Do the Integers Exist?
The Unknowability of Arithmetic Consistency

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It is an article of faith for most mathematicians that Peano's axioms for arithmetic are consistent, perhaps because (in a kind of Platonist view) they simply state truths concerning a set of objects, the integers, that somehow exist, even though they constitute an infinite collection. A well-known (but extreme) statement of this point of view is Kurt Gödel's 1961 "The modern development of the foundations of mathematics in the light of philosophy"; see his *Collected Works*, volume III, Oxford University Press, 1981. Taken in an extreme form, the opposed "nominalist" position holds that, since we can have no direct physical experience with any such infinite collection, or even with its remoter members, the most that can be said is that extensive work with these axioms, including studies of their relationship to other axiom families considered likely to be consistent, demonstrates their consistency experimentally. If this latter position is accepted wholeheartedly, one ought to ask how thorough the search for an inconsistency implicit in this work really is, how it can be made more thorough, and how thorough a search might be possible in the best of cases. This paper will argue that truly comprehensive search for an inconsistency in any set of axioms is impossible. This will be done by demonstrating that an assumption distinctly weaker than the assumption of absolute consistency accounts for all the logical and practical experience with integers that we have had or can have.

Either of two kinds of work with integers, reasoning or computation, might lead to detection of an inconsistency in Peano's axioms (which we can imagine to be taken in their narrowest predicate form). We might find such an inconsistency either (1) by some chain of reasoning from the Peano axioms that proves a pair of statements which directly negate each other, or (2) by running a computer program of which some property (e.g., nonhalting) has been proven using Peano's axioms, and finding that it fails to have this property (e.g., halts).

In considering case (1), it is useful to measure the complexity of proofs in a slightly unusual way. A proof is simply a set of statements S, each of which is either an axiom, a definition, or a consequence of some finite collection of prior statements ($S_1, S_2, \ldots, S_n$) by some rule of inference. Call the statements ($S_1, S_2, \ldots, S_n$) needed to prove each step S its *basis*. As a proof, possibly a very long one, develops, one must retain a statement S appearing in it only as long as S is needed as the basis for some subsequent inference step. Once S has been used in the last such step, it can be thrown away. In practice this means that to prove even

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a relatively "advanced" theorem T, one need only retain the chain of definitions and prior theorems leading up to T, plus the statements belonging to the proof of whatever theorem T′ in the chain of reasoning is currently being demonstrated. Once a theorem has been proven, the statements used to prove it can be thrown away. The proof can therefore be conducted without ever having to remember more than some amount A of intermediate text at any one point. For each theorem T, take the proof that minimizes this A and call the minimum of all such A the *memory size* Mem_size(T) of T. In this sense, we can speak of theorems of memory size 5 pages, 10 pages, 20 pages, etc.

The author's work with a proof verifier indicates that Mem_size(T) is not likely to be very large for any theorem T in the published mathematical literature. For example, statement and proof of the Cauchy integral theorem should require retention (at any moment) of no more than 20 pages of theorems and definitions, and no proof longer than 600 lines, i.e., 10 pages, should be needed. Thus we estimate the memory size of this theorem to be no more than 30 pages, and probably a good deal less. This should be true of any theorem in the mathematical literature. So it is not unreasonable to assume that no theorem of memory size greater than, let us say, 100 pages has ever been published, or even worked out in detail by its discoverer.

We can therefore say that all our mathematical experience to date is consistent with the assumption that the Peano axioms are inconsistent, but that the size of the minimal inconsistency is more than, let us say, 100 pages. If this is the case, then mathematicians reasoning from such axioms will never encounter an inconsistency, since an infeasibly long and intricate proof would be needed to establish it. Shorter proofs, which is to say all the proofs that have thus far appeared or are likely to appear in the mathematical literature, could never do so. It is easily understood that in this assumed situation the community of mathematicians, noting that the language which they have been able to use confidently and consistently for centuries might grow to assume that the objects assumed in the language had some real (though necessarily abstract) existence, and that the axioms were consistent because at bottom they simply asserted truths about these objects.

How, if at all, can we tell that we do not find ourselves in this assumed situation?

Note that there must certainly exist theorems of arbitrarily large memory size. For it is a well-known consequence of the unsolvability of the decision problem that the size of a theorem's proof cannot be bounded by any recursive function of the size of its statement. On the other hand, if the memory size Mem_size(T) of a theorem is B (in bits), the number of configurations that can enter into any of its inferences is at most 2^{B}, and so (if for convenience we assume that the inference rule to be used at each step and the location of the statements referenced is held as part of this configuration) at most 2^{B} distinct proof steps are possible within memory size B. But in a nonredundant proof the same configuration will never recur twice, since it contains all the information carried forward in the course of the proof, i.e., between two occurrences of the same configuration no progress will have been made. Thus the maximum length of the minimal proof of a theorem

T of memory size Mem_size(T) is at most $2^{\text{Mem\_size(T)}}$, from which it follows that Mem_size(T) cannot be bounded by any recursive function of the size of the statement of T. This leaves open the possibility that the memory size of even a very small provable statement, for example "false," should be quite large, which is the possibility considered above.

It would seem therefore that the collective intellectual work of mathematicians cannot be regarded as constituting a comprehensive search for an inconsistency in Peano's axioms. Indeed, we can characterize whatever such search there has been as rather causal. In the first place, the assumption that something does not exist is poor psychological ground for mounting an extensive and challenging search for that thing. Secondly, we can be quite certain that only a microscopic part of the enormous space of proofs of memory size (let us say) 10 pages has till now been explored.

Note also that prior discussion of this point may often have involved an implicit assumption that the proof of a contradiction, if one exists, should be relatively short. The best-known previous example of this sort of thing is Russell's well-known observation that Frege's initial axiomatization of set theory was inconsistent. Concerning this the *Stanford Encyclopedia of Philosophy* website remarks:

> Russell wrote to Gottlob Frege with news of his paradox on June 16, 1902. The paradox was of significance to Frege's logical work since, in effect, it showed that the axioms Frege was using to formalize his logic were inconsistent.... Immediately appreciating the difficulty the paradox posed, Frege added to the Grundgesetze a hastily composed appendix discussing Russell's discovery. In the appendix Frege observes that the consequences of Russell's paradox are not immediately clear. For example, "Is it always permissible to speak of the extension of a concept, of a class? And if not, how do we recognize the exceptional cases? Can we always infer from the extension of one concept's coinciding with that of a second, that every object which falls under the first concept also falls under the second? These are the questions," Frege notes, "raised by Mr. Russell's communication." Because of these worries, Frege eventually felt forced to abandon many of his views about logic and mathematics.

In this particular case it would seem that the restrictions on set formation embodied in standard Zermelo-Fraenkel set theory remove the contradiction that troubled Frege. But Russell's proof of a contradiction is very short. Can we be certain that a system modification which removes this very direct, relatively obvious contradiction removes all others, no matter how intricate their proof might be?

Could the failure of computational rather than deductive search for an inconsistency yield stronger evidence that the Peano axioms are consistent? To find an inconsistency in this way, execution of a computer program that has been proven correct but then fails to behave as it provably must would reveal a contradiction. However, it is easy to see that a strengthened version of the supposition advanced above can also rule out comprehensive discovery of such cases. Execution of a computer program, if conducted in perfect physical agreement with a binary model of a computer of known instruction set, can be viewed as mechanical evaluation of some recursive function f (the computer code) for some integer value n (the input), followed by evaluation of some recursive predicate P for which the value P(f(n)) is provably "true," i.e., (FORALL n | P(f(n))) can be proven. Discovery of any actual integer n_0 for which $P(f(n_0))$ is then actually "false" then constitutes an inconsistency found by computation. A simple example might be, for example, discovery of an integer (perhaps one having a binary representation more than 30 pages long) for which n^2 mod 11 evaluated to 10. This is a bit different from the situation considered above, since such a code, together with its input, can easily be longer than the memory size of the largest feasible proof. But in fact few computer programs have actually been proven correct, and fewer still have then been tested at all comprehensively. Moreover, if we extend our preceding supposition to cover texts too large to be held in the physical memory of any computer, the skeptical argument given previously still applies.

Note that consideration of extreme cases that might behave differently from the way in which ordinary experience suggests is commonplace in physics, even though heretical in mathematics. No one would argue that the complete absence from daily experience of black holes means that none exist, or that the vanishingly small effect of relativistic laws on objects moving at ordinary speeds means that such effects are fictitious.

We now turn to examination of a very distinguished view directly opposed to that advanced above, namely, the "Realist" or "Platonist" view maintained by Gödel in the article already cited. Gödel begins by arranging foundational opinions in a roughly linear arrangement running from extreme versions of what we have called the Platonist viewpoint to various more or less radical kinds of nominalism:

> I believe that the most fruitful principle for gaining an overall view of the possible world-views will be to divide them up according to the degree and the manner of their affinity to or, respectively, turning away from metaphysics (or religion). In this way we immediately obtain a division into two groups: skepticism, materialism and positivism stand on one side, spiritualism, idealism and theology on the other.... [T]he development of philosophy since the Renaissance has by and large gone from right to left.... Particularly in physics, this development has reached a peak in our own time, in that, to a large extent, the possibility of knowledge of the objectivisable states of affairs is denied, and it is asserted that we must be content to predict results of observations. This is really the end of all theoretical science in the usual sense (although this

> predicting can be completely sufficient for practical purposes such as making television sets or atom bombs).... It would truly be a miracle if this (I would like to say rabid) development had not also begun to make itself felt in the conception of mathematics.

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In these remarks Gödel clearly positions himself somewhere toward the Platonist end of his scale, in opposition to the presumably misguided "Zeitgeist," noting that "mathematics, by its nature as an a priori science, always has, in and of itself, an inclination toward the right, and, for this reason, has long withstood the spirit of the time [Zeitgeist] that has ruled since the Renaissance." He views the skeptical attitude toward the actual existence of even transfinite sets as an instance of this same lamentable empiricism:

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> Finally... its hour struck: in particular, it was the antinomies of set theory, contradictions that allegedly appeared within mathematics, whose significance was exaggerated by skeptics and empiricists and which were employed as a pretext for the leftward upheaval. I say "allegedly" and "exaggerated" because, in the first place, these contradictions did not appear within mathematics but near its outermost boundary toward philosophy, and secondly, they have been resolved in a manner that is completely satisfactory.... Such arguments are, however, of no use against the spirit of the time, and so ... many or most mathematicians denied that mathematics, as it had developed previously, represents a system of truths; rather, they acknowledged this only for a part of mathematics (larger or smaller, according to their temperament) and retained the rest at best in a hypothetical sense, namely, one in which the theory properly asserts only that from certain assumptions (not themselves to be justified), we can justifiably draw certain conclusions. They thereby flattered themselves that everything essential had really been retained....
>
> Now one may view the whole development of empirical science as a systematic and conscious extension of what the child does when it develops in the first direction. The success of this procedure is indeed astonishing and far greater than one would expect a priori: after all, it leads to the entire technological development of recent times.... [I]t turns out that in the systematic establishment of the axioms of mathematics, new axioms, which do not follow by formal logic from those previously established, again and again become evident. It is not at all excluded by the negative results mentioned earlier that nevertheless every clearly posed mathematical yes-or-no question is solvable in this way.

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Though occasionally obscure, these remarks seem to rely on the idea that human reason is and of itself can serve as an engine capable of establishing at least

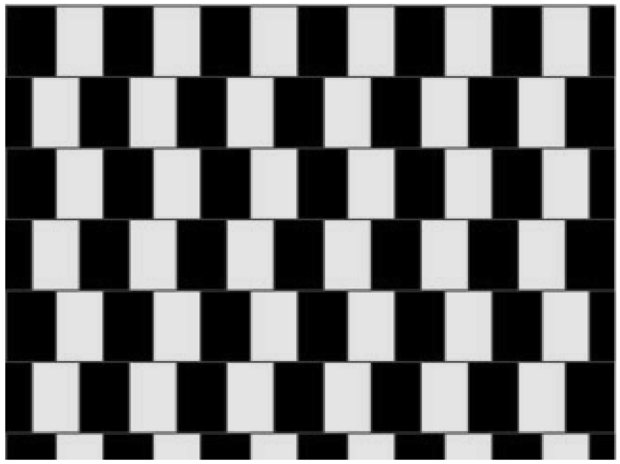

FIGURE 1. The “café wall” illusion.

mathematical truths definitively, and that these truths can be recognized, at least by judges having Gödel’s level of acuity, by the feeling that they are “completely satisfactory.” But this traditional view separates reason from the individual and collective biological limitations of the real brains that actually support it. Many observations, of which Figure 1 is an amusing but trenchant example, show the insupportability of this position. On viewing this figure, the visual brain, arguably a part of the human mind that is particularly well adapted to reality, sees it wrongly. A definite and stable perception of slant is experienced. But if a straight edge is held against the apparently slanted lines in the figure, it is readily seen that they are perfectly horizontal. This misperception seems to be common to all, or at any rate the great majority of the human population, and would seem to be a genetically specified fault in the human visual system. If vision can misbehave in this way, it is hard to believe that refined individual judgments, and even shared judgments concerning infinite cardinalities, or for that matter very large integers, are to be trusted absolutely.

A more cautious view than that of Gödel, but one tending in the same Realist or Platonist direction, which holds that the statements of mathematics are truths about some partly but progressively comprehended class of ideal objects, can be stated as follows: Of all formally plausible and apparently consistent sets of axioms (for example, statements asserting the existence of various kinds of very large cardinal numbers) not provable from assumptions presently accepted, a growing and coherent collection, this view predicts, will prove to be particularly rich in consequences, including consequences for questions that can be stated in currently accepted terms but not settled. These new axioms may be taken to hint at an underlying truth. The negatives of these progressively discovered axioms will prove to be unfruitful and so will gradually die out as dead ends.

The considerations advanced above undercut this view also. If a logical system is inconsistent, any proposition can be proven in it. We may therefore suspect that as such a system, already inconsistent but only having inconsistencies of some relatively large memory size Mem_size(T), is made more inconsistent; e.g., by adding new axioms or methods of deduction that reduce Mem_size(T) for the minimal inconsistency T, it will grow more fruitful, that is, will allow proofs of more abundant and interesting results than formerly. This shows that in the situation we have assumed, richness of consequence may be less a sign of truth than of impending collapse.

What Would It Mean If Peano's Axioms Were Found to Be Inconsistent?

If inconsistencies were found to result from Peano's axioms, the traditional view of mathematical knowledge as something absolutely "analytic," i.e., entirely independent of physical experience, would be undercut. ("Analytic knowledge is, classically, tautological 'knowledge' that is true by analysis of language: e.g., 'a black cat is black.' In this sense, analytic knowledge is prior to empirical observation.") Mathematics would then descend from the pedestal of absolute certainty on which it now stands, and join the other sciences in their empirical realm. Though this would add a new level of recognized uncertainty to our ongoing attempt to understand the universe, the organizing role of mathematics in science might change very little provided that the proof of the shortest such inconsistency was long and complex (in the quantitative sense described earlier in this paper). It is not clear that an *absolutely* rigorous derivation of a scientific fact from equations known to give only an approximate representation of physical reality is much more valuable scientifically than a *highly* rigorous derivation of the same fact from the same equations. Mathematics serves science by establishing the equivalence of alternate formulations of the same equations, principles, and theories, and by predicting properties of computations. It would retain both roles, as long as only arguments and computations below some limiting, perhaps very comfortable, size were admitted.

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