# Weekly Meeting

Topic: Construction of Optimal Designs under $\Sigma$-pattern

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# Notations

Consider a symmetric OA, $\mathrm{OA}(n, m, s^p, t)$, $s$ denotes the base level.

$\mathbb{Z}_s = \{0, 1, \ldots, s-1\}$ and $\mathbb{Z}_{s^p} = \{0, 1, \ldots, s^p - 1\}$ are collections of level settings.

1. For $x \in \mathbb{Z}_{s^p}$, let $f_i(x) = \lfloor x/sp - i \rfloor \bmod s$, which represents the $i$th dight of $x$ in the base-$s$ numeral system.
2. Define $\rho(x) = p + 1 - \min\{i : f_i(x) \neq 0, i = 1, \ldots, p\}$ if $x \neq 0$ and $\rho(0) = 0$. $\rho(x)$ represents the number of digits needed to express $x$ in the base-$s$ numeral system (after eliminating all the leading zeros).

# Notations

For $x \in \mathbb{Z}_{2^3}$,

| $x$ | $f_1(x)$ | $f_2(x)$ | $f_3(x)$ | $\rho(x)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 |
| 2 | 0 | 1 | 0 | 2 |
| 3 | 0 | 1 | 1 | 2 |
| 4 | 1 | 0 | 0 | 3 |
| 5 | 1 | 0 | 1 | 3 |
| 6 | 1 | 1 | 0 | 3 |
| 7 | 1 | 1 | 1 | 3 |

# Notations

3. Define an inverse inner product between $u, x \in \mathbb{Z}_{s^p}$ by
   $\langle u, x \rangle = f_p(u) f_1(x) + \cdots f_1(u) f_p(x) = \sum_{i=1}^{m} f_{p-i+1}(u) f_i(x)$.
4. For $u, x \in \mathbb{Z}_{s^p}$, define $\chi_u(x) = \xi^{\langle u, x \rangle}$ where $\xi = \exp(2\pi \mathrm{i}/s)$ with $\mathrm{i} = (-1)^{1/2}$.

**Extend to m dimensions**

For $u = (u_1, \ldots, u_m) \in \mathbb{Z}_{s^p}^m$ and $x = (x_1, \ldots, x_m) \in \mathbb{Z}_{s^p}^m$,

- $\rho(u) = \sum_{i=1}^{m} \rho(u_i)$.
- $\chi_u(x) = \prod_{i=1}^{m} \chi_{u_i}(x_i)$.

# Characteristics

Let $D$ be a design with $n$ runs, $m$ columns and $s^p$ levels. For any $u \in \mathbb{Z}_{s^p}^m$, define $\chi_u(D) = \sum_{x \in \mathrm{D}} \chi_u(x)$, where $x$ is a row of $D$.

**Prop.**

The set of all $\chi_u(D)$ fully characterizes the properties of $D$ and uniquely determines $D$ once given by $n$ runs, $m$ columns and $s^p$ levels.

# Space-filling Pattern

For $j = 0, \ldots, mp$, define

$$S_j(D) = n^{-2} \sum_{\rho(u)=j} |\chi_u(D)|^2,$$

where the summation is over all $u \in \mathbb{Z}_{s^p}^m$ with $\rho(u) = j$.

The vector $(S_1(D), \cdots, S_{mp}(D))$ is called the space-filling pattern.

# Space-filling Pattern

**Prop. 1**

If $S_j(D_1) = S_j(D_2)$ for $j = 1, \ldots, l$ and $S_{l+1}(D_1) < S_{l+1}(D_2)$, then $D_1$ is more space-filling than $D_2$. Design $D_1$ is the most space-filling if there is no other design that is more space-filling than $D_1$.

**Prop. 2**

If the first $j$ elements of the space-filling pattern are zeros, the general strong orthogonal array achieves stratification on any $s^j$ grids from projection.

e.g. If $S_2(D) = 0$, then both $s^2$ grids on 1-dim and $s \times s$ on 2-dim grids achieve stratification.

# Stratification Pattern

For $u = (u_1, \ldots, u_m) \in \mathbb{Z}_{s^p}^m$, define $wt(\mathbf{u})$ by the number of non-zeroes in the vector $\mathbf{u}$. Then, define stratification pattern by a $m \times mp$ matrix with

$$\xi_{i,j}(D) = n^{-2} \sum_{wt(\mathbf{u})=i} \sum_{\rho(\mathbf{u})=j} |\chi_u(D)|^2$$

Generally, $\xi_{i,j} = 0$ if $j < i$ or $j > ip$.

Minimizing $D$ sequentially by

$$(\xi_{1,1}(D), \xi_{1,2}(D), \xi_{2,2}(D), \ldots, \xi_{1,p}(D), \ldots, \xi_{p,p}(D) \ldots, \xi_{2,p+1}(D), \ldots, \xi_{p+1,p+1}(D), \ldots, \xi_{m,mp}(D))$$

will give us a design with optimal stratification pattern.

# Stratification Pattern

Simply put, stratification pattern **prioritize dimensions over grids** while minimizing.

**Example (notations of $i$ and $j$ are switched)**

Design $D_1$

| $j$ | $i$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 0 | 0 | 0 | · | · | · | · | · | · |
| 2 | · | 0 | 0 | 9 | 6 | 6 | · | · | · |
| 3 | · | · | 1 | 0 | 3 | 6 | 12 | 12 | 8 |
| Sum | 0 | 0 | 1 | 9 | 9 | 12 | 12 | 12 | 8 |

# $\Sigma$-pattern

With the similar settings of stratification patterns, instead we minimize $D$ by

$$(\xi_{1,1}(D), \ldots, \xi_{1,p}(D), \xi_{2,2}(D), \ldots, \xi_{2,2p}(D), \ldots, \xi_{m,m}(D), \ldots, \xi_{m,mp}(D)).$$

Simply put, $\Sigma$-pattern **prioritize grids over dimensions** while minimizing.

# $\Sigma$-pattern

**Example (notations of $i$ and $j$ are switched)**

Design $D_1$

| $j$ | $i$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 0 | 0 | 0 | . | . | . | . | . | . |
| 2 | . | 0 | 0 | 9 | 6 | 6 | . | . | . |
| 3 | . | . | 1 | 0 | 3 | 6 | 12 | 12 | 8 |
| Sum | 0 | 0 | 1 | 9 | 9 | 12 | 12 | 12 | 8 |