离散数学习题课

第一讲 —— 集合论

Outline

- Some methods for solving in set theory problems
  - Examples
- Important concepts
  - The world of boxes
  - Connectives
  - Tricky logic
- References
- Resources

Why set theory & logic?

- A formal language
- A way of expressing and checking mathematical results
- So:
  - Proofs in set theory should be strict and formal
- Two formal ways of proving set theory results:
  - Set identities (集合恒等式，cf. 教材p.99-100)
  - Logic (cf. "常见逻辑定理.pdf")

Example 1 (Set identities)

已知： $A \cap B = A \cap C$ 且 $\sim A \cap B = \sim A \cap C$

证明： $B = C$

Proof:
$$
\begin{aligned}
B &= E \cap B \\
&= (A \cup \sim A) \cap B \\
&= (A \cap B) \cup (\sim A \cap B) \\
&= (A \cap C) \cup (\sim A \cap C) \\
&= (A \cup \sim A) \cap C \\
&= E \cap C \\
&= C
\end{aligned}
$$

Q.E.D.

Example 2 (Logic)

证明：对任意集合A, B，有

$$A = B \iff P(A) = P(B)$$

Proof: "$\Longrightarrow$":

If $A = B$, then for any S,

$$\begin{aligned} S \in P(A) &\iff S \subseteq A \\ &\iff S \subseteq B \\ &\iff S \in P(B) \end{aligned}$$

Therefore, $A = B \implies P(A) = P(B)$

Example 2 (cont.)

"$\Longleftarrow$":

If $P(A) = P(B)$, then for any x,

$$\begin{aligned} x \in A &\iff \{x\} \subseteq A \\ &\iff \{x\} \in P(A) \\ &\iff \{x\} \in P(B) \\ &\iff \{x\} \subseteq B \\ &\iff x \in B \end{aligned}$$

Thus we have,

$$P(A) = P(B) \implies A = B$$

Q.E.D.

Summary of definitions

$$
\begin{aligned}
A \subseteq B &\iff \forall x(x \in A \rightarrow x \in B) \\
A = B &\iff A \subseteq B \land B \subseteq A \\
&\iff \forall x(x \in A \leftrightarrow x \in B) \\
A = \emptyset &\iff \neg\exists x(x \in A) \\
&\iff \forall x(x \notin A) \\
x \in A \cap B &\iff x \in A \land x \in B \\
x \in A \cup B &\iff x \in A \lor x \in B \\
x \in A - B &\iff x \in A \land x \notin B \\
x \in \sim A &\iff x \notin A \\
x \in P(A) &\iff x \subseteq A
\end{aligned}
$$

$$\boxed{x \notin A \underset{\triangle}{=} \neg x \in A}$$

The world of "boxes"

- Everything is a set
  - One way to encode "natural numbers":

$$0 = \emptyset$$
$$1 = \{0\} = \{\emptyset\}$$
$$2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$$
$$\cdots\cdots$$
$$n + 1 = \{0, 1, 2, \cdots, n\}$$

- Therefore, $\cup\{a, b\} = a \cup b$

$$\cup\{1, 2, 3\} = 1 \cup 2 \cup 3$$

Connectives

- "If ... then ...":
  - "$\rightarrow$"与"$\supset$"意义完全相同，是命题联结词"蕴含"的两种不同写法，用于将两个命题连接成一个复合命题
  - "$\Longrightarrow$"是自然语言中"如果......则......"的简记，用于表达作者的断言和结论。
  - 二者的关系是：说"$P \Longrightarrow Q$"等同于断言：

    "$P \rightarrow Q$ 为永真命题"

Connectives (cont.)

- "If and only if":
  - "$\leftrightarrow$"与"$\equiv$"意义完全相同，是命题联结词"等价"的两种不同写法，用于将两个命题连接成一个复合命题
  - "$\iff$"是自然语言中"当且仅当"(有时写成 iff)的简记，用于表达作者的断言和结论。
  - 二者的关系是：说"$P \iff Q$"等同于断言：

"$P \leftrightarrow Q$ 为永真命题"

Tricky logic

- A correct proof of

$$A \subseteq B \implies A \cap C \subseteq B \cap C$$

Proof: For any x,

$$\begin{aligned} x \in A \cap C &\iff x \in A \wedge x \in C \\ &\implies x \in B \wedge x \in C \\ &\iff x \in B \cap C \end{aligned}$$

Q.E.D.

Tricky logic (cont.)

- A false "proof" of the (incorrect) conclusion:

$$A \subseteq B \implies {\sim}A \cup C \subseteq {\sim}B \cup C$$

Proof: For any x,

$$\begin{aligned} x \in {\sim}A \cup C &\iff \neg x \in A \vee x \in C \\ &\iff x \in A \to x \in C \\ &\implies x \in B \to x \in C \\ &\iff \neg x \in B \vee x \in C \\ &\iff x \in {\sim}B \cup C \end{aligned}$$

Tricky logic (cont.)

- In fact, if $A \implies B$, then we have

$$A \wedge C \implies B \wedge C$$
$$A \vee C \implies B \vee C$$
$$C \rightarrow A \implies C \rightarrow B$$

- But

$$\neg A \wedge C \not\Rightarrow \neg B \wedge C$$
$$\neg A \vee C \not\Rightarrow \neg B \vee C$$
$$A \rightarrow C \not\Rightarrow B \rightarrow C$$

Example 3

- Let $\langle a, b\rangle = \{\{\{a\}, \emptyset\}, \{\{b\}\}\}$, show that

$$\langle a, b\rangle = \langle c, d\rangle \iff a = c \wedge b = d$$

Proof: "$\Longleftarrow$": Trivial.

"$\Longrightarrow$":

$$\begin{aligned} & \langle a, b\rangle = \langle c, d\rangle \\ \iff & \{\{\{a\}, \emptyset\}, \{\{b\}\}\} = \{\{\{c\}, \emptyset\}, \{\{d\}\}\} \end{aligned}$$

注意到，$\{\{a\}, \emptyset\}$ 和 $\{\{c\}, \emptyset\}$ 为2元集，$\{\{b\}\}$ 和 $\{\{d\}\}$ 为1元集。

Example 3 (cont.)

- 因此，

$$
\begin{aligned}
&\langle a, b\rangle = \langle c, d\rangle \\
\Longleftrightarrow\ & \{\{\{a\},\emptyset\},\{\{b\}\}\} = \{\{\{c\},\emptyset\},\{\{d\}\}\} \\
\Longrightarrow\ & \{\{a\},\emptyset\} = \{\{c\},\emptyset\} \wedge \{\{b\}\} = \{\{d\}\} \\
\Longrightarrow\ & \cup\cup\{\{a\},\emptyset\} = \cup\cup\{\{c\},\emptyset\} \wedge \cup\cup\{\{b\}\} = \cup\cup\{\{d\}\} \\
\Longrightarrow\ & \cup(\{a\}\cup\emptyset) = \cup(\{c\}\cup\emptyset) \wedge \cup\{b\} = \cup\{d\} \\
\Longleftrightarrow\ & \cup\{a\} = \cup\{c\} \wedge b = d \\
\Longleftrightarrow\ & a = c \wedge b = d
\end{aligned}
$$

Q.E.D.

Example 4

- Show that $\langle x, y\rangle = \{x, \{y\}\}$ is not a proper definition for *ordered pairs*.

  Proof:

  Let $a = b = \{\emptyset\}, c = \{\{\emptyset\}\}, d = \emptyset$, then

  $$\langle a, b\rangle = \{\{\emptyset\}, \{\{\emptyset\}\}\} = \{\{\{\emptyset\}\}, \{\emptyset\}\} = \langle c, d\rangle$$

  But $a \neq c$.

  Q.E.D

References

- Discrete mathematics and its applications, Fifth Edition. Kenneth H. Rosen. 机械工业出版社, 2003，影印版. (翻译版：《离散数学及其应用》. 袁崇义、屈婉玲、王捍贫、刘田译，机械工业出版社，2007年)
- 《离散数学教程》，耿素云、屈婉玲、王捍贫编著，北京大学出版社，2002
- 《数学基础》，汪芳庭编著，科学出版社，2001
- 《公理集合论导引》，张锦文著，科学出版社，1991

Resources

Lecture slides & other resources:

http://lamda.nju.edu.cn/xiaoxp/math08

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Thank you

Any questions?