离散数学习题课

第十一讲 ——群论（三）

Homomorphisms

- Fundamental theorem on homomorphisms:

  Given two groups G and H and a group homomorphism $f : G \to H$, let $\varphi : G \to G/K$ be the natural surjective homomorphism.

  For any $K \trianglelefteq G$, if $K \subseteq \ker f$, then there exists a unique homomorphism $h : G/K \to H$ such that $f = \varphi \circ h$.

Homomorphisms (cont.)

- First isomorphism theorem

If G, H are groups, $f: G \to H$ is a homomorphism, then

(1) $\ker f \trianglelefteq G$

(2) $G/\ker f \cong f(G)$

(3) $f(G) \leq H$

Homomorphisms (cont.)

- Second isomorphism theorem

  If G is a group, $H \leq G$, $K \trianglelefteq G$, then

  (1) $HK \leq G$

  (2) $K \trianglelefteq HK$

  (3) $H \cap K \trianglelefteq H$

  (4) $\mathrm{HK/K} \cong H/(H \cap K)$

Homomorphisms (cont.)

- Third isomorphism theorem

If $N, K \trianglelefteq G$, $N \subseteq K$, then

(1) $N \trianglelefteq K$

(2) $K/N \trianglelefteq G/N$

(3) $(G/N)/(K/N) \cong G/K$

Homomorphisms (cont.)

Comments:

- Let $\varphi : G_1 \to G_2$ be a homomorphism, then

$$(1)\ H \le G_1 \implies \varphi(H) \le \varphi(G_1) \le G_2$$

$$(2)\ H \trianglelefteq G_1 \implies \varphi(H) \trianglelefteq \varphi(G_1)$$

$$(3)\ |G_1| < \infty \implies |\varphi(G_1)| = [G_1 : \ker\varphi] \mid |G_1|$$

$$(4)\ |G_2| < \infty \implies |\varphi(G_1)| \mid |G_2|$$

Automorphisms

Definition:

Let G be a group, the set of all endomorphisms on G is

$$\operatorname{End} G = \{f \mid f : G \to G \wedge f \text{ is a homomorphism}\}$$

Let G be a group, the set of all automorphisms on G is

$$\operatorname{Aut} G = \{f \mid f : G \to G \wedge f \text{ is an isomorphism}\}$$

Let G be a group, the set of all inner automorphisms on G is

$$\operatorname{Inn} G = \{f_a \mid a \in G \wedge \forall x \in G(f_a(x) = axa^{-1})\}$$

Comments:

- $\langle \operatorname{End} G, \circ \rangle$ is a monoid; $\langle \operatorname{Aut} G, \circ \rangle$ is a group;

$$\operatorname{Inn} G \trianglelefteq \operatorname{Aut} G.$$

Symmetric groups

Definition:

The set $S_n = \{f \mid f : \mathbb{Z}_n \xrightarrow[onto]{1-1} \mathbb{Z}_n\}$ forms a group (called <u>symmetric group</u>) under function compositions.

Comments:

- For every finite group G with $|G| = n$, there is a group $H \leq S_n$, such that $G \cong H$.
- $\langle S_n, \circ \rangle$ contains <u>all kinds</u> of groups of order n.
- Note that $|S_n| = n!$

Problems

1. Show that, for any group G,

$$G/Z(G) \cong \operatorname{Inn} G$$

2. For any group G, let $\varphi : G \to G, \forall x \in G, \varphi(x) = x^{-1}$, show that

$$G \text{ is abelian} \iff \varphi \text{ is an automorphism}$$

3. Let G be a group, $H \leq G, N \trianglelefteq G$, show that

$$(|H|, [G:N]) = 1 \implies H \leq N$$

Thank you

Any questions?