Discrete Probability

"Life is a school of probability — Walter Bagehot"

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任务: 破坏与建设

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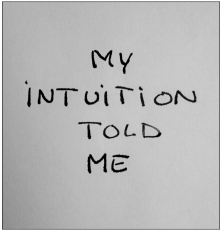
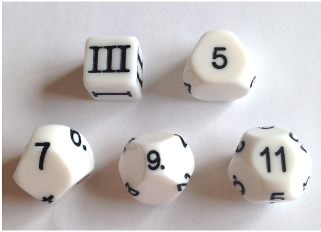
"...and the many *paradoxes* *show clearly that we, as humans,* *lack a well grounded intuition* *in this matter."*

— "The Art of Probability", Richard W. Hamming





". . . and the many paradoxes show clearly that we, as humans, lack a well grounded intuition in this matter."

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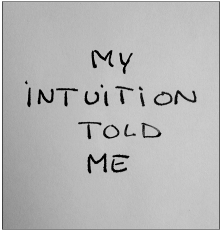

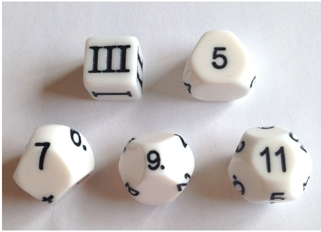

*"When called upon to judge probability, people actually judge something else and **believe** they have judged probability."*

— "Thinking, Fast and Slow", Daniel Kahneman


J
A
Jim Sturgess
Kate Bosworth
Laurence Fishburne
y Kevin Spacey
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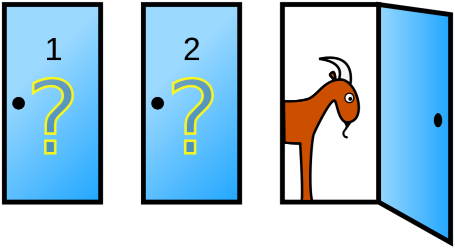
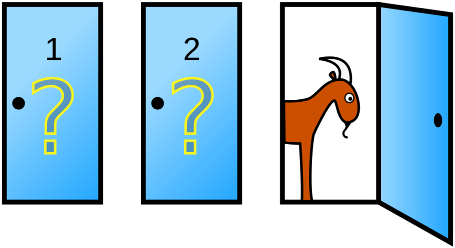
The Monty-Hall Problem



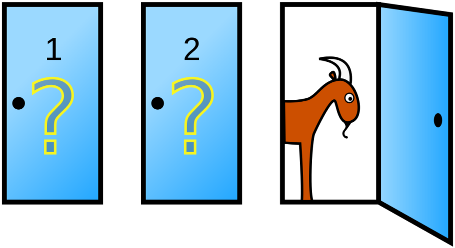
The Monty-Hall Problem



You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)

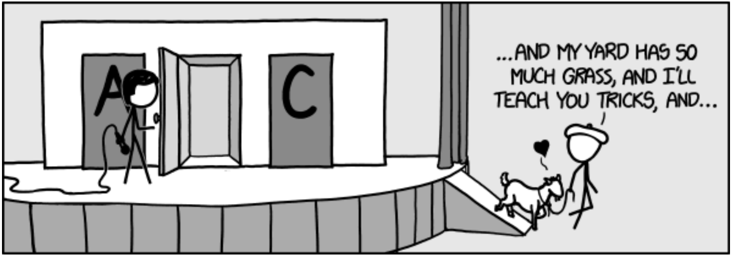
The Monty-Hall Problem



You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)

Q : Do you want to switch to door 2?



"...and my yard has so much grass, and I'll teach you tricks, and ..."

C_i : The car is behind door i $(i = 1, 2, 3)$

$$\boxed{\Pr\{C_i\} = \frac{1}{3}}$$

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ASSUMPTION: The car is initially hidden randomly behind the doors.

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C_i : The car is behind door i $(i = 1, 2, 3)$

$$\boxed{\Pr\{C_i\} = \frac{1}{3}}$$

ASSUMPTION: The car is initially hidden randomly behind the doors.

Y_1 : you initially pick door 1

$$\boxed{\Pr\{Y_1\} = \frac{1}{3}}$$

ASSUMPTION: Your initial choice is random.

I_3 : I open door 3 AND happen to reveal a goat

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ASSUMPTION: I know what's behind the doors.

I_3 : I open door 3 AND happen to reveal a goat

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ASSUMPTION: I never open the door you initially picked.

I_3 : I open door 3 AND happen to reveal a goat

ASSUMPTION: I know what's behind the doors.

ASSUMPTION: I never open the door you initially picked.

ASSUMPTION: If you initially pick the car, then I open a door randomly.

I_3 : I open door 3 AND happen to reveal a goat

ASSUMPTION: I know what's behind the doors.

ASSUMPTION: I never open the door you initially picked.

ASSUMPTION: If you initially pick the car, then I open a door randomly.

ASSUMPTION: I always open a door to reveal a goat and never the car.

I_3 : I open door 3 AND happen to reveal a goat

ASSUMPTION: I know what's behind the doors.

ASSUMPTION: I never open the door you initially picked.

ASSUMPTION: If you initially pick the car, then I open a door randomly.

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$$\boxed{\Pr\{C_2 \mid I_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}}$$

$$= \frac{\Pr\{I_3 \mid C_2, Y_1\} \Pr\{C_2 \mid Y_1\} \Pr\{Y_1\}}{\Pr\{I_3 \mid Y_1\} \Pr\{Y_1\}}$$

$$
\begin{aligned}
\Pr\{C_2 \mid I_3, Y_1\} &= \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}} \\
&= \frac{\Pr\{I_3 \mid C_2, Y_1\} \Pr\{C_2 \mid Y_1\} \Pr\{Y_1\}}{\Pr\{I_3 \mid Y_1\} \Pr\{Y_1\}} \\
&= \frac{\Pr\{I_3 \mid C_2, Y_1\} \Pr\{C_2 \mid Y_1\}}{\Pr\{I_3 \mid Y_1\}}
\end{aligned}
$$

$$
\begin{aligned}
\Pr\{C_2 \mid I_3, Y_1\} &= \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}} \\
&= \frac{\Pr\{I_3 \mid C_2, Y_1\}\Pr\{C_2 \mid Y_1\}\Pr\{Y_1\}}{\Pr\{I_3 \mid Y_1\}\Pr\{Y_1\}} \\
&= \frac{\Pr\{I_3 \mid C_2, Y_1\}\Pr\{C_2 \mid Y_1\}}{\Pr\{I_3 \mid Y_1\}}
\end{aligned}
$$

$$
\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{3\Pr\{I_3 \mid Y_1\}}
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$\Pr\{I_3 \mid Y_1\}$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{3\Pr\{I_3 \mid Y_1\}}$$

$$\begin{aligned}\Pr\{I_3 \mid Y_1\} &= \Pr\{I_3 \mid C_1, Y_1\}\Pr\{C_1 \mid Y_1\} \\ &+ \Pr\{I_3 \mid C_2, Y_1\}\Pr\{C_2 \mid Y_1\} \\ &+ \Pr\{I_3 \mid C_3, Y_1\}\Pr\{C_3 \mid Y_1\}\end{aligned}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{3\Pr\{I_3 \mid Y_1\}}$$

$$\begin{aligned}
\Pr\{I_3 \mid Y_1\} &= \Pr\{I_3 \mid C_1, Y_1\}\Pr\{C_1 \mid Y_1\} \\
&\quad + \Pr\{I_3 \mid C_2, Y_1\}\Pr\{C_2 \mid Y_1\} \\
&\quad + \Pr\{I_3 \mid C_3, Y_1\}\Pr\{C_3 \mid Y_1\} \\
&= \frac{1}{3}\Big(\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}\Big)
\end{aligned}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{3 \Pr\{I_3 \mid Y_1\}}$$

$$\begin{aligned}
\Pr\{I_3 \mid Y_1\} &= \Pr\{I_3 \mid C_1, Y_1\} \Pr\{C_1 \mid Y_1\} \\
&+ \Pr\{I_3 \mid C_2, Y_1\} \Pr\{C_2 \mid Y_1\} \\
&+ \Pr\{I_3 \mid C_3, Y_1\} \Pr\{C_3 \mid Y_1\} \\
&= \frac{1}{3}\Big(\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\} \Big)
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$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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It depends on how I choose the door to open!

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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$$\Pr\{C_1 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_1, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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$$\Pr\{C_1 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_1, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\frac{\Pr\{C_2 \mid I_3, Y_1\}}{\Pr\{C_1 \mid I_3, Y_1\}} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\}}$$

Q : Switching *vs.* Randomly Choosing

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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$$\Pr\{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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$\boxed{\textit{Always Switch!}}$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

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$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr\{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

Always Switch!

$$\frac{\Pr\{C_2 \mid I_3, Y_1\}}{\Pr\{C_1 \mid I_3, Y_1\}} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

Assumption: I know.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

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Assumption: I know.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

Assumption: I don't know either.

Opens one randomly and happens to reveal a goat.

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

ASSUMPTION: I DON'T KNOW EITHER.

ASSUMPTION: I KNOW.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

Opens one randomly and happens to reveal a goat.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{2}$$

Monty Hall problem (wiki)

The Boy/Girl Puzzle

Both Girls (CS Problem $5.3 - 12$)

Mr. and Mrs. Smith have two children of different ages, what is the probability that they has two girls,

(a) given that one of the children is a girl?

(b) given that the older child is a girl?

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\}$

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
&= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}}
\end{aligned}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$
\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
&= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} \\
&= \frac{1/4}{3/4} = \frac{1}{3}
\end{aligned}
$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\ &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} \\ &= \frac{1/4}{3/4} = \frac{1}{3}\end{aligned}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
&= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} \\
&= \frac{1/4}{3/4} = \frac{1}{3}
\end{aligned}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\}}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\ &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} \\ &= \frac{1/4}{3/4} = \frac{1}{3}\end{aligned}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

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$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

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$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

Suppose that the older child is a girl:

Suppose that the older child is a girl:

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr\{G_1 \mid G_1 \vee G_2\}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

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$$\Pr\{G_1 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge (G_1 \vee G_2)\}}{\Pr\{G_1 \vee G_2\}}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

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$$\Pr\{G_1 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge (G_1 \vee G_2)\}}{\Pr\{G_1 \vee G_2\}} = \frac{\Pr\{G_1\}}{\Pr\{G_1 \wedge G_2\}} = \frac{2}{3}$$


Easy!


Easy!


KEEP
CALM
AND
WE ARE NOT
FINISHED YET

Both Girls (CS Problem $5.3 - 12$)

Mr. and Mrs. Smith have two children of different ages,
what is the probability that they has two girls,

(a) given that one of the children is a girl?

Both Girls (CS Problem $5.3 - 12$)

Mr. and Mrs. Smith have two children of different ages,
what is the probability that they has two girls,

(a) given that one of the children is a girl?

Q : **How** do you know that "one of the children is a girl"?

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(I) I KNOW them well and I tell you that "one of the children is a girl".

Q : **How** do you know that "one of the children is a girl"?

(I) I KNOW them well and I tell you that "one of the children is a girl".

(II) I DON'T KNOW them. I just open a room door and see a girl.

Q : **How** do you know that “one of the children is a girl”?

(II) g : I DON'T KNOW them. I just open a room door and see a girl.

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$$\Pr\{G_1 \wedge G_2 \mid g\} = \frac{\{G_1 \wedge G_2 \wedge g\}}{\Pr\{g\}}$$

Q : **How** do you know that "one of the children is a girl"?

(II) g : I DON'T KNOW them. I just open a room door and see a girl.

$$\Pr\{G_1 \land G_2 \mid g\} = \frac{\{G_1 \land G_2 \land g\}}{\Pr\{g\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

After-class Exercise:

A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?

Boy or Girl paradox (wiki)

The Independent

$$\Pr(EF) = \Pr(E) \times \Pr(F) \quad (E \perp F)$$

The INDEPENDENT

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The INDEPENDENT

$$\Pr(EF) = \Pr(E) \times \Pr(F) \quad (E \perp F)$$

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$$Q : E \perp F \implies F \perp E\,?$$

The Independent

$$\Pr(EF) = \Pr(E) \times \Pr(F) \quad (E \perp F)$$

$$\Pr(E) = \Pr(E \mid F)$$

$$Q : E \perp F \implies F \perp E\,?$$

$$Q : E \perp F \implies E \perp F^c\,?$$

The INDEPENDENT

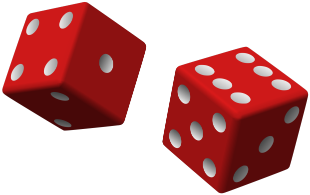
$$\Pr(EF) = \Pr(E) \times \Pr(F) \quad {\color{red}(E \perp F)}$$

$$\Pr(E) = \Pr(E \mid F)$$

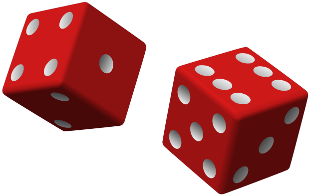
$${\color{red}Q : E \perp F \implies F \perp E\,?}$$

$${\color{red}Q : E \perp F \implies E \perp F^c\,?}$$

$${\color{blue}(E = EF \cup EF^c)}$$

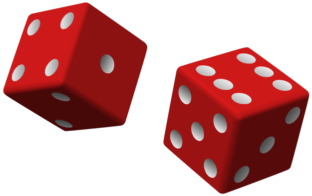


$$(d_1, d_2)$$



$$(d_1, d_2)$$

$$E : d_1 + d_2 = 6 \qquad F : d_1 = 4$$



$$(d_1, d_2)$$

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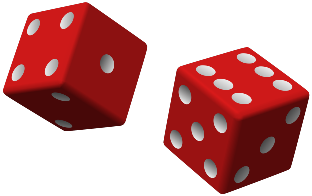
$$Q : E \perp F\,?$$



$$(d_1, d_2)$$

$$E : d_1 + d_2 = 7 \qquad F : d_1 = 4$$

$$Q : E \perp F\,?$$

$$Q : E \perp F \wedge E \perp G \implies E \perp (FG)\,?$$

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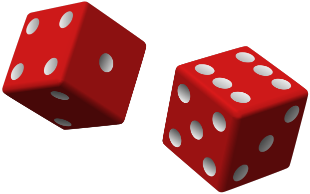
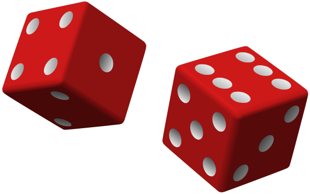


$$(d_1, d_2)$$

$$Q : E \perp F \wedge E \perp G \implies E \perp (FG)\,?$$



$$(d_1, d_2)$$

$$E : d_1 + d_2 = 7$$

$$F : d_1 = 4 \qquad G : d_2 = 3$$

Definition (Expectation)

$$\mathbb{E}[X] = \sum_x x \Pr(X = x)$$

Definition (Expectation)

$$\mathbb{E}[X] = \sum_x x \Pr(X = x)$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

Theorem (The Law of Total Expectation)

Let X be a random variable defined on a sample space Ω.
Let $E_1, E_2, \cdots, E_n$ be a partition of Ω.

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \Pr(E_i)$$

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Theorem (The Law of Total Probability)

Let X be a random variable defined on a sample space Ω.
Let $E_1, E_2, \cdots, E_n$ be a partition of Ω.

$$\Pr[X = x] = \sum_{i=1}^{n} \Pr[X = x \mid E_i] \Pr(E_i)$$

Coin Flipping Problem

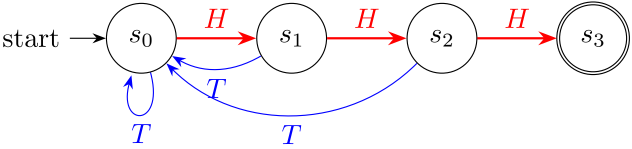
X : # of tosses to get 3 consecutive heads (HHH)

$$\mathbb{E}[X]$$

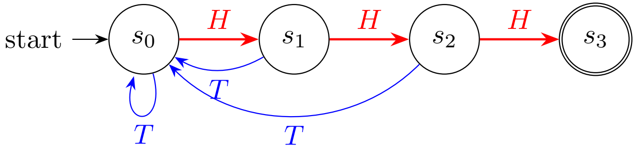
Coin Flipping Problem

X : # of tosses to get 3 consecutive heads (HHH)

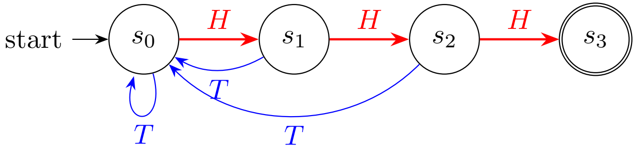
$$\mathbb{E}[X]$$



$\boxed{X : \# \text{ of tosses to get } HHH}$



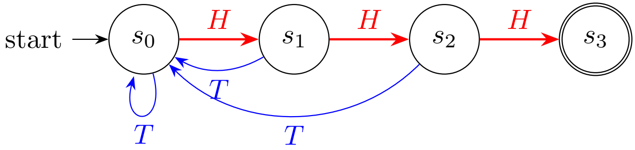
$\boxed{X : \# \text{ of tosses to get } HHH}$



Conditioning on the first 3 tosses

$$T, \quad HT, \quad HHT, \quad HHH$$

$\boxed{X : \# \text{ of tosses to get } HHH}$



Conditioning on the first 3 tosses

$$T, \quad HT, \quad HHT, \quad HHH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

$\boxed{X : \# \text{ of tosses to get } HHT}$

$$T, \quad HT, \quad HHH, \quad HHT$$

$$\boxed{X : \#\ \text{of tosses to get}\ HHT}$$

$$T, \quad HT, \quad HHH, \quad HHT$$

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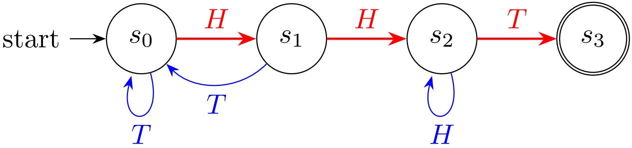
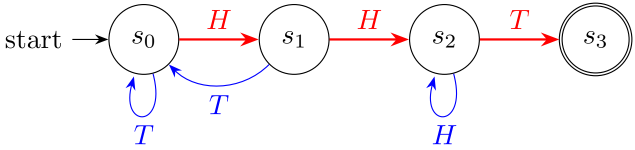
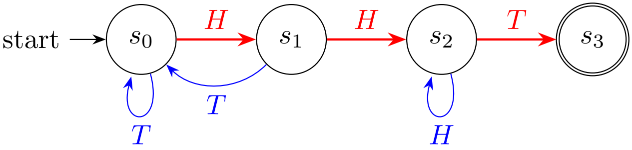
$$T, \quad HT, \quad HHH, \quad HHT$$

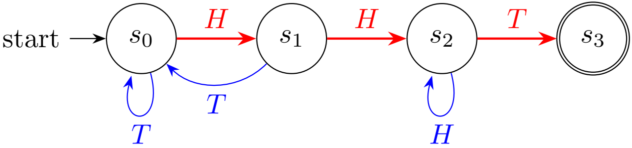
$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X]+1) + \frac{1}{4}(\mathbb{E}[X]+2) + \frac{1}{8}(\mathbb{E}[X]+3) + \frac{1}{8} \times 3$$




start
s_0
s_1
s_2
s_3
H
H
T
T
T
H

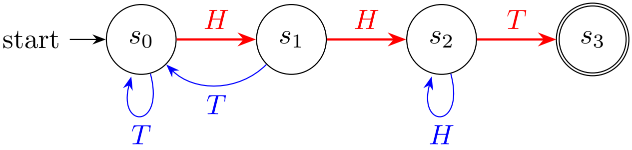


$\boxed{S_i : \text{ Expected number of tosses from state } s_i \text{ to reach state } s_n}$



$\boxed{S_i : \text{ Expected number of tosses from state } s_i \text{ to reach state } s_n}$

$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$
$$S_1 = \frac{1}{2}(1 + S_0 + 1 + S_2)$$
$$S_2 = \frac{1}{2}(1 + S_2 + 1 + S_3)$$
$$S_3 = 0$$



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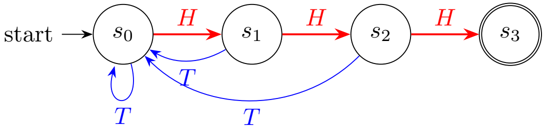
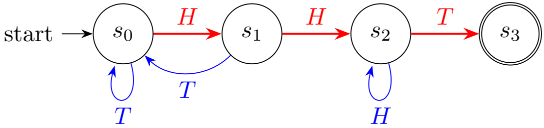
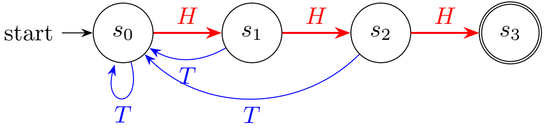
$$S_3 = 0$$

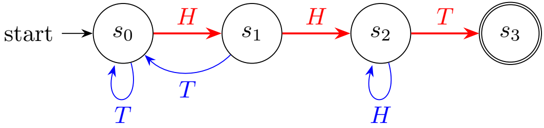
$$S_0 = 8$$

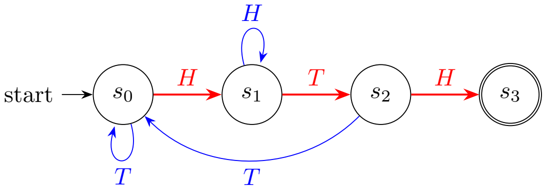




$$\mathbb{E}[X_{HHH}] = 14 > \mathbb{E}[X_{HHT}] = 8$$


start
s_0
s_1
s_2
s_3
H
H
H
T
T
T


start
s_0
s_1
s_2
s_3
H
H
T
T
T
H


start
s_0
s_1
s_2
s_3
H
H
T
H
T
T

$$A : A = \sum_{k=1}^{n} 2^{k-1} [A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)$$

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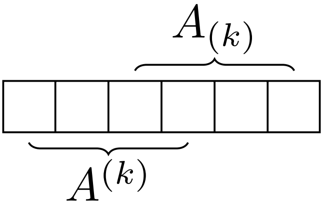
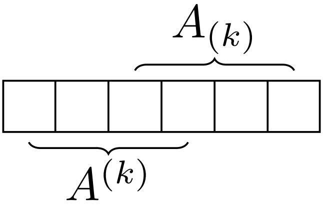


$$A : A = \sum_{k=1}^{n} 2^{k-1} [A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)$$



$$A = THTTH \qquad \mathbb{E}[X_A] = 2(2^1 + 2^4) = 36$$

$$\boxed{A : A = \sum_{k=1}^{n} 2^{k-1}[A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)}$$

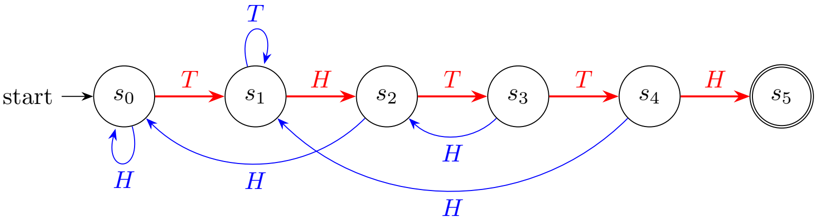
$$\boxed{A : A = \sum_{k=1}^{n} 2^{k-1}[A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)}$$

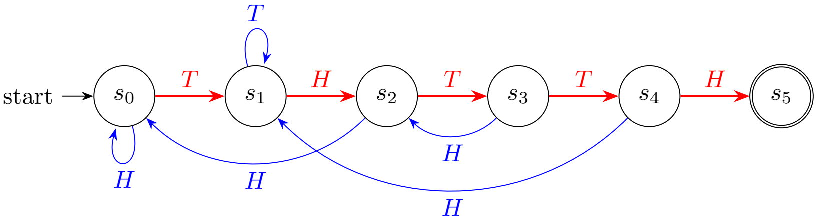
$$\mathbb{E}[X_{H^n}] = 2(2^0 + 2^1 + 2^{n-1}) = 2(2^n - 1)$$

$$\boxed{A : A = \sum_{k=1}^{n} 2^{k-1}[A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)}$$

$$\mathbb{E}[X_{H^n}] = 2(2^0 + 2^1 + 2^{n-1}) = 2(2^n - 1)$$

$$\mathbb{E}[X_{H^{n-1}T}] = 2(2^{n-1}) = 2^n$$


start
s_0
s_1
s_2
s_3
s_4
s_5
T
H
T
T
H
T
H
H
H
H



$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$

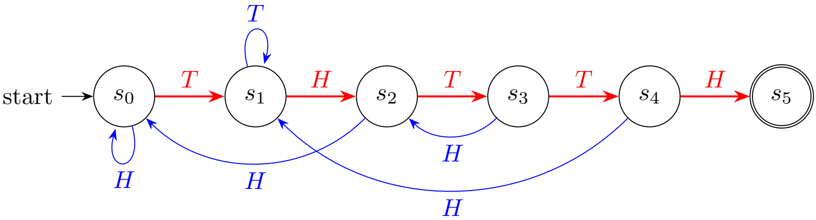
$$S_1 = \frac{1}{2}(1 + S_1 + 1 + S_2)$$

$$S_2 = \frac{1}{2}(1 + S_0 + 1 + S_3)$$

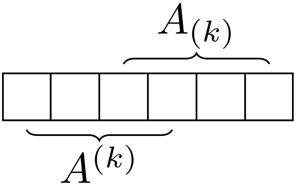
$$S_3 = \frac{1}{2}(1 + S_2 + 1 + S_4)$$

$$S_4 = \frac{1}{2}(1 + S_1 + 1 + S_5)$$

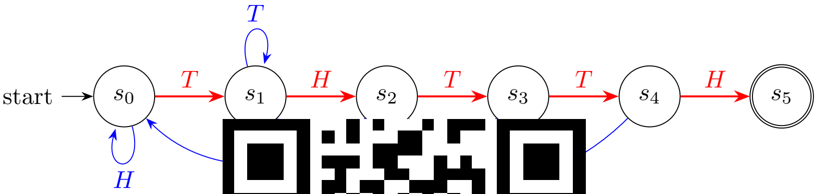
$$S_5 = 0$$



$$
\begin{aligned}
S_0 &= \frac{1}{2}(1 + S_0 + 1 + S_1) \\
S_1 &= \frac{1}{2}(1 + S_1 + 1 + S_2) \\
S_2 &= \frac{1}{2}(1 + S_0 + 1 + S_3) \\
S_3 &= \frac{1}{2}(1 + S_2 + 1 + S_4) \\
S_4 &= \frac{1}{2}(1 + S_1 + 1 + S_5) \\
S_5 &= 0
\end{aligned}
$$



$$2\sum_{k=1}^{n} 2^{k-1}[A^{(k)} = A_{(k)}]$$



$S_0 = \frac{1}{2}(1 +$

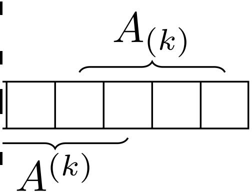
$S_1 = \frac{1}{2}(1 +$

$S_2 = \frac{1}{2}(1 +$

$S_3 = \frac{1}{2}(1 + S_2 + 1 + S_4)$

$S_4 = \frac{1}{2}(1 + S_1 + 1 + S_5)$

$S_5 = 0$



$$2\sum_{k=1}^{n} 2^{k-1}[A^{(k)} = A_{(k)}]$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_{x} x \Pr(X = x \mid E)$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

Definition (Conditional Expectation on a Random Variable)

$$\mathbb{E}[X \mid Y = y] = \sum_x x \Pr(X = x \mid Y = y)$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

Definition (Conditional Expectation on a Random Variable)

$$\mathbb{E}[X \mid Y = y] = \sum_x x \Pr(X = x \mid Y = y)$$

Notation:

$$\mathbb{E}[X \mid Y](y) = \mathbb{E}[X \mid Y = y]$$

Definition (Conditional Expectation on an Event)

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Notation:

$$\mathbb{E}[X \mid Y](y) = \mathbb{E}[X \mid Y = y]$$

Theorem

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] = \sum_y \mathbb{E}[X \mid Y = y] \Pr(Y = y)$$


Thank
You!

Your opinon Matters

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