

Discrete Probability

"Life is a school of probability — Walter Bagehot"

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June 07, 2018



Two Extra Tasks

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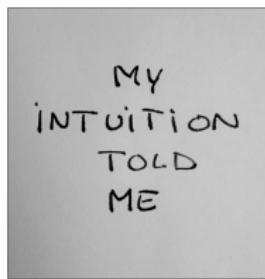
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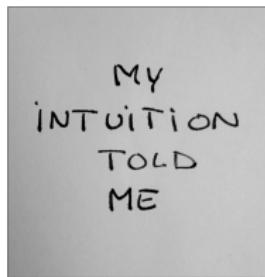


Q : What is probability?



*"...and the many **paradoxes** show clearly that we, as humans, lack a well grounded intuition in this matter."*

— “*The Art of Probability*”, Richard W. Hamming



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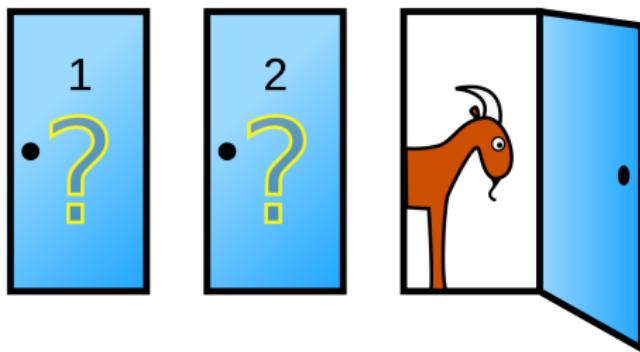
*“When called upon to judge probability, people actually judge something else and **believe** they have judged probability.”*

— “*Thinking, Fast and Slow*”, Daniel Kahneman

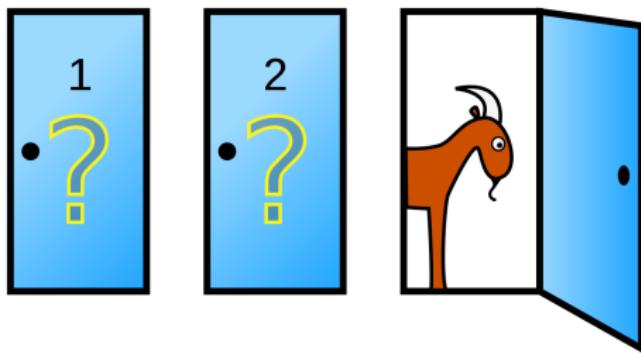


Let us calculate [calculemus].

The Monty-Hall Problem



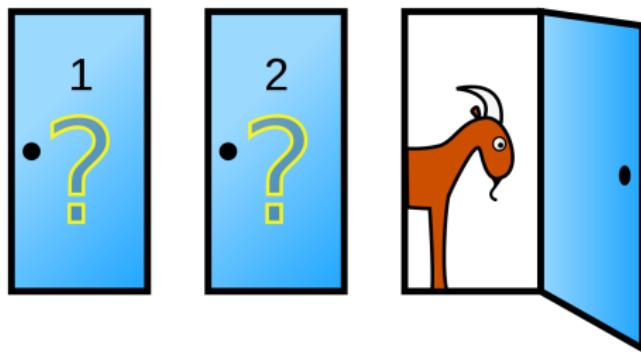
The Monty-Hall Problem



You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)

The Monty-Hall Problem



You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)

Q : Do you want to switch to door 2?

C_i : The car is behind door i ($i = 1, 2, 3$)

$$\Pr \{C_i\} = \frac{1}{3}$$

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ASSUMPTION: The car is initially hidden randomly behind the doors.

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Y_1 : you initially pick door 1

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C_i : The car is behind door i ($i = 1, 2, 3$)

$$\Pr \{C_i\} = \frac{1}{3}$$

ASSUMPTION: The car is initially hidden randomly behind the doors.

Y_1 : you initially pick door 1

$$\Pr \{Y_1\} = \frac{1}{3}$$

ASSUMPTION: Your initial choice is random.

I_3 : I open door 3 **and** happen to reveal a goat

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ASSUMPTION: I know what's behind the doors.

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ASSUMPTION: If you initially pick the car, then I open a door randomly.

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ASSUMPTION: If you initially pick the car, then I open a door randomly.

ASSUMPTION: I always open a door to reveal a goat and never the car.

I_3 : I open door 3 **and** happen to reveal a goat

ASSUMPTION: I know what's behind the doors.

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ASSUMPTION: I always open a door to reveal a goat and never the car.

$$\Pr \{C_2 \mid I_3, Y_1\}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{C_2, I_3, Y_1\}}{\Pr \{I_3, Y_1\}}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{C_2, I_3, Y_1\}}{\Pr \{I_3, Y_1\}} = \frac{\Pr \{I_3, Y_1 | C_2\} \Pr \{C_2\}}{\Pr \{I_3 | Y_1\} \Pr \{Y_1\}}$$

$$\begin{aligned}\Pr \{C_2 \mid I_3, Y_1\} &= \frac{\Pr \{C_2, I_3, Y_1\}}{\Pr \{I_3, Y_1\}} = \frac{\Pr \{I_3, Y_1 \mid C_2\} \Pr \{C_2\}}{\Pr \{I_3 \mid Y_1\} \Pr \{Y_1\}} \\ &= \frac{\Pr \{I_3, Y_1 \mid C_2\}}{\Pr \{I_3 \mid Y_1\}}\end{aligned}$$

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$$\Pr \{I_3, Y_1 \mid C_2\} = \Pr \{I_3 \mid C_2, Y_1\} \Pr \{Y_1 \mid C_2\}$$

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$$\begin{aligned}\Pr \{I_3, Y_1 \mid C_2\} &= \Pr \{I_3 \mid C_2, Y_1\} \Pr \{Y_1 \mid C_2\} \\ &= \frac{1}{3} \Pr \{I_3 \mid C_2, Y_1\}\end{aligned}$$

$$\begin{aligned}\Pr \{C_2 | I_3, Y_1\} &= \frac{\Pr \{C_2, I_3, Y_1\}}{\Pr \{I_3, Y_1\}} = \frac{\Pr \{I_3, Y_1 | C_2\} \Pr \{C_2\}}{\Pr \{I_3 | Y_1\} \Pr \{Y_1\}} \\ &= \frac{\Pr \{I_3, Y_1 | C_2\}}{\Pr \{I_3 | Y_1\}}\end{aligned}$$

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$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{3 \Pr \{I_3 | Y_1\}}$$

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It depends on how I choose the door to open!

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It depends on how I choose the door to open!

$$\Pr \{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

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$$\boxed{\Pr \{C_2 \mid I_3, Y_1\} = \frac{2}{3}}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\} + \Pr \{I_3 \mid C_3, Y_1\}}$$

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$$\Pr \{C_1 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_1, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

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$$\Pr \{C_2 | I_3, Y_1\} > \Pr \{C_1 | I_3, Y_1\} \iff \Pr \{I_3 | C_2, Y_1\} > \Pr \{I_3 | C_1, Y_1\}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

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Q : Switching vs. Choosing between the two remaining doors randomly?

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{1}{1+q}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

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$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr \{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

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$$\boxed{\Pr \{C_2 \mid I_3, Y_1\} > \Pr \{C_1 \mid I_3, Y_1\} \iff \Pr \{I_3 \mid C_2, Y_1\} > \Pr \{I_3 \mid C_1, Y_1\}}$$

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$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

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Always Switch!

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

$$\Pr \{C_2 | I_3, Y_1\} > \Pr \{C_1 | I_3, Y_1\} \iff \Pr \{I_3 | C_2, Y_1\} > \Pr \{I_3 | C_1, Y_1\}$$

ASSUMPTION: I KNOW.

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{2}{3}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

$$\Pr \{C_2 | I_3, Y_1\} > \Pr \{C_1 | I_3, Y_1\} \iff \Pr \{I_3 | C_2, Y_1\} > \Pr \{I_3 | C_1, Y_1\}$$

ASSUMPTION: I DON'T KNOW EITHER.

ASSUMPTION: I KNOW.

Opens one randomly that happens not to reveal the car.

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{2}{3}$$

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$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

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Opens one randomly that happens not to reveal the car.

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

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Monty Hall problem (wiki)

The Boy/Girl Puzzle



Both Girls (CS Problem 5.3 – 12)

Mr. and Mrs. Smith have two children of different ages,
what is the probability that they has **two girls**,

- (a) given that **one of the children** is a girl?
- (b) given that **the older child** is a girl?



G_1 : the older child is a girl

G_2 : the younger child is a girl

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1 \vee G_2\}}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

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$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$

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Both Girls (CS Problem 5.3 – 12)

Mr. and Mrs. Smith have two children of different ages,
what is the probability that they has **two girls**,

- (a) given that **one of the children** is a girl?

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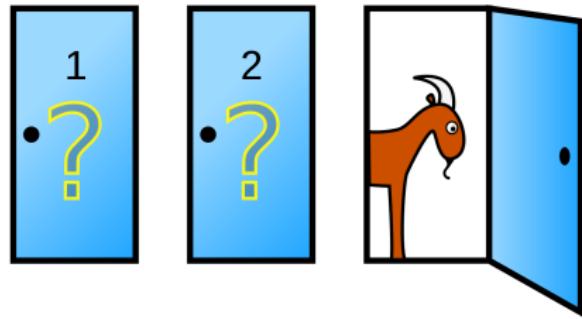
- (I) I **KNOW** them well and I tell you that “one of the children is a girl”.

Q : How do you know that “one of the children is a girl”?



- (I) I **KNOW** them well and I tell you that “one of the children is a girl”.
- (II) I **DON'T KNOW** them. I just open a room door and see a girl.

The Monty-Hall Problem Comes Back



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After-class Exercise:

A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?





Boy or Girl paradox (wiki)



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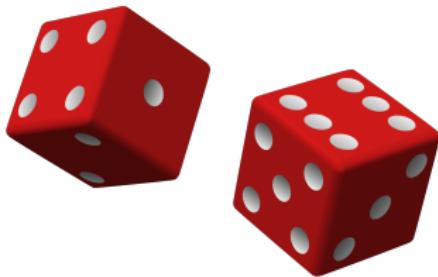
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$$(E = EF \cup EF^c)$$



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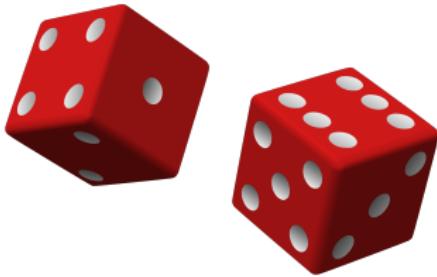
$Q : E \perp F \wedge E \perp G \implies E \perp (FG) ?$

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$$(d_1, d_2)$$

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$$F : d_1 = 4 \quad G : d_2 = 3$$

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Let X be a discrete random variable that takes on **only nonnegative integer values** \mathbb{N} .

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Proof.

$$\sum_{j=1}^{\infty} \sum_{i=1}^j \Pr(X = j) = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \Pr(X = j)$$



Searching an Unsorted Array (CLRS Problem 5 – 2 (f))

```
1: procedure DETERMINISTIC-SEARCH( $A[1 \cdots n]$ ,  $x$ )
2:    $i \leftarrow 1$ 
3:   while  $i \leq n$  do
4:     if  $A[i] = x$  then
5:       return true
6:      $i \leftarrow i + 1$ 
7:   return false
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(e)

$$\exists! i : A[i] = x$$

(f)

$$\exists!_k i : A[i] = x$$

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$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$



Summation by parts ([Abel transformation](#); wiki)

How Did I (an ant) Evaluate this Summation:

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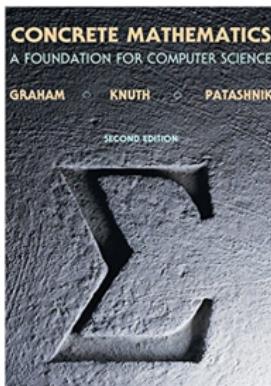
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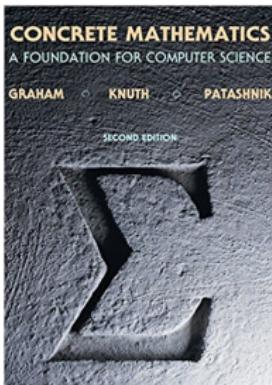
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Chapter 5: Binomial Coefficients

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$$r \binom{r-1}{k-1} = k \binom{r}{k}$$

$$\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

$$\begin{aligned}
\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} &= \sum_{i=0}^{n-k} (i+1) \binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k} ((n+1) - (n-i)) \binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k} (n+1) \binom{n-i-1}{k-1} - \sum_{i=0}^{n-k} (n-i) \binom{n-i-1}{k-1} \\
&= (n+1) \sum_{i=0}^{n-k} \binom{n-i-1}{k-1} - k \sum_{i=0}^{n-k} \binom{n-i}{k} \\
&= (n+1) \sum_{m=k-1}^{n-1} \binom{m}{k-1} - k \sum_{m=k}^n \binom{m}{k} \\
&= (n+1) \binom{n}{k} - k \binom{n+1}{k+1} = \binom{n+1}{k+1}
\end{aligned}$$

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Inversions (CLRS Problem 5.2 – 5)

$A[1 \cdots n]$ of n distinct numbers

(i, j) is an **inversion** of $A : i < j \wedge A[i] > A[j]$

$X : \#$ of inversions in A

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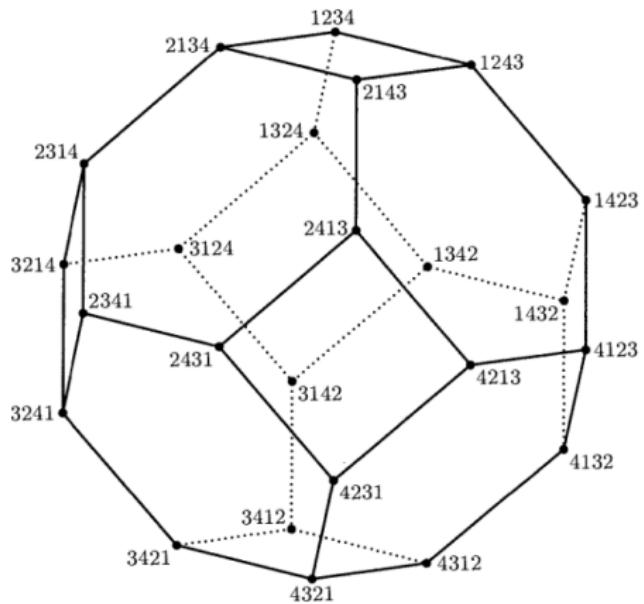
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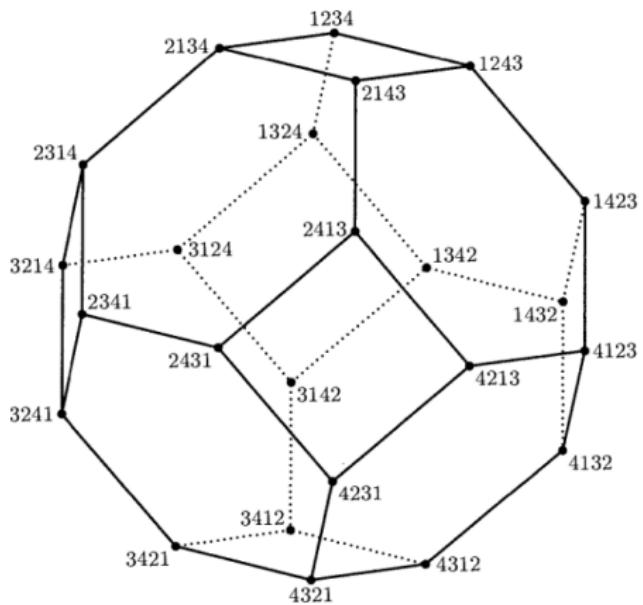
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$$\langle 3214 \rangle \sim \langle 4123 \rangle$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

$$\left(\mathbb{E}[X] = \sum_x x \Pr(X = x) \right)$$

Theorem (CS Theorem 5.23))

Let X be a random variable defined on a sample space Ω .

Let E_1, E_2, \dots, E_n be a **partition** of Ω .

$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X \mid E_i] \Pr(E_i)$$

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Proof.

By definition.

$$\sum_x x \sum_{i=1}^n \Pr(X = x, E_i) = \sum_{i=1}^n \sum_x x \Pr(X = x, E_i)$$



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A : \$1.00; Repeat

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$$4 * A + 1 * Q = \$8.00$$

In-class Exercise: Coin Pattern (Provided by Yifan Pei)



X : # of tosses to get 3 consecutive heads (HHH)

$$\mathbb{E}[X]$$

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T , HT , HHT , HHH

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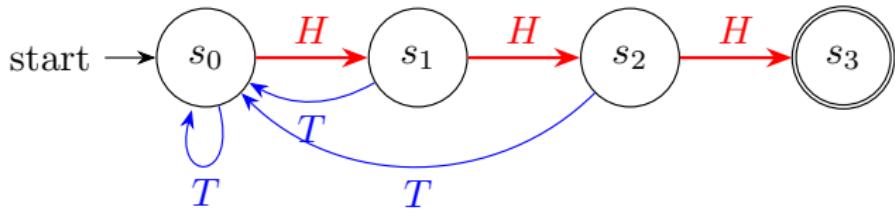
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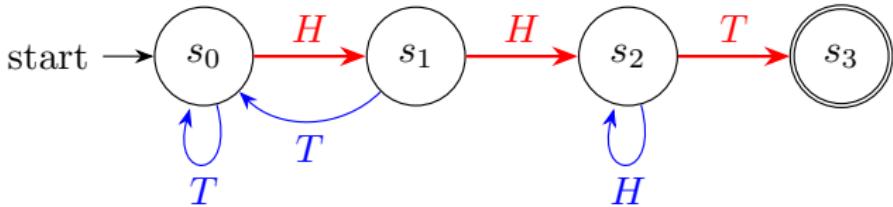
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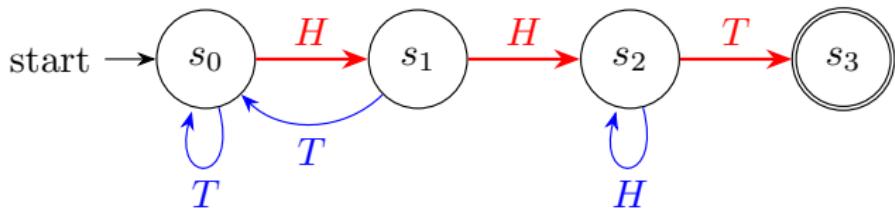
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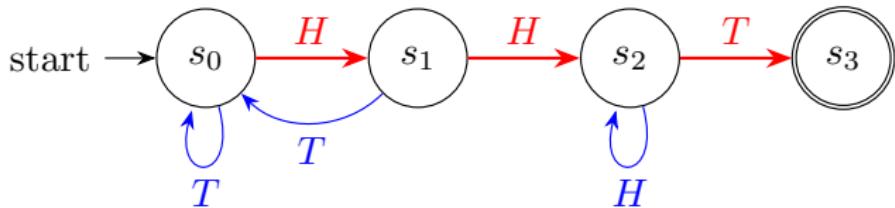
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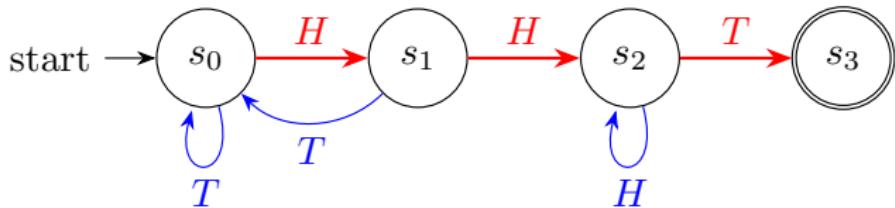
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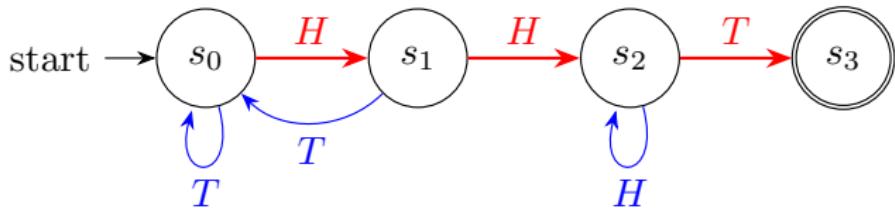
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$$S_3 = 0$$



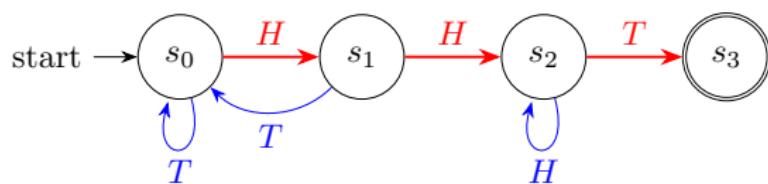
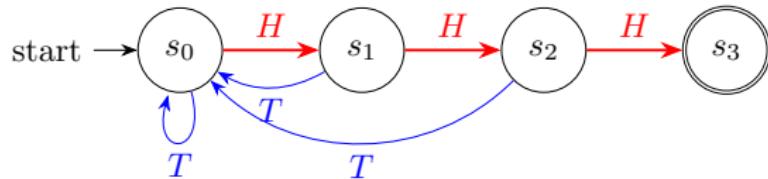
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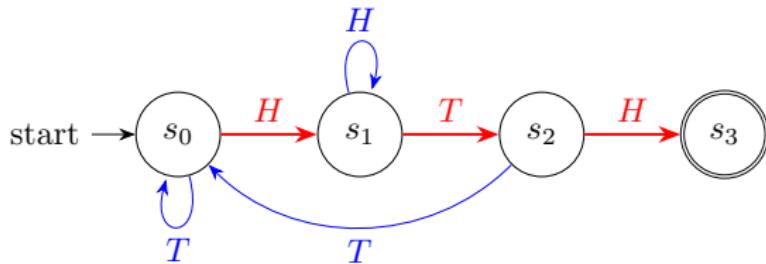
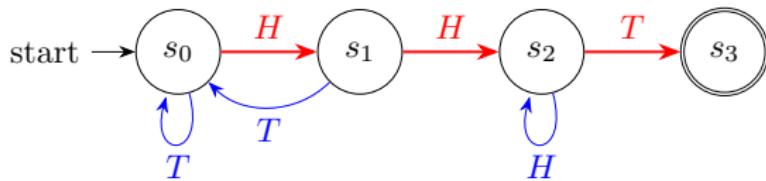
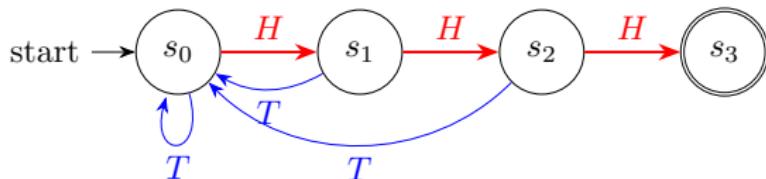
$$S_1 = \frac{1}{2}(1 + S_0 + 1 + S_2) \quad S_0 = 8$$

$$S_2 = \frac{1}{2}(1 + S_2 + 1 + S_3)$$

$$S_3 = 0$$

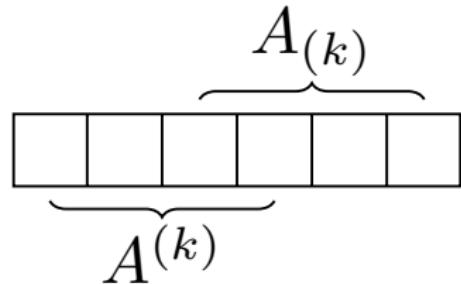


$$\mathbb{E}[X_{HHH}] = 14 > \mathbb{E}[X_{HHT}] = 8$$

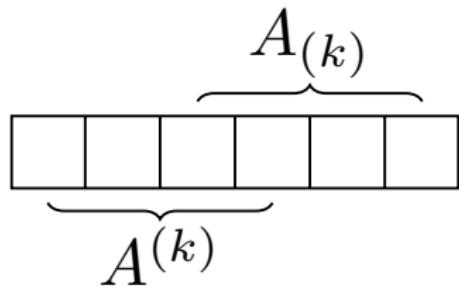


$$A : A = \sum_{k=1}^n 2^{k-1} [A^{(k)} = A_{(k)}] \quad \mathbb{E}[X_A] = 2(A : A)$$

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$$A = THHTTH \quad \mathbb{E}[X_A] = 2(2^1 + 2^4) = 36$$

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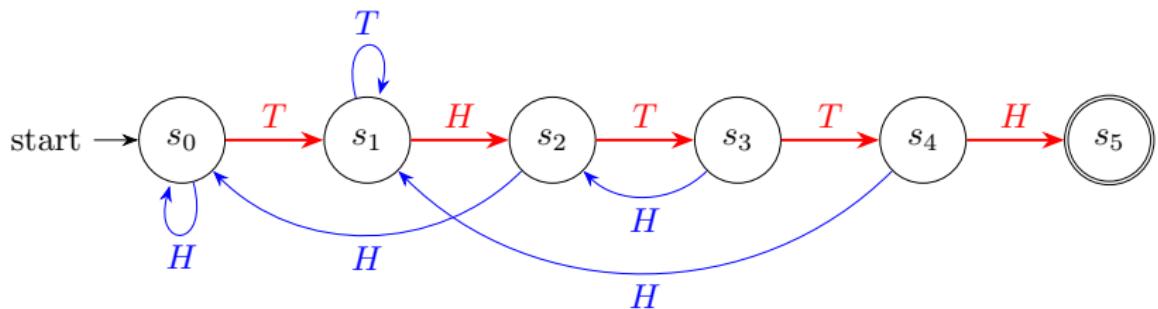
$$\mathbb{E}[X_{H^n}] = 2(2^0 + 2^1 + 2^{n-1}) = 2(2^n - 1)$$

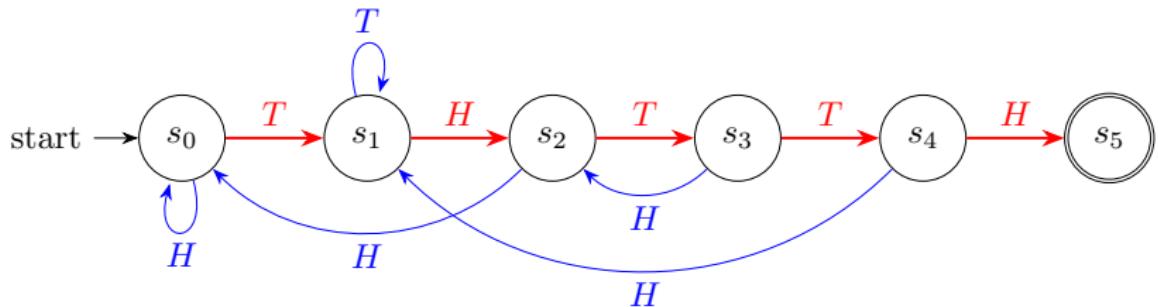
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$$\mathbb{E}[X_{H^{n-1}T}] = 2(2^{n-1}) = 2^n$$





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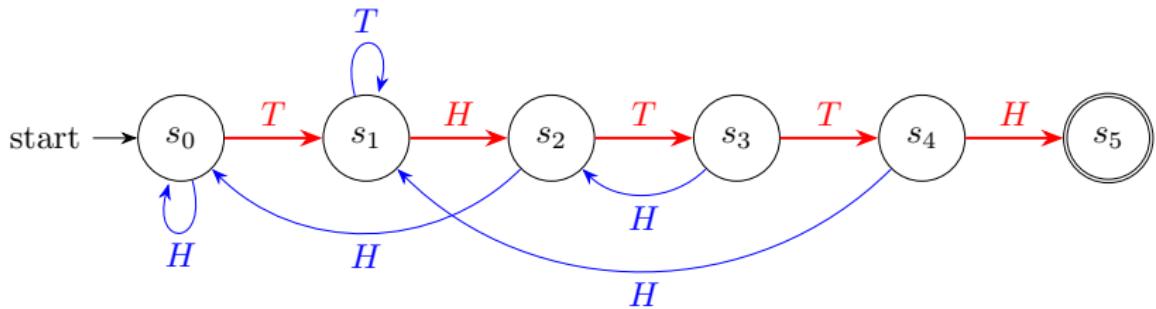
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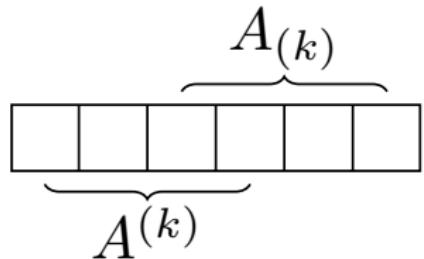
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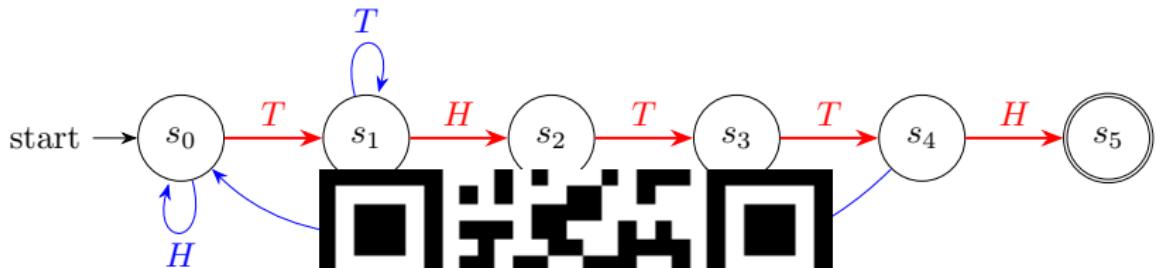
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$$2 \sum_{k=1}^n 2^{k-1} [A^{(k)} = A_{(k)}]$$



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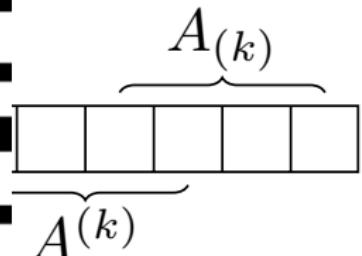
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Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X | E] = \sum_x x \Pr(X = x | E)$$

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Definition (Conditional Expectation on a Random Variable)

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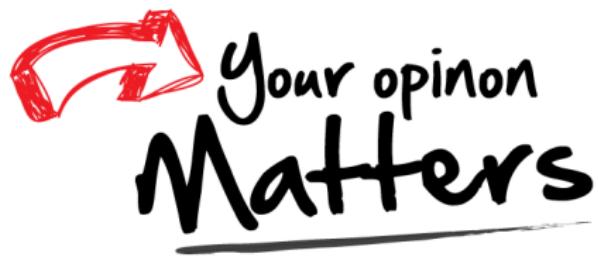
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Theorem

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X | Y]] = \sum_y \mathbb{E}[X | Y = y] \Pr(Y = y)$$

Thank You!



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