离散数学习题课

第九讲 ——期中测验回顾

Set theory

Q1: Prove or disprove: For any sets A, B,

(1) $A \subseteq B \implies \cup A \subseteq \cup B$

(2) $\emptyset \neq A \subseteq B \implies \cap B \subseteq \cap A$

(3) $A \neq \emptyset \implies \cap A \subseteq \cup A$

(1) Proof: For any x,

$$
\begin{aligned}
x \in \cup A &\iff \exists y (y \in A \wedge x \in y) \\
&\implies \exists y (y \in B \wedge x \in y) \quad (\because A \subseteq B) \\
&\iff x \in \cup B
\end{aligned}
$$

Q.E.D

Set theory (cont.)

(2) Proof: Since $\emptyset \neq A$, we have $\exists x \in A$, and since $A \subseteq B$, we have $x \in B$. Thus $B \neq \emptyset$, both $\cap A, \cap B$ are well defined.

For any x,

$$
\begin{aligned}
x \in \cap B &\iff \forall y(y \in B \to x \in y) \\
&\iff \forall y(\neg y \in B \vee x \in y) \\
&\implies \forall y(\neg y \in A \vee x \in y) \quad (\because \sim B \subseteq \sim A) \\
&\iff \forall y(y \in A \to x \in y) \\
&\iff x \in \cap A
\end{aligned}
$$

Q.E.D.

Set theory (cont.)

(3) Proof: For any $x \in \cap A$,

$$\because A \neq \emptyset$$

$$\therefore \exists S(S \in A)$$

$$\because x \in \cap A$$

$$\therefore \forall y(y \in A \to x \in y)$$

$$\therefore \exists S(S \in A \land x \in S)$$

$$\therefore x \in \cup A$$

Q.E.D.

Cartesian products

Q2: Prove or disprove: For any sets A, B, C, D,

(1) $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$

(2) $(A \times C) \cup (B \times D) = (A \cup B) \times (C \cup D)$

(1) Proof: For any x, y,

$$\begin{aligned}
& \langle x, y\rangle \in (A \times C) \cap (B \times D) \\
\Longleftrightarrow\ & x \in A \wedge y \in C \wedge x \in B \wedge y \in D \\
\Longleftrightarrow\ & x \in A \wedge x \in B \wedge y \in C \wedge y \in D \\
\Longleftrightarrow\ & x \in A \cap B \wedge y \in C \cap D \\
\Longleftrightarrow\ & \langle x, y\rangle \in (A \cap B) \times (C \cap D)
\end{aligned}$$

Q.E.D

Cartesian products (cont.)

(2) Disproof: Let $A = D = \emptyset, B = C = \{1\}$, then

$$\begin{aligned}(A \times C) \cup (B \times D) &= \emptyset \\ &\neq \{\langle 1, 1 \rangle\} \\ &= (A \cup B) \times (C \cup D)\end{aligned}$$

Q.E.D

In fact,

$$\begin{aligned}(A \times C) \cup (B \times D) &\subseteq (A \cup B) \times (C \cup D) \\ &= (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)\end{aligned}$$

Relations

Q3: Let $R \subseteq A \times A$ be a reflexive relation on A, show that, if $\forall x, y, z \in A((xRy \wedge xRz) \to yRz)$, then R is an equivalence.

Proof:

(1) Reflexivity is given by premises.

(2) For any $a, b \in A$,

$$aRb \implies aRb \wedge aRa \implies bRa$$

So R is symmetric.

Relations (cont.)

(3) For any $a, b, c \in A$,

$$aRb \wedge bRc \implies bRa \wedge bRc \implies aRc$$

So R is transitive.

Thus R is an equivalence.

Q.E.D

Functions

Q4: Prove or disprove: For any sets A, B and any function $f : A \to B$,

$(1)\ \forall X, Y \subseteq A, f(X \cap Y) = f(X) \cap f(Y)$

$(2)\ \forall X, Y \subseteq B, f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

where $f^{-1}(X) = \{a \mid a \in A \wedge f(a) \in X\}$.

(1) Disproof:

Let $A = B = \mathbb{R}, \forall x \in \mathbb{R}, f(x) = x^2, X = \{1\}, Y = \{-1\}$

Then $f(X \cap Y) = \emptyset \neq \{1\} = f(X) \cap f(Y)$.

Q.E.D.

Functions (cont.)

(2) Proof: For any a,

$$\begin{aligned} a \in f^{-1}(X \cap Y) &\iff a \in A \wedge f(a) \in X \cap Y \\ &\iff a \in A \wedge f(a) \in X \wedge f(a) \in Y \\ &\iff a \in A \wedge a \in A \wedge f(a) \in X \wedge f(a) \in Y \\ &\iff a \in A \wedge f(a) \in X \wedge a \in A \wedge f(a) \in Y \\ &\iff a \in f^{-1}(X) \wedge a \in f^{-1}(Y) \\ &\iff a \in f^{-1}(X) \cap f^{-1}(Y) \end{aligned}$$

Q.E.D.

Cardinal numbers

Q5: $\operatorname{card}(\mathbb{R}-\mathbb{Q})=?$

Solution 1: (relies on the Axiom of Choice)

Let $\lambda=\operatorname{card}(\mathbb{R}-\mathbb{Q})$, note that $\lambda>0, \operatorname{card}\mathbb{Q}=\aleph_0$, so

$$\begin{aligned}\aleph &= \operatorname{card}\mathbb{R}=\operatorname{card}((\mathbb{R}-\mathbb{Q})\cup\mathbb{Q}) \\ &= \operatorname{card}(\mathbb{R}-\mathbb{Q})+\operatorname{card}\mathbb{Q}=\max\{\lambda,\aleph_0\}\in\{\lambda,\aleph_0\}\end{aligned}$$

But $\aleph\neq\aleph_0$, so $\aleph=\lambda=\operatorname{card}(\mathbb{R}-\mathbb{Q})$.

Cardinal numbers (cont.)

Solution 2: Clearly, $\mathrm{card}(\mathbb{R}-\mathbb{Q}) \leq \mathrm{card}\,\mathbb{R} = \aleph$.

Let $f:(0,1) \to \mathbb{R}-\mathbb{Q}, \forall x \in \mathbb{R}$,

$$f(x) = \begin{cases} x+\sqrt{2}, & x \in \mathbb{Q} \\ x, & x \notin \mathbb{Q} \end{cases}$$

Then f is 1-1.

So, we have $\aleph = \mathrm{card}(0,1) \leq \mathrm{card}(\mathbb{R}-\mathbb{Q})$.

By Cantor-Bernstein-Schöder theorem, we have

$$\mathrm{card}(\mathbb{R}-\mathbb{Q}) = \aleph$$

Cardinal numbers (cont.)

Proof 3: Clearly $\operatorname{card}(\mathbb{R}-\mathbb{Q}) \le \operatorname{card}\mathbb{R} = \aleph$.

Let $f: \mathbb{R} \to \mathbb{R}-\mathbb{Q}, \forall x \in \mathbb{R}, f(x) = [x].x_1\pi_1x_2\pi_2\ldots x_n\pi_n\ldots$

where $[x]$ is the integral part of x, and x_i, π_i are the *i*-th digit after the decimal point of x and π respectively. (Here, as usual, we pick the unique decimal representation of any $x \in \mathbb{R}$, i.e. for $x = 0.3\dot{9} = 0.4\dot{0}$, we pick $0.4\dot{0}$ to be the representation of x)

Then f is 1-1. So we have $\aleph = \operatorname{card}\mathbb{R} \le \operatorname{card}(\mathbb{R}-\mathbb{Q})$.

Thus we have $\operatorname{card}(\mathbb{R}-\mathbb{Q}) = \aleph$.

Group theory

Q6: Let G be a group, $L = \{aba^{-1}b^{-1} \mid a, b \in G\}$,

$H = \{c_1c_2\cdots c_n \mid n \in \mathbb{N}^+ \land \forall i \in \mathbb{N}(1 \leq i \leq n \to c_i \in L))\}$

Show that

(1) $H \leq G$;

(2) $H \trianglelefteq G$;

(3) G/H is abelian;

(4) For any $N \trianglelefteq G$, if G/N is abelian, then $H \subseteq N$.

Group theory (cont.)

(1) Proof:

$$\because e = eee^{-1}e^{-1} \in L \subseteq H$$

$$\therefore H \neq \emptyset$$

Note that, for any $c = aba^{-1}b^{-1} \in L, c^{-1} = bab^{-1}a^{-1} \in L,$

Thus, for all $x, y \in H,$

$$x = c_1 c_2 \cdots c_n, y = d_1 d_2 \cdots d_m (c_i, d_j \in L)$$

$$xy^{-1} = c_1 c_2 \cdots c_n d_m^{-1} \cdots d_2^{-1} d_1^{-1} \in H$$

Therefore, $H \leq G.$

Q.E.D.

Group theory (cont.)

(2) Proof:

Note that, for any $c = aba^{-1}b^{-1} \in L, x \in G$

$$xcx^{-1} = xax^{-1}xbx^{-1}xa^{-1}x^{-1}xb^{-1}x^{-1}$$

$$= (xax^{-1})(xbx^{-1})(xax^{-1})^{-1}(xax^{-1})^{-1} \in L$$

Thus, for all $x \in G, h = c_1c_2\cdots c_n \in H,$

$$xhx^{-1} = xc_1x^{-1}xc_2x^{-1}\cdots xc_nx^{-1} \in H$$

Therefore, $H \trianglelefteq G.$

Q.E.D.

Group theory (cont.)

(3) Proof:

For any $Ha, Hb \in G/H$

$$
\begin{aligned}
Ha \circ Hb &= Hab \\
&= Hba \qquad (\because ab(ba)^{-1} = aba^{-1}b^{-1} \in H) \\
&= Hb \circ Ha
\end{aligned}
$$

Therefore, G/H is abelian.

Q.E.D.

Group theory (cont.)

(4) Proof:

For any $N \trianglelefteq G$, if G/N is abelian, then for any $a, b \in G$,

$$Na \circ Nb = Nb \circ Na$$

i.e. $ab(ba)^{-1} = aba^{-1}b^{-1} \in N$

Thus we have $L \subseteq N$.

Since N is a group, by its closure, we have $H \subseteq N$.

Q.E.D.

Thank you

Any questions?