离散数学习题课

第三讲 —— 二元关系

Outline

- More about set theory
  - Characteristic functions
  - Solutions to selected exercises
- Binary relations
  - Definitions
  - Operations
  - Properties

Characteristic functions

For any set A, the characteristic function of A is defined as:

$$F_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

And it's easy to show that:

$$A \subseteq B \iff \forall x(F_A(x) \leq F_B(x))$$

$$A = B \iff \forall x(F_A(x) = F_B(x))$$

$$A = \emptyset \iff \forall x(F_A(x) = 0)$$

$$A = E \iff \forall x(F_A(x) = 1)$$

Characteristic functions (cont.)

More importantly, we have:

$$F_{A\cap B}(x) = F_A(x)F_B(x)$$

$$F_{A\cup B}(x) = F_A(x) + F_B(x) - F_A(x)F_B(x)$$

$$F_{A\oplus B}(x) = F_A(x) + F_B(x) - 2F_A(x)F_B(x)$$

$$F_{A-B}(x) = F_A(x) - F_A(x)F_B(x)$$

$$F_{\sim A}(x) = 1 - F_A(x)$$

- This provides an algebraic way of solving set theory problems

Characteristic functions (cont.)

Example: Find a necessary and sufficient condition for

$$A - B = A$$

Solution:

$$\begin{aligned} A - B = A &\iff \forall x(F_A(x) - F_A(x)F_B(x) = F_A(x)) \\ &\iff \forall x(F_A(x)F_B(x) = 0) \\ &\iff \forall x(F_{A \cap B}(x) = 0) \\ &\iff A \cap B = \emptyset \end{aligned}$$

Solutions

- P104. Ex.8(3)

$$\mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

- P108. Ex.42

$$
\begin{array}{lllll}
(1) & (A-B)\cup(A-C)=A & \iff & A\cap B\cap C=\emptyset \\
(2) & (A-B)\cup(A-C)=\emptyset & \iff & A\subseteq B\cap C \\
(3) & (A-B)\cap(A-C)=\emptyset & \iff & A\subseteq B\cup C \\
(4) & (A-B)\cap(A-C)=A & \iff & A\cap(B\cup C)=\emptyset
\end{array}
$$

Solutions (cont.)

- P109. Ex.44(1)

$$\mathcal{A} = \{A_i \mid A_i \text{ 为实数区间 } (-\frac{1}{i}, \frac{1}{i}) \wedge i \in \mathbb{Z}^+\}$$

$$\begin{aligned}
\cup\mathcal{A} &= (-1, 1) \\
&= \{x \mid x \in (-1, 1)\} \\
&= \{x \mid x \in \mathbb{R} \wedge -1 < x < 1\} \\
&\neq \{(-1, 1)\}
\end{aligned}$$

$$\cap\mathcal{A} = \{0\} \neq \emptyset$$

Solutions (cont.)

- P109. Ex.45

$$
\begin{aligned}
\mathcal{A} &= \{\{\emptyset\}, \{\{\emptyset\}\}\} \\
\mathcal{P}(\mathcal{A}) &= \{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\} \\
\mathcal{P}(\cup\mathcal{A}) &= \mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \\
\cup\mathcal{P}(\mathcal{A}) &= \cup\{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\} \\
&= \emptyset \cup \{\{\emptyset\}\} \cup \{\{\{\emptyset\}\}\} \cup \{\{\emptyset\}, \{\{\emptyset\}\}\} \\
&= \{\{\emptyset\}, \{\{\emptyset\}\}\} \\
&= \mathcal{A}
\end{aligned}
$$

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Ordered pairs

- Definition:

  For any sets x, y, let
  $$\langle x, y\rangle = \{\{x\}, \{x, y\}\}$$
  then, for any sets a, b, c, d, we have
  $$\langle a, b\rangle = \langle c, d\rangle \iff a = c \wedge b = d$$

- Comments:
  - "Everything is a set"
  - $\langle x, y\rangle$ can be considered as a "function" or "operator" that maps two sets x, y to a new set

Cartesian Products

- Definition:

  For any sets A, B, let

$$A \times B = \{\langle a, b\rangle \mid a \in A \wedge b \in B\}$$
$$= \{\{\{a\}, \{a, b\}\} \mid a \in A \wedge b \in B\}$$

- Comments:
  - For any set A, we have $A \times \emptyset = \emptyset \times A = \emptyset$
  - For any finite sets A, B, we have $|A \times B| = |A| \cdot |B|$

Binary relations

- Definition:
  - 设R为一集合，若$R \subseteq A \times B$，则称R为“从A到B的二元关系”(a binary relation from A to B)
  - 设R为一集合，若$R \subseteq A \times A$，则称R为“A上的二元关系”(a binary relation on A)
- Comments:
  - 若R是从A到B的二元关系，则R是$A \cup B$上的二元关系
  - 对任何集合A, B，空关系（即空集）都是从A到B的二元关系

Operations on relations

- Notes:
  - For any binary relation $R \subseteq A \times B$, generally

$$I_{\operatorname{dom} R} \subseteq R \circ R^{-1} \neq I_A$$

$$I_{\operatorname{ran} R} \subseteq R^{-1} \circ R \neq I_B$$

  - For any binary relation $R \subseteq A \times A$, generally

$$R^n \circ R^{-1} \neq R^{n-1}$$

$$R^n = R^m \not\Rightarrow R^{n-1} = R^{m-1}$$

(see P.119 Example 7.8)

Properties of relations

- 自反性：

For any binary relation $R \subseteq A \times A$,

$$
\begin{aligned}
& R \text{ is reflexive} \\
\Longleftrightarrow\ & \forall x (x \in A \rightarrow \langle x, x \rangle \in R) \\
\Longleftrightarrow\ & I_A \subseteq R \\
\Longleftrightarrow\ & M_R \text{ 的主对角线全为 } 1 \\
\Longleftrightarrow\ & G_R \text{ 中每个顶点都有环}
\end{aligned}
$$

Properties of relations (cont.)

- 反自反性：

  For any binary relation $R \subseteq A \times A$,

$$
\begin{aligned}
& R \text{ is irreflexive} \\
\iff & \forall x(x \in A \to \langle x, x\rangle \notin R) \\
\iff & I_A \cap R = \emptyset \\
\iff & M_R \text{ 的主对角线全为 } 0 \\
\iff & G_R \text{ 中每个顶点都无环}
\end{aligned}
$$

Properties of relations (cont.)

- 对称性：

For any binary relation $R \subseteq A \times A$,

$$
\begin{aligned}
& R \text{ is symmetric} \\
\Longleftrightarrow\ & \forall x \forall y (\langle x, y \rangle \in R \to \langle y, x \rangle \in R) \\
\Longleftrightarrow\ & R = R^{-1} \\
\Longleftrightarrow\ & M_R \text{ 为对称矩阵} \\
\Longleftrightarrow\ & G_R \text{ 中的边都成对出现}
\end{aligned}
$$

Properties of relations (cont.)

- 反对称性：

For any binary relation $R \subseteq A \times A$,

$$
\begin{aligned}
& R \text{ is antisymmetric} \\
\Longleftrightarrow\ & \forall x \forall y (\langle x, y \rangle \wedge \langle y, x \rangle \in R \rightarrow x = y) \\
\Longleftrightarrow\ & R \cap R^{-1} \subseteq I_A \\
\Longleftrightarrow\ & \text{若 } i \neq j\text{，则 } r_{ij} \text{ 与 } r_{ji} \text{ 中至多有一个为 } 1 \\
\Longleftrightarrow\ & G_R \text{ 中没有成对出现的边}
\end{aligned}
$$

Properties of relations (cont.)

- 传递性：

For any binary relation $R \subseteq A \times A$,

$$
\begin{aligned}
& R \text{ is transitive} \\
\iff & \forall x \forall y \forall z (\langle x, y \rangle \in R \land \langle y, z \rangle \in R \to \langle x, z \rangle \in R) \\
\iff & R \circ R \subseteq R \\
\iff & M_R \cdot M_R \le M_R \\
\iff & G_R \text{ 中若存在从顶点 } u \text{ 到顶点 } v \text{ 的通路，} \\
& \text{则存在从顶点 } u \text{ 到顶点 } v \text{ 的边}
\end{aligned}
$$

Problems

1. For any binary relation $R \subseteq A \times A$, show that

$$R \text{ is reflexive and transitive } \implies R^2 = R$$

2. Let $R, S \subseteq A \times A$ be both symmetric relations, show that

$$R \circ S \text{ is symmetric } \iff R \circ S = S \circ R$$

3. Let A be any finite set with $|A| = n$, how many binary relations can be defined on A, which are:

   a) Reflexive $= 2^{n(n-1)}$ b) Irreflexive $= 2^{n(n-1)}$

   c) Symmetric $= 2^{n(n+1)/2}$

   d) Anti-symmetric $= 2^n 3^{n(n-1)/2}$

   e) Reflexive and symmetric $= 2^{n(n-1)/2}$

Problems (cont.)

4. For any binary relation $R \subseteq A \times B$, show that

$$\cup \cup R = \operatorname{fld} R$$

5. Let $R \subseteq A \times A$ be a reflexive relation, show that

$$R \text{ is an equivalence}$$
$$\iff \forall x \forall y \forall z (\langle x, y \rangle \in R \wedge \langle x, z \rangle \in R \rightarrow \langle y, z \rangle \in R)$$

Thank you

Any questions?