TRENDS IN LOGIC - STUDIA LOGICA LIBRARY

Metamathematics of Fuzzy Logic

Petr Hájek

SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.

METAMATHEMATICS OF FUZZY LOGIC

TRENDS IN LOGIC
Studia Logica Library

VOLUME 4

SCOPE OF THE SERIES

Trends in Logic is a bookseries covering essentially the same area as the journal *Studia Logica* – that is, contemporary formal logic and its applications and relations to other disciplines. These include artificial intelligence, informatics, cognitive science, philosophy of science, and the philosophy of language. However, this list is not exhaustive, moreover, the range of applications, comparisons and sources of inspiration is open and evolves over time.

The titles published in this series are listed at the end of this volume.

PETR HÁJEK

Institute of Computer Science,
Academy of Sciences of the Czech Republic,
Prague, Czech Republic

METAMATHEMATICS OF FUZZY LOGIC

SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 978-1-4020-0370-7 ISBN 978-94-011-5300-3 (eBook)
DOI 10.1007/978-94-011-5300-3

Printed on acid-free paper

To my wife Marie,
to my daughter Marie,
to my son Jonáš,
to my grandson Jonáš
with love.

CONTENTS

PREFACE

This book presents a systematic treatment of deductive aspects and structures of fuzzy logic understood as many valued logic *sui generis*. Some important systems of real-valued propositional and predicate calculus are defined and investigated. The aim is to show that fuzzy logic as a logic of imprecise (vague) propositions does have well developed formal foundations and that most things usually named "fuzzy inference" can be naturally understood as logical deduction.

There are two main groups of intended readers. First, logicians: they can see that fuzzy logic is indeed a branch of logic and may find several very interesting open problems. Second, equally important, researchers involved in fuzzy logic applications and soft computing. As a matter of fact, most of these are not professional logicians so that it can easily happen that an application, clever and succcessful as it may be, is presented in a way which is logically not entirely correct or may appear simple-minded. (Standard presentations of the logical aspects of fuzzy controllers are the most typical example.) This fact would not be very important if only the *bon ton* of logicians were harmed; but it is the opinion of the author (who is a mathematical logician) that a better understanding of the strictly logical basis of fuzzy logic (in the usual broad sense) is very useful for fuzzy logic appliers since if they know better what they are doing, they may hope to do it better. Still more than that: a better mutual understanding between (classical) logicians and researchers in fuzzy logic promises to lead to deeper cooperation and new results.

The book has been developed from a series of lectures which I held first in the Institute of Computer Science of the Academy of Sciences of the Czech Republic (for postgraduate students) and then also at the Technical University of Vienna, Austria. My paper [75] was the first draft of the contents of the book. Later I had the opportunity to give tutorials on fuzzy logic at the Helena Rasiowa minisemester at Warsaw in winter 1996, for Italian PhD students at Ravello in spring 1997 and at the IFSA congress in Prague, as well as the FUZZ-IEEE congress at Barcelona in summer 1997. There is written material that appeared as ref. [81].

I am indebted to my collaborators Ivan Kramosil, Ms. Dagmar Harmancová, Milan Daniel, David Coufal and David Švejda for stimulating comments and cooperation. I have extremely enjoyed long-term cooperation with my Spanish (Catalan) colleagues F. Esteva and L. Godo. I have learned from the work of and contact with S. Gottwald, D. Mundici, U. Höhle, M. Baaz,

J. Paris, J. Shepherdson, Ms. Isabel Ferreirim and several other colleagues. Finally, the criticism of the anonymous referee was helpful in the final stage of writing this book. The text was typed in LaTex by Ms. Iva Šindelková, J. Oliverius and Ms. I. Baranovská. My Tex experts were A. Štědrý and R. Neruda. My thanks go to all of them.

I also recognize partial support by the COST Action 15 and the grant No A1030601/1966 of the Grant Agency of the Academy of Sciences of the Czech Republic; this support has been crucial for attending conferences, meeting people and buying relevant books.

Finally let me mention two related monographs in preparation: Cignoli, D'Ottaviano and Mundici [30] and Gottwald [66] (English version of his [67].) The overlap with the present book is small and the books complement each other.

Prague, September 1997.

CHAPTER ONE

PRELIMINARIES

1.1. Introduction

(1) *Fuzzy logic is popular.* The number of papers dealing, in some sense, with fuzzy logic and its applications is immense, and the success in applications is evident, in particular in fuzzy control. From numerous books written on this subject we mention at least [224], [68], [134]. As is stated in the introduction to [134], in 1991 there were about 1400 papers dealing with fuzzy systems. Naturally, in this immense literature the quality varies; a mathematician (logician) browsing in it is sometimes bothered by papers that are mathematically poor (and he/she may easily overlook those that are mathematically excellent). This should not lead to a quick rejection of the domain. Let us quote Zadeh, the inventor of fuzzy sets ([134], Preface): "Although some of the earlier controversies regarding the applicability of fuzzy logic have abated, there are still influential voices which are critical and/or skeptical. Some take the position that anything that can be done with fuzzy logic can be done equally well without it. Some are trying to prove that fuzzy logic is wrong. And some are bothered by what they perceive to be exaggerated expectations. That may well be the case but, as Jules Verne had noted at the turn of the century, scientific progress is driven by exaggerated expectations."

I mention a recent paper [49] whose author claimed to prove that fuzzy logic is impossible (his "proof" was based on an evident misunderstanding; nevertheless, it had lead to a big discussion, mainly contained in a special volume [105]). To get insight into the domain let us first ask three questions: what is logic, what is fuzziness and what meaning(s) does the term "fuzzy logic"have?

(2) *Logic studies the notion(s) of consequence.* It deals with propositions (sentences), sets of propositions and the relation of consequence among them. The task of formal logic is to represent all this by means of well-defined logical calculi admitting exact investigation. Various calculi differ in their definitions of sentences and notion(s) of consequence (propositional logics, predicate logics, modal propositional/predicate logics, many-valued propositional/predicate logics etc.). Often a logical calculus has *two* notions of consequence: syntactical (based on a notion of proof) and semantical (based on a notion of truth); then the natural questions of soundness (does provability imply truth?) and completeness (does truth imply provability?) pose themselves.

(3) *Fuzziness is imprecision (vagueness); a fuzzy proposition may be true to some degree.* The word "crisp" is used as meaning "non-fuzzy". Standard examples of fuzzy propositions use a *linguistic variable* [214] as, for example, *age* with possible values *young, medium, old* or similar. The sentence "The patient is young" is true to some degree – the lower the age of the patient (measured e.g. in years), the more the sentence is true. *Truth of a fuzzy proposition is a matter of degree.*

I recommend to everybody interested in fuzzy logic that they sharply distinguish fuzziness from uncertainty as a degree of belief (e.g. probability). Compare the last proposition with the proposition "The patient will survive next week". This may well be considered as a crisp proposition which is either (absolutely) true or (absolutely) false; but we do not know which is the case. We may have some *probability* (chance, degree of belief) that the sentence is true; but probability is *not* a degree of truth. (Note that Section 4 in Chapter 8 is devoted to a discussion of the relation between fuzziness and probability; see also point (8) below.)

(4) *The term "fuzzy logic" has two different meanings – wide and narrow.* This is a very useful distinction, made by Zadeh; we again quote from [134], Preface: "In a narrow sense, fuzzy logic, FLn, is a logical system which aims at a formalization of approximate reasoning. In this sense, FLn is an extension of multivalued logic. However, the agenda of FLn is quite different from that of traditional multivalued logics. In particular, such key concepts in FLn as the concept of a linguistic variable, canonical form, fuzzy if-then rule, fuzzy quantification and defuzzification, predicate modification, truth qualification, the extension principle, the compositional rule of inference and interpolative reasoning, among others, are not addressed in traditional systems. This is the reason why FLn has a much wider range of applications than traditional systems. In its wide sense, fuzzy logic, FLw, is fuzzily synonymous with the fuzzy set theory, FST, which is the theory of classes with unsharp boundaries. FST is much broader than FLn and includes the latter as one of its branches." Let me add a comment: even if I agree with Zadeh's distinction between many-valued logic and fuzzy logic in the narrow sense, I consider formal calculi of many-valued logic (including non-"traditional" ones, of course) to be the kernel or base of fuzzy logic in the narrow sense and the task of explaining things Zadeh mentions by means of these calculi to be a very promising task (not yet finished).

(5) *Fuzzy logic (in the narrow sense) is worth studying.* We shall understand FLn as a logic with a *comparative notion of truth*: sentences may be compared according to their truth values. Our main aim is to elaborate in detail strictly logical properties of the most important many-valued logics whose set of truth values is the unit interval [0,1] (both propositonal and

predicate logics) and then discuss some applications from the logical point of view.

The reader coming from a logician's perspective will (hopefully) be pleased by seeing that the logical systems of many-valued logic relevant to fuzzy logic have a depth and beauty comparable with that of classical logic. The theoretically-minded reader working in soft computing will (hopefully) be pleased by seeing that fuzzy propositional calculus as well as fuzzy predicate calculus are powerful tools in analyzing topics of Zadeh's agenda, such as fuzzy modus ponens, compositional rule of inference, fuzzy functions and fuzzy control, fuzzy generalized quantifiers and modalities (like "many", "probably" and others). We shall stress the expressive power of fuzzy logic as well as its *deductive power*: surprisingly, much of the agenda of FLn can be understood as a truth-preserving *deduction.* To give just one example, let us mention properties of fuzzy control expressible and provable in an axiomatic theory FC over the basic fuzzy predicate calculus. Furthermore, we shall analyze some paradoxes concerning the notion of truth (the liar's paradox, the undefinability of truth in arithmetic) and show that fuzzy logic offers new and non-trivial insights into them.

(6) *Where do the truth values come from?* First, why the unit interval [0,1]? And second, what does it mean that the truth degree of a proposition is 0.7? Leaving aside any philosophical discussion, I offer as a typical example *questionnaires:* you may be asked, e.g., *Do you like Haydn?* and you have to select one of four possible answers: *absolutely yes, more or less, rather not, absolutely not.* Or you may have another scale with 11 possibilities, etc. We shall assume that the truth degrees are linearly ordered, with 1 as maximum (absolutely yes) and 0 as minimum (absolutely no). Thus truth degrees will be coded by (some) reals. And even if logics of finitely many truth degrees can be developed (and have an extensive literature) we choose not to exclude any real number from the set of truth degrees. We shall always take the set [0,1] with its natural (standard) linear order; which other structure might be considered depends on the concrete logical system. On the other hand, we shall be naturally led to more abstract systems (algebras) of truth values, not necessarily linearly ordered.

Turning to the second subquestion, what it means that the truth degree of a proposition is 0.7, we have to distinguish, as in the classical logic, between the case of an atomic proposition and a compound proposition. Let us take the former. On the propositional level, we just work with truth evaluations of atomic propositions as given: if you are asked the question above (about liking Haydn), *you* select your answer and *you* know what it means. (This is the same situation as if you have only two choices, yes and no.) Imagine we have two questions: (i) Do you like Haydn? and (ii) Are you old? If you answer "absolutely yes" to the first and "more or less" to the second, then this does *not* mean that you like Haydn more than you are old; it just means that the truth degree you assigned to (i) is greater than the truth degree you assigned to (ii). (Again, as in classical logic: imagine you have answered "yes" to (i) and "no" to (ii).) Thus, on the level of propositional calculi the assignment of truth degrees to atomic propositions is not analyzed further. (In predicate calculi such assignment is given by the notion of a *model* – as in the classical logic.)

(7) *Most many-valued logics are truth-functional.* This means that the truth degree of a compound formula, built from its compounds using a logical connective (implication, conjunction, etc.) is a function of the truth degrees of the compounds – the truth function of the connective. (The situation is more complicated for quantifiers.) This answers the second subquestion from the previous paragraph for compound formulas – and there is nothing wrong in truth functionality if properly understood. For example, if we take the maximum as the truth function of disjunction, then saying that (under some circumstances) the truth degree of $\varphi \vee \psi$ is 0.7 we say nothing more (and nothing less) than that the maximum of the truth degrees of φ and of ψ is 0.7, thus we impose some constraints on the respective truth degrees. Observe that this is exactly the same situation as in classical logic: in saying that $\varphi \vee \psi$ is true (has the value 1) we say only that at least one of φ, ψ is true, thus constraining the truth values of φ, ψ (exclude the pair $\langle 0, 0 \rangle$).

(8) *The frequentist's temptation.* Needless to say, there are real-valued functions on formulas that are not truth-functional, e.g. probability: the probability of $\varphi \vee \psi$ is not a function of the probabilities of φ and of ψ. Thus probability is not directly grasped by truth-functional systems of many-valued logics; as mentioned above, probability is a particular measure of degrees of *belief*, and beliefs (mostly) are not truth functional. There have been attempts to explain non-extremal truth degrees as some relative frequencies. For example, take the sentence "Sagrada Familia is beautiful" and ask n people "Is Sagrada Familia beautiful?" allowing them to say only "yes" or "no". Imagine 70% of them say "yes". Can you take 0.7 to be the truth degree of our sentence? This causes problems, because "beautiful" in your question was two-valued (yes-no, say, "beautiful$_2$") whereas in your sentence you deal with fuzzy "beautiful" ("beautiful$_f$", say). Strictly speaking, 0.7 can be perfectly taken to be the truth degree of the (fuzzy) sentence "Many people consider Sagrada Familia to be beautiful$_2$" or "The relative frequency of answers 'yes' to the question if Sagrada Familia is beautiful (with the possibility of answers 'yes' or 'no') is HIGH". Compare this with the sentence "The patient is YOUNG": this is the same structure.

(9) *Mathematical expertise assumed.* The reader is assumed to be able

to follow mathematical arguments and to have at least a basic mathematical knowledge of undergraduate level (for example to know most important properties of continuous functions of one or several real arguments). The reader is further assumed to have at least a partial experience with the classical (Boolean) propositional logic; some knowledge of predicate calculus is very helpful. But we do not assume any deep knowledge of mathematical logic or algebra. For the reader's convenience, basic facts about classical logic and algebra are surveyed in the rest of the present chapter, with reference to monographs where the reader may find details if required. (Note that several results of the classical logic summarized below in this chapter will later be proved as corollaries of results on many valued systems.) Note also that there will be three survey sections in other chapters: Chapter 6 Sec. 1 (computational complexity and arithmetical hierarchy), Chapter 8 Sec. 1 (generalized quantifiers in Boolean logic) and Chapter 8 Sec. 2 (modal logics). The experienced reader may just skip these surveys, returning to them if necessary.

(10) *Things to come.* Chapters 2 – 4 deal with fuzzy propositional calculi. Chapter 2 elaborates on the approach to fuzzy propositional calculi, based on the notion of a t-norm (as semantics of conjunction) and the corresponding implication. It turns out that there are three basic t-norms and hence three corresponding logics, named Łukasiewicz logic, Gödel logic and product logic; the first one is presented in Chapter 3, the second and third in Chapter 4. Chapter 5 deals with fuzzy predicate calculi and Chapter 6 with the complexity of fuzzy propositional calculi and problems of undecidability of fuzzy predicate calculi. Further chapters are devoted to an analysis of various points from the agenda of fuzzy logic (Zadeh's generalized modus ponens and fuzzy control in Chapter 7, fuzzy quantifiers and modalities in Chapter 8). Chapter 9 contains miscellanea, among them an analysis of the liar's paradox as well as concluding remarks.

I hope that the book shows the following:

– *Fuzzy logic is neither a poor man's logic nor poor man's probability.* Fuzzy logic (in the narrow sense) is a reasonably deep theory.

– *Fuzzy logic is a logic.* It has its syntax and semantics and notion of consequence. It is a study of consequence.

– *There are various systems of fuzzy logic, not just one.* We have one basic logic (BL) and three of its most important extensions: Łukasiewicz logic, Gödel logic and the product logic.

– *Fuzzy logic in the narrow sense is a beautiful logic, but is also important for applications:* it offers foundations.

1.2. A SURVEY OF BOOLEAN PROPOSITIONAL LOGIC

1.2.1 In Boolean (classical) propositional logic,[1] propositions are true or false. One takes "true" and "false" for two *truth values*. Almost always one identifies the truth value "true" with the number 1 and "false" with the number 0. One works with *propositional variables* $p_1, p_2, \ldots$ (finitely or infinitely many). A *truth evaluation* (or just *evaluation*) is a mapping e assigning to each propositional variable p its truth value $e(p)$. *Formulas* are the formal counterpart of the intuitive notion of a proposition. They are built from propositional variables and two propositional constants $\overline{0}, \overline{1}$ using *connectives*: *implication* $\rightarrow$, *conjunction* $\wedge$ (in Boolean logic this may be alternatively denoted by &), *disjunction* $\vee$, *equivalence* $\equiv$ and *negation* $\neg$. The definition of a formula reads as follows: Propositional variables and propositional constants are formulas; if φ, ψ are formulas then $(\varphi \rightarrow \psi), (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \equiv \psi), \neg\varphi$ are formulas. There are no other formulas.

The *principle of truth functionality* says that the truth-values of compounds of a formula uniquely determine the truth value of a compound formula. This is achieved by defining the *truth functions* of connectives as follows:

	$(-)$
1	0
0	1

$\Rightarrow$	1	0
1	1	0
0	1	1

$\cap$	1	0
1	1	0
0	0	0

$\cup$	1	0
1	1	1
0	1	0

$\Leftrightarrow$	1	0
1	1	0
0	0	1

(Here $(-)$ is the truth function of $\neg$, $\Rightarrow$ of $\rightarrow$ etc.) Using this, each evaluation e extends uniquely to an evaluation of all formulas (denoted also by e) as follows:

$$\begin{aligned}
e(\neg\varphi) &= (-)e(\varphi),\\
e(\varphi \rightarrow \psi) &= (e(\varphi) \Rightarrow e(\psi)),\\
e(\varphi \wedge \psi) &= (e(\varphi) \cap e(\psi)),\\
e(\varphi \vee \psi) &= (e(\varphi) \cup e(\psi)),\\
e(\varphi \equiv \psi) &= (e(\varphi) \Leftrightarrow e(\psi)).
\end{aligned}$$

This may be visualised by constructing a table like this.

[1] There are many monographs on classical mathematical logic, e.g. [189, 48].

p	q	$p \to q$	$(p \to q) \to q$
1	1	1	1
1	0	0	1
0	1	1	1
0	0	1	0

Each row corresponds to one possible evaluation of variables occurring in a given formula, each column gives the truth values of the formula naming the column in dependence of the respective evaluations of variables.

1.2.2 A formula φ is a *tautology* if $e(\varphi) = 1$ for each evaluation φ (thus φ is identically true). Formulas φ, ψ are *semantically equivalent* if $e(\varphi) = e(\psi)$ for each e. Note that φ, ψ are semantically equivalent iff $(\varphi \equiv \psi)$ is a tautology. Also observe that if φ and ψ are semantically equivalent then so are $\neg\varphi$ and $\neg\psi$; also $(\varphi \to \chi)$ and $(\psi \to \chi)$; also $(\chi \to \varphi)$ and $(\chi \to \psi)$; and similarly for other connectives.

We shall present some examples of tautologies. They are well known and easy to verify by constructing the corresponding truth tables.

Lemma 1.2.3 The following formulas are Boolean tautologies, for each φ, ψ:

$$\neg\varphi \equiv (\varphi \to \overline{0}),$$

$$\overline{1} \equiv (\overline{0} \to \overline{0}),$$

$$(\varphi \wedge \psi) \equiv \neg(\varphi \to \neg\psi),$$

$$(\varphi \vee \psi) \equiv ((\varphi \to \psi) \to \psi),$$

$$(\varphi \equiv \psi) \equiv ((\varphi \to \psi) \wedge (\psi \to \varphi)).$$

Corollary 1.2.4 In Boolean propositional calculus, each formula is semantically equivalent to a formula built from propositional variables and the built constant $\overline{0}$ using only the connective $\to$.

Clearly, the equivalences of the previous lemma show how to successively eliminate negation, $\overline{1}$, conjunction, disjunction and equivalence.

Lemma 1.2.5 The following formulas are Boolean tautologies, for each φ, ψ, χ:

$$\varphi \to (\psi \to \varphi) \qquad (Bool1)$$

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \qquad (Bool2)$$

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi) \qquad (Bool3)$$

We take these formulas for axioms of the deductive system $Bool$ of the Boolean logic. Due to 1.2.4 we restrict ourselves to formulas built from propositional variables and $\overline{0}$ using only $\rightarrow$. Other connectives are understood as abbreviations, in particular $\neg\varphi$ is $\varphi \rightarrow \overline{0}$.

Definition 1.2.6 *Axioms* of $Bool$ are the formulas $(Bool1)$, $(Bool2)$, $(Bool3)$ (for any formulas φ, ψ, χ). The *deduction rule* is *modus ponens*: from φ and $\varphi \rightarrow \psi$ infer ψ.

A *proof* in $Bool$ is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that each φ_i either is an axiom of $Bool$ or follows from some preceding $\varphi_j, \varphi_k (j, k < i)$ by modus ponens. A formula is *provable* (notation: $\vdash \varphi$) if it is the last member of a proof in $Bool$.

Lemma 1.2.7 *(Soundness.)* Each formula provable in $Bool$ is a Boolean tautology. (This is because axioms are tautologies and modus ponens preserves tautologicity.)

Definition 1.2.8 A *theory* is a set of formulas, called *special axioms* of the theory. A *proof* in a theory T is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that each φ_i either is an axiom of $Bool$ or is special axiom of T or follows from some preceding φ_j, φ_k by modus ponens. φ is *provable* in T (notation $T \vdash \varphi$) if it is the last member of a proof. An evaluation e is a *model* of T if $e(\varphi) = 1$ for each $\varphi \in T$ (all special axioms are true in e).

Lemma 1.2.9 (*Strong soundness.)* If $T \vdash \varphi$ then φ is true in each model of T (i.e. whenever e is a model of T then $e(\varphi) = 1$).

Theorem 1.2.10 *Deduction theorem.* Let T be a theory, φ, ψ formulas.

$$T \cup \{\varphi\} \vdash \psi \text{ iff } T \vdash (\varphi \rightarrow \psi).$$

Note that $T \cup \{\varphi\} \vdash \psi$ means the existence of a proof that may use all axioms of T and also the axiom φ, and whose last member is ψ. $T \vdash (\varphi \rightarrow \psi)$ means the existence of a proof that may use all axioms of T (but not φ) and whose last member is $(\varphi \rightarrow \psi)$.

Theorem 1.2.11 (*Completeness.*) (1) For each formula φ, φ is provable in $Bool$ iff it is a Boolean tautology.

(2) *(Strong completeness.)* Let T be a theory, φ a formula. $T \vdash \varphi$ iff φ is true in each model of T.

We shall not present here proofs of the deduction and completeness theorem. The reader may find them in most textbooks of logic. Moreover, we obtain the proofs of these theorems on *Bool* as corollaries of theorems on more general logics. Section 3 of Chapter 4 contains details.

*

Lemma 1.2.12 The following formulas are Boolean tautologies:

$$(\varphi \wedge \psi) \equiv (\psi \wedge \varphi) \qquad (\varphi \vee \psi) \equiv (\psi \vee \varphi)$$

$$(\varphi \wedge (\psi \wedge \chi)) \equiv ((\varphi \wedge \psi) \wedge \chi) \qquad (\varphi \vee (\psi \vee \chi)) \equiv ((\varphi \vee \psi) \vee \chi)$$

(commutativity and associativity of conjunction and disjunction).

1.2.13 This shows that we may freely speak about the disjunction $\bigvee_{i=1}^{n} \varphi_i$ (or conjunction $\bigwedge_{i=1}^{n} \varphi_i$) of finitely many formulas $\varphi_1, \ldots, \varphi_n$: their order and bracketting are immaterial. It is technically suitable to allow $n = 0$, i.e. empty conjunction and disjunction: $\bigvee \emptyset$ is $\overline{0}$ and $\bigwedge \emptyset$ is $\overline{1}$. Moreover, $\bigvee_{i=1}^{1} \varphi_i$ is φ_1 and $\bigwedge_{i=1}^{1} \varphi_i$ is also φ_1.

A *literal* is a formula of the form p_i or $\neg p_i$ where p_i is a propositional variable. An *elementary conjunction of length n* is a formula $\bigwedge_{i=1}^{n} L_i$ where L_i are literals and the variable of L_i is p_i. For example if $n = 3$ then $p_1 \wedge \neg p_2 \wedge p_3$ is an elementary conjunction.

Lemma 1.2.14 *(Normal form.)* Each formula not containing any propositional variables except $p_1, \ldots, p_n$ is semantically equivalent to a disjunction of finitely many elementary conjunctions.

The proof is easy: the number of elementary conjunctions in the normal form of φ equals to the number of values 1 in the column of φ in the truth table of φ. If there are no 1's then the normal form is $\bigvee \emptyset$, i.e. $\overline{0}$. For example, for the formula $(\varphi \to \psi) \to \psi$ (i.e. $\varphi \vee \psi$) see the table above; the normal form is

$$(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \vee q).$$

Remark 1.2.15 Dually we may define elementary disjunctions and show that each formula is semantically equivalent to a conjunction of finitely many elementary disjunctions.

1.3. Boolean predicate calculus

In propositional calculus, atomic formulas have no structure, they are just propositional variables. In predicate calculus atomic formulas do have structure: each atom consists of a *predicate* and some *terms* forming the arguments of the predicate. *Object variables* are particular terms; variables may be quantified using *quantifiers* $\forall$ – "for all" and $\exists$ –"there is". Exact definitions follow:

Definition 1.3.1 A *predicate language* consists of a non-empty set of *predicates*, each together with a positive natural number — its *arity* – and a (possibly empty) set of *object constants*. Predicates are mostly denoted by $P, Q, R, \ldots$, constants by $c, d, \ldots$ *Logical symbols* are *object variables* $x, y, \ldots$, the *connective* $\rightarrow$, *truth constants* $\bar{0}, \bar{1}$ and the *quantifier* $\forall$. Other connectives $(\wedge, \vee, \neg, \equiv)$ are defined as in the propositional calculus; the *existential quantifier* $\exists$ is defined as $\neg\forall\neg$. *Terms* are object variables and object constants.

Atomic formulas have the form $P(t_1, \ldots, t_n)$ where P is a predicate of arity n and $t_1, \ldots, t_n$ are terms. If φ, ψ are formulas and x is an object variable then $\varphi \rightarrow \psi$, $(\forall x)\psi$, $\bar{0}$, $\bar{1}$ are formulas; each formula results from atomic formulas by iterated use of this rule.

Let $\mathcal{J}$ be a predicate language. A *structure* $\mathbf{M} = \langle M, (r_P)_P, (m_c)_c \rangle$ *for* $\mathcal{J}$ has a non-empty *domain* M, for each n-ary predicate P an n-ary relation $r_P \subseteq M^n$ on M (associating to each n-tuple $(m_1, \ldots, m_n)$ of elements of M) and for each object constant c, m_c is an element of M).

Example 1.3.2 $\mathcal{J}$ has one binary predicate D (read: divides) and one object constant $\underline{1}$. $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$; $m_{\underline{1}} = 1$ and r_D is given by the following matrix.

	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	0	1	0	1	0	1	0	1	0
3	0	0	1	0	0	1	0	0	1
4	0	0	0	1	0	0	0	1	0
5	0	0	0	0	1	0	0	0	0
6	0	0	0	0	0	1	0	0	0
7	0	0	0	0	0	0	1	0	0
8	0	0	0	0	0	0	0	1	0
9	0	0	0	0	0	0	0	0	1

Definition 1.3.3 Let $\mathcal{J}$ be a predicate language and $\mathbf{M}$ a structure for $\mathcal{J}$. An $\mathbf{M}$-*evaluation* of *object variables* is a mapping v assigning to each object variable x an element $v(x) \in M$. Let $v,\ v'$ be two evaluations. $v \equiv_x v'$ means that $v(y) = v'(y)$ for each variable y distinct from x.

The value of a term given by M, v is defined as follows: $\|x\|_{\mathbf{M},v} = v(x)$; $\|c\|_{\mathbf{M},v} = m_c$. We define the truth value $\|\varphi\|_{\mathbf{M},v}$ of a formula. $\Rightarrow$ denotes the truth function of the implication.

$$\|P(t_1,\dots,t_n)\|_{\mathbf{M},v} = r_P(\|t_1\|_{\mathbf{M},v},\dots,\|t_n\|_{\mathbf{M},v});$$
$$\|\varphi \to \psi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M},v} \Rightarrow \|\psi\|_{\mathbf{M},v};$$
$$\|\bar{0}\|_{\mathbf{M},v} = 0; \quad \|\bar{1}\|_{\mathbf{M},v} = 1;$$
$$\|(\forall x)\varphi\|_{\mathbf{M},v} = \min\{\|\varphi\|_{\mathbf{M},v'} | v \equiv_x v'\}.$$

The last line obviously means that $\|(\forall x)\varphi\|_{\mathbf{M},v}$ is 1 iff for all $v' \equiv_x v$, $\|\varphi\|_{\mathbf{M},v'}$ is 1; otherwise $\|(\forall x)\varphi\|_{\mathbf{M},v}$ is 0.

Definition 1.3.4 *Free and bound variables* of a formula are defined as follows:

- No variable is free/bound in a truth constant.
- If φ is atomic, say $P(t_1,\dots,t_n)$, then x is free in φ if x is one of $t_1,\dots,t_n$; no variable is bound in φ.
- A variable x is free in $\varphi \to \psi$ if it is free in φ or free in ψ; x is bound in $\varphi \to \psi$ if it is bound in φ or bound in ψ.
- A variable x is bound in $(\forall x)\varphi$ and is not free in $(\forall x)\varphi$. For any variable y distinct from x, y is free/bound in $(\forall x)\varphi$ if it is free/bound in φ.

Remark 1.3.5 (1) For example, take $(\forall x)(\exists y)P(x,y) \rightarrow (\exists y)P(x,y)$. Here y is bound and not free, but x is both free and bound.

(2) Observe that if x is not free in φ then the value $||\varphi||_{\mathbf{M},v}$ does not depend on $v(x)$, i.e. if $v \equiv_x v'$ then $||\varphi||_{\mathbf{M},v} = ||\varphi||_{\mathbf{M},v'}$.This justifies the following convention: If φ is a formula with free variables $x, \ldots, y$, $\mathbf{M}$ is a model and $a, \ldots, b \in M$ then $||\varphi||_{\mathbf{M}}[a, \ldots, b]$ stands for $||\varphi||_{\mathbf{M},v}$ for any v such that $v(x) = a, \ldots, v(y) = b$.

Definition 1.3.6 We define the result of *substitution* of a term t for a variable x in φ, denoted by $\varphi(x/t)$.

- If φ is atomic then $\varphi(x/t)$ results from φ by replacing all occurrences of x in φ, by t. $\overline{0}(x/t) = \overline{0}$ (and similarly for $\overline{1}$).
- $(\varphi \rightarrow \psi)(x/t)$ is $\varphi(x/t) \rightarrow \psi(x/t)$
- $[(\forall x)\varphi](x/t)$ is $(\forall x)\varphi$ (no change); for each variable y distinct from x, $[(\forall y)\varphi](x/t)$ is $(\forall y)[\varphi(x/t)]$. (For example, substituting c for x into the formula in 1.3.5 (1) we get $(\forall x)(\exists y)P(x,y) \rightarrow P(c,y)$.)

Similarly we may define the (intuitively clear) notion of a subformula:

- Each formula φ is a subformula of itself.
- If φ is a subformula of ψ or of χ then φ is a subformula of $\psi \rightarrow \chi$. If φ is a subformula of ψ then φ is a subformula of $(\forall x)\psi$ (x any variable).

We may distinguish several *occurrences* of a variable x in a formula φ, some being free in φ and some bound. In our example $(\forall x)(\exists y)(P(x,y) \rightarrow P(x,y))$, the variable x occurs three times; the first two occurences are bound and the third one is free.[2]

When substituting a variable y for x into φ, the unwanted situation may happen that a free occurrence of x in φ becomes a bound occurrence of y. For example, take $(\exists y)P(x,y)$ (i.e. $(\forall x)(P(x,y) \rightarrow \overline{0}) \rightarrow \overline{0}$) and observe that $[(\exists y)P(x,y)](x/y)$ is $(\exists y)P(y,y)$. To eliminate such "pathologies" one makes the following definition:

Definition 1.3.7 A variable y is *substitutable for x in φ* if no subformula of φ of the form $(\forall x)\psi$ contains an occurrence of x free in φ. A constant is substitutable for any variable in any formula.

Definition 1.3.8 (1) Let φ be a formula of a language $\mathcal{J}$ and let $\mathbf{M}$ be a structure for $\mathcal{J}$. The *truth value* of φ in $\mathbf{M}$ is

$$||\varphi||_{\mathbf{M}} = \min\{||\varphi||_{\mathbf{M},v} \mid v \ \ \mathbf{M}\text{-evaluation}\}.$$

(Thus $||\varphi||_{\mathbf{M}}$ is 1 iff $||\varphi||_{\mathbf{M},v} = 1$ for all evaluations v, otherwise it is 0.)

[2] To get a formal definition it is useful to think of formulas as being finite sequences of symbols.

(2) A formula φ of a language $\mathcal{J}$ is a *tautology* if $\|\varphi\|_{\mathbf{M}} = 1$ for each structure **M**, i.e. $\|\varphi\|_{\mathbf{M},v} = 1$ for each structure **M** and each **M**-valuation of object variables.

Definition 1.3.9 The *logical axioms* of the Boolean predicate calculus are the axioms of the Boolean propositional calculus (with the present notion of a formula) plus the following *logical axioms on quantifiers*:

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$)

($\forall$2) $(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ (x not free in ν)

The *deduction rules* are *modus ponens* (see above) and *generalization* (from φ infer $(\forall x)\varphi$).

This completes the definition of the Boolean predicate calculus; we shall denote it by $Bool\forall$.

Given this, the notions of proof, provability, theory, proof/provability in a theory over $Bool\forall$ are defined in the obvious way; in particular, a proof in a theory T is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that for each i, φ_i is a logical axiom of $Bool\forall$ or an axiom of T or φ_i follows from some preceding members of the sequence by one of the deduction rules.

Definition 1.3.10 Let T be a theory over $Bool\forall$, let **M** be a structure for the language of T. **M** is a *model* of T if all axioms of T are true in **M**, i.e. $\|\varphi\|_M = 1$ in each $\varphi \in T$.

Theorem 1.3.11 (*Strong completeness*) (1) For each formula φ, $Bool\forall$ proves φ iff φ is a tautology.

(2) For each φ and each theory T, $T \vdash_{Bool} \varphi$ (i.e. T proves φ over $Bool\forall$ iff $\|\varphi\|_{\mathbf{M}} = 1$ for each model of T.

This is Gödel's celebrated completeness theorem; its proof can be found in any textbook of logic and we get it (by a big detour) as a corollary of a more general completeness result (cf. 5.2.10).

Definition 1.3.12 A theory T is *contradictory* if for some φ, T proves φ and T proves $\neg\varphi$. T is *consistent* if it is not contradictory.

Corollary 1.3.13 T is consistent iff T has a model.

We shall now survey some examples of theories and their models. The examples concern the notions of equality and order.

Example 1.3.14 The (Boolean, first-order) *theory of preorder* has one binary predicate $\leq$ and the following axioms $x \geq x$ (reflexivity), $x \leq y \to (y \leq z \ \to x \leq z)$ (transitivity).

The theory of *linear preorder* has in addition the axiom $x \leq y \vee y \leq x$ (dichotomy).

Models of the theory of (linearly) ordered sets are called (linearly) ordered sets. Each such model consists of a set M and a binary relation $r_{\leq}$ satisfying the axioms. We often take the liberty of writing $\leq$ instead of $r_{\leq}$; thus the fact that $(M, \leq)$ is a preorder means that $a \leq a$ for each $a \in M$ and, for $a, b, c \in M$, whenever $a \leq b$ and $b \leq c$ then $a \leq c$. Linearity means that $a \leq b$ or $b \leq a$ for each $a, b \in M$.

Definition 1.3.15 The *axioms of equality* for a binary predicate = are

$$x = x \quad (reflexivity)$$

$$x = y \to (y = z \to x = z) \quad (transitivity)$$

$$x = y \to y = x \quad (symmetry).$$

(M, r) is a model of these axioms iff r is an *equivalence* on M, i.e. a reflexive, symmetric and transitive relation.

Let J be a predicate language containing the binary predicate =. The *axioms of equality* for =, J are the above axioms (reflexivity, transitivity, symmetry) plus the following *congruence axioms*: for each n-ary predicate P (distinct from =), $(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \to (P(x_1, \ldots, x_n) \equiv P(y_1, \ldots, y_n))$.

A theory T with the language J is called a *theory with the equality* = if T proves all axioms of equality for =,J.

A model **M** of T has *absolute equality* if it interprets = absolutely, i.e. $r_=$ is $\langle (a, a) | a \in M \rangle$.

One easily shows that, T being a theory with equality =, and $\varphi(x_1, \ldots, x_n)$ a formula,

$$T \vdash \bigwedge_{i=1}^{n} (x_i = y_i) \to (\varphi(x_1, \ldots, x_n) \equiv \varphi(y_1, \ldots, y_n)).$$

Example 1.3.16 *Continuing 1.3.14.* In the theory of a preorder we may *define* an equality predicate = by

$$x = y \equiv (x \leq y \wedge y \leq x).$$

With this =, axioms of equality for = and $\leq$ are easily provable. Futhermore, we may define *strict preorder* $\leq$ by

$$x < y \equiv (x \leq y \wedge y \not\leq x)$$

(where $y \not\leq x$ stands for $\neg(y \leq x)$).

A model $\mathbf{M}$ of the theory of preorder is an *ordered set* if $\mathbf{M}$ interprets the equality = above as identity, i.e. whenever $a \leq b$ and $b \leq a$ then a coincides with b. Similarly for a *linearly ordered set.*

1.4. FUNCTION SYMBOLS; VARIETIES OF ALGEBRAS

Here we survey Boolean predicate logic with function symbols (like +). In this logic we have formulas like $x + y = y + x$. On the one hand, this logic fully reduces to the logic *Bool*$\forall$ of the previous section; on the other hand, formulas like the above (equalities of terms) are extremely useful in defining various useful classes of algebras.

Definition 1.4.1 A *predicate language $\mathcal{I}$ with function symbols* consists of a non-empty set of predicates $P, Q, \ldots$ (each with its arity, a (possibly empty) set of object constants $c, d, \ldots$ and a (non-empty) set of *function symbols* $F, G, \ldots$, each having a positive natural number as its *arity*. *Terms* are defined as following object variables and object constants are terms; if F is an n-ary function symbol and $t_1, \ldots, t_n$ are terms then $F(t_1, \ldots, t_n)$ is a term; there are no other terms. Formulas are defined exactly as in Definition 1.3.1. Free and bound variables are defined similarly as in 1.3.4, only saying: if y is atomic, $P(t_1, \ldots t_n)$, then x is free in φ if it *occurs* in one $t_1, \ldots, t_n$. A *structure* for $\mathcal{I}$ has the form

$$\mathbf{M} = \langle M, (r_p)_{p \text{ predicate}}, (m_c)_{c \text{ constant}}, (f_F)_{F \text{ funct. symb.}} \rangle,$$

where each f_F is an n-ary operation on M (F n-ary, i.e. a mapping $f_F : M^n \to M$).

Given M and an evaluation v of object variables, the value $||t||_{\mathbf{M},v}$ of t given by $\mathbf{M}, v$ is defined as follows:
$||x||_{\mathbf{M},v} = v(x)$ for each variable x,
$||c||_{\mathbf{M},v} = m_c$ for each constant c,
$||F(t_1, \ldots, t_n)||_{\mathbf{M},v} = f_F(||t_1||_{\mathbf{M},v}, \ldots ||t_n||_{\mathbf{M},v})$.

Having this, the value $||\varphi||_{\mathbf{M},v}$ of a formula φ is defined exactly as in 1.3.3.

Remark 1.4.2 There are some useful *façons de parler*:

First if F is binary then we often write xFy instead of $F(x, y)$. Second, observing that if $t(x_1, \ldots, x_n)$ is a term containing no variables besides

$x_1, \dots, x_n$ then the value $\|t\|_{\mathbf{M},v}$ depends only on the values $v(x_1), \dots, v(x_n)$ of these variables, we may write, for $a_1, \dots, a_n \in M$,

$$t(a_1, \dots, a_n)$$

to mean the value of t in $\mathbf{M}$ for any valuation with $v(x_i) = a_i \quad (i = 1, \dots, m)$. For example if t is $(x + y) + z$, M is clear from the context and $a, b, c \in M$ then $(a + b) + c$ obviously means the value of t in M for $v(x) = a, \quad v(y) = b$ and $v(z) = c$.

Definition 1.4.3 Logical axioms are as in 1.3.9, with the definition of substitutability modified as follows:
A term t is substitutable for x into φ if for each variable y occurring in t, no subformula of φ of the form $(\forall y)\psi$ contains an occurrence of x free in φ.

Remark 1.4.4 (1) Thus e.g. $x + y$ is not substitutable for x into $(\exists y)P(x, y)$.
(2) Note that for each φ, x is substitutable for x in φ.

Definition 1.4.5 The *equality axioms* for a predicate = and a language $\mathcal{I}$ with function symbols are those of 1.3.15 (reflexivity, symmetry, transitivity, congruence for predicates) plus the following congruence axiom for each n-ary function symbol F of $\mathcal{I}$:

$$(\bigwedge_{i=1}^{n} x_i = y_i) \longrightarrow F(x_1, \dots, x_n) = F(y_1, \dots, y_n).$$

A theory T in the language $\mathcal{I}$ (over *Bool*$\forall$) is a *theory with the equality* = if it proves all equality axioms for =, $\mathcal{I}$.

Example 1.4.6 *The theory of semigroups.* The language consists of the binary predicate = and the binary function symbol $*$. Axioms are equality axioms for $=, *$ plus the axiom of *associativity*:

$$x * (y * z) = (x * y) * z.$$

Example 1.4.7 *Arithmetic.* Language: binary predicates $=$, $\leq$, constant $\underline{0}$, unary function symbol S (successor), binary function symbols $+, *$ (addition, multiplication) axioms of *Robinson arithmetic* QA: equality axioms plus

$$Sx \neq \underline{0}$$

$$Sx = Sy \rightarrow x = y$$

$$y \neq \underline{0} \rightarrow (\exists x)(y = Sx)$$

$$x + Sy = S(x + y)$$
$$x + \underline{0} = x$$
$$x * Sy = (x * y) + x$$
$$x * \underline{0} = \underline{0}$$
$$x \leq y \equiv (\exists z)(z + x = y)$$

Peano arithmetic PA: Robinson arithmetic with the additional axiom scheme of induction for each $\varphi(x)$:

$$(\varphi(x/0) \wedge (\forall x)(\varphi(x) \to \varphi(x/Sx))) \to (\forall x)\varphi(x).$$

The *standard model* of both PA and QA is the structure

$$\mathbf{N} = \langle N, 0_N, S_N, +_N, *_N, \leq_N \rangle$$

where N is the set of natural numbers, 0_N is zero, $+_N$ and $*_N$ are addition and multiplication of natural numbers, $S_N(m) = m + 1$ for each m and $\leq_N$ is the natural ordering of N (e.g. $3 <_N 7$, etc.). = is interpreted absolutely. Using the convention above we write e.g. $+, *$ instead of $+_N$,$*_N$,etc.[3]

$*$

Next we shall indicate that function symbols are redundant and may be replaced by predicates. (Therefore we have good reasons to develop the fuzzy predicate logic as a logic without function symbols.) On the other hand, using function symbols is useful in particular when studying classes of algebras; this will also be shown in this section.

Definition 1.4.8 Let $\mathcal{I}$ be a predicate language with function symbols and equality =. A formula φ is without *compound terms* if each function symbol F (n-ary) occurring in φ occurs in the context

$$y = F(t_1, \ldots, t_n)$$

where y is a variable and t_i are variables or constants.

Remark 1.4.9 (1) One can define a transformation assigning to each φ a formula $\varphi^{\#}$ without compound terms and for each theory T over $\mathcal{I}$ with equality=, $T \vdash \varphi \equiv \varphi^{\#}$. For example let φ be $(x + y) + z = x + (y + z)$; then $\varphi^{\#}$ may be

$$(\exists u, v, w, q)(u = x + y \wedge v = u + z \wedge w = y + z \wedge q = x + w \wedge v = q).$$

[3] For more information on metamathematics of arithmetic see [91].

Note that in T this formula is equivalent to

$$(\forall u, v, w, q)((u = x + y \wedge v = u + z \wedge w = y + z \wedge q = x + w) \rightarrow v = q).$$

(2) Having this one may construct a new language $\mathcal{I}'$ replacing each n-ary function symbol F of $\mathcal{I}$ by a new (n+1)-ary predicate F'; for each T-formula φ let $(\varphi^{\#})'$ be the formula resulting from $\varphi^{\#}$ by replacing each subformula of the form $y = F(t_1, \dots, t_n)$ (t_i variables or constants) by $F'(t_1, \dots, t_n, y)$. Let T' be the theory over $\mathcal{I}'$ with the following axioms:

- equality axioms for = and $\mathcal{I}'$,
- $(\varphi^{\#})'$ for each axiom φ of T distinct from the equality axioms
- functionality axioms for the new predicates:

$$(F'(x_1, \dots, x_n, y) \wedge F'(x_1, \dots, x_n, z)) \rightarrow y = z.$$

(3) It is not difficult to show that T' is equivalent to T in the following sense: each T'-formula has the form $(\varphi^{\#})'$ for some T-formula φ; and for each T-formula φ,

$$T \vdash \varphi \text{ iff } T' \vdash (\varphi^{\#})'.$$

Example 1.4.10 The following is the function-free counterpart of Robinson's arithmetic QA: language - predicates $=, \leq, S$ (binary), A, B (ternary), constant $\underline{0}$. Axioms: equality axioms for this language, functionality axioms for S, A, B, and

$$S(x, y) \rightarrow y \neq \underline{0}$$

$$(S(x, u) \wedge S(y, u)) \rightarrow x = y$$

$$y \neq \underline{0} \rightarrow (\exists x) S(x, y)$$

$$(S(y, u) \wedge A(x, u, v) \wedge A(x, y, z) \wedge S(z, w)) \rightarrow v = w$$

$$A(x, \underline{0}, x)$$

$$((S(y, u) \wedge B(x, u, v) \wedge B(x, y, z) \wedge A(z, x, w)) \rightarrow v = w$$

$$B(x, \underline{0}, \underline{0})$$

$$x \leq y \equiv (\exists z) A(z, x, y).$$

PA – the induction schema becomes

$$[\varphi(x/\underline{0}) \wedge (\forall x, y)((S(x, y) \wedge \varphi(x)) \rightarrow \varphi(x/y))] \rightarrow (\forall x)\varphi(x).$$

We shall use these forms of arithmetic occasionally in the book.

*

1.4.11 *Algebras* are structures of the form $\langle M, f_1, \ldots, f_n \rangle$ where f_i are operations on M. Such an algebra is naturally a structure for a predicate language $\mathcal{I}$ having the equality predicate = (interpreted absolutely) and function symbols $F_1, \ldots F_n$ of corresponding arities. Many important classes of algebras are defined as classes of all models of a theory T over $\mathcal{I}$ with the equality = (see the example 1.4.6 – semigroups). In particular, the axioms of F (other than the equality axioms) may be just some *atomic* formulas, i.e. *identities* $t = s$ (for some terms t, s). This leads us to the following definition:

Definition 1.4.12 Let $\mathcal{I}$ be a language $(=, F_1, \ldots, F_n)$ with n function, let $\mathcal{K}$ be a class of structures for $\mathcal{I}$ (interpreting = as identity). $\mathcal{K}$ is a *variety* if there is a set T of identities (atomic formulas of $\mathcal{I}$) such that $\mathcal{K}$ is the class of all structures $\mathbf{M}$ for $\mathcal{I}$ (interpreting = identically) such that all identities from T are true in $\mathbf{M}$.

Remark 1.4.13 Thus semigroups form a variety. We shall investigate the varieties of lattices, Boolean algebras and Abelian groups in subsequent sections.

Definition 1.4.14 Let $\mathbf{M}_1 = \langle M_1, f_1, \ldots, f_n \rangle$, $\mathbf{M}_2 = \langle g_1, \ldots, g_n \rangle$ be structures for the language $\mathcal{I}$ as in 1.4.12 (with absolute identity).

(1) $\mathbf{M}_1$ is a *subalgebra* of $\mathbf{M}_2$ if $M_1 \subseteq M_2$ and for each $i = 1, \ldots, n$, f_i is the restriction of g_i to $M_1^{ar(f_i)}$ (hence M_1 is closed under all g_i's).

(2) $\mathbf{M}_1$ is a *homomorphic image* of $\mathbf{M}_2$ if there is a mapping h of M_2 onto M_1 commuting with the operations, i.e. for each $i = 1, \ldots, n$, and any $a_1, \ldots, a_k \in M$, $h(f_i(a_1, \ldots, a_k) = g_i(h(a_1), \ldots, h(a_k))$ (where k is the arity of f_i and g_i).

(3) Now let $I \neq \emptyset$ be a set and for each $\lambda \in I$ let $\mathbf{M}_\lambda = \langle M_\lambda, f_{1\lambda}, \ldots, f_{n\lambda} \rangle$ be a structure for $\mathcal{I}$. The *direct product* $\prod_{\lambda \in I} \mathbf{M}_\lambda$ is the algebra $\mathbf{M} = \langle M, f_1, \ldots, f_n \rangle$ where M is the set of all functions a whose domain is I and for each $\lambda \in I$, $a(\lambda) \in M_\lambda$ (selectors). The operations are defined coordinatewise, i.e. for $a_1, \ldots, a_n \in M$ and f_i k-ary, $f_i(a_1, \ldots, a_k) = b$ iff for each $\lambda \in I, b(\lambda) = f_{i\lambda}(a_1(x), \ldots, a_n(x))$. Note that in general the non-emptiness of $\mathbf{M}$ follows from the set theoretical axiom of choice.

(4) An algebra $\mathbf{M}$ is a *subdirect product* of a system $\{\mathbf{M}_\lambda | \lambda \in I\}$ of algebras if $\mathbf{M}$ is a subalgebra of $\prod_{\lambda \in I} \mathbf{M}_\lambda$.

Theorem 1.4.15 *Birkhoff's theorem* (see e.g. [69]). A class $\mathcal{K}$ of structures for $\mathcal{I}$ is a variety iff it is closed under subalgebras, homomorphic images and direct products, i.e. $\mathcal{K}$ contains with each $\mathbf{M}$ all its subalgebras and homomorphic images and will each system $\langle \mathbf{M}_\lambda | \lambda \in I\rangle$ of elements of $\mathcal{K}$ contains its direct product $\prod_{\lambda \in I} \mathbf{M}_\lambda$.

Note that it is easy to prove that a variety is closed under these three things; the converse is much deeper but we shall not need this difficult part.

1.5. Lattices and Boolean algebras

Here we present main notions concerning lattices and Boolean algebras and survey their most important properties. We shall give (almost) no proofs; the reader may consult Chapter XIV of the monograph [128] if necessary.

Definition 1.5.1 The language of lattices has two binary function symbols $\cap$ and $\cup$ (meet and joint) plus equality =, (interpreted as identity). An algebra $\mathbf{L} = \langle L, \cap, \cup\rangle$ is a *lattice* if the following identities are true in $\mathbf{L}$:

$$
\begin{array}{rrl}
x \cap x = x & x \cup x = x & \text{(idempotence)} \\
x \cap y = y \cap x & x \cup y = y \cup x & \text{(commutativity)} \\
x \cap (y \cap z) = (x \cap y) \cap z & x \cup (y \cup z) = (x \cup y) \cup z & \text{(associativity)} \\
x \cap (x \cup y) = x & x \cup (x \cap y) = x & \text{(absorption)}
\end{array}
$$

Remark 1.5.2 (1) Evidently, lattices form a variety.

(2) To get an example of a lattice take a linearly ordered set $\langle L, \leq\rangle$ and put $x \cap y = \min(x, y)$, $x \cup y = \max(x, y)$. Verify that this is a lattice.

(3) Using the easy part of Birkhoff's theorem, produce other examples as direct products of some lattices.

(4) Each set X gives us the lattice whose domain L is the power set (set of all subsets) of X, $\cap$ is set theoretic intersection and $\cup$ is set theoretic union.

Lemma 1.5.3 Let $\mathbf{L} = \langle L, \cap, \cup\rangle$ be a lattice.

(1) For any $a, b \in L$, $a \cap b = a$ iff $a \cup b = b$.

(2) Define $x \leq y$ to mean $x \cap y = x$ (equivalently, $x \cup y = y$).
Then $\langle L, \leq\rangle$ is an ordered set, $x \cap y = x$ is the infimum of x, y (greatest lower bound) and $x \cup y$ is the supremum of x, y, i.e. the following is true in $\mathbf{L}$:

$$x \cap y \leq x, \quad x \cap y \leq y, \quad (\forall z)((z \leq x \wedge z \leq y) \to z \leq x \cap y);$$

$$x \leq x \cup y, \quad y \leq x \cup y, \quad (\forall z)((x \leq z \wedge y \leq z) \to x \cup y \leq z).$$

(3) On the other hand, if $\langle L, \leq\rangle$ is an ordered set in which each pair of elements has its supremum and its infimum then putting $x \cap y = \inf(x, y)$, $x \cup y = \sup(x, y)$ we get an algebra $\langle L, \cap, \cup\rangle$ which is a lattice.

Definition 1.5.4 For each lattice $\mathbf{L} = \langle L, \cap, \cup \rangle$, the ordering $\leq$ from the preceding lemma is called the *ordering determined by* $\mathbf{L}$ or just *the ordering of* $\mathbf{L}$. We shall identify $\langle L, \cap, \cup \rangle$ with $\langle L, \cap, \cup, \leq \rangle$.

Lemma 1.5.5 In each lattice $\mathbf{L}$, $\cap$ and $\cup$ are non-decreasing w.r.t. $\leq$, i.e. the following is true in $\mathbf{L}$:

$$(x_1 \leq x_2 \wedge y_1 \leq y_2) \rightarrow (x_1 \cap y_1 \leq x_2 \cap y_2),$$
$$(x_1 \leq x_2 \wedge y_1 \leq y_2) \rightarrow (x_1 \cup y_1 \leq x_2 \cup y_2).$$

Definition 1.5.6 A lattice $\mathbf{L}$ is *distributive* if the identity

$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$$

is true in $\mathbf{L}$.

Lemma 1.5.7 $\mathbf{L}$ is distributive iff the identity

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$$

is true in $\mathbf{L}$; in other words, the two forms of the distributivity axiom (distributivity of $\cap$ w.r.t. $\cup$, distributivity of $\cup$ w.r.t.$\cap$) are equivalent in each lattice.

Remark 1.5.8 (1) If $\mathbf{L}$ is linearly ordered (i.e. the ordering $\leq$ of $\mathbf{L}$ is linear) then $\mathbf{L}$ is distributive. Note that the linearly ordered lattice [0,1] (real unit interval with min, max and the standard order of reals) will play a central role in our investigations.

(2) Note that $\mathbf{L}$ is linearly ordered iff the formula $(x \cap y = x) \vee (x \cap y = y)$ is true in $\mathbf{L}$. This is a disjunction of two atomic formulas. It is easy to prove that linearity cannot be expressed by a system of atomic formulas (identities), i.e. the class of linearly ordered lattices is not a variety. To show this just observe that the direct product of two linearly ordered lattices need not be linearly ordered.

Definition 1.5.9 The *language of Boolean algebras* is the extension of the language $\cap, \cup, =$ of lattices by two constants 0,1 (least and greatest element, bottom and top) and one unary operation $-$ of complement. An algebra $\mathbf{L} = \langle L, \cap, \cup, 0, 1, - \rangle$ is a *Boolean algebra* if $\langle L, \cap, \cup \rangle$ is a distributive lattice, 0 is the least element of $\mathbf{L}$, 1 its greatest element and $-$ is complementation i.e. in addition to the identities defining distributive lattices, the following are true in $\mathbf{L}$:

$$x \cap 0 = 0, \quad x \cup 1 = 1,$$
$$x \cup -x = 1,$$
$$x \cap -x = 0.$$

Remark 1.5.10 (1) Thus Boolean algebras form a variety.

(2) The most obvious example is that of all subsets of a given set (of 1.5.2(4)): Here 0 is $\emptyset$ (empty subset) 1 is X (the whole set) and, for $Y \subseteq X$, $-Y$ is $X - Y$ (set-theoretical complement).

(3) *Lindenbaum algebra of classes of formulas.* Let T be a theory over $Bool$ (or over $Bool\forall$). For each formula φ, let $[\varphi]_T$ be the set of all ψ such that $T \vdash \varphi \equiv \psi$. Let L be the set of all $[\varphi]_T$ (classes of equivalent formulas) and define $0 = [\bar{0}]_T$, $1 = [\bar{1}]_T$, $[\varphi]_T \cap [\psi]_T = [\varphi \wedge \psi]_T$, $[\varphi]_T \cup [\psi]_T = [\varphi \vee \psi]_T$, $-[\varphi]_T = [\neg\varphi]_T$. These are sound definitions (not depending on the choice of the representative φ of a class $[\varphi]_T$) and this $\langle L, \cap, \cup, -, 0, 1\rangle$ is a Boolean algebra. This method of algebras of classes of equivalent formulas is very fruitful also in other logics (but then one gets different algebras, not Boolean ones).

Definition 1.5.11 Let $\mathbf{L}$ be a lattice and $X \subseteq L$ (possibly infinite subset). The *supremum* of X is an element $a \in L$ which is the least upper bound of X, i.e. $b \leq a$ for all $b \in X$ and whenever $b \leq c$ for all $b \in X$ then $a \leq c$. We write $\sup X$ for the supremum of X. Similarly, a is the infimum of X if it is the greatest lower bound.

Remark 1.5.12 It is possible that X has no supremum; but if it exists it is uniquely determined. Similarly for infimum. A lattice $\mathbf{L}$ is *complete* if each $X \subseteq L$ has its sup and inf. Note that the real interval [0,1] is a complete lattice.

1.6. Ordered Abelian Groups

We already defined *semigroups* in 1.4.6 and *linearly ordered sets* in 1.3.14, 1.3.16. We shall investigate structures having both a semigroup operation and a linear order, related in some way. The structures of ultimate interest in this section are ordered Abelian groups. We shall present their definition and basic properties. They will play an important role in the characterization of algebras of some logics. Everything except results related to the Gurevich-Kokorin theorem can be found in Fuchs's monograph [54]. We shall give references concerning the last named theorem later.

Definition 1.6.1 (1) A semigroup $\langle G, *\rangle$ is *Abelian* (or *commutative*) if it satisfies the axiom of commutativity $x * y = y * x$ (for all x, y).

Remark 1.6.2 It is usual to denote the operation of an Abelian semigroup by + (or another symbol resembling addition) rather than $*$. Thus from now on, we shall denote Abelian semigroups by $\langle G, +\rangle$ or $\langle G, +_G\rangle$ and similar.

Definition 1.6.3 (1) An element $o \in G$ is the *zero* element of the Abelian semigroup $\langle G, + \rangle$ if $x + o = x$ for each $x \in G$. (Evidently, each Abelian semigroup has at most one zero element.)

(2) Let $\langle G, + \rangle$ be an Abelian semigroup with a zero element o, let $x, y \in G$. The element y is the *inverse* of x if $x + y = o$. (Evidently, each x has at most one inverse.)

(3) $\langle G, + \rangle$ satisfies *cancellation* if for each $x, y, z \in G$, $x + z = y + z$ implies $x = y$. (Thus the formula $(\forall x, y, z)(x + z = y + z \rightarrow x = y)$ is true in $\langle G, + \rangle$.)

Definition 1.6.4 (1) A *linearly ordered Abelian semigroup* is a structure $\langle G, +, \leq \rangle$ such that $\langle G, + \rangle$ is an Abelian semigroup, $\langle G, \leq \rangle$ is a linearly ordered set and the following *monotonicity axiom* is true in $\langle G, +, \leq \rangle$:
$x \leq y \rightarrow (x + z \leq y + z)$.

(2) An *Abelian group* is a structure $\langle G, +, 0, - \rangle$ (where $+$ is a binary operation, $0 \in G$ and $-$ is a unary operation) such that $\langle G, + \rangle$ is an Abelian semigroup, 0 is its zero element and $-$ is the operation of inverse, i.e. $x + -x = 0$ for each x.

(3) A *linearly ordered Abelian group* (in brief, an o-group) is a structure $\langle G, +, 0, -, \leq \rangle$ such that $\langle G, +, 0, - \rangle$ is an Abelian group and $\langle G, +, \leq \rangle$ is a linearly ordered Abelian semigroup.

Remark 1.6.5 Thus $\langle G, +, 0, -, \leq \rangle$ is an o-group iff the following axioms are true in if:

$$x + (y + z) = (x + y) + z$$
$$x + y = y + x$$
$$x + \underline{0} = x$$
$$x + -x = \underline{0}$$
$$x \leq y \vee y \leq x$$
$$(x \leq y \wedge y \leq z) \rightarrow x \leq z$$
$$(x \leq y \wedge y \leq x) \rightarrow x = y$$
$$x \leq y \rightarrow (x + z \leq y + z)$$

The first four axioms are axioms of an Abelian group, the following three are axioms of linear preorder, the last is the monotonicity axiom (= is interpreted absolutely).

Since the zero element 0 and the operation of inverse are definable from the group addition + we often present an o-group brifely as $\langle G, +, \leq \rangle$.

Example 1.6.6 Let Re be the set of all real numbers, + the addition of reals, $*$ the multiplication of reals. Let N, Z denote the set of natural numbers and integers respectively. (We assume $N \subseteq Z \subseteq Re$ and use $+, *$ also for addition and multiplication restricted to N or Z.) Let $\leq$ be the standard linear order on Re $(N\ ,Z)$.

(i) $\mathbf{N} = (N, +, \leq)$ is an ordered Abelian semigroup with zero element 0 but no non-zero element has an inverse.

(ii) $\mathbf{Z} = (Z, +, \leq)$ and $\mathbf{Re} = (Re, +, \leq)$ are o-groups; Z is a subgroup of $\mathbf{Re}$ (in the obvious sense). These groups are called the *additive o-group of integers* and *reals* respectively.

(iii) $(N, *, \leq)$ is a linearly ordered Abelian semigroup whose zero element is 1; no element different from 1 has an inverse.

Lemma 1.6.7 (1) If $\mathbf{G} = \langle G, +, \leq \rangle$ is a linearly ordered Abelian semigroup with cancellation then $\mathbf{G}$ satisfies strict monotonicity:
$x + z < y + z$ implies $x < y$.

(2) Each o-group $\mathbf{G} = \langle G, +, 0, -, \leq \rangle$ satisfies cancellation.

(3) Each o-group satisfies *subtraction*: for each x, y there is a unique z such that $x + z = y$.
(Very easy proofs)

Definition 1.6.8 The *positive cone* of an o-group $\mathbf{G} = \langle G, +, \leq \rangle$ is the set $\{x \in G | 0 \leq x\}$, endowed with the restrictions of $+, \leq$.

Lemma 1.6.9 A linearly ordered Abelian semigroup $\mathbf{S} = \langle S, +, 0, \leq \rangle$ is the positive cone of an o-group iff it satisfies the following:

- positive subtraction: for each $x \leq y$ there is a unique z such that $x + z = y$,
- positive ordering: $x \leq x + y$ for each x, y.

To verify that the positive cone of an o-group satisfies the two conditions is immmediate. Conversely, given $\mathbf{S}$ satisfying the two conditions one may construct $\mathbf{G}$ exactly as integers are constructed from natural numbers.

Definition 1.6.10 (1) An o-group $\mathbf{G}$ is *discrete* if for each $x \in G$ there is a smallest $y \in G$ such that $x < y$ (upper neighbour).

(2) $\mathbf{G}$ is *dense* if for each $x, y \in G$ such that $x < y$ there is a $z \in G$ such that $x < z$ and $z < y$.

Lemma 1.6.11 (1) There is a trivial o-group having just one element.

(2) Each non-trivial o-group is infinite.

(3) Each non-trivial o-group is either discrete or dense.

(Note that if $x \neq 0$ then $x > 0$ or $-x > 0$. Assume the former; then $x, \quad x+x \quad x+x+x, \ldots nx(= x+\ldots+x, n$ times) are pairwise different thanks to strict monotonicity.)

Definition 1.6.12 An o-group $G = \langle G, +, 0, -, \leq \rangle$ is *Archimedean* if for each pair x, y of positive elements $(x > 0, y > 0)$ there is a natural number n such that $nx \geq y$.

Example 1.6.13 (1) The additive groups of reals and of integers are Archimedean.

(2) The following group $Z \times Z$ (lexicographic product of Z, Z) is not Archimedean: the domain is $Z \times Z$ (ordered pairs); $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$, $(m_1, n_1) \leq (m_2, n_2)$ iff $m_1 < m_2$ or $[m_1 = m_2$ and $n_1 \leq n_2]$. (Define zero and inverse explicitly.)

Observe that if $x = (0, 1)$ and $y = (1, 0)$ then for each n, $nx = (0, n) < (1, 0) = y$.

Theorem 1.6.14 *Hölder's theorem.* An o-group **G** is Archimedean iff it has an isomorphic embedding into the additive o-group **Re** of reals, i.e. there is a one-one mapping f of G into Re such that, for each $x, y \in G$,

$$f(x +_G y) = f(x) + f(y)$$

$$x \leq_G y \text{ iff } f(x) \leq f(y)$$

($+_G, \leq_G$ are the operation and the ordering of G; +, $\leq$ are those of **Re**).

Note that the two conditions imply the following

$$f(0_G) = 0,$$

$$f(-_G x) = -_G f(x).$$

This is a non-trivial theorem; for a proof see [54].

Definition 1.6.15 An o-group **G** is *partially embeddable* into **Re** if for each finite $X \subseteq G$ there is a finite $Y \subseteq Re$ and a one-one mapping f of X onto Y which is a partial isomorphism, i.e. for each $x, y, z \in G$,

$$z = x +_G y \text{ iff } f(z) = f(x) + f(y)$$

$$x \leq_G y \text{ iff } f(x) \leq f(y).$$

Theorem 1.6.16 Each o-group is partially embeddable into **Re**.

A full proof of this theorem may be found in [86], Lemma 7.3.20. The theorem is a direct consequence of the following theorem of Gurevich and Kokorin [70].

Theorem 1.6.17 Let $\varphi(x, \dots, y)$ be a quantifier-free formula in the language of o-groups. If the formula $(\forall x, \dots, y)\varphi(x, \dots, y)$ is true in the o-group $\mathbf{Re}$ then it is true in all o-groups.

Indeed, if $x_1, \dots, x_n$ are evaluated by all the elements $a_1, \dots, a_n$ of X and $\varphi(x_1, \dots, x_n)$ is the conjuction of all formulas $x_i = x_y + x_z$, $x_i \leq x_y$, $x_i \neq x_y + x_z$, $\neg(x_i \leq x_y)$, satisfied in G then $(\exists x_i, \dots, x_n)\ \varphi(x_1, \dots, x_n)$ is true in G, hence $(\forall x_i, \dots)\neg\varphi$ is not true in all o-groups, hence not true in $\mathbf{Re}$ and $(\exists x_1, \dots)\varphi$ is true in $\mathbf{Re}$. Thus some $b_1, \dots, b_n \in Re$, satisfy φ in $\mathbf{Re}$. Put $f(a_i) = b_i$.

Remark 1.6.18 We shall need the following easy generalization of 1.6.16: we may assume that **G** is an o-group with some additional operations $F_1, \dots$ $\dots, F_k$ definable by open formulas from the group operation and ordering, i.e. there are open formulas φ_i such that

$$z = F_i(x, \dots, y) \equiv \varphi_i(x, \dots, y, z)$$

is true in G and in $\mathbf{Re}$. The partial isomorphism f may be found in such a way that it preserves all F_i.

Indeed, the sketched proof of 1.6.17 from the Gurevich-Kokorin theorem generalizes immediately to the present case (and so does the direct proof of 1.6.16 in [86]).

Example 1.6.19 As a very simple example on may show that the operation of inverse is preserved, i.e. if $x, y \in X$ then $y = -_G x$ iff $f(y) = -f(x)$ since $y = -x$ is equivalent to $x + y = 0$ (and $z = 0$ is equivalent to $z + z = z$).

CHAPTER TWO

MANY-VALUED PROPOSITIONAL CALCULI

2.1. CONTINUOUS T-NORMS AND THEIR RESIDUA

Now we are going to start our work with many-valued logic. We have already made several choices. First, as indicated in the title of the present chapter, we shall deal with *propositional logic*. Second, we take the real unit interval $[0, 1]$ for our set of truth values, 1 being absolute truth, 0 absolute falsity. The natural ordering $\leq$ of reals will play a very important role; thus our truth values are linearly ordered, the ordering is dense and complete (each set of truth values has its supremum and infimum). In some basic considerations we shall deal with lattice valued logics too.

The third choice we have made for the main part of the book is *truth functionality*: we shall deal with logical calculi in which each connective c (say, binary) has a truth function $f_c : [0, 1]^2 \rightarrow [0, 1]$ (thus $f_c(x, y) \in [0, 1]$ for each $x, y \in [0, 1]$) determining, for any pair of formulas φ, ψ, the truth degree of the compound formula $c(\varphi, \psi)$ (or $\varphi c \psi$ if you prefer) from the truth degrees of φ and of ψ. (Similarly for connectives of other arity.) This is a very common and technically useful assumption.

When choosing truth functions of connectives we shall obey very strictly the principle saying that each many-valued logic must be a generalization of classical two-valued logic, i.e. for the values $0, 1$ the truth functions must behave classically. For example, if we call a connective & conjunction its truth function $*$ must satisfy, besides other things, the equalities $1 * 1 = 1$, $1 * 0 = 0 * 1 = 0 * 0 = 0$. Similarly for other connectives of classical logic.

The last choice we make is: we start our investigation of good candidates for truth functions of connectives by formulating our requirements on a truth function for the *conjuction*. We shall see that a proper choice of the semantics of a conjunction determines a whole propositional calculus.

Our intuitive understanding of the conjunction is as follows: a large truth degree of $\varphi \& \psi$ should indicate that both the truth degree of φ and the truth degree of ψ is large, without any preference between φ and ψ. Thus it is natural to assume that the truth function of conjunction is non-decreasing in both arguments, 1 its unit element and 0 its zero element. These requirements are met by the following definition:

Definition 2.1.1 A *t-norm*[4] is a binary operation $*$ on $[0,1]$ (i.e. $t : [0,1]^2 \to [0,1]$) satisfying the following conditions:

(i) $*$ is commutative and associative, i.e., for all $x, y, z \in [0,1]$,

$$\begin{aligned} x * y &= y * x, \\ (x * y) * z &= x * (y * z), \end{aligned}$$

(ii) $*$ is non-decreasing in both arguments, i.e.

$$\begin{aligned} x_1 \leq x_2 \quad &\text{implies} \quad x_1 * y \leq x_2 * y, \\ y_1 \leq y_2 \quad &\text{implies} \quad x * y_1 \leq x * y_2, \end{aligned}$$

(iii) $1 * x = x$ and $0 * x = 0$ for all $x \in [0,1]$.[5]

$*$ is a *continuous t-norm* if it is a t-norm and is a continuous mapping of $[0,1]^2$ into $[0,1]$ (in the usual sense).

Example 2.1.2 The following are our most important examples of continuous t-norms:

(i) *Łukasiewicz t-norm*: $x * y = \max(0, x + y - 1)$,

(ii) *Gödel t-norm:* $x * y = \min(x, y)$,

(iii) *Product t-norm:* $x * y = x \cdot y$ (product of reals).

It is elementary to verify conditions (i)-(iii) from 2.1.1; for comments on the names see Historical Remarks.

Note that the dual notion of a *t-conorm* (replace (iii) in 2.1.1 by (iii′): $1 * x = 1$ and $0 * x = x$) will not play any important role in this book. This is because conjunction and disjunction do not have any dual relation to the implication.

2.1.3 Now let us discuss the implication. In two-valued logic, the implication $\varphi \to \psi$ is true iff the truth-value of φ is less than or equal to the truth-value of ψ. Let us generalize by saying that a large truth-value of $\varphi \to \psi$ should indicate that the truth value of φ is *not too much larger* than the truth value of ψ. This leads us to require (besides the principle of classical behaviour

[4] See Historical Remarks.

[5] It is easy to show that the condition $0 * x = 0$ is redundant.

on $0, 1)$ that a truth function $x \Rightarrow y$ of implication should be non-increasing in x and non-decreasing in y. Moreover we shall require soundness of *fuzzy modus ponens*: from (a lower bound of) the truth degree x of φ and (a lower bound of) the truth degree $x \Rightarrow y$ of $(\varphi \rightarrow \psi)$ one should be able to compute a lower bound of the truth degree y of ψ. The operation computing the lower bound for y should clearly be non-decreasing in both arguments (the more true the antecedent is the more true the succedent is) and similarly we can argue that 1 is the unit and 0 the neutral element of the operation in question. It may be difficult to justify commutativity or associativity; nevertheless it is useful to take just a t-norm $*$ (i.e. a truth function of conjunction), which gives

IF $a \leq x$ and $b \leq x \Rightarrow y$, THEN $a * b \leq y$,

thus in particular, taking $a = x$ and writing z instead of b we get

IF $z \leq x \Rightarrow y$, THEN $x * z \leq y$.

On the other hand, we would like to define $x \Rightarrow y$ as large as possible (to make the rule powerful); thus whenever $x * z \leq y$, then z is a possible candidate for $x \Rightarrow y$ (in the sense that the inference of $a * b \leq y$ from $a \leq x$ and $b \leq x \Rightarrow y$ would be sound); thus we may require, conversely,

IF $x * z \leq y$ THEN $z \leq x \Rightarrow y$,

and hence

$x * z \leq y$ IFF $z \leq (x \Rightarrow y)$.

Then it follows that $x \Rightarrow y$ is the *maximal* z satisfying $x * z \leq y$.

Lemma 2.1.4 Let $*$ be a continuous t-norm. Then there is a unique operation $x \Rightarrow y$ satisfying, for all $x, y, z \in [0,1]$, the condition $(x * z) \leq y$ iff $z \leq (x \Rightarrow y)$, namely $x \Rightarrow y = \max\{z \mid x * z \leq y\}$.

Proof: For each $x, y \in [0,1]$, let $(x \Rightarrow y) = \sup\{z \mid x * z \leq y\}$. Let, for a fixed z, $f(x) = x * z$; f is continuous and non-decreasing and hence commutes with sups. Thus

$$x * (x \Rightarrow y) = x * \sup\{z \mid x * z \leq y\} = \sup\{x * z \mid x * z \leq y\} \leq y.$$

Hence $x \Rightarrow y = \max\{z \mid x * z \leq y\}$. Uniqueness is obvious. Note that it would suffice to assume that $*$ is left continuous. But we shall work with continuous t-norms. $\square$

Definition 2.1.5 The operation $x \Rightarrow y$ from 2.1.4 is called the *residuum* of the t-norm. (We shall learn more about residuation below.)

Lemma 2.1.6 For each continuous t-norm $*$ and its residuum $\Rightarrow$

(i) $x \leq y$ iff $(x \Rightarrow y) = 1$.

(ii) $(1 \Rightarrow x) = x$

Proof: Obvious. □

Lemma 2.1.7 .[6]

(1) If $x \leq y$, then $x = y * (y \Rightarrow x)$.

(2) If $x \leq u \leq y$ and u is idempotent then $x * y = x$.

Proof:

(1) Let $f(z) = z * y$; f is continuous on $[0,1]$, $f(0) = 0$ and $f(1) = y$. Thus for some z with $0 \leq z \leq 1, f(z) = x$. For the maximal z satisfying $x = z * y$ we get $z = y \Rightarrow x$.

(2) First assume $u = y$. Then $x = u * (u \Rightarrow x)$, $x * u = u * (u \Rightarrow x) * u = u * (u \Rightarrow x) = x$. Now let $u \leq y$.Then $x * y \geq x * u = x$ and obviously $x * y \leq x$, thus $x * y = x$.

□

Theorem 2.1.8 The following operations are residua of the three t-norms of 2.1.2: For $x \leq y$, $x \Rightarrow y = 1$. For $x > y$,

(i) *Łukasiewicz implication*: $x \Rightarrow y = 1 - x + y$

(ii) *Gödel implication*: $x \Rightarrow y = y$

(iii) *Goguen implication*: $x \Rightarrow y = y/x$ (residuum of product conjunction).

Proof: Assume $x > y$.

(i) Then $x * z = y$ iff $x + z - 1 = y$ iff $z = 1 - x + y$; thus $1 - x + y = \max\{z \mid x * z \leq y\}$.

(ii) Then $x * z = y$ iff $\min(x, z) = y$ iff $z = y$.

(iii) Similarly; $x.z = y$ iff $z = y/x$ (since $x > 0$).

□

---

[6] Cf. [98] 2.6.

Definition 2.1.9 In the sequel we shall consider the unit interval [0,1] as an algebra endowed with the operations min and max, a fixed t-norm $*$ and its residuum $\Rightarrow$, as well as the elements $0, 1$. This algebra will be denoted by $L(*)$. We shall use $\cap$ and $\cup$ to denote min and max.

Note that the ordering $\leq$ is definable: $x \leq y$ iff $x \cap y = x$. We show that min and max are definable from $*$ and $\Rightarrow$.

Lemma 2.1.10 [7] For each continuous t-norm $*$, the following identities are true in $L(*)$:

(i) $x \cap y = x * (x \Rightarrow y)$,

(ii) $x \cup y = ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$.

Proof:

(i) If $x \leq y$ then $x \Rightarrow y = 1$ and $x * (x \Rightarrow y) = x$; if $x > y$ then $x * (x \Rightarrow y) = y$ by 2.1.7.

(ii) Assume $x \leq y$; then $(x \Rightarrow y) = 1$ and $(x \Rightarrow y) \Rightarrow y = (1 \Rightarrow y) = y$. Furthermore, $y \leq (y \Rightarrow x) \Rightarrow x$ since $y * (y \Rightarrow x) \leq x$ (by residuation); thus $((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x) = y$. The case $y \leq x$ is completely symmetric.

□

Remark 2.1.11 Observe that Łukasiewicz implication is continuous but Gödel and Goguen are not; but it is easy to show that the residuum of each continuous t-norm is left continuous in the first (antecedent) variable and right continuous in the second (succedent) variable.

Definition 2.1.12 The residuum $\Rightarrow$ defines its corresponding unary operation of *precomplement* $(-)x = (x \Rightarrow 0)$ (future truth function of negation).

Lemma 2.1.13 The following operations are precomplements of our three distinguished t-norms:

(i) Łukasiewicz negation $(-)x = 1 - x$,

(ii) Gödel negation $(-)0 = 1$, $(-)x = 0$ for $x > 0$.

---

[7] Cf. [98] 2.5.

(iii) The precomplement given by Goguen implication coincides with Gödel negation.

Proof: by elementary computation. □

In the rest of this section we show that the three t-norms above are fundamental continuous t-norms, i.e. each continuous t-norm is a combination of them (in a sense we are going to make precise).

In the sequel, $*$ will denote a continuous t-norm; we shall investigate the commutative ordered semigroup $\langle [0,1], *, \leq \rangle$. Recall that an element x is *idempotent*, if $x * x = x$. Thus both 0 and 1 are idempotents; a t-norm may and may not have other idempotents. (For example if $*$ is minimum, then each element is idempotent.) An element x is *nilpotent* if there is an n such that $x * \ldots * x$ (n factors) equal 0. We write x^{*n} for $x * \ldots * x$ with n factors.

Definition 2.1.14 A continuous t-norm is *Archimedean* if it has no idempotents except 0 and 1. An Archimedean t-norm is *strict* if it has no nilpotent elements except 0; otherwise it is *nilpotent.*

Remark 2.1.15 Observe that for each continuous t-norm the set E of all its idempotents is a closed subset of [0,1] and hence its complement is a union of a set $\mathcal{I}_{\text{open}}(E)$ of countably many non-overlapping open intervals. Let $[a, b] \in \mathcal{I}(E)$ iff $(a, b) \in \mathcal{I}_{\text{open}}(E)$ (the corresponding closed intervals, contact intervals of E). For $I \in \mathcal{I}(E)$ let $(*|I)$ be the restriction of $*$ to I^2. The following theorem characterizes all continuous t-norms.

Theorem 2.1.16 [8] If $*, E, \mathcal{I}(E)$ are as above, then

(i) for each $I \in \mathcal{I}(E)$, $(*|I)$ is isomorphic either to the product t-norm (on [0,1]) or to Łukasiewicz t-norm (on [0,1]).

(ii) If $x, y \in [0, 1]$ are such that there is no $I \in \mathcal{I}(E)$ with $x, y \in I$, then $x * y = \min(x, y)$.

Proof: It is easy to see that for each $I \in \mathcal{I}(E)$, $I = (a, b)$, the linear mapping f sending $[a, b]$ increasingly to $[0, 1]$ makes the restriction of $*$ to $[a, b]$ an Archimedean continuous t-norm. Indeed, for $a < x < b$ we get $a * x = a$ and $b * x = x$ by 2.1.7. If $x < y$ andx, y are not from the same interval $I \in \mathcal{I}(E)$ then there is an a, $x \leq a \leq y$, a idempotent; thus by 2.1.7, $x * y = x$. To

[8] See [143].

prove our theorem it suffices to show that each strict Archimedean continuous t-norm is isomorphic to the product t-norm and each nilpotent Archimedean continuous t-norm is isomorphic to Łukasiewicz t-norm. This will be done in the rest of this section. □

Until the end of the section we assume $*$ to be an Archimedean continuous t-norm, i.e. having no idempotents except $0, 1$.

Lemma 2.1.17 For each positive $x < 1$,

$$\lim_{n\to\infty} x^{*n} = 0.$$

If $*$ is nilpotent then each $x < 1$ is nilpotent. If $1 > x^{*n} > 0$ then $m > n$ implies $x^{*m} < x^{*n}$.

Proof: Since the sequence x^{*n} is non-increasing (and bounded by 0 from below), $\bar{x} = \lim_n x^{*n}$ exists; moreover, $\bar{x}$ is an idempotent:

$$\bar{x} * \bar{x} = \lim_n x^{*n} * \lim_m x^{*m} = \lim_{m,n\to\infty} x^{*(m+n)} = \bar{x}.$$

Thus $\bar{x} = 0$. If $x^{*m} = x^{*n}$ put $y = x^{*n}, z = x^{*(m-n)}; y = y * z = y * z^{*k}$ for all k, hence $y = y * 0 = 0$. If $0 < z$ is nilpotent and $0 < x < z$ then x is also nilpotent; if $z < x < 1$ then for some $m > 0$, $x^{*m} < z$ (since $\lim_m x^{*m} = 0$) thus if $z^{*n} = 0$ then $x^{*mn} = 0$ too. □

Remark 2.1.18 This lemma can be used to show that a continuous t-norm $*$ is Archimedean iff for each $0 < x, y < 1$ there is an n such that $x^{*n} \leq y$. This justifies the name "Archimedean".

Lemma 2.1.19 For each positive $x < 1$ and each positive n, there is a unique y such that $y^{*n} = x$.

Proof: Assume $n > 1$. The existence follows from continuity of the function $f(y) = y^{*n}$ (since $f(0) = 0$ and $f(1) = 1$). Clearly if $y^{*n} = x$ then $0 < x < y < 1$. Now let $x < z < y$ and $z^{*n} = y^{*n}$; by 2.1.7, $z = y * t$ for some t, thus $y^{*n} = z^{*n} = y^{*n} * t^{*n} = y^{*n} * t^{*(kn)}$ for each $k > 0$; but by 2.1.17, $\lim_k t^{*(kn)} = 0$ and, by continuity, $x = y^{*n} = y^{*n} * 0 = 0$, a contradiction. □

Definition 2.1.20 For each $x \in [0, 1]$, $x^{*\frac{1}{n}}$ is the unique $y \in [0, 1]$ with $y^n = x$. For a rational number $r = \frac{m}{n}$,

$$x^{*r} = (x^{*\frac{1}{n}})^{*m}$$

Lemma 2.1.21 (1) If $\frac{m}{n} = \frac{m'}{n'}$ then $x^{*\frac{m}{n}} = x^{*\frac{m'}{n'}}$.

(2) $x^{*r} * x^{*s} = x^{*(r+s)}$ for all $x \in [0,1]$, r, s positive rational.

(3) If $x > 0$ then

$$\lim_n x^{*\frac{1}{n}} = 1.$$

Proof:

(1) We may assume $m' = km, n' = kn$ (common denominator). Then

$$x^{*r} = (x^{*\frac{1}{kn}})^{*km} = ((x^{*\frac{1}{kn}})^{*k})^{*m} = (x^{*\frac{1}{n}})^{*m} = x^{*\frac{m}{n}}.$$

(2) $r = \frac{m}{n}, s = \frac{k}{n}$; then

$$x^{*r} * x^{*s} = (x^{*\frac{1}{n}})^{*m} * (x^{*\frac{1}{n}})^{*k} = (x^{*\frac{1}{n}})^{*(m+k)} = x^{*(r+s)}.$$

(3) If $x > 0$ then the sequence $\{x^{*\frac{1}{n}} \mid n\}$ is increasing and its limit is an idempotent; thus the limit is 1.

□

Lemma 2.1.22 (1) If $*$ is strict, then $\langle [0,1], * \rangle$ is isomorphic to $\langle [0,1], *_\Pi \rangle$ where $x *_\Pi y = x \cdot y$ (product).

(2) if $*$ is nilpotent, then $\langle [0,1], * \rangle$ is isomorphic to $\langle [\frac{1}{4}, 1], *_{CP} \rangle$ where $x *_{CP} y = \max(\frac{1}{4}, x \cdot y)$ (product cut at $\frac{1}{4}$).

Proof: We construct an isomorphism between dense subsets of the respective algebras. For each rational number r with $0 \leq r \leq \infty$, let $c_r = (\frac{1}{2})^r = \frac{1}{2^r}$ (in the sense of exponentiation of real numbers). Clearly the set $c = \{c_r \mid r\}$ is a dense subset of $[0,1], c_r < c_s$ for $r > s$ and $c_r.c_s = c_{r+s}$ for all r, s. If $*$ is strict, let $d = \frac{1}{2}$; if it is nilpotent let d be the maximal x such that $x * x = 0$. Clearly d exists and $0 < d < 1$. Now let $d_r = d^{*r}$ (exponentiation in the sense of $*$, cf. 2.1.20). We have $d_r * d_s = d_{r+s}$, for all r, s. If $*$ is strict, then $r > s$ implies $d_r < d_s$ for all r, s; if $*$ is nilpotent, then the implication holds for $2 \geq r > s$ (note that $d_2 = 0$) whereas for $r \geq 2, d_r = 0$. Indeed, assume $\infty > r > s > 0$, let m_1, m_2, n be such that $r = \frac{m_1}{n}, r_2 = \frac{m_2}{n}$ and let $x = d^{*\frac{1}{n}}$; then $d_r = x^{*m_1}, d_s = x^{*m_2}, 0 < x < 1, m_1 > m_2$ and hence if $x^{*m_2} > 0$, then $x^{*m_1} < x^{*m_2}$ (by 2.1.17). For $*$ nilpotent observe that $d_r = 0$ iff $r \geq 2$, i.e. iff $c_r \leq \frac{1}{4}$. Now it remains to prove that the set $D = \langle d_r \mid d_r > 0 \rangle$ is dense in $[0, 1]$. Let $0 < x < 1$. We shall approximate x from above by elements d_r where r has the form $\frac{m}{2^n}$. For typographical reasons, write $r(m, n)$ for $\frac{m}{2^n}$; take an n_0 such $d_{r(1,n_0)} \geq x$ (remembering that $\lim_n d_{r(1,n)} = 1$) and let, for $n \geq n_0$, r_n be $r(m, n)$ for the largest m such that $d_{r(m,n)} \geq x$ (remembering that for fixed n, $\lim_m d_{r(m,n)} = 0$). Observe $d_{r(m+1,n)} = d_{r(m,n)} * d_{r(1,n)} < x \leq d_{r_n}$. Going to the limit and recalling again

$$\lim_n d_{r(1,n)} = 1$$

we get $\lim_n (d_{r_n} * d_{r(1,n)}) = \lim_n d_{r_n} = x$. □

Lemma 2.1.23 The ordered semigroup $\langle [\frac{1}{4}, 1], *_{CP} \rangle$ is isomorphic to $\langle [0, 1], *_{Ł} \rangle$ where $*_{Ł}$ is Łukasiewicz t-norm $x *_{Ł} y = \max(0, x + y - 1)$.

Proof: For $x \in [0, 1]$ let $f(x) = 2^{2(x-1)}$; then f maps [0,1] increasingly to $[\frac{1}{4}, 1]$ and induces on [0,1] Łukasiewicz t-norm:

$$x * y = f^{-1}(f(x).f(y)) = \frac{1}{2} \log(2^{2(x-1)} \cdot 2^{2(y-1)}) + 1 = x + y - 1$$

if $x + y - 1 \geq 0$, otherwise $x * y = 0$. □

2.2. THE BASIC MANY-VALUED LOGIC

If we fix a continuous t-norm $*$ we fix a propositional calculus (whose set of truth values is $[0, 1]$): $*$ is taken for the truth function of the (strong) conjunction &, the residuum $\Rightarrow$ of $*$ becomes the truth function of the implication. In more details, we have the following

Definition 2.2.1 The *propositional calculus $PC(*)$ given by* $*$ has propositional variables $p_1, p_2, \ldots$, connectives $\&$, $\rightarrow$ and the truth constant $\bar{0}$ for 0. *Formulas* are defined in the obvious way: each propositional variable is a formula; $\bar{0}$ is a formula; if φ, ψ are formulas, then $\varphi \& \psi$, $\varphi \rightarrow \psi$ are formulas. Further connectives are defined as follows:

$$\begin{array}{rll} \varphi \wedge \psi & \text{is} & \varphi \& (\varphi \rightarrow \psi), \\ \varphi \vee \psi & \text{is} & ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi), \\ \neg \varphi & \text{is} & \varphi \rightarrow \bar{0}, \\ \varphi \equiv \psi & \text{is} & (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi). \end{array}$$

An *evaluation of propositional variables* is a mapping e assigning to each propositional variable p its truth value $e(p) \in [0, 1]$. This extends uniquely to the evaluation of all formulas as follows:

$$e(\bar{0}) = 0,$$
$$e(\varphi \to \psi) = (e(\varphi) \Rightarrow e(\psi)),$$
$$e(\varphi \& \psi) = (e(\varphi) * e(\psi)).$$

Lemma 2.2.2 For any formulas φ, ψ

$$\begin{aligned} e(\varphi \wedge \psi) &= \min(e(\varphi), e(\psi)), \\ e(\varphi \vee \psi) &= \max(e(\varphi), e(\psi)), \end{aligned}$$

Proof: This follows immediately from 2.1.10. □

Definition 2.2.3 A formula φ is a 1-tautology of $PC(*)$ if $e(\varphi) = 1$ for each evaluation e.

Thus a 1-tautology is a formula that is absolutely true under any evaluation. In the present section we are going to choose some formulas that are 1-tautologies of every $PC(*)$ (for any continuous t-norm $*$) for our axioms and develop a logic that is a common base of all the logics $PC(*)$. The first goal will be to prove several important formulas in our basic logic. The logic is sound, thus each provable formula will be a 1-tautology of each $PC(*)$ and thus also a tautology of the classical two-valued logic. Needless to say, for different t-norms t_1, t_2, the sets of 1-tautologies of $PC(t_1)$ and of $PC(t_2)$ may well be different sets.

Definition 2.2.4 The following formulas are axioms of the basic logic BL:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(A2) $(\varphi \& \psi) \to \varphi$

(A3) $(\varphi \& \psi) \to (\psi \& \varphi)$

(A4) $(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$

(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$

(A5b) $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$

(A7) $\bar{0} \to \varphi$

The *deduction rule* of BL is modus ponens. Given this, the notions of a *proof* and of a *provable formula* in BL are defined in the obvious way (cf. 1.2.6).

Remark 2.2.5 Let us comment on these axioms. (A1) is transitivity of implication. (A2) says that &-conjunction implies its first element and (A3) says that &-conjunction is commutative. (A4) expresses the commutativity of $\wedge$-conjunction. (A5) expresses residuation. (A6) is a variant of the proof by cases: if χ follows from $\varphi \to \psi$ then if χ also follows from $\psi \to \varphi$ then χ. Finally (A7) says that $\bar{0}$ implies everything (*ex falso quodlibet*).[9]

Lemma 2.2.6 All axioms of BL are 1-tautologies in each $PC(*)$. If φ and $\varphi \to \psi$ are 1-tautologies of $PC(*)$ then ψ is also a 1-tautology of $PC(*)$. Consequently, each formula provable in BL is a 1-tautology of each $PC(*)$.

Proof: (A2)–(A4) and (A7) are obviously 1-tautologies. To verify (A1) we have to prove

$$1 \leq (x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)),$$

i.e., using residuation three times

$$(x \Rightarrow y) * (y \Rightarrow z) * x \leq z\,;$$

but this follows from the fact that $x * (x \Rightarrow y) = \min(x, y) \leq y$ and similarly $y * (y \Rightarrow z) \leq z$. (A5) expresses residuation — the following are equivalent.

$$\begin{aligned} &t \leq x \Rightarrow (y \Rightarrow z), \\ &t * x \leq (y \Rightarrow z), \\ &t * x * y \leq z, \\ &t \leq (x * y) \Rightarrow z. \end{aligned}$$

To verify (A6) observe that $[x \Rightarrow y = 1$ or $(y \Rightarrow x) = 1]$ and that $1 \Rightarrow y = y$ (since $z \leq 1 \Rightarrow y$ iff $z \leq y$). For modus ponens observe that if $x = 1$ and $x \Rightarrow y = 1$ then necessarily $y = 1$ (since $1 \Rightarrow y = y$ as above). □

We shall now verify provability of several groups of formulas in BL. Note that the connectives $\wedge \vee \neg \equiv$ are defined as above.

Lemma 2.2.7 [10] BL proves the following properties of implication:

[9] Our BL is stronger than Höhle's monoidal logic [98].

[10] Cf. [98] 3.1.

(1) $\varphi \to (\psi \to \varphi)$

(2) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$

(3) $\varphi \to \varphi$

Proof:
(1) By (A2), $\vdash (\varphi \& \psi) \to \varphi$;
by (A5), $\vdash ((\varphi\&\psi) \to \varphi) \to (\varphi \to (\psi \to \varphi))$. Thus $\vdash \varphi \to (\psi \to \varphi)$ by modus ponens.
(2) By (A1), $\vdash ((\psi\&\varphi) \to (\varphi\&\psi)) \to (((\varphi\&\psi) \to \chi) \to (\psi\&\varphi) \to \chi))$
thus $\vdash ((\varphi\&\psi) \to \chi) \to (\psi\&\varphi) \to \chi)$. Applying (A5) twice we get the following provable chain of implications.

$$\vdash [\varphi \to (\psi \to \chi)] \to [(\varphi\&\psi) \to \chi] \to [(\psi\&\varphi) \to \chi] \to [\psi \to (\varphi \to \chi)].$$

(3) Applying (2) to (1) we get $\vdash \psi \to (\varphi \to \varphi)$; take any axiom for ψ and apply modus ponens. □

Lemma 2.2.8 BL proves the following properties of strong conjunction:

(4) $(\varphi\&(\varphi \to \psi)) \to \psi$

(5) $\varphi \to (\psi \to (\varphi\&\psi))$

(6) $(\varphi \to \psi) \to ((\varphi\&\chi) \to (\psi\&\chi))$

(7) $((\varphi_1 \to \psi_1)\&(\varphi_2 \to \psi_2)) \to ((\varphi_1\&\varphi_2) \to (\psi_1\&\psi_2))$

(8) $(\varphi\&\psi)\&\chi \to \varphi\&(\psi\&\chi)$, $(\varphi\&(\psi\&\chi)) \to ((\varphi\&\psi)\&\chi)$.

Proof:

(4) BL $\vdash (\varphi \to \psi) \to (\varphi \to \psi)$ by (3), hence
BL $\vdash \varphi \to ((\varphi \to \psi) \to \psi)$ by(2) and
BL $\vdash (\varphi\&(\varphi \to \psi)) \to \psi$ by (A5).

(5) BL $\vdash (\varphi\&\psi) \to (\varphi\&\psi)$ by (3), thus
BL $\vdash \varphi \to (\psi \to (\varphi\&\psi))$ by (A5).

(6) BL $\vdash (\varphi\&(\varphi \to \psi)) \to \psi$ and
BL $\vdash \psi \to (\chi \to (\psi\&\chi))$, thus
BL $\vdash (\varphi\&(\varphi \to \psi)) \to (\chi \to (\psi\&\chi))$ by (A1). Thus
BL $\vdash \varphi \to ((\varphi \to \psi) \to (\chi \to (\psi\&\chi)))$,
BL $\vdash \varphi \to (\chi \to ((\varphi \to \psi) \to (\psi\&\chi)))$,
BL $\vdash (\varphi\&\chi) \to ((\varphi \to \psi) \to (\psi\&\chi))$,
BL $\vdash (\varphi \to \psi) \to ((\varphi\&\chi) \to (\psi\&\chi))$.

(7) BL $\vdash (\varphi_1 \to \psi_1) \to ((\varphi_1 \& \varphi_2) \to (\psi_1 \& \varphi_2))$,
BL $\vdash ((\varphi_1 \to \psi_1) \& (\varphi_2 \to \psi_2)) \to ((\varphi_1 \& \varphi_2) \to (\psi_1 \& \varphi_2)) \& (\varphi_2 \to \psi_2)$,
BL $\vdash [((\varphi_1 \& \varphi_2) \to (\psi_1 \& \varphi_2)) \& (\varphi_2 \to \psi_2)] \to$
$[((\varphi_1 \& \varphi_2) \to (\psi_1 \& \varphi_2)) \& ((\psi_1 \& \varphi_2) \to (\psi_1 \& \psi_2))]$,
BL $\vdash [((\varphi_1 \& \varphi_2) \to (\psi_1 \& \varphi_2)) \& ((\psi_1 \& \varphi_2) \to (\psi_1 \& \psi_2))] \to [(\varphi_1 \& \varphi_2) \to$
$\to (\psi_1 \& \psi_2)]$.
Thus BL $\vdash (\varphi_1 \to \psi_1) \to ((\varphi_2 \to \psi_2) \to ((\varphi_1 \& \varphi_2) \to (\psi_1 \& \psi_2)))$.

(8) For each δ, the following chain of implications is provable in BL by repeated use of (A5), (A1):
$[((\varphi \& \psi) \& \chi) \to \delta] \to [(\varphi \& \psi) \to (\chi \to \delta)]$
$\to [(\varphi \to (\psi \to (\chi \to \delta)))] \to$
$[(\varphi \to ((\psi \& \chi) \to \delta)] \to [(\varphi \& (\psi \& \chi)) \to \delta]$

This elegant proof is from [2]. □

Lemma 2.2.9 BL proves the following properties of min-conjunction:

(9) $(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \psi) \to \psi$, $(\varphi \& \psi) \to (\varphi \wedge \psi)$

(10) $(\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$

(11) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$

(12) $((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$

Proof:

(9) BL $\vdash (\varphi \wedge \psi) \to \varphi$ by the definition of $\wedge$ and by (A2);
BL $\vdash (\varphi \wedge \psi) \to \psi$ is just (4).
BL $\vdash (\varphi \& \psi) \to (\varphi \& (\varphi \to \psi))$ follows by (1) and (6).

(10) BL $\vdash (\varphi \to \psi) \to (\varphi \to (\varphi \& (\varphi \to \psi)))$ follows from
BL $\vdash (\varphi \& (\varphi \to \psi)) \to (\varphi \& (\varphi \to \psi))$ using (A5) and (2).

(11) This is just (A4).

(12) Observe the following:
BL $\vdash (\psi \to \chi) \to (\psi \to (\psi \wedge \chi))$ and
BL $\vdash (\psi \to (\psi \wedge \chi)) \to [((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))]$,
thus
BL $\vdash (\psi \to \chi) \to (12)$; similarly,
BL $\vdash (\chi \to \psi) \to (12)$, therefore
BL $\vdash (12)$ using (the first time) the axiom (A6).

□

Lemma 2.2.10 BL proves the following properties of max-disjunction:

(13) $\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$, $(\varphi \vee \psi) \to (\psi \vee \varphi)$,

(14) $(\varphi \to \psi) \to ((\varphi \vee \psi) \to \psi)$,

(15) $(\varphi \to \psi) \vee (\psi \to \varphi)$,

(16) $((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$.

Proof:

(13) BL $\vdash (\varphi \vee \psi) \to (\psi \vee \varphi)$ is obvious from the definition of $\vee$ and from (11). To prove $\varphi \to (\varphi \vee \psi)$, observe BL $\vdash \varphi \to ((\varphi \to \psi) \to \psi)$ and BL $\vdash \varphi \to ((\psi \to \varphi) \to \varphi)$; thus BL $\vdash \varphi \to (\varphi \vee \psi)$ by (12).

(14) By the definition of $\vee$, BL $\vdash (\varphi \vee \psi) \to ((\varphi \to \psi) \to \psi)$; apply (2).

(15) We have BL $\vdash (\varphi \to \psi) \to$ (15), BL $\vdash (\psi \to \varphi) \to$ (15), thus BL $\vdash$ (15) using (A6).

(16) Observe: BL $\vdash (\varphi \to \psi) \to ((\varphi \vee \psi) \to \psi)$ and $((\varphi \vee \psi) \to \psi) \to [((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)]$, thus BL $\vdash (\varphi \to \chi) \to$ (16); similarly, BL $\vdash (\psi \to \varphi) \to$ (16) and hence BL $\vdash$ (16) by (A6).

□

Corollary 2.2.11 BL proves

(12′) $((\varphi \to \psi) \& (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$,

(16′) $((\varphi \to \chi) \& (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$.

Lemma 2.2.12 BL proves the following properties of negation.

(17) $\varphi \to (\neg\varphi \to \psi)$, in particular, $\varphi \to \neg\neg\varphi$ and $(\varphi \& \neg\varphi) \to \bar{0}$.

(18) $(\varphi \to (\psi \& \neg\psi)) \to \neg\varphi$

(18′) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

(18″) $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$

Proof:

(17) BL $\vdash \varphi \to ((\varphi \to \bar{0}) \to \bar{0})$ and BL $\vdash \bar{0} \to \psi$, thus
BL $\vdash ((\varphi \to \bar{0}) \to \bar{0}) \to ((\varphi \to \bar{0}) \to \psi)$, hence
BL $\vdash \varphi \to ((\varphi \to \bar{0}) \to \psi)$.
Thus BL $\vdash \varphi \to (\neg\varphi \to \bar{0})$, which gives BL $\vdash \varphi \to \neg\neg\varphi$ and BL $\vdash (\varphi \& \neg\varphi) \to \bar{0}$ using (A5).

(18) BL $\vdash (\psi \& (\psi \to \bar{0})) \to \bar{0}$, thus
BL $\vdash (\varphi \to (\psi \& \neg\psi)) \to (\varphi \to \bar{0})$.

(18′) Evidently, BL proves $(\varphi \to \psi) \to ((\psi \to 0) \to (\varphi \to 0))$. (18″) follows from (18′) and (17).

□

Definition 2.2.13 $\bar{1}$ stands for $\bar{0} \to \bar{0}$.

Lemma 2.2.14 BL proves the following:

(19) $\bar{1}$,

(20) $\varphi \to (\bar{1} \& \varphi)$,

(20′) $(\bar{1} \to \varphi) \to \varphi$.

Proof:

(19) Trivial by (3).

(20) BL $\vdash \bar{1} \to (\varphi \to (\bar{1} \& \varphi))$; thus BL $\vdash$ (20) by (19).

(20′) BL$\vdash \bar{1} \to ((\bar{1} \to \varphi) \to \varphi)$; use (19).

□

Lemma 2.2.15 BL proves the following additional properties of $\wedge$, $\vee$:

(21) $(\varphi \wedge (\psi \wedge \chi)) \to ((\varphi \wedge \psi) \wedge \chi)$
$((\varphi \wedge \psi) \wedge \chi) \to (\varphi \wedge (\psi \wedge \chi))$ (associativity of $\wedge$),

(22) analogous associativity for $\vee$,

(23) $\varphi \to \varphi \wedge (\varphi \vee \psi)$
$(\varphi \vee (\varphi \wedge \psi)) \to \varphi$

Proof:

(21) BL $\vdash (\varphi\wedge(\psi\wedge\chi)) \to \gamma$ for γ being $\varphi, \psi\wedge\chi, \psi, \chi, (\varphi\wedge\psi), ((\varphi\wedge\psi)\wedge\chi)$ using $(12')$.

Similarly for the other direction and dually for (22) using $(16')$.

(23): BL $\vdash (\varphi \wedge (\varphi \wedge \psi)$by (13), thus
BL $\vdash \varphi \to (\varphi \wedge (\varphi \vee \psi))$ by (12).

□

Now we prove some properties of equivalence:

Lemma 2.2.16 BL proves

(24) $\varphi \equiv \varphi, \quad (\varphi \equiv \psi) \to (\psi \equiv \varphi), \quad ((\varphi \equiv \psi)\&(\psi \equiv \chi)) \to (\varphi \equiv \chi),$

(25) $(\varphi \equiv \psi) \to (\varphi \to \psi), \quad (\varphi \equiv \psi) \to (\psi \to \varphi)$

(26) $(\varphi \equiv \psi) \to ((\varphi\&\chi) \equiv (\psi\&\chi)),$

(27) $(\varphi \equiv \psi) \to ((\varphi \to \chi) \equiv (\psi \to \chi)),$

(28) $(\varphi \equiv \psi) \to ((\chi \to \varphi) \equiv (\chi \to \psi)),$

(29) $(\varphi \equiv \psi) \to ((\varphi \to \psi) \wedge (\psi \to \varphi)).$

Proof: (24)-(28) are immediate consequences of the preceding provabilities. For example to prove (26) we reason as follows:
BL $\vdash (\varphi \equiv \psi) \to ((\varphi \to \psi)\&(\psi \to \varphi)),$
BL $\vdash ((\varphi \to \psi)\&(\psi \to \varphi)) \to (((\varphi\&\chi) \to (\psi\&\chi))\&((\psi\&\chi) \to (\psi\&\chi)))$
(using (6) and (7)); thus we get (26) applying transitivity of implication. To prove (29) observe that the implication $\to$ follows from (9).
Conversely, BL $\vdash ((\varphi \to \psi) \wedge (\psi \to \varphi)) \to (\psi \to \varphi)$, thus
BL $\vdash (\varphi \to \psi) \to (((\varphi \to \psi) \wedge (\psi \to \varphi)) \to (\psi \equiv \varphi)),$
similarly BL $\vdash (\psi \to \varphi) \to (((\varphi \to \psi) \wedge (\psi \to \varphi)) \to (\psi \equiv \varphi)),$
thus BL (29) by axiom (A6). □

Definition 2.2.17 A *theory* over BL is a set of formulas. A *proof* in a theory T is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas whose each member is either an axiom of BL or a member of T (special axiom) or follows from some preceding members of the sequence using the deduction rule modus ponens.

$T \vdash \varphi$ means that φ is *provable* in T, i.e. is the last member of of a proof in T. The following is a variant of the *deduction theorem*:

Theorem 2.2.18 Let T be a theory and let φ, ψ be formulas. $T \cup \{\varphi\} \vdash \psi$ iff there is an n such that $T \vdash \varphi^n \rightarrow \psi$ (where again φ^n is $\varphi \& \ldots \& \varphi, n$ factors).

Proof: First note that we may impose any bracketting on $\varphi \& \ldots \& \varphi$ thanks to commutativity and associativity of & (and properties of $\rightarrow$ and $\equiv$). Thus if $n > 1$ and $T \vdash \varphi^n \rightarrow \psi$ then $T \vdash (\varphi \& \varphi^{n-1}) \rightarrow \psi$, $T \vdash \varphi \rightarrow (\varphi^{n-1} \rightarrow \psi)$, hence $T \cup \{\varphi\} \vdash \varphi^{n-1} \rightarrow \psi$. Replacing this we get $T \cup \{\varphi\} \vdash \varphi \rightarrow \psi$ and hence $T \cup \{\varphi\} \vdash \psi$.

Conversely assume $T \cup \{\varphi\} \vdash \psi$ and let $\gamma_1, \ldots, \gamma_k$ be a corresponding $T \cup \{\varphi\}$-proof of ψ. We prove by induction that, for each $j = 1, \ldots, k$, there is an n_j such that $T \vdash \varphi^{n_j} \rightarrow \gamma_j$. This is clear for γ_j being an axiom of BL or of $T \cup \{\varphi\}$. If γ_j results by modus ponens from previous members $\gamma_i, (\gamma_i \rightarrow \gamma_j)$ then, by the induction hypothesis we assume $T \vdash \varphi^n \rightarrow \gamma_i$, $T \vdash \varphi^m \rightarrow (\gamma_i \rightarrow \gamma_j)$, thus by (7), $T \vdash (\varphi^n \& \varphi^m) \rightarrow (\gamma_i \& (\gamma_i \rightarrow \gamma_j))$, thus $T \vdash \varphi^{n+m} \rightarrow \gamma_j$ (cf. (4)). This completes the proof. □

Remark 2.2.19 Note that in general we can not prove the deduction theorem in the form 1.2.10 valid for classical logic; we shall see below that only one of our logics (Gödel logic) will have the classical deduction theorem. Also note that the formula $\varphi^n \rightarrow \psi$ may be replaced by $\varphi \rightarrow (\varphi \rightarrow \ldots \rightarrow (\varphi \rightarrow \psi) \ldots)$ (n copies of φ).

Definition 2.2.20 A theory T is *contradictory* (or *inconsistent*) if $T \vdash \bar{0}$; otherwise it is *consistent*.

Lemma 2.2.21 T is inconsistent iff $T \vdash \varphi$ for each φ.

Proof: If T proves each formula, then it proves $\bar{0}$. Conversely, if $T \vdash \bar{0}$ then $T \vdash \varphi$ since $T \vdash \bar{0} \rightarrow \varphi$ (axiom A7). □

Lemma 2.2.22 If $T \cup \{\varphi\}$ is inconsistent, then for some n, $T \vdash \neg(\varphi^n)$.

Proof: If $T \cup \{\varphi\} \vdash \bar{0}$, then, by the deduction theorem, there is an n such that $T \vdash \varphi^n \rightarrow \bar{0}$. □

Lemma 2.2.23 [11] BL proves the following distributive laws:

[11] Cf. [98] 2.6.

(30) $\varphi\&(\psi \vee \chi) \equiv (\varphi\&\psi) \vee (\varphi\&\chi)$
$\varphi\&(\psi \wedge \chi) \equiv (\varphi\&\psi) \wedge (\varphi\&\chi)$

(31) $(\varphi \wedge (\psi \vee \chi)) \equiv ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
$(\varphi \vee (\psi \wedge \chi)) \equiv ((\varphi \vee \psi) \wedge (\varphi \vee \chi))$

Proof:

(30) (a) Let us first prove the first formula.
BL $\vdash ((\varphi\&\chi) \vee (\psi\&\chi)) \to ((\varphi \vee \psi)\&\chi)$ using (13) and (16) in 2.2.10; we prove the converse implication. This means to prove
BL $\vdash ((\varphi \vee \psi)\&\chi) \to [((\varphi\&\chi) \to (\psi\&\chi)) \to (\psi\&\chi)]$
and the same formula with φ, ψ in [...] exchanged (by the definition of $\vee$). After obvious transformations we have to prove
BL $\vdash ((\varphi\&\chi) \to (\psi\&\chi)) \to [((\varphi \vee \psi)\&\chi) \to (\psi\&\chi)]$.
i.e. BL $\vdash (\varphi \to (\chi \to (\psi\&\chi))) \to [(\varphi \vee \psi) \to (\chi \to (\psi\&\chi))]$;
but this follows from BL $\vdash \psi \to (\chi \to (\psi\&\chi))$ and (16') in 2.2.10.

(b) Now let us prove the second formula.
$\vdash (\varphi \wedge \psi)\&\chi \to ((\varphi\&\chi) \wedge (\psi\&\chi))$ by (9) and (12) in 2.2.8; we prove the converse,
i.e. $\vdash ((\varphi\&\chi) \wedge (\psi\&\chi)) \to ((\varphi \wedge \psi)\&\chi)$, i.e.
$\vdash [(\varphi\&\chi)\&((\varphi\&\chi) \to \psi\&\chi)] \to ((\varphi \wedge \psi)\&\chi)$,or
$\vdash ((\varphi\&\chi) \to (\psi\&\chi)) \to [(\varphi\&\chi) \to ((\varphi \wedge \psi)\&\chi)]$.
Denote the last formula by (*); we prove
$\vdash (\varphi \to \psi) \to (*)$ and $\vdash (\psi \to \varphi) \to (*)$ (this suffices by axiom (A6)).
BL $\vdash (\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$ by the definition of $\wedge$,
BL $\vdash (\varphi \to (\varphi \wedge \psi)) \to ((\varphi\&\chi) \to ((\varphi \wedge \psi)\&\chi))$ (by 2.2.8 (6)), thus
BL $\vdash (\varphi \to \psi) \to$ (anything $\to ((\varphi\&\chi) \to ((\varphi \wedge \psi)\&\chi)))$; on the other hand,
BL $\vdash (\psi \to \varphi) \to ((\psi\&\chi) \to ((\varphi \wedge \psi)\&\chi))$,thus
BL $\vdash (\psi \to \varphi) \to ([(\psi\&\chi) \to (\psi\&\chi)] \to [(\psi\&\chi) \to ((\varphi \wedge \psi)\&\chi)]$.)
Using (A6) we get (*). (30) is proved.

(31) Since BL $\vdash (\varphi \wedge \psi) \to (\varphi \wedge (\psi \vee \chi))$ and similarly $(\varphi \wedge \chi) \to (\varphi \wedge (\psi \vee \chi))$ we get BL $\vdash ((\varphi \wedge \psi) \vee (\varphi \wedge \chi)) \to (\varphi \wedge (\psi \vee \chi))$. We prove the converse implication by applying the definition of $\wedge$ and the distributing of & with respect to $\vee$ (see (30)):
BL $\vdash ((\psi \vee \chi) \wedge \varphi) \equiv [(\psi \vee \chi)\&((\psi \vee \chi) \to \varphi)] \equiv [(\psi\&((\psi \vee \chi) \to \varphi)) \wedge (\chi\&((\psi \vee \chi) \to \varphi))]$;
in BL, the last formula implies $[\psi\&(\chi \to \varphi)] \wedge [\chi\&(\chi \to \varphi)]$, i.e.

$(\psi \wedge \varphi) \wedge (\chi \wedge \varphi)$.
It remains to prove
$\mathrm{BL} \vdash ((\varphi \vee \psi) \wedge (\varphi \vee \chi)) \equiv (\varphi \vee (\psi \wedge \chi))$.
But this can be obtained from the dual distributivity as is usual in lattice theory: the following chain of equivalences is provable:
$[(\varphi \vee \psi) \wedge (\varphi \vee \chi)] \equiv [((\varphi \vee \psi) \wedge \varphi) \vee ((\varphi \vee \psi) \wedge \chi)] \equiv [\varphi \vee ((\varphi \vee \chi) \wedge \chi)] \equiv [\varphi \vee (\varphi \wedge \chi) \vee (\psi \wedge \chi)] \equiv [\varphi \vee (\psi \wedge \chi)]$.

□

Lemma 2.2.24 BL proves:

(32) $(\varphi \vee \psi) \& (\varphi \vee \psi) \rightarrow ((\varphi \& \varphi) \vee (\psi \& \psi))$
$(\varphi \wedge \psi) \& (\varphi \wedge \psi) \rightarrow ((\varphi \& \varphi) \wedge (\psi \& \psi))$

(33) $(\varphi \rightarrow \psi)^n \vee (\psi \rightarrow \varphi)^n$, for each n.

Proof:

(32) We write again φ^2 for $\varphi \& \varphi$ etc.; we have to prove $(\varphi \vee \psi)^2 \equiv (\varphi^2 \vee \psi^2)$. Now, by (30), $\vdash (\varphi \vee \psi)^2 \equiv (\varphi^2 \vee \psi^2 \vee \varphi \& \psi)$, thus it suffices to prove $\vdash (\varphi \& \psi) \rightarrow (\varphi^2 \vee \psi^2)$ to get $(\varphi^2 \vee \psi^2 \vee \varphi \& \psi) \equiv \varphi^2 \vee \psi^2$. Prove by cases $(\varphi \rightarrow \psi), (\psi \rightarrow \varphi)$: $\vdash (\varphi \rightarrow \psi) \rightarrow (\varphi \& \psi \rightarrow \psi^2)$, thus $\vdash (\varphi \rightarrow \psi) \rightarrow (\varphi \& \psi \rightarrow \varphi^2 \vee \psi^2)$, and dually, $\vdash (\varphi \rightarrow \psi) \rightarrow (\varphi \& \psi \rightarrow \varphi^2 \vee \psi^2)$, which gives the result.

(33) First show that (15) and (5) give $\vdash [(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)]^2$. By (32) we get $(\varphi \rightarrow \psi)^2 \vee (\psi \rightarrow \varphi)^2$. Iterate and use $((\varphi \rightarrow \psi)^{n+1} \vee (\psi \rightarrow \varphi)^{n+1}) \rightarrow ((\varphi \rightarrow \psi)^n \vee (\psi \rightarrow \varphi)^n)$, which comes easily from $\vdash \varphi^{n+1} \rightarrow \varphi^n$ and monotonicity of $\vee$ (see (22)).

□

To close this section we show that BL proves de Morgan laws.

Theorem 2.2.25 (34) $(\neg\varphi \wedge \neg\psi) \equiv \neg(\varphi \vee \psi)$,

(35) $(\neg\varphi \vee \neg\psi) \equiv \neg(\varphi \wedge \psi)$.

Proof:

(34) Since BL $\vdash (\varphi \wedge \psi) \to \varphi$ we have BL $\vdash \neg\varphi \to \neg(\varphi \wedge \psi)$. Similarly, BL $\vdash \neg\psi \to \neg(\varphi \wedge \psi)$; thus BL $\vdash (\neg\varphi \vee \neg\psi) \to \neg(\varphi \wedge \psi)$ (see (16′)).
Conversely, BL $\vdash (\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$ (see (10)), thus BL $\vdash (\varphi \to \psi) \to (\neg(\varphi \wedge \psi) \to \neg\varphi)$ and BL $\vdash (\varphi \to \psi) \to (\neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi))$.
Similarly, BL $\vdash (\psi \to \varphi) \to (\neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi))$, thus applying axiom (A6) we get BL $\vdash \neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi)$.

(35) BL $\vdash \varphi \to (\varphi \vee \psi)$, thus BL $\vdash \neg(\varphi \vee \psi) \to \neg\varphi$; analogously, BL $\vdash \neg(\varphi \vee \psi) \to \neg\psi$ and hence BL $\vdash \neg(\varphi \vee \psi) \to (\neg\varphi \wedge \neg\psi)$.
Conversely, we show BL $\vdash ((\neg\varphi \wedge \neg\psi)\&(\varphi \vee \psi)) \to \bar{0}$, which gives BL $\vdash (\neg\varphi \wedge \neg\psi) \to \neg(\varphi \vee \psi)$. First work with the assumption $(\varphi \to \psi)$. BL $\vdash (\varphi \to \psi) \to ((\varphi \vee \psi) \equiv \psi)$ (see (14)), analogously using BL $\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ we get BL $\vdash (\varphi \to \psi) \to ((\neg\varphi \wedge \neg\psi) \equiv \neg\psi)$ (see (10)), thus
BL $\vdash (\varphi \to \psi) \to ((\neg\varphi \wedge \neg\psi)\&(\varphi \vee \psi) \equiv (\psi\&\neg\psi))$ and $\psi\&\neg\psi$ may be replaced by $\bar{0}$. Analogously we get
BL $\vdash (\psi \to \varphi) \to ((\neg\varphi \wedge \neg\psi)\&(\varphi \vee \psi) \equiv (\varphi\&\neg\varphi))$, thus using (A6) we get
BL $\vdash (\neg\varphi \wedge \neg\psi)\&(\varphi \vee \psi) \to \bar{0}$, which completes the proof. □

2.3. Residuated Lattices; a Completeness Theorem

2.3.1 In the two preceding sections, we studied continuous t-norms as candidates for truth functions of the conjunction, corresponding residua as truth functions of the implication and showed definability of other truth functions (the corresponding negation, minimum and maximum). For each fixed continuous t-norm $*$ we get a corresponding propositional calculus $PC(*)$; we formulated logical axioms that are 1-tautologies in each $PC(*)$, defined provability and showed many formulas to be provable in the basic logic BL. The logic is sound: each provable formula is a 1-tautology of each $PC(*)$.

Now we are in the proper place to start an *algebraization* of BL. We shall introduce a variety of algebras called *BL-algebras* and show that

(i) for each t-norm $*$, the unit interval $[0, 1]$ endowed with the truth functions of connectives is a linearly ordered BL-algebra,

(ii) BL is sound even for each linearly ordered BL-algebra, i.e. each provable formula is a 1-tautology over each such lattice (in an obvious meaning explicitly stated below),

(iii) the set of all formulas, factorized by provable equivalence (i.e. the set of classes of provably equivalent formulas), endowed with operations given by connectives, is a BL-algebra (not linearly ordered), and

(iv) a formula which is a tautology over all linearly ordered BL-algebras is a tautology over all BL-algebras (linearly ordered or not).

This will give us the desired completeness theorem. BL-algebras will be defined as particular residuated lattices.

Granted this, the theorem 2.3.16 on subdirect product will immediately give us a completeness theorem, stating that the implication in (ii) can be reverted. This is the task of the present section. In the next three sections we shall investigate calculi based on three particular choices of a t-norm (cf. 2.1.5) and exhibit completeness theorems for them.

Definition 2.3.2 A *residuated lattice*[12] is an algebra

$$(L, \cap, \cup, *, \Rightarrow, 0, 1)$$

with four binary operations and two constants such that

(i) $(L, \cap, \cup, 0, 1)$ is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering $\leq$),

(ii) $(L, *, 1)$ is a commutative semigroup with the unit element 1, i.e. $*$ is commutative, associative, $1 * x = x$ for all x.

(iii) $*$ and $\Rightarrow$ form an adjoint pair, i.e.

(1) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$ for all x, y, z.

Definition 2.3.3 A residuated lattice $(L, \cap, \cup, *, \Rightarrow, 0, 1)$ is a *BL-algebra* iff the following two identities hold for all $x, y \in L$:

(2) $x \cap y = x * (x \Rightarrow y)$

(3) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

(The last axiom will be called the axiom of *prelinearity* from reasons that become apparent later.)

Lemma 2.3.4 In each BL-algebra, the following hold in each x, y, z:

---

[12] See [34].

(4) $x * (x \Rightarrow y) \leq y$ and $x \leq (y \Rightarrow (x * y))$;

(5) $x \leq y$ implies $x * z \leq y * z$, $(z \Rightarrow x) \leq (z \Rightarrow y)$, $(y \Rightarrow z) \leq (x \Rightarrow z)$;

(6) $x \leq y$ iff $x \Rightarrow y = 1$;

(7) $(x \cup y) * z = (x * z) \cup (y * z)$;

(8) $x \cup y = ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$

Proof:

(4) $x * (x \Rightarrow y) \leq y$ follows from $(x \Rightarrow y) \leq (x \Rightarrow y)$ by the adjointness condition (iii) in 2.3.2; similarly $x \leq y \Rightarrow (x*y)$ follows from $(x*y) \leq (x * y)$.

(5) Assume $x \leq y$; by (4), $y \leq (z \Rightarrow y * z)$, thus $x \leq (z \Rightarrow y * z)$ and $x * z \leq y * z$ by adjointness.

Furthermore, assuming $x \leq y$ we have $z * (z \Rightarrow x) \leq x \leq y$, thus $(z \Rightarrow x) \leq (z \Rightarrow y)$, and also $x * (y \Rightarrow z) \leq y * (y \Rightarrow z) \leq z$, thus $(y \Rightarrow z) \leq (x \Rightarrow z)$.

(6) $x \leq y$ iff $1 * x \leq y$ iff $1 \leq (x \Rightarrow y)$ iff $1 = (x \Rightarrow y)$.

(7) Since $x \leq x \cup y$ we get $x * z \leq (x \cup y) * z$. Similarly, $y * z \leq (x \cup y) * z$; thus $(x*z) \cup (y*z) \leq (x \cup y)*z$. Conversely, $x*z \leq (x*z) \cup (y*z)$, thus $x \leq (z \Rightarrow [(x*z) \cup (y*z)])$; similarly, $y \leq (z \Rightarrow [(x*z) \cup (y*z)])$, hence $(x \cup y) \leq z \Rightarrow [(x*z) \cup (y*z)]$ and therefore $(x \cup y)*z \leq (x*z) \cup (y*z)$.

(8) $[((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)] =$
$= [\cdots] * ((x \Rightarrow y) \cup (y \Rightarrow x)) = ([\cdots] * (x \Rightarrow y)) \cup ([\cdots] * (y \Rightarrow x)) \leq$
$\leq [((x \Rightarrow y) \Rightarrow y) * (x \Rightarrow y)] \cup [((y \Rightarrow x)] \Rightarrow x) * (y \Rightarrow x)] \leq y \cup x =$
$x \cup y$.
On the other hand,
$(x \Rightarrow y) * (x \cup y) = (x * (x \Rightarrow y)) \cup (y * (x \Rightarrow y))) \leq y \cup y = y$,
thus $x \cup y \leq (x \Rightarrow y) \Rightarrow y$,
similarly $x \cup y \leq (y \Rightarrow x) \Rightarrow x$,
and hence $x \cup y \leq ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$.

□

Definition 2.3.5 A residuated lattice $(L, \cap, \cup, *, \Rightarrow, 0, 1)$ is *linearly ordered* if its lattice ordering is linear, i.e. for each pair x, y
$x \cap y = x$ or $x \cap y = y$ (equivalently, $x \cup y = x$ or $x \cup y = y$).

Note that the class of linearly ordered residuated lattices is not a variety (not closed under direct products).

Lemma 2.3.6 A linearly ordered residuated lattice is a BL-algebra iff the identity $x \cap y = x * (x \Rightarrow y)$ is true in it.

Proof: Just check that the proof of 2.3.3 (2) (for t-norms) in 2.1.10 works. Prelinearity is evident. □

Remark 2.3.7 (i) Show that the condition (2) $(x \cap y = x * (x \Rightarrow y))$ is satisfied provided the linearly ordered residuated lattice is *divisible*, i.e. for each x, y such that $x > y$ there is a z such that $y = x * z$.

(ii) Clearly each continuous t-norm determines a BL-algebra on the unit interval $[0, 1]$ with its standard linear ordering.

Definition 2.3.8 Let **L**=$(L, \cap, \cup, *, \Rightarrow, 0, 1)$ be a BL-algebra. In an analogy to 2.2.2 we define an **L**-evaluation of propositional variables to be any mapping e assigning to each propositional variable p an element $e(p)$ of **L**. This extends in the obvious way to an evaluation of all formulas using the operations on **L** as truth functions, i.e.

$$\begin{aligned} e(\bar{0}) &= 0 \\ e(\varphi \to \psi) &= e(\varphi) \Rightarrow e(\psi) \\ e(\varphi \& \psi) &= e(\varphi) * e(\psi) \end{aligned}$$

(and hence $e(\varphi \wedge \psi) = e(\varphi) \cap e(\psi), e(\varphi \vee \psi) = e(\varphi) \cup e(\psi), e(\neg\varphi) = e(\varphi) \Rightarrow 0$.)
A formula φ is an **L**-tautology if $e(\varphi) = 1$ for each **L**-evaluation e.

Theorem 2.3.9 The logic BL is sound with respect to BL-tautologies: if φ is provable in BL, then φ is an **L**-tautology for each BL-algebra **L**. More generally, if T is a theory over BL and T proves φ, then, for each BL-algebra **L** and each **L**-evaluation e of propositional variables assigning the value 1 to all the axioms of T we have $e(\varphi) = 1$.

Proof: We have to show that all axioms of BL are **L**-tautologies and also that the definition of $x \cup y$ from $\Rightarrow$ is an **L**-tautology. All axioms except (A6) immediately verify as in 2.2.6. We verify (A6):
$((x \Rightarrow y) \Rightarrow z) * ((y \Rightarrow x) \Rightarrow z) =$

$[((x \Rightarrow y) \Rightarrow z) * ((y \Rightarrow x) \Rightarrow z] * ((x \Rightarrow y) \cup (y \Rightarrow x)) =$
$[\cdots] * (x \Rightarrow y) \cup [\cdots] * (y \Rightarrow x) \leq$
$\leq [((x \Rightarrow y) \Rightarrow z) * (x \Rightarrow y)] \cup [((y \Rightarrow x) \Rightarrow z) * (y \Rightarrow x)] \leq z \cup z = z.$
Finally, tautologicity of the definition of $x \cup y$ was proved as (8) above. □

Lemma 2.3.10 The class of all BL-algebras is a variety of algebras.

Proof: As we know from chapter 1, the class of all lattices is a variety; also the conditions on $0, 1$ are expressible by identities $(x \cap 1 = x, x \cap 0 = 0)$. Similarly the conditions on $*$ (commutativity, associativity, 1 is a unit) are identities. We verify that the adjointness condition is expressed by the following:

(9) $x \cap (y \Rightarrow (x * y)) = x$,

(10) $((x \Rightarrow y) * x) \cup y = y$,

(11) $(x \Rightarrow (x \cup y)) = 1$,

(12) $((z \Rightarrow x) \Rightarrow (z \Rightarrow (x \cup y))) = 1$,

(13) $(x \cap y) * z = (x * z) \cap (y * z)$.

First observe that they are true in each BL-algebra: (9) is equivalent to $x \leq y \Rightarrow (x * y)$ and (10) to $(x \Rightarrow y) * x \leq y$ – see (4) above.

(11) follows from $x \leq x \cup y$ by (6). (12) is a particular case of the fact $y \leq z$ implies $(x \Rightarrow y) \leq (x \Rightarrow z)$ [$\Rightarrow$ is non-decreasing in the second argument — indeed, if $y \leq z$, $u \leq (x \Rightarrow y)$ implies $x * u \leq y$ implies $x * u \Rightarrow z$ implies $u \leq (x \Rightarrow z)$].
(13) is one of distributivities; it follows from the corresponding provability 2.2.23 (30) by 2.3.9

Conversely, replace in the definition of a BL-algebra the adjointness condition by (9)–(13); we prove the adjointness.
(14) $x \leq y$ implies $x * z \leq y * z$.
Indeed, if $x \leq y$, then $x = x \cap y$, thus $x * z = (x \cap y) * z = (x * z) \cap (y * z)$ by (13), hence $x * z \leq y * z$.
(15) $x \leq y$ iff $(x \Rightarrow y) = 1$.
On the one hand, $x \leq y$ implies $(x \cup y) = y$, hence $x \Rightarrow y = 1$ by (12).
On the other hand, if $(x \Rightarrow y) = 1$, then $x \cap y = x * (x \Rightarrow y) = x * 1 = x$, thus $x \leq y$.
(16) $x \leq y$ implies $(z \Rightarrow x) \leq (z \Rightarrow y)$.
This follows directly from (12) since $x \leq y$ gives $y = x \cup y$.
(17) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$.

Using (14)–(16), reason as follows: $z \leq x \Rightarrow (x * z)$, thus if $x * z \leq y$, then $z \leq x \Rightarrow y$. Similarly, if $z \leq x \Rightarrow y$ then $x * z \leq x * (x \Rightarrow y) \leq y$. This proves adjointness. □

We shall now show that classes of provably equivalent formulas form a BL-algebra.

Definition 2.3.11 Let T be a fixed theory over BL. For each formula φ, let $[\varphi]_T$ be the set of all formulas ψ such that $T \vdash \varphi \equiv \psi$ (formulas T-provably equivalent to φ). L_T is the set of all the classes $[\varphi]_T$. We define

$$\begin{array}{rcl} 0 & = & [\bar{0}]_T, \\ 1 & = & [\bar{1}]_T, \\ [\varphi]_T * [\psi]_T & = & [\varphi \& \psi]_T, \\ [\varphi]_T \Rightarrow [\psi]_T & = & [\varphi \to \psi]_T, \\ [\varphi]_T \cap [\psi]_T & = & [\varphi \wedge \psi]_T, \\ [\varphi]_T \cup [\psi]_T & = & [\varphi \vee \psi]_T, \end{array}$$

(This is correct due to the provabilities (26)-(28) in 2.2.16.) This algebra is denoted by $\mathbf{L}_T$.

Lemma 2.3.12 $\mathbf{L}_T$ is a BL-algebra.

Proof: For lattice properties of $\cap, \cup$ see the provabilities (9), (13), (21), (22), (23) and, for idempotence, (11) or (16); for semigroup properties of $*$ see axioms (A3) and provabilities (8), (20). Now observe that the lattice ordering $\leq$ satisfies the following:

$$[\varphi]_T \leq [\psi]_T \text{ iff } \quad T \vdash \varphi \to \psi.$$

Indeed, if $T \vdash \varphi \to \psi$ then $T \vdash \varphi \equiv (\varphi \wedge \psi)$ (by obvious proof), thus $[\varphi]_T = [\varphi]_T \cap [\psi]_T$ and $[\varphi]_T \leq [\psi]_T$. Conversely, if $[\varphi]_T \leq [\psi]_T$, thus $T \vdash \varphi \equiv (\varphi \wedge \psi)$ then $T \vdash \varphi \to \psi$ (since over BL, $T \vdash \varphi \wedge \psi \to \psi$).

We show the adjointness property.
$[\chi]_T \leq [\varphi]_T \Rightarrow [\psi]_T$ iff $T \vdash \chi \to (\varphi \to \psi)$ iff $T \vdash (\chi \& \varphi) \to \psi$ iff $[\chi \& \varphi]_T \leq [\psi]_T$. Thus $\mathbf{L}_T$ is a residuated lattice.

The validity of conditions (2), (3) in the definition of a BL-algebra follows from the definition of the conjunction $\wedge$ and from the provability of (15). This completes the proof. □

Now we shall show how filters on residuated lattices determine homomorphisms and characterize homomorphisms to linearly ordered lattices.

Definition 2.3.13 Let $\mathbf{L} = (L, \cap, \cup, *, \Rightarrow, 0, 1)$ be a residuated lattice. A *filter* on $\mathbf{L}$ is a non-empty set $F \subseteq L$ such that for each $x, y \in L$,

$$a \in F \quad \text{and} \quad b \in F \quad \text{implies} \quad a * b \in F,$$
$$a \in F \quad \text{and} \quad a \leq b \quad \text{implies} \quad b \in F.$$

F is a *prime filter* iff for each $x, y \in L$,

$$(x \Rightarrow y) \in F \quad \text{or} \quad (y \Rightarrow x) \in F.$$

Lemma 2.3.14 Let $\mathbf{L}$ be a BL-algebra and let F be a filter. Put

$$x \sim_F y \quad \text{iff} \quad (x \Rightarrow y) \in F \quad \text{and} \quad (y \Rightarrow x) \in F.$$

Then

(i) $\sim_F$ is a congruence and the corresponding quotient algebra $\mathbf{L}/\sim_F$ is a BL-algebra.

(ii) $\mathbf{L}/\sim_F$ is linearly ordered iff F is a prime filter.

Proof: To show that $\sim_F$ is an equivalence we just observe that $\sim_F$ is transitive: this follows from the fact that the formula $((\varphi \to \psi) \& (\psi \to \chi)) \to (\varphi \to \chi)$ is a 1-tautology over $\mathbf{L}$, thus $((x \Rightarrow y) * (y \Rightarrow z)) \leq (x \Rightarrow z)$ for each x, y, z; if $(x \Rightarrow y), (y \Rightarrow z) \in F$ then $(x \Rightarrow z) \in F$. Thus we may define equivalence classes $[x]_F = \{y \mid x \sim_F y\}$. One verifies that $\sim_F$ is a congruence, i.e. preserves the operation, by imitating the proof of 2.3.12, i.e. proving $[x]_F \leq [y]_F$ iff $(x \Rightarrow y) \in F$ and verifying that $[x]_F = [y]_F$ implies $[x * z]_F = [y * z]_F$, $[x \Rightarrow z]_F = [y \Rightarrow z]_F$ and $[z \Rightarrow x]_F = [z \Rightarrow y]_F$. Thus we may induce the operations $*, \Rightarrow$ (and $\cap, \cup$) on the set $L/\sim_F$ of equivalence classes setting $[x]_F * [y]_F = [x * y]_F$ etc.; the mapping assigning to each x its class $[x]_F$ is then a homomorphism and $L/\sim_F$ under the induced operation is a BL-algebra (since BL-algebras form a variety of algebras and hence is closed under homomorphisms: a homomorphism preserves validity of identities).

Now assume F to be a prime filter and let $x, y \in L$; then either $(x \Rightarrow y) \in F$ and, then $[x]_F \leq [y]_F$ or $(y \Rightarrow x) \in F$ and, then $[y]_F \leq [x]_F$; $\leq$ is linear. Conversely, if $\mathbf{L}/\sim_F$ is linearly ordered and $x, y \in L$, then either $[x]_F \leq [y]_F$ and $(x \Rightarrow y) \in F$ or $[y]_F \leq [x]_F$ and $(y \Rightarrow x) \in F$; F is a prime filter. $\square$

Lemma 2.3.15 Let $\mathbf{L}$ be a BL-algebra and let $a \in L, a \neq 1$. Then there is a prime filter F on $\mathbf{L}$ not containing a.

Proof: There are filters not containing a, e.g. $F_0 = \{1\}$. We shall show that if F is any filter not containing a and $x, y \in L$ are such that $(x \Rightarrow y) \notin F$ and $(y \Rightarrow x) \notin F$, then there is a filter $F' \supseteq F$ not containing a but containing either $(x \Rightarrow y)$ or $(y \Rightarrow x)$. Note that the least filter F' containing F as a subset and z as an element is $F' = \{u \mid (\exists\, v \in F)(\exists\, n \,\text{natural})(v * z^n \leq u)\}$. Indeed, if $F'' \supseteq F$ is a filter and $z \in F$ then for each $v \in F$ and n natural, $v * z^n \in F''$; on the other hand, F' itself is a filter since it is obviously closed under $*$ and contains with each z all $z' \geq z$.

Thus assume $(x \Rightarrow y) \notin F$, $(y \Rightarrow x) \notin F$ and let F_1, F_2 be the smallest filters containing F as a subset and $(x \Rightarrow y), (y \Rightarrow x)$ respectively as an element. We claim that $a \notin F_1$ or $a \notin F_2$. Assume the contrary; then for some $v \in F$ and natural n, $v * (x \Rightarrow y)^n \leq a$ and $v * (y \Rightarrow x)^n \leq a$, thus $a \geq v*(x \Rightarrow y)^n \cup v*(y \Rightarrow x)^n = v*((x \Rightarrow y)^n \cup (y \Rightarrow x)^n) = v*1 = v$, thus $a \in F$, a contradiction. (We use here the provabilities 2.2.24 (33).) Thus $a \notin F_1$ or $a \notin F_2$.

Now if $\mathbf{L}$ is countable (which will be our case in the proof of completeness), then we may arrange all pairs (x, y) from L^2 into a sequence $\{(x_n, y_n) \mid n \text{ natural }\}$, put $F_0 = \{1\}$ and having constructed F_n such that $a \notin F_n$ we take $F_{n+1} \supseteq F_n$ such that $a \notin F_{n+1}$ according to our construction; if possible we take F_{n+1} such that $(x_n \Rightarrow y_n) \in F_{n+1}$, if not, we take that with $(y_n \Rightarrow x_n) \in F_{n+1}$. Our desired prime filter is the union

$$\bigcup_n F_n.$$

If $\mathbf{L}$ is uncountable, then one has to use the axiom of choice and work similarly with a transfinite sequence of filters. □

Lemma 2.3.16 Each BL-algebra is a subalgebra of the direct product of a system of linearly ordered BL-algebras.

Proof: Let $\mathcal{U}$ be the system of all prime filters on $\mathbf{L}$. For $F \in \mathcal{U}$ let $\mathbf{L}_F = \mathbf{L}/F$ and let

$$\mathbf{L}^* = \prod_{F \in \mathcal{U}} \mathbf{L}_F.$$

$\mathbf{L}^*$ is the direct product of linearly ordered residuated lattices $\{\mathbf{L}_F \mid F \in \mathcal{U}\}$ of $\mathbf{L}^*$. For $x \in \mathbf{L}$ let $i(x)$ be the element $\{[x]_F | F \in \mathcal{U}\}$ of $\mathbf{L}^*$. Clearly this embedding preserves operations; it remains to show that it is one-one. If

$x, y \in F$ and $x \neq y$ then $x \not\leq y$ or $y \not\leq x$. Assume the former; then $(x \Rightarrow y) \neq 1$ in $\mathbf{L}$. By 2.3.15 let F be an prime filter on $\mathbf{L}$ not containing $(x \Rightarrow y)$; then in $\mathbf{L}/F$, $[x]_F \not\leq [y]_F$, hence $[x]_F \neq [y]_F$ and therefore $i(x) \neq i(y)$. □

Definition 2.3.17 Associate with each formula φ of BL a term $\varphi^\bullet$ of the language of residuated lattices by replacing the connectives $\rightarrow, \&, \wedge, \vee, \bar{0}, \bar{1}$ by function symbols and constants $\Rightarrow, *, \cap, \cup, 0, 1$ respectively and replacing each propositional variable p_i by a corresponding object variable x_i.

Lemma 2.3.18 (1) Each formula which is an $\mathbf{L}$-tautology for all linearly ordered BL-algebras is an $\mathbf{L}$-tautology for *all* BL-algebras.

(2) φ is an $\mathbf{L}$-tautology iff the identity $\varphi^\bullet = 1$ is true in $\mathbf{L}$.

Proof:

(1) follows from (2) and the subdirect product representation;

(2) is evident since the value of the term $\varphi^\bullet$ given by an evaluation e is just $e_{\mathbf{L}}(\varphi)$.

□

Theorem 2.3.19 *(Completeness)* BL is complete, i.e. for each formula φ the following there things are equivalent:

(i) φ is provable in BL,

(ii) for each linearly ordered BL-algebra $\mathbf{L}$, φ is an $\mathbf{L}$-tautology;

(iii) for each BL-algebra $\mathbf{L}$, φ is an $\mathbf{L}$- tautology.

Proof: The implications from (i) to (ii) and from (ii) to (iii) have been established. Thus assume (iii) and prove (i). To this end recall 2.3.12 saying, among other things, that the algebra $\mathbf{L}_{BL}$ of classes of equivalent formulas of BL is a BL-algebra; thus, a φ satisfying (iii) is an $\mathbf{L}_{BL}$-tautology. In particular, let $e(p_i) = [p_i]_{BL}$ for all propositional variables; then $e(\varphi) = [\varphi]_{BL} = [1]_{BL}$, thus BL $\vdash \varphi \equiv 1$, hence BL $\vdash \varphi$. □

We shall generalize this completeness theorem as follows:

Definition 2.3.20 (1) An *axiom schema* given by a formula $\Phi(p_1, \ldots, p_n)$ is the set of all formulas $\Phi(\varphi_1, \ldots, \varphi_n)$ resulting by the substitution of φ_i for $p_i (i = 1, \ldots, n)$ in $\Phi(p_1, \ldots, p_n)$.

(2) A logical calculus $\mathcal{C}$ is a *schematic extension* of BL if it results from BL by adding some (finitely or infinitely many) axiom schemata to its axioms. (The deduction rule remains modus ponens.)

(3) Let $\mathcal{C}$ be a schematic extension of BL and let $\mathbf{L}$ be a BL-algebra. $\mathbf{L}$ is a *$\mathcal{C}$-algebra* if all axioms of $\mathcal{C}$ are $\mathbf{L}$-tautologies.

Remark 2.3.21 Note that axioms of BL are given by axiom schemata. Note furthermore that each schematic extension of BL has the *substitution property*: if φ is an axiom and ψ results from φ by a substitution (of formulas for propositional variables), then ψ is also an axiom.

Theorem 2.3.22 *(Completeness)* Let $\mathcal{C}$ be a schematic extension of BL and let φ be a formula. The following are equivalent:

(i) $\mathcal{C}$ proves φ,

(ii) φ is an $\mathbf{L}$-tautology for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$,

(iii) φ is an $\mathbf{L}$-tautology for each $\mathcal{C}$-algebra $\mathbf{L}$.

Proof: The implication (i) $\Rightarrow$ (ii) is soundness and is proved as usual. (ii) $\Rightarrow$ (iii) follows by the lemma 2.3.16 and its proof: an arbitrary $\mathcal{C}$-algebra is embedded into a direct product of its linearly ordered factor algebras — and the factor algebras are also $\mathcal{C}$-algebras. To prove (iii) $\Rightarrow$ (i) observe that the algebra $\mathbf{L}_{\mathcal{C}}$ of classes of mutually $\mathcal{C}$-equivalent formulas is itself a $\mathcal{C}$-algebra: if $\Phi(\varphi_1, \dots, \varphi_n)$ is an instance of the axiom schema $\Phi(p_1, \dots, p_n)$ and $e(p_i) = [\psi_i]_{\mathcal{C}}$ then $e(\Phi(\varphi_1, \dots, \varphi_n)) = [\Phi(\varphi_1', \dots, \varphi_n')]_{\mathcal{C}}$ where φ_i' results from φ_i by substituting ψ_i for p_i, thus $\Phi(\varphi_1', \dots, \varphi_n')$ is also an instance of the schema and therefore $[\Phi(\varphi_1', \dots, \varphi_n')]_{\mathcal{C}} = [1]_{\mathcal{C}}$. □

Remark 2.3.23 We know that each continuous t-norm on $[0, 1]$ determines a BL-algebra (see 2.3.7). Call algebras determined by continuous t-norms *t-algebras* and call a formula φ a *t-tautology* if it is an $\mathbf{L}$-tautology for each t-algebra $\mathbf{L}$. Clearly, if BL$\vdash \varphi$ then φ is a t-tautology. We have the following interesting problem (of t-completeness of BL). Is each t-tautology provable in BL? Thus is BL a complete axiomatization of the intersection of all logics given by continuous t-norms? Or can one find a formula which is a t-tautology but is unprovable in BL, thus for some BL-algebra $\mathbf{L}$ that is not a t-algebra, φ is not an $\mathbf{L}$-tautology?

This problem is a subject of intensive research. Presently an extension of BL by two (not too simple) axioms is known which is complete with respect to t-tautologies; but it is unknown whether the additional axioms are provable in BL or not.

2.4. SOME ADDITIONAL TOPICS

We shall discuss two additional topics. A strong completeness theorem, relating provability in a theory to truth in all models of the theory, and extension of the language of basic propositional logic by a connective $\triangle$ which is two-valued (has value 1 for 1 and 0 otherwise). This connective proves to be useful and we shall meet it from time to time in further chapters. We show that the completeness theorems obtained till now hold also for the enriched logic. Throughout the section we work with a fixed schematic extension $\mathcal{C}$ of BL as a given logic. Generalizing 2.2.17 in the obvious way we get

Definition 2.4.1 (1) A *theory* over $\mathcal{C}$ is a set T of formulas; elements of T are *axioms* of T. $T \vdash \varphi$ (or, more precisely, $T \vdash_{\mathcal{C}} \varphi$) means that φ is *provable* in T, i.e. there is a $\mathcal{C}$-*proof* of φ in T (a sequence each of whose members either is an axiom of $\mathcal{C}$ or an element of T or follows from some preceding members by modus ponens).

(2) Let $\mathbf{L}$ be a $\mathcal{C}$-algebra. An $\mathbf{L}$-evaluation e is an $\mathbf{L}$-*model* of T if $e_{\mathbf{L}}(\alpha) = 1_{\mathbf{L}}$ for each axiom $\alpha \in T$.

(3) T is *complete* if for each pair φ, ψ of formulas, $T \vdash (\varphi \to \psi)$ or $T \vdash (\psi \to \varphi)$.

Lemma 2.4.2 (1) T is complete iff the $\mathcal{C}$-algebra $\mathbf{L}_T$ is linearly ordered.

(2) If T is a theory and $T \nvdash \varphi$, then there is a consistent complete supertheory $T' \supseteq T$ such that $T' \nvdash \varphi$.

Proof: For $\mathbf{L}_T$ see 2.3.11–2.3.12 (they readily generalize for theories over $\mathcal{C}$). The proof of (1) is a direct analogy of the proof of 2.3.14 (ii); and that of (2) to the proof of 2.3.15. For the reader's convenience we sketch the proof of (2).

Let (α_n, β_n) be a sequence of all pairs of formulas; put $T_0 = T$ and having $T_n \nvdash \varphi$ observe that $T \cup \{\alpha_n \to \beta_n\} \nvdash \varphi$ or $T \cup \{\beta_n \to \alpha_n\} \nvdash \varphi$ (otherwise for some k, $T \vdash (\alpha_n \to \beta_n)^k \vee (\beta_n \to \alpha_n)^k \vdash \varphi$, thus $T \vdash \varphi$). Take T_{n+1} to be $T \cup \{\alpha_n \to \beta_n\}$ if this theory does not prove φ, $T_{n+1} = T \cup \{\beta_n \to \alpha_n\}$ otherwise; put

$$T' = \bigcup_n T_n.$$

□

Theorem 2.4.3 (Strong completeness.) Let T be a theory over $\mathcal{C}$ and let φ be a formula. Then the following are equivalent:

(i) $T \vdash_{\mathcal{C}} \varphi$;

(ii) For each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of T, $e_{\mathbf{L}}(\varphi) = 1_{\mathbf{L}}$.

(iii) For each $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model e of T, $e_{\mathbf{L}}(\varphi) = 1_{\mathbf{L}}$.

Proof: Soundness is clear: by 2.3.21 all axioms of $\mathcal{C}$ are true in all $\mathcal{C}$-models of T, axioms of T are true in all models of T by the definition of a model; and formulas true in a model are evidently closed under modus ponens (see 2.3.4 (8).

Conversely, assume $T \nvdash \varphi$, and let $T' \supseteq T$ be complete and $T' \nvdash \varphi$. Thus setting, for each ψ, $e(\psi) = [\psi]_T$ we get an $\mathbf{L}_T$-model of T in which $e(\varphi) < 1_{\mathbf{L}_T}$; $\mathbf{L}_T$ is a linearly ordered $\mathcal{C}$-algebra. □

*

Definition 2.4.4 In each linearly ordered BL-algebra $\mathbf{L}$ introduce the operation $\triangle$ by postulating $\triangle(1) = 1$, $\triangle(a) = 0$ for $a \neq 1$. Expand the language of BL by the corresponding new unary connective (denoted also by $\triangle$).

We shall exhibit an axiom system for the enriched language and isolate a variety of (expanded) BL-algebras for which we have completeness.

Definition 2.4.5 The language of the basic logic $\mathrm{BL}_\triangle$ is the language of BL expanded by the connective $\triangle$. *Axioms* of $\mathrm{BL}_\triangle$ are those of BL plus[13]

($\triangle$1) $\triangle\varphi \vee \neg\triangle\varphi$

($\triangle$2) $\triangle(\varphi \vee \psi) \rightarrow (\triangle\varphi \vee \triangle\psi)$

($\triangle$3) $\triangle\varphi \rightarrow \varphi$

($\triangle$4) $\triangle\varphi \rightarrow \triangle\triangle\varphi$

($\triangle$5) $\triangle(\varphi \rightarrow \psi) \rightarrow (\triangle\varphi \rightarrow \triangle\psi)$.

Deduction rules of $\mathrm{BL}_\triangle$ are modus ponens and *generalization*: from φ derive $\triangle\varphi$.

Definition 2.4.6 A $\triangle$-*algebra* is a structure $\mathbf{L} = \langle L, *, \Rightarrow, \sqcap, \sqcup, 0, 1, \triangle\rangle$ which is a BL-algebra expanded by an unary operation $\triangle$ in which the following formulas are true:

[13] The $\triangle$-axioms are from [8].

$$\triangle x \cup (-) \triangle x = 1$$
$$\triangle(x \cup y) \leq \triangle x \cup \triangle y$$
$$\triangle x \leq x$$
$$\triangle x \leq \triangle \triangle x$$
$$(\triangle x) * (\triangle(x \Rightarrow y)) \leq \triangle y$$
$$\triangle 1 = 1$$

Remark 2.4.7 (1) The axioms evidently resemble modal logic with $\triangle$ as necessity; but in the axiom on $\triangle(\varphi \vee \psi)$, $\triangle$ behaves as possibility rather than necessity.

(2) The definitions 2.3.7 of **L**-evaluation, **L**-tautology etc. generalize trivially to $\mathrm{BL}_\triangle$ and $\triangle$-algebras.

Lemma 2.4.8 (1) Linearly ordered $\triangle$-algebras are exactly algebras from 2.4.4.

(2) The class of all $\triangle$-algebras is a variety of algebras.

(3) If $\mathrm{BL}_\triangle$ proves φ, then φ is an **L**-tautology for each linearly ordered $\triangle$-algebra and hence for each $\triangle$-algebra.

Proof:

(1) One easily verifies that a linearly ordered BL-algebra expanded by $\triangle$ such that $\triangle 1 = 1$ and $\triangle a = 0$ for $a \neq 1$ satisfies the axioms of 2.4.6; on the other hand if a $\triangle$-algebra is linearly ordered then the first axiom says $\max(\triangle x, (-) \triangle x) = 1$, thus either $x = 1$ and $\triangle x = 1$ or $x < 1$, then $\triangle x \leq x < 1$, hence $(-) \triangle x = 1$ and therefore $\triangle x = 0$. ($a \to 0 = 1$ implies $1 \leq a \to 0$, hence $1 * a \leq 0$, $a = 0$).

(2) is evident from the definition ($\leq$ is a lattice ordering, i.e. $a \leq b$ is $a \cap b = a$).

(3) is also easy: for the linearly ordered case prove by induction of proofs. (Cf. 2.2.6, concerning necessitation: if $a = 1$, then $\triangle a = 1$ by the last axiom $\triangle 1 = 1$. Then use preservation by subdirect products as in 2.3.18.)

□

Remark 2.4.9 Before we prove completeness let us mention some useful properties of $\triangle$.

(1) The formula $\triangle(\neg\varphi)$ defines in each linearly ordered $\triangle$-algebra Gödel negation, i.e. $\triangle(x \Rightarrow 0)$ is 1 for $x = 0$ and is 0 otherwise. Indeed, if $\triangle(x \Rightarrow 0) = 1$ then $x \Rightarrow 0 = 1$, hence $1 \leq x \Rightarrow 0$, $x = 0$; thus $\triangle((-)x) = 1$ iff $x = 0$, hence for $x > 0$ we get $\triangle((-)x) = 0$. Setting $\triangledown(\varphi) = \triangle(\neg \triangle (\neg\varphi))$ we get double Gödel negation: it has value 1 for a positive value of φ, and value 0 for φ having the value 0. More generally, the formula $\psi \vee \triangle(\varphi \to \psi)$ evidently defines Gödel implication in $BL_\triangle$. Thus $BL_\triangle$ extends Gödel logic **G**. We shall prove more on this embedding in Chapter 4.

(2) The mapping I assigning to each formula φ of classical (Boolean) logic the formula $I(\varphi)$ which results by replacing each variable p by $\triangle(p)$ is obviously a faithful embedding of Boolean logic into $BL_\triangle$: φ is a Boolean tautology iff $BL_\triangle$ proves $I(\varphi)$.

Definition 2.4.10 Let $\mathcal{C}$ be a schematic extension of BL; the calculus $\mathcal{C}_\triangle$ is defined in an analogy to BL as an extension of $\mathcal{C}$ by the connective $\triangle$, the axioms $\triangle1$–$\triangle5$ from 2.4.6 and by the deduction rule of necessitation. $\mathcal{C}_\triangle$-algebras are expansions of $\mathcal{C}$-algebras that are $\triangle$-algebras (i.e. BL-algebras expanded by $\triangle$, satisfying identities given by $\mathcal{C}$ and the identities of 2.4.6).

Lemma 2.4.11 $\mathcal{C}_\triangle$ proves the following:

(1) $\triangle\varphi \equiv \triangle(\varphi\&\varphi)$,

(2) $\triangle\varphi \equiv \triangle\varphi\& \triangle \varphi$,

(3) $\triangle\varphi \to \varphi^n$ (for each n),

(4) $\triangle(\varphi\&\psi) \equiv \triangle\varphi\& \triangle \psi$.

Proof: (1) $\mathcal{C}_\triangle$ proves $(\varphi\&\varphi) \to \varphi$, thus $\triangle((\varphi\&\varphi) \to \varphi)$ (by generalization), thus $\triangle(\varphi\&\varphi) \to \triangle\varphi$; conversely,
$\mathcal{C}_\triangle \vdash \triangle\varphi \to (\triangle\varphi \to \triangle(\varphi\&\varphi))$
since it proves $\varphi \to (\varphi \to (\varphi\&\varphi))$,
$\mathcal{C}_\triangle \vdash \neg \triangle \varphi \to (\triangle\varphi \to \triangle(\varphi\&\varphi))$
since $\neg\alpha \to (\alpha \to \text{anything})$ is provable, cf. 2.2.12 (18), thus $\mathcal{C}_\triangle \vdash (\triangle\varphi \vee \neg \triangle \varphi) \to (\triangle\varphi \to \triangle(\varphi\&\varphi))$,
$\mathcal{C}_\triangle \vdash \triangle\varphi \to \triangle(\varphi\&\varphi)$ by $(\triangle1)$.

The proof of (2) is similar but simpler. From (1) we get $\mathcal{C}_\triangle \vdash \triangle\varphi \to \triangle\varphi^n$ for each n, so that we get (3) by $(\triangle3)$. For (4) use $\triangle(\varphi\&\psi) \to (\triangle(\varphi\&\psi)\&\triangle(\varphi\&\psi)) \to (\triangle\varphi\& \triangle \psi))$; the converse is obvious. □

Theorem 2.4.12 (Completeness.)

(1) The following are equivalent:

(i) $\mathcal{C}_\triangle \vdash \varphi$

(ii) φ is an **L**-tautology for each linearly ordered $\mathcal{C}_\triangle$-algebra

(iii) φ is an **L**-tautology for each $\mathcal{C}_\triangle$-algebra.

(2) Let T be a theory over $\mathcal{C}_\triangle$. The following are equivalent:

(i) $T \vdash_{\mathcal{C}_\triangle} \varphi$

(ii) For each linearly ordered $\mathcal{C}_\triangle$-algebra **L**, φ is true in each **L**-model of T,

(iii) the same for *all* $\mathcal{C}_\triangle$-algebras **L**.

Proof: This is again a variant of preceding completeness proofs; we indicate the important point. This concerns the notion of a *filter* for $\mathcal{C}_\triangle$-algebras. The necessary modification of the definition of a filter on a $\mathcal{C}_\triangle$-algebra results by adding the condition

$$x \in F \text{ implies } \triangle x \in F$$

to 2.3.13. Then carefully read 2.3.14; show that for a filter F, $x \sim_F y$ implies $\triangle x \sim_F \triangle y$ (since $(x \Rightarrow y) \in F$ implies $\triangle(x \Rightarrow y) \in F$ and hence $(\triangle x \Rightarrow \triangle y) \in F$). Thus $L/\sim_F$ is a $\mathcal{C}_\triangle$-algebra for each filter; it is linearly ordered if F is a prime filter.

Then inspect 2.3.15 — construction of a prime filter F on **L** not containing an $a < 1$. Observe that given a filter F the least filter containing F as a subset and z as an element is

$$F' = \{u \mid (\exists v \in F)(u \geq v * \triangle z)\}.$$

(Indeed, if $u_i \geq v_i * \triangle z$ then $u_1 * u_2 \geq v_1 * v_2 * \triangle z * \triangle z = v_1 * v_2 * \triangle z$; if $u \geq v * \triangle z$, then $\triangle u \geq \triangle(v * \triangle z) = \triangle v * \triangle \triangle z = \triangle v * z$.) Now if F_1, F_2 extend F by $(x \Rightarrow y), (y \Rightarrow x)$ respectively and $a \in F_1, a \in F_2$, then for some $v \in F$,
$a \geq (v * \triangle(x \Rightarrow y)) \vee (v * \triangle(y \Rightarrow x)) = v * (\triangle(x \Rightarrow y) \vee \triangle(y \Rightarrow x)) \geq v * \triangle((x \Rightarrow y) \vee (y \Rightarrow x)) = v * \triangle 1 = v * 1 = v$ and $a \in F$. Thus the construction of a prime filter not containing a is O.K. In this way we get the subdirect product representation 2.3.16 and the rest is routine. □

Remark 2.4.13 Note that in $\mathcal{C}_\triangle$ the deduction theorem 2.2.18 fails, since $\varphi \vdash \triangle\varphi$ but for each n, $\nvdash \varphi^n \to \triangle\varphi$. (Show as a small exercise using the theorem on intermediate values of a continuous function that, given a t-norm $*$ and a natural $n \geq 1$,there is an $x < 1$ such that $x^n > 0$.) But we get the following form:

Theorem 2.4.14 Let T be a theory over $\mathcal{C}_\triangle$and let φ, ψ be formulas. $T \cup \{\varphi\} \vdash \psi$ iff $T \vdash \triangle\varphi \to \triangle\psi$.

Proof: Assuming $T \vdash \triangle\varphi \to \triangle\psi$ we prove the following formulas in $T \cup \{\varphi\}$: $\varphi, \triangle\varphi$ (generalization), $\triangle\psi$ (using $\triangle\varphi \to \triangle\psi$), ψ (by $\triangle$3).

Conversely, if $\varphi_1, \ldots, \varphi_n$ is a $(T \cup \{\varphi\})$–proof of ψ, then show $T \vdash \triangle\varphi \to \triangle\varphi_i$ by induction. We check the deduction rules. If $T \vdash \triangle\varphi \to \triangle\varphi_j$ and $T \vdash \triangle\varphi \to \triangle(\varphi_i \to \varphi_j)$, then $T \vdash (\triangle\varphi)^2 \to \triangle\varphi_j$ (see the proof of 2.2.18); but by 2.4.11 (2), $T \vdash (\triangle\varphi)^2 \equiv \triangle\varphi$, thus $T \vdash \triangle\varphi \to \triangle\varphi_i$. This verifies modus ponens. Now assume $T \vdash \triangle\varphi \to \triangle\varphi_j$; then $T \vdash \triangle(\triangle\varphi \to \triangle\varphi_j)$, $T \vdash \triangle\triangle\varphi \to \triangle\varphi_j$ (by $\triangle$5), thus $T \vdash \triangle\varphi \to \triangle\varphi_j$ (by $\triangle$4). This verifies generalization. □

CHAPTER THREE

ŁUKASIEWICZ PROPOSITIONAL LOGIC

In this chapter we are going to investigate the propositional logic given by Łukasiewicz t-norm (and the corresponding Łukasiewicz implication), and some of its extensions. Clearly, we want to find a complete axiomatization. It turns out that it is enough to add just one 1-tautology of this logic to the axioms of BL, namely the following axiom $(\neg\neg)$ of double negation[14]

$$\neg\neg\varphi \to \varphi.$$

We shall call the theory BL $+(\neg\neg)$ *Łukasiewicz propositional logic* and denote it by Ł. We first derive Łukasiewicz's original famous set of axioms (Ł1) - (Ł4) from Ł and then make a digression showing that all axioms of Ł are provable from (Ł1) to (Ł4).

This will be done in Sec. 1. In Sec. 2 we shall study BL-algebras satisfying the condition given by $(\neg\neg)$ and show that these are algebras known as MV-algebras. To get completeness of Łukasiewicz logic with respect to 1-tautologies over [0,1] it will be sufficient to show that each such 1-tautology is a 1-tautology over all linearly ordered MV-algebras. This will be obtained from a characterization of linearly ordered MV-algebras with the help of ordered Abelian groups due to Chang and the Gurevich-Kokorin theorem.

Section 3 is devoted to the extension of Łukasiewicz logic by truth constants for rational truth values; this extension is extremely important since it enables us to deduce partially true consequences from partially true premisses, which seems to be very proper for fuzzy logic. We shall obtain some variants of the completeness theorem for theories in this extended logic (called Rational Pavelka Logic).

3.1. Getting Łukasiewicz logic

We investigate the propositional calculus PC($*_{Ł}$) where $*_{Ł}$ is the Łukasiewicz t-norm taken as the truth function of conjunction, i.e.

$$x * y = \max(0, x + y - 1).$$

[14] My original treatment added the axiom (Ł4); the present elegant axiomatization was conjectured by Nicola Olivetti and proved by Agata Ciabattoni and Fernando Cicalese.

Lemma 3.1.1 (1) Ł $\vdash \neg\neg\varphi \equiv \varphi$,

(2) Ł $\vdash (\varphi \to \psi) \equiv (\neg\psi \to \neg\varphi)$,

(3) Ł $\vdash (\varphi \to \psi) \equiv \neg(\varphi\&\neg\psi)$.

(4) Ł$\vdash ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

Proof:

(1) Obvious.

(2) Is proved easily from 2.2.12(18′) and (1).

(3) BL $\vdash \varphi \to (\neg\psi \to \neg(\varphi \to \psi))$ since
BL $\vdash \varphi \to ((\varphi \to \psi) \to \psi)$; hence
BL $\vdash (\varphi\&\neg\psi) \to \neg(\varphi \to \psi)$. On the other hand,
BL $\vdash \varphi \to (\neg\psi \to (\varphi\&\neg\psi))$,
BL $\vdash \varphi \to (\neg(\varphi\&\neg\psi) \to \neg\neg\psi)$,
BL $\vdash \neg(\varphi\&\neg\psi) \to (\varphi \to \neg\neg\psi)$,
Ł $\vdash \neg(\varphi\&\neg\psi) \to (\varphi \to \psi)$,
Ł $\vdash \neg(\varphi \to \psi) \to (\varphi\&\neg\psi)$.

(4) BL $\vdash (\neg\varphi\&(\neg\varphi \to \neg\psi)) \to (\neg\psi\&(\neg\psi \to \neg\varphi))$ (by A4),
Ł $\vdash (\neg\varphi\&(\psi \to \varphi)) \to (\neg\psi\&(\varphi \to \psi))$,
Ł $\vdash \neg((\varphi \to \psi)\&\neg\psi) \to \neg((\psi \to \varphi)\&\neg\varphi)$,
Ł $\vdash ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

□

Corollary 3.1.2 For Łukasiewicz implication,

$$((x \Rightarrow y) \Rightarrow y) = \max(x, y).$$

Definition 3.1.3 The following are *Łukasiewicz's axioms:*[15]

(Ł1) $\varphi \to (\psi \to \varphi)$,

(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,

(Ł4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

---

[15] See [127] and Historical Remarks.

Lemma 3.1.4 Ł proves (Ł1),(Ł2),(Ł3), (Ł4).

Proof: For (Ł1) see 2.2.7; (Ł2) is axiom (A1). For (Ł3) see 3.1.1(2) and for (Ł4) see 3.1.1(4). □

*

The main task of the rest of this section is to show that Łukasiewicz axioms prove axioms of BL. The reader uninterested in this result may skip the rest of the section *except for* 3.1.11–3.1.13 when we define a *strong disjunction* and show some of its properties. The reader in a hurry will read this as information on the logic Ł (i.e. BL + ($\neg\neg$)).

Definition 3.1.5 We shall be slightly more specific on the system investigated: the only basic connective is an implication $\rightarrow$, then there is the truth constant $\bar{0}$. Negation and conjunction are defined as follows:

$$\begin{array}{rcl} \neg\varphi & \text{is} & \varphi \rightarrow \bar{0}, \\ \varphi \& \psi & \text{is} & \neg(\varphi \rightarrow \neg\psi). \end{array}$$

(Further connectives will be defined later.) Axioms are (Ł1)–(Ł4); this system is denoted by Ł$'$.

To prove axioms of BL we shall have to develop Ł$'$ a little.

Lemma 3.1.6 The following formulas are provable in Ł$'$:

(1) $\varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi)$

(2) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$

(3) $\varphi \rightarrow \varphi$

(4) $\bar{0} \rightarrow \varphi$

(5) $\neg\neg\varphi \rightarrow \varphi$

(6) $(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$

(7) $\varphi \rightarrow \neg\neg\varphi$.

Proof:

(1) $Ł' \vdash \varphi \to ((\psi \to \varphi) \to \varphi)$ by (Ł1), thus $Ł' \vdash \varphi \to ((\varphi \to \psi) \to \psi)$ by (Ł4) and (Ł2).

(2) First, $Ł' \vdash (\varphi \to (\psi \to \chi)) \to (((\psi \to \chi) \to \chi) \to (\varphi \to \chi))$ by (Ł2) (transitivity); also $Ł' \vdash (\psi \to ((\psi \to \chi) \to \chi)) \to [(((\psi \to \chi) \to \chi) \to (\varphi \to \chi)) \to (\psi \to (\varphi \to \chi))]$ from transitivity and $Ł' \vdash (\psi \to ((\psi \to \chi) \to \chi)$ by (1); thus $Ł' \vdash [\ldots]$. But from the first provability in this paragraph and from the last one we get $Ł' \vdash (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$ again by transitivity.

(3) From $Ł' \vdash \varphi \to (\psi \to \varphi)$ and from (2) we get $Ł' \vdash \psi \to (\varphi \to \varphi)$; take an axiom for ψ and get $Ł' \vdash \varphi \to \varphi$.

(4) We have $Ł' \vdash \bar{0} \to \bar{0}$, i.e. $Ł' \vdash \neg\bar{0}$, thus $Ł' \vdash \neg\varphi \to \neg\bar{0}$, $Ł' \vdash \bar{0} \to \varphi$ by (Ł3).

(5) $Ł'$ proves $\neg\neg\varphi \to [(\varphi \to \bar{0}) \to \bar{0}] \to [(\bar{0} \to \varphi) \to \varphi] \to \varphi$; the middle implication by (Ł4), the last by (4).

(6) $Ł' \vdash (\varphi \to \neg\psi) \to (\neg\neg\varphi \to \neg\psi)$ by (5) and transitivity; and $Ł' \vdash (\neg\neg\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$ by (Ł3).

(7) $Ł' \vdash \neg\varphi \to \neg\varphi$, thus $Ł'\ \varphi \to \neg\neg\varphi$ by (6).

□

Definition 3.1.7 In the rest of this section, $\varphi \leftrightarrow \psi$ stands for the pair of formulas $(\varphi \to \psi), (\psi \to \varphi)$; $Ł' \vdash \varphi \leftrightarrow \psi$ means $Ł' \vdash (\varphi \to \psi)$ and $Ł' \vdash (\psi \to \varphi)$. We could introduce the connective $\equiv$ of equivalence but proving its properties would be unnecessarily losing time; after we show that $Ł'$ proves all axioms of BL we shall have $\equiv$ introduced in BL at our disposal.

Remark 3.1.8 Note that using transitivity we may show that $Ł' \vdash (\varphi \leftrightarrow \psi)$ implies $Ł' \vdash (\varphi \to \chi) \leftrightarrow (\psi \to \chi)$ and also $Ł' \vdash (\chi \to \varphi) \leftrightarrow (\chi \to \psi)$ (thus also $Ł' \vdash (\neg\varphi \leftrightarrow \neg\psi)$); hence provably equivalent subformulas can be replaced by each other.

Lemma 3.1.9 $Ł'$ proves the axioms (A1)–(A3), (A5) and (A7) of BL as well as the axiom $(\neg\neg)$.

Proof: (A1) is (Ł2): (A7) was proved in 3.1.6 (4). (A2): in the presence of (A3) (see below) it suffices to prove $Ł' \vdash (\varphi\&\psi) \to \psi$. Now $Ł' \vdash \neg\psi \to$

$(\varphi \rightarrow \neg\psi)$, thus $Ł' \vdash \neg(\varphi \rightarrow \neg\psi) \rightarrow \neg\neg\psi$, $Ł' \vdash (\varphi \& \psi) \rightarrow \psi$.
(A3): $Ł' \vdash (\varphi\&\psi) \rightarrow \neg(\varphi \rightarrow \neg\psi) \rightarrow \neg(\psi \rightarrow \neg\varphi) \rightarrow (\psi\&\varphi)$ (using 3.1.6 (6)).
We prove (A5): $Ł' \vdash [\varphi \rightarrow (\psi \rightarrow \chi)] \leftrightarrow [\varphi \rightarrow (\neg\chi \rightarrow \neg\psi)] \leftrightarrow [\neg\chi \rightarrow (\varphi \rightarrow \neg\psi)] \leftrightarrow [\neg\chi \rightarrow \neg(\varphi\&\psi)] \leftrightarrow [(\varphi\&\psi) \rightarrow \chi]$. For $(\neg\neg)$ see 3.1.6 (5). This completes the proof. □

*

Remark 3.1.10 Before we show $Ł' \vdash$ (A4),(A6) we shall prove in $Ł'$ (and hence in Ł) some few things on defined connectives $\vee, \wedge$ and on a new connective $\veebar$ of strong disjunction, dual to &. Its existence is typical of $Ł'$ and possible due to the provable equivalence $\varphi \leftrightarrow \neg\neg\varphi$.

Definition 3.1.11 New connectives $\wedge, \vee$ and $\veebar$ are introduced in $Ł'$ as follows:

$$\begin{array}{rll} \varphi \wedge \psi & \text{stands for} & \varphi\&(\varphi \rightarrow \psi), \\ \varphi \vee \psi & \text{stands for} & (\varphi \rightarrow \psi) \rightarrow \psi, \\ \varphi \veebar \psi & \text{stands for} & \neg\varphi \rightarrow \psi. \end{array}$$

Remark 3.1.12 In Ł, $\varphi \wedge \psi$ was defined as here; Ł *proves* $(\varphi \vee \psi) \leftrightarrow ((\varphi \rightarrow \psi) \rightarrow \psi)$; and the definition of $\varphi\veebar\psi$ is new. Note that (Ł4) just expresses the commutativity of $\vee$. Observe also that the truth function $\oplus$ corresponding to the connective $\veebar$ satisfies

$$x \oplus y = \min(1, x + y).$$

Indeed, $x \oplus y = [(x \Rightarrow 0) \Rightarrow y] = [(1 - x) \Rightarrow y]$; thus if $x + y \leq 1$, then $1 - x \geq y$ and $x \oplus y = 1 - (1 - x) + y = x + y$; if $x + y \geq 1$, then $1 - x \leq y$ and $x \oplus y = 1$.

Lemma 3.1.13 $Ł'$ proves the following properties of $\wedge$, $\vee$, $\veebar$ (note that (8)–(11) are de Morgan rules):

(8) $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$

(9) $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$

(10) $\neg(\varphi\&\psi) \leftrightarrow (\neg\varphi\veebar\neg\psi)$

(11) $\neg(\varphi\veebar\psi) \leftrightarrow (\neg\varphi\&\neg\psi)$

(12) $\psi \rightarrow (\varphi\veebar\psi)$

(13) $(\varphi\veebar\psi) \rightarrow (\psi\veebar\varphi)$

(14) $(\varphi\underline{\vee}(\psi\underline{\vee}\chi)) \leftrightarrow ((\varphi\underline{\vee}\psi)\underline{\vee}\chi)$

(15) $(\varphi \wedge \psi) \leftrightarrow (\varphi\underline{\vee}\neg\psi)\&\psi$

(16) $(\varphi \vee \psi) \leftrightarrow (\varphi\&\neg\psi)\underline{\vee}\psi$

(17) $\varphi\underline{\vee}\neg\varphi$

(18) $((\varphi\&\neg\psi)\underline{\vee}\psi) \leftrightarrow (\varphi\underline{\vee}(\psi\&\neg\varphi))$

(19) $((\varphi\underline{\vee}\neg\psi)\&\psi) \leftrightarrow (\varphi\&(\psi\underline{\vee}\neg\varphi))$

Proof: The proofs are easy applications of basic $\leftrightarrow$-equivalences we already have proved (and could be left to the reader as exercises). We always present a chain of provable $\leftrightarrow$-equivalences.

(8) $\neg(\varphi \wedge \psi) \leftrightarrow \neg(\varphi\&(\varphi \rightarrow \psi)) \leftrightarrow \neg(\varphi\&\neg(\varphi\&\neg\psi)) \leftrightarrow (\varphi \rightarrow (\varphi\&\neg\psi)) \leftrightarrow (\neg(\varphi\&\neg\psi) \rightarrow \neg\varphi) \leftrightarrow ((\varphi \rightarrow \psi) \rightarrow \neg\varphi) \leftrightarrow ((\neg\psi \rightarrow \neg\varphi) \rightarrow \neg\varphi) \leftrightarrow (\neg\psi \vee \neg\varphi) \leftrightarrow (\neg\varphi \vee \neg\psi)$.
(Note that from (8) we easily get commutativity of $\wedge$ in Ł$'$: we get $(\varphi \wedge \psi) \leftrightarrow \neg(\neg\varphi \vee \neg\psi) \leftrightarrow \neg(\neg\psi \vee \neg\varphi) \leftrightarrow (\psi \wedge \varphi)$.)

(9) $\neg(\varphi \vee \psi) \leftrightarrow \neg((\varphi \rightarrow \psi) \rightarrow \psi) \leftrightarrow \neg(\neg\psi \rightarrow \neg(\varphi \rightarrow \psi)) \leftrightarrow (\neg\psi\&(\varphi \rightarrow \psi)) \leftrightarrow (\neg\psi\&(\neg\psi \rightarrow \neg\varphi)) \leftrightarrow (\neg\psi \wedge \neg\varphi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$.

(10) $\neg(\varphi\&\psi) \leftrightarrow \neg(\psi\&\varphi) \leftrightarrow (\psi \rightarrow \neg\varphi) \leftrightarrow (\neg\neg\varphi \rightarrow \neg\psi) \leftrightarrow (\neg\varphi\underline{\vee}\neg\psi)$.

(11) $\neg(\varphi\underline{\vee}\psi) \leftrightarrow \neg(\neg\varphi \rightarrow \psi) \leftrightarrow (\neg\varphi\&\neg\psi)$

(12) $\psi \rightarrow (\neg\varphi \rightarrow \psi) \rightarrow (\varphi\underline{\vee}\psi)$

(13) $(\varphi\underline{\vee}\psi) \leftrightarrow (\neg\varphi \rightarrow \psi) \leftrightarrow (\neg\psi \rightarrow \varphi) \leftrightarrow (\psi\underline{\vee}\varphi)$

(14) $(\varphi\underline{\vee}(\psi\underline{\vee}\chi)) \leftrightarrow (\neg\varphi \rightarrow (\neg\psi \rightarrow \chi)) \leftrightarrow ((\neg\varphi\&\neg\psi) \rightarrow \chi) \leftrightarrow (\neg(\varphi\underline{\vee}\psi) \rightarrow \chi) \leftrightarrow ((\varphi\underline{\vee}\psi)\underline{\vee}\chi)$

(15) $(\varphi \wedge \psi) \leftrightarrow (\psi \wedge \varphi) \leftrightarrow (\psi\&(\psi \rightarrow \varphi)) \leftrightarrow ((\neg\varphi \rightarrow \neg\psi)\&\psi) \leftrightarrow ((\varphi\underline{\vee}\neg\psi)\&\psi)$ (For commutativity of $\wedge$ see the remark in (8).)

(16) $(\varphi \vee \psi) \leftrightarrow ((\varphi \rightarrow \psi) \rightarrow \psi) \leftrightarrow (\neg(\varphi\&\neg\psi) \rightarrow \psi) \leftrightarrow ((\varphi\&\neg\psi)\underline{\vee}\psi)$.

(17) $\varphi\underline{\vee}\neg\varphi$ is $\neg\varphi \rightarrow \neg\varphi$.

(18) and (19) follows immediately from commutativity of $\vee$ and $\wedge$ and from expressions (15),(16).

□

Lemma 3.1.14 Ł′ proves the following:

(20) $(\varphi \vee \varphi) \rightarrow \varphi$

(21) $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$

Proof:

(20) Ł′ proves $((\varphi \vee \varphi) \rightarrow \varphi) \leftrightarrow (((\varphi \rightarrow \varphi) \rightarrow \varphi) \rightarrow \varphi) \leftrightarrow ((\varphi \rightarrow (\varphi \rightarrow \varphi) \rightarrow (\varphi \rightarrow \varphi))$ (the last equivalence follows by (Ł4)) and the last formula is provable since Ł′ $\vdash (\varphi \rightarrow \varphi)$.

(21) By repeated use of transitivity we get Ł′ $\vdash (\varphi \rightarrow \chi) \rightarrow (((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\chi \rightarrow \psi) \rightarrow \psi))$, thus Ł′ $\vdash (\varphi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow (\psi \vee \chi))$. Similarly, Ł′ $\vdash (\psi \rightarrow \chi) \rightarrow ((\psi \vee \chi) \rightarrow (\chi \vee \chi))$, thus Ł′ $\vdash (\psi \rightarrow \chi) \rightarrow ((\psi \vee \chi) \rightarrow \chi)$. From this we get Ł′ $\vdash (\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$ by repeated use of transitivity. (The pattern is as follows: For appropriate $\alpha, \beta, \gamma, \delta, \varepsilon$, we have proved Ł′ $\vdash \alpha \rightarrow \beta$, $\gamma \rightarrow \delta$, $\beta \rightarrow (\delta \rightarrow \varepsilon)$; we want to show $\vdash \alpha \rightarrow (\gamma \rightarrow \varepsilon)$. We proceed as follows: Ł′ $\vdash (\alpha \rightarrow \beta) \rightarrow [(\beta \rightarrow (\delta \rightarrow \varepsilon)) \rightarrow (\alpha \rightarrow (\delta \rightarrow \varepsilon))]$, thus Ł′ $\vdash \alpha \rightarrow (\delta \rightarrow \varepsilon))$ by double modus ponens. This means Ł′ $\vdash \delta \rightarrow (\alpha \rightarrow \varepsilon)$ and since Ł′ $\vdash (\gamma \rightarrow \delta) \rightarrow [(\beta \rightarrow (\alpha \rightarrow \varepsilon)) \rightarrow (\gamma \rightarrow (\alpha \rightarrow \varepsilon))]$ we get Ł′ $\vdash \gamma \rightarrow (\alpha \rightarrow \varepsilon)$, thus Ł′ $\vdash \alpha \rightarrow (\gamma \rightarrow \varepsilon)$.)

□

Theorem 3.1.15 Ł′ proves $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$.

Proof: We shall start with the formula $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ and repeatedly make $\leftrightarrow$-equivalent transformations using the expression of $\vee, \rightarrow$ by $\underline{\vee}$ and &, commutativity and associativity of $\underline{\vee}$, &, de Morgan rules and similar; in particular, (18) will be repeatedly used. Finally we arrive at a formula of the form $(\varphi \underline{\vee} \neg\varphi \underline{\vee} \ldots)$ which is Ł′-provable.

The proof consists of a sequence of equivalent transformations below. We use some abbreviations, e.g. α abbreviates $(\varphi \& \neg\psi) \underline{\vee} \varphi$; these abbreviations are easy to detect.

$(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi) \leftrightarrow$
$\leftrightarrow ((\varphi \rightarrow \psi) \& \neg(\psi \rightarrow \varphi)) \underline{\vee} (\psi \rightarrow \varphi) \leftrightarrow$
$\leftrightarrow ((\varphi \rightarrow \psi) \& \psi \& \neg\varphi) \underline{\vee} \neg\psi \underline{\vee} \varphi \leftrightarrow$
$\leftrightarrow (((\varphi \rightarrow \psi) \& \neg\varphi) \vee \neg\psi) \underline{\vee} \varphi \leftrightarrow$

$\leftrightarrow ((\varphi \to \psi)\&\neg\varphi\underline{\vee}(\neg\psi\&\neg((\varphi \to \psi)\&\neg\varphi))\underline{\vee}\varphi \leftrightarrow$
$\leftrightarrow ((\varphi \to \psi)\&\neg\varphi)\underline{\vee}\varphi\underline{\vee}(\neg\psi\&((\varphi\&\neg\psi)\underline{\vee}\varphi)) \leftrightarrow$
$\leftrightarrow ((\varphi \to \psi) \vee \varphi)\underline{\vee}(\neg\psi\&\alpha) \leftrightarrow$
$\leftrightarrow (\varphi\&\neg(\varphi \to \psi))\underline{\vee}(\varphi \to \psi)\underline{\vee}(\neg\psi\&\alpha) \leftrightarrow$
$\leftrightarrow \beta\underline{\vee}\neg\varphi\underline{\vee}\psi\underline{\vee}(\neg\psi\&\alpha) \leftrightarrow$
$\leftrightarrow \beta\underline{\vee}\neg\varphi\underline{\vee}\alpha\underline{\vee}(\psi\&\neg\alpha) \leftrightarrow$
$\leftrightarrow \neg\varphi\underline{\vee}\alpha\underline{\vee}\gamma \leftrightarrow$
$\leftrightarrow \neg\varphi\underline{\vee}\varphi\underline{\vee}(\varphi\&\neg\psi)\underline{\vee}\gamma \leftrightarrow$
$\leftrightarrow (\varphi\underline{\vee}\neg\varphi)\underline{\vee}\delta.$ □

Corollary 3.1.16 Ł$'$ proves the axioms (A4) and (A6) of BL.

Proof: For (A4) see 3.1.13 (8) (proof) and the definition of $\wedge$. We prove (A6). By 3.1.14 (21),
Ł$' \vdash ((\varphi \to \psi) \to \chi) \to [((\psi \to \varphi) \to \chi) \to (((\varphi \to \psi) \vee (\psi \to \varphi)) \to \chi)]$,
Ł$' \vdash [((\varphi \to \psi) \to \chi)\&((\psi \to \varphi) \to \chi)] \to [((\varphi \to \psi) \vee (\psi \to \varphi)) \to \chi]$;
but by 3.1.15,
Ł$' \vdash [((\varphi \to \psi) \to \chi)\&((\psi \to \varphi) \to \chi)] \to \chi$,
i.e. we get (A6). □

We have completed our proof of the fact that Ł$'$ proves all the axioms of BL. From now on we know that Ł and Ł$'$ are equivalent theories (prove the same formulas); thus Ł will denote either of them. Besides this, we have introduced a new connective (strong disjunction) and proved several of its properties.

3.2. MV-ALGEBRAS; A COMPLETENESS THEOREM

3.2.1 As we have seen, Łukasiewicz propositional calculus Ł can be understood as a schematic extension of the basic logic BL by the schema $\neg\neg\varphi \to \varphi$. Thus 2.3.22 gives us completeness with respect to Ł-algebras, i.e. BL-algebras in which the identity $x = ((x \Rightarrow 0) \Rightarrow 0)$ is valid. Ł-algebras have been broadly investigated under another name, which we adopt:

Definition 3.2.2 An *MV-algebra* is a BL-algebra in which the identity $x = ((x \Rightarrow 0) \Rightarrow 0)$ is valid (MV stands for "many-valued").

Remark 3.2.3 Recall that each formula φ determines the corresponding term $\varphi^\bullet$ of the language of residuated lattices; the completeness above implies that

(i) Ł $\vdash \varphi$ iff the identity $\varphi^\bullet = 1$ is valid in each (linearly ordered) MV-algebra;

(ii) Ł $\vdash \varphi \equiv \psi$ iff the identity $\varphi^\bullet = \psi^\bullet$ is valid in each linearly ordered MV-algebra. (Note that in an MV-algebra $((x \Rightarrow y) = 1$ and $(y \Rightarrow x) = 1)$ iff $(x \leq y$ and $y \leq x)$ iff $x = y$.)

Our definition of an MV-algebra is natural in the context of residuated lattices; but there are equivalent simpler definitions. We elaborate one such definition based on Łukasiewicz's original axioms (Ł1–Ł4). To prevent any confusion we shall call the algebras satisfying the new definition Wajsberg algebras. (After we show their relation to MV algebras we shall forget the new name.)

Definition 3.2.4 A *Wajsberg algebra*[16] is an algebra $\mathbf{A} = \langle A, \Rightarrow, 0\rangle$ in which the following identities are valid:
Put $(-)x = x \Rightarrow 0$, $1 = (0 \Rightarrow 0)$. Then

(W1) $(1 \Rightarrow y) = y$,

(W2) $(x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)) = 1$,

(W3) $(((-)x \Rightarrow (-)y) \Rightarrow (y \Rightarrow x)) = 1$,

(W4) $((x \Rightarrow y) \Rightarrow y) = ((y \Rightarrow x) \Rightarrow x)$.

Lemma 3.2.5 The following is true in each Wajsberg algebra:

(i) $(x \Rightarrow x) = 1$

(ii) $x = y$ iff $(x \Rightarrow y) = 1$ and $(y \Rightarrow x) = 1$

(iii) $(x \Rightarrow 1) = 1$

(iv) $(x \Rightarrow (y \Rightarrow x)) = 1$

(v) $((x \Rightarrow y) \Rightarrow y) \Rightarrow ((y \Rightarrow x) \Rightarrow x) = 1$

Proof:

(i) $1 \Rightarrow 1 = 1$ by (W1); thus, again by (W1), $(x \Rightarrow x) = [1 \Rightarrow (x \Rightarrow x)] =$
$= [1 \Rightarrow 1] \Rightarrow ([1 \Rightarrow x] \Rightarrow [1 \Rightarrow x]) = 1$
by (W2) (transitivity).

[16] See [52].

(ii) The implication from left to right follows by (i); conversely, if $(x \Rightarrow y) = (y \Rightarrow x) = 1$ then $x = (1 \Rightarrow x) = ((y \Rightarrow x) \Rightarrow x) = ((x \Rightarrow y) \Rightarrow y) = (1 \Rightarrow y) = y$.

(iii) $x \Rightarrow 1 = x \Rightarrow (x \Rightarrow x) =$
$= (1 \Rightarrow x) \Rightarrow ((1 \Rightarrow x) \Rightarrow x) =$
$= (1 \Rightarrow x) \Rightarrow ((x \Rightarrow 1) \Rightarrow 1) =$
$= (1 \Rightarrow x) \Rightarrow ((x \Rightarrow 1) \Rightarrow (1 \Rightarrow 1)) = 1$ by transitivity (W2).

(iv) $x \Rightarrow (y \Rightarrow x) = [1 \Rightarrow (x \Rightarrow (y \Rightarrow x))] =$
$= 1 \Rightarrow ((1 \Rightarrow x) \Rightarrow (y \Rightarrow x)) =$
$= (y \Rightarrow 1) \Rightarrow ((1 \Rightarrow x) \Rightarrow (y \Rightarrow x)) = 1$
(by (iii) and transitivity (W2)).

(v) follows by (W4) and (ii).

□

Lemma 3.2.6 For arbitrary formulas φ, ψ and each Wajsberg-algebra $\mathbf{A}$,

(i) Ł $\vdash \varphi$ implies that the identity $\varphi^\bullet = 1$ is valid in $\mathbf{A}$;

(ii) Ł $\vdash \varphi \equiv \psi$ implies that the identity $\varphi^\bullet = \psi^\bullet$ is valid in $\mathbf{A}$.

Proof:

(i) Check by the induction on proofs observing that if $x = 1$ and $x \Rightarrow y = 1$, then $(1 \Rightarrow y) = 1$, thus $y = 1$.

(ii) follows immediatelly by 3.2.5 (ii).

□

Theorem 3.2.7 (i) If $\mathbf{A}$ is an MV-algebra then $\langle A, \Rightarrow, 0\rangle$ is a Wajsberg-algebra.

(ii) If $\mathbf{A} = \langle A, \Rightarrow, 0\rangle$ is a Wajsberg-algebra and if $*, \cap, \cup, 1$ are defined in the obvious way, i.e.

$$\begin{aligned} x * y &= (-)(x \Rightarrow (-)y), \\ x \cap y &= x * (x \Rightarrow y), \\ x \cup y &= (x \Rightarrow y) \Rightarrow y, \end{aligned}$$

then $\mathbf{A}' = \{A, \cap, \cup, *, \Rightarrow, 0, 1\}$ is an MV-algebra.

(In other words, the restriction of an MV-algebra to $\Rightarrow$, 0 is a Wajsberg-algebra; each Wajsberg-algebra expands to an MV-algebra.)

Proof:

(i) follows from the fact that counterparts of the axioms (Ł1–Ł4) are provable and so are $(1 \Rightarrow x) = x$ and $((x \Rightarrow y) \Rightarrow y) = ((y \Rightarrow x) \Rightarrow x)$.

(ii) follows by observing that MV-algebras are characterized by finitely many identities and that the corresponding formulas are Ł provable; thus if $\mathbf{A}$ is a Wajsberg algebra and $\mathbf{A}'$ is its expansion by the obvious definitions of $*, \cap, \cup$, then the identities in question are valid in $\mathbf{A}'$. Thus from now on we may identify Wajsberg algebras with MV-algebras.

□

*

Now our interest turns to linearly ordered MV-algebras. Our aim is to prove a completeness theorem for Ł with respect to 1-tautologies over the "standard" MV-algebra [0,1] with truth functions of connectives. To this end we prove a famous characterization of linearly ordered MV-algebras by (linearly) ordered Abelian (semi)groups. Recall Section 1.6 when we discussed ordered Abelian groups and semigroups .

Definition 3.2.8 Let $\mathbf{G} = \langle G, +_G, \leq_G \rangle$ be a linearly ordered Abelian group (o-group) and let $e \in G$, $0 <_G e$ be a positive element. $MV(\mathbf{G}, e)$ is the algebra $\mathbf{A} = \langle A, \Rightarrow, 0_G \rangle$ whose domain $\mathbf{A}$ is the interval $[0, e]_G = \{g \in G \mid 0 \leq_G g \leq_G e\}$, $x \Rightarrow y = e$ if $x \leq_G y$ and $x \Rightarrow y = e - x + y$ otherwise.

Lemma 3.2.9 $MV(\mathbf{G}, e)$ is a linearly ordered MV-algebra.

Proof: Check that the four identities in the definition 3.2.4 (of an Wajsberg-algebra) are valid in $MV(\mathbf{G}, e)$ (cf. 3.1.4). Then define $*, \cap, \cup$ and observe that for $x, y \in [0, e]_G$,

$$x \cap y = \min(x, y)$$
$$x \cup y = \max(x, y)$$

so that the ordering $\leq$ induced by $\cap$ (or $\cup$) coincides with $\leq_G$. Thus $MV(\mathbf{G}, e)$ is a linearly ordered MV-algebra. □

Theorem 3.2.10 For each linearly ordered MV-algebra $\mathbf{A}$ there is a linearly ordered Abelian group $\mathbf{G}$ and a positive element $e \in G$ such that $\mathbf{A} = MV(\mathbf{G}, e)$.

Before we elaborate the proof of this theorem (below in this section) let us show how it gives the desired completeness theorem for Łukasiewicz logic.

Lemma 3.2.11 (1) If an identity $\sigma = \tau$ in the language of MV-algebras is valid in the standard MV-algebra $[0,1]$ with truth functions, then it is valid in each linearly ordered MV-algebra.

(2) Consequently, if a formula φ is a 1-tautology over the standard MV-algebra, then φ is an **A**-tautology for each linearly ordered MV-algebra **A**.

(3) More generally, if T is a finite theory and φ is true in each $[0,1]_Ł$-model of T, then for each linearly ordered MV-algebra **A**, φ is true in each **A**-model of T.

Proof: Recall the Gurevich-Kokorin theorem 1.6.17 saying that a $\forall$-sentence of ordered Abelian groups is true in the additive o-group of reals iff it is true in all o-groups. By 1.6.18 we know that the same is true if we introduce new operations by *open* definitions. In particular, expand the theory of o-groups by the ternary operation $x \Rightarrow_e y$ defined as follows:

$$x \Rightarrow_e y = e \text{ if } x \leq y, \text{ otherwise } x \Rightarrow_e y = e - x + y.$$

(Clearly, for $e > 0$ this just defines the operation $\Rightarrow$ in $MV(G, e)$.) Now to each term σ of MV-algebras (assume it is constructed from variables using $0, \Rightarrow$ only) we associate a term σ_e^* of o-groups putting $x_{i_e}^* = x_i$, $0_e^* = 0$, $(\sigma_1 \Rightarrow \sigma_2)_e^* = (\sigma_1)_e^* \Rightarrow_e (\sigma_2)_e^*$. Now let **A** be an MV-algebra and σ, τ terms such that $\sigma = \tau$ is not valid in **A**, i.e. for some $\mathbf{a} = a_1, \dots, a_n \in \mathbf{A}$, $\mathbf{A} \models \sigma(a) \neq \tau(a)$. Let **G** be an o-group such that $\mathbf{A} = MV(\mathbf{G}, e)$ for an appropriate $e \in G$; thus $\mathbf{G} \models \sigma_e^*(\mathbf{a}) \neq \tau_e^*(\mathbf{a})$, $\mathbf{G} \models 0 \leq \mathbf{a} \leq e$. By Gurevich-Kokorin, there are reals $e > 0$, $0 < a_1, \dots, a_n < e$ such that $Re \models \sigma_e^*(\mathbf{a}) \neq \tau_e^*(\mathbf{a})$; by dividing by e we get $b_1, \dots, b_n$ such that $Re \models \sigma_1^*(\mathbf{b}) \neq \tau_1^*(\mathbf{b})$, hence the standard MV-algebra over $[0,1]$ satisfies $\sigma(\mathbf{b}) \neq \tau(\mathbf{b})$.

The proof of (3) is obtained in a similar way, just observing that **A** is a linearly ordered MV-algebra and **a** is a tuple of its elements such that $\mathbf{A} \models \sigma_1(\mathbf{a}) = \tau_1(\mathbf{a}), \dots, \mathbf{A} \models \sigma_n(\mathbf{a}) = \tau_n(\mathbf{a})$ but $\mathbf{A} \models \sigma(\mathbf{a}) \neq \tau(\mathbf{a})$, then the Gurevich-Kokorin theorem gives a tuple **b** of elements of $[0,1]$ such that

$$[0,1]_Ł \models \sigma_1(\mathbf{b}) = \tau_1(\mathbf{b}), \dots, \sigma_n(\mathbf{b}) = \tau_n(\mathbf{b}), \ \sigma(\mathbf{b}) \neq \tau(\mathbf{b}).$$

Thus if $T = \{\varphi_1, \dots, \varphi_n\}$ and φ is not true in an **A**-model μ of T (**A** an MV-algebra, μ an **A**-evaluation φ containing propositional variables $p_1, \dots, p_n$) then for $a_i = e(p_i)$, $\mathbf{a} = (a_1, \dots, a_n)$ we get $\varphi_1^*(\mathbf{a}) = 1_A \dots \varphi_n^*(\mathbf{a}) = 1_A$, $\varphi^*(\mathbf{a}) \neq 1_A$. The above gives $\mathbf{b} = (b_1, \dots, b_n)$ such that $\varphi_i^*(\mathbf{b}) = 1$ $(i = 1, \dots, n)$ and $\varphi^*(\mathbf{b}) \neq 1$, thus any $[0,1]$ evaluation μ' such that $\mu'(p_i) = b_i$ is a $[0,1]_Ł$-model of T in which φ is not 1-true. □

Corollary 3.2.12 Let $[0,1]_Ł$ denote the standard MV-algebra on $[0,1]$ with truth functions of Łukasiewicz logic.

(1) A formula φ is a 1-tautology of Łukasiewicz logic Ł (i.e. a $[0,1]_Ł$-tautology) iff it is an **A**-tautology for each linearly ordered MV-algebra **A**.

(2) Let T be a finite theory over Ł. The following are equivalent.

(2i) φ is true in each $[0,1]_Ł$-model of T,

(2ii) for each linearly ordered MV-algebra **A**, φ is true in each **A**-model of T.

(3) The claims (1), (2) remain valid if Ł is replaced by $Ł_\triangle$, $[0,1]_Ł$ by $[0,1]_{Ł,\triangle}$ and MV-algebras by MV-algebras with $\triangle$.

Theorem 3.2.13 (Completeness.)

(1) A formula φ is provable in Łukasiewicz logic Ł iff it is a 1-tautology of Łukasiewicz logic.

(2) Let T be a finite theory over Ł, φ a formula. T proves φ over Łukasiewicz logic iff φ is true in each model of T (in the sense of Ł).

(3) The claims (1), (2) remain valid if Ł is extended by the connective $\triangle$ ($\triangle(1) = 1$, $\triangle(x) = 0$ otherwise) and by the corresponding axioms ($\triangle1$)–($\triangle5$).

Proof: This follows immediately from 2.4.3, 2.4.12 by the preceding corollary. □

Remark 3.2.14 It is easy to show that we cannot omit the assumption of finiteness of T. For example, take variables p, q and

$$T = \{np \to q \mid n \text{ natural}\} \cup \{\neg p \to q\}(\text{where } np \text{ is } p \veebar \ldots \veebar p)$$

thus the axioms are

$$p \to q, (p \veebar p) \to q, (p \veebar p \veebar p) \to q, \ldots, \neg p \to q.$$

Obviously, q is true in each model of T: if $e(p) = 0$ the last axiom gives $e(q) = 1$, if $e(p) > \frac{1}{n}$, the the axiom $np \to q$ does the job. But for each finite subtheory $T_0 \subseteq T$ we can find a model e of T_0 (with $e(p)$ sufficiently small positive) in which $e(q) < 1$. (One expresses this by saying that the semantic consequence is not compact.)

On the other hand, any T-proof uses only finitely many axioms of T, thus if a formula is provable in T it is true in all models of each sufficiently large finite subtheory of T. Thus T does not prove q.

$*$

The rest of this section elaborates a proof of Theorem 3.2.10. For this purpose we first introduce an operation in MV-algebras (corresponding to the connective $\veebar$) and prove some of its properties (cf. Remark 3.1.12).

Definition 3.2.15 In each MV-algebra **A**, define

$$x \oplus y = (-)((-)x * (-)y) = ((-)x \Rightarrow y).$$

Lemma 3.2.16 The following holds in each MV-algebra.

$$(-)(x * y) = (-)x \oplus (-)y \tag{1}$$

$$(-)(x \oplus y) = (-)x * (-)y \tag{2}$$

$$x \leq x \oplus y \tag{3}$$

$$x \oplus y = y \oplus x \tag{4}$$

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z \tag{5}$$

$$x \cap y = (x \oplus (-)y) * y \tag{6}$$

$$x \cup y = (x * (-)y) \oplus y \tag{7}$$

$$x \oplus (-)x = 1 \tag{8}$$

$$(x * (-)y) \oplus y = x \oplus (y * (-)x) \tag{9}$$

$$(x \oplus (-)y) * y = x * (y \oplus (-)x) \tag{10}$$

Proof: This follows from the provability of formulas (10)–(19) in 3.1.13 □

Lemma 3.2.17 The following holds in each linearly ordered MV-algebra:

(11) If $x \oplus y < 1$, then $y < (-)x$ and $x * y = 0$

(12) If $x \oplus y = 1$ and $x * y = 0$, then $x = (-)y$

(13) If $x \oplus y = x \oplus z < 1$ or if $x * y = x * z > 0$, then $y = z$

(14) If $x \oplus y = x \oplus z$ and $x * y = x * z$, then $y = z$

(15) If $x \oplus y = x$, then $x = 1$ or $y = 0$

(16) If $x \oplus y = 1$ and $x \oplus z < 1$, then $(x * y) \oplus z = (x \oplus z) * y$.

Proof: (11) By the assumption, $((-)x \Rightarrow y) < 1$, thus $(-)x \not\leq y$, hence $y < (-)x$; $x * (-)x = 0$ (e.g. by 2.2.12), thus $x * y = 0$.
(12) The assumption gives $((-)x \Rightarrow y) = 1$ and $(y \Rightarrow (-)x) = 1$, thus $(-)x = y$.
(13) Since $x \oplus y$, $x \oplus z < 1$ we get $(-)x > y$, $(-)x > z$, thus $(-)x \cap y = y$, $(-)x \cap z = z$. But, then

$$y = y \cap (-)x = (y \oplus x) * (-)x = (z \oplus x) * (-)x = z \cap x = z.$$

If $x*y = x*z > 0$, then $(-)x \oplus (-)y = (-)x \oplus (-)z < 1$, thus $(-)y = (-)z$ and $y = z$.
(14) follows from (13) and (12).
To prove (15) observe that if $x \oplus y = x = x \oplus 0$ and $x < 1$, then $y = 0$ by (13).
Finally we prove (16). We have $(-)y \leq x$, thus

$$\begin{array}{rclcl} (-)y \oplus (x * y) \oplus z & = & ((-)y \cup x) \oplus z & = & x \oplus z, \\ (-)y \oplus (y * (x \oplus z)) & = & (-)y \cup (x \oplus z) & = & x \oplus z, \end{array}$$

thus $(x * y) \oplus z = y * (x \oplus z)$ by cancellation (see (12)). □

We are now going to prove Theorem 3.2.10 using the original method of Chang [25].

Definition 3.2.18 Let $\mathbf{A} = \langle A, \cap, \cup, *, \Rightarrow, 0, 1 \rangle$ be a linearly ordered MV-algebra, let $\oplus$ and $(-)$ have the obvious meaning. We associate with A the structure $\mathbf{G} = OG(\mathbf{A})$ defined as follows: $\mathbf{G} = \langle G, +, \leq \rangle$ where G is the set of pairs (m, x) where m is an integer and $x \in A$, with each $(m, 1)$ identified with $(m + 1, 0)$ (pedantically, G is the Cartesian product $Z \times A$, Z being the set of integers, factorized by the obvious equivalence).

$(m, x) + (n, y) = (m + n, x \oplus y)$ if $x \oplus y < 1$,
$(m, x) + (n, y) = (m + n + 1, x * y)$ otherwise,
$(m, x) \leq (n, y)$ if $(m, x) = (n, y)$ or $m < n$ or $[m = n$ and $x < y]$

(Caution: Recall that $(m, 1) = (m + 1, 0)$.)

Theorem 3.2.19 For each linearly ordered MV-algebra A, the structure $\mathbf{G} = OG(\mathbf{A})$ is an o-group and A is isomorphic to $MV(\mathbf{G}, (0, 1))$.

Proof: First observe that the operation + and the ordering $\leq$ respect the equality imposed to our ordered pairs. Clearly, + is commutative, (0,0) is the neutral element and for each (m, x), its inverse is $(1 - m, (-)x)$. Also monotonicity is easy to check. The only property whose verification is rather laborious is associativity. Let $a = (m, x), b = (n, y), c = (k, z)$ be given; we shall investigate 6 cases.

Case 1. $x \oplus y \oplus z < 1$. Then $a+(b+c) = (a+b)+c = (m+n+k, x \oplus y \oplus z)$. In all remaining cases, $x \oplus y \oplus z = 1$.[17]

Case 2. $x \oplus y < 1, y \oplus z < 1, y \oplus z < 1$ (but $x \oplus y \oplus z = 1$, thus $z \geq (-)(x \oplus y)$ and $x \geq (-)(y \oplus z)$). Then $z \oplus x = (z \vee (-)(x \oplus y)) \oplus x = (z * (x \oplus y)) \oplus (-)(x \oplus y) \oplus x = (x \oplus y) * z \oplus ((-)x * (-y)) \oplus x = (x \oplus y) * z \oplus x * y \oplus (-)y = (x \oplus y) * z \oplus (-)y$ since $x \oplus y < 1$ and thus $x * y = 0$. (We use Lemma 3.2.16 freely.)

On the other hand , $z \oplus x = z \oplus (x \vee (-)(y \oplus z)) = z \oplus x * (y \oplus z) \oplus (-)(y \oplus z) = x * (y \oplus z) \oplus z \oplus ((-)y * (-)z) = x * (y \oplus z) \oplus y * z \oplus (-)y = x * (y \oplus z) \oplus (-)y$ since $y \oplus z = 0$.

We have proved $(x \oplus y) * z \oplus (-)y = x * (y \oplus z) \oplus (-y) = z \oplus x < 1$; thus by 3.2.17 (13) we get $(x \oplus y) * z = x * (y \oplus z)$, hence $a + (b + c) = (m + n + k, x * (y \oplus z)) = (m + n + k, (x \oplus y) * z) = (a + b) + c$.

Case 3. $x \oplus y < 1, y \oplus z < 1, x \oplus z = 1$.Then using 3.2.17 (16) we get $x * (y \oplus z) = y \oplus x * z = (x \oplus y) * z$ and hence $a + (b + c) = (a + b) + c$ as in Case 2.

Case 4. $x \oplus y < 1, y \oplus z = 1$. Then $(a+b)+c = (m+n+k+1, (x \oplus y) * z)$, $a + (b + c) = (m, x) + (n + k + 1, y * z) = (m + n + k + 1, x \oplus y * z)$ since $x \oplus y * z \leq x \oplus y < 1$ By (18) we get $(x \oplus y) * z = x \oplus y * z$.

Case 5. $x \oplus y = 1, y \oplus z < 1$. This case is fully analogous to the preceding case.

Case 6. $x \oplus y = y \oplus z = 1$ Now $(a + b) + c = (m + n + 1, x * y) + (k, z)$, $a + (b + c) = (m, x) + (n + k + 1, y * z)$.

Subcase 6(i). $x * y \oplus z = 1$. We have to prove $x \oplus y * z = 1$. Then $a+(b+c) = (a+b)+c = (m+n+k+1, x*y*z)$. Indeed, $x*y \oplus z = 1$ implies $z \geq (-)(x * y) = (-)x \oplus (-)y$; furthermore, $x \oplus y = 1$ implies $y \geq (-)x$, thus $1 = x \oplus (-)x = x \oplus ((-)x \wedge y) = x \oplus y * ((-)x \oplus (-)y) \leq x \oplus y * z$.

Subcase 6(ii). $x * y \oplus z < 1$. Then $x \oplus y * z < 1$ (otherwise Case 1 would apply to z, y, x and would get $x * y \oplus z = 1$) and we get $a + (b + c) = (m + n + k + 1, x \oplus y * z) = (m + n + k + 1, x * y \oplus z) = (a + b) + c$. This **G** is an o-group. It remains to observe that the mapping i associating with each $x \in A$ the pair $(0, x)$ is an isomorphism of A onto $MV(\mathbf{G}, (0, 1))$. This completes the proof. □

[17] In expressions containing $\oplus$ and $*$ we use the same convention concerning brackets as is usual in expressions with + and $\cdot$, thus $x * y \oplus z$ stands for $(x * y) \oplus z$ etc.

We have finished the presentation of Chang's representation of linearly ordered MV-algebras. This also completes the proof of the completeness theorem 3.2.13 of Łukasiewicz logic with respect to 1-tautologies of the standard MV-algebra of truth functions over $[0, 1]$.

3.3. RATIONAL PAVELKA LOGIC

Until now we have been interested in absolute truth, i.e. in 1-tautologies and in what formulas have the value 1 if axioms of a theory have the value 1; we succeeded to axiomatize this. But one may ask: can we also prove *partially* true conclusions from *partially* true premisses? (This seems to fit well into the intuitive notion of a fuzzy logic.) We shall show that for Łukasiewicz logic it is indeed possible. The key observation is that, for any evaluation e, if $e(\varphi) = r$, then, for any formula ψ, $e(\psi) \geq r$ iff $e(\varphi \to \psi) = 1$. Thus it appears useful to introduce, for each *rational* $r \in [0, 1]$, a *truth constant* $\bar{r}$ — a special formula whose truth value under each evaluation is r. (We already have truth constants $\bar{0}, \bar{1}$; why not have $\overline{0.7}$? We could introduce $\bar{r}$ for each real number from $[0, 1]$, but this would make our language uncountable.) Thus we shall have $e(\psi) \geq r$ iff $e(\bar{r} \to \psi) = 1$; $e(\psi) \leq r$ iff $e(\psi \to \bar{r}) = 1$. The formalism follows.

Definition 3.3.1 The language of Rational Pavelka Logic, RPL,[18] results from the language of Ł by adding truth constants $\bar{r}$ for each rational r. Each truth constant is a formula; formulas are built from propositional variables and truth constants using connectives $\to, \neg$ (as well defined connectives of Ł: $\&, \veebar, \wedge, \vee, \equiv$).

An evaluation e of propositional variables by reals from $[0, 1]$ extends to an evaluation of all formulas using truth functions of Ł extended by the clause $e(\bar{r}) = r$ for each e, r.

The *axioms* of RPL are the axioms of Ł (thus (Ł1)–(L4)) plus the following bookkeeping axioms for truth constants:

$$\begin{array}{rcl} (\bar{r} \to \bar{s}) & \equiv & \overline{r \Rightarrow s} \\ \neg\bar{r} & \equiv & \overline{1 - r} \end{array}$$

(e.g. $\overline{0.8} \to \overline{0.7} \equiv \overline{0.9}$, $\neg\overline{0.4} \equiv \overline{0.6}$). The deduction rule is modus ponens. A *theory* is a set of formulas — special axioms. Proofs and provability are as usual; $T \vdash \varphi$ means that T proves φ (φ is provable in T). An evaluation e is a *model* of T if $e(\alpha) = 1$ for all axioms of T.

[18] See [163, 74].

A *graded formula* is a pair (φ, r) where φ is a formula and r a rational element of $[0, 1]$; it is just another notation for the formula $(\bar{r} \to \varphi)$.

Lemma 3.3.2 The following is a derived deduction rule in RPL:

$$\frac{(\varphi, r), (\varphi \to \psi, s)}{(\psi, r * s)},$$

i.e. whenever a theory T proves (φ, r) and $(\varphi \to \psi, s)$ it proves $(\psi, r * s)$ where $*$ is Łukasiewicz t-norm.

Proof: Compare this with the discussion following 2.1.15. If $T \vdash \bar{r} \to \varphi$ and $T \vdash \bar{s} \to (\varphi \to \psi)$ then $T \vdash (\bar{r} \& \bar{s}) \to (\varphi \& (\varphi \to \psi))$, thus $T \vdash \overline{r * s} \to \psi$. □

Remark 3.3.3 Note that the deduction theorem 2.2.18 remains valid for RPL: $T \cup \{\varphi\} \vdash \psi$ iff for some n, $T \vdash \varphi^n \to \psi$.

Two main notions will be introduced now. The idea is that the truth of the axioms of T may fail to guarantee that φ is true but it may guarantee that φ is not too false, i.e. may entail that the truth value of φ must be at least r. In other words, truth of T may imply truth of $(\bar{r} \to \varphi)$. Similarly, for provability. Our definitions follow.

Definition 3.3.4 Let T be a theory over RPL, φ a formula.

(1) The *truth degree* of φ over T is $\|\varphi\|_T = \inf\{e(\varphi) \mid e \text{ is a model of } T\}$.

(2) The *provability degree* of φ over T is $|\varphi| = \sup\{r \mid T \vdash (\varphi, r)\}$.

Theorem 3.3.5 (Completeness.) For each theory T and each formula φ, the provability degree equals the truth degree, i.e.

$$|\varphi|_T = \|\varphi\|_T.$$

We shall elaborate a proof. First some useful notions.

Remark 3.3.6 Recall definitions 2.2.20 of a consistent theory T $(T \nvdash \bar{0})$ and 2.4.1 of a complete theory T ($T \vdash (\varphi \to \psi)$ or $T \vdash (\psi \to \varphi)$ for each pair (φ, ψ)). Recall also 2.4.2 (2) implying that each consistent theory T has a consistent complete supertheory T' (the proof is not violated by the presence of truth constants).

Lemma 3.3.7 If T does not prove $(\bar{r} \to \varphi)$ then $T \cup \{\varphi \to \bar{r}\}$ is consistent.

Proof: Assume $T \cup \{\varphi \to \bar{r}\}$ inconsistent, thus $T \cup \{\varphi \to \bar{r}\} \vdash \bar{0}$, and, by the deduction theorem, there is an n such that $T \vdash (\varphi \to \bar{r})^n \to \bar{0}$. Recall 2.2.24 (33): $T \vdash (\varphi \to \bar{r})^n \vee (\bar{r} \to \varphi)^n$, thus $T \vdash \bar{0}^n \vee (\bar{r} \to \varphi)^n$, $T \vdash \bar{0} \vee (\bar{r} \to \varphi)^n$, $T \vdash (\bar{r} \to \varphi)^n$, a contradiction. □

Lemma 3.3.8 Let T be consistent and complete.

(1) For each φ, $|\varphi|_T = \sup\{r | T \vdash \bar{r} \to \varphi\} = \inf\{s | T \vdash \varphi \to \bar{s}\}$.

(2) The provability degree commutes with connectives, i.e.

$$|\neg\varphi|_T = 1 - |\varphi|_T, \quad |\varphi \to \psi|_T = |\varphi|_T \Rightarrow |\psi|_T.$$

Thus the evaluation $e(p_i) = |p_i|_T$ is a model of T.

Proof:

(1) Since for each r either $T \vdash \varphi \to \bar{r}$ or $T \vdash \bar{r} \to \varphi$ it suffices to show that $T \vdash \bar{r} \to \varphi$ and $T \vdash \varphi \to \bar{s}$ implies $r \leq s$. Assume $r > s$; we would get $T \vdash \bar{r} \to \bar{s}$, i.e. $T \vdash \overline{r \Rightarrow s}$, but $r \Rightarrow s < 1$, thus T would be inconsistent since for some n, $(r \Rightarrow s)^n = 0$.

(2) We just omit the index T. $|\neg\varphi| = \sup\{t | T \vdash \bar{t} \to \neg\varphi\} = \sup\{t | T \vdash \varphi \to 1 - \bar{t}\} = \sup\{1 - s | T \vdash \varphi \to \bar{s}\} = 1 - \inf\{s | T \vdash \varphi \to \bar{s}\} = 1 - |\varphi|$.

$|\varphi| \Rightarrow |\psi| = \inf\{r | T \vdash \varphi \to \bar{r}\} \Rightarrow \sup\{s | T \vdash \bar{s} \to \psi\} =$
$= \sup\{r \Rightarrow \sup\{s | T \vdash \bar{s} \to \psi\} | T \vdash \varphi \to \bar{r}\} =$
$= \sup\{r \Rightarrow s | T \vdash \varphi \to \bar{r}, T \vdash \bar{s} \to \psi\} \leq \sup\{t | T \vdash \bar{t} \to (\varphi \to \psi)\} =$
$= |\varphi \to \psi|$ (since $T \vdash \overline{r \Rightarrow s} \to (\bar{r} \to \bar{s})$, thus if $T \vdash \varphi \to \bar{r}$ and $T \vdash \bar{s} \to \psi$, then $T \vdash \overline{r \Rightarrow s} \to (\varphi \to \psi)$). Note that we have heavily used the continuity of $\Rightarrow$.

Conversely, assume $(|\varphi| \Rightarrow |\psi|) < t < t' < |\varphi \to \psi|$ for some rational t, t'. Express t as $r \Rightarrow s$ for some $r < |\varphi|$, $s > |\psi|$; then $T \vdash \bar{r} \to \varphi$, $T \vdash \psi \to \bar{s}$, thus $T \vdash (\varphi \to \psi) \to \overline{(\bar{r} \to \bar{s})}$, $T \vdash (\varphi \to \psi) \to \bar{t}$, $T \vdash t' \to (\varphi \to \psi)$, hence $T \vdash \bar{t'} \to \bar{t}$. $T \vdash \overline{t' \Rightarrow t}$, but $t' \Rightarrow t < 1$ and thus T is inconsistent. We have proved $|\varphi| \Rightarrow |\psi| \geq |\varphi \to \psi|$, which completes the proof.

3.3.9 (Proof of completeness of RPL.) Showing $|\varphi|_T \leq \|\varphi\|_T$ is routine (soundness). To prove the converse one has to show that for each $r < \|\varphi\|$, $T \vdash \bar{r} \to \varphi$. This is to show that if T does not prove $\bar{r} \to \varphi$, then $r \geq \|\varphi\|$. But if $T \not\vdash \bar{r} \to \varphi$, then $T \cup \{\varphi \to \bar{r}\}$ is consistent by 3.3.7, thus has a consistent complete extension T' by 3.3.6 and, by 3.3.8, the evaluation e defined by $e(p_i) = |p_i|_T$ is a model of T' and $e(\varphi \to \bar{r}) = 1$, thus $e(\varphi) \leq r$. This completes the proof. □

*

In the rest of this section we shall discuss three additional topics: (1) we show a formulation of compactness true for RPL (and for Ł), (2) we reduce RPL to Ł and gain a variant of strong completeness for RPL and, finally, (3) show how RPL may be extended by some connectives (product conjunction), but not by some others (as $\triangle$ or a non-continuons implication).

We showed in the preceding section that the usual formulation of compactness (for each T, if φ is true in each model of T, then there is a finite $T_0 \subseteq T$ such that φ is true in each model of T_0) *fails* for Ł and hence fails for RPL. In Boolean logic this a equivalent to say that for each T, if each finite $T_0 \subseteq T$ has a model, then T has a model. But these two formulations are *not* equivalent for Ł and (since $|\varphi|_T = 0$ is not the same as $|\varphi|_T < 1$!); and in fact, the latter formulation is true for RPL:

Theorem 3.3.10 Let T be a theory over RPL. If each finite $T_0 \subseteq T$ has a model, then T has a model.

Proof: Assume T has no model; then, by the proof of the completeness theorem, T is inconsistent, hence a finite subtheory $T_0 \subseteq T$ is inconsistent and hence T_0 has no models. □

We turn to our second additional topic. We show that if desired we can get rid of truth constants, replacing them by some special axioms over Ł. (In fact, it is more elegant to keep the constants; the reduction is useful for some future proofs.)

Lemma 3.3.11 Associate with each rational $r \in [0, 1]$ a propositional variable p_r. For each r there is a formula φ_r of Ł (i.e. not containing truth constants) built from p_r and at most two other propositional variables p_s, p_t such that for each evaluation e, $e(\varphi_r) = e(\varphi_s) = e(\varphi_t) = 1$ iff $e(p_r) = r$, $e(p_s) = s$ and $e(p_t) = t$.

Proof: Let φ_0 be $p_0 \equiv p_0 \& \neg p_0$, φ_1 be $p_1 \equiv p_1 \underline{\vee} \neg p_1$ and $\varphi_{\frac{1}{2}}$ be $p_{\frac{1}{2}} \equiv \neg p_{\frac{1}{2}}$. Let $n \geq 2$; then let $\varphi_{\frac{1}{2n}}$ be $np_{\frac{1}{2n}} \equiv p_{\frac{1}{2}}$ (recall that $n\varphi$ is $\varphi \underline{\vee} \ldots \underline{\vee} \varphi$, n times). For arbitrary $r = \frac{m}{n} = \frac{2m}{2n}$ let φ_r be $p_r \equiv (2m)p_{\frac{1}{2n}}$. The rest is obvious. □

Definition 3.3.12 Let φ_r be as in the preceding proof. Let φ be a formula not containing any variable p_r, let φ^* result from φ by replacing any truth constant $\bar{r}$ by the variable p_r and let $\hat{T}$ be the theory in Ł whose axioms are all the axioms φ_r.

Lemma 3.3.13 (1) Let $q_1, \ldots$ be variables distinct from all p_r. Each evaluation e of the q-variables extends uniquely to the evaluation $\hat{e}$ of p_r's which is a model of $\hat{T}$; for each formula φ built from q-variables, $e(\varphi) = \hat{e}(\varphi^*)$.

(2) For each theory T in the q-variables, let $T^{**} = T^* \cup \hat{T}$ where $T^* = \{\alpha^* | \alpha \in T\}$. For each q-formula φ, $T \vdash_{RPL} \varphi$ iff $T^{**} \vdash_{Ł} \varphi^*$; thus theories in RPL reduce to theories in Ł.

Proof: (1) is evident; we prove (2). To show that if φ is provable in T (over RPL), then φ^* is provable in T^{**} it is enough to observe that the $*$-translations of bookkeeping axioms are provable in $\hat{T}$. But this follows by finite strong completeness; e.g. $\neg p_{0.3} \equiv p_{0.7}$ is true in each model of $\varphi_{0.3}, \varphi_{0.7}, \varphi_{\frac{1}{2}}, \varphi_{\frac{1}{20}}$ (see the definitions of these formulas) and hence the latter formulas prove $\neg p_{0.3} \equiv p_{0.7}$ over Ł.

Conversely, if $T^{**} \vdash_{Ł} \varphi^*$ then take a corresponding proof $\varphi_1, \ldots, \varphi_n$ and substitute $\bar{r}$ for p_r at each occurence. Check that each φ_i is T-provable over RPL: axioms of T^* go back to axioms of T and axioms of $\hat{T}$ become RPL-provable after our substitution. □

We shall use our (cumbersome) analysis to prove the following strong completeness theorem for RPL:

Theorem 3.3.14 [19] Let T be a finite theory over RPL, let φ be a formula. If φ is true in all models of T, then $T \vdash \varphi$.

Proof: This is because using the above techniques we can construct a *finite* subtheory $\hat{T}_0$ of $T^* \cup \hat{T}$ such that φ^* is true in all models of $\hat{T}_0$ (only truth constants occurring in T, φ have to have counterparts in $\hat{T}_0$); hence, by the finite strong completeness for Ł, $\hat{T}_0 \vdash_{Ł} \varphi^*$ and thus $T \vdash \varphi$. □

Corollary 3.3.15 If T is as above and $\|\varphi\|_T = r$ (rational) then $T \vdash (\varphi, r)$, thus $\|\varphi\|_T$ is in fact *maximal* r such that $T \vdash \bar{r} \to \varphi$.

[19] See [93].

We can improve this by showing the following:

Theorem 3.3.16 If T is a finite theory over RPL and φ is a formula then $\|\varphi\|_T$ is rational; thus for $r = \|\varphi\|_T$ we have $T \vdash \bar{r} \to \varphi$.

Proof: This follows by the preceding from the following lemma. □

Lemma 3.3.17 Assume T finite, φ a formula. Then there is an evaluation e taking only rational values and such that e is a model of T and $e(\varphi) = \|\varphi\|_T$.

Proof: Since a full proof requires some familiarity with polyhedra and simplexes we first present a proof for the particular case that T and φ involves just two propositional variables; then we sketch a full proof for the reader knowing the necessary notions and properties. (or for the reader willing to learn them, e.g. from [223]). Such a reader may of course skip the particular proof for two variables.

Note that each condition $kx+hy+q \geq 0$ (k, h, q integers, $k \neq 0$ or $h \neq 0$), defines a closed half-plane (call it an r-half plane). The unit square $[0, 1]^2$ is the intersection of four r-halfplanes ($x \geq 0, y \geq 0, -x+1 \geq 0, -y+1 \geq 0$). We shall consider systems of r-halfplanes containing the four just mentioned; for the sake of this proof call the set defined as the intersection of such a system an r-set.

Claim 1. Each r-set is a subset of the unit square which is empty, or a singleton $\{(x, y)\}$ with x, y rational, or a segment whose endpoints have rational coordinates or a convex polygon whose vertices have rational coordinates.

This is true for the starting unit square; and it is easily verified that if an r-set has the property stated in Claim 1 then its intersection with an r-halfplane has the same property.

Claim 2. For each $(x, y) \in [0, 1]^2$ and each formula φ, there is an r-set A_φ containing (x, y) such that on A_φ the truth function of φ is linear with integer coefficients (let $e_{u,v}(p) = u$, $e_{u,v}(q) = v$; the function in question is $e_{u,v}(\varphi)$).

Indeed, for φ atomic $A_\varphi = [0, 1]^2$; for any φ, ψ, $A_{\neg\varphi} = A_\varphi$, and if $e_{x,y}(\varphi) \leq e_{x,y}(\psi)$ then $A_{\varphi\to\psi} = A_\varphi \cap A_\psi \cap \{(u, v) | e_{u,v}(\psi) - e_{u,v}(\varphi) \geq 0\}$ (the last set in the intersection being an r-halfplane); similarly for $e_{x,y}(\varphi) \geq e_{x,y}(\psi)$.

Claim 3. For each $(x, y) \in [0, 1]^2$ and each φ such that $e_{x,y}(\varphi) = i$ where $i = 0$ or 1, there is a r-set B_φ, containing (x, y) such that $e_{u,v}(\varphi) = i$ constantly on B_φ. If $e_{x,y}(\varphi) \neq 0, 1$ we put $B_\varphi = \emptyset$.

Generalize the proof of Claim 2: for $y = i$ $B_\varphi = [0, 1] \times \{i\}$, analogously for $x = i$. For any φ, $B_{\neg\varphi} = B_\varphi$. If $e_{x,y}(\varphi \to \psi) = 1$ then $B_{\varphi\to\psi} =$

$A_{\varphi \to \psi} \cap \{(u,v) | e_{u,v}(\varphi) \leq e_{u,v}(\psi)\}$; if $e_{x,y}(\varphi \to \psi) = 0$ then $e_{x,y}(\varphi) = 1$, $e_{x,y}(\psi) = 0$ and $B_{\varphi \to \psi} = B_\varphi \cap B_\psi$.

Now assume that T has just one axiom α (take the conjunction of all axioms). If T has a model (x,y) such that $e_{x,y}(\alpha) = 1$ then the set of all such models is compact and hence contains a model (x,y) in which $e_{x,y}(\varphi)$ has the least value among all models of α. Take $A_\varphi \cap B_\alpha$; this is an r-set containing (x,y). We claim that (x,y) may be taken rational.

This is clear if $A_\varphi \cap B_\alpha$ is a singleton. If $A_\varphi \cap B_\alpha$ is a segment with rational endpoints then either $e_{u,v}(\varphi)$ is constant on the segment and either endpoint is a model of α with minimal value of φ. or $e_{u,v}(\varphi)$ is not constant and then it is either increasing or decreasing on the segment and thus one endpoint has the desired properties.

Finally if $A_\varphi \cap B_\alpha$ is a polygon then one of the vertices gives the minimal value of φ. Indeed, if (x,y) is a point in the interior of $A_\varphi \cap B_\alpha$ then take any segment containing (x,y) and having endpoints on the boundary of $A_\varphi \cap B_\alpha$. Show that the minimum of $e_{u,v}(\varphi)$ on our segment is taken in one of the endpoints. Finally if (x,y) is on the boundary but not a vertex take the corresponding two neighboured vertices and the segment having them as endpoints. Show that the minimum of $e_{u,v}(\varphi)$ on the segment is taken in one of the endpoints. Thus the minimum over $A_\varphi \cap B_\alpha$ is taken in one of the vertices and therefore is rational.

This was the proof for two variables; now we present a general proof (which is a generalization of the present proof) for the reader knowing the necessary notions and facts[20].

Observe that the truth functions of Ł are continuous and piecewise linear with integer coefficients and so is the truth function of each formula of Ł. Note that we may get rid of truth constants in T and φ adding finitely many axioms (see above). Let $p_1, \ldots, p_m$ be propositional variables occurring in T and φ; let $\alpha_1, \ldots, \alpha_h$ be axioms of T.

Call the truth functions of $\alpha_1, \ldots, \alpha_h$ (as functions of m variables) $f_1, \ldots$ $\ldots, f_h$ and denote the truth function of φ by g. Let us split their domain into a complex of m-dimensional polyhedra $P_1, \ldots, P_k$ such that each function is linear over each P_i. Indeed, by a suitable generalization of the triangulation process for a polygon, where no new vertices are added, we can assume that each P_j is in fact an m-simplex. The faces of the simplexes $P_1, ..., P_k$ can safely be thought of as zerosets of linear polynomials with integer coefficients, whence the vertices of each P_j are solutions of systems of n linear equations with integer coefficients. Thus these vertices are rational points.

[20] Thanks are due to Professor D. Mundici for communicating this proof to me. For an extremely deep investigaton of Łukasiewicz propositional logic and MV-algebras see [30].

Upon restriction to P_j, each f_i, as well as g, will reach its maximum and its minimum on some vertex of P_j.

The set A where all the f_i equal 1 is compact; thus there is a point in A in which g is minimal. Our aim is to show that there is such a point with rational coordinates. A is a union of some simplexes, say $Q_1, ..., Q_m$ in the above collection $P_1, ..., P_k$. If g is not constantly equal to 1 over A (i.e., if a formula corresponding to g does not logically follow from the formulas corresponding to the f_i) then the minimum of g over A is attained at the vertex v of some such m-simplex Q_k. By the above discussion the coordinates of v are rationals. □

*

We turn to our last additional topic — expanding the language. As an example we show that RPL may be expanded by the product conjunction.

Definition 3.3.18 RPL($\odot$) stands for RPL whose language results from that of RPL by adding a new binary connective $\odot$, interpreted as the product conjunction. The axioms are extended by the following:

$$(\varphi \to \psi) \to ((\varphi \odot \chi) \to (\psi \odot \chi))$$
$$(\varphi \to \psi) \to ((\chi \odot \varphi) \to (\chi \odot \psi))$$
$$\bar{r} \odot \bar{s} \equiv \overline{r.s} \text{ (for all rational } r, s \in [0,1]).$$

Theorem 3.3.19 (Completeness.) For each theory T over RPL($\odot$), $|\varphi|_T = \|\varphi\|_T$.

Proof: Soundness ($|\varphi|_T \leq \|\varphi\|_T$) is obvious. To prove $|\varphi|_T \leq \|\varphi\|_T$ just read the proof of 3.3.5 for the expanded logic. The crucial point is 3.3.8 (2): we have to prove $|\varphi \odot \psi| \equiv |\varphi|.|\psi|$. This is proved analogously to the case of $\to$ (again using continuity and monotonicity). Indeed, $|\varphi|.|\psi| = \sup\{r | T \vdash \bar{r} \to \varphi\}.\sup\{s | T \vdash \bar{s} \to \psi\} = \sup\{r.s | T \vdash \bar{r} \to \varphi, T \vdash \bar{s} \to \psi\} = |\varphi \odot \psi|$. (All three axioms above used.)

Conversely, assume $|\varphi|.|\psi| < t < t' < |\varphi \odot \psi|$; let $t = r.s$, $r > |\varphi|$, $s > |\psi|$, then $T \vdash \varphi \to \bar{r}$, $T \vdash \psi \to \bar{s}$, $T \vdash \varphi \odot \psi \to \bar{t}$, but $T \vdash \overline{t'} \to \varphi \odot \psi$, thus $T \vdash \bar{t} \to \overline{t'}$, $T \vdash \overline{t \Rightarrow t'}$, $t \Rightarrow t' < 1$, thus T is inconsistent, a contradiction. □

Remark 3.3.20 Continuity was crucial; e.g. one cannot have this style of completeness when adding the non-continuous implication of product logic (cf. 2.1.8) or the connective $\triangle$. Example: Let $T = \{\bar{r} \to p | r < 1\}$; then $\| \triangle p\|_T = 1$, but under any finitary axiomatization (each proof uses only finitely many axioms), $| \triangle p|_T = 0$ (verify easily).

Example 3.3.21 Let us close this section by an example (or exercise). We are going to discuss the notion of a large natural number. Using crisp logic we obtain a paradox (called the sorites paradox). There are two assumptions:

(A) One million is a large number. (Instead one million you may substitute any other number you consider to be large.)

(B) If a large number is diminished by 1 then the result is again large.

But then applying (B) one million times you get that 0 is a large number, which is not what you want.

We may immediately object that largeness is a fuzzy notion and each number is large in a certain degree. Let p_i be a sentence "i is large". Then even if we admit that $p_{1000000}$ is true (truth degree 1) we shall say that (B) is only partially true. Let us discuss various possibilities of expressing this in Ł (or RPL). We work with p_i for $i = 0, \ldots, 10^6$; T_0 is the theory $\{p_i \to p_{i+1} | i < 10^6\}$ ("if i is large then so is $i + 1$").

(1) Let $r = 0.999999$; let T_1 be the extension of T_0 by $p_{1000000}$ ("one million is large"), $\neg p_0$ ("0 is not large"), and by each axiom $\bar{r} \to (p_{i+1} \to p_i)$ for $i < 10^6$. The last axiom is true iff the truth degree of $(p_{i+1} \to p_i)$ is at least 0.999999; thus an evaluation e is a model of T_1 if $e(p_{1000000}) = 1$, $e(p_0) = 0$ and for each $i < 10^6$, $e(p_{i+1}) \leq e(p_i) + 0.000001$; thus $e(p_i) = 0.000001i$. Now take $r < 0.999999$ (e.g. $r = 0.999$); what, then, are models of our T_1?

(2) Now let T_2 be the extension of T_1 by $\neg p_0$ and for each $i < 10^6$, $(p_{i+1} \& p_{i+1}) \to p_i$. Here $p_{i+1} \& p_{i+1}$ (or p_{i+1}^2 in brief) may be read "$(i + 1)$ is very large" thus we assume that if $(i + 1)$ is very large then i is large. If e is a model of T_2 and we put $x_i = e(p_i)$ then we get $2x_{i+1} - 1 \leq x_i$, thus $x_{i+1} \leq x_i + \frac{1-x_i}{2}$. Thus the maximal model of T_2 has $x_0 = 0, x_1 = \frac{1}{2}, x_2 = \frac{3}{4}, x_3 = \frac{7}{8}, \ldots x_i = (2^i - 1)/2^i, \ldots$, i.e. we have never $x_i = 1$ (but of course letting i go to infinity we get $\lim_{i \to \infty} x_i = 1$).

(3) Alternatively, we may use $p_i \underline{\vee} p_i$ as saying "i is more or less large". Let T_3 be T_0 extended by $p_{1000000}$ and for each $i < 10^6$, by $p_{i+1} \to 2p_i$. Each model e of T_2 satisfies $x_{i+1} \leq 2x_i$; then $x_0 = 0$ is impossible. For example taking $x_0 = 0.001$ and choosing x_i for $i > 0$ as large as possible we get the sequence

$0.001, 0.002, 0.004, 0.008, 0.016, 0.032, 0.064, \ldots, 0.512, 1, 1, \ldots.$

Similarly, for each rational $r = \frac{m}{n}$, we may modify T_3 to $\hat{T}_3$ by postulating $np_{i+1} \to mp_i \quad (i < 10^6)$. Then $x_i = \min(1, x_0 \cdot r^i)$ and the corresponding evaluation is a model of $\hat{T}_3$ iff $x_0 \cdot r^{1000000} \geq 1$.

We shall continue this example in Ch. 4 for product logic and in Ch. 5 for predicate calculus.

PRODUCT LOGIC, GÖDEL LOGIC (AND BOOLEAN LOGIC)

4.1. PRODUCT LOGIC

We are going to investigate the second of the three most important propositional calculi, namely PC($*_\Pi$) where $*_\Pi$ is the product t-norm; we shall call this logic just the *product logic* and denote it by Π. Recall that the corresponding implication is Goguen and the corresponding negation is Gödel negation (cf. 2.1.11,2.1.17). We present an axiom system (extension of BL by two axioms) and show its completeness[21] by relating linearly ordered Π-algebras — called product algebras — to ordered Abelian groups, similarly as in the case of MV-algebras (but now the proof is much simpler than in the case of MV-algebras). We also show that Łukasiewicz logic Ł has a faithful interpretation in Π.[22] We shall close the section by discussing some additional topics. *Convention:* In this section $\rightarrow$ (without any subscript) will be Goguen implication; the product conjunction will be denoted by $\odot$.

Definition 4.1.1 The *axioms* of Π are those of BL plus

(Π1) $\neg\neg\chi \rightarrow ((\varphi \odot \chi \rightarrow \psi \odot \chi) \rightarrow (\varphi \rightarrow \psi))$,

(Π2) $\varphi \wedge \neg\varphi \rightarrow \bar{0}$.

Lemma 4.1.2 The axioms are 1-tautologies over the algebra $[0,1]_\Pi$ of the truth functions.

Proof:

(Π1) Let e be an evaluation; if $e(\chi) = 0$ then $e(\neg\neg\chi) = 0$ and $e(\neg\neg\chi \rightarrow$ anything$)= 1$. If $e(\chi) > 0$ then $e(\neg\neg\chi) = 1$, and *either* $e(\varphi \odot \chi) \leq e(\psi \odot \chi)$, thus $e(\varphi).e(\chi) \leq e(\psi).e(\chi)$ and $e(\varphi) \leq e(\psi)$, hence $e((\varphi \odot \chi) \rightarrow (\psi \odot \chi)) = e(\varphi \rightarrow \psi) = 1$, *or* $e(\varphi \odot \chi) > e(\psi \odot \chi)$ then $e(\varphi) > e(\psi)$ and $e((\varphi \odot \chi) \rightarrow (\psi \odot \chi)) = e(\varphi \rightarrow \psi) = \frac{e(\psi)}{e(\varphi)}$.

(Π2) Since in Π $\neg$ is Gödel negation, either $e(\varphi)$ or $e(\neg\varphi)$ must be 0.

□

[21] See [80].

[22] See [9].

Lemma 4.1.3 Π proves the following formulas:

(1) $\neg(\varphi \odot \psi) \to \neg(\varphi \wedge \psi)$

(2) $(\varphi \to \neg\varphi) \to \neg\varphi$

(3) $\neg\varphi \vee \neg\neg\varphi$

Proof:

(1) The following are equivalent forms of the formula to be proved:
$((\varphi \odot \psi) \to 0) \to ((\varphi \wedge \psi) \to 0)$,
$[(\varphi \to (\psi \to 0)) \odot (\varphi \wedge \psi)] \to 0$,
$[(\varphi \to \neg\psi) \odot (\varphi \wedge \psi)] \to 0$.
Now, we following chains of implications are provable:
$[(\varphi \to \neg\psi) \odot (\varphi \wedge \psi)] \to [(\varphi \to \neg\psi) \odot \varphi] \to \neg\psi$,
$[(\varphi \to \neg\psi) \odot (\varphi \wedge \psi)] \to [(\varphi \to \neg\psi) \odot \psi] \to \psi$,
$[(\varphi \to \neg\psi) \odot (\varphi \wedge \psi)] \to [\psi \wedge \neg\psi] \to 0$.
(We have used various provabilities of BL, see (9), (4), axiom (A2), (11) and finally axiom (Π2).)

(2) We have $\neg(\varphi \odot \varphi) \to \neg\varphi$ by(1), thus $(\varphi \odot \varphi \to \bar{0}) \to (\varphi \to \bar{0})$,
$(\varphi \to (\varphi \to \bar{0})) \to (\varphi \to \bar{0})$, thus $(\varphi \to \neg\varphi) \to \neg\varphi$.

(3) The following implications are provable:
$(\neg\varphi \to \neg\neg\varphi) \to \neg\neg\varphi$ (by (2) here),
$\neg\neg\varphi \to (\neg\varphi \vee \neg\neg\varphi)$ (by BL), thus
$(\neg\varphi \to \neg\neg\varphi) \to (\neg\varphi \vee \neg\neg\varphi)$.
On the other hand,
$(\neg\neg\varphi \to \neg\varphi) \to (\neg\neg\varphi \to \neg\neg\neg\varphi)$ (by BL), thus
$(\neg\neg\varphi \to \neg\neg\neg\varphi) \to \neg\neg\neg\varphi$ (by (2) here),
$\neg\neg\neg\varphi \to \neg\varphi$ (by BL), thus
$(\neg\neg\varphi \to \neg\varphi) \to \neg\varphi$,
$(\neg\neg\varphi \to \neg\varphi) \to (\neg\varphi \vee \neg\neg\varphi)$.
We get $(\neg\varphi \vee \neg\neg\varphi)$ applying axiom (A6) to $\neg\varphi$, $\neg\neg\varphi$, $\neg\varphi \vee \neg\neg\varphi$.

$\square$

Lemma 4.1.4 The axiom (Π2) can be equivalently replaced by each of the following formulas:

$$\neg(\varphi \odot \varphi) \to \neg\varphi, \; (\varphi \to \neg\varphi) \to \neg\varphi, \; \neg\varphi \vee \neg\neg\varphi.$$

Proof: We have just seen that all three formulas are provable in Π. We prove that each of them, together with BL+(Π1), proves (Π2).

(1) Take $(\varphi \to \neg\varphi) \to \neg\varphi$ together with BL+(Π1). We have the following chain of provable implications:
$(\varphi \wedge \neg\varphi) \to [\varphi \odot (\varphi \to \neg\varphi)] \to [\varphi \odot \neg\varphi] \to 0$.

(2) Now take $\neg(\varphi \odot \varphi) \to \neg\varphi$; we get $(\varphi \odot \varphi \to 0) \to (\varphi \to 0)$, and hence $(\varphi \to (\varphi \to 0)) \to \varphi \to 0$, which is $(\varphi \to \neg\varphi) \to \neg\varphi$. This was (1).

(3) Finally take $\neg\varphi \vee \neg\neg\varphi$. Then the following are provable:
$\neg\neg\varphi \to (((\varphi \odot \varphi) \to (\varphi \odot \bar{0})) \to (\varphi \to \bar{0}))$ (axiom (Π1)),
$\neg\varphi \to$ (anything $\to (\varphi \to \bar{0})$),
thus $(\neg\neg\varphi \vee \neg\varphi) \to ((\varphi \odot \varphi \to \bar{0}) \to (\varphi \to \bar{0}))$
(observing that $\bar{0}$ is equivalent to $\varphi \odot \bar{0}$ in BL), hence we get $(\varphi \odot \varphi \to \bar{0}) \to (\varphi \to \bar{0})$ i.e. $\neg(\varphi \odot \varphi) \to \neg\varphi$; this was (2). □

Definition 4.1.5 Following the general approach we define a Π-*algebra* (or *product algebra*) to be a BL-algebra satisfying

$$(-)(-)z \leq ((x * z \Rightarrow y * z) \Rightarrow (x \Rightarrow y)),$$
$$x \cap (-)x = 0.$$

Remark 4.1.6 Trivially, the class of all product algebras is a variety and Π is sound with respect to product algebras, i.e. each formula provable on Π is a **L**-tautology for each product algebra **L**.

Lemma 4.1.7 The following holds in each linearly ordered product algebra:

(1) if $x > 0$ then $(-)x = 0$,

(2) if $z > 0$ then $x * z = y * z$ implies $x = y$,

(3) if $z > 0$ then $x * z < y * z$ implies $x < y$.

Proof:

(1) $0 = x \cap -x = \min(x, -x)$, hence if $x > 0$ then $-x = 0$.

(2) If $z > 0$ then $-(-z) = 1$, thus if $x * z \leq y * z$ then $(x * z) \Rightarrow (y * z) = 1$ and $x \Rightarrow y = 1$, hence $x \leq y$. Thus $x * z = y * z$ implies $x = y$. On the other hand, evidently $x \leq y$ implies $x * z \leq y * z$, thus if $x * z < y * z$ (i.e. $x * z \leq y * z$ and *not* $x * z \leq y * z$ then $x < y$. This gives (3).

□

Theorem 4.1.8 Let $\mathbf{L} = \langle L, *, \Rightarrow, \cap, \cup, 0_L, 1_L\rangle$ be a linearly ordered product algebra. Then there is a linearly ordered Abelian group $\mathbf{G} = = \langle G, +_G, 0_G, \leq_G\rangle$ such that $L - \{0\} = Neg_G = \{g \in G | g \leq_G 0_G\}$ such that, for all $g, h \in L - \{0\}$,

$$0_G = 1_L$$
$$g +_G h = g * h$$
$$g \leq_G h \text{ iff } g \leq h$$

Furthermore, for $g \geq h$, $g \Rightarrow h = h -_G g$.

Proof: Observe that $L - \{0_L\}$ is a linearly ordered commutative semigroup and 1_L is its greatest and neutral element. (It is closed under $*$ due to 4.1.3 (1), which gives: $x * y = 0$ implies $\min(x, y) = 0$.) Observe further that $L - \{0_L\}$ satisfies "dual subtraction": for each $0_L < x \leq y$ the equation $x * z = x$ has a solution (namely $y \Rightarrow x$; recall that $y * (y \Rightarrow x) = \min(x, y)$) and this solution is unique due to 4.1.7 (2). Hence, by 1.6.9 , $L - \{0_L\}$ (with $*$ and $\leq$) is the non-positive part of a unique (up to an isomorphism) linearly ordered Abelian group $\mathbf{G}$. □

Definition 4.1.9 For each linearly ordered Abelian group $\mathbf{G}$ let $\Pi(\mathbf{G})$ be the algebra $\mathbf{L} = \langle L, *, \Rightarrow, \cap, \cup, 0_L, 1_L\rangle$ where $L = Neg_G \cup \{-\infty\}$, where $-\infty$ is a new element, less than all $x \in Neg_\infty$,
$x * y = x +_G y$ for $x, y \in Neg_G$,
$(-\infty) * x = x * -\infty = (-\infty)$ for $x \in L$,
$x \Rightarrow y = 1$ for $x \leq y$, $x, y \in L$,
$x \Rightarrow y = y -_G x$ for $x > y$, $x, y \in L - \{-\infty\}$,
$x \Rightarrow -\infty = -\infty$ for $x > -\infty$;
$x \cap y = \min(x, y)$, $x \cup y = \max(x, y)$,
$0_L = -\infty$, $1_L = 0_G$.

Lemma 4.1.10 Under the notation of the previous definition, $\Pi(\mathbf{G})$ is a linearly ordered product algebra and the group corresponding to $\Pi(\mathbf{G})$ by 4.1.8 is just $\mathbf{G}$.

Proof: by elementary checking. □

Corollary 4.1.11 If an identity $\tau = \sigma$ in the language of product algebras is valid in the standard product algebra $[0, 1]_\Pi$ with truth functions then it is valid in all linearly ordered product algebras.

Proof: as in 3.2.11. □

Remark 4.1.12 This representation evidently extends to the representation of linearly ordered $\Pi_\triangle$-algebras, i.e. product algebras with the projection added ($\triangle(1) = 1$, $\triangle(x) = 0$ otherwise). Thus we immediately get the following.

Theorem 4.1.13 (Completeness.)

(1) A formula φ is provable in the product logic Π iff it is a 1-tautology of the product logic.

(2) Let T be a finite theory over Π, φ a formula. T proves φ over the product logic iff it is true in each model of T (in the sense of Π).

(3) The claims (1)–(2) remain valid if Π is extended by the connective $\triangle$ and by the corresponding axioms ($\triangle$1)–($\triangle$5), together with the rule of necessitation.

*

We now show that Łukasiewicz logic Ł has a faithful interpretation in Π. The main idea is that of 2.1.23: Łukasiewicz conjunction on $[0,1]$ is isomorphic to restricted product $\max(a, x \cdot y)$ on $[0,1]$ for each $0 < a < 1$. More than that:

Lemma 4.1.14 For each a such that $0 < a < 1$, the standard MV-algebra on $[0,1]$ is isomorphic to the algebra $\langle [a,1], *_a, \Rightarrow_a, \cap_a, \cup_a, 0_a, 1_a \rangle$ where, for all $a \leq x$, $y \leq 1$,

$$\begin{aligned}
&x *_a y = \max(a, x \cdot y),\\
&x \Rightarrow_a y = x \Rightarrow_\Pi y, \text{ (Goguen implication)},\\
&\cap_a, \cup_a \text{ are min and max on } [a,1],\\
&0_a = a,\ 1_a = 1.
\end{aligned}$$

Proof: The isomorphism is
$f_a(x) = a^{1-x}\ (= (\frac{1}{a})^{x-1})$; $f_a^{-1}(y) = 1 - \log_a y$. We compute:
$f_a(x).f_a(y) \geq a$ iff $a^{2-x-y} \geq a$ iff $2 - x - y \leq 1$ (note $a < 1$!) iff $x + y - 1 \geq 0$. If this holds then
$f_a(x + y - 1) = a^{-x-y+2} = a^{1-x}.a^{1-y} = f_a(x).f_a(y)$.
For $x \geq y$, $f_a(1 - x + y) = a^{x-y} = a^{1-y}/a^{1-x} = f_a(y)/f_a(x) = f_a(y) \Rightarrow_\Pi f_a(x)$. The rest is evident. □

Definition 4.1.15 Let p_0 be a propositional variable. For each formula φ of Ł not containing p_0, define its translation φ^Π as follows: $(\bar{0})^\Pi$ is p_0, for each propositional variable q different from p_0, q^Π is $q \vee p_0$; $(\varphi \to \psi)^\Pi$ is $(\varphi^\Pi \to \psi^\Pi)$; $(\varphi \& \psi)^\Pi$ is $p_0 \vee (\varphi^\Pi \odot \psi^\Pi)$.

Lemma 4.1.16 (1) Under the previous notation, φ is a 1-tautology of Ł iff $\neg\neg p_0 \to \varphi^\Pi$ is a 1-tautology of Π.

(2) Furthemore, let T be a theory and let $T^\Pi = \{\neg\neg p_0\} \cup \{\alpha^\Pi | \alpha \in T\}$. Then φ is true in all Ł-models of T iff φ^Π is true in all Π-models of T^Π.

Proof: e runs over evaluations, φ over formulas not containing the variable p_0. For each e such that $0 < e(p_0 < 1$ we define its *shift* $e^\uparrow$ and its *cut* $e^\to$ as follows: $e^\uparrow(p_0) = e^\to(p_0) \equiv e(p_0) = a$; for each variable q different from p_0, $e^\uparrow(q) = f_a(e(q))$, $e^\to(q) = \max(e(q), a)$. The following properties are evident from the preceding (still assuming $0 < a < 1$):

(i) $e_{Ł}(\varphi) = e^\uparrow_\Pi(\varphi^\Pi)$,
(ii) $e_\Pi(\varphi^\Pi) = e^\to_\Pi(\varphi^\Pi)$
(iii) there is an evaluation $\hat{e}$ such that $e^\to = \hat{e}^\uparrow$.

Now let φ be an Ł-tautology. If $e(p_0) = 0$ then $e(\neg\neg p_0) = 0$ and $e(\neg\neg p_0 \to \varphi^\Pi) = 1$ If $e(p_0) = 1$ then $e(\varphi^\Pi) = 1$ for each φ. Thus let $0 < e(p_0) < 1$. Then $e_\Pi(\varphi^\Pi) = e^\to_\Pi(\varphi^\Pi) = e^\uparrow_\Pi(\varphi^\Pi) = \hat{e}_{Ł}(\varphi) = 1$. Thus the formula $\neg\neg p_0 \to \varphi^\Pi$ is a Π-tautology.

Conversely let $\neg\neg p_0 \to \varphi^\Pi$ be a Π-tautology; take an arbitrary e and assume $0 < e(p_0) < 1$ (this does not affect $e_{Ł}(\varphi)$). Now $e_{Ł}(\varphi) = e^\uparrow_\Pi(\varphi^\Pi) = 1$ since $e^\uparrow_\Pi(\neg\neg p_0) = e^\uparrow(\neg\neg p_0 \to \varphi^\Pi) = 1$. This completes the proof of (1). The proof of (2) is similar. □

Corollary 4.1.17 The mapping assigning the formula $\neg\neg p_0 \to \varphi^\Pi$ to each formula φ not containing p_0 is a faithful embedding of Ł into Π: $\vdash_{Ł} \varphi$ iff $\vdash_\Pi \neg\neg p_0 \to \varphi^\Pi$ for each φ. Moreover, if T^Π is as above then $T \vdash_{Ł} \varphi$ iff $T^\Pi \vdash_\Pi \varphi^\Pi$.

Proof: This is an immediate consequence of the preceding lemma and the completeness theorem. □

Corollary 4.1.18 Π does not satisfy strong completeness for infinite theories. Immediate from Example 3.2.14 and the faithful embedding.

Remark 4.1.19 The reader may show as an easy exercise that linearly ordered MV-algebras are exactly all algebras of the form $\mathrm{MV}'(\mathbf{L}, a)$ where $\mathbf{L}$ is a linearly ordered product algebra, $a \in \mathbf{L}$ and $0_{\mathbf{L}} < a < 1_{\mathbf{L}}$, the domain of $\mathrm{MV}'(\mathbf{L}, a)$ is the interval $[a, 1_{\mathbf{L}}]$ and the operations are defined analogously to 4.1.15.

Lemma 4.1.20 Let T be a theory over Π.

(1) If $T \cup \{\varphi\}$ is inconsistent then $T \vdash \neg\varphi$.

(2) For each φ, at least one of the theories $T \cup \{\varphi\}$, $T \cup \{\neg\varphi\}$ is consistent.

Proof:

(1) If $T \cup \{\varphi\}$ is inconsistent then for some n, $T \vdash \varphi^n \to 0$; but then $T \vdash \varphi \to 0$ by iterated use of 4.1.3 ($\Pi \vdash (\varphi \odot \varphi \to 0) \to (\varphi \to 0)$).

(2) Assume both theories are inconsistent, then $T \vdash \neg\varphi$ and $T \vdash \neg\neg\varphi$, thus $T \vdash \neg\varphi \odot \neg\neg\varphi$, and $T \vdash \bar{0}$; by 2.2.12.

□

Corollary 4.1.21 For each consistent theory T (over Π) and each formula φ, there is a model e of T such that $e(\varphi) = 1$ or there is a model e of T such that $e(\varphi) = 0$. Consequently, there is no formula φ defining a non-extremal truth constant (e.g. $\frac{1}{2}$). This contrasts with Łukasiewicz logic (cf. 3.3.11).

Remark 4.1.22 A direct analogy of Rational Pavelka logic over Π is impossible: the corresponding Completeness theorem fails badly due to the discontinuity of Goguen implication. Indeed, take the system analogous to RPL but over Π, let $T = \{p \to \bar{r} | r \text{ positive}\}$, $\varphi = p \to \bar{0} (= \neg p)$. Then evidently $\|\varphi\|_T = 1$; but on the other hand, $T \nvdash p \to \bar{0}$ since then $p \to \bar{0}$ would be provable from a finite subtheory T_0 of T but this is impossible: if r_0 is the smallest number r such that the axiom $p \to \bar{r}$ is in T_0 then T_0 has a trivial model e with $e(p) = r_0$. Moreover, $|\varphi|_{T_0} = 0$ since if for some $s > 0$ T proved $\bar{s} \to (p \to \bar{0})$ then $T \vdash p \to (\bar{s} \to \bar{0})$, thus $T \vdash p \to \bar{0}$ by the definition of Goguen implication. (Exercise: but show that over RPL, $|\varphi|_T$ would be 1.)

The question remains if there is another way of proving partially true conclusions in Π. One simple (lazy) way is as follows:

Definition 4.1.23 Let r be a non-extremal rational number ($0 < r < 1$); enrich $\Pi_\triangle$ by the single truth constant $\bar{r}$ and axioms saying that $\bar{r}$ is non-extremal:

$$\neg\neg\bar{r}, \quad \neg\triangle\bar{r}.$$

The semantics is as in $\Pi_\triangle$, with $e(\bar{r}) = r$ for each evaluation e. (Note that this gives infinitely many formulas with constant semantics, namely $\bar{r}^n = \bar{r} \odot \ldots \odot \bar{r}$; you may call then $\overline{r^n}$.) Call this calculus $\Pi_\triangle(\bar{r})$.

Lemma 4.1.24 Let T be a theory over $\Pi_\triangle(\bar{r})$ and φ a formula of $\Pi_\triangle(\bar{r})$; let p_0 be a variable not occuring in T, φ and for each formula α not containing p_0, let $\alpha^\sharp$ result from α by replacing $\bar{r}$ by p_0. Let $T^\sharp$ be the theory (over $\Pi_\triangle$) such that $T^\sharp = \{\neg\neg p_0\} \cup \{\alpha^\sharp | \alpha \in T\}$. Then φ is true in each model of T (over $\Pi_\triangle(\bar{r})$) iff φ is true in each model of $T^\sharp$ (over $\Pi_\triangle$).

Proof: The implication ($\Leftarrow$) is obvious; to prove the converse just observe that for any non-extremal r, s ($0 < r,\ s < 1$) there is an automorphism f of $[0, 1]_\Pi$ such that $f(r) = s$ ($f(x) = x^d$ for an appropriate real d); thus if $\varphi(\bar{r})$ is true in all models of $T(\bar{r})$ then $\varphi(\bar{s})$ is true in all models of $T(\bar{s})$ and hence $\varphi(p_0)$ is true in all models of $T(p_0)$. □

Corollary 4.1.25 (Completeness.)

(1) A formula φ is provable in $\Pi_\triangle(\bar{r})$ iff it is a 1-tautology of $\Pi_\triangle(\bar{r})$.

(2) Let T be a finite theory over $\Pi_\triangle(\bar{r})$; a formula φ is provable in T over $\Pi_\triangle(\bar{r})$ iff it is true in all $\Pi_\triangle(\bar{r})$-models of T.

Remark 4.1.26 (1) Note that over $\Pi_\triangle$, T may be consistent whereas both $T \cup \{\varphi\}, T \cup \{\neg\varphi\}$ may be inconsistent. For example, let $T = \{\neg\neg p, \neg\triangle p\}$; then e is a model of T iff $0 < e(p) < 1$, but $(T, p) \vdash \triangle p$ and $(T, \neg p) \vdash \neg p$ which shows that both theories are inconsistent (since T proves $\neg\triangle p$ and $\neg\neg p$).

(2) Note also that over $\Pi_\triangle$, $T \cup \{\varphi\}$ is inconsistent iff $T \vdash \neg\triangle\varphi$ (Exercise.)

Example 4.1.27 Recall Example 3.3.21 on large numbers. Let us discuss the same situation in Π (probably with an added truth constant $\bar{r}$ with $0 < r < 1$. Again we have propositional variables $p_0, \ldots, p_{1000000}$ and $T_0 = \{p_i \to p_{i+1} | i < 10^6\}$.

(1) T_1 is the extension of T_0 by $p_{1000000}$ and $\bar{r} \to (p_{i+1} \to p_i)$ (for $i < 10^6$), i.e. by $\bar{r} \odot p_{i+1} \to p_i$. T_1 proves $(\bar{r})^i \odot p_i \to p_0$, i.e. $\overline{r^i} \odot p_i \to p_0$, thus if e is a model of T_1, $e(p_i) = x_i$ then $x_i \leq x_0 \cdot (\frac{1}{r})^i$. Hence either $x_0 = 0$ and then all $x_i = 0$ (unwanted!) or $x_0 > 0$ and the maximal model satisfies $x_i = \min(1, x_0 \cdot (\frac{1}{r})^i)$. This e is a model of T_1 iff $(\frac{1}{r})^{1000000} > x_0$. (We imagine r slightly less than 1, thus $\frac{1}{r}$ slightly bigger than 1.)

(2) Now let T_2 result from T_0 by adding the axioms $(p_{i+1})^2 \to p_i$. Let e be a model of T_2 and $x_i = e(p_i)$ as above; then $x_{i+1}^2 \leq x_i$ (here the square is in the sense of reals!). Thus in a non-trivial case, $x_0 > 0, x_{1000000} < 1$ and for the maximal model e with $x_0 = z$ we get $x_i = z^{2^{-i}}$. The sequence of x_i's for $i \to \infty$ has the limit 1, but $x_i < 1$ for all i. Compute (some) members of the sequence of x_i's for various choices of z.

4.2. GÖDEL LOGIC

The last (but not least) of our three most important logics given by a continuous t-norm is Gödel logic G interpreting & as minimum. We shall subject it to the same analysis as the previous two.

Definition 4.2.1 The axiom system of G is the extension of the axiom system of BL by the single axiom

(G) $\qquad \varphi \to (\varphi \& \varphi)$

stating (together with $(\varphi\&\varphi) \to \varphi$) the idempotence of &.

Lemma 4.2.2 G proves $(\varphi\&\psi) \equiv (\varphi \wedge \psi)$.

Proof: Clearly, BL $\vdash (\varphi\&\psi) \to (\varphi\wedge\psi)$; on the other hand, BL $\vdash (\varphi\wedge\psi) \to \varphi$, BL $\vdash (\varphi \wedge \psi) \to \psi$, hence BL $\vdash [(\varphi \wedge \psi)\&(\varphi \wedge \psi)] \to (\varphi\&\psi)$ and G $\vdash (\varphi \wedge \psi) \to [(\varphi \wedge \psi)\&(\varphi \wedge \psi)]$. □

Thus we may get rid in G of one conjunction, say &; thus we may present Gödel logic equivalently as follows:

Definition 4.2.3 The axiom system G′ has the connectives $\wedge, \to, 0$ and axioms (A1)-(A3), (A5)-(A7) of BL, with & replaced by $\wedge$ plus the axiom (G4):

$$\varphi \to (\varphi \wedge \varphi).$$

Other connectives are defined as follows:

$$\neg\varphi \text{ is } \varphi \to \bar{0},\ (\varphi \equiv \psi) \text{ is } (\varphi \to \psi) \wedge (\psi \to \varphi),$$
$$(\varphi \vee \psi) \text{ is } ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi).$$

We have to verify that the definition of $\wedge$ from & gives nothing new here:

Lemma 4.2.4 (1) G$'$ proves $(\varphi \wedge (\varphi \to \psi)) \to (\psi \wedge (\psi \to \varphi))$.
(2) G$'$ proves $(\varphi \wedge \psi) \equiv \varphi \wedge (\varphi \to \psi)$.

Proof: Clearly, if BL proves a formula α (using only connectives $\to, \&, \bar{0}$) then G$'$ proves the result α' of replacing each & by $\wedge$. Thus checking that (A4) was not used in the proofs of 2.2.7, 2.2.8, we get G$' \vdash (\varphi \wedge (\varphi \to \psi)) \to \psi$ (by (4) there), G$' \vdash (\varphi \wedge (\varphi \to \psi)) \to (\psi \to \varphi)$ (by (1), (A2) and (A1)), and hence $(\varphi \wedge (\varphi \to \psi) \to (\psi \wedge (\psi \to \varphi))$ by (7), using the axiom (G4) for the formula $\varphi \wedge (\varphi \to \psi)$. This proves our (1).

To get our (2) observe that G$' \vdash \psi \to (\varphi \to \psi)$, G$' \vdash (\varphi \wedge \psi) \to (\varphi \wedge (\varphi \to \psi))$. Conversely, G$' \vdash (\varphi \wedge (\varphi \to \psi)) \to \psi$, thus G$'$ proves $[\varphi \wedge (\varphi \to \psi)] \to [\varphi \wedge \varphi \wedge (\varphi \to \psi)] \to [\varphi \wedge \psi]$. $\square$

Corollary 4.2.5 G and G$'$ are equivalent in the sense that G $\vdash \alpha$ iff G$' \vdash \alpha'$ (where α' results from α by identifying $\&, \wedge$). Thus in the sequel we shall not distinguish between G and G$'$.

Gödel logic is closely related to a famous logical system called *intuitionist logic*; we shall describe it precisely.

Definition 4.2.6 The *intuitionist logic* I has the connectives $\to, \wedge, \vee, \neg$ and the following axioms:

(I1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(I2) $\varphi \to (\varphi \vee \psi)$

(I3) $\psi \to (\varphi \vee \psi)$

(I4) $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)))$

(I5) $(\varphi \wedge \psi) \to \varphi$

(I6) $(\varphi \wedge \psi) \to \psi$

(I7) $(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to (\varphi \wedge \psi)))$

(I8) $(\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi)$

(I9) $((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

(I10) $(\varphi \wedge \neg\varphi) \to \psi$

(I11) $(\varphi \to (\psi \wedge \neg\psi)) \to \neg\psi$

Lemma 4.2.7 G proves all axioms of I.

Proof: In fact, BL proves (I1)–(I7) and (I10), (I11) as they stand and (I8), (I9) with & instead of $\wedge$. □

Remark 4.2.8 Conversely, it can be shown that I extended by the axiom (A6) (or, alternatively, by $(\varphi \to \psi) \vee (\psi \to \varphi)$) proves all axioms of G plus $(\varphi \vee \psi) \equiv ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$ (the definition of $\vee$ in BL). Indeed, $\mathrm{I} \vdash \varphi \to ((\varphi \to \psi) \to \psi)$, $\mathrm{I} \vdash \varphi \to$ (anything $\to \varphi$), similarly for ψ; thus
$\mathrm{I} \vdash (\varphi \vee \psi) \to [((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)]$. Conversely,
$\mathrm{I} \vdash (\varphi \to \psi) \to [((\varphi \to \psi) \to \psi) \to \psi]$, thus
$\mathrm{I} \vdash (\varphi \to \psi) \to [((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)] \to (\varphi \vee \psi)$;
similarly, $\mathrm{I} \vdash (\psi \to \varphi) \to [\ldots] \to (\varphi \to \psi)$, thus, by (A6), $\mathrm{I} \vdash [\ldots] \to (\varphi \vee \psi)$.

Hence G is just the extension of I by (A6). *Caution:* in I the connective $\vee$ is *not* definable from the others.

Lemma 4.2.9 G proves the formula

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \psi)).$$

Proof: This is because evidently
$\mathrm{BL} \vdash (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to ((\varphi \& \varphi) \to \chi)$. □

Theorem 4.2.10 (1) G has the classical deduction theorem: for each theory T over G and formulas φ, ψ,

$$T \cup \{\varphi\} \vdash \psi \quad \text{iff} \quad T \vdash (\varphi \to \psi).$$

(2) G is the only logic PC($*$) having the classical deduction theorem; i.e. if PC($*$) has the classical deduction theorem then $*$ is minimum.

Proof:

(1) follows from the deduction theorem 2.2.18 for BL, observing that in G for each n φ^n is equivalent to φ. (An alternative proof is by observing that the proof of the deduction theorem in classical logic works here; but this is almost the same proof.)

(2) Assume PC($*$) has the classical deduction theorem; since evidently $\{\varphi\} \vdash \varphi\&\varphi$ (thanks to $\vdash \varphi \to (\varphi \to (\varphi\&\varphi))$), we get $\vdash \varphi \to (\varphi\&\varphi)$, which means that $*$ is idempotent. Thus let $x, y \in [0,1]$ and assume $x \leq y$; then $x * y \geq x * x = x$ and obviously $x * y \leq x * 1 = x$, thus $x * y = x$.

□

Lemma 4.2.11 G proves $(\varphi \to \neg\varphi) \to \neg\varphi$.

Proof: Indeed, G proves

$$[\varphi \to (\varphi \to \bar{0})] \to [(\varphi \wedge \varphi) \to \bar{0}] \to [\varphi \to \bar{0}].$$

□

$*$

Definition 4.2.12 BL-algebras satisfying the identity $x * x = x$ (i.e. with idempotent multiplication) are called *G-algebras.*

Remark 4.2.13 For the reader knowing the notion of a Heyting algebra we mention that a G-algebra is just a Heyting algebra satisfying prelinearity: $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

To get the completeness theorem we need some lemmas.

Lemma 4.2.14 Let H be a linearly ordered G-algebra.

(1) For each $x, y,\ x > y$ implies $(x \Rightarrow y) = y$.

(2) Each subset of H containing 0_H and 1_H is a subalgebra.

Proof:

(1) $z \leq (x \Rightarrow y)$ implies $x \cap z \leq y$; thus if $x > y$ we get $x \cap z < x$, $x \cap z = z$, thus $z \leq y$. Hence $(x \Rightarrow y) \leq y$; conversely $y \leq (x \Rightarrow y)$ in each residuated lattice.

(2) follows: in a linearly ordered H, $x \cap y = x$ or $x \cap y = y$, similarly for $\cup$, $(x \Rightarrow y) = 1$ or $(x \Rightarrow y) = y$.

□

Corollary 4.2.15 (1) If H_1, H_2 are two finite linearly ordered G-algebras of the same cardinality then they are isomorphic.

(2) Each at most countable linearly ordered G-algebra is isomorphic to a subalgebra of the standard linearly ordered G-algebra $[0,1]_G$; moreover, it is isomorphic to a subalgebra of the G-algebra of rational elements of $[0,1]_G$.

Proof:

(1) They are isomorphic as linearly ordered sets; but the linear order determines all the operations.

(2) Again this follows immediately from the fact that each countable linear order can be isomorphically embedded to rationals from $[0,1]$.

□

Lemma 4.2.16 If an identity $\tau = \sigma$ in the language of G-algebras is valid in the standard Heyting algebra $[0,1]_G$ of truth functions then it is valid in all linearly ordered G-algebras.

Proof: Let $\tau = \sigma$ be violated by $a_1, \ldots, a_n \in H$;. thus it is violated in $H_1 = \{0, a_1, \ldots, a_n, 1\}$ as a subalgebra of H. Take an isomorphic copy H_2 which is a subalgebra of $[0,1]_G$; $\tau = \sigma$ is violated in H_2 and hence in $[0,1]_G$. □

This can be used for a completeness theorem as in Ł and Π; but here we may do better — we get a strong completeness for *arbitrary*, not necessarily finite, theories.

Theorem 4.2.17 *(Completeness.)*[23]

(1) A formula φ of Gödel logic G is provable in G iff it is a 1-tautology of G.

(2) For each theory T over G and each formula φ, T proves φ (over G) iff φ is true in each model of T (over G).

(3) This generalizes for $G_\triangle$ — the extension of G by the connective $\triangle$ and the axioms ($\triangle$1–$\triangle$5).

[23] See [46].

Proof: Soundness is routine; to prove completeness we show that if $T \nvdash \varphi$ then there is a model e of T *over the rationals* from $[0,1]$ such that $e(\varphi) < 1$. By 2.4.3, T has a model e over $\mathbf{L}_{\hat{T}}$ (where $\hat{T}$ is a completion of T) such that $e(\varphi) < 1_{\hat{T}}$; by 4.2.15, $\mathbf{L}_{\hat{T}}$ can be isomorphically embedded into the rationals from $[0,1]$ (as a Heyting algebra), which gives the result. This proves (1),(2). To get (3) use 2.4.12 instead of 2.4.3. □

*

We shall now discuss the behaviour of Gödel logic with respect to partial truth.

Theorem 4.2.18 For each theory T over G, each formula φ and each rational r such that $0 < r \leq 1$, $T \vdash \varphi$ iff each evaluation e such that $e(\alpha) \geq r$ for each axiom $\alpha \in T$ satisfies $e(\varphi) \geq r$ (i.e. if e makes all axioms r-true then it makes φ r-true).

Proof: Assume $T \vdash \varphi$ and $e(\alpha) \geq r$ for each $\alpha \in T$; if $\varphi_1, \ldots, \varphi_n$ is a proof of φ then show $e(\varphi_i) \geq \alpha$ by induction, observing that if $e(\varphi_i) \geq \alpha$ and $e(\varphi_i \to \varphi_j) \geq \alpha$ then $e(\varphi_j) \geq \alpha \wedge \alpha = \alpha$ (cf. 2.1.3).

Conversely, if r_0 is such that $0 < r_0 < 1$ and each e making T r_0-true makes φ r_0-true then for *each* $0 < r < 1$, each e making T r-true makes φ r-true (just take any monotone one-one mapping of $[0,1]$ onto itself such that $i(r) = r_0$ and observe that assigning to each e a valuation e' such that $e'(p) = i(e(p))$ we map the set of all evaluations onto itself, and for each formula ψ, $e(\psi) \geq r$ iff $e'(\psi) \geq r_0$. Thus for each $r < 1$ we get: if e makes T r-true then it makes φ r-true. It follows easily that this must also hold for $r = 1$ and hence $T \vdash \varphi$ by the above completeness theorem. □

It is easy to show that the theorem generalizes for $G_\triangle$.

Remark 4.2.19 Observe that Lemma 4.1.20 and corollary 4.1.21 also holds for G instead of Π. Thus if $T \cup \{\varphi\}$ is inconsistent then $T \vdash \neg\varphi$; and if T is consistent then $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ is also consistent. Thus no formula defines a non-extremal truth-constant over G. Over $G_\triangle$ we may use finitely many rational truth constants, as the next theorem shows.

Definition 4.2.20 Fix finitely many rationals $0 < r_1 < \ldots < r_n < 1$. The logic $G_\triangle(r_1, \ldots, r_n)$ has the language with $\triangle, \bar{r_1}, \ldots, \bar{r_n}$;

$$
\begin{aligned}
\bar{r_i} \wedge \bar{r_j} &\equiv \overline{\min(r_i, r_j)}, \\
\bar{r_i} \to \bar{r_j} &= \bar{1} \text{ for } i \leq j, \\
\bar{r_i} \to \bar{r_j} &= \bar{r_j} \text{ for } j < i, \\
\neg\neg\bar{r_i}, & \quad \neg \triangle r_i.
\end{aligned}
$$

Semantics over $[0,1]_G$ is obvious, if we postulate $e(\bar{r_i}) = r_i,\ i = 1, \ldots, n.$

Theorem 4.2.21 Let T be a theory over $G_\triangle(r_1, \ldots, r_n)$, φ a formula. $T \vdash \varphi$ iff $e(\varphi) = 1$ for each model e of T.

Proof: Let T' consist of the axioms of T *plus* all additional axioms of $G_\triangle(r_1, \ldots, r_n)$ (not being axioms of $G_\triangle$). Over $G_\triangle$, consider $\bar{r_i}$ just as new formulas. By 4.2.17, there is a complete extension $\hat{T}$ of T and a model e of T over $\mathbf{L}_{\hat{T}}$ such that $e(\varphi) < 1$. Moreover, $0 < e(r_1) < \ldots < e(r_n) < 1$ thanks to the bookkeeping axioms. Now you may clearly embed $L_{\hat{T}}$ isomorphically into rationals from $[0,1]$ by an isomorphism ι such that $\iota(\bar{r_1}) = r_1, \ldots, \iota(\bar{r_n}) = r_n$. This produces the desired model over $[0,1]_G$. □

Remark 4.2.22 But we cannot have Pavelka-style completeness (provability degree = satisfiability degree) for infinite theories over RPG due to the non-continuity of Gödel implication. Let $T = \{p \to \bar{r} | r > 0\}$; we show $\|\neg p\|_T = 1$, $|\neg p|_T = 0$. Indeed, $T \not\vdash p \to \bar{0}$ by compactness. Moreover, if $r > 0$ and $T \vdash \bar{r} \to (p \to \bar{0})$ then $T \vdash p \to (\bar{r} \to \bar{0})$, $T \vdash p \to \overline{r \Rightarrow 0}$ and hence $T \vdash p \to \bar{0}$ (note $r \Rightarrow 0 = 0$ for > 0). Thus $\sup\{r | T \vdash \bar{r} \to \neg p\} = 0$.

4.3. Appendix: Boolean logic

Having developed our basic propositional logic as well as its three most important strengthenings Ł, Π and G, let us now ask how this relates to the classical (Boolean, two-valued) propositional logic.

Definition 4.3.1 BL2 is the extension of BL by the single axiom

$$\varphi \vee \neg\varphi$$

Note that this axiom is known under the name *tertium non datur*; BL2 stands for Basic logic with tertium non datur.

Lemma 4.3.2 BL2 proves $\varphi \to \varphi\&\varphi$, i.e. BL2 extends G.

Proof: Clearly,
BL$\vdash \varphi \to (\varphi \to (\varphi\&\varphi))$ and
BL$\vdash \neg\varphi \to (\varphi \to (\varphi\&\varphi))$, thus
BL$\vdash (\varphi \vee \neg\varphi) \to (\varphi \to (\varphi\&\varphi))$, hence
BL2 $\vdash \varphi \to (\varphi\&\varphi)$. □

Remark 4.3.3 Thus BL2 extends intuitionistic logic, proves tertium nod datur and is sound for evaluations by zeros and ones. This implies that BL2 is a sound and complete axiomatization of classical propositional calculus *Bool* by properties of intuitionistic and classical logic known to experts. For the reader's convenience, we provide BL2-proofs of the usual axioms of *Bool* (cf. 1.2.5, 1.2.6) and show how completeness of BL2 w.r.t. $\{0, 1\}$-evaluations quickly follows from what we already have made.

Lemma 4.3.4 BL2 proves the following formulas:

(1) $\varphi \to (\psi \to \varphi)$

(2) $\neg\neg\varphi \to \varphi$

(3) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

(4) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

Proof: (1) is provable in BL and (3) is provable in G. To prove (2) observe the following:
BL$\vdash \varphi \to (\neg\neg\varphi \to \varphi)$ by (1),
BL$\vdash \neg\varphi \to (\neg\neg\varphi \to \varphi)$ (cf. 2.2.12), thus
BL$\vdash (\varphi \vee \neg\varphi) \to (\neg\neg\varphi \to \varphi)$ and hence
BL2$\vdash \neg\neg\varphi \to \varphi$.

Thus BL2$\vdash \neg\neg\varphi \equiv \varphi$ and (4) follows by 2.2.12(18′) by removing double negations. □

Lemma 4.3.5 (i) *Bool* proves the following formulas:

$$\varphi \to \varphi \tag{1}$$

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \tag{2}$$

$$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) \tag{3}$$

$$\varphi \to \neg\neg\varphi \tag{4}$$

$$\neg\neg\varphi \to \varphi \tag{5}$$

$$(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi) \tag{6}$$

$$\overline{0} \to \varphi \tag{7}$$

$$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi) \tag{8}$$

$$\neg\varphi \to (\varphi \to \psi) \tag{9}$$

$$\varphi \to (\neg\psi \to \neg(\varphi \to \psi)) \tag{10}$$

$$(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi) \tag{11}$$

$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi) \tag{12}$$

(ii) $Bool$ has the deduction theorem (cf. 1.2.10):
$(T \cup \{\varphi\}) \vdash_{Bool} \psi$ iff $T \vdash_{Bool} (\varphi \to \psi)$.

Proof:
(1) $(\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$ $(Bool2)$
$\varphi \to ((\varphi \to \varphi) \to \varphi)$ $(Bool1)$
$(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$ (modus ponens)
$\varphi \to (\varphi \to \varphi)$ $(Bool1)$
$\varphi \to \varphi$ (modus ponens)
Having (1) we prove the deduction theorem: If $T \vdash (\varphi \to \psi)$ then trivially $(T \cup \{\varphi\}) \vdash \psi$ (prolong a T-proof of $(\varphi \to \varphi)$ by φ, ψ). Conversely, given a $(T \cup \{\varphi\})$-proof $\alpha_1, \ldots, \alpha_n$ of ψ, show by induction that $T \vdash \varphi \to \alpha_i$ (for α_i a logical axiom or a T-axiom using $(Bool1)$, for α_i being φ using our (1) and for α_i resulting by modus ponens using $(Bool2)$).

Now we prove the formulas (2)-(12) with the help of the deduction theorem.

(2) $(\varphi \to \psi), (\psi \to \chi), \psi \vdash \chi$ (by modus ponens), thus $(\varphi \to \psi), (\varphi \to \chi) \vdash \varphi \to \chi, (\varphi \to \psi) \vdash (\psi \to \chi) \to (\varphi \to \chi)$, and finally $\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$.

(3) Similarly using the Deduction theorem.

(4) $\varphi, \varphi \to \overline{0} \vdash \overline{0}$, thus $\vdash \varphi \to ((\varphi \to \overline{0}) \to \overline{0})$, i.e. $\vdash \varphi \to \neg\neg\varphi$.

(5) $\vdash \neg\varphi \to \neg\neg\neg\varphi$ by (4), thus $\vdash \neg\neg\varphi \to \varphi$ by $(Bool3)$.

(6) Use $(\varphi \to \psi), (\varphi \to (\psi \to \overline{0})), \varphi \vdash \overline{0}$ and the Deduction theorem.

(7) $\vdash \overline{0} \to \overline{0}$ by (1), thus $\vdash \neg\overline{0}$, $\vdash \neg\varphi \to \neg\overline{0}$ by $(Bool1)$, and hence $\vdash \overline{0} \to \varphi$ by $(Bool3)$.

(8) $\vdash (\varphi \to \psi) \to ((\psi \to \overline{0}) \to (\varphi \to \overline{0}))$ by (2).

(9) $\vdash (\varphi \to \overline{0}) \to (\varphi \to \psi)$ by (2), (3) and (7)

(10) $\vdash \varphi \to ((\varphi \to \psi) \to \psi)$ by (3) and (1); then $\vdash \varphi \to (\neg\psi \to \neg(\varphi \to \psi))$ by (8) and (2).

(11) $(\varphi \to \psi), (\neg\varphi \to \psi), \neg\psi \vdash \psi$ by (8), thus
$(\varphi \to \psi), (\neg\varphi \to \psi), \neg\psi \vdash \overline{0}$ (since $\ldots \psi \to \overline{0})$), then
$(\varphi \to \psi), (\neg\varphi \to \psi) \vdash \neg\neg\psi$ and
$\vdash (\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi)$ by (5) and Deduction.

(12) $\varphi, (\varphi \to \psi) \to \psi \vdash (\psi \to \varphi) \to \varphi$ by $(Bool1)$;
$\psi, (\varphi \to \psi) \to \psi \vdash (\psi \to \varphi) \to \varphi$ since $\psi, (\psi \to \varphi) \vdash \varphi$;
$\neg\varphi, \neg\psi, (\varphi \to \psi) \to \psi \vdash \psi, \neg\psi$ by (9), thus $\ldots \vdash \overline{0}$ and

$\neg\varphi, \neg\psi, (\varphi \to \psi) \to \psi \vdash (\psi \to \varphi) \to \varphi$ by (7).
The second and last line give, by (11),
$\neg\varphi, (\varphi \to \psi) \to \psi \vdash (\psi \to \varphi) \to \varphi$,
which with the first line, using again (11), gives
$(\varphi \to \psi) \to \psi \vdash (\psi \to \varphi) \to \varphi$. □

Corollary 4.3.6 BL2 is equivalent to the classical propositional calculus *Bool* as described in 1.2.5, 1.2.6.

Proof: The items (1), (3), (4) above are just the axioms of *Bool*. The definition of $\wedge$ in *Bool*, $(\varphi \wedge \psi) \equiv \neg(\varphi \to \neg\psi)$ is provable in BL2 by the same proof as in 3.1.1 and so is the definition of $\vee$ from $\wedge$ (i.e. from &) since BL2 proves the law of double negation and therefore extends Ł. Thus BL2 extends *Bool*. Conversely, by 4.3.5, *Bool* proves all the axioms of the original Łukasiewicz's system Ł′ (see 3.1.3) and hence proves all the axioms of BL (see 3.1.9, 3.1.16). Finally, *Bool* proves $\varphi \vee \neg\varphi$ (i.e. $(\varphi \to \neg\varphi) \to \neg\varphi)$) by 4.3.5 (6) and (1). □

Lemma 4.3.7 BL2-algebras are just Boolean algebras. Thus there are exactly two linearly ordered BL2-algebras: the (degenerate) one-element algebra and the two-element Boolean algebra $\{0, 1\}$.

Proof: BL2-algebras are particular G-algebras, satisfying *tertium non datur*, thus distributive complementary lattices, i.e. Boolean algebras. If a Boolean algebra has a non-extremal element a (different from 0 and 1) then a and $(-)a$ are incomparable, hence the algebra is not linearly ordered. □

Theorem 4.3.8 (Completeness.) BL2 proves φ iff φ is a $\{0, 1\}$-tautology. More generally, for each theory T over BL2, $T \vdash \varphi$ iff φ is true in each $\{0, 1\}$-model of T.

Proof: Since there is just one non-trivial linearly ordered BL2-algebra, the theorem is a direct consequence of the completeness theorem 2.4.3 for schematic extensions of BL. □

To close this minisection, we show that the union of any two of our three logics Ł, G, Π gives BL2.

Theorem 4.3.9 (1) Ł ∪ G is equivalent to BL2.

(2) Ł ∪ Π is equivalent to BL2.

(3) G ∪ Π is equivalent to BL2.

Proof: We already know that BL2 extends G and Ł and it is easy to show that it extends Π. Indeed, BL2 proves $(\varphi \wedge \neg\varphi) \to \bar{0}$ since G proves it; and since BL2 proves $\neg\neg\varphi \equiv \varphi$, BL2-provability of the second axiom of Π reduces to
BL2$\vdash \chi \to (((\varphi \wedge \chi) \to (\psi \wedge \chi)) \to (\varphi \to \psi))$, i.e.
BL2$\vdash \chi \to (\varphi \to (((\varphi \wedge \chi) \to (\psi \wedge \chi)) \to \psi))$, i.e.
BL2$\vdash (\varphi \wedge \chi) \to (((\varphi \wedge \chi) \to (\psi \wedge \chi)) \to \psi)$, which is evident.

Let us prove the converse. (1) G proves the first two axioms of classical logic and Ł proves the third.

(2) Π proves $\neg\varphi \vee \neg\neg\varphi$ and Ł proves $\neg\neg\varphi \equiv \varphi$, thus their union proves tertium non datur.

(3) $\Pi \vdash \neg\neg\varphi \to (((\bar{1}\&\varphi) \to (\varphi\&\varphi)) \to (\bar{1} \to \varphi))$ and G$\vdash (\bar{1}\&\varphi) \to (\varphi\&\varphi)$, thus G∪Π proves $\neg\neg\varphi \to \varphi$. Thus continue as in (2). □

CHAPTER FIVE

MANY-VALUED PREDICATE LOGICS

We are now ready to start our investigation of fuzzy predicate logics (or first-order logics, quantification logics). We shall develop logics broadly analogous to the classical predicate logic; in particular, we shall deal only with two quantifiers, $\forall$ and $\exists$ (universal and existential). Generalized quantifiers will be studied in later chapters. In Section 1 we shall develop the predicate counterpart BL$\forall$ of our basic propositional logic BL; in Section 2 we prove a rather general completeness theorem for predicate logics (with respect to semantics over residuated lattices). Sections 3 and 4 are devoted to Gödel and Łukasiewicz predicate logics respectively; we show that Gödel predicate logic has a recursive axiomatization that is complete with respect to the semantics over [0,1], whereas for Łukasiewicz we only present a variant of Pavelka logic. (We show in the next chapter that Łukasiewicz does not have a recursive complete axiomatization.) We close Sec. 4 with some for remarks on the predicate product logic. Sec. 5 discusses many-sorted calculi and Sec. 6 introduces and studies similarity (fuzzy equality). This notion will be crucial for our analysis of fuzzy control in Chap. 7.

5.1. THE BASIC MANY-VALUED PREDICATE LOGIC

Definition 5.1.1 A *predicate language* consists of a non-empty set of *predicates*, each together with a positive natural number — its *arity* and a (possibly empty) set of *object constants*. Predicates are mostly denoted by $P, Q, R, \ldots$, constants by $c, d, \ldots$ *Logical symbols* are *object variables* $x, y, \ldots$, *connectives* $\&, \rightarrow$, *truth constants* $\bar{0}, \bar{1}$ and *quantifiers* $\forall, \exists$. Other connectives $(\wedge, \vee, \neg, \equiv)$ are defined as in the previous chapters. *Terms* are object variables and object constants.

Atomic formulas have the form $P(t_1, \ldots, t_n)$ where P is a predicate of arity n and $t_1, \ldots, t_n$ are terms. If φ, ψ are formulas and x is an object variable then $\varphi \rightarrow \psi$, $\varphi\&\psi$, $(\forall x)\psi$, $(\exists x)\varphi$, $\bar{0}$, $\bar{1}$ are formulas; each formula results from atomic formulas by iterated use of this rule.

Let $\mathcal{J}$ be a predicate language and let $\mathbf{L}$ be a linearly ordered BL-algebra. An $\mathbf{L}$-*structure* $\mathbf{M} = \langle M, (r_P)_P, (m_c)_c\rangle$ *for* $\mathcal{J}$ has a non-empty *domain* M, for each n-ary predicate P a $\mathbf{L}$-fuzzy n-ary relation $r_P : M^n \rightarrow \mathbf{L}$ on M (associating to each n-tuple $(m_1, \ldots, m_n)$ of elements of M the degree

$r_P(m_1, \ldots, m_n) \in \mathbf{L}$ of the membership of $(m_1, \ldots, m_n)$ to the fuzzy relation) and for each object constant c, m_c is an element of M.

Example 5.1.2 $\mathcal{J}$ has one binary predicate *like* and one object constant $Mary$. $\mathbf{L}$ is the lattice $[0,1]_Ł$. $M = \{1,2,3\}$, $m_{Mary} = 1$, r_{like} is given by the following matrix.

	1	2	3
1	1	0.3	0.7
2	0.9	0.9	0
3	0.9	0.1	0.8

Definition 5.1.3 Let $\mathcal{J}$ be a predicate language and $\mathbf{M}$ an $\mathbf{L}$-structure for $\mathcal{J}$. An $\mathbf{M}$-*evaluation!of object variables* is a mapping v assigning to each object variable x an element $v(x) \in M$. Let v, v' be two evaluations. $v \equiv_x v'$ means that $v(y) = v'(y)$ for each variable y distinct from x.

The value of a term given by $\mathbf{M}, v$ is defined as follows: $\|x\|_{\mathbf{M},v} = v(x)$; $\|c\|_{\mathbf{M},v} = m_c$. We define the *truth value* $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ of a formula. Clearly, $\Rightarrow$ and $*$ denote the operations of $\mathbf{L}$.

$$\|P(t_1, \ldots, t_n)\|^{\mathbf{L}}_{\mathbf{M},v} = r_P(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v});$$
$$\|\varphi \to \psi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} \Rightarrow \|\psi\|^{\mathbf{L}}_{\mathbf{M},v};$$
$$\|\varphi \& \psi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} * \|\psi\|^{\mathbf{L}}_{\mathbf{M},v};$$
$$\|\bar{0}\|_{\mathbf{M},v} = 0; \ \ \|\bar{1}\|_{\mathbf{M},v} = 1;$$
$$\|(\forall x)\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'} | v \equiv_x v'\};$$
$$\|(\exists x)\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \sup\{\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'} | v \equiv_x v'\}$$

provided the infimum/supremum exists in the sense of $\mathbf{L}$; otherwise the truth value of the formula in question is undefined.

The structure $\mathbf{M}$ is $\mathbf{L}$-*safe* if all the needed infima and suprema exist, i.e. $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ is defined for all φ, v.

Example 5.1.4 In particular, each finite structure (with finite domain) is safe. Verify in the trivial example above that $\|(\forall x)like(x, Mary)\|^{\mathbf{L}}_{\mathbf{M},v} = 0.9$ (independently from v); what is $\|(\exists x)\neg like(Mary, x)\|^{\mathbf{L}}_{\mathbf{M},v}$? (Recall that in the example we work with $[0,1]_Ł$ — the standard MV-algebra.)

Remark 5.1.5 (1) You will certainly find all these definitions to be rather direct generalizations of the coresponding definitions in the classical predicate calculus. Indeed, if $\mathbf{L}$ is the two-element Boolean algebra then we just give the classical definitions.

(2) We shall not introduce function symbols, even if this is possible; but either we can introduce crisp semantics of them (and this is not particularly interesting) or we would have to discuss fuzzy equality. This would be interesting but we shall not go into it now. (But see Sec. 5.6.)

(3) We shall not repeat the definitions of free and bound variables of a formula neither of substitutability of a term for a variable in a formula (we only roughly recall that a constant is always substitutable; and a variable y is substitutable into φ for x if the substitution does not make any free occurence of x in φ into a bound occurence of y). For precise definitions see 1.3.6, 1.3.7.

Definition 5.1.6 (1) Let φ be a formula of a language $\mathcal{J}$ and let $\mathbf{M}$ be a safe $\mathbf{L}$-structure for $\mathcal{J}$. The *truth value* of φ in $\mathbf{M}$ is

$$||\varphi||^{\mathbf{L}}_{\mathbf{M}} = \inf\{||\varphi||^{\mathbf{L}}_{\mathbf{M},v} \,|\, v \ \ \mathbf{M}\text{-evaluation}\}.$$

(2) A formula φ of a language $\mathcal{J}$ is an $\mathbf{L}$-*tautology* if $||\varphi||_{\mathbf{M}} = 1_{\mathbf{L}}$ for each safe $\mathbf{L}$-structure $\mathbf{M}$, i.e. $||\varphi||_{\mathbf{M},v} = 1$ for each safe $\mathbf{L}$-structure $\mathbf{M}$ and each $\mathbf{M}$-valuation of object variables.

Definition 5.1.7 (1) The following are *logical axioms on quantifiers:*

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$)
($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$)
($\forall$2) $(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ (x not free in ν)
($\exists$2) $(\forall x)(\varphi \to \nu) \to ((\exists x)\varphi \to \nu)$ (x not free in ν)
($\forall$3) $(\forall x)(\varphi \vee \nu) \to ((\forall x)\varphi \vee \nu)$ (x not free in ν)

(2) Let $\mathcal{C}$ be a schematic extension of the basic propositional logic BL. We associate with $\mathcal{C}$ the corresponding predicate calculus $\mathcal{C}\forall$ (over a given predicate language $\mathcal{J}$) by taking as logical axioms

- all formulas resulting from the axioms of $\mathcal{C}$ by substituting arbitrary formulas of $\mathcal{J}$ for propositional variables, and
- the axioms $(\forall 1), (\forall 2), (\forall 3), (\exists 1), (\exists 2)$ for quantifiers

and taking as deduction rules

- modus ponens (from φ, $\varphi \to \psi$ infer ψ) and
- generalization (from φ infer $(\forall x)\varphi$).

(3) Given this, the notions of proof, provability, theory, proof/provability in a theory over $\mathcal{C}\forall$ are obvious.

Remark 5.1.8 In particular, we shall be interested in the basic fuzzy predicate logic BL∀ and three stronger logics: Ł∀ (Łukasiewicz), G∀ (Gödel), Π∀ (product). For example, axioms of BL∀ are (A1)–(A7) of 2.2.4 (with φ, ψ, χ being formulas of predicate logic) and the present five quantifier axioms (∀1)–(∃2).

Lemma 5.1.9 The axioms (∀1),(∀2),(∀3), (∃1),(∃2), are **L**-tautologies for each linearly ordered BL-algebra **L**.

Proof: To verify (∀1), (∃1) show that if y is substitutable for x to φ then $\|\varphi(y)\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi(x)\|^{\mathbf{L}}_{\mathbf{M},v''}$ where $v'' \equiv_x v$ and $v''(x) = v(y)$ (Caution: a pedantic proof shows that the assumption of substitutability is indispensable.)

$$\|(\forall x)\varphi(x)\|^{\mathbf{L}}_{\mathbf{M},v} = \inf_{v' \equiv_x v} \|\varphi(x)\|^{\mathbf{L}}_{\mathbf{M},v'} \leq \|\varphi(y)\|^{\mathbf{L}}_{\mathbf{M},v''} \leq$$
$$\leq \sup_{v'} \|\varphi(x)\|^{\mathbf{L}}_{\mathbf{M},v'} = \|(\exists x)\varphi(x)\|^{\mathbf{L}}_{\mathbf{M},v}.$$

We consider $(\forall x)(\nu \to \varphi(x)) \to (\nu \to (\forall x)\varphi(x))$. We have to show

$$\inf_w(\|\nu\|_w \Rightarrow \|\varphi\|_w) \leq (\|\nu\|_v \Rightarrow \inf_w \|\varphi\|_w)$$

(indices $\mathbf{M}, \mathbf{L}$ deleted, w runs over all evaluations $\equiv_x v$). Put $\|\nu\|_v = \|\nu\|_w = a$ (note that x is not free in ν!), $\|\varphi\|_w = b_w$; thus we must prove $\inf_w(a \Rightarrow b_w) \leq (a \Rightarrow \inf_w b_w)$. We even prove equality. On the one hand, $\inf_w b_w \leq b_w$, thus $a \Rightarrow b_w \geq a \Rightarrow \inf_w b_w$ for each w, thus $\inf_w(a \Rightarrow b_w) \geq (a \Rightarrow \inf_w b_w)$. On the other hand if $z \leq (a \Rightarrow b_w)$ for each w, then $z * a \leq b_w$ for each w, $z * a \leq \inf_w b_w$, $z \leq (a \Rightarrow \inf_w b_w)$. Thus $(a \Rightarrow \inf_w b_w)$ is the infimum of all $(a \Rightarrow b_w)$.

Similarly, we verify $\inf_w(a_w \Rightarrow b) = (\sup_w a_w \Rightarrow b)$. Indeed, $\sup_w a_w \geq a_w$, thus $(\sup_w a_w \Rightarrow b) \leq (a_w \Rightarrow b)$, hence $(\sup_w a_w \Rightarrow b) \leq \inf_w(a_w \Rightarrow b)$. If $z \leq a_w \Rightarrow b$ for all w then $a_w \leq (z \Rightarrow b)$ for all w, $\sup_w a_w \leq (z \Rightarrow b)$, $z \leq (\sup_w a_w \Rightarrow b)$; thus $\sup_w a_w \Rightarrow b$ is the infimum.

Finally we verify (∀3); we even prove $\inf_w(a \vee b_w) = a \vee \inf_w b_w$. Indeed, $a \leq a \vee b_w$, thus $a \leq \inf_w(a \vee b_w)$; similarly, $\inf_w b_w \leq \inf_w(a \vee b_w)$, thus $a \vee \inf_w b_w \leq \inf_w(a \vee b_w)$. Conversely, let $z \leq a \vee b_w$ for all w; we prove $z \leq a \vee \inf_w b_w$.

Case 1. Let $a \leq \inf_w b_w$. Then $z \leq b_w$ for each w, $z \leq \inf_w b_w$ and $z \leq a \vee \inf_w b_w$.

Case 2. Let $a > \inf_w b_w$. Then for some w_0, $a \geq b_{w_0}$, thus $z \leq a$ and $z \leq a \vee \inf_w b_w$. □

Lemma 5.1.10 (Soundness of deduction rules)

(1) For arbitrary formulas φ, ψ, safe **L**-structure **M** and evaluation v,

$$\|\psi\|^{\mathbf{L}}_{\mathbf{M},v} \geq \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} * \|\varphi \to \psi\|^{\mathbf{L}}_{\mathbf{M},v};$$

thus in particular, if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi \to \psi\|^{\mathbf{L}}_{\mathbf{M},v} = 1_{\mathbf{L}}$ then $\|\psi\|^{\mathbf{L}}_{\mathbf{M},v} = 1_{\mathbf{L}}$.

(2) Consequently,

$$\|\psi\|^{\mathbf{L}}_{\mathbf{M}} \geq \|\varphi\|^{\mathbf{L}}_{\mathbf{M}} * \|\varphi \to \psi\|^{\mathbf{L}}_{\mathbf{M}},$$

thus if φ, $\varphi \to \psi$ are $1_{\mathbf{L}}$-true in **M** then ψ is $1_{\mathbf{L}}$-true in **M**.

(3) $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = \|(\forall x)\varphi\|^{\mathbf{L}}_{\mathbf{M}}$;
thus if φ is $1_{\mathbf{L}}$-true in **M** then $(\forall x)\varphi$ is.

Proof:

(1) is just as in propositional calculus (from the basic property of residuation). To prove (2) put $\|\varphi\|_w = a_w$, $\|\psi\|_w = b_w$, $\inf_w a_w = a$. We have to prove $\inf_w(a_w \Rightarrow b_w) \leq \inf_w a_w \Rightarrow \inf_w b_w$. Observe the following:

$$\inf(a_w \Rightarrow b_w) \leq a_w \Rightarrow b_w \leq a \Rightarrow b_w, \text{ thus}$$
$$\inf(a_w \Rightarrow b_w) \leq \inf_w(a \Rightarrow b_w).$$

It remains to prove $\inf_w(a \Rightarrow b_w) \leq a \Rightarrow \inf b_w$, but this is exactly as in 5.1.9.

(3) is evident from the definition of $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}}$ (in fact, if $x, \dots y$ are the free variables of φ then $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = \|(\forall x) \dots (\forall y)\varphi\|^{\mathbf{L}}_{\mathbf{M}}$.

□

Definition 5.1.11 Let $\mathcal{C}$ be a schematic extension of BL, let T be a theory over $\mathcal{C}$, let **L** be a linearly ordered $\mathcal{C}$-algebra and **M** a safe **L**-structure for the language of T. **M** is an **L**-*model* of T if all axioms of T are $1_{\mathbf{L}}$-true in **M**, i.e. $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1_{\mathbf{L}}$ in each $\varphi \in T$.

Theorem 5.1.12 (Soundness of provability.) Let $\mathcal{C}$ be a schematic extension of BL, let T be a theory in the language of T over $\mathcal{C}\forall$, let φ be a formula of T. If $T \vdash \varphi$ (φ is provable in T) then $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1_{\mathbf{L}}$ for each linearly ordered $\mathcal{C}$-algebra **L** and each **L**-model **M** of T.

Proof: This follows by the obvious induction on the length of a proof. □

Remark 5.1.13 (1) Note that if there is a proof of φ (in T, over $\mathcal{C}$) not using the rule of generalization then we also have the following variant of soundness: If $\mathbf{L}$ is a linearly ordered $\mathcal{C}$-algebra, $\mathbf{M}$ a safe $\mathbf{L}$-structure and v an $\mathbf{M}$-evaluation of object variables such that $\|\alpha\|^{\mathbf{L}}_{M,v} = 1_{\mathbf{L}}$ for each axiom $\alpha \in T$ then $\|\varphi\|^{\mathbf{L}}_{M,v} = 1_{\mathbf{L}}$.

(2) In the rest of this section we shall show several formulas with quantifiers to be in BL$\forall$. Note that the axiom ($\forall$3) will not be used.

*

Theorem 5.1.14 Let φ be an arbitrary formula, ν a formula not containing x freely. Then BL$\forall$ proves the following:

(1) $(\forall x)(\nu \to \varphi) \equiv (\nu \to (\forall x)\varphi)$

(2) $(\forall x)(\varphi \to \nu) \equiv ((\exists x)\varphi \to \nu)$

(3) $(\exists x)(\nu \to \varphi) \to (\nu \to (\exists x)\varphi)$

(4) $(\exists x)(\varphi \to \nu) \to ((\forall x)\varphi \to \nu)$

Proof: The implications $\to$ in (1), (2) are axioms.

(1) $\vdash (\forall x)\varphi \to \varphi$ by ($\forall$1), thus
$\vdash (\nu \to (\forall x)\varphi) \to (\nu \to \varphi)$ by transitivity. Generalize:
$\vdash (\forall x)[(\nu \to (\forall x)\varphi) \to (\nu \to \varphi)]$, hence
$\vdash (\nu \to (\forall x)\varphi) \to (\forall x)(\nu \to \varphi)$ by ($\forall$2).

(2) $\vdash \varphi \to (\exists x)\varphi$,
$\vdash ((\exists x)\varphi \to \nu) \to (\varphi \to \nu)$,
generalizing and applying ($\forall$2) we get
$\vdash ((\exists x)\varphi \to \nu) \to (\forall x)(\varphi \to \nu)$

(3) $\vdash (\nu \to \varphi) \to (\nu \to (\exists x)\varphi)$,
generalize and apply ($\exists$2):
$\vdash (\exists x)(\nu \to \varphi) \to (\nu \to (\exists x)\varphi)$

(4) $\vdash (\varphi \to \nu) \to ((\forall x)\varphi \to \nu)$, thus by ($\exists$2),
$\vdash (\exists x)(\varphi \to \nu) \to ((\forall x)\varphi \to \nu)$.

□

Remark 5.1.15 The converse implications in (3), (4) are *not* provable in BL. We shall see later that neither of them is a tautology of G$\forall$; the converse of (3) is, but the converse of (4) is not a tautology of $\Pi\forall$; and both converses are tautologies of Ł$\forall$.

Theorem 5.1.16 For arbitrary formulas φ, ψ, BL$\forall$ proves the following:

(5) $(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$

(6) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$

(7) $((\forall x)\varphi \& (\exists x)\psi) \to (\exists x)(\varphi \& \psi)$

Proof:

(5) From $\vdash (\forall x)(\varphi \to \psi) \to (\varphi \to \psi)$ and $\vdash (\forall x)\varphi \to \varphi$ we get, using transitivity,
$\vdash (\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to \psi)$.
Generalizing and applying ($\forall$2) twice we get
$\vdash (\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$.

(6) Analogously, we get
$\vdash (\forall x)(\varphi \to \psi) \to (\varphi \to (\exists x)\psi)$,
from which we get
$\vdash (\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$
using ($\forall$2) and ($\exists$2).

(7) Generalize in
$\vdash \varphi \to (\psi \to (\varphi \& \psi))$
and then apply (5); you get
$\vdash (\forall x)\varphi \to (\forall x)(\psi \to (\varphi \& \psi))$
and using (6) you get
$\vdash ((\forall x)\varphi \to ((\exists x)\psi) \to (\exists x)(\varphi \& \psi))$

□

Theorem 5.1.17 If y is substitutable for x in $\varphi(x)$ then BL$\forall$ proves

(8) $(\forall x)\varphi(x) \equiv (\forall y)\varphi(y)$ and $(\exists x)\varphi(x) \equiv (\exists y)\varphi(y)$.

Proof: From $\vdash (\forall x)\varphi(x) \to \varphi(y)$ we get $(\forall x)\varphi(x) \to (\forall y)\varphi(y)$ by generalization and ($\forall$2). We get $(\forall y)\varphi(y) \to (\forall x)\varphi(x)$ in the same way. The proof of $\vdash (\exists x)\varphi(x) \equiv (\exists y)\varphi(y)$ is analogous. □

Theorem 5.1.18 For arbitrary φ and for ν not containing x freely, BL$\forall$ proves

(9) $(\exists x)(\varphi \& \nu) \equiv ((\exists x)\varphi \& \nu)$,

(10) $(\exists x)(\varphi \& \varphi) \equiv ((\exists x)\varphi \& (\exists x)\varphi)$.

Proof:

(9) $\vdash (\varphi \& \nu) \to ((\exists x)\varphi \& \nu)$ (using ($\exists$1)); generalize and use ($\exists$2) to get $\vdash (\exists x)(\varphi \& \nu) \to ((\exists x)\varphi \& \nu)$. Conversely observe that $\vdash (\forall x)(\nu \to \nu)$ gives $\vdash \nu \to (\forall x)\nu$ by ($\forall$1); thus
$\vdash ((\exists x)\varphi \& \nu) \to ((\exists x)\varphi \& (\forall x)\nu)$,
which gives
$\vdash ((\exists x)\varphi \& \nu) \to (\exists x)(\varphi \& \nu)$
by (7).

(10) Write Φ for $\varphi(x) \to (\exists x)\varphi(x)$ (an instance of ($\exists$1); then
$\vdash (\Phi \& \Phi) \to (\varphi(x) \& \varphi(x)) \to (\exists x)\varphi(x) \& (\exists x)\varphi(x))$.
Delete $\Phi \& \Phi$ by modus ponens, generalize and apply $\exists$2; you get
$\vdash (\exists x)(\varphi(x) \& \varphi(x)) \to ((\exists x)\varphi(x) \& (\exists x)\varphi(x))$.
This can be written as
$\vdash (\exists x)\varphi^2(x) \to ((\exists x)\varphi(x))^2$.
Conversely, observing the propositional tautology $(p \& q) \to (p^2 \vee q^2)$ we have
$\vdash (\varphi(x) \& \varphi(y)) \to (\varphi^2(x) \vee \varphi^2(y))$,
$\vdash (\varphi(x) \& \varphi(y)) \to ((\exists x)\varphi^2(x) \vee (\exists y)\varphi^2(y))$ (by ($\exists$1)),
$\vdash (\varphi(x) \& \varphi(y)) \to ((\exists z)\varphi^2(z) \vee (\exists z)\varphi^2(z))$ (by (8)),
$\vdash (\forall y)(\forall x)[(\varphi(x) \& \varphi(y)) \to (\exists z)\varphi^2(z)]$
$\vdash (\forall y)(\forall x)[\varphi(x) \to (\varphi(y) \to (\exists z)\varphi^2(z))]$
$\vdash (\exists x)\varphi(x) \to (\forall y)(\varphi(y) \to (\exists z)\varphi^2(z))$
$\vdash (\exists x)\varphi(x) \to ((\exists y)\varphi(y) \to (\exists z)\varphi^2(z))$
$\vdash (\exists x)\varphi(x) \to ((\exists x)\varphi(x) \to (\exists x)\varphi^2(x))$
$\vdash ((\exists x)\varphi(x))^2 \to (\exists x)\varphi^2(x)$.

□

Remark 5.1.19 The reader may ask if the formula $(\forall x)(\varphi \& \nu) \equiv ((\forall x)\varphi \& \nu)$ (where ν does not contain x freely) is provable in BL$\forall$. We are unable to answer this question; we only show that the formula in question is a $1_{\mathbf{L}}$-tautology for each linearly *densely* ordered BL-algebra $\mathbf{L}$ (in particular, in each such lattice over $[0,1]$, thus e.g. $[0,1]_{Ł}$, $[0,1]_{G}$, $[0,1]_{\Pi}$).

Indeed, let $\mathbf{L}$ be such that $\inf_w a_w$ and $\inf_w(a_w * b)$ exist; obviously, $((\inf_w a_w) * b) \leq (a_w * b)$ for each w, thus $(\inf_w a_w) * b \leq \inf_w(a_w * b)$. To prove the converse inequality we have to show that $(\inf_w a_w) * b$ is the infimum of all $a_w * b$, i.e. if $z \leq a_w * b$ for all w then $z \leq (\inf_w a_w) * b$. Assume $z > (\inf_w a_w) * b$. Since $z \leq a_w * b$ for all w we get $z \leq b$ (note that here we use the assumption that the domain of each structure is non-empty) hence $z = z \cap b = b * (b \to z)$. From this we get (comparing with $z > b * \inf_w a_w$) $\inf_w a_w < b \to z$, hence for some w_0 we have $a_{w_0} \leq b \to z$; thus $a_{w_0} * b \leq z$, which together $z \leq a_{w_0} * b$ gives $a_{w_0} * b = z$. Thus we get $\inf_w(a_w * b) = z$; but z was arbitrary such that $(\inf_w a_w) * b < z \leq \inf_w(a_w * b)$. Hence if the ordering $\leq$ is dense we get a contradiction.

Theorem 5.1.20 BL$\forall$ proves the following:

(11) $(\exists x)\varphi \to \neg(\forall x)\neg\varphi$

(12) $\neg(\exists x)\varphi \equiv (\forall x)\neg\varphi$

Proof:

(11) $\vdash (\exists x)\varphi \to ((\forall x)\neg\varphi \to (\exists x)(\varphi\&\neg\varphi))$ by (7); but $\vdash (\varphi\&\neg\varphi) \to \bar{0}$, thus $\vdash (\forall x)((\varphi\&\neg\varphi) \to \bar{0})$ and $\vdash (\exists x)(\varphi\&\neg\varphi) \to \bar{0}$; hence $\vdash (\exists x)\varphi \to ((\forall x)\neg\varphi \to \bar{0})$.

(12) $\vdash \neg(\exists x)\varphi(x)\&\varphi(x) \to \neg(\exists x)\varphi(x)\&(\exists x)\varphi(x)$ thus,
$\vdash (\neg(\exists x)\varphi(x)\&\varphi(x)) \to \bar{0}$,
$\vdash \neg(\exists x)\varphi(x) \to (\varphi(x) \to \bar{0})$; generalize and apply ($\forall$2) to get
$\vdash \neg(\exists x)\varphi(x) \to (\forall x)(\varphi(x) \to \bar{0})$.
The converse implication follows from (11) by BL.

□

Lemma 5.1.21 BL$\forall$ proves the following:

(13) $(\exists x)(\nu \wedge \varphi) \equiv (\nu \wedge (\exists x)\varphi)$,

(14) $(\exists x)(\nu \vee \varphi) \equiv (\nu \vee (\exists x)\varphi)$.

(15) $(\forall x)(\nu \wedge \varphi) \equiv (\nu \wedge (\forall x)\varphi)$

Here again ν is a formula not containing x freely.

Proof:

(13) BL$\forall$ proves $(\exists x)(\nu \wedge \varphi) \rightarrow (\exists x)\nu \rightarrow \nu$ and $(\exists x)(\nu \wedge \varphi) \rightarrow (\exists x)\varphi$; thus BL$\forall \vdash (\exists x)(\nu \wedge \varphi) \rightarrow (\nu \wedge (\exists x)\varphi$.

Conversely, BL$\forall$ proves $(\nu \rightarrow \varphi(x)) \rightarrow (\nu \rightarrow (\nu \wedge \varphi(x))) \rightarrow (\nu \rightarrow (\exists x)(\nu \wedge \varphi(x)) \rightarrow ((\nu \wedge (\exists x)\varphi(x)) \rightarrow (\exists x)(\nu \wedge \varphi(x))),$
$(\varphi(x) \rightarrow \nu) \rightarrow (\varphi(x) \rightarrow (\nu \wedge \varphi(x)) \rightarrow (\varphi(x) \rightarrow (\exists x)(\nu \wedge \varphi(x))) \rightarrow ((\nu \wedge (\exists x)\varphi(x) \rightarrow (\exists x)(\nu \wedge \varphi(x))),$
thus we get BL$\forall \vdash (\nu \wedge (\exists x)\varphi(x)) \rightarrow (\exists x)(\nu \wedge \varphi(x))$.

(14) BL$\forall$ proves $\nu \rightarrow (\exists x)(\nu \vee \varphi(x))$, $(\exists x)\varphi(x) \rightarrow (\exists x)(\nu \vee \varphi(x))$, thus BL$\forall \vdash (\nu \vee (\exists x)\varphi(x)) \rightarrow (\exists x)(\nu \vee \varphi(x))$. Conversely, BL$\forall$ proves $(\exists x)(\nu \vee \varphi(x)) \rightarrow (\exists x)(\nu \vee (\exists x)\varphi(x)) \rightarrow (\nu \vee (\exists x)\varphi(x))$.

(15) Evidently, BL$\forall \vdash (\forall x)(\nu \wedge \varphi) \rightarrow \nu$, BL$\forall \vdash (\forall x)(\nu \wedge \varphi) \rightarrow (\forall x)\varphi$, thus BL$\forall \vdash (\forall x)((\nu \wedge ((\forall x)\varphi) \rightarrow (\nu \wedge (\forall x)\varphi$. Conversely, BL$\forall \vdash [\nu \wedge (\forall x)\varphi] \rightarrow [(\forall x)(\nu \wedge (\forall x)\varphi)] \rightarrow [(\forall x)(\nu \wedge \varphi)]$.

□

Corollary 5.1.22 BL$\forall$ proves the following:

(16) $(\exists x)(\varphi \vee \psi) \equiv ((\exists x)\varphi \vee (\exists x)\psi)$,

(17) $(\forall x)(\varphi \wedge \psi) \equiv ((\forall x)\varphi \wedge (\forall x)\psi)$.

Proof: (16) Evidently, BL$\forall \vdash (\exists x)\varphi \rightarrow (\exists x)(\varphi \vee \psi)$ and BL$\forall \vdash (\exists x)\psi \rightarrow (\exists x)(\varphi \vee \psi)$, which gives the implication $\leftarrow$. Conversely, BL$\forall$ proves $[(\exists x)(\varphi \vee \psi)] \rightarrow [(\exists x)(\varphi \vee (\exists x)\psi)] \rightarrow [(\exists x)\varphi \vee (\exists x)\psi]$ by (14).

(17) Here $\rightarrow$ is easy; conversely, BL$\forall$ proves $[((\forall x)\varphi \wedge (\forall x)\psi)] \rightarrow [(\forall x)(\varphi \wedge (\forall x)\psi)] \rightarrow [(\forall x)(\varphi \wedge \psi)]$ by (15). □

To close this section we prove the deduction theorem for $C\forall$.

Theorem 5.1.23 Let T be a theory over $C\forall$ and let φ, ψ be closed formulas of the language of T. Then $(T \cup \{\varphi\}) \vdash \psi$ iff there is an n such that $T \vdash \varphi^n \rightarrow \psi$.

Proof: The proof is an extension of the proof of the analogous deduction theorem for propositional logic 2.2.18: we have to discuss the case of generalization. Thus assume $T \vdash \varphi^n \rightarrow \gamma_j$ and let γ_i be $(\forall x)\gamma_j$; then $T \vdash (\forall x)(\varphi^n \rightarrow \gamma_j)$ and since φ is closed it follows that $T \vdash \varphi^n \rightarrow (\forall x)\gamma_j$, thus $T \vdash \varphi^n \rightarrow \gamma_i$ by the axiom ($\forall$2). □

5.2. COMPLETENESS

Recall 2.3.20–2.4.3 where we introduced schematic extensions $\mathcal{C}$ of BL and proved their completeness with respect to evaluations in linearly ordered $\mathcal{C}$-algebras. In the preceding section we introduced a predicate logic counterpart $\mathcal{C}\forall$ of each schematic $\mathcal{C}$; now we are going to investigate its completeness. First let us make some definitions (a fixed schematic extension $\mathcal{C}$ of BL is supposed to be given).

Definition 5.2.1 Let T be a theory over $\mathcal{C}$.

(1) T is *consistent* if there is a formula φ unprovable in T.

(2) T is *complete* if for each pair φ, ψ of closed formulas, $T \vdash (\varphi \to \psi)$ or $T \vdash (\psi \to \varphi)$. (Cf. 2.4.1.)

(3) T is *Henkin* if for each closed formula of the form $(\forall x)\varphi(x)$ unprovable in T there is a constant c in the language of T such that $\varphi(c)$ is unprovable in T.

Lemma 5.2.2 T is inconsistent iff $T \vdash \bar{0}$.

Proof: Obvious. □

Lemma 5.2.3 T is complete iff for each pair φ, ψ of closed formulas such that $T \vdash \varphi \vee \psi$, T proves φ or T proves ψ.

Proof: If the condition holds apply it to $(\varphi \to \psi) \vee (\psi \to \varphi)$: this formula is provable in BL, hence in T, thus $T \vdash (\varphi \to \psi)$ or $T \vdash (\psi \to \varphi)$. Conversely, assume T complete and $T \vdash \varphi \vee \psi$. Either $T \vdash \varphi \to \psi$ and then $T \vdash (\varphi \vee \psi) \to \psi$, thus $T \vdash \psi$, or $T \vdash \psi \to \varphi$ and then similarly $T \vdash \varphi$. □

Remark 5.2.4 (1) Observe that in the case that $\mathcal{C}\forall$ is the Boolean predicate calculus T is complete in the sense of 5.2.1 iff for each closed φ, $T \vdash \varphi$ or $T \vdash \neg\varphi$. Indeed, if T is complete then $T \vdash \varphi$ or $T \vdash \neg\varphi$ since $T \vdash \varphi \vee \neg\varphi$; conversely, if the condition holds then $T \vdash (\varphi \to \psi)$ or $T \vdash \neg(\varphi \to \psi)$, but in the latter case $T \vdash (\psi \to \varphi)$.

(2) Furthemore, for classical logic and a complete theory T, the condition for T to be Henkin is clearly equivalent to the following: for each closed formula $(\exists x)\varphi(x)$, if $T \vdash (\exists x)\varphi(x)$ then for some constant c, $T \vdash \varphi(c)$. This will also be the case for Ł$\forall$ as we shall see later.

Definition 5.2.5 For each theory T over $\mathcal{C}\forall$ we let $\mathbf{L}_T$ be the algebra of classes of T-equivalent closed formulas (*caution*: do not overlook that we restrict ourselves to closed formulas) with the usual operations ($[\varphi]_T \Rightarrow [\psi]_T = [\varphi \to \psi]_T$ etc.). Clearly, $\mathbf{L}_T$ is a $\mathcal{C}$-algebra.

Lemma 5.2.6 (1) If T is complete then $\mathbf{L}_T$ is linearly ordered.

(2) If T is Henkin then for each formula $\varphi(x)$ with just one free variable x,

$$[(\forall x)\varphi]_T = \inf_c[\varphi(c)]_T,$$
$$[(\exists x)\varphi]_T = \sup_c[\varphi(c)]_T$$

(c running over all constants of T).

Proof:

(1) is obvious since $[\varphi]_T \leq [\psi]_T$ iff $T \vdash (\varphi \to \psi)$.

We prove (2). Clearly, $[(\forall x)\varphi(x)]_T \leq [\varphi(c)]_T$ for each c, thus $[(\forall x)\varphi(x)]_T \leq \inf_c[\varphi(c)]_T$. To prove that $[(\forall x)\varphi(x)]_T$ is the infimum of all $[\varphi(c)]_T$, assume $[\gamma]_T \leq [\varphi(c)]_T$ for each c; we have to prove $[\gamma]_T \leq [(\forall x)\varphi(x)]_T$ (which means that $[(\forall x)\varphi(x)]_T$ is the greatest lower bound of all $[\varphi(c)]_T$). But if $[\gamma]_T \not\leq [(\forall x)\varphi(x)]_T$ then $T \nvdash \gamma \to (\forall x)\varphi(x)$, thus $T \nvdash (\forall x)(\gamma \to \varphi(x))$, thus by the Henkin property, there is a constant c such that $T \nvdash \gamma \to \varphi(c)$, thus $[\gamma]_T \not\leq [\varphi(c)]_T$, a contradiction.

Similarly, $[\varphi(c)]_T \leq [(\exists x)\varphi(x)]_T$ for each c. Assume $[\varphi(c)]_T \leq [\gamma]_T$ for each c; we prove $[(\exists x)\varphi(x)]_T \leq [\gamma]_T$. Indeed, if $[(\exists x)\varphi(x)]_T \not\leq [\gamma]_T$ then $T \nvdash (\exists x)\varphi(x) \to \gamma$, thus $T \nvdash (\forall x)(\varphi(x) \to \gamma)$ and for some c, $T \nvdash \varphi(c) \to \gamma$, thus $[\varphi(c)]_T \not\leq [\gamma]$, a contradiction. This completes the proof.

□

Lemma 5.2.7 For each theory T and each closed formula α, if $T \nvdash \alpha$ then there is a complete Henkin supertheory $\hat{T}$ of T such that $\hat{T} \nvdash \alpha$.

Proof: First observe that if T' is an extension of T, $T' \nvdash \alpha$, and (φ, ψ) is a pair of closed formulas then either $(T' \cup \{\varphi \to \psi\}) \nvdash \alpha$ or $(T' \cup \{\psi \to \varphi\}) \nvdash \alpha$; This is proved easily using the deduction theorem (5.1.20). (Indeed, if T', $(\varphi \to \psi) \vdash \alpha$ and T', $(\psi \to \varphi) \vdash \alpha$ for some n, $T' \vdash (\varphi \to \psi)^n \to$

α, $T' \vdash (\psi \to \varphi)^n \to \alpha$, thus $T' \vdash ((\varphi \to \psi)^n \vee (\psi \to \varphi)^n) \to \alpha$ and $T \vdash \alpha$.) Put $T'' = T' \cup \{\varphi \to \psi\}$ in the former case and $T'' = T' \cup \{\psi \to \varphi\}$ in the latter; T'' is the extension of T' deciding (φ, ψ) and keeping α unprovable.

We shall construct $\hat{T}$ in countably many stages. First extend the language $\mathcal{J}$ of T to $\mathcal{J}'$ adding new constants $c_0, c_1, c_2, \ldots$ In the construction we have to decide each pair (φ, ψ) of closed $\mathcal{J}'$-formulas and ensure the Henkin property for each closed $\mathcal{J}'$-formula of the form $(\forall x)\chi(x)$. These are countably many tasks and may be enumerated by natural numbers (e.g. in even steps we shall decide all pairs (φ, ψ), in odd ones process all formulas $(\forall x)\chi(x)$ — or take any other enumeration).

Put $T_0 = T$ and $\alpha_0 = \alpha$; then $T_0 \nvdash \alpha_0$. Assume T_n, α_n have been constructed such that T_n extends T_0, $T_n \vdash \alpha \to \alpha_n$, $T_n \nvdash \alpha_n$; we construct T_{n+1}, α_{n+1} in such a way that $T_{n+1} \vdash \alpha_n \to \alpha_{n+1}$, $T_{n+1} \nvdash \alpha_{n+1}$ and T_{n+1} fulfils the n-th task.

Case 1 n-th task is deciding (φ, ψ). Let T_{n+1} be the extension of T_n deciding (φ, ψ) and keeping α_n unprovable; put $\alpha_{n+1} = \alpha_n$.

Case 2 n-th task is processing $(\forall x)\chi(x)$. First let c be one of the new constants not occuring in T_n.

Subcase (a) $T_n \nvdash \alpha_n \vee \chi(c)$, thus $T_n \nvdash (\forall x)\chi(x)$. Put $T_{n+1} = T_n$, $\alpha_{n+1} = \alpha_n \vee \chi(c)$.

Subcase (b) $T_n \vdash \alpha_n \vee \chi(c)$, thus $T_n \vdash \alpha_n \vee \chi(x)$ by the standard argument (in the proof of $\alpha_n \vee \chi(c)$ replace c by a new variable x throughout). Hence $T_n \vdash (\forall x)(\alpha_n \vee \chi(x))$ and, using the axiom ($\forall 3$) for the first time, $T_n \vdash \alpha_n \vee (\forall x)\chi(x)$. Thus $T_n \cup \{(\forall x)\chi(x) \to \alpha_n\} \vdash \alpha_n$ so that $T_n \cup \{\alpha_n \to (\forall x)\chi(x)\}$ does not prove α_n but it does prove $(\forall x)\chi(x)$. Thus put $T_{n+1} = T_n \cup \{\alpha_n \to (\forall x)\chi(x)\}$ and $\alpha_{n+1} = \alpha_n$.

Now let $\hat{T}$ be the union of all T_n. Then clearly $\hat{T}$ is complete and $\hat{T} \nvdash \alpha$ (since for all n, $\hat{T} \nvdash \alpha_n$). We show that $\hat{T}$ is Henkin. Let $\hat{T} \nvdash (\forall x)\chi(x)$ and let $(\forall x)\chi(x)$ be processed in step n. Then $T_{n+1} \nvdash (\forall x)\chi(x)$, thus subcase (a) applies and $\hat{T} \nvdash \alpha_{n+1}$, α_{n+1} being $\alpha_n \vee \chi(c)$. Hence $\hat{T} \nvdash \chi(c)$. This completes the proof. □

Lemma 5.2.8 For each complete Henkin theory T and each closed formula α unprovable in T there is a linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$ and $\mathbf{L}$-model $\mathbf{M}$ of T such that $\|\alpha\|_{\mathbf{M}}^{\mathbf{L}} < 1_{\mathbf{L}}$.

Proof: Take M to be the set of all constants of the language of T; $m_c = c$ for each such constant. Let $\mathbf{L}$ be the lattice of classes of T-equivalent closed formulas, i.e. put $[\varphi]_T = \{\psi | T \vdash \varphi \equiv \psi\}$, $[\varphi]_T * [\psi]_T = [\varphi \& \psi]_T$, $[\varphi]_T \Rightarrow [\psi]_T = [\varphi \to \psi]_T$. Using the methods of Section 2.3 prove that $\mathbf{L}$ is a $\mathcal{C}$-algebra and is linearly ordered (since $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$ for each pair (φ, ψ)).

For each predicate P of arity n, let $r_P(c_1, \ldots, c_n) = [P(c_1, \ldots, c_n)]_T$; this completes the definition of $\mathbf{M}$. It remains to prove $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = [\varphi]_T$ for each closed φ. Then for each axiom φ of T we have $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = [\varphi]_T = [1]_T = 1_{\mathbf{L}}$, but $\|\alpha\|^{\mathbf{L}}_{\mathbf{M}} = [\alpha]_T \neq [1]_T = 1_{\mathbf{L}}$. For atomic closed φ the claim follows by definition; the induction step for connectives is obvious. We handle the quantifiers. Let $(\forall x)\varphi(x)$, $(\exists x)\varphi(x)$ be closed. Then, by the induction hypothesis,

$$\|(\forall x)\varphi(x)\|^{\mathbf{L}}_{\mathbf{M}} = \inf_c \|\varphi(c)\|^{\mathbf{L}}_{\mathbf{M}} = \inf_c [\varphi(c)]_T = [(\forall x)\varphi(x)]_T,$$
$$\|(\exists x)\varphi(x)\|^{\mathbf{L}}_{\mathbf{M}} = \sup_c \|\varphi(c)\|^{\mathbf{L}}_{\mathbf{M}} = \sup_c [\varphi(c)]_T = [(\exists x)\varphi(x)]_T.$$

Here we use lemma 5.2.6 and the fact that in our $\mathbf{M}$, each element c of M is the meaning of a constant (namely itself); this gives $\|(\forall x)\varphi(x)\|^{\mathbf{L}}_{\mathbf{M}} = \inf_c \|\varphi(c)\|^{\mathbf{L}}_{\mathbf{M}}$ and the dual for $\exists$. □

Theorem 5.2.9 *(Completeness.)* Let $\mathcal{C}\forall$ be the predicate calculus given by a schematic extension $\mathcal{C}$ of BL, let T be a theory over $\mathcal{C}\forall$ let φ be a formula of the language of T. T proves φ iff for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$ and each safe $\mathbf{L}$-model $\mathbf{M}$ of T, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1_{\mathbf{L}}$.

Proof: This is a direct corollary of the preceding two lemmas. □

Remark 5.2.10 Note that this theorem immediately gives the classical strong completeness of Boolean predicate logic since if $\mathcal{C}$ is *Bool* (equivalently, BL2), then the only non-trivial linearly ordered $\mathcal{C}$-algebra (Boolean algebra) is the two-element Boolean algebra.

*

We shall now present some simple consequences of the completeness theorem, concerning extension of theories.

Definition 5.2.11 (1) Let $\mathcal{J}_1$ be a language, $\mathcal{J}_2$ a richer language (resulting by adding some new predicates and/or constants); let T_1 be a theory in $\mathcal{C}\forall$ over $\mathcal{J}_1$, $T_2 \supseteq T_1$ its extension in $\mathcal{C}\forall$ over $\mathcal{J}_2$. T_2 is a *conservative extension* of T_1 if for each formula φ of $\mathcal{J}_1$, $T_2 \vdash \varphi$ implies $T_1 \vdash \varphi$ (a formula of the poorer language provable in the richer theory is provable in the poorer theory).

(2) An $\mathbf{L}$-structure $\mathbf{M}_2$ for $\mathcal{J}_2$ is an *expansion* of an $\mathbf{L}$-structure $\mathbf{M}_1$ for $\mathcal{J}_\infty$ if $\mathbf{M}_1$ and $\mathbf{M}_2$ have the same domain and the same interpretation of predicates and constants of $\mathcal{J}_1$. ($\mathbf{M}_2$ results from $\mathbf{M}_1$ by adding the interpretation of new symbols.)

Lemma 5.2.12 Let $\mathcal{J}_1 \subseteq \mathcal{J}_2$ be languages, T_i theories in $\mathcal{C}\forall$ over $\mathcal{J}_C$, $T_1 \subseteq T_2$. If each safe model of $\mathbf{M}_1$ has an expansion to a safe model of T_2 then T_2 is a conservative extension of T_1.

Proof: Let φ be a (closed) $\mathcal{J}_1$-formula and let $T_1 \nvdash \varphi$. Then by 5.2.8 there are $\mathbf{L}, \mathbf{M}_1$ such that $\mathbf{M}_1$ is a safe $\mathbf{L}$-model of T_1 and $||\varphi||^{\mathbf{L}}_{M_1} < 1_L$. Now take an expansion $\mathbf{M}_2$ of $\mathbf{M}_1$ which is a safe $\mathbf{L}$ model of T_2. Clearly, $||\varphi||^{\mathbf{L}}_{M_2} < 1_L$, thus φ is unprovable in T_2. □

Remark 5.2.13 We can use the last lemma to quickly show that increasing the language of a theory T without adding any new axiom is a conservative extension: expand each safe model $\mathbf{M}$ of T over $\mathcal{J}_1$ to a safe model of T over $\mathcal{J}_2$ by interpreting all new constants by an arbitrary $m \in M$ and all new predicates e.g. the empty relation of the respective arity: $r(m_1, \ldots, m_n) = 0_L$ for all arguments. Observe that this is a safe $\mathbf{L}$-model of T over $\mathcal{J}_2$.

Definition 5.2.14 Let T be a theory in $\mathcal{C}\forall$ over a language $\mathcal{J}_1$.

(1) Let P be an n-ary predicate not in $\mathcal{J}_1$ and $\varphi(x_1, \ldots, x_n)$ a formula of $\mathcal{J}_1$ with n free variables $x_1, \ldots, x_n$. The *definition* of P by φ is the formula $(\forall x_1, \ldots, x_n)(P(x_1, \ldots, x_n) \equiv \varphi(x_1, \ldots, x_n))$.

(2) Let c be a constant not in $\mathcal{J}_1$ and let $(\exists x)\varphi(x)$ be a closed formula of $\mathcal{J}_1$ provable in T. The *witnessing of φ by c in T* is the formula $\varphi(c)$.

Theorem 5.2.15 (1) If T_2 results from T_1 by adding a definition of P then T_2 is a conservative extension of T_1.

(2) If T_2 results from T_1 by adding witnessing of $\varphi(x)$ (provided $T_1 \vdash (\exists x)\varphi(x)$), then T_2 is a conservative extension of T_1.

Proof: Use lemma 5.2.12. (1) Given a safe $\mathbf{L}$-model of T_1, expand it by r_P defined as follows:

$$r_P(m_1, \ldots, m_n) = ||\varphi||^{\mathbf{L}}_{M,v}$$

for an arbitrary v such that $v(x_i) = m_i$, $(i = 1, \ldots, n)$; an easy proof by induction shows that $(\mathbf{M}, r_P)$ is a safe $\mathbf{L}$-model. Indeed, for any formula ψ

containing P construct the formula ψ^* resulting from ψ by replacing each atom $P(t_1,\ldots,t_n)$ by $\varphi(t_1,\ldots,t_n)$, assuming that variables bound in φ do not occur in ψ; then show that $||\psi||^{\mathbf{L}}_{M,v} = ||\psi^*||^{\mathbf{L}}_{M,v}$ for each r.

(2) Expand $\mathbf{M}$ by m_c where m_c is any element such that for $v(x) = m$, $||\varphi||^{\mathbf{L}}_{M,v} = 1$. Again it is easy to show that $(\mathbf{M}, m_c)$ is $\mathbf{L}$-safe. Now ψ^* results from ψ by replacing c by a variable z not occuring in ψ. Then show that $||\psi||^{\mathbf{L}}_{M,v} = ||\psi^*||^{\mathbf{L}}_{M,v^*}$ where $v^* \equiv_z v$ and $v^*(z) = m_c$.) □

5.3. Axiomatizing Gödel Logic

In this short section we show that tautologies of G∀ (over all linearly ordered G-algebras) coincide with tautologies over the standard G-algebra $[0,1]_{\mathrm{G}}$ of truth functions; thus G∀ is recursively axiomatized. Note that we shall show in the next chapter that a similar result for Ł∀ and Π∀ is impossible; but in the next section we present a Pavelka-style axiomatization for Łukasiewicz predicate logic with truth constants.

We shall continue to assume throughout that all languages considered are at most countable (so that the set of all formulas is countable and so is the lattice $\mathbf{L}_T$ for each theory T).

Lemma 5.3.1 Let $\mathbf{L}$ be a countable linearly ordered G-algebra. Then there is a countable densely linearly ordered G-algebra $\mathbf{L}'$ such that $\mathbf{L} \subseteq \mathbf{L}'$ and the identical embedding of $\mathbf{L}$ into $\mathbf{L}'$ preserves all infinite suprema and infima existing in $\mathbf{L}$.

Proof: $\mathbf{L}'$ results from $\mathbf{L}$ by inserting a copy of the rational open unit interval into each "hole" (a pair (x, y) of elements of $\mathbf{L}$ such that y is a successor of x – there is no $z \in \mathbf{L}$ with $x < z < y$). In more detail, for $u \in \mathbf{L}$ put

$$C_u = \{(u,0)\} \text{ if } u \text{ has no successor in } \mathbf{L},$$

$$C_u = \{(u,r) | 0 \leq r < 1, r \text{ rational}\} \text{ otherwise.}$$

Let $\mathbf{L}' = \bigcup\{C_u | u \in L\}$ and for $(u,r),(v,s) \in L'$ put $(u,r) \leq (v,s)$ iff $u <_L v$ or ($u = v$ and $r \leq s$) (lexicographic product). Clearly $(L', \leq)$ is a densely ordered set with a least and greatest element; the mapping $f(u) = (u,0)$ embeds $\mathbf{L}$ isomorphically to $\mathbf{L}'$. (Call the image $\mathbf{L}_1$.) If $X \subseteq L_1$ and $y = \sup X$ in L_1 then there are two possibilities:
Case 1. $y \in X$, then $y = \max X$ in $\mathbf{L}_1$ and also in $\mathbf{L}$.
Case 2. $y \notin X$, then for each $z \in \mathbf{L}_1, z < y$, there is a $z' \in L_1$ such that $z < z' < y$. Assume now $t \in L' - L_1, t < y$. Then $t = (u,r)$ for an $u \in L$ and $r > 0$, thus $(u,0) \in L_1$, $(u,0) < y$ and for some v, $(v,0) \in L_1$,$(u,0) < (v,0) < y$; but then $t = (u,r) < (v,0) < y$. □

Lemma 5.3.2 Let $\mathbf{L}$ be a countable densely linearly ordered G-algebra. There is an isomorphism f of $\mathbf{L}$ onto the G-algebra $Q \cap [0,1]_G$ of rationals from $[0,1]$. As an embedding into $[0,1]_G$ f preserves all sups and infs existing in $\mathbf{L}$.

Proof: It is well known that any two countable densely linearly ordered sets are isomorphic (as ordered sets); let f be an order isomorphism of $\mathbf{L}$ onto $Q \cap [0,1]_G$ (Q is the set of all rational numbers). Then f is also an isomorphism of $\mathbf{L}$ and $Q \cap [0,1]_G$ as G-algebras since the operations $*$ (min) and $\Rightarrow$ are definable from the order. Thus f is an isomorphic embedding of $\mathbf{L}$ into $[0,1]_G$ as Heyting algebras. And since $Q \cap [0,1]$ is dense in $[0,1]$ f preserves infinite infima and suprema. Indeed, if $A \subseteq [0,1] \cap Q$ and $r \in Q$ is such that $r = \sup A$ in $[0,1] \cap Q$ then clearly $r = \sup A$ in $[0,1]$. $\square$

Theorem 5.3.3 *(Completeness)*[24] Let T be a theory over G$\forall$. $T \vdash \varphi$ iff $\|\varphi\|_{\mathbf{M}} = 1$ for each $[0,1]_G$-model $\mathbf{M}$ of T. ($\|\varphi\|_{\mathbf{M}}$ stands for $\|\varphi\|_{\mathbf{M}}^{[0,1]_G}$.)

Proof: If $T \vdash \varphi$ then $\|\varphi\|_{\mathbf{M}} = 1$ for each $[0,1]_G$-model of T by soundness; conversely, if $T \nvdash \varphi$ then there is a countable dense linearly ordered G-algebra $\mathbf{L}$ and an $\mathbf{L}$-model $\mathbf{M}$ of T such that $\|\varphi\|_{\mathbf{M}} < 1$. By the preceding lemma we may assume that $\mathbf{L}$ is in fact $[0,1]_G$ (or, if we want, $[0,1]_G \cap Q$). This is completes the proof. $\square$

Corollary 5.3.4 The following are equivalent:

(i) $T \vdash \varphi$ (pedantically, $T \vdash_{G\forall} \varphi$),

(ii) $\|\varphi\|_{\mathbf{M}} = 1$ for each $[0,1]_G$-model of T,

(iii) For each $[0,1]_G$-structure $\mathbf{M}$ and each $r \in [0,1]$, if $\|\alpha\|_{\mathbf{M}} \geq r$ for each axiom $\alpha \in T$ then $\|\varphi\|_{\mathbf{M}} \geq r$,

(iv) For each $[0,1]_G$-structure $\mathbf{M}$ there is an axiom $\alpha \in T$ such that $\|\alpha\|_{\mathbf{M}} \leq \|\varphi\|_{\mathbf{M}}$.

Proof: Clearly, (iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$ (i); and (i)$\Rightarrow$(iv) is soundness. Let $\alpha_1, \dots, \alpha_n \in T$ be all axioms of T used in a proof of φ; we claim that for each $\mathbf{M}$,

$$\min(\|\alpha_1\|_{\mathbf{M}}, \dots, \|\alpha_n\|_{\mathbf{M}}) \leq \|\varphi\|_{\mathbf{M}}.$$

[24] [103] and [195] are predecessors of this theorem; [193] has a completeness theorem very similar to ours.

This is verified by induction on the proof of φ. □

Recall 5.1.19 where we discussed the formula $(\forall x)(\varphi(x)\&\nu) \equiv$ $\equiv ((\forall x)\varphi(x)\&\nu)$; we now prove this formula (with $\wedge$ instead of &) in G∀ (in fact, in BL∀).

Lemma 5.3.5 BL∀ proves

$$(\forall x)(\varphi(x) \wedge \nu) \equiv ((\forall x)\varphi(x) \wedge \nu)$$

(ν not containing x freely).

Proof: Evidently, $\vdash ((\forall x)\varphi(x) \wedge \nu) \to (\forall x)\varphi(x)$ and $\vdash (\forall x)(\varphi(x) \wedge \nu) \to \nu$; thus we get the implication $\to$. Conversely, $\vdash ((\forall x)\varphi(x)\wedge\nu) \to (\varphi(x)\wedge\nu)$ (by (∀1) and properties of $\wedge$); generalizing and using (∀2) we get

$$\vdash ((\forall x)\varphi(x) \wedge \nu) \to (\forall x)(\varphi(x) \wedge \nu).$$

□

To close this section recall 5.1.14 where we proved the following formulas, assuming that ν does not contain x freely:

$$(\forall x)(\nu \to \varphi) \equiv (\nu \to (\forall x)\varphi), \quad (\forall x)(\varphi \to \nu) \equiv ((\exists x)\varphi \to \nu),$$

$$(\exists x)(\nu \to \varphi) \to (\nu \to (\exists x)\varphi), \quad (\exists x)(\varphi \to \nu) \to ((\forall x)\varphi \to \nu).$$

Now we prove that in Gödel logic this cannot be strengthened.

Lemma 5.3.6 The formulas

$$(\nu \to (\exists x)\varphi) \to (\exists x)(\nu \to \varphi), \quad ((\forall x)\varphi \to \nu) \to (\exists x)(\varphi \to \nu)$$

are not 1-tautologies of Gödel predicate logic.

Proof:

(1) Assume $0 < y < 1$ and let $\{x_n | n \in N\}$ be a sequence of positive numbers strictly less than y and such that $\sup x_n = y$. Then

$$\sup(y \Rightarrow x_n) = \sup x_n = y < 1,$$
$$y \Rightarrow \sup x_n = y \to y = 1.$$

Thus if $M = N$ (the set of natural numbers) $r_P(n) = x_n$, $r_Q(n) = y$ for all n, $m_c = 0$, and $\nu = Q(c)$ we get $\|\nu \to (\exists x)\varphi(x)\|_{\mathbf{M}} = 1$ but $\|(\exists x)(\nu \to \varphi(x))\|_{\mathbf{M}} = y < 1$.

(2) Similarly, let x_n be positive, $\inf_n x_n = 0 = y$; then $x_n \Rightarrow y = 0$, $\sup_n(x_n \Rightarrow y) = 0$, but $(\inf_n x_n) \Rightarrow y = 0 \Rightarrow 0 = 1$.

□

5.4. ŁUKASIEWICZ AND PRODUCT PREDICATE LOGIC

The main goal of this section is to present a predicate logic version of RPL — Rational Pavelka logic. This is because Łukasiewicz logic is not axiomatizable in the usual sense — there is no recursive system of axioms and deduction rules for which provability would equal 1-tautologicity over [0,1]. We shall prove this (and much more) in the next chapter. The main thing we have is Pavelka-style completeness, saying, besides other things, that 1-tautologies are exactly all formulas with provability degree 1. (This has some variants to be mentioned below.) For product logic we have only the completeness of Section 2, (w.r.t. safe models over all linearly ordered product algebras) — looking for a sort of completeness relative to models over $[0,1]_\Pi$ remains as a promising research task. We show how the embeddability result of Łukasiewicz into product logic extends to predicate calculus. (This gives immediately that predicate product logic is not axiomatizable (in the above sense.)

Let us investigate Łukasiewicz predicate logic Ł∀. First we present one provability.

Lemma 5.4.1 Let φ be a formula. The logic Ł∀ proves

$$(\exists x)\varphi \equiv \neg(\forall x)\neg\varphi$$

Proof: By 5.1.20, BL∀ $\vdash (\exists x)\varphi \to \neg(\forall x)\neg\varphi$ and BL∀ $\vdash \neg(\exists x)\varphi \to (\forall x)\neg\varphi$. Now Ł $\vdash (\neg\alpha \to \beta) \to (\neg\beta \to \alpha)$ (see 3.1.1 (1), (2)), thus Ł∀ $\vdash \neg(\forall x)\neg\varphi \to (\exists x)\varphi$. □

Remark 5.4.2 It follows that $\exists$ is definable in Ł∀ from $\forall$. Thus an alternative presentation of Ł∀ is to allow only connectives $\neg, \to$ and the quantifier $\forall$ (taking $\bar{0}, \&, \veebar, \wedge, \vee, \exists$ as defined symbols), take axioms (Ł1)–(Ł4) for the propositional calculus and (∀1),(∀2) for predicate calculus; (∃1), (∃2) and (∀3) become provable as the following chains of equivalences and implications show:
$(\varphi(t) \to (\exists x)\varphi(x)) \equiv (\varphi(t) \to \neg(\forall x)\neg\varphi(x)) \equiv ((\forall x)\neg\varphi(x) \to \neg\varphi(t))$;
$(\forall x)(\varphi \to \nu) \equiv (\forall x)(\neg\nu \to \neg\varphi) \equiv (\neg\nu \to (\forall x)\neg\varphi) \equiv (\neg(\forall x)\neg\varphi \to \nu) \equiv ((\exists x)\varphi \to \nu)$.

$(\forall x)(\varphi(x) \vee \nu) \to (\forall x)((\nu \to \varphi(x)) \to \varphi(x)) \to$
$\to [(\forall x)(\nu \to \varphi(x)) \to (\forall x)\varphi(x)] \to$
$\to [(\nu \to (\forall x)\varphi(x)) \to (\forall x)\varphi(x)] \to [((\forall x)\varphi(x)) \vee \nu]$. (As always, ν is a formula not containing x freely. Note that we have used 5.1.14, but this was proved in BL$\forall$ without using ($\forall$3).)

We now present a predicate logic variant of Rational Pavelka logic and prove Pavelka-style completeness. Then we obtain some further results for Ł$\forall$.

Definition 5.4.3 An MV-algebra **A** is said to *contain the rational unit interval* if the MV-algebra $Q \cap [0,1]_{Ł}$ of rational numbers from $[0,1]$ (with the usual truth functions) is a subalgebra of **A**. (More generally, **A** may contain an isomorphic copy of $Q \cap [0,1]_{Ł}$; but then we may assume, without loss of generality, that **A** contains the unit interval, i.e. that the isomorphism is just identity.)

Definition 5.4.4 *The logic RPL$\forall$ – Rational Pavelka predicate logic.* Extend (in 5.1.1) logical symbols by truth constants $\bar{r}$ for each rational $r \in [0,1]$ and extend the definition of formulas by adding the clause saying: for each $r, \bar{r}$ is a formula. In 5.1.3, add the condition

$$\|\bar{r}\|^{\mathbf{L}}_{M,v} = r,$$

assuming that **L** contains the rational unit interval.

The logic RPL$\forall$ (over a predicate language $\mathcal{J}$) has formulas, **L**-structures and truth definition as above (**L** containing the rational unit interval), axioms are:

– axioms of RPL (with the present notion of a formula), i.e. (Ł1)–(Ł4) and the bookkeeping axioms, of 3.3.1
– and the axioms ($\forall$1),($\forall$2) as above.

The deduction rules are modus ponens and generalization.

Definition 5.4.5 A *theory* over RPL$\forall$ is (of course) a set of formulas — special axioms of T. We may assume all special axioms to be closed. In an analogy to 3.3.4 we define

(1) the *truth degree* of φ over T to be

$$\|\varphi\|_T = \inf\{\|\varphi\|_{\mathbf{M}} \mid M \text{ a model of } T\}$$

("model" meaning "model over the standard MV-algebra $[0,1]_{Ł}$" and $\|\varphi\|_{\mathbf{M}}$ meaning $\|\varphi\|^{[0,1]_{Ł}}_{\mathbf{M}}$);

(2) the *provability degree* of φ in T to be

$$|\varphi|_T = \sup\{r|\, T \vdash (\bar{r} \to \varphi)\}.$$

Lemma 5.4.6 (Soundness.) For each theory T over RPL$\forall$ and each formula φ, $|\varphi|_T \leq \|\varphi\|_T$.

Proof: This immediately follows from the usual formulation of soundness saying that if $T \vdash \psi$ then $\|\psi\|_{\mathbf{M}} = 1$ for each model $\mathbf{M}$ of φ, cf. 5.1.12 □

Lemma 5.4.7 If a theory T over RPL$\forall$ is consistent then it has a model over the standard MV-algebra $[0, 1]_{\text{Ł}}$.

Proof: An easy inspection of 5.2.7 shows that that lemma is also valid for any consistent theory T over RPL$\forall$: there is a consistent complete Henkin extension $\hat{T}$ of T. This gives a model of T over an MV-algebra $\mathbf{L}$ containing the rational unit interval as in 5.2.9; but we want a model over $[0, 1]_{\text{Ł}}$. To this end we take M to be the set of all constants of $\hat{T}$ and interpret each constant by itself (as in 5.2.8) but modify the interpretation of predicates as follows:

$$r_P(c_1, \ldots, c_n) = |P(c_1, \ldots, c_n)|_{\hat{T}} = \sup\{r|\, \hat{T} \vdash \bar{r} \to P(c_1, \ldots, c_n)\}.$$

Then inspect Lemma 3.3.8 to show that, for each closed formula φ,

$$|\varphi|_{\hat{T}} = \sup\{r|\, \hat{T} \vdash (\bar{r} \to \varphi)\} = \inf\{\bar{r}|\, \hat{T} \vdash (\varphi \to \bar{r})\}$$

and that the provability degree commutes with connectives and quantifiers, i.e.

$$|\neg\varphi|_{\hat{T}} = 1 - |\varphi|_{\hat{T}},\ |\varphi \to \psi|_{\hat{T}} = |\varphi|_{\hat{T}} \Rightarrow |\psi|_{\hat{T}},$$
$$|(\forall x)\varphi|_{\hat{T}} = \inf_c |\varphi(c)|_{\hat{T}} \text{ (c varying over all constants)}.$$

For connectives the proof is as in 3.3.8; for quantifiers we have to present a proof now.

Clearly, $\hat{T} \vdash \bar{r} \to (\forall x)\varphi(x)$ implies $\hat{T} \vdash \bar{r} \to \varphi(c)$ for each c; thus $|(\forall x)\varphi(x)|_{\hat{T}} \leq \inf_c |\varphi(c)|_{\hat{T}}$. Conversely, if $r < \inf_c |\varphi(c)|_{\hat{T}}$, i.e. $\hat{T} \vdash \bar{r} \to \varphi(c)$ for all c, then $\hat{T} \vdash \bar{r} \to (\forall x)\varphi(x)$; otherwise we would have $\hat{T} \nvdash \bar{r} \to (\forall x)\varphi(x)$, $\hat{T} \nvdash (\forall x)(\bar{r} \to \varphi(x))$ and hence, by the Henkin property, $\hat{T} \nvdash \bar{r} \to \varphi(c)$ for some c — a contradiction. Thus if $r < \inf_c |\varphi(c)|_{\hat{T}}$ then $r \leq |(\forall x)\varphi(x)|_{\hat{T}}$, which gives, together with the above, $|(\forall x)\varphi(x)|_{\hat{T}} = \inf_c |\varphi(c)|_{\hat{T}}$.

This shows that $\|\varphi\|_{\mathbf{M}} = |\varphi|_{\hat{T}}$ for each closed formula φ; thus $\mathbf{M}$ is a model of T. □

Remark 5.4.8 We can use the preceding proof to show that if an MV-algebra A contains the rational unit interval then the mapping $f(a) = \sup\{r | r$ rational, $r \leq a\}$ is a homomorphism of A into $[0,1]_{Ł}$ preserving all sups (and infs) existing in A. The interested reader may elaborate details.

Lemma 5.4.9 If $T \nvdash \bar{r} \to \varphi$ then the theory $T \cup \{\varphi \to \bar{r}\}$ is consistent.

Proof: See lemma 3.3.7. □

Theorem 5.4.10 (*Completeness.*) For each theory T over RPL∀ and for each formula φ,

$$|\varphi|_T = \|\varphi\|_T,$$

i.e. the provability degree of φ equals the truth degree of φ.

Proof: Since we have soundness, $|\varphi|_T \leq \|\varphi\|_T$, it remains to prove $\|\varphi\|_T \leq |\varphi|_T$. For this purpose let $|\varphi|_T < r$; it is enough to show that there is a model $\mathbf{M}$ of T such that $\|\varphi\|_{\mathbf{M}} \leq r$. But since $|\varphi|_T < r$, T does not prove $\bar{r} \to \varphi$, hence by the preceding lemma, $T \cup \{\varphi \to \bar{r}\}$ is consistent and by 5.4.7 has a model $\mathbf{M}$; then $\|\varphi \to \bar{r}\|_{\mathbf{M}} = 1$, thus $\|\varphi\|_{\mathbf{M}} \leq r$ and hence $\|\varphi\|_T \leq r$. □

In particular, φ is true in each model of T iff for each $r < 1$, $T \vdash \bar{r} \to \varphi$. We can get a similar characterization not mentioning truth constants.

Theorem 5.4.11 For each theory T and formula φ, φ is 1-true in each model of T iff for each natural $n \geq 1$,

$$T \vdash \varphi \underline{\vee} \varphi^n$$

Proof: By soundness, $T \vdash \varphi \underline{\vee} \varphi^n$ implies that for each model $\mathbf{M}$ of T, $\|\varphi\|_{\mathbf{M}} \geq \frac{n}{n+1}$. (Indeed, let $\|\varphi\|_{\mathbf{M}} = x$. Then either $x^n = 0$ and $x + x^n = x + 0 = 1$, thus $x = 1$, or otherwise $x^n = nx - (n-1)$ and $1 \leq x + nx - n + 1$, hence $x \geq \frac{n}{n+1}$.) Thus if for each n, $T \vdash \varphi \underline{\vee} \varphi^n$ then for each model $\mathbf{M}$ of T, $\|\varphi\|_{\mathbf{M}} = 1$.

Conversely assume that for some n, $T \nvdash \varphi \underline{\vee} \varphi^n$, i.e. $T \nvdash \neg\varphi \to \varphi^n$. Then the theory $T \cup \{\varphi^n \to \neg\varphi\}$ is consistent (by the same argument as in 3.3.7), hence it has a model $\mathbf{M}$. Then $\|\varphi^n \to \neg\varphi\|_{\mathbf{M}} = 1$, thus $\|\varphi^n\|_{\mathbf{M}} \leq \|\neg\varphi\|_{\mathbf{M}}$, hence $\|\varphi\|_{\mathbf{M}} < 1$. (If $\|\varphi\|_{\mathbf{M}} = 1$ then $\|\varphi^n\| = 1$ and $\|\neg\varphi\|_{\mathbf{M}} = 1$, which is absurd.) □

We shall come back to Łukasiewicz logic and show that the same theorem holds for it.

*

First we show some further formulas to be provable in Ł∀. We are going prove the formula $(\forall x)(\varphi \& \nu) \equiv ((\forall x)\varphi \& \nu)$ (ν not containing x freely). Recall that by 5.1.19 we know that it is a tautology over $[0,1]_{Ł}$; thus we shall know that it is a 1-tautology over each linearly ordered MV-algebra.

Lemma 5.4.12 Ł∀ proves

$$(\forall x)(\alpha(x) \vee \beta) \equiv (((\forall x)\alpha(x)) \vee \beta),$$

β not containing x freely.

Proof: Evidently, $\vdash (\forall x)\alpha(x) \to (\forall x)(\alpha(x) \vee \beta)$ and $\vdash \beta \to (\forall x)(\alpha(x) \vee \beta)$; thus $\vdash ((\forall x)\alpha(x) \vee \beta) \to (\forall x)(\alpha(x) \vee \beta))$. The converse implication (the axiom (∀3)) has been proved in 5.4.2. □

Lemma 5.4.13 The following are propositional 1-tautologies of Ł:

(i) $(\neg q \to p) \to [((p \& q) \underline{\vee} \neg q) \equiv p]$,

(ii) $(p \to q) \to [((p \underline{\vee} \neg q) \& q \equiv p]$.

Proof: Recall that by 3.1.13, in Ł, $(p \& q)\underline{\vee}\neg q$ is equivalent to $p \vee \neg q$ and $(p\underline{\vee}\neg q)\&q$ is equivalent to $p \wedge q$; thus we claim

$$Ł \vdash (\neg q \to p) \to ((p \vee \neg q) \equiv p),$$
$$Ł \vdash (p \to q) \to ((p \wedge q) \equiv p),$$

which is obvious. □

Lemma 5.4.14 Ł∀ proves $(\forall x)(\varphi(x) \& \nu) \equiv ((\forall x)\varphi(x) \& \nu)$, ν not containing x freely.

Proof: $\leftarrow$ is easy; use $\vdash ((\forall x)\varphi \& \nu) \to (\varphi \& \nu)$, generalize and shift ∀.

Conversely, let $(\forall x)(\varphi(x) \& \nu)$ be γ; we prove $\gamma \to ((\forall x)\varphi \& \nu)$. We have

(1) $\vdash (\forall x)(\gamma \to (\varphi(x) \& \nu))$.
Now, by 5.4.13 (i) and (1),

(2) $\vdash (\neg \nu \to \varphi(x)) \to [(\gamma \underline{\vee} \neg \nu) \to \varphi(x)]$,

(3) $\vdash (\varphi(x) \to \neg\nu) \to ((\varphi(x)\&\nu) \to \bar{0})$, thus

(4) $\vdash (\varphi(x) \to \neg\nu) \to (\gamma \to \bar{0})$.
Thus by (2) and (4)

(5) $\vdash [(\gamma\underline{\vee}\neg\nu) \to \varphi(x)] \vee (\gamma \equiv \bar{0})$.
Generalize and use 5.4.12:

(6) $\vdash (\forall x)[(\gamma\underline{\vee}\neg\nu) \to \varphi(x)] \vee (\gamma \equiv \bar{0})$,
thus

(7) $\vdash [(\gamma\underline{\vee}\neg\nu) \to (\forall x)\varphi(x)] \vee (\gamma \equiv \bar{0})$.
But since $\vdash (\gamma \to \nu)$ by (1), we can use 5.4.13 (ii) and get

(8) $\vdash [\gamma \to ((\forall x)\varphi(x)\&\nu)] \vee \gamma \equiv \bar{0}$,
thus we get $\vdash \gamma \to ((\forall x)\varphi(x)\&\nu)$ as desired.

□

Lemma 5.4.15 Ł$\forall$ proves the following:

(i) $(\nu \to (\exists x)\varphi) \to (\exists x)(\nu \to \varphi)$,

(ii) $((\forall x)\varphi \to \nu) \to (\exists x)(\varphi \to \nu)$.

Proof:

(i) Ł$\forall$ proves the following chain of implications (the first using 5.4.14):
$\neg(\exists x)(\nu \to \varphi) \to (\forall x)(\nu\&\neg\varphi) \to [\nu\&(\forall x)\neg\varphi] \to$
$\to \neg[\nu \to \neg(\forall x)\neg\varphi] \to \neg[\nu \to (\exists x)\varphi]$. Thus Ł$\forall$ proves
$\neg(\exists x)(\nu \to \varphi) \to \neg[\nu \to (\exists x)\varphi]$ and hence it proves
$[\nu \to (\exists x)\varphi] \to (\exists x)(\nu \to \varphi)$.

(ii) The following is provable, using the preceding provability (i):
$((\forall x)\varphi \to \nu) \to (\neg\nu \to \neg(\forall x)\varphi) \to (\neg\nu \to (\exists x)\neg\varphi) \to$
$\to (\exists x)(\neg\nu \to \neg\varphi) \to (\exists x)(\varphi \to \nu)$.

□

Lemma 5.4.16 For each natural $n \geq 1$,

(1) Ł$\forall \vdash (\exists x)\varphi^n \equiv ((\exists x)\varphi)^n$,

(2) Ł$\forall \vdash (\exists x)n\varphi \equiv n((\exists x)\varphi)$.

Proof: (1) is proved in the same way as 5.1.18 (10) (the case $n = 2$). We prove (2) also for $n = 2$, letting the reader generalize for arbitrary n. From $\vdash \varphi \to (\exists x)\varphi$ we get

$\vdash (\varphi \veebar \varphi) \to ((\exists x)\varphi \veebar (\exists x)\varphi)$ (thus $\vdash 2\varphi \to 2(\exists x)\varphi$),

and by generalization and 5.1.13 (2) we get

$\vdash (\exists x)2\varphi \to 2(\exists x)\varphi$.

Conversely, $\vdash (\exists x)\varphi \to (\exists x)\varphi$ gives $\vdash (\exists x)((\exists x)\varphi \to \varphi)$ by 5.4.15 (ii). Moreover,

$\vdash ((\exists x)\varphi \to \varphi)^2 \to (2(\exists x)\varphi \to 2\varphi)$ by propositional Ł[25],

generalizing and applying 5.1.16 (6) we get

$\vdash (\exists x)((\exists x)\varphi \to \varphi)^2 \to (\exists x)(2(\exists x)\varphi \to 2\varphi)$

and by modus ponens, 5.1.18 and 5.1.14 (3) we get

$\vdash 2(\exists x)\varphi \to (\exists x)2\varphi$. □

Lemma 5.4.17 Let T be a theory over Ł∀, let $(\exists x)\varphi(x)$ be a closed formula in the language of T, let c be a new constant (not in the language of T). If $T \vdash (\exists x)\varphi(x)$ then the theory $T' = T \cup \{\varphi(c)\}$ is consistent (moreover, T' is a conservative extension of T).

Proof: Let γ be closed and let $T' \vdash \gamma$, thus for some n, $T \vdash (\varphi(c))^n \to \gamma$. Replacing c by a variable y and inspecting the last T-proof we get $T \vdash (\varphi(y)^n \to \gamma)$, thus by generalizing and moving $\forall$ to the antecedent as $\exists$, we get $T \vdash (\exists y)(\varphi(y))^n \to \gamma$, thus $T \vdash ((\exists y)\varphi(y))^n \to \gamma$. But since $T \vdash (\exists y)\varphi(y)$ we get $T \vdash ((\exists y)\varphi(y))^n$ and hence $T \vdash \gamma$. □

Definition 5.4.18 A theory T is *maximal* if for each closed φ, $T \nvdash \varphi$ implies that $T \cup \{\varphi\}$ is inconsistent (i.e. for some n, $T \vdash \varphi^n \to 0$, thus $T \vdash \neg(\varphi^n)$).

Lemma 5.4.19 If T is maximal then T is complete.

Proof: Assume T maximal and φ, ψ closed. If $T \nvdash (\varphi \to \psi)$ and $T \nvdash (\psi \to \varphi)$ then $T \vdash (\varphi \to \psi)^n \to \bar{0}$ and $T \vdash (\psi \to \varphi)^n \to \bar{0}$ (for a suitable n), hence $T \vdash ((\varphi \to \psi)^n \vee (\psi \to \varphi)^n) \to \bar{0}$ and thus $T \vdash \bar{0}$; T is inconsistent), a contradiction. □

Definition 5.4.20 A linearly ordered MV-algebra is *Archimedean* if for each $a > 0$ there is a natural $n \geq 1$ such that $na = 1$.

Lemma 5.4.21 Each Archimedean linearly ordered MV-algebra **A** is isomorphic to a subalgebra of the standard MV-algebra $[0, 1]_{Ł}$.

[25] Show that $(p \to q)^2 \to (2p \to 2q)$ is a tautology.

Proof: Construct the Chang linearly ordered Abelian group $\mathbf{G}$ such that (for a positive e, $\mathbf{A}$ is $\mathrm{MV}(\mathbf{G}, e)$ (see 3.2.9 and its proof). It is easy to show that $\mathbf{G}$ is an Archimedean o-group, due to archimedicity of $\mathbf{A}$. Indeed, for each positive $b \in G$, there is an n such that $ne > b$ — this follows directly from the construction. Thus if $a \geq e$ we get $na > b$; if $0 < a < e$ then for some m we get $ma = e$) in $\mathbf{A}$, thus $ma \geq e$ in G and $(mn)a \geq b$. By Hölder theorem 1.6.14, $\mathbf{G}$ is isomorphic to a subgroup of the additive group of reals — and we may assume that the isomorphic embedding ι sends e to 1, thus $\iota(e) = 1$. Thus ι makes $\mathbf{A}$ a subalgebra of $[0, 1]_{Ł}$. □

Lemma 5.4.22 If T is a consistent theory (over Ł∀) then T has a maximal consistent Henkin extension.

Proof: Put $T = T_0$ and let $\varphi_0, \varphi_1, \ldots$ be a sequence enumerating all closed formulas. Assume T_n has been constructed and construct T_{n+1}.

(Case 1) If $T_n \cup \{\varphi_n\}$ is consistent put $T_{n+1} = T_n \cup \{\varphi_n\}$.

(Case 2) If $T_n \cup \{\varphi_n\}$ is inconsistent and φ_n does not have the form $(\forall x)\beta(x)$ then put $T_{n+1} = T_n$; if φ_n is $(\forall x)\beta(x)$ then take an m such that $T \vdash \neg(\varphi_n^m)$, i.e. $T \vdash m(\exists x)\neg\beta(x)$, consequently $T \vdash (\exists x)m(\neg\beta(x))$ (by 5.4.16 (2)). Take a new constant c and put $T_{n+1} = T_n \cup \{m(\neg\beta(c))\}$. Finally put

$$\hat{T} = \bigcup_n T_n.$$

Clearly, $\hat{T}$ is a consistent maximal extension of T. If $\hat{T} \not\vdash (\forall x)\beta(x)$ and $(\forall x)\beta(x) = \varphi_n$ then $T_n \cup \{\varphi_n\}$ was inconsistent (Case 2) and we put $m(\neg\beta(c))$ into T_{n+1} and hence into $\hat{T}$; since $\hat{T}$ is consistent, $\hat{T}$ does not prove $\beta(c)$. We have shown $\hat{T}$ to be Henkin.

□

Lemma 5.4.23 If T is a consistent maximal Henkin theory (over Ł∀) then T has a model over the standard MV-algebra $[0, 1]_{Ł}$.

Proof: By 5.2.8 T has a model over $\mathbf{L}_T$ — the linearly ordered MV-algebra of classes of equivalent formulas. We show that $\mathbf{L}_T$ is Archimedean; this will give the result using 5.4.21.

Indeed, let $[\varphi]_T > 0_{L_T}$, i.e. $T \not\vdash \varphi \rightarrow 0$. By maximality of T, $T \cup \{\neg\varphi\}$ is inconsistent, thus for some n, $T \vdash \neg(\neg\varphi)^n$, $T \vdash n(\neg\neg\varphi)$, $T \vdash n\varphi$. This gives $n[\varphi]_T = 1_{L_T}$, which is archimedicity.

To get the statement of the lemma, one has to strengthen Lemma 5.4.21: the isomorphism of an Archimedean linearly ordered MV-algebra **A** to a subalgebra of the standard MV-algebra must be proven to preserve all infinite sups and infs existing in **A**. But this can be done as follows: Either **A** is discretely ordered and then the statement is trivial, or it is densely ordered and then it suffices to show that its image $\mathbf{A}'$ – a subalgebra of the standard MV-algebra – is dense (not only in itself but) in [0,1]. But this is clear: since $\mathbf{A}'$ has no least positive element, elements of $\mathbf{A}'$ have an accumulation point, hence differences of elements of $\mathbf{A}'$ (that themselves are elements of $\mathbf{A}'$) are arbitrarily small. And knowing that $\mathbf{A}'$ is dense in [0,1] it is easy to show that if X is a subset of $\mathbf{A}'$ and x is its sup (inf) in $\mathbf{A}'$ then x is $\sup(X)$ $(\inf(X))$ also in the sense of [0,1] (cf. 5.3.2). □

Theorem 5.4.24 [26] Each consistent theory over Ł∀ has a model over $[0,1]_{Ł}$.

Proof: Immediate from 5.4.22 and 5.4.23. □

Theorem 5.4.25 *(Completeness.)*[27] Let T be a theory over Ł∀; let φ be a closed formula φ is 1-true in each model of T over $[0,1]_{Ł}$ iff for each $n \geq 1$, T proves $\varphi\underline{\vee}\varphi^n$. (Cf. 5.4.11.)

Proof: From 5.4.24 exactly as in 5.4.11. □

Remark 5.4.26 Let φ be a closed formula of Ł∀. φ is provable in Ł∀ iff it is provable in RPL∀. This was recently proved and will appear as [90].

We shall now characterize 1-tautologies of Ł∀ using finitely valued Łukasiewicz logics. We shall use this result in Chapter 7.

Definition 5.4.27 . Let, for a moment, $\mathbf{L}_n (n \geq 2)$ be the subalgebra of the standard MV-algebra $[0,1]_{Ł}$ with the domain $\{0, \frac{1}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$ (n-element MV-algebra).

Clearly, each $[0,1]_{Ł}$-tautology is a $\mathbf{L}_n$-tautology for each n. We are going to show the converse: if for each n, φ is a $\mathbf{L}_n$-tautology then φ is a $[0,1]_{Ł}$-tautology. We need some preparation.

[26] See [13].
[27] See [95].

Definition 5.4.28 (1) Assume a fixed language with n predicates $P, \cdots, Q$. Let $\mathbf{M} = (M, r_P, \cdots, r_Q)$, $\mathbf{M}' = (M, r'_P, \cdots, r'_Q)$ be two models with the same domain. Put
$d(r_P, r'_P) = \sup\{|r_P(a) - r'_P(a)| \, |a \in M^{ar(P)}\}$ (*ar* being the arity), analogously for other predicates;
$d(\mathbf{M}, \mathbf{M}') = d(r_P, r'_P) + \cdots + d(r_Q, r'_Q)$.
(Clearly, d is a metric on the set of all models with the domain M.)

(2) Define the complexity of a formula φ as follows: for atomic φ, $\tau(\varphi) = 0$; $\tau(\bar{0}) = 0$; $\tau(\varphi \to \psi) = \max(\tau(\varphi), \tau(\psi)) + 1$; $\tau((\forall x)\varphi) = \tau(\varphi)$. (Recall that other connectives are definable.)

Lemma 5.4.29 [28] Let $\varepsilon > 0$, let $\mathbf{M}, \mathbf{M}'$ be as above and let, let φ be a formula. If $d(\mathbf{M}, \mathbf{M}') < \varepsilon/2^{\tau(\varphi)}$ then for each evaluation v
$| \, \|\varphi\|_{\mathbf{M},v} - \|\varphi\|_{\mathbf{M}',v}| < \varepsilon$.

Proof: Assume that $\mathbf{M}'$ differs from $\mathbf{M}$ only by the interpretation of P (thus $d(\mathbf{M}, \mathbf{M}') = d(r_P, r'_P)$. (The general result then follows by n-fold use of this particular case, using the triangle inequality.)

We show by induction that $| \, \|\varphi\|_{\mathbf{M},v} - \|\varphi\|_{\mathbf{M},v}| \leq d(r_P, r'_P) \cdot 2^{\tau(\varphi)}$.

The assertion is obvious for atomic φ and for φ being $\bar{0}$. Assume the assertion for φ and ψ and observe that
$|(x \Rightarrow y) - (x' \Rightarrow y')| \leq |x - x'| + |y - y'|$.

Thus $| \, \|\varphi \to \psi\|_{\mathbf{M},v} - \|\varphi \to \psi\|_{\mathbf{M}',v}\|| \leq | \, \|\varphi\|_{\mathbf{M},v} - \|\varphi\|_{\mathbf{M}',v}| + | \, \|\psi\|_{\mathbf{M},v} - \|\psi\|_{M',v}| \leq d(r_P, r'_P) \cdot (2^{\tau(\varphi)} + 2^{\tau(\psi)}) \leq$
$\leq d(r_P, r'_P) \cdot 2^{\max(\tau(\varphi),\tau(\psi))+1} = d(r_P, r'_P) \cdot 2^{\tau(\varphi \to \psi)}$.

Finally we investigate $(\exists x)\varphi$. Since $\|(\exists x)\varphi\|_{\mathbf{M},v} = \sup_{w \equiv_x v} \|\varphi\|_{\mathbf{M},w}$ we get $| \, \|(\exists x)\varphi\|_{\mathbf{M},v} - \|(\exists x)\varphi\|_{\mathbf{M}',v}| \leq \sup_{w \equiv_x v} |\|\varphi\|_{\mathbf{M},w} - \|\varphi\|_{\mathbf{M}',w}| \leq d(r_P, r'_P) \cdot 2^{\tau(\varphi)} = d(r_P, r'_P) \cdot 2^{\tau((\exists x)\varphi)}$.

Thus if $d(r_P, r'_P) \leq \varepsilon/2^{\tau(\varphi)}$ we get $|\|\varphi\|_{\mathbf{M},v} - \|\varphi_{\mathbf{M}',v}| < \varepsilon$. □

Theorem 5.4.30 A formula φ of $L\forall$ is a $[0,1]_{\text{Ł}}$-tautology iff for each $n \geq 2$, φ is a $\mathbf{L}_n$-tautology.

Proof: Assume φ is not a $[0,1]_{\text{Ł}}$-tautology. Thus there is a model $\mathbf{M}$ and an evaluation v such that $\|\varphi\|_{M,v} < 1 - \varepsilon$ for some $\varepsilon > 0$. Let k be such that

$$\frac{1}{k} < \frac{\varepsilon}{n \cdot 2^{\tau(\varphi)}}.$$

[28] See [68].

For each $a \in M^{ar(P)}$, let $r'_P(a)$ be a value $\frac{i}{k}$ where i is such that $\frac{i}{k}$ is nearest to $r_P(a)$. Thus $d(r_P, r'_P) \leq \varepsilon/(n \cdot 2^{\tau(\varphi)})$. Similarly for other predicates; we get a structure $\mathbf{M}' = (M, r'_P, \ldots, r'_Q)$ such that $d(\mathbf{M}, \mathbf{M}') < \varepsilon/2^{\tau(\varphi)}$, thus $| \, \|\varphi\|_{\mathbf{M},v} - \|\varphi\|_{\mathbf{M}',v}| < \varepsilon$, and consequently $\|\varphi\|_{\mathbf{M}',v} < 1$. $\mathbf{M}'$ is a $\mathbf{L}_k$-structure. This completes the proof. □

*

In the rest of this section we investigate the product logic $\Pi\forall$. Recall that its axioms are the axioms of the product propositional logic (BL plus $\Pi 1, \Pi 2$) and the axioms $(\forall 1)$, $(\forall 2)$, $(\forall 3)$, $(\exists 1)$, $(\exists 2)$ for quantifiers. Recall also 5.1.14 and 5.1.15 where we proved in BL$\forall$ two equivalences and two implications on "moving quantifiers over an implication"; we know that in Ł$\forall$, both implications can be converted (and are provable in Ł$\forall$) but none of the converted implications is a G$\forall$-tautology.

Lemma 5.4.31 Assume x not free in ν.

(1) The formula $(\nu \to (\exists x)\varphi) \to (\exists x)(\nu \to \varphi)$ is a 1-tautology of $\Pi\forall$.

(2) The formula $((\forall x)\varphi \to \nu) \to (\exists x)(\varphi \to \nu)$ is not a 1-tautology of $\Pi\forall$.

Proof:

(1) Take a $[0,1]_\Pi$-model $\mathbf{M}$ and a fixed evaluation w_0; let w vary over all evaluations $\equiv_x w_0$. Put $\|\nu\|_{w_0} = a$, $\|\varphi\|_w = b_w$. Then $\|\nu \to (\exists x)\varphi\|_{w_0} = a \Rightarrow \sup_w b_w = \sup_w(a \Rightarrow b_w)$ since the Goguen implication is left continuous (in fact, it is continuous in all points except $(0,0)$); thus $\|\nu \to (\exists x)\varphi\|_{w_0} = \|(\exists x)(\nu \to \varphi)\|_{w_0}$.

(2) The counter-example for G$\forall$ in 5.3.6 (2) also works for $\Pi\forall$.

□

Remark 5.4.32 Note that by 5.1.19 the formula $(\forall x)(\varphi \& \nu) \equiv ((\forall x)\varphi \& \nu)$ is a 1-tautology (over $[0,1]_\Pi$). It appears to be an open problem whether this formula and the formula from 5.4.31 (1) is provable in $\Pi\forall$.

We now extend the embedding of Ł into Π to an embedding of Ł$\forall$ into $\Pi\forall$. We just generalize 4.1.15–4.1.17.

Definition 5.4.33 Given a predicate language, extend it by a new unary predicate Q and a new constant c_0 (thus $Q(c_0)$ is a new closed formula). For each Ł∀-formula not containing Q, c_0, define its translation φ^{Π} as follows:
$(\bar{0})^{\Pi}$ is $Q(c_0)$;
$(P(t_1, \ldots, t_n))^{\Pi}$ is $P(t_1, \ldots, t_n) \vee Q(c_0)$ for each atomic formula $P(t_1, \ldots$ $\ldots, t_n)$;
$(\varphi \rightarrow \psi)^{\Pi}$ is $\varphi^{\Pi} \rightarrow \psi^{\Pi}$; $(\varphi \& \psi)^{\Pi}$ is $Q(c_0) \vee (\varphi^{\Pi} \odot \psi^{\Pi})$;
$((\forall x)\varphi)^{\Pi}$ is $(\forall x)(\varphi^{\Pi})$.

Lemma 5.4.34 (1) Under the previous notation, φ is a 1-tautology of Ł∀ iff $\neg\neg Q(c) \rightarrow \varphi^{\Pi}$ is a 1-tautology of Π∀.

(2) Furthemore, let T be a theory and let $T^{\Pi} = \{\neg\neg Q(c_0)\} \cup \{\alpha^{\Pi} | \alpha \in T\}$. Then φ is true in all Ł∀-models of T iff φ^{Π} is true in all Π∀-models of T^{Π} (all models over $[0,1]_{Ł}$, $[0,1]_{\Pi}$ respectively).

Proof: Let $f_a(x) = a^{1-x}$ as in 4.1.14. For each $[0,1]$-model $\mathbf{M} =$ $= \langle M, (r_P)_P, (m_c)_c \rangle$ define a new $[0,1]$-model of the extended language by a letting $m_{c_0} \in M$ be arbitrary (but fixed), $r_Q(m_{c_0}) = a$, $r_Q(m) = 0$ for $m \in M, m \neq m_c$;

$$\mathbf{M}'_a = \langle M, r_Q, (r'_P)_P, (m_c)_c, m_{c_0} \rangle,$$

where $r'_P(m_1, \ldots) = \max(a, r_P(m_1, \ldots))$,

$$f_a(\mathbf{M}) = \langle M, r_Q, (r''_P)_P, (m_c)_c, m_{c_0} \rangle,$$

where $r''_P(m_1, \ldots, m_n) = f_a(r_P(m_1, \ldots, m_n))$.

Then show, for each evaluation w of object variables, that

$$\begin{aligned} f_a(\|\varphi\|^{[0,1]_{Ł}}_{M,w}) &= \|\varphi^{\Pi}\|^{[0,1]_{\Pi}}_{f_a(M),w}; \\ \|\varphi^{\Pi}\|^{[0,1]_{\Pi}}_{M,w} &= \|\varphi^{\Pi}\|^{[0,1]_{\Pi}}_{M'_a,w}. \end{aligned}$$

This gives the conclusion as in 4.1.16. □

Corollary 5.4.35 The mapping assigning to each formula φ not containing Q, c_0, the formula

$$\neg\neg Q(c_0) \rightarrow \varphi^{\Pi}$$

is a faithful embedding of Ł∀ into Π∀ with respect to 1-tautologicity: φ is an Ł∀-tautology (over $[0,1]_{Ł}$) iff $\neg\neg Q(c_0) \rightarrow \varphi^{\Pi}$ is a Π∀-tautology (over $[0,1]_{\Pi}$). (The reader may formulate a generalization for truth in models of theories.)

5.5. MANY-SORTED FUZZY PREDICATE CALCULI

In this short section we describe an inessential but useful modification of the systems developed. The aim is to have a means to distinguish objects of various *sorts* like points and lines in geometry or pressures, temperatures and colours in some applied field. We shall carefully define all necessary notions and then inspect the previous parts of this chapter, verifying that everything remains true after very minor modifications. We shall make substantial use of many-sorted logic later on when analyzing fuzzy inference.

Definition 5.5.1 Paraphrasing Definition 5.1.1, we define a many sorted *language* as given by the following: a non-empty finite set of *sorts*; each non-empty finite sequence of sorts is a *type*; for each sort s_i an infinite set of *variables* of the sort s_i, a non-empty (possibly infinite) set of *predicates*, each having a type, and a (possibly empty, possibly infinite) set of *constants*, each having a sort. (As always, all infinite sets are assumed countable.) Having such a language, one defines terms to be variables and constants; *atomic formulas* have the form $P(t_1, \dots, t_n)$, P being a predicate of a type $\langle s_1, \dots, s_n \rangle$ and each t_i having the sort s_i. *Formulas* are built from atomic formulas and truth constants $\bar{0}, \bar{1}$ using connectives and quantifiers by the same rules as in 5.1.1.

Given a language $\mathcal{J}$ and a BL-algebra $\mathbf{L}$, a $\mathbf{L}$-*structure*

$$\langle (M_s)_{s \text{ sort}}, (r_P)_{P \text{ predicate}}, (m_c)_{c \text{ constant}} \rangle$$

consists of the following (cf. 5.1.3).

- for each sort s, a non-empty domain M_s of objects of this sort,
- for each predicate P of a type $\langle s_1, \dots, s_n \rangle$ a fuzzy relation r_P on $M_{s_1} \times \dots \times M_{s_n}$, i.e. a mapping r_P associating with each tuple $\langle m_1, \dots, m_n \rangle$ such that $m_i \in M_{s_i}$ for $i = 1, \dots, n$, a truth value $r_P(m_1, \dots, m_n) \in \mathbf{L}$,
- for each constant c of a sort s, an element $m_c \in M_s$.

A *valuation* v of object variables assigns to each variable x of a sort s an element $v(x) \in M_s$. The definition of $\|\varphi\|^{\mathbf{L}}_{M,v}$ remains the same. We modify the definition of *substitutability* of t for x into φ by adding the condition saying that t and x must be of the same sort (cf. 5.1.5).

Remark 5.5.2 The definition of an $\mathbf{L}$-*tautology* is the same as in 5.1.6 (and so is the notion of a *safe* interpretation). The axioms of msBL$\forall$ are as axioms of BL$\forall$ (with the new notion of formulas and substitutability). The facts 5.1.9–5.1.12 remain as they stand; and so do the proofs of (1)–(12).

The definition 5.2.1 is trivially modified by demanding that if $T \not\vdash (\forall x)\varphi(x)$ then there is a constant c of the same sort as x such that $T \not\vdash \varphi(c)$. In 5.2.6 we shall now have c running over all constants of T having the same sort as x. Thus we get the following theorem, analogous to 5.2.7–5.2.9:

Theorem 5.5.3 Let $msC\forall$ be a many-sorted predicate calculus given by a schematic extension $\mathcal{C}$ of BL and by a many-sorted predicate language $\mathcal{J}$.

(1) For each theory T and each closed formula α, if $T \not\vdash \alpha$ then there is a complete Henkin supertheory $\hat{T}$ of T such that $\hat{T} \not\vdash \alpha$.

(2) For each complete Henkin theory T and each closed formula α unprovable in T there is a linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$ and a safe $\mathbf{L}$-model $\mathbf{M}$ of T such that $\|\alpha\|_{\mathbf{M}} < 1_{\mathbf{L}}$.

(3) (Completeness) T proves φ iff for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$ and each safe $\mathbf{L}$-model $\mathbf{M}$ of T, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1_{\mathbf{L}}$.

*

We similarly paraphrase Section 3 and 4 of the present chapter; but let us proceed more quickly, letting the reader to check details. Thus we get strong completeness for msG∀:

Theorem 5.5.4 (Completeness of msG∀) Let T be a theory over msG∀ and let φ be a formula. T proves φ iff for each $[0,1]_{\mathrm{G}}$-model $\mathbf{M}$ of T, $\|\varphi\|_{\mathbf{M}} = 1$.

The rest of Section 3 generalizes as it stands. Finally we define the many-sorted rational Pavelka logic msRPL∀ in the obvious way and get the following (cf. 5.4.7–5.4.14).

Theorem 5.5.5 (1) If a theory T over msRPL∀ is consistent then it has a model over the standard MV-algebra $[0,1]_{\text{Ł}}$.

(2) For each theory T over msRPL∀ and for each formula φ,

$$|\varphi|_T = \|\varphi\|_T,$$

i.e. the provability degree of φ equals to the truth degree of φ.

And the rest of Sec. 4 generalizes easily. This completes our inspection.

5.6. SIMILARITY AND EQUALITY

Similarity is fuzzified equality[29]; we shall use the sign $\approx$ for similarity and read the formula $x \approx y$ "x is similar to y". The axioms characterizing similarity are direct analogies of the classical axioms of equality.

Definition 5.6.1 The following are axioms of similarity:

$$(\forall x)(x \approx x) \qquad \text{(reflexivity)},$$

$$(\forall x, y)(x \approx y \to y \approx x) \qquad \text{(symmetry)},$$

$$(\forall x, y, z)((x \approx y \& y \approx z) \to x \approx z) \qquad \text{(transitivity)}.$$

Lemma 5.6.2 If $\langle M, r\rangle$ is a model of the axioms of similarity (over a BL-algebra $\mathbf{L}$) then the fuzzy relation r is a *similarity* (or a *fuzzy equivalence*), i. e. for each a, b, $c \in M$,

$$r(a, a) = 1,$$

$$r(a, b) = r(b, a),$$

$$r(a, b) * r(b, c) \leq r(a, c).$$

Proof: Obvious. □

Example 5.6.3 Let us discuss similarities in our three favorite logics, with respect to their standard algebras of truth values

(1) Ł∀. Let r be a similarity on M and put $\rho(u, v) = 1 - r(u, v)$ for u, $v \in M$. Then $\rho(u, v)$ is a (pseudo)metric (reflexive, symmetric, $\rho(u, u) = 0$, triangle inequality: $\rho(u, w) \leq \rho(u, v) + \rho(v, w)$). Moreover, $\rho(u, v) \leq 1$ for each u, v. Conversely, each pseudometric on M with $\rho(u, v) \leq 1$ for all u, v determines a similarity.

(2) Π∀. Let r be a similarity on M and first assume that $r(u, v) > 0$ for all u, $v \in M$. Put $\rho(u, v) = -\log r(u, v)$. Then $\rho(u, v) \in [0, +\infty)$ is a metric on M (since $\rho(u, v) = -\log r(u, v) \leq -\log(r(u, w) \cdot r(w, v)) = -\log r(u, w) + (-\log r(w, v)) = \rho(u, w) + \rho(w, v)$). In the general case $r(u, v) = 0$ is not excluded define a relation $G \subseteq M \times M$ as follows: $(u, v) \in G$ iff $r(u, v) > 0$. G is an equivalence and determines its equivalence classes. On each class, ρ defines a metric; for elements of different components $\rho(u, v) = +\infty$. Conversely, each partition of M into disjoint

[29] See Zadeh [212], Trillas and Valverde [202], Höhle [97], Ruspini [178], Ovchinnikov [157] and Kruse et al. [118, 119].

sets, each bearing a metric and defining $\rho(u,v) = +\infty$ for u, v from different sets defines a similarity in the sense of $\Pi\forall$.

(3) $G\forall$. Again let r be a similarity. Here we have $r(u,w) \geq \min(r(u,v), r(v,w))$; thus for each α, the crisp relation $r^\alpha = \{(u,v) | r(u,v) \geq \alpha\}$ is a crisp equivalence. Moreover, $\alpha \leq \beta$ implies $r^\beta \subseteq r^\alpha$ and $r^0 = M \times M$. Conversely, each system $\{r^\alpha | \alpha \in [0,1]\}$ of crisp equivalences on M satisfying the last two conditions determines a fuzzy equivalence on M in the sense of $G\forall$.

Lemma 5.6.4 Let $x \approx^k y$ abbreviate $(x \approx y) \& \cdots \& (x \approx y)$ (k times). (Formally, $x \approx^0 y$ is $\bar{1}$.) Over BL$\forall$, the theory of similarity with the axioms 5.6.1 proves the same axioms with $\approx$ replaced by $\approx^k$. Consequently, if r is the similarity (over **L**) then so is the relation r^k such that $r^k(u,v) = r(u,v) * \cdots * r(u,v)$ (k times).

Proof: Clearly, over BL$\forall$, $x \approx y \vdash x \approx^k y$, $(x \approx y) \to (y \approx x) \vdash (x \approx^k y) \to (y \approx^k x)$ and, finally, $(x \approx y \& y \approx z) \to x \approx z \vdash (x \approx^k y \& y \approx^k z) \to (x \approx^k z)$. (We repeatedly use the fact that BL proves the formula

$$((p_1 \to q_1) \& \cdots \& (p_k \to q_k)) \to ((p_1 \& \cdots \& p_k) \to (q_1 \& \cdots \& q_k))$$

□

Definition 5.6.5 (1) The *congruence axiom* (or *extensionality axiom*) for a predicate P of arity n with respect to $\approx$ is the formula

$$(x_1 \approx y_1 \& \cdots \& x_n \approx y_n) \to (P(x_1, \cdots, x_n) \equiv P(y_1, \cdots, y_n)).$$

(2) A fuzzy relation $s : M^n \to [0,1]$ is *extensional* w. r. t. a similarity r on M iff, for each $u_1, \cdots, u_n, v_1, \cdots, v_n \in M$,

$$r(u_1, v_1) * \cdots * r(u_n, v_n) * s(u_1, \cdots, u_n) \leq s(v_1, \cdots, v_n).$$

Lemma 5.6.6 Let $\mathbf{M} = \langle M, r, s \rangle$ be an interpretation of the predicates $\approx, P$ ($r, \approx$ binary, s, P n-ary) which is a model of the axioms of similarity. $\mathbf{M}$ is a model of the congruence axiom for P iff s is extensional w. r. t. r.

Proof: If s is extensional and $r(\mathbf{u}, \mathbf{v})$ abbreviates $r(u_1, v_1) * \cdots * r(u_n, v_n)$ then we have $r(\mathbf{u}, \mathbf{v}) \leq s(\mathbf{u}) \Rightarrow s(\mathbf{v})$, $r(\mathbf{u}, \mathbf{v}) \leq s(\mathbf{v}) \Rightarrow s(\mathbf{u})$, thus $r(\mathbf{u}, \mathbf{v}) \leq [(s(\mathbf{u}) \Rightarrow s(\mathbf{v})) \wedge (s(\mathbf{v}) \Rightarrow s(\mathbf{u}))]$, thus the axiom of congruence is 1-true. The converse is obvious. □

Definition 5.6.7 We define the *syntactic degree* of each formula. $\mathrm{dg}(\varphi) = 1$ if φ is atomic; $\mathrm{dg}(\varphi) = 0$ if φ is a truth constant. $\mathrm{dg}((\forall x)\varphi) = \mathrm{dg}((\exists x)\varphi) = \mathrm{dg}(\neg\varphi) = \mathrm{dg}(\varphi)$, $\mathrm{dg}(\varphi \to \psi) = \mathrm{dg}(\varphi \& \psi) = \mathrm{dg}(\varphi) + \mathrm{dg}(\psi)$. Furthermore, $\mathrm{dg}(\varphi \wedge \psi) = \mathrm{dg}(\varphi \vee \psi) = \max(\mathrm{dg}(\varphi), \mathrm{dg}(\psi))$. (Caution: $\wedge$ and $\vee$ are expressible by $\&$ and $\to$; but if you use the clauses for $\wedge$ and $\vee$ you get a syntactic degree less than that given by using only $\&, \to$. The only use is for the following lemma.)

Lemma 5.6.8 Let T be a theory (over BL$\forall$) containing the similarity axioms for $\approx$ and the congruence axioms for $P_1, \ldots, P_n$ w.r.t $\approx$. Let φ be a formula built from the predicates $P_1, \cdots, P_n$ and possibly $\approx$, let $k = \mathrm{dg}(\varphi)$. Let $x_1, \cdots, x_n$ be variables including all free variables of φ and let y_i be substituable for x_i in φ $(i = 1, \cdots, n)$. Then

$$T \vdash (x_1 \approx^k y_1 \& \ldots \& x_n \approx^k y_n) \to (\varphi(x_1, \cdots, x_n) \equiv \varphi(y_1, \cdots, y_n)).$$

Proof: The assertion is true for φ atomic by definition (and is vacuous for truth constants). Let φ, ψ be formulas for which the assertion is true; put $\mathrm{dg}(\varphi) = k$, $\mathrm{dg}(\psi) = h$, let $\mathbf{x} \approx^k \mathbf{y}$ stand for $x_1 \approx^k y_1 \& \cdots \& x_n \approx^k y_n$. Then

$$T \vdash [\mathbf{x} \approx^k \mathbf{y} \to \varphi(\mathbf{x}) \equiv \varphi(\mathbf{y})] \& [\mathbf{x} \approx^h \mathbf{y} \to \psi(\mathbf{x}) \equiv \psi(\mathbf{y})],$$

$$T \vdash (\mathbf{x} \approx^k \mathbf{y} \& \mathbf{x} \approx^h \mathbf{y}) \to [(\varphi(\mathbf{x}) \equiv \varphi(\mathbf{y})) \& (\psi(\mathbf{x}) \equiv \psi(\mathbf{y}))]$$

$$T \vdash \mathbf{x} \approx^{k+h} \mathbf{y} \to [(\varphi(\mathbf{x}) \& \psi(\mathbf{x})) \equiv (\varphi(\mathbf{y}) \& \psi(\mathbf{y}))],$$

as well as

$$T \vdash \mathbf{x} \approx^{k+h} \mathbf{y} \to [(\varphi(\mathbf{x}) \to \psi(\mathbf{x})) \equiv (\varphi(\mathbf{y}) \to \psi(\mathbf{y}))]$$

(Cf. 2.2.10 (7), 2.2.16 (26).)
Moreover, let $k \leq h$; then

$$T \vdash (\mathbf{x} \approx^h \mathbf{y}) \to [(\varphi(\mathbf{x}) \equiv \varphi(\mathbf{y})) \wedge (\psi(\mathbf{x}) \equiv \psi(\mathbf{y}))]$$

and hence

$$T \vdash (\mathbf{x} \approx^h \mathbf{y}) \to [(\varphi(\mathbf{x}) \wedge \psi(\mathbf{x})) \equiv (\varphi(\mathbf{y}) \wedge \psi(\mathbf{y}))]$$

as well as

$$T \vdash (\mathbf{x} \approx^h \mathbf{y}) \to [(\varphi(\mathbf{x}) \vee \psi(\mathbf{x})) \equiv (\varphi(\mathbf{y}) \vee \psi(\mathbf{y}))].$$

(Cf. 2.2.12 (11), 2.2.16 (16).)

Now assume only $T \vdash (\mathbf{x} \approx^k \mathbf{y}) \rightarrow (\varphi(\mathbf{x}) \equiv \varphi(\mathbf{y}))$; then evidently $T \vdash (\mathbf{x} \approx^k \mathbf{y}) \rightarrow (\neg\varphi(\mathbf{x}) \equiv \neg\varphi(\mathbf{y}))$. Moreover, write $\mathbf{x}' \approx^k \mathbf{y}'$ for $x_1 \approx^k y_1 \& \cdots \& x_{n-1} \approx^k y_{n-1}$ (thus $\mathbf{x} \approx^k \mathbf{y}$ is $\mathbf{x}' \approx^k \mathbf{y}' \& x_n \approx^k y_n$). Our assumption implies (substituting x_n for y_n), $T \vdash \mathbf{x}' \approx^k \mathbf{y}' \rightarrow (\varphi(\mathbf{x}', x_n) \equiv \varphi(\mathbf{y}', x_n))$, thus $T \vdash \mathbf{x}' \approx^k \mathbf{y}' \rightarrow ((\forall x_n)\varphi(\mathbf{x}', x_n) \equiv (\forall x_n)\varphi(\mathbf{y}', x_n))$ by usual handling quantifiers. This completes the proof. □

*

We shall now be interested in crisp similarities (crisp equivalences). For this purpose we shall use the following lemma, useful in various other places in the book, notably in Chap. 9 Sec. 3 (liar's paradox).

Lemma 5.6.9 Let r_P be the interpretation of a predicate P of arity n in a structure $\mathbf{M}$. The relation r_P is crisp (i.e. its range is included in $\{0, 1\}$) iff the formula $(\forall x_1 \ldots x_n)(P(x_1, \ldots, x_n) \vee \neg P(x_1, \ldots, x_n))$ is 1-true in $\mathbf{M}$.

Proof: Clearly, if r_P is crisp then the above formula is 1-true. On the other hand, if $0 < r_P(\mathbf{a}) < 1$ (in a given $\mathbf{L}$, then $(r_P(\mathbf{a}) \Rightarrow 0) < 1$, thus if $v(x_i) = a_i$ then $\|P(x_1, \ldots) \vee \neg P(x_1, \ldots)\|_{\mathbf{M},v} < 1$ and hence
$\|(\forall x_1, \ldots)(P(x_1, \ldots) \vee \neg P(x_1, \ldots)\|_{\mathbf{M}} < 1$. □

Definition 5.6.10 Let $\mathcal{J}$ be a language whose predicates are $=, P_1, \ldots, P_n$ ($=$ binary). The *equality axioms for* $\mathcal{I}$, $=$ are

- similarity axioms for $=$,
- congruence axioms for $P_1, \ldots, P_n$ w.r.t $=$,
- the crispness axiom $(\forall x, y)(x = y \vee \neg(x = y))$ for $=$.

Remark 5.6.11 (1) Note that the congruence axiom for $=$, i.e.
$(x_1 = x_2 \& y_1 = y_2) \rightarrow (x_1 = y_1 \equiv x_2 = y_2)$, follows by transitivity.

(2) Observe also that $x =^k y$ is provably equivalent to $x = y$ due to crispness:

$$\mathrm{BL}\forall \vdash x = y \rightarrow (x = y \rightarrow x =^2 y),$$
$$\mathrm{BL}\forall \vdash x \neq y \rightarrow (x = y \rightarrow x =^2 y),$$
$$\mathrm{BL}\forall \vdash (x = y \vee x \neq y) \rightarrow (x = y \rightarrow x =^2 y).$$

Thus 5.6.8 has the following corollary:

Corollary 5.6.12 Let T be a theory (over BL$\forall$) containing the equality axioms for the language of T and $=$; let $\varphi(x_1, \ldots, x_n)$ be a formula of T. Then

$$T \vdash (x_1 = y_1 \& \ldots \& x_n = y_n) \rightarrow (\varphi(x_1, \ldots, x_n) \equiv \varphi(y_1, \ldots, y_n)).$$

In the rest of the section we present the role of similarity relations for defining fuzzy sets from crisp examples. We show that if X is a crisp subset of M and r is a similarity on M then there is a smallest fuzzy set s which is extensional w.r.t. r and $s(m) = 1$ for all $m \in X$ (s is the fuzzy set of elements similar to elements from X). Moreover, we show that s is definable in the corresponding language and the assumption that X is crisp is superfluous.

Theorem 5.6.13 Let T be a theory (over BL$\forall$) containing axioms of similarity for $\approx$, let $P, \hat{P}$ be n-ary predicates of T and let the following formula be provable in T:

$$\hat{P}(x_1, \ldots, x_n) \equiv (\exists y_1, \ldots, y_n)(x_1 \approx y_1 \& \ldots \& x_n \approx y_n \& P(y_1, \ldots, y_n)).$$

Then T proves the congruence axiom for $\hat{P}$ and proves $(\forall x \ldots)(P(x, \ldots) \to \hat{P}(x, \ldots))$; moreover, if $Congr(Q, \approx)$ is the congruence axiom for another n-ary predicate Q then T proves

$$[Congr(Q, \approx) \& (\forall x, \ldots)(P(x, \ldots) \to Q(x, \ldots)] \to$$

$$\to (\forall x, \ldots)(\hat{P}(x, \ldots) \to Q(x, \ldots)).$$

(This formula evidently says that if Q is extensional and contains P then it also contains $\hat{P}$.)

Proof: For simplicity assume $n = 1$. Thus T proves the formula $\hat{P}(x) \equiv (\exists y)(y \approx x \& P(y))$. Consequently,

$$T \vdash (\exists y)(y \approx x \& P(y) \to \hat{P}(x)),$$

$$T \vdash (\forall y)(y \approx x \& P(y) \to \hat{P}(x)),$$

$$T \vdash x \approx x \& P(x) \to \hat{P}(x),$$

$$T \vdash P(x) \to \hat{P}(x).$$

Furthemore, T proves the following chain of implications: $[(\hat{P}(x) \& x \approx z)] \to [(\exists y)(y \approx x \& P(y)) \& x \approx z)] \to [\to (\exists y)(y \approx z \& P(y))] \to \hat{P}(z)$. (Note that we have used 5.1.18 (9).) Thus T proves the congruence axiom.

Finally, let Q be as above; write $Congr$ for $Congr(Q, \approx)$ and $P \subseteq Q$ for $(\forall x)(P(x) \to Q(x))$. Then

$$T \vdash P \subseteq Q \to [(x \approx y \& P(y)) \to (x \approx y \& Q(y))],$$

$$T \vdash Congr \to [(x \approx y \& Q(y)) \to Q(x)],$$

$$T \vdash (P \subseteq Q\&\mathit{Congr}) \to (\forall x)(\forall y)((x \approx y\&P(y)) \to Q(x)),$$

$$T \vdash (P \subseteq Q\&\mathit{Congr}) \to (\forall x)(\hat{P}(x) \to Q(x)).$$

This completes the proof. □

Note that each unary predicate defines a similarity as follows.

Lemma 5.6.14 Let T be a theory containing a unary predicate P and a binary predicate $\approx$ and proving

$$(\forall x, y)(x \approx y \equiv (P(x) \equiv P(y))).$$

Then T proves similarity axioms for $\approx$.

Proof: Easy; just use BL$\forall$ $\vdash$ $[(P(x) \equiv P(y))\&(P(y) \equiv P(z))] \to$ $\to (P(x) \equiv P(z))$. □

Remark 5.6.15 Note that our investigation generalizes immediately to the many-sorted case in such a way that each sort s may have its similarity (equality) predicate of the type (s, s). A variant having one similarity predicate across all sorts is also possible but needs some changes in our basic definitions (of a language, etc.)

Example 5.6.16 To give simple examples of theories in fuzzy predicate calculus, let us reexamine our examples concerning large numbers 3.3.21, 4.1.27. Consider a theory T_0 with one binary predicate S (successor), the equality predicate =, a constant $\bar{0}$ for zero and a unary predicate L (large). Axioms are as follows:

crispness axioms for S and =, i.e.
reflexivity, symmetry and transitivity of =;
congruence for S, L:
$(x_1 = x_2\&y_1 = y_2) \to (S(x_1 y_1) \equiv S(x_2, y_2))$,
$x_1 = x_2 \to (L(x_1) \equiv L(x_2))$
functionality for S:
$(S(x, y_1)\&S(x, y_2)) \to y_1 = y_2$
Axioms of successor (cf. 1.4.10)
$\neg S(x, \bar{0})$
$(S(x_1, y)\&S(x_2, y)) \to x_1 = x_2$
$y \neq \bar{0} \to (\exists x)S(x, y)$.
Monotonicity for L: $(S(x, y)\&L(x)) \to L(y)$.

A *standard* model of T_0 is the set N of natural numbers with absolute equality , S interpreted by the successor relation $\{(m,n)|n = m+1\}$ and some (fuzzy) interpretation of L, ($r_L : N \to [0,1]$, non-decreasing). Note the existence of non-standard models, e.g. a copy of the standard model together with a disjoint copy of integers with the successor relation and some interpretation of L. More formally:

$M = \{(i,m)|(i = 0 \text{ and } m \in N) \text{ or } (i = 1 \text{ and } m \in Z)\}$ (where Z is the set of integers); $r_S = \{((i,m),(i,n))|n = m+1\}$.

Now look at how we may express our assumption on L:
T_1 is T_0 plus $(\bar{r}\&S(x,y)\&L(y)) \to L(x)$ (think of $r = 0.999999$). Note that in Ł∀ this is equivalent to $(S(x,y)\&L(y)) \to (L(x) \vee \overline{1-r})$.

A standard model of T_0 is a model of T_1 over Ł∀ iff for each $n \in N$, $r_L^{n+1} \leq r_L(n) + (1-r)$; for example, $r_L(n) = 0.000001n$ for $n \leq 10^6$, $r_L(n) = 1$ for $n \geq 10^6$.

T_2 is T_0 plus $S(x,y)\&L^2(y) \to L(x)$. A standard model of T_2 over Ł∀ satisfies, for each n, $r_L(n+1) \leq r_L(n) + \frac{1-r_L(n)}{2}$; thus if $r_L(n) < 1$ for some n then for all n. Note that non-standard model of T_0 above is a model of T_2 e.g. if $r_L(0,0) = 0$, $r_L(0,n+1) = r_L(0,n) + \frac{1-r_L(0,n)}{2}$, $r_L(1,k) = 1$ for all k. (All non-standard elements are large in degree 1.) T_3 is T_0 plus $S(x,y)\&L(y) \to 2L(x)$ etc. Discuss the models of T_3 over Ł∀. Discuss the models of T_1, T_2 over Π∀.

Note that we shall investigate less simple examples of theories in Ch. VII (theories of fuzzy control, FC) and Ch. IX Sec. 3 (arithmetic with a fuzzy truth predicate).

CHAPTER SIX

QUESTIONS OF COMPUTATIONAL COMPLEXITY AND UNDECIDABILITY

The present chapter is devoted to an analysis of computational complexity of fuzzy propositional calculi and to the determination of the degree of undecidability of fuzzy predicate calculi. The results are important but will be only rarely used in other chapters, so the reader unfamiliar with the theory of (polynomial) complexity and/or arithmetical hierarchy may skip the chapter as a whole. On the other hand, in Section 1 we summarize all necessary material so that the non-expert in complexity or arithmetical hierarchy may also read the chapter and understand the results. In Section 2 we study the computational complexity of Ł, G, Π (propositional Łukasiewicz, Gödel and product logic) and show, among other things, that their sets of 1-tautologies $TAUT_1^{Ł}$, $TAUT_1^{G}$, $TAUT_1^{\Pi}$ are all co-NP-complete. In Section 3 we study the corresponding predicate calculi Ł∀, G∀, Π∀. We show that the set $TAUT_1^{G\forall}$ of Gödel predicate logic is Σ_1-complete, $TAUT_1^{Ł\forall}$ is Π_2-complete and $TAUT_1^{\Pi\forall}$ is Π_2-hard (thus not in Σ_2; it remains open if it is in Π_2 or is still more complex). Hence all these predicate calculi are undecidable.

6.1. Preliminaries on complexity and arithmetical hierarchy

In the present section we are going to survey basic facts on polynomial computational complexity and recursive (un)decidability. All necessary details are available in standard monographs on complexity theory (as e.g. [158] and on recursion theory (e.g. [173]). The reader may also consult [91]. The first part of the section deals with complexity and the second with undecidability and arithmetical hierarchy. They may be read independently.

6.1.1 The notion of a multitape *Turing machine* in supposed to be known: it has finitely many *tapes*, one being the *input tape*, finite *alphabet*, finitely many *internal states* and a finite set of *instructions*. Given a *word* (finite sequence of elements of the alphabet) the machine puts the word in its input tape and starts to work in discrete steps. In each step the machine either stops or changes the content of the tapes. This is done in dependence on the actual content of the tapes, the actual internal state and the set of instructions. The mechanism M is *deterministic* if the next content of the tapes (and the next internal state) is uniquely determined by all these; or it may have finitely many possibilities from which it can choose – then it is *nondeterministic*. Any sequence of successive steps determined by the machine is a *computation* with the input word given. It may be finite (stop after finitely many steps) or infinite (the machine runs forever). We ignore the details: either the reader knows them or he/she just gets only preliminary information. M is *total* if each computation is finite.

6.1.2 Each deterministic Turing machine M defines a function f_M assigning to each word x understood as the input a word $f_M(x)$ uniquely determined by the last content of a tape called *output tape* – if the computation determined by x is finite; otherwise $f_M(x)$ is undefined. This easily extends to a function from N^n into N (N is the set of all natural numbers) by coding numbers e.g. by their dyadic representations and using an additional symbol as a marker. A function from N^n into N is *partial recursive* if it is computed by a deterministic Turing machine; it is *recursive* if it is partial recursive and total. (This may be taken as a definition; if you prefer another definition you get this as a theorem.)

6.1.3 Let M be a Turing machine (deterministic or not) and let e be one of its internal states called *accepting* state. The *language accepted by* M, e is the set of all words x such that there is a finite computation with input x such that when stopping the machine is in the accepting internal state.

6.1.4 A machine M *runs in polynomial time* if there is a polynomial $p(n)$ such that for each word x of length n, each computation with the input x stops after $\leq p(n)$ steps. P is the class of all languages L accepted by a deterministic Turing machine running in polynomial time, similarly for NP and non-deterministic. Clearly, P $\subseteq$ NP; note that the problem whether P $=$ NP is one of most famous open problems of computational complexity. A language is in co-NP if its complement is in NP.

A language L is NP-*complete* if it is in NP and each $L' \in$ NP is polynomially reducible to L, i.e. there is a function f computed by a deterministic Turing machine running in polynomial time such that for each word x in the alphabet of L', $x \in L'$ iff $f(x) \in L$. Similarly for co-NP.

6.1.5 Let SAT^{Bool} be the set of all satisfiable formulas φ if the Boolean propositional calculus (i.e. there is an evaluation e of propositional variables such that $e(\varphi) = 1$). Note that for complexity considerations we understand formulas of propositional calculus as words in the finite alphabet $p, 0, 1, \rightarrow, \neg$, [possibly also $\wedge, \vee$]; the n-th propositional variable is coded as the word consisting of p followed by the dyadic representation of n (e.g. p_5 is $p101$).

Theorem 6.1.6 (*Cook's theorem.*) SAT^{Bool} is NP-complete.

Remark 6.1.7 Let $TAUT^{Bool}$ be the set of all tautologies of Boolean propositional logic. Clearly, $TAUT^{Bool}$ is co-NP complete since

$$\varphi \in TAUT^{Bool} \text{ iff } (\neg\varphi \notin SAT^{Bool}).$$

Finally we present a fact on the computational complexity of linear programming; we shall use it in determining the complexity of Łukasiewicz logic.

Definition 6.1.8 The LP-*problem.* Assume the following to be given: a rational matrix A having m rows and n columns, a rational column vector b, a rational row vector c and a rational number d. (Thus $A = (a_{kl})_{k=1,\ldots,m}^{l=1,\ldots,n}$, $b = (b_1, \ldots, b_n)$, $c = (c_1, \ldots, c_m)$, a_{kl}, b_l, c_k all rational.)

(A, b, c, d) is a *particular* LP-*problem*. The *size* of (A, b, c, d) is the number of bits necessary to represent all these rationals as fractions of dyadic numbers (using some few additional markers as /, comma, end-of-line etc., details unimportant).

The general LP-problem reads: given A, b, c, d as above, does the system $Ax \leq b$, $cx > d$ have a solution, i.e. is there a vector $x = (x_1, \ldots, x_m)$ such that

$$\sum_k a_{kl}x_k \leq b_l \text{ for each } l,$$
$$\sum_k c_k x_k > d.$$

Theorem 6.1.9 The general LP-problem is in NP.

For a proof and a detailed exposition see [182]): (see p. 121, Corollary 10.1a). Note that more is known (and can be found in [182]): there is a function f in P assigning to each (A, b, c, d) a solution or saying that no solution exists, i.e. the LP-problem is in P. But we shall not need this stronger result. On the other hand, we may assume that $x_1, \ldots, x_n$ vary over $[0, 1]$ since this is achieved by adding just new inequalities $x_i \leq 1$, $-x_i \leq 0$.

*

6.1.10 Our next topic is *arithmetical hierarchy.* It concerns sets of natural numbers (or things coded by natural numbers) definable by formulas of arithmetic. Recall 1.4.7 where we defined the language of arithmetic with function symbols for successor, addition and multiplication and predicates $=, \leq$. In 1.4.10 we defined its counterpart without function symbols, with the binary predicate S ($(S(x,y))$ saying "y is the successor of x") and ternary predicates A, B for addition and multiplication. Recall also the *standard model* $\mathbf{N}$ of the language of arithmetic – the structure of natural numbers. We shall use the same symbol $\mathbf{N}$ whether it is the structure of natural numbers with operations or with the corresponding relations. The reader may consult [91] for a more detailed survey (Chapter 0 and 1).

We shall formulate our survey in such a way that it is meaningful in both presentations of arithmetic – with function symbols or without. (When used in fuzzy logic we shall need the version without function symbols.)

Definition 6.1.11 (1) *Bounded quantifiers* $(\forall x \leq y)$, $(\exists x \leq y)$ are defined as abbreviations: $(\forall x \leq y)\varphi$ stands for $(\forall x)(x \leq y \rightarrow \varphi)$ and $(\exists x \leq y)\varphi$ stands for $(\exists x)(x \leq y \wedge \varphi)$. A formula is *bounded* if all quantifiers occuring in it are bounded.

(2)

A formula is	if it has the form
Σ_1	$(\exists u)\psi$
Π_1	$(\forall u)\psi$
Σ_2	$(\exists u)(\forall v)\psi$
Π_2	$(\forall u)(\exists v)\psi$

for some bounded formula ψ. For each $n > 0$ we may define Σ_n formulas as formulas of the form $(\exists u_1)(\forall u_2)\ldots\psi$ with n alternating quantifiers and similarly Π_n; but we shall need only those in the table above.

(3) A set $A \subseteq N$ is Σ_i (Π_i respectively) if there is a Σ_i-formula (Π_i-formula) $\varphi(x)$ with one free variable defining A in $\mathbf{N}$, i.e. for each n,
$n \in A$ iff $||\varphi||_{\mathbf{N}}[n] = 1$.

Similarly for a relation, i.e. a subset of $N^k, k > 1$. The set A is Δ_i if it is both Σ_i and Π_i, i.e. has two definitions, one being a Σ_i-formula and the other a Π_i-formula.

6.1.12 (1) $A \subseteq N$ is a recursive set iff it is Δ_1. (This is in fact a theorem; the definition was presented in 6.1.2).

(2) $A \subseteq N$ is *recursively enumerable* if it is Σ_1. (Take this for a definition.)

Definition 6.1.13 $A \subseteq N$ is *recursively reducible* to $B \subseteq N$ if there is a recursive function f such that $A = \{n | f(n) \in B\}$. (Compare this with 6.1.4). A is *Σ_i-complete* if A is a Σ_i-set and each Σ_i-set is recursively reducible to A. Similarly for Π_i-complete.

Lemma 6.1.14 (1) For each $i > 0$ there is a Σ_i-complete set and a Π_i-complete set. A Σ_1-complete set is not recursive; the same for a Π_i-complete set. A Σ_{i+1}-complete set is neither in Σ_i nor in Π_i; the same for a Π_{i+1}-complete set.

(2) On the other hand, each Σ_i-set and each Π_i-set is obviously both Σ_{i+1} and Π_{i+1}; each recursive set is both Σ_1 and Π_1.

6.1.15 We mentioned above the coding of numbers by words; conversely, we can code words of a finite alphabet by numbers, e.g. if the alphabet has n numbers we can understand them as non-zero digits in the $(n + 1)$-adic representation of numbers. Given this we have the following classical result of logic:

Lemma 6.1.16 Let $\mathcal{I}$ be a predicate language having at least one at least binary predicate. Then the set of all formulas of $\mathcal{I}$ provable in $Bool\forall$ is a Σ_1-complete set.

Similarly, the set of all formulas provable in the arithmetic QA is Σ_1-complete; the same for PA.

Lemma 6.1.17 There is a Σ_1-relation $C \subseteq N^2$ such that, if we define $C_m = \{n|(m,n) \in C\}$ then the set $Fin = \{m|C_m \text{ is finite}\}$ is a Σ_2-complete set (thus $N - Fin$ is a Π_2-complete set).

Lemma 6.1.18 Let us investigate Robinson arithmetic QA. It has finitely many axioms so that we may replace them by one – the conjuction of all of them – and call it also QA. We have the standard model **N** of QA; but it is a result of mathematical logic that QA also has *non-standard models* different from **N**: in each model of QA, all formulas provable in QA are true, but there is a model **M** of QA and formula φ true in **N** but false in **M**. The same for each recursively axiomatized consistent theory $T \supseteq QA$ (i.e. T as the set of special axioms is a recursive set). This is Gödel's celebrated first incompleteness theorem.

For each model **M** of QA there is a unique isomorphic embedding of **N** onto an initial sequent of **M**; i.e. **M** has a zero (interpretation of $\underline{0}$), the successor of zero, the succcessor of the succcessor of zero etc.; this gives a copy of **N** – standard elements of **M**. All nonstandard elements (if any) are only behind the standard ones.

The n-th standard element (copy of n) is defined in each **M** by the formula

$$(\exists x_0, \ldots, x_n)(x_0 = \underline{0} \wedge \bigwedge_{i=1}^{n} S(x_i, x_{i+1}) \wedge x_n = x);$$

we call it $num_n(x)$. We can define, inside QA, the n-th *numeral* to be a constant $\underline{n}$ by postulating $num_n(\underline{n})$. [30]

Lemma 6.1.19 (Σ_1-*completeness of arithmetic*) (1) Each closed Σ_1-formula true in **N** is provable in QA and hence true in each model of QA.

(2) More generally if $\varphi(x, \ldots, y)$ is Σ_1, $m, \ldots, n \in N$ and $\|\varphi\|_{\mathbf{N}}[m, \ldots, n] = 1$ then for each model M of QA, $\|\varphi\|_{\mathbf{M}}[m', \ldots, n'] = 1$ where $m', \ldots, n'$ are isomorphic copies of $m, \ldots, n$ (interpretations of $\underline{m}, \ldots, \underline{n}$).

6.2. Complexity of fuzzy propositional calculi

Definition 6.2.1 Let $\mathcal{C}$ stand for Ł, G, or Π.

$$\begin{aligned} SAT_1^{\mathcal{C}} &= \{\varphi \mid \text{for some } [0,1]\text{-evaluation } e,\ e_{\mathcal{C}}(\varphi) = 1\}, \\ SAT_{pos}^{\mathcal{C}} &= \{\varphi \mid \text{for some } [0,1]\text{-evaluation } e,\ e_{\mathcal{C}}(\varphi) > 0\}, \\ TAUT_1^{\mathcal{C}} &= \{\varphi \mid \text{for each } [0,1]\text{-evaluation } e,\ e_{\mathcal{C}}(\varphi) = 1\}, \\ TAUT_{pos}^{\mathcal{C}} &= \{\varphi \mid \text{for each } [0,1]\text{-evaluation } e,\ e_{\mathcal{C}}(\varphi) > 0\}. \end{aligned}$$

Clearly, SAT_1 stands for 1-satisfiable, SAT_{pos} for positively satisfiable and similarly $TAUT$ for tautologies.

Definition 6.2.2 Let $p_1, \ldots, p_n$ be propositional variables and $I \subseteq \{p_1, \ldots, p_n\}$. I is understood as the set of variables evaluated by 0. We define, for each formula φ built from $p_1, \ldots, p_n$, its translation φ^I as follows:[31]

$$\begin{aligned} \bar{0}^I = \bar{0}, &\quad \bar{1}^I = \bar{1}, \\ p_i^I = \bar{0} \text{ if } i \in I, &\quad p_i^I = p_i \text{ otherwise;} \\ (\bar{0} \to \varphi)^I &= \bar{1} \text{ for all } \varphi, \\ (\varphi \to \bar{0})^I &= \bar{0} \text{ if } \varphi^I \text{ is not } \bar{0}, \\ (\varphi \to \psi)^I &= \varphi^I \to \psi^I \text{ otherwise} \\ (\varphi \& \psi)^I &= \bar{0} \text{ if at least one of } \varphi, \psi \text{ is } \bar{0}; \\ (\varphi \& \psi)^I &= \varphi^I \& \psi^I \text{ otherwise.} \end{aligned}$$

[30] When dealing with function symbols we write $x_{i+1} = S(x_i)$ and may understand $\underline{n}$ to be the term $S(S(S \ldots (\underline{0}) \ldots))$ (n copies of S).

[31] See [9].

Lemma 6.2.3 Let $\mathcal{C}$ be G or Π. For each formula φ, and set $I \subseteq \{p_1, \ldots, p_n\}$, either φ^I is $\bar{0}$ or φ^I is a formula not containing $\bar{0}$. For each evaluation e such that $e(p_i) = 0$ iff $p_i \in I$, $e_{\mathcal{C}}(\varphi) = e_{\mathcal{C}}(\varphi^I)$. Moreover $e_{\mathcal{C}}(\varphi^I) = 0$ iff φ^I is $\bar{0}$.

Proof: easy by induction. □

Remark 6.2.4 Recall that in 6.1.7 we defined SAT^{Bool} to be the set of all formulas φ satisfiable in Boolean (two-valued) logic, i.e. such that form some $\{0,1\}$-evaluation v ($v(p_i) = 1$ or $v(p_i) = 0$), $v(\varphi) = 1$. Similarly $TAUT^{Bool}$.

Lemma 6.2.5 The following are equivalent:

(1) $\varphi \in SAT^{\mathrm{G}}_{pos}$;

(2) $\varphi \in SAT^{\Pi}_{pos}$;

(3) for some I, φ^I is not $\bar{0}$;

(4) $\varphi \in SAT^{Bool}$.

Proof: Evidently, (4) implies both (1) and (2). We prove (1)$\Rightarrow$(3), (2)$\Rightarrow$(3) and (3) $\Rightarrow$ (4).

Let $\mathcal{C}$ be G or Π, let $\varphi \in SAT^{\mathcal{C}}_{pos}$. Thus for an evaluation e and for $I = \{p_i | \, e(p_i) = 0\}$ we get $e_{\mathcal{C}}(\varphi^I) = e_{\mathcal{C}}(\varphi) > 0$ and hence φ^I is not $\bar{0}$.

Now assume (3) and let $v(p_i) = 0$ for $p_i \in I$, $v(p_i) = 1$ otherwise. Since φ^I does not contain $\bar{0}$ we get $v(\varphi^I) = 1$ and hence $v(\varphi) = 1$ (compute $v(\varphi)$, $v(\varphi)^I$ using Boolean truth tables or, which is the same, in G or Π — they coincide for values $0, 1$). □

Corollary 6.2.6 The sets

$$SAT^{\mathrm{G}}_1,\ SAT^{\mathrm{G}}_{pos},\ SAT^{\Pi}_1,\ SAT^{\Pi}_{pos},\ SAT^{Bool}$$

are all equal; hence they are NP-complete.

*

We now turn to the problem of tautologicity.

Lemma 6.2.7 $TAUT^{\mathrm{G}}_{pos} = TAUT^{\Pi}_{pos} = TAUT^{Bool}$, thus all there sets are co-NP-complete.

Proof: Clearly, $TAUT^{Bool} \supseteq TAUT^{\mathcal{C}}_{pos}$ for $\mathcal{C} = G, \Pi$. On the other hand, recall that Boolean logic in axiomatized by BL+$(\varphi \vee \neg\varphi)$. Now for $\mathcal{C} = G, \Pi$, $(\varphi \vee \neg\varphi) \in TAUT^{\mathcal{C}}_{pos}$ and $TAUT^{\mathcal{C}}_{pos}$ is closed under modus ponens (since $*$ has no zero divisors in G, Π). This gives the result. (Note: for Ł, $TAUT^{Ł}_{pos}$ is not closed under modus ponens: $(p \vee \neg p)$ belongs to $TAUT^{Ł}_{pos}$, but $(p \vee \neg p)^2$ not.) □

Lemma 6.2.8 Let $\mathcal{C}$ be G or Π. For each formula φ, let $\varphi^{\neg\neg}$ be the formula resulting from φ by replacing each propositional variable p_i by its double negation $\neg\neg p_i$. Then $\varphi \in TAUT^{Bool}$ iff $\varphi^{\neg\neg} \in TAUT^{\mathcal{C}}_1$.

Proof: Clearly, for each e, $e_{\mathcal{C}}(\neg\neg p_i)$ is 1 or 0. And if $e'(p_i) = (-)(-)e(p_i)$ then $e(\varphi^{\neg\neg}) = e'(\varphi)$. □

Corollary 6.2.9 For $\mathcal{C}$ being G or Π, $TAUT^{\mathcal{C}}_1$ is co-NP-hard.

Theorem 6.2.10 [32] $TAUT^{\mathrm{G}}_1$ is co-NP.

Proof: To show that $TAUT^{\mathrm{G}}_1$ is in co-NP observe that $\varphi \notin TAUT^{\mathrm{G}}_1$ iff there is an evaluation e of $p_1, \ldots, p_n$ (variables of φ) taking only values in the set $D_n = \{0, \frac{1}{n+1}, \ldots, \frac{n}{n+1}, 1\}$ such that $e_G(\varphi) < 1$. Indeed, the set $\{e(p_1), \ldots, e(p_n)\}$ contains at most n elements different from 0,1. And there is a trivial order isomorphism f of $[0,1]$ sending $e(p_1), \ldots, e(p_n)$ into D_n. And in $[0,1]_G$ each order isomorphism is an isomorphism of the G-algebra $[0,1]_G$.

Thus an algorithm showing $\varphi \notin TAUT^{\mathrm{G}}_1$ guesses an evaluation e, $e(p_i) \in D_n$, and computes the truth value. This shows that the complement of $TAUT^{\mathrm{G}}_1$ is in NP and thus $TAUT^{\mathrm{G}}_1$ is co-NP-complete. □

We shall obtain co-NP-completeness of $TAUT^{\Pi}_1$ later, as a consequence of our investigation of Ł (and of our embedding of Ł into Π). Note that for $\mathcal{C} = G$ or Π, $\varphi \in TAUT^{\mathcal{C}}_{pos}$ iff $\neg\varphi \notin SAT^{\mathcal{C}}_{pos}$.

$*$

Remark 6.2.11 Let us consider Ł. Recall that in Ł, $e(\varphi) = 1$ iff $e(\neg\varphi) = 0$; likewise, $e(\varphi) = 0$ iff $e(\neg\varphi) = 1$. Thus

$$\varphi \in TAUT^{Ł}_1 \quad \text{iff} \quad (\neg\varphi) \notin SAT^{Ł}_{pos},$$
$$\varphi \in TAUT^{Ł}_{pos} \quad \text{iff} \quad (\neg\varphi) \notin SAT^{Ł}_1.$$

[32] See [9].

Lemma 6.2.12 (1) $SAT^{Bool} \subset SAT_1^{Ł} \subset SAT_{pos}^{Ł}$;

(2) $TAUT^{Bool} \supset TAUT_{pos}^{Ł} \supset TAUT_1^{Ł}$.
All these inclusions are strict.

Proof: Non-strict inclusions are obvious from definitions; strictness is showed by simple examples. Indeed,

$$(p \wedge \neg p) \in SAT_{pos}^{Ł} - SAT_1^{Ł},$$
$$(p \equiv \neg p) \in SAT_1^{Ł} - SAT^{Bool},$$
$$(p \vee \neg p)\&(p \vee \neg p) \in TAUT^{Bool} - TAUT_{pos}^{Ł},$$
$$(p \vee \neg p) \in TAUT_{pos}^{Ł} - TAUT_1^{Ł}.$$

□

Lemma 6.2.13 $SAT_1^{Ł}$ and $SAT_{pos}^{Ł}$ are NP-hard; thus $TAUT_1^{Ł}$ and $TAUT_{pos}^{Ł}$ are co-NP-hard.

Proof: For each formula φ, let φ^*, φ^{**} be the formulas

$$(p_1 \vee \neg p_1)\& \dots \&(p_n \vee \neg p_n)\&\varphi$$
$$(p_1 \vee \neg p_1)^2\& \dots \&(p_n \vee \neg p_n)^2\&\varphi^2$$

respectively (α^2 is $\alpha\&\alpha$). Thus φ^{**} is equivalent to $(\varphi^*)^2$. Clearly, both $lh(\varphi^*)$ and $lh(\varphi^{**})$ depend polynomially on $lh(\varphi)$ (the length of φ). First observe that

$$\varphi \in SAT^{Bool} \text{ iff } \varphi^* \in SAT_1^{Ł}$$

(since $e(p \vee \neg p) = 1$ implies that $e(p) = 1$ or $e(p) = 0$). Second we show, assuming that φ contains only connectives $\wedge, \vee, \neg$,

$$\varphi \in SAT^{Bool} \text{ iff } \varphi^{**} \in SAT_{pos}^{Ł}.$$

To prove this observe first that $e((p \vee \neg p)^2) > 0$ is equivalent to $e(p) \neq \frac{1}{2}$. Furthemore, if $e(p_i) \neq \frac{1}{2}$ for all $i = 1, \dots, n$ and $e'(p_i) = 1$ iff $e(p_i) > \frac{1}{2}$, $e'(p_i) = 0$ iff $e(p_i) < \frac{1}{2}$ then for each φ containing only connectives $\wedge, \vee, \neg$ we get $e'(\varphi) = 1$ iff $e(\varphi) > \frac{1}{2}$ iff $e(\varphi^2) > 0$. (Verify this by an easy induction on the construction of φ.) Thus if $e(\varphi^{**}) > 0$ then $e(p_i) > 0$ for all i and $e(\varphi^2) > 0$, and $e'(\varphi) = 1$. □

Lemma 6.2.14 $SAT_{pos}^{Ł}$ is in NP and so is $SAT_1^{Ł}$.

The proof is postponed to the end of this section.

Corollary 6.2.15 $SAT^{Ł}_{pos}$, $SAT^{Ł}_{1}$ are NP-complete. Thus $TAUT^{Ł}_{1}$ and $TAUT^{Ł}_{pos}$ are co-NP-complete. [33]

Lemma 6.2.16 $TAUT^{\Pi}_{1}$ is co-NP-complete.

Proof: It remains to prove that $TAUT^{\Pi}_{1}$ is in co-NP, which in turn is equivalent to the statement saying that the set $\{\varphi |$ for some e, $e_{\Pi}(\varphi) < 1\}$ is in NP. Recall the embedding of Łukasiewicz logic into Π (cf. 4.1.14) and the mapping φ^I from 6.2.2. The following are equivalent:
$(\exists e)(e_{\Pi}(\varphi) < 1)$,
$(\exists I \subseteq \{p_1, \dots, p_n\})(\exists e'$ positive$)$ $(e'_{\Pi}(\varphi^I) < 1)$
$(e'(p_i) > 0$ for p_i occuring in $\varphi^I)$,
$(\exists I)(\exists e''$ positive$)$ $(e''_{Ł}(\varphi^I) < 1)$ (thanks to the embedding),

$$(\exists I)\left(\bigwedge_{p_i \notin I} p_i \wedge \neg(\varphi^I) \in SAT^{Ł}_{pos}\right).$$

This reduces polynomially the complement of $TAUT^{\Pi}_{1}$ to $SAT^{Ł}_{pos}$ and hence $TAUT^{\Pi}_{1}$ is in co-NP. □

Theorem 6.2.17 Summarizing, we have the following:

(1) $SAT^{G}_{1} = SAT^{G}_{pos} = SAT^{\Pi}_{1} = SAT^{\Pi}_{pos} = SAT^{Bool}$ is NP-complete.

(2) $TAUT^{G}_{pos} = TAUT^{\Pi}_{pos} = TAUT^{Bool}$, is co-NP-complete.

(3) $TAUT^{G}_{1}$, $TAUT^{\Pi}_{1}$, $TAUT^{Bool}$ are pairwise distinct and all co-NP complete.

(4) $SAT^{Ł}_{pos} \supset SAT^{Ł}_{1} \supset SAT^{Bool}$, all NP-complete.

(5) $TAUT^{Ł}_{1} \subset TAUT^{Ł}_{pos} \subset TAUT^{Bool}$, all co-NP-complete.

---

[33] See [150]; for the following lemma see [9]

*

6.2.18 We now present a proof of $SAT_1^{Ł}$, $SAT_{pos}^{Ł}$ being in NP (this is the only thing remaining to be proved).[34] We shall need the following definition: a *particular MIP-problem* (mixed integer programming), is a tuple (A, b, c, d, k) where (A, b, c, d) is a particular LP-problem, cf. 6.1.8 and k represents the additional condition saying that $x_k, \ldots, x_n$ must be Boolean (zeros or ones). It is immediately seen that the general MIP-problem is in NP: given (A, b, c, d, k) guess $x_k, \ldots, x_n$ nondeterministically (zeros or ones) and then solve the resulting particular LP-problem.

We shall construct functions f_1, f_{pos}, both in P, such that for each formula φ, $f_1(\varphi)$ is a particular MIP-problem which has a solution iff $\varphi \in SAT_1^{Ł}$; similarly for f_{pos} and $SAT_{pos}^{Ł}$. The following is the key lemma:

Lemma 6.2.19 For any formulas φ, ψ, any $i \in [0,1]$ and any evaluation e,

(1) $e(\varphi \to \psi) \geq i$ iff there are $i_1, i_2 \in [0,1]$ such that

$$\begin{aligned}
&e(\varphi) \leq i_1,\\
&e(\psi) \geq i_2,\\
&i + i_1 - i_2 = 1.
\end{aligned}$$

(2) $e(\varphi \to \psi) \leq i$ iff there are $i_1, i_2 \in [0,1]$ and $y \in \{0,1\}$ such that

$$\begin{array}{rcl}
e(\varphi) & \geq & i_1,\\
e(\psi) & \leq & i_2,\\
y - i & \leq & 0,\\
y + i_1 & \leq & 1,\\
y & \leq & i_2,\\
y + i + i_1 - i_2 & = & 1.
\end{array}$$

Proof:

(1) Assume the condition; then either $e(\varphi) \leq e(\psi)$ and $e(\varphi \to \psi) = 1 \geq i$ or $e(\varphi) \geq e(\psi)$, $e(\varphi \to \psi) = 1 - e(\varphi) + e(\psi) \geq 1 - i_1 + i_2 = i$. Conversely, let $e(\varphi \to \psi) \geq i$. Assume $i < 1$. If $z \geq e(\varphi)$ then

[34] The first proof is due to Mundici [150]; our proof follows Hähnle's [71].

$(z \Rightarrow e(\psi)) \leq e(\varphi \to \psi)$; thus if there is an $i_1 \geq e(\varphi)$ such that $(i_1 \Rightarrow e(\psi)) = i$ then put $i_2 = e(\psi)$; you have $e(\varphi) \leq i_1$, $e(\psi) = i_2$ and $i = 1 - i_1 + i_2$. If $(1 \Rightarrow e(\psi)) = e(\psi) > i$ then put $i_1 = 1$, $i_2 = i$; you have $e(\varphi) \leq i_1$, $e(\psi) \geq i_2$, $i + i_1 - i_2 = i + 1 - i = 1$. Finally if $i = 1$ and $e(\varphi \to \psi) = 1$ then $e(\varphi) \leq e(\psi)$; put $i_1 = i_2 = e(\varphi)$.

(2) Assume the condition and let $y = 0$; we get $e(\varphi) \geq i_1$, $e(\psi) \leq i_2$, $i = 1 - i_1 + i_2 \leq 1$, $i_2 \leq i_1$, thus $(e(\varphi) \Rightarrow e(\psi)) \leq 1 - i_1 + i_2 = i$. Now assume the condition and let $y = 1$, then we get $i = 1$, $i_1 = 0$, $i_2 = 1$, $e(\varphi) \geq 0$, $e(\psi) \leq 1$, which is true for all φ, ψ; and also $e(\varphi \to \psi) \leq 1$ is always true. Thus the condition implies $e(\varphi \to \psi) \leq i$ in both cases for y. Conversely, assume $e(\varphi \to \psi) \leq i$. If $i = 1$ then choose $y = 1$, $i_1 = 0$, $i_2 = 1$ and check that all the conditions are satisfied. If $i < 1$ then choose $y = 0$; you have $1 - e(\varphi) + e(\psi) \leq i$, thus for some $i_1 \geq e(\varphi)$ and $i_2 \leq e(\psi)$ you get $1 - i_1 + i_2 = i$.

□

6.2.20 (Construction). Let us work with finite lists of conditions of the following kinds: at most one condition $i_0 > 0$, where i_0 is a variable ranging over $[0, 1]$; conditions of the forms $e(\varphi) \leq i$, $e(\varphi) \geq i$ where φ is a formula and i a variable ranging over $[0, 1]$; linear inequalities whose left hand side is a linear polynomial in some variables $i_1, \ldots$ over $[0, 1]$ and some variables $y_1, \ldots$ over $\{0, 1\}$ with integer coefficients, the inequality sign is $\leq$ and the right hand side is an integer. For testing $\varphi \in SAT_1^{Ł}$ the starting list consists just of $e(\varphi) \geq 1$; for $\varphi \in SAT_{pos}^{Ł}$ it consists of $(e(\varphi) \geq i_0, i_0 > 0)$. Using lemma 6.2.19 we may replace each such list containing at least one condition on $e(\alpha \to \beta)$ by another, which results by replacing this condition by some conditions on $e(\alpha)$, $e(\beta)$ and some new linear inequalities with some new variables: at most 5 new inequalities (an equality expressed as two inequalities) with at most 2 new real variables and 1 boolean variable. This reduction preserves satisfiability: there is a choice of values of propositional variables and of the i's and y's satisfying all the conditions of the starting list iff the same is true for the resulting list.

Iterating this we process each subformula of φ exactly once (distinct occurences of the same formula as a, subformula in φ understood as different subformulas; conditions $e(p_k) \geq i$ and $e(p_k) \leq i$ are replaced by $j_k \geq i$, $j_k \leq i$ where j_k is a new variable over $[0, 1]$ corresponding to $e(p_k)$. We finally get a list containing no conditions on formulas, thus consisting only of inequalities; thus we get a particular MIP-problem which has a solution iff the starting list is satisfiable. In the case of starting list $e(\varphi) \geq 1$ satisfiabily means that $\varphi \in SAT_1^{Ł}$; in the case of the list $(e(\varphi) \geq i_0, i_0 > 0)$ this means $\varphi \in SAT_{pos}^{Ł}$.

Thus we have reduced both $SAT_1^{\text{Ł}}$ and $SAT_{pos}^{\text{Ł}}$ to the general MIP problem; a pedantic analysis shows that the reduction is achieved in deterministic polynomial time (and the size of the constructed MIP problem is polynomial in the size of the starting formula). This completes the proof of Lemma 6.2.14 saying that both $SAT_1^{\text{Ł}}$ and $SAT_{pos}^{\text{Ł}}$ are in NP.

6.3. UNDECIDABILITY OF FUZZY LOGICS

In this section we shall investigate the position of the set of 1-tautologies of Ł∀, G∀, Π∀ respectively in the arithmetical hierarchy. We denote these sets by $TAUT^{\text{Ł}\forall}$, $TAUT^{\mathrm{G}\forall}$, $TAUT^{\Pi\forall}$ respectively. We assume that a predicate language $\mathcal{I}$ is fixed throughout and that K is recursive in the usual sense, so that e.g. the set of all formulas, proofs is recursive etc. (This is in particular the case if $\mathcal{I}$ is finite, i.e. has finitely many predicates and object constants.) The results are as follows: $TAUT^{\mathrm{G}\forall}$ is Σ_1-complete, $TAUT^{\text{Ł}\forall}$ is Π_2-complete (and so is $TAUT^{RPL\forall}$ — Pavelka predicate logic); $TAUT^{\Pi\forall}$ is Π_2-hard (and it is an open problem if it is in Π_2). Thus G∀ is as undecidable as the classical predicate calculus; Ł∀ and Π∀ are more undecidable (are not recursively axiomatizable). We start with a simple lemma.

Lemma 6.3.1 The logics G∀, Π∀ are undecidable, i.e. the sets $TAUT^{\mathrm{G}\forall}$, $TAUT^{\Pi\forall}$ are not recursive.

Proof: For G and Π extend the definition 6.2.8 to the predicate calculus ($\varphi^{\neg\neg}$ results by attaching double negation to each atomic formula). Clearly, φ is a Boolean tautology iff $\varphi^{\neg\neg}$ is in $TAUT^{\mathrm{G}\forall}$ iff $\varphi^{\neg\neg}$ is in $TAUT^{\Pi\forall}$. □

We shall prove undecidability of Ł∀ (and much more) below.

Corollary 6.3.2 $TAUT^{\mathrm{G}\forall}$ is a Σ_1-complete set.

Proof: Indeed, $TAUT^{\mathrm{G}\forall}$ is Σ_1-hard by the preceding lemma; on the other hand, $TAUT^{\mathrm{G}\forall}$ is in Σ_1 since by the Completeness Theorem 5.3.3, $\varphi \in$ $\in TAUT^{\mathrm{G}\forall}$ iff φ is provable in G∀. □

Lemma 6.3.3 $TAUT^{\text{Ł}\forall}$ is in Π_2.

Proof: This follows immediately from the (Pavelka style) completeness of RPL∀: $\varphi \in TAUT^{\text{Ł}\forall}$ iff the provability degree $|\varphi|_{\mathrm{RPL}\forall}$ of φ equals 1, i.e.

$$(\forall r < 1 \text{ rational})(\exists d)(d \text{ is a RPL-proof of } (\bar{r} \to \varphi)).$$

Clearly, this condition is Π_2. □

Theorem 6.3.4 $TAUT^{Ł\forall}$ is a Π_2-complete set.

This needs a rather complicated proof[35]; before we start it we show the following consequence:

Lemma 6.3.5 $TAUT^{\Pi\forall}$ is Π_2-hard (thus not in Σ_2).

Proof: We recursively reduce $TAUT^{Ł\forall}$ to $TAUT^{\Pi\forall}$ using the embedding of Ł into Π. Extend the given language by a new constant c and a new unary predicate Q; this gives a new closed formula $Q(c)$. For each formula φ of the original language define φ^* as follows:

$$\begin{array}{l} P(x_1,\ldots)^* \text{ is } Q(c) \vee P(x_1,\ldots) \\ 0^* = Q(c),\ \bar{1}^* = \bar{1}, \\ (\varphi \to \psi)^* = (\varphi^* \to \psi^*), \\ (\varphi \& \psi)^* = Q(c) \vee (\varphi^* \odot \psi^*), \\ ((\forall x)\varphi)^* = (\forall x)(\varphi^*). \end{array}$$

Put φ^{**} to be $\neg\neg Q(c) \to \varphi^*$. Now it is easy to show that $\varphi \in TAUT^{Ł\forall}$ iff $\varphi^{**} \in TAUT^{\Pi\forall}$ in an analogy to 4.1.16 (For each model $\mathbf{M} = = (M, (r_P)_P, (m_c)_c)$ define $\mathbf{M} \circ f_a$ to be $(M, (r'_P)_P, (m_c)_c)$ where $r'(u,\ldots,v) = f_a(r(u,\ldots,v))$; further let $\mathbf{M}^{\uparrow a}$ be the model $(M, (r''_P)_P, (m_c))$ where $r''_P(u,\ldots,v) = \max(a, r(u,\ldots,v))$. Then prove

$$\begin{array}{l} f_a(\|\varphi\|^{Ł\forall}_{\mathbf{M},v}) = \|\varphi^*\|^{\Pi\forall}_{\mathbf{M}\circ f_a,v}, \\ \|\varphi^*\|^{\Pi\forall}_{\mathbf{M},v} = \|\varphi^*\|^{\Pi\forall}_{\mathbf{M}^{\uparrow a},v}, \end{array}$$

thus

$$\begin{array}{rcl} f_a(\|\varphi\|^{Ł\forall}_{\mathbf{M}}) & = & \|\varphi^*\|^{\Pi\forall}_{\mathbf{M}\circ f_a}, \\ \|\varphi^*\|^{\Pi\forall}_{\mathbf{M}} & = & \|\varphi^*\|^{\Pi\forall}_{\mathbf{M}^{\uparrow a}} \end{array}$$

as in the proof of 4.1.16.) □

Theorem 6.3.6 The set of all formulas of Ł∀ true in all *finite* models is Π_1.

---

[35] Due to Ragaz [167].

Proof: This is because by the proof of Theorem 5.4.30 a formula is true in all finite models iff it is true in all rational-valued finite models (all relations r_P take only rational values). □

*

Now we elaborate the proof of $TAUT^{Ł\forall}$ being Π_2-complete.

Definition 6.3.7 (1) Call a formula *classical* if all connectives it contains are among $\wedge, \vee, \neg$.

(2) A model $\mathbf{M}$ is *predefinite* if for each classical formula φ and each evaluation v, $\|\varphi\|_{\mathbf{M},v} \neq \frac{1}{2}$.

(3) For an n-ary predicate P, $\delta(P)$ is the formula $[(\forall x_1, \ldots, x_n)(P(x_1, \ldots, x_n) \vee \neg P(x_1, \ldots, x_n))]^2$.

Lemma 6.3.8 A model $\mathbf{M}$ is predefinite iff $\|\delta(P)\|_{\mathbf{M}} > 0$ for each predicate P.

Proof: If for some P $\|\delta(P)\|_{\mathbf{M}} = 0$ then $\|(\forall \mathbf{x})(P(\mathbf{x}) \vee \neg P(\mathbf{x})\|_{\mathbf{M}} \leq \frac{1}{2}$, thus $\|(\forall \mathbf{x})(P(\mathbf{x}) \vee \neg P(\mathbf{x}))\|_{\mathbf{M}} = \frac{1}{2}$ (since $\|P(\mathbf{x}) \vee \neg P(\mathbf{x})\|_{\mathbf{M},v} \geq \frac{1}{2}$ for all v), which means that $\mathbf{M}$ is not predefinite. On the other hand, assume $\|\delta(P)\|_{\mathbf{M}} > 0$ for all P and let φ be a classical formula containing no predicates except $P_1, \ldots, P_n$; let $t = \min(\|\delta(P_1)\|_{\mathbf{M}}, \ldots, \|\delta(P_n)\|_{\mathbf{M}})$. This means that for $i = 1, \ldots, n$, $\|(\forall \mathbf{x})(P(\mathbf{x}) \vee \neg P(\mathbf{x}))\|_{\mathbf{M}} \geq (1+t)/2$ (since $((1+t)/2) * ((1+t)/2) = t$). It follows that for each $\mathbf{a} \in M^n$, $r_P(\mathbf{a}) > \frac{1}{2} + \frac{t}{2}$ or $r_P(a) < \frac{1}{2} - \frac{t}{2}$. This extends to an arbitrary classical formula φ built from $P_1, \ldots, P_n$: for each v, $\|\varphi\|_{\mathbf{M},v} > \frac{1}{2} + \frac{t}{2}$ or $\|\varphi\|_{\mathbf{M},v} < \frac{1}{2} - \frac{t}{2}$. Hence $\mathbf{M}$ is predefinite. □

Definition 6.3.9 For each predefinite structure $\mathbf{M} = (M, (r_P)_P, (m_c)_c)$ the corresponding Boolean structure is $\mathbf{M}_{/2} = (M, (r'_P)_P, (m_c)_c)$ where $r'_P(\mathbf{a}) = 1$ iff $r_P(\mathbf{a}) > \frac{1}{2}$, otherwise $r'_P(\mathbf{a}) = 0$.

Lemma 6.3.10 Let $\mathbf{M}$ be predefinite, φ a classical formula. Then

$$\|\varphi\|_{\mathbf{M}/2} = 1 \text{ iff } \|\varphi\|_{\mathbf{M}} > \tfrac{1}{2} \text{ iff } \|\varphi^2\|_{\mathbf{M}} > 0.$$

Proof: This is clear for atomic formulas and follows by an easy induction for all classical formulas (elaborate the induction step for $\neg, \wedge, \forall$ using the estimate Ł from 6.3.8). □

Lemma 6.3.11 The logic Ł$\forall$ is undecidable.

Proof: Generalizing 6.2.13 show that, for each classical closed formula φ containing predicates $P_1, \ldots, P_n$, φ is satisfiable in classical predicate logic, iff the formula $\delta(P_1)\& \cdots \&\delta(P_n)\&\varphi^2$ is positively satisfiable for Ł$\forall$ (there is a model **M** in which the above formula has a positive value). Thus φ is a Boolean tautology iff

$$[\neg(\delta(P_1)\& \cdots \&\delta(P_n)\&(\neg\varphi)^2)] \in TAUT^{\text{Ł}\forall}, \text{ i.e.}$$

$$[(\delta(P_1)\& \cdots \&\delta(P_n)) \rightarrow 2\varphi] \in TAUT^{\text{Ł}\forall}.$$

□

Definition 6.3.12 Let $\gamma(x, y)$ be a Σ_1-formula defining in **N** the relation C from 6.1.17, i.e. for each $m, n \in N$,

$$(m, n) \in C \quad \text{iff} \quad \|\gamma\|_{\mathbf{N}}[m, n] = 1.$$

Let $Predef$ stand for $\delta(=) \wedge \delta(S) \wedge \delta(A) \wedge \delta(B)$ (expressing predefiniteness of the interpretation of arithmetical language) and let, for each m, γ^*_m stand for the formula

$$Predef \wedge (QA)^2 \wedge (\forall x, y, z)(2num_m(z) \wedge 2\gamma(z, x) \wedge 2\gamma(z, y) \wedge 2(x \neq y) \rightarrow$$

$$\rightarrow (U(x) \not\equiv U(y)))$$

where U is a new unary predicate.

Remark 6.3.13 To understand γ^*_m imagine a crisp ($\{0, 1\}$-valued) model **M** of it: then all squarings and doublings are superfluous and **M** is a model of QA in which, for any two elements a, b such that both the pair (m, a) and the pair (m, b) satisfies γ, $\|U\|_{\mathbf{M}}[a] \neq \|U\|_{\mathbf{M}}[b]$; thus exactly one of a, b satisfies U. Thus in this case (of a *crisp* model of γ^*_m), C_m has at most two elements since for each $n \in C_m$ we have $\|\gamma\|_{\mathbf{N}}[m, n] = 1$ and hence (γ being Σ_1), $\|\gamma\|_{\mathbf{M}}[m', n'] = 1$ and $\|num_m(m')\|_{\mathbf{M}} = 1$ for m', n' being the isomorphic copies of m, m in **M**.

Lemma 6.3.14 Under the above relation, C_m is finite iff $\gamma^*_m \in SAT^{\text{Ł}\forall}_{pos}$.

Proof: First assume C_m finite, $C_m = \{n_1, \ldots, n_k\}$. Take **N** and expand it by defining $r_U(n_i) = i/k$, $r_U(j) = 0$ for j distinct from $n_1, \ldots, n_k$. Verify easily that for the resulting model **M**, $\|\gamma_m^*\|_{\mathbf{M}} \geq \frac{1}{k}$. Indeed, $\|Predef\|_{\mathbf{M}} = 1$, $\|QA\|_{\mathbf{M}} = 1$. Take $a, b \in N$ and assume that the value $\|2num_m(u) \wedge 2\gamma(u,x) \wedge 2\gamma(u,y) \wedge 2(x \neq y)\|_{\mathbf{M}}[m, a, b]$ is positive (otherwise there is nothing to prove). This means that the values of $num_m(u), \gamma(u,x), \gamma(u,y), x \neq y$ are $> \frac{1}{2}$ and hence $= 1$ (since in **M** everything except r_U is crisp). But then $a \neq b$ and $a, b \in C_m$; thus for some $i, j \leq k$ we have $a = n_i, b = n_j, i \neq j$ and $\|U(x) \not\equiv U(y)\|_{\mathbf{M}}[n_i, n_j] = |\frac{i}{k} - \frac{j}{k}| \geq \frac{1}{k}$. Thus $\|\gamma_m^*\|_{\mathbf{M}} \geq \frac{1}{k}$.

Conversely, let C_m be infinite, $C_m = \{n_i\}_{i=1}^{\infty}$. We show that γ_m^* is not positively satisfiable. Assume it is, let $\|\gamma_m^*\|_{\mathbf{M}} = t > 0$. Delete r_U from **M**; you get a predefinite model $\mathbf{M}'$ of the language of QA, hence $\mathbf{M}'' = \mathbf{M}'/_2$ is a model of QA. We may assume that **N** is an initial segment of $\mathbf{M}''$. Since $\|\gamma\|_{\mathbf{N}}[m, n_i] = 1$ for $i = 1, 2, \ldots$, we have $\|\gamma\|_{\mathbf{M}''}[m, n_i] = 1$, thus $\|\gamma\|_{\mathbf{M}'}[m, n_i] > \frac{1}{2}$. Hence $\|2\gamma\|_{\mathbf{M}'}[m, n_i] = 1$. For $i \neq j$ we get $\|2(x \neq y)\|_{\mathbf{M}'}[n_i, n_j] = 1$. Come back to **M** (returning r_U). Since $\|\gamma_m^*\|_{\mathbf{M}} = t$ we get $\|U(x) \not\equiv U(y)\|_{\mathbf{M}}[n_i, n_j] \geq t$, thus putting $\|U\|_{\mathbf{M}}[n_i] = t_i$ we get $|t_i - t_j| \geq t$ for $i \neq j$. But this is a clear contradiction, because i, j runs over all natural numbers and t_i, t_j, t are positive reals. This completes the proof. □

Corollary 6.3.15 The set $SAT_{pos}^{\text{Ł}\forall}$ is Σ_2-complete and the set $TAUT_1^{\text{Ł}\forall}$ is Π_2-complete.

We have not yet discussed $TAUT_{pos}^{\text{Ł}\forall}$. The next theorem contains the answer (and much more).

Definition 6.3.16 Let, for each r, $TAUT_{(r,1]}^{\text{Ł}\forall}$ be the set of all formulas φ such that $\|\varphi\|_{\mathbf{M}} > r$ for all **M**; similarly, $TAUT_{[r,1]}^{\text{Ł}\forall}$ is the set of all φ such that $\|\varphi\|_{\mathbf{M}} \geq r$ for all **M**.

Theorem 6.3.17 [36]

(1) For each $r < 1$, $TAUT_{(r,1]}^{\text{Ł}\forall}$ is a Σ_1-complete set.

(2) For each $r > 0$, $TAUT_{[r,1]}^{\text{Ł}\forall}$ is a Π_2-complete set.

[36] (1) is due to Mostowski [145]; Chang proved the set in (2) not to be recursively enumerable – see [12].

Proof:

(1) Let $r > 0$; if $\varphi \in TAUT^{Ł\forall}_{(r,1]}$ then $\varphi \to \bar{r}$ has no model and therefore is contradictory (cf. 5.4.9); thus $(\varphi \to \bar{r}) \vdash \bar{0}$. Conversely, if $(\varphi \to \bar{r}) \vdash \bar{0}$ then $(\varphi \to \bar{r})$ has no model. Clearly $(\varphi \to \bar{r}) \vdash \bar{0}$ is a Σ_1 condition; thus $TAUT^{Ł\forall}_{(r,1]}$ is in Σ_1. We reduce $TAUT^{Bool\forall}$ to $TAUT^{Ł\forall}_{(r,1]}$: φ is a tautology of Boolean predicate logic (written using $\wedge, \vee, \neg$) iff $Predef \to 2\varphi$ is in $TAUT^{Ł\forall}_{(0,1]}$ (as above), iff $\bar{r}\underline{\vee}(Predef \to 2\varphi)$ is in $TAUT^{Ł\forall}_{(r,1]}$.

(2) $\varphi \in TAUT^{Ł\forall}_{[r,1]}$ iff $(\bar{r} \to \varphi) \in TAUT^{Ł\forall}_{1}$; this reduces $TAUT^{Ł\forall}_{[r,1]}$ to $TAUT^{Ł\forall}_{1}$. Conversely, $\varphi \in TAUT^{Ł\forall}_{1}$ iff $(\varphi \& \bar{r}) \in TAUT^{Ł\forall}_{[r,1]}$; this gives the converse reduction (assuming $r > 0$). Hence $TAUT^{Ł\forall}_{1}$ and $TAUT^{Ł\forall}_{[r,1]}$ are recursively equivalent.

□

Remark 6.3.18 It is easy to check that in the last theorem Ł∀ may be replaced by RPL∀.

CHAPTER SEVEN

ON APPROXIMATE INFERENCE

In chapters 2 – 6 we developed and investigated real-valued (or lattice-valued) propositional and predicate calculi based on the notion of a (continuous) t-norm and its residuum. Now our understanding of these calculi is reasonably advanced: we have obtained several completeness theorems etc. But now we must ask: how does this relate to fuzzy logic (in the narrow sense, obviously)?

Let us again quote from Zadeh [221]:

"In a narrow sense, fuzzy logic (FLn) is a logical system which aims at a formalization of approximate reasoning. As such, it is rooted in multivalued logic but its agenda is quite different from that of traditional multivalued logical systems, e.g., Łukasiewicz logic. In this connection, it should be noted that many of the concepts which account for the effectiveness of fuzzy logic as a logic of approximate reasoning are not a part of traditional multivalued logical systems. Among these are the concepts of a linguistic variable, canonical form, fuzzy rule, fuzzy graph, fuzzy quantifiers and such modes of reasoning as interpolative reasoning, syllogistic reasoning and dispositional reasoning."

In this chapter and also in some further chapters we shall investigate some items from Zadeh's agenda of questions or topics typical of fuzzy logic in this meaning: without denying that they do not belong to the "classical" questions of many-valued logic we shall try to show that these topics admit a strictly logical analysis in the sense of formal logic. It will turn out that much of approximate inference may be presented as *deduction* in a suitably chosen logic, i.e. in a truth-preserving way. This does not mean that we claim to *reduce* fuzzy logic to many valued logic; but it does mean that the role of many-valued logic as a base (foundation) of fuzzy logic is much greater and more important than one would guess at the beginning. It goes without saying that fuzzy logic indeed has aspects that are not grasped by classical many-valued logic (as control aspects of fuzzy control); but, I repeat, the *deductive* aspect (with its corresponding semantical counterpart) is very important.

Section 1 analyses Zadeh's compositional rule of inference and two particular cases of it: *generalized modus ponens* and a (less known) *generalized conjunctive rule*. We also comment on some dangers in certain popular but questionable uses of fuzzy logic in expert systems. Section 2 analyzes logical aspects of fuzzy control. Section 3 presents an alternative formalization of the two generalized rules and the corresponding alternative formalization of fuzzy control.

7.1. THE COMPOSITIONAL RULE OF INFERENCE

7.1.1 We start with the notion of a *variate*.[37] A variate is given by its *name* X and its *domain* D. X is just a symbol; D is a non-empty set. Examples are: age with the domain of integers ≤ 120 (say), temperature (with some domain), etc. Fuzzy logic notoriously uses expressions of the form "X is A" where A is (the name of) a fuzzy subset of D, e. g. "the age is high". These expressions typically occur in *fuzzy rules*, to be analyzed later.

It is not automatically clear how this fits into our formalism of predicate calculus, and, in fact, we shall present two ways of doing it. In this and the next section, we shall proceed as follows: having n variates $(X_1, D_1), \cdots,$ $\cdots (X_n, D_n)$ we understand the D's as domains of a many-sorted structure interpreting a predicate language; fixed fuzzy subsets of a domain interpret some unary predicates. Besides we may have predicates of higher arity and their interpretation in our many-sorted structure. The most important thing comes now: the name of a variate is taken to be an *object constant*, interpreted in each situation as the *actual* value of the variate. The expression "X is A" becomes an atomic closed formula $A(X)$ (once again: A is a unary predicate, X is an object constant). A typical rule "IF X is A THEN Y is B" may be interpreted as $A(X) \rightarrow B(Y)$. (Caution: this is not the only possible reading of a "fuzzy IF – THEN rule", as we shall see later.)

Summarizing: We have understood an n-tuple of variates $(X_i, D_i)_{i=1}^n$ as determining a language $\mathcal{I}$ with sorts $s_1, \cdots, s_n$ and with object constants $X_1, \cdots, X_n$, X_i of the sort s_i. Besides our language may contain arbitrary predicates of any type and any other object constants. Any $\mathcal{I}$-structure

$$\langle (D_i)_i, , (r_P)_{P\,pred.}, m_{X_1}, \cdots, m_{X_n} \cdots \rangle$$

(where the dots stand for the interpretation of additional constants, if any) is understood as a fuzzy structure over the given variates, with X_i denoting the actual value of the i-th variate.

Given this, we may start our analysis of Zadeh's compositional rule of inference.[38]

7.1.2 The *compositional rule of inference* in its traditional formulation can be stated as follows:

From "X is A" *and* "(X, Y) is R" *infer* "Y is B" if for all $v \in D_Y$,

$$r_B(v) = \sup_{u \in D_X} (r_A(u) * r_R(u, v)).$$

---

[37] Similarly to [86] we reserve the term "variable" for variables of a logical calculus (propositional variable, object variable) and use the term "variate" for what several people call variable.

[38] See [213].

where $*$ is a continuous t-norm. The relation r_B is sometimes called the *composition* of r_A and r_R, or the *image* of r_A under the relation r_R.

7.1.3 We ask: what does this mean? What is inferred? Is this any *deduction*?

First observe that the rule is *semantical*: given any structure $\mathbf{D}$ of domains of variates, it formulates a semantical condition on the fuzzy set r_B interpreting the predicate B in terms of r_A (fuzzy set) and r_R (fuzzy relation). Typically, no more explanation is given. The rule is stated more or less as self-evident.

But observe that in fact the definition of r_B in terms of r_A and r_R is expressible in BL$\forall$: The condition above just means that the formula $(\forall y)(B(y) \equiv (\exists x)(A(x) \,\&\, R(x,y))$ is 1-true in $\mathbf{D}$. Call the last formula $Comp$ (or $Comp(A,R,B)$, the composition); thus the rule demands (assumes) $\|Comp\|_{\mathbf{D}} = 1$.

Lemma 7.1.4 Under the present notation,

$$\text{BL}\forall \vdash Comp \to ((A(X)\&R(X,Y)) \to B(Y)).$$

Consequently, for each structure $\mathbf{D}$ such that $\|Comp\|_{\mathbf{D}} = 1$, $\|A(X)\&R(X,Y)\|_{\mathbf{D}} \leq \|B(Y)\|_{\mathbf{D}}$

Proof: By 5.1.14,
BL$\forall \vdash (\forall y)([(\exists x)(A(x)\&R(x,y)) \to B(y)] \to (\forall x)[(A(x)\&R(x,y)) \to B(y)])$ and
BL$\forall \vdash (\forall y)(\forall x)((A(x)\&R(x,y)) \to B(y)) \to [(A(X)\&R(X,Y)) \to B(Y)]$.
Thus
BL$\forall \vdash Comp \to ((A(X)\&R(X,Y)) \to B(Y))$
by transitivity of implication. The rest is evident. □

We also show that $Comp$ is the best condition making the above inference possible.

Lemma 7.1.5 Let B' be a unary predicate of the same sort as B. Then
BL$\forall \vdash Comp \,\&\, (\forall y)(\forall x)((A(x)\&R(x,y)) \to B'(y)) \to (\forall y)(B(y) \to B'(y))$
(thus in each model $\mathbf{D}$, if $Comp$ is true then r_B is the smallest fuzzy subset of D_Y making the inference of the composition rule sound).

Proof: Evidently,
BL$\forall \vdash (\forall y)(\forall x)((A(x)\&R(x,y)) \to B'(y)) \to$

$\rightarrow (\forall y)[(\exists x)(A(x)\&R(x,y)) \rightarrow B'(y)]$,
but assuming $Comp$,the formula $(\exists x)(A(x)\&R(x,y))$ is equivalent to $B(y)$, thus we get the desired implication.
Proof: □

We shall now discuss two important particular cases. For this purpose we replace the atomic formula $R(x,y)$ by an arbitrary formula $\varphi(x,y)$.

Corollary 7.1.6 (1) Let $Comp$ be the formula
$(\forall y)(B(y) \equiv (\exists x)(A(x) \,\&\, \varphi(x,y)))$. Then
$\mathrm{BL}\forall \vdash (Comp \,\&\, A(X) \,\&\, \varphi(X,Y)) \rightarrow B(Y)$.

Proof: Evident modifications of the above. □

7.1.7 Now we shall consider Zadeh's Generalized Modus Ponens as a particular case of the Compositional Inference rule. To this end let us slightly change notation: we replace A by A^*, B by B^* and then take $\varphi(x,y)$ to be $A(x) \rightarrow B(y)$ for some predicates A,B. Then 7.1.4 gives the following theorem.

Theorem 7.1.8 Let $Comp_{MP}$ be the formula

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x) \,\&\, (A(x) \rightarrow B(y)))).$$

Then BL∀ proves

$$(Comp_{MP} \,\&\, A^*(X) \,\&\, (A(X) \rightarrow B(Y))) \rightarrow B^*(Y).$$

Remark 7.1.9 (1) This may be visualized as a deduction rule:

$$\frac{Comp_{MP}, A^*(X), A(X) \rightarrow B(Y)}{B^*(Y)}$$

The obvious reading is: if $Comp_{MP}$, $A^*(X), A(X) \rightarrow B(Y)$ are 1-true (in a given structure **D**) then $B^*(Y)$ is 1-true. But 7.1.8 gives more:

$$\|Comp_{MP} \,\&\, A^*(X) \,\&\, (A(X) \rightarrow B(Y))\|_{\mathbf{D}} \leq \|B^*(Y)\|_{\mathbf{D}},$$

in particular, if $Comp_{MP}$ is 1-true (B^* is defined as above), $A^*(X)$ is r-true and $A(X) \rightarrow B(Y)$ is s-true then $B^*(Y)$ is at least $(r * s)$ - true (where $*$ is the truth function of &).

(2) We stress that we have shown *provability in* BL∀. Thus the above rule may be read for implication and conjunction in Ł∀, G∀, Π∀ and any other predicate calculus given by a continuous t-norm.

(3) Furthermore, observe that you may replace & by $\wedge$ in 7.1.8 and make the obvious modifications, e.g. if the (modified) $Comp_{MP}$ is 1-true, $A^*(X)$ is r-true and $A(X) \to B(Y)$ is s-true then $B^*(Y)$ is $\min(r, s)$-true.

(4) Let us mention that the use of A, A^*, B, B^* should suggest that A^* is similar to A in some sense - and then $Comp_{MP}$ should say that B^* is similar to B in some other sense.[39] But be careful: If A, B, A^* are interpreted by crisp (0, 1 - valued) subsets of the respective domains then the interpretation of B^* is also crisp and

(i) either $r_{A^*} \subseteq r_A$ and $r_{A^*} \neq \emptyset$ and $r_{B^*} = r_B$,

(ii) or $r_{A^*} \subseteq r_A$ and $r_{A^*} = \emptyset$ and $r_{B^*} = \emptyset$,

(iii) or r_{A^*} is not a subset of r_A and then $r_{B^*} = D_Y$ (the full set). (This can be formulated and proved in BL$\forall$ - exercise.) See Figure 7.1.

(5) In general, if $Comp_{MP}$ is defined as in 7.1.8 then
$Comp_{MP} \vdash (\forall y)[((\exists x)(A^*(x) \,\&\, \neg A(x)) \to B^*(y)]$.
Thus for each $v \in D_Y, r^*_B(v) \geq \sup_{u \in D_X}(r_{A^*}(u) * (-)r_A(u))$.

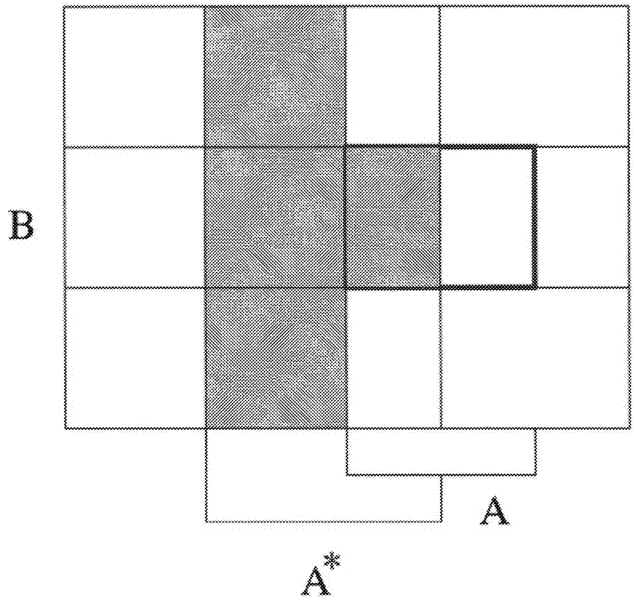

Figure 1. The grey domain is the set of pairs satisfying $A^*(x)\&(A(x) \to B(y))$.

(6) Observe that, under the present notation, and over BL$\forall$,
$Comp_{MP} \vdash (\exists x)A^*(x) \to (\forall y)(B(y) \to B^*(y))$.
Indeed, the following formulas are provable:

[39] The notorious example is: If the colour is red then the tomato is ripe; the colour is very red – what follows?

$B(y) \to (A(x) \to B(y))$,
$A^*(x) \,\&\, B(y)) \to (A^*(x) \,\&\, (A(x) \to B(y))$,
$(\exists x)(A^*(x) \,\&\, B(y)) \to (\exists x)(A^*(x) \,\&\, (A(x) \to B(y))$,
$(\exists x)(A^*(x) \,\&\, B(y)) \to B^*(y)$,
$[(\exists x)A^*(x) \,\&\, B(y)] \to B^*(y)$,
$(\exists x)A^*(x) \to (B(y) \to B^*(y))$ (by 5.1.18, (9)),
$(\exists x)A^*(x) \to (\forall y)(B(y) \to B^*(y))$.
Thus in particular, if for some $u \in D_X$, $r_{A^*}(u) = 1$, then for all $v \in D_Y$, $r_B(v) \leq r_{B^*}(v)$.

*

7.1.10 Now let us have predicates A, A^* (of the same sort), B, B^* (of the same sort) and take $A(x) \,\&\, B(y)$ for $\varphi(x,y)$. (Note that we could also take $A(x) \wedge B(y)$.) We automatically get a BL$\forall$- provable tautology and hence a sound deduction rule. We shall ask what it means and how it relates to classical logic. In the next section we shall use our analysis of Generalized Modus Ponens and the present Generalized Conjunctive Rule to an analysis of the logical background of fuzzy controllers. As above, 7.1.4 gives the following

Theorem 7.1.11 Let $Comp_{CR}$ be

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x) \,\&\, A(x) \,\&\, B(y))).$$

Then BL$\forall$ proves

$$Comp_{CR} \,\&\, A^*(x) \,\&\, A(x) \,\&\, B(y) \to B^*(y)$$

Remark 7.1.12 (1) As above, we get a sound deduction rule

$$\frac{Comp_{CR}, A^*(X), A(X) \,\&\, B(Y)}{B^*(Y)},$$

Soundness means: for each domain structure $\mathbf{D}$,

$$\|Comp_{CR}, A^*(X) \,\&\, A(X) \,\&\, B(Y)\|_{\mathbf{D}} \leq \|B^*(Y)\|_{\mathbf{D}}.$$

In the language of "fuzzy rules": From "X is A and Y is B" and "X is A^*" infer "Y is B^*". This is not a frequently used "fuzzy rule", but we get it naturally from the Compositional rule.

(2) There are at least two particular cases in the classical (Boolean) case. First, the usual rule for conjunction, second a rule of contradiction:

$$\frac{A(X) \,\&\, B(Y)}{B(Y)} \qquad \frac{\neg A(X), A(X) \,\&\, B(Y)}{\neg B(Y) \,\&\, B(Y)}$$

(3) In general, for A, B, A^* crisp, we get the picture for $A^*(x) \,\&\, A(x) \,\&$ $\&\, B(y)$ given in Fig. 7.2:

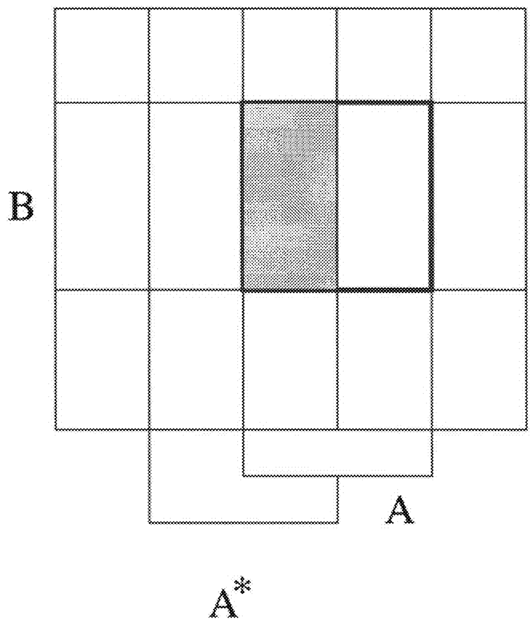

Figure 2.

(i) either $r_A \cup r_{A^*} \neq \emptyset$ (for some u, $r_A(u) = r_{A^*}(u) = 1$) and $r_B = r_{B^*}$
(ii) or $r_A \cup r_{A^*} = \emptyset$ and then $r_{B^*} = \emptyset$ ($r_{B^*}(v) = 0$ for all $v \in D_Y$).

(4) In full generality, the following is provable over BL$\forall$: (for the present meaning of $Comp_{CR}$):

(i) $Comp_{CR} \vdash (\forall y)(B^*(y) \to B(y))$,

(ii) $Comp_{CR} \vdash (\forall y)(B^*(y) \to (\exists x)(A(x) \,\&\, (A^*(x)))$.

Indeed, assuming $Comp_{CR}$ we prove
$B^*(y) \to (\exists x)(A^*(x) \,\&\, A(x) \,\&\, B(y)) \to [(\exists x)(A^*(x) \,\&\, A(x)) \,\&\, B(y)]$ (by 5.1.18 (9)).

From (ii) we also get
$Comp_{CR} \vdash (\exists y)B^*(y) \to (\exists x)(A(x) \,\&\, A^*(x))$.

and hence
$Comp_{CR} \vdash \neg(\exists x)(A(x) \,\&\, A^*(x) \to \neg(\exists y)B^*(y)$.

(5) We again stress that all provabilities are over BL$\forall$. These are valid over

Ł∀, G∀, Π∀ and any other t-norm logic. Moreover, the reader may check that everything remains valid if we replace & by $\wedge$ throughout.

Example 7.1.13 We shall present here a critical analysis of the use of the compositional rule of inference in some fuzzy expert systems like CADIAG-2 (developed by K.P. Adlassnig and his group; see [3, 4, 5, 117] or, for a short description , [115]; for a detailed analysis see [32]). We have three sets: M – the set of patients, S – the set of symptoms and D - the set of diagnoses. Further we have three fuzzy binary relations: $P : (M \times S) \to [0,1]$ expressing for each pair $(p, s), p \in M, s \in S$, how much p has the symptoms s, further $R : (S \times D) \to [0,1]$ expressing for each $s \in S$ and $d \in D$ how much s confirms d, and $P' : (M \times D) \to [0,1]$ expressing how much a patient p has the diagnosis d. Thus we have a structure with three domains M, S, D and three binary relations. Let $Has, Conf, Has'$ be binary predicates naming P, R, P' respectively; thus $Has(x, s)$ says "x has the symptom s", similarly $Has'(x, d)$; and $Conf(s, d)$ says "s confirms d". Using the compositional rule (for an arbitrary but fixed x) one defines

$$Diag(x, d) \equiv (\exists s)(Has(x, s)\&Conf(s, d)).$$

This defines over Ł∀ or other logic a relation $C : (P \times D) \to [0,1]$, expressing for each p and d how much d is confirmed for p.

So far so good; but *what does this mean*? Clearly, the answer depends on the definition of the meaning of $Conf$, i.e. of the relation R. Here our *warning comes:* In CADIAG and similar systems, one defines R from some data. $R(s, d)$ is taken to be the relative frequency $Fr(d|s)$ of presence of d among objects (patients) having d (for implicity, symptoms and diagnoses are assumed to be crisp). What does $Diag(x, d)$ mean in this case? This is difficult to say; but one thing is clear. Let p be a patient and let $s_1, \ldots, s_k$ be the symptoms he has. We would be interested in knowing, or at least estimating $Fr(d|s_1 \ldots s_k)$ – the relative frequency of d among object having $s_1 \ldots, s_k$ – as a possible estimate of the value of $Has'(x, d)$. But it must be clearly said that $Diag$ does *not* estimate this relative frequency; $Diag$ just defines $\max(Fr(d|s_1), \ldots, Fr(d(s_k))$. And observe that it can happen, for example, that $(k = 2)$, $Fr(d|s_1) = Fr(d|s_2) = 0.9$ but $Fr(d|s_1, s_2) = 0.2$. (See the following frequency table)

s_1	s_2	d	
1	1	1	1
1	0	1	44
0	1	1	44
0	0	1	5
1	1	0	4
1	0	0	1
0	1	0	1
0	0	0	3

Note that it does not help to allow S to contain conjunctions of symptoms – see [32] for details.

Let us offer one possible interpretation of $Conf$ with desirable properties (without any claim that this is the only right interpretation). Assume the relations P, P' interpreting Has, Has' to be fuzzy and let

$$Conf(s,d) \equiv (\forall x)(Has(x,s) \rightarrow Has'(x,d)).$$

Thus the truth degree of $Conf(s,d)$ is the minimum, over all patients x, of the truth degrees of the implication $Has(x,s) \rightarrow Has'(x,d)$. *Caution.* For example, in Ł$\forall$, $\|Conf(s,d)\| = 0.9$ means that for each patient, $\|Has(x,s)\| \leq \min(1, Has'(x,d)\| + 0.1)$. (All truth values in the given structure.) Then for each patient p, the following formulas are true:

$(Has(p,s)\&Conf(s,d)) \rightarrow Has'(p,d)$, thus

$(\exists s)(Has(p,s)\&Conf(s,d)) \rightarrow Has'(p,d)$,

i.e. $Diag(p,d) \rightarrow Has'(p,d)$. Consequently, the truth degree of $Diag(p,d)$ is a lower bound of the degree in which p has the diagnosis d. This is surely pleasing; the question remains if e.g. the physician can supply knowledge necessary to evaluate $Conf$ in the present meaning.

7.1.14 In the rest of this section we shall discuss the question of what are "the present (actual) values of variables X, Y" and sketch a formalism for this. The present discussion will be used in an example in Ch. 8 and may be skipped now if desired.

We can imagine a new domain M of objects (situations, time elements, persons) interpreting a new sort of variables; denote the new variables by $z, z_1, \cdots$. We assume that for each object $m \in M$, each variate X takes a value $f_X(m) \in D_X$ (temperature, colour, etc. *of* m). Thus X becomes a *function symbol*; $X(z)$ is a term. Fixing an $m_0 \in M$ as a meaning of a new constant c for *actual object*, $X(c), Y(c)$ are terms for the values of the corresponding variates for the actual object.

Definition 7.1.15 Formally, this leads to a many-sorted many-valued calculus whose language consists of a sort of s_0 objects, sorts $s_1, \cdots, s_n$ for domains of variates,

- unary function symbols $X_1, \cdots, X_n$ for variates and an object constant c for "the actual object" (possibly other constants),
- variables for each sort,
- predicates of various types.

The language is called the *ground language*. A *structure* for this language has the form

$$\mathbf{M} = \langle M, D_{X_1}, \cdots, D_{X_n}, f_1, \cdots, f_n, (r_P)_{P\,predicate}, m_c, \cdots \rangle$$

(dots for possible meanings for other constants) where each f_i maps M into D_{X_i}. $\mathbf{M}$ is called a *ground structure.*

Terms are variables, constants and expressions $X_i(z)$ where X_i is the name of the i-th variate and z is a variable of the sort s_0. Everything else is as usual;

$$\|X_i(z)\|_{M,v} = f_i(v(z)).$$

7.1.16 We may now formulate axioms like $(\forall z)(A(X(z)) \to B(Y(z))$, saying "for each object z (situation etc.), if the value of the variate X (on the object z, in the situation z etc.) is A (big, etc) then the value of Y (on z) is B". Then e.g. the Generalized Modus Ponens with the old condition $Comp_{MP}$ saying $(\forall y)(B^*(y) \equiv (\exists x)(A^*(x) \,\&\, A(x) \to B(y))$ can be formulated as

$$Comp_{MP} \to (\forall z)((A^*(X(z)) \,\&\, (A(X(z)) \to B(Y(z))) \to B^*(Y(z))$$

and shown to be provable in the corresponding obvious modification of BL$\forall$ (with a limited use of function symbols). You may derive various corollaries, e.g.

$$\frac{Comp_{MP}, (\forall z)(A(X(z)) \to B(Y(z)), A^*(X(c))}{B^*(Y(c))}$$

saying that if B^* is defined as $Comp_{MP}$ demands, if for all situations, [X is A implies Y is B] and if in the actual situation X is A^* then in the actual situation Y is B^*. Soundness of the rule guarantees e.g. that if $Comp_{MP}$ and [for all z, X is A implies Y is B] are true (i.e. 1-true) in a given ground structure and [X is A^*] is r-true in the actual situation then [Y is B^*] is at least r-time in this situation.

This appears to be the the way that the Generalized Modus Ponens is actually used.

7.2. FUZZY FUNCTIONS AND FUZZY CONTROLLERS

7.2.1 Fuzzy control is apparently the most broadly used application of fuzzy logic. Various books explaining fuzzy control, written by non-logicians, suffer by logical mismatch caused by the fact that "fuzzy IF-THEN rules" are presented as implications but then used to construct a fuzzy relation having little to do with any implication, at least at first glance (the relation is defined by a disjunction of conjunctions). Attempts to call e.g. the min-conjunction a "Mamdani implication" (see e.g. [35]) must be strictly rejected since we insist that the fuzzy truth function of a connective must behave classicaly for extremal values $0, 1$ – and this is not the case for minimum as implication. It has slowly become clear that fuzzy control deals with *approximation of functions* on the basis of pieces of fuzzy information of the kind "for arguments approximately equal c_i the image is approximately equal to d_i".[40]

It is illuminating to analyze the crisp situation. Assume we have two domains M_1, M_2 and a crisp, possibly partial, function f from M_1 to M_2. Moreover, let us have distinct elements $(u_1, v_1), \ldots, (u_n, v_n) \in M_1 \times M_2$ such that for each $i = 1, \ldots, n$, $f(u_i) = v_i$. Let us have a two-sorted language with equality (denoted $=$ for both domains) and a binary predicate F interpreted by f, let $\mathbf{M} = \langle M_1, M_2, f, =_1, =_2 \rangle$ where $=_i$ is identity on M_i, x-variables range on M_1, y-variables on M_2. The fact that f is a partial mapping is expresed by the sentence $(\forall x, y_1, y_2)((F(x, y_1) \wedge F(x, y_2)) \to y_1 = y_2)$. Let c_i be the constants for u_i, and d_i for v_i respectively.

(1) The formula

$$\bigwedge_i F(c_i, d_i)$$

just expresses the fact that $f(u_i) = v_i$; it is true in $\mathbf{M}$.

(2) The formula

$$\bigwedge_i ((x = c_i) \to (y = d_i))$$

defines a relation $r \subseteq M_1 \times M_2$ whose restriction to $\{u_1, \ldots, u_n\}$ coincides with the restriction of f to $\{u_1, \ldots, u_n\}$ and containing all pairs (u, v) where u is distinct from all $u_1, \ldots, u_n$ and $v \in M_2$; thus $f \subseteq r$.

(3) The formula

$$\bigvee_i (x = c_i \wedge y = d_i)$$

defines a relation $s \subseteq M_1 \times M_2$ which is the restriction of f to $\{u_1, \ldots, u_n\}$; i.e. no pair (u, v) with u distinct from all $u_1, \ldots, u_n$ belongs to s. Thus $s \subseteq f$.

[40] For analyses of IF-THEN rules see [43, 118, 119, 155, 198]. Our presentation is a free elaboration of Kruse et al. [118] Sec. 4.4-4.5; and Godo and Hájek [61, 63, 62]; but our notion of a fuzzy function seems to be new.

Compare this, in the fuzzy case, with the deduction rules of the last section. We shall develop a general theory of fuzzy functions and "partial knowledge" on them and then apply it to describe (the logical aspect of) fuzzy control. We shall systematically develop the theory in BL$\forall$ (showing various statements to be provable); thus this will give, in particular, sound results for any t-norm logic $\mathcal{C}\forall$. To simplify matters, we shall deal only with unary functions (having one argument); a generalization to functions of several variables is easy.

After having discussed fuzzy functions we shall investigate the general logical structure of fuzzy controllers, not using fuzzy functions. The hurrying reader, not interested in fuzzy functions, may skip to 7.2.17.

Definition 7.2.2 Let T be a theory with a binary predicate F of a type (t_1, t_2), let $\approx_i$ be a similarity predicate in T for the sort t_i. (We shall write $\approx$ both for $\approx_1$ and $\approx_2$ without any danger of confusion.) We say that F *defines a (partial) fuzzy function* in T *with respect to* $\approx$ if T proves the following:

$$(x \approx x' \,\&\, y \approx y') \to (F(x, y) \equiv F(x', y')),$$

$$(F(x, y) \,\&\, F(x, y')) \to y \approx y'.$$

The first formula is the congruence axiom (cf. 5.6.5); the second says that any two images of x are similar.

Lemma 7.2.3 Let F define a partial fuzzy function in T w.r.t. $\approx$. Let c, d be constants such that $T \vdash F(c, d)$.

(1) Then $T \vdash (x \approx c \,\&\, F(x, y)) \to y \approx d$.

(2) Moreover, if $A(x)$ is the formula $x \approx c$ and $B(y)$ is the formula given by the condition $Comp$ of the compositional rule of inference from F and A (cf. 7.1.6 (2)), i. e. $B(y)$ is $(\exists x)(x \approx c \,\&\, F(x, y))$ then $T \vdash (B(y) \equiv y \approx d)$. (Thus the compositional rule transforms $x \approx c$ and $F(x, y)$ to $y \approx d$.)

Proof: (1) In T, $x \approx c \,\&\, F(x, y)$ implies $F(c, y)$ which, due to the provability of $F(c, d)$ gives $y \approx d$.

(2) Clearly, $T \vdash y \approx d \to [c \approx c \,\&\, F(c, y)] \to [(\exists x)(x \approx c \,\&\, F(x, y))]$. On the other hand, $T \vdash (\exists x)(x \approx c \,\&\, F(x, y)) \to y \approx d$ follows from (1). $\square$

Definition 7.2.4 Let r_i be a similarity on M_i (i = 1, 2). A fuzzy relation $s : (M_1 \times M_2) \to [0, 1]$ is a *fuzzy mapping* from M_1 into M_2 *w.r.t.* r_1, r_2 if s is extensional, i. e. for all $x, x' \in M_1, y, y' \in M_2$,

$$r_1(x, x') * r_2(y, y') * s(x, y) \leq s(x', y')$$

and functional, i. e.

$$s(x,y) * s(x,y') \leq r_2(y,y')$$

Lemma 7.2.5 Let F define a function w.r.t. $\approx$ in T. If $\langle M_1, M_2, r_1, r_2, s\rangle$ is a model of T then s is a fuzzy mapping from M_1 into M_2 w.r.t. r_1, r_2.

Proof: Obvious. □

Example 7.2.6 (1) Assume that s is a fuzzy mapping from M_1 into M_2 w.r.t. r_1, r_2 and let r_1, r_2, s be crisp (0, 1-valued). Then r_i is an equivalence on M_i and if we factorize, i. e. put $M_i' = M_i/r_i$ (in more details: $[u]_1 = \{u' \in M_1 | r_1(u,u') = 1\}$, analogously $[v]_2$ then putting $f([u]_1) = [v]_2$ iff $s(u,v) = 1$ we get a crisp mapping from M_1' into M_2' – see Fig. 7.3.

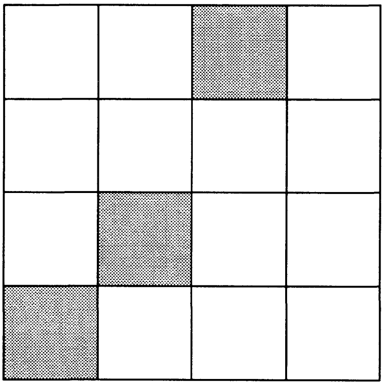

Figure 3.

(2) Now let r_i be similarities on M_i $(i = 1,2)$ and let f be a crisp partial mapping from M_1 into M_2. Take the least extensional relation s containing f (cf. 5.6.13). Our question is under which conditions s is a fuzzy mapping from M_1 into M_2. The condition is that f respects the similarities r_1, r_2, i. e., for all $x_1, x_2 \in M_1$,

$$r_1(x_1,x_2) \leq r_2(f(x_1), f(x_2)).$$

Indeed, consider $\mathbf{M} = \langle M_1, M_2, r_1, r_2, f, s\rangle$ and let $\approx_1, \approx_2, F, \hat{F}$ be names of r_1, r_2, f, s; thus the formula

$$(\forall \cdots)(\hat{F}(x,y) \equiv (\exists x', y')(x \approx x' \,\&\, y \approx y' \,\&\, F(x',y')) \qquad (*)$$

is 1-true in **M**. Our condition reads

$$(\forall \cdots)((x_1 \approx x_2 \mathbin{\&} F(x_1, y_1) \mathbin{\&} F(x_2, y_2)) \rightarrow y_1 \approx y_2) \qquad (**)$$

Clearly, if the axioms of a fuzzy function are 1-true for $\approx_i$, $\hat{F}$ in **M** then (**) is 1-true. Conversely, assume (*) 1-true; we have to show that the formula

$$(\forall \cdots)((\hat{F}(x, y_1) \mathbin{\&} \hat{F}(x, y_2)) \rightarrow y_1 \approx y_2)$$

is 1-true. Let T contain axioms of similarity for $\approx_i$ and (*).

$T \vdash [x \approx x' \mathbin{\&} y_1 \approx y' \mathbin{\&} x \approx x'' \mathbin{\&} y_2 \approx y'' \mathbin{\&} F(x', y') \mathbin{\&} F(x'', y'')] \rightarrow y_1 \approx y_2$ (since the left-hand side implies $x' \approx x'' \mathbin{\&} F(x', y') \mathbin{\&} F(x'', y'') \mathbin{\&} y_1 \approx y' \mathbin{\&} y_2 \approx y''$, which in turn implies $y' \approx y'' \mathbin{\&} y_1 \approx y' \mathbin{\&} y_2 \approx y''$ (by (**)), and this implies $y_1 \approx y_2$). Thus

$$T \vdash (\exists x', y', x'', y'')[\cdots] \rightarrow y_1 \approx y_2$$

(cf. 5.1.14 (2))),
$T \vdash [(\exists x', y')(x \approx x' \mathbin{\&} y_1 \approx y' \mathbin{\&} F(x', y')) \mathbin{\&}$
$(\exists x'', y'')(x \approx x'' \mathbin{\&} y_2 \approx y'' \mathbin{\&} F(x'', y''))] \rightarrow y_1 \approx y_2$ (cf. 5.1.18 (9)), which gives

$$T \vdash (\hat{F}(x, y_1) \mathbin{\&} \hat{F}(x, y_2)) \rightarrow y_1 \approx y_2.$$

See Fig. 7.4.

(3) We give an example of the meaning of the previous condition for Łukasiewicz logic. Assume $M_i = [a_i, b_i]$ are real intervals; let for $u, v \in m_i$, $r_i(u, v) = \max(0, 1 - c_i|u - v|)$. Then a sufficient condition for a mapping f of M_1 into M_2 is to satisfy the Lipschitz condition

$$|f(x_1) - f(x_2)| \leq \varepsilon |x_1 - x_2|$$

for an $\varepsilon \leq c_2/c_1$.

*

Our next task is to investigate the situation described as follows: there is a fuzzy mapping s from M_1 into M_2 (w.r.t. r_1, r_2), that is not at our disposal as a whole; but we know finitely many examples u_i, v_i ($i = 1, \ldots, n$) such that $s(u_i, v_i) = 1$, i. e. if F names s, c_i name u_i and d_i name v_i then $F(c_i, d_i)$ is 1-true in $\mathbf{M} = \langle M_1, M_2, r_1, r_2, s, u_i, v_i \rangle$. It follows immediately that each formula

$$x \approx c_i \mathbin{\&} F(x, y) \rightarrow y \approx d_i$$

is 1-true; and this resembles an "IF- THEN rule"

IF x is similar to c_i THEN y is similar to d_i.

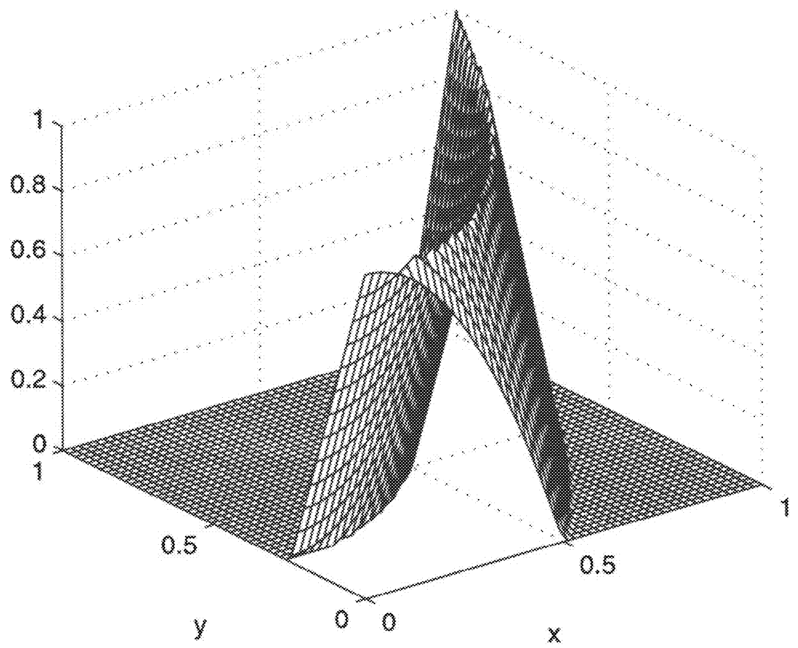

Figure 4. A fuzzy mapping given by the crisp function $y = x^2$.

What more can we say?

Definition 7.2.7 We say that, in a theory T, F *defines a $\approx$-function with examples* (c_i, d_i) $(i = 1, \ldots n)$ if F defines a fuzzy function in T w.r.t. a similarity $\approx$ and for $i = 1, \ldots, n$, T proves $F(c_i, d_i)$.

Repeating once more, the definitions say that the following formulas are provable in T: Similarity axioms for $\approx$,
extensionality of F: $(F(x,y) \,\&\, x \approx x' \,\&\, y \approx y') \to F(x',y')$,
functionality of F: $(F(x,y) \,\&\, F(x,y')) \to y \approx y')$,
examples of F: $\bigwedge_{i=1}^{n} F(c_i, d_i)$.
Let us agree that in the sequel $A_i(x)$ will stand for $x \approx c_i$ and $B_i(y)$ for $y \approx d_i$, unless stated otherwise.

Theorem 7.2.8 Let T be a theory over BL$\forall$ and assume that in T, F defines a $\approx$-function with examples (c_i, d_i) $(i = 1, \cdots, n)$. Then T proves the following formulas:

$$F(x,y) \to \bigwedge_i (A_i(x) \to B_i(y)),$$

$$\bigvee_i (A_i(x) \,\&\, B_i(y)) \to F(x,y).$$

Proof: (1) $T \vdash (F(x,y) \,\&\, x \approx c_i) \to F(c_i,y)$ (from extensionality),
$T \vdash F(c_i,y) \to y \approx d_i$ (from functionality and $T \vdash F(c_i,d_i)$).
Thus $T \vdash (F(x,y) \,\&\, A_i(x)) \to B_i(y)$,
$T \vdash F(x,y) \to (A_i(x) \to B_i(y))$,
$T \vdash F(x,y) \to \bigwedge_i (A_i(x) \to B_i(y))$. (cf. 7.2.3 (1).)
(2) $T \vdash (x \approx c_i \,\&\, y \approx d_i) \to F(x,y)$ from extensionality, thus
$T \vdash (A_i(x) \,\&\, B_i(y)) \to F(x,y)$,
$T \vdash \bigvee_i (A_i(x) \,\&\, B_i(y)) \to F(x,y)$. □

Remark 7.2.9 Given predicates A_i, B_i, we let $RULES(x,y)$ stand for the formula

$$\bigwedge_i (A_i(x) \to B_i(y))$$

and $MAMD(x,y)$ (resembling the name Mamdani, see his [131, 132, 133]) for the formula

$$\bigvee_i (A_i(x) \& B_i(y))$$

We shall prove various results on the relation of these two formulas. In particular, Theorem 7.2.8 says that under the assumptions made,

$$T \vdash MAMD(x,y) \to F(x,y) \to RULES(x,y).$$

Lemma 7.2.10 Let $T, F, \approx, c_i, d_i, A_i, B_i$ be as above and let $T \vdash$ $\vdash MAMD(x,y) \equiv \bigvee_i (A_i(x) \,\&\, B_i(y))$. Then $MAMD$ defines in T a $\approx$-function with examples (c_i, d_i).

Proof: Extensionality:
$T \vdash (x \approx c_i \,\&\, y \approx d_i) \to ((x' \approx x \,\&\, y' \approx y) \to (x' \approx c_i \,\&\, y' \approx d_i))$,
$T \vdash (A_i(x) \,\&\, B_i(y)) \to ((x' \approx x \,\&\, y' \approx y) \to (A_i(x') \,\&\, B_i(y')))$,
$T \vdash (A_i(x) \,\&\, B_i(y)) \to ((x' \approx x \,\&\, y' \approx y) \to MAMD(x',y'))$,
$T \vdash \bigvee_i (A_i(x) \,\&\, B_i(y)) \to ((x' \approx x \,\&\, y' \approx y) \to MAMD(x',y'))$,
$T \vdash (MAMD(x,y) \,\&\, x' \approx x \,\&\, y' \approx y) \to MAMD(x',y')$.
Functionality:
$T \vdash MAMD(x,y) \to F(x,y)$; thus
$T \vdash (MAMD(x,y) \,\&\, MAMD(x,y')) \to (F(x,y) \,\&\, F(x,y'))$, hence
$T \vdash MAMD(x,y) \,\&\, MAMD(x,y')) \to y \approx y'$, by the functionality of F.
Examples: Clearly, $T \vdash A_i(c_i) \,\&\, B_i(d_i)$,hence $T \vdash MAMD(c_i,d_i)$. □

Remark 7.2.11 Thus the formula $MAMD(x,y)$, i. e. $\bigvee_i(A_i(x)\,\&\,B_i(y))$ defines in T the least $\approx$-function with examples (c_i,d_i). *Caution*: The formula $RULES(x,y)$, i. e. $\bigwedge_i(A_i(x) \rightarrow B_i(y))$ (with our fixed assumptions, $A_i(x)$ is $x = c_i$ etc.) need not define a $\approx$-function! This can be seen already in the crisp case: for x non-equivalent to any of $c_1,\ldots,c_n$, our formula gives no restriction to the value of y. In more details, if T is as above and $T \vdash RULES(x,y) \equiv \bigwedge_i(A_i(x) \rightarrow B_i(y))$ then
$T \vdash (\bigwedge_i \neg(A_i(x)) \rightarrow RULES(x,y)$ (since $T \vdash \neg A_i(x) \rightarrow (A_i(x) \rightarrow B_i(y)))$. see Fig.7.5.

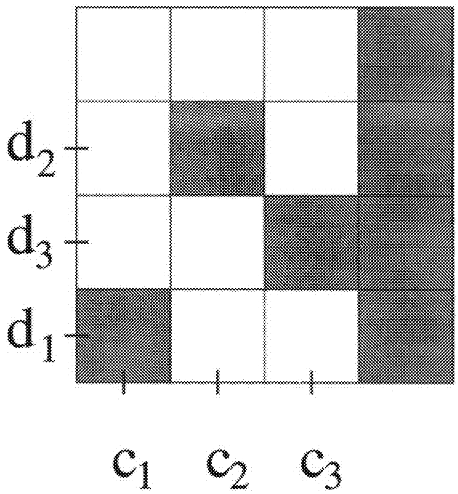

Figure 5.

Thus keeping our assumptions on T we may ask under which conditions the two formulas, $RULES(x,y)$ and $MAMD(x,y)$ are equivalent. The following lemma gives the answer:

Lemma 7.2.12 Let $T, F, \approx, c_i, d_i, A_i, B_i$ be as above, let $MAMD(x,y)$ stand for $\bigvee_i(A_i(x)\,\&\,B_i(y))$ (i. e. for $\bigvee_i(x \approx c_i \,\&\, y \approx d_i)$) and let $RULES(x,y)$ stand for $\bigwedge_i(A_i(x) \rightarrow B_i(y))$ i. e. for $\bigwedge_i(x \approx c_i \rightarrow y \approx d_i)$. Then

$$T \vdash (\bigvee_i A_i^2(x)) \rightarrow (MAMD(x,y) \equiv RULES(x,y)).$$

Proof: $T \vdash (A_i(x) \rightarrow B_i(y)) \rightarrow [(A_i(x)\,\&\,A_i(x)) \rightarrow (A_i(x)\,\&\,B_i(y))]$,
$T \vdash A_i^2(x) \rightarrow [(A_i(x) \rightarrow B_i(y)) \rightarrow (A_i(x)\,\&\,B_i(y))]$,

$T \vdash A_i^2(x) \to [\bigwedge_i(A_i(x) \to B_i(y)) \to \bigvee_i(A_i(x) \,\&\, B_i(y))]$,
$T \vdash \bigvee_i A_i^2(x) \to [MAMD(x,y) \equiv RULES(x,y)]$ (by 7.2.9; recall that in propositional calculus, $p \equiv q$ is equivalent to $(p \to q) \wedge (q \to p)$, cf. 2.2.16 (29)). □

Corollary 7.2.13 Under the present notation,
$T \cup \{(\forall x)(\bigvee_i A_i(x))\} \vdash (\forall x, y)(MAMD(x,y) \equiv RULES(x,y))$.
(Note that if $\mathbf{M}$ is a model of the theory in question then for each $u \in M$ there is an i such that if c_i denotes u_i in $\mathbf{M}$, u is similar to u_i in degree 1.)

Proof: This follows from facts on the propositional calculus: we know that $\bigvee_i p_i \vdash (\bigvee_i p_i)^2$ (since $q \to (q \to q^2)$ is BL-provable); and $(\bigvee_i p_i)^2 \vdash \bigvee_i(p_i^2)$ by 2.2.24. □

What we have done up to now may be described (or interpreted) as follows: We have two domains M_1, M_2 (you could write D_X, D_Y instead), similarities r_1, r_2 on M_1, M_2 respectively and a partial fuzzy mapping s from M_1 to M_2, thus a model $\mathbf{M} = \langle M_1, M_2, r_1, r_2, s\rangle$. We introduce the language $\approx_1, \approx_2, F$ and assume we have n examples (u_i, v_i) named (c_i, d_i) such that $F(c_i, d_i)$ is 1-true in $\mathbf{M}$ (i. e. $s(u_i, v_i) = 1$). This can be expressed by saying "F sends c_i to d_i", or, "F sends (x similar to c_i) to (y similar to d_i)", or "IF x is similar to c_i (and $F(x,y)$) THEN y is similar to d_i". We know that in each model $\mathbf{M}$ as above the formula $(\forall x, y)(F(x,y) \to \bigwedge_i((x \approx c_i) \to (y \approx d_i)))$ is 1-true, thus s is a subrelation of the fuzzy relation defined by $\bigwedge_i(x \approx c_i \to y \approx d_i)$ (which itself need not be a $\approx$ mapping); on the other hand, the formula $\bigvee_i(x \approx c_i) \,\&\, y \approx d_i)$ defines in $\mathbf{M}$ a $\approx$-fuzzy mapping h which is a subrelation of s and satisfies $h(u_i, v_i) = 1$.

Lemma 7.2.12 says that for each $u \in M_1$, $v \in M_2$, the degree in which u satisfies $\bigvee_i(x \approx c_i)^2$ (i. e. which u is *very similar* to an u_i, $i = 1, \ldots, n$) is a lower bound for the degree in which (u, v) satisfies $MAMD(x,y) \equiv RULES(x,y)$.

We should ask the following: What if we just have M_i, similarities r_i and (potential) examples (u_i, v_i)? What must be assumed to be sure that there is a fuzzy mapping s (w.r.t. r_i) such that $s(u_i, v_i) = 1$? The following lemma gives the answer.

Lemma 7.2.14 Let T be a theory with two sorts and similarity predicates $\approx_1, \approx_2$ of the respective sorts; let $c_1, \ldots, c_n$ be constants of the first sort and $d_1, \ldots, d_n$ constants of the second sort. If $T \vdash c_i \approx c_j \to d_i \approx d_j$ for each i, j (indices at $\approx$ deleted) and $T \vdash MAMD(x,y) \equiv \bigvee_i(x \approx c_i \,\&\, y \approx d_i)$

then $MAMD$ defines a $\approx$-function in T and $T \vdash MAMD(c_i, d_i)$ for $i = 1, \ldots, n$.

Proof: *Extensionality* as above.
Functionality: T proves the following chain of implications.
$[MAMD(x, y_1) \,\&\, MAMD(x, y_2)] \rightarrow$
$[(\bigvee_i x \approx c_i \,\&\, y_1 \approx d_i) \,\&\, \bigvee_j (x \approx c_j \,\&\, y_2 \approx d_j)] \rightarrow$
$[\bigvee_{i,j}(x \approx c_i \,\&\, x \approx c_j \,\&\, y_1 \approx d_i \,\&\, y_2 \approx d_j)] \rightarrow$
$[\bigvee_{i,j}(c_i \approx c_j \,\&\, y_1 \approx d_i \,\&\, y_2 \approx d_j)] \rightarrow$
$[\bigvee_{i,j}(d_i \approx d_j \,\&\, y_1 \approx d_i \,\&\, y_2 \approx d_j)] \rightarrow y_1 \approx y_2$.
Examples: Obviously, $T \vdash (c_i \approx c_i \,\&\, d_i \approx d_i)$, thus $T \vdash MAMD(c_i, d_i)$.
□

*

Proof: Let us be still more modest: let us have M_1, M_2 and fuzzy subsets r_{A_i} of M_1, r_{B_i} of M_2. We ask under which conditions we may assume

- similarities s_1 on M_1 and s_2 on M_2 with respect to which r_{A_i}, r_{B_i} are extensional,
- elements $u_1, \ldots, u_n \in M_1$, $v_1, \ldots, v_n \in M_2$ such that such that r_{A_i} are "fuzzy singletons given by u_i with respect to s_1" and similarly for r_{B_i}, v_i, s_2,
- a s_1, s_2-fuzzy mapping r_F "sending u_i to v_i".

We shall answer these questions.

Lemma 7.2.15 [41] Let T be a theory, A_i unary predicates of the same sort $(i = 1, \ldots, n)$.

(1) Define a binary predicate $\approx$ as follows:

$$(\forall x, x')(x \approx x' \equiv \bigwedge_i (A_i(x) \equiv A_i(x'))).$$

The resulting extension T' of T is conservative, $\approx$ is a similarity in T' and T' proves all A_i to be extensional.

(2) Add new constants c_i and axioms $(\forall x)(A_i(x) \equiv x \approx c_i))$. The resulting theory T'' is a conservative extension of T' iff T' proves all formulas

$$(\exists x) A_i(x),$$

$$(\exists x)(A_i(x) \,\&\, A_j(x)) \rightarrow (\forall x)(A_i(x) \equiv A_j(x)).$$

[41] See [118] 4.13.

Proof: (1) T' is a conservative extension of T by 5.2.15; the proof that in T' $\approx$ is a similarity making all A_i extensional is an easy variant of 5.6.14.

(2) First assume T'' to be a conservative extension of T'; then it suffices to prove the above formulas in T''.

$T'' \vdash c_i \approx c_i$ thus $T'' \vdash (\exists x)(x \approx c_i)$ and $T'' \vdash (\exists x)A_i(x)$. Furthermore, $T'' \vdash (\exists x)(A_i(x) \,\&\, A_j(x)) \equiv (c_i \approx c_j)$ (since $T'' \vdash c_i \approx c_j \rightarrow (c_i \approx c_i \,\&\, c_i \approx c_j)$ and $T'' \vdash (x \approx c_i \,\&\, x \approx c_j) \rightarrow c_i \approx c_j$); and $T'' \vdash c_i \approx c_j \rightarrow (x \approx c_i \equiv x \approx c_j)$. Thus $T'' \vdash (\exists x)(A_i(x)\&A_j(x)) \rightarrow (\forall x)(A_i(x) \equiv A_j(x))$.

Conversely, assume that T' proves the above formulas. Then we may extend T' conservatively by all axioms $A_i(c_i)$; call the resulting theory T'''. $T''' \vdash x \approx c_i \rightarrow A_i(x)$ immediately from the definition of $\approx$; on the other hand, $T''' \vdash A_i(x) \rightarrow (A_i(c_i) \equiv A_i(x))$ (since $A_i(c_i)$ is provable),
$T''' \vdash (A_i(x) \,\&\, A_j(x)) \rightarrow (\forall z)(A_i(z) \equiv A_j(z))$,
$T''' \vdash A_i(x) \rightarrow (A_j(x) \rightarrow (A_i(c_i) \equiv A_j(c_i)))$,
$T''' \vdash A_i(x) \rightarrow (A_j(x) \rightarrow A_j(c_i))$ (since $T''' \vdash A_i(c_i)$)
and also
$T''' \vdash (A_i(c_i)\&A_j(c_i) \rightarrow (A_i(x) \equiv A_j(x))$
$T''' \vdash A_j(c_i) \rightarrow (A_i(x) \rightarrow A_j(x))$
thus all together,
$T''' \vdash A_i(x) \rightarrow \bigwedge_j(A_j(x) \equiv A_j(c_i))$, hence
$T''' \vdash A_i(x) \rightarrow x \approx c_i$.
Thus T''' is stronger than T'' (in fact equivalent to T'') and hence T'' is a conservative extension of T'. □

Theorem 7.2.16 Let T be a theory, A_i unary predicates of one sort, B_i unary predicates of another sort. Assume

$$T \vdash (\exists x)A_i(x), \qquad T \vdash (\exists y)B_i(y),$$

$$T \vdash (\exists x)(A_i(x)\&A_j(x)) \rightarrow (\forall x)(A_i(x) \equiv A_j(x)),$$

$$T \vdash (\exists y)(B_i(y)\&B_j(y)) \rightarrow (\forall y)(B_i(y) \equiv B_j(y)).$$

Add definitions $x_1 \approx x_2 \equiv \bigwedge_i(A_i(x_1) \equiv A_i(x_2))$, $y_1 \approx y_2 \equiv \bigwedge_i(B_i(y_1) \equiv B_i(y_2))$, new constants c_i, d_i and axioms
$A_i(x) \equiv x \approx c_i, \qquad B_i(y) \equiv y \approx d_i$.
Finally add the definition

$$MAMD(x, y) \equiv \bigvee_i A_i(x)\&B_i(y).$$

The resulting theory T^M is a conservative extension of T and $\approx_1, \approx_2$ are similarities.

$MAMD$ defines in T^M a fuzzy mapping w.r.t. $\approx_1, \approx_2$ with the examples (c_i, d_i) iff

$$T \vdash (\exists x)(A_i(x) \& A_j(x)) \to (\exists y)(B_i(y) \& B_j(y)).$$

Proof: We only put things together. Adding $\approx_i, c_i, d_i$ and the axioms concerning them to T is conservative and the extension proves similarity axioms by the preceding lemma. Also recall that $c_i \approx c_j$ is equivalent to $(\exists x)(A_i(x)\&A_j(x))$ and similarly for $d_i \approx d_j$. Thus the fact that $MAMD(x, y)$ defines a fuzzy mapping follows by 7.2.14. □

*

7.2.17 After having discussed fuzzy functions at large, let us ask what we can say about the (logical) principles of fuzzy control in general, without relating it to the notion of similarity. (We again restrict ourselves to just one "input" variate X, a generalization to more input variates being easy.) The heartof the matter is as follows: We have n rules "IF X is A_i THEN Y is B_i" $r_i = 1, \cdots, n$, A_i unary predicates of the same sort, B_i predicates, all of the same sort (possibly different from the former sort). Let us write the rules as $A_i(x) \to B_i(y)$ cf. 7.1.1). We use A_i and B_i to define a binary predicate $MAMD$ by

$$(\forall x, y)(MAMD(x, y) \equiv \bigvee_i (A_i(x) \& B_i(y))) \qquad (Mamd)$$

and, given another unary predicate A^* of the first sort, define a B^* from A^*, via the compositional rule of inference, i. e.

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x) \& MAMD(x, y))). \qquad (B*)$$

Given a model $\mathbf{M} = \langle D_X, D_Y, r_{A_i}, r_{B_i} \rangle$ this defines a functional associating to each fuzzy subset r_{A^*} of D_X the corresponding fuzzy subset r_{B^*} of D_Y. (Note that in fuzzy control this is used to define a crisp mapping of D_X into D_Y: one first uses a *fuzzification* operation, associating to each $u \in D_X$ a fuzzy set r_{A^*} ("approximately u"), then applies the functional to get r_{B^*} and finally applies a *defuzzification* procedure converting the fuzzy set r_{B^*} into a crisp output v. We shall disregard the operations of fuzzification and defuzzification.)

Our question now reads: *is there any logic here*? Let us try a positive answer, as general as possible. To this end we shall make the above formulas axioms of a theory of fuzzy control:

Definition 7.2.18 FC is a two-sorted theory having unary predicates $A_1, \ldots,$ $\ldots A_n, A^*$ of sort 1, unary predicates $B_1, \ldots, B_n, B^*$ of sort 2 a binary predicate $MAMD$ of the type $\langle 1, 2\rangle$. The axioms are the formulas (Mamd), (B*) above (defining $MAMD$ from A_i, B_i and defining B^* from $A^*, MAMD$). In addition, FC has two constants: X of sort 1 and Y of sort 2.

Theorem 7.2.19 FC proves the following (over BL$\forall$):

$$[\bigwedge_i (A_i(X) \to B_i(Y)) \& \bigvee_i (A_i)^2(X)] \to (A^*(X) \to B^*(Y)):$$

7.2.20 Before we prove the theorem let us discuss its meaning. It says that, under the asumptions as how B^* is obtained, if the current value of the variate X (denoted by the constant X) satisfies, together with the current value the variate Y, all the rules $A_i(X) \to B_i(Y)$ *and* (sharp and) X satisfies $\bigvee_i {A_i}^2(X)$ then $A^*(X)$ implies $B^*(Y)$. In particular, assume $[\cdots]$ to be 1-true in **M**. Then $\|A_i(X)\|_M \leq \|B_i(Y)\|_M$ for all i and $\|A_i(X)\|_M = 1$ for at least one i. The conclusion is $\|A^*(X)\|_M \leq \|B^*(Y)\|_M$.

But this is not all. Assume the value of the antecedent $[\cdots]$ to be $\geq r$, i.e. the rules are *sufficiently* true and X *sufficiently* satisfies one of A_i's. The conclusion is that $\|B^*(Y)\|_M$ is *not much less* than $\|A^*(X)\|_M$. For example, if the rules are 1-true then $\|B^*(Y)\|_M \geq \|A^*(X)\|_M * \|\bigvee A_i^2(X))\|_M$ ($*$ being the interpretation of &). We shall come back to this discussion in a moment.

7.2.21 *Proof of* 7.2.19.
FC$\vdash$ $(A_i(X) \& (A_i(X) \to B_i(Y))) \to B_i(Y)$, thus
FC$\vdash$ $(A_i^2(X) \& \bigwedge_i (A_i(X) \to B_i(Y))) \to (A_i(X) \& B_i(Y))$, thus
FC$\vdash$ $[A_i^2(X) \& \bigwedge_i (A_i(X) \to B_i(Y)) \& A^*(X)] \to A^*(X) \& MAMD(X, Y)$.
Consequently,
FC$\vdash$ $(A_i^2(X) \& RULES(X, Y)) \to (A^*(X) \to$
$\to (\exists x)(A^*(x) \& MAMD(x, Y)))$,
which gives the result by the definition of B^*.

Now let us see what happens if we assume A^* to be equivalent to A_i:

Theorem 7.2.22 FC proves (over BL$\forall$) the following:

$$[(\forall x)(A^*(x) \equiv A_i(x)) \& (\exists x) A_i^2(x)] \to (\forall y)(B_i(y) \to B^*(y)),$$

$$[(\forall x)(A^*(x) \equiv A_i(x)) \& (\forall x)(\bigwedge_{i \neq j} \neg(A_i(x) \& A_j(x)))] \to$$

$$\to (\forall y)(B^*(y) \to B_i(y)).$$

Proof: (i)
FC$\vdash (A_i^2(x)\&(A_i(x) \equiv A^*(x))\&B_i(y)) \to A^*(x)\&A_i(x)\&B_i(y)$,
FC$\vdash [(\exists x)(A_i^2(x)\&(A_i(x) \equiv A^*(x))\&B_i(y)] \to$
$\to (\exists x)(A^*(x)\& \bigvee_j(A_j(x)\&B_j(y)))$,
FC$\vdash [(\exists x)A_i^2(x)\&(\forall x)(A_i(x) \equiv A^*(x)] \to (\exists x)(A_i^2(x)\&(A_i(x) \equiv A^*(x)))$,
FC$\vdash [(\exists x)A_i^2(x)\&(\forall x)(A_i(x) \equiv A^*(x))\&B_i(y)] \to$
$\to (\exists x)(A^*(x)\&MAMD(x,y))$,
FC$\vdash [(\exists x)A_i^2(x)\&(\forall x)(A_i(x) \equiv A^*(x))] \to (B_i(y) \to B^*(y))$.
Now we prove (ii).
Write $Dsjnt(A_i)$ for $(\forall x) \bigwedge_{j\neq i} \neg(A_i(x)\&A_j(x))$,
$Equiv(A_i, A^*)$ for $(\forall x)(A^*(x) \equiv A_i(x))$.
FC$\vdash (B^*(y)\&Equiv(A^*, A_i)) \to (\exists x)(A_i(x)\& \bigvee_j(A_j(x)\&B_j(y)))$,
FC$\vdash Dsjnt(A_i) \to [(A_i(x)\& \bigvee_j(A_j(x)\&B_j(y)) \to A_i^2(x)\&B_i(y)]$
(since $A_i(x)\&A_j(x)\&B_j(y)$ implies $\bar{0}$ for $i \neq j$), i. e.
FC$\vdash [Dsjnt(A_i)\&Equiv(A^*, A)\&B^*(y)] \to (\exists x)(A_i^2(x)\&B_i(y))$,
FC$\vdash [Dsjnt(A_i)\&Equiv(A^*, A)] \to (B^*(y) \to B_i(y))$,
which gives the result by generalizing (by $(\forall y)$) and moving $(\forall y)$. □

Remark 7.2.23 (1) Again read the formulas as true in a model – first with the antecedent 1-true and then with the antecedent *sufficiently true*. We see that

(i) if $A^*(x)$ is sufficiently near to A_i and A_i is (sufficiently) non-empty then B_i is sufficiently included in B^*;

(ii) if A_i is sufficiently disjoint from all the other A_j's and A^* is sufficiently near to A_i then B^* is sufficiently included in B_i. Obviously, these are fuzzy readings; the precise meaning is given by the formulas proved and may be expressed in greater detail again as an exercise.

(2) Let us repeat once more that instead of antecedent of the form $A_i(X)$ we could investigate $A_{i1}(X_1)\&\ldots\&A_{ik}(X_k)$ or $A_{i1}(X_1) \wedge \ldots \wedge A_{ik}(X_k)$; this brings no problems but is more cumbersome.

(3) On the other hand, replacing $\&$ by $\wedge$ in the definition on $MAMD$ does bring additional problems (unless your logic is $G\forall$ – Gödel). We shall not go into them here.

7.3. AN ALTERNATIVE APPROACH TO FUZZY RULES

7.3.1 Up to now, we have worked with two variates X, Y with domains D_X, D_Y respectively; syntactically, we had just two sorts, predicates A_i, A^*

of the first sort and B_i, B^* of the second. X and Y were understood as *object constants* (for the actual value of the respective variate) and we had *rules* of the form $A_i(X) \to B_i(Y)$ saying "if the actual value of X is A_i then the actual value of Y is B_i". X and Y thus denoted possibly unknown but crisp elements of the respective domains.

Let us now try to be still more fuzzy and let X, Y denote *fuzzy subsets* of the respective domains, giving some vague information on the actual values of our variates. Syntactically this means that X and Y become *unary predicates* of the respective sorts. Then it is natural to formalize the assertion "X is A_i" to be just the formula $(\forall x)(X(x) \to A_i(x))$ (briefly, $X \subseteq A_i$). Indeed, the formula is 1-true if for each element m of D_X, the degree in which m satisfies X is a lower bound of the degree in which m satisfies A_i. (The reader may state in words the meaning of $X \subseteq A_i$ if this formula is r-true.) We shall reconsider the generalized modus ponens and the inference in fuzzy control in this new setting.[42]

Definition 7.3.2 (1) If X, A are unary predicates of the same sort then $X \subseteq A$ stands for $(\forall x)(X(x) \to A(x))$. Similarly, if φ, ψ are formulas with exactly one free variable x then $\varphi \subseteq \psi$ stands for $(\forall x)(\varphi(x) \to \psi(x))$

(2) Given A^*, A, B, B^{**} of the obvious sort, $Comp_{MPA}$ (alternative composition for generalized modus ponens) stands for the formula

$$(\forall y)(B^{**}(y) \equiv [(\forall x)(A^*(x) \to A(x)) \to B(y)]))$$

or, briefly,

$$B^{**}(y) \equiv [(A^* \subseteq A) \to B(y)].$$

Remark 7.3.3 Recall $Comp_{MP}$, i. e. the formula

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x) \& (A(x) \to B(y)))$$

and observe that, over BL$\forall$, $Comp_{MP}, Comp_{MPA} \vdash B^* \subseteq B^{**}$. Indeed, assuming B^*, B^{**} to be defined by $Comp_{MP}, Comp_{MPA}$ respectively, we can prove the following:

$$B^{**}(y) \equiv ((\forall x)(A^*(x) \to A(x)) \to B(y)),$$

$$B^{**}(y) \equiv (\exists x)((A^*(x) \to A(x)) \to B(y)),$$

$$B^*(y) \equiv (\exists x)(A^*(x) \& (A(x) \to B(y))),$$

$$[A^*(x) \& (A(x) \to B(y))] \to [(A^*(x) \to A(x)) \to B(y)]$$

42 Cf. [63, 62].

(note that the last formula is equivalent to the provable formula

$$[A^*(x)\&(A^*(x) \to A(x))\&(A(x) \to B(y))] \to B(y)\,),$$

$$(\exists x)(A^*(x)\&(A(x) \to B(y)) \to (\exists x)((A^*(x) \to A(x)) \to B(y)),$$

$$B^*(y) \to B^{**}(y).$$

Theorem 7.3.4 Let $Comp_{MPA}$ be as in 7.3.2, i. e. $(\forall y)(B^{**}(y) \equiv (A^* \subseteq A) \to B(y)))$. Then BL$\forall$ proves

$$[Comp_{MPA}\&(X \subseteq A^*)\&((X \subseteq A) \to (Y \subseteq B))] \to Y \subseteq B^{**}$$

Proof: Observe that it suffices to prove, in BL$\forall$, the formula

$$[(X \subseteq A^*)\&((X \subseteq A) \to (Y \subseteq B))] \to Y \subseteq [(A^* \subseteq A) \to B] \qquad (1)$$

Indeed, having (1) we get
$[(X \subseteq A^*)\&((X \subseteq A) \to (Y \subseteq B))\&Comp_{MPA} \to (Y \subseteq [(A^* \subseteq A) \to B]\&[(A^* \subseteq A) \to B] \subseteq B^{**}$ which gives the result by (provable) transitivity of $\subseteq$. Thus we prove the formula (1).

First observe that, by our axiom on quantifiers, the following chain of implications is provable:

$$\begin{aligned} &[(\forall x)(X(x) \to A(x)) \to (\forall y)(Y(y) \to B(y))] \to \\ \to\quad &(\forall y)[(\forall x)(X(x) \to A(x)) \to (Y(y) \to B(y))] \to \\ &\to (\forall y)[Y(y) \to ((\forall x)(X(x) \to A(x)) \to B(y))]. \end{aligned} \qquad (2)$$

on the other hand, by the (provable) properties of implication,

$$(X(x) \to A^*(x)) \to [((X(x) \to A(x)) \to B(y)) \to$$
$$\to ((A^*(x) \to A(x)) \to B(y))],$$

thus

$$X \subseteq A^* \to [(\exists x)((X(x) \to A(x)) \to B(y)) \to$$
$$\to (\exists x)((A^*(x) \to A(x)) \to B(y))],$$

and

$$X \subseteq A^* \to [((\forall x)(X(x) \to A(x)) \to B(y)) \to$$
$$\to ((\forall x)(A^*(x) \to A(x)) \to B(y))];$$

in short,

$$X \subseteq A^* \to ([(X \subseteq A) \to B] \subseteq [(A^* \subseteq A) \to B]) \qquad (3)$$

The implications in (2) prove

$$((X \subseteq A) \to (Y \subseteq B)) \to (Y \subseteq [(X \subseteq A) \to B]); \qquad (4)$$

and (3) and (4) give

$$[(X \subseteq A \to Y \subseteq B) \,\&\, (X \subseteq A^*)] \to$$
$$\to (Y \subseteq [X \subseteq A) \to B] \,\&\, ([(X \subseteq A) \to B] \subseteq [(A^* \subseteq A) \to B]);$$

by transitivity of $\subseteq$ we get our formula (1); this completes the proof of the theorem. □

Remark 7.3.5 (1) This remark is analogous to 7.1.9: we may visualize the result as a rule

$$\frac{Comp_{MPA}, X \subseteq A^*, (X \subseteq A) \to (Y \subseteq B)}{Y \subseteq B^{**}}$$

Thus if the assumptions are 1-true in a structure $\mathbf{M}$ then so is the conclusion. But again , let us stress that 7.3.4 gives more:

$$\|Comp_{MPA}\&(X \subseteq A^*)\&((X \subseteq A) \to (Y \subseteq B))\|_{\mathbf{M}} \leq \|Y \subseteq B^{**}\|_{\mathbf{M}}$$

(2) Observe that taking A for A^* $Comp_{MPA}$ becomes equivalent to $B \subseteq B^{**}$, thus we get the trivial rule (modus ponens)

$$\frac{X \subseteq A \to Y \subseteq B, X \subseteq A}{Y \subseteq B}$$

as a particular case.

(3) More generally, assume $\mathbf{M}$ to be a model; iff $\|A^* \subseteq A\|_{\mathbf{M}} = 1$ then in $\mathbf{M}$, $Comp_{MPA}$ is equivalent to $B \subseteq B^{**}$, if $\|A^* \subseteq A\|_{\mathbf{M}} = r < 1$ then $\|Comp_{MPA}\|_{\mathbf{M}} = 1$ iff for each m from the common domain of Y, B, B^{**}, $r_{B^{**}}(m) \geq \min(1, r_B(m) + 1 - r)$.

(4) Furthermore, we show that the rule in (1) becomes ill (non-sound) if we replace $Comp_{MPA}$ (the alternative composition for modus ponens) by $Comp_{MP}$ (see 7.3.3) and B^{**} by B^*. We exhibit the simple example of a structure in which $Comp_{MP}, X \subseteq A^*, X \subseteq A \to Y \subseteq B$ are 1-true but $Y \subseteq B^*$ is not. Let $D_X = \{x_0, x_1\}, D_Y = \{y_0\}$. (The example works in any of the logics $L\forall, G\forall, \Pi\forall$.)

The following tables give the interpretation of X, A, A^* and Y, B, B^* :

	X	A	A^*
x_0	1	1	1
x_1	$\frac{1}{2}$	0	$\frac{1}{2}$

	Y	B	B^*
y_0	1	$\frac{1}{2}$	$\frac{1}{2}$

One trivially verifies $\|Comp_{MP}\| = \|X \subseteq A^*\| = 1$, $\|X \subseteq A\| = \|Y \subseteq B\| = \frac{1}{2}$, thus $\|(X \subseteq A) \to (Y \subseteq B)\| = 1$, but $\|Y \subseteq B^*\| = \frac{1}{2}$. Note also $\|(\exists x)X(x)\| = 1$.

Thus the rule

$$\frac{Comp_{MP}, (\exists x)X(x), (X \subseteq A^*), (X \subseteq A) \to (Y \subseteq B)}{Y \subseteq B^*}$$

is not sound (in $Ł\forall, G\forall, \Pi\forall$). The reader may show as an exercise that the last rule *is* sound in the Boolean logic $Bool\forall$.

(5) Finally, observe that if $\approx$ a similarity predicate in T and T proves A, B, A^*, B^* to be extensional (i.e. proves congruence axioms for them) and for some constants c, d, the theory T proves $(\forall x)(X(x) \equiv x \approx c)$ and $(\forall y)(Y(y) \equiv y \approx d)$ i.e. X, Y define fuzzy singletons) then $T \vdash (X \subseteq A) \equiv A(c)$, $T \vdash (X \subseteq A^*) \to A^*(c)$, $T \vdash (Y \subseteq B) \equiv Y(d)$ etc. and $T \vdash (Comp_{MPA} \& (X \subseteq A^*) \& ((X \subseteq A) \to Y \subseteq B))] \to (Y \subseteq B^*)$. Thus we get, in this particular case, again the result of 7.1.8.

*

Let us now turn to fuzzy control. We just restrict ourselves to one result (relating the Mamdani formula to the rules in our new sense).

Theorem 7.3.6 Let A_i, A^*, X be unary predicates of one sort, B_i, B^*, Y unary predicates of another sort. Let $MAMD(x, y)$ stand for the formula $\bigvee_i (A_i(x) \& B_i(y))$ (as above) and let $Comp_{MAMD}$ be the formula

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x) \& MAMD(x, y))$$

Then BL$\forall$ proves

$$[Comp_{MAMD} \& X \subseteq A^* \& \bigwedge_i ((X \subseteq A_i) \to Y \subseteq B_i)) \&$$

$$\& (\exists x)X^2(x) \& \bigvee_i (X \subseteq A_i)^2] \to Y \subseteq B^*.$$

Remark 7.3.7 Before we prove the theorem let us comment on its meaning. The inference pattern of the Mamdani-like fuzzy control mechanism can be formulated as

"if X is A^*, B^* is defined from A_i, B_i using the Mamdani formula, and Y corresponds to X then Y is B^*".

We have analyzed this pattern as a sound deduction rule (or, better, as a provable implication) in the preceding section. The question is what it means that Y corresponds to X. In the theorem this is understood as the assumption that the pair (X, Y) satisfies all the rules $(\bigwedge_i((X \subseteq A_i) \to (Y \subseteq B_i))$. Alternatively we replace this assumption by the assumption that (X, Y) satisfies an analogon of the Mamdani formula $\bigvee_i(x \subseteq A_i \& Y \subseteq B_i)$. We shall do this in the next lemma and then show how this gives our Theorem.

Lemma 7.3.8 Under the assumptions of 7.3.6, BL$\forall$ proves

$$[Comp_{MAMD} \& X \subseteq A^* \& (\exists x)\, X^2(x) \& \bigvee_i (X \subseteq A_i \& Y \subseteq B_i)] \to Y \subseteq B^*$$

Proof: BL$\forall$ proves
$[(\exists x)(X(x) \& X(x) \& (X \subseteq A^*) \& (X \subseteq A_i)] \to (\exists x)(A^*(x) \& A_i(x))$, thus
$(X_i \subseteq A_i \& Y \subseteq B_i) \to$
$\to [(X \subseteq A^* \& (\exists x(X^*2(x) \& Y(y)) \to (\exists x)(A^*(x) \& A_i(x) \& B_i(y))]$, and
$(\exists x)(A^*(x) \& A_i(x) \& B_i(y)) \to (\exists x)(A^*(x) \& \bigvee_i(A_i(x) \& B_i(y))$, thus
$\bigvee_i((X \subseteq A_i) \to [((X \subseteq A^*) \& (\exists x) X^2(x) \& Y(y)) \to$
$\to (\exists x)(A^*(x) \& MAMD(x, y))]$ thus
$[Comp_{MAMD} \& X \subseteq A^* \& (\exists x) X^2(x) \& \bigvee_i (X \subseteq A_i \& Y \subseteq B_i)] \to$
$\to (Y(y) \to B^*(y))$,
which gives the result. □

7.3.9 *Proof of* 7.3.6. The theorem follows from our lemma by observing the provability of

$$[\bigvee_i (X \subseteq A_i)^2 \& \bigwedge_i (X \subseteq A_i \to Y \subseteq B_i)] \to \bigvee_i ((X \subseteq A_i) \& (Y \subseteq B_i))$$

(which is an easy exercise in propositional calculus).

CHAPTER EIGHT

GENERALIZED QUANTIFIERS AND MODALITIES

We are going to study, from the logical point of view, expressions as "usually φ", "φ is probable", "for many x, φ" and similar. We shall handle them both as generalized quantifiers and as modalities. This should not look unnatural since in general modalities can be viewed as hidden quantifiers. We shall try to show how the theory of generalized quantifiers and modal logic can be applied to the above expressions (stressed by Zadeh as items of the specific agenda of fuzzy logic in the narrow sense) and that they admit "classical" logical analysis. We shall offer a logical apparatus, precise definitions, some results and various problems: here much study and development is still necessary. To be able to use and generalize the results and approaches of Boolean logic, we shall have to review and summarize several notions and facts on generalized quantifiers, some modal logics, and also on measures of uncertainty (that will be used to define various quantifiers and modalities). Thus we shall have *two* preliminary sections in this chapter: Section 1 on generalized quantifiers (with an appendix on uncertainty measures) and Section 2 on modal logics, both sections in the frame of Boolean (two-valued) logic. Generalized quantifiers in fuzzy logic and fuzzy modal logics are studied in Sections 3, 4.

8.1. GENERALIZED QUANTIFIERS IN BOOLEAN LOGIC; MEASURES OF UNCERTAINTY

8.1.1 We stress again that this is a survey section, dealing with two-valued logic. The primitive quantifier is $\forall$ (universal); the existential quantifier $\exists$ is definable as $\neg\forall\neg$. We may also define binary quantifiers applicable to two formulas, e.g. $(\mathrm{Incl}\, x)(\varphi, \psi)$ may stand for $(\forall x)(\varphi \to \psi)$ (φ is included in ψ). We may read this formula also as "all x satisfying φ satisfy ψ" and denote by $(\forall x|\varphi)\psi$. In the particular case of a binary atomic φ, e.g. φ being $x < y$, we write $(\forall x|x < y)\psi$ also as $(\forall x < y)\psi$ (and call this a formula with a bounded quantifier). Similarly we may define $(\mathrm{NDisjnt}\, x)(\varphi, \psi)$ to be

$(\exists x)(\varphi \& \psi)$ and also write $(\exists x|\varphi)\psi$ or $(\exists x < y)\psi$. A quantifier always *binds* a variable (or more variables).[43]

8.1.2 *Towards a general definition.* The above quantifiers were definable, i.e. may be considered to be only abbreviations. We shall deal with quantifiers extending the expressive capability in an essential way. We describe three syntactic types of quantifiers (and it will be apparent how to define infinitely many other types). These will be unary quantifiers (giving formulas of the form $(qx)\varphi$), unary parametric quantifiers (giving $(qx|y)\varphi$) and binary (giving $(qx)(\varphi, \psi)$). Our above examples are of these types.

What determines the truth value of a quantified formula, say a formula resulting from φ by applying a quantifier q binding a variable x? Clearly, the relevant object is

$$Sat_{\mathbf{M},v}(\varphi, x) = \{m \in M| \, \|\varphi\|_{\mathbf{M},v(x/m)} = 1\},$$

i.e. the set of all m satisfying φ in M (the meaning of variables different from φ being given by v). The truth value of $(qx)\varphi$ in M is a function of this set. Thus there is a family $\mathcal{F} \subseteq \mathcal{P}(M)$ of subsets of M such that $\|(qx)\varphi\|_{\mathbf{M},v} = 1$ iff $Sat_{\mathbf{M},v}(\varphi, x) \in \mathcal{F}$; otherwise $\|(qx)\varphi\|_{\mathbf{M},v} = 0$.

For example, if q is $\forall$ then $\mathcal{F}$ has just one element, namely M. If q is $\exists$ then $\mathcal{F}$ consists of all non-empty subsets of M.

If q is binary then we have an $\mathcal{F} \subseteq (\mathcal{P}(M) \times \mathcal{P}(M))$, e.g. for $(\forall x|\varphi)\psi$ we have $\mathcal{F} = \{(X, Y)|X \subseteq Y \subseteq M\}$ and for $(\exists x|\varphi)\psi$ we have $\mathcal{F} = \{(X, Y)|X, Y \subseteq M \text{ and } X \cap Y \neq \emptyset\}$.

We are ready for a general definition.

Definition 8.1.3 A (Boolean) predicate calculus with a *unary generalized quantifier* q is given by the following things:

- a predicate language $\mathcal{I}$
- a class $\mathcal{K}$ of structures of the form $(\mathbf{M}, \mathcal{F})$ where $\mathbf{M}$ is an interpretation of $\mathcal{I}$ and $\mathcal{F}$ is a set of subsets of the domain M of $\mathbf{M}$. Elements of $\mathcal{K}$ are called *models* of the calculus.

Logical symbols are those of predicate calculus plus q. The definition of a formula is extended by the clause "if φ is a formula and x is a variable then $(qx)\varphi$ is a formula (and the notion of a bound variable is changed correspondingly).

[43] The pioneering paper on generalized quantifiers is [144]; for the present state of the art see [120].

The definition of satisfaction is extended by the clause

$$\|(qx)\varphi\|_{(\mathbf{M},\mathcal{F}),v} = 1 \text{ iff } Sat_{(M,\mathcal{F}),v}(\varphi, x) \in \mathcal{F}$$

A formula φ is *true* in $(\mathbf{M}, \mathcal{F})$ iff $\|\varphi\|_{(M,\mathcal{F}),v} = 1$ for all v. φ is a *tautology* if it is true in each model.

The reader will produce with ease the corresponding definitions of a calculus with a binary quantifier $((Sat_{(M,\mathcal{F}),v}(\varphi, x), Sat_{(M,\mathcal{F}),v}(\psi, x)) \in \mathcal{F})$ and will be able to extend to an n-ary quantifier if desired.

Note that we shall often write $\mathbf{M}$ instead of $(\mathbf{M}, \mathcal{F})$ if $\mathcal{F}$ is clear from the context.

8.1.4 We shall now offer the reader various examples of (types of) calculi with generalized quantifiers and information on relevant references. First we discuss *cardinality quantifiers*. These may be considered to be generalized existential quantifiers: for each cardinal number κ we have the quantifier $(\exists^\kappa x)$ such that $(\exists^\kappa x)\varphi$ means "there are at least κ objects such that φ". The models of the calculus are in one-one correspondence with interpretations $\mathbf{M}$ of the corresponding language; for each such $\mathbf{M}$, $\mathcal{F}_{\mathbf{M}}$ is the set of all subsets of M having cardinality at least κ. Let us discuss some particular cases.

(i) κ is finite (think of $\kappa = 4$). If the calculus has an equality predicate $=$ (interpreted absolutely) then $(\exists^\kappa x)$ is definable: $(\exists^4 x)\varphi$ is equivalent to $(\exists x_1, x_2, x_3, x_4)(\bigwedge_{i=1}^4 \varphi(x_i) \& \bigwedge\{x_i \neq x_j | i \neq j, 1 \leq i, j \leq 4\}$. Thus if all interpretations of the language are taken as models, the calculus is axiomatized by the usual axioms of the Boolean predicate logic with equality.

(ii) κ is $\aleph_0$ (the countable cardinality). This calculus is not recursively axiomatizable (see 8.1.10). This fact is related to (and easily follows from) the known fact that the set of all formulas of the classical predicate calculus true in all *finite* models is not recursively enumerable. (Note that φ is true in all finite models iff the formula $\neg(\exists^{\aleph_0} x)(1) \rightarrow \varphi$ is a tautology of the present calculus.)

(iii) κ is $\aleph_1$ (the first uncountable cardinality). There is an elegant recursive axiom system complete for this logic [205], [107].

8.1.5 Note that the quantifiers $\exists^\kappa$ for infinite κ make sense only if the calculus has (finite and) infinite models. But we may have calculi having only finite models (by definition, i.e. the class $\mathcal{K}$ of models is *defined* to contain only finite models of a certain kind). We shall discuss calculi of this type (having great importance in database theory and data analysis) later in this section.

8.1.6 Now let us investigate quantifiers in Boolean logic given by numerical characteristics of subsets of a given set. We shall restrict ourselves to *quantifiers given by measures of belief* (probability, possibility, Dempster-Shafer belief functions). The necessary definitions are contained in the Appendix of this section. Note that if $\mathbf{M}$ is an interpretation of a language $\mathcal{I}$ and μ is a real-valued function defined on (some or all) subsets of M then we may use μ at least in two ways to define a quantifier.

(i) *Threshold quantifiers*. Take an α from the range of μ and define $\mathcal{F}$ to be the set of all $X \subseteq M$ such that ($\mu(X)$ is defined and) $\mu(X) \geq \alpha$. Thus
$\|(qx)\varphi\|_{(\mathbf{M},\mathcal{F}),v} = 1$ iff $\mu(Sat_{\mathbf{M},v}(\varphi, x)) \geq \alpha$
$\|(qx)\varphi\|_{(\mathbf{M},\mathcal{F}),v} = 0$ iff $\mu(Sat_{\mathbf{M},v}(\varphi, x)) < \alpha$
and is undefined otherwise. Call $\mathbf{M}$ *safe* for μ if all necessary sets are in the domain of μ, thus satisfaction of all formulas is defined. For safe models our definition agrees with that using $\mathcal{F}$. (Note that we could define $\mathcal{F}$ using sharp inequality $\mu(x) > \alpha$ as well; this gives a different quantifier.)

(ii) *Comparative quantifiers*. Introduce a binary quantifier $\lhd$ (write $\varphi \lhd_x \psi$ instead of $(\lhd\, x)(\varphi, \psi)$ and define $\mathcal{F}$ to be the set of all pairs (X, Y) where $X, Y \subseteq M$, $\mu(X)$ and $\mu(Y)$ is defined and $\mu(X) \leq \mu(Y)$. Then $\varphi \lhd_x \psi$ says "ψ is at least as much believed as φ" and for safe models the definition using $\mathcal{F}$ applies. Note that both possibilistic measures and Dempster-Shafer measures are total, i.e. each structure is safe; but, as commonly known, a σ-additive probability is defined just on a σ-algebra of subsets of M and there may be non-measurable subsets.

Probabilistic threshold quantifiers have been extensively studied; Raković [170] is an excellent survey and so is Keisler's [108] (older). Since σ-additivity is stressed these approaches use logical systems admitting infinite countable conjunctions $\bigwedge_{i=1}^{\infty} \varphi_i$ as formulas or deduction rules with infinitely many assumptions.

Little seems to be known about possibilistic and Dempster-Shafer belief quantifiers (but see the next section on modal logics).

8.1.7 We mention also *quantifiers given by* a *similarity* (as studied in Chap. 5 Sec. 6).

These are threshold quantifiers giving formulas of the form $(\forall x\, \mathrm{sim}_\alpha y)\varphi$, $(\exists x\, \mathrm{sim}_\alpha y)\varphi$. Evidently, if a model $\mathbf{M}$ and an evaluation v are given and $v(y) = b$ then $Sat_{\mathbf{M},v}(x\, \mathrm{sim}_\alpha y, x)$ consists of all $a \in M$ whose degree of similarity to b is at least α.

We shall meet similarities again in the next section.

8.1.8 Furthermore, let us mention that all we have said generalizes for *many sorted calculi*: a quantifier and the sort and the corresponding system $\mathcal{F}$ is a system of subsets for the corresponding domain (if q is non-parametric unary;

obvious modification for other kinds). This will be very important for our treatment of modal logic.

8.1.9 We are going to comment on *finite model theory*. As mentioned above, this means a study of predicate calculi (with or without generalized quantifiers) whose class of models consists of all finite interpretations of the given language (i.e. interpretations whose domain is a finite set). Ref. [73] is one of early papers on the subject, for a modern treatment see e.g. [130, 47]. Trahtenbrot [201] was the first to prove that for a sufficiently rich language the predicate calculus with finite models cannot be recursively axiomatized. The final result (probably proved by various authors, is as follows):

Theorem 8.1.10 Let $\mathcal{J}$ be a language containing at least one at least binary predicate, let $\mathcal{K}$ be the class of all finite interpretations of $\mathcal{J}$. The set of tautologies of the predicate calculus with the language $\mathcal{J}$, classical quantifiers and $\mathcal{K}$ as the class of models is Π_1-complete (hence not recursively enumerable).

See [16] for a detailed treatment of Trakhtenbrot's theorem. The very rough idea of a proof is to simulate initial segments of the structure of natural numbers by finite models satisfying appropriate axioms and show, in this way, that the set of all formulas having a model is Σ_1-complete. We mention in passing one more property of classical predicate calculus:

Theorem 8.1.11 Let $\mathcal{J}$ be any language and φ a formula of $\mathcal{J}$ (with classical quantifiers). If for each natural $n > 0$ there is a model $\mathbf{M}$ of cardinality $\geq n$ in which φ is true (i.e. $\|\varphi\|_{\mathbf{M}} = \bar{1}$) then there is an infinite model $\mathbf{M}$ with $\|\varphi\|_{\mathbf{M}} = \bar{1}$.

This follows by completeness of classical predicate logic and definability of quantifiers $(\exists^n x)$: if for each n the theory with axioms φ and $(\exists^n x)\bar{1}$ is constant then so is $\{\varphi\} \cup \{(\exists^n x)\bar{1} | n \in N\}$.

8.1.12 Let us now study quantifiers given by frequencies. Given a finite model $\mathbf{M}$ and a formula $\varphi(x)$ with one free variable x (or with more variables, all except x being interpreted by an evaluation v), we have two complementary subsets of M: that of objects satisfying φ (i.e. $Sat_{\mathbf{M},v}(\varphi, x)$) and that of object satisfying $\neg\varphi$. The cardinalities of these two sets may be denoted by $Fr_{\mathbf{M},v}(\varphi(x))$ and $Fr_{\mathbf{M},v}(\neg\varphi(x))$ or simply $Fr_{\mathbf{M}}(\varphi)$ and $Fr_{\mathbf{M}}(\neg\varphi)$ (if v, x are clear from the context) or also $a_{\mathbf{M}}, b_{\mathbf{M}}$. Needless to say, $Fr_{\mathbf{M}}(\varphi) + Fr_{\mathbf{M}}(\neg\varphi)$ is the cardinality of $\mathbf{M}$.

If we have two formulas $\varphi(x)$ and $\psi(x)$ we get four frequencies: $Fr_{\mathbf{M}}(\varphi\&\psi), Fr_{\mathbf{M}}(\varphi\&\neg\psi), Fr_{\mathbf{M}}(\neg\varphi\&\psi), Fr_{\mathbf{M}}(\neg\varphi\&\neg\psi)$. We often denote them $a_{\mathbf{M}}, b_{\mathbf{M}}, c_{\mathbf{M}}, d_{\mathbf{M}}$ respectively and represent them as a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

8.1.13 A *unary statistic* is a real-valued function $t(a,b)$ defined for all pairs (a,b) of natural numbers such that $a+b \geq 1$. A *binary statistic* is a real-valued function $t(a,b,c,d)$ defined for all quadruples (a,b,c,d) of natural numbers such that $a+b+c+d \geq 1$.

Caution: This term is borrowed from statistics but as such does not have any a priori relation to probability theory: it is just a numerical characteristics given by frequencies.

Example 8.1.14 Unary: relative frequency $t(a,b) = a/(a+b)$ for $a+b > 0$; $t(0,0) = 1$, say.

Binary: cross product

$$\frac{ad}{ad+bc}.$$

More sophisticated statistics are used in statistical hypothesis testing; but we shall not go into this.

8.1.15 Clearly statistics may be used to define threshold quantifiers in finite modal calculi. Ref. [85] contains much material about such quantifiers (called *observational*). We recall some definitions. Observe that the definitions formalize intuitive notions of measures of association or measures of high relative frequency. If normalized between $0,1$ they may be taken as possible semantics of the fuzzy quantifiers meaning "φ, ψ are associated" or "ψ is highly frequent in φ". This will be analyzed in more detail in Section 3 of this chapter.

Definition 8.1.16 (1) A binary statistic $t(a,b,c,d)$ is *associational* if $a \leq a', b \geq b', c \geq c'$ and $d \leq d'$ implies $t(a,b,c,d) \leq t(a',b',c',d')$. (2) t is *multitudinal*[44] if $a \leq a'$ and $b \geq b'$ implies $t(a,b,c,d) \leq t(a',b',c',d')$ (i.e. t does not depend on c,d).

Definition 8.1.17 A quantifier q in a given finite model predicate calculus is *associational* (multitudinal) if there is an associational (multitudinal) statistic $t(a,b,c,d)$ and a number α such that for each model $\mathbf{M}$, pair of formulas φ, ψ and the corresponding frequencies $a_{\mathbf{M}}, b_{\mathbf{M}}, c_{\mathbf{M}}, d_{\mathbf{M}}$, $\|(qx)\varphi\|_{\mathbf{M},v} = 1$ iff $t(a_{\mathbf{M}}, b_{\mathbf{M}}, c_{\mathbf{M}}, d_{\mathbf{M}}) \geq \alpha$ (alternatively, $> \alpha$).

[44] In [85] and other literature on the GUHA method multitudinal quantifiers are called *implicational*; but I find the term "multitudinal" (referring to "many") more appropriate.

Definition 8.1.18 A statistic is *symmetric* if $t(a, b, c, d) = t(a, c, b, d)$ for each a, b, c, d. A threshold quantifier is *symmetric* if it is given by a symmetric statistics.

Remark 8.1.19 We usually denote a multitudinal quantifier by $\sqsupset$; $\varphi \sqsupset_x \psi$ can be read"*many* objects x satisfying φ satisfy ψ." (Caution: this is a crisp, non-fuzzy "many": $\varphi \sqsupset_x \psi$ is either true or false. The fuzzy "many" will be discussed in Sec. 3.

A symmetric associational quantifier may be denoted by $\sim$; $(\sim x)(\varphi, \psi)$ (or $\varphi \sim_x \psi$, if desired) is read "φ and ψ are *associated*".

We shall offer two examples of tautologies, but no systematic attempt to find a complete axiomatization is made.

Lemma 8.1.20 If $\sqsupset$ is a multitudinal quantifier then

$$((\varphi \wedge \psi) \sqsupset_x \chi) \to (\varphi \sqsupset_x (\neg\psi \vee \chi))$$

is a tautology.

(This should resemble the propositional Boolean tautology $((\varphi \& \psi) \to \chi) \to (\varphi \to (\neg\psi \vee \chi))$; the converse implication is also a Boolean tautology, but our formula with $\sqsupset$, if $\to$ converted, is not more a tautology.)

Proof: Let m_{ijk} be the number of objects m such that $\|\varphi\|_{\mathbf{M},m} = i$, $\|\psi\|_{\mathbf{M},m} = j$, $\|\chi\|_{\mathbf{M},m} = k$ $(i, j, k \in \{0, 1\}$). Then the frequencies a, b from the four-fold table given by $\varphi \wedge \psi$ and χ are m_{111}, m_{110} whereas the corresponding frequencies a', b' given by $\varphi, \neg\psi \vee \chi$ are $m_{111} + m_{101} + m_{100}$, m_{110}. Thus $t(a, b) \leq t(a', b')$. Note that this proves much more than stated; we shall recall this when presenting fuzzy quantifiers on crisp finite structures. □

Lemma 8.1.21 If $\sim$ is a symmetric associational quantifier then

$$[(\varphi \sim_x \psi) \wedge (\forall x)((\varphi \wedge \psi) \to \chi)] \to ((\varphi \wedge \chi) \sim_x \psi)$$

is a tautology.

Proof: similar: if a, b, c, d are the four frequencies given by φ, ψ and a', b', c', d' those given by $(\varphi \wedge \chi), \psi$ and if $(\forall x)((\varphi \wedge \psi) \to \chi)$ is true in $\mathbf{M}$ then $a = a', b \geq b', c \geq c', d \leq d'$ thus $t(a, b, c, d) \leq t(a', b', c', d')$ as desired. □

8.1.22 See [85] for more information on observational quantifiers (given by statistics).

*

Appendix: measures of belief – a short survey.

Definition 8.1.23 A *field of subsets* of a set $M \neq \emptyset$ is a system $\mathcal{F}$ of subsets of M containing M and $\emptyset$ and closed under union $\cup$ and complement $-$. (Then $\mathcal{F}$ is a Boolean algebra with respect to set-theoretical operations.) $\mathcal{F}$ is a σ-field if for each sequence $\{X_n | n \in N\}$ of elements of $\mathcal{F}$, its union $\bigcup_{n \in N} X_n$ is an element of $\mathcal{F}$. A *belief measure* on $(M, \mathcal{F})$ is a function $\mu : \mathcal{F} \to [0, 1]$ such that

$$\mu(\emptyset) = 0, \qquad \mu(M) = 1,$$

$$X, Y \in \mathcal{F} \text{ and } X \subseteq Y \text{ implies } \mu(X) \leq \mu(Y).$$

(Note that this is sometimes called a *fuzzy measure* but we shall avoid this term.)

We shall survey definitions and basic properties of probability, possibility and Dempster-Shafer belief functions.[45]

8.1.24 A belief measure μ on $(M, \mathcal{F})$ is a *finitely additive probability* if $X, Y \in \mathcal{F}$ and $X \cap Y = \emptyset$ implies $\mu(X \cup Y) = \mu(X) + \mu(Y)$. μ is a *σ-additive probability* (or simply *probability*) if its domain $\mathcal{F}$ is a σ-field and for each sequence $\{X_n | n \in N\}$ of pairwise disjoint elements of $\mathcal{F}$ (i.e., $X_i \cap X_j = \emptyset$ for $i \neq j$), $\mu(\bigcup_{n \in N} X_n) = \Sigma_{n \in N} \mu(X_n)$.

Lemma 8.1.25 (1) Let μ be a finitely additive probability on $(M, \mathcal{F})$. Then for each finite sequence $X_0, \ldots, X_n$ of pairwise disjoint elements of $\mathcal{F}$,

$$\mu(\bigcup_{i=0}^{n} X_i)) = \Sigma_{i=1}^{n} \mu(X_i).$$

(2) Let μ be a probability on $(M, \mathcal{F})$ and let $\{X_n | n \in N\}$ be a sequence of elements of $\mathcal{F}$. If the sequence is increasing (i.e. $X_i \subseteq X_{i+1}$ for each i) then $\mu(\bigcup_{i \in N} X_i) = \sup_{i \in N} \mu(X_i)$. If it is decreasing ($X_i \supseteq X_{i+1}$ for each i) then $\mu(\bigcap_{i \in N} X_i) = \inf_{i \in N} \mu(X_i)$.

[45] The classical monograph on possibility theory is [40], that on belief functions is [186]. For the state of the art of belief function theory see [188].

Proof: Obvious. □

Lemma 8.1.26 Let M be finite and let $\mathcal{F}$ be the set $\mathcal{P}(M)$ of all subsets of M.

(1) Each finitely additive probability on $(M, \mathcal{F})$ is σ-additive.

(2) Each probability μ on $(M, \mathcal{F})$ is uniquely given by its values of singletons: if $\mu_0 : M \to [0,1]$ is such that $\Sigma_{m \in M} \mu_0(m) = 1$ then there is a unique probability μ on $(M, \mathcal{F})$ such that $\mu(\{m\}) = \mu_0(m)$ for each $m \in M$.

Proof: obvious. □

Remark 8.1.27 Note that the assertion (2) from the preceding lemma generalizes to the case that M is countable, $\mathcal{F} = \mathcal{P}(M)$ and μ is a σ-additive measure.

Definition 8.1.28 Let $\mathcal{F} = \mathcal{P}(M)$. A *possibility* on M is a belief measure $\Pi : \mathcal{F} \to [0,1]$ such that for any system $\{X_\alpha | \alpha \in I\}$ of subsets of M (I any non - empty index set),

$$\Pi(\bigcup_{\alpha \in I} X_\alpha) = \sup_{\alpha \in I} \Pi(X_\alpha).$$

A *necessity* is a belief measure $\mathcal{N} : \mathcal{F} \to [0,1]$ such that for any $\{X_\alpha | \alpha \in I\}$ as above,

$$\mathcal{N}(\bigcap_{\alpha \in I} X_\alpha) = \inf_{\alpha \in I} \mathcal{N}(X_\alpha).$$

Lemma 8.1.29 (1) If Π is a possibility on M then the function $\mathcal{N}$ defined as $\mathcal{N} = 1 - \Pi(M - X)$ is a necessity measure on M. Conversely, each necessity $\mathcal{N}$ defines a possibility $\Pi(X) = 1 - \mathcal{N}(M - X)$.

(2) Each possibility is uniquely determined by its values on singletons: if $\pi : M \to (0,1)$ is a normalized fuzzy subset of M (i.e. $\sup_{m \in M} \pi(m) = 1$) then

$$\Pi(X) = \sup_{m \in X} \pi(m)$$

is a possibility. (And of course , if Π is a possibility and $\pi(m) = \Pi(\{m\})$ then π is a normalized fuzzy subset of M determining Π by the above formula.)

Proof: obvious □

Definition 8.1.30 (1) Let $f : \mathcal{P}(M_1) \to \mathcal{P}(M_2)$ (for each $X \subseteq M_1, f(X) \subseteq M_2$). f is a $\cap$-*homomorphism* (or *meet homomorphism*) if $f(M_1) = M_2$, $f(\emptyset) = \emptyset$ and $f(X \cap Y) = f(X) \cap f(Y)$ for all $X, Y \subseteq M_1$.

A function $bel : \mathcal{P}(M) \to [0,1]$ is a *Dempster-Shafer belief function* (briefly, a DS-belief function or just belief function) if there is a finitely additive probability μ on some $(M', \mathcal{F}'$ and a $\cap$ - homomorphism h mapping $\mathcal{P}(M)$ into $\mathcal{F}'$ such that $bel(X) = \mu(h(X))$ for each $X \subseteq M$. (We say that h *allocates* the probability $\mu(h(X))$ to X – see [187]).

Remark 8.1.31 Show that the $\cap$ - homomorphism is monotone, i.e. $X \subseteq Y$ implies $h(X) \subseteq h(Y)$; thus $X \subseteq Y$ implies $bel(X) \leq bel(Y)$. Also $bel(\emptyset) = 0, bel(M) = 1$, thus bel is a belief measure.

Definition 8.1.32 If bel is a DS-belief function on M then the function $pl(x) = 1 - bel(M - X)$ is the corresponding *plausibility*.

We mention without a proof the following characterization of belief functions (see [187]):

Lemma 8.1.33 A function $bel : \mathcal{P}(M) \to [0,1]$ is a DS-belief function iff $bel(\emptyset) = 0, bel(M) = 1$ and bel is monotone of order ∞, i.e. for each $n \geq 1$, each $A \subseteq M, A_1, \ldots, A_n \subseteq M, A_i \subseteq A$ for $i = 1, \ldots, n$,

$$bel(A) \geq \sum\{(-1)^{|I|+1} bel(\bigcap_{i \in I} A_i) | \emptyset \neq I \subseteq \{1, \ldots, n\}\}.$$

For *finite* M DS-functions have a nice alternative characterization:

Lemma 8.1.34 Let M be finite. A function $bel : \mathcal{P}(M) \to [0,1]$ is a DS - belief function iff there is a function $m : \mathcal{P}(M) \to [0,1]$ (called a *basic belief assignment*) such that $m(\emptyset) = 0$, $\sum\{m(X)|X \subseteq M\} = 1$ and, for each $X \subseteq M$,

$$bel(X) = \sum_{Y \subseteq X} m(Y).$$

Proof: Hint: If bel is given by h and μ define $m(\{x\}) = bel(\{x\}) = \mu(h(\{x\}))$ for each singleton; if $m(X)$ has been defined for all $Y \subseteq M$ with $|Y| = k$ define, for $X \subseteq M, |X| = k+1, m(X) = bel(X) - \sum\{m(Y)|Y \subset X, |Y| \leq k\}$.

Conversely, if bel is defined by the basic belief assignment m, let $M_1 = \mathcal{P}(M), \mathcal{F}_1 = \mathcal{P}(\mathcal{P}(M))$ and let, for $Z \subseteq \mathcal{P}(M), \mu(Z) = \sum\{m(Y)|Y \in Z\}$. For $X \subseteq M$ let $h(X) = \{Y|Y \subseteq X\}$. $\square$

Definition 8.1.35 Let m be a basic belief assignment on $\mathcal{P}(M)$. $F \subseteq M$ is a *focal element* if $m(F) > 0$.

Theorem 8.1.36 Let M be finite.
(1) $\mu : \mathcal{P}(M) \to [0, 1]$ is a probability iff it is a DS-belief function whose all focal elements are singletons.
(2) $\mathcal{N} : \mathcal{P}(M) \to [0, 1]$ is a necessity iff it is a DS-belief function whose focal elements are linearly ordered by inclusion.

The reader may prove this theorem as a (moderately difficult) exercise.

Remark 8.1.37 There are lots of monographs on probability theory. For possibility theory, the "classical" book is [40], for belief functions [186], [187]. The literature is numerous; see e.g. [192]. Recall our sharp distinction between degrees of truth and degrees of belief; the latter play only an auxiliary role in our considerations of fuzzy logic, in relation to generalized quantifiers (and modal logic).

8.2. Two-Valued Modal Logics

8.2.1 This is a second survey section of this chapter. Classical modalities are *necessity* $\Box$ and *possibility* $\Diamond$. We can extend the propositional calculus as well as the predicate calculus by these modalities and generalize the notion of a formula by adding the formation rule "if φ is a formula then $\Box\varphi$ and $\Diamond\varphi$ are formulas." They are usually read "necessarily φ" and "possibly φ" respectively; but there are various other ($\Box\varphi$ may be understood as"φ is known", "φ is believed", "always in the future φ will hold", "φ is provable" etc; this leads to various semantics of modal logic. We shall mainly deal with modal propositional logic but shall also present some information on modal predicate logic. We shall stress the fact that modal logic (propositional and predicate) can be seen as a sublogic of predicate logic (possibly with many sorts and generalized quantifiers). This will be useful in introducing new modalities.[46]

Definition 8.2.2 A (two-valued) *Kripke model* of a propositional language $p_1, p_2, \ldots$ (propositional variables) is a structure $\mathbf{K} = \langle W, e, r\rangle$ where $W \neq \emptyset$ is a set of *possible worlds* (or *states*), e is a mapping associating with each $w \in W$ and each variable p_i the truth value (0 or 1) of p_i in w and $r \subseteq W \times W$ is a-binary relation on W. If $(u, w) \in r$ we say that w is *accessible* for u. We

[46] Ref. [104] is a good monograph on modal logic. See also [55, 56].

write $||p_i||_{\mathbf{K},w}$ for $e(w, p_i)$ (as defined in $\mathbf{K}$) and extend this to all formulas, defining $||\varphi \rightarrow \psi||_{\mathbf{K},w}$, $||\varphi \wedge \psi||_{\mathbf{K},w}$, $||\neg\varphi||_{\mathbf{K},w}$ from $||\varphi||_{\mathbf{K},w}$, $||\psi||_{\mathbf{K},w}$ by truth functions (truth tables, e.g. $||\varphi \wedge \psi||_{\mathbf{K},w} = 1$ iff $||\varphi||_{\mathbf{K},w} = ||\psi||_{\mathbf{K},w} = 1$ etc.) and defining $||\Box\varphi||_{\mathbf{K},w} = 1$ iff $||\varphi||_{\mathbf{K},v} = 1$ for each v such that $(w, v) \in r$ (i.e. $\Box\varphi$ is true in w iff φ is true in all worlds accessible for w.) $||\Diamond\varphi||_{\mathbf{K},w} = 1$ iff $||\varphi||_{\mathbf{K},v} = 1$ for some v accessible for w. Thus $\Diamond\varphi$ is equivalent to $\neg\Box\neg\varphi$.

Definition 8.2.3 A *semantics* of modal logic is given by a class $\mathcal{K}$ of Kripke structures for a given propositional language. The definition in particular determines properties of the accessibility relation r and possibly also the cardinality of W (our finite structures, or also infinite ones etc). A $\mathcal{K}$- tautology is a formula true in each world of each structure $\mathbf{K} \in \mathcal{K}$.

Example 8.2.4 (1) $\mathcal{K}_{S5}$ is the class of all Kripke structures with $r = W \times W$ (i.e. any world is accessible for any world). For structures from $\mathcal{K}_{S5}$ the relation r is uniquely given by W and therefore is superfluous; we may present models of this class just as pairs (W, e). Note that in such a model the value $||\varphi||_{\mathbf{K},w}$ is independent of w : $||\Box\varphi||_K = 1$ iff $||\varphi||_{\mathbf{K},v} = 1$ for all $v \in W$. $\mathcal{K}_{S5}$ will be the semantics of a logic S5 called the *logic of knowledge*.

(2) $\mathcal{K}_{KD45}$ is the class of all Kripke structures $\mathbf{K} = (W, e, r)$ such that for some $\emptyset \neq W_0 \subseteq W$, $r = W \times W_0$. Worlds from W_0 are called *believable* worlds; w is accessible for u iff w is believable. Also here $||\Box\varphi||_{\mathbf{K},w}$ is independent from w. KD45 (historical notation) is called the *logic of belief.*

(3) There are very many *tense logics* (logics of time); we restrict ourselves just to one example.[47] Our $\mathcal{K}_{tense}$ consists of all Kripke structures $\mathbf{K} = (W, e, r)$ where r is a reflexive linear preorder, i.e. r is reflexive, transitive and dichotomic: for each $u, v \in W$, $(u, v) \in r$ or $(v, u) \in r$. In the corresponding logic we shall investigate two necessities: G (for all worlds from now on) and H (for all worlds up to now). The semantics is:
$||G\varphi||_{\mathbf{K},w} = 1$ iff $||\varphi||_{\mathbf{K},v} = 1$ for all v such that $(w, v) \in r$,
$||H\varphi||_{\mathbf{K},w} = 1$ iff $||\varphi||_{\mathbf{K},v} = 1$ for all v such that $(v, w) \in r$.

The corresponding possibilities are F and P; $F\varphi$ is equivalent to $\neg G\neg\varphi$, $P\varphi$ to $\neg H\neg\varphi$ (in some world from now on, in some world up to now).

The reader surely expects that we shall select some $\mathcal{K}$-tautologies as axioms and produce some deductive system complete for the semantics given by the chosen $\mathcal{K}$. This will indeed be the case; but before we go into this it will (we hope) be illuminating to clarify the relation of modal logic to predicate calculus. The key observation is that a modality is in fact a hidden quantifier on possible worlds. Let us make it explicit.

---

[47] Pioneering papers are those of Gabbay [57, 58]. For a good survey of tense logics see [19].

Definition 8.2.5 Associate with a given propositional language $\mathcal{I}$ a predicate language $\mathcal{I}^*$ having for each p_i a unary predicate P_i and, in addition, a binary predicate R. For each modal formula φ and each object variable x, define $\varphi^*(x)$ as follows:

- $p_i^*(x)$ is $P_i(x)$; $*$ commutes with connectives (i.e. $(\varphi\&\psi)^*$ is $\varphi^*\&\psi^*$ etc.);
- given $\varphi^*(x)$ and object variable y not occuring in $\varphi^*(x)$ let $(\Box\varphi)^*(x)$ be the formula $(\forall y)(R(x,y) \rightarrow \varphi^*(y))$ (where $\varphi^*(y))$ results from $\varphi^*(x)$ by replacing all (free) occurences of x by y).

Finally, associate with each Kripke model $\mathbf{K} = (W, e, r)$ an interpretation $\mathbf{K}^*$ of our predicate language such that
$K^* = W$,
$r_{P_i}(w) = e(w, p_i)$ for each i,
$r_R = r$.

Lemma 8.2.6 Under the above notation,

$$\|\varphi\|_{\mathbf{K},w} = \|\varphi^*\|_{\mathbf{K}^*,w}$$

for each φ and $\mathbf{K}$ (the left hand value being computed by the rules of modal logic, the right hand according to predicate logic). Moreover, the mapping $*$ of Kripke models is a one-one mapping of all Kripke structure for $\mathcal{I}$ onto all interpretations of $\mathcal{I}^*$.

Proof: routine $\Box$

Remark 8.2.7 Observe that in the above examples, the condition on the relation r was expressible by a formula Φ of predicate calculus with classical quantifiers (and no condition has been made on the cardinality of the models). Whenever this is the case, the above definition and lemma shows that φ is a tautology of $\mathcal{K}$ iff φ^* is true in all models of Φ (in the sense of the predicate calculus), thus, by Gödel's completeness theorem, φ is a tautology of $\mathcal{K}$ iff $\Phi \vdash \varphi^*$ (φ^* is provable from Φ in the predicate calculus). We can conclude that the set of $\mathcal{K}$-tautologies is recursively enumerable. But this does not give us any elegant axiomatization of $\mathcal{K}$-tautologies, to find one is another task. We shall present complete axiomatizations of our above examples.

Definition 8.2.8 The *deduction rules* of a two-valued modal logic (with the modality $\Box$, $\Diamond$ being defined as $\neg\Box\neg$) are modus ponens and necessitation: from φ deduce $\Box\varphi$.

Logical axioms consist of axiom of the underlying (Boolean) propositional calculus plus some *modal axioms*. In particular:

(1) S5 - *the logic of knowledge.* Modal axioms

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi),$$

$$\Box\varphi \to \Box\Box\varphi,$$

$$\Diamond\varphi \to \Box\Diamond\varphi,$$

$$\Box\varphi \to \varphi$$

(2) KD45 - *the logic of belief*: As S5, with the last axiom ($\Box\varphi \to \varphi$) replaced by $\Box\varphi \to \Diamond\varphi$.

(3) RTL - *reflexive tense logic*. Recall that we have two necessities G, H (always from now on, always up to now) and two corresponding possibilities F, P. Modal axioms:

$$\begin{array}{rl} G(\varphi \to \psi) \to (G\varphi \to G\psi) & H(\varphi \to \psi) \to (H\varphi \to H\psi) \\ PG\varphi \to \varphi & FH\varphi \to \varphi \\ F\varphi \to G(P\varphi \vee F\varphi) & P\varphi \to H(P\varphi \vee F\varphi) \\ G\varphi \to GG\varphi & H\varphi \to HH\varphi \\ G\varphi \to \varphi & H\varphi \to \varphi \end{array}$$

We have two necessitations; "from φ deduce $G\varphi$" and "from φ deduce $H\varphi$".

Lemma 8.2.9 For all three examples (S5, KD45, RTL), each axiom is a tautology with respect to the corresponding class of models; and deduction rules preserve tautologicity.

Proof: mechanical checking; in particular, observe that the axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ (called K) is true in *each* Kripke model; $\Box\varphi \to \Box\Box\varphi$ is true if r is transitive etc. $\Box$

Theorem 8.2.10 *(Completeness).* (1) The above logics S5, KD45, RTL are *complete*, i.e. a formula is a tautology (with respect to the corresponding class of models) iff it is provable from the axioms.

(2) Moreover, all these three logics have the *finite model property*: if there is a model $\mathbf{K}$ and its world w such that $||\varphi||_{\mathbf{K},w} = 1$ then there is a finite model $\mathbf{K}$ (i.e. where set W of possible worlds is finite) and its world w such that $||\varphi||_{\mathbf{K},w} = 1$.

Theorem 8.2.11 In RTL we may define a modality $\Box$: $\Box\varphi$ stands for $(G\varphi \wedge H\varphi)$. With this definition, RTL is a conservative extension of S5: a formula of S5 is provable in S5 iff it is provable in RTL.

These are non-trivial results and we shall not attempt to provide proofs (see for example [104], [19]). But we shall use common techniques of these proofs in proving completeness of some fuzzy modal logics.

Remark 8.2.12 Let us also mention that S5 and KD45 are sound and complete with respect to broader classes. Let $\mathcal{K}'_{S5}$ be the class of all Kripke structures $\mathbf{K} = (W, e, r)$ where r is an equivalence, i.e. reflexive, symmetric and transitive relation. Let $\mathcal{K}'_{\mathbf{K}D45}$ be the class of Kripke models $\mathbf{K} = (W, e, r)$ such that r is transitive, serial (each world has at least one accessible world) and Euclidean (if u, v are accessible for w then they are accessible for each other). It is easy to check soundness; and completeness follows trivially from the preceding completeness, since $\mathcal{K}_{S5} \subseteq \mathcal{K}'_{S5}$ and the same for KD45.[48]

*

8.2.13 We shall now comment on the (two-valued) predicate modal logic. *Formulas* are built from predicates, variables, constants, connectives, modalities and quantifiers in the standard way. (We may take $\rightarrow, \neg, \forall, \Box$ to be primitive and consider the other connectives/quantifier/modality) as defined.)

8.2.14 A (two-valued) *predicate Kripke model* has the form

$$\mathcal{K} = \langle W, M, (r_{P,w})_{P,w}, (m_c)_c, r \rangle,$$

where $W \neq 0 \neq M$, W is the set of possible worlds, M a common domain of objects, for each predicate P of arity n and each $w \in W$, $r_{P,w}$ is an n-ary relation on M (interpretation of P in the world w), for each constant c, $m_c \in M$, and r is a binary relation on W.

The specific assumption we make here is that the set of objects is fixed, independently of possible worlds. This admits generalizations, but we shall disregard them.

In particular, for S5 the relation r is $W \times W$ and hence superfluous; models have the form $\mathcal{K} = \langle W, M, (r_{P,w})_{P,w}, (m_c)_c \rangle$. They could be easily seen as two-sorted structures in predicate calculus; if P is n-ary then P^* is $(n+1)$-ary, n arguments being of sort M (objects) and one of the sort W (possible world).

[48] For more information on the belief logic KD45 see [207].

8.2.15 Given a Kripke model $\mathbf{K} = \langle W, M, (r_P)_P, (m_c)_c \rangle$ and an evaluation σ of object variables by elements of M, we define the *value* $\|\varphi\|_{\mathbf{K},\sigma}[w]$ of a formula φ in a world w under the evaluation σ. Our definition is standard by induction on the complexity of formulas.

$$\|P(x_1,\ldots,x_n)\|_{\mathbf{K},\sigma}[w] = 1 \text{ iff } \langle \sigma(x_1),\ldots,\sigma(x_n), w\rangle \in r_{P,w}$$

($P(x_1,\ldots,x_n)$ is satisfied if the tuple of values of $x_1,\ldots,x_n$ is in the relation interpreting P in the world w);

$$\begin{aligned}
\|\neg\varphi\|_{\mathbf{K},\sigma}[w] = 1 &\quad \text{iff} \quad \|\varphi\|_{\mathbf{K},\sigma}[w] = 0,\\
\|\varphi \to \psi\|_{\mathbf{K},\sigma}[w] = 1 &\quad \text{iff} \quad \|\varphi\|_{\mathbf{K},\sigma}[w] = 0 \text{ or } \|\psi\|_{\mathbf{K},\sigma}[w] = 1,\\
\|\Box\varphi\|_{\mathbf{K},\sigma}[w] = 1 &\quad \text{iff} \quad \|\varphi\|_{\mathbf{K},\sigma}[v] = 1 \text{ for all } v \in W,\\
\|(\forall x)\varphi\|_{\mathbf{K},\sigma}[w] = 1 &\quad \text{iff} \quad \|\varphi\|_{\mathbf{K},\sigma'}[w] = 1 \text{ for all evaluations } \sigma'\\
\text{differing} &\\
&\quad \text{from } \sigma \text{ at most in the value of } x.
\end{aligned}$$

8.2.16 A formula φ is *true* in K if $\|\varphi\|_{\mathbf{K},\sigma}[w] = 1$ for all σ, w. Furthermore, φ is a *tautology* (of predicate Boolean S5) if φ is true in each Kripke model (of predicate Boolean S5).

One can easily verify that all axioms of propositional S5 are tautologies of predicate Boolean S5; moreover, the usual axioms for quantifiers in predicate logic, i.e.
$(\forall x)\varphi(x) \to \varphi(t)$ (if t is free for x in φ)
$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$ (if x is not free in χ)
are tautologies of our logic. Modus ponens and necessitation preserve tautologicity; and so does *generalization* (from φ deduce $(\forall x)\varphi$). Thus, with the obvious notion of proof and provability, this logic is sound: each provable formula is a tautology. Moreover, we have completeness:

8.2.17 A formula φ is provable in predicate Boolean S5 iff it is a tautology of S5.

8.2.18 In particular, the following formula, called the *Barcan formula*, is provable in our logic:
(B) $\qquad \Box(\forall x)\varphi \equiv (\forall x)\Box\varphi.$
Thus $\forall$ commutes with $\Box$.

8.2.19 For predicate Boolean KD45 we change the notion of a Kripke model by adding a non-empty set $W_0 \subseteq W$ of *believable worlds*:

$\mathbf{K} = \langle W, M, (r_{P,w})_{P,w}, (m_c)_c, W_0 \rangle$ and change the definition of satisfaction as follows:

$$\|\Box\psi\|_w = 1 \text{ iff } \|\psi\|_v = 1 \text{ for all } v \in W_0.$$

The corresponding axiom system of predicate Boolean KD45 consists of axioms of predicate calculus, axioms of KD45 *plus* the Barcan formula. One can show that this system is *sound and complete* with respect to tautologies of predicate Boolean KD45, i.e. formulas true in each Kripke model $\mathbf{K} = \langle W, M, (r_P)_P, W_0 \rangle$ in the sense of our modified definition of satisfaction.

*

8.2.20 Up to now, our modalities were generalized quantifiers on possible worlds and they were all of the same form: $\Box\varphi$ ment "in all worlds accessible for the given world, φ holds". Let us now use our investigation of generalized quantifiers to introduce modalities given by measures of belief (threshold or comparative) or by similarities. Then we shall investigate modalities "With probability $\geq p$, φ" etc.

Definition 8.2.21 A (two-valued) *Kripke model* with *a belief measure* has the form $\mathbf{K} = \langle W, e, \mu \rangle$ where $W \neq \emptyset$, e is an evaluation $(e(w,p) \in \{0,1\})$ for $w \in W$, p propositional variable and μ is a belief measure on a field of subsets of W. Given an $\alpha \in [0,1]$ we may define a modality $\Box_\alpha$ interpreted in $\mathbf{K}$ as follows:
$\|\Box_\alpha\varphi\|_{\mathbf{K},w} = 1$ iff $(Sat_{\mathbf{K}}(\varphi) = \{v \mid \|\varphi\|_{\mathbf{K},v} = 1\}$ is in the domain of μ and) $\mu(Sat_{\mathcal{K}}(\varphi)) \geq \alpha$.

Remark 8.2.22 Let $\mathcal{K}_{prob}, \mathcal{K}_{poss}, \mathcal{K}_{DS}$ be the class of all Kripke models with a probability, possibility, or DS-belief function respectively. It would be very interesting to have reasonable axiomatizations of modal logics with one of these classes as semantics and with one or several threshold modalities $\Box_\alpha$ (for one or several α). For the modality (modalities) "α-probably φ" the work on probabilistic quantifiers mentioned in the preceding section would be useful. For necessity some attempts were made by Dubois and Prade and their school [41]. Note that if $\Box_\alpha$ means "the necessity of φ is $\geq \alpha$" then $\Diamond_\alpha\varphi$ means "the possibility of φ is $\geq 1 - \alpha$". Similarly for DS - belief functions. This appears to be a rather promising field of research, with few results to date. But we shall not go into more details since the present survey of modalities in Boolean logic is primarily a preparation for our investigation of modalities in fuzzy logic.

8.2.23 We now turn to comparative quantifiers: we continue to work with the classes $\mathcal{K}_{prob}, \mathcal{K}_{poss}, \mathcal{K}_{DS}$; in each structure $\mathbf{K} = \langle W, e, \mu \rangle$ we may evaluate formulas of the form $\varphi \lhd \psi$ such that $\|\varphi \lhd \psi\|_{\mathbf{K}} = 1$ iff ($\mu(\varphi), \mu(\psi)$ are defined and) $\mu(\varphi) \leq \mu(\psi)$. Not much is known on the modality comparing probabilities; but we quickly describe systems with modalities comparing possibilities and DS - belief functions.

Definition 8.2.24 The *comparative possibilistic modal propositional logic* (CPMPL, see [14]) has formulas built from propositional variables, logical connectives and one binary modality $\lhd$ (if φ, ψ are formulas then $\varphi \lhd \psi$ is a formula). Models are Kripke structures $\mathbf{K} = \langle W, e, \pi \rangle$ where π is a possibility on W (once more: $\pi : W \to [0,1], \sup_{w \in W} \pi(w) = 1$). Semantics is standard: $\|\varphi \lhd \psi\|_{\mathbf{K}} = 1$ iff $\pi(\varphi) \leq \pi(\psi)$ (possibility of φ is less than or equal to the possibility of ψ). We define a new modality $\Box$: $\Box\varphi$ stands for $(\neg\varphi \lhd \bar{0})$. Thus $\|\Box\varphi\|_{\mathbf{K}} = 1$ iff φ holds in all worlds of positive possibility.

Definition 8.2.25 The *axioms* of CPMPL are

- axioms of Boolean propositional calculus,
- $(\varphi \lhd \psi) \to ((\psi \lhd \chi) \to (\varphi \lhd \chi))$ (transitivity)
- $(\varphi \lhd \psi) \vee (\psi \lhd \varphi)$ (linearity)
- $\neg(\bar{1} \lhd \bar{0})$ (non-triviality)
- $(\varphi \lhd \psi) \to ((\varphi \vee \chi) \lhd (\psi \vee \chi))$ (disjunction)
- $(\varphi \lhd \psi) \to \Box(\varphi \lhd \psi)$ (boxing 1)
- $\neg(\varphi \lhd \psi) \to \Box\neg(\varphi \lhd \psi)$. (boxing 2)

Deduction rules are modus ponens and necessitation: from $\varphi \to \psi$ infer $\varphi \lhd \psi$.

Theorem 8.2.26 CPMPL is complete with respect to the given semantics, i.e. CPMPL$\vdash \varphi$ iff φ holds in each world of each $\mathbf{K} \in \mathcal{K}_{poss}$.

A *proof* is obtained by combining the completeness proof in [14] with that of [94].

Theorem 8.2.27 CPMPL is a conservative extension of KD45, i.e. a formula of KD45 is provable in KD45 iff it is provable in CPMPL.

(Proof easy.)

Definition 8.2.28 CPMPL$^+$ is CPMPL extended by the axiom $\Box\varphi \to \varphi$ (thus $(\neg\varphi \lhd \bar{0}) \to \varphi$). $\mathcal{K}^+_{poss}$ is the class of possibilistic Kripke models $\mathbf{K} = \langle W, e, \pi \rangle$ with a positive π, i.e. $\pi(w) > 0$ for all $w > 0$.

Theorem 8.2.29 CPMPL$^+$ is complete with respect to $\mathcal{K}^+_{poss}$. Furthermore, CPMPL$^+$ is a conservative extension of S5.

(Proof by a modification of the preceding.)

Remark 8.2.30 (1) We may define a translation of CPMPL-formulas into formulas of the reflexive tense logic RTL as follows: $p^\#$ is p, $\#$ commutes with connectives, $(\varphi \lhd \psi)^\#$ is $\Box(\varphi^\# \to F\psi^\#)$. One easily shows that this is a faithful embedding of CPMPL$^+$ into RTL, i.e. a CPMPL$^+$ - formula φ is provable in CPMPL$^+$ iff $\varphi^\#$ is provable in RTL. (See [14] again.)

(2) We can dually develop a comparative *necessity* logic CNMPL with the same class $\mathcal{K}_{poss}$ (or $\mathcal{K}^+_{poss}$) of models and the same formulas but interpreting $\varphi \lhd \psi$ as saying $\mathcal{N}(\varphi) \leq \mathcal{N}(\psi)$. Then $\Box\varphi$ is to be defined as $\bar{1} \lhd \varphi$ and the disjunction axiom has to be changed to a conjunction axiom $(\varphi \lhd \psi) \to ((\varphi \wedge \chi) \lhd (\psi \wedge \chi))$. We leave the details to a diligent reader. (Compare also with the following axiomatization for DS-belief functions, recalling that necessity is a particular DS-belief function.)

Theorem 8.2.31 The *comparative belief function modal logic.* CBMPL has the same language as CPMPL i.e. the only binary modality $\lhd$; models are elements of $\mathcal{K}_{DS}$, i.e. structure $\mathbf{K} = \langle W, e, bel\rangle$ where bel is a belief function ($bel : \mathcal{P}(W) \to [0,1]$). Thus $\|\varphi \lhd \psi\|_K = 1$ iff $bel(\varphi) \leq bel(\psi)$ (pedantically, $bel\{w \mid \|\varphi\|_{\mathbf{K},w} = 1\} \leq bel\{w \mid \|\psi\|_{\mathbf{K},w} = 1\}$, as always).

Axioms of CBMPL result from the axioms of CPMPL by replacing the disjunction axiom by the following two:

- $\Box(\varphi \to \psi) \to (\varphi \lhd \psi)$
- $[\Box(\varphi \to \psi) \wedge \Box\neg(\psi \wedge \chi)] \to [((\psi \vee \chi) \lhd (\varphi \vee \chi)) \to (\psi \lhd \varphi)]$

The remaining axioms on $\lhd$ are transitivity, linearity, non-triviality and the two boxing axioms.

Deduction rules are modus ponens and necessities "from $\varphi \to \psi$ infer $\varphi \lhd \psi$" (as in CPMPL). (Note: alternatively you could take the necessitation "from φ infer $\Box\varphi$".)

Theorem 8.2.32 (1) CBMPL is complete, i.e. CBMPL$\vdash \varphi$ iff φ holds in each world of each model $\mathbf{K} \in \mathcal{K}_{DS}$.

(2) CBMPL extends KD45 conservatively, i.e. a KD45 - formula is provable in KD45 iff it is provable in CBMPL.

For a *proof* see [94].

8.2.33 To close this section we shall discuss Kripke models with a *similarity*: $\mathcal{K} = \langle W, e, r \rangle$ where $r : W \times W \rightarrow [0,1]$ is a similarity. This enables us to define modalities $\Box_\alpha$ that are *parametric* quantifiers on possible worlds: $\Box_\alpha \varphi$ says, in a world w: "φ is true in all worlds at least α-similar to me." Thus r gives a system of relations

$$(w, v) \in r_\alpha \quad \text{iff} \quad r(w, v) \geq \alpha.$$

These modalities were investigated in detail by Esteva and Godo [50]; we comment on their results.

It is easy to verify that the following formulas are valid in each Kripke model $K \in \mathcal{K}^*_{sim}$ where $\mathcal{K}^*_{sim}$ is the class of all Kripke models with a similarity with respect to the t-norm $*$ as above:
$\Box_\alpha(\varphi \rightarrow \psi) \rightarrow (\Box_\alpha \varphi \rightarrow \Box_\alpha \psi)$,
$\Box_\alpha \varphi \rightarrow \varphi$
$\varphi \rightarrow \Box_\alpha \Diamond_\alpha \varphi$
$\Box_{\alpha \times \beta} \varphi \rightarrow \Box_\alpha \Box_\beta \varphi$,
$\Box_\alpha \varphi \rightarrow \Box_\beta \varphi \qquad$ (for $\beta \leq \alpha$).

In fact, Esteva and Godo investigate a different definition of modalities, i.e.
$\|\Diamond_\alpha \varphi\|_{\mathbf{K},w} = 1$ iff $\sup\{r(w, w') | \, \|\varphi\|_{\mathbf{K},w'} = 1\} \geq \alpha$,
which gives $\|\Box_\alpha \varphi\|_{\mathbf{K},w} = 1$ iff $\sup\{r(w, w') | \, \|\varphi\|_{\mathbf{K},w'} = 0\} < \alpha$.
This is equivalent to the above definition for the case that the range of r is finite and $\alpha > 0$. Apparently no complete system of axioms for the logic of $\mathcal{K}^*_{sim}$ is known; but the paper of Esteva and Godo contains completeness result for reasonable subclasses of $\mathcal{K}^*_{sim}$ (e.g. $\mathcal{K}^{*G}_{ssim}$ – the class of similarity models whose similarity is *strong*, i.e. $r(w, w') < 1$ for $w \neq w'$, and the range of r is included in a fixed finite $G \subseteq [0,1]$, G containing 0 and 1).

8.2.34 Concluding remark. Let us stress that in the previous and the present section we have dealt with *Boolean*, two-valued logics and studied generalized quantifiers and modalities in them. Thus values of formulas were always 0 or 1. We did use some real-valued entities (probabilities, possibilities/necessities, DS-belief functions/plausibilities) as well as various statistics on finite models to define quantifiers and modalities; but we obtained two-valuedness by comparing a real value assigned to a formula either with a threshold or with the value of another formula. In the next two sections we shall elaborate a theory of generalized quantifiers and modalities in fuzzy logic.

8.3. Fuzzy Quantifiers and Modalities

From now on we shall study fuzzy quantifiers and modalities.[49] The present section is devoted to quantifiers definable in fuzzy logic from the "classical" quantifiers $\forall$, $\exists$, their relation to generalized possibility theory (possibilities and necessities of fuzzy events) as well as to fuzzy modal logics generalizing S5. We start with some general definitions.

Definition 8.3.1 Let $\mathcal{I}$ be a predicate language, $\mathbf{L}$ a linearly ordered BL-algebra, $\mathbf{M}$ an $\mathbf{L}$-structure for $\mathcal{I}$ and φ a formula such that for each evaluation v of object variables in M, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ is defined. Then, for each $m \in M$,

$$Sat^{\mathbf{L}}_{\mathbf{M},v}(\varphi, x)(m) = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v(x/m)};$$

Thus $Sat^{\mathbf{L}}_{\mathbf{M},v}(\varphi, x)$ is an $\mathbf{L}$-fuzzy subset of M assigning to each $m \in M$ the degree in which m satisfies $\varphi(x)$ (all other variables being evaluated by v).

Remark 8.3.2 Observe that if $\mathbf{L}$ is the two-element Boolean algebra then this definition reduces to 8.1.2. We shall mostly be interested in the case of $\mathbf{L}$ being a t-algebra, notably $[0,1]_{Ł}$, $[0,1]_G$, $[0,1]_\Pi$. Now we shall generalize 8.1.3.

Definition 8.3.3 A $\mathcal{C}$-fuzzy predicate calculus with a unary generalized quantifier q is given by the following things:

- a predicate language $\mathcal{I}$
- a class $\mathcal{K}$ of structures of the form $(\mathbf{M}, \mathbf{L}, \mathcal{F})$ where $\mathbf{M}$ is an $\mathbf{L}$-interpretation of $\mathcal{I}$, $\mathbf{L}$ is a $\mathcal{C}$-algebra and $\mathcal{F}$ is a mapping defined for some fuzzy subsets f of M and assigning to each such f (for which $\mathcal{F}(f)$ is defined) an element $\mathcal{F}(f) \in \mathbf{L}$.

Again here we shall often write $\mathbf{M}$ instead of $(\mathbf{M}, \mathbf{L}, \mathcal{F})$ if $\mathbf{L}$ and $\mathcal{F}$ will be clear from the context.

Elements of $\mathcal{K}$ are called *models* of the calculus. Logical symbols are those of the predicate calculus plus the symbol q. The definition of formulas is extended by the clause "if φ is a formula and x a variable then $(qx)\varphi$ is a formula". The definition 5.1.3 of the truth value of a formula is extended by the clause

$$\|(qx)\varphi\|^{\mathbf{L}}_{\mathbf{M}} = \mathcal{F}(Sat^{\mathbf{L}}_{\mathbf{M}}(\varphi, x)),$$

where for each $m \in M$, $Sat^{\mathbf{L}}_{\mathbf{M}}(\varphi, x)(m) = \|\varphi\|^{L}_{(M,\mathcal{F}),v(x/m)}$. In words, to get the truth value of $(qx)\varphi$, one takes the fuzzy subset f of M such that

[49] Preliminary results were obtained by Thiele et al. [197, 199, 200].

for each m, $f(m)$ is the truth degree of φ for m as the value of x (all other variables being evaluated using v); the desired truth degree is $\mathcal{F}(f)$.

Note also that $\|(qx)\varphi\|^L_{(M,\mathcal{F}),v}$ may be undefined if the fuzzy set f (i.e. $Sat^{\mathbf{L}}_{\mathbf{M}}(\varphi, x)$) is not in the domain of $\mathcal{F}$. We call $(\mathbf{M}, \mathcal{F})$ *safe* if $\|\varphi\|^L_{(M,\mathcal{F})}$ is defined for each φ.

8.3.4 The definition of a binary quantifier is fully analogous, $\mathcal{F}$ is now a mapping associating to some pairs (f, g) $\mathbf{L}$-fuzzy subsets of M a value $\mathcal{F}(f, g) \in \mathbf{L}$; and we put

$$\|(qx)(\varphi, \psi)\|^L_{(M,\mathcal{F}),v} = \mathcal{F}(Sat^{\mathbf{L}}_{\mathbf{M}}(\varphi, x), Sat^{\mathbf{L}}_{\mathbf{M}}(\psi, x)).$$

Example 8.3.5 Let us discuss the quantifiers $(\forall x|\varphi)\psi$ and $(\exists x|\varphi)\psi$. As in Boolean logic, $(\forall x|\varphi)\psi$ stands for $(\forall x)(\varphi \to \psi)$ and $(\exists x|\varphi)\psi$ stands for $(\exists x)(\varphi \& \psi)$. We may read these formulas "all φ's are ψ's" and "some φ's are ψ's" as was usual in the syllogisms. Observe that the corresponding $\mathcal{F}$'s are as follows: given $\mathbf{L}$ and M, and $f, g : M \to \mathbf{L}$, for the restricted $\forall$ we have

$$\mathcal{F}(f, g) = \inf_{m \in M} (f(m) \Rightarrow g(m)),$$

and for restricted $\exists$

$$\mathcal{F}(f, g) = \sup_{m \in M} (f(m) * g(m)).$$

Lemma 8.3.6 For the quantifiers just defined, the following formulas are $\mathbf{L}$-tautologies for each $\mathbf{L}$ (i.e. the formulas have the truth value $1_{\mathbf{L}}$ for each safe model:

$$(\forall x|\varphi)\psi \to \neg(\exists x|\varphi)\neg\psi \tag{1}$$

$$(\exists x|\varphi)\psi \to \neg(\forall x|\varphi)\neg\psi \tag{2}$$

$$((\forall x|\varphi)\psi \& (\forall x|\varphi)(\psi \to \chi)) \to (\forall x|\varphi^2)\chi \tag{3}$$

Proof: Show that BL$\forall \vdash ((\forall x)(\varphi \to \psi)\&(\exists x)(\varphi \& \neg\psi)) \to (\exists x)(\psi \& \neg\psi)$, $(\exists x)(\psi \& \neg\psi) \to 0$, thus it proves $(\forall x)(\varphi \to \psi) \to ((\exists x)(\varphi \& \neg\psi) \to \bar{0})$, which is (1). The proof of (2) in BL$\forall$ is the same.

We prove (3) in BL$\forall$:
$((\varphi \to \psi)\&(\varphi \to (\psi \to \chi))) \to ((\varphi \& \varphi) \to \chi)$ by propositional calculus, from which the result follows by generalizing and distributing $\forall$. □

Caution: The converse implications of (1) and (2) are provable in Ł$\forall$ but not in BL$\forall$ (find counterexamples).

*

8.3.7 Let us now show how restricted classical quantifiers can be used to define quantitative necessity and possibility in fuzzy logic. To this end assume that our language contains a specific unary predicate π (for possibility) and investigate, over BL$\forall$, the theory T_{poss} with the only axiom $(\exists x)\pi(x)$. **M** is an **L**-model of T_{poss} if $\|(\exists x)\pi(x)\|_{\mathbf{M}} = 1$, i.e. if $\sup_{m \in M} r_\pi(m) = 1_{\mathbf{L}}$. (Think e.g. of $[0,1]_{\text{Ł}}$ etc.) Define, for each formula φ with just one free variable,

$$\Pi_{\mathbf{M}}(\varphi) = \|(\exists x|\pi)\varphi\|_{\mathbf{M}},$$
$$\mathcal{N}_{\mathbf{M}}(\varphi) = \|(\forall x|\pi)\varphi\|_{\mathbf{M}}.$$

$\Pi_{\mathbf{M}}(\varphi)$ is the *possibility* of φ (in **M**), $\mathcal{N}_{\mathbf{M}}(\varphi)$ the *necessity* of φ.

Lemma 8.3.8 Assume **L** linearly ordered, **M** is model of T_{poss}, φ, ψ formulas with one free variable.

$$\Pi_{\mathbf{M}}(\varphi \vee \psi) = \max(\Pi_{\mathbf{M}}(\varphi), \Pi_{\mathbf{M}}(\psi)), \tag{1}$$
$$\mathcal{N}_{\mathbf{M}}(\varphi \wedge \psi) = \min(\mathcal{N}_{\mathbf{M}}(\varphi), \mathcal{N}_{\mathbf{M}}(\psi)). \tag{2}$$

Proof: (1) The following chain of equivalences is provable in BL$\forall$: $(\exists x|\pi)(\varphi \vee \psi) \equiv (\exists x)(\pi \& \varphi) \vee (\pi \& \psi)) \equiv [(\exists x|\pi)\varphi \vee (\exists x|\pi)\psi]$ by 2.2.23, 5.1.22.

(2) The proof is fully analogous. □

Remark 8.3.9 We see that our definition of numerical possibility/necessity of formulas of fuzzy logic has some basic properties of possibilities of crisp formulas. But some care is necessary:

(1) Unlike the crisp case, we may have $\mathcal{N}_{\mathbf{M}}(\varphi) > 0$ and $\mathcal{N}_{\mathbf{M}}(\neg\varphi) > 0$ (take $[0,1]_{\text{Ł}}$ and assume that in $Sat_{\mathbf{M}}(\varphi, x)$ constantly equals $\frac{1}{2}$).

(2) Unlike the crisp case, we may have $\Pi_{\mathbf{M}}(\varphi) < 1$ and $\Pi_{\mathbf{M}}(\neg\varphi) < 1$ (the same example).

We shall now investigate necessity and possibility in the frame of fuzzy modal propositional calculus.

*

Our aim is now to study the modal logics that are fuzzy analogues of the logic S5. Such an analogue is specified by the underlying propositional logic $\mathcal{C}$ and the BL-algebra(s) allowed as the structure(s) of truth degrees. Given this we have the usual notion of tautologies and may ask whether they are axiomatizable and, if so, how.

Definition 8.3.10 Let $\mathcal{C}$ be a fuzzy propositional logic (schematic extension of BL). *Formulas* of the logic $S5(\mathcal{C})$ are built from propositional variables, connectives $(\&, \rightarrow)$, the truth constant $\bar{0}$ and the two modalities $\Box, \Diamond$. Let $\mathbf{L}$ be a $\mathcal{C}$-algebra. A Kripke $\mathbf{L}$-model of $S5(\mathcal{C})$ has the form $\mathbf{K} = \langle W, e, \mathbf{L}\rangle$ (or just $\langle W, e\rangle$ if $\mathbf{L}$ is clear from the context) where W is a non-empty set of possible worlds and e is an $\mathbf{L}$-evaluation of pairs (w, p) where $w \in W$ and p is a propositional variable, i.e. $e : (W \times Prop) \rightarrow \mathbf{L}$, $Prop$ being the set of all propositional variables. The truth degree $\|\varphi\|^{\mathbf{L}}_{\mathbf{K},w}$ is defined in the obvious way.

$$\|p\|^{\mathbf{L}}_{\mathbf{K},w} = e(w,p), \qquad \|\bar{0}\|^{\mathbf{L}}_{\mathbf{K},w} = 0_{\mathbf{L}}.$$

$$\|\varphi\&\psi\|^{\mathbf{L}}_{\mathbf{K},w} = \|\varphi\|^{\mathbf{L}}_{\mathbf{K},w} * \|\psi\|^{\mathbf{L}}_{\mathbf{K},w},$$

analogously for $\rightarrow$,

$$\|\Box\varphi\|^{\mathbf{L}}_{\mathbf{K},w} = \inf_{v\in W} \|\varphi\|^{\mathbf{L}}_{\mathbf{K},v}, \qquad \|\Diamond\varphi\|^{\mathbf{L}}_{\mathbf{K},w} = \sup_{v\in W} \|\varphi\|^{\mathbf{L}}_{\mathbf{K},v}$$

(if this inf/sup exists, otherwise indefined).

Definition 8.3.11 We associate with our propositional language a predicate language (with only unary predicates) and associate with each modal propositional formula φ its translation φ^* and with each Kripke $\mathbf{L}$-model $\mathbf{K}$ of $S5(\mathcal{C})$ its counterpart $\mathbf{K}^*$ literally as in Definition 8.2.5. In more detail, p_i^* is $P_i(x)$, $*$ commutes with connectives, $(\Box\varphi)^*$ is $(\forall x)\varphi^*$, $(\Diamond\varphi)^*$ is $(\exists x)\varphi^*$. For each Kripke $\mathbf{L}$-structure $K = (W, e)$, we define a predicate $\mathbf{L}$-structure $K^* = (W, (r_{P_i})_i)$ where $r_{P_i}(w) = e(p_i, w)$. We call K $\mathbf{L}$-*safe* if K^* is $\mathbf{L}$-safe in the sense of predicate calculus. A formula φ is an $\mathbf{L}$-*tautology* of $S5(\mathcal{C})$ if $\|\varphi\|^{L}_{\mathbf{K},w} = 1_{\mathbf{L}}$ for each safe Kripke $\mathbf{L}$-model of $S5(\mathcal{C})$. Finally, φ is a $S5(\mathcal{C})$-*tautology*, if for each $\mathcal{C}$-algebra $\mathbf{L}$, φ is a $\mathbf{L}$-tautology for $S5(\mathcal{C})$. The class of all safe Kripke $\mathbf{L}$-models in an arbitrary $\mathcal{C}$-algebra may be denoted by $\mathcal{K}_{S5(\mathcal{C})}$.

Lemma 8.3.12 For a safe $\mathbf{K}$ and each modal formula φ,

$$\|\varphi\|^{\mathbf{L}}_{\mathbf{K},w} = \|\varphi^*\|^{\mathbf{L}}_{\mathbf{K}^*,w}.$$

(Strictly speaking, on the right hand side the w should be replaced by an evaluation of object variables assigning w to x and otherwise arbitrary, but this is an obvious *façon de parler*.)

The mapping $*$ of $\mathbf{L}$-Kripke models is a one-one mapping of all $\mathbf{L}$-Kripke structures for our propositional language $\mathcal{I}$ onto all $\mathbf{L}$-structures for the predicate language $\mathcal{I}^*$. (Proof obvious.)

Corollary 8.3.13 Given $\mathcal{C}$ and $\mathbf{L}$, a formula φ is an $\mathbf{L}$-tautology of $S5(\mathcal{C})$ iff φ^* is an $\mathbf{L}$-tautology of $\mathcal{C}\forall$.

Corollary 8.3.14 For $\mathcal{C}$ = BL, Ł, G, Π, the set of all S5($\mathcal{C}$)-tautologies is recursively enumerable.

Proof: Due to the respective completeness theorems for $\mathcal{C}\forall$, φ is a S5($\mathcal{C}$)-tautology iff $\mathcal{C}\forall$ proves φ^*. Observe that the mapping $*$ is recursive and reduces the set of all S5($\mathcal{C}$)-tautologies to the Σ_1-set of all sentences provable in $\mathcal{C}\forall$. □

Remark 8.3.15 By a trick (due to Craigh) we may use the above to produce a recursive set of axioms for S5($\mathcal{C}$) complete with respect to proofs using modus ponens as the only deduction rule. Indeed, given a usual encoding of formulas and proofs by natural numbers (Gödel numbering), we put
$Ax(S5(\mathcal{C})) = \{\bigwedge_{i=1}^{n} \varphi \mid$ for some $d < n$, d is a $\mathcal{C}\forall$-proof of $\varphi^*\}$.
Here $\bigwedge_{i=1}^{n} \varphi$ stands for $\varphi \wedge \cdots \wedge \varphi$ (n conjuncts, min-conjunction). Now take elements of $Ax(S5(\mathcal{C}))$, together with all axioms of $\mathcal{C}$, for your axioms. The only rule you need is "from $\bigwedge_{i=1}^{n} \varphi$ derive $\bigwedge_{i=1}^{n-1} \varphi$" ($n > 1$) – and this is clearly a derived rule of $\mathcal{C}$.

Needless to say, this is an extremely artificial and ugly axiomatization. One would like to have an elegant axiom system, complete with respect to modus ponens and necessitation. Unfortunately, this remains a research task.

Remark 8.3.16 Note that axioms of BL$\forall$ give us natural tautologies of S5(BL) (and hence of each S5($\mathcal{C}$)), namely (ν stands for a propositional combination of formulas of the form $\Box\alpha$, $\Diamond\alpha$):

($\Box$1) $\Box\varphi \to \varphi$

($\Diamond$1) $\varphi \to \Diamond\varphi$

($\Box$2) $\Box(\nu \to \varphi) \to (\nu \to \Box\varphi)$

($\Diamond$2) $\Box(\varphi \to \nu) \to (\Diamond\varphi \to \nu)$

($\Box$3) $\Box(\varphi \vee \nu) \to (\Box\varphi \vee \nu)$

Check that from these axioms, using modus ponens and necessitation, you may prove modal analoga of all formulas in Chapter 5 Sections 1, 2 except for (10). (10) can be taken as a new axiom:

($\Diamond$3) $(\Diamond\varphi \,\&\, \Diamond\varphi) \equiv \Diamond(\varphi\&\varphi)$

Unfortunately , the completeness proof in 5.2.9 cannot be automatically used for modal logic (or for the predicate calculus with just one variable), thus we have the following

Problem. Is the system of axioms of $\mathcal{C}$, together with the axioms ($\Box$1) – ($\Box$3), ($\Diamond$1) – ($\Diamond$3), modus ponens and necessitation, complete for S5($\mathcal{C}$)? (For Ł see remarks below.)

Remark 8.3.17 (1) Tautologies for a concrete $\mathcal{C} =$ Ł, G, Π may be obtained from formulas proved in $\mathcal{C}\forall$ in the corresponding section of Chapter 5. For example, $\Diamond\varphi \equiv \neg\Box\neg\varphi$ is an S5(Ł)-tautology, but it is *not* an S5(G)-tautology, neither a S5(Π)-tautology.

(2) This has concerned S5($\mathcal{C}$)-tautologies, i.e. Kripke **L**-models for arbitrary $\mathcal{C}$-algebras **L**. Clearly, for $\mathcal{C}$ being Ł, G, Π we are interested in $[0,1]_{\mathcal{C}}$-tautologies of S5($\mathcal{C}$), i.e. tautologies with respect to the respective standard truth algebra. Let us investigate them now.

First we investigate $[0,1]_G$-tautologies of S5(G), briefly S5($[0,1]_G$)-tautologies.

Theorem 8.3.18 The set of all S5($[0,1]_G$)-tautologies is recursively enumerable.

Proof: This is because, by our interpretation and by the completeness theorem 5.3.3 , S5($[0,1]_G$)-tautologies coincide with S5(G)-tautologies and thus with formulas φ such that $G\forall \vdash \varphi^*$. $\Box$

Definition 8.3.19 (1) A modal fuzzy calculus with the truth set $[0,1]$ has the *finite model property* if each formula 1-true in all finite Kripke models (models with a finite set of possible worlds) is 1-true in all Kripke models.

(2) The calculus in question has the *finite rational model property* if each formula 1-true in all Kripke models (W,e) with W finite and e rational-valued is 1-true in all Kripke models.

Lemma 8.3.20 S5($[0,1]_G$) does not have the finite model property.

Proof: It is easy to show that the formula

$$(\Box p \to \Diamond q) \to \Diamond(\Box p \to q)$$

is true in each finite $[0,1]_G$-Kripke model $\mathbf{K} = (W,e)$: Let $\|\Box p\|_{\mathbf{K}} = y$, $\|q\|_{\mathbf{K},w} = x_w$; then since W is finite, $\|\Diamond q\|_{\mathbf{K}} = \max_{w\in W} x_w = x_{w_0}$,

$||\Box p \to \Diamond q||_{\mathbf{K}} = y \Rightarrow x_{w_0}$. Furthermore, $||\Diamond(\Box p \to q)||_{\mathbf{K}} = \max_{w \in W}(y \Rightarrow x_w) = (y \Rightarrow \max_{w \in W} x_w) = (y \Rightarrow x_{w_0})$. On the other hand, the example in 5.3.6 directly yields a Kripke model in which the above formula is not 1-true.
□

8.3.21 Problem. Is the set of $S5([0,1]_G)$ - tautologies recursive, i.e. is logic decidable?

Theorem 8.3.22 $S5([0,1]_{Ł})$ has the finite rational model property.

Proof: Let φ be a modal formula and let $\mathbf{K}$ be a Kripke model over $[0,1]_{Ł}$ such that $||\varphi||_{\mathbf{K},w} < 1$ for some w. Assume $p_1, \ldots, p_n$ are the propositional variables of φ. Then $||\varphi^*||_{\mathbf{K}^*,w} < 1$ and, by 5.4.30 there is a k and a model $\mathbf{K}_1^*$ of Ł∀ such that $\mathbf{K}_1^*$ takes values only from $\{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$ and $||\varphi^*||_{\mathbf{K}_1^*,w} < 1$. This gives us a Kripke model $\mathbf{K}_1$ with all truth values in $\{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$ and such that $||\varphi||_{\mathbf{K}_1,w} < 1$; $\mathbf{K}_1 = (W, e_1)$. Let $C(v) = \langle e_1(v,p_1), \ldots, e_1(v,p_n)\rangle$ (the card of v). Each $C(v)$ is an element of $\{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}^n$ and the latter set is finite. Let W_2 be a finite subset of W such that for each $v \in W$ there is a $v' \in W_2$ with $C(v) = C(v')$. It is easy to show that $\mathbf{K}_2 = (W_2, e_1)$ is the desired finite rational-valued model. (Show by induction that $||\psi||_{\mathbf{K}_2,v} = ||\psi||_{\mathbf{K}_1,v}$ for each formula ψ and each v); since $||\varphi||_{\mathbf{K}_1,w} < 1$ we get $||\varphi||_{\mathbf{K}_2,w} < 1$.) □

Corollary 8.3.23 The set of all $S5([0,1]_{Ł})$-tautologies is in Π_1.

Proof: Let f be the function deciding for each finite rational-valued Kripke model $\mathbf{K} = \langle W, e\rangle$ (where W is an initial segment of integers) and for each modal formula φ whether φ is 1-true in $\mathbf{K}$ (if yes then $f(\mathbf{K},\varphi) = 1$, if not then $f(\mathbf{K},\varphi) = 0$). Clearly, f is recursive. Moreover, φ is a $S5([0,1]_{Ł})$-tautology iff for all $\mathbf{K}$, $f(\mathbf{K},\varphi) = 1$; thus the set of all such φ is Π_1. □

*

Remark 8.3.24 We shall show that the set of all $S5([0,1]_{Ł})$-tautologies is recursive, i.e. that the logic is decidable. We shall do this by showing, similarly as in propositional calculus (and differently from Ł∀), that $S5([0,1]_{Ł})$-tautologies coincide with S5(Ł)-tautologies, i.e. that a formula is $[0,1]_{Ł}$-tautology iff it is a $\mathbf{L}$-tautology for each linearly ordered MV-algebra $\mathbf{L}$. Unfortunately we shall not prove completeness with respect to any concrete axiom system. Note that Rutledge in his dissertation [179] presents such a complete axiom system (he considers predicate calculus with unary predicates and just one object variable but this is evidently equivalent to modal logic through our translation $*$). But his proof is extremely demanding and rather long; we shall not present it here.

Lemma 8.3.25 Let T be a theory over Ł∀, let $(\forall x)\varphi$ be a sentence and let c be a new constant. Let T' be the extension of T by the constant c and the axiom $\varphi(c) \to (\forall x)\varphi(x)$. Then T' is a conservative extension of T.

Proof: Let α be a closed formula not containing c. Let $T' \vdash \alpha$. Then for some n,
$T \vdash (\varphi(c) \to (\forall x)\varphi(x))^n \to \alpha$ (by Deduction theorem),
$T \vdash (\varphi(y) \to (\forall x)\varphi(x))^n \to \alpha$ (for a new variable y),
$T \vdash (\exists y)(\varphi(y) \to (\forall x)(\varphi(x))^n \to \alpha$ (by generalization and (∃2)),
$T \vdash [(\exists y)(\varphi(y) \to (\forall x)\varphi(x))]^n \to \alpha$ (see 5.1.18(10)),
$T \vdash [(\forall y)\varphi(y) \to (\forall x)\varphi(x)]^n \to \alpha$ (see 5.4.15(ii)),
$T \vdash \alpha$ (by 5.1.17(8)). □

Corollary 8.3.26 Each theory T over Ł∀ has a conservative extension T' with possibly infinitely many new constants such that for each sentence $(\forall x)\varphi(x)$ there is a constant c such that $T' \vdash (\forall x)\varphi(x) \equiv \varphi(c)$. Moreover; for each sentence $(\exists x)\varphi(x)$, there is a constant d such that $T' \vdash (\exists x)\varphi(x) \equiv \varphi(d)$.

Remark 8.3.27 (1) In each model $\mathbf{M}$ of T',

$$\|(\forall x)\varphi\|_{\mathbf{M},v} = \min\{\|\varphi\|_{\mathbf{M},w} | v \equiv_x w\} = \|\varphi(c)\|_{\mathbf{M},v}$$

and similarly for ∃ and max (inf and sup are attained).

(2) T' is Henkin in the sense of Definition 5.2.1.

Lemma 8.3.28 If φ is not an S5(Ł)-tautology then there is a finite $[0,1]_{Ł}$-structure $\mathbf{K}_1$ such that $\|\varphi\|_{\mathbf{K}_1} < 1$.

Proof: By the assumption, if φ is not an S5(Ł) tautology then Ł∀ $\nvdash \varphi^*$; by the previous lemma, $T_0 \nvdash \varphi^*$ where T_0 is a conservative extension of Ł∀ as in the Corollary above. Thus there is an $\mathbf{L}$-model $\mathbf{K}^*$ of T_0 in which $\|\varphi^*\|_{\mathbf{K}^*} < 1$, $\mathbf{L}$ being an MV-algebra, linearly ordered. Note that φ^* contains just one variable x; we may assume φ^* closed. Let C be a set containing for each subformula of φ of the form $(\forall x)\psi$ or $(\exists x)\psi$ a corresponding witnessing constant c (so that $T_0 \vdash (\forall x)\psi(x) \equiv \psi(c)$, $T_0 \vdash (\exists x)\psi(x) \equiv \psi(c)$ respectively). Clearly, C is finite; let W_0 be the set of meanings of constants from C, i.e. $W_0 = \{m_c | c \in C\}$. Let $\mathbf{K}_0^*$ be the substructure of $\mathbf{K}^*$ with the domain W_0. Let v vary over evaluations of x by elements of W_0 and show, by induction, that $\|\alpha\|_{\mathbf{K}_0^*,v} = \|\alpha\|_{\mathbf{K}^*,v}$. (Trivial for atoms and connectives; for quantifiers use witnessing constants.) Thus $\|\varphi\|_{\mathbf{K}_0^*} = \|\varphi\|_{\mathbf{K}^*} < 1$.

Now recall the local embedding theorem 3.2.11 for linearly ordered MV-algebras; using it we may replace the finite $\mathbf{L}$-structure $\mathbf{K}_0$ by a corresponding $[0,1]_Ł$-structure $\mathbf{K}_1$ which results by replacing each value $e(p,w)$ by its isomorphic image $e'(p,w) = f(e(p,w))$. Here f embeds the set of all values $\|\alpha\|_{\mathbf{K}_0^*,v}$ (α a subformula of φ, v evaluates x by one of the elements of W_0) – which is a finite subset of $\mathbf{L}$ – into $[0,1]_Ł$. This completes the proof. □

Corollary 8.3.29 $S5([0,1]_Ł)$ is decidable (and so is S5(Ł)).

Proof: By 8.3.28, the set of all $S5([0,1]_Ł$-tautologies equals to the set of all S5(Ł)-tautologies and thus by 8.3.14 is Σ_1. By 8.3.23 it is Π_1; thus it is recursive (see 6.1.12). □

*

We shall now present a Pavelka-style axiomatization for $S5([0,1]_Ł)$.[50] It is sufficiently elegant, but only expresses the set of all tautologies as a Π_2-set (even if we know it is recursive).

Definition 8.3.30 The logic of the logic RPL□ results from that of RPL by adding the modality □ (◇ is defined as ¬□¬). Kripke models are as above; $\|\bar{r}\|_{\mathbf{K},v} = r$. *Axioms* of RPL□ are

- axioms of RPL (with the new notion of a formula)
- axioms of S5 (with the new notion of a formula)
- Fitting style axioms:

$$(\bar{r} \to \Box\varphi) \equiv \Box(\bar{r} \to \varphi),$$
$$(\bar{r} \to \Diamond\varphi) \equiv \Diamond(\bar{r} \to \varphi).$$

Last axiom:

$$(\Diamond\varphi \,\&\, \Diamond\varphi) \equiv \Diamond(\varphi \,\&\, \varphi)$$

[50] See [82].

Deduction rules: Modus ponens and generalization: from φ infer $\Box\varphi$.
A *proof* is a sequence of formulas $\varphi_1, ..., \varphi_n$ such that for each $j = 1, ...n$, φ_j is an axiom or follows from some previous ones by one of the deduction rules.
This completes the definition of the logic.

Lemma 8.3.31 *(Soundness).* Each provable formula is a $[0,1]_Ł$-tautology.

Definition 8.3.32 The *provability degree* of φ is $|\varphi| = \sup\{r \mid \vdash (\overline{r} \to \varphi)\}$. The *truth degree* of φ is $||\varphi|| = \inf\{||\varphi||_{\mathbf{K}} \mid K$ Kripke model and $w \in W\}$.

Lemma 8.3.33 *(Strong soundness).* $|\varphi| \leq ||\varphi||$, i.e. whenever $\vdash \overline{r} \to \varphi$ then $||\varphi||_{\mathbf{K},w} \geq r$ for each $\mathbf{K}, w$.

The easy *proof* is left to the reader as an exercise.

Lemma 8.3.34 MVS5 proves the following duals of Fitting-style axioms:

$$(\Diamond\varphi \to \overline{r}) \equiv \Box(\varphi \to \overline{r}),$$
$$(\Box\varphi \to \overline{r}) \equiv \Diamond(\varphi \to \overline{r}).$$

Proof: We show the provability of the former formula. This is done by the following chain of provable equivalences: $(\Diamond\varphi \to \overline{r}) \equiv (\neg\Box\neg\varphi \to \overline{r}) \equiv (\overline{1-r} \to \Box\neg\varphi) \equiv \Box(\overline{1-r} \to \neg\varphi) \equiv \Box(\varphi \to \overline{r})$. The latter formula is proved in the same manner. $\Box$

Theorem 8.3.35 *Completeness Theorem.* For each φ, $|\varphi| = ||\varphi||$. In particular, φ is a 1-tautology iff, for each $r < 1$, $\vdash \overline{r} \to \varphi$.

We elaborate a proof. It is enough to prove that if for some $s < r, \nvdash \overline{s} \to \varphi$ then there is a "counterexample" model $\mathbf{K} = \langle W, e\rangle$ such that for some $w \in W, ||\ \varphi\ ||_w < r$. This can be equivalently expressed as follows: for each $s < 1$, if $\nvdash \overline{s} \to \varphi$ then $||\varphi||_w < 1$ for some $\mathbf{K} = \langle W, e\rangle$ and $w \in W$. Indeed, let $t < 1$ be such that $s = t\&r$. If $\nvdash \overline{s} \to \varphi$ then $\nvdash \overline{t} \to (\overline{r} \to \varphi)$, thus $||\overline{r} \to \varphi|| < 1$ and hence $||\varphi|| < r$. We apply the usual construction of a "canonical model".

We shall use the following definitions: Let Θ be the set of all formulas provable on our logic, $\Theta = \{\varphi \mid \vdash \varphi\}$. A *theory* is a set T of formulas containing all elements of Θ and closed under modus ponens. (Caution: closedness under necesitation is not requested; theories correspond to truth in one possible world.) T is *complete* if for each r, $(\overline{r} \to \varphi) \in T$ or $(\varphi \to \overline{r}) \in T$.

T is *consistent* if $(\overline{r} \to \overline{0}) \notin T$ for each $r > 0$ (cf 3.3.7). *B-formulas* are formulas resulting from truth constants and from boxed formulas (beginning by $\Box$) using connectives. (Note that if φ is a B-formula then for each $\mathbf{K} = \langle W, e\rangle$, $\| \varphi \|_w$ is independent from w.)

Lemma 8.3.36 If $\not\vdash \overline{s} \to \psi$ then there is a theory T_0 such that $T_0 \cup \{\psi \to \overline{s}\}$ is consistent and for each B-formula φ and each r, $(\overline{r} \to \varphi) \in T_0$ or $(\varphi \to \overline{r}) \in T_0$ (T_0 is complete for B-formulas).

Proof: If $\not\vdash \overline{s} \to \psi$ then $\overline{s} \to \psi \notin \Theta$ and Θ is a theory. It follows that the closure Θ' of $\Theta \cup \{\psi \to \overline{s}\}$ under modus ponens is a consistent theory – the proof is exactly as in [74] . Having this we may arrange all B-formulas into a sequence $\beta_0, \beta_1, \ldots$ and apply the completion technique of 2.4.2 (quoted lemma) to get a desired T_0. $\Box$

Lemma 8.3.37 For T_0 as above, if $(\Diamond\psi) \in T_0$ then there is a consistent complete $T' \supseteq T_0$ such that $\psi \in T'$.

Proof: Let T_1 be the closure of $T_0 \cup \{\psi\}$ under modus ponens. It is enough to show that T_1 is consistent. Note that by the last axiom, $\Diamond(\psi^m) \in T_0$ for each $m > 0$.

Assume T_1 is inconsistent; thus by 2.2.13 , for some $m > 0$, T_0 proves $\psi^m \to \overline{0}$, thus T_0 proves $\neg\psi^m$ (by a modus-ponens proof, no generalization!). Hence there is a proof of $\neg\psi^m$ from finitely many elements $\delta_1, ..., \delta_n$ of T_0; by the above deduction theorem again, for some natural k $\vdash (\delta_1 \& .... \& \delta_n)^k \to \neg\psi^m$, thus $\vdash \delta_1^k \to (\ldots \to (\delta_n^k \to \neg\psi^m)$ and δ_i are either logically provable or are B-formulas; thus for each $i, T_0 \vdash \Box\delta_i^k$. On the other hand, $\vdash \Box\delta_1^k \to (\ldots \to (\Box\delta_n^k \to \Box\neg\psi^m)\ldots)$ thus using modus ponens we get $\Box\neg\psi^m \in T_0$, which is a contradiction. This proves the lemma, using the fact that each consistent theory has a consistent complete extension.

8.3.38 Construction of the model. Let $\{T_w \mid w \in W\}$ be the system of all consistent complete extensions of T_0, indexed by a set W; define $e(p, w) = |p|_{T_w} = \sup\{r \mid (\overline{r} \to p) \in T_w\}$; put $\mathbf{K} = \langle W, e\rangle$. Let w_0 be an element of W such that $(\psi \to \overline{s}) \in T_{w_0}$ (ψ, s as at the beginning of the proof).

We prove, for each α and w, that $|\alpha|_{T_w} = \|\alpha\|_{T_w}$.

This is clear for α atomic and the induction step for connectives is proved as in 3.3.8. It remains to prove the induction step for $\Box$. On the one hand, $|\Box\alpha|_{T_w} = \sup\{r \mid (\overline{r} \to \Box\alpha) \in T_w\} = \sup\{r \mid \Box(\overline{r} \to \alpha) \in T_w\} \leq$

$\sup\{r \mid (\overline{r} \to \alpha) \in T_w\} = |\alpha|_{T_w}$; on the other hand, if $r > |\Box\alpha|_{T_w}$ then $T \not\vdash (\overline{r} \to \Box\alpha)$, $T_0 \not\vdash (\overline{r} \to \Box\alpha)$, $T_0 \vdash (\Box\alpha \to \overline{r})$, $T_0 \vdash (\neg\overline{r} \to \Diamond\neg\alpha)$, thus $T_0 \vdash \Diamond(\neg\overline{r} \to \neg\alpha)$, thus there is a $T_{w'}$ such that $T_{w'} \vdash \neg\overline{r} \to \neg\alpha$, $T_{w'} \vdash \alpha \to r$, thus $|\alpha|_{T_{w'}} \leq r$. This gives $\inf_w |\alpha|_{T_w} \leq |\Box\alpha|_{T_w}$ and together we have $\inf_w |\alpha|_{T_w} = |\Box\alpha|_{T_w} = |\Box\alpha|_{T_0}$.

Now $\|\Box\alpha\| = \inf_w \|\alpha\|_{T_w} = \inf_w |\alpha|_{T_w} = |\Box\alpha|_{T_w}$, as we claimed.

Completeness follows: we have assumed $s < 1$ and $\not\vdash \overline{s} \to \psi$ and constructed a model $\mathbf{K} = \langle W, e\rangle$ and $w_0 \in W$ such that $\|\psi\|_{w_0} \leq s$. □

Remark 8.3.39 (1) We can say rather little about $S5([0,1]_\Pi)$, S5 over the product logic. Similarly as $S5([0,1]_G)$, $S5([0,1]_\Pi)$ does *not* have the finite model property; this can be shown using the example in 5.3.6

(2) *Problem*: Is this logic axiomatizable or even decidable?

*

We shall now discuss the fuzzification of the logic of belief, KD45. We shall present no definitive results, just some observations.

Definition 8.3.40 Given a propositional logic $\mathcal{C}$, *formulas* of the fuzzy modal belief logic (call it FMBEL($\mathcal{C}$)) are built from infinitely many modalities $\Box_n$, $\Diamond_n$ $(n \geq 1)$. *Kripke models* have the form $\mathbf{K} = (W, e, \pi)$ where π is a normalized non-empty fuzzy subset of W, i.e. $\pi : W \to [0,1]$ and $\sup_{w\in W} \pi(w) = 1$. Semantics of modalities is as follows ($\pi^n(v)$ stands for $\pi(v) * \cdots * \pi(v)$ – n times)

$$\|\Box_n\varphi\|^{\mathbf{L}}_{\mathbf{K},w} = \inf_{v\in W}(\pi^n(v) \Rightarrow \|\varphi\|^{\mathbf{L}}_{\mathbf{K},v}),$$

$$\|\Diamond_n\varphi\|^{\mathbf{L}}_{\mathbf{K},w} = \sup_{v\in W}(\pi^n(v) * \|\varphi\|^{\mathbf{L}}_{\mathbf{K},v}).$$

Definition 8.3.41 (1) We define a translation x of FMBEL$\mathcal{C}$-formulas into formulas of predicate logic (modifying 8.3.11 and 8.2.5). Each propositional variable p_i is attached the atomic formula $P_i(x) = p_i^*$; $*$ commutes with connectives; for a fixed new unary predicate Q,

$$(\Box_n\varphi)^* = (\forall x)(Q^n(x) \to \varphi^*),$$

$$(\Diamond_n\varphi)^* = (\exists x)(Q^n(x) \& \varphi^*).$$

(Here $Q^n(x)$ stands for $Q(x) \& \cdots \& Q(x)$ – n times.)

(2) Similarly we define an interpretation of FMBEL($\mathcal{C}$)-formulas in S5($\mathcal{C}$)-formulas as follows: $p_i^\sharp = p_i$. $\sharp$ commutes with connectives, and for a fixed new propositional variable q,

$$(\Box_n\varphi)^\sharp = \Box(q^n(x) \to \varphi^\sharp),$$

$$(\Diamond^n\varphi)^\sharp = \Diamond(q^n\&\varphi^\sharp).$$

(Here q^n is $q\&\cdots\&q$ – n times.) We generalize in the obvious manner definition 8.3.10 of an **L**-tautology for FMBEL($\mathcal{C}$) and an FMBEL($\mathcal{C}$)-tautology.

Lemma 8.3.42 (1) φ is a FMBEL($\mathcal{C}$)-tautology iff $\varphi^\sharp$ is true in each safe model **M** of $\mathcal{C}\forall$ such that $\|(\exists x)Q(x)\|_{\mathbf{M}} = 1$.

(2) φ is a FMBEL($\mathcal{C}$)-tautology iff $\varphi^\sharp$ is 1-true in each Kripke model **K** of S5($\mathcal{C}$) such that $\|\Diamond_q\|_{\mathbf{K}} = 1$.

Proof obvious.

Lemma 8.3.43 The following analogues of axioms KD45 are FMBEL($\mathcal{C}$)-tautologies (for each $\mathcal{C}$):

$$\Box_n\varphi \to (\Box_m(\varphi \to \psi) \to \Box_{n+m}\psi) \tag{1}$$

$$\Box_n\varphi \equiv \Box_m\Box_n\varphi \tag{2}$$

$$\Diamond_n\varphi \equiv \Box_m\Diamond_n\varphi \tag{3}$$

$$\Box_n\varphi \to \Diamond_n\varphi \tag{4}$$

$$\Box_n\varphi \to \Box_{n+1}\varphi, \qquad \Diamond_{n+1}\varphi \to \Diamond_n\varphi \tag{5}$$

Proof: Possibly the quickest way is to consider the corresponding formulas of the predicate calculus,

(1) By propositional calculus
BL$\forall \vdash (Q \to \varphi) \to ((Q' \to (\varphi \to \psi)) \to (Q\&Q') \to \psi)$, thus
BL$\forall \vdash (\forall x)(Q^n \to \varphi) \to [(\forall x)(Q^m \to (\varphi \to \psi)) \to (\forall x)((Q^n\&Q^m) \to \psi)]$
and, of course BL$\forall \vdash Q^n\&Q^m = Q^{n+m}$.

(2) On the one hand,
BL$\forall \vdash (\forall x)(Q^n \to \varphi) \to (Q^m \to (\forall x)(Q^n \to \varphi))$;
generalizing and shifting the outmost $\forall$, we get $(\Box_n\varphi \to \Box_m\Box_n\varphi)^*$. Conversely,
BL$\forall \vdash (\forall x)(Q^m \to (\forall x)(Q^n \to \varphi)) \to [(\exists x)Q^m \to (\forall x)(Q^n \to \varphi)]$.
Let T be the theory with the only axiom $(\exists x)Q$ (over BL$\forall$)). Then

$T \vdash (\exists x)Q^m, T \vdash (\forall x)(Q^m \to (\forall x)(Q^n \to \varphi)) \to (\forall x)(Q^n \to \varphi)$, i.e. $T \vdash (\Box_m \Box_n \varphi \to \Box_n \varphi)^*$. This suffices by 8.3.20.

(3) The proof is similar and uses 5.1.18 (9)

(4) Let T be as above; $T \vdash (\exists x)(Q^n \& Q^n)$. Thus $T \vdash [(\forall x)(Q^n \to \varphi) \& (\exists x)(Q^n \& Q^n)] \to (\exists x)(Q^n \& Q^n \& (Q^n \to \varphi))$, which gives $T \vdash (\forall x)(Q^n \to \varphi) \to (\exists x)(Q^n \& \varphi)$, i.e. $T \vdash (\Box_n \varphi \to \Diamond_n \varphi)^*$.

(5) Clear from BL$\forall \vdash (\forall x)(Q^{n+1} \to Q^n)$. □

8.3.44 *Problem.* Find an elegant axiomatization for FMBEL with models over all $\mathcal{C}$-algebras). For models over $[0, 1]$ the situation is similar as for S5.

8.4. On "probably" and "many"

8.4.1 Introduction In the last section we considered quantifiers and modalities that were, in one way or another, definable from the universal and existential quantifier. In the present and next section we shall focus our attention on modalities and quantifiers related to probability or to relative frequence. We may use probability (relative frequence) to define fuzzy modalities and fuzzy quantifiers in various natural ways, in accordance with the use of "probably", "many" in the natural language ("Probably it will rain tomorrow"; "Many students pass the exam without any problems".) In particular, we may have calculi with crisp or fuzzy models, [both finite and infinite] or [only finite] models, with unary or binary quantifiers (unconditional or conditional probability) etc. We shall just make a choice of some few (typical and important) systems and shall try to present some *results* on them (not just a panopticum of definitions). In particular, in the present section, we shall investigate

- the modality *probably* on crisp models,
- the modality *many* on crisp finite models.

In this context we shall present fuzzy multitudinal quantifiers (for finite crisp models).

In the next section, we shall deal with

- the quantifier *probably* on fuzzy models,
- the quantifier *many* on fuzzy finite models.

Throughout both sections, we shall formulate various open problems and research topics.

First we shall assume Kripke models like in 8.2.15, i.e. of the form (W, e, μ) where $W \neq \emptyset$ is a set of possible worlds, $e(p, w) \in \{0, 1\}$ for each propositional variable and μ is a finitely additive probability on a field of subsets of W such that for each p, the set $\{w \in W | e(p) = 1\}$ is measurable (is in the domain of μ). We call the class of all such models $\mathcal{K}_{prob}$.

Definition 8.4.2 Let $\mathcal{C}$ be a propositional logic given by a continuous t-norm $*$. The *fuzzy probability modal logic* over $\mathcal{C}$ (shortly, the *fuzzy probability logic*, FP($\mathcal{C}$)) has the connectives and truth constants $\&, \rightarrow, \wedge, \vee, \neg, \bar{0}, \bar{1}$ of $\mathcal{C}$, one unary modality P (for "probably") and two kinds of formulas:

- *Boolean formulas* resulting from propositional variables and $\bar{0}, \bar{1}$ using only connectives $\wedge, \vee, \neg$, and
- *modal formulas* resulting from atoms of the form $P\varphi$, φ any Boolean formula and truth constants $\bar{0}, \bar{1}$ using all the connectives of $\mathcal{C}$.

For the special case of $\mathcal{C}$ being RPL we modify this definition by adding truth constants $\bar{r}$ (r rational from $(0, 1)$) to atomic modal formulas.

Models are structures from $\mathcal{K}_{prob}$. For each Boolean formula φ, the truth value $\|\varphi\|_{\mathbf{K},w}$ ($w \in W$) is defined in the obvious way and is 0 or 1. Furthermore,

$$\|P\varphi\|_{\mathbf{K}} = \mu(\{w \in W | \, \|\varphi\|_{\mathbf{K},w} = 1\})$$

and $\|\Phi\|_{\mathbf{K}}$ for a non-atomic modal formulas is defined using the truth functions of $\mathcal{C}$ (over the BL-algebra given by the t-norm $*$).

Remark 8.4.3 (1) Observe the distinction between φ and $P\varphi$: φ is a Boolean formula and, given $\mathbf{K}$, has its probability (probability of the set of all $w \in W$ in which φ is true). This probability is taken to be the truth degree of $P\varphi$, the formula saying "φ is probable."

(2) Even if we have defined the probability logic for any t-norm $*$, we shall restrict ourselves to Łukasiewicz logic Ł (or RPL) since its truth functions are ell suited to expressing basic properties of probability. From now on, FP stands for the fuzzy probabilistic modal logic FP(Ł) or FP(RPL).[51]

Definition 8.4.4 The following are *axioms* of FP:

- axioms of Boolean propositional logic *Bool* for Boolean formulas,
- axioms of Ł (or of RPL) for modal formulas,
- axioms on P:

$$P\varphi \rightarrow (P(\varphi \rightarrow \psi) \rightarrow P\psi), \qquad (FP1)$$

$$P(\neg\varphi) \equiv \neg P(\varphi), \qquad (FP2)$$

[51] See [79].

$$P(\varphi \vee \psi) \equiv [(P\varphi \to P(\varphi \wedge \psi)) \to P\psi] \qquad (FP3)$$

for arbitrary Boolean formulas φ, ψ.

Deduction rules are modus ponens and necessitation: "from φ deduce $P\varphi$" (φ Boolean).

Lemma 8.4.5 (Soundness). The axioms (FP1)-(FP3) are 1-true in each model $\mathbf{K} \in \mathcal{K}_{prob}$.

Proof: Given $\mathbf{K} = (W, e, \mu)$, write $\mu(\varphi) = \mu(\{w \in W \mid \|\varphi\|_{\mathbf{K},w} = 1\})$ for each Boolean formula φ. To verify $\|(FP1)\|_{\mathbf{K}} = 1$ we show $\mu(\varphi \to \psi) \leq (\mu(\varphi) \Rightarrow \mu(\psi))$. Put $\mu(\varphi \wedge \psi) = a, \mu(\varphi \wedge \neg\psi) = b, \mu(\neg\varphi \wedge \psi) = c, \mu(\neg\varphi \wedge \neg\psi) = d$; then $\mu(\varphi \to \psi) = 1 - b$ whereas $(\mu(\varphi) \Rightarrow \mu(\psi)) = 1$ (if $\mu(\varphi) \leq \mu(\psi)$) or $(\mu(\varphi) \Rightarrow \mu(\psi)) = 1 - (a + b) + (a + c) = 1 - b + c$. In either case, $\mu(\varphi) \Rightarrow \mu(\psi) \geq \mu(\varphi \to \psi)$.

$\|FP2\|_{\mathbf{K}} = 1$ trivially follows from $\mu(\neg\varphi) = 1 - \mu(\varphi)$. Let us verify $\|FP3\|_{\mathbf{K}} = 1$ keeping the above notation. We have $\mu(\varphi \vee \psi) = a+b+c$ and $(\mu(\varphi) \Rightarrow \mu(\varphi \wedge \psi)) \Rightarrow \mu(\psi) = [((a + b) \Rightarrow a) \Rightarrow (a + c)] = ((1 - b) \Rightarrow (a+c)) = a+b+c$; thus $\|P(\varphi \vee \psi)\|_{\mathbf{K}} = \|(P\varphi \to P(\varphi \wedge \psi)) \to P(\psi)\|_{\mathbf{K}}$. The rest is evident. □

Definition 8.4.6 In the present context a *modal* theory is a theory whose all axioms are modal formulas.

Corollary 8.4.7 Let T be a modal theory over FP. If $T \vdash \Phi$ then $\|\Phi\|_{\mathbf{K}} = 1$ for each FP-model $\mathbf{K}$ of T.

Remark 8.4.8 We shall prove Pavelka-style completeness for FP over Pavelka logic RPL. To this end recall the *provability degree* of Φ over T.

$$|\Phi|_T = \sup\{r | T \vdash \bar{r} \to \Phi\}$$

and the *truth degree* of Φ over T.

$$\|\Phi\|_T = \inf\{\|\Phi\|_{\mathbf{K}} | \mathbf{K} \text{ FP-model of } T\}.$$

Here Φ is any modal formula; provability is understood as provability in the sense of the modal logic FP(RPL) and models are understood also in the sense of FP(RPL), i.e. crisp Kripke models with probability.

Theorem 8.4.9 *(Completeness.)* For each modal theory T over FP(RPL) and each modal formula Φ,

$$|\Phi|_T = \|\Phi\|_T.$$

Proof: The inequality $|\Phi|_T \leq \|\Phi\|_T$ follows directly from soundness 8.4.7. To prove the converse we represent the modal part of T as a theory T^* over the propositional logic RPL.

For each Boolean formula φ, let p_φ be a propositional variable corresponding to the formula $P\varphi$. We write $p_\varphi = (P\varphi)^*$ and define $\bar{r}^* = \bar{r}$, $(\Phi \to \Psi)^* = \Phi^* \to \Psi^*$ and $(\Phi \& \Psi)^* = \Phi^* \& \Psi^*$. Let T^* be a theory (over RPL) whose axioms are

- atoms p_φ for φ being a Boolean tautology,
- formulas of the form $(FP1)^*, \ldots, (FP3)^*$ (for (FPi) see 8.4.4)
- formulas α^* for $\alpha \in T$.

First observe that for each modal formula Φ,

$$T \vdash_{FP(RPL)} \Phi \qquad \text{iff} \qquad T^* \vdash_{RPL} \Phi^*.$$

Indeed, each RPL-proof of Φ^* in T^* has the form $\Psi_1^*, \ldots, \Psi_n^*$ where Ψ_i are modal formulas. The sequence $\Psi_1, \ldots, \Psi_n$ is converted into an FP(RPL)-proof in T by adding, for each Ψ_i of the form p_φ (φ Boolean tautology) a proof of φ in Boolean propositional logic and then applying generalization to obtain p_φ.

Conversely, a FP(RPL)-proof $\Psi_1, \ldots, \Psi_n$ of Φ in T is made to an RPL-proof of Φ^* in T^* by deleting all Boolean formulas and taking for each modal Ψ_i the formula Ψ_i^*. (Note that, by the definition of a FP(RPL)-proof, each Boolean formula provable in a model theory T is a Boolean tautology.)

Now by 3.3.5, $|\Phi|_T = |\Phi^*|_{T^*} = \|\Phi^*\|_{T^*}$. Thus it remains to prove $\|\Phi^*\|_{T^*} = \|\Phi\|_T$, i.e. that for each model E of T^* (evaluation of propositional variables) there is a model $\mathbf{K}$ of T such that $E(\Phi^*) = \|\Phi\|_{\mathbf{K}}$ (and consequently, each $\mathbf{K}$ determines an E but this is easy).

Thus let us have such an E. Let W be the set of all evaluations w of Boolean propositional variables of T (i.e. $w(p) = 0$ or 1 for each Boolean p); let $e(p, w) = w(p)$ and for each Boolean φ put

$$\mu(\varphi) = \mu(\{w | w(\varphi) = 1\}) = E(p_\varphi).$$

(In words: the probability of the set of all worlds satisfying φ is set to be the truth degree of the atom p_φ in the model E of T^*). This defines our $\mathbf{K}$.

We have to prove that μ is indeed a finitely additive probability in the Boolean algebra of sets of the above form. Clearly, $\mu(W) = \mu(\bar{1}) = E(p_{\bar{1}}) = 1$. Since $E(p_{\neg\varphi}) = 1 - E(p_\varphi)$ (due to the axiom $(FP2)^*$) we get $\mu(\emptyset) = \mu(\bar{0}) = 0$. Now take arbitrary Boolean φ, ψ and put $a = \mu(\varphi \vee \psi), b = \mu(\varphi), c = \mu(\psi), d = \mu(\varphi \wedge \psi)$. By $(FP3)^*$, $a = ((b \Rightarrow d) \Rightarrow c)$ and $b \geq d$ (since $(\varphi \wedge \psi) \to \varphi$ is a Boolean tautology, hence $E(p_{(\varphi\wedge\psi)\to\varphi}) = 1$ and by $(FP1)^*$, $E(p_{\varphi\wedge\psi}) \leq E(p_\varphi)$). Hence $a = ((1 - b + d) \Rightarrow c)$. Furthermore,

$\psi \to (\varphi \to (\varphi \wedge \psi))$ is a Boolean tautology, hence, by a similar argument, $c = E(p_\psi) \le E(p_\varphi \to p_{\varphi\wedge\psi}) = 1-b+d$. Summarizing, $a = ((1-b+d) \Rightarrow c) = 1-(1-b+d)+c = b+c-d$, i.e. $\mu(\varphi\vee\psi) = \mu(\varphi)+\mu(\psi)-\mu(\varphi\wedge\psi)$, which is finite additivity. This completes the proof. □

Example 8.4.10 As an example of a proof in FP we show that FP$\vdash$ $P(\varphi) \equiv (P(\varphi \wedge \psi)\underline{\vee}P(\varphi \wedge \neg\psi))$. Indeed

$$\mathrm{FP} \vdash P(\varphi) \equiv P((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi)),$$

$$\mathrm{FP} \vdash P((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi)) \equiv$$

$$\equiv P(\varphi \wedge \psi) \to P((\varphi \wedge \psi) \wedge (\varphi \wedge \neg\psi))) \to P(\varphi \wedge \neg\psi)$$

(by (FP3)),

$$\mathrm{FP} \vdash P((\varphi \wedge \psi) \wedge (\varphi \wedge \neg\psi)) \equiv \bar{0}$$

(since FP $\vdash P(\bar{1})$, FP$\vdash P(\bar{1}) \equiv \bar{1}$, FP$\vdash \neg P(\bar{1}) \equiv \bar{0}$, FP$\vdash P(\bar{0}) \equiv \bar{0}$), thus

$$\mathrm{FP} \vdash (P(\varphi \wedge \psi) \to P((\varphi \wedge \psi) \wedge (\varphi \wedge \neg\psi))) \equiv \neg P(\varphi \wedge \psi),$$

$$\mathrm{FP} \vdash P(\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi)) \equiv \neg P(\varphi \wedge \psi) \to P(\varphi \wedge \neg\psi),$$

$$\mathrm{FP} \vdash P(\varphi) \equiv P(\varphi \wedge \psi)\underline{\vee}P(\varphi \wedge \neg\psi).$$

Remark 8.4.11 (1) Analogously, we may develop a fuzzy logic of necessities FN (with crisp models); the axioms are

$$\mathcal{N}(\varphi \to \psi) \to (\mathcal{N}\varphi \to \mathcal{N}\psi) \qquad (FN1)$$

$$\neg\mathcal{N}(\bar{0}) \qquad (FN2)$$

$$\mathcal{N}(\varphi \wedge \psi) \equiv (\mathcal{N}\varphi \wedge \mathcal{N}\psi). \qquad (FN3)$$

The reader may verify completeness as an exercise.

(2) Is FP(Ł) complete in the usual sense, i.e. is FP(Ł) + Φ equivalent to $\|\Phi\|_{\mathbf{K}} = 1$ for all models **K**? Is it strongly complete for finite theories? We shall show that a careful analysis of our presentation yields a positive answer. Recall 3.3.14 according to which we do have strong completeness (in the usual sense) for *finite* theories over RPL. The problem is that if we start with a finite modal theory over FP(RPL) then the corresponding theory T^* over RPL is infinite since it contains infinitely many instances of $(FP1)^* - (FP3)^*$. Our aim is to show that they can be replaced by finitely many axioms.

Definition 8.4.12 In the sequel, T is a fixed finite modal theory over FP(RPL); we fix n Boolean propositional variables $q_1, \dots, q_n$ containing at least all variables in T. In each Boolean formula φ built from $q_1, \dots, q_n$, $NF(\varphi)$ is the disjunctive normal form of φ (1.2.14); note that there are 2^n different normal forms. For each modal formula Φ, its normal translation Φ^*_{NF} results from Φ^* by replacing each atom p_φ by $p_{NF(\varphi)}$ (φ varies over Boolean formulas from the q's).

T^*_{NF} is the set of all Ψ^*_{NF} where Ψ is an axiom of T. Note that T^*_{NF} is *finite* (since T is finite and there are only finitely many formulas of the form $(FPi)^*_{NF}$).

Lemma 8.4.13 (1) $T^* \vdash \Phi^*$ iff $T^*_{NF} \vdash \Phi^*_{NF}$ (both provabilities over RPL).

(2) $\|\Phi^*_{NF}\|_{T^*_{NF}} = 1$ iff $\|\Phi^*\|_{T^*} = 1$

Proof: (1) Note that if φ, ψ are equivalent in Boolean propositional logic then $T^* \vdash p_\varphi \equiv p_\psi$. If $\Psi^*_1, \dots, \Psi^*_n$ is a RPL-proof in T^* (where all Boolean variables in the Ψ's are among $q_1, \dots, q_n$) then $\Psi^*_{1,NF}, \dots, \Psi^*_{n,NF}$ is aproof in T^*_{NF}. Conversely observe that T^* proves each axiom of T^*_{NF}; thus if $T^*_{NF} \vdash \Phi^*_{NF}$ then $T^* \vdash \Phi^*_{NF}$ since $T^* \vdash \Phi^* \equiv \Phi^*_{NF}$.

(2) Let E be an evaluation of all p_φ-variables which is a model of T^* such that $E(\Phi^*) < 1$. Then E restricted to $p_{NF(\varphi)}$-variables is a model of T^*_{NF} such that $E(\Phi^*_{NF}) < 1$. Conversely, if E_0 is a model of T^*_{NF} such that $E_0(\Phi^*_{NF}) < 1$ then extend E_0 to an evaluation of all p_φ-variables putting $E(p_\varphi) = E_0(p_{NF(\varphi)})$ and observe that E is a model of T^* with $E(\Phi^*) = E_0(\Phi^*_{NF}) < 1$. (In particular, $E(\Psi^*) = 1$ for Ψ being (FP1), (FP2), (FP3).)
□

Theorem 8.4.14 Under the present notation

$$T \vdash \Phi \quad \text{iff} \quad \|\Phi\|_T = 1$$

(i.e. Φ is provable in T iff Φ 1-true in all Kripke models of T).

Proof: By the preceding lemma, by 3.3.14 and by the above, $T \vdash \Phi$ iff $T^* \vdash \Phi^*$ iff $T^*_{NF} \vdash \Phi^*_{NF}$ iff $\|\Phi^*_{NF}\|_{T^*_{NF}} = 1$ iff $\|\Phi^*\|_{T^*} = 1$ iff $\|\Phi\|_T = 1$.
□

*

Now let us turn to the modality $Many$ in the present setting. This means: we consider finite Kripke models and probability is replaced by relative frequency.

Definition 8.4.15 The fuzzy modal logic of "many" FMany(RPL) with crisp models has the same formulas as the corresponding probabilistic logic FP(RPL) (the formula $P\varphi$ being read "for many worlds, φ" or "usually φ"). Models are finite models $\mathbf{K} = (W, e)$ of the modal logic S5. Satisfaction of $P\varphi$ is defined as follows

$$\|P\varphi\|_{\mathbf{K}} = \frac{Fr_{\mathbf{K}}(\varphi)}{card(W)},$$

i.e. the truth value of "usually φ" is the relative frequency of worlds satisfying φ among all worlds in W.
Axioms are those of FP(RPL); in particular, we have the axioms (FPi), $i = 1, 2, 3$.

Theorem 8.4.16 *(Completeness).* Let T be a finite model theory over FMany(RPL), Φ a modal formula. Then

$$T \vdash \Phi \qquad \text{iff } \|\Phi\|_T = 1$$

($\|\Phi\|_T = 1$ meaning that Φ is 1-true in all finite Kripke models of T).

Proof: Soundness is trivial (since it follows from the soundness of the logic FP(RPL)). On the other hand, if $T \not\vdash \Phi$ then by 8.4.14 and by the proof of 8.4.9, there is a RPL-model E of T^* such that $E(\Phi^*) < 1$. Furthermore apply 3.3.17 to get such an E taking only rational values. Now inspect the part of the proof of 8.4.9 where E is used to produce a Kripke model $\mathbf{K} = (W, e, \mu)$. Let $q_1, \ldots, q_n$ be all Boolean variables in T, Φ and let $r_1, \ldots, r_{2^n}$ be the μ-measures of the 2^n elementary conjunctions of $q_1, \ldots, q^n$. Clearly $\sum_{i=1}^{2^n} r_i = 1$ and μ is fully determined by the r_i's. Let k be the common denominator of these rationals; i.e. $r_i = a_i/k$. Now produce a Kripke model of S5 $\mathbf{K}' = (W', e')$ such that in $\mathbf{K}'$ exactly a_i objects satisfy the i-th elementary conjunction. Observe that for each Boolean φ, $Fr_{\mathbf{K}'}(\varphi) = \mu_{\mathbf{K}}(\varphi)$, thus $\mathbf{K}'$ is a model of T and $\|\Phi\|_{\mathbf{K}'} = \|\Phi\|_{\mathbf{K}} < 1$ ($\mathbf{K}'$ is a model of FMany, $\mathbf{K}$ a model of FP). This completes the proof. □

*

Let us now turn to binary modalities corresponding to conditional probability and relative frequence. We continuous to work with crisp (Boolean) interpretation of propositional variables. The underlying fuzzy logic is RPL$\odot$ – RPL extended by the product conjunction (see 3.3.18 – 3.3.19). In this logic we may express basic properties of the above notions e.g.

$$\|\bar{r} \odot P(\varphi) \to P(\varphi \wedge \psi)\|_{\mathbf{K}} = 1$$

iff the conditional probability $\mu(\psi|\varphi)$ is at least r. Similarly, the formula $P(\varphi \wedge \psi) \equiv P(\varphi) \odot P(\psi)$ expresses independence of φ, ψ; and one can easily write formulas expressing conditional independence.

But we may directly introduce $P(\varphi|\psi))$ as a binary modality.

Definition 8.4.17 The modal logic FP(RPL$\odot$) has the same class $\mathcal{K}_{prob}$ of models as FP(RPL); the language is extended by the connective $\oplus$ and the binary modality $P(-I-)$. The definition of formulas is extended by the two clauses

- if φ, ψ are Boolean formulas then $P(\varphi|\psi)$ is a modal formula,
- if Φ, Ψ are modal formulas then $\Phi \odot \Psi$ is a modal formula.

The axioms are the following (cf. 8.4.4): axioms of Boolean logic for Boolean formulas, axioms of RPL$\odot$, axiom (FP1) - (FP3) for P and the axiom

$$P(\varphi|\psi) \odot P(\psi) = P(\varphi \wedge \psi). \qquad (FP4)$$

Deduction rules: modus ponens and necessitation (from φ infer p_φ).

The definition of truth values of formulas is extended by the following clauses:

- $\|\varphi \odot \psi\|_{\mathbf{K},w} = \|\varphi\|_{\mathbf{K},w} \cdot \|\psi\|_{\mathbf{K},w}$
- $\|P(\varphi|\psi)\|_{\mathbf{K}} = \frac{\|P(\varphi \wedge \psi\|_{\mathbf{K}}}{\|P(\psi)\|_{\mathbf{K}}}$ if $\|P(\psi)\|_{\mathbf{K}} \neq \emptyset$, otherwise $\|P(\varphi|\psi)\|_{\mathbf{K}}$ is arbitrary (0, say).

Theorem 8.4.18 (Soundness and completeness.) (FP4) is 1-true in each model $\mathbf{K} \in \mathcal{K}_{prob}$. For each modal theory T over FP(RPL$\odot$) and each modal formula Φ,

$$\|\Phi\|_T = |\Phi|_T.$$

Proof: Check the proof of 8.4.9, define $P(\varphi|\psi)^*$ to be a propositional variable $p_{\varphi,\psi}$. Replace RPL by RPL$\odot$ and FP(RPL) by FP(RPL$\odot$). Refer to the completeness theorem 3.3.19. □

Remark 8.4.19 (1) Recall 8.4.14 – the strong completeness theorem for FP(RPL) (for a finite modal theory T, $T \vdash \Phi$ iff Φ is true in all models of T). That theorem depends heavily on Theorem 3.3.14 (strong completenes for finite theories over RPL) which, in turn, depends on our reduction of RPL to Ł. To get a strong completeness for FP(RPL$\odot$) one would have to exhibit a reduction of RPL$\odot$ to an extension of Ł by $\odot$. To elaborate this remains an open *problem*.

(2) Now let us turn to finite models and relative frequencies: we shall investigate the binary modality $Many$.

Definition 8.4.20 The modal logic FMany(RPL$\odot$) has the same class of models as FMany(RPL) (i.e. finite models of S5). The language of FMany(RPL) (i.e. that of FP(RPL)) is expanded by the connective $\odot$ and binary modality $\sqsupset$. The formula $\psi \sqsupset \varphi$ is read "many objects satisfying ψ satisfy φ" (or just "many ψ's are φ's"). The semantics is

$$\|\psi \sqsupset \varphi\|_{\mathbf{K}} = Fr_{\mathbf{K}}(\varphi|\psi) = Fr_{\mathbf{K}}(\varphi \& \psi)/Fr_{\mathbf{K}}(\psi)$$

if $Fr_{\mathbf{K}}(\psi) > 0$ (frequence in **K**), otherwise $\|\psi \sqsupset \varphi\|_{\mathbf{K}} = 0$. Axioms and rules are as in FP(RPL$\odot$), with $P(\varphi|\psi)$ replaced by $\psi \sqsupset \varphi$.

Remark 8.4.21 Clearly, the deductive system of FMany(RPL$\odot$) is sound. An attempt to prove Pavelka-style completeness (in an analogy to 8.4.16) – by showing that if T is a finite theory over FMany(RPL$\odot$) and Φ a formula unprovable in T then there is a FMany(RPL$\odot$)-model **K** of T such that $\|\Phi\|_{\mathbf{K}} < 1$ – fails because we do not have an analogy of 3.3.17 for RPL$\odot$. Thus the *problem* of completeness of FMany(RPL$\odot$) seems to be open. On the other hand, we have the following (more or less evident) theorem.

Theorem 8.4.22 The set of 1-tautologies of FMany(RPL$\odot$) is a Π_1-set (and so is the set of 1-tautologies of FMany(RPL)).

Proof: Just observe that finite (Boolean) models of S5 are finite objects easily coded by natural numbers in such a way that the function assigning to each model **K** the value $\|\Phi\|_{\mathbf{K}}$ (in the sense of FMany(RPL$\odot$)) is recursive. Φ is a 1-tautology iff for all **K**, $\|\Phi\|_{\mathbf{K}} = 1$. □

Remark 8.4.23 *From modalities to quantifiers.* In our survey sections on generalized quantifiers and modalities in Boolean logic speaking on probabilistic threshold quantifiers, given by statistics (see 8.1.6 – 8.1.22); in 8.2.29 we only mentioned corresponding modalities. But in the present section, we have dealt with (many-valued) *modalities* related to probability and frequencies. We have elaborated calculi with crisp interpretation of propositional variables as an important intermediate class of calculi (between fully Boolean and fully fuzzy ones). Thus we had two kinds of formulas: Boolean and modal (the latter being fuzzy – many-valued). Clearly, one may develop, in parallel, *predicate* calculi with crisp interpretation of predicates and generalized quantifiers
$(Px)\varphi$ (where $\|(Px)\varphi\|_{\mathbf{M},v} = \mu(\{m \in M|\ \|\varphi\|_{\mathbf{M},v(x/m)} = 1\}$
and the corresponding calculi with finite models. In fact we did use such calculi with unary predicates and just one object variable – they are equivalent to respective modal calculi. Note that such and similar calculi seem to be useful in databases with crisp data but fuzzy queries and in data mining. To illustrate this we present the notion of many-valued *multitudinal quantifiers*.

Definition 8.4.24 A *predicate calculus* $\mathcal{C}\forall$ $\sqsupset$-fincrisp with finite crisp models and a fuzzy multitudinal quantifier $\sqsupset$ is given by the following:

- a propositional fuzzy logic $\mathcal{C}$,
- a predicate language $\mathcal{I}$,
- a multitudinal statistic f (cf. 8.1.16) with range included in $[0, 1]$. (Thus f assigns to each four-fold table (a, b, c, d) a real $f(a, b, c, d) \in [0, 1]$ such that $a' \geq a$ and $b' \leq b$ imply $f(a', b', c', d') \geq f(a, b, c, d)$.)

Boolean formulas are built from atomic formulas using connectives $(\wedge, \vee, \neg$, say) and $\forall, \exists$. If φ, ψ are Bolean formulas and x is a variable then $\varphi \sqsupset_x \psi$ using connectives and $\forall, \exists$.

Semantics is clear for Boolean formulas, $\|\varphi \sqsupset_x \psi\|_{\mathbf{M},v} = f(a, b)$ where $a = Fr_{\mathbf{M},v}(\varphi \wedge \psi, x)$ (number of object in $\mathbf{M}$ satisfying $\varphi(x) \wedge \psi(x)$, all other variables being fixed by v) analogously for b and $\varphi \wedge \neg\psi$. Needless to say, f is the statistic introduced above. Semantics of other (general) formulas is as in $\mathcal{C}\forall$.

Example 8.4.25 (1) Relative frequence: $f(a, b) = a/(a + b)$ (and $= 0$ if $a = b = 0$). This corresponds to the modality "many" above.

(2) The statistic $LIMPL_\alpha$ used in GUHA (α small positive e.g. $\alpha = 0.05$)

$$LIMPL_\alpha(a, b) = \max\{p| \quad \Sigma_{i=a}^{a+b} \binom{a+b}{i} p^i (1-p)^{r-i} \leq \alpha\}.$$

This relates to statistical hypothesis testing: $LIMPL(a, b)$ is the maximal p such that on the base of the data we reject, on significance level α, the null hypothesis $P(\psi|\varphi) \leq p$. See [85] for details and for a proof that $LIMPL_\alpha$ is implicational. Note that $LIMPL_\alpha(a, b)$ is always strictly less than 1 (but can be arbitrarily near to 1). The formula $\varphi \sqsupset_x^\alpha \psi$ may be read "it is likely that the conditional provability $P(\psi|\varphi)$ is high" (on significance level α).

Theorem 8.4.26 If $\sqsupset$ is a multitudinal fuzzy quantifier then the formula

$$((\varphi \wedge \psi) \sqsupset_x \chi) \to (\varphi \sqsupset_x (\neg\psi \vee \chi))$$

is a 1-tautology.

Proof: See the proof of 8.1.20; it even proves the present theorem. □

8.5. MORE ON "PROBABLY" AND "MANY"

We now switch to the general case of the quantifier "probably" in a calculus with fuzzy models (i.e. predicates are interpreted by fuzzy relations as usual). We shall elaborate rational Pavelka predicate logic with integrals as a generalization of our investigations earlier in this section. Note that the results (a completeness theorem) mainly show a direction of necessary research and cannot be considered to be final results. Nevertheless, the results below contribute (we hope) to our understanding of how one can speak on probability in fuzzy logic and, more than that, how one can *deduce* fuzzy statements containing the quantifier "probably" in an extension of RPL, complete in a (weak) sense.

Definition 8.5.1 We extend the Lukasiewicz-Pavelka rational predicate logic RPL$\forall$ by a new quantifier $\int$ (integral, read "probably")[52] and extend the definition of a formula by the clause saying that if φ is a formula and x a variable then $\int \varphi \mathrm{d}x$ is a formula; x is closed in $\int \varphi \mathrm{d}x$. The intended semantics is as follows: We consider a structure

$$\mathbf{M} = \langle M, (r_P)_{P\,\text{predicate}}, (m_c)_{c\,\text{constant}}, \mu \rangle$$

where M is a non-empty *countable* or *finite* domain, r_P, m_c interpret predicates and constants as usual (over the real unit interval $[0, 1]$) and μ is a probability measure on M, i.e. a function assigning to each singleton $m \in M$ a real $\mu(m) \in [0, 1]$ such that $\sum_{m \in M} \mu(m) = 1$. This extends to an arbitrary subset $A \subseteq M$: $\mu(A) = \sum_{m \in A} \mu(m)$. More generally, if $A \subseteq M^n$ then $\mu(A) = \sum\{\mu(m_1) \cdot \mu(m_2) \dots \mu(m_n) | \langle m_1, \dots, m_n \rangle \in A\}$ (product measure). $\mathbf{M}$ is called a *probabilistic model* for the given language.

For each function $f : M \to [0, 1]$, the *integral* $\int f \mathrm{d}\mu$ is defined as $\sum_{m \in M} f(m) \cdot \mu(m)$, written also $\int f(x)\mathrm{d}x$. It can be also called the *expected value* of f or the *probability* of the fuzzy set f. Observe that if f is two-valued, $range(f) \subseteq \{0, 1\}$ then $\int f \mathrm{d}x = \mu(f^{-1}(1)) = \mu(\{m | f(m)\} = 1)$ (probability of the set whose characteristic function is f).

Given $\mathbf{M}$ we extend the definition of the value $\|\varphi\|_{\mathbf{M},v}$ of $\varphi \in \mathbf{M}$ and the evaluation v as follows:

$$\| \textstyle\int \varphi \mathrm{d}x \|_{\mathbf{M},v} = \textstyle\int Sat_{\mathbf{M},v}(\varphi, x)\mathrm{d}x$$

where $Sat_{\mathbf{M},v}(\varphi, x)$ is the function assigning to each $m \in M$ the truth-value $\|\varphi\|_{\mathbf{M},v(m/x)}$ ($v(m/x)$ means, of course, the evaluation v' coinciding with v for all arguments except x, and such that $v'(x) = m$). Thus: all arguments except x are fixed, x runs over m.

[52] See [218] for Zadeh's definition of the probability of a fuzzy event.

Lemma 8.5.2 The following formulas are 1-true in each probabilistic model:

(μ1) $\int \nu \mathrm{d}x \equiv \nu$ for ν not containing x freely,

(μ2) $\int(\neg\varphi)\mathrm{d}x \equiv \neg \int \varphi \mathrm{d}x$,

(μ3) $\int(\varphi \to \psi)\mathrm{d}x \to (\int \varphi \mathrm{d}x \to \int \psi \mathrm{d}x)$,

(μ4) $\int(\varphi \underline{\vee} \psi)\mathrm{d}x \equiv ((\int \varphi \mathrm{d}x \to \int(\varphi \& \psi)\mathrm{d}x) \to \int \psi \mathrm{d}x)$,

(μ5) $(\forall x)\varphi \to \int \varphi \mathrm{d}x$,

(μ6) $\int(\int \varphi \mathrm{d}x)\mathrm{d}y \equiv \int(\int \varphi \mathrm{d}y)\mathrm{d}x$ (Fubini).

Proof: Write f_φ instead of $Sat_{\mathbf{M},v}(\varphi, x)$ if there is no danger of misunderstanding. If ν does not contain x freely then f_ν is constant (and equal to $\|\nu\|_{\mathbf{M},v}$), which proves (μ1) to be a 1-tautology. Similarly, $f_{\neg\varphi} = 1 - f_\varphi$, thus $\| \int \neg\varphi \mathrm{d}x \|_{\mathbf{M},v} = \int(1 - f_\varphi)\mathrm{d}x = \int \bar{1} \mathrm{d}x - \int f_\varphi \mathrm{d}x = 1 - \int f_\varphi \mathrm{d}x = \|\neg \int \varphi \mathrm{d}x\|_{\mathbf{M},v}$. This is ($\mu$2).

Let us now verify (μ4). First, $(\varphi \underline{\vee} \psi) \equiv (\varphi \to (\varphi \& \psi)) \to \psi$ is a 1-tautology (as one easily verifies). Thus, for each argument m, $f_{(\varphi \underline{\vee} \psi)} = (f_\varphi \Rightarrow f_{\varphi\&\psi}) \Rightarrow f_\psi = [(1 - f_\varphi + f_{\varphi\&\psi}) \Rightarrow f_\psi]$ since $f_\varphi \geq f_{\varphi\&\psi}$; furthermore $1 - f_\varphi + f_{\varphi\&\psi} \geq 1 - f_\varphi + f_\varphi + f_\psi - 1 = f_\psi$, hence $[\cdots] = 1 - (1 - f_\varphi + f_{\varphi\&\psi}) + f_\psi = f_\varphi + f_\psi - f_{\varphi\&\psi}$. This gives $\int f_{\varphi\underline{\vee}\psi} \mathrm{d}x = \int f_\varphi \mathrm{d}x + \int f_\psi \mathrm{d}x - \int f_{\varphi\&\psi} \mathrm{d}x = (\int f_\varphi \mathrm{d}x \Rightarrow \int f_{\varphi\&\psi} \mathrm{d}x) \Rightarrow \int f_\psi \mathrm{d}x$, hence $\| \int(\varphi \underline{\vee} \psi)\mathrm{d}x \|_{\mathbf{M},v} = \|(\int \varphi \mathrm{d}x \to \int(\varphi\&\psi)\mathrm{d}x) \to \int \psi \mathrm{d}x\|_{\mathbf{M},v}$. This proves ($\mu$4).

Now return to (μ3), observing that if $\|(\forall x)(\alpha \equiv \beta)\|_{\mathbf{M},v} = 1$ then obviously $\| \int \alpha \mathrm{d}x \|_{\mathbf{M},v} = \| \int \beta \mathrm{d}x \|_{\mathbf{M},v}$. Thus $\| \int(\varphi \to \psi)\mathrm{d}x \| = \| \int(\neg\varphi \underline{\vee} \psi)\mathrm{d}x \| = \|(\int \neg\varphi \mathrm{d}x \to \int(\neg\varphi\&\psi)\mathrm{d}x) \to \int \psi \mathrm{d}x\| = \|(\int(\varphi \underline{\vee} \neg\psi)\mathrm{d}x \to \int \varphi \mathrm{d}x) \to \int \psi \mathrm{d}x\| \leq \| \int \varphi \mathrm{d}x \to \int \psi \mathrm{d}x\|$. (We have deleted the indices M, r.) The last inequality follows from $\| \int \varphi \mathrm{d}x \| \leq \|\chi \to \int \varphi \mathrm{d}x\|$ for χ being $\int \varphi \underline{\vee} \neg\psi)\mathrm{d}x$.

(μ5):Put $\xi = \|(\forall x)\varphi(x)\|$; then for each m, $\xi \leq f_\varphi(m)$, thus $\xi = \int \xi \mathrm{d}x \leq \int f_\varphi \mathrm{d}x$, hence $\|(\forall x)\varphi(x)\| \leq \| \int \varphi \mathrm{d}x \|$.

Finally we prove (μ6). This follows immediately from the fact that if $f : M^2 \to [0,1]$ then $\int(\int f(x,y)\mathrm{d}x)\mathrm{d}y = \sum_{a\in M}(\sum_{b\in M} f(a,b)\mu(b))\mu(a) =$
$= \sum_{a,b\in M} f(a,b)\mu(a)\mu(b) = \sum_{b\in M}(\sum_{a\in M} f(a,b)\mu(a))\mu(b) =$
$= \int\int f(x,y)\mathrm{d}y\mathrm{d}x$. This completes the proof. □

Remark 8.5.3 Try to read the formulas, reading $\int \varphi \mathrm{d}x$ as "probably φ" For example (μ3) reads "If $(\varphi \to \psi)$ is probable then if φ is probable then ψ is probable". Note that (μ4) is a generalization of finite additivity, see below.

Definition 8.5.4 The logic RPL$\forall\int$ (*rational Pavelka predicate logic with integrals*) has the language as above, the axioms of RPL$\forall$ (with the present notion of a formula) and the axioms (μ1) – (μ6); the deduction rules are modus ponens and generalization (as in RPL$\forall$, no new deduction rule).

Lemma 8.5.5 The following are derived deduction rules:

$$\frac{\varphi}{\int \varphi \mathrm{d}x} \qquad \frac{\varphi \to \psi}{\int \varphi \mathrm{d}x \to \int \psi \mathrm{d}x}$$

Proof: (1) If $\vdash \varphi$ then $\vdash (\forall x)\varphi$ and hence $\vdash \int \varphi \mathrm{d}x$ by (μ5). (2) If $\vdash \varphi \to \psi$ then $\vdash (\forall x)(\varphi \to \psi)$, thus $\vdash \int(\varphi \to \psi)\mathrm{d}x$ and hence $\vdash \int \varphi \mathrm{d}x \to \int \psi \mathrm{d}x$ by (μ3). □

Lemma 8.5.6 (Deduction theorem). Let T be a theory, φ and ψ formulas, φ closed. $T \cup \{\varphi\} \vdash \psi$ iff fore some n, $T \vdash \varphi^n \to \psi$.

Proof: Just check the usual proof. □

Definition 8.5.7 Let M be a non-empty set and let $\mathcal{F}$ be an algebra of $[0,1]$-fuzzy subsets of M containing each rational constant function with the value $r \in [0,1]$ and closed under $\Rightarrow$ (if $f, g \in \mathcal{F}$ and $h(a) = f(a) \Rightarrow g(a)$ for all $a \in M$ then $h \in \mathcal{F}$). Note that then $\mathcal{F}$ is closed under $\neg, \&, \wedge, \vee, \underline{\vee}$.

A *weak integral* on $\mathcal{F}$ is a mapping $\mathbf{I}$ associating to each $f \in \mathcal{F}$ its "integral" $\mathbf{I}f\mathrm{d}x \in [0,1]$ and satisfying the following ($f, g \in \mathcal{F}$, k_r is the constant function with the value r, $h : M \times M \to [0,1]$ is such that for each $a, b \in M$, the functions $f_b(x) = h(x,b)$ and $g_a(y) = h(a,y)$ are both in $\mathcal{F}$:

$$\mathbf{I}k_r\mathrm{d}x = r,$$

$$\mathbf{I}(1-f)\mathrm{d}x = 1 - \mathbf{I}f\mathrm{d}x,$$

$$\mathbf{I}(f \Rightarrow g)\mathrm{d}x \leq (\mathbf{I}f\mathrm{d}x \Rightarrow \mathbf{I}g\mathrm{d}x),$$

$$\mathbf{I}(f \oplus g)\mathrm{d}x = \mathbf{I}f\mathrm{d}x + \mathbf{I}g\mathrm{d}x - \mathbf{I}(f\&g)\mathrm{d}x,$$

$$\mathbf{I}(\mathbf{I}h\mathrm{d}x)\mathrm{d}y = \mathbf{I}(\mathbf{I}h\mathrm{d}y)\mathrm{d}x \text{ if both sides defined.}$$

Definition 8.5.8 A *weak probabilistic model* is a structure

$$\mathbf{M} = \langle M, (r_P)_P, (m_c)_c, \mathbf{I}\rangle$$

where M is countable or finite, $\langle M, (r_P)_P, (m_c)_c\rangle$ interprets the given language over $[0,1]$ and $\mathbf{I}$ is a weak integral on an algebra $\mathcal{F}$ of fuzzy subsets of M. Define truth values of formulas extending the usual definition by

$$\| \int \varphi \mathrm{d}x \|_{\mathbf{M},v} = \mathbf{I} Sat_{\mathbf{M},v}(\varphi, x)\mathrm{d}x)$$

if $Sat_{\mathbf{M},v}(\varphi, x)$ is in $\mathcal{F}$ (otherwise undefined). $\mathbf{M}$ is *safe* if all the truth values are defined.

Remark 8.5.9 Observe that a safe weak probabilistic model defines a (finitely additive) probability on definable subsets of M: if f_φ is two-valued and $A \subseteq M$ is the set whose characteristic function is f_φ then put $P(A) = \mathbf{I} f_\varphi \mathrm{d}x$. It follows easily, using the axiom ($\mu 4$), that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (and $P(M) = 1, P(\emptyset) = 0$), so that P is indeed a finitely additive probability. (But we do not claim more.)

8.5.10 Now check the proof of Theorem 5.2.7 saying that each consistent theory T such that $T \not\vdash \alpha$ (α a closed formula) has a consistent Henkin complete extension $\hat{T}$ not proving α: everything goes through. Our task is to construct a model of T from the theory $\hat{T}$. We shall not construct a probabilistic model but only a weak probabilistic model.

Definition 8.5.11 Let $\hat{T}$ be a consistent complete Henkin theory. The weak probabilistic *model* $\mathbf{M}$ *corresponding to* $\hat{T}$ is defined as follows: M is the set of all constants, $m_c = c$, $r_P(c_1 \cdots c_n) = |P(c_1 \cdots c_n)|_{\hat{T}}$; for each formula $\varphi(x)$ with just one free variable, $f_\varphi(c) = |\varphi(c)|_{\hat{T}}$ and $\mathbf{I} f_\varphi \mathrm{d}x = |\int \varphi \mathrm{d}x|_{\hat{T}}$.

Lemma 8.5.12 The preceding definition is sound, i.e. $\mathbf{I}$ is a mapping.

Proof: If $f_\varphi = f_\psi$, i.e. $|\varphi(c)|_{\hat{T}} = |\psi(c)|_{\hat{T}}$ for each c then $|(\forall x)(\varphi(x) \equiv \psi(x))|_{\hat{T}} = 1$, (see below) hence $|\int \varphi \mathrm{d}x \equiv \int \psi \mathrm{d}x|_{\hat{T}} = 1$ and $|\int \varphi \mathrm{d}x|_{\hat{T}} = |\int \psi \mathrm{d}x|_{\hat{T}}$. We elaborate the step stating that if $|\varphi(c)|_{\hat{T}} = |\psi(c)|_{\hat{T}}$ for all c, then $|(\forall x)(\varphi(x) \equiv \psi(x))|_{\hat{T}} = 1$. Let r be rational and ε rational small positive such that $r < |\varphi(c)|_{\hat{T}} = |\psi(c)|_{\hat{T}} < (r + \varepsilon)$. (If $|\varphi(c)|_{\hat{T}} = 0$ take $r = 0$; if $|\varphi(c)|_{\hat{T}} = 1$ take $r + \varepsilon = 1$.) Then, since $\hat{T}$ is complete,

$$\hat{T} \vdash \varphi(c) \to \overline{(r + \varepsilon)}, \qquad \hat{T} \vdash \bar{r} \to \psi(c),$$

thus $\hat{T} \vdash \varphi(c) \to (\bar{r} \underline{\vee} \bar{\varepsilon})$, $\hat{T} \vdash ((\overline{(1-\varepsilon)} \& \varphi(c)) \to \bar{r}$, $\hat{T} \vdash \overline{(1-\varepsilon)} \to (\varphi(c) \to \bar{r})$, hence $\hat{T} \vdash \overline{(1-\varepsilon)} \to (\varphi(c) \to \psi(c))$ for each $\varepsilon > 0$ and each c. Since $\hat{T}$ is Henkin this gives $\hat{T} \vdash \overline{(1-\varepsilon)} \to (\forall x)(\varphi(x) \to \psi(x))$ for each $\varepsilon > 0$; similarly $\hat{T} \vdash \overline{(1-\varepsilon)} \to (\forall x)(\psi(x) \to \varphi(x))$ for each $\varepsilon > 0$ and hence $|(\forall x)(\varphi(x) \equiv \psi(x)|_{\hat{T}} = 1$. Thus $\mathbf{I} f_\varphi \mathrm{d}x$ is uniquely determined. □

Lemma 8.5.13 Under the above notation, $\|\varphi\|_{\mathbf{M}} = |\varphi|_{\hat{T}}$ for each closed formula φ.

Proof: We extend the proof of the corresponding theorem 5.4.10 for RPL$\forall$; we investigate the induction step for $\int \varphi(x)\mathrm{d}x$. By definition, $\| \int \varphi \mathrm{d}x\|_{\mathbf{M}} = \int f_{\varphi}\mathrm{d}x = | \int \varphi \mathrm{d}x|_{\hat{T}}$ so that the step is obvious. □

Corollary 8.5.14 *(Completeness.)* Let T be a theory over RPL$\forall \int$ and φ a formula. Then $|\varphi|_T = \|\varphi\|_T$; in particular, $|\varphi|_T = 1$ (over RPL$\forall \int$) iff φ is 1-true in all safe weak probabilistic models of T. (of course, $|\varphi|_T$ is the provability degree of φ in T, i.e. $\{r | T \vdash (\bar{r} \rightarrow \varphi)$ over $RPL\forall \int\})$.

Remark 8.5.15 The question if we can improve this completeness by replacing countable weak probabilistic models by countable probabilistic models needs further investigation. Ref. [101] could be relevant. Apparently, to enforce countable additivity we would have to extend our language by countably infinite conjunctions/disjunctions or by infinitary deduction rules (with infinitely many assumptions). We shall not go into this. Instead, we shall make several comments that should make the quantifier $\int$ better understood (and might be useful in deciding the completeness problem). We assume that our language contains an equality predicate $=$ interpreted absolutely (i.e. $r_{=}(a,b) = 1$ iff $a = b$, $r_{=}(a,b) = 0$ otherwise). Then we may speak on the measure of a singleton: $\| \int (y = x)\mathrm{d}y\|_{\mathbf{M},v} = \mu(a)$ where $v(x) = a$.

(1) The following rule is sound over probabilistic (at most countable) models:

$$\frac{(\forall x)(\int ((y = x) \& \varphi(y))\mathrm{d}y \rightarrow \bar{0})}{\int \varphi(y)\mathrm{d}y \rightarrow \bar{0}}$$

Indeed, the assumption is 1-true in **M** iff, for each $b \in M$,$\mu(b) \cdot \|\varphi(y)\|_{\mathbf{M}}[b] = 0$. (The truth value of φ in **M** for any evaluation assigning b to y.) Then clearly $(\int \varphi \mathrm{d}y \rightarrow \bar{0})$ is 1-true.

(2) But the implication

$$(\forall x)(\int ((y = x) \& \varphi(y))\mathrm{d}y \rightarrow \bar{0}) \rightarrow (\int \varphi(y)\mathrm{d}y \rightarrow \bar{0})$$

or, equivalently,

$$(8.1) \quad \int \varphi(y)\mathrm{d}y \rightarrow (\exists x)(\int ((y = x) \& \varphi(y))\mathrm{d}y)$$

is *not* necessarily true in each probabilistic (countable) model; imagine $\|\varphi\|_{\mathbf{M}}$ crisp and $\|\varphi\|_{\mathbf{M}}[a_i] = 1$ for at least two elements $a_1 \neq a_2$ of positive measure: $\mu(a_1) \neq 0 \neq \mu(a_2)$. Then the formula (8.1) is not 1-true.

(3) Observe the formula

(8.2) $(\forall x)(\int(y = x)\mathrm{d}y \to \bar{0})$

This formula is 1-true in a weak probabilistic $\mathbf{M}$ iff for each $a \in M, \mu(a) = 0$ (the finitely additive measure given by the model). Thus μ is then not σ-additive and $\mathbf{M}$ is not a probabilistic model.

(4) The negation of the formula (8.2), i.e. the formula $(\exists x)(\int(y = x)\mathrm{d}y)$ is 1-true in a weak probabilistic model $\mathbf{M}$ iff there is an $a \in \mathbf{M}$ such that $\mu(a) = 1$. Take care to distinguish this from the fact that $\|(8.1)\|_{\mathbf{M}} < 1$ which means that for some $a \in M, \mu(a) > 0$.

(5) If $\mathbf{M}$ is a countably infinite probabilistic model then $\|\exists x)(\int(y = x)\mathrm{d}y \to 0\| = 1$ since $\inf_{a\in M}\mu(a) = 0$, thus $\sup_{a\in M}(1 - \mu(a)) = 1$. (Note that all $\mu(a)$ may be positive!)

(6) We could express more by introducing the triangle connective $\triangle$ (cf. 2.4.5) – but then the proof of completeness does not work any more. For example, $\|(\exists x)\triangle(\int(y = x)\mathrm{d}y \to 0)\|_{\mathbf{M}} = 1$ iff there is an a with $\mu(a) = 0$.

(7) Let us give an example of a weak probabilistic model which is not a probabilistic model. An *ultrafilter* on N is a system $\mathcal{U}$ of subsets of N such that $\mathcal{U} \neq \emptyset$, $\emptyset \notin \mathcal{U}$, ($A \subseteq B$ and $A \in \mathcal{U}$ implies $B \subseteq \mathcal{U}$) and ($A \in \mathcal{U}$ iff $(N - A) \notin \mathcal{U}$) for all $A, B \subseteq N$. An ultrafilter $\mathcal{U}$ is *non-principal* if all elements of $\mathcal{U}$ are infinite. (Existence follows from the set-theoretical axiom of choice.) Put $\mu(A) = 1$ if $A \in \mathcal{U}$, otherwise $\mu(A) = 0$. Observe that μ is a finitely additive (but not σ additive) probability on N (defined for all subsets of N). Let $\mathbf{M} = (N, (r_P)_P, (m_c)_c, \mu)$ be such that μ is our present two-valued probability and assume that each r_P takes only *finitely many* values (the range of r_P is a finite subset of [0,1]). Then for each formula φ, the range of $\|\varphi\|_{\mathbf{M},v}$ (with varying v) is also finite. Define, for a finitely valued function $f : M \to [0, 1]$, $\mathbf{I}f\mathrm{d}x = \alpha$ iff the set $\{n \in N | f(n) = \alpha\}$ is in $\mathcal{U}$. You may check that this is a weak probabilistic model.

(8) The reader with a more advanced knowledge of the metamathematics of arithmetic can easily show a theory T in the language extended by Δ whose models are isomorphic to the standard model $\mathbf{N}$ of QA with the probability μ such that $\mu(n) = \frac{1}{2^{n+1}}$ for each n (write an axiom saying that there are no elements of measure 0). T can be taken as finitely axiomatized. This shows that the set of all formulas of the extended language 1-true in all models of T is not axiomatizable (even not arithmetical).

(9) On the other hand, the *problem* remains of whether the set of all formulas of the original language of RPL$\forall\int$ in all (countable) probabilistic models satisfies Pavelka style completeness ($|\varphi|_T = \|\varphi\|_T$) with respect to all *probabilistic* models (with a possibly enriched set of logical axioms). No direct answer seems to be contained in the literature.

*

Remark 8.5.16 We may investigate the corresponding probabilistic modal logic with the modality $\int$ and axioms like

$$\int(\int \varphi) \equiv \int \varphi$$

(more generally $\int \psi \equiv \psi$ for ψ being a propositional combination of formulas beginning by $\int$ and truth constants),

$$\int(\varphi \to \psi) \to (\int \varphi \to \int \psi),$$

$$\int \neg\varphi \equiv \neg \int \varphi,$$

$$\int(\varphi\underline{\vee}\psi) \equiv [(\int \varphi \to \int(\varphi\&\psi)) \to \int \psi]$$

Remark 8.5.17 Since 1-tautologies of RPL$\forall \int$ not containing the quantifier $\int$ are exactly all 1-tautologies of Łukasiewicz predicate logic Ł$\forall$ it immediately follows from results of Chapter 6 (6.3.4) that 1-tautologies of RPL$\forall \int$ are not recursively axiomatizable; moreover the set of these formulas is Π_2-hard. *Problem.* Is the set in Π_2, i.e. is it Π_2-complete?

*

Let us turn to the calculus with the quantifier "probably" and *finite fuzzy models*. Then we get rid of problems with σ-additivity and have relative frequency as a prominent particular case (the quantifier "many").

Definition 8.5.18 The calculus RPL$\forall \int$-fin has the same language, formulas, axioms and deduction rules as RPL$\int \forall$. *Models* are structures

$$\langle M, (r_P)_P, m_c)_c, \mu\rangle$$

where M is a non-empty finite set, r_P and m_c are as usual, and μ is a probability on M, i.e. a function mapping of M into $[0,1]$ such that $\sum_{m \in M} \mu(m) = 1$.

Note that $\| \int \varphi \mathrm{d}x\|_{\mathbf{M},r} = \sum_{m \in M} \mu(m) \cdot Sat_{\mathbf{M},v}(\varphi, x)(m)$ (weighted sum of truth values of φ for x running over elements $m \in M$).

Remark 8.5.19 (1) Evidently the deductive system is sound, i.e. each provable formula is a 1-tautology. It was a *problem* whether RPL$\forall \int$-fin is recursively axiomatizable (it is not); but we shall at least show, generalizing Corollary 6.3.6, that the set of all tautologies of RPL$\forall \int$-fin is in Π_1.

(2) Observe that in a finite model as above, μ behaves as the interpretation of a unary fuzzy predicate. Then we modify the definition of the distance of two models $\mathbf{M}, \mathbf{M}'$ with the same domain 5.4.28 as follows:

Definition 8.5.20 Let $\mathbf{M}, \mathbf{M}'$ be two models of RPL$\forall\int$-fin with the same finite domain $M = \{m_1, \ldots, m_n\}$. Let the language in question have predicates $P, \ldots, Q$ and let μ, μ' be the probabilities of $\mathbf{M}, \mathbf{M}'$. We set

$$d(r_P, r_{P'}) = \sup\{|r_P(a) - r_{P'}(a)| \quad | \quad a \in M^{ar(P)}\}, \ldots$$

$$d(\mu, \mu') = m_P\{|\mu(a) - \mu(a)| \quad | \quad a \in M\},$$

$$d(\mathbf{M}, \mathbf{M}') = d(r_P, r_{P'}) + \ldots + d(r_Q, r_{Q'}) + d(\mu, \mu').$$

Lemma 8.5.21 There is a function $U(\varphi, \varepsilon, n)$ associating with each formula φ of RPL$\forall\int$-fin, each $\varepsilon > 0$ and each natural $n > 0$ a positive real number $U(\varphi, \varepsilon, n)$ such that if $\mathbf{M}, \mathbf{M}'$ are two models with the same domain M of cardinality n and $d(\mathbf{M}, \mathbf{M}') < U(\varphi, \varepsilon, n)$ then $| \, \|\varphi\|_{\mathbf{M},v} - \|\varphi\|_{\mathbf{M}',v}| < \varepsilon$.

Proof: Adapt the proof of 5.4.29: if φ does not contain $\int$ then $U(\varphi, \varepsilon, n) = \varepsilon/2^{\tau(\varphi)}$. The induction steps are as in 5.4.29, we only have to discuss $\int$. Let φ be $\int \psi \mathrm{d}x$ and let $C(\varepsilon) > 0$ be such that if $|x-x'|+|y-y'| < C(\varepsilon)$ then $|xy - x'y'| < \varepsilon$ (such $C(\varepsilon)$ exists since product is uniformly continuous on $[0, 1]^2$). Let $u(\varphi, \varepsilon, n) = U(\psi, C(\varepsilon/n), n)$; we may assume $U(\psi, C(\varepsilon/n), n) \leq C(\varepsilon/n)$. If $d(\mathbf{M}, \mathbf{M}') < U(\varphi, \varepsilon, n)$ and we put $Sat_{\mathbf{M},v}(\psi, x) = f_\psi, Sat_{\mathbf{M}',v}(\psi, x) = f'_\psi$ then, for each $m \in M$, $|f_\psi(m) - f'_\psi(n)| < C(\varepsilon/n)$, $|\mu(m) - \mu'(m)| < C(\varepsilon/n)$, thus $|f_\psi(m) \cdot \mu(m) - f'_\psi(m) \cdot \mu'(m)| < \varepsilon/n$ and $| \quad \|\int \psi \mathrm{d}x\|_{\mathbf{M},v} - \|\int \psi \mathrm{d}x\|_{\mathbf{M}',v}| = |\sum_m f_\psi(m) \cdot \mu(m) - \sum_m f'_\psi(m) \cdot \mu'(m)| \leq \sum_m |f_\psi(m) \cdot \mu(m) - f'_\psi(m) \cdot \mu'(m)| \leq n \cdot (\varepsilon/n) = \varepsilon$. □

Theorem 8.5.22 The set of all 1-tautologies of RPL$\forall\int$-fin (i.e. all formulas 1-true in all finite probabilistic models) is Π_1.

Proof: Using the preceding lemma we may easily show that if $\varphi, \mathbf{M}, v$ are such that $\|\varphi\|_{\mathbf{M},v} < 1$ ($\mathbf{M}$ of cardinality n) then there is a $\mathbf{M}'$ rational-valued with the same domain such that $\|\psi\|_{\mathbf{M}',v} < 1$. (Cf. the proof of 5.4.29.) □

Definition 8.5.23 The calculus RPL$\forall Many$ has the same language as RPL$\forall\int$-fin but models are just finite models of the underlying predicate language $\mathbf{M} = \langle M, (r_P)_P, (m_c)_c \rangle$. Each such model determines the corresponding probability with equiprobability elements $\mu(m) = \frac{1}{n}$ where the cardinality of M is n, thus for $A \subseteq M$, $\mu(A) = card(A)/n$ (relative frequence). Thus

$$\|\int \varphi \mathrm{d}x\|_{\mathbf{M},v} = \textstyle\sum_{m \in M} \|\varphi\|_{\mathbf{M},v}.$$

$\int \varphi \mathrm{d}x$ is read "many objects satisfy φ".

Theorem 8.5.24 RPL$\forall Many$ has the same 1-tautologies as RPL$\forall \int$-fin.

Proof: We have to show that if $\|\varphi\|_{\mathbf{M},v} < 1$ for some finite probabilistic $\mathbf{M}$ then there is a model $\mathbf{M}'$ of RPL$\forall Many$ (with equiprobable elements) and a v' such that $\|\varphi\|_{\mathbf{M}',v'} < 1$. By the last theorem we may assume $\mathbf{M}$ to be rational valued; let k be the least common denominator of all the finitely many values (of interpretations of predicates occuring in φ). To produce M' replace each element $m_j \in M$ with $\mu(m) = \frac{i}{k}$ by i elements $m_1^j, \ldots, m_i^j$. This produces a set M'' of cardinality k and a mapping f of M' to M with $f(m_1^j) = \ldots = f(m_i^j) = m_j$. Now define, for each tuple $(m', \ldots, m'')$ of elements of M', $r'_P(m', \ldots, m'') = r_P(f(m'), \ldots, f(m''))$, i.e. induce the model $\mathbf{M}'$ by $\mathbf{M}$ and f. Verify by induction that for each φ and each $\mathbf{M}'$-evaluation v, $\|\psi\|_{\mathbf{M}',v'} = \|\psi\|_{\mathbf{M},v}$ where $v(y) = f(v'(y))$ for each variable y. This completes the proof. $\square$

*

Now we only add some remarks.

Remark 8.5.25 (1) We could now turn to the binary fuzzy quantifier $\sqsupset$ (many φ's are ψ's). There are several competing definitions in the literature, notably [210], [199]; unfortunately these papers contain few results on (non)-axiomatizability or arithmetical complexity. To investigate various candidates (and
classes of candidates) for the binary fuzzy modality "many" – i.e. fuzzy multitudinal quantifiers - from a strictly logical point of view, remains a very promising *problem*.

(2) There are other classes of fuzzy quantifiers (and modalities) deserving logical analysis, as *cardinality quantifiers* (for finite models, see [39]), *temporal quantifiers* (cf. [200]) and others. Let us mention also the following: our interpretation of $\forall$ and $\exists$ is based on inf and sup, i.e. infinite $\wedge$ and $\vee$. It is also possible to investigate, at least in Ł$\forall$, quantifiers based on infinite analysis of & and $\veebar$ (in $\Pi\forall$ we do not have a dual for $\odot$). See [199] for some hints.

(3) In parallel to fuzzy probabilistic logic one may develop fuzzy possibilistic and fuzzy belief function logic. For comparative fuzzy possibilistic logic see [84, 83]; to construct a fuzzy possibilistic (belief function) logic analogous to our RPL$\forall \int$ seems to be an open problem.

Example 8.5.26 We shall comment on fuzzy quantifiers in the GUHA method of hypothesis formation. This method, the main principle of which was formulated in 1965, is a method of explorary data analysis or, to use a fashionable term, *data mining*. The principle consists in generating automatically all hypotheses that are interesting on the basis of given data or, more concretely, all statements of a certain formal language expressing interesting associations, dependences etc. and true on the data. We shall be very sketchy; for details see [85], [92] and references therein.

The *data* form a rectangular matrix whose rows correspond to objects and columns to variates. The element in the i-th row and j-th column is the value of the j-th variate for the i-th object. Assume the domain of the j-th variate is D_j and let M be the set of objects; then the data may be represented as a structure

$$(M, D_1, \dots, D_n, f_1, \dots, f_n)$$

where $f_j : M \rightarrow D_j$ associates with each object x the value of the j-th variate for x. (See 7.1.14.) We introduce the names X_j of variates as function symbols.

With each D_j we associate a finite number of subsets $a_{j1}, \dots, a_{jk_j}$ (crisp or fuzzy) and introduce unary predicates $A_{j1}, \dots, A_{jk_j}$ to name them. (Think e.g. of X_j to name temperature and A_{j1}, A_{j2}, A_{j3} to name the sets of temperatures ≤ 36.5, between 36.5 and 37.5 and above 37.5 (crisp) – or the fuzzy sets of low, medium and high temperatures. The latter resembles Zadeh's linguistic variables.) We have just one object variable x. Atomic formulas have the form $A_{jk}(X_j(x))$ (read: for x, the value of X_j is A_{jk}) and are called *literals*. Conjunctions of finitely many literals having pairwise different variates are called *elementary conjunctions*. *Hypotheses* (or *relevant sentences*) are some formulas of the form $(qx)(\varphi(x), \psi(x))$ where φ, ψ are elementary conjunctions and q is a quantifier (modality).

The *input* of a GUHA procedure consists of (1) the data matrix

$$(M, (D_j)_j, \dots (f_j)_j)$$

with domains indicated (D_j may be reals, integers, a finite set of integers, $\{0, 1\}$ etc.), (2) for each variate X_j, the meanings $a_{j1}, \dots$ of the predicates for X_{jk}, (3) furthermore parameters determining the semantics of the quantifier $\sim$, syntactic restrictions for φ, ψ (variates that may occur, maximum number of conjuncts, etc). The *output* is the list of all sentences $q(\varphi, \psi)$ satisfying the syntactic restrictions, true (or *sufficiently* true) in the data and strong – not immediately following from preceding ones via some simple deduction rules. Typically, q is a multitudinal quantifier or an associational quantifier ($(qx)(\varphi, \psi)$ says "many x having φ have ψ" and "φ, ψ are associated" respectively).

The case of crisp meaning of predicates and fuzzy quantifiers was repeatedly implemented and used; for examples of quantifiers see 8.4.24, 8.1.16. (Note that in the original literature one speaks of a crisp quantifier given by a statistic; $\|(qx)(\varphi, \psi)\| = 1$ iff the value of the corresponding statistic is larger than or equal to a threshold. Alternatively, one may speak of a fuzzy quantifier and identify $\|(qx)(\varphi, \psi)\|$ with the value of the statistic, cf. 8.1.6.)

The case of fuzzy predicates in GUHA (i.e. work with linguistic variables) remains to be elaborated (and implemented).

CHAPTER NINE

MISCELLANEA

This chapter is devoted to three mutually unrelated topics showing three directions of further development of fuzzy logic. (Needless to say, several other directions are possible.) In Section 1 we present a rather strong fuzzy logic, based on the work of Takeuti and Titani, and containing Łukasiewicz, Gödel and product predicate logics Ł$\forall$, G$\forall$, $\Pi\forall$ as its sublogics. We show completeness with respect to a non-finitary notion of provability. In Section 2 we show how to develop fuzzy logic that is not necessarily truth-functional. This section is based on work by Pavelka. Section 3 is based on recent work by Hájek, Paris and Shepherdson and discusses the Liar paradox in the frame of fuzzy logic. The final Section 4 contains some conclusions.

9.1. TAKEUTI-TITANI FUZZY LOGIC

We have developed three important fuzzy predicate logics Ł$\forall$, G$\forall$, $\Pi\forall$, based on an analysis of continuous t-norms as truth functions of the conjunction and their residua as truth functions of the implication. The choice of a continuous t-norm has automatically given the corresponding choice of the truth function of negation. This was a clean and fruithful approach; nevertheless, the reader may feel not fully satisfied since in the practice of informal fuzzy logic one sometimes likes to combine e.g. Gödel and product conjunction with Łukasiewicz negation (in particular when one wants to have a duality between t-norms and t-conorms). Logical research in this direction appears to be possible; we demonstrate this by elaborating a logic having three conjunctions and three implications and extending each of Ł$\forall$, G$\forall$, $\Pi\forall$. The presentation is heavily dependent on the paper [196] of Takeuti and Titani and we shall call the logic TT$\forall$ (Takeuti-Titani predicate logic). But we have to stress the fact that our system is far from identical with that of [196]. Nevertheless, the basic idea and the method of proof of completeness are taken from the work of Takeuti and Titani, which justifies our name. It should also be clear from the beginning that since TT$\forall$ extends Ł$\forall$ we cannot expect recursive axiomatization: TT$\forall$ will have an infinitary deduction rule (with countably many premisses).

Definition 9.1.1 The *logical symbols* of TT$\forall$ are as follows:

- connectives – three conjunctions $\&, \wedge, \odot$ (called Łukasiewicz, Gödel and product conjunction respectively), three implications $\rightarrow_{\text{Ł}}, \rightarrow_G, \rightarrow_\Pi$ (Łukasiewicz, Gödel and Goguen implication),
- truth constants $\bar{r}$ for each rational $r \in [0,1]$;
- quantifiers $\forall, \exists$;
- object variables.

The connectives have their usual semantics over $[0,1]$.

A *language* $\mathcal{I}$ is given by predicates (with arities) and object constants. The notion of a structure for $\mathcal{I}$ is as in BL$\forall$ (cf. 5.1.1); but we restrict ourselves to $[0,1]$-structures. Thus a *structure* for TT$\forall$ has the form $\mathbf{M} = \langle M, (r_P)_P, (m_c)_c \rangle$ where $M \neq \emptyset$, $r_P : M^{ar(P)} \rightarrow [0,1]$ for each predicate P and $m_c \in M$ for each constant c.

Atomic formulas have the form $P(t_1, \ldots, t_n)$ where P is a predicate of arity n and $t_1, \ldots, t_n$ are terms or have the form $\bar{r}$ (truth constant). If φ, ψ are formulas then $\varphi \& \psi$, $\varphi \wedge \psi$, $\varphi \odot \psi$, $\varphi \rightarrow_{\text{Ł}} \psi$, $\varphi \rightarrow_G \psi$, $\varphi \rightarrow_\Pi \psi$ are formulas; if φ is a formula and x a variable then $(\forall x)\varphi$, $(\exists x)\varphi$ are formulas. We introduce the following abbreviatons:
$\neg\varphi$ is $\varphi \rightarrow_{\text{Ł}} \bar{0}$,
$\dot{\neg}\varphi$ is $\varphi \rightarrow_G \bar{0}$
$\varphi \underline{\vee} \psi$ is $\neg\varphi \rightarrow_{\text{Ł}} \psi$
$\varphi \vee \psi$ is $(\varphi \rightarrow_{\text{Ł}} \psi) \rightarrow_{\text{Ł}} \psi$,
$\triangle\varphi$ is $\dot{\neg}\neg\varphi$,
$\varphi \equiv_{\text{Ł}} \psi$ is $(\varphi \rightarrow_{\text{Ł}} \psi)\&(\psi \rightarrow_{\text{Ł}} \varphi)$,
similarly for $\equiv_G$, $\equiv_\Pi$.

Convention: $\rightarrow$ will denote any of $\rightarrow_{\text{Ł}}, \rightarrow_G, \rightarrow_\Pi$ (but the same in all occurences in a formula); similarly for $\equiv$.

The definition of the *truth value* $\|\varphi\|_{\mathbf{M},v}$ (where v is an M-evaluation of object variables) is the obvious modification of 5.1.3; for the reader's convenience, we display it in full.

$$\|P(t_1, \ldots, t_n)\|_{\mathbf{M},v} = r_P(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v})$$

$$\|\bar{r}\|_{\mathbf{M},v} = r,$$

$$\|\varphi \& \psi\|_{\mathbf{M},v} = \max(0, \|\varphi\|_{\mathbf{M},v} + \|\psi\|_{\mathbf{M},v} - 1),$$

$$\|\varphi \wedge \psi\|_{\mathbf{M},v} = \min(\|\varphi\|_{\mathbf{M},v}, \|\psi\|_{\mathbf{M},v}),$$

$$\|\varphi \odot \psi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M},v} \cdot \|\psi\|_{\mathbf{M},v},$$

$$\|\varphi \rightarrow \psi\|_{\mathbf{M},v} = 1 \quad \text{iff} \quad \|\varphi\|_{\mathbf{M},v} \leq \|\psi\|_{\mathbf{M},v}; \quad \text{otherwise}$$

$$\|\varphi \rightarrow_{\text{Ł}} \psi\|_{\mathbf{M},v} = 1 - \|\varphi\|_{\mathbf{M},v} + \|\psi\|_{\mathbf{M},v},$$

$$\|\varphi \to_G \psi\|_{\mathbf{M},v} = \|\psi\|_{\mathbf{M},v},$$
$$\|\varphi \to_\Pi \psi\|_{\mathbf{M},v} = \|\psi\|_{\mathbf{M},v} / \|\varphi\|_{\mathbf{M},v};$$
$$\|(\forall x)\varphi\|_{\mathbf{M},v} = \inf\{\|\varphi\|_{\mathbf{M},v'} | v \equiv_x v'\};$$
$$\|(\exists x)\varphi\|_{\mathbf{M},v} = \sup\{\|\varphi\|_{\mathbf{M},v'} | v \equiv_x v'\}.$$

Remark 9.1.2 One easily checks that this gives the following rules for the defined connectives:

$$\|\neg\varphi\|_{\mathbf{M},v} = 1 - \|\varphi\|_{\mathbf{M},v};$$
$$\|\dot{\neg}\varphi\|_{\mathbf{M},v} = 1 \quad \text{for} \quad \|\varphi\|_{\mathbf{M},v} = 0,$$
$$\|\dot{\neg}\varphi\|_{\mathbf{M},v} = 0 \quad \text{otherwise}$$
$$\| \triangle \varphi\|_{\mathbf{M},v} = 1 \quad \text{for} \|\varphi\|_{\mathbf{M},v} = 1,$$
$$\| \triangle \varphi\|_{\mathbf{M},v} = 0 \quad \text{otherwise, (cf.2.4.4),}$$
$$\|\varphi \underline{\vee} \psi\|_{\mathbf{M},v} = \min(1, \|\varphi\|_{\mathbf{M},v} + \|\psi\|_{\mathbf{M},v})$$
$$\|\varphi \vee \psi\|_{\mathbf{M},v} = \max(\|\varphi\|_{\mathbf{M},v}, \|\psi\|_{\mathbf{M},v}),$$
$$\|\varphi \equiv \psi\|_{\mathbf{M},v} = \min(\|\varphi \to \psi\|_{\mathbf{M},v}, \|\psi \to \varphi\|_{\mathbf{M},v})$$

($\equiv$ being the equivalence corresponding to $\to$).

Definition 9.1.3 *Logical axioms* of TT$\forall$ are as follows:

– axioms of BL for each of the pairs $(\&, \to_Ł), (\wedge, \to_G), (\odot, \to_\Pi)$ (e. g. (A2) is $(\varphi\&\psi) \to_Ł \varphi$, $(\varphi \wedge \psi) \to_G \varphi$, $(\varphi \odot \psi) \to_\Pi \psi$, etc.)
– additional axioms of Ł, G, Π:

$$\neg\neg\varphi \to_Ł \varphi,$$

$$\varphi \to_G (\varphi \wedge \varphi),$$
$$\dot{\neg}\dot{\neg}\chi \to_\Pi (((\varphi \odot \chi) \to_\Pi (\psi \odot \chi)) \to_\Pi (\varphi \to_\Pi \psi)),$$
$$\varphi \wedge \dot{\neg}\varphi \to \bar{0}$$

– axioms for connectives from different groups:

$$\dot{\neg}\varphi \equiv (\varphi \to_\Pi \bar{0})$$

$$(\varphi \wedge \psi) \equiv (\varphi\&(\varphi \to_Ł \psi)), \quad (\varphi \wedge \psi) \equiv (\varphi \odot (\varphi \to_\Pi \psi))$$
$$(\varphi \vee \psi) \equiv (((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi))$$
$$(\varphi\&\psi) \to (\varphi \odot \psi), \quad (\varphi \odot \psi) \to (\varphi \wedge \psi)$$
$$(\varphi \to_G \psi) \to (\varphi \to_\Pi \psi), \quad (\varphi \to_\Pi \psi) \to (\varphi \to_Ł \psi)$$

– bookkeeping for truth constants:

$$(\bar{r}\&\bar{s}) \equiv \overline{r *_{Ł} s}, \quad (\bar{r} \rightarrow_{Ł} \bar{s}) \equiv \overline{r \Rightarrow_{Ł} s}$$

and similarly for $(\wedge, \rightarrow_G, *_G, \Rightarrow_G)$, $(\odot, \rightarrow_\Pi, *_\Pi, \Rightarrow_\Pi)$ (e.g. $(\bar{r} \odot \bar{s}) \equiv \overline{r \cdot s}$, $(\bar{r} \rightarrow_{Ł} \bar{s}) \equiv \overline{\min(1, 1 - r + s)}$ etc.)

– axioms for $\triangle$ (cf. 2.4.4), i.e.

$$\triangle\varphi \vee \neg \triangle \varphi,$$

$$\triangle(\varphi \vee \psi) \rightarrow (\triangle\varphi \vee \triangle\psi),$$

$$\triangle\varphi \rightarrow \varphi,$$

$$\triangle\varphi \rightarrow \triangle \triangle \varphi,$$

$$\triangle(\varphi \rightarrow \psi) \rightarrow (\triangle\varphi \rightarrow \triangle\psi),$$

$$\triangle(\varphi \rightarrow_{Ł} \psi) \rightarrow \triangle(\varphi \rightarrow_G \psi),$$

– axioms for quantifiers (∀1), (∀2), (∀3), (∃1), (∃2) (cf.5.1.7).

Deduction rules are modus ponens (for any our three implications), generalization, the $\triangle$-necessitation: from φ derive $\triangle\varphi$ and finally the following infinitary rule:

$$\frac{\{\gamma \vee (\varphi \rightarrow (\psi \underline{\vee} \bar{r})) | r > 0\}}{\gamma \vee (\varphi \rightarrow \psi)}$$

This means: if for each rational positive $r \in [0, 1]$ the formula $\gamma \vee (\varphi \rightarrow (\psi \vee \bar{r}))$ is provable then $\gamma \vee (\varphi \rightarrow \psi)$ is provable. In particular, taking γ to be $\bar{0}$ and φ to be $\bar{1}$, we get: if for each positive r, $\psi \underline{\vee} \bar{r}$ is provable, then ψ is provable.

We could define proofs as some possibly infinite countable trees; but it may be simpler not to define proofs at all and only define what provability means.

A *theory* over TT∀ is a set of closed formulas. The set $Cn_{tt}(T)$ of all formulas *provable* in T is the smallest set T' containing T as a subset, containing all axioms of TT∀ and closed under all deduction rules (i.e. such that if all premisses of the rule are in T' then so is the conclusion of the rule). A theory T is *consistent* if $\bar{0} \notin Cn_{tt}(T)$. $T \vdash \varphi$ means $\varphi \in Cn_{tt}(T)$.

Lemma 9.1.4 (*Soundness*). The logic TT∀ is sound, i.e. if **M** is a model of T and $T \vdash \varphi$ then $\|\varphi\|_{\mathbf{M},v} = 1$ for each M-valuation v.

Proof: It follows from our investigations of BL∀, Ł∀, G∀, Π∀ and of the connective $\triangle$ that all axioms except [those for connectives from different groups and the last $\triangle$-axiom] are 1-tautologies and all rules except the infinite rule are truth-preserving (if all assumptions are 1-true in $\mathbf{M}$ then so is the conclusion). To verify the remaining axioms is an easy exercise, e.g. for the conjunctions we have

$$\max(0, x+y-1) \leq x \cdot y \leq \min(x, y)$$

which gives 1-truth of $(\varphi \& \psi) \to (\varphi \odot \psi) \to (\varphi \wedge \psi)$; knowing that the negation of the product logic is the same as that of Gödel logic gives 1-truth of $\dot{\neg}\varphi \equiv (\varphi \to_\Pi \bar{0})$ etc.

Soundness of the infinite rule follows immediately from the fact that (even if Gödel and Goguen implication are not continuous) all three implications are continuous from above in the second argument i.e. $x \Rightarrow y = \lim_{\varepsilon>0, \varepsilon\to 0} x \Rightarrow (y+\epsilon)$. □

Lemma 9.1.5 Let i, j run over Ł, G, Π.

(1) Then

$$\vdash \triangle(\varphi \to_i \psi) \equiv \triangle(\varphi \to_j \psi)$$

$$\vdash \triangle(\varphi \equiv_i \psi) \equiv \triangle(\varphi \equiv_j \psi)$$

(2) The following rule is a derived rule of TT∀:

$$\frac{\varphi \to_i \psi}{\varphi \to_j \psi},$$

i.e. if $T \vdash (\varphi \to_i \psi)$ then $T \vdash (\varphi \to_j \psi)$.

(3) Similarly,

$$\frac{\gamma \vee (\varphi \to_i \psi)}{\gamma \vee (\varphi \to_j \psi)},$$

(i.e. if $T \vdash \gamma \vee (\varphi \to_i \psi)$ then $T \vdash \gamma \vee (\varphi \to_j \psi)$).

Proof: (1) For (i, j), being (G, Π) or $(\Pi, Ł)$ we have
$\vdash (\varphi \to_i \psi) \to (\varphi \to_j \psi)$, thus, by $\triangle$-necessitation,
$\vdash \triangle[(\varphi \to_i \psi) \to (\varphi \to_j \psi)]$, thus, by $\triangle$-axioms
$\vdash \triangle(\varphi \to_i \psi) \to \triangle(\varphi \to_j \psi)$.
For (i, j) being $(Ł, G)$ see the last $\triangle$-axiom.

(2) If $T \vdash \varphi \to_i \psi$ then $T \vdash \triangle(\varphi \to_i \psi)$, thus $T \vdash \triangle(\varphi \to_j \psi)$ by (1) and hence $T \vdash (\varphi \to_j \psi)$ by a $\triangle$-axiom.

(3) is proved similarly using the provability of $\triangle(\gamma \vee (\varphi \to_i \psi)) \to (\triangle\gamma \vee \triangle(\varphi \to_i \psi))$. □

Corollary 9.1.6 If $T \vdash \varphi \equiv_i \psi$ then $T \vdash \varphi \equiv_j \psi$. $T \vdash \neg\varphi$ iff $T \vdash \dot{\neg}\varphi$.

Remark 9.1.7 (1) These results are in fact not surprising assuming completeness (to be proved): provability of $\varphi \rightarrow_i \psi$ means that $\varphi \rightarrow_i \psi$ is true in all models, i.e. $\|\varphi\| \leq \|\psi\|$, whatever i is. But be careful: this does not mean that one may arbitrarily replace $\rightarrow_i$ by $\rightarrow_j$ in formulas; we only say that $T \vdash \varphi \rightarrow_i \psi$ is the same as $T \vdash \varphi \rightarrow_j \psi$. Even this is very useful.

(2) We have chosen the axioms of TT$\forall$ in a form suitable for remembering and there can be redundant axioms. The reader may care to try to identify redundant axioms as an exercise.

(3) The following lemma shows in which sense we can replace a subformula by an equivalent one.

Lemma 9.1.8 Let φ, ψ be formulas with the same free variables and let $\Phi(\psi)$ be formula resulting from a formula $\Phi(\varphi)$ by replacing the subformula φ by ψ.

(1) $\vdash \Delta(\varphi \equiv \psi) \rightarrow (\Phi(\varphi) \equiv \Phi(\psi))$

(2) Thus if $T \vdash \varphi \equiv \psi$ then $T \vdash \Phi(\varphi) \equiv \Phi(\psi)$.

Proof: (1) Recall that $\vdash \Delta(\varphi \equiv_i \psi) \rightarrow \Delta(\varphi \equiv_j \psi)$ for each i, j. The assertion is trivial if $\Phi(\varphi)$ is φ or $\Phi(\varphi)$ does not contain φ as a subformula. To prove the induction step for a connective, say $\rightarrow_Ł$, we assume

$$\vdash \Delta(\varphi \equiv \psi) \rightarrow (\Phi_1(\varphi) \equiv \Phi_1(\psi)), \qquad \vdash \Delta(\varphi \equiv \psi) \rightarrow (\Phi_2(\varphi) \equiv \Phi_2(\psi))$$

and by the above we may understand $\equiv$ as $\equiv_Ł$, thus we have

$$\vdash (\Delta(\varphi \equiv_Ł \psi) \& \Delta\, (\varphi \equiv_Ł \psi)) \rightarrow$$

$$\rightarrow [(\Phi_1(\varphi) \equiv_Ł \Phi_1(\psi)) \& (\Phi_2(\varphi) \equiv_Ł \Phi_2(\psi))],$$

$$\vdash [\ldots] \rightarrow ((\Phi_1(\varphi) \rightarrow \Phi_2(\varphi)) \equiv_Ł (\Phi_1(\psi) \rightarrow_Ł \Phi_2(\psi))),$$

$$\vdash \Delta\alpha \rightarrow (\Delta\alpha \& \Delta\, \alpha) \qquad \text{(see 2.4.11), thus}$$

$$\vdash \Delta(\varphi \equiv_Ł \psi) \rightarrow ((\Phi_1 \rightarrow_Ł \Phi_2)(\varphi) \equiv_Ł (\Phi_1 \rightarrow_Ł \Phi_2)(\psi)).$$

Similarly for other connectives. The induction step for quantifiers is easy. □

Lemma 9.1.9 Let T be a theory over TT$\forall$ and c a constant not occuring in the axioms of T; let φ be a formula possibly containing c. If $T \vdash \varphi$ then for all but finitely many variables x, $T \vdash \varphi(c/x)$ (where $\varphi(c/x)$ is the result of substituting x for c in φ).

Proof: Let $Cn'(T)$ be the set of all formulas α such that $T \vdash \alpha$ and for all but finitely many variables x, $T \vdash \alpha(c/x)$. One easily shows that $Cn'(T)$ contains all axioms of TT, all axioms of T, and is closed under deduction rules. (For example check modus ponens: if $\alpha \in Cn'(T)$ call a variable x *exceptional* if $T \not\vdash \alpha(c/x)$. Clearly if $\alpha \in Cn'(T)$ and $(\alpha \to \beta) \in Cn'(T)$ then each variable x exceptional for β is either exceptional for α or for $(\alpha \to \beta)$; thus the set of variables exceptional for β is finite and $\beta \in Cn'(T)$.) Thus $Cn'(T) = Cn_{tt}(T)$ (since $Cn_{tt}(T)$ is the smallest set containing logical axioms, axioms of T and closed under deduction rules). This completes the proof. □

*

Theorem 9.1.10 Completeness. For each theory T over TT$\forall$ and each formula φ, $T \vdash \varphi$ iff $\|\varphi\|_{\mathbf{M},v} = 1$ for all models of T and all valuations v.

9.1.11 The proof is elaborated in the rest of this section. For this purpose let T be a fixed theory and φ a formula such that $T \not\vdash \varphi$. Furthermore let c_n, n natural, be constants not occuring in T and let $\alpha_0, \alpha_1, \ldots$ be a sequence containing all (closed) formulas of the enriched language. Remember our proof of Lemma 5.2.7, crucial for the proof of completeness of BL$\forall$: in each step we made some formula(s) provable in a future theory $\hat{T} \supseteq T$ and guaranteed some formula(s) to be unprovable in $\hat{T}$, among them the formula φ. The idea of Takeuti and Titani is that the second half is sufficient: we shall construct, in countably many steps, a set Φ of formulas such that for any finite subset $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Phi$, the disjunction $\gamma_1 \vee \ldots \vee \gamma_n$ is unprovable in T. In particular, for each closed formula γ we may construct the real $\inf\{r \in [0,1] | (\bar{r} \to \gamma) \in \Phi\}$; this will be the truth degree $\|\gamma\|_{\mathbf{M}}$ in the model we shall construct from our Φ.

First, since $T \not\vdash \varphi$ we may take a $r_{-1} > 0$ such that $T \not\vdash \varphi \underline{\vee} \overline{r_{-1}}$ (due to the infinitary deduction rule). Put $\Phi_0 = \{\varphi, \varphi \underline{\vee} \overline{r_{-1}}\}$. Clearly $T \not\vdash \bigvee \Phi_0$ (for a finite set Γ of formulas $\bigvee \Gamma$ is the disjunction of the members of Γ).

Now let Φ_k have been constructed such that $T \not\vdash \bigvee \Phi_k$. If $T \vdash \bigvee \Phi_k \vee \alpha_k$ then let $\Phi_{k+1} = \Phi_k$. Otherwise there is an $r_k > 0$ such that $T \not\vdash \bigvee \Phi_k \vee (\alpha_k \underline{\vee} \overline{r_k})$; let $\Phi'_{k+1} = \Phi_k \cup \{\alpha_k, \alpha_k \underline{\vee} \overline{r_k}\}$. Clearly, $T \not\vdash \bigvee \Phi'_{k+1}$.
If α_k has the form $(\forall x)\beta(x)$ take a constant c from the list of new constants such that c does not occur in Φ'_{k+1}; by 9.1.9, $T \not\vdash \bigvee \Phi'_{k+1} \vee \beta(c)$. (For $\gamma = \bigvee \Phi'_{k+1}$, if $T \vdash \gamma \vee \beta(c)$ then $T \vdash \gamma \vee \beta(y)$ for each non-exceptional y, thus $T \vdash (\forall y)(\gamma \vee \beta(y))$ and $T \vdash \gamma \vee (\forall y)\beta(y)$ by ($\forall$3); now for a suitable non-exceptional y, $T \vdash (\forall x)\beta(x) \equiv (\forall y)\beta(y)$ and $T \vdash \gamma \vee (\forall x)\beta(x)$, a contradiction.) Let Φ_{k+1} be $\Phi'_{k+1} \cup \{\beta(c)\}$.

If α_k has the form $\beta \to \gamma$ (for one of our three implications) then for a suitable $r'_k > 0, T \not\vdash \bigvee \Phi'_{k+1} \vee (\beta \to (\gamma \underline{\vee} \overline{r'_k}))$ (due to the infinitary rule); let $\Phi_{k+1} = \Phi'_{k+1} \cup \{\beta \to (\gamma \underline{\vee}\, \overline{r'_k})\}$. Again, $T \not\vdash \bigvee \Phi_{k+1}$.

Finally, put $\Phi = \bigcup_k \Phi_k$. In the next series of lemmas we shall prove several properties of Φ.

Lemma 9.1.12 $(\beta \vee \gamma) \in \Phi$ iff $\beta \in \Phi$ and $\gamma \in \Phi$, for each β, γ.

Proof: Let $\beta = \alpha_k, \gamma = \alpha_l, (\beta \vee \gamma) = \alpha_m, n > \max(k, l, m)$. If $(\beta \vee \gamma) \in \Phi$ then $(\beta \vee \gamma) \in \Phi_n, T \not\vdash \bigvee \Phi_n, T \not\vdash \bigvee \Phi_n \vee (\beta \vee \gamma)$, hence $T \not\vdash \bigvee \Phi_k \vee \beta$ and $T \not\vdash \bigvee \Phi_l \vee \gamma$. Therefore $\beta \in \Phi_{k+1}$ and $\gamma \in \Phi_{l+1}$.

Conversely, if $\beta \in \Phi$ and $\gamma \in \Phi$ then $\beta \in \Phi_{k+1}, \gamma \in \Phi_{l+1}$ thus $\beta, \gamma \in \Phi_n$ and $T \not\vdash \bigvee \Phi_n$. Hence $T \not\vdash \bigvee \Phi_m \vee (\beta \vee \gamma)$ and $(\beta \vee \gamma) \in \Phi_{m+1}$. □

Lemma 9.1.13 $\beta \notin \Phi$ iff there exists $\gamma \in \Phi$ such that $T \vdash \beta \vee \gamma$.

Proof: If $\beta \notin \Phi$ then $T \vdash \bigvee \Phi_k \vee \beta$; take $\gamma = \bigvee \Phi_k$. On the other hand, if $\beta \in \Phi$ and $\gamma \in \Phi$ then $(\beta \vee \gamma) \in \Phi$ by the preceding lemma and hence $T \not\vdash (\beta \vee \gamma)$. □

Lemma 9.1.14 If $T \vdash \gamma \vee (\alpha \to \beta), \gamma, \beta \in \Phi$ then $\alpha \in \Phi$. (Here again $\to$ stands for any of $\to_{Ł}, \to_G, \to_\Pi$.)

Proof: Assume $T \vdash \gamma \vee (\alpha \to \beta), \gamma, \beta \in \Phi$ and $\alpha \notin \Phi$. Then by the last lemma there is an $\alpha' \in \Phi$ such that $T \vdash \alpha \vee \alpha'$.
Now $T \vdash \alpha \to (\gamma \vee \beta)$, thus $T \vdash \alpha \to (\alpha' \vee \gamma \vee \beta)$; also $T \vdash \alpha' \to (\alpha' \vee \gamma \vee \beta)$, hence $T \vdash (\alpha \vee \alpha') \to (\alpha' \vee \gamma \vee \beta)$ and $T \vdash \alpha' \vee \gamma \vee \beta$. But this contradicts the fact that $\alpha', \gamma, \beta \in \Phi$. □

Lemma 9.1.15 The following are mutually equivalent:

$$(\beta \wedge \gamma) \in \Phi, (\beta \& \gamma) \in \Phi, (\beta \odot \gamma) \in \Phi, [\beta \in \Phi \text{ or } \gamma \in \Phi].$$

Proof: If $\beta \in \Phi$ then $(\beta \& \gamma) \in \Phi$ follows from $T \vdash (\beta \& \gamma) \to_{Ł} \beta$ (i.e. $T \vdash \bar{1} \vee [(\beta \& \gamma) \to_{Ł} \beta]$) by the last lemma. Similarly for $\gamma \in \Phi$ and other conjunctions.

Conversely, if $\beta \notin \Phi$ and $\gamma \notin \Phi$ then for some $\beta', \gamma' \in \Phi$ we have $T \vdash \beta \vee \beta', T \vdash \gamma \vee \gamma'$, thus $T \vdash (\beta \vee \beta') \& (\gamma \vee \gamma')$, hence

$$T \vdash (\beta \& \gamma) \vee (\beta \& \gamma') \vee (\beta' \& \gamma) \vee (\beta' \& \gamma')$$

(cf. 2.2.23 (30)). Thus the last formula is not in Φ; since $(\beta\&\gamma'), (\beta'\&\gamma)$, $(\beta'\&\gamma') \in \Phi$, we conclude $(\beta\&\gamma) \in \Phi$.

The same proof works for $\wedge, \odot$ too. □

Lemma 9.1.16 If $(\alpha \to \beta) \in \Phi$ and $(\beta \to \gamma) \in \Phi$ then $(\alpha \to \gamma) \in \Phi$.

Proof: Since $T \vdash (\alpha \to \beta) \vee (\beta \to \alpha)$ we get

$$T \vdash (\alpha \to \gamma) \to [(\alpha \to \beta) \vee (\beta \to \gamma)].$$

Since $(\alpha \to \beta) \vee (\beta \to \gamma) \in \Phi$ by 9.1.12, we get $(\alpha \to \gamma)$ by 9.1.14. □

Lemma 9.1.17 For i, j ranging over Ł, G, Π,

$$(\beta \to_i \gamma) \in \Phi \qquad \text{iff} \qquad (\beta \to_j \gamma) \in \Phi.$$

Proof: If $(\beta \to_i \gamma) \notin \Phi$ then there is an $\alpha \in \Phi$ such that $T \vdash \alpha \vee (\beta \to_i \gamma)$; thus by 9.1.5 (3), $T \vdash \alpha \vee (\beta \to_j \gamma)$ and $(\beta \to_j \gamma) \notin \Phi$. □

Lemma 9.1.18 If $(\alpha \to \alpha') \notin \Phi$ and $(\beta \to \beta') \notin \Phi$ then

$$(\alpha\&\beta) \to (\alpha'\&\beta') \notin \Phi \tag{i}$$

(and the same for $\wedge, \odot$),

$$(\alpha' \to \beta) \to (\alpha \to \beta') \notin \Phi. \tag{ii}$$

Proof: (i) If $(\alpha \to \alpha'), (\beta \to \beta') \notin \Phi$ then $(\alpha \to \alpha')\&(\beta \to \beta') \notin \Phi$ by 9.1.15; since

$$T \vdash [(\alpha \to \alpha')\&(\beta \to \beta')] \to [(\alpha\&\beta) \to (\alpha'\&\beta')],$$

we get the result using 9.1.14.

(ii) is proved in the same way. □

Lemma 9.1.19 For $r < 1, \bar{r} \in \Phi$.

Proof: Assume not, then for some $\gamma \in \Phi, T \vdash \gamma \vee \bar{r}$, $T \vdash \Delta(\gamma \vee \bar{r})$, $T \vdash \Delta\gamma \vee \Delta\bar{r}$. Thus it suffices to show $T \vdash \Delta\bar{r} \to \bar{0}$ to get $T \vdash \Delta\gamma, T \vdash \gamma$, a contradiction.

Indeed, $T \vdash \Delta\bar{r} \equiv \dot{\neg}\bar{r} \equiv \dot{\neg}\overline{(1-r)} \equiv \bar{0}$, since $1 - r > 0$. This completes the proof. □

Lemma 9.1.20 For each formula α,

$$\inf\{r|(\bar{r} \to \alpha) \in \Phi\} = \sup\{r'|(\alpha \to \overline{r'}) \in \Phi\}.$$

Proof: If $(\bar{r} \to \alpha) \in \Phi$ and $(\alpha \to \overline{r'}) \in \Phi$ then $(\bar{r} \to \overline{r'}) \in \Phi$ by 9.1.16, thus $T \vdash \bar{r} \to \overline{r'}$ and hence $r > r'$. Therefore $\inf\{r|(\bar{r} \to \alpha) \in \Phi\} \geq \sup\{r'|(\alpha \to \overline{r'}) \in \Phi\}$. We want to prove equality. Assume not; take r, r' such that

$$\inf\{r|(\bar{r} \to \alpha \in \Phi\} > r > r' > \sup\{r'|(\alpha \to \overline{r'}) \in \Phi\};$$

Thus for this r, r', $(\bar{r} \to \alpha) \notin \Phi$, $(\alpha \to \overline{r'}) \notin \Phi$; consequently, $(\bar{r} \to \overline{r'}) \notin \Phi$ (by 9.1.18) and hence $\bar{s} \notin \Phi$ for $s = (r \Rightarrow r')$. But $s < 1$ and thus $s \in \Phi$ by 9.1.19, a contradiction. □

Definition 9.1.21 The model $\mathbf{M}$ given by Φ is defined as follows:

- The domain M consists of all constants of the extended language; each such constant c, $m_c = c$ (c denotes itself).
- For each n-ary predicate P and constants $c, \ldots, d$,

$$r_P(c, \ldots, d) = \inf\{r|(\bar{r} \to P(c, \ldots, d)) \in \Phi\}$$

Lemma 9.1.22 For each closed formula α,

$$\|\alpha\|_{\mathbf{M}} = \inf\{r|(\bar{r} \to \alpha) \in \Phi\}$$

Proof: The assertion is immediate for α atomic closed directly by the definition of r_P. For s rational in $[0, 1]$, $\|\bar{s}\|_{\mathbf{M}} = s$. It follows from our investigation that $(\bar{r} \to \bar{s}) \in \Phi$ iff $r > s$; thus $s = \inf\{r|(\bar{r} \to \bar{s}) \in \Phi\}$. We shall verify the induction step for conjunction(s), implication(s) and quantifiers.

Assume the assertion for β, γ, i.e. $\|\beta\|_{\mathbf{M}} = \inf\{r|(\bar{r} \to \beta) \in \Phi\}$ and similarly for γ. We omit the subscript $\mathbf{M}$. If $r_1 < \|\beta\|$ and $r_2 < \|\gamma\|$ then $(\overline{r_1} \to \beta) \notin \Phi$ and $(\overline{r_2} \to \gamma) \notin \Phi$, thus, by 9.1.18, $((\overline{r_1} \& \overline{r_2} \to (\beta \& \gamma))) \notin \Phi$ and $(\overline{r_1 * r_2} \to (\beta \& \gamma)) \notin \Phi$ where $*$ is the corresponding t-norm.
Sublemma 1. $\|\beta\| * \|\gamma\| \leq \inf\{r|(\bar{r} \to (\beta \& \gamma)) \in \Phi\}$.
This is trivial if $\|\beta\| = 0$ or $\|\gamma\| = 0$, thus assume $\|\beta\|, \|\gamma\| > 0$. Then, thanks to the continuity and monotonicity of $*$,

$$\|\beta\| * \|\gamma\| = \sup_{r_1 < \|\beta\|} \sup_{r_2 < \|\gamma\|} (r_1 * r_2) \leq \inf\{r|(\bar{r} \to (\beta \& \gamma)) \in \Phi\},$$

since by the above for each such r_1, r_2, $(\overline{r_1 * r_2} \to (\beta \& \gamma)) \notin \Phi$.

Sublemma 2. $\|\beta\| * \|\gamma\| \geq \sup\{r|((\beta\&\gamma) \to \bar{r}) \in \Phi\}$.

First assume $\|\beta\|, \|\gamma\| < 1$. If $r_1 > \|\beta\|$ and $r_2 > \|\gamma\|$ then $(\beta \to \overline{r_1}), (\gamma \to \overline{r_2}) \notin \Phi$, thus $((\beta\&\gamma) \to \overline{r_1 * r_2}) \notin \Phi$. Then similarly as above we get

$$\|\beta\| * \|\gamma\| = \inf_{r_1 > \|\beta\|} \inf_{r_2 \geq \|\gamma\|} (r_1 * r_2) \geq \sup\{r|((\beta\&\gamma) \to \bar{r}) \in \Phi\}.$$

If e.g. $\|\beta\| = 1$ then $\|\beta\| * \|\gamma\| = \|\gamma\| = \sup\{r|(\gamma \to \bar{r}) \in \Phi\} \geq \sup\{r|((\beta\&\gamma) \to \bar{r}) \in \Phi\}$ (since $\vdash (\gamma \to \bar{r}) \to ((\beta\&\gamma) \to \bar{r})$, thus $((\beta\&\gamma) \to \bar{r}) \in \Phi$ implies $(\gamma \to \bar{r}) \in \Phi$).

Since $\|\beta\&\gamma\| = \|\beta\| * \|\gamma\|$ we get, using 9.1.20,

$$\|\beta\&\gamma\| = \inf\{r|(r \to (\beta\&\gamma)) \in \Phi\}.$$

Observe that the same proof works for $\wedge$ and $\odot$. We turn to the implication, continuing to assume the assertion for β, γ.

Sublemma 3. If $\|\beta\| > \|\gamma\|$ then

$$\|\beta\| \Rightarrow \|\gamma\| \leq \inf\{\bar{r}|(r \to (\beta \to \gamma)) \in \Phi\}.$$

Recall that $\Rightarrow$ is non-increasing in the first variable, non-decreasing in the second, and if $x > y$ then $\Rightarrow$ is *continuous* in the point (x, y). Thus we may proceed analogously as above. First assume $\|\beta\| < 1, \|\gamma\| > 0$. If $r_1 > \|\beta\|$ and $r_2 < \|\gamma\|$ then $(\beta \to \overline{r_1} \notin \Phi, (\overline{r_2} \to \gamma) \notin \Phi$, thus $((\overline{r_1} \to \overline{r_2}) \to (\beta \to \gamma)) \notin \Phi$ by 9.1.18 (ii). Hence $\|\beta\| \Rightarrow \|\gamma\| = \sup_{r_1 > \|\beta\|} \sup_{r_2 < \|\gamma\|} (r_1 \Rightarrow r_2) \leq \inf\{r|(\bar{r} \to (\beta \to \gamma)) \in \Phi\}$.

Sublemma 4. If $\|\beta\| > \|\gamma\|$ then

$$\|\beta\| \Rightarrow \|\gamma\| \geq \sup\{r|((\beta \to \gamma) \to \bar{r}) \in \Phi\}.$$

Our assumption implies $\|\beta\| > 0$ and $\|\gamma\| < 1$. Similarly as above we get

$$\|\beta\| \Rightarrow \|\gamma\| = \inf_{r_1 < \|\beta\|} \inf_{r_2 > \|\gamma\|} (r_1 \Rightarrow r_2) \geq \sup\{r|((\beta \to \gamma) \to \bar{r}) \in \Phi\}.$$

Sublemma 5. If $\|\beta\| \leq \|\gamma\|$ then $\|\beta\| \Rightarrow \|\gamma\| = \inf\{r|(\bar{r} \to (\beta \to \gamma)) \in \Phi\} = 1$.

Here we can encounter non-continuity, thus special care is necessary. We show $\inf\{r|(\bar{r} \to (\beta \to \gamma)) \in \Phi\} = 1$. First assume $\|\gamma\| = 1$. Then for all $r < 1$, $(r \to \gamma) \notin \Phi$ and therefore $(\bar{r} \to (\beta \to \gamma)) \notin \Phi$ (since $\vdash (\bar{r} \to \gamma) \to (\bar{r} \to (\beta \to \gamma))$). Hence $\inf\{r|(\bar{r} \to (\beta \to \gamma)) \in \Phi\} = 1$.

Now assume $\|\gamma\| < 1$. We claim that for each r, $(\bar{r} \to (\beta \to \gamma)) \notin \Phi$. If $\bar{r} \to (\beta \to \gamma) \in \Phi$ for some r then $((\beta \to \gamma) \in \Phi$ by 9.1.14, and hence by the construction of Φ, $(\beta \to (\gamma \underline{\vee} \overline{r_k})) \in \Phi$ for some r_k. Take an r' such that $\|\gamma\| < r' < \|\gamma\| + r_k = \|\gamma \underline{\vee} \overline{r_k}\|$. Then $((\gamma \underline{\vee} \overline{r_k} \to \overline{r'}) \in \Phi$ and thus, by 9.1.16, $(\beta \to \overline{r'}) \in \Phi$, hence $\|\beta\| > r' \geq \|\gamma\|$ which contradicts our assumption $\|\beta\| \leq \|\gamma\|$. Thus $\inf\{r|(\bar{r} \to (\beta \to \gamma)) \in \Phi\} = \inf \Phi = 1$.

Thus we have verified the induction step for implications. We still have to handle quantifiers. Let $\beta(x)$ be a formula with one free variable and assume the assertion for each instance $\beta(c)$. (This is a good inductive assumption: for a natural notion of complexity of a formula we prove our assertion by induction of the complexity of the formula – and all instances $\beta(c)$ of $\beta(x)$ have the same complexity.)

Sublemma 6.

$$\|(\forall x)\beta(x)\| = \inf\{r|(\bar{r} \to (\forall x)\beta(x)) \in \Phi\}.$$

Clearly,

$$\|(\forall x)\beta(x)\| = \inf_{c \in M} \|\beta(c)\|,$$

let $\|(\forall x)\beta(x)\| = z$. If $r < z$ then for each constant c, $r < \|\beta(c)\|$, hence $(\bar{r} \to \beta(c)) \notin \Phi$. By the construction of Φ we conclude that $(\forall x)(\bar{r} \to \beta(x)) \notin \Phi$, thus $(\bar{r} \to (\forall x)\beta(x)) \notin \Phi$.

If $r > z$ then for some c, $r > \|\beta(c)\|$, thus $(\bar{r} \to \beta(c)) \in \Phi$ and hence $(\bar{r} \to (\forall x)\beta(x)) \in \Phi$. Consequently, $z = \inf\{r \mid \bar{r} \to (\forall x)\beta(x)) \in \Phi\}$.

Sublemma 7.

$$\|(\exists x)\beta(x)\| = \inf\{r|(\bar{r} \to (\exists x)\beta(x)) \in \Phi\}.$$

Let now $z = \|(\exists x)\beta(x)\|$. If $r < z$ then for some c, $r < \|\beta(c)\|$, thus $(\beta(c) \to \bar{r}) \in \Phi$, thus $((\exists x)\beta(x) \to \bar{r}) \in \Phi$ (again by 9.1.14).

If $r > z$ then $(\beta(c) \to \bar{r}) \notin \Phi$ for all c, thus, by the construction of Φ, $(\forall x)(\beta(x) \to \bar{r}) \notin \Phi$ and hence $((\exists x)\beta(x) \to \bar{r}) \notin \Phi$ (since $\vdash (\forall x)(\beta(x) \to \bar{r}) \to ((\exists x)\beta(x) \to \bar{r})$ - axiom($\exists$2)). Thus $z = \sup\{r|((\exists x)\beta(x) \to \bar{r}) \in \Phi\}$. This completes the proof of the lemma. $\square$

9.1.23 *Proof of the completeness theorem.* Recall our φ such that $T \not\vdash \varphi$. Since $\varphi \in \Phi$, there is an $r_k > 0$ such that $(\varphi \underline{\vee} \overline{r_k}) \in \Phi$, thus $((\overline{(1 - r_k)} \to \varphi) \in \Phi$. (Take $\to_{Ł}$ for $\to$.) Hence $\|\varphi\| \leq 1 - r_k < 1$.

On the other hand, for each $\alpha \in T$ and each r, $T \vdash \bar{r} \to \alpha$. Thus for each r, $(\bar{r} \to \alpha) \notin \Phi$ and $\|\alpha\| = \inf\{r|(\bar{r} \to \alpha) \in \Phi\} = 1$. Hence $\mathbf{M}$ is a model of T in which φ is not true. This proves the completeness theorem of TT$\forall$.

9.2. AN ABSTRACT FUZZY LOGIC

In this section we present, more or less in passing, some selected material from Pavelka's [163] Part I. The approach is rather abstract and the main thing to stress is that it covers also logical systems that are not truth-functional; formulas are treated as abstract objects, possibly without any structure. On the other hand, it gives some light to truth-functional systems: we shall see, among other things, the origin of rational Pavelka logic RPL.

Definition 9.2.1 In the sequel, $\mathbf{L}$ is a complete lattice (each subset of $\mathbf{L}$ has its infimum and supremum). We do not assume $\mathbf{L}$ to be linearly ordered, but our favorite example will be the ordered unit interval $[0,1]$. $Form$ is a non-empty set; its elements are called *formulas*. $\mathbf{L}^{Form}$ denotes the set of all mappings $f : Form \to \mathbf{L}$; in other words, all $\mathbf{L}$-fuzzy subsets of $Form$.

Definition 9.2.2 An $\mathbf{L}$-*semantics* for $Form$ is a non-empty subset $\mathcal{S} \subseteq \mathbf{L}^{Form}$. Each element $E \in \mathcal{S}$ is called an (admissible) evaluation. For each $\varphi \in Form$, $\|\varphi\|_E = E(\varphi)$ is the truth value of φ in the evaluation E. An element $X \in \mathbf{L}^{Form}$ (not necessarily in $\mathcal{S}$) may be understood as a *theory*; $E \in \mathcal{S}$ is a *model of* X if for each $\varphi \in Form$, $E(\varphi) \geq X(\varphi)$. (Thus the fuzzy set X assigns to each formula a lower bound for its truth degree.)

$$\|\varphi\|_X = \inf\{\|\varphi\|_E | E \text{ model of } X\}$$

is the truth degree of φ in the theory X.

Example 9.2.3 (1) For each propositional calculus $\mathcal{C}$ given by a continuous t-norm, the predicate calculus $\mathcal{C}\forall$ determines the above notion as follows: $Form$ is the set of all formulas of $\mathcal{C}\forall$, $L \in [0,1]$. $E \in \mathcal{S}$ iff there is an interpretation $\mathbf{M}$ of the language of $\mathcal{C}\forall$ such that $E(\varphi) = \|\varphi\|_{\mathbf{M}}$ for each φ. Similarly for $RPL\forall$.

(2) Let $Form$ be the set of all formulas of Boolean propositional logic with propositional variables $p_1, \ldots, p_n$ and let $\mathcal{S}$ be the set of all probabilities on $Form$. In more details, $E \in \mathcal{S}$ if there is a probabilistic Kripke model $\mathbf{K} = \langle W, e, \mu \rangle \in \mathcal{K}_{prob}$ (cf.8.2.22) such that for each formula φ, $E(\varphi) = \mu\{w \in W | \, \|\varphi\|_{K,v} = 1\}$. Thus here we may afford what was forbidden in truth-functional systems: we take the probability of φ and understand it as the truth value of φ. Similarly for such other belief measures as possibilities/necessities, belief/plausibility functions.

Definition 9.2.4 (1) Let $Form, \mathbf{L}$ be as above. An n-ary *fuzzy deduction rule* for $(Form, \mathbf{L})$ consists of two (crisp) functions R_1, R_2. R_1 is a partial mapping from $Form^n$ into $Form$ (i.e. the domain of R_1 is a subset of $Form^n$

and for each $(\varphi_1, \ldots, \varphi_n)$ in the domain, $R_1(\varphi_1, \ldots, \varphi_n) \in Form$) and R_2 is a total mapping of $\mathbf{L}^n$ into $\mathbf{L}$. The rule may be visualized as follows:

$$\frac{\varphi_1, \ldots, \varphi_n}{R_1(\varphi_1 \ldots \varphi_n)} \quad (\frac{a_1, \ldots, a_n}{R_2(a_1, \ldots, a_n)})$$

The intuitive meaning is: if each φ_i is at least a_i-true then the formula $R_1(\varphi_1, \ldots, \varphi_n)$ is at least $R_2(a_1, \ldots, a_n)$ true.

(2) R_2 *preserves non-empty suprema* if for each $i = 1, \ldots, n$ and each non-empty $U \subseteq \mathbf{L}, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in \mathbf{L}$,

$$R_2(a_1, \ldots, a_{i-1}, \sup U, a_{i+1}, \ldots a_n) =$$

$$= \sup_{a_i \in U} R_2(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n).$$

(3) A fuzzy set $X \in \mathbf{L}^{Form}$ is *closed* under the rule (R_1, R_2) if for each $(\varphi_1, \ldots, \varphi_n) \in dom(R_1)$

$$X(R_1(\varphi_1, \ldots, \varphi_n)) \geq R_2(X(\varphi_1), \ldots, X(\varphi_n))$$

(In other words, if X demands the assumptions φ_i to be at least a_i-true for $i = 1, \ldots, n$ then it demands the conclusion $R_1(\varphi, \ldots, \varphi_n)$ to be at least $R_2(a_1, \ldots, a_n)$-true.)

X is *closed* under a non-empty *set* $\mathcal{R}$ of *rules* if it is closed under each member of $\mathcal{R}$.

(4) A fuzzy *deductive system* for $Form, \mathbf{L}$ consists of a fuzzy set $A : Form \to \mathbf{L}$ called the fuzzy set of *logical axioms* and of a non-empty set $\mathcal{R}$ of fuzzy deduction rules, each preserving non-empty suprema; X is *closed* under (A, R) iff $X \geq A$ (i.e. for each φ, $X(\varphi) \geq A(\varphi)$) and X is closed under R.

Lemma 9.2.5 (1) If (R_1, R_2) is a deduction rule preserving non-empty suprema then R_2 is non-decreasing in each argument.

(2) Under the same assumption, let $\mathcal{X}$ be a non-empty set of fuzzy theories, each closed under (R_1, R_2) and let $X = \inf \mathcal{X}$, i.e. for each φ let $X(\varphi) = \inf\{Y(\varphi) | Y \in \mathcal{X}\}$. Then X is closed under (R_1, R_2).

Proof: (1) Let $a \leq b, a, b \in \mathbf{L}$; then $a \cup b = b$ and

$$R_2(\ldots b \ldots) = R_2(\ldots a \ldots) \cup R_2(\ldots b \ldots) \geq R_2(\ldots a \ldots).$$

(2) Evidently, $X(R_1(\varphi_1, \ldots)) =$

$$\inf\{\varphi(R_1(\varphi_1, \ldots)) | \varphi \in \mathcal{X}\} \geq \inf\{R_2(Y(\varphi_1) \ldots Y(\varphi_n) | Y \in \mathcal{X}\}$$

due to the closedness of each $Y \in \mathcal{X}$. Now since for each such Y, $Y(\varphi_i) \geq X(\varphi_i)$ for each i, we get from our (1)

$$R_2(Y(\varphi_1), \ldots Y(\varphi_n)) \geq R_2(X(\varphi_1, \ldots, X(\varphi_n)),$$

thus

$$\inf_Y\{R_2(Y(\varphi_1), \ldots Y(\varphi_n))\} \geq R_2(X(\varphi_1), \ldots, X(\varphi_n))$$

and hence

$$X(R_1(\varphi_1, \ldots \varphi_n)) \geq R_2(X(\varphi_1), \ldots, X(\varphi_n)).$$

This proves that X is closed under (R_1, R_2). □

Theorem 9.2.6 Let $(A, \mathcal{R})$ be a fuzzy deduction system for $(Form, \mathbf{L})$. Then for each fuzzy theory $X : Form \to \mathbf{L}$ there is a uniquely determined least theory $Y : Form \to \mathbf{L}$ containing X (i.e. $Y \geq X$) and closed under $(A, \mathcal{R})$.

Proof: Let $\mathcal{W}$ be the set of all $Z : Form \to \mathbf{L}$ containing X and closed under $(A, \mathcal{R})$. Note that $\mathcal{W} \neq \emptyset$ since the Z such that $Z(\varphi) = 1_{\mathbf{L}}$ for all φ is in $\mathcal{W}$. Put

$$Y(\varphi) = \inf\{Z(\varphi) | Z \in \mathcal{W}\};$$

by the above, Y contains X and is closed under $(A, \mathcal{R})$. Clearly Y is the least theory with this property. □

Definition 9.2.7 (1) The unique Y in the preceding theorem is called the *deductive closure* of X with respect to $(A, \mathcal{R})$ and denoted by $Cn_X^{A,\mathcal{R}}$.

(2) A *weighted proof* from X on the base of $(A, \mathcal{R})$ is a finite sequence of pairs

$$(\varphi_1, w_1), \ldots, (\varphi_k, w_k)$$

where for each i $\varphi_i \in Form$ and $w_i \in \mathbf{L}$ and , moreover, at least one of the following conditions holds:

- $A(\varphi_i) = w_i$ (φ_i treated as a logical axiom)
- $X(\varphi_i) = w_i$ (φ_i treated as a special axiom)
- for some n-ary rule $(R_1, R_2) \in R$ there are $i_1, \ldots, i_n < i$ such that $R_1(\varphi_{i_1}, \ldots, \varphi_{i_n}) = \varphi_i$ and $R_2(w_{i_1}, \ldots, w_{i_n}) = w_i$

(3) The *provability degree* of φ from X in (A, R) is

$$|\varphi|_X^{(A,\mathcal{R})} = \sup\{w | \text{ there is a weighted proof of } (\varphi, w)\}.$$

Clearly, the proofs are from X in (A, R); saying "a proof of (φ, w)" we mean that (φ, w) is the last member of the proof.

Theorem 9.2.8 Let (A, R) be a deductive system for $(Form, \mathbf{L})$. Then for each $\varphi \in Form$ and $X \in \mathbf{L}^{Form}$,

$$|\varphi|_X^{(A,\mathcal{R})} = Cn_X^{A,\mathcal{R}}(\varphi).$$

Proof: On the one hand, if $(\varphi_1, w_1), \ldots, (\varphi_n, w_n)$ is a weighted proof from X in (A, R) and Y is closed under (A, R) and $Y \geq X$ then it follows easily by induction on the length of the proof that $Y(\varphi_i) \geq w_i$ for each i. For example, assume φ_i results from $\varphi_j, \varphi_k (j, k < i)$ by a binary rule (R_1, R_2), i.e. $\varphi_i = R_1(\varphi_j, \varphi_k)$ and $w_i = R_2(w_j, w_k)$. By the induction assumption, $Y(\varphi_j) \geq w_j, Y(\varphi_k) \geq w_k$; thus, by closedness of Y, $Y(\varphi_i) \geq R_2(w_j, w_k) = w_i$. This completes the induction step. Thus $|\varphi|_X^{A,\mathcal{R}} \leq$ $\leq Cn_X^{A,\mathcal{R}}(\varphi)$.

To prove the converse inequality it suffices to show that $|\varphi|_X^{(A,\mathcal{R})}$ understood as a fuzzy theory (assigning to each formula φ the value $|\varphi|_X^{(A,\mathcal{R})}$) is closed under $(A, \mathcal{R})$. Here we need preservation of suprema. For simplicity, delete all indices. First, clearly $|\varphi| \geq A(\varphi)$, $|\varphi| \geq X(\varphi)$ (thanks to trivial one-element proofs of axioms). Now let $\varphi = R_1(\varphi_1, \ldots \varphi_n)$; we have to prove $R_2(|\varphi_1|, \ldots, |\varphi_n|) \leq |\varphi|$.

Now $R_2(|\varphi_1|, \ldots |\varphi_n|) =$
$= R_2(\sup\{a_1 | X \vdash (\varphi_1, a_1)\}, \ldots, \sup\{a_n | X \vdash (\varphi_n, a_n)\} =$
$= \sup\{R_2(a_1, \ldots, a_n) | X \vdash (\varphi_i, a_i) \quad \text{for} \quad i = 1, \ldots n\} \leq |\varphi|$.
Here $X \vdash (\varphi, a_i)$ obviously means that there is a proof of (φ, a_i) from X in $(A, \mathcal{R})$. The last inequality holds since for each choice of $a_1, \ldots a_n$ such that $X \vdash (\varphi_i, a_i)$ for all i, we get $X \vdash (\varphi, R_2(a_1, \ldots a_n))$ by concatenating the proofs and applying the rule. □

Definition 9.2.9 Let $\mathcal{S}$ be a semantics for $(Form, \mathbf{L})$ and let $(A, \mathcal{R})$ be a deduction system for $(Form, \mathbf{L})$. Call , for a moment, $(Form, \mathbf{L}, \mathcal{S}, A, \mathcal{R})$ an *abstract fuzzy logic* and denote it by $\mathcal{L}$.

(1) $\mathcal{L}$ is *sound* if for each φ and X,

$$|\varphi|_X^{A,\mathcal{R}} \leq \|\varphi\|_X^{\mathcal{S}}$$

(equivalently, whenever $X \vdash (\varphi, w)$ then $\|\varphi\|_X^{\mathcal{S}} \geq w$).

(2)L is *Pavelka-complete* if $|\varphi|_X^{A,\mathcal{R}} = \|\varphi\|_X^{\mathcal{S}}$ for each φ, X.

Remark 9.2.10 Here we stop our "visit" with the old seminal Pavelka's work. Needless to say, much of the material presented in his [163] Parts II, III has been extremely influential at several places in previous chapters. But it should be stressed that the original presentation in these last-mentioned papers was repeatedly and substantially simplified over the course of time.

Things presented in this section may be further developed in at least two directions:

(1) First, finding natural Pavelka-complete abstract fuzzy logics for particular semantics, e.g. for probabilities on logical formulas. See [59] for such an investigation.

(2) Second, following Pavelka, investigating abstract fuzzy logics given by a continuous t-norm, with the language extended by truth constants. Pavelka always deals with truth constants for all reals from $[0, 1]$. We have seen in the case of RPL∀, i.e. extended Ł∀, that rational constants are sufficient. This is true in general; the diligent reader may elaborate details. For any logic $\mathcal{C}\forall$ given by a continuous t-norm we get a sound abstract fuzzy logic.

As far as completeness is concerned, we have Pavelka-completeness for $RPL\forall$ and it is an easy exercise that the same is true for each t-norm isomorphic to Łukasiewicz. We showed that we cannot have Pavelka-completeness for G∀, nor for Π∀, and the general result is: if the truth-function $\Rightarrow$ of implication in $\mathcal{C}\forall$ is not continuous then there is no $(A, \mathcal{R})$ making the logic Pavelka-complete. One easily shows, using the characterization of continuous t-norm by Mostert and Shields (see 2.1.17), that the only t-norms with continuous residuum are those isomorphic to Łukasiewicz. (Hint: if $*$ is not isomorphic to $*_{Ł}$ then either it is isomorphic to product and its residuum is Goguen, which is not continuous in $(0, 0)$, or $*$ has at most one internal idempotent u, $0 < u < 1$; then $\Rightarrow$ is not continuous in (u, u).) This analysis also shows how to prove Pavelka incompleteness of such fuzzy deductive system, in an analogy to 4.1.22.

This fact just shows the limitation of this approach: even if very general, it fails to yield Pavelka-complete axiomatization e.g. for G∀ and Π∀, i.e. We cannot incorporate truth-constants as easily as in Ł∀. We showed, on the one hand, some limited possibilities (cf. 4.1.23, 4.2.22); on the other hand, in the previous section we presented a more general logic $TT\forall$ with an infinitary deduction rule, for which we do have completeness.

9.3. On the Liar Paradox

One of the most well-known "paradoxes" of logical semantics consists in the fact that a sentence asserting its own falsity (I say: "The sentence I am just stating is false.") can be neither true (since it would then be false) nor false (since it would then be true). Thus in a two-valued logic this sentence cannot have any truth value. In the context of fuzzy logic it is natural to ask how many-valuedness can change the situation. Zadeh [216] (see also [116]) investigated the liar paradox in the context of possibility theory and among other things came to the (natural) conclusion that the truth-value of the liar sentence is $\frac{1}{2}$. Interestingly, a similar investigation was done by Skolem [190] in the context of paradoxes of set theory.

A deeper analysis of the paradox needs technical means that enable discussion of sentences like the above as parts of a formal deductive system. Such means are at our disposal in the metamathematics of arithmetic, as initiated by Gödel's celebrated incompleteness theorems. I shall survey the necessary things and refer to [91] for all necessary details. The expert in metamathematics of arithmetic will easily be able to fill in all detaills; on the other hand, the presentation here shouldalso give the non-expert reader enough information on the results obtained and some feeling for the methods of proof.

The whole section is based on the recent paper [89].

9.3.1 Recall Robinson's arithmetic QA described in Section 6.1., thus a weak crisp arithmetic with predicates $=$ (equality), S (successor), A (addition), B (multiplication), numeral $\underline{n}$ for each natural $n \in N$. Recall that we have identified formulas with some natural numbers (formulas are numbers), thus in particular for each formula φ we have the corresponding numeral $\underline{\varphi}$. The following is the famous *diagonal lemma*, which is the essence of Gödel's incompleteness result and of self-reference in arithmetic:

Lemma 9.3.2 Let $T \supseteq$ QA be a theory (one-sorted, in Boolean logic, possibly with a language richer than that of QA). For each formula $\psi(x)$ with one free variable x there is a closed formula φ such that $T \vdash \varphi \equiv \psi(\underline{\varphi})$.

The formula φ is usually paraphrased as saying "I have the property ψ."

Definition 9.3.3 The theory QATr (QA with a truth predicate Tr) has the language of QA extended by a unary predicate Tr and axioms of QA plus the following *dequotation schema*:

$$\varphi \equiv Tr(\underline{\varphi})$$

where φ is any closed formula of the language of QATr, i.e. Tr may occur inside φ.

Remark 9.3.4 The term "dequotation" is used by philosophers and relates to the notion of truth illustrated by the celebrated example of Tarski:

The sentence "it's snowing" is true iff it's snowing.

Theorem 9.3.5 As a theory in Boolean logic, QATr is contradictory.

Proof: We construct the liar sentence: apply the diagonal lemma to $\neg Tr(x)$. We get QATr $\vdash \lambda \equiv \neg Tr(\underline{\lambda})$, thus, by dequotation, QATr $\vdash \lambda \equiv \neg\lambda$, which is contradictory over $Bool\forall$ (e.g. since $Bool\forall$ extends G$\forall$, thus proves $(\lambda \to \neg\lambda) \to \neg\lambda$, QA$Tr$ proves $\neg\lambda$ as well as $\neg\neg\lambda$, i.e. λ).

Before we turn to the question of how this is in fuzzy logic, let us state one more definition and make one more remark. □

Remark 9.3.6 We recall and slightly modify the definition 1.4.10 of Peano arithmetic:

Peano arithmetic PA is the extension, in Boolean logic, of QA by the following additional inference rule:

$$\frac{\varphi(\underline{0}), (\forall x, y)((\varphi(x) \wedge S(x, y)) \to \varphi(y))}{(\forall x)\varphi(x)}.$$

This is the rule of *induction*: if you have proved that $\underline{0}$ satisfies φ and if x satisfies φ then also the successor of x satisfies φ then you may conclude that all x satisfy φ.

Recall that the *standard model* of arithmetic is the crisp structure of natural numbers, i.e. **N** with $=$ interpreted as equality, S interpreted as successor ($r_S(m, n) = 1$ iff $n = m + 1$, otherwise $= 0$), analogously A is interpreted as addition of natural numbers ($r_A(m, n, k) = 1$ iff $k = m + n$, otherwise 0) and B as multiplication. Each $\underline{n}$ is interpreted by n.

Peano arithmetic is the most famous axiomatic arithmetic. Note that over $Bool\forall$, the deduction rule of induction may be replaced by the induction axiom schema:

$$[\varphi(\underline{0}) \wedge (\forall x, y)((\varphi(x) \wedge S(x, y)) \to \varphi(y))] \to (\forall x)\varphi(x).$$

But in fuzzy logic we have to work with the rule (since the induction schema is not generally sound in fuzzy logic). QA is our *weak* arithmetic and PA our *strong* arithmetic; needless to say, they may be replaced by various related systems. PATr has the clear meaning. Obviously, it is a contradictory theory.

Our question now reads: *can we keep arithmetic* (as a theory of successor, addition and multiplication) crisp but *endow it with a fuzzy truth predicate? Can we have then dequotation?* This is the main problem to be answered in this section.

Definition 9.3.7 Let $\mathcal{C}\forall$ be a fuzzy predicate calculus. (1) QA($\mathcal{C}\forall$) is the theory in $\mathcal{C}\forall$ with the language of QA, whose axioms are

– crispness axioms for $=, S, A, B$, i.e.

$$(\forall x, y)(x = y \vee \neg(x = y))$$

$$(\forall x, y, z)(A(x, y, z) \vee \neg A(x, y, z))$$

and similarly for S, B,
– all the axioms of QA.

PA($\mathcal{C}\forall$) is defined analogously.

(2) QA($\mathcal{C}\forall$)Tr is the extension, over $\mathcal{C}\forall$, of QA($\mathcal{C}\forall$) by a new unary predicate Tr and by the dequotation scheme

$$\varphi \equiv Tr(\underline{\varphi})$$

for each sentence φ of the language of QA($\mathcal{C}\forall$)Tr. PA($\mathcal{C}\forall$)Tr is defined analogously.

Lemma 9.3.8 QA($\mathcal{C}\forall$) proves φ over $\mathcal{C}\forall$ iff QA$\vdash$ φ over $Bool\forall$.

Proof: $Bool\forall$ proves the additional axioms of QA($\mathcal{C}\forall$) and is stronger than $\mathcal{C}\forall$; thus QA($\mathcal{C}\forall$) $\vdash \varphi$ implies QA$\vdash$ φ. Conversely, one easily shows by induction that the crispness axiom $(\forall x \ldots)(\alpha(x \ldots) \vee \neg\alpha(x \ldots))$ is provable in QA($\mathcal{C}\forall$) for each α in the language of QA. (Induction step for $\forall$: if we can prove $(\forall x)(\alpha(x) \vee \neg\alpha(x))$ then also $(\forall x)(\alpha(x) \vee \neg(\forall x)\alpha(x))$ and hence $(\forall x)\alpha(x) \vee \neg(\forall x)\alpha(x)$ by Axiom ($\forall$3). Similarly, for $\exists$.) Thus QA$\vdash$ φ implies QA($\mathcal{C}\forall$) $\vdash \varphi$. □

Theorem 9.3.9 The theories QA(G$\forall$)Tr, QA($\Pi\forall$)Tr are contradictory, i.e. they do not allow a truth predicate with dequotation for (crisp) arithmetic.

We shall prove this theorem later in this section.

Thus of our favourite theories only Ł$\forall$ still resists. We shall formulate two results – one negative and one positive. (Later we add a third negative result.) But let us stop for one definition.

Definition 9.3.10 PA(Ł$\forall$)Tr_2 is PA(Ł$\forall$)Tr extended by the rule of induction for all formulas (possibly containing Tr).

Theorem 9.3.11 (1) The standard model $\mathbf{N}$ cannot be expanded by any fuzzy relation r_{Tr} such that $(\mathbf{N}, r_{Tr})$ would be a model of QA(Ł$\forall$)Tr (*a fortiori*, of PA(Ł$\forall$)Tr, PA(Ł$\forall$)Tr_2, etc). Thus there is no standard model of full dequotation.

(2) On the other hand, the theory PA(Ł$\forall$)Tr_2 is consistent over Ł$\forall$ and hence has a model. Each model of PA(Ł$\forall$)Tr_2 is non-standard, i.e. its crisp (arithmetical) part is not isomorphic to $\mathbf{N}$.

We shall prove this theorem below. First we shall have to recall the technique of arithmetization of metamathematics. The hurrying reader may skip the rest; but it is of some general interest that we shall prove our (1) by a rather natural fuzzy variant of the liar paradox. We construct a sentence that may be paraphrased as saying "I am at least a little false" and leads to a paradox that might be called "the modest liar paradox": it is absolutely true iff it is at least a little false. On this informal level it is admittedly not too clear; thus let us go into a formalization.

*

9.3.12 (Survey of arithmetization of metamathematics.) We already know that for us formulas are particular numbers and the set of all formulas is defined in **N** by a formula $Form$ of PA. We get the following: φ is a formula iff PA$\vdash$ $Form(\underline{\varphi})$. More than that, reasonable logical operations are definable by such formulas and their properties are formally provable. For example, let us take the operation assigning to each pair φ, ψ of formulas their implication $(\varphi \to \psi)$. There is a formula $Impl(x, y, z)$ of PA such that

– PA$\vdash$ $(\forall x, y)((Form(x) \wedge Form(y)) \to (\exists! z)(Form(z) \wedge$ $\wedge Impl(x, y, z)))$ (here $(\exists! x)$ means, as usual, "there is a unique x such that..."); thus in PA, $Impl$ defines a function assigning to each pair of formulas-in-the-sense-of-PA another such formula, and
– for each pair φ, ψ of (actual) formulas,

$$\text{PA} \vdash Impl(\underline{\varphi}, \underline{\psi}, z) \equiv (z = \underline{\varphi \to \psi}).$$

In particular, in **N** $Impl$ defines just the operation we started with. Informally we shall denote, inside PA, the unique z satisfying $Impl(x, y, z)$ by $x \dot{\to} y$. Then the last formula can be written as

$$\text{PA} \vdash \underline{\varphi} \dot{\to} \underline{\psi} = \underline{\varphi \to \psi}.$$

The same holds for other connectives and similar operations; in particular for substitution $Subst(\varphi, t)$ of a term t for the first variable in a formula φ,

$$PA \vdash Subst'(\underline{\varphi}, \underline{t}) = \underline{Subst(\varphi, t)}.$$

Note that all this take place in the crisp arithmetic PA or, equivalently, in PA(Ł$\forall$). The expert knows and the non-expert will believe that our informal use of function symbols $\underline{\dot{\vee}}$, $\dot{\times}$ etc. is tolerable and can be eliminated, but simplifies the presentation. Next we show that Gödel's diagonal lemma generalizes for PA$(\mathcal{C}\forall)Tr$:

Lemma 9.3.13 For each formula $\varphi(x)$ of PA$(\mathcal{C}\forall)Tr$ with exactly one variable x there is a sentence (closed formula) ψ such that

$$\text{PA}(\mathcal{C}\forall)Tr \vdash \psi \equiv \varphi(\underline{\psi}).$$

Proof: We use the notation above; in addition, inside PA$(\mathcal{C}\forall)Tr$, $\dot{x}$ denotes the x-th (formalized) numeral. We just check the usual proof. Let $\chi(x)$ be $\psi(Sub\dot{s}t(x, \dot{x}))$, i.e. ψ applied to the result of substituting the x-th numeral for the first variable of x. Let φ be $\chi(\underline{\chi})$, i.e. $Subst(\chi, \underline{\chi})$. Then

$$\text{PA}(\mathcal{C}\forall)Tr \vdash \varphi \equiv \chi(\underline{\chi}) \equiv \psi(Sub\dot{s}t(\underline{\chi}, \dot{\underline{\chi}}) \equiv \psi(\underline{Subst(\chi, \underline{\chi})}) \equiv \psi(\underline{\varphi}).$$

$\square$

9.3.14 *Proof of 9.3.9.* For each $\mathcal{C}\forall$ we can construct the usual liar formula (using the diagonal lemma for QA$(\mathcal{C}\forall)Tr$, see below), i.e. we get a sentence λ such that our theory proves $\lambda \equiv \neg\lambda$. But this is contradictory over both G$\forall$ and $\Pi\forall$ since they both prove $(\lambda \to \neg\lambda) \to \neg\lambda$ (and hence $(\neg\lambda \to \lambda) \to \neg\neg\lambda$).

Remark 9.3.15 In the rest of the section we shall deal with PA$(\text{Ł}\forall)Tr$ (and its strengthenings).We shall use the dot notation for the following operations:
strong disjunction $\varphi\underline{\vee}\psi$: PA$\vdash$ $\underline{\varphi}\dot{\underline{\vee}}\underline{\psi} = \underline{\varphi\underline{\vee}\psi}$;
iterated strong disjunction $n\varphi$ (i.e. $\varphi\underline{\vee}\ldots\underline{\varphi}$, n times):

$$\text{PA} \vdash \underline{n}\dot{\times}\underline{\varphi} = \underline{n\varphi};$$

Definition 9.3.16 The *modest liar formula* is the formula λ (existing by the diagonal lemma) such that

$$\text{PA}(\text{Ł}\forall)Tr \vdash \lambda \equiv (\exists x)Tr(x\dot{\times}\dot{\neg}\bar{\lambda}).$$

(Thus λ says: for some x, the x-th multiple of my negation is true.)

Now we are ready to prove Theorem 9.3.11.

9.3.17 *Proof of 9.3.11.*

Proof: (1) Suppose on the contrary that $\mathbf{M}$ was an expansion of $\mathbf{N}$ to a model of PA$(\text{Ł}\forall)Tr$. If $\|\lambda\|_{\mathbf{M}} = 1$, then $\|\neg\lambda\|_{\mathbf{M}} = 0$ and for each k, $\|k \times \neg\lambda\|_{\mathbf{M}} = 0$, so $\|Tr(\underline{k}\dot{\times}\dot{\neg}\underline{\lambda})\|_{\mathbf{M}} = 0$ and $\|(\exists x)Tr(x\dot{\times}\dot{\neg}\underline{\lambda})\|_{\mathbf{M}} = 0$, giving $\|\lambda\|_{\mathbf{M}} = 0$, a contradiction.

Thus it must be the case that $\|\lambda\|_{\mathbf{M}} < 1$. But then $\|\neg\lambda\|_{\mathbf{M}} > 0$, say $\|\neg\lambda\|_{\mathbf{M}} \geq \frac{1}{k}$ for some k. Hence $\|k \times \neg\lambda\|_{\mathbf{M}} = 1$, $\|Tr(\underline{k}\dot{\times}\dot{\neg}\underline{\lambda})\|_{\mathbf{M}} = 1$, so $\|(\exists x)Tr(x\dot{\times}\dot{\neg}\underline{\lambda})\|_{\mathbf{M}} = 1$ and $\|\lambda\|_{\mathbf{M}} = 1$, a contradiction. Therefore there is no such $\mathbf{M}$.

(2) We show that each finite subtheory of PA(Ł∀)Tr_2 has a (standard) model. Let $\varphi_1, \ldots, \varphi_m$ be sentences of PA(Ł∀)Tr_2. Define $\|Tr(\underline{n})\|_{\mathbf{M}} = 0$ for each $n \neq \varphi_1, \ldots, \varphi_m$; let $\|Tr(\underline{\varphi_i})\|_{\mathbf{M}} = x_i$ (to be determined shortly). We denote by $\mathbf{M}(\mathbf{x})$ the model given by $\mathbf{x} = (x_1, \ldots, x_m)$. By 5.4.29, $\|\varphi_i\|_{\mathbf{M}(\mathbf{x})}$ is a continuous function, $f_i(\mathbf{x})$ say, of $\mathbf{x}$. Obviously, $f_i : [0,1]^m \to [0,1]$. By Brouwer's fixed point theorem ([18],[191]), there is a fixed point $e_1, \ldots, e_m$ such that for each i, $e_i = f_i(\mathbf{e})$. Thus $\mathbf{M}(\mathbf{e})$ is the desired model: it satisfies PA(Ł∀), $\varphi_i \equiv Tr(\underline{\varphi_i})$ and, by standardness, the sentences true in $\mathbf{M}$ are closed under the induction rule.

Thus PA(Ł∀)Tr_2 is consistent, i.e. does not prove falsity $\underline{0}$.

Therefore PA(Ł∀)Tr_2 has a model by 5.4.24. □

Remark 9.3.18 It follows from the dequotation that for concrete formulas φ, ψ, Tr commutes with connectives, i.e.

$$\text{PA(Ł}\forall)Tr \vdash Tr(\underline{\varphi \to \psi}) \equiv (Tr(\underline{\varphi}) \to Tr(\underline{\psi})),$$

$$Tr(\underline{\varphi \& \psi}) \equiv (Tr(\underline{\varphi}) \& Tr(\underline{\psi})),$$

and therefore also

$$\text{PA(Ł}\forall)Tr \vdash Tr(\underline{\varphi \veebar \psi}) \equiv (Tr(\underline{\varphi}) \veebar Tr(\underline{\psi}))$$

etc. The corresponding formal axioms

$$Tr(x \dot{\to} y) \equiv (Tr(x) \to Tr(y)) \tag{$*$}$$

$$Tr(x \dot{\&} y) \equiv (Tr(x) \& Tr(y)) \tag{$**$}$$

are NOT provable in PA(Ł∀)Tr_2, moreover:

Theorem 9.3.19 PA(Ł∀)Tr_2 extended by the axioms $(*)$, $(**)$ is inconsistent.

The interested reader may consult the proof of this in [89].

9.4. Concluding remarks

Let us now summarize what we have achieved, recall some open problems and indicate some further directions of research.

9.4.1 *Propositional calculus.* We have developed logical systems with semantics based on a continuous t-norm as the truth function of the conjunction & and the residuum of the t-norm as the truth function of the implication $\rightarrow$. We have formulated the basic logic BL, sound for all those systems, and shown that BL is rather rich in consequences. We have introduced a variety of BL-algebras and shown that BL is sound and strongly complete for semantics based on all BL-algebras. We have developed the logics Ł, G, Π extending BL (Łukasiewicz, Gödel, product logic) and shown their completeness with respect to the corresponding standard semantics given by the corresponding particular continuous t-norm. (Strong standard semantics for G; for Ł and Π only strong standard completeness for finitely axiomatized theories.) We have investigated several variants and extensions, notably Rational Pavelka logic RPL, extending Ł by rational truth constants. It satisfies Pavelka-style completeness (provability degree equals truth degree). We have determined the computational complexity of the set of 1-tautologies, positive tautologies, 1-satisfiable and positively satisfiable formulas for each of the three logics and shown that everything is as expected (NP-complete, co-NP-complete).

9.4.2 *Predicate calculus.* For each predicate language $\mathcal{I}$ and each BL-algebra $\mathbf{L}$ we have $\mathbf{L}$-interpretations $\mathbf{M}$ of $\mathcal{I}$; predicates are interpreted as fuzzy relations on a given domain M and constants as some elements of M. An interpretation is safe if the truth value $\|\varphi\|_{\mathbf{M},v}$ is defined (by Tarski-style conditions) for each formula φ and each evaluation v of object variables. We have five axioms for quantifiers; together with the axioms of BL they form the axiom system of BL∀ (basic predicate logic). Similarly for Ł∀, G∀, Π∀. The logic BL∀ is strongly complete with respect to safe $\mathbf{L}$-interpretations over arbitrary BL-algebras $\mathbf{L}$. Standard strong completeness (with respect to interpretations over the standard semantics) holds for G∀; by contrast, neither Ł∀ nor Π∀ has a recursive axiomatization that would be complete with respect to the corresponding standard semantics. Rational Pavelka predicate logic RPL∀ has Pavelka-like strong completeness. This shows that the set of all its 1-tautologies (over the standard semantics) is a Π_2 set in the sense of the arithmetical hierarchy; moreover, it is Π_2-complete. Several other results on the position of various important sets of formulas in the arithmetical hierarchy were obtained. We have presented the many-sorted variant of all these systems and also the axiomatic theory of similarity (fuzzy equality).

9.4.3 *Applications to approximate reasoning and fuzzy control.* We formalized the popular expression "X is A" as $A(X)$ (A unary predicate, X constant); and later as $X \subseteq A$, i.e. $(\forall x)(X(x) \rightarrow A(x))$, where now X, A are predicates). An "IF-THEN RULE" becomes an implication $A(X) \rightarrow B(Y)$. First we investigated Zadeh's generalized modus ponens (and similar things).

We presented it in a purely syntactic form, as an implication

$$[(A(X) \to B(Y)\&A^*(X)\&Comp] \to B^*(Y)$$

provable in BL∀, where *Comp* is a formula defining B^* from A, B, A^*. Here and on similar places we stressed the fuzzy reading: the formula says much more than just "if [...] is true then $B^*(Y)$ is true too" namely "the degree of truth of [...] is a lower bound of the degree of truth of $B^*(Y)$".

Then we turned to systems of "IF-THEN rules". We first based our formalism on the notion of a fuzzy function F (using the notion of a similarity) and n examples $F(c_i, d_i)$ (c_i arguments, d_i images). $A_i(x)$ said $x \sim c_i$, $B_i(y)$ said $y \sim d_i$. We defined Mamdani's formula $MAMD$ as $\bigvee(A_i(x)\&B_i(y))$ and the system $RULES$ of rules as $\bigwedge(A_i(x) \to B_i(y))$. We showed the inclusion $MAMD \subseteq F \subseteq RULES$ (also to be read fuzzily!). We showed that $\bigvee_i A_i^2(x)$ (saying that x very much satisfies some A_i) implies $MAMD \equiv RULES$. We defined a theory *FC* (fuzzy control) over BL∀ containing A_i, B_i, A^*, B^* and having several axioms, among them a definition of B^* from the other predicates. We showed things like

$$FC \vdash (RULES\& \bigvee A_i^2(x)\&A^*(x)) \to B^*(y)$$

$$FC \vdash (A^* = A_i\&A_i^2 \neq \emptyset) \to B_i \subseteq B^*$$

$$FC \vdash (A^* = A_i\&A_i \text{ pairwise disjoint}) \to B^* \subseteq B_i$$

(Here we are rather informal; e.g. $A^* = A_i$ stands for $(\forall x)(A^*(x) \equiv A_i(x))$ etc. Read fuzzily!) This shows that fuzzy control indeed has logical aspects.

9.4.4 *Generalized quantifiers and modalities.* We understood modalities as hidden quantifiers. Four types of calculi were investigated, in dependence on the semantics: crisp or fuzzy models, and finite models or model of both finite and infinite cardinality. For all four types we investigated analogs of the logics S5 (of knowledge), KD45 (of belief) and then the logics with the modality /quantifiers "probably" (read "many" in the case of finite models). We focused our attention on the question of complete axiomatizability.

9.4.5 *Miscellanea.* We developed the Takeuti-Titani system TT∀ with three conjunctions and three implications (the system has an infinitary deduction rule), Pavelka's non-truth-functional fuzzy logics, and finally we presented an analysis of the liar's paradox and undefinability of truth in arithmetic. This last investigation has shown that fuzzy logic can be well understood as a branch of "philosophical logic"; it does not make the old questions and paradoxes trivial but brings new insight and understanding.

The reader is advised to have a look at the conclusion of Ch. 1 Sec. 1 and see if the claims made for fuzzy logic are now justified.

Let us now turn to open problems and topics of future research. First note that the index contains an entry "problem" referring to pages containing some open problems. Here we list both problems occuring explicitly in the text together with some other problems and topics.

9.4.6 *Propositional calculus.*

(1) Is BL complete with respect to t-tautologies? (See 2.3.23 and [72].) If not find an elegant axiomatization BLt of the set of all t-tautologies.

(2) Characterize the intersection of our logics Ł, G, Π (i.e. formulas provable in all these three logics) as well as the intersection of any two of them.[53] (It is clear that the intersection of G and Π is stronger than BL - e.g. $(\varphi \wedge \neg\varphi) \rightarrow \overline{0}$ is provable both in G and in Π but not in BL.)

(3) Investigate logics given by a particular t-norm different from the three famous ones (Łukasiewicz, Gödel, product). This is currently the topic of a Master's thesis.

(4) Investigate the extension of basic logic with a new negation satisfying the double negation axiom (intended semantics: any t-algebra expanded by the involutive negation $1 - x$), i.e. the logic of continuous t-norms and t-conorms. Note that the forthcoming paper [51] contains important results of this kind.

(5) Investigate logics with implications that are not R-implications (residua of continuous t-norms), e.g. so-called S-implications: one has a continuous t-norm $*$ a continuous involutive negation n (i.e. satisfying $n(n(x)) = x$ and decreasing) and the corresponding t-conorm $x \oplus y = n(n(x) * n(y))$. One defines $x \Rightarrow y$ as $n(x) \oplus y$. Note that such a logic may have no 1-tautologies if $x * y$ is $\min(x, y)$ and $n(x)$ is $1 - x$ then the evaluation giving $\frac{1}{2}$ to all variables gives $\frac{1}{2}$ to all formulas. This is not inviting; but note that this logic does have positive tautologies (never taking the value 0) and these are exactly all tautologies of classical propositional calculus. See [204], [112], [20] for details.

9.4.7 *Predicate calculus*

(6) A formula φ is a $[0,1]_{Ł}$-tautology of Łukasiewicz predicate logic Ł∀ iff $|\varphi|_{RPL} = 1$; but thanks to the non-axiomatizability of Ł∀ there are $[0,1]_{Ł}$-tautologies of Ł∀ not provable in Ł∀. Give a natural example.

(7) Is the set of all $[0,1]_{\Pi}$-tautologies of Π∀ a Π_2-set? (cf. the beginning of Ch. 6 Sec. 3)

[53] This question was asked by the anonymous referee.

(8) Let φ, ν be formulas, x a variable not free in ν. Is the formula $(\forall x)(\varphi \& \nu) \equiv ((\forall x)\varphi \& \nu)$ provable in Π∀ or even in BL∀? It is provable in G∀ and Ł∀ (see 5.1.19, 5.3.5,5.4.14,5.4.32).

(9) Investigate predicate calculi corresponding to propositional calculus mentioned above in (3),(4) as well as various subsystems of the logic TT∀ (see Ch. 9 Sec. 1).

9.4.8 *Approximate reasoning.* We have two rather general problems.

(10) Investigate deductive properties of $MAMD$ with & replaced by $\wedge$ (cf. 7.2.23).

(11) Apply the framework of Ch. 7 to investigate the theory FC (of fuzzy control) over logics stronger than BL∀ - i.e. Ł∀, G∀, Π∀. What specific results can one obtain? (Admittedly, this is a long-term task.)

Modal logics

(12) Do the axioms $(\Box 1) - (\Box 3)$, $(\Diamond 1) - (\Diamond 3)$ completely axiomatize the modal logic $\mathrm{S5}(\mathcal{C})$? See 8.3.16.

(13) Is the logic $\mathrm{S5}([0,1]_G)$ decidable (8.3.21)? And what can be said about $S5([0,1]_\Pi)$ (8.3.39)?

(14) Investigate the fuzzy logic of belief FMBEL, fuzzifying KD45 (see 8.3.43).

9.4.9 *Generalized quantifiers.*

(15) Do we have Pavelka-like strong completeness for $FP(RPL\odot)$ (see 8.4.19)? The same question for FMany$(RPL\odot)$ (8.4.21).

(16) Is the fuzzy logic $RPL\forall \int$ Pavelka complete with respect to probabilistic (not just weak probabilistic) models? See 8.5.15.

(17) Is the set of [0,1]-tautologies of $RPL\forall \int$ in Π_2? See 8.5.17.

(18) Is $RPL\forall \int$ -fin axiomatizable? See 8.5.19.(Solved: It is not.)

(19) Investigate fuzzy multitudinal quantifiers (see 8.5.25).

(20) Investigate an alternative semantics of $\forall, \exists$ based on infinitely iterated t-form diffrent from min (t-conorm diffrent from max). See 8.5.25.

9.4.10 *Comparative truth.* As mentioned above our investigation of the liar's paradox in Sec. 3 of the present chapter appears to show the importance of fuzzy logic as philosophical logic. This book offers no philosophical discussion (since the author is not a philosopher);[54] but hopefully offers foundations for a philosophical discussion on fuzzy logic as a formal system of deduction

[54] The reader may be interested in some informal discussion on fuzzy logic presented in [88], [78]. The latter paper is a reaction to an earlier criticism of fuzzy logic by Parikh, see [159, 160].

under vagueness and on a comparative notion of truth. Such a discussion, based on the formal results presented here, remains a topic for further research. The paper [135] may be relevant; see also the discussion in [22]. To close let me call the reader's attention to two quotations from Aristotle [7] tempting us to ask whether Aristotle was the first fuzzy logician.[55]

(1) *Categories Ch. 8, 10b.* "Qualities admit of variation of degree. Whiteness is predicated of one thing in a greater or less degree than of another. This is also the case with reference to justice. Moreover, one and the same thing may exhibit a quality in a greater degree than it did before: if a thing is white, it may become whiter."

(2) *Metaphysics Bk. 4, Ch. 4, 1008.* "Again, however much all things may be 'so and not so', still there is a more and a less in the nature of things; for we should not say that two and three are equally even, nor is he who thinks four things are five equally wrong with him who thinks that they are thousand. If then they are not equally wrong, obviously one is less wrong and therefore more right. If then that which has more of any quality is nearer the norm, there must be some truth to which the more true is nearer."

[55] I am grateful to Ms. K. Bendová for calling my attention to the first quotation; the second is found in [22].

CHAPTER TEN

HISTORICAL REMARKS

Here I refer to important works concerning the topics of the present book. Even if I have tried to be as complete as possible, I cannot hope to have covered everything; I apologize for all omissions. On the other hand, an attempt to collect all publications concerning fuzzy logic (in both broad and narrow senses) would lead to a special publication; note that e.g. the book of Klir and Yuan [115] contains 1731 references! We thus only select references that are relevant to fuzzy logic in the narrow sense.

10.1. Until the forties.

Even if there are some papers in the XIX century that are more or less relevant to many-valued logic, as [164] and [129] and, as Łukasiewicz mentions in [126], even Aristotle admitted that statements about the future may be neither true nor false, the real beginning of many-valued logic is the work of Jan Łukasiewicz; the first mention of a three-valued logic is in his speech published as [123]; this was elaborated in [124][56]. The first occurence of an infinitely valued logic seems to be in [125]; this is further discussed in [127], where the authors also refer to Łukasiewicz's conjecture that the set of five axioms (our (Ł1) to (Ł4) plus the prelinearity axiom) is complete for 1-tautologies of the infinitely-valued (Łukasiewicz) logic. Wajsberg [208] proves completeness of an axiom system of the three-valued Łukasiewicz logic and claims to have a completeness proof for the infinitely valued one; but his proof has been never published. In [126] Łukasiewicz discusses philosophical problems of his many-valued logic; it is interesting that his motivation for the third truth value was *modal*, i.e. he understood it as "possible"; he defined the formula "possibly φ" ($M\varphi$ in his symbolism) to be $\neg\varphi \to \varphi$ and verified some classical formulas like $\varphi \to M\varphi$. This may appear naive today since we know that modality is not a truth-functional connective; but do not forget that Kripke models came only 40 years later. Another remarkable thing is that the only connectives Łukasiewicz uses are implication and negation; in particular, what we call "Łukasiewicz's conjunction" seems never to appear explicitly in his writings. But note that Frink [53] uses the term "Łukasiewicz arithmetical conjunction" for what we call Łukasiewicz conjunction.

[56] See [17] for English versions of Łukasiewicz's works.

Moisil began the algebraic study of Łukasiewicz many-valued logics in the early forties. The work of Moisil (including his contributions to intuitionism and residuated lattices) is collected in the volume [142].[57]

Independently of Łukasiewicz, Post investigated finitely-valued calculi [166], apparently without any philosophical aims, but with an interest in functional completeness.

Heyting [96] introduced a three-valued propositional calculus related to intuitionistic logic. Gödel [60] has an infinite hierarchy of finitely-valued systems; his aim is to show that there is no finitely valued propositional calculus that is sound and complete for intuitionistic logic. What we call Gödel logic is just the infinite-valued version of his systems.

We mention two papers not related to logic but important for historical reasons. Menger [137] defines t-norms (in a way different from the presently used definition); and Dilworth and Ward [34] is an early paper on residuated lattices.

Finally we mention three papers of lesser importance for us, just since they are rather old: Bochvar [15], Reichenbach [171] and, surprisingly, Zich's paper in Czech [222].

10.2. THE FIFTIES

McNaughton's paper [136] characterizes functions definable by formulas of infinitely valued Lukasiewicz logic (as continuous mappings of $[0,1]^n$ to $[0,1]$ that are piecewise linear with integer coefficients). The paper [143] contains the first proof (known to the author) of the characterization of continuous t-norms (our 2.1.16), in a more general context of semigroups on a compact manifold. The representation of Łukasiewicz's t-norm by restricted product in [143] is crucial for our 2.1.22 (2) and through this for our embedding of Łukasiewicz into product logic (4.1.16). The year 1958 was very fruitful: Rose and Rosser [177] gave the first (published) proof of the completeness of Łukasiewicz propositional logic (this was preceded by [175], [176]) and Chang started a series of papers [24] (proving the axiom of prelinearity to be redundant as an axiom of Łukasiewicz logic; the same result appears to be obtained by Meredith [138]), [23] (introducing the notion of an MV-algebra), [25] (giving an algebraic completeness proof of Łukasiewicz propositional logic).

In 1959, Dummett publishes his [46], proving completeness of infinite-valued Gödel logic.

[57] I am grateful to R. Cignoli and Ms. L. Iturrioz for calling my attention to the work of Moisil.

Let us mention some other papers: Kleene [111] uses a three-valued logic in his discussion of recursive functions; Church [29] uses some finitely valued logics in proofs of independence of axioms of systems of two-valued logic. Skolem [190] works with infinitely valued set theory (!). Rose [174] has an eight-valued logic and Takekuma [194] a nine-valued.

10.3. THE SIXTIES

The series of Chang's works continued with joint works with Keisler [26, 27]. The latter is a book on general model theory of infinitely-valued predicate calculi satisfying certain conditions of continuity (e.g. Łukasiewicz predicate calculus). Scarpellini [181] showed that the latter calculus is not recursively axiomatizable; Hay [95] gave a non-finitary syntactic characterization of of its 1-tautologies. See also Mostowski [145], Belluce-Chang [13][58], Belluce [12]. Let us also mention Rutledge [180] as an early paper on infinitely valued predicate calculi. Horn [103, 102] presented a completeness result for Gödel predicate logic; he explicitly formulated the axiom that we call($\forall 3$).

The notion of t-norms, in the presently used meaning, was introduced by Schweizer and Sklar in [183]; Ling [122] (re)proved the characterization of continuous t-norms.

1965 is the year when fuzzy sets were born: Zadeh published his [211]. Goguen [65] seems to be the first paper on fuzzy logic; note that in that paper, residuated lattices (under the name *closgs*) are introduced in the context of fuzzy logic (or, as Goguen nicely says, the logic of inexact concepts). In [64] Goguen studied fuzzy sets with values in a lattice.

The monograph of Rasiowa and Sikorski [169] contains much material on algebraic structures relevant for many-valued logics, notably on algebras we (and others) call Heyting algebras; Rasiowa and Sikorski prefer the name pseudo-boolean algebras. I mention also Ackermann's monograph [1]. Ref. [165] is on the deduction theorem.

10.4. THE SEVENTIES

In 1970, selected works of Łukasiewicz appeared [17] edited by Borkowski. Rasiowa [168] worked with an infinite valued predicate calculus. Scott [185] discussed possible meaning of many truth values and gave an alternative completeness proof of Łukasiewicz logic.

[58] Shepherdson proved that the soundness theorem as presented in this paper is incorrect

Of Zadeh's publications we mention [214] (on linguistic variables) and [215] (on fuzzy logic). In his [213] he introduces generalized modus ponens and compositional rule of inference. Chang and Lee [28] discuss the propositional calculus with max-disjunction, min-conjunction and the negation $1 - x$ and call it fuzzy logic.

Pavelka's series of papers [163] is a very important contribution to the syntax and semantics of fuzzy logic, starting on a very abstract level and ending by a completeness theorem for his system of graded Łukasiewicz-style logic with real truth constants. The work is pioneering in many aspects; the only disadvantage is the fact that the author did not know of the work on completeness of Łukasiewicz propositional logic which made his system unnecessarily complicated.

Also note that in 1979 Rodriguez [172] introduced the notion of Wajsberg algebras. Johnstone [106] discusses the axiom of prelinearity in the context of deMorgan rules. Lee [121] it the first paper on resolution in fuzzy logic.

10.5. The Eighties

Ragaz's dissertation [167] contained many results on arithmetical complexity of many-valued (Łukasiewicz) logic, notably the result on the Π_2-completeness of 1-tautologies of Łukasiewicz predicate calculus, improving Scarpellini's result. Font et al. [52] elaborates the theory of Wajsberg algebras. The paper of Takeuti and Titani [195] presented a very important contribution to Gödel logic (even if the name Gödel is not mentioned); our presentation of the axiomatizability of Gödel logic is an elaboration of [195]. I note that Professor H. Ono has informed me on Takano's [193] where the author proved strong completeness of Gödel predicate logic with an axiom system equivalent to ours, namely intuitionist logic, prelinearity and ($\forall$3). Mundici's [149] started a series of his papers related to MV-algebras and Łukasiewicz logic; here he introduced the notion of a good sequence, crucial for his proof of Chang's representation of linearly ordered MV-algebras as intervals in linearly ordered Abelian groups. We refer also to [150] where Mundici proves the set of all 1-tautologies of Łukasiewicz propositional calculus to be co-NP-complete.

Of the extensive work of Dubois and Prade we refer to [36, 37, 38]; from the work of Bandler and Kohout [10, 11]. Zadeh discussed the notion of fuzzy logic in [217, 219]. Alsina, Trillas and Valverde [6, 203] discussed fuzzy connectives. Mukaidono's [146, 147] are basic papers for fuzzy Prolog (see also [148].

Gottwald's monograph [67] has long been the basic work on many-valued logics. Other important monographs: Schweizer and Sklar [184]; Klir and Folger [114]; for the history of Polish logic Woleński [209]. Finally I mention

Novák's [153], his first paper on Pavelka-style predicate calculus (see the next subsection).

10.6. THE NINETIES

Of Zadeh's writings see [220, 221]. Dubois and Prade [41, 42] offer a broad survey of the understanding of fuzzy logic in the fuzzy community. Also the monograph Klir-Yuan [115] brings detailed information. Of the numerous literature on fuzzy IF-THEN rules let us cite [44]. There are several papers on fuzzy logic programming; see e.g. Klawonn-Kruse [110], Vojtáš-Paulík [206].

From the work of Mundici and his group we refer to [151], [31] (our proof of Chang's representation theorem is based on the latter paper) and to the book in preparation by R. Cignoli, I. d'Ottaviano and D. Mundici [30]. In [152], Mundici studies analogues of probability on MV-algebras.

Novák's attempt [153] to develop Pavelka-style predicate calculus suffered from a mistake, improved in [154], but the system was similarly complicated as Pavelka's approach. I obtained a simplification of Pavelka's propositional logic (by eliminating irrational truth constants, making use of Łukasiewicz's axioms and by some small tricks) in [74] and of its predicate logic version in [77] (see also [75]). Independently, Novák presented a simplification of his system in [156]

Höhle's paper [98] was very inspiring in our development of the Basic logic BL. Proofs of several formulas in Chapter 2 Section 2 have their counterparts in [98]. See also his [99, 100]. The propositional product logic was axiomatized in [80]; [79] discusses the relation of fuzzy logic to probability. I also refer to [83, 84] for logics of both vagueness and uncertainty (many-valued modal logics). Baaz's paper [8] is a very valuable contribution to Gödel logic; I have made extensive use of his $\triangle$-connective in Chapters 2, 3. Hähnle's [71] proves the NP-completenes of satisfiability in Łukasiewicz logic using Mixed Interger Programming; our presentation is based on his.

Hájek-Kohout [87] elaborate [10] in relation with multitudinal quantifiers (see also [76]). The theory of t-norms is developed e.g. in Butnariu-Klement [21], De Baets-Mesiar [33], Mesiar [139, 141, 140]; the forthcoming monograph [113] by Klement, Mesiar and Papp will be extremely informative.

Gottwald's second monograph [68] pays much attention to fuzzy control and relational equations but also contains material about t-norms and corresponding logics; it is well written and can be fully recommended (caution: Proposition 1.27 is wrong, see here 5.3.6).

The monograph [118] by Kruse *et al.* (English translation [119]) contains, among other things, an analysis of fuzzy control in terms of similarity relations; our presentation is partially dependent on theirs. See also Klawonn-Castro [109].

Paris's monograph [162] concentrates on probabilistic theories of belief but has a chapter on "truth-functional belief" containing basic information on continous t-norms and related connectives. Its criticism of truth-functional belief is fully justified; we have stressed repeatedly that for us fuzzy logic is a logic of degrees of truth, not of degrees of belief. For a recent discussion on fuzzy logic (in the narrow sense) see [88] and [161].

The second paper by Takeuti and Titani on fuzzy logic [196] is the base of our Chapter 9 Section 1. Baaz *et al.* [9] study the embedding of Łukasiewicz logic into product logic.

Dubois, Prade and Yager published an excellent anthology [45] of important older papers on fuzzy sets and fuzzy logic. Klir and Yuan published collected papers by L.Zadeh [116].

REFERENCES

1. ACKERMANN, R. J. *An Introduction to Many-valued Logics*. Dover, New York, 1967.
2. ADILLON, R. J., AND VERDÙ, V. Product logic and the deduction theorem. Math. Preprint Series 232, Universitat de Barcelona, 1997.
3. ADLASSNIG, K.-P. Approaches to computer-assisted medical diagnosis. *Computers in Biology and Medicine 15* (1985), 315–335.
4. ADLASSNIG, K.-P. Fuzzy set theory in medical diagnosis. *IEEE Transaction on Systems, Man, and Cybernetics SMC-16*, 2 (1986), 260–265.
5. ADLASSNIG, K.-P. Update on CADIAG-2: a fuzzy medical expert system for general internal medicine. In *Progress in Fuzzy Sets and Systems* (1990), W. H. Janko, M. Roubens, and H.-J. Zimmermann, Eds., Kluwer Academic Publishers, pp. 1–6.
6. ALSINA, C., TRILLAS, E., AND VALVERDE, L. On some logical connectives for fuzzy set theory. *Journal of Mathematical Analysis and Applications 93*, 1 (1983), 15–26.
7. ARISTOTLE. *The Basic Works of Aristotle, edited and with an introduction by Richard McKeon*. Random House, New York, 1941.
8. BAAZ, M. Infinite-valued Gödel logics with 0-1-projections and relativizations. In *GÖDEL'96 - Logical Foundations of Mathematics, Computer Science and Physics; Lecture Notes in Logic 6* (1996), P. Hájek, Ed., Springer-Verlag, pp. 23–33.
9. BAAZ, M., HÁJEK, P., KRAJÍČEK, J., AND ŠVEJDA, D. Embedding logics into product logic. *Studia Logica* (to appear).
10. BANDLER, W., AND KOHOUT, L. J. Semantics of implication operators and fuzzy relational products. *Int. Journal of Man-Machine Studies 12* (1980), 89–116.
11. BANDLER, W., AND KOHOUT, L. J. Unified theory of multiple-valued logical operators in the light of the checklist paradigm. In *Proc. of the 1984 IEEE Conference on Systems, Man and Cybernetics* (New York, 1984), IEEE, pp. 356–364.
12. BELLUCE, L. P. Further results on infinite valued predicate logic. *J. Symb. Logic 29* (1964), 69–78.
13. BELLUCE, L. P., AND CHANG, C. C. A weak completeness theorem on infinite valued predicate logic. *Jn. Symb.Logic 28* (1963), 43–50.
14. BENDOVÁ, K., AND HÁJEK, P. Possibilistic logic as a tense logic. In *Qualitative Reasoning and Decision Technologies (Proceedings of QUARDET'93)* (Barcelona, 1993), N. P. Carreté et al., Eds., CIMNE, pp. 441–450.
15. BOCHVAR, D. A. Ob odnom trehznačnom isčislenii i ego primenenii k analizu klassičeskogo rasširennogo funkcional'nogo isčislenia (On a three-valued

calculus and its application to the analysis of the extended functional calculus, in Russian). *Matem. Sbornik n.s. 4* (1938), 287–308.

16. BÖRGER, E., GRÄDEL, E., AND GUREVICH, Y. *The Classical Decision Problem.* Springer-Verlag, 1997.
17. BORKOWSKI, L. *J. Łukasiewicz: Selected Works.* Studies in logic and the foundations of Mathematics. North-Holland Publ. Comp. Amsterdam and PWN Warszawa, 1970.
18. BROUWER, L. E. J. Über Abbildungen von Mannigfaltigkeiten. *Math. Ann. 71* (1910), 97 – 115.
19. BURGESS, J. P. Basic tense logic. In *Handbook of Philosophical Logic, Vol. II* (Dordrecht, 1984), F. G. D. Gabbay, Ed., Reidel, pp. 89–133.
20. BUTNARIU, D., KLEMENT, E. P., AND ZAFRANY, S. On triangular norm-based propositional fuzzy logics. *Fuzzy Sets and Systems 69* (1995), 241–255.
21. BUTNARIU, D., AND KLEMENT, P. *Triangular Norm-Based Measures and Games with Fuzzy Coalitions.* Kluwer, 1993.
22. CASARI, E. Conjoining and disjoining at different levels. In *Logic and Scientific Methods, Volume One of the Tenth International congress of Logic, Methodology and Philosophy of Science* (Florence, 1997), M. L. dalla Chiara et al., Ed., Kluwer, pp. 261–288.
23. CHANG, C. C. Algebraic analysis of many-valued logics. *Trans. A.M.S. 88* (1958), 467–490.
24. CHANG, C. C. Proof of an axiom of Łukasiewicz. *Trans. A.M.S. 87* (1958), 55–56.
25. CHANG, C. C. A new proof of the completeness of the Łukasiewicz's axioms. *Trans. A.M.S. 93* (1959), 74–80.
26. CHANG, C. C., AND KEISLER, H. J. Model theories with truth values in an uniform space. *Bull. Amer. Math. Soc. 68* (1962), 107–109.
27. CHANG, C. C., AND KEISLER, H. J. *Continuous Model Theory.* Princeton Univ. Press, 1966.
28. CHANG, C. L., AND LEE, R. C. T. Some properties of fuzzy logic. *Information and Control 19* (1975), 417–431.
29. CHURCH, A. *Introduction to Metamathematics I.* Princeton Univ. Press, 1956.
30. CIGNOLI, R., D'OTTAVIANO, I. M. L., AND MUNDICI, D. Algebraic foundations of many-valued reasoning, book in preparation.
31. CIGNOLI, R., AND MUNDICI, D. An elementary proof of Chang's completeness theorem for the infinite-valued calculus of Łukasiewicz. *Studia Logica 58* (1997), 79–97.
32. DANIEL, M., HÁJEK, P., AND NGUYEN, P. H. CADIAG-2 and MYCIN-like systems. *Artificial Intelligence in Medicine 9* (1997), 241–259.
33. DE BAETS, B., AND MESIAR, R. Residual implicators with respect to continuous t-norms. In *Proceedings of EUFIT'96* (Aachen, Germany, 1996), pp. 27–31.
34. DILWORTH, R. P., AND WARD, M. Residuated lattices. *Trans. A.M.S. 45* (1939), 335–354.

35. DRIANKOV, D., HELLENDOORN, H., AND REINFRANK, M. *An Introduction to Fuzzy Control.* Springer-Verlag, Berlin Heidelberg, 1993.
36. DUBOIS, D., AND PRADE, H. *Fuzzy Sets and Systems: Theory and Applications.* Academic Press, New York, 1980.
37. DUBOIS, D., AND PRADE, H. Fuzzy logics and the generalized modus ponens revisited. *Cybernetics and Systems 15* (1984), 293–331.
38. DUBOIS, D., AND PRADE, H. Evidence measures based on fuzzy information. *Automatica 31* (1985), 547–562.
39. DUBOIS, D., AND PRADE, H. Fuzzy cardinality and the modelling of imprecise quantification. *Fuzzy Sets and Systems 16* (1985), 199–230.
40. DUBOIS, D., AND PRADE, H. *Possibility Theory.* Plenum Press, New York, 1988.
41. DUBOIS, D., AND PRADE, H. Fuzzy sets in approximate reasoning, part 1: Inference with possibility distribution. *Fuzzy Sets and Systems 40* (1991), 143–202.
42. DUBOIS, D., AND PRADE, H. Fuzzy sets in approximate reasoning, part 2: Logical approaches. *Fuzzy Sets and Systems 40* (1991), 203–244.
43. DUBOIS, D., AND PRADE, H. Fuzzy rules in knowledge-based systems – modelling gradedness, uncertainty and preference. In *An Introduction to Fuzzy Logic Applications in Intelligent Systems*, R. R. Yager and L. A. Zadeh, Eds. Kluwer, 1992.
44. DUBOIS, D., AND PRADE, H. Possibility theory as a basis for preference propagation in automated reasoning. In *IEEE Int. Conf. on Fuzzy Systems FUZZ-IEEE'92* (San Diego, 1992), pp. 821–832.
45. DUBOIS, D., PRADE, H., AND YAGER, R. R., Eds. *Readings in Fuzzy Sets for Intelligent Systems* (1993), Morgan Kaufman Publishers.
46. DUMMETT, M. A propositional calculus with denumerable matrix. *J. Symb. Logic 24* (1959), 97–106.
47. EBBINGHAUS, H. D., AND FLUM, J. *Finite Model Theory.* Perspectives in Math. Logic. Springer-Verlag, 1995.
48. EBBINGHAUS, H. D., FLUM, J., AND THOMAS, W. *Mathematical Logic.* Springer-Verlag, New York, 1984.
49. ELKAN, C. The paradoxical success of fuzzy logic. In *Proceedings of the Eleventh National Conference on Artificial Intelligence AAAI'93* (1993), MIT Press, pp. 203–244.
50. ESTEVA, F., GARCIA, P., GODO, L., AND RODRIGUEZ, R. A modal account of similarity-based reasoning. *International Journal of Approximate Reasoning* (1997). To appear.
51. ESTEVA, F., GODO, L., AND HÁJEK, P. Residuated fuzzy logics with an involutive negation. Submitted.
52. FONT, J., RODRIGUEZ, A. J., AND TORRENS, A. Wajsberg algebras. *Stochastica 8* (1984), 5–31.
53. FRINK, O. New algebras of logic. *Amer. Math. Monthly 45* (1938), 210–219.
54. FUCHS, L. *Partially Ordered Algebraic Structures.* Pergamon Press, Oxford, 1963.

55. GABBAY, D., AND GUENTHNER, F. *Handbook of Philosophical Logic, Vol. II.* Reidel, Dordrecht, 1994.
56. GABBAY, D., AND GUENTHNER, F. *Handbook of Philosophical Logic, Vol. III.* Reidel, Dordrecht, 1996.
57. GABBAY, D. M. Tense logics with discrete moments of time. *Journ. Philos. Logic 1* (1972), 35–45.
58. GABBAY, D. M. Model theory of tense logics. *Annals of Pure and Applied Logic 8* (1975), 185–236.
59. GERLA, G. Inferences in probability logic. *Artificial Intelligence 70* (1994), 33–52.
60. GÖDEL, K. Zum intuitionistischen Aussagenkalkül. *Anzeiger Akademie der Wissenschaften Wien, Math. - naturwissensch. Klasse 69* (1932), 65–66. Also in Ergebnisse eines matematischen Kolloquiums 4 (1933), 40.
61. GODO, L., AND HÁJEK, P. On deduction in Zadeh's fuzzy logic. In *IPMU'96 Information Processing and Management of Uncertainty in Knowledge-Based Systems* (Granada, Spain, 1996), pp. 991–996.
62. GODO, L., AND HÁJEK, P. Fuzzy inference as deduction. *Int. Journal of Applied Non-Classical Logic* ((to appear)).
63. GODO, L., AND HÁJEK, P. A note on fuzzy inference as deduction. In *Proc. Linz Seminar 1997 on fuzzy logic* (Linz, Austria, to appear), D. Dubois et al., Ed.
64. GOGUEN, J. A. L-fuzzy sets. *J. Math. Anal. and Applications 18* (1967), 145–174.
65. GOGUEN, J. A. The logic of inexact concepts. *Synthese 19* (1968-9), 325–373.
66. GOTTWALD, S. *Many-valued Logic.* revised English version of Mehrwertige Logik (to appear).
67. GOTTWALD, S. *Mehrwertige Logik.* Akademie-Verlag, Berlin, 1988.
68. GOTTWALD, S. *Fuzzy Sets and Fuzzy Logic.* Vieweg, Wiesbaden, 1993.
69. GRÄTZER, G. *Universal Algebra.* Springer, Berlin, Heidelberg, New York, 1979.
70. GUREVICH, Y. S., AND KOKORIN, A. I. Universal equivalence of ordered Abelian groups (in Russian). *Algebra i logika 2* (1963), 37–39.
71. HÄHNLE, R. Many-valued logic and mixed integer programming. *Annals of Mathematics and Artificial Intelligence 12*, 3,4 (Dec. 1994), 231–264.
72. HÁJEK, P. BL-logic and BL-algebras. submitted.
73. HÁJEK, P. Generalized quantifiers and finite sets. In *Proc. Set Theory and Hierarchy Theory '74.* Politechnika Wrocłav, 1977, pp. 91–104.
74. HÁJEK, P. Fuzzy logic and arithmetical hierarchy. *Fuzzy Sets and Systems 73*, 3 (1995), 359–363.
75. HÁJEK, P. Fuzzy logic from the logical point of view. In *SOFSEM'95: Theory and Practice of Informatics; Lecture Notes in Computer Science 1012* (Milovy, Czech Republic, 1995), M. Bartošek, J. Staudek, and J. Wiedermann, Eds., Springer-Verlag, pp. 31–49.

76. HÁJEK, P. A remark on Bandler–Kohout products of relations. *International Journal of General Systems 25*, 2 (1996), 165–166.
77. HÁJEK, P. Fuzzy logic and arithmetical hierarchy II. *Studia Logica 58* (1997), 129–141.
78. HÁJEK, P. Ten questions and one problem on fuzzy logic. *Annals of Pure and Applied logic* (to appear).
79. HÁJEK, P., ESTEVA, F., AND GODO, L. Fuzzy logic and probability. In *Uncertainty in Artificial Intelligence; Proceedings of the Eleventh Conference (1995)* (Montreal, Canada, 1995), P. Besnard and S. Hanks, Eds., pp. 237–244.
80. HÁJEK, P., ESTEVA, F., AND GODO, L. A complete many-valued logic with product-conjunction. *Archive for Math. Logic 35* (1996), 191–208.
81. HÁJEK, P., AND GODO, L. Deductive systems of fuzzy logic. In *Fuzzy Structures Current Trends*, R. Mesiar and B. Riečan, Eds., vol. 13 of *Tatra Mountains Mathematical Publications*. Math. Inst. Slovak Acad. Sci. Bratislava, 1997, pp. 35–68.
82. HÁJEK, P., AND HARMANCOVÁ, D. A many-valued modal logic. In *IPMU'96 Information Processing and Management of Uncertainty in Knowledge-Based Systems* (Granada, Spain, 1996), pp. 1021–1024.
83. HÁJEK, P., HARMANCOVÁ, D., ESTEVA, F., GARCIA, P., AND GODO, L. On modal logics for qualitative possibility in a fuzzy setting. In *Uncertainty in Artificial Intelligence; Proceedings of the Tenth Conference* (Seattle, WA, 1994), R. López de Mántaras and D. Poole, Eds., pp. 278–285.
84. HÁJEK, P., HARMANCOVÁ, D., AND VERBRUGGE, R. A qualitative fuzzy possibilistic logic. *International Journal of Approximate Reasoning 12* (1994), 1–19.
85. HÁJEK, P., AND HAVRÁNEK, T. *Mechanizing Hypothesis Formation (Mathematical Foundations for a General Theory).* Springer-Verlag, Berlin-Heidelberg-New York, 1978.
86. HÁJEK, P., HAVRÁNEK, T., AND JIROUŠEK, R. *Uncertain Information Processing in Expert Systems.* CRC Press, Boca Raton, Florida, 1992.
87. HÁJEK, P., AND KOHOUT, L. Fuzzy implications and generalized quantifiers. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 4*, 3 (1996), 225–233.
88. HÁJEK, P., AND PARIS, J. A dialogue on fuzzy logic. *Soft Computing 1* (1997), 3–5.
89. HÁJEK, P., PARIS, J., AND SHEPHERDSON, J. The liar paradox and fuzzy logic, 1997. submitted.
90. HÁJEK, P., PARIS, J., AND SHEPHERDSON, J. Rational Pavelka predicate logic is a conservative extension of Łukasiewicz predicate logic. Submitted.
91. HÁJEK, P., AND PUDLÁK, P. *Metamathematics of First-order Arithmetic.* Springer-Verlag, 1993. 460 pp.
92. HÁJEK, P., SOCHOROVÁ, A., AND ZVÁROVÁ, J. GUHA for personal computers. *Computational Statistics & Data Analysis 19* (1995), 149–153.

93. HÁJEK, P., AND ŠVEJDA, D. A strong completeness theorem for finite theories in Łukasiewicz's logic. *Tatra Mountains Mathematical Journal 12* (1997), 213–219.
94. HARMANEC, D., AND HÁJEK, P. A qualitative belief logic. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 2*, 2 (1994), 227–236.
95. HAY, L. Axiomatization of the infinite-valued predicate calculus. *Journ. Symb. Logic 28* (1963), 77–86.
96. HEYTING, A. Die formalen Regeln der intuitionistischen Logik. In *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*, Math. Kl. Berlin, 1930, pp. 42–56.
97. HÖHLE, U. The Poincaré paradox and the cluster problem. In *Trees and Hierarchical Structures, Lect. Notes in Biomathematics 84*, A. Dress and A. v. Haeseler, Eds. Springer, 1990, pp. 117–124.
98. HÖHLE, U. Commutative residuated monoids. In *Non-classical Logics and their Applications to Fuzzy Subsets (A handbook of the mathematical foundations of the fuzzy set theory)*, U. Höhle and P. Klement, Eds. Kluwer, Dordrecht, 1995.
99. HÖHLE, U. Enriched MV-algebras. *Mathware & Soft Computing 2* (1995), 167–180.
100. HÖHLE, U. On the fundamentals of fuzzy set theory. *J. Math. Analysis and Applications 201* (1996), 786–826.
101. HOOVER, D. Probability logic. *Ann. Math. Logic 14* (1978), 287–313.
102. HORN, A. Free L-algebras. *J. Symb. Logic 34* (1969), 475–480.
103. HORN, A. Logic with truth values in a linearly ordered Heyting algebra. *J. Symb. Logic 34* (1969), 395–408.
104. HUGHES, G., AND CRESSWELL, M. *An Introduction to Modal Logic*. Methuen, London, 1968.
105. IEEE Expert, August 1994. (special issue "A fuzzy logic symposium").
106. JOHNSTONE, P. T. Conditions related to de Morgan's law. In *Applications of Sheaves. Lecture Notes in Math. 753* (1979), M. Fourman et al., Ed., Springer-Verlag, pp. 479–491.
107. KEISLER, J. Logic with the quantifier "there are uncountably many". *Annals Math. Log. 1* (1970), 1–93.
108. KEISLER, U. J. Probability quantifiers. In *Model-Theoretic Logics*, J. Barwise and S. Feferman, Eds. Springer-Verlag, New York, 1985, pp. 539–556.
109. KLAWONN, F., AND CASTRO, J. L. Similarity in fuzzy reasoning. *Mathware & Soft Computing 2* (1995), 197–228.
110. KLAWONN, F., AND KRUSE, R. A Łukasiewicz logic based Prolog. *Mathware & Soft Computing 1* (1994), 5–29.
111. KLEENE, S. C. *Introduction to metamathematics*. D. van Nostrand Company, Inc., 1952.
112. KLEMENT, E. P. Operations on fuzzy sets and fuzzy numbers related to triangular norms. In *Proceedings of the Eleventh International Symposium on Multiple-Valued Logic* (Oklahoma, 1981), IEEE, New York, pp. 218–225.

113. KLEMENT, P., MESIAR, R., AND PAPP, L. *Triangular norms.* in preparation.
114. KLIR, G. J., AND FOLGER, T. A. *Fuzzy Sets, Uncertainty, and Information.* Prentice-Hall, Englewood Cliffs, NJ, 1988.
115. KLIR, G. J., AND YUAN, B. *Fuzzy Sets and Fuzzy Logic: Theory and Applications.* Prentice Hall PTR, Upper Saddle River, NJ, 1995.
116. KLIR, G. J., AND YUAN, B., Eds. *Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers by Lotfi A. Zadeh.* World Scientific, Singapore, 1996.
117. KOLARZ, G., AND ADLASSNIG, K.-P. Problems in establishing the medical expert systems CADIAG-1 and CADIAG-2 in rheumatology. *Journal of Medical Systems 10*, 4 (1986), 395–405.
118. KRUSE, R., GEBHART, J., AND KLAWONN, F. *Fuzzy–Systeme.* B. E. Teubner, Stuttgart, 1993.
119. KRUSE, R., GEBHART, J., AND KLAWONN, F. *Foundations of Fuzzy Systems.* John Willey, 1994.
120. KRYNICKI, M., MOSTOWSKI, M., AND SZCZERBA, L. W., Eds. *Quantifiers: Logics, Models and Computing*, vol. I Surveys. Kluwer Academic Publishers, 1995.
121. LEE, R. C. T. Fuzzy logic and the resolution principle. *Jour. Assoc. Computing Machinery 19* (1972), 109–119.
122. LING, C. H. Representation of associative functions. *Publ. Math. Debrecen 12* (1965), 182–212.
123. ŁUKASIEWICZ, J. Farewell lecture at Warsaw University. Warszawa, March 7 1918.
124. ŁUKASIEWICZ, J. O logice trówartosciowej (on three-valued logic). *Ruch filozoficzny 5* (1920), 170–171.
125. ŁUKASIEWICZ, J. Zagadnienia prawdy (the problems of truth). In *Księga pamiątkowa XI zjazdu lekarzy i przyrodników polskich* (1922), pp. 84–85,87.
126. ŁUKASIEWICZ, J. Philosophische Bemerkungen zu mehrwertigen Systemen der Aussagenlogik. *Comptes Rendus de la Siciete des Sciences et des Letters de Varsovie, cl.iii 23* (1930), 51–77.
127. ŁUKASIEWICZ, J., AND TARSKI, A. Untersuchungen über den Aussagenkalkül. *Comptes Rendus de la Siciete des Sciences et des Letters de Varsovie, cl. iii 23* (1930), 1–21.
128. MAC LANE, S., AND BIRKHOFF, G. *Algebra.* The Macmillan Company, 1967.
129. MACCOLL, H. The calculus of equivalent statements. *Proc. London Math. Society* (1896-7), 156–183.
130. MAKOWSKY, J. A., AND PNUELI, Y. B. Computable quantifiers and logics over finite structures. In Krynicki *et al.* [120].
131. MAMDANI, E. H. Application of fuzzy algorithms for the control of a simple dynamic plant. In *Proc IEEE* (1974), pp. 121–158.
132. MAMDANI, E. H. Advances in the lingvistic synthesis of fuzzy controllers. *Int. J. Man-Machine Studies 8* (1976), 669.
133. MAMDANI, E. H. Application of fuzzy logic to approximate reasoning using linguistic synthesis. *IEEE Trans. Computer 26* (1977), 1182–1191.

134. MARKS-II, R. J. Fuzzy logic technology and applications, 1994. IEEE Technical Activities Board.
135. MARTIN, J. N. Existence, negation and abstraction in the neoplatonic hierarchy. *History and Philosophy of Logic 16* (1995), 169–196.
136. MCNAUGHTON, R. A theorem about infinite-valued sentential logic. *Journ. Symb. Logic. 16* (1951), 1–13.
137. MENGER, K. Statistical metrics. *Proc. Nat. Acad. Sci. U.S.A. 8* (1942), 535–537.
138. MEREDITH, C. A. The dependence of an axiom of Łukasiewicz. *Trans. A.M.S. 87* (1958), 54.
139. MESIAR, R. Compensatory operators based on triangular norms and conorms. In *Proceedings of EUFIT'95* (Aachen, Germany, August 1995), pp. 131–135.
140. MESIAR, R. Fuzzy sets, difference posets, and MV-algebras. In *Fuzzy Logic and Soft Computing* (Singapore, 1995), B. Bouchon-Meunier, R. Yager, and L. Zadeh, Eds., World Scientific, pp. 345–352.
141. MESIAR, R. On some constructions of new triangular norms. *Mathware & Soft Computing 2*, 1 (1995).
142. MOISIL G. C. *Essais sur les logiques non-chrysippiennes.* Editions de l'Academie de la R. S. de Roumaine, Bucharest, 1972.
143. MOSTERT, P. S., AND SHIELDS, A. L. On the structure of semigroups on a compact manifold with boundary. *Annals of Math. 65* (1957), 117–143.
144. MOSTOWSKI, A. On a generalization of quantifiers. *Fundamenta mathematicae 44* (1957), 12–36.
145. MOSTOWSKI, A. Axiomatizability of some many valued predicate calculi. *Fund. Math. 50* (1961), 165–190.
146. MUKAIDONO, M. Fuzzy inference in resolution style. In *Fuzzy Sets and Possibility Theory*, R. Yager, Ed. Pergamon, New York, 1982, pp. 224–231.
147. MUKAIDONO, M. Towards a fuzzy Prolog. In *27th National Congress of the Information Processing Society of Japan* (Japanese, 1983), 1N-2, pp. 289–290.
148. MUKAIDONO, M., SHEN, Z. L., AND DING, L. Y. Fundamentals of fuzzy Prolog. *Int. Journal of Approximate Reasoning 3* (1989), 179–193.
149. MUNDICI, D. Interpretation of AF C^*-algebras in Łukasiewicz's sentential calculus. *Journal of Functional Analysis 65* (1986), 15–63.
150. MUNDICI, D. Satisfiability in many-valued sentential logic is NP-complete. *Theoretical Computer Science 52* (1987), 145–153.
151. MUNDICI, D. Ulam's game, Łukasiewicz logic and AF C^*-algebras. *Fund. Informaticae 18* (1993), 151–161.
152. MUNDICI, D. Averaging the truth-value in Łukasiewicz logic. *Studia Logica 55* (1995), 113–127.
153. NOVÁK, V. First-order fuzzy logic. *Studia Logica 46* (1987), 87–109.
154. NOVÁK, V. On the syntactico-semantical completeness of first-order fuzzy logic I, II. *Kybernetika 26* (1990), 47–26, 134–152.

155. NOVÁK, V. Fuzzy control from the logical point of view. *Fuzzy Sets and Systems 66* (1994), 159–173.
156. NOVÁK, V. A new proof of completeness of fuzzy logic and some conclusions for approximate reasoning. In *Proc. Int. Conference FUZZ-IEEE/IFES'95* (Yokohama, 1995), pp. 1461–1468.
157. OVCHINNIKOV, S. Similarity relations, fuzzy partitions and fuzzy orderings. *Fuzzy Sets and Systems 40* (1991), 107–126.
158. PAPADIMITRIOU, C. H. *Computational Complexity*. Addison Wesley, 1994.
159. PARIKH, R. A test for fuzzy logic. *SIGACT NEWS 22*, 3 (1991), 49–50.
160. PARIKH, R. Vagueness and utility: the semantics of common nouns. *Linguistics and Philosophy 17* (1994), 521–535.
161. PARIS, J. A semantics for fuzzy logic. *Soft Computing 1*, 3 (1997), 143–147.
162. PARIS, J. B. *The uncertain reasoner's companion – a mathematical perspective*. Cambridge University Press, 1994.
163. PAVELKA, J. On fuzzy logic I, II, III. *Zeitschr. f. Math. Logik und Grundl. der Math. 25* (1979), 45–52, 119–134, 447–464.
164. PEIRCE, C. S. On the algebra of logic: A contribution to the theory of notation. *Amer. J. Math. 7* (1885), 180–202.
165. POGORZELSKI, W. A. The deduction theorem for Łukasiewicz many valued propositional calculi. *Studia Logica 15* (1964), 7–23.
166. POST, E. L. Introduction to a general theory of elementary propositions. *Amer. Journ. of Math. 43* (1921), 163–185.
167. RAGAZ, M. E. Arithmetische Klassifikation von Formelnmengen der unendlichwertigen Logik. ETH Zürich, 1981. Thesis.
168. RASIOWA, H. On generalized Post algebras of order ω^+ and ω^+-valued predicate calculi. *Bull. Acad. Polon. Sci. Ser. Math., Astr., Phys. 21* (1973), 209–219.
169. RASIOWA, H., AND SIKORSKI, R. *The Mathematics of Metamathematics*. Panstwowe Wydawnictwo Naukowe, Warszawa, 1963.
170. RAŠKOVIĆ, M., AND DORDEVIĆ, R. *Probability quantifiers and operators*. Series in pure and applied mathematics. Vesta, Belgrade, 1996.
171. REICHENBACH, H. Philosophical Foundations of Quantum Mechanics. University of California Press Berkeley, Los Angeles, 1944.
172. RODRIGUEZ, A. J. El álgebra de Wajsberg de las partes difusas de un conjunto. *Rev. Univ. Santadei 2* (1979), II 927–929.
173. ROGERS JR., H. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, 1967.
174. ROSE, A. Eight-valued geometry. *Proc. London Math. Soc. Ser. 3 2* (1952), 30–44.
175. ROSE, A. A Gödel theorem for an infinite-valued erweiterter Aussagenkalkül. *Zeitschr. Math. Log. Grundlagen Math. 1* (1955), 89–90.
176. ROSE, A. Some formalizations of $\aleph_0$-valued propositional calculi. *Zeitschr. Math. Log. Grundlagen Math. 2* (1956), 204–209.
177. ROSE, A., AND ROSSER, J. B. Fragments of many-valued statement calculi. *Trans. A.M.S. 87* (1958), 1–53.

178. Ruspini, E. H. On the semantics of fuzzy logic. *International Journal of Approximate Reasoning 5* (1991), 45–88.
179. Rutledge, J. D. A preliminary investigation of the infinitely-valued predicate calculus. Cornell University, 1959. Thesis.
180. Rutledge, J. D. On the definition of an infinitely-valued predicate calculus. *J. Symb. Logic 25* (1960), 212–216.
181. Scarpellini, B. Die Nichtaxiomatisierbarkeit des unendlichwertigen Prädikatenkalküls von Łukasiewicz. *J. Symb. Log. 27* (1962), 159–170.
182. Schrijver, A. *Theory of Linear and Integral Programming.* John Willey & Sons, 1986.
183. Schweizer, B., and Sklar, A. Statistical metric spaces. *Pacific J. Math. 10* (1960), 313–334.
184. Schweizer, B., and Sklar, A. *Probabilistic Metric Spaces.* North Holland, Amsterdam, 1983.
185. Scott, D. Completeness and axiomatizability in many-valued logic. In *Proceedings of the Tarski symposium* (1974), L. Henkin *et al.*, Eds., Proceedings of symposia in pure and applied mathematics Vol XXV, American Math. Soc.
186. Shafer, G. *A Mathematical Theory of Evidence.* Princeton University Press, Princeton, New Jersey, 1976.
187. Shafer, G. Allocations of probability. *The Annals of Probability 7*, 5 (1979), 827–839.
188. Shafer, G. Perspectives on the theory and practice of belief functions. *International Journal of Approximate Reasoning 4*, 5/6 (1990), 323–362.
189. Shoenfield, J. R. *Mathematical Logic.* Addison-Wesley, 1967.
190. Skolem, T. A. Mengenlehre gegründet auf einer Logik mit unendlich vielen Wahrheitswerten. *Sitzungsberichte Berliner Math. Gesellschaft 58* (1957), 41–56.
191. Smart, D. R. *Fixed Point Theorems.* Cambridge Univ. Press, 1974.
192. Smets, P., Mamdani, E. H., Dubois, D., and Prade, H., Eds. *Non-Standard Logics for Automated Reasoning* (London, 1988), Academic Press.
193. Takano, M. Another proof of the strong completeness of the intuitionistic fuzzy logic. *Tsukuba Journal of Mathematics 11*, 1 (1987), 101–105.
194. Takekuma, R. On a nine-valued propositonal calculus. *J. Comp. Syst 1* (1954), 225–228.
195. Takeuti, G., and Titani, S. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *J. Symb. Logic 49* (1984), 851–866.
196. Takeuti, G., and Titani, S. Fuzzy logic and fuzzy set theory. *Arch. Math. Logic 32* (1992), 1–32.
197. Thiele, H. On the definition of modal operatos in fuzzy logic. In *Proc. 23. Int. Symp. on multiple-valued logic* (Sacramento, California, 1993), pp. 62 – 67.
198. Thiele, H. Investigation of IF-THEN rule bases by methods of mathematical logic. In *Proc. Int. Joint Conf. of the Fourth IEEE Int. Conf. on Fuzzy Systems and Second Int. Fuzzy Engineering Symp.* (Yokohama, 1995), pp. 1391 – 1396.

199. THIELE, H. On fuzzy quantifiers. In *Fuzzy Logic and its Applications to Engineering, Information Ssciences and Intelligent Systems*, Bien and Min, Eds. Kluwer, 1995.

200. THIELE, H., AND KALENKA, S. On fuzzy temporal logic. In *Proceedings FUZZ-IEEE'93* (San Francisco, California, 1993), IEEE Neural Networks Council, pp. 1027 – 1032.

201. TRAKHTENBROT, B. A. Impossibility of an algorithm for the decision problem on finite classes. *Doklady Akademii Nauk SSSR 70* (1950), 569–572. in Russian.

202. TRILLAS, E., AND VALVERDE, L. An inquiry on t-indistinguishability operator. In *Aspects of Vagueness*, H. Skala et al., Ed. Reidel, Dordrecht, 1984, pp. 231–256.

203. TRILLAS, E., AND VALVERDE, L. On implication and indistinguishability in the setting of fuzzy logic. In *Management Decision Support Systems Using Fuzzy Sets and Possibility Theory*, J. Kacprzyk and R. R. Yager, Eds. Verlag TÜV, Rheinland, Köln, 1985, pp. 198–212.

204. TRILLAS, E., AND VALVERDE, L. On mode and implication in approximate reasoning. In *Approximate Reasoning in Expert Systems*, M. M. Gupta, A. Kandel, W. Bandler, and J. B. Kiszka, Eds. North Holland, Amsterdam, 1985, pp. 157–166.

205. VAUGHT, R. The completeness of logic with the added quantifier "there are uncountably many". *Fundamenta mathematicae 54* (1964), 303–304.

206. VOJTÁŠ, P., AND PAULÍK, L. Soundness and completeness of non-classical SLD resolution. In *Proc. ELP'96 Leipzig, Lect. Notes in Comp. Sci.* (Leipzig, 1996), R. Dyckhoff et al., Ed., Springer Verlag, pp. 298–301.

207. VOORBRAAK, F. As Far as I Know. Epistemic Logic and Uncertainty. Department of Philosophy Utrecht University, 1993. dissertation.

208. WAJSBERG, M. Beiträge zum Metaaussagenkalkül I. *Monatshefte für Mathematik und Physik 42* (1935), 221–242.

209. WOLEŃSKI, J. *Filozoficzna szkola Lwowsko-Warszawska (The philosophical school of Lwow and Warsaw, in Polish).* Państwowe wydawnictwo naukowe, Warsaw, 1985.

210. YAGER, R. R. Quantified statements and database fuzzy querying. In *Fuzziness in Database Management Systems*, P. Bosc and J. Kacprzyk, Eds. Physica Verlag, 1995, pp. 275–308.

211. ZADEH, L. A. Fuzzy sets. *Information and Control 8*, 3 (1965), 338–353.

212. ZADEH, L. A. Similarity relations and fuzzy orderings. *Information Sciencies 3* (1971), 177–200.

213. ZADEH, L. A. On the analysis of large scale systems. In *Systems approaches and environment processes* (Göttingen, 1974), H. Gabbinger, Ed., Vanderhoek and Ruprecht, pp. 23–37.

214. ZADEH, L. A. The concept of a lingustic variable and its application to approximate reasoning. *Information Sciences 8* (1975), 199–245, 301–357.

215. ZADEH, L. A. Fuzzy logic and approximate reasoning. *Synthese 30* (1975), 407–428.

216. ZADEH, L. A. Liar's paradox and truth-qualification principle. ERL Memorandum M79/34, University of California, Berkeley, 1979.
217. ZADEH, L. A. The role of fuzzy logic in the management of uncertainty in expert systems. *Fuzzy Sets and Systems 11* (1983), 199–227.
218. ZADEH, L. A. Is probability theory sufficient for dealing with uncertainty in AI: A negative view. In *Uncertainty in Artificial Intelligence* (1986), L. N. Kanal and J. F. Lemmer, Eds., Elsevier Science Publishers B.V. North Holland, pp. 103–115.
219. ZADEH, L. A. *Fuzzy Logic*, vol. 1. IEEE Computer, 1988.
220. ZADEH, L. A. The birth and evolution of fuzzy logic. *Int. J. of General Systems 17* (1990), 95–105.
221. ZADEH, L. A. Preface. In *Fuzzy Logic Technology and Applications* (1994), R. J. Marks-II, Ed., IEEE Technical Activities Board.
222. ZICH, O. V. Výrokový počet s komplexními hodnotami (Sentential calculus with complex values). *Česká mysl 34* (1938), 189–196.
223. ZIEGLER, G. M. *Lectures on polytopes*, vol. 152 of *Graduate Texts in Mathematics*. Springer-Verlag, 1995.
224. ZIMMERMANN, H.-J. *Fuzzy Set Theory and Its Applications*, second ed. Kluwer Academic Publisher, Boston/Dordrecht/London, 1991.

INDEX

TRENDS IN LOGIC

1. G. Schurz: *The Is-Ought Problem.* An Investigation in Philosophical Logic. 1997 ISBN 0-7923-4410-3
2. E. Ejerhed and S. Lindström (eds.): *Logic, Action and Cognition.* Essays in Philosophical Logic. 1997 ISBN 0-7923-4560-6
3. H. Wansing: *Displaying Modal Logic.* 1998 ISBN 0-7923-5205-X
4. P. Hájek: *Metamathematics of Fuzzy Logic.* 1998 ISBN 0-7923-5238-6

KLUWER ACADEMIC PUBLISHERS – DORDRECHT / BOSTON / LONDON