

Discrete Probability

"Life is a school of probability — Walter Bagehot"

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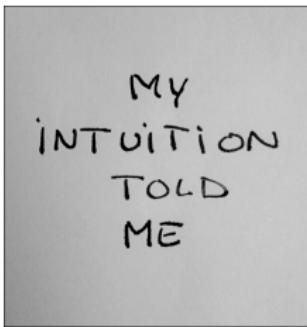


任务：破坏与建设



“...and the many paradoxes show clearly that we, as humans, lack a well grounded intuition in this matter.”

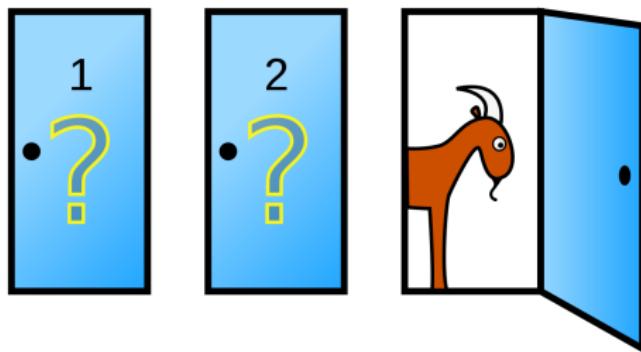
— “*The Art of Probability*”, Richard W. Hamming



*“When called upon to judge probability, people actually judge something else and **believe** they have judged probability.”*

— “*Thinking, Fast and Slow*”, Daniel Kahneman

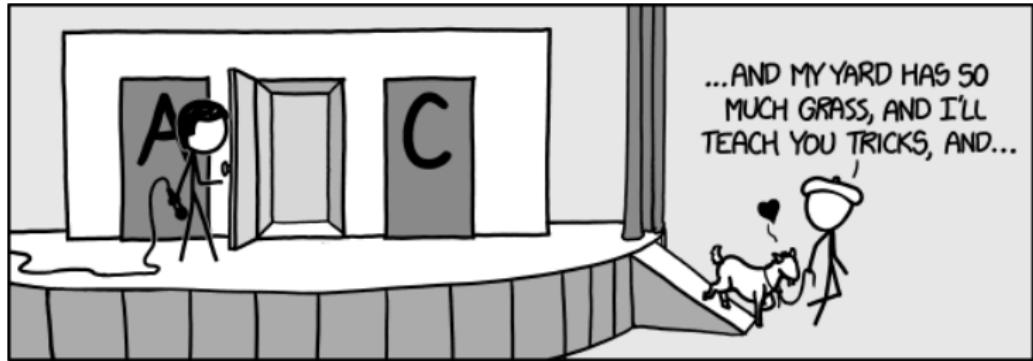
The Monty-Hall Problem



You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)

Q : Do you want to switch to door 2?



*“...and my yard has so much grass,
and I'll teach you tricks, and ...”*

C_i : The car is behind door i ($i = 1, 2, 3$)

$$\Pr \{C_i\} = \frac{1}{3}$$

ASSUMPTION: The car is initially hidden randomly behind the doors.

Y_1 : you initially pick door 1

$$\Pr \{Y_1\} = \frac{1}{3}$$

ASSUMPTION: Your initial choice is random.

I_3 : I open door 3 **AND** happen to reveal a goat

ASSUMPTION: I know what's behind the doors.

ASSUMPTION: I never open the door you initially picked.

ASSUMPTION: If you initially pick the car, then I open a door randomly.

ASSUMPTION: I always open a door to reveal a goat and never the car.

$$\Pr \{C_2 \mid I_3, Y_1\}$$

$$\begin{aligned}\Pr \{C_2 | I_3, Y_1\} &= \frac{\Pr \{C_2, I_3, Y_1\}}{\Pr \{I_3, Y_1\}} = \frac{\Pr \{I_3, Y_1 | C_2\} \Pr \{C_2\}}{\Pr \{I_3 | Y_1\} \Pr \{Y_1\}} \\ &= \frac{\Pr \{I_3, Y_1 | C_2\}}{\Pr \{I_3 | Y_1\}}\end{aligned}$$

$$\begin{aligned}\Pr \{I_3, Y_1 | C_2\} &= \Pr \{I_3 | C_2, Y_1\} \Pr \{Y_1 | C_2\} \\ &= \frac{1}{3} \Pr \{I_3 | C_2, Y_1\}\end{aligned}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{3 \Pr \{I_3 | Y_1\}}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{3 \Pr \{I_3 \mid Y_1\}}$$

$$\begin{aligned}\Pr \{I_3 \mid Y_1\} &= \Pr \{I_3 \mid C_1, Y_1\} \Pr \{C_1 \mid Y_1\} \\&\quad + \Pr \{I_3 \mid C_2, Y_1\} \Pr \{C_2 \mid Y_1\} \\&\quad + \Pr \{I_3 \mid C_3, Y_1\} \Pr \{C_3 \mid Y_1\} \\&= \frac{1}{3} \left(\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\} + \Pr \{I_3 \mid C_3, Y_1\} \right)\end{aligned}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\} + \Pr \{I_3 \mid C_3, Y_1\}}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\} + \Pr \{I_3 \mid C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr \{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{I_3 \mid C_3, Y_1\} = 0$$

$$\boxed{\Pr \{C_2 \mid I_3, Y_1\} = \frac{2}{3}}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

$$\boxed{\Pr \{I_3 | C_3, Y_1\} = 0}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

$$\Pr \{C_1 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_1, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

$$\frac{\Pr \{C_2 | I_3, Y_1\}}{\Pr \{C_1 | I_3, Y_1\}} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\}}$$

Q : Switching vs. Randomly Choosing



$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\} + \Pr \{I_3 \mid C_3, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr \{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

Always Switch!

$$\frac{\Pr \{C_2 \mid I_3, Y_1\}}{\Pr \{C_1 \mid I_3, Y_1\}} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\}}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

ASSUMPTION: I DON'T KNOW EITHER.

ASSUMPTION: I KNOW.

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

$$\Pr \{I_3 | C_3, Y_1\} = 0$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{2}{3}$$

Opens one randomly and happens to reveal a goat.

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_2, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_3, Y_1\} = 0$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{1}{2}$$



Monty Hall problem (wiki)

The Boy/Girl Puzzle



Both Girls (CS Problem 5.3 – 12)

Mr. and Mrs. Smith have two children of different ages,
what is the probability that they has **two girls**,

- (a) given that **one of the children** is a girl?
- (b) given that **the older child** is a girl?



G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1 \vee G_2\}} \\&= \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1\} + \Pr \{G_2\} - \Pr \{G_1 \wedge G_2\}} \\&= \frac{1/4}{3/4} = \frac{1}{3}\end{aligned}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1\} = \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$

$$\Pr \{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$

WLOG

..... stands for

Without Loss Of Generality



Abbreviations.com

Suppose that the older child is a girl:

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \Pr \{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr \{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr \{G_1 \mid G_1 \vee G_2\}} \Pr \{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr \{G_1 \mid G_1 \vee G_2\} = \frac{\Pr \{G_1 \wedge (G_1 \vee G_2)\}}{\Pr \{G_1 \vee G_2\}} = \frac{\Pr \{G_1\}}{\Pr \{G_1 \wedge G_2\}} = \frac{2}{3}$$



Both Girls (CS Problem 5.3 – 12)

Mr. and Mrs. Smith have two children of different ages,
what is the probability that they has **two girls**,

- (a) given that **one of the children** is a girl?

Q : How do you know that “one of the children is a girl”?

Q : How do you know that “one of the children is a girl”?



- (I) I **KNOW** them well and I tell you that “one of the children is a girl”.
- (II) I **DON'T KNOW** them. I just open a room door and see a girl.

Q : How do you know that “one of the children is a girl”?

(II) *g* : I DON'T KNOW them. I just open a room door and see a girl.

$$\Pr \{G_1 \wedge G_2 \mid g\} = \frac{\{G_1 \wedge G_2 \wedge g\}}{\Pr \{g\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

After-class Exercise:

A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?





Boy or Girl paradox (wiki)



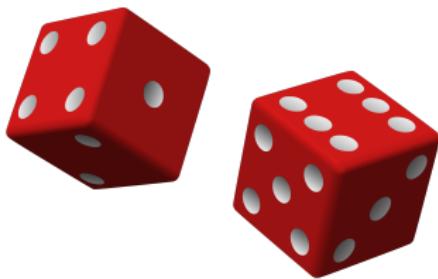
$$\Pr(EF) = \Pr(E) \times \Pr(F) \quad (E \perp F)$$

$$\Pr(E) = \Pr(E \mid F)$$

$$Q : E \perp F \implies F \perp E ?$$

$$Q : E \perp F \implies E \perp F^c ?$$

$$(E = EF \cup EF^c)$$



$$(d_1, d_2)$$

$$E : d_1 + d_2 = 6 \quad F : d_1 = 4$$

$$E : d_1 + d_2 = 7 \quad F : d_1 = 4$$

$$Q : E \perp F ?$$

$$Q : E \perp F \wedge E \perp G \implies E \perp (FG) ?$$



$$(d_1, d_2)$$

$$E : d_1 + d_2 = 7$$

$$F : d_1 = 4 \qquad G : d_2 = 3$$

Definition (Expectation)

$$\mathbb{E}[X] = \sum_x x \Pr(X = x)$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X | E] = \sum_x x \Pr(X = x | E)$$

Theorem (The Law of Total Expectation)

Let X be a random variable defined on a sample space Ω .

Let E_1, E_2, \dots, E_n be a **partition** of Ω .

$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X | E_i] \Pr(E_i)$$

Theorem (The Law of Total Probability)

Let X be a random variable defined on a sample space Ω .

Let E_1, E_2, \dots, E_n be a **partition** of Ω .

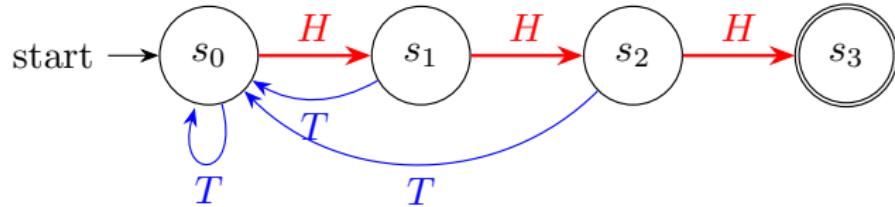
$$\Pr[X = x] = \sum_{i=1}^n \Pr[X = x | E_i] \Pr(E_i)$$

Coin Flipping Problem

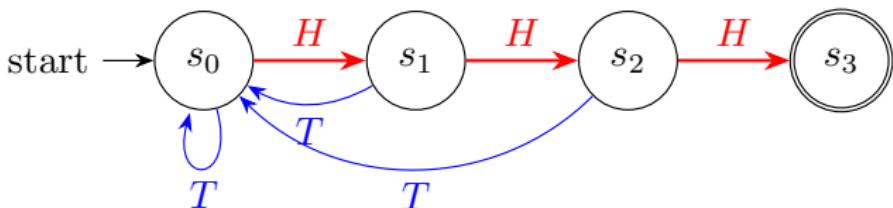


$X : \# \text{ of tosses to get 3 consecutive heads } (HHH)$

$$\mathbb{E}[X]$$



$X : \# \text{ of tosses to get } HHH$



Conditioning on the first 3 tosses

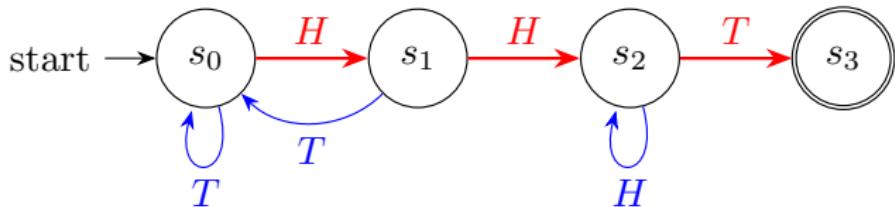
T, HT, HHT, HHH

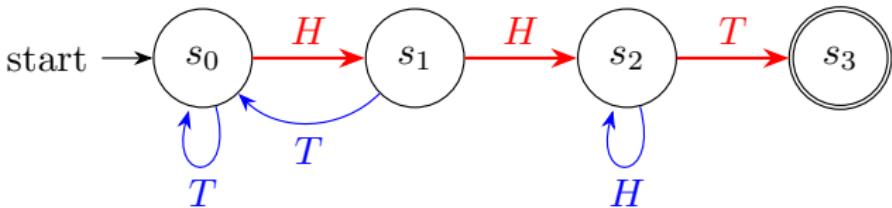
$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

$X : \# \text{ of tosses to get } HHT$

$T, \quad HT, \quad HHH, \quad HHT$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$





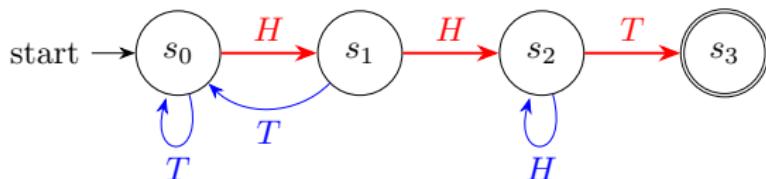
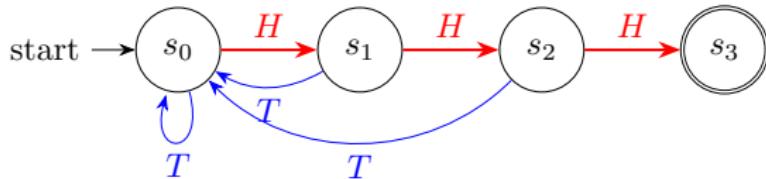
S_i : Expected number of tosses from state s_i to reach state s_n

$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$

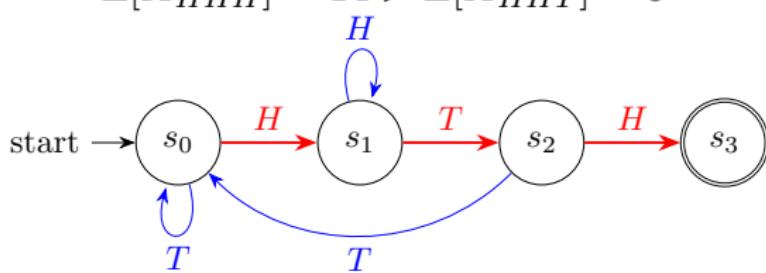
$$S_1 = \frac{1}{2}(1 + S_0 + 1 + S_2) \quad S_0 = 8$$

$$S_2 = \frac{1}{2}(1 + S_2 + 1 + S_3)$$

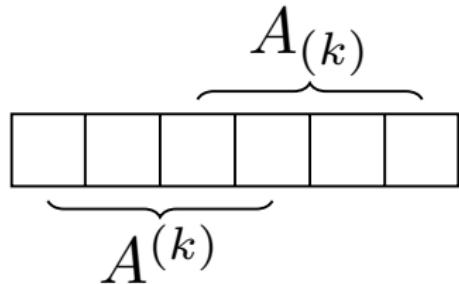
$$S_3 = 0$$



$$\mathbb{E}[X_{HHH}] = 14 > \mathbb{E}[X_{HHT}] = 8$$



$$A : A = \sum_{k=1}^n 2^{k-1} [A^{(k)} = A_{(k)}] \quad \mathbb{E}[X_A] = 2(A : A)$$

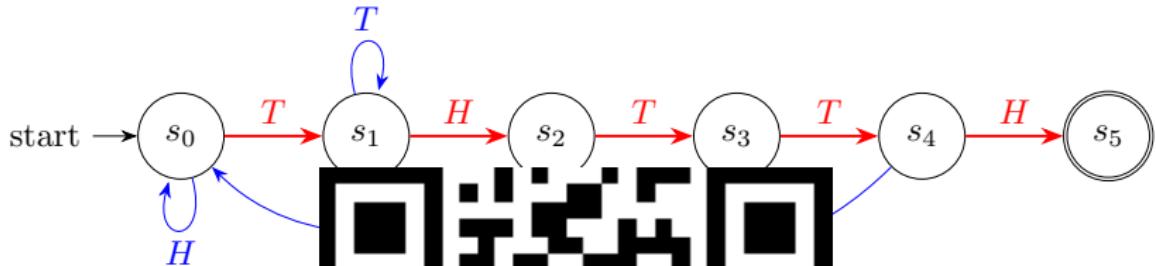


$$A = THHTTH \quad \mathbb{E}[X_A] = 2(2^1 + 2^4) = 36$$

$$A : A = \sum_{k=1}^n 2^{k-1} [A^{(k)} = A_{(k)}] \quad \mathbb{E}[X_A] = 2(A : A)$$

$$\mathbb{E}[X_{H^n}] = 2(2^0 + 2^1 + 2^{n-1}) = 2(2^n - 1)$$

$$\mathbb{E}[X_{H^{n-1}T}] = 2(2^{n-1}) = 2^n$$



$$S_0 = \frac{1}{2}(1 + S_1)$$

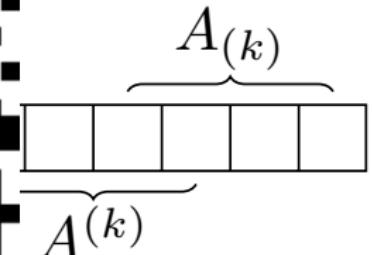
$$S_1 = \frac{1}{2}(1 + S_2)$$

$$S_2 = \frac{1}{2}(1 + S_3)$$

$$S_3 = \frac{1}{2}(1 + S_2 + 1 + S_4)$$

$$S_4 = \frac{1}{2}(1 + S_1 + 1 + S_5)$$

$$S_5 = 0$$



$$2 \sum_{k=1}^n 2^{k-1} [A^{(k)} = A_{(k)}]$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

Definition (Conditional Expectation on a Random Variable)

$$\mathbb{E}[X \mid Y = y] = \sum_x x \Pr(X = x \mid Y = y)$$

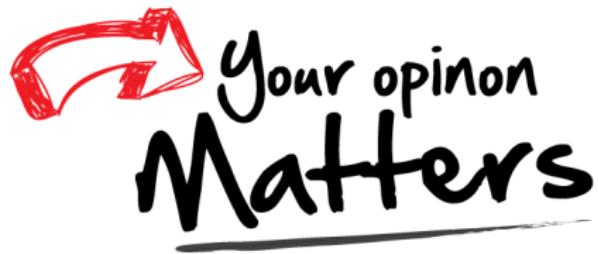
Notation:

$$\mathbb{E}[X \mid Y](y) = \mathbb{E}[X \mid Y = y]$$

Theorem

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] = \sum_y \mathbb{E}[X \mid Y = y] \Pr(Y = y)$$

Thank You!



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