离散数学习题课

第四讲 —— 函数

Definitions and notations

- Definition:

  Let $f \subseteq A \times B$ be a relation from A to B, f is called *a function from A to B* if and only if:

  $$\operatorname{dom} f = A \ \wedge \ \forall x, y, z(\langle x, y\rangle \in f \wedge \langle x, z\rangle \in f \rightarrow y = z)$$

- Notation:
  - "f is a function from A to B" is denoted as $f : A \rightarrow B$
  - The set of all functions from A to B can be denoted as

    $$A_B \text{ or } B^A \text{ or } A \rightarrow B$$

    i.e. $$A_B = B^A = A \rightarrow B = \{f \mid f : A \rightarrow B\}$$

Properties

- For any finite sets A, B,

$$\left|B^A\right| = |B|^{|A|}$$

- Let A be any nonempty set, then

$$\emptyset^\emptyset = A^\emptyset = \{\emptyset\}, \quad \emptyset^A = \emptyset$$

  That is to say, for any set X, we have

$$\emptyset : \emptyset \to X$$

  But if $X \neq \emptyset$, there will be no function f satisfying

$$f : X \to \emptyset$$

Injective and surjective

Definition:

For any $f : A \to B$,

- f is called a <u>1-1 function</u> or an <u>injection</u> if

$$\forall x \forall y (f(a) = f(b) \to a = b)$$

- f is called an <u>onto function</u> or a <u>surjection</u> if

$$\text{ran}\, f = B$$

i.e. $\forall y (y \in B \to \exists x (x \in A \land f(x) = y))$

- f is called an <u>one-one correspondence</u> or a <u>bijection</u> if f is both injective and surjective

Injective and surjective (cont.)

Comments:

- Let A, B be <u>finite</u> sets, with $|A| = |B|$, then for all $f : A \to B$, we have

$$f \text{ is injective} \iff f \text{ is surjective} \iff f \text{ is bijective}$$

- Let $f : A \to B, g : B \to C$ be two functions,
  - If $f \circ g$ is injective, then f is injective
  - If $f \circ g$ is surjective, then g is surjective

Operations

- If $f : A \to B$ is an injection, then f^{-1} is a bijection from $\operatorname{ran} f$ to A
- If $f : A \to B$ is a bijection, then f^{-1} is a bijection from B to A
- If $f : A \to A$ is a bijection, then for any positive integer n, we can define

$$f^{-n} = (f^{-1})^n$$

in this case, for any $m, n \in \mathbb{Z}$, we have

$$f^m \circ f^n = f^{m+n}, \quad (f^m)^n = f^{mn}$$

Inverses

Definition: (supposing $A \neq \emptyset$)

For any functions $f : A \to B, g : B \to A,$

if $f \circ g = I_A,$ then f is called a left inverse of $g,$ and g is called a right inverse of f.

Comments:

- Inverses are not always unique
- If f has both a left inverse and a right inverse, then the left inverse and the right inverse must be the same function, and this function is called the (unique) inverse of f.

Inverses (cont.)

Comments:

- $f : A \to B$ has left inverses if and only if f is surjective
- $f : A \to B$ has right inverses if and only if f is injective
- $f : A \to B$ has an inverse if and only if f is bijective
- If $f : A \to B$ is bijective, then f^{-1} is its unique inverse

Problems

1. Show that

$$A^B = B^A \iff A = B$$

2. Show that, for any $f : A \to A$, if $f^n = I_A$ for some $n \in \mathbb{N}^+$ then f is bijective.
3. Let $f, g : A \to B$ be two functions, show that

$$f \subseteq g \implies f = g$$

Thank you

Any questions?