离散数学习题课

第八讲 ——部分知识点总结

Notes on cardinalities

For (possibly infinite) sets A, B,

- It's <u>inadequate</u> to say:

  $$\operatorname{card}(A \cup B) = \operatorname{card} A + \operatorname{card} B - \operatorname{card}(A \cap B)$$

- It's <u>incorrect</u> to say:

  $$\operatorname{card}(A \cup B) = \operatorname{card} A + \operatorname{card} B$$

- What we can show is:

  $$\operatorname{card}(A \cup B) \leq \operatorname{card} A + \operatorname{card} B$$

- We <u>cannot</u> assume that:

  $$A \approx n \quad \text{or} \quad A = \{a_1, a_2, \cdots, a_n\}$$

Notes on cardinalities (cont.)

For (possibly infinite) sets A, B,

- It's <u>inadequate</u> to assume (unless you proved something):

  $$A \cap B = \emptyset$$

- Note that:

  $$\text{card}\, A \leq \aleph_0 \not\Longrightarrow \exists n \in \mathbb{N}(A \approx n)$$

- Be careful that:

$$\begin{aligned} A \approx B &\Longleftrightarrow \exists f : A \xrightarrow[onto]{1-1} B \\ &\not\Longrightarrow \forall f \in B^A (f : A \xrightarrow[onto]{1-1} B) \end{aligned}$$

Notes on cardinalities (cont.)

For any <u>finite</u> sets A, B, if $A \approx B,$ then

- For any $f : A \to B,$

  f is injective $\iff$ f is surjective $\iff$ f is bijective

- As a result, we have

$$A \approx B \wedge A \subseteq B \implies A = B$$

If the <u>Axiom of Choice</u> holds, then, for any <u>infinite</u> set $A,$

$$\mathbb{N} \preccurlyeq\cdot A \quad \text{and} \quad A \approx A \times A$$

- As a result, for any <u>cardinal numbers</u> $\kappa, \lambda,$

$$\kappa \neq 0 \wedge \lambda \geq \aleph_0 \implies \kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$$

Some results on groups

Let G be a group, for any $H, K \subseteq G, x \in G$

(1) $K \leq H \leq G \implies K \leq G$

(2) $H \leq G \wedge K \leq G \wedge K \subseteq H \implies K \leq H$

(3) $H \leq G \wedge |G| = n < \infty \implies |H| \,\big|\, n \wedge [G : H] \,\big|\, n$

(4) $H \leq G \wedge K \leq G \implies H \cap K \leq G$

(5) $\langle a \rangle = \{a^k \mid k \in \mathbb{Z}\} \leq G$

(6) $|a| = |\langle a \rangle| \,\big|\, |G|$, when $|G| < \infty$

(7) $a^k = e \implies |a| \,\big|\, k \not\Longrightarrow |a| = k$

Notes on subgroups

- To show $H \leq G$, we need to show that:

$$(1)\ H \subseteq G$$

$$(2)\ H \neq \emptyset$$

$$(3)\ \forall a, b \in H, ab \in H \wedge a^{-1} \in H$$

$$\text{or}\quad \forall a, b \in H, ab^{-1} \in H$$

$$\text{or}\quad |H| < \infty \wedge \forall a, b \in H, ab \in H$$

- Note:
  - Must show that $H \neq \emptyset$
  - It's not enough just to show that $\forall a, b \in H (ab \in H)$

Notes on normal subgroups

Let G be a group, $H \le G$

$$H \trianglelefteq G \iff \forall x \in G(xH = Hx)$$

$$\iff \forall x \in G(xHx^{-1} = H)$$

$$\iff \forall x \in G, h \in H(xhx^{-1} \in H)$$

Let G be a group, $H \le G$, then

$(1)\ \forall x \in G(H \approx xHx^{-1})$

$(2)\ |H| = n < \infty \wedge |\{K \mid K \le G \wedge |K| = n\}| = 1 \implies H \trianglelefteq G$

$(3)\ G \text{ is abelian} \implies H \trianglelefteq G$

$(4)\ K \trianglelefteq H \trianglelefteq G \;\not\Longrightarrow\; K \trianglelefteq G$

Notes on quotient groups

Let G be a group, $N \trianglelefteq G$

- Note that $Na = Nb \not\Rightarrow a = b$, therefore:
- Be careful to say: $f(Na) = Ka$, because you need to prove: $\forall a, b \in G(Na = Nb \to Ka = Kb)$
- $G/N = \{Na \mid a \in G\}$ is a "set of sets", the results of the operations on it are also sets.
- $\langle G/N, \circ \rangle$ is a group (called quotient group) on G/N, where $\forall Na, Nb \in G/N$,

$$Na \circ Nb = Nab$$

Some results in number theory

For any $m, n \in \mathbb{N}$

(1) $(m,n) = 1 \iff \exists x, y \in \mathbb{Z}(xm + yn = 1)$

(2) $(m,n) = \min\{d \mid d = xm + yn \wedge d \geq 1 \wedge x, y \in \mathbb{Z}\}$

Thank you

Any questions?