Discrete Mathematics Cheat Sheet

Set Theory

$B \subseteq A \Leftrightarrow \forall x(x \in B \rightarrow x \in A)$, $\varnothing = \{x \mid x \neq x\}$, $\mathcal{P}(A) = \{x \mid x \subseteq A\}$
$A \cap B = \{x \mid x \in A \wedge x \in B\}$, $A \cup B = \{x \mid x \in A \vee x \in B\}$
$A - B = \{x \mid x \in A \wedge x \notin B\}$, $A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
$\cup\mathcal{A} = \{x \mid \exists z(z \in \mathcal{A} \wedge x \in z)\}, \cap\mathcal{A} = \{x \mid \forall z(z \in \mathcal{A} \rightarrow x \in z\}$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), A \cup (A \cap B) = A \cap (A \cup B) = A$
[De-Morgan]$A - (B \cup C) = (A - B) \cap (A - C)$, $A - (B \cap C) = (A - B) \cup (A - C)$
$\sim (B \cup C) = \sim B \cap \sim C$, $\sim (B \cap C) = \sim B \cup \sim C$
$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A \Leftrightarrow A \oplus B = \varnothing \Leftrightarrow A - B = B - A = \varnothing$
Character function: $f_{A\cap B} = f_A f_B$, $f_{A\cup B} = f_A + f_B - f_A f_B$, $f_{A-B} = f_A - f_A f_B \ldots$

Binary Relations

$\langle x, y \rangle = \langle u, v \rangle \Leftrightarrow x = u \wedge y = v$, $A \times B = \{\langle x, y \rangle \mid x \in A \wedge y \in B\}$
$A \times (B \cup C) = (A \times B) \cup (A \times C)$
R' is the p closure of $R \Leftrightarrow p(R') \wedge R \subseteq R' \wedge (\forall p(R'') \rightarrow R' \subseteq R'')$
$r(R) = R \cup I_A, s(R) = R \cup R^{-1}, t(R) = R \cup R^2 \cup R^3 \cup \ldots$
R is equivalence $\Leftrightarrow$ R is reflexive, symmetric and transitive.
$[x]_R = \{y \mid y \in A \wedge xRy\}$, $\forall x, y \in A((xRy \rightarrow [x] = [y]) \wedge (\neg xRy \rightarrow [x] \cap [y] = \varnothing))$
$A/R = \{[x]_R \mid x \in A\}$ (a partition of A).

Functions

Function: $\forall x \in \mathrm{dom}F(\exists! y \in \mathrm{ran}F \wedge xFy)$, $f : A \rightarrow B \Leftrightarrow \mathrm{dom}f = A \wedge \mathrm{ran}f \subseteq B$
f: injection (1-1) $\Leftrightarrow \forall x \forall y(f(x) = f(y) \rightarrow x = y)$
f: surjection (onto) $\Leftrightarrow \forall y(y \in B \rightarrow \exists x(x \in A \wedge f(x) = y))$
f: bijection (one-one correspondence) $\Leftrightarrow$ f is injection and surjection.
$f^{-1} \circ f = I_B, f \circ f^{-1} = I_A$

Cardinalities

$A \approx B \Leftrightarrow$ exists surjection $f : A \rightarrow B$, $A \preceq B \Leftrightarrow$ exists injection $f : A \rightarrow B$
[Cantor-Bernstein-Schröder] $A \preceq B \wedge B \preceq A \Leftrightarrow A \approx B$
$f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} : f(\langle m, n \rangle) = (m + n + 1)(m + n)/2 + m$
$\mathrm{card}\mathbb{N} = \aleph_0$, $\mathrm{card}\mathbb{R} = \aleph$. A is countable $\Leftrightarrow$ $\mathrm{card}A \preceq \aleph_0$

Semi-group, Monoid and Group

$\langle S, \circ \rangle$ is a semi-group $\Leftrightarrow \forall x, y \in S(xy\circ \in S) \wedge \forall x \forall y \forall z((x \circ y) \circ z = x \circ (y \circ z))$
semi-group $\langle S, \circ \rangle$ is a monoid $\Leftrightarrow \exists e \in S(\forall x(e \circ x = x \circ e = x))$
monoid $\langle S, \circ \rangle$ is a group $\Leftrightarrow \forall x(\exists x^{-1}(x \circ x^{-1} = x^{-1} \circ x = e))$
$|a| = \min\{k | a^k = e\}$, $ab = ac \Rightarrow b = c, ba = ca \Rightarrow b = c$, $\langle a \rangle = \{a^k | k \in \mathbb{Z}\}$
$H \leq G \Leftrightarrow \forall a, b \in H(ab^{-1} \in H)$,
[Lagrange] $H \leq G \Rightarrow |G| = |H| \cdot [G : H] \Rightarrow \forall a \in G(|a| \mid n \wedge a^n = e)$
$N \trianglelefteq G \Leftrightarrow \forall a \in G(Na = aN) \Leftrightarrow \forall g \in G, \forall n \in N(gng^{-1} \in N) \Leftrightarrow gNg^{-1} = N$
$G/N = \{Ng \mid g \in G\}, Na \circ Nb = Nab$