离散数学习题课

第十二讲——偏序与格

Partial orders

Definition:

A relation $R \subseteq A \times A$ is called a <u>partial order</u> on A if R is

a) reflexive,

b) transtive, and

c) anti-symmetric

$\langle A, R \rangle$ is called a <u>partially ordered set</u>, or <u>poset</u>.

Comments:

- For $x, y \in A$, x and y are not always <u>comparable</u>.

Elements in posets

Definition: Let $\langle A, R\rangle$ be a poset, $B \subseteq A$, $x \in A$, then

(1) x is called a <u>minimal element</u> of B if

$$x \in B \wedge \forall y \in B(yRx \rightarrow x = y);$$

(2) x is called a <u>maximal element</u> of B if

$$x \in B \wedge \forall y \in B(xRy \rightarrow x = y);$$

(3) x is called an <u>upper bound</u> of B if

$$\forall y \in B(yRx);$$

(4) x is called a <u>lower bound</u> of B if

$$\forall y \in B(xRy);$$

Elements in posets (cont.)

Definition: Let $\langle A, R\rangle$ be a poset, $B \subseteq A$, $x \in A$, then

(5) x is called the <u>greatest element</u> of B if

$x \in B$ AND x is an upper bound of B;

(6) x is called the <u>least element</u> of B if

$x \in B$ AND x is a lower bound of B;

(7) x is called the <u>supremum</u> of B if

x is the least upper bound of B;

(8) x is called the <u>infimum</u> of B if

x is the greatest lower bound of B.

Elements in posets (cont.)

Comments:

- The greatest/least element (if exists) must be the supremum/infimum
- The greatest/least element (if exists) must be maximal/minimal
- The supremum/infimum (if exists) must be unique
- The supremum/infimum (if exists) of B is not necessarily in B
- If B is finite, then B always have maximal elements and minimal elements

Lattices

Definition:

A poset $\langle L, \preccurlyeq \rangle$ is called a lattice if

$$\forall a, b \in L,\ \{a, b\} \text{ has both supremum and infimum.}$$

A algrebraic system $\langle L, \wedge, \vee \rangle$ is called a lattice if

$\forall a, b, c \in L,$

(1) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$;

(2) $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$;

(3) $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

The operators $\wedge$ and $\vee$ are called meet and join respectively.

Lattices (cont.)

Comments:

- If $\langle L, \wedge, \vee \rangle$ is a lattice, then there is a unique corresponding lattice $\langle L, \preccurlyeq \rangle$, and vice versa
- If $\langle L, \wedge, \vee \rangle$ is a lattice, then $\forall a \in L,$

$$a \wedge a = a \vee a = a$$

- If $\langle L, \wedge, \vee \rangle$ is a lattice, $\emptyset \neq S \subseteq L$, and $\forall x, y \in S,$

$$x \wedge y \in S \text{ and } x \vee y \in S$$

then $\langle S, \wedge, \vee \rangle$ is a <u>sublattice</u> of $\langle L, \wedge, \vee \rangle$.

The principle of duality

The principle:

设 f 是含有格中元素以及符号 $=, \preccurlyeq, \succcurlyeq, \vee, \wedge$ 的命题，若 f 对一切格为真，则 f 的对偶命题 f^* 也对一切格为真。

Notes:

- f <u>must not</u> contain other symbols
- f must hold for <u>every lattice</u>

Properties of lattices

Let L be a lattice, for any $a, b, c, d \in L$,

(1) $a \wedge b \preccurlyeq a \preccurlyeq a \vee b$;

(2) $a \wedge b \preccurlyeq b \preccurlyeq a \vee b$;

(3) $a \preccurlyeq c$ and $b \preccurlyeq d \implies a \wedge b \preccurlyeq c \wedge d$;

(4) $a \preccurlyeq c$ and $b \preccurlyeq d \implies a \vee b \preccurlyeq c \vee d$;

(5) $a \vee (b \wedge c) \preccurlyeq (a \vee b) \wedge (a \vee c)$;

(6) $a \wedge (b \vee c) \succcurlyeq (a \wedge c) \vee (a \wedge c)$;

(7) $a \preccurlyeq b \iff a \wedge b = a \iff a \vee b = b$

Homomorphisms of lattices

Definition:

Let $\langle L, \wedge, \vee \rangle, \langle L', \wedge', \vee' \rangle$ be lattices, $\varphi : L \to L'$ is called a homomorphism from L to L' if and only if $\forall x, y \in L$,

$$\varphi(x \vee y) = \varphi(x) \vee' \varphi(y) \text{ and } \varphi(x \wedge y) = \varphi(x) \wedge' \varphi(y)$$

Comments:

- If $\varphi : L \to L'$ is a homomorphism, then $\forall x, y \in L$,

$$x \preccurlyeq y \implies \varphi(x) \preccurlyeq' \varphi(y)$$

- For any bijection $f : L \xrightarrow[onto]{1-1} L'$, f is a isomorphism if and only if

$$\forall x, y \in L(x \preccurlyeq y \leftrightarrow f(x) \preccurlyeq' f(y))$$

Problems

1. Let L be a lattice, show that, for all $a, b, c \in L$,

$$a \preccurlyeq c \iff a \vee (b \wedge c) \preccurlyeq (a \vee b) \wedge c$$

2. Let L be a lattice, and for all $a, b, c \in L$, we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Show that, for all $a, b, c \in L$,

$(1)\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c);$

$(2)\ a \preccurlyeq c \implies a \vee (b \wedge c) = (a \vee b) \wedge c$

Problems (cont.)

3. Let L be a lattice, show that, for all $a, b, c, d \in L$,

   $(1)\ (a \wedge b) \vee (c \wedge d) \preccurlyeq (a \vee c) \wedge (b \vee d)$

   $(2)\ (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \preccurlyeq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

4. A lattice L is called a <u>modular lattice</u> if $\forall a, b, c \in L$,

   $$a \preccurlyeq c \implies a \vee (b \wedge c) = (a \vee b) \wedge c$$

   Show that, a lattice L is a modular lattice if and only if $\forall a, b, c \in L$,

   $$a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$$

Thank you

Any questions?