Specification, Implementation, and Complexity of Replicated Data Types with Composite Operations

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Abstract

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1 Introduction

2 Preliminaries

2.1 Observed-Remove Set (`OR-Set`)

$$\mathcal{F}_{\texttt{orset}}(\texttt{rd}, E, \mathsf{op}, \mathsf{vis}, \mathsf{ar}) = \{a \mid \exists e \in E.\mathsf{op}(e) = \texttt{add}(a) \quad (1)$$

$$\land \big(\forall f \in E.\mathsf{op}(f) = \texttt{rm}(a) \implies \neg(e \xrightarrow{\mathsf{vis}} f)\big)\}. \quad (2)$$

3 Replicated Data Types with Composite Operations

3.1 Specification

We consider a composite operation of a replicated data type τ in the form of $C = A \oplus B$, where A, B, and C are different objects of type τ.

Q : Generalize to different data types?

Following [?], we specify the semantics of a composite operation $A \oplus B$ of a replicated data type τ by a function $\mathcal{F}_\tau$ that determines the return value of $\oplus$ based on prior operations performed on the two objects involved (i.e., A and B). However, $\mathcal{F}_\tau$ for a composite operation $\oplus$ takes as parameters two, not one as in [?], *operation contexts*, one on each object involved.

Note: Partial operation context [?]

▶ **Definition 1** (Product of Operation Contexts)**.** Consider two operation contexts for the same replicated data type τ:

$$\mathcal{C}_A = (E_A, \mathsf{op}_A, \mathsf{vis}_A, \mathsf{ar}_A) \tag{3}$$
$$\mathcal{C}_B = (E_B, \mathsf{op}_B, \mathsf{vis}_B, \mathsf{ar}_B) \tag{4}$$

The product $\mathcal{C} = \mathcal{C}_A \times \mathcal{C}_B$ of $\mathcal{C}_A$ and $\mathcal{C}_B$ is also an operation context defined as $\mathcal{C} = (E, \mathsf{op}, \mathsf{vis}, \mathsf{ar})$, where

- $E = E_A \times E_B$
- $\mathsf{op} = \mathsf{op}_A \sqcup \mathsf{op}_B$
- $\mathsf{vis} = \mathsf{vis}_A \times \mathsf{vis}_B$
- $\mathsf{ar} = \mathsf{ar}_A \times \mathsf{ar}_B$

▶ **Definition 2.**

$$\mathcal{F}_\tau(\oplus, \mathcal{C}_A, \mathcal{C}_B) = \mathcal{F}_\tau(\oplus, \mathcal{C}_A \times \mathcal{C}_B) \tag{5}$$

4 Replicated Counter with Composite Operations

We consider the replicated counter data type with composite operations including addition $(+)$ and subtraction $(-)$.

5 Replicated Set with Composite Operations

We consider the replicated set data type with composite operations including union $(\cup)$, intersection $(\cap)$, difference $(\setminus)$, and symmetric difference $(\otimes)$.

5.1 Specification

General method for set union, set intersection, and set difference: There are four $\cdots$

$$\mathcal{F}_{\texttt{orset}}(A \setminus B, \mathcal{C}_A, \mathcal{C}_B) \tag{6}$$
$$= \{a \mid \exists (e, e') \in E_A \times E_B. \Big(\big(\texttt{op}(e) = \texttt{add}(a) \land \texttt{op}(e') = \texttt{rm}(a)\big) \tag{7}$$
$$\land \forall (f, f') \in E_A \times E_B. \big((\texttt{op}(f) = \texttt{add}(a) \land \texttt{op}(f') = \texttt{add}(a)) \lor (\texttt{op}(f) = \texttt{rm}(a) \land \texttt{op}(f') = \texttt{add}(\tag{8}$$
$$\implies \neg\big((e, e') \xrightarrow{\mathsf{vis}} (f, f')\big)\Big)\}. \tag{9}$$

▶ **Example 3** (Specification for Composite Operations of Replicated Set)**.**

5.2 Protocol

5.2.1 Methodology

Op-based

We always assume that $A \neq \emptyset$, $A' \neq \emptyset$, and $(\mathbb{A} \cup \mathbb{R}) \cap (\mathbb{A}' \cup \mathbb{R}') = \emptyset$. Note that for any replica state $(a, \mathbb{A}, \mathbb{R})$, we have $\mathbb{R} \subseteq \mathbb{A}$.

$Q : \emptyset$ issues

The goal is to find $\Box$ and $\Diamond$ such that

$$\cdots \iff \Box \subseteq \Diamond$$

$$
\begin{array}{lcl}
U & : & \text{universe of elements} \\
I & : & \text{set of identifiers} \\
\mathbb{A}, \mathbb{R} & \in & \text{SUBSET } \mathcal{I} \\
\Sigma & = & \mathcal{U} \times \mathbb{A} \times \mathbb{R} \\
\sigma_0 & = & \lambda r.\, \emptyset
\end{array}
$$

Figure 1 Implementation of replicated set with composite operations.

5.2.2 Set Union ($\cup$)

The goal is to find $\Box$ and $\Diamond$ such that

$$(\mathbb{R} \subset \mathbb{A}) \lor (\mathbb{R}' \subset \mathbb{A}') \iff \Box \subset \Diamond.$$

▶ **Theorem 4** (Set Union). *Given that $\mathbb{R} \subseteq \mathbb{A}$ and $\mathbb{R}' \subseteq \mathbb{A}'$, we have*

$$(\mathbb{R} \subset \mathbb{A}) \lor (\mathbb{R}' \subset \mathbb{A}') \iff (\mathbb{R} \cup \mathbb{R}') \subset (\mathbb{A} \cup \mathbb{A}').$$

▶ **Theorem 5** (Set Union (Alternative)). *Given that $\mathbb{R} \subseteq \mathbb{A}$ and $\mathbb{R}' \subseteq \mathbb{A}'$, we have*

$$(\mathbb{R} \subset \mathbb{A}) \lor (\mathbb{R}' \subset \mathbb{A}') \iff (\mathbb{A} \times \mathbb{R}') \cap (\mathbb{R} \times \mathbb{A}') \subset (\mathbb{A} \times \mathbb{A}')$$

Q : How to choose? Another way:

$$(\mathbb{R} \subset \mathbb{A}) \lor (\mathbb{R}' \subset \mathbb{A}'$$

Another way: IF THEN ELSE

5.2.3 Set Intersection ($\cap$)

The goal is to find $\Box$ and $\Diamond$ such that

$$(\mathbb{R} \subset \mathbb{A}) \land (\mathbb{R}' \subset \mathbb{A}') \iff \Box \subset \Diamond.$$

▶ **Theorem 6** (Set Intersection). *Given that $\mathbb{R} \subseteq \mathbb{A}$ and $\mathbb{R}' \subseteq \mathbb{A}'$, we have*

$$(\mathbb{R} \subset \mathbb{A}) \land (\mathbb{R}' \subset \mathbb{A}') \iff (\mathbb{A} \times \mathbb{R}') \cup (\mathbb{R} \times \mathbb{A}') \subset (\mathbb{A} \times \mathbb{A}').$$

5.2.4 Set Difference ($\setminus$)

The goal is to find $\Box$ and $\Diamond$ such that

$$(\mathbb{R} \subset \mathbb{A}) \land (\mathbb{R}' \not\subset \mathbb{A}') \iff \Box \subset \Diamond.$$

▶ **Theorem 7** (Set Difference). *Given that $\mathbb{R} \subseteq \mathbb{A}$ and $\mathbb{R}' \subseteq \mathbb{A}'$, we have*

$$(\mathbb{R} \subset \mathbb{A}) \land (\mathbb{R}' \not\subset \mathbb{A}') \iff (\mathbb{R} \times \mathbb{A}') \subset (\mathbb{A} \times \mathbb{R}').$$

5.2.5 Set Symmetric Difference ($\otimes$)

The goal is to find $\Box$ and $\Diamond$ such that

$$\big((\mathbb{R} \subset \mathbb{A}) \land (\mathbb{R}' \not\subset \mathbb{A}')\big) \lor \big((\mathbb{R} \not\subset \mathbb{A}) \land (\mathbb{R}' \subset \mathbb{A}')\big) \iff \Box \subset \Diamond.$$

By the identity $A \otimes B = (A \cup B) \setminus (A \cap B)$ and Theorems 4, 6, and 7, we have

▶ **Theorem 8** (Set Symmetric Difference). *Given that $\mathbb{R} \subseteq \mathbb{A}$ and $\mathbb{R}' \subseteq \mathbb{A}'$, we have*

$$
\begin{aligned}
&\big((\mathbb{R} \subset \mathbb{A}) \land (\mathbb{R}' \not\subset \mathbb{A}')\big) \lor \big((\mathbb{R} \not\subset \mathbb{A}) \land (\mathbb{R}' \subset \mathbb{A}')\big) \\
&\quad \iff (\mathbb{R} \cup \mathbb{R}') \times (\mathbb{A} \cup \mathbb{A}') \subset (\mathbb{A} \cup \mathbb{A}') \times \big((\mathbb{A} \times \mathbb{R}') \cup (\mathbb{R} \cup \mathbb{A}')\big)
\end{aligned}
$$

It fails to use Theorem 4 here.

By the identity $A \otimes B = (A \setminus B) \cup (B \setminus A)$ and Theorems 5 and 7, we have

▶ **Theorem 9** (Set Symmetric Difference (Alternative)). *Given that $\mathbb{R} \subseteq \mathbb{A}$ and $\mathbb{R}' \subseteq \mathbb{A}'$, we have*

$$\big((\mathbb{R} \subset \mathbb{A}) \land (\mathbb{R}' \not\subset \mathbb{A}')\big) \lor \big((\mathbb{R} \not\subset \mathbb{A}) \land (\mathbb{R}' \subset \mathbb{A}')\big)$$
$$\iff (\mathbb{A} \times \mathbb{R}') \times (\mathbb{R}' \times \mathbb{A}) \cap (\mathbb{R} \times \mathbb{A}') \times (\mathbb{A}' \times \mathbb{R}) \subset (\mathbb{A} \times \mathbb{R}') \times (\mathbb{A}' \times \mathbb{R})$$

6 Related Work

[?] [?]

7 Conclusion and Future Work

References