

## RESEARCH STATEMENT OF HENG MA

My research focuses on probability theory, with a particular interest in probabilistic models that incorporate tree and graph structures. These models provide powerful tools for analyzing and understanding both the structure and evolution of many real-world systems, with applications in various fields such as statistical mechanics, computer science, biology, and social science. Let me begin with a quick overview of my doctoral research and future plans, followed by more detailed discussions in the next two sections. My work falls into two main subjects:

- (a) **Random trees and graphs:** Random graphs serve as foundational models for understanding real-world network properties such as connectivity, degree distribution, and diameter. However, in practice, we often have access only to local observations, making the question of whether global structures can be reconstructed from these local information highly significant. We addressed this question in [22] for a central model of random graphs, the sparse Erdős–Rényi graph, and provided algorithms for the recovery process. I plan to continue this investigation to dense Erdős–Rényi graphs. Preferential attachment trees are essential models in sorting algorithms and network science. I am interested in investigating the deviation probabilities of their height profile—the count of nodes at a given distance from the root—and the maximal displacement within their spatial embeddings. Additionally I aim to study the the number of non-leaf vertices in a growing network with a redirection-based attachment rule that uses a redirection-based attachment rule relying only on local structural information.
- (b) **Branching random walks:** Branching processes model diverse scenarios, from the extinction of family names to nuclear fission. A spatial embedding, the branching random walk (BRW), better captures natural phenomena like viral spread in tissues or paths in polymeric networks. Mathematically, BRWs share properties with models in the log-correlated fields universality class. My work in this area involves understanding rare events, such as when an unusually large number of particles reach a high level [20], and when certain normalized partition functions grow unexpectedly large [19]. We also conduct a finer analysis on the fractal dimensions of BRWs in non-Euclidean spaces [42], and explore how the extreme values of the process are affected when branching rates and movements of particles depend on their types [46, 47]. I aim to study the front of particles within given directions in  $\mathbb{R}^d$ .

### 1. RANDOM TREES AND GRAPHS

**1.1. Recovery threshold of sparse Erdős–Rényi graphs.** The shotgun assembly problems aim for recovering a global structure from local observations and has broad applications, including DNA sequencing [2] and neural network recovery [40]. A mathematical framework is as follows:

Consider a graph  $\mathcal{G}$ —either fixed or random, possibly with random labeling of the vertices. For  $r \geq 1$ , we observe the (rooted) $r$ -neighborhood  $N_r(v)$  of each vertex  $v$ , defined as the subgraph induced by all vertices within a distance  $r$  from  $v$ . Crucially, vertex names are invisible except for  $v$  itself (see Figure 1). Can we recover  $\mathcal{G}$  from the collection  $\{N_r(v) : v \in \mathcal{G}\}$ ? As the amount of information increases with the depth  $r$ , the goal is to determine the recovery threshold  $r^*$ .

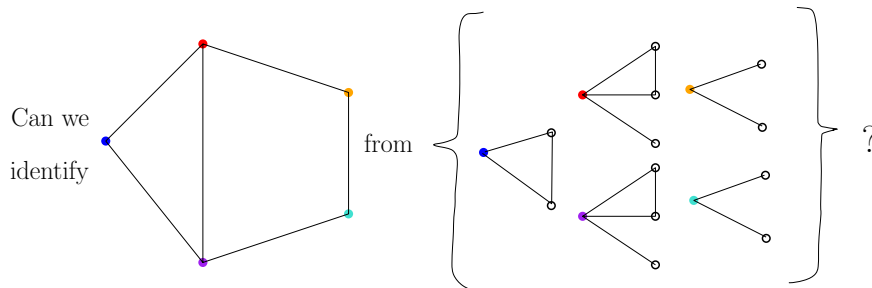


FIGURE 1. An Illustration for the shotgun assembly problem with  $r = 1$ .

A natural first step is to explore this threshold in the well-studied Erdős–Rényi graphs. An Erdős–Rényi graph  $\mathcal{G}(n, p_n)$  on  $n$  vertices is generated by independently connecting each pair of vertices with probability  $p_n$ . For graphs with polynomially growing average degree (dense regime), [28, 32, 38, 49] determined whether recovery is possible from neighborhoods of depths  $1, 2, 3, \dots$ . In contrast, for graphs in sparse regime where the average degree remains constant, [49] showed that the shotgun threshold is of order  $\log n$ , except in the critical case (average degree equal to one), where only a polynomial upper bound was obtained.

Together with Jian Ding and Yiyang Jiang [22], we determined its sharp asymptotics of the shotgun threshold for the sparse Erdős–Rényi graphs, establishing a connection to the isomorphic probability for Poisson–Galton–Watson trees.

**Theorem.** Fix  $\lambda > 0$ . Let  $\mathcal{G}_n \sim \mathcal{G}(n, \lambda/n)$ . Define  $\gamma_\lambda$  as the probability that two independent Galton–Watson trees with offspring distribution  $\text{Poisson}(\lambda)$  are rooted isomorphic.<sup>1</sup> Let  $C_\lambda := 1/\log(\frac{1}{\lambda^2\gamma_\lambda})$ . Then for any fixed  $\epsilon_0 > 0$ , the following assertions hold:

- (1) For  $r \leq (1 - \epsilon_0)C_\lambda \log n$ , with high probability, the shotgun problem is non-identifiable: there exists a non-isomorphic graph to  $\mathcal{G}_n$  that shares the same empirical neighborhood profile.
- (2) For  $r \geq (1 + \epsilon_0)C_\lambda \log n$ , with high probability,  $\mathcal{G}_n$  can be recovered using a polynomial-time algorithm.

**Future plans:** For a dense Erdős–Rényi graph  $\mathcal{G}_n \sim \mathcal{G}(n, n^{-\alpha})$  with  $\alpha \in (0, 1)$ , it has been shown in [28, 32] that given depth-1 neighborhoods,  $\mathcal{G}_n$  is identifiable for  $0 < \alpha < 1/2$  but non-identifiable for  $\alpha > 1/2$ . However, no efficient recovery algorithm has been founded for  $1/3 \leq \alpha < 1/2$ . I am interested in studying whether computation-information gap exists for  $\alpha \in [1/3, 1/2)$ . Additionally, [28, 38] showed that given depth-2 neighborhoods,  $\mathcal{G}_n$  is identifiable for  $\alpha < 2/3 + \delta_0$  with some small constant  $\delta_0 > 0$ , but non-identifiable for  $\alpha > 3/4$ . I aim to address this gap. Further, I am also interested in studying the shotgun assembly problem in other graph models, such as random geometric graphs and random regular graphs.

### 1.2. Future Project: Profile and spatial embedding of linear preferential attachment trees.

Preferential attachment trees are foundational models in both data structures and complex networks. In such trees, each new vertex attaches to an existing vertex chosen at random, with a probability proportional to  $\beta$  times the vertex’s out-degree plus one. Notable cases include the uniform recursive tree ( $\beta = 0$ ), the binary search tree ( $\beta = -1/2$ ) and the plane oriented recursive tree ( $\beta = 1$ ). An essential way to characterize the shape of a tree is through its height profile  $(L_n(k))_{k \geq 0}$ , where  $L_n(k)$  represents the number of nodes at distance  $k$  from the root. Previous studies [18, 25, 52] have shown that  $L_n(\lfloor \lambda \log n \rfloor)$ , normalized by its mean, converges to certain random variables over specific ranges of  $\lambda$ ; the exact rate of convergence for rational  $\beta$  was established in [39]. My first goal is to extend this result to general values of  $\beta$  by analyzing the associated Crump–Mode–Jagers branching process. Next, I aim to study the large deviation principle for the normalized profile,  $\ln L_n(\lfloor \lambda \log n \rfloor) / \ln n$ , drawing on insights from a related problem in branching Brownian motion (see §2.1). Additionally, I am interested in examining the profile of random split trees [21, 35], which generalize linear preferential attachment trees and other important tree models.

Beyond this, I intend to study spatial embeddings of these trees, where each new vertex is positioned in  $\mathbb{R}^d$  according to i.i.d. random vectors relative to its parent’s location. This model can be interpreted as a random walk indexed by preferential attachment trees, following [8], and as a Pólya urn scheme with infinitely many colors, as studied in [5, 13, 36]. The latter work addresses the scaling limit of the empirical measure for spatial trees in the uniform attachment case ( $\beta = 0$ ). I believe similar results hold for general values of  $\beta$ . Moreover, motivated by research on the maximum of branching random walks, I am particularly interested in studying the maximal displacement of the spatial tree and comparing it with that of BRWs.

**1.3. Future project: Non-leaf vertices in random friend trees.** Saramäki and Kaski [51] proposed a redirection-based network growth model that, unlike the global-information-dependent preferential attachment rule above, relies only on local information to produce skewed degree distributions. In

<sup>1</sup>That is, there exists a bijection between their vertices that preserves the edges and the root.

this model, each new vertex simply selects a starting vertex uniformly at random and connects to the endpoint of a random walk of length  $\ell$ . When  $\ell = 1$ , this process produces the random friend tree (RFT) [3]. Cannings and Jordan [17] showed that, the RFT on  $n$  vertices contains  $n - o(n)$  leaves almost surely. A conjecture in [41] suggests that the number of non-leaf vertices,  $X_n^{\geq 2}$ , grows as  $n^\mu$  with  $\mu \approx 0.566$ . This was partially verified in [3], which showed that  $X_n^{\geq 2}$  lies between  $n^c$  and  $n^C$  with high probability for constants  $0 < c < C < 1$ . Together with Shankar Bhamidi, we are interested in further studying the asymptotics of the number of non-leaf vertices.

## 2. BRANCHING RANDOM WALKS AND THEIR VARIANTS

A branching random walk (BRW) begins with one particle located at some vertex in the state space  $G$ . At every time  $n \geq 1$ , each particle in generation  $n - 1$  dies and gives birth to an independent random number of offspring, following a common distribution. Each offspring then independently takes a step relative to its parent, according to a random walk (RW) on  $G$ , which is referred to as the underlying RW of the BRW.

A branching Brownian motion (BBM) is a continuous-time version of the BRW on  $\mathbb{R}$ , where particles perform Brownian motion and undergo dyadic branching at rate one.

**2.1. BBM conditioned on large level sets.** A fundamental aspect of understanding how particles in a BBM spread through space is to examine the sizes of level sets. For each  $y \geq 0$ , the  $y$ -**level set** at time  $t$  consists of all particles alive at time  $t$  that are positioned above level  $y$ . Let  $\mathcal{L}_t(y)$  denote the size of the  $y$ -level set. The typical behavior of these level sets is well-established: Results of Bramson [15] implies that almost surely, at sufficiently large time  $t$ , the  $\sqrt{2}t$ -level set is empty. While Biggins [10] and Glenz, Kistler and Schmidt [29] proved that for any  $x \in [0, \sqrt{2})$ ,  $\mathcal{L}_t(xt)$ , normalized by its mean  $\mathbb{E}[\mathcal{L}_t(xt)] \asymp \frac{1}{\sqrt{t}} e^{(1-\frac{x^2}{2})t}$ , converges a.s. to a positive limit  $W_\infty(x)$  (which is defined in §2.2).

With this law of large numbers established, it is natural to investigate the large deviation probabilities. Aïdékon, Hu, and Shi [24] studied the exponential decay rate of the probability that the level sets are unusually large. Specifically, for  $x > 0$  and  $a \in ((1-x^2/2)_+, 1)$ , they showed that  $\mathbf{P}(\mathcal{L}_t(xt) \geq e^{at}) = \exp\{-I(x, a)t + o(t)\}$  with rate function  $I(x, a) = \frac{x^2}{2(1-a)} - 1$ . Building on this result, two natural questions arise:

- (i) What is the exact order of  $\mathbf{P}(\mathcal{L}_t(xt) \geq e^{at})$ ?
- (ii) What is the typical behavior of the BBM under the conditional law  $\mathbf{P}(\cdot \mid \mathcal{L}_t(xt) \geq e^{at})$ ?

Together with Xinxin Chen [20], we addressed Question (i) by showing that  $\mathbf{P}(\mathcal{L}_t(xt) \geq ye^{at}) \sim Cy^{-\frac{2}{\theta^2}} t^{-\frac{1}{\theta^2}} e^{-I(x,a)t}$  where  $\theta$  and  $C$  are constants depending only on  $x$  and  $a$ . As for the second question, we demonstrated that, conditionally on  $\mathcal{L}_t(xt) \geq e^{at}$ , the most recent common ancestor of two particles, selected independently and uniformly from the  $xt$ -level set at time  $t$ , branched at a random time  $r \approx pt + c_1 \mathcal{N} \sqrt{t}$ <sup>2</sup> and was positioned around  $bpt + c_2 \mathcal{N} \sqrt{t}$ , where  $p \in (0, 1)$ ,  $b > \sqrt{2}$  and  $c_i$  are constants depending on  $x, a$ ; and  $\mathcal{N}$  denotes a standard Gaussian random variable. Additionally, the maximum of the BBM at time  $t$ , namely  $M_t$ , behaves as  $M_t \approx [bp + \sqrt{2}(1-p)]t + c_3 \mathcal{N} \sqrt{t}$ . In contrast to the unconditioned BBM, where  $r = O_{\mathbf{P}}(1)$  and  $M_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O_{\mathbf{P}}(1)$ , this behavior highlights a phenomenon of entropy repulsion.

**Future plans:** In [29], it was conjectured that a law of large numbers, analogous to that for the  $xt$ -level sets of BBM, holds for all models within the BBM universality class (i.e., log-correlated fields). This conjecture has been proven for the 2-dim GFF [12], the local times of 2-dim RW/BM [1, 37], and a random model of the Riemann-zeta function [4]. My long-term goal is to study large deviation probabilities for these models.

**2.2. BRW conditioned on large martingale limits.** Let  $(V(u) : u \in \cup_{n \geq 0} \mathcal{T}_n)$  be a BRW on  $\mathbb{R}$ , where  $\mathcal{T}_n$  is the set of particles in generation  $n$ , and  $V(u)$  denotes the position of particle  $u$ . The normalized partition function  $(W_n(\beta))_{n \geq 0}$  of the Gibbs-Boltzmann distribution  $(\frac{1}{Z_n(\beta)} e^{\beta V(u)} : u \in \mathcal{T}_n)$ , where  $W_n(\beta) := Z_n(\beta) / \mathbb{E}[Z_n(\beta)]$ , is a nonnegative martingale provided  $\mathbb{E}[Z_1(\beta)] < \infty$ . The martingale limit  $W_\infty(\beta)$  contains important information about the BRW, such as the growth of level sets discussed

<sup>2</sup>The law of this quantity  $r$  is known as the overlap distribution.

in §2.1. Furthermore,  $W_\infty(\beta)$  corresponds to the total mass of the Mandelbrot cascade measure and provides valuable insights into the Gaussian multiplicative chaos measure. The fundamental question of whether  $W_\infty(\beta) = 0$  a.s. was addressed by Biggins [9]. When  $W_\infty(\beta)$  is non-trivial, Liu [45] studied the tail probability  $\mathbf{P}(W_\infty(\beta) > x)$  using random difference equations.

Together with Xinxin Chen and Loïc de Reph  lis [19], we establish the joint tail distribution of  $W_\infty(\beta)$  and the global minimum of the BRW. This enables us to examine the typical behavior of the BRW viewed from the minimum, conditioned on  $W_\infty(\beta)$  being large. As a byproduct, we derive the right tail of the limit of the derivative martingale  $D_n(\beta) = -\frac{\partial}{\partial \beta} W_n(\beta)$ .

**2.3. Multifractal Spectrum of BRW on free groups.** RWs and BRWs have been extensively studied in Euclidean spaces. However, it has been discovered in [8, 26] that BRWs on nonamenable graphs exhibit a transient phase not observed in their Euclidean counterparts. Specifically, a BRW with mean offspring  $r \in (1, R]$ , where  $R^{-1} \in (0, 1)$  is the spectral radius of the underlying RW<sup>3</sup>, survives forever with positive probability, but eventually vacates every compact subset of the state space.

Interesting questions arise when the state space  $\Gamma$  has a nontrivial boundary, such as when  $\Gamma$  is a (nonelementary) hyperbolic group. In this case, the **limit set**  $\Lambda_r$ , defined as the random subset of  $\partial\Gamma$  (the boundary of  $\Gamma$  endowed with the visual metric) consisting of those points to which particle trajectories in the BRW converge, is a proper random subset of  $\partial\Gamma$ . Several studies have explored the Hausdorff dimension of  $\Lambda_r$  for different state spaces  $\Gamma$  ([16, 23, 33, 43, 44, 53]), finding that the dimension of  $\Lambda_r$  is no larger than half the dimension of  $\partial\Gamma$ . However, from a fractal geometry perspective, to describe the dynamics of a random fractal like  $\Lambda_r$ , a single exponent (the fractal dimension) is not fine enough; instead, a continuous spectrum of exponents (known as the multifractal spectrum) is needed. Roughly speaking, the multifractal spectrum captures the spatial heterogeneity of fractal patterns.

We aim to conduct a multifractal analysis of the limit set  $\Lambda_r$  in the transient regime  $r \in (1, R]$ . To simplify the analysis, we focus exclusively on a symmetric nearest-neighbor BRW on a free group  $\mathbb{F}$  of rank  $d \geq 2$ . For each  $\omega$  in the limit set  $\Lambda_r$ , results in [34] guarantee that there exists a unique sequence  $(t_n)_{n \geq 0}$  of particles in the BRW, where each  $t_{n+1}$  is a child of  $t_n$ , such that  $\lim_{n \rightarrow \infty} V(t_n) = \omega$ . We propose using the rate of escape for the walk  $(V(t_n))_{n \geq 1}$  to describe the degree of singularity around the point  $\omega = \lim_{n \rightarrow \infty} V(t_n)$  in the fractal  $\Lambda_r$ . The question of multifractal analysis then can be described as follows: Can we determine the Hausdorff dimension of the subfractal  $\Lambda_r(\alpha)$ , which consists of the points in  $\Lambda_r$  with singularity  $\alpha$ , for each  $\alpha \in [0, 1]$ ? Precisely,  $\Lambda_r(\alpha) := \{\omega \in \partial\Gamma : \exists (t_n)_{n \geq 0} \in \partial\mathcal{T}, \lim V(t_n) = \omega \text{ and } \lim |V(t_n)|/n = \alpha\}$ .

Together with Shuwen Lai and Longmin Wang, we addressed this question in [42]. Let  $L^*$  denote the rate function of the large deviation principle for  $(|Z_n|/n)_{n \geq 1}$ , where  $Z_n$  is the underlying RW of the BRW.

**Theorem.** *Let  $r \in (1, R]$ . Almost surely, for any  $\alpha \in [0, 1]$ ,  $\Lambda_r(\alpha) \neq \emptyset$  if and only if  $L^*(\alpha) \leq \ln r$ . In this case, the Hausdorff dimension of  $\Lambda_r(\alpha)$  is given by*

$$\dim_{\text{H}} \Lambda_r(\alpha) = \frac{\ln r - L^*(\alpha)}{\alpha}. \quad 4$$

*Consequently, there is  $\alpha(r) \in [0, 1]$  (see Figure 2) such that*

$$\dim_{\text{H}} \Lambda_r(\alpha(r)) = \dim_{\text{H}} \Lambda_r \text{ and } \dim_{\text{H}} \Lambda_r(\alpha) < \dim_{\text{H}} \Lambda_r, \forall \alpha \neq \alpha(r).$$

Beyond the multifractal analysis of the limit set  $\Lambda_r$ , a deeper question was explored in [42]: For simple BRWs, we obtain the dimensions of the sets  $\Lambda_r(\alpha, \beta) \subset \Lambda_r$  which consist of each point  $\omega \in \Lambda_r$  to which particle trajectory  $(V(t_n))_{n \geq 0}$ , with the set of average escape rates  $\{|V(t_n)|/n : n \geq 1\}$  having accumulation points  $[\alpha, \beta]$ , converge. Additionally, similar to results in [50], we derive the dimensions of the level sets  $E(\alpha, \beta)$  of infinite branches in the boundary of the underlying Galton-Watson tree, along which the averages of the BRW have  $[\alpha, \beta]$  as the set of accumulation points.

<sup>3</sup>That is,  $R^{-1} = \limsup_{n \rightarrow \infty} P_n(x, y)$  where  $P_n$  is the  $n$ -step transition probability for the RW.

<sup>4</sup>Here,  $\alpha = 0$  is permissible only if  $r = R$ , in which case  $L(0) = \ln R$  and the expression  $\frac{\ln R - L^*(0)}{0}$  should be read as  $\lim_{\alpha \downarrow 0} \frac{L^*(0) - L^*(\alpha)}{\alpha} = -(L^*)'(0)$ .



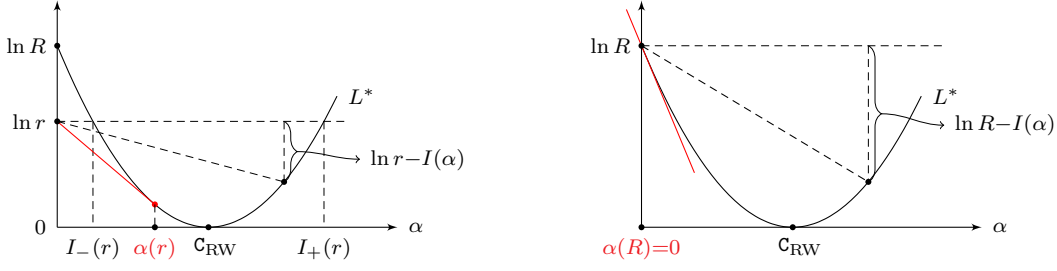


FIGURE 2. Illustration for  $\alpha(r)$  for the subcritical case  $1 < r < R$  (left) and the critical case  $r = R$  (right). Notably,  $\alpha(R) = 0$ !

**Future Plans:** We believe that similar properties should hold for all nearest-neighbor symmetric BRWs on general hyperbolic groups. I am interested in extending these results, starting with specific state spaces such as Fuchsian groups. Additionally, the range of BRWs with mean offspring one on general graphs, such as random trees and scale-free networks, is also of particular interest [14], but our current knowledge in this area is quite limited.

**2.4. Phase Transition of the extreme value of two-type BBM.** One of the most intriguing problems in BBM is understanding its extreme value. However, when the system involves two types of particles with distinct branching mechanisms and diffusion coefficients, how does this affect the extreme value?

Consider a two-type (reducible) BBM, where type 1 particles diffuse with coefficient  $\sigma^2$ , branching at rate  $\beta$  into two type 1 offspring, while also generating type 2 particles at rate  $\alpha$ . Type 2 particles follow standard BBM and do not produce Type 2 particles. It was studied in [6, 7, 11] for the maximum (and extremal processes) of the two-type BBM and in [30, 31] for the corresponding F-KPP PDEs system. A phase transition occurs, based solely on the parameters  $(\beta, \sigma^2)$  (see Figure 3): for  $(\beta, \sigma^2)$  in certain domain  $\mathcal{C}_I$  (resp.  $\mathcal{C}_{II}$ ), type 1 (resp. type 2) particles dominate, with the leading order of the maximum matching that of a single-type system. But for  $(\beta, \sigma^2)$  in certain domain  $\mathcal{C}_{III}$ , an **anomalous spreading** phenomenon arises, where the leading order of the maximum of two-type BBM exceeds that of a BBM with only type 1 or type 2 particles. However, when the parameters  $(\beta, \sigma^2)$  are on the boundaries between these three domains, the second order of the maximum is unclear, except for  $(\beta, \sigma^2) = (1, 1)$  which was addressed in [6].

Together with Yan-Xia Ren [46], we complete the the phase diagram of the two-type BBM, by determining the asymptotic order of the maximum for parameters  $(\beta, \sigma^2)$  that lie on the boundaries between  $\mathcal{C}_I, \mathcal{C}_{II}, \mathcal{C}_{III}$  (see the left panel of Figure 3). We also proved the convergence of extremal process. As an interesting by-product, a **double jump** occurs in the maximum of the two-type reducible BBM when the parameters  $(\beta, \sigma^2)$  cross the boundary of the anomalous spreading region  $\mathcal{C}_{III}$ , and only single jump occurs when the parameters cross the boundary between  $\mathcal{C}_I, \mathcal{C}_{II}$ .

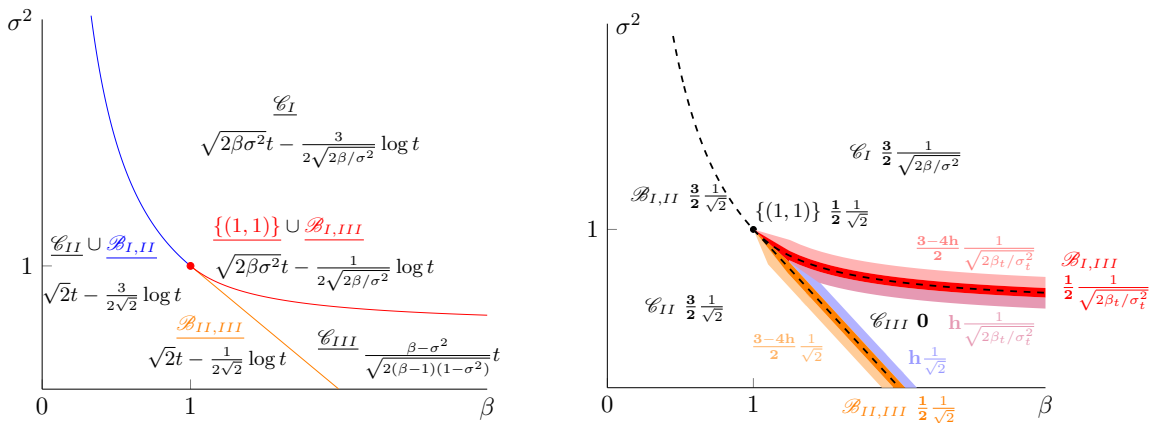


FIGURE 3. Phase diagram for the maximum of the two-type BBM. The coefficients shown on the right correspond to the  $\log t$  order, with  $h$  running over  $(0, 1/2]$ .

Additionally, in [47], we further investigate this double jump phenomenon by studying a two-type BBM where parameters  $(\beta_t, \sigma_t^2)$  depend on the time horizon  $t$ . We show that when these parameters approach the boundary of the anomalous spreading region  $\mathcal{C}_{III}$  in a suitable manner, the order of the maximum can interpolate smoothly between different surrounding regimes (see the right panel of Figure 3).

**2.5. Future Project: How far can BBM in  $\mathbb{R}^d$  spread in given directions?** Consider a BBM evolving in  $\mathbb{R}^d$ . Mallein [48] showed that the maximal displacement of the BBM grows asymptotically as  $\sqrt{2}t + \frac{d-4}{2\sqrt{2}} \log t + O_{\mathbf{P}}(1)$  as time  $t$  becomes large. On the other hand, Gärtner’s analysis of the F-KPP PDEs [27] shows that the maximal displacement of particles within a unit distance of the  $x$ -axis grows as  $\sqrt{2}t - \frac{d+2}{2\sqrt{2}} \log t + O_{\mathbf{P}}(1)$ . This difference, for  $d \geq 2$ , suggests that the front of particles observed in a fixed direction is slightly closer to the origin than the overall front of all particles.

Together with Zhenyao Sun, we aim to identify the positions of other fronts in the BBM whose distances from the origin lie between  $\sqrt{2}t - \frac{d+2}{2\sqrt{2}} \log t$  and  $\sqrt{2}t + \frac{d-4}{2\sqrt{2}} \log t$ .

## REFERENCES

- [1] Y. Abe and M. Biskup, *Exceptional points of two-dimensional random walks at multiples of the cover time*, Probab. Theory Relat. Fields **183** (2022), no. 1-2, 1–55 (English).
- [2] Guy Bresler Abolfazl Motahari and David Tse, *Information theory of DNA shotgun sequencing*, IEEE Transactions on Information Theory **59** (2013), no. 10, 6273–6289.
- [3] L. Addario-Berry, S. Briend, L. Devroye, S. Donderwinkel, C. Kerriou, and G. Lugosi, *Random friend trees*, 2024.
- [4] L.-P. Arguin, L. Hartung, and N. Kistler, *High points of a random model of the riemann-zeta function and gaussian multiplicative chaos*, Stochastic Processes and their Applications **151** (2022), 174–190.
- [5] Antar Bandyopadhyay and Debleena Thacker, *Pólya urn schemes with infinitely many colors*, Bernoulli **23** (2017), no. 4B, 3243–3267.
- [6] M. Belloum, *The extremal process of a cascading family of branching brownian motion*, 2022.
- [7] M. Belloum and B. Mallein, *Anomalous spreading in reducible multitype branching Brownian motion*, Electronic Journal of Probability **26** (2021), no. none, 1–39.
- [8] I. Benjamini and Y. Peres, *Markov chains indexed by trees*, Ann. Probab. **22** (1994), no. 1, 219–243. MR1258875
- [9] J. D. Biggins, *Martingale convergence in the branching random walk*, Journal of Applied Probability **14** (1977), no. 1, 25–37.
- [10] ———, *Growth rates in the branching random walk*, Z. Wahrscheinlichkeitstheor. Verw. Geb. **48** (1979), 17–34 (English).
- [11] ———, *Spreading speeds in reducible multitype branching random walk*, The Annals of Applied Probability **22** (2012), no. 5, 1778–1821.
- [12] M. Biskup and O. Louidor, *On intermediate level sets of two-dimensional discrete Gaussian free field*, Annales de l’Institut Henri Poincaré, Probabilités et Statistiques **55** (2019), no. 4, 1948–1987.
- [13] Arthur Blanc-Renaudie, *A.S. convergence for infinite colour Pólya urns associated with stable random walks*, 2023. arXiv:2304.04542.
- [14] Ignacio Bordeu, Saoirse Amarteifio, Rosalba Garcia-Millan, Benjamin Walter, Nanxin Wei, and Gunnar Pruessner, *Volume explored by a branching random walk on general graphs*, Scientific reports **9** (2019), no. 1, 15590.
- [15] M. Bramson, *Maximal displacement of branching brownian motion*, Communications on Pure and Applied Mathematics **31** (1978), no. 5, 531–581, available at <https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160310502>.
- [16] E. Candellero, L. A. Gilch, and S. Müller, *Branching random walks on free products of groups*, Proc. Lond. Math. Soc. (3) **104** (2012), no. 6, 1085–1120. MR2946082
- [17] C. Cannings and J. Jordan, *Random walk attachment graphs*, Electronic Communications in Probability **18** (2013), no. none, 1–5.
- [18] B. Chauvin, T. Klein, J.-F. Marckert, and A. Rouault, *Martingales and profile of binary search trees*, Electron. J. Probab. **10** (2005), 420–435 (English). Id/No 12.
- [19] X. Chen, L. de Raphélis, and H. Ma, *Branching random walk conditioned on large martingale limit*, 2024. arXiv:2408.05538.
- [20] X. Chen and H. Ma, *Branching brownian motion conditioned on large level sets*, 2024. arXiv:2409.16104.
- [21] Luc Devroye, *Universal limit laws for depths in random trees*, SIAM J. Comput. **28** (1998), no. 2, 409–432 (English).
- [22] J. Ding, Y. Jiang, and H. Ma, *Shotgun threshold for sparse erdős-rényi graphs*, IEEE Transactions on Information Theory **69** (2023), no. 11, 7373–7391.
- [23] M. Dussaule, L. Wang, and W. Yang, *Branching random walks on relatively hyperbolic groups*, 2024. arXiv:2211.07213, to appear in Ann. Probab.
- [24] Y. Hu E. Aïdékon and Z. Shi, *Large deviations for level sets of a branching Brownian motion and Gaussian free fields*, J. Math. Sci., New York **238** (2019), no. 4, 348–365 (English).

- [25] Michael Fuchs, Hsien-Kuei Hwang, and Ralph Neininger, *Profiles of random trees: Limit theorems for random recursive trees and binary search trees*, *Algorithmica* **46** (2006), no. 3-4, 367–407 (English).
- [26] N. Gantert and S. Müller, *The critical branching Markov chain is transient*, *Markov Process. Relat. Fields* **12** (2006), no. 4, 805–814 (English).
- [27] J. Gärtner, *Location of wave fronts for the multi-dimensional  $k$ - $p$ - $p$  equation and brownian first exit densities*, *Mathematische Nachrichten* **105** (1982), no. 1, 317–351.
- [28] J. Gaudio and E. Mossel, *Shotgun assembly of Erdos-Renyi random graphs*, *Electron. Commun. Probab.* **27** (2022), Paper No. 5, 14.
- [29] C. Glenz, N. Kistler, and M. A. Schmidt, *High points of branching Brownian motion and McKean’s Martingale in the Bovier-Hartung extremal process*, *Electron. Commun. Probab.* **23** (2018), no. none, 1–12.
- [30] M. Holzer, *Anomalous spreading in a system of coupled fisher-kpp equations*, *Physica D: Nonlinear Phenomena* **270** (2014), 1–10.
- [31] ———, *A proof of anomalous invasion speeds in a system of coupled fisher-kpp equations*, *Discrete and Continuous Dynamical Systems* **36** (2016), no. 4, 2069–2084.
- [32] H. Huang and K. Tikhomirov, *Shotgun assembly of unlabeled erdos-renyi graphs*. Preprint, arXiv 2108.09636.
- [33] I. Hueter and S.P. Lalley, *Anisotropic branching random walks on homogeneous trees*, *Probab. Theory Relat. Fields* **116** (2000), no. 1, 57–88 (English).
- [34] T. Hutchcroft, *Non-intersection of transient branching random walks*, *Probab. Theory Relat. Fields* **178** (2020), no. 1-2, 1–23 (English).
- [35] Svante Janson, *Random recursive trees and preferential attachment trees are random split trees*, *Comb. Probab. Comput.* **28** (2019), no. 1, 81–99 (English).
- [36] ———, *A.s. convergence for infinite colour Pólya urns associated with random walks*, *Ark. Mat.* **59** (2021), no. 1, 87–123 (English).
- [37] A. Jęgo, *Planar Brownian motion and Gaussian multiplicative chaos*, *The Annals of Probability* **48** (2020), no. 4, 1597–1643.
- [38] T. Johnston, G. Kronenberg, A. Roberts, and A. Scott, *Shotgun assembly of random graphs*, 2023.
- [39] Zakhar Kabluchko, Alexander Marynych, and Henning Sulzbach, *General Edgeworth expansions with applications to profiles of random trees*, *Ann. Appl. Probab.* **27** (2017), no. 6, 3478–3524 (English).
- [40] S. Keshri, E. Pnevmatikakis, A. Pakman, B. Shababo, and L. Paninski, *A shotgun sampling solution for the common input problem in neural connectivity inference*. preprint, arXiv:1309.3724.
- [41] P. L. Krapivsky and S. Redner, *Emergent network modularity*, *Journal of Statistical Mechanics: Theory and Experiment* **2017** (2017jul), no. 7, 073405.
- [42] S. Lai, H. Ma, and L. Wang, *Multifractal spectrum of branching random walks on free groups*, 2024. arXiv:2409.01346.
- [43] S. Lalley and T. Sellke, *Hyperbolic branching Brownian motion*, *Probab. Theory Related Fields* **108** (1997), no. 2, 171–192. MR1452555
- [44] T. M. Liggett, *Branching random walks and contact processes on homogeneous trees*, *Probab. Theory Relat. Fields* **106** (1996), no. 4, 495–519 (English).
- [45] Q. Liu, *On generalized multiplicative cascades*, *Stochastic Processes and their Applications* **86** (2000), no. 2, 263–286.
- [46] H. Ma and Y.-X. Ren, *Double jump in the maximum of two-type reducible branching brownian motion*, Accepted in *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*. arXiv:2305.09988.
- [47] ———, *From 0 to 3: Intermediate phases between normal and anomalous spreading of two-type branching brownian motion*, 2024.
- [48] B. Mallein, *Maximal displacement in the  $d$ -dimensional branching Brownian motion*, *Electron. Commun. Probab.* **20** (2015), no. none, 1–12.
- [49] E. Mossel and N. Ross, *Shotgun assembly of labeled graphs*, *IEEE Transactions on Network Science and Engineering* **6** (2019), no. 2, 145–157.
- [50] A. Najmeddine and B. Julien, *Hausdorff and packing spectra, large deviations, and free energy for branching random walks in  $\mathbb{R}^d$* , *Comm. Math. Phys.* **331** (2014), no. 1, 139–187. MR3231998
- [51] J. Saramäki and K. Kaski, *Scale-free networks generated by random walkers*, *Physica A: Statistical Mechanics and its Applications* **341** (2004), 80–86.
- [52] Eva-Maria Schopp, *A functional limit theorem for the profile of  $b$ -ary trees*, *Ann. Appl. Probab.* **20** (2010), no. 3, 907–950 (English).
- [53] V. Sidoravicius, L. Wang, and K. Xiang, *Limit set of branching random walks on hyperbolic groups*, *Comm. Pure Appl. Math.* **76** (2023), no. 10, 2765–2803. MR4630601