Lagrangian Optimization

- For optimization with constraints
  - Equality constraints
  - Inequality constraint

1. Lagrange Optimization with one Equality constraint.

Use typical math notation.

Problem: Find an extremum of $f(\underline{x})$ ($=J(w)$ for us), subject to constraint:

$\underbrace{g(\underline{x})=0}$ → new constraint

Solution:

1. set up Lagrangian function

$$L(\underline{x},\lambda)=f(\underline{x})+\lambda g(\underline{x})$$

$f(\underline{x})$: original function to minimize $f(x)$

$g(\underline{x})$ → new constraint $g(\underline{x})=0$

Note: when $g(x)=0$ (constraint satisfied) $L(\underline{x},\lambda)=f(\underline{x})$

2. Lagrange method $\nabla_{\underline{x},\lambda} L(\underline{x},\lambda)=\underline{0}$

$$\Rightarrow (*) \begin{cases} \nabla_x L(\underline{x},\lambda)=\nabla_x f(\underline{x})+\nabla_x \lambda g(\underline{x})=0 \\ \nabla_\lambda L(\underline{x},\lambda)=g(\underline{x})=0 \end{cases}$$

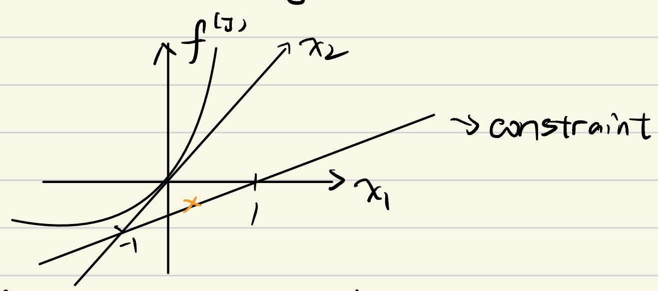
If x has d dimension $\Rightarrow \begin{cases} d+1 & \text{unknowns} \\ d+1 & \text{equations} \end{cases}$

Example of Lagrangian Optimization with 1 Equality Constraint.

Let $f(\underline{x})=\|x\|_2^2=x_1^2+x_2^2$ $(d=2)$

Minimize $f(\underline{x})$ subject to (s.t.) constraint: $x_2-x_1=-1$

$\Rightarrow g(\underline{x})=x_2-x_1+1=0$



Lagrange method:

Let $L(\underline{x},\lambda)=f(\underline{x})+\lambda g(\underline{x})$

$=\|x\|_2^2+\lambda(x_2-x_1+1)$

$$\nabla_{\underline{x},\lambda} L(\underline{x},\lambda)=\underline{0}$$

$$\Rightarrow \begin{cases} \nabla_x L=\begin{bmatrix} 2x_1-\lambda \\ 2x_2+\lambda \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \leftarrow \begin{matrix} ① \\ ② \end{matrix} \\ \nabla_\lambda L=x_2-x_1+1=0 \leftarrow ③ \end{cases}$$

$$\therefore \begin{cases} x_1=\frac{1}{2} \\ x_2=-\frac{1}{2} \\ \lambda=1 \end{cases}$$

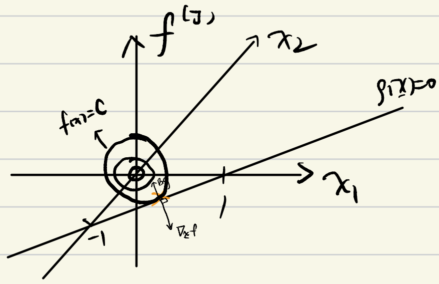
$$\therefore x^*=\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \quad \lambda=1.$$

Intuitive explanation of Lagrange example.

In x_1x_2-plane. $f(x) = \|x\|_2^2$

$f(\underline{x}) = d_E^2(\underline{0}, \underline{x}) = r^2$

(f = constant) is a circle, about origin.



Increase r until circle tangent to $g(x)=0$

$\rightarrow$ point x^*

$\therefore$ at x^*, we have

$$\left\{\begin{array}{l} \boxed{\nabla_x f(x)} \perp \boxed{f(x)=C} \leftarrow \nabla_x f(x)=0 \\ \boxed{\nabla_x g(x)} \perp \boxed{g(x)=0} \leftarrow \nabla_x g(x)=0 \\ f(x)=C \text{ is tangent to } g(x)=0 \end{array}\right.$$

$\therefore$ $\nabla_x f(x^*) \parallel \nabla_x g(x^*)$

$\therefore$ $\nabla_{\underline{x}} f(\underline{x}) = \pm\lambda \nabla_{\underline{x}} g(\underline{x})$ at $\underline{x} = x^*$ $\quad$ } λ comes from here

$\therefore$ $\nabla_x f(x) \pm \lambda \nabla_x g(x) = 0$

$\longrightarrow$ choose $+$ sign

$$\nabla_x (f(x) + \lambda g(x)) = 0 \Rightarrow \left\{\begin{array}{l} \nabla_x (L(x,\lambda)) = \underline{0} \\ \nabla_\lambda L(x,\lambda) = \underline{0} \end{array}\right.$$

2. Lagrangian Optimization with Multiple Equality Constraints

Find min of $f(x)$ s.t. $g_i(x) = 0$ $\quad i = 1, 2, \cdots$ $\quad R < d$

$$\boxed{\begin{array}{l} L(x,\lambda) = f(x) + \sum_{i=1}^{R} \lambda_i g_i(x) \\ \nabla_{x,\lambda} L(x,\lambda) = 0 \end{array}}$$

λ_i = Lagrangian multiplier

$\Rightarrow d+R$ eqns, $d+R$ unknowns.

3. Lagrangian Optimization with 1 Inequality constraint.

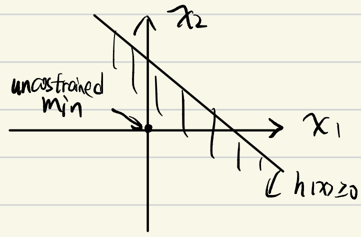
Extension of the equality constraint case

Find min of $f(x)$ s.t. $h(x) \geq 0$

Let:

$$L(x,\mu) = f(x) - \mu h(x)$$

Find extremum of L over $\underline{x}$ and μ.



<u>2 cases:</u>

(a) unconstrained min is in region $h \geq 0$

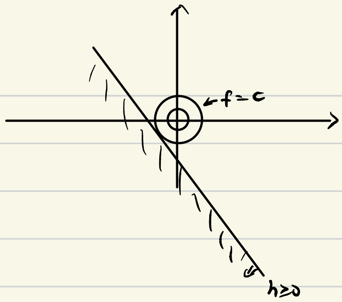
$\Rightarrow$ constrained min = unconstrained min

$\nabla_x f(x) = 0$

$\Rightarrow$ can solve using $L(x,\mu)$ with $\mu = 0$

(b) unconstrained min is outside region $h \geq 0$

$\Rightarrow$ constrained min is on $h(\underline{x}) = 0$ curve
same as equality constrained case



$L(\underline{x}, \mu)$ with $\mu \neq 0$
$\nabla_{\underline{x}, \mu} L = \underline{0}$

$\Rightarrow \nabla_{\underline{x}} f = \mu \nabla_{\underline{x}} h \quad , \mu > 0$

$$\boxed{\begin{array}{l} L(\underline{x}, \mu) = f(\underline{x}) - \mu h(\underline{x}) \text{ with } h \geq 0 \text{ (constraint)} \\ \text{In case (a): } \mu = 0, \; h(\underline{x}^*) > 0 \\ \text{In case (b): } \mu > 0, \; h(\underline{x}^*) = 0 \end{array}}$$

in both case, $\mu h(\underline{x}^*) = 0$

Summary: $\min f(\underline{x})$ s.t. $h(\underline{x}) \geq 0$

$L(\underline{x}, \mu) = f(\underline{x}) - \mu h(\underline{x})$
$\nabla_{\underline{x}, \mu} L(\underline{x}, \mu) = \underline{0}$

$$\left.\begin{array}{l} h(\underline{x}^*) \geq 0 \\ \mu \geq 0 \\ \mu h(\underline{x}^*) = 0 \end{array}\right\} \text{KKT conditions}$$

General case: multiple Equalities & multiple Inequalities constraints

$$\boxed{\begin{array}{l} \min f(\underline{x}) \text{ s.t. } g_i(\underline{x}) = 0 \;\; \forall i, \; h_j(\underline{x}) \geq 0 \;\; \forall j. \\ L(\underline{x}, \lambda, \mu) = f(\underline{x}) + \sum_{i=1}^{R} \lambda_i g_i(\underline{x}) - \sum_{i=1}^{R'} \mu_i h_j(\underline{x}) \\ \nabla_{\underline{x}, \lambda, \mu} L(\underline{x}, \lambda, \mu) = \underline{0} \qquad \text{Require: } \mu_j \geq 0 \;\; \forall j \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \mu_j h_j(\underline{x}^*) = 0 \;\; \forall j \\ \qquad\qquad\qquad\qquad\qquad \text{Given } g_i(\underline{x}^*) = 0 \;\; \forall i \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad h_j(\underline{x}^*) \geq 0 \;\; \forall j \end{array}}$$