第2次离散数学习题课

集合论(1)
集合代数&二元关系

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March 14, 2011

Outline

1. Review
   - Set Theory
     - Algebra of Sets(Optional)
     - Counting — Principle of Inclusion-Exclusion
   - Binary Relation
     - Binary Relation
     - Equivalence Relation
2. Problem Set
   - Set Theory
     - Algebra of Sets
     - Counting — Principle of Inclusion-Exclusion
   - Binary Relation
3. Applications and Extension(Optional)
   - Relational Database
   - Cantor Set
   - Russell's Paradox and Axiomatic Set Theory

集合论的先驱发展

萌芽(*Euclid*):

空间乃位置点之无限堆积。

对无穷集合的思考(*Galileo*):

$$\lambda_1 : 1, 2, 3, \cdots, n, \cdots$$

$$\lambda_2 : 1, 2, 3, \cdots, n, \cdots$$

Figure: "Dialogues Concerning Two New Science"

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$$\lambda_1 : 1, 2, 3, \cdots, n, \cdots$$

$$\lambda_2{}^2 : 1^2, 2^2, 3^2, \cdots, n^2, \cdots$$

$$\lambda^* : 1^{100}, 2^{100^{100}}, \cdots, n^{100^{100^{\cdots}}}, \cdots$$

Figure: "Dialogues Concerning Two New Science"

集合论的先驱发展

“从有限推进到无限，乃是Cantor的不朽功绩。”

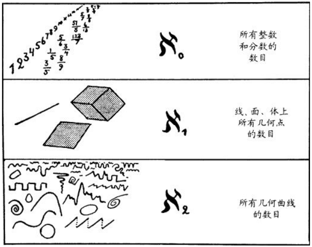

Figure: From “One,Two,Three,...,Infinity”

Figure: Georg Cantor(1845-1918) 缔造集合论

集合基本概念

"吾人直观或思维之对象，如为相异而确定之物，其总括之全体 即谓之集合，其组成此集合之物谓之集合之元素。" — *Cantor*

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"这是在用莫名定义莫名。" — *Hausdorff*

(P_{96} 第3(3)题:)

$$N - \{1, 2\}$$

集合的运算— $\cup$(*union*), $\cap$(*intersection*), $\sim$ (*complement*)

集合运算律: (compare P_{93} with P_{18})

Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

De Morgan Law

$$A - (B \cup C) = (A - B) \cap (A - C)$$

$$A - (B \cap C) = (A - B) \cup (A - C)$$

Complement

$$A - B = A \cap \sim B = A - (A \cap B)$$

$$A \cap (B - A) = \emptyset$$

$$A \cup (B - A) = A \cup B$$

集合的运算— $\oplus$(symmetric difference)

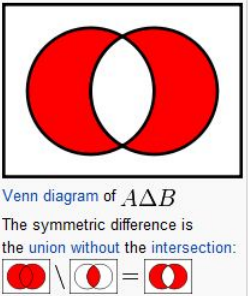
对称差等价定义:

1. "Union of both relative complements"

$$A \oplus B = (A - B) \cup (B - A)$$

2. "Union of two sets, minus their intersection"

$$A \oplus B = (A \cup B) - (A \cap B)$$



Venn diagram of $A \Delta B$

The symmetric difference is the union without the intersection:

集合的运算— $\oplus$(symmetric difference)

对称差运算性质:

Commutative $A \oplus B = B \oplus A$

Associative $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

Cancellative $A \oplus B = A \oplus C \Rightarrow B = C$

$$A \oplus \emptyset = A$$

$$A \oplus E = \sim A$$

$$A \oplus A = \emptyset$$

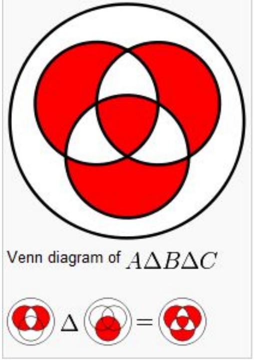


Figure: Venn diagram of $A \oplus B \oplus C$

集合的运算— 幂集(Power Set)

定义:

$$P(A) = \{x \mid x \subseteq A\}$$

Theorem

$$\mid A \mid = n \in N \rightarrow \mid P(A) \mid = 2^n$$

Proof

Relation to binomial theorem.

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n} = 2^n \quad \square$$

Q: What happens if A is an infinite set?

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Q: What happens if A is an infinite set?

A: 欲知后事如何，请听下回分解。

广义并&广义交

广义并:

$$\cup A = \{x \mid \exists z(z \in A \wedge x \in z)\}$$

广义交:

$$A \neq \emptyset, \cup A = \{x \mid \forall z(z \in A \rightarrow x \in z)\}$$

Q: What happens if $A = \emptyset$ in arbitrary intersection?

广义并&广义交

广义并:

$$\cup A = \{x \mid \exists z(z \in A \land x \in z)\}$$

广义交:

$$A \neq \emptyset, \cup A = \{x \mid \forall z(z \in A \to x \in z)\}$$

Q: What happens if $A = \emptyset$ in arbitrary intersection?

A: $\cap \emptyset = E$

容斥原理

$$|A \cup B| = |A| + |B| - |A \cap B|$$

容斥原理

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \ldots \\ +(-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

容斥原理

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \ldots$$
$$+(-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Complement Form:

U: universe object

A_i: specific property to avoid

$\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_n$: objects without any of the properties.

$$|\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_n| = |U| - \sum |A_i| + \sum |A_i \cap A_j|$$
$$+ \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|$$

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有序对(Ordered Pair)

有序对:

$$\langle x, y\rangle = \langle u, v\rangle \Leftrightarrow x = u \& y = v.$$

如何用集合定义有序对:

1. $\langle x, y\rangle = \{x, y\}$
2. $\langle x, y\rangle = \{x, \{y\}\}$
3. $\langle x, y\rangle = \{\{\{x\}, \emptyset\}, \{\{y\}\}\}$
4. $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$

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2. $\langle x, y\rangle = \{x, \{y\}\}$ ×
3. $\langle x, y\rangle = \{\{\{x\}, \emptyset\}, \{\{y\}\}\}$ √
4. $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ √

笛卡尔积(The Cartesian Product)

定义:

$$A \times B = \{\langle a, b\rangle \mid a \in A \wedge b \in B\}$$

性质(Distributive Law): (P_{130} 第4题)

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

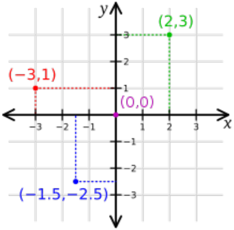


Figure: Cartesian coordinate system

二元关系及其运算

定义:

若$R \subseteq A \times B$,则称R为从A到B的二元关系.

关系的运算

1. **Inverse:** $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$
2. **Composition:**
   1. **Associative Law:** $(F \circ G) \circ H = F \circ (G \circ H)$
   2. **Distributive Law over Union:** $F \circ (G \cup H) = F \circ G \cup F \circ H$
   3. Distributive Law over Intersection: $F \circ (G \cap H) \subseteq F \circ G \cap F \circ H$ (课本P_{109}定理7.4,P_{132}第18题)

二元关系的性质

二元关系的重要性质:

reflexive: $(\forall x \in A)(xRx)$

irreflexive: $(\forall x \in A)(\neg xRx)$

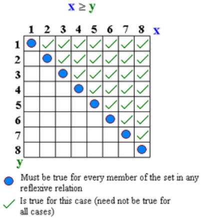
symmetric: $(\forall x, y \in A)(xRy \to yRx)$

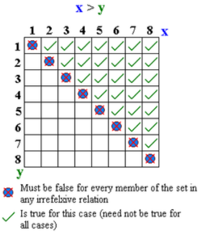
antisymmetric: $(\forall x, y \in A)(xRyRx \to x = y)$

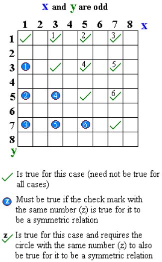
transitive: $(\forall x, y, z \in A)(xRyRz \to xRz)$

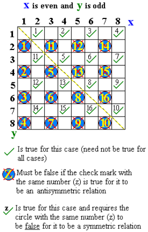
二元关系的性质

二元关系可以使用关系矩阵表示:









二元关系的性质

二元关系也是集合!(P_{118}表7.2 and P_{116}例题7.13)

reflexive: $I_A \subseteq R$

irreflexive: $R \cap I_A = \emptyset$

symmetric: $R = R^{-1}$

antisymmetric: $R \cap R^{-1} \subseteq I_A$

transitive: $R \circ R \subseteq R$

等价关系(Equivalence Relation)

$\sim$ is an equivalence relation:

1. $x \sim x$
2. $x \sim y \rightarrow y \sim x$
3. $x \sim y \sim z \rightarrow x \sim z$

Q: Why ?

等价关系(Equivalence Relation)

$\sim$ **is an equivalence relation:**

1. $x \sim x$
2. $x \sim y \to y \sim x$
3. $x \sim y \sim z \to x \sim z$

Q: Why ?

A: The simplest and the commonest.

等价关系(Equivalence Relation)

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3. $x \sim y \sim z \to x \sim z$

Q: Why ?

A: The simplest and the commonest.

Equivalence Class:

$$[x]_R = \{y \mid y \in A \wedge xRy\}$$

Quotient Set:

$$A/R = \{[x]_R \mid x \in A\}$$

Q: Why Quotient Set?

Equivalence Class and Partition

Equivalence class holds some interesting properties:

1. $[x] \neq \emptyset$
2. $xRy \rightarrow [x] = [y]$
3. $\neg(xRy) \rightarrow [x] \cap [y] = \emptyset$
4. $\cup\{[x] \mid x \in A\} = A$

Partition: $\pi \subseteq P(A)$

1. $\emptyset \notin \pi$
2. $\forall x \forall y (x, y \in \pi \wedge x \neq y \rightarrow x \cap y = \emptyset)$
3. $\cap \pi = A$

There is an one-to-one correspondence between equivalence relation and partition.

Outline

集合运算律

1: 请证明补集之唯一性(Optional)。

Theorem:

令A, B为E的任意子集，则$B = \sim A \Leftrightarrow A \cup B = E \& A \cap B = \emptyset$

tips:

$$
\begin{aligned}
& B = B \cap E = B \cap (A \cup \sim A) = (B \cap A) \cup (B \cap \sim A) \\
& = \emptyset \cup (B \cap \sim A) = (A \cap \sim A) \cap (B \cap \sim A) \\
& = \sim A \cap (A \cup B) = \sim A \cap E = \sim A.
\end{aligned}
$$

集合运算律

2: 请证明以下命题等价:

1. $A \subseteq B$
2. $A \cup B = B$
3. $A \cap B = A$
4. $A - B = \emptyset$

Q: 为什么可以采用循环证明?

集合运算律

2: 请证明以下命题等价:

1. $A \subseteq B$
2. $A \cup B = B$
3. $A \cap B = A$
4. $A - B = \emptyset$

Q: 为什么可以采用循环证明?

A: $\Leftrightarrow$ is an equivalence relation.

集合运算律

3: 请证明$\cap$关于$-$是可分配的:

Theorem:

$$A \cap (B - C) = (A \cap B) - (A \cap C)$$

集合运算律

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Theorem:

$$A \cap (B - C) = (A \cap B) - (A \cap C)$$

练习:

请问$\cup$关于$-$是可分配的吗?

集合运算律

4: 请证明$\cap$关于$\oplus$是可分配的:

Theorem:

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

tips:

- $B \oplus C = (B - C) \cup (C - B)$
- $A \cap (B - C) = (A \cap B) - (A \cap C)$

集合运算律

4: 请证明$\cap$关于$\oplus$是可分配的:

Theorem:

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

tips:

- $B \oplus C = (B - C) \cup (C - B)$
- $A \cap (B - C) = (A \cap B) - (A \cap C)$

练习:

请问$\cup$关于$\oplus$是可分配的吗?

集合运算律

5: 请解答如下与幂集相关的题目:

1. $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$ (课本P_{100}第36题& P_{101}第44题)

集合运算律

5: 请解答如下与幂集相关的题目:

1. $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$ (课本P_{100}第36题& P_{101}第44题)
2. $P(A) \cap P(B) = P(A \cap B)$ (课本P_{101}第45(1)题)

集合运算律

5: 请解答如下与幂集相关的题目:

1. $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$ (课本P_{100}第36题& P_{101}第44题)
2. $P(A) \cap P(B) = P(A \cap B)$ (课本P_{101}第45(1)题)
3. 字母集合$|A| = n$,自然数集合$|B| = m$,求$P(A) \cap P(B)$ (课本P_{98}第12(1)题)

集合运算律

5: 请解答如下与幂集相关的题目:

1. $A \subseteq B \Leftrightarrow P(A) \subseteq P(B)$ (课本P_{100}第36题& P_{101}第44题)
2. $P(A) \cap P(B) = P(A \cap B)$ (课本P_{101}第45(1)题)
3. 字母集合$|A| = n$,自然数集合$|B| = m$,求$P(A) \cap P(B)$ (课本P_{98}第12(1)题)
4. $P(\bigcap A_i) = \bigcap P(A_i)$ (2010年期中测试题)

容斥原理

6: 求Euler函数ϕ:

(P_{91}例6.6,2001年期中测试题)

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

$$\phi(n) = n \prod_{i=1}^{k} (1 - \frac{1}{p_k}).$$

Outline

有序对

7: 如何定义三元组(Optional)?

We have:

$$\langle x, y \rangle = \{\{x\}, \{x, y\}\}$$

then:

$$\langle x, y, z \rangle = \{\{x\}, \{x, y\}, \{x, y, z\}\}$$

有序对

7: 如何定义三元组(Optional)?

We have:

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}$$

then:

$$\langle x, y, z\rangle = \{\{x\}, \{x, y\}, \{x, y, z\}\}$$

Q: Is the definition OK?

Tips: Consider $\langle x, y, x\rangle$ *and* $\langle x, y, y\rangle$

有序对

7: 如何定义三元组(Optional)?

We have:

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}$$

then:

$$\langle x, y, z\rangle = \{\{x\}, \{x, y\}, \{x, y, z\}\}$$

Q: Is the definition OK?

Tips: Consider $\langle x, y, x\rangle$ *and* $\langle x, y, y\rangle$

A: $\langle x_1, x_2, x_3\rangle = \langle\langle x_1, x_2\rangle, x_3\rangle$.

Operation over Binary Relation

8: 请证明如下运算性质:

$R_1 \subseteq A \times B, R_2 \subseteq A \times B$

1. $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$ (P_{132}第20(1)题)
2. $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$ (P_{132}第20(2)题)
3. $(\sim R)^{-1} = \sim (R^{-1})$
4. $(R_1 - R_2)^{-1} = R_1^{-1} - R_2^{-1}$

Properties of Binary Relation

9: 请证明如下命题: (P_{118}表7.2)

$$R, S \text{ are symmetric, so are } R^{-1}, R \cap S, R \cup S, \text{ and } R - S.$$

tips:

- R is symmetric $\Leftrightarrow R = R^{-1}$.
- $(\sim R)^{-1} = \sim (R^{-1})$.
- $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$.

Equivalence Relation

10: 请证明如下定义的关系为等价关系，并给出商群。

1. $\langle a, b\rangle \sim \langle c, d\rangle \Leftrightarrow a + d = b + c$
   (where $a < b, a, b \in N$)(P_{133}第36题,作业补充题)
2. $\langle a, b\rangle \sim \langle c, d\rangle \Leftrightarrow ad = bc$
3. $A = P(X), C \subseteq X, \forall x, y \in A, xRy \Leftrightarrow x \oplus y \subseteq C$ (P_{133}第32(5)题)

Equivalence Relation

11: Counting partitions on a set with n elements(Optional)

try:

- $\left\{ {n \atop 0} \right\} = 0$
- $\left\{ {n \atop 1} \right\} = 1$
- $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$
- $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$
- $\left\{ {n \atop n} \right\} = 1$

Equivalence Relation

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- $\left\{ n \atop n-1 \right\} = \binom{n}{2}$
- $\left\{ n \atop n \right\} = 1$

Recurrence relation:

$$\left\{ n \atop r \right\} = r\left\{ n-1 \atop r \right\} + \left\{ n-1 \atop r-1 \right\}$$

Equivalence Relation

11: Counting partitions on a set with n elements(Optional)

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- $\left\{ n \atop 2 \right\} = 2^{n-1} - 1$
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- $\left\{ n \atop n \right\} = 1$

Recurrence relation:

$$\left\{ n \atop r \right\} = r\left\{ n-1 \atop r \right\} + \left\{ n-1 \atop r-1 \right\}$$

Bell number:

$$B_n = \sum_{r=0}^{n} \left\{ n \atop r \right\} (n \geq 1)$$

Outline

Relational Database(关系数据库)

LName	FName	City	Age	Salary
Smith	John	3	35	$280
Doe	Jane	1	28	$325
Brown	Scott	3	41	$265
Howard	Shemp	4	48	$359
Taylor	Tom	2	22	$250

Figure: Person table

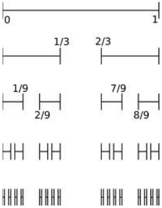
Concept and Operator:

- *relation* $\to$ *table*, *tuple* $\to$ *row*
- Union, Intersection, Difference, Cartesian product.
- Select(subset), Project, $\cdots$
  "select person.LName from person where person.city=3."

http://www.seas.upenn.edu/~zives/03f/cis550/codd.pdf

Outline

Cantor Set



What are removed?

$$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots=\frac{1}{3}\sum_{n=0}^{\infty}\frac{2^n}{3^n}=1.$$

And what remains?

Just as many "points" as there were before we began!

(from `http://personal.bgsu.edu/~carother/cantor/Cantor1.html`)

Outline

Russell's Paradox

When we meet infinity...

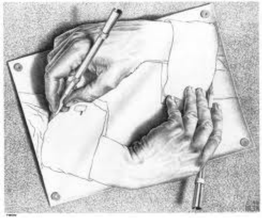


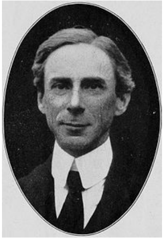
Figure: Drawing Hands by M.C.Escher

When we meet paradox...

$$R = \{x \mid x \notin x\}.$$

Q: Is R a set?

A: $R \in R \Leftrightarrow R \notin R$



Axiomatic Set Theory

If you are encouraged by the paradox,

or you are annoyed and begin to lose sleep night after night,

please refer to this article: .

`http://mplab.ucsd.edu/tutorials/settheory.pdf`

Axiomatic Set Theory

If you are encouraged by the paradox,

or you are annoyed and begin to lose sleep night after night,

please refer to this article or a doctor immediately.

http://mplab.ucsd.edu/tutorials/settheory.pdf

That's the end. Thank you.

Figure: Bring Up a Question