离散数学习题课

第五讲 —— 基数

Natural numbers

- Inductive sets:
  - A set S is called *inductive* if and only if

$$\emptyset \in S \land \forall x(x \in S \to x^+ \in S)$$

- Let $\mathbb{N} = \cap\{v \mid v \text{ is an inductive set}\}$
- Actually,

$$\mathbb{N} = \{\emptyset, \emptyset^+, \emptyset^{++}, \cdots\}$$

- It's easy to see that, for any $n \in \mathbb{N}$,

$$n \in n^+ \in \mathbb{N}$$

Transitive sets (review)

- A set S is called *transitive*, if and only if for any sets A, B,

$$A \in B \wedge B \in S \implies A \in S$$

- The following propositions are equivalent:

(1) S is a transitive set;

(2) $\cup S \subseteq S$;

(3) For any set A, $A \in S \implies A \subseteq S$;

(4) $S \subseteq \mathcal{P}(S)$.

More about transitive sets

- For any set A,

$$A \text{ is transitive} \iff \mathcal{P}(A) \text{ is transitive}$$

$$A \text{ is transitive} \implies \cup(A^+) = A$$

$$A \text{ is transitive} \implies A^+ \text{ is transitive}$$

- For any set $\mathscr{A}$, if all elements in $\mathscr{A}$ are transitive, then

(1) $\cup\mathscr{A}$ is transitive

(2) If $\mathscr{A} \neq \emptyset$, then $\cap\mathscr{A}$ is transitive

- Every natural number is transitive
- $\mathbb{N}$ is transitive

Results on natural numbers

For any $m, n \in \mathbb{N}$,

(1) $\forall x(x \in n \to x \subseteq n)$

(2) $m^+ \in n^+ \iff m \in n$

(3) $n \notin n$

(4) $n \neq 0 \implies \emptyset \in n$

(5) $m \in n \vee m = n \vee n \in m$

(6) $\forall x(x \subset n \to x \not\approx n)$

(7) $\cup(n^+) = n$

Extension of "counting"

- Intuition:
  - Two sets have the "same" number of elements if and only if there is a bijection between them
- Definition:

$$
\begin{aligned}
\operatorname{card} A = \operatorname{card} B &\iff A \approx B \\
&\iff \text{There is a bijection from } A \text{ to } B \\
\operatorname{card} A \leq \operatorname{card} B &\iff A \preccurlyeq\cdot\; B \\
&\iff \text{There is a injection from } A \text{ to } B \\
\operatorname{card} A < \operatorname{card} B &\iff A \prec\cdot\; B \iff A \preccurlyeq\cdot\; B \land A \not\approx B
\end{aligned}
$$

Some comments

- A "paradox":
  - An infinite set can be equipollent to its proper subset
- Other properties of "counting" are preserved:

For any $n \in \mathbb{N}$, and any set A,

$$|A| = n \iff \operatorname{card} A = n \iff A \approx n$$

For any sets A, B, C,

$$\operatorname{card} A = \operatorname{card} A$$

$$\operatorname{card} A \leq \operatorname{card} B \wedge \operatorname{card} B \leq \operatorname{card} A \implies \operatorname{card} A = \operatorname{card} B$$

$$\operatorname{card} A \leq \operatorname{card} B \wedge \operatorname{card} B \leq \operatorname{card} C \implies \operatorname{card} A \leq \operatorname{card} C$$

Some important results

- Some important results about cardinalities:
  - For any set A,
$$A \prec\cdot \mathcal{P}(A) \approx 2^A$$
  - $\mathbb{N} \approx \mathbb{Z} \approx \mathbb{Q} \prec\cdot \mathbb{R} \approx (0,1) \approx [0,1] \approx \mathcal{P}(\mathbb{N}) \approx 2^{\mathbb{N}}$
  - For any infinite set A,
$$A \approx A \times A$$
$$\mathbb{N} \preccurlyeq\cdot A$$

Finite sets

Definition:

A set A is called finite if and only if

$$\exists n(n \in \mathbb{N} \land A \approx n)$$

Comments:

- No finite set can be equipollent to its proper subset
- Subsets of a finite set are also finite
- The unions and Cartesian products of finitely many finite sets are also finite

Countable sets

Definition:

A set A is called *countable* (or *enumerable*) if and only if

$$A \preccurlyeq\cdot \mathbb{N}$$

Comments:

- A countable set is either a finite set or equipollent to $\mathbb{N}$
- A set is a countable infinite set if and only if it can be described as $\{a_1, a_2, \cdots, a_n, \cdots\}$
- Subsets of a countable set are also countable
- Unions and Cartesian products of finitely many countable sets are also countable

Operations of cardinal numbers

- Let $\kappa = \operatorname{card} K, \lambda = \operatorname{card} L$ be two cardinal numbers, define:

$$(1)\ \kappa + \lambda = \operatorname{card}(K \cup L) \quad (\text{requiring } K \cap L = \emptyset)$$

$$(2)\ \kappa \cdot \lambda = \operatorname{card}(K \times L)$$

$$(3)\ \kappa^{\lambda} = \operatorname{card}(K^{L})$$

- Such definitions are well defined (i.e. do not depend on the choice of K and L)

Properties of operations

For any cardinal number κ and natural number $n \in \mathbb{N}$,

(1) $\kappa + 0 = \kappa$ (2) $\kappa \cdot 0 = 0$ (3) $\kappa \cdot 1 = \kappa$

(4) $\kappa^0 = 1$ (5) $0^\kappa = 0 (\kappa \neq 0)$ (6) $\kappa + \kappa = 2 \cdot \kappa$

(7) $\kappa^1 = \kappa$ (8) $n + 1 = n^+$

For any cardinal numbers κ, λ, μ,

(1) $\kappa + \lambda = \lambda + \kappa$ (2) $\kappa \cdot \lambda = \lambda \cdot \kappa$

(3) $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$ (4) $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$

(5) $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$ (6) $\kappa^{\lambda + \mu} = \kappa^\lambda \cdot \kappa^\mu$

(7) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$ (8) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$

Problems

1. Let A, B be finite sets, show that
$$A \approx B \wedge A \subseteq B \implies A = B$$
2. Let A, B be finite sets with $A \approx B$, show that for all $f : A \to B$,
$$f \text{ is injective } \iff f \text{ is surjective } \iff f \text{ is bijective}$$
3. Show that
$$\mathrm{card}(\mathbb{R} - \mathbb{Q}) = \aleph$$
4. Show that, for any $n \in \mathbb{N}$,
$$n \prec\cdot \mathbb{N}$$

Problems (cont.)

5. Show that, for any set A,

$$A \text{ is infinite } \implies \mathcal{P}(A) \text{ is uncountable}$$

6. Show that, for any cardinal numbers $\kappa, \lambda,$

$$\kappa \neq 0 \wedge \lambda \geq \aleph_0 \implies \kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$$

Thank you

Any questions?