RESEARCH STATEMENT OF HENG MA

My research focuses on probability theory, with a particular interest in probabilistic models that incorporate tree and graph structures. These models provide powerful tools for analyzing and understanding both the structure and evolution of many real-world systems, with applications in various fields such as statistical mechanics, computer science, biology, and social science. Let me begin with a quick overview of my doctoral research and future plans, followed by more detailed discussions in the next two sections. My work falls into two main subjects:

(a) **Random trees and graphs**: Random graphs serve as foundational models for understanding real-world network properties such as connectivity, degree distribution, and diameter. However, in practice, we often have access only to local observations, making the question of whether global structures can be reconstructed from these local information highly significant. We addressed this question in [20] for a central model of random graphs, the sparse Erdős–Rényi graph, and provided algorithms for the recovery process. I plan to continue this investigation to dense Erdős-Rényi graphs. Preferential attachment trees are essential models in sorting algorithms and network science. I am interested in investigating the large deviation probabilities of their heights. I am also interested in studying the maximal displacement of random walks indexed by preferential attachment trees.

(b) **Branching random walks**: Branching processes model diverse scenarios, from the extinction of family names to nuclear fission. A spatial embedding, the branching random walk (BRW), better captures natural phenomena like viral spread in tissues or paths in polymeric networks. Mathematically, BRWs share properties with models in the log-correlated fields universality class. My work in this area involves understanding rare events, such as when an unusually large number of particles reach a high level [19], and when certain normalized partition functions grow unexpectedly large [18]. We also conduct a finer analysis on the fractal dimensions of BRWs in non-Euclidean spaces [39], and explore how the extreme values of the process are affected when branching rates and movements of particles depend on their types [43,44]. I aim to study the front of particles within given directions in $\mathbb{R}^d$. For a long-time goal, I am interested in studying the extremal values and intermediate level sets of log-correlated fields, as well as corresponding large deviation problems.

1. Random Trees and Graphs

1.1. Recovery threshold of sparse Erdős–Rényi graphs.

The shotgun assembly problems aim for recovering a global structure from local observations and has broad applications, including DNA sequencing [2] and neural network recovery [37]. A mathematical framework is as follows:

Consider a graph $\mathcal{G}$—either fixed or random, possibly with random labeling of the vertices. For $r \geq 1$, we observe the (rooted)r-neighborhood $\mathsf{N}_r(v)$ of each vertex v, defined as the subgraph induced by all vertices within a distance r from v. Crucially, vertex names are invisible except for v itself (see Figure 1). Can we recover $\mathcal{G}$ from the collection $\{\mathsf{N}_r(v) : v \in \mathcal{G}\}$? As the amount of information increases with the depth r, the goal is to determine the recovery threshold r^*.

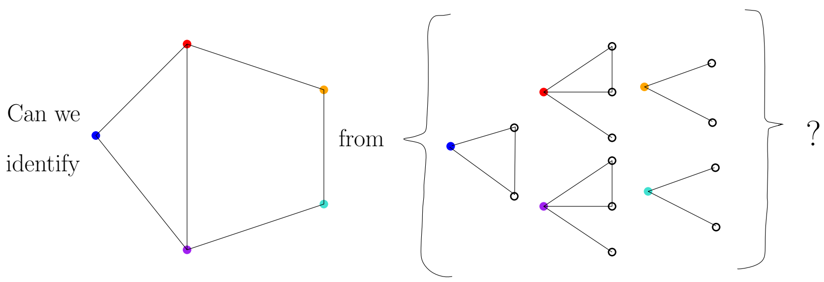

Figure 1. An Illustration for the shotgun assembly problem with $r = 1$.

A natural first step is to explore this threshold in the well-studied Erdős–Rényi graphs. An Erdős–Rényi graph $\mathcal{G}(n, p_n)$ on n vertices is generated by independently connecting each pair of vertices with probability p_n. For graphs with polynomially growing average degree (dense regime), [26,30,35,46] determined whether recovery is possible from neighborhoods of depths $1, 2, 3, \cdots$. In contrast, for graphs in sparse regime where the average degree remains constant, [46] showed that the shotgun threshold is of order $\log n$, except in the critical case (average degree equal to one), where only a polynomial upper bound was obtained.

Together with Jian Ding and Yiyang Jiang [20], we determined its sharp asymptotics of the shotgun threshold for the sparse Erdős–Rényi graphs, establishing a connection to the isomorphic probability for Poisson–Galton–Watson trees.

Theorem. *Fix $\lambda > 0$. Let $\mathcal{G}_n \sim \mathcal{G}(n, \lambda/n)$. Define γ_λ as the probability that two independent Galton–Watson trees with offspring distribution $\mathrm{Poisson}(\lambda)$ are rooted isomorphic.* Let $\mathsf{C}_\lambda := 1/\log(\frac{1}{\lambda^2\gamma_\lambda})$. Then for any fixed $\epsilon_0 > 0$, the following assertions hold:*

- *(1) For $r \le (1-\epsilon_0)\mathsf{C}_\lambda \log n$, with high probability, the shotgun problem is non-identifiable: there exists a non-isomorphic graph to $\mathcal{G}_n$ that shares the same empirical neighborhood profile.*
- *(2) For $r \ge (1+\epsilon_0)\mathsf{C}_\lambda \log n$, with high probability, $\mathcal{G}_n$ can be recovered using a polynomial-time algorithm.*

Future plans: For a dense Erdős-Rényi graph $\mathcal{G}_n \sim \mathcal{G}(n, n^{-\alpha})$ with $\alpha \in (0,1)$, it has been shown in [26, 30] that given depth-1 neighborhoods, $\mathcal{G}_n$ is identifiable for $0 < \alpha < 1/2$ but non-identifiable for $\alpha > 1/2$. However, no efficient recovery algorithm has been found for $1/3 \le \alpha < 1/2$. I am interested in studying whether computation-information gap exists for $\alpha \in [1/3, 1/2)$. Additionally, [26, 35] showed that given depth-2 neighborhoods, $\mathcal{G}_n$ is identifiable for $\alpha < 2/3 + \delta_0$ with some small constant $\delta_0 > 0$, but non-identifiable for $\alpha > 3/4$. I aim to address this gap. Further, I am also interested in studying the shotgun assembly problem in other graph models, such as random geometric graphs and random regular graphs.

1.2. **Future Project: On preferential attachment trees.** Preferential attachment trees (PATs) are foundational models in both data structures and complex networks. In such trees, each new vertex attaches to an existing vertex v with a probability proportional to $f(\deg^+(\mathrm{v}))$, where $\deg^+(v)$ denote the current out-degree of vertex v. Key examples are linear PATs, where $f(d) = \beta d + 1$ is called linear PATs, including models such as the uniform recursive tree ($\beta = 0$), the plane oriented recursive tree ($\beta = 1$) and the binary search tree ($\beta = -1/2$).

An essential way to characterize the shape of a tree is through its height profile $(L_n(k))_{k\ge 0}$, where $L_n(k)$ represents the number of nodes at distance k from the root. Previous studies [17, 23, 49] have shown that $L_n(\lfloor \lambda \log n \rfloor)$, normalized by its mean, converges to certain random variables over specific ranges of λ; the exact rate of convergence for linear PATs was established in [36]. In an ongoing project, I aim to extend this result to random trees with general preferential rules f, by analyzing the associated Crump—Mode-–Jagers branching process (which can be viewed as a branching random walk with only positive displacement).

Next, I plan to investigate the large deviation properties of the height H_n of a linear PAT on n nodes, where H_n is the largest k s.t. $L_n(k) \ge 1$. While the law of large numbers for H_n is well understood—[48] showed that $H_n/\ln n \to C(\beta)$ almost surely for some constant $C(\beta) > 0$—its large deviation principle remains open. I aim to establish that, at least for uniform attachment trees, $\mathbf{P}(H_n > \alpha \ln n) \sim n^{-I(\alpha)+o(1)}$ for $\alpha > C(\beta)$ and $-\ln \mathbf{P}(H_n \le \alpha \ln n) \sim n^{-I(\alpha)+o(1)}$ for $\alpha < C(\beta)$.

Beyond this, I intend to study spatial version of PATs, where each new vertex is positioned in $\mathbb{R}^d$ according to i.i.d. random vectors relative to its parent's location. This model can be interpreted as a random walk indexed by PATs, following [7], and as a Pólya urn scheme with infinitely many colors, as studied in [4, 12, 33]. The latter work addresses the scaling limit of the empirical measure for spatial trees in the uniform attachment case ($\beta = 0$). Motivated by research on the maximum of BRWs, I am interested in studying the maximal displacement of the spatial tree and comparing it with that of BRWs.

2. Branching Random Walks and Their Variants

A branching random walk (BRW) begins with one particle located at some vertex in the state space G. At every time $n \ge 1$, each particle in generation $n-1$ dies and gives birth to an independent random number of offspring, following a common distribution. Each offspring then independently takes a step relative to its parent, according to a random walk (RW) on G, which is referred to as the underlying RW of the BRW.

*That is, there exists a bijection between their vertices that preserves the edges and the root.

A branching Brownian motion (BBM) is a continuous-time version of the BRW on $\mathbb{R}$, where particles perform Brownian motion and undergo dyadic branching at rate one.

2.1. **BBM conditioned on large level sets.** A fundamental aspect of understanding how particles in a BBM spread through space is to examine the sizes of level sets. For each $y \geq 0$, the y-**level set** at time t consists of all particles alive at time t that are positioned above level y. Let $\mathscr{L}_t(y)$ denote the size of the y-level set. The typical behavior of these level sets is well-established: Results of Bramson [15] implies that almost surely, at sufficiently large time t, the $\sqrt{2}t$-level set is empty. While Biggins [9] and Glenz, Kistler and Schmidt [27] proved that for any $x \in [0, \sqrt{2})$, $\mathscr{L}_t(xt)$, normalized by its mean $\mathbf{E}[\mathscr{L}_t(xt)] \asymp \frac{1}{\sqrt{t}} e^{(1-\frac{x^2}{2})t}$, converges a.s. to a positive limit $W_\infty(x)$ (which is defined in §2.2).

With this law of large numbers established, it is natural to investigate the large deviation probabilities. Aïdékon, Hu, and Shi [22] studied the exponential decay rate of the probability that the level sets are unusually large. Specifically, for $x > 0$ and $a \in ((1-x^2/2)_+, 1)$, they showed that $\mathbf{P}\left(\mathscr{L}_t(xt) \geq e^{at}\right) = \exp\{-I(x,a)t + o(t)\}$ with rate function $I(x,a) = \frac{x^2}{2(1-a)} - 1$. Building on this result, two natural questions arise:

(i) What is the exact order of $\mathbf{P}\left(\mathscr{L}_t(xt) \geq e^{at}\right)$?
(ii) What is the typical behavior of the BBM under the conditional law $\mathbf{P}\left(\cdot \mid \mathscr{L}_t(xt) \geq e^{at}\right)$?

Together with Xinxin Chen [19], we addressed Question (i) by showing that $\mathbf{P}\left(\mathscr{L}_t(xt) \geq ye^{at}\right) \sim Cy^{-\frac{2}{\theta^2}} t^{-\frac{1}{\theta^2}} e^{-I(x,a)t}$ where θ and C are constants depending only on x and a. As for the second question, we demonstrated that, conditionally on $\mathscr{L}_t(xt) \geq e^{at}$, the most recent common ancestor of two particles, selected independently and uniformly from the xt-level set at time t, branched at a random time $r \approx pt + c_1\mathcal{N}\sqrt{t}$ [†] and was positioned around $bpt + c_2\mathcal{N}\sqrt{t}$, where $p \in (0,1), b > \sqrt{2}$ and c_i are constants depending on x, a; and $\mathcal{N}$ denotes a standard Gaussian random variable. Additionally, the maximum of the BBM at time t, namely M_t, behaves as $M_t \approx [bp + \sqrt{2}(1-p)]t + c_3\mathcal{N}\sqrt{t}$. In contrast to the unconditioned BBM, where $r = O_{\mathbf{P}}(1)$ and $M_t = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + O_{\mathbf{P}}(1)$, this behavior highlights a phenomenon of entropy repulsion.

Future plans: In [27], it was conjectured that a law of large numbers, analogous to that for the xt-level sets of BBM, holds for all models within the BBM universality class (i.e.,log-correlated fields). This conjecture has been proven for the 2-dim GFF [11], the local times of 2-dim RW/BM [1,34], and a random model of the Riemann-zeta function [3]. My long-term goal is to study large deviation probabilities for these models.

2.2. **BRW conditioned on large martingale limits.** Let $(V(u) : u \in \cup_{n\geq 0}\mathcal{T}_n)$ be a BRW on $\mathbb{R}$, where $\mathcal{T}_n$ is the set of particles in generation n, and $V(u)$ denotes the position of particle u. The normalized partition function $(W_n(\beta))_{n\geq 0}$ of the Gibbs-Boltzmann distribution $(\frac{1}{Z_n(\beta)} e^{\beta V(u)} : u \in \mathcal{T}_n)$, where $W_n(\beta) := Z_n(\beta)/\mathbf{E}[Z_n(\beta)]$, is a nonnegative martingale provided $e^{\psi(\beta)} := \mathbf{E}[Z_1(\beta)] < \infty$. The martingale limit $W_\infty(\beta)$ contains important information about the BRW, such as the growth of level sets discussed in §2.1. Furthermore, $W_\infty(\beta)$ corresponds to the total mass of the Mandelbrot cascade measure and provides valuable insights into the Gaussian multiplicative chaos measure. The fundamental question of whether $W_\infty(\beta) = 0$ a.s. was addressed by Biggins [8], showing that it is the case iff $\beta > 0$ strictly smaller than some critical value β_c. When $W_\infty(\beta)$ is non-trivial, Liu [42] studied the tail probability $\mathbf{P}(W_\infty(\beta) > x)$ using random difference equations.

It's also interesting to investigate the properties of the derivative of $W_n(\beta)$, which form the (sub-critical) derivative martingale $D_n(\beta) := \frac{\partial}{\partial\beta}W_n(\beta)$. In GMC theory, these (subcritical) derivative martingales play a crucial role in constructing quantum Mabuchi theory [38]. Unlike in the critical case, where $D_\infty(\beta_c)$ is negative almost surely with a heavy-tailed distribution, the subcritical derivative martingale limit $D_\infty(\beta)$ is highly asymmetric: The right tail $D_\infty(\beta)$ is expected to behave similarly to that of W_∞ up to logarithmic corrections, while in [38], the left-tail behavior of $D_\infty(\beta_c)$ for general GMC measures was conjectured to follow $-\ln \mathbf{P}(D_\infty(\beta) < -x) \asymp x^{\gamma(\beta)}$ as $x \to \infty$, and in the Gaussian BRW setting, [13] established this asymptotic in the L^4 phase.

Together with Xinxin Chen and Loïc de Raphélis [18], we establish the joint tail distribution of $W_\infty(\beta)$ and the global minimum of the BRW $\mathbf{M} := \min_{u\in\mathcal{T}}\{V(u) - \psi(\beta)|u|\}$. This enables us to

[†] The law of this quantity r is known as the overlap distribution.

examine the typical behavior of the BRW viewed from the minimum conditioned on $W_\infty(\beta)$ being unusually large. As a byproduct, we derive the right tail of the limit of the derivative martingale $D_\infty(\beta)$, confirming that its right tail behaves similarly to that of $W_\infty(\beta)$ up to logarithmic corrections. Without loss of generality we assume $\beta = 1$, $\psi(1) = 0$, $\psi'(1) < 0$:

Theorem. *Assume that $\psi(\kappa) = 0$ for some $\kappa > 1$. Under mild moment conditions, the following hold:*

(i) There is a non-increasing and continuous function γ on $(0,\infty)$ with $\gamma(0+) > 0$ such that

$$\mathbf{P}\left(W_\infty \geq ax, \exp(-\mathbf{M}) \geq x\right) \sim \gamma(a)x^{-\kappa}, \text{ as } x \to \infty.$$

(ii) There exists a constant $c_D \in (0,\infty)$ such that

$$\mathbf{P}(D_\infty \geq x) \sim c_D \frac{(\ln x)^\kappa}{x^\kappa} \text{ as } x \to \infty.$$

(iii) Conditionally on $W_\infty \geq x$, the following convergence in law holds as $x \to \infty$:

$$\left(\mathbf{M} + \ln x, \sum_{u\in\mathcal{T}} \delta_{V(u)-\mathbf{M}}, \frac{W_\infty}{x}, \frac{D_\infty}{x\ln x},\right) \Rightarrow \left(\ln \widehat{Z} - U, \widehat{\mathcal{E}}_\infty, e^U, \frac{\psi'(\kappa) - \psi'(1)}{\psi'(\kappa)} e^U\right)$$

where U has exponential distribution with mean κ^{-1} independent of $(\widehat{\mathcal{E}}_\infty, \widehat{Z})$.

2.3. **Multifractal Spectrum of BRW on free groups.** RWs and BRWs have been extensively studied in Euclidean spaces. However, it has been discovered in [7, 24] that BRWs on nonamenable graphs exhibit a transient phase not observed in their Euclidean counterparts. Specifically, a BRW with mean offspring $r \in (1, R]$, where $R^{-1} \in (0,1)$ is the spectral radius of the underlying RW[‡], survives forever with positive probability, but eventually vacates every compact subset of the state space.

Interesting questions arise when the state space Γ has a nontrivial boundary, such as when Γ is a (nonelementary) hyperbolic group. In this case, the **limit set** Λ_r, defined as the random subset of $\partial\Gamma$ (the boundary of Γ endowed with the visual metric) consisting of those points to which particle trajectories in the BRW converge, is a proper random subset of $\partial\Gamma$. Several studies have explored the Hausdorff dimension of Λ_r for different state spaces Γ ([16, 21, 31, 40, 41, 50]), finding that the dimension of Λ_r is no larger than half the dimension of $\partial\Gamma$. However, from a fractal geometry perspective, to describe the structure of a random fractal like Λ_r, a single exponent (the fractal dimension) is not fine enough; instead, a continuous spectrum of exponents (known as the multifractal spectrum) is needed. Roughly speaking, the multifractal spectrum captures the spatial heterogeneity of fractal patterns.

We aim to conduct a multifractal analysis of the limit set Λ_r in the transient regime $r \in (1, R]$. To simplify the analysis, we focus exclusively on a symmetric nearest-neighbor BRW on a free group $\mathbb{F}$ of rank $d \geq 2$. For each ω in the limit set Λ_r, results in [32] guarantee that there exists a unique sequence $(t_n)_{n\geq 0}$ of particles in the BRW, where each t_{n+1} is a child of t_n, such that $\lim_{n\to\infty} V(t_n) = \omega$. We propose using the rate of escape for the walk $(V(t_n))_{n\geq 1}$ to describe the degree of singularity around the point $\omega = \lim_{n\to\infty} V(t_n)$ in the fractal Λ_r. The question of multifractal analysis then can be described as follows: Can we determine the Hausdorff dimension of the subfractal $\Lambda_r(\alpha)$, which consists of the points in Λ_r with singularity α, for each $\alpha \in [0,1]$? Precisely, $\Lambda_r(\alpha) := \{\omega \in \partial\Gamma : \exists\, (t_n)_{n\geq 0} \in \partial\mathcal{T}, \lim V(t_n) = \omega \text{ and } \lim |V(t_n)|/n = \alpha\}$.

Together with Shuwen Lai and Longmin Wang, we addressed this question in [39]. Let L^* denote the rate function of the large deviation principle for $(|Z_n|/n)_{n\geq 1}$, where Z_n is the underlying RW of the BRW.

Theorem. *Let $r \in (1, R]$. Almost surely, for any $\alpha \in [0,1]$, $\Lambda_r(\alpha) \neq \emptyset$ if and only if $L^*(\alpha) \leq \ln r$. In this case, the Hausdorff dimension of $\Lambda_r(\alpha)$ is given by*

$$\dim_{\mathrm{H}} \Lambda_r(\alpha) = \frac{\ln r - L^*(\alpha)}{\alpha}. \text{[§]}$$

Consequently, there is $\alpha(r) \in [0,1]$ (see Figure 2) such that

$$\dim_{\mathrm{H}} \Lambda_r(\alpha(r)) = \dim_{\mathrm{H}} \Lambda_r \text{ and } \dim_{\mathrm{H}} \Lambda_r(\alpha) < \dim_{\mathrm{H}} \Lambda_r, \forall\, \alpha \neq \alpha(r).$$

‡That is, $R^{-1} = \limsup_{n\to\infty} P_n(x,y)$ where P_n is the n-step transition probabiliy for the RW.

§Here, $\alpha = 0$ is permissible only if $r = R$, in which case $L(0) = \ln R$ and the expression $\frac{\ln R - L^*(0)}{0}$ should be read as $\lim_{\alpha\downarrow 0} \frac{L^*(0) - L^*(\alpha)}{\alpha} = -(L^*)'(0)$.

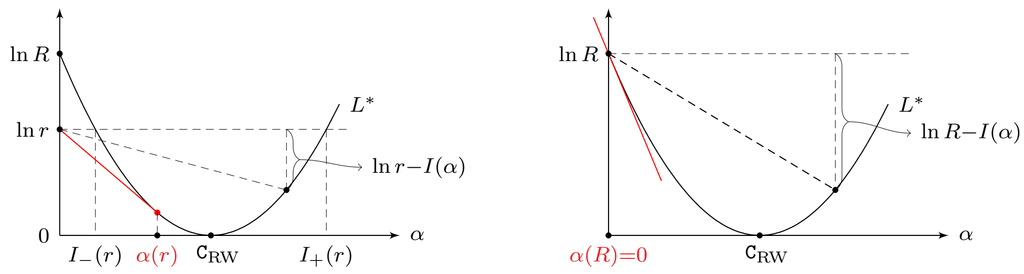

FIGURE 2. Illustration for $\alpha(r)$ for the subcritical case $1 < r < R$ (left) and the critical case $r = R$ (right). Notably, $\alpha(R) = 0$!

Beyond the multifractal analysis of the limit set Λ_r, a deeper question was explored in [39]: For simple BRWs, we obtain the dimensions of the sets $\Lambda_r(\alpha, \beta) \subset \Lambda_r$ which consist of each point $\omega \in \Lambda_r$ to which particle trajectory $(V(t_n))_{n\geq 0}$, with the set of average escape rates $\{|V(t_n)|/n : n \geq 1\}$ having accumulation points $[\alpha, \beta]$, converge. Additionally, similar to results in [47], we derive the dimensions of the level sets $E(\alpha, \beta)$ of infinite branches in the boundary of the underlying Galton-Watson tree, along which the averages of the BRW have $[\alpha, \beta]$ as the set of accumulation points.

Future Plans: We believe that similar properties should hold for all nearest-neighbor symmetric BRWs on general hyperbolic groups. I am interested in extending these results, starting with specific state spaces such as Fuchsian groups. Additionally, the range of BRWs with mean offspring one on general graphs, such as random trees and scale-free networks, is also of particular interest [14], but our current knowledge in this area is quite limited.

2.4. **Phase Transition of the extreme value of two-type BBM.** One of the most intriguing problems in BBM is understanding its extreme value. However, when the system involves two types of particles with distinct branching mechanisms and diffusion coefficients, how does this affect the extreme value?

Consider a two-type (reducible) BBM, where type 1 particles diffuse with coefficient σ^2, branching at rate β into two type 1 offspring, while also generating type 2 particles at rate α. Type 2 particles follow standard BBM and do not produce Type 2 particles. It was studied in [5, 6, 10] for the maximum (and extremal processes) of the two-type BBM and in [28, 29] for the corresponding F-KPP PDEs system. A phase transition occurs, based solely on the parameters (β, σ^2) (see Figure 3): for (β, σ^2) in certain domain $\mathscr{C}_I$ (resp. $\mathscr{C}_{II}$), type 1 (resp. type 2) particles dominate, with the leading order of the maximum matching that of a single-type system. But for (β, σ^2) in certain domain $\mathscr{C}_{III}$, an **anomalous spreading** phenomenon arises, where the leading order of the maximum of two-type BBM exceeds that of a BBM with only type 1 or type 2 particles. However, when the parameters (β, σ^2) are on the boundaries between these three domains, the second order of the maximum is unclear, except for $(\beta, \sigma^2) = (1, 1)$ which was addressed in [5].

Together with Yan-Xia Ren [43], we complete the phase diagram of the two-type BBM, by determining the asymptotic order of the maximum for parameters (β, σ^2) that lie on the boundaries between $\mathscr{C}_I, \mathscr{C}_{II}, \mathscr{C}_{III}$ (see the left panel of Figure 3). We also proved the convergence of extremal process. As an interesting by-product, a **double jump** occurs in the maximum of the two-type reducible BBM when the parameters (β, σ^2) cross the boundary of the anomalous spreading region $\mathscr{C}_{III}$, and only single jump occurs when the parameters cross the boundary between $\mathscr{C}_I, \mathscr{C}_{II}$.

Additionally, in [44], we further investigate this double jump phenomenon by studying a two-type BBM where parameters (β_t, σ_t^2) depend on the time horizon t. We show that when these parameters (β_t, σ_t^2) approach the boundary of the anomalous spreading region $\mathscr{C}_{III}$ in a suitable manner, the order of the maximum can interpolate smoothly between different surrounding regimes. For instance, if (β_t, σ_t^2) lies in $\mathscr{C}_I$ and is approximately t^{-H} away from the boundary $\mathscr{B}_{I,III}$ between $\mathscr{C}_I$ and $\mathscr{C}_{III}$, the renormalized coefficient of the log-correction term $C_2(\beta_t, \sigma_t^2)\sqrt{2\beta_t/\sigma_t^2}$ equals $\frac{3-4\min\{H,1/2\}}{2}$ which interpolates continuously from $\frac{3}{2}$ to 0 as H runs over from $(0, \infty)$. Figure 3 illustrates other cases.

2.5. **Future Project: How far can BBM in $\mathbb{R}^d$ spread in given directions?** Consider a BBM evolving in $\mathbb{R}^d$. Mallein [45] showed that the maximal displacement of the BBM grows asymptotically as $\sqrt{2}t + \frac{d-4}{2\sqrt{2}} \log t + O_{\mathbf{P}}(1)$ as time t becomes large. On the other hand, Gärtner's analysis of the F-KPP PDEs [25] shows that the maximal displacement of particles within a unit distance of the x-axis grows

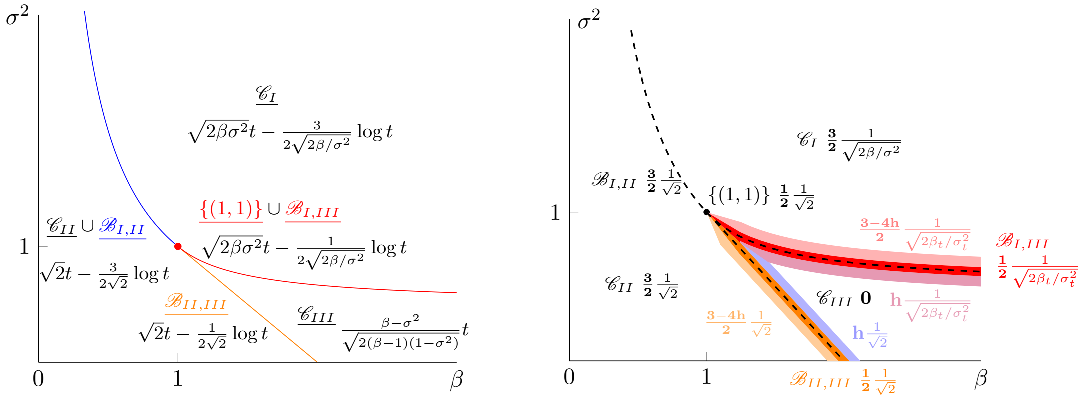

FIGURE 3. Phase diagram for the maximum of the two-type BBM. The coefficients shown on the right correspond to the $\log t$ order, and the colored area shows parameters (β_t, σ_t^2) at a distance t^{-H} from $\partial\mathscr{C}_I \cap \partial\mathscr{C}_{II}$, where H ranges over $(0,\infty)$ and $h := H \wedge \frac{1}{2}$.

as $\sqrt{2}t - \frac{d+2}{2\sqrt{2}}\log t + O_{\mathbf{P}}(1)$. This difference, for $d \geq 2$, suggests that the front of particles observed in a fixed direction is slightly closer to the origin than the overall front of all particles.

Together with Zhenyao Sun, we aim to identify the positions of other fronts in the BBM whose distances from the origin lie between $\sqrt{2}t - \frac{d+2}{2\sqrt{2}}\log t$ and $\sqrt{2}t + \frac{d-4}{2\sqrt{2}}\log t$.

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