Weekly Meeting

Topic: Construction of Optimal Designs under Σ-pattern

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Notations

Consider a symmetric OA, $\mathrm{OA}(n, m, s^p, t)$, s denotes the base level.

$\mathbb{Z}_s = \{0, 1, \ldots, s-1\}$ and $\mathbb{Z}_{s^p} = \{0, 1, \ldots, s^p - 1\}$ are collections of level settings.

1. For $x \in \mathbb{Z}_{s^p}$, let $f_i(x) = \lfloor x / s^{p-i} \rfloor \bmod s$, which represents the ith dight of x in the base-s numeral system.
2. Define $\rho(x) = p + 1 - \min\{i : f_i(x) \neq 0, i = 1, \ldots, p\}$ if $x \neq 0$ and $\rho(0) = 0$. $\rho(x)$ represents the number of digits needed to express x in the base-s numeral system (after eliminating all the leading zeros).

Notations

For $x \in \mathbb{Z}_{2^3}$,

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$\rho(x)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	2
3	0	1	1	2
4	1	0	0	3
5	1	0	1	3
6	1	1	0	3
7	1	1	1	3

Notations

3. Define an inverse inner product between $u, x \in \mathbb{Z}_{s^p}$ by $\langle u, x\rangle = f_p(u)f_1(x) + \cdots f_1(u)f_p(x) = \sum_{i=1}^{m} f_{p-i+1}(u)f_i(x)$.
4. For $u, x \in \mathbb{Z}_{s^p}$, define $\chi_u(x) = \xi^{\langle u,x\rangle}$ where $\xi = \exp(2\pi\mathrm{i}/s)$ with $\mathrm{i} = (-1)^{1/2}$.

Extend to m dimensions

For $u = (u_1, \ldots, u_m) \in \mathbb{Z}_{s^p}^m$ and $x = (x_1, \ldots, x_m) \in \mathbb{Z}_{s^p}^m$,

- $\rho(u) = \sum_{i=1}^{m} \rho(u_i)$.
- $\chi_u(x) = \prod_{i=1}^{m} \chi_{u_i}(x_i)$.

Characteristics

Let D be a design with n runs, m columns and s^p levels. For any $u \in \mathbb{Z}_{s^p}^m$, define $\chi_u(D) = \sum_{x \in \mathrm{D}} \chi_u(x)$, where x is a row of D.

Prop.

The set of all $\chi_u(D)$ fully characterizes the properties of D and uniquely determines D once given by n runs, m columns and s^p levels.

Space-filling Pattern

For $j = 0, \ldots, mp$, define

$$S_j(D) = n^{-2} \sum_{\rho(u)=j} |\chi_u(D)|^2,$$

where the summation is over all $u \in \mathbb{Z}_{s^p}^m$ with $\rho(u) = j$.

The vector $(S_1(D), \cdots, S_{mp}(D))$ is called the space-filling pattern.

Space-filling Pattern

Prop. 1

If $S_j(D_1) = S_j(D_2)$ for $j = 1, \ldots, l$ and $S_{l+1}(D_1) < S_{l+1}(D_2)$, then D_1 is more space-filling than D_2. Design D_1 is the most space-filling if there is no other design that is more space-filling than D_1.

Prop. 2

If the first j elements of the space-filling pattern are zeros, the general strong orthogonal array achieves stratification on any s^j grids from projection.

e.g. If $S_2(D) = 0$, then both s^2 grids on 1-dim and $s \times s$ on 2-dim grids achieve stratification.

Stratification Pattern

For $u = (u_1, \ldots, u_m) \in \mathbb{Z}_{s^p}^m$, define $wt(\mathbf{u})$ by the number of non-zeroes in the vector $\mathbf{u}$. Then, define stratification pattern by a $m \times mp$ matrix with

$$\xi_{i,j}(D) = n^{-2} \sum_{wt(\mathbf{u})=i} \sum_{\rho(\mathbf{u})=j} |\chi_u(D)|^2$$

Generally, $\xi_{i,j} = 0$ if $j < i$ or $j > ip$.

Minimizing D sequentially by

$(\xi_{1,1}(D), \xi_{1,2}(D), \xi_{2,2}(D), \ldots, \xi_{1,p}(D), \ldots, \xi_{p,p}(D) \ldots, \xi_{2,p+1}(D), \ldots, \xi_{p+1,p+1}(D), \ldots, \xi_{m,mp}(D))$

will give us a design with optimal stratification pattern.

Stratification Pattern

Simply put, stratification pattern **prioritize dimensions over grids** while minimizing.

Example (notations of i and j are switched)

Design D_1

j	i								
	1	2	3	4	5	6	7	8	9
1	0	0	0	·	·	·	·	·	·
2	·	0	0	9	6	6	·	·	·
3	·	·	1	0	3	6	12	12	8
Sum	0	0	1	9	9	12	12	12	8

Σ-pattern

With the similar settings of stratification patterns, instead we minimize D by

$$(\xi_{1,1}(D), \ldots, \xi_{1,p}(D), \xi_{2,2}(D), \ldots, \xi_{2,2p}(D), \ldots, \xi_{m,m}(D), \ldots, \xi_{m,mp}(D)).$$

Simply put, Σ-pattern **prioritize grids over dimensions** while minimizing.

Σ-pattern

Example (notations of i and j are switched)

Design D_1

j	i								
	1	2	3	4	5	6	7	8	9
1	0	0	0	.	.	.	.	.	.
2	.	0	0	9	6	6	.	.	.
3	.	.	1	0	3	6	12	12	8
Sum	0	0	1	9	9	12	12	12	8

TODO

- Read SOA strength 2+, implement an algorithm if an design A can be used to construct SOA 2+