离散数学习题课

第十讲 ——群论（二）

Morphism

- “抽象代数(Abstract Algebra)的研究对象是代数结构(Algebraic Structures)，并且是通过研究保持运算的映射（称为“态射”(morphism)）来研究代数结构的。”

Homomorphism

Definition:

Let G, G' be groups. A <u>homomorphism</u> $\varphi : G \to G'$ is any function satisfying the rule:

$$\varphi(ab) = \varphi(a)\varphi(b)$$

Comments:

- A homomorphism $\varphi : G \to G'$ certainly satisfies

  $(1)\ \varphi(e) = e';\qquad (2)\ \varphi(a^{-1}) = (\varphi(a))^{-1}$

- Let $\ker\varphi = \{x \mid x \in G \wedge \varphi(x) = e'\}$, then

  $(1)\ \ker\varphi \trianglelefteq G;\qquad (2)\ G/\ker\varphi \cong \varphi(G)$

Isomorphism

Definition:

Let G, G' be groups, $\varphi : G \to G'$ is any homomorphism,

- If φ is surjective, then φ is called an epimorphism
- If φ is injective, then φ is called a monomorphism
- If φ is bijective, then φ is called an isomorphism

Comments:

- If two groups are isomorphic (to each other), they have the same structure (only with different names)
- For any homomorphism $\varphi : G \to G'$,

$$\varphi \text{ is injective } \iff \ker \varphi = \{e\}$$

Cyclic groups

Definition:

A group G is called cyclic if $\exists a \in G$,

$$\langle a \rangle = \{a^k \mid k \in \mathbb{Z}\} = G$$

such an a is called a <u>generator</u> of G.

Comments:

- Generators are not unique
- Every finite cyclic group of order n is isomorphic to $\langle \mathbb{Z}_n, +_n \rangle$.
- Every infinite cyclic group is isomorphic to $\langle \mathbb{Z}, + \rangle$.

Cyclic groups (cont.)

Properties:

- Cyclic groups are all abelian
- Subgroups and quotient groups of a cyclic group are also cyclic
- Let G be a finite cyclic group of order n, then for any $d \mid n$, G has a unique subgroup of order d.
- Let $G = \langle a \rangle$ be a cyclic group, then every subgroup of G takes the form $H = \langle a^k \rangle$.

Problems

1. Let G be a group, show that, for any $H \leq G$, and for any $N, K \trianglelefteq G$,

   (1) $HK \leq G$

   (2) $NK \trianglelefteq G$

   (3) $H \cap K \trianglelefteq H$

2. Let G be a group, $N \trianglelefteq G, |N| = 2,$ show that

   $$N \subseteq Z(G) = \{a \mid a \in G \wedge \forall x \in G(ax = xa)\}$$

3. Let G be a group, $N \trianglelefteq G, K \leq G,$ show that

   $$\langle N \cup K \rangle = NK$$

Problems (cont.)

4. Let G be a group, $Z(G) = \{a \mid a \in G \wedge \forall x \in G(ax = xa)\}$ be its center. Show that, for any $H \leq Z(G)$,

   (1) $H \trianglelefteq G$;

   (2) If G/H is cyclic, then G is abelian.

5. Let G be a group, with $|G| = 2p$, where p is an odd prime. Show that, if there is a $N \trianglelefteq G$, such that $|N| = 2$, then G is cyclic.

6. Show that, every finite group of order p^2 is abelian, where p is any prime.

Thank you

Any questions?