

# Weekly Meeting

Topic: Construction of Optimal Designs under  $\Sigma$ -pattern

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# Notations

Consider a symmetric OA,  $\text{OA}(n, m, s^p, t)$ ,  $s$  denotes the base level.

$\mathbb{Z}_s = \{0, 1, \dots, s - 1\}$  and  $\mathbb{Z}_{s^p} = \{0, 1, \dots, s^p - 1\}$  are collections of level settings.

1. For  $x \in \mathbb{Z}_{s^p}$ , let  $f_i(x) = \lfloor x/s^{p-i} \rfloor \bmod s$ , which represents the  $i$ th digit of  $x$  in the base- $s$  numeral system.
2. Define  $\rho(x) = p + 1 - \min\{i : f_i(x) \neq 0, i = 1, \dots, p\}$  if  $x \neq 0$  and  $\rho(0) = 0$ .  $\rho(x)$  represents the number of digits needed to express  $x$  in the base- $s$  numeral system (after eliminating all the leading zeros).

# Notations

For  $x \in \mathbb{Z}_{2^3}$ ,

$x$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$\rho(x)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	2
3	0	1	1	2
4	1	0	0	3
5	1	0	1	3
6	1	1	0	3
7	1	1	1	3

# Notations

3. Define an inverse inner product between  $u, x \in \mathbb{Z}_{s^p}$  by

$$\langle u, x \rangle = f_p(u)f_1(x) + \cdots f_1(u)f_p(x) = \sum_{i=1}^m f_{p-i+1}(u)f_i(x).$$

4. For  $u, x \in \mathbb{Z}_{s^p}$ , define  $\chi_u(x) = \xi^{\langle u, x \rangle}$  where  $\xi = \exp(2\pi i/s)$  with  $i = (-1)^{1/2}$ .

## Extend to m dimensions

For  $u = (u_1, \dots, u_m) \in \mathbb{Z}_{s^p}^m$  and  $x = (x_1, \dots, x_m) \in \mathbb{Z}_{s^p}^m$ ,

- $\rho(u) = \sum_{i=1}^m \rho(u_i)$ .
- $\chi_u(x) = \prod_{i=1}^m \chi_{u_i}(x_i)$ .

# Characteristics

Let  $D$  be a design with  $n$  runs,  $m$  columns and  $s^p$  levels. For any  $u \in \mathbb{Z}_{s^p}^m$ , define  $\chi_u(D) = \sum_{x \in D} \chi_u(x)$ , where  $x$  is a row of  $D$ .

## **Prop.**

The set of all  $\chi_u(D)$  fully characterizes the properties of  $D$  and uniquely determines  $D$  once given by  $n$  runs,  $m$  columns and  $s^p$  levels.

# Space-filling Pattern

For  $j = 0, \dots, mp$ , define

$$S_j(D) = n^{-2} \sum_{\rho(u)=j} |\chi_u(D)|^2,$$

where the summation is over all  $u \in \mathbb{Z}_{s^p}^m$  with  $\rho(u) = j$ .

The vector  $(S_1(D), \dots, S_{mp}(D))$  is called the space-filling pattern.

# Space-filling Pattern

## Prop. 1

If  $S_j(D_1) = S_j(D_2)$  for  $j = 1, \dots, l$  and  $S_{l+1}(D_1) < S_{l+1}(D_2)$ , then  $D_1$  is more space-filling than  $D_2$ . Design  $D_1$  is the most space-filling if there is no other design that is more space-filling than  $D_1$ .

## Prop. 2

If the first  $j$  elements of the space-filling pattern are zeros, the general strong orthogonal array achieves stratification on any  $s^j$  grids from projection.

e.g. If  $S_2(D) = 0$ , then both  $s^2$  grids on 1-dim and  $s \times s$  on 2-dim grids achieve stratification.

# Stratification Pattern

For  $u = (u_1, \dots, u_m) \in \mathbb{Z}_{s^p}^m$ , define  $wt(\mathbf{u})$  by the number of non-zeroes in the vector  $\mathbf{u}$ . Then, define stratification pattern by a  $m \times mp$  matrix with

$$\xi_{i,j}(D) = n^{-2} \sum_{wt(\mathbf{u})=i} \sum_{\rho(\mathbf{u})=j} |\chi_u(D)|^2$$

Generally,  $\xi_{i,j} = 0$  if  $j < i$  or  $j > ip$ .

Minimizing  $D$  sequentially by

$$(\xi_{1,1}(D), \xi_{1,2}(D), \xi_{2,2}(D), \dots, \xi_{1,p}(D), \dots, \xi_{p,p}(D) \dots, \xi_{2,p+1}(D), \dots, \xi_{p+1,p+1}(D), \dots, \xi_{m,mp}(D))$$

will give us a design with optimal stratification pattern.



# Stratification Pattern

Simply put, stratification pattern **prioritize dimensions over grids** while minimizing.

**Example (notations of  $i$  and  $j$  are switched)**

Design $D_1$									
$j$	$i$								
	1	2	3	4	5	6	7	8	9
1	0	0	0	.	.	.	.	.	.
2	.	0	0	9	6	6	.	.	.
3	.	.	1	0	3	6	12	12	8
Sum	0	0	1	9	9	12	12	12	8

# $\Sigma$ -pattern

With the similar settings of stratification patterns, instead we minimize  $D$  by

$$(\xi_{1,1}(D), \dots, \xi_{1,p}(D), \xi_{2,2}(D), \dots, \xi_{2,2p}(D), \dots, \xi_{m,m}(D), \dots, \xi_{m,mp}(D)).$$

Simply put,  $\Sigma$ -pattern **prioritize grids over dimensions** while minimizing.

# $\Sigma$ -pattern

**Example (notations of  $i$  and  $j$  are switched)**

Design $D_1$									
$j$	$i$								
	1	2	3	4	5	6	7	8	9
1	0	0	0	.	.	.	.	.	.
2	.	0	0	9	6	6	.	.	.
3	.	.	1	0	3	6	12	12	8
Sum	0	0	1	9	9	12	12	12	8

# TODO

- Read SOA strength 2+, implement an algorithm to check if a design  $A$  can be used to construct SOA 2+