# Menger's Theorem

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*In its best-known version, Menger's theorem states that the maximum number of pairwise internally disjoint paths between a given pair of non-adjacent vertices in a graph equals the minimum number of vertices whose deletion disconnects the pair. Thus, the maximum number of pairwise internally disjoint paths that connect a given pair of vertices is a local measure that indicates how well a given pair of vertices is connected. The connectivity of a graph is the minimum number of vertices whose deletion disconnects the graph and can be expressed in terms of the connectivity between pairs of vertices. We survey some classical results from this field and highlight some advances in closely related areas.*

## 1 Introduction

Since a given pair of vertices in a graph may be connected by many paths, it is natural to ask how one should measure the connectedness between them. One way of doing this is to determine the largest number of such paths that are pairwise 'independent' of one another in sharing no other vertices. Another way of measuring their connectedness is to determine the smallest number of vertices whose deletion from the graph destroys every path between this pair. Menger's elegant theorem [56] states that for each pair of non-adjacent vertices these two measures are equal. An equivalent formulation of this theorem states that if $V$ and $W$ are nonempty sets of vertices in a graph, then the maximum number of internally disjoint $V$–$W$ paths equals the minimum number of vertices whose deletion destroys every such path.

In Section 2 we give two proofs of Menger's theorem, one for each formulation and then discuss some edge versions in Section 3. We then proceed to the global measure of 'connectivity of a graph' and Whitney's version of Menger's theorem [73]; where the connectivity of a non-trivial, non-complete graph is the smallest number of vertices whose deletion produces a disconnected graph. In Section 4 we discuss mixed versions of Menger's theorem. We look at the problem of disconnecting a pair of vertices in a graph by deletion of both vertices and edges and the relationship of this with the problem of finding a largest collection of edge-disjoint paths between a pair of vertices, some sub-collection of which is internally disjoint.

Graphs with the same connectivity may have vastly differing degrees of connectedness properties. Motivated by Menger's theorem, in Section 5 we consider another global measure, called the 'mean connectivity', defined as the average of the maximum number of internally disjoint paths connecting pairs of vertices. Section 6 is devoted to 'Menger-type' results for paths of bounded length between a pair of vertices. In particular, we discuss relationships between the maximum number of internally disjoint paths of a constrained length connecting a pair of vertices and the minimum number of vertices whose deletion destroys all such paths between this pair of vertices.

We conclude the chapter with two sections that survey 'Menger-type' results for sets of more than two vertices.

Throughout most of the chapter, we focus on vertex measures, but usually, as with Menger's theorem itself, there is a corresponding edge measure, and when appropriate, we also discuss these. Unless specifically stated otherwise, we assume that all graphs are simple, finite, and undirected.

# 2 Vertex-connectivity

### Menger's Theorem

Let $v$ and $w$ be two non-adjacent vertices in a graph $G$. A set $S$ of vertices is a *$v$–$w$ separating set* if $v$ and $w$ lie in different components of $G - S$; that is, if every $v$–$w$ path contains a vertex in $S$. The minimum order of a $v$–$w$ separating set is called the *$v$–$w$ connectivity* and is denoted by $\kappa(v, w)$.

For any two vertices $v$ and $w$, a collection of $v$–$w$ paths is called *internally disjoint* if the paths are pairwise disjoint except for the vertices $v$ and $w$. The maximum number of internally disjoint $v$–$w$ paths is denoted by $\mu(v, w)$. Since each path in such a set must contain a different vertex from every $v$–$w$ separating set, it is clear that $\mu(v, w) \leq \kappa(v, w)$.

As an example, consider the graph in Fig. 1. It is easy to verify that both $\kappa(v, w)$ and $\mu(v, w)$ are 3. That the two parameters are equal in this case is no coincidence, and the fact that this holds in general is the content of one version of Menger's theorem.

**Theorem 2.1** *If $v$ and $w$ are non-adjacent vertices in a graph $G$, then the maximum number of internally disjoint $v$–$w$ paths equals the minimum number of vertices in a $v$–$w$ separating set.*

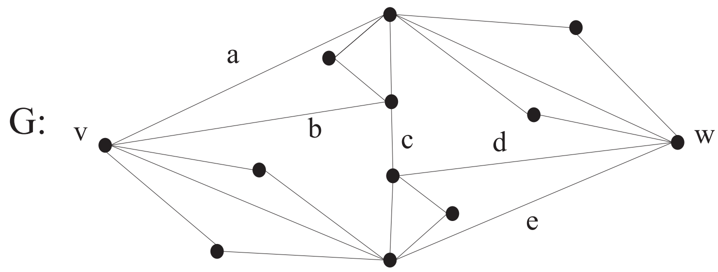


Figure 1: A graph with $v$–$w$ connectivity 3 and $v$–$w$ edge-connectivity 5

Menger discovered his result in the course of research he was conducting in the theory of curves in point-set topology (see [57]). However, as König [42] pointed out, Menger's original proof contained a significant gap. The first flawless proof was provided by Noebeling, and appeared in another paper by Menger [58]. It uses an earlier result of König on bipartite graphs [40]. Other early proofs were given by König [41] and Hajós [33], since when numerous other proofs have been given (see [55] for an extensive list). We give two proofs here, the first of which uses induction and follows one by McCuaig [55]. Our second proof is for an equivalent version of the theorem, and uses an algorithmic approach. It is due to Ore [65], and bears a strong resemblance to the max-flow min-cut algorithm of Ford and Fulkerson [24] (see the Introduction to this book) from which Menger's theorem can also be deduced.

**Proof.** Let $v$ and $w$ be a pair of non-adjacent vertices in a graph $G$. As observed earlier $\mu(v,w) \leq \kappa(v,w)$ since a $v$–$w$ separator must contain at least one vertex from each of the paths in any collection of internally disjoint $v$–$w$ paths.

We now show that $\mu(v,w) \geq \kappa(v,w)$. Let $k = \kappa(v,w)$. Then no set of fewer than $k$ vertices separates $v$ and $w$. We proceed to show, by induction on $k$, that if $\kappa(v,w) \geq k$, then $\mu(v,w) \geq k$. (Thus in particular $\kappa(v,w) = k$ implies that $\mu(v,w) \geq k$, which is the desired result.) If $k = 1$, then there is a $v$–$w$ path. Assume thus that $k \geq 1$ and that if $\kappa(v,w) \geq k$ that $\mu(v,w) \geq k$. Assume further that $v$ and $w$ are non-adjacent vertices in $G$ with $\kappa(v,w) \geq k+1$.

By the induction hypothesis, there are $k$ internally disjoint $v$–$w$ paths $P_1, P_2, \ldots, P_k$. Since the collection of vertices that follow $v$ on these paths (there are $k$ of these) do not separate $v$ and $w$, there is a $v$–$w$ path $P$ whose initial edge is not on any $P_i$. Let $x$ be the first vertex after $v$ on $P$ that belongs to some $P_i$. Let $P_{k+1}$ be the $v$–$x$ subpath of $P$. Assume that $P_1, P_2, \ldots, P_{k+1}$ have been chosen in such a way that the distance from $x$ to $w$ in $G - v$ is a minimum. If $x = w$, then we have the desired collection of $k+1$ internally disjoint paths. Assume therefore that $x \neq w$.

Again, by the induction hypothesis, there are $k$ internally disjoint $v$–$w$ paths $Q_1, Q_2, \ldots, Q_k$ in $G - x$. Assume that these paths have been chosen so that a minimum number of edges not on any of the paths $P_i$ are used. Let $H$ be the graph consisting of the paths $Q_1, Q_2, \ldots, Q_k$ together with the vertex $x$. Choose some $P_j$ for $1 \le j \le k+1$, whose initial edge is not in $H$. Let $y$ be the first vertex on $P_j$ after $v$ which is in $H$. If $y = w$, then we have the desired collection of $k+1$ internally disjoint $v$–$w$ paths. So assume $y \ne w$.

If $y = x$, then let $R$ be the shortest $x$–$w$ path in $G - v$. Let $z$ be the first vertex on $R$ that is on some $Q_i$. Then the distance in $G - v$ from $z$ to $w$ is less than the distance from $x$ to $w$. This contradicts our choice of $P_1, P_2, \ldots, P_{k+1}$. So $y \ne x$.

If $y$ is on some $Q_i$ for $1 \le i \le k$, then the $v$–$y$ subpath of $Q_t$ has an edge in $B$. Otherwise, two paths from among $P_1, P_2, \ldots, P_{k+1}$ intersect at a vertex other than $v$, $w$ or $x$. If we replace the $v$–$y$ subpath of $Q_i$ by the $v$–$y$ subpath of $P_j$, we get a collection of $k$ internally disjoint $v$–$w$ paths in $G - x$ that uses fewer edges from $B$ than $Q_1, Q_2, \ldots, Q_k$ do, which is a contradiction. ■

Before presenting the second proof of Menger's theorem we introduce an equivalent formulation of Menger's theorem. Assume that $V$ and $W$ are sets of vertices in a graph $G$. Then a *$V$–$W$ path* is a path from some $v \in V$ to some $w \in W$ that passes through no other vertex of $V$ or $W$. If a vertex $x$ belongs to both $V$ and $W$, then $x$ is itself a $V$–$W$ path. We say a set $S$ of vertices *separates* $V$ and $W$, if every $V$–$W$ path contains a vertex of $S$, and that $S$ is an *$V$–$W$ separating set*. In particular, both $V$ and $W$ are themselves $V$–$W$ separating sets. The following theorem is equivalent to Menger's Theorem.

**Theorem 2.2** *Let $V$ and $W$ be sets of vertices in a graph $G$. For any positive integer $k$, there are $k$ (pairwise) disjoint $V$–$W$ paths in $G$ if and only if every $V$–$W$ separating set contains at least $k$ vertices.*

To see that this theorem is equivalent to Theorem 2.1, suppose first that Theorem 2.1 holds. Let $V$ and $W$ be sets of vertices in $G$. Introduce two new vertices $v$ and $w$ and join $v$ to every vertex of $V$ and $w$ to every vertex of $W$, and let $H$ be the resulting graph. Then $v$ and $w$ are not adjacent in $H$ and every collection of internally disjoint $v$–$w$ paths in $H$ corresponds to a collection of disjoint $V$–$W$ paths in $G$ and vice versa. Moreover, a set $S$ is a $v$–$w$ separator in $H$ if and only if it is a $V$–$W$ separating set in $G$. Hence Theorem 2.1 implies Theorem 2.2.

Suppose now that Theorem 2.2 holds. Let $v$ and $w$ be two non-adjacent vertices in a graph $G$. Let $V$ be the set of neighbours of $v$ and let $W$ be the set of neighbours of $w$. As before, every pair of internally disjoint $v$–$w$ paths in $G$ corresponds to a pair of disjoint $V$–$W$ paths in $G$ and vice versa. Moreover, $S$ is a $v$–$w$ separator if and only if it is an $V$–$W$ separating set. Hence Theorem 2.2 implies Theorem 2.1.

We now sketch a second proof of Menger's theorem by proving Theorem 2.2.

**Proof.** Let $V$ and $W$ be sets of vertices of a graph $G$. As before, it is obvious that if there are $k$ internally disjoint $V$–$W$ paths in $G$, then every $V$–$W$ separating set must have at least $k$ vertices. For the converse, suppose that every $V$–$W$ separating set has at least $k$ vertices. We show that then there must exist $k$ disjoint $V$–$W$ paths. We assume that $V \cap W = \emptyset$; the result then follows when $V \cap W \neq \emptyset$, since every vertex in both $V$ and $W$ is itself a $V$–$W$ path. Begin with a collection $\pi_l$ of $l$ disjoint $V$–$W$ paths $P_1, P_2, \ldots, P_l$ (possibly $l = 0$). We now describe how this collection of $l$ paths can be used to construct a collection of $l+1$ disjoint $V$–$W$ paths, if there is such a collection. For this purpose we define a trail $Q : u_0, e_0, u_1, e_1, \ldots, e_{k-1}, u_k$ to be an *alternating trail* with respect to $\pi_l$ if the following three conditions are satisfied:

(i) If $e_i$ belongs to a path $P_j$ in $\pi_l$, then $P_j$ traverses $e_i$ in the opposite direction to $Q$.
(ii) If $u_r = u_s$ with $r \neq s$, then $u_r$ lies on some $P_i$ (that is, if a vertex of $Q$ appears more than once on $Q$, then it lies on a path in $\pi_l$).
(iii) If $u_i$ lies on some $P_j$, then at least one of $e_{i-1}$ and $e_i$ belongs to $Q$.

An *augmenting $V$–$W$ trail with respect to* $\pi_l$ is an alternating trail from a vertex $v \in V$ to a vertex $w \in W$, with neither $v$ nor $w$ on any $P_i$.

Note, that if $Q$ is such trail, then every vertex of $Q$ that is not on any $P_i$ appears exactly once on $Q$, and, by (iii), every vertex of $Q$ that lies on some $P_i$ appears at most twice on $Q$. Let $H$ be the union of the $l$ paths $P_i$. If $u_r = u_s$ for some $r, s$ with $1 \leq r < s \leq k$, then either $e_{r-1}$ and $e_s$ are in $H$ and $e_r$ and $e_{s-1}$ are not in $H$, or $e_r$ and $e_{s-1}$ are in $H$ and $e_{r-1}$ and $e_s$ are not in $H$.

Using this observation, one can show that if $Q$ is an augmenting $V$–$W$ trail with respect to $\pi_l$, then $G$ has $l+1$ disjoint $V$–$W$ paths. Indeed the subgraph consisting of the union of $Q$ and the paths $P_1, P_2, \ldots, P_l$ with the edges on both $Q$ and some $P_i$ deleted consists of $l+1$ disjoint $V$–$W$ paths, which we denote $\pi_{l+1}$ (see Fig. 2 for an illustration).

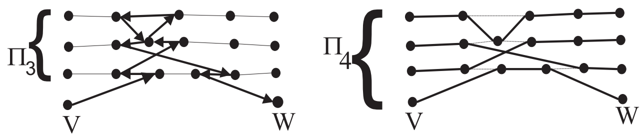


Figure 2: Augmenting $\pi_3$ along an Augmenting $V$–$W$ Trail to obtain $\pi_4$.

We continue in this manner until arriving at a collection $\pi_L$ of $L$ disjoint $V$–$W$ paths for which there is no augmenting $V$–$W$ trail. For each path $P$ in $\pi_L$, let $v_P$ be the last vertex of $P$ that lies on some alternating trail that begins with a vertex in $V$ but does not lie on any path in $\pi_L$ (if such a trail exists), otherwise, let $v_P$ be the first vertex of $P$. Then it can be shown that the set $X$ consisting of the $L$ vertices $v_P$ for $P \in \pi_L$ is a $V$–$W$ separating set. Hence $L = |X| \geq k$. ■

We note that Menger's theorem also holds for digraphs and multigraphs. Both of the proofs given here can easily be adapted to digraphs, and the existence of multiple edges does not change the proofs.

We now turn our attention briefly to infinite graphs. For sets of vertices $V$ and $W$ in an infinite graph $G$, $V$–$W$ paths and separating sets are defined as for finite graphs. Erdös showed if there are at least $k$ vertices in any $V$–$W$ separating set, then there are at least $k$ disjoint $V$–$W$ paths in $G$, a result published in König [41]. Menger's theorem also holds for infinite cardinals, but Erdös proposed a better choice for an extension of the theorem to infinite graphs when he made the following classic conjecture.

**Conjecture 2.1** *Let $V$ and $W$ be sets of vertices in an infinite graph $G$. Then $G$ contains a set $\pi$ of disjoint $V$–$W$ paths and an $V$–$W$ separating set $S$ that are in one-to-one correspondence, where each vertex of $S$ lies on exactly one path in $\pi$, and each path in $\pi$ contains exactly one vertex of $S$.*

Podewski and Steffens [67] made some progress on this conjecture when they showed it to be true for countable graphs that contain no infinite paths. Aharoni strengthened their result when he showed the conjecture to be true for all graphs that contain no infinite path [1] and for all countable graphs [2].

**Vertex connectivity**

The *connectivity* $\kappa(G)$ of a graph $G$ is the smallest number of vertices whose deletion from $G$ produces a disconnected or trivial graph. Clearly, a complete graph cannot be disconnected by deleting vertices, but all other graphs can. It is not difficult to see that in any case $\kappa(G) = \min\{\kappa(v,w) | v, w \in V(G)\}$.

A graph $G$ for which $\kappa(G) \geq k$ is said to be *$k$-connected*. The first characterization of $k$-connected graphs was given by Whitney [73] in 1932. He proved this result independently of Menger's theorem, but with Menger's theorem this result can be obtained in a straightforward manner.

**Theorem 2.3** *A graph $G$ is $k$-connected if and only if every pair of vertices is connected by $k$ internally disjoint paths.*

# 3 Edge-connectivity

The vertex versions of Menger's theorem discussed in Section 2 have edge analogues that we briefly describe here. Let $v$ and $w$ be two vertices in a graph $G$. A set $S$ of edges is a *$v$–$w$ edge-separating* set if $v$ and $w$ lie in different components of $G - S$; that is, if every $v$–$w$ path contains an edge of $S$. The minimum cardinality of a $v$–$w$ edge-separating set is the *$v$–$w$ edge-connectivity* and is denoted by $\lambda(v,w)$.

The maximum number of edge-disjoint $v$–$w$ paths in $G$ is denoted by $\nu(v,w)$. Since each such path must contain an edge from every $v$–$w$ edge-separating set, $\nu(v,w) \leq \lambda(v,w)$. For example, consider the graph of Fig 1. It is easy to see that $\nu(v,w)$ and $\lambda(v,w)$ are both 5. That the two parameters are equal in this case is again not a mere coincidence, and the fact that this holds in general is the local edge version of Menger's theorem.

**Theorem 3.1** *For any vertices $v$ and $w$ in a graph $G$, $\nu(v, w) = \lambda(v, w)$.*

One may well ask whether there always exists a system of $\nu(v, w)$ edge-disjoint paths that contains a system of $\mu(v, w)$ internally disjoint $v$–$w$ paths. Beineke and Harary [4] showed that this need not be the case. For the graph of Fig. 1, $\mu(v, w) = 3$ and $\nu(v, w) = 5$, but no set of three internally disjoint $v$–$w$ paths is contained in a set of five edge-disjoint $v$–$w$ paths. To see this, note that every set of three internally disjoint $v$–$w$ paths contains all five edges $a, b, c, d, e$ of a minimal $v$–$w$ edge-separating set and thus cannot be extended to five edge-disjoint $v$–$w$ paths.

If $v$ and $w$ are not adjacent, then both deg $v$ and deg $w$ may exceed $\kappa(v, w)$ by an arbitrarily large amount. Take, for example, two complete graphs $K_n$ and join one vertex from each copy by an edge. The resulting graph has minimum degree $n - 1$ and $\kappa(v, w) = 1$ for every pair of non-adjacent vertices $v$ and $w$ in the resulting graph. However, Mader in [50] and [51] proved the following:

**Theorem 3.2** *Every non-null graph has adjacent vertices $v$ and $w$ for which $\mu(v, w) = min\ \{deg\ v,\ deg\ w\}$.*

An immediate consequence of the above theorem is that there exist vertices $v$ and $w$ such that $\mu(v, w)=\ \nu(v, w)=\ \lambda(v, w)=\ \min\{\deg\ v, \deg\ w\}$. We note that this theorem is not true for multigraphs, since a multigraph formed from a cycle by doubling every edge does not satisfy the theorem. However, it is true that every multigraph $M$ has adjacent vertices $v$ and $w$ for which $\nu(v, w) = \min\{\deg\ v, \deg\ w\}$ (see [54]).

The *edge-connectivity* $\lambda(G)$ of a non-trivial graph $G$ is the smallest number of edges whose deletion produces a disconnected graph, while that of the trivial graph is defined to be 0. It is not difficult to see that $\lambda(G) = \min\{\lambda(v, w) : v, w \in V(G)\}$. A graph $G$ is *$l$-edge-connected* if $\lambda(G) \geq l$ . The following is a global edge version of Menger's theorem.

**Theorem 3.3** *A graph is $l$-edge-connected if and only if every two vertices are connected by at least $l$ edge-disjoint paths.*

# 4 Mixed connectivity

In this section we consider the problem of disconnecting pairs of vertices by permitting the removal of both vertices and edges. We also look at optimal notions for this concept and relate it to the existence of a combination of edge-disjoint and internally disjoint paths.

**Connectivity pairs**

In their 1967 paper, Beineke and Harary [4] considered the problem of disconnecting a graph by deleting both vertices and edges. In a graph $G$ with vertices $v$ and $w$, a set $S$ of vertices and a set $T$ of edges form a *$v$–$w$ disconnecting pair* if $v$ and $w$ belong to different components of $G - (S \cup T)$. The vertices $v$ and $w$ are *$(k, l)$-connected* if there is no disconnecting pair of $s$ vertices and $t$ edges with $s < k$ and $t \leq l$ or $s \leq k$ and $t < l$. The pair of integers $(k, l)$ is a *$v$–$w$ connectivity pair* if $v$ and $w$ are neither $(k + 1, l)$- nor $(k, l + 1)$-connected. Beineke and Harary claimed to prove a mixed version of Menger's theorem, but Mader [54] pointed out a gap in their proof. Hence, the following remains a conjecture.

**Conjecture 4.1** *If vertices $v$ and $w$ are $(k, l)$-connected in graph $G$, then $G$ has a system of $k + l$ edge-disjoint $v$–$w$ paths, of which $k$ are internally disjoint.*

The available evidence seems to indicate that the conjecture is true, but it has resisted all attempts at a proof. If it is true, then the graph of Fig. 3 illustrates that it is in some sense best possible. The pair (2, 2) is a connectivity pair for vertices $v$ and $w$, and there are four edge-disjoint $v$–$w$ paths (but no more) of which two are internally disjoint. This graph also serves to illustrate that $k$ internally disjoint $v$–$w$ paths in a system of $k + l$ edge-disjoint paths cannot always be chosen in such a way that each of these paths is internally disjoint from the remaining $k + l - 1$ paths.

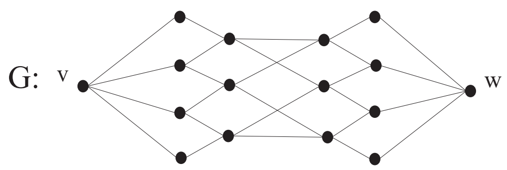


Figure 3: Connectivity pairs and edge-disjoint paths

Even if true, the Beineke-Harary conjecture is not as strong as one might have hoped. If $(k, l)$ is a connectivity pair for $v$ and $w$, there can be more than $k + l$ edge-disjoint $v$–$w$ paths of which more than $k$ are internally disjoint. For example, $(2, 1)$ is a $v$–$w$ connectivity pair in the graph $G$ of Fig. 4. So, by the conjecture, there are three edge-disjoint $v$–$w$ paths; but in fact there are five such paths, with three internally disjoint.

There are some interesting open questions related to this conjecture. For example, if $v$ and $w$ are vertices in a graph $G$ and $k$ is an integer with $0 \leq k \leq \kappa(v, w)$, what is the maximum number of edge-disjoint $v$–$w$ paths of which $k$ are internally disjoint? The graph of Fig. 1 shows that this number need not be $\lambda(v, w)$. A follow-up problem is to determine how difficult is it to compute the second coordinate in a $v$–$w$ connectivity pair $(k, l)$.

If $G$ is a graph and $\kappa = \kappa(G)$, then it is readily seen that for each $k$, $0 \leq k \leq \kappa$, there is a unique connectivity pair $(k, l_k)$. Thus the connectivity

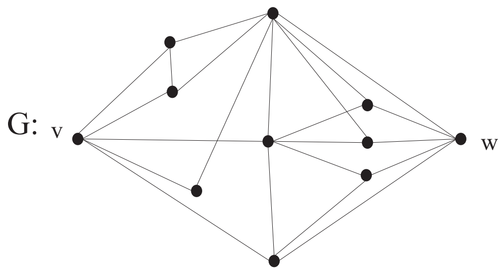


Figure 4: Three internally disjoint paths among five edge-disjoint paths

pairs of a graph determine a function $f$ from the set $\{0, 1, \ldots, \kappa\}$ into the non-negative integers such that $f(\kappa) = 0$. This function is called the *connectivity function* of $G$. It is not difficult to show that $f$ is a strictly decreasing function. Beineke and Harary [4] showed that these two conditions characterize these functions.

**Theorem 4.1** *Suppose $\kappa$ is a positive integer. A function $f$ from $\{0, 1, \ldots, \kappa\}$ into the non-negative integers is the connectivity function of a graph if and only if $f$ is decreasing and $f(\kappa) = 0$.*

### Edge-disjoint skeins

Another mixed version of connectivity was considered by Egawa, Kaneko, and Matsumoto [16]. Recall that a *$v$–$w$ $k$-skein* is a collection of $k$ internally disjoint $v$–$w$ paths. Menger's theorem states that non-adjacent vertices $v$ and $w$ in a graph $G$ are $k$-connected if and only if $G$ has a $v$–$w$ $k$-skein. Let $k$ and $r$ be positive integers and $v$ and $w$ distinct vertices of a graph $G$. Then $v$ and $w$ are *$[k, r]$-joined* if, for every set $S$ of vertices (not containing $v$ or $w$) and every set $T$ of edges with $r|S| + |T| < kr$, the vertices $v$ and $w$ are in the same component of $G - (S \cup T)$. The following mixed version of Menger's theorem was established by Egawa, Kaneko and Matsumoto.

**Theorem 4.2** *Let $k$ and $l$ be positive integers and $v$ and $w$ vertices of a graph $G$. Then $v$ and $w$ are $[k, r]$-joined if and only if $G$ contains $r$ pairwise edge-disjoint $v$–$w$ $k$-skeins.*

Extending these concepts, we define a graph $G$ to be *$[k, r]$-joined* if every pair of vertices is *$[k, r]$-joined* – that is, for any set $S$ of $s$ vertices and any set $T$ of $t$ edges with $rs + t < kr$ the graph $G - (S \cup T)$ is connected. Thus a graph is $[k, 1]$-joined if and only if it is $k$-connected and $[1, r]$-joined if and only if it is $r$-edge-connected. As Kaneko and Ota [38] pointed out, the next result is a consequence of Theorem 4.2 and extends Whitney's characterization of $k$-connected graphs.

**Theorem 4.3** *A graph $G$ of order at least $k+1$ is $[k, r]$-joined if and only if every pair of vertices $v$ and $w$ are joined by $r$ pairwise edge-disjoint $k$-skeins.*

Dirac [14] established the following connection between $k$-connected graphs and cycles.

**Theorem 4.4** *In a $k$-connected graph, every collection of $k$ vertices lie on a common cycle.*

The following generalization of Dirac's theorem was conjectured by Egawa, Kaneko and Matsumoto [16] and again Kaneko and Ota [38].

**Conjecture 4.2** *In every $[k, l]$-joined graph, every collection of $k$ vertices lies on $l$ pairwise edge-disjoint cycles.*

Enomoto and Kaneko [17] established a link between the concepts of pairs of vertices being $[k, l]$-joined and connectivity pairs. Using Theorem 4.3, they showed that the Beineke-Harary conjecture holds for certain pairs $(k, l)$. They also made the following conjecture.

**Conjecture 4.3** *If, in graph $G$, $(k, l)$ is a connectivity pair for vertices $v$ and $w$, then $G$ contains $k+l$ edge-disjoint $v$–$w$ paths of which $k+1$ are internally disjoint.*

# 5 Average connectivity

The connectivity of a graph is a worst-case measure and as such often does not distinguish between graphs that obviously have different degrees of connectedness. For example, for $n \geq 4$, if $G$ is the graph obtained from the complete graph $K_{n-1}$ by adding an end-vertex and $T$ is any tree of order $n$, then they have the same order and the same connectivity, but $G$ appears much more connected than $T$. Several measures of reliability, including toughness, binding number and integrity, have been introduced as more sensitive measures of reliability. However, computing these parameters appears to be a difficult problem. An average measure inspired by Menger's theorem is the topic of this section, and it can be computed in polynomial time (see Chapter 13 for good algorithms for computing connectivity).

### Average vertex connectivity

From Menger's theorem, we know that the connectivity $\kappa(v, w)$ between two non-adjacent vertices $v$ and $w$ in a graph $G$ is the maximum number of internally disjoint $v$–$w$ paths. For the purposes of this section, we extend this to cover all pairs of vertices, and thus define, for adjacent vertices $v$ and $w$, $\kappa(v,w)$ also to be the maximum number of internally disjoint $v$–$w$ paths.

The *average connectivity* $\bar{\kappa}(G)$ of a graph $G$ is the average of the connectivities of all pairs of vertices of $G$; that is,

$$\bar{\kappa}(G) = \frac{\sum_{v,w} \kappa(v,w)}{\binom{n}{2}}.$$

Since $\kappa(v,w) \leq \min\{\deg\, v, \deg\, w\}$ for all pairs of vertices $v$ and $w$ in a graph $G$, it is not difficult to see that if $G$ has degrees $d_1 \geq d_2 \geq \ldots \geq d_n$, then

$$\overline{\kappa}(G) \leq \frac{\sum_{i=1}^{n}(i-1)d_i}{\binom{n}{2}}. \tag{1}$$

Using this observation and the fact that the average degree $\overline{d}(G)$ is $\frac{2m}{n}$, Beineke, Oellermann and Pippert [5] established the upper bound in the next result. The lower bound was established by Dankelmann and Oellermann [12] by counting the number of 'short paths' that are guaranteed to connect pairs of vertices.

**Theorem 5.1** *Let $G$ be a graph on $n$ vertices and $m$ edges with $m \geq n$, and let $r = 2m - n\lfloor \overline{d}(G) \rfloor$. Then*

$$\frac{\overline{d}^2}{n-1} \leq \overline{\kappa}(G) \leq \overline{d}(G) - \frac{r(n-r)}{n(n-1)}.$$

*Moreover, these bounds are sharp.*

Wang and Kleitman [72] considered the problem of constructing a graph with the largest possible connectivity with a given degree sequence. Asano [3] investigated complexity issues related to this problem. The problem of finding the largest average connectivity of a graph with a given degree sequence is open. A related problem is to determine for which graphical degree sequences there is a graph for which equality holds in (1).

Dankelmann and Oellermann [12] found bounds on the average connectivity of several families of graphs, including planar and outerplanar graphs, and Cartesian products of graphs. For planar and Cartesian products of graphs these bounds are sharp and for outerplanar graphs the established bound is asymptotically sharp for maximal outerplanar graphs.

Mader [49] posed another interesting question in asking about the greatest connectivity of a subgraph in a graph with a given number of edges. He showed that for every positive integer $t$ there is a number $g(t)$ such that every graph $G$ of sufficiently large order $n$ and with more than $g(t)(n-t+1)$ edges contains a $t$-connected subgraph. Furthermore, there are infinitely many graphs $G$ with $g(t)(n-t+1)$ edges and no $t$-connected subgraph.

In the same paper Mader showed that

$$\frac{3t}{2} - 2 \leq g(t) < (t-1)(\frac{2+\sqrt{2}}{2}) \qquad (2)$$

and conjectured the following.

**Conjecture 5.1** *There is an integer $n_0$ such that if $G$ is a graph of order $n \geq n_0$ and at least $\frac{3t}{2} - 2$ edges, then $G$ contains a $t$-connected subgraph.*

For sufficiently large orders, equation (2) can be used to show that every graph with average connectivity at least $k$ has a subgraph of connectivity at least $\frac{k}{2+\sqrt{2}} + 1$, and if Mader's conjecture is true, then this can be improved to $\frac{k+4}{3}$. An upper bound is provided by the graph $(2K_r) + \overline{K}_r$, whose average connectivity is asymptotically equal to $\frac{4}{9}(4r-3)$, and for which the largest connectivity of a subgraph is $r$. For sufficiently large $r$, this is approximately $\frac{9}{16}(k + \frac{4}{3})$.

## Uniformly connected graphs

We now turn to graphs with equal connectivity and average connectivity. A graph $G$ is *uniformly $k$-connected* if $\kappa(G) = \overline{\kappa}(G) = k$. A graph $G$ with connectivity $k$ is *critically $k$-connected* if $\kappa(G - v) < k$ for every vertex $v$ and is *minimally $k$-connected* if $\kappa(G - e) < k$ for every edge $e$. Some necessary conditions for a graph to be uniformly $k$-connected was given by Beineke, Oellermann and Pippert [5].

**Theorem 5.2** *Every uniformly $k$-connected graph is minimally $k$-connected if $k \geq 1$ and critically $k$-connected if $k \geq 2$.*

These conditions are not sufficient as there are graphs that are both minimally and critically $k$-connected but not uniformly $k$-connected. For example, for $r \geq 4$, let $G$ be the graph obtained from the $2r$-cycle $u_1u_2 \dots u_{2r}u_1$ by adding two new vertices $v$ and $w$ and joining $v$ to each $u_i$ with an even subscript and $w$ to each $u_i$ with an odd subscript. Then $G$ is both minimally and critically 3-connected but not uniformly 3-connected since $\kappa(v, w) = r > 3$. Using similar constructions but with the cycle replaced by circulants of sufficiently large order and connectivity $k - 1$, graphs can be constructed that are both minimally and critically $k$-connected but not uniformly $k$-connected for $k > 3$.

It is easy to see that the uniformly 0-, 1- and 2-connected graphs are the null graphs, the trees and the cycles, respectively, and the uniformly $(n-1)$- and $(n-2)$-connected graphs of order $n$ are the complete graphs and the complete graphs minus a 1-factor, respectively. Moreover, for $k \geq 3$, a graph is uniformly $k$-connected if and only if does not contain a subdivision of $K_2 + \overline{K}_k$. However, finding nontrivial characterizations of uniformly $k$-connected graphs remains an open problem.

### Average edge-connectivity

Analogous to the connectivity between a pair of vertices, the *edge-connectivity* $\lambda(v,w)$ between $v$ and $w$ is the maximum number of edge-disjoint $v$–$w$ paths in $G$. The *average edge-connectivity* of $G$, denoted $\overline{\lambda}(G)$, is defined to be the average edge-connectivity between pairs of vertices of $G$; that is,

$$\bar{\lambda}(G) = \frac{\sum_{v,w} \lambda(v,w)}{\binom{n}{2}}.$$

Much less is known about the average edge-connectivity than the average connectivity itself. We give here a few bounds on this parameter and state some open problems and conjectures. It is well-known that for all vertices $v$ and $w$ in a graph $G$, $\kappa(v,w) \le \lambda(v,w) \le \min\{\deg\, v, \deg\, w\}$. Thus

$$\overline{\kappa}(G) \le \overline{\lambda}(G) \le \overline{d}(G).$$

From Theorem 5.1 it follows that if $G$ is a graph of order $n$ and average degree $\overline{d}$, then

$$\frac{\overline{d}^2}{n-1} \le \overline{\lambda}(G) \le \overline{d}.$$

Much research has focused on conditions that guarantee equality of the edge-connectivity and the minimum degree of a graph. Some of these conditions are given by Chartrand [8], Lesniak [44], Plesník [66] and Volkmann [71]. For example, Chartrand showed that if the minimum degree of an $n$-vertex graph is at least $n/2$, then its edge-connectivity equals its minimum degree. It turns out that these four conditions imply something even stronger, namely, that $\lambda(v,w) = \min\{\deg\, v,\, \deg\, w\}$ for all vertices $v$ and $w$, thereby implying that the average edge-connectivity is as large as possible for these graphs. Edmonds [15] proved that if $\mathbf{D}$ is the degree sequence of a connected graph, then there is a graph $G$ with $\mathbf{D}$ as its degree sequence and $\lambda(G) = \delta(G)$. Thus, if $\mathbf{D}$ is the degree sequence of a connected graph, then it can be realized by a graph whose edge-connectivity is as large as possible. It is natural to ask a similar question with regards to the average edge-connectivity, namely, if $\mathbf{D}$ is the degree sequence of a connected graph, can it be realized by a graph for which $\lambda(v,w) = \min\{\deg\, v, \deg\, w\}$ for all pairs of vertices $v$ and $w$? If there is a graph $G$ with degree sequence $\mathbf{D}$ and $\lambda(v,w) = \min\{\deg\, v, \deg\, w\}$ for all pairs of vertices $v$ and $w$, then $\mathbf{D}$ is called *optimal.* Dankelmann and Oellermann [13] proved the following result about such sequences for multigraphs.

**Theorem 5.3** *Let $\boldsymbol{D} = d_1 \ge d_2 \ge \ldots \ge d_n$ be the degree sequence of a multigraph with $r$ the number of terms equal to* 1. *Then $\boldsymbol{D}$ is optimal if and only if $r \le d_1 - d_2$ or $r = n-1$.*

In order for a graphical sequence $\mathbf{D} = d_1 \ge d_2 \ge \ldots \ge d_n$ in which $r$ terms are 1 to be optimal it is necessary that $d_1 - r \ge d_2$. Indeed, $\mathbf{D}$ is optimal if and only if the sequence obtained from $\mathbf{D}$ by replacing $d_1$ with $d_1 - r$ and deleting all $r$ terms equal to 1 is graphical. Fricke, Oellermann and Swart [25] conjectured the following.

**Conjecture 5.2** *Every degree sequence with no 0s or 1s is optimal.*

As was done for the average connectivity, it is natural to ask for the largest edge-connectivity of a subgraph in a graph with given average edge-connectivity. We make a few observations related to this problem. Mader [47] showed that if $t$ is a positive integer and if $G$ is a graph of order $n \geq t$ and at least $(t-1)n - \binom{t}{2}$ edges, then $G$ contains a $t$-edge-connected subgraph. With a little algebra, it can be shown that if $l = 2s$ is an even integer, then a graph $G$ with average edge-connectivity at least $l$ satisfies the hypotheses of Mader's result with $t = s+1$. Hence, $G$ contains an $(\frac{l}{2}+1)$-edge-connected subgraph. However, we believe that this lower bound can be improved significantly. As to upper bounds for this parameter, the graph in Fig. **??**, due to Jacques Verstraëte, has average edge-connectivity $\frac{103}{34}$, but has no 3-edge-connected subgraph. Hence, in general, if $\bar{\lambda}(G) \geq l$ for some integer $l$, the largest edge-connectivity of a subgraph may be less than $l$.

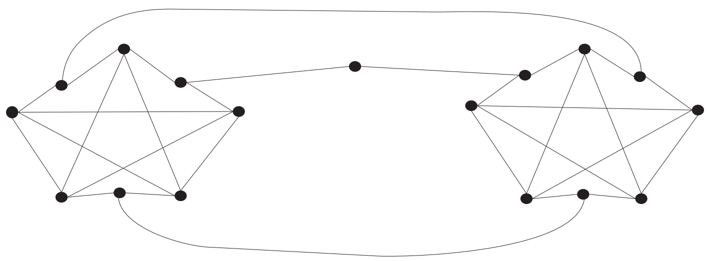

Figure 5: A Graph with average edge-connectivity greater than 3

Another question to which it would be interesting to have an answer is when do all pairs of vertices have the same edge-connectivity; that is, which graphs have equal edge-connectivity and average edge-connectivity.

### Average connectivity of digraphs

As one would expect, the *average connectivity* $\overline{\kappa}(D)$ of a digraph $D$ is the average over all ordered pairs $(v, w)$ of vertices of the maximum number $\kappa(v, w)$ of internally disjoint directed $v$–$w$ paths; that is,

$$\bar{\kappa}(D) = \frac{\sum_{v,w} \kappa(v, w)}{n(n-1)},$$

where $n$ is the order of $D$. Clearly, $\kappa(v, w) \leq \min\{d^+(v), d^-(w)\}$, from which it follows that the average connectivity is bounded by the average out-degree; that is, for a digraph with $n$ vertices and $m$ arcs,

$$\bar{\kappa}(D) \leq \frac{m}{n(n-1)}.$$

Rather than look at other results for digraphs similar to those for graphs, we turn to another type of problem. An *orientation* of a graph $G$ is the result of assigning a direction to each edge of $G$. A major focus of work on the average connectivity of digraphs has been the maximum average connectivity among the orientations of a given graph. Let $\overline{\kappa}_{max}(G)$ and $\overline{\kappa}_{min}(G)$ denote the maximum and minimum average connectivity among the orientations of $G$.

Our starting point for this discussion is the class of trees. For a tree $T$, the problem of finding $\overline{\kappa}_{max}(T)$ is equivalent to finding an orientation of $T$ that maximizes the number of pairs of vertices $v$ and $w$ for which there exists a directed path from $v$ to $w$. The *centroid* of a tree of order $n$ consists of those vertices for which no component of $T - v$ has more than $\frac{1}{2}n$ vertices. A basic result in this area is that the centroid of every tree consists either of a single vertex or a pair of adjacent vertices.

Let $v$ be a vertex of tree $T$. Recall that a *branch at* $v$ is the subtree of $T$ induced by $v$ and those vertices whose path from $v$ begins with a specified edge. An orientation $D$ of $T$ is *$v$-based* if every branch at $v$ is oriented as either an in-tree or an out-tree rooted at $v$. Note that then $D$ consists of an in-tree at $v$ and an out-tree at $v$ having only $v$ in common. The following result is due to Henning and Oellermann [36].

**Theorem 5.4** *If $D$ is an orientation of a tree $T$ that achieves the maximum average directed connectivity, then $D$ is a $v$-based orientation at a centroid vertex $v$, with the difference in the orders of the in- and out-trees at $v$ as small as possible.*

Henning and Oellermann also established bounds on the maximum average connectivity of trees.

**Theorem 5.5** *If $T$ is a tree of order at least 3, then*

$$\frac{2}{9} < \overline{\kappa}_{max}(T) \leq \frac{1}{2}.$$

The bounds in this theorem are sharp in that the upper bound is attained by oriented paths (and only paths), and there are orientations of trees whose average connectivity gets arbitrarily close to $\frac{2}{9}$. The minimum average connectivity of an orientation of a tree $T$ of order $n$ can readily be seen to be $\frac{1}{n}$ by considering $T$ as a bipartite graph and orienting the edges from one partite set to the other. Indeed, this argument shows that for any bipartite graph $G$ with $n$ vertices and $m$ edges, $\overline{\kappa}_{min}(G) = \frac{m}{n(n-1)}$.

The values of $\overline{\kappa}_{max}(G)$ and $\overline{\kappa}_{min}(G)$ for regular complete multipartite graphs were found by Henning and Oellermann [36], [35]. These have tournaments as a special case and bounds on the average connectivity are given in the next theorem.

**Theorem 5.6** *It $T$ is a tournament of order $n$, then*

$$\frac{n+1}{6} \leq \overline{\kappa}(T) \leq \begin{cases} \dfrac{n-1}{2} & \text{for } n \text{ odd} \\[2ex] \dfrac{2n^2-5n+4}{4(n-1)} & \text{for } n \text{ even.} \end{cases}$$

*Moreover the upper bound is sharp and the lower bound is attained if and only if $T$ is a transitive tournament.*

Obviously $\overline{\kappa}_{max}(G)$ can be no greater than $\overline{\kappa}(G)$, but it seems unlikely that equality ever holds for non-trivial graphs. It would be interesting to have good bounds on $\overline{\kappa}_{max}(G)$ in terms of $\overline{\kappa}(G)$.

# 6 Menger results for paths of bounded length

Some paths in a graph may be so long that they are of little practical value. This observation has led to the investigation of connectivity being restricted to paths no longer than some prescribed distance. This section focuses on parameters analogous to $\kappa$ and $\mu$ for paths of bounded length.

**Short paths**

Following Lovász, Neumann-Lara and Plummer [46], for any integer $d \geq 2$ and non-adjacent vertices $v$ and $w$ in a graph $G$, we let $\kappa_d(v, w)$ be the minimum number of vertices in $G$ whose deletion destroys all $v$–$w$ paths of length at most $d$. Further, for any positive integer $d$ and any vertices $v$ and $w$ in $G$, we let $\mu_d(v, w)$ denote the maximum number of internally disjoint $v$–$w$ paths of length at most $d$. Obviously, for $d \geq 2$, $\mu_d(v, w) \leq \kappa_d(v, w)$, and there are of course values of of $d$, such as $d = n - 1$ when $G$ has order $n$, for which equality holds. However, equality does not always hold, as the graph in Fig. 6 shows. Here, $\mu_5(v, w) = 1$ and $\kappa_5(v, w) = 2$. Lovász, Neumann-Lara and Plummer [46], Entringer, Jackson and Slater [18] and Hartman and Rubin [34] independently showed that equality also holds at the low end of the range of path-lengths.

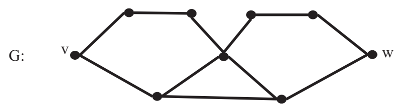


Figure 6: Destroying paths of bounded length

**Theorem 6.1** *If $d$ is the distance between vertices $v$ and $w$ in graph $G$, then $\kappa_d(v, w) = \mu_d(v, w)$.*

We now turn to the question of upper bounds for $\kappa_d(v,w)$ in terms of $\mu_d(v,w)$. It is convenient to look at this in the form of the ratio

$$\rho_d(v,w) = \frac{\kappa_d(v,w)}{\mu_d(v,w)}.$$

The following results were established in [46].

**Theorem 6.2** *If $v$ and $w$ are non-adjacent vertices in a graph $G$, then*

(a) $\rho_2(v,w) = \rho_3(v,w) = \rho_4(v,w) = 1$;

(b) $\rho_5(v,w) = 2$;

(c) *for* $d \geq 6$, $\rho_d(v,w) \leq \lfloor \frac{d}{2} \rfloor$.

It is not known, for general $d$, how close $\rho_d(v,w)$ can get to $\frac{1}{2}d$. Let $\rho_d(G) = \max\{\rho_d(v,w)\}$ over all pairs of non-adjacent vertices $v$ and $w$ in graph $G$, and let

$$\rho^*(d) = \sup_G\{\rho_d(G)\},$$

where the supremum is taken over all graphs $G$ for which $\rho_d(G)$ is defined. Chung [11] obtained bounds of this invariant.

**Theorem 6.3** *For $d \geq 6$,*

$$\frac{d+1}{3} \leq \rho^*(d) \leq \frac{d}{2}.$$

Having established the lower bound in the above theorem, Chung [11] conjectured that the lower bound in the theorem is close to being exact.

**Conjecture 6.1** $\rho^*(d) = \frac{d}{3} + O(1)$

We observed that $\mu_d(v,w) \leq \kappa_d(v,w)$ and that this inequality may be strict. Intuitively one may expect though that for a fixed $d$ there is a sufficiently large $d_m$ such that $\kappa_d(v,w) \leq \mu_{d_m}(v,w)$. This question was answered in the affirmative in [46].

**Theorem 6.4** *Let $v$ and $w$ be nonadjacent vertices of a graph $G$ and $d$ and $k$ positive integers. Then there is a constant $c(d,k)$ such that if $\kappa_d(v,w) \geq k$, then $\mu_{c(d,k)}(v,w) \geq k$.*

Let $c_{min}(d,k)$ be the smallest value for such a constant $c(d,k)$. An upper bound on $c_{min}(d,k)$ was established in [46] and improved by Pyber and Tuza [68] to

$$c_{min}(d,k) < \binom{d+k-2}{k}\left(1 + \frac{k}{d+k-2}\right).$$

They also showed by construction that $c(d,k) \geq \lfloor \frac{d}{k} - 1 \rfloor^k$.

### Menger path systems

Up to now, we have focused more on the connectivity parameter $\kappa_d(v, w)$ than on the path parameter $\mu_d(v, w)$, but now we shift our focus. For positive integers $d$ and $k$, a $(d, k)$-*path-system* (or a $(d, k)$-*skein*) is a collection of $k$ internally disjoint paths of length at most $d$ joining a pair of vertices. A graph is said to be $(d, k)$-*Mengerian* if there is a $(d, k)$-path-system between every pair of its vertices; that is, if $\mu_d(v, w) \geq k$ for all $v$ and $w$ in $G$. In introducing this idea, Ordman [64] related it to computer networks and distributed processing. Faudree, Jacobson, Ordman, Schelp and Tuza [23] found a variety of conditions, involving parameters such as the minimum degree, the number of edges and the connectivity, that guarantee this Mengerian property, one example of which is the following result.

**Theorem 6.5** *If $G$ is a graph of order $n$ with connectivity greater than $k - 1 + \frac{n-k}{d}$, then $G$ is $(d, k)$-Mengerian.*

Other conditions for a graph to be $(d, k)$-Mengerian were given by Faudree, Gould, Lesniak and Schelp ([21] and [22]). Path systems for edge-disjoint paths were studied by Fathnezhad [19]. An extensive survey of results on Menger path systems is included in [20].

### Long paths

Montejano and Neumann-Lara [59] investigated the opposite topic, focusing on long paths rather than short ones. For $d \geq 2$ and non-adjacent vertices $v$ and $w$ in a graph $G$, let $\kappa'_d(v, w)$ denote the minimum number of vertices in $G - \{v, w\}$ whose deletion destroys all $v$–$w$ paths of length at least $d$ and let $\mu'_d(v, w)$ denote the maximum number of internally disjoint $v$–$w$ paths of length at least $d$. The quantity that Montejano and Neumann-Lara were interested in was a lower bound for the maximum-sized skein of paths joining vertices with 'long-path connectivity' $\kappa'$ at least $k$. More precisely, given a graph $G$, for $d \geq 2$ and $k \geq 0$, let $\psi_d(G, k) = \min\{\mu'_d(v, w) : \kappa'_d(v, w) \geq k\}$ and let $\psi_d(k) = \inf_G\{\psi_d(G, k)\}$. The following theorem combines results of Mader (see Hager [31]) for the case $d = 3$ with general bounds of Montejano and Neumann-Lara [59] and Hager [31].

**Theorem 6.6**

(a) *For $k \geq 0, \psi_2(k) = k$ and $\psi_3(k) = \lceil \frac{k}{2} \rceil$.*

(b) *For $k \geq 0$ and $d \geq 4$,*

$$\left\lceil \frac{k}{3d-5} \right\rceil \leq \psi_d(k) \leq \left\lceil \frac{k}{d-1} \right\rceil.$$

# 7 Connectivity of sets

The beauty of Menger's theorem lies in the way it relates separating sets with paths joining a pair of vertices. In this section we consider extensions of these concepts to sets of more than two vertices. Our story begins with collections of paths in a graph joining (independent) sets of vertices into a connected subgraph, and in Section 8 moves on to trees joining unrestricted sets of vertices.

### Total separation

Let $G$ be a graph, and let $A$ be an independent set of vertices with $|A| \geq 2$. A subset $S$ of $V - A$ is a *total $A$-separating set* if the vertices of $A$ are all in different components of $G - S$, and the *total $A$-connectivity* $\kappa(A)$ is the minimum cardinality of a total $A$-separating set. An *$A$-path* is a path whose only vertices in $A$ are its endpoints. We let $\mu(A)$ denote the maximum cardinality of a set of internally disjoint $A$-paths. Clearly $\mu(A) \leq \kappa(A)$, and, from Menger's theorem, we know that equality holds when $|A| = 2$. However, this need not be the case for larger sets. In fact, $\mu(A)$ can be as small as half of $\kappa(A)$, as the following example shows. For $r \geq 1$, let $G$ be the graph obtained from $K_{2r+1}$ by adding an end-vertex at each of its vertices and let $A$ be the set of end-vertices. Then $\mu(A) = r$ and $\kappa(A) = 2r$ (see Fig. 7 for the case $r = 1$). It was conjectured by Gallai [26] and proved by Mader [53] that this is the extreme.

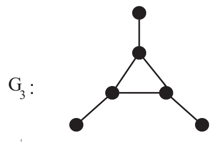


Figure 7: Paths between pairs of vertices in a set

**Theorem 7.1** *If $A$ is an independent set of vertices in a graph, then $\frac{1}{2}\kappa(A) \leq \mu(A) \leq \kappa(A)$, and these bounds are sharp.*

The upperbound is attained, for example, by the complete bipartite graph $K_{r,r}$ where $A$ is one of the partite sets.

Analogous to the edge-connectivity of a graph $G$, there is also an edge version of total connectivity. For a set of vertices $A$ with $|A| \geq 2$, a set $F$ of edges is a *total edge-$A$-separating set* if the vertices of $A$ are all in different components of $G - F$. The *total edge-$A$-connectivity* $\lambda(A)$ is the minimum cardinality of a total edge-$A$-separating set. The number $\nu(A)$ is the maximum number of paths between pairs of vertices in $A$ that are pairwise edge-disjoint. Mader [52] established results similar to Theorem 7.1.

**Theorem 7.2** *If $A$ is any set of vertices in a graph, then $\frac{1}{2}\lambda(A) \le \nu(A) \le \lambda(A)$ and these bounds are sharp.*

The sharpness of the lower bound follows from the complete bipartite graph $K_{2r+1,2r+1}$ with $A$ being one of the partite sets, since then $\lambda(A) = 2r(2r+1)$ and $\nu(A) = r(2r+1)$. The upper bound holds for any complete graph with $A$ the entire vertex set.

**Disconnecting a graph into more than two components**

There are also global parameters related to $A$-paths. Just as the connectivity of a non-complete graph is the minimum connectivity of a pair of vertices in the graph, the *$r$-connectivity* $\kappa_r(G)$ of a graph $G$ with independence number at least $r$ is the minimum value of $\kappa(A)$ where $A$ is a set of $r$ independent vertices and that of a graph of order $n$ and independence number less than $r$ is $n - r$; that is, $\kappa_r(G)$ is the smallest number of vertices whose removal produces a graph with at least $r$ components or a graph with fewer than $r$ vertices. This concept was first introduced by Chartand, Kapoor, Lesniak and Lick [10], who also defined the *connectivity sequence* of a graph $G$ of order $n$ as the sequence $(\kappa_2(G), \kappa_3(G), \ldots, \kappa_n(G))$. One of their results gives criteria for a sequence of non-negative integers to be the connectivity sequence of some graph.

**Theorem 7.3** *A sequence $r_2, r_3, \ldots, r_n$ of nonnegative integers is the connectivity sequence of a graph of order $n$ if and only if there is an integer $a$ such that $r_2 \le r_3 \le \ldots \le r_a \le r_{a+1}$ and $r_{a+i} = n-(a+i)+1$ for $i = 1, 2, \ldots, n-a$. Moreover, $a$ is the independence number of the graph realizing this as its connectivity sequence.*

There is naturally a counterpart for edge-deletion: the *$r$-edge-connectivity* $\lambda_r(G)$ of a graph $G$ is the minimum number of edges whose removal results in a graph with at least $r$ components. As in the vertex case, $\lambda_2(G) = \lambda(G)$. This concept was introduced by Boesch and Chen [6] and was studied further by Goldsmith [27], [28] and Goldsmith, Manvel and Faber [29]. The focus of these papers is on heuristics and bounds for approximating the $r$-edge-connectivity.

Much of the other work on $r$-connectivity concerns degree conditions that guarantee lower bounds for this parameter, see [62], [63] and [69]. The next result was established in [10].

**Theorem 7.4** *Suppose $G$ is a graph of order $n$ and independence number at least $r$. If*

$$\delta(G) \ge \frac{n + (r-1)(k-2)}{r},$$

*then $\kappa_r(G) \ge k$.*

With $r = 2$ this result gives the well-known minimum degree condition guaranteeing $k$-connectedness in a graph (see [9]). This minimum degree condition also guarantees a large number of 'short' internally disjoint paths (see [62]).

**Theorem 7.5** *Let $G$ be a graph of order $n \geq 2$ and $r \geq 3$ and $k$ integers with $1 \leq k \leq n - r + 1$. If*

$$\delta(G) \geq \frac{n + (r-1)(k-2)}{r},$$

*then for each set $A$ of $r$ vertices there exist at least $k$ internally disjoint $A$-paths each of length at most 2.*

Whitney [73] showed that $\kappa(G) \leq \lambda(G)$ for any graph $G$. It seems natural to seek extensions for Whitney's inequality, namely that $\kappa_r(G) \leq \lambda_r(G)$ for any graph $G$. However, it turns out that the proposed inequality fails even for $r = 3$. Take, for example, the graph $G$ obtained from $K_{l,l} \cup K_2$, $l \geq 3$, by adding an edge between the two components. Then $\lambda_3(G) = 2$ and $\kappa_3(G) = l$.

# 8 Connecting with trees

In 1961, Tutte [70] and Nash-Williams [60] independently proved a classic result on packing a graph with trees. (An analogue for covering a graph with trees was published three years later by Nash-Williams [61].) In order to state their result, it is convenient to have some additional terminology and notation. The *boundary* $\partial S$ of a set $S$ of vertices in a graph $G$ is the set of edges that join a vertex in $S$ to a vertex not in $S$. More generally, given a collection of pairwise disjoint sets $\mathcal{P} = \{U_1, U_2, \ldots, U_t\}$ of the vertices of $G$, $|\mathcal{P}|$ is the number of sets in $\mathcal{P}$, and the *boundary* $\partial\mathcal{P}$ of $\mathcal{P}$ is the union of the boundaries of the elements of $\mathcal{P}$.

**Theorem 8.1** *A graph $G$ has $r$ edge-disjoint spanning trees if and only if, for each partition $\mathcal{P}$ of its vertex set $V$, $|\partial\mathcal{P}| \geq r(|\mathcal{P}|-1)$.*

The following corollary can readily be deduced from this result.

**Corollary 8.2** *Every $2r$-edge-connected graph has a set of $r$ edge-disjoint spanning trees.*

While this result is similar in appearance to Menger's theorem, there is also a deeper connection. It concerns separation that is less restricted than the total version considered in Section 7, and involves connecting sets of vertices with trees rather than paths. We begin with a result of Keijsper and Schrijver [39].

Given a graph $G$ with $S \subseteq V$, a set $F$ of edges of $G$ is an *$S - \overline{S}$-connector* if every component of the subgraph of $G$ with edge-set $F$ meets both $S$ and $\overline{S}$ (thus $F$ must span $G$). A *subpartition* $\mathcal{P}$ of a set $X$ is a collection of pairwise disjoint nonempty subsets of $X$. Keijsper and Schrijver proved the following extension of the theorem of Tutte and Nash-Williams.

**Theorem 8.3** *Let $S$ be a set of vertices in graph $G$. Then $G$ has $r$ edge-disjoint $S - \overline{S}$-connectors if and only if $|\partial \mathcal{P}| \geq r|\mathcal{P}|$ for each subpartition $\mathcal{P}$ of $S$ or $\overline{S}$.*

In the case where $S$ is a singleton, an $S$-connector is a connected spanning subgraph of $G$, and hence Theorem 8.1 is a special case of Theorem 8.3. Indeed, Keijsper and Schrijver used the result of Nash-Williams and Tutte to establish their own result. Their proof also yields a polynomial algorithm for finding $r$ edge-disjoint $S$-connectors when they exist.

Given a set $A$ of vertices in a graph $G$, a subgraph of $G$ is called an $A$-*tree* if it is a tree containing the vertices in $A$ and if each of its end-vertices is in $A$. Just as with most of the concepts we have considered, there are both vertex and edge versions, but here, primarily for historical reasons, we begin with the edge results.

### Edge-disjoint trees

Kriesell [43] considered the case where $2 < |A| < n$ further. A set of vertices $A$ is $l$-*edge-connected* if $\lambda(v, w) \geq l$ for all vertices $v$ and $w$ in $A$. Kriesell showed that for any positive integers $r$ and $s$ there is a smallest number $l = f_s(r)$ such that for every $l$-edge-connected set $A$ of order at most $s$, there is a collection of $r$ edge-disjoint $A$-trees. He settled the case of $s = 3$; for larger values of $s$ the problem remains open.

**Theorem 8.4** *Suppose $G$ is a graph and $A$ a set of three vertices of $G$ that is $\lfloor \frac{4k+1}{3} \rfloor$-edge-connected. Then $G$ has a set of $k$ edge-disjoint $A$-trees. Furthermore, this value is sharp.*

Kriesell also conjectured an analogue to the Corollary to Theorem 8.1.

**Conjecture 8.1** *If $A$ is a $2k$-edge-connected set of vertices in a graph $G$, then $G$ has a set of $k$ edge-disjoint $A$-trees.*

Note that the hypothesis of the conjecture does not include the assumption that $G$ itself be $2k$-edge-connected. The following theorem, also due to Kriesell [43], gives the minimum edge-connectivity that guarantees that a graph have a specified number of edge-disjoint spanning trees.

**Theorem 8.5** *For $n, k \geq 2$, every $(2k + 1 - \lceil \frac{2k+2}{n} \rceil)$-edge-connected graph of order $n$ has $k$ edge-disjoint spanning trees. Furthermore, this value is sharp.*

### Internally disjoint trees

We now turn to the vertex analogue, in which the $A$-trees are required to be disjoint except for the vertices in $A$; parallel to other usage, we call such $A$-trees *internally disjoint*. For a set $A$ of independent vertices in a graph $G$, let $\mu_\tau(A)$ be the maximum number of trees in any set of internally disjoint $A$-trees in $G$. For $k, t \geq 2$, let $\beta(t,k)$ be the minimum value of $\mu_\tau(A)$ taken over all graphs $G$ of order at least $t$ and all $k$-connected sets $A$ of $t$ independent vertices in $G$. It follows from Menger's theorem that $\beta(2,k) = k$, and exact values are also known for $t = 3$ and 4. However, for larger values of $t$, only bounds are known. The results in the following theorem are due to Hind and Oellermann [37].

**Theorem 8.6**

(a) *For $k \geq 2$, $\beta(2,k) = k$, $\beta(3,k) = \lfloor \frac{3k+1}{4} \rfloor$, and $\beta(4,k) = \lfloor \frac{2k}{3} \rfloor$.*

(b) *For $t \geq 5$ and $k \geq 2$,*

$$\left\lceil \left(\tfrac{2}{3}\right)^{t-2} k \right\rceil \leq \beta(t,k) \leq \left\lfloor \tfrac{1}{t-1} \left\lceil \tfrac{tk}{2} \right\rceil \right\rfloor.$$

We note that, for $t = 2, 3$, and 4, the upper bound in part (b) equals the value in part (a), and Hind and Oellermann [37] conjectured that this is always an equality. The upper bound was established by constructing a multigraph $H$ with $t$ vertices and no more than $\left\lfloor \frac{1}{t-1} \left\lceil \frac{tk}{2} \right\rceil \right\rfloor$ edge-disjoint spanning trees. If $G$ is the result of subdividing each edge of $H$ once and $A$ is the set of vertices in $H$, the result then follows from the observation that a family of internally disjoint $A$-trees in $G$ corresponds to a family of edge-disjoint $A$-trees in $H$. The lower bound can be established using induction and the fact that if a set $A$ of vertices in a graph $G$ is $k$-connected, then so is every subset of $A$ of order at least 2.

Hager [30] studied sets of internally disjoint $A$-trees with additional structural properties and in a global setting. For positive integers $t$ and $k$, a graph $G$ of order at least $t + k$ is $(t,k)$-*pendant-tree-connected* if, for each set $A$ of $t$ vertices, there are $k$ internally disjoint $A$-trees in each of which every vertex of $A$ is an end-vertex (also known as a pendant vertex). For a graph $G$, the *$t$-th pendant-tree-connectivity* $\tau_t(G)$ is the largest integer $k$ for which $G$ is $(t,k)$-pendant-tree-connected. Let $\tau(t,k) = 1 +$ max $\{\kappa(G)\colon\ \tau_t(G) < k\}$. It is not hard to see that for all $t$, $\tau(t,1) = t$. The following theorem contains some of Hager's results [30].

**Theorem 8.7**

(a) *For $k \geq 2$, $\tau(1,k) = \tau(2,k) = k$.*

(b) *For $k \geq 2$ and $t \geq 3$, $t + k + 1 \leq \tau(t,k) \leq 2^{k(t+1)}$.*

The lower bound was established by a recursive construction. In establishing the upper bound, it was first shown that $\tau_t(G) \geq k$ for every $kt$-connected graph containing a subdivision of the complete graph $K_{k(t+1)}$. This gives the upper bound when combined with the result of Mader [48] that every graph of order $n$ with at least $2^{r-1}n$ edges contains a subdivision of $K_r$.

Hager [30] also established some relationships between $k$-linked graphs and the pendant-tree-connectivity.

**Theorem 8.8**

(a) *If $G$ is a graph with minimum degree $\delta(G) \geq 2k(t-1)+1$ for which $G-v$ is $k(t-1)$-linked for each vertex $v$, then $\tau_t(G) \geq k$.*

(b) *If $\tau_{2k}(G) \geq k$, then $G$ is $k$-linked.*

Dirac [14] showed that, in a $k$-connected graph, every set of $k+1$ vertices lie on a common path, a result that motivated Hager [32] to ask how many internally disjoint $A$-paths exist for a given set $A$ of vertices in a graph $G$. This question is most interesting when $G$ satisfies extra conditions, often involving connectivity. Similarly, one can seek conditions on $G$ such that, for each set $A$ of $t$ vertices, there are at least $k$ internally disjoint $A$-paths. A graph $G$ with at least max $\{t, k+1\}$ vertices is called $(t,k)$-*path-connected* if every set $A$ of $t$ vertices has $k$ internally disjoint $A$-paths. The *t-th path-connectivity number* $\pi_t(G)$, is the largest integer $k$ for which $G$ is $(t,k)$-path-connected; thus, $\pi_2(G) = \kappa(G)$. Hager [32] established the following result on connectivity that ensures $t$-path-connectivity of a given size.

**Theorem 8.9**

(a) *If $G$ is $3k$-connected, then $\pi_4(G) \geq k$.*

(b) *If $G$ is $\lceil \frac{9k}{2} \rceil$-connected, then $\pi_5(G) \geq k$.*

(c) *If $G$ is $2^{t-2}k$-connected, where $t \geq 6$, then $\pi_t(G) \geq k$.*

Analogous to the function $\tau(t,k)$ discussed above, we define, for $t \geq 2$ and $k \geq 1$, $\pi(t,k)$=1 + $\max\{\kappa(G) : \pi_t(G) < k\}$. Hager [32] established the results in the next theorem.

**Theorem 8.10**

(a) *For $k \geq 2$, $\pi(2,k) = k$, $\pi(3,k) = 2k$ and $\pi(4,k) = 3k$.*

(b) *For $t \geq 5$ and $k \geq 2, \pi(t,k) \geq k(t-1)$.*

Note that the upper bound in (b) is the correct value for the cases $t = 2, 3$ and 4. Hager conjectured this to always be the case.

**Conjecture 8.2** *For $t \geq 5$ and $k \geq 2$, $\pi(t, k) = k(t-1)$.*

# References


[1] R. Aharoni, Menger's theorem for graphs containing no infinite paths, *Europ. J. Combin.* **4** (1983), 201–204.

[2] R. Aharoni, Menger's theorem for countable graphs, *J. Combin. Theory (B)* **43** (1987), 303–313.

[3] T. Asano, An $O(n \log \log n)$ time algorithm for constructing a graph of maximum connectivity with presecribed degrees, *J. Comp. Sys. Sciences* **51** (1995), 503–510.

[4] L. W. Beineke and F. Harary, The connectivity function of a graph, *Mathematika,* **14** (1967), 197–202.

[5] L. W. Beineke, O. R. Oellermann and R. E. Pippert, The average connectivity of a graph, *Discrete Math.* **252** (2002), 31–45.

[6] F. T. Boesch and S. Chen, A generalization of line connectivity and optimally invulnerable graphs, *SIAM J. Appl. Math.* **34** (1978), 657–665.

[7] S. M. Boyles and G. Exoo, A counterexample on a conjecture on paths of bounded lengths, *J. Graph Theory* **6** (1982), 205–209.

[8] G. Chartrand, A graph theoretic approach to a communications problem, *SIAM J. Appl. Math.* **14** (1996), 778–781.

[9] G. Chartrand and F. Harary, Graphs with presecribed connectivities, *Theorey of Graphs: Proceedings of the Colloquium Held at Tihany, Hungary.* Akadémiai Kiadói Budapest (1968).

[10] G. Chartrand, S. F. Kapoor, L. Lesniak and D. R. Lick, Generalized connectivity in graphs, *Bull. Bombay Math. Colloq.* **2** (1984), 1–6.

[11] F. R. K. Chung, Problem on Short Menger Path Systems, *Finite and Infinites Sets* (ed. A. Hajnal, L. Lovász and V. T. Sós), Colloquia Mathematica Societatis János Bolyai, **37** (1984), 873.

[12] P. Dankelmann and O. R. Oellermann, Bounds on the average connectivity of a graph. *Discr. Appl. Math.* **129** (2003), 305–318.

[13] P. Dankelmann and O. R. Oellermann, Degree sequences of optimally edge-connected multigraphs, *ARS Combin.* **77** (2005) 161–168.

[14] G. A. Dirac, In abstrakten Graphen vorhandene vollständige 4-Graphen und ihre Unterleitungen, *Math. Nachr.* **22** (1960), 61–85.

[15] J. Edmonds, Existence of $k$-edge connected ordinary graphs with prescribed degrees, *J. Res. Nat. Bur. Std.-B, Mathematics and Mathematical Physics* **68B** (1964).

[16] Y. Egawa, A. Kaneko and M. Matsumoto, Note: A mixed version of Menger's theorem, *Combinatorica* **11** (1991) 71–74.

[17] H. Enomoto and A. Kaneko, The condition of Beineke and Harary on edge-disjoint paths some of which are openly disjoint, *Tokyo J. Math.* **17** (1994), 355–357.

[18] R. Entringer, D. E. Jackson and P. J. Slater, Geodetic connectivity of graphs, *IEEE Trans. Circuits Systems* **24** (1977), 460–463.

[19] F. Fathnezhad, *Generalized Degree and Connectivity Conditions That Imply Edge Menger Path Systems*, Ph. D. Thesis, Memphis State University, 1992.

[20] R. J. Faudree, Some strong variations of connectivity, *Combinatorics, Paul Erdős is eighty, Bolyai Soc. Math. Stud.* **1** (1993), 125–144.

[21] R. J. Faudree, R. J. Gould and L. Lesniak, Generalized degrees and Menger path systems, *Discr. Appl. Math.* **38** (1992), 1–13.

[22] R. J. Faudree, R. J. Gould and R. H. Schelp, Menger path systems, *J. Combin. Math. Combin. Comput.* **6** (1989), 9–21.

[23] R. J. Faudree, M. S. Jacobson, E. T. Ordman, R. H. Schelp and Z. Tuza, Menger's theorem and short paths, *J. Combin. Math. Combin. Comput.* **2** (1987), 235–253.

[24] L. R. Ford Jr. and D. R. Fulkerson, Maximal flow through a network, *Canad. J. Math.* **8** (1956), 399–404.

[25] G. Fricke, O. R. Oellermann and H. C. Swart, The average edge-connectivity and degree conditions, preprint.

[26] T. Gallai, Maximum-Minimum Sätze and verallgemeinerte Faktoren von Graphen, *Acta Math. Acad. Sci. Hungar.* **12** (1961), 131–173.

[27] D. L. Goldsmith, On the second-order edge-connectivity of a graph, *Congr. Numer.* **29** (1980), 479–484.

[28] D. L. Goldsmith, On the $n$th order connectivity of a graph. *Congr. Numer.* **32** (1981), 375–382.

[29] D. L. Goldsmith, B. Manvel and V. Faber, Separation of graphs into three components by removal of edges, *J. Graph Theory* **4** (1980), 213–218.

[30] M. Hager, Pendant tree-connectivity, *J. Combin. Theory (B)* **38** (1985), 179–189.

[31] M. Hager, A Mengerian theorem for paths of length at least three, *J. Graph Theory* **10** (1986), 533–540.

[32] M. Hager, Path-connectivity in graphs, *Discrete Math.* **59** (1986), 53–59.

[33] G. Hajós, Zum Mengerschen Graphensatz, *Acta Lit. Sci. Szeged* **7** (1934), 44–47.

[34] J. Hartman and I. Rubin, On diameter stability of graphs, *Theory and Applications of Graphs* (ed. Y. Alavi and D.R. Lick), Springer Lecture Notes, 1976, 247–254.

[35] M. A. Henning and O. R. Oellermann, The average connectivity of regular multipartite tournaments, *Australas. J. Combin.* **23** (2001), 101–113.

[36] M. A. Henning and O. R. Oellermann, The average connectivity of a digraph. *Discr. Appl. Math.* **140** (2004), 143–153.

[37] H. R. Hind and O. R. Oellermann, Menger-type results for three or more vertices, *Congressus Numerantium* **113** (1996), 179–204.

[38] A. Kaneko and K. Ota, On minimally $(n, \lambda)$-connected graphs, *J. Combin. Theory (B)* **80** (2000), 156–171.

[39] J. Keijsper and A. Schrijver, On packing connectors, *J. Combin. Theory (B)* **73** (1998), 184–188.

[40] D. König, Graphok és matrixok, *Matematikai és Fizikai Lapok* **38** (1931), 116–119.

[41] D. König, Über trennende Knotenpunkte in Graphen, *Acta Litt. Sci. Szeged* **6** (1933), 155–179.

[42] D. König, *Theorie der endlichen und unendlichen Graphen*, Chelsea (1936).

[43] M. Kriesell, Edge-disjoint trees containing some given vertices in a graph, *J. Combin. Theory (B)* **88** (2003), 53–65.

[44] L. Lesniak, Results on the edge-connectivity of graphs. *Discrete Math.* **8** (1974), 351–354.

[45] L. Lovász, On some connectivity properties of eulerian graphs, *Acta Math. Acad. Sci. Hungar.* **28** (1976), 129–138.

[46] L. Lovász, V. Neumann-Lara and M. Plummer, Mengerian theorems for paths of bounded length, *Periodica Math. Hung.* **9** (1978), 269–276.

[47] W. Mader, Minimale $n$-fach kantenzusammenhängende Graphen, *Math. Ann.* **191** (1971), 21–28.

[48] W. Mader, Hinreichende Bedingungen für die Existenz von Teilgraphen, die zu einem vollständigen Graphen homöomorph sind, *Math. Nachr.* **53** (1972), 145–150.

[49] W. Mader, Existenz $n$-fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte, *Abh. Math. Sem. Univ. Hamburg* **37** (1972), 86–97.

[50] W. Mader, Grad und lokaler Zusammenhang in endlichen Graphen, *Math. Ann.* **205** (1973), 9–11.

[51] W. Mader, Ecken mit starken Zusammenhangseigenschaften in endlichen Graphen, *Math. Ann.* **216** (1975), 123–126.

[52] W. Mader, Über die Maximalzahl kantendisjunkter $A$-Wege, *Arch. Math.* **30** (1978), 325–336.

[53] W. Mader, Über die Maximalzahl kreuzungsfreier $H$-Wege, *Arch. Math.* **31** (1978), 387–402.

[54] W. Mader, Connectivity and edge-connectivity in finite graphs, *Surveys in Combinatorics. London Math. Soc. Lecture Notes* **38** (1979), 66–95.

[55] W. McCuaig, A simple proof of Menger's theorem, *J. Graph Theory* **8** (1984), 427–429.

[56] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.* **10** (1927), 96–115.

[57] K. Menger, On the origin of the $n$-arc theorem, *J. Graph Theory* **5** (1981), 341–350.

[58] K. Menger, *Kurventheorie*, Teubner, 1932.

[59] L. Montejano and V. Neumann-Lara, A variation os Menger's theorem for long paths, *J. Combin. Theory (B)* **36** (1984), 213–217.

[60] C. St. J. A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, *J. London Math. Soc.* **36** (1961), 445–450.

[61] C. St. J. A. Nash-Williams, Decomposition of finite graphs into forests, *J. London Math. Soc.* **39** (1964), 12.

[62] O. R. Oellermann, On the $l$-connectivity of a graph, *Graphs and Combin.* **3** (1987), 285–299.

[63] O. R. Oellermann, A note on the $l$-connectivity function of a graph, *Congr. Numer.* **60** (1987), 181–188.

[64] E. T. Ordman, Fault-tolerant networks and graph connectivity, *J. Combin. Math. Combin. Comput.* **1** (1987), 191–205.

[65] O. Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. Publ. XXXVIII, American Mathematical Society, 1962.

[66] J. Plesník, Critical graphs of given diameter, *Acta. Fac. Rerum Natur. Univ. Commenian Math.* **30** (1975), 71–93.

[67] K. P. Podewski and K. Steffens, Über Translationen und der Satz von Menger in unendlichen Graphen, *Acta. Math. Acad. Sci. Hungar.* **30** (1977), 69–84.

[68] L. Pyber and Z. Tuza, Menger-type theorems with restrictions on path lengths, *Discrete Math.* **120** (1993), 161–174.

[69] E. Sampathkumar, Connectivity of a graph - a generalization. *J. Combin. Info. Syst. Sciences* **9** (1984), 71–78.

[70] W. T. Tutte, On the problem of decomposing a graph into $n$ connected factors, *J. London Math. Soc.* **36** (1961), 221–230.

[71] L. Volkmann, Edge-connectivity in $p$-partite graphs, *J. Graph Theory* **13** (1989), 1–6.

[72] D. L. Wang and D. J. Kleitman, On the existence of $n$-connected graphs with prescribed degrees, *Networks* **3** (1973), 225–239.

[73] H. Whitney, Congruent graphs and the conectivity of graphs, *Amer. J. Math.* **54** (1932), 150–168.