Consensus Numbers

魏恒峰

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这是一个关于“宿管”的专题

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日常工作：解决学生之间的矛盾，让学生达成一致意见

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日常工作：解决学生之间的矛盾，让学生达成一致意见

(Consensus!)

年度考核:

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Q : 如何考量宿管解决学生矛盾的能力?

年度考核:

Q : 你最多能处理几个学生之间的矛盾?

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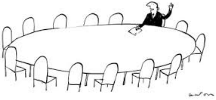

"It looks like we have a consensus."

年度考核:

Q : 你最多能处理几个学生之间的矛盾?


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Consensus Number

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$\# = 2$

Consensus Number

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$\# = 5$

Consensus Number

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Consensus Number

年度考核:

Q : 你最多能处理几个学生之间的矛盾?

$\# = 2$

$\# = 5$

$\# = 1$

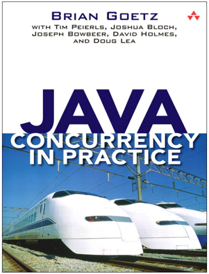
$\# = \infty$

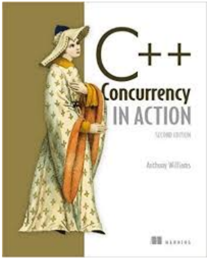
Consensus Number

Let's Move on to the Technical Parts.

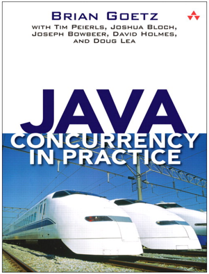
Concurrent Programming?

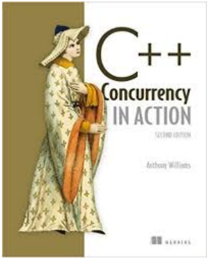
Synchronization





Synchronization Primitives





Synchronization Primitives

`Lock` `BlockingQueue` `Phaser`

`Semaphore` `ConcurrentMap` `Barrier`

```
class AtomicInteger: // java.util.concurrent
  get()               // read
  set(int newVal)     // write

  getAndIncrement()
  getAndDecrement()
  getAndSet(int newVal)

  compareAndSet(int expect, int update)
```

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  get()              // read
  set(int newVal)    // write

  getAndIncrement()
  getAndDecrement()
  getAndSet(int newVal)

  compareAndSet(int expect, int update) // atomically
```

CAS

(compareAndSet   compareAndSwap)

CAS

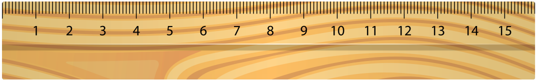
(compareAndSet   compareAndSwap)

```
// ''appears to'' be atomic
compareAndSet(int expect, int update) {
  old = this.val

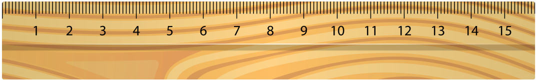
  if (old == expect)
    this.val = update

  return old
}
```

(Relative) *Power* of Various Synchronization Primitives



Consensus Number: n-thread consensus problems



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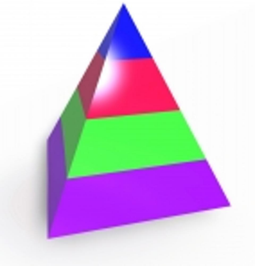

$$1, \quad 2, \quad \cdots, \quad n, \quad \cdots \quad \infty$$

Definition (Consensus Number)

The *consensus number* of a class C of synchronization primitive is the largest n for which C *solves*[a] n-thread consensus.

If no largest n exists, the consensus number is said to be *infinite*.

[a] defined later

The Beautiful Idea of "Consensus Numbers" (Maurice Herlihy@TOPLAS'1991)

Wait-Free Synchronization

MAURICE HERLIHY
Digital Equipment Corporation

A *wait-free* implementation of a concurrent data object is one that guarantees that any process can complete any operation in a finite number of steps, regardless of the execution speeds of the other processes. The problem of constructing a wait-free implementation of one data object from another lies at the heart of much recent work in concurrent algorithms, concurrent data structures, and multiprocessor architectures. First, we introduce a simple and general technique, based on reduction to a consensus protocol, for proving statements of the form, "there is no wait-free implementation of X by Y." We derive a hierarchy of objects such that no object at one level has a wait-free implementation in terms of objects at lower levels. In particular, we show that atomic read/write registers, which have been the focus of much recent attention, are at the bottom of the hierarchy: they cannot be used to construct wait-free implementations of many simple and familiar data types. Moreover, classical synchronization primitives such as *test&set* and *fetch&add*, while more powerful than *read* and *write*, are also computationally weak, as are the standard message-passing primitives. Second, nevertheless, we show that there do exist simple universal objects from which one can construct a wait-free implementation of any sequential object.

Theorem (Consensus Numbers)

Consensus Number	Object
1	read/write registers
2	test&set, swap, fetch&add, queue, stack
$\vdots$	$\vdots$
$2n-2$	n-register assignment
$\vdots$	$\vdots$
∞	memory-to-memory move and swap, augmented queue, compare&swap, fetch&cons, sticky byte

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∞ : CAS

1 : Atomic read/write register

2 : Queue, getAndSet

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∞ : CAS

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Consensus

$$d \leftarrow \textsc{Decide}(v)$$

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Definition (The Consensus Problem)

- **Consistent** All (non-faulty) threads must *agree* on the same value.
- **Valid** The common decision value must be the value *proposed* by some thread.
- **Wait-free** Each (non-faulty) thread must decide in a *finite* number of steps.

On the Nature of Progress

Maurice Herlihy[1,⋆] and Nir Shavit[2]

[1] Brown University Computer Science
herlihy@cs.brown.edu
[2] MIT CSAIL
shanir@csail.mit.edu

Wait-free (very informally):
(Must finish in a finite number of steps.)

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$\cdots$ *regardless of* the execution speeds of other threads

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Wait-free (very informally):
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⋯ *regardless of* the execution speeds of other threads

⋯ *eventually* finishes when it runs *solo*

We next focus on the *binary* consensus problem.

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(“宿舍空调开还是不开?”)

Consensus Protocol

Definition (Consensus Protocol (Informally))

A *consensus protocol* is a system of n *threads* that communicate through a set of *shared objects*.

Propose

$$\{0,1\}^n$$

Communicate

invocation response

Decide

$$\{0,1\}$$

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A *consensus protocol* is a system of n *threads* that communicate through a set of *shared objects*.

Propose

$\{0,1\}^n$

Communicate

invocation response

Decide

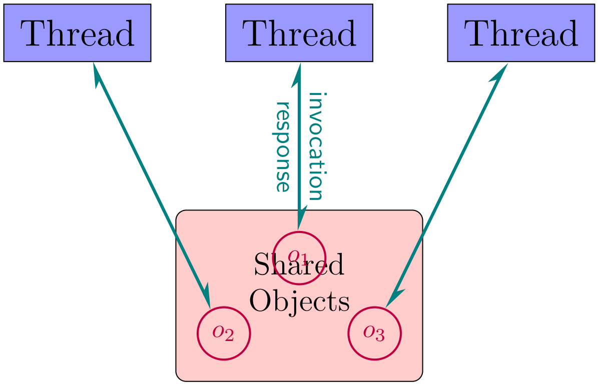
$\{0,1\}$

It *correctly* solves the n-thread consensus problem:

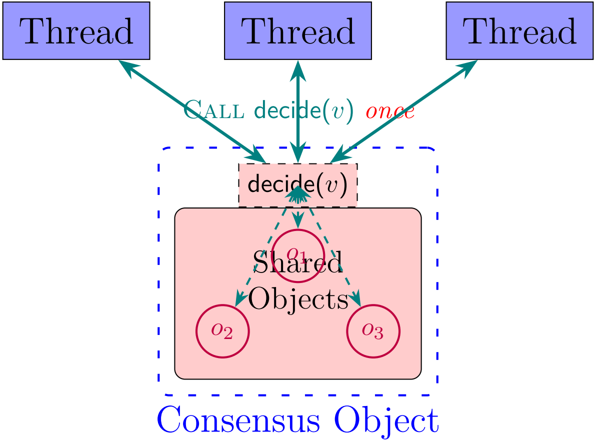
Consistent: $\cdots$

Valid: $\cdots$

Wait-free: $\cdots$



Threads communicate through shared objects.




```
interface Consensus {
  int decide(int val);
}
```

Definition ("Solves")

A *class C* of synchronization primitive *solves* n-thread consensus if there exists a *consensus protocol* using

(i) any number of objects of class C

(ii) any number of atomic read/write registers

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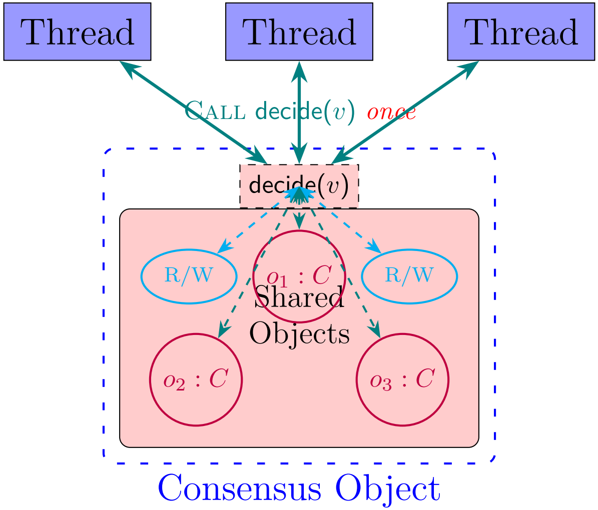
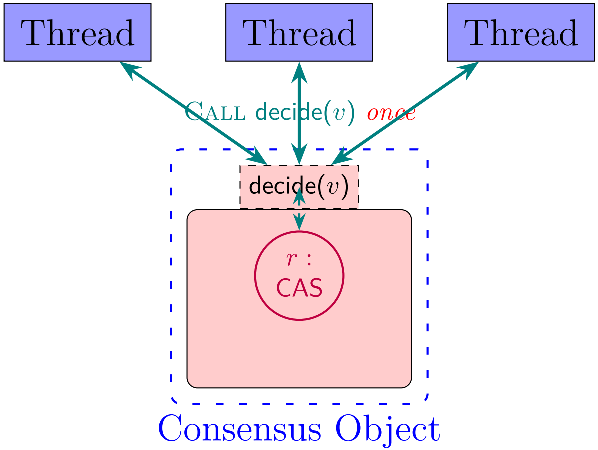
(ii) any number of atomic read/write registers

(iii) initialized to any state



A class C solves n-thread consensus.

n-Consensus Protocol Using (a Single) CAS



```
class CASConsensus implements Consensus {
  // shared CAS object
  CAS r = new CAS(⊥); // initialized

  int decide(int val) {
    first = r.compareAndSet(⊥, val);

    if (first == ⊥)
      return val;     // I won
    else
      return first;   // I lose
  }
}
```

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```

The decision value is established by the thread that *succeeds*.

Theorem (Power of CAS)

A CAS register providing the compareAndSet() method can solve the consensus problem for any number of threads.

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Theorem (Consensus Number of CAS)

A CAS register providing the compareAndSet() method has the consensus number ∞.

B
前方高能
-WARNING-

Theorem (Consensus Number of Atomic Registers)

Atomic read/write registers have consensus number 1.

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Theorem ("Weakness" of Atomic Registers)

Atomic read/write registers cannot solve 2*-thread consensus.*

Impossibility Results

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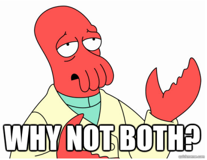

Protocol

$$\mathcal{P} = \{\text{All executions of this protocol}\}$$

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Execution

$$e : \sigma_0 \xrightarrow{o_1} \sigma_1 \xrightarrow{o_2} \cdots \xrightarrow{o_{n-1}} \sigma_n$$

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State

$$\sigma_i : \text{ states of individual threads} + \text{states of shared objects}$$

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State

σ_i : states of individual threads + states of shared objects

Operation

o_i : a method call to some shared object

$$e = \sigma_0 \xrightarrow{o_1} \sigma_1 \xrightarrow{o_2} \cdots \xrightarrow{o_{n-1}} \sigma_n$$

(i) Sequential execution: alternates matching invocations and responses.

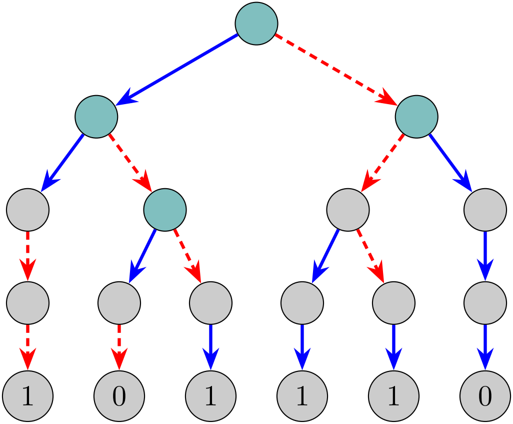
(ii) Only interleaved at the granularity of complete method calls.

Theorem

A consensus protocol is correct iff all its sequential executions are correct.

Modeling $\mathcal{P}$ as a Finite Execution "Tree"

(Binary Consensus for 2 Threads)



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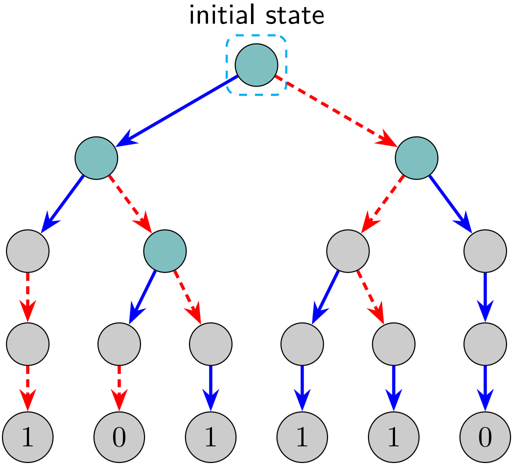

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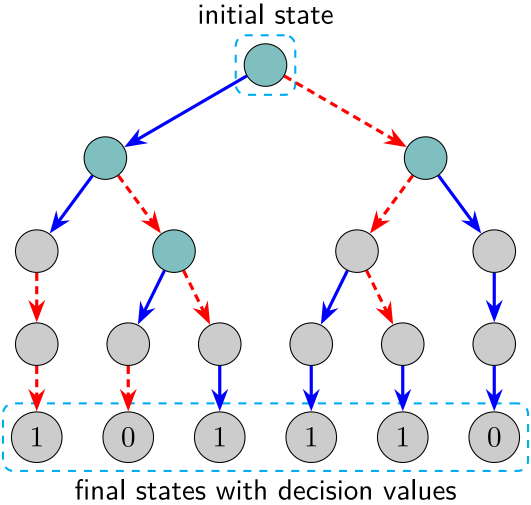

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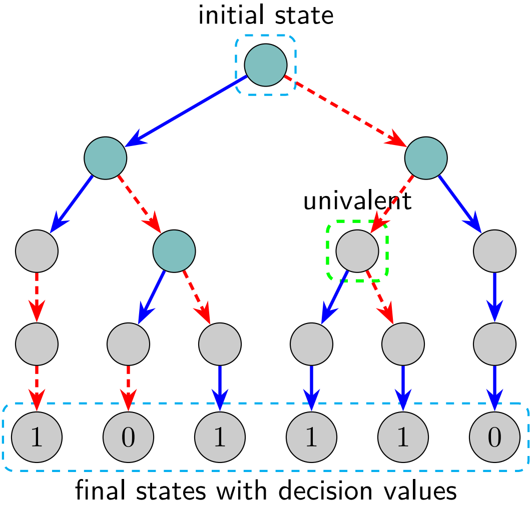

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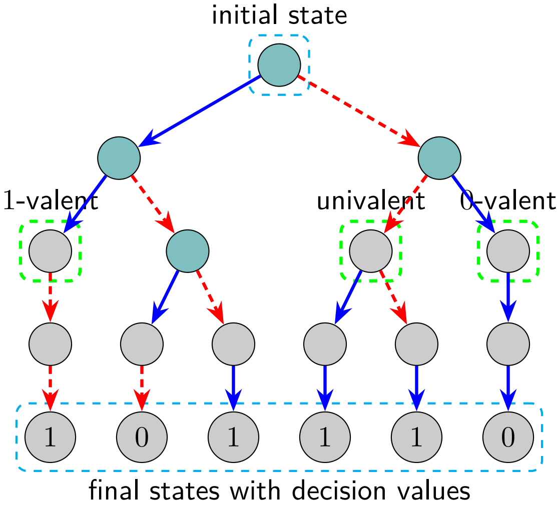

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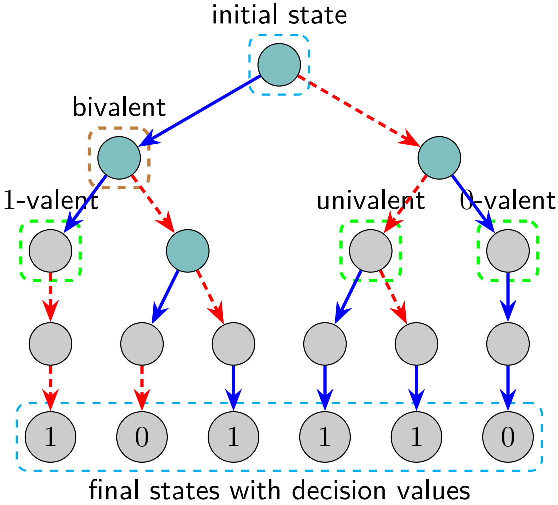

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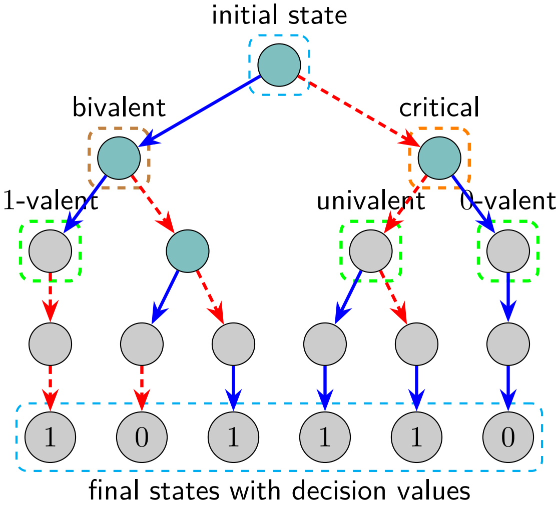

Theorem (Bivalent Initial State)

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$$(A, B, O) : (0, 0, *) \quad (1, 1, *) \quad (0, 1, *) \quad (1, 0, *)$$

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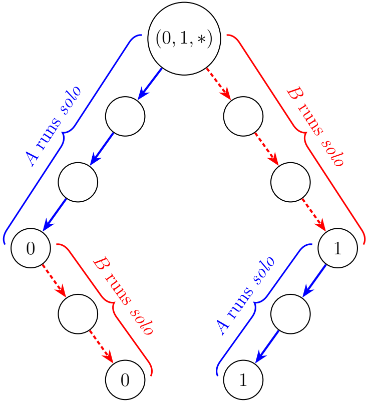

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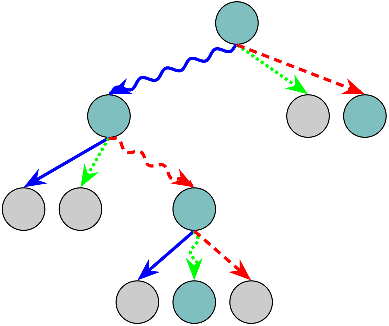
By Contradiction: Wait-free!



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- A's next step carries $\mathcal{P}$ to a 0-valent state
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Q : What are the next steps of two threads A and B?

□

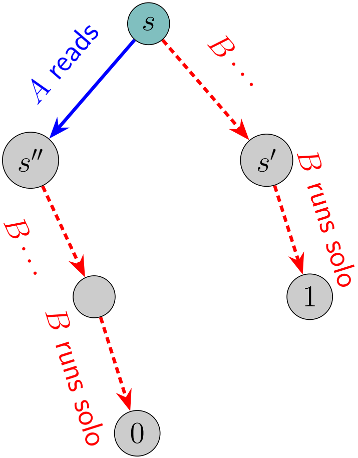
r_i : read/write register

From the critical state s:

CASE 1: A reads r

CASE 2: A writes r_0; B writes r_1

CASE 3: A writes r; B writes r



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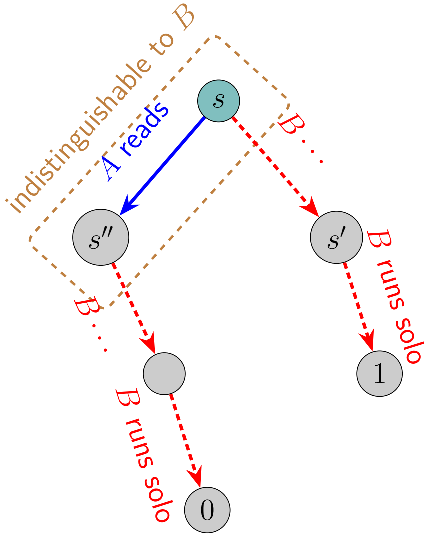

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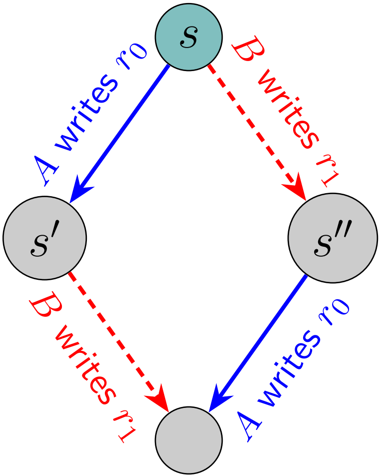
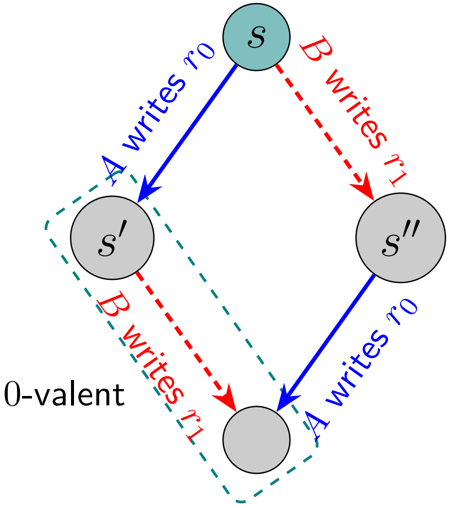


CASE 2: A writes r_0; B writes r_1



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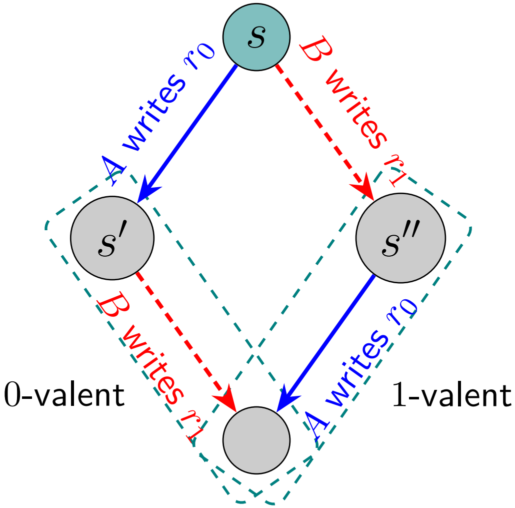

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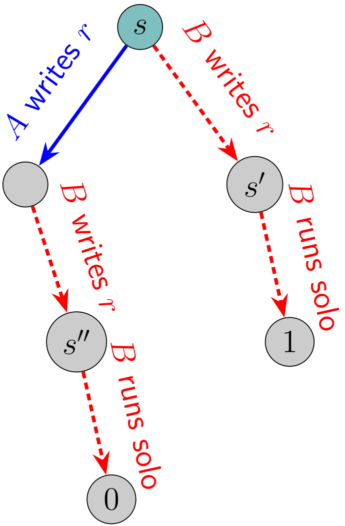
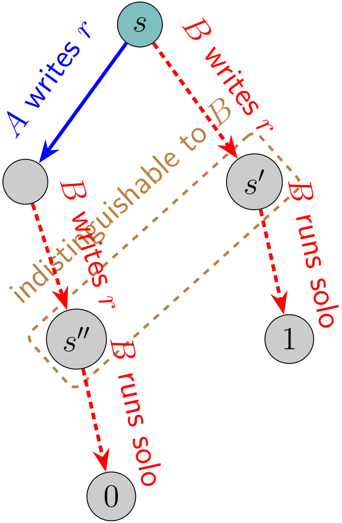


CASE 3: A writes r; B writes r



CASE 3: A writes r; B writes r

Theorem ("Weakness" of Atomic Registers)

It is impossible to construct a wait-free implementation of any object with consensus number ≥ 2 using atomic read/write registers.

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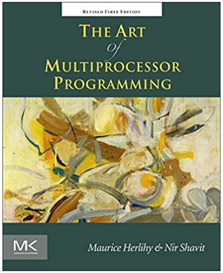
It is impossible to construct a wait-free implementation of any object with consensus number ≥ 2 using atomic read/write registers.

"$\cdots$ is perhaps one of the most striking impossibility results in Computer Science."




2

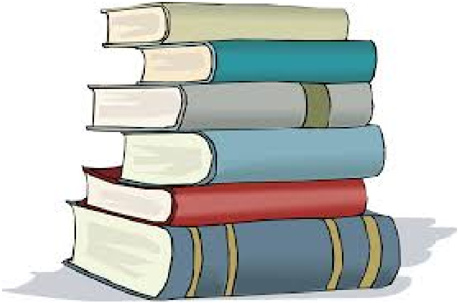
Theorem (Consensus Number 2)

The data types FIFO queue, stack, set, list, and priority queue all have consensus number 2.

Theorem (Consensus Number 2 (More))

The synchronization primitives get&set and get&add have consensus number 2.

References



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Wait-Free Synchronization

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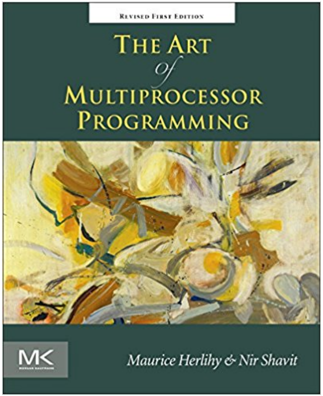
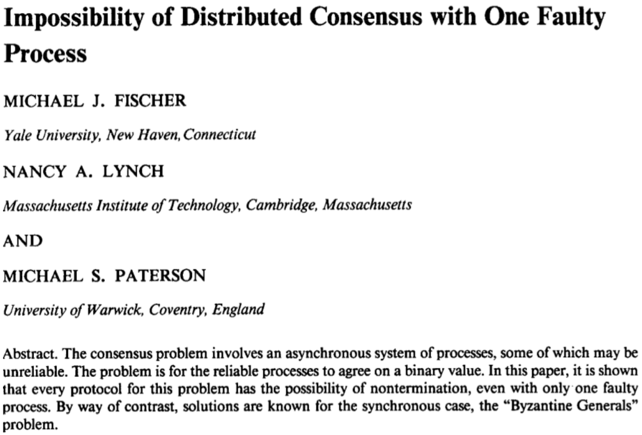
A *wait-free* implementation of a concurrent data object is one that guarantees that any process can complete any operation in a finite number of steps, regardless of the execution speeds of the other processes. The problem of constructing a wait-free implementation of one data object from another lies at the heart of much recent work in concurrent algorithms, concurrent data structures, and multiprocessor architectures. First, we introduce a simple and general technique, based on reduction to a consensus protocol, for proving statements of the form, “there is no wait-free implementation of X by Y.” We derive a hierarchy of objects such that no object at one level has a wait-free implementation in terms of objects at lower levels. In particular, we show that atomic read/write registers, which have been the focus of much recent attention, are at the bottom of the hierarchy: they cannot be used to construct wait-free implementations of many simple and familiar data types. Moreover, classical synchronization primitives such as *test&set* and *fetch&add*, while more powerful than *read* and *write*, are also computationally weak, as are the standard message-passing primitives. Second, nevertheless, we show that there do exist simple universal objects from which one can construct a wait-free implementation of any sequential object.

“The Art of Multiprocessor Programming” (Maurice Herlihy & Nir Shavit@2008)



The Idea of "Valency Argument" (FLP@JACM'1985)



Impossibility of Distributed Consensus with One Faulty Process

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Abstract. The consensus problem involves an asynchronous system of processes, some of which may be unreliable. The problem is for the reliable processes to agree on a binary value. In this paper, it is shown that every protocol for this problem has the possibility of nontermination, even with only one faulty process. By way of contrast, solutions are known for the synchronous case, the "Byzantine Generals" problem.


Thank You!