Weekly Meeting

Topic: Construction Algorithm for SOA of strength 3 with property α

Presenter: Heng-Tse Chou @ NTHU STAT

Date: Jul. 5, 2024

Goal

- We want to find a metric that can measure, or quantify how close a SOA of strength 3 is to have property α, that is, stratification on $s^2 \times s^2$ grids in all 2-dims for $s = 3$.
- This metric should help us in constructing SOA of strength 3 with the desired property.

Minimum moment aberration

(from Xu 2003, but with our notations)

With n runs, m factors and s levels, the p-th power moment of D is defined by $K_p = (n(n-1)/2)^{-1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} (\delta(x_i, x_j))^p$,

where $\delta(x_i, x_j) = \sum_{k=1}^{m} 1(x_{ik} = x_{jk})$.

$m - \delta(x_i, x_j)$ is known as the Hamming distance between x_i and x_j.

Minimum moment aberration

(from Xu 2003)

The power moments measure the similarity among runs.

The first and second power moments measure the average and variance of the similarity among runs.

Minimizing the power moments makes runs be as dissimilar as possible.

Minimum moment aberration

These lower bounds can be used to check if a design is OA.

- $K_1(D) \geq (m(n-s))/((n-1)s)$, with equality iff D is $\mathrm{OA}(1)$.
- $K_2(D) \geq (nm(n+s-1)-(ms)^2)/((n-1)s^2)$, with equality iff D is $\mathrm{OA}(2)$.
- $K_3(D) \geq (nm(m^2+3ms+s^2-3m-3s+2)-(ms)^3)/((n-1)s^3)$, with equality iff D is $\mathrm{OA}(3)$.

J_2-optimality

(Xu 2000)

Now let $\delta_{i,j}(D) = \sum_{k=1}^{m} w_k \delta(x_{ik}, x_{jk})$, where $\delta(x, y) = 1$ if $x = y$ and 0 otherwise.

Define $J_2(D) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} [\delta_{i,j}(D)]^2$.

A design is called J_2-optimal if it minimizes J_2.

J_2-optimality

For a mixed level design D with each column having s_k levels,

$$L(m) = 2^{-1}\left[(\sum_{k=1}^{m} ns_k^{-1}w_k)^2 + (\sum_{k=1}^{m}(s_k-1)(ns_k^{-1}w_k)^2) - n(\sum_{k=1}^{m} w_k)^2\right].$$

We have $J_2(D) \geq L(m)$, where the equality holds iff D is an OA of strength 2.

The J_2-optimality is a special case of the minimum moment aberration.

The construction algorithm

The algorithm is aimed to generate mixed level OA and NOA.

Main idea: sequentially add columns to an existing design.

(1) Consider adding a column $c = (c_1, \ldots, c_n)'$ to D.

Let D_+ be the new $n \times (m+1)$ design.

If c has s_p levels and weight w_p, then $\delta_{i,j}(D_+) = \delta_{i,j}(D) + \delta_{i,j}(c)$,

where $\delta_{i,j}(c) = w_p \delta(c_i, c_j)$.

The construction algorithm

Moreover,

$$J_2(D_+) = J_2(D) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\delta_{i,j}(D)\delta_{i,j}(c) + (n/2)w_p^2(s_p/n - 1)$$

if the added column is balanced.

The construction algorithm

(2) Consider switching a pair of symbols in the added column.

Suppose the symbols in rows a and b of the added column are distinct, i.e., $c_a \neq c_b$.

If these two symbols are exchanged, then all $\delta_{i,j}(c)$ are unchanged except that $\delta_{a_j}(c) = \delta_{j,a}(c)$ and $\delta_{b,j}(c) = \delta_{j,b}(c)$ are switched for $j \neq a, b$.

The construction algorithm

Hence, $J_2(D_+)$ is reduced by $2S(a,b)$, where

$$S(a,b) = -\sum_{1 \leq j \neq a,b \leq n} \left[\delta_{a,j}(D) - \delta_{b,j}(D)\right]\left[\delta_{a,j}(c) - \delta_{b,j}(c)\right]$$

For each candidate column, the algorithm searches all possible interchanges and makes an interchange which reduces J_2 most.

The construction algorithm

The algorithm is given as follows:

1. For $p = 1, \ldots, n$, compute the lower bound $L(p)$.
2. Specify an initial design D with two columns: $(0, \ldots, 0, 1, \ldots, 1, \ldots, s_1 - 1, \ldots, s_1 - 1)$ and $(0, \ldots, s_2 - 1, 0, \ldots, s_2 - 1, \ldots, 0, \ldots, s_2 - 1)$. Compute $\delta_{i,j}(D)$ and $J_2(D)$. If $J_2(D) = L(2)$, let $m_0 = 2$ and $T = T_1$; otherwise, let $m_0 = 0$ and $T = T_2$.

The construction algorithm

3. For $p = 3, \ldots, m$, do the following:

(a) Randomly generate a balanced s_p-level column c. Compute $J_2(D_+)$. If $J_2(D_+) = L(p)$, go to (d).

(b) For all pairs of row a and b with distinct symbols, compute $S(a, b)$. Choose a pair of rows with largest $S(a, b)$ and exchange the symbols in rows a and b of column c. Reduce $J_2(D_+)$ by $2S(a, b)$. If $J_2(D_+) = L(p)$, go to (d); otherwise, repeat this procedure until no further improvement is made.

The construction algorithm

(c) Repeat (a) and (b) T times and choose a column c that produces the smallest $J_2(D_+)$.

(d) Add column c as the p-th column of D, let $J_2(D) = J_2(D_+)$ and update $\delta_{i,j}(D)$. If $J_2(D_+) = L(p)$, let $m_0 = p$; otherwise, let $T = T_2$.

The construction algorithm

4. Return the final $n \times m$ design D, of which the first m_0 columns form an OA.

TODO

- Reproducing the construction algorithm, making sure it works.
- Applying minimum moment aberration and the construction algorithm for SOA of strength 3 with property α.