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1 |----- MODULE Voting -----|
  | This is a high-level algorithm in which a set of processes cooperatively choose a value.
6 | EXTENDS Integers
  |-----|
8 | CONSTANTS
9   | Value,      | The set of choosable values.
10  | Acceptor,    | A set of processes that will choose a value.
11  | Quorum      | The set of “quorums”, where a “quorum” is a “large enough” set of acceptors.
13 | ASSUME QuorumAssumption  $\triangleq$ 
14   |  $\wedge \forall Q \in \textit{Quorum} : Q \subseteq \textit{Acceptor}$ 
15   |  $\wedge \forall Q1, Q2 \in \textit{Quorum} : Q1 \cap Q2 \neq \{\}$ 
17 | THEOREM QuorumNonEmpty  $\triangleq \forall Q \in \textit{Quorum} : Q \neq \{\}$ 
  |-----|
19 | Ballot  $\triangleq \textit{Nat}$  | The set of “ballot numbers”.
20 |-----|
  | Each acceptor can cast one or more votes, where each vote cast by an acceptor has the form  $\langle b, v \rangle$ 
  | indicating that the acceptor has voted for value  $v$  in ballot  $b$ .
26 | VARIABLES
27   | votes,      | votes[ $a$ ]: the set of votes cast by acceptor  $a$ 
28   | maxBal     | maxBal[ $a$ ]: a ballot number.
29   |             | Acceptor  $a$  will cast further votes only in ballots numbered  $\geq \textit{maxBal}[a]$ .
31 | TypeOK  $\triangleq$ 
32   |  $\wedge \textit{votes} \in [\textit{Acceptor} \rightarrow \text{SUBSET} (\textit{Ballot} \times \textit{Value})]$ 
33   |  $\wedge \textit{maxBal} \in [\textit{Acceptor} \rightarrow \textit{Ballot} \cup \{-1\}]$ 
34 |-----|
35 | VotedFor( $a, b, v$ )  $\triangleq \langle b, v \rangle \in \textit{votes}[a]$  | Acceptor  $a$  has voted for  $v$  in ballot  $b$ .
37 | ChosenAt( $b, v$ )  $\triangleq \langle b, v \rangle$  is chosen if a quorum of acceptors have voted for it.
38   |  $\exists Q \in \textit{Quorum} :$ 
39     |  $\forall a \in Q : \textit{VotedFor}(a, b, v)$ 
41 | chosen  $\triangleq$  | The set of values that have been chosen.
42   |  $\{v \in \textit{Value} : \exists b \in \textit{Ballot} : \textit{ChosenAt}(b, v)\}$ 
43 |-----|
44 | DidNotVoteAt( $a, b$ )  $\triangleq$  | The acceptor  $a$  did not vote (for any value) at ballot  $b$ .
45   |  $\forall v \in \textit{Value} : \neg \textit{VotedFor}(a, b, v)$ 
47 | CannotVoteAt( $a, b$ )  $\triangleq$  | The acceptor  $a$  cannot vote (for any value) at ballot  $b$ .
48   |  $\wedge \textit{DidNotVoteAt}(a, b)$ 
49   |  $\wedge \textit{maxBal}[a] > b$ 
51 | NoneOtherChoosableAt( $b, v$ )  $\triangleq$  | ChosenAt( $b, w$ ) is not and can never become true for any  $w \neq v$ .
52   |  $\exists Q \in \textit{Quorum} :$ 
53     |  $\forall a \in Q : \textit{VotedFor}(a, b, v) \vee \textit{CannotVoteAt}(a, b)$ 

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55 THEOREM *ChoosableThm* \triangleq
56 $\forall b \in \text{Ballot}, v \in \text{Value} :$
57 $\text{ChosenAt}(b, v) \Rightarrow \text{NoneOtherChoosableAt}(b, v)$
58

59 *SafeAt*(b, v) \triangleq No value other than v has been or can ever be chosen in any ballot $< b$.
60 $\forall c \in 0 \dots (b - 1) : \text{NoneOtherChoosableAt}(c, v)$
62 THEOREM *AllSafeAtZero* $\triangleq \forall v \in \text{Value} : \text{SafeAt}(0, v)$
63

64 *OneVote* $\triangleq \forall a \in \text{Acceptor}, b \in \text{Ballot}, v, w \in \text{Value} :$
65 $\text{VotedFor}(a, b, v) \wedge \text{VotedFor}(a, b, w) \Rightarrow (v = w)$
67 *OneValuePerBallot* \triangleq
68 $\forall a1, a2 \in \text{Acceptor}, b \in \text{Ballot}, v1, v2 \in \text{Value} :$
69 $\text{VotedFor}(a1, b, v1) \wedge \text{VotedFor}(a2, b, v2) \Rightarrow (v1 = v2)$
71 THEOREM *OneValuePerBallot* \Rightarrow *OneVote*
72

73 *VotesSafe* $\triangleq \forall a \in \text{Acceptor}, b \in \text{Ballot}, v \in \text{Value} :$
74 $\text{VotedFor}(a, b, v) \Rightarrow \text{SafeAt}(b, v)$
76 THEOREM *VotesSafeImpliesConsistency* \triangleq
77 $\wedge \text{TypeOK}$
78 $\wedge \text{VotesSafe}$
79 $\wedge \text{OneVote}$
80 $\Rightarrow \vee \text{chosen} = \{\}$
81 $\vee \exists v \in \text{Value} : \text{chosen} = \{v\}$
82

83 *ShowsSafeAt*(Q, b, v) \triangleq
84 $\wedge \forall a \in Q : \text{maxBal}[a] \geq b$
85 $\wedge \exists c \in -1 \dots (b - 1) :$
86 $\wedge (c \neq -1) \Rightarrow \exists a \in Q : \text{VotedFor}(a, c, v)$
87 $\wedge \forall d \in (c + 1) \dots (b - 1), a \in Q : \text{DidNotVoteAt}(a, d)$
89 THEOREM *ShowsSafety* \triangleq
90 $\text{TypeOK} \wedge \text{VotesSafe} \wedge \text{OneValuePerBallot} \Rightarrow$
91 $\forall Q \in \text{Quorum}, b \in \text{Ballot}, v \in \text{Value} :$
92 $\text{ShowsSafeAt}(Q, b, v) \Rightarrow \text{SafeAt}(b, v)$
93

94 *Init* \triangleq
95 $\wedge \text{votes} = [a \in \text{Acceptor} \mapsto \{\}]$
96 $\wedge \text{maxBal} = [a \in \text{Acceptor} \mapsto -1]$
98 *IncreaseMaxBal*(a, b) \triangleq Acceptor a is allowed to increase $\text{maxBal}[a]$ to a ballot number.
99 $\wedge b > \text{maxBal}[a]$
100 $\wedge \text{maxBal}' = [\text{maxBal} \text{ EXCEPT } ![a] = b]$
101 $\wedge \text{UNCHANGED votes}$

103 $VoteFor(a, b, v) \triangleq$ Acceptor a votes for v in ballot b .
104 $\wedge \maxBal[a] \leq b$ The acceptor cannot cast a vote in a ballot less than $\maxBal[a]$
105 $\wedge \forall vt \in votes[a] : vt[1] \neq b$ to maintain *OneValuePerBallot*
106 $\wedge \forall c \in Acceptor \setminus \{a\} :$
107 $\quad \forall vt \in votes[c] : (vt[1] = b) \Rightarrow (vt[2] = v)$
108 $\wedge \exists Q \in Quorum : ShowsSafeAt(Q, b, v)$ to maintain *VotesSafe*
109 $\wedge votes' = [votes \text{ EXCEPT } ![a] = @ \cup \{(b, v)\}]$
110 $\wedge \maxBal' = [\maxBal \text{ EXCEPT } ![a] = b]$
111 \mid
112 $Next \triangleq$
113 $\quad \exists a \in Acceptor, b \in Ballot :$
114 $\quad \vee IncreaseMaxBal(a, b)$
115 $\quad \vee \exists v \in Value : VoteFor(a, b, v)$
117 $Spec \triangleq Init \wedge \Box [Next]_{\langle votes, \maxBal \rangle}$
118 \mid
119 $Inv \triangleq TypeOK \wedge VotesSafe \wedge OneValuePerBallot$
121 THEOREM *Invariance* $\triangleq Spec \Rightarrow \Box Inv$
122 \mid
123 $C \triangleq$ INSTANCE *Consensus* WITH $Value \leftarrow Value, chosen \leftarrow chosen$
125 THEOREM $Spec \Rightarrow C!Spec$
126 $\langle 1 \rangle 1. Inv \wedge Init \Rightarrow C!Init$
127 $\langle 1 \rangle 2. Inv \wedge [Next]_{\langle votes, \maxBal \rangle} \Rightarrow [C!Next]_{chosen}$
128 $\langle 1 \rangle 3.$ QED
129 $\quad \langle 2 \rangle 1. \Box Inv \wedge \Box [Next]_{\langle votes, \maxBal \rangle} \Rightarrow \Box [C!Next]_{chosen}$
130 \quad BY $\langle 1 \rangle 2$ and temporal reasoning
131 $\quad \langle 2 \rangle 2. \Box Inv \wedge Spec \Rightarrow C!Spec$
132 \quad BY $\langle 2 \rangle 1, \langle 1 \rangle 1$
133 $\quad \langle 2 \rangle 3.$ QED
134 \quad BY $\langle 2 \rangle 2, Invariance$
135 \mid