离散数学习题课

第二讲 —— 集合论部分习题解析

Trivia

- Lecture notes:

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- Assignments:
  - Please write both your <u>name</u> and <u>student number</u> on the <u>cover</u> of the exercise books
- Solving additional problems are encouraged

Uniqueness of elements

P104. Ex.8(4)

$$\mathcal{P}(\{\{1,2\},\{2,1,1\},\{2,1,1,2\}\}) =?$$

By definition, we have

$$A=\{1,2\}=\{2,1,1\}=\{2,1,1,2\}$$

Thus,

$$\begin{aligned}\{\{1,2\},\{2,1,1\},\{2,1,1,2\}\} &= \{A,A,A\}\\ &= \{A\}\\ &= \{\{1,2\}\}\end{aligned}$$

Uniqueness of elements(cont.)

Solution:

Since

$$\begin{aligned}\{\{1,2\},\{2,1,1\},\{2,1,1,2\}\} &= \{\{1,2\},\{1,2\},\{1,2\}\} \\ &= \{\{1,2\}\}\end{aligned}$$

So

$$\begin{aligned}\mathcal{P}(\{\{1,2\},\{2,1,1\},\{2,1,1,2\}\}) &= \mathcal{P}\{\{1,2\}\} \\ &= \{\emptyset,\{\{1,2\}\}\}\end{aligned}$$

Uniqueness of elements(cont.)

Also note that,

$$\begin{aligned}\{\{1,2\}\} = \{\{2,1,1\}\} = \{\{2,1,1,2\}\} &= \{\{1,2\},\{2,1,1\}\} \\ &= \{\{1,2\},\{2,1,1,2\}\} = \{\{2,1,1\},\{2,1,1,2\}\} \\ &= \{\{1,2\},\{2,1,1\},\{2,1,1,2\}\}\end{aligned}$$

And be careful that

$$\begin{aligned}|\{\{1,2\},\{2,1,1\},\{2,1,1,2\}\}| &= 1 \\ |\mathcal{P}(\{\{1,2\},\{2,1,1\},\{2,1,1,2\}\})| &= 2\end{aligned}$$

Power sets

Here

$$\mathcal{P}(\{\{1,2\}\}) = \{\emptyset, \{\{1,2\}\}\}$$
$$\neq \{\emptyset, \{1,2\}\}$$

Note that, for any set S,

$$\emptyset \in \mathcal{P}(S),\ \ S \in \mathcal{P}(S)$$

Also, $|\{\{1,2\}\}| = 1$, while $|\{1,2\}| = 2$.

Power sets (cont.)

P105. Ex.12(1)

设 A 是 $n(n \geq 1)$ 元集，其元素为英文字母，B 是 m 元集，其元素为自然数，求 $\mathcal{P}(A) \cap \mathcal{P}(B)$.

Note that,

$$\because \quad \emptyset \subseteq A \wedge \emptyset \subseteq B$$

$$\therefore \quad \emptyset \in \mathcal{P}(A) \wedge \emptyset \in \mathcal{P}(B)$$

$$\emptyset \in \mathcal{P}(A) \cap \mathcal{P}(B)$$

Therefore,

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset\} \neq \emptyset$$

Expression simplification

Ex. Simplify the expression:

$$(A - B) \cup (A \cap B)$$

Solution:

$$\begin{aligned}(A - B) \cup (A \cap B) &= (A \cap {\sim}B) \cup (A \cap B)\\ &= A \cap ({\sim}B \cup B)\\ &= A \cap E\\ &= A\end{aligned}$$

Equivalent conditions

P108. Ex.41

Let A, B be sets, find necessary and sufficient conditions for the following equations:

(1) $A \cup B = A$

(2) $A - B = A$

(3) $A - B = B$

(4) $A - B = B - A$

(5) $A \oplus B = A$

(6) $A \oplus B = \emptyset$

Equivalent conditions (cont.)

Some useful results:

$$\begin{aligned} X = Y &\iff X \oplus Y = \emptyset \\ &\iff X \subseteq Y \land Y \subseteq X \\ &\iff X - Y = \emptyset \land Y - X = \emptyset \\ &\implies X \subseteq Y \end{aligned}$$

$$\begin{aligned} X \subseteq Y &\iff X \cap Y = X \\ &\iff X \cup Y = Y \\ &\iff X - Y = \emptyset \end{aligned}$$

Equivalent conditions (cont.)

(2) $A - B = A$

Solution: $$\begin{aligned} A - B = A &\implies (A - B) \cap B = A \cap B \\ &\iff A \cap {\sim}B \cap B = A \cap B \\ &\iff \emptyset = A \cap B \end{aligned}$$

And, if $A \cap B = \emptyset$, then

$$\begin{aligned} A &= (A - B) \cup (A \cap B) \\ &= (A - B) \cup \emptyset \\ &= A - B \end{aligned}$$

Therefore, $A - B = A \iff A \cap B = \emptyset$

Equivalent conditions (cont.)

(4) $A - B = B - A$

Solution:

$$
\begin{aligned}
A - B = B - A \implies & A \cap (A - B) = A \cap (B - A) \\
\iff & A \cap A \cap \sim B = A \cap B \cap \sim A \\
\iff & A \cap \sim B = \emptyset \\
\iff & A - B = \emptyset \\
\iff & A \subseteq B
\end{aligned}
$$

Similarly, we have $A - B = B - A \implies B \subseteq A$

Equivalent conditions (cont.)

Thus, we have

$$A - B = B - A \implies A = B$$

And it's trivial to show that

$$A = B \implies A - B = B - A$$

Therefore, we conclude that

$$A - B = B - A \iff A = B$$

Counting

For any sets $A, B,$

$$|A-B| = |A| - |A \cap B|$$

Proof:

$$\begin{aligned} |A| &= |(A-B) \cup (A \cap B)| \\ &= |A-B| + |A \cap B| - |(A-B) \cap (A \cap B)| \\ &= |A-B| + |A \cap B| - |A \cap {\sim}B \cap A \cap B| \\ &= |A-B| + |A \cap B| - |\emptyset| \\ &= |A-B| + |A \cap B| \end{aligned}$$

Q.E.D.

Rule of substitution

- 置换规则(Rule of substitution):

设 $\Phi(A)$ 是含公式 A 的公式，用公式 B 置换 $\Phi(A)$ 中的 A，得到公式 $\Phi(B)$。若 $B \Leftrightarrow A$，则 $\Phi(B) \Leftrightarrow \Phi(A)$。

- Compare:

If for any $\vec{x}$, we have $f(\vec{x}) = g(\vec{x})$, then for any function h, we have $h(f(\vec{x})) = h(g(\vec{x}))$.

More about power sets

For any sets A, B, C,

$$(1)\ \cup\mathcal{P}(A) = A$$

$$(2)\ \mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$$

$$(3)\ \mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$$

$$(4)\ C \subseteq B \wedge B \in \mathcal{P}(A) \rightarrow C \in \mathcal{P}(A)$$

More about power sets (cont.)

(1) $\cup\mathcal{P}(A) = A$

Proof: For any set x,

$$
\begin{aligned}
x \in A &\iff x \in A \wedge A \subseteq A \\
&\iff x \in A \wedge A \in \mathcal{P}(A) \\
&\implies \exists t(x \in t \wedge t \in \mathcal{P}(A)) \\
&\iff x \in \cup\mathcal{P}(A)
\end{aligned}
$$

Thus we have,

$$A \subseteq \cup\mathcal{P}(A)$$

More about power sets (cont.)

On the other hand,

$$\begin{aligned} x \in \cup\mathcal{P}(A) &\iff \exists t(x \in t \wedge t \in \mathcal{P}(A)) \\ &\iff \exists t(x \in t \wedge t \subseteq A) \\ &\implies \exists t(x \in A) \\ &\implies x \in A \end{aligned}$$

Which means

$$\cup\mathcal{P}(A) \subseteq A$$

To conclude, we have

$$\cup\mathcal{P}(A) = A$$

Q.E.D.

More about power sets (cont.)

(2) $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$

Proof: For any set S,

$$
\begin{aligned}
& S \in \mathcal{P}(A \cap B) \\
\iff & S \subseteq A \cap B \\
\iff & \forall x(x \in S \to x \in A \cap B) \\
\iff & \forall x(x \in S \to x \in A \wedge x \in B) \\
\iff & \forall x(\neg x \in S \vee (x \in A \wedge x \in B)) \\
\iff & \ldots\ldots
\end{aligned}
$$

More about power sets (cont.)

$$\ldots\ldots$$

$$\Longleftrightarrow \forall x(\neg x \in S \vee (x \in A \wedge x \in B))$$

$$\Longleftrightarrow \forall x((\neg x \in S \vee x \in A) \wedge (\neg x \in S \vee x \in B)$$

$$\Longleftrightarrow \forall x((x \in S \to x \in A) \wedge (x \in S \to x \in B))$$

$$\Longleftrightarrow \forall x(x \in S \to x \in A) \wedge \forall x(x \in S \to x \in B)$$

$$\Longleftrightarrow S \subseteq A \wedge S \subseteq B$$

$$\Longleftrightarrow S \in \mathcal{P}(A) \wedge S \in \mathcal{P}(B)$$

$$\Longleftrightarrow S \in \mathcal{P}(A) \cap \mathcal{P}(B)$$

Q.E.D.

Subtlety of quantifiers

For any predicates $P(x), Q(x)$, we have

$$\forall x P(x) \land \forall x Q(x) \iff \forall x (P(x) \land Q(x))$$

Yet

$$\forall x P(x) \lor \forall x Q(x) \implies \forall x (P(x) \lor Q(x))$$

Similarly,

$$\exists x (P(x) \land Q(x)) \implies \exists x P(x) \land \exists x Q(x)$$

$$\forall x (P(x) \to Q(x)) \implies \forall x P(x) \to \forall x Q(x)$$

$$\forall x (P(x) \to Q(x)) \implies \exists x P(x) \to \exists x Q(x)$$

More about power sets (cont.)

- Properties of sets:

$$
\begin{aligned}
&a) \quad A \subseteq B \wedge B \subseteq S \implies A \subseteq S \\
&b) \quad A \in B \wedge B \subseteq S \implies A \in S \\
&c) \quad A \subseteq B \wedge B \in S \not\Longrightarrow A \in S \\
&d) \quad A \in B \wedge B \in S \not\Longrightarrow A \in S
\end{aligned}
$$

- $\mathcal{P}(X)$ is an example of S for which

$$A \subseteq B \wedge B \in S \implies A \in S$$

Transitive sets

- A set S is called *transitive*, if and only if for any sets $A, B,$

$$A \in B \wedge B \in S \implies A \in S$$

- The following propositions are equivalent:

$$\begin{aligned}
&(1)\ S \text{ is a transitive set;}\\
&(2)\ \cup S \subseteq S;\\
&(3)\ \text{For any set } A,\ A \in S \implies A \subseteq S;\\
&(4)\ S \subseteq \mathcal{P}(S).
\end{aligned}$$

Transitive sets (cont.)

- Examples of transitive sets:

$$A = \emptyset$$

$$B = \{\emptyset\}$$

$$C = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$$

$$D = 3 = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

$$\mathbb{N} = \{0, 1, 2, \cdots\}$$

- In fact, each natural number (by von Neumann's definition) is a transitive sets

Problems

1. For any sets A, B, C, show that

$$A \cap C \subseteq B \cap C \wedge A - C \subseteq B - C \iff A \subseteq B$$

2. For any sets A, B, C, show that

$$A \cap C = B \cap C \wedge A \cup C = B \cup C \iff A = B$$

3. For any sets A, B, show that

$$A \subseteq B \iff \sim B \subseteq \sim A \iff \sim A \cup B = E$$

$$\iff A - B \subseteq B \iff A - B \subseteq \sim A$$

4. Find 5 examples of sets A, such that $\cup A = A$.

Thank you

Any questions?