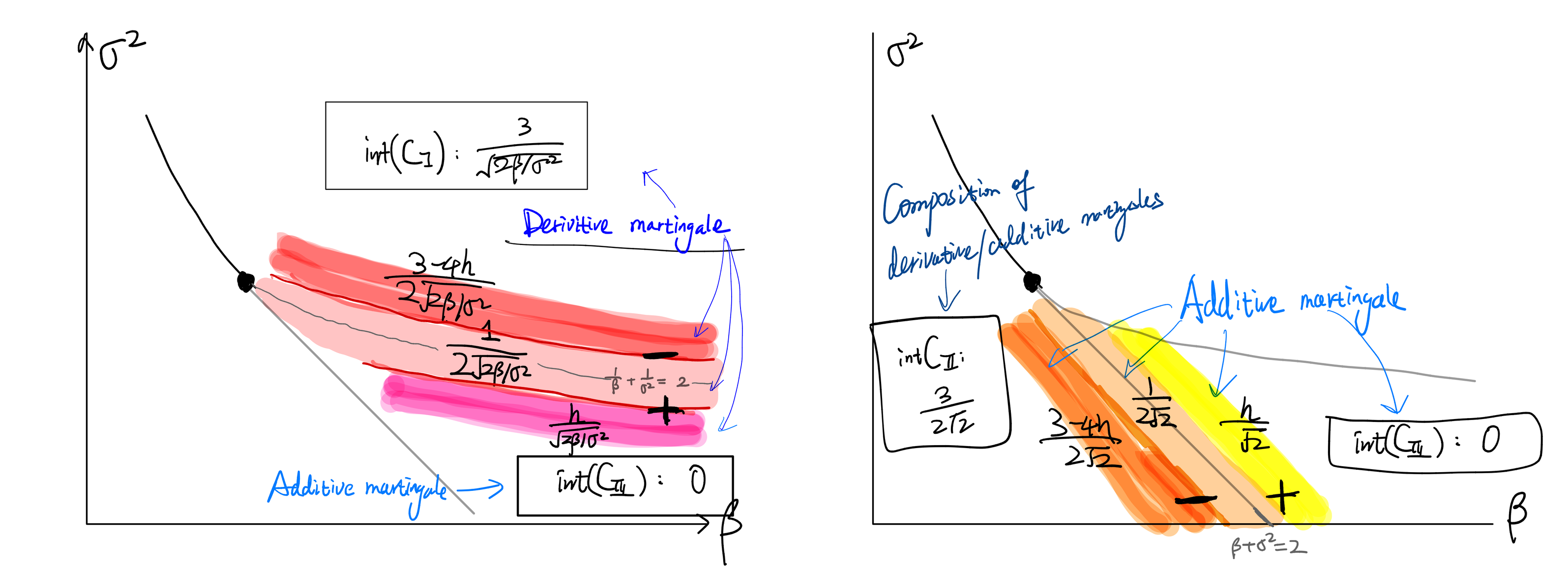


Figure 4. Phase diagram of two-type BBM. Here we explain the left one. In the inner phase $(h \geq \frac{1}{2})$, everything is as in the boundary case. In the northwest regime, the correct centering of the maximum and the limiting distribution is the same as a single-type 1 BBM. In the regime in between $(0 < h < \frac{1}{2})$, the order of the maximum **interpolates smoothly** between the surrounding regimes. In the southeast anomalous spreading regime, the correct centering of the maximum has no log term. The limiting distribution is similar to the one for BBM but the martingale appearing is different. In the regime, with $\beta_t^{-1} + \sigma_t^{-2} = 2 + t^{-h}, 0 < h < \frac{1}{2}$, the order of the maximum **interpolates smoothly** between the anomalous spreading regime and the boundary case. Notice that in the three middle regimes the limiting distribution coincides up to constant shift and the martingale is always the derivative martingale.

Remark: Similar results are ontained for the case $\frac{1}{\beta_t}+\frac{1}{\sigma_t^2} = \beta_t+\sigma_t^2 = 2+\frac{1}{t^h}$, the case $\beta_t = \sigma_t^2 = 1-\frac{1}{t^h}$ and the case $\frac{1}{\beta_t} = \frac{1}{\sigma_t^2} = 1 - \frac{1}{t^h}$. However in these three case, $\min\{h,\frac{1}{2}\}$ becomes $\min\{h,1\}$ and $\min\{3-4h,\frac{1}{2}\}$ becomes $\min\{3-2h,1\}$, so the critical window is of order $1/t$.

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