Problem Solving

2-11 Heap & Heapsort

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Institute of Computer Software

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Contents

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Heaps

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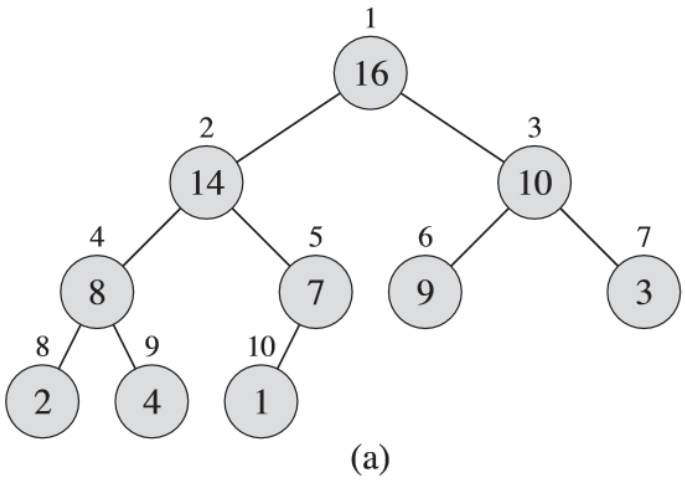
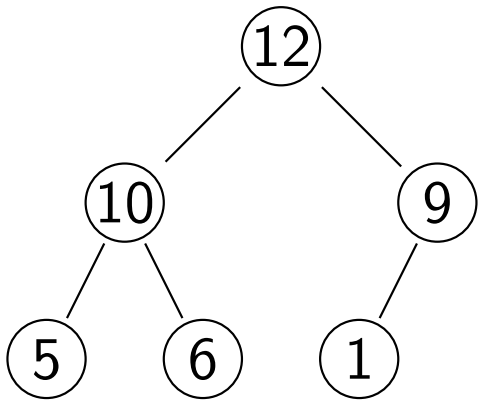
- The tree is completely filled on all levels except possibly the lowest



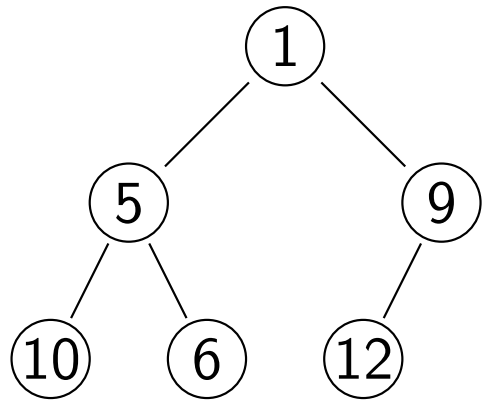
(a)

Heaps: Max-heap VS Min-heap



Max-heap property

$$A[Parent(i)] \geq A[i]$$



Min-heap property

$$A[Parent(i)] \leq A[i]$$

Heaps: Storage

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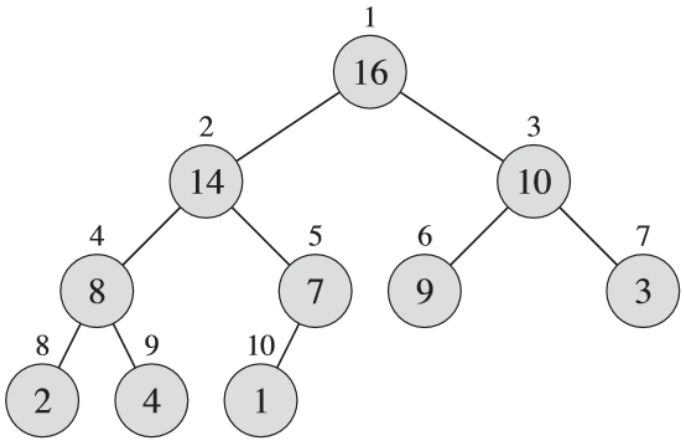

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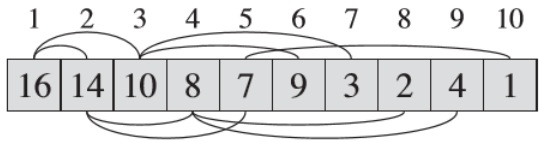

(b)

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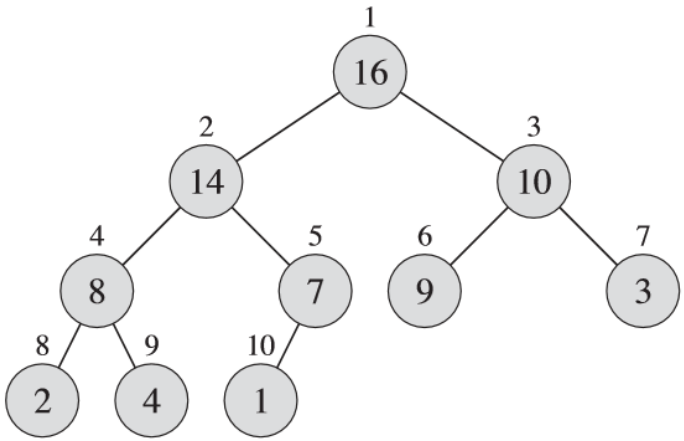

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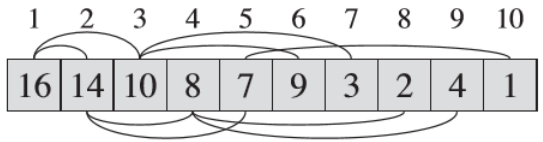

(b)

PARENT(i)
1 **return** $\lfloor i/2 \rfloor$

LEFT(i)
1 **return** $2i$

RIGHT(i)
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- Easy to index

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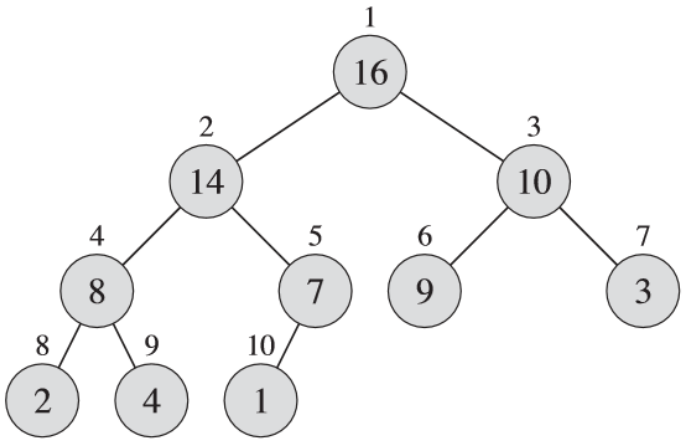

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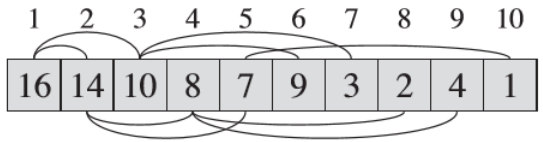

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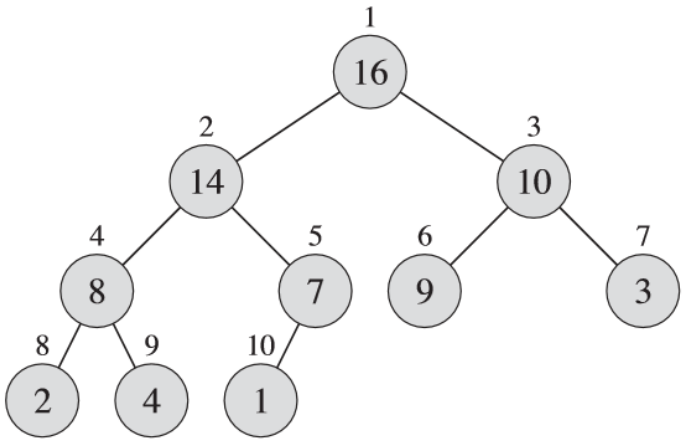

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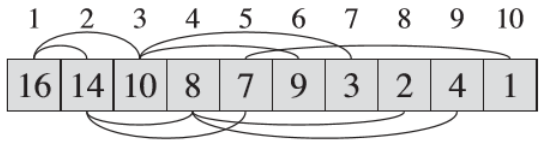

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- Easy to index
- Save memory
- Better cache locality

Heaps: Height

The height of a node

- The number of edges on the longest simple downward path from the node to a leaf.

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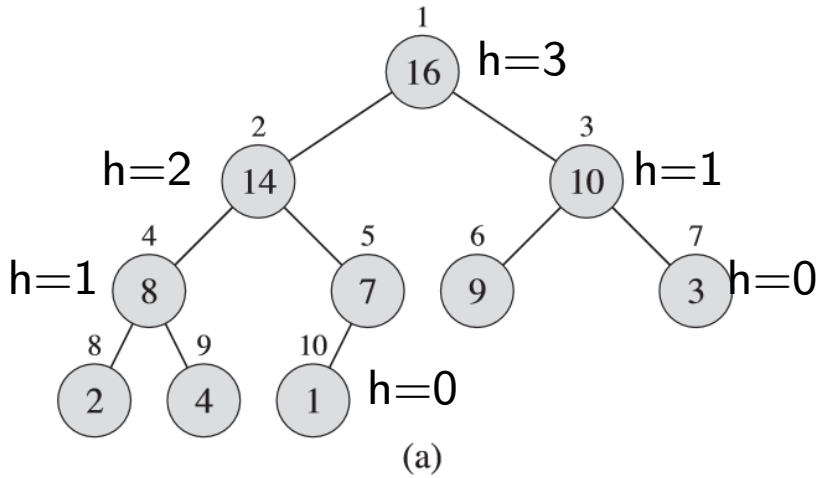
The height of a heap

- The height of its root, $\Theta(\lg n)$.
- A heap of n elements is based on a **nearly complete binary tree**



Heaps: basic operations

- The MAX-HEAPIFY procedure, which runs in $O(\lg n)$ time, is the key to maintaining the max-heap property.
- The BUILD-MAX-HEAP procedure, which runs in linear time, produces a max-heap from an unordered input array.
- The HEAPSORT procedure, which runs in $O(n \lg n)$ time, sorts an array in place.
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Maintaining the heap property: MAX-HEAPIFY

Q-2: Can you explain the process of MAX-HEAPIFY

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MAX-HEAPIFY(A, i)
 1  l = LEFT(i)
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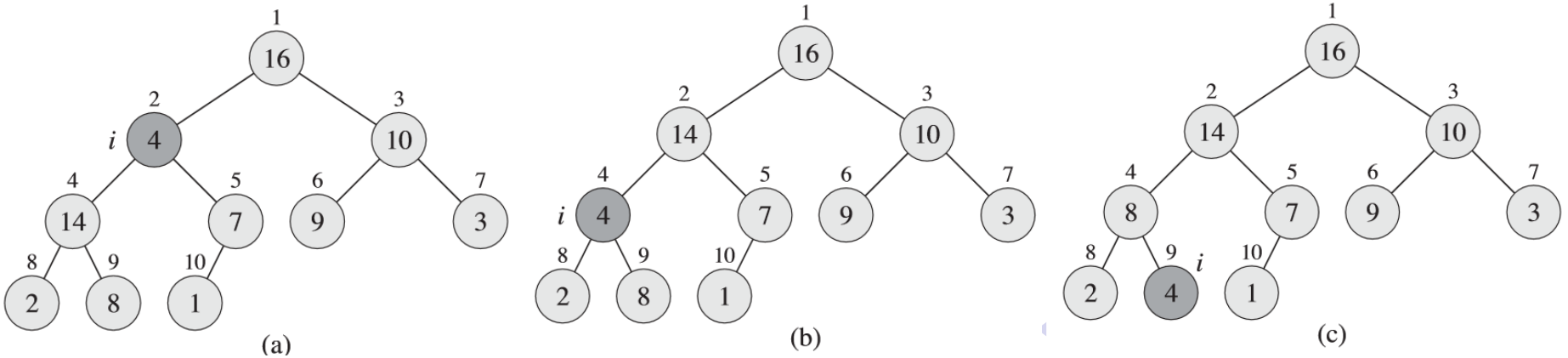

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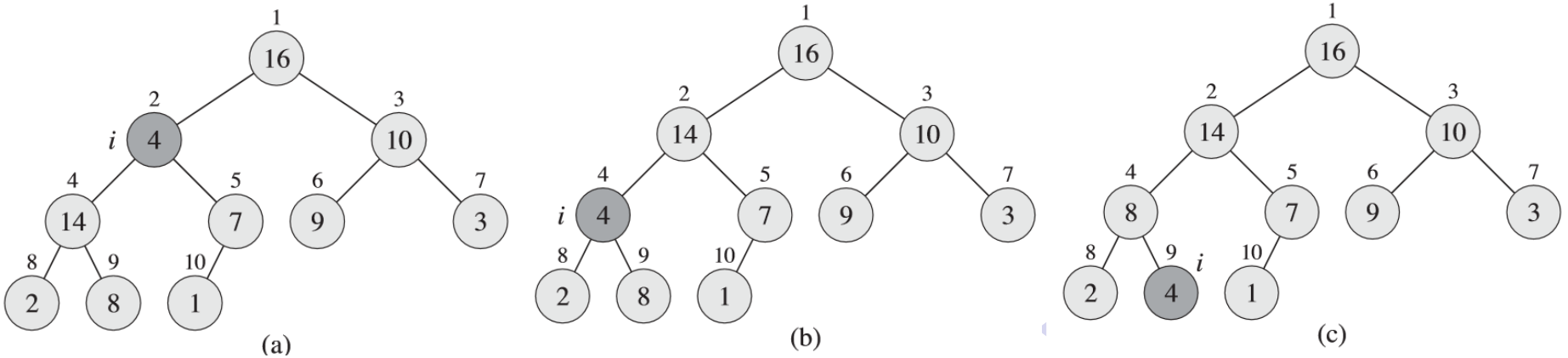
Pre-condition: *Left*(i) and *Right*(i) are maxheaps.

Post-condition: the tree rooted at i is a maxheap.



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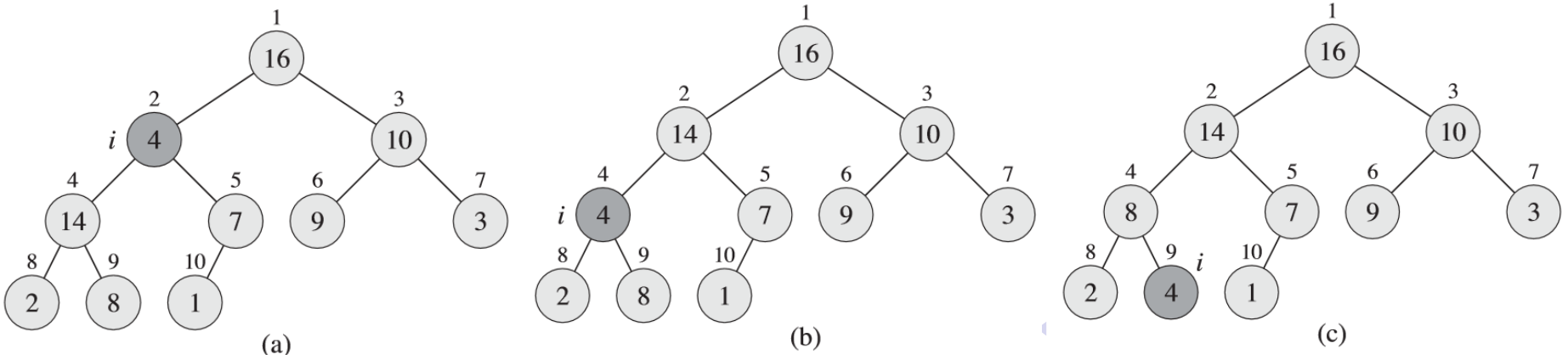

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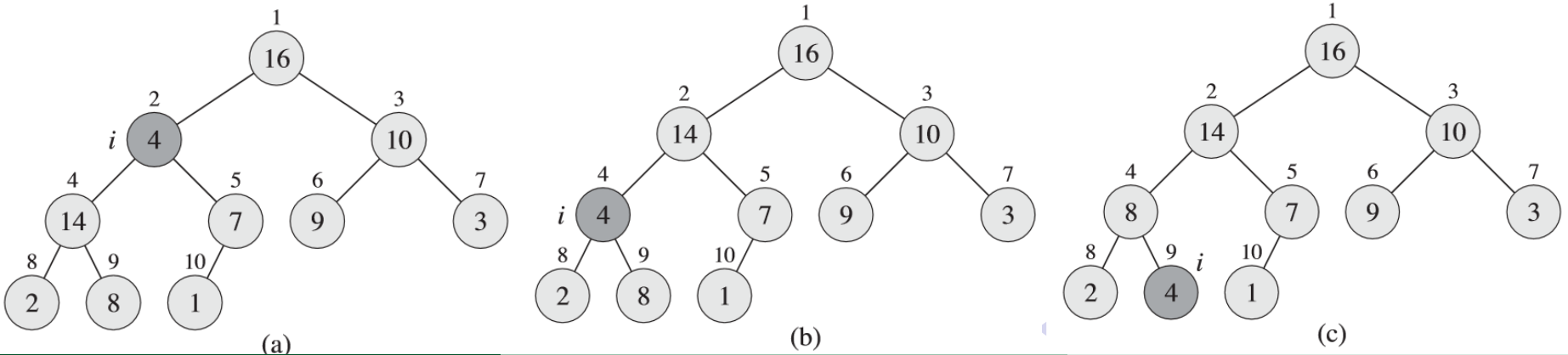

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Worst-case for MAX-HEAPIFY

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$$T(n) \leq T(2n/3) + \Theta(1) = O(\lg n) = O(h)$$

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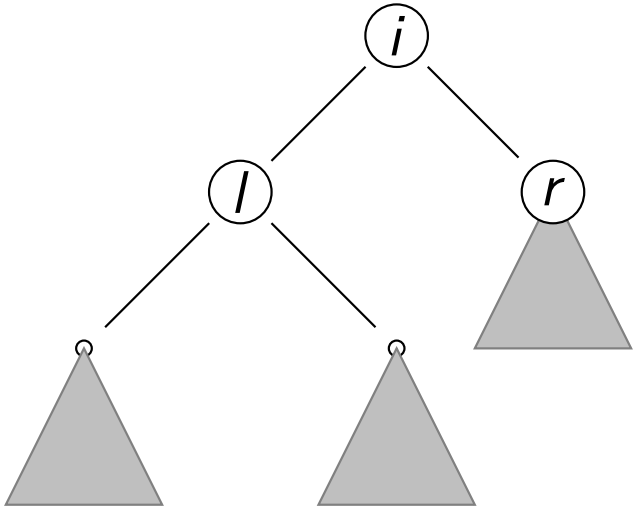

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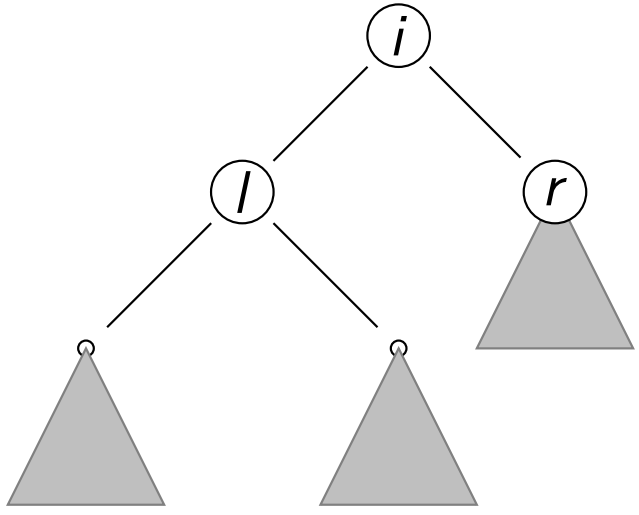

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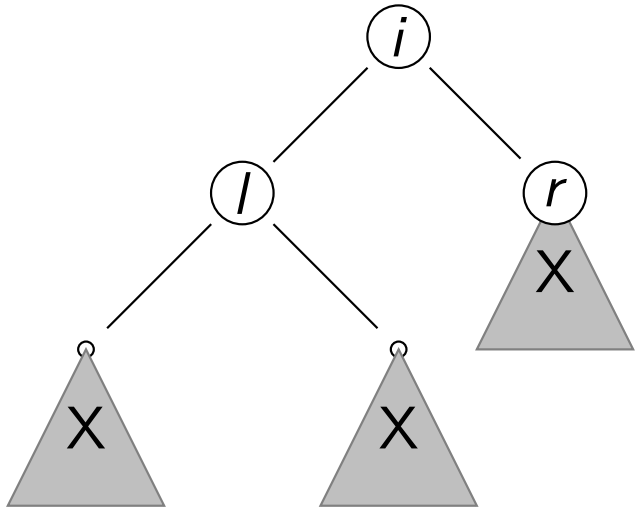

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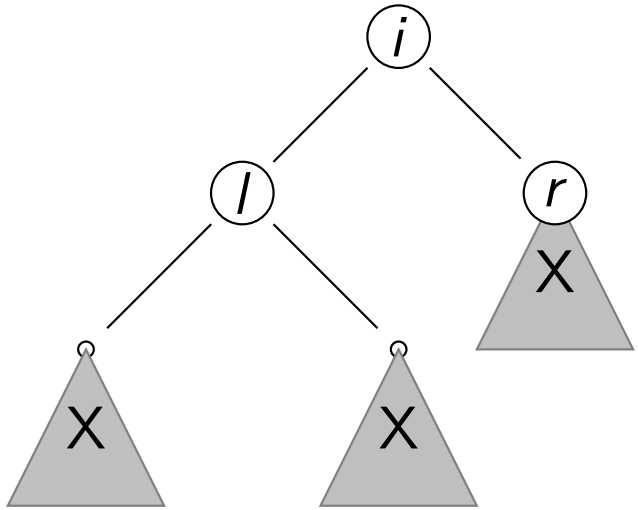

The subtree rooted at l is full

- Total number of nodes $n = 3X + 2$

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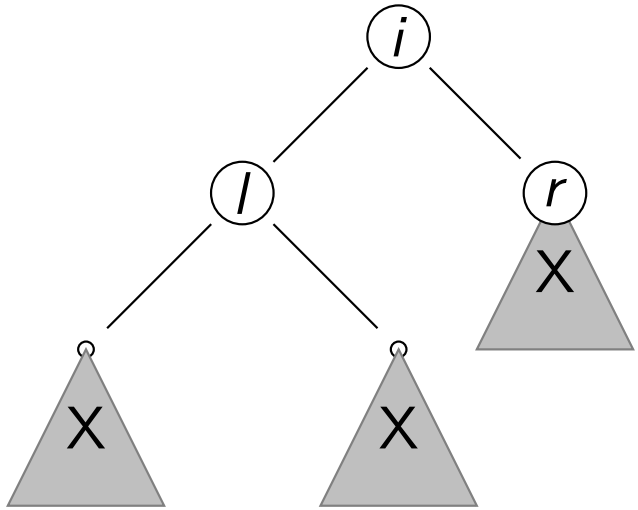

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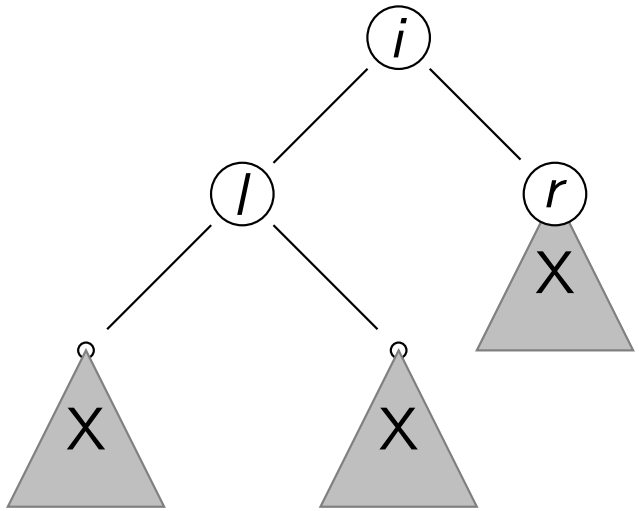

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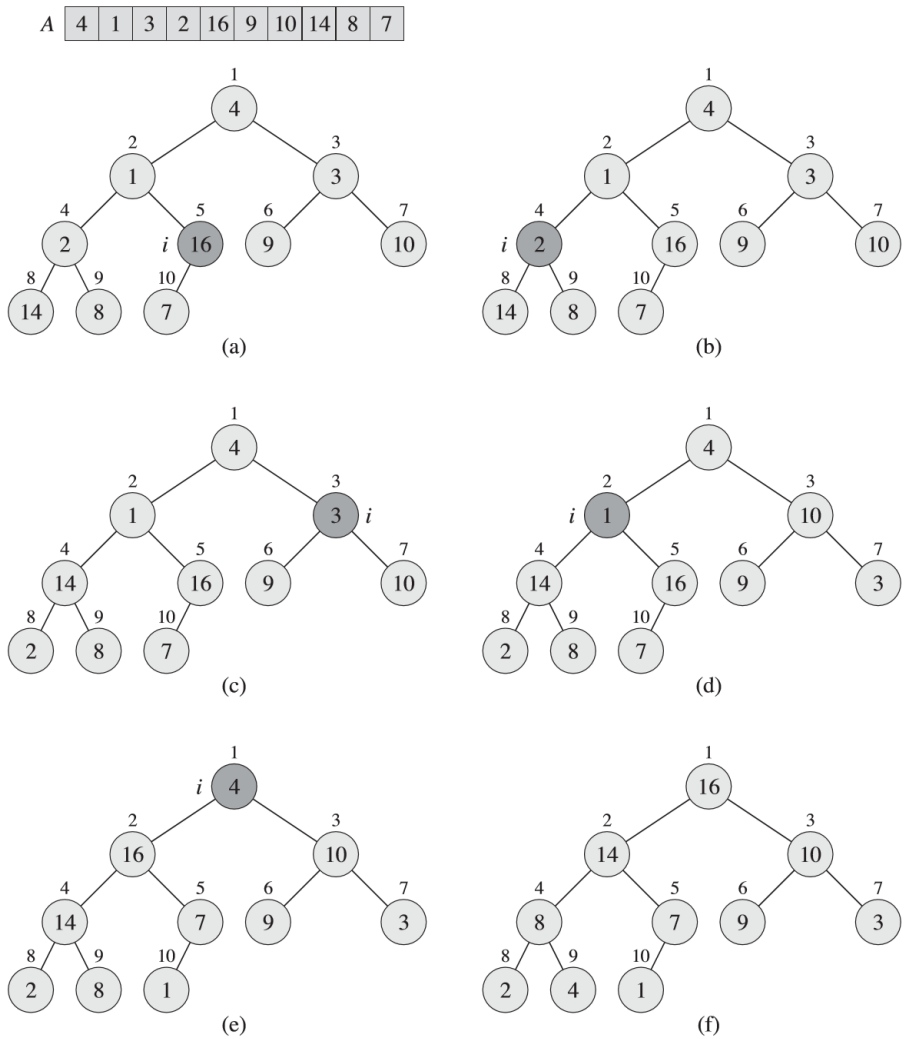
- $\frac{2X+1}{3X+2} < 2/3$

Building a heap: BUILD-MAX-HEAP

Q-3: Can you explain the process of BUILD-MAX-HEAP

BUILD-MAX-HEAP(A)

1 $A.heap\text{-}size = A.length$
2 **for** $i = \lfloor A.length/2 \rfloor$ **downto** 1
3 MAX-HEAPIFY(A, i)



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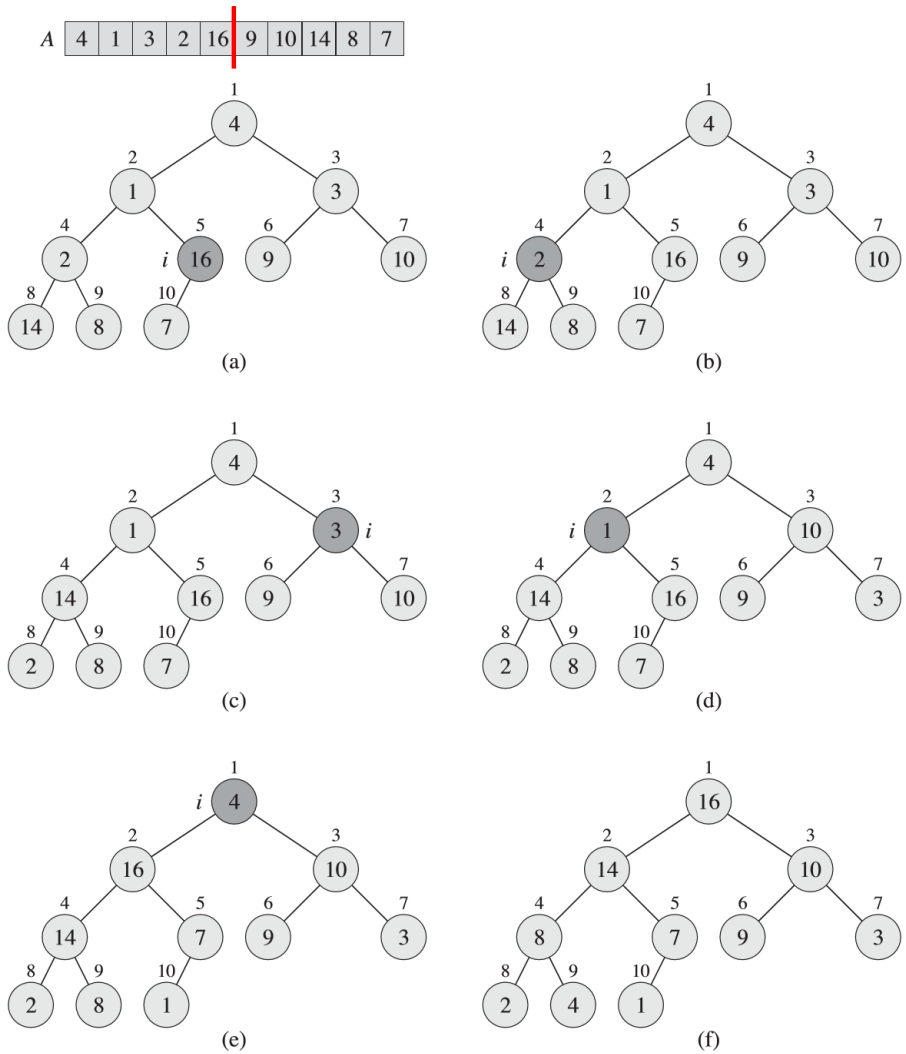

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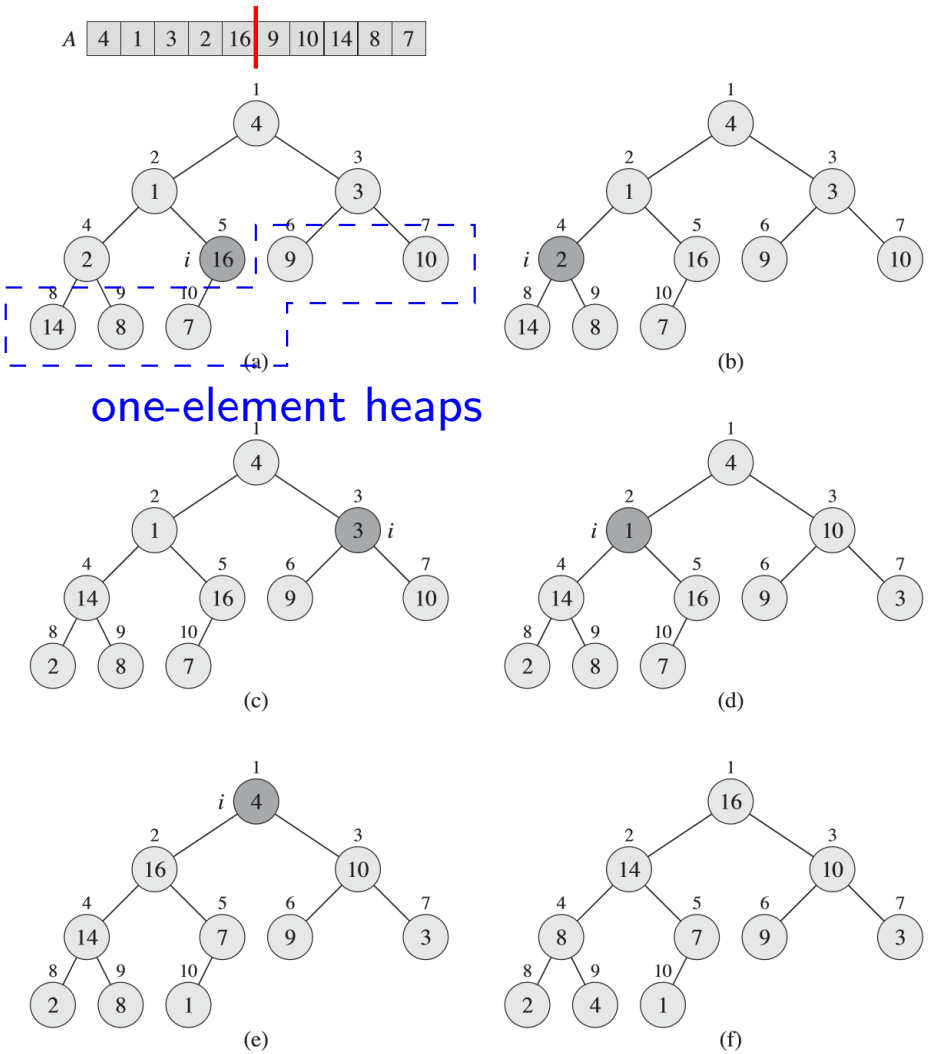

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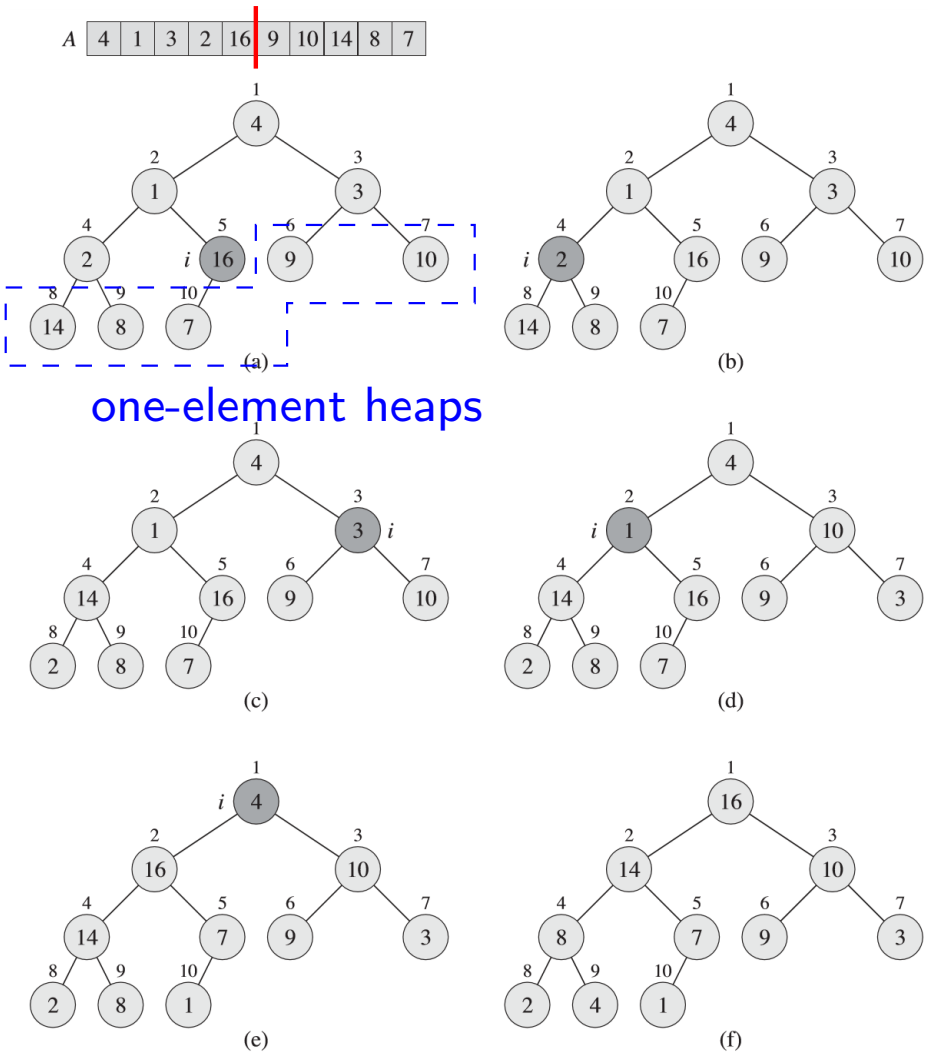
```

Q-4: Can you prove the correctness of BUILD-MAX-HEAP?



Correctness of BUILD-MAX-HEAP

Invariant

At the start of each iteration of the for loop of lines 2-3, each node $i+1, i+2, \cdots, n$ is the root of a max-heap.

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Initialization: Prior to the first iteration of the loop, $i = \lfloor n/2 \rfloor$. Each node $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$ is a leaf and is thus the root of a trivial max-heap.

Maintenance: To see that each iteration maintains the loop invariant, observe that the children of node i are numbered higher than i. By the loop invariant, therefore, they are both roots of max-heaps. This is precisely the condition required for the call MAX-HEAPIFY(A, i) to make node i a max-heap root. Moreover, the MAX-HEAPIFY call preserves the property that nodes $i+1, i+2, \ldots, n$ are all roots of max-heaps. Decrementing i in the **for** loop update reestablishes the loop invariant for the next iteration.

Termination: At termination, $i = 0$. By the loop invariant, each node $1, 2, \ldots, n$ is the root of a max-heap. In particular, node 1 is.

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Running time of BUILD-MAX-HEAP

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BUILD-MAX-HEAP(A)
1  A.heap-size = A.length
2  for i = ⌊A.length/2⌋ downto 1
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- Each call to MAX-HEAPIFY costs $O(\lg n)$
- At most $O(n)$ calls
- Thus, $O(n \lg n)$

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Q-5: Can you give a better one?

Running time of Build-Max-Heap

A tighter linear upper bound

Running time of BUILD-MAX-HEAP

A tighter linear upper bound

- An n-element heap has height $\lfloor \lg n \rfloor$.

Running time of BUILD-MAX-HEAP

A tighter linear upper bound

- An n-element heap has height $\lfloor \lg n \rfloor$.
- At most $\lceil n/2^{h+1} \rceil$ nodes of any height h.

Running time of BUILD-MAX-HEAP

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Running time of BUILD-MAX-HEAP

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Running time of BUILD-MAX-HEAP

A tighter linear upper bound

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- At most $\lceil n/2^{h+1} \rceil$ nodes of any height h.
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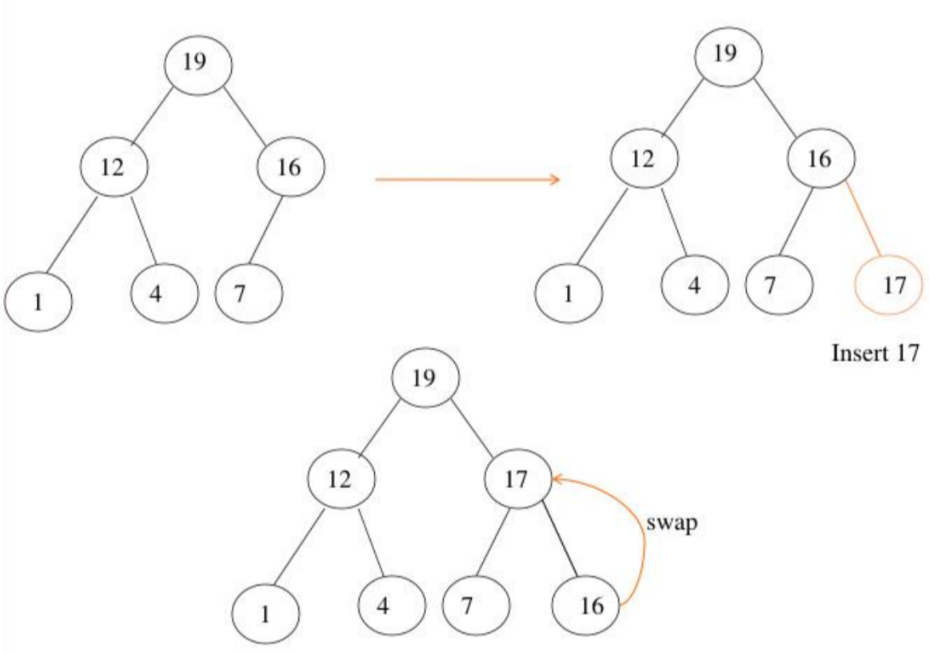
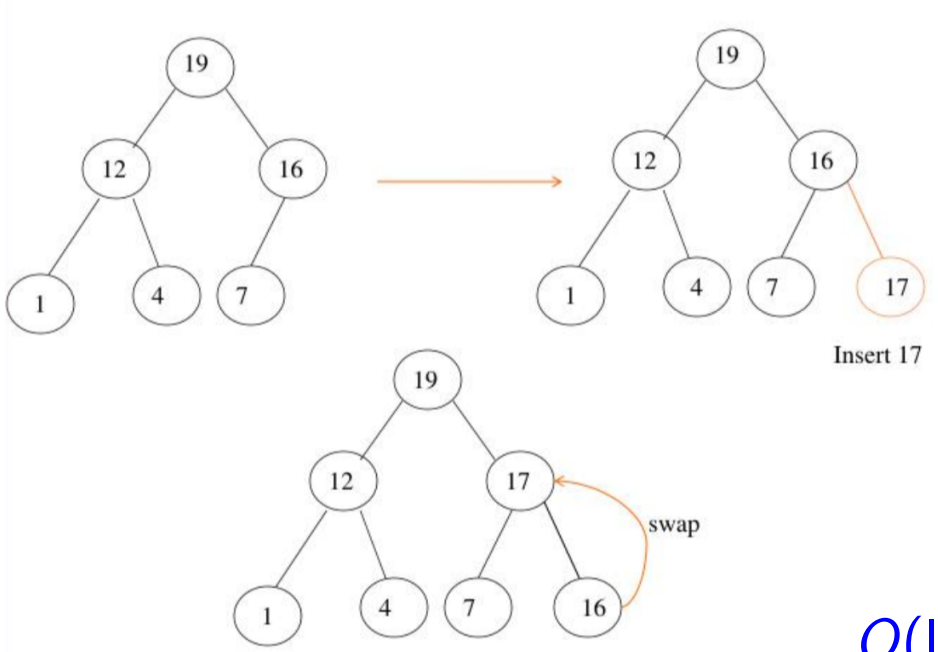
$$\sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \leq \sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2$$

Inserting an element into a Heap



- **Step-1**: Add the new element to the end of the heap
- **Step-2**: Compare the new element to its parent, if it is greater than its parent, swap the two elements
- **Step-3**: Repeat step-2 until the new element is smaller than its parent or it is the root.

Inserting an element into a Heap



$O(\lg n)$

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Build heap with insertion?

- Insert $A[1, .., n]$ to a heap one by one.
- Complexity?

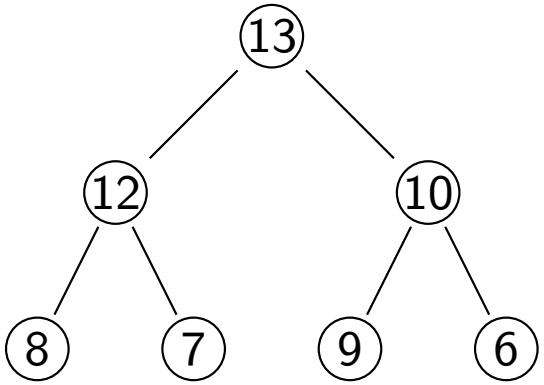
Deleting an element from a Heap

Assume that we try to delete a node i

Deleting an element from a Heap

Assume that we try to delete a node i

- **Step-1**: Copy the value of the last node to node i
- **Step-2**: Remove the last node
- **Step-3**: Call MAX-HEAPIFY on node i



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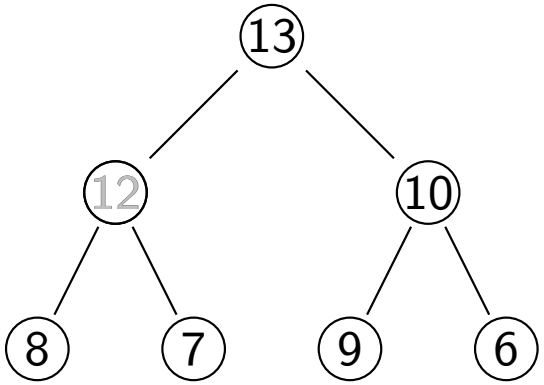

Delete 12

Deleting an element from a Heap

Assume that we try to delete a node i

- **Step-1**: Copy the value of the last node to node i
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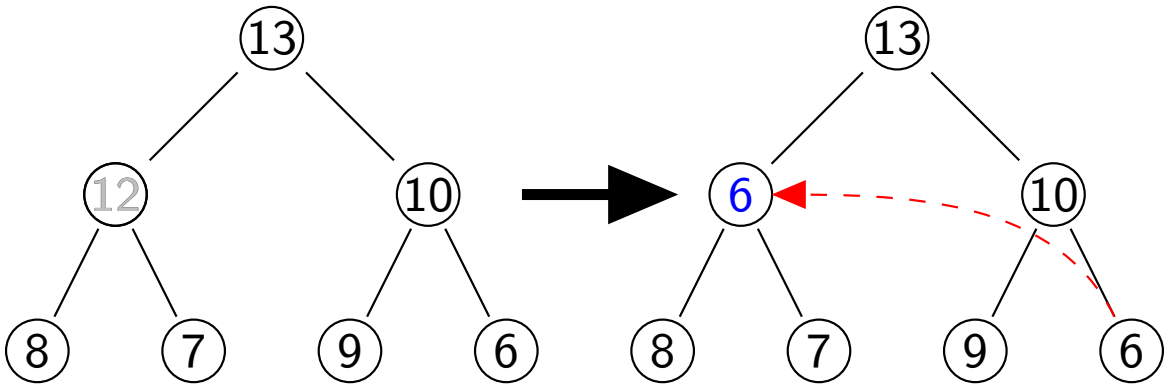

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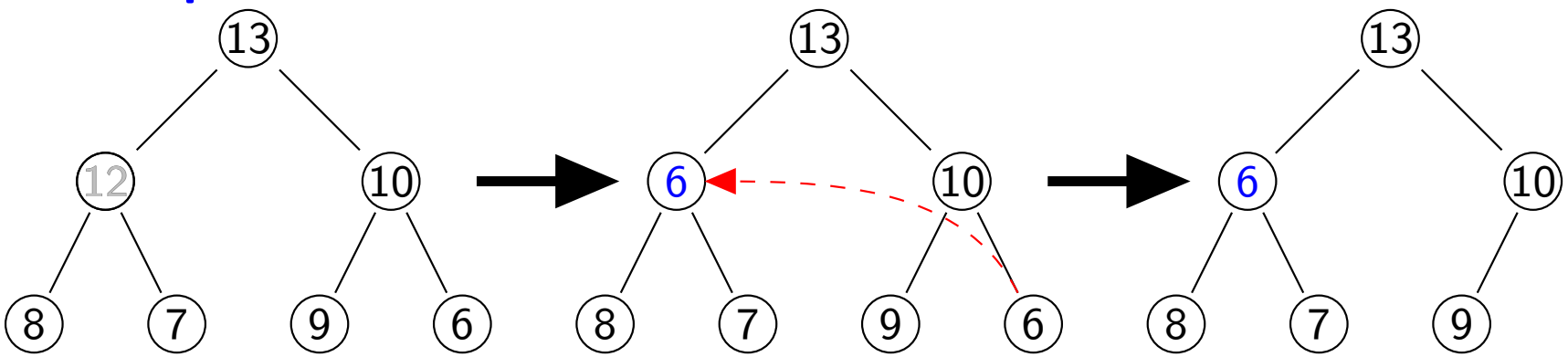

Delete 12

Copy 6 to the node to be deleted

Delete the last node

Contents

Heapsort

```
HEAPSORT(A)
1  BUILD-MAX-HEAP(A)
2  for i = A.length downto 2
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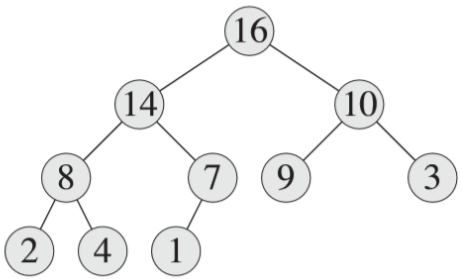
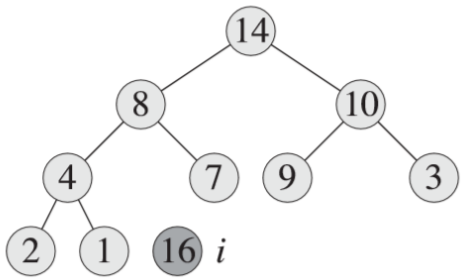


(a)



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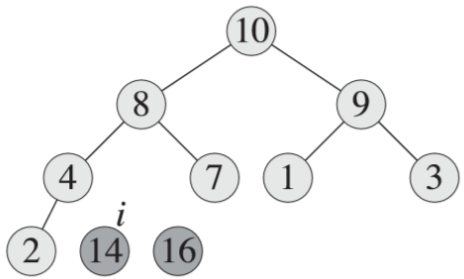
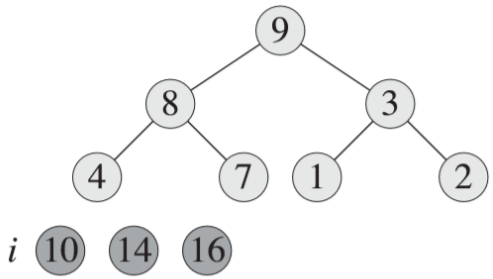
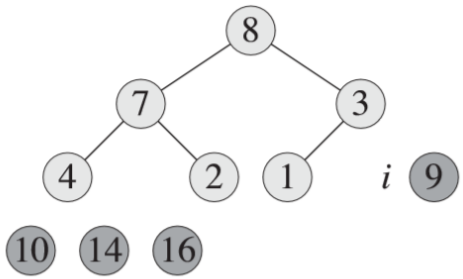


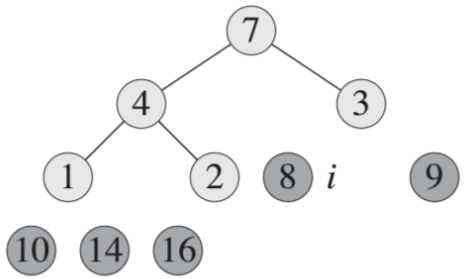
(c)



(d)



(e)



(f)

Heapsort: Correctness

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1  BUILD-MAX-HEAP(A)
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Q-6: How to prove the correctness of HEAPSORT?

Heapsort: Correctness

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```

Q-6: How to prove the correctness of HEAPSORT?

Loop Invariant (Exercise 6.4-2)

At the start of each iteration of the for loop of lines 2-5,

- the subarray $A[1..i]$ is a max-heap containing the i smallest elements of $A[1..n]$,
- the subarray $A[i+1..n]$ contains the $n-i$ largest elements of $A[1..n]$, sorted.

In-place sorting

In-place sorting algorithms

Algorithms require $O(1)$ extra space and sorting is said to be happened in-place, or for example, within the array itself.

In-place sorting

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- Bubble-sort ✓
- Insertion-sort ✓
- Heapsort ✓
- Mergesort ✓
- Quicksort✗

Review Quicksort

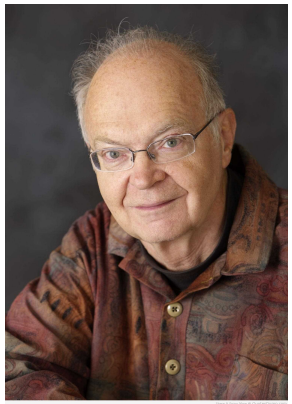
Q-7: Why is Quicksort more efficient in practice?

Review QuickSort

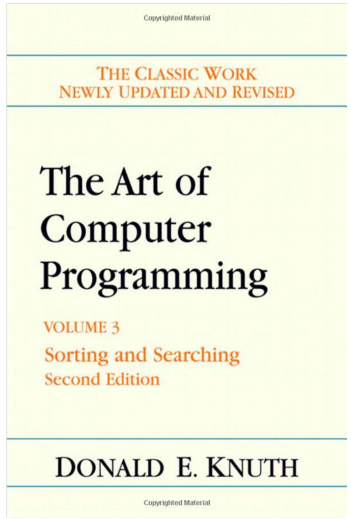
Q-7: Why is QuickSort more efficient in practice?

Based on a **fix computer model**!

- QuickSort: $11.667(n+1)\ln n - 1.74n - 18.74$
- MergeSort: $12.5n \ln n$
- HeapSort: $16n \ln n + 0.01n$
- InsertionSort: $2.25n^2 + 7.75n - 3 \ln n$



Donald Knuth



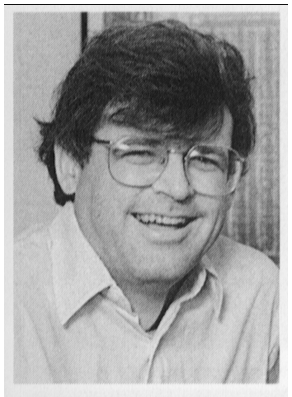
https://www-cs-faculty.stanford.edu/~knuth/taocp.html

Review QUICKSORT

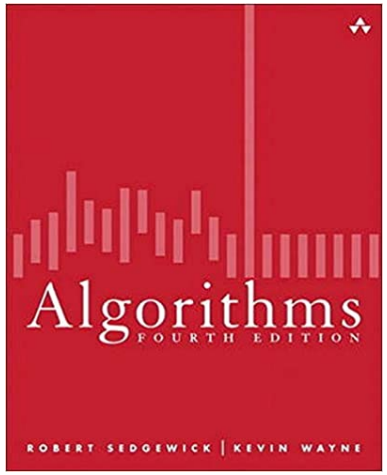
Q-7: Why is QUICKSORT more efficient in practice?

Analyze **abstract basic operations**! #swap & #comparison

- QUICKSORT: $2n \ln n$ comparisons and $\frac{1}{3} n \ln n$ swaps on average
- MERGESORT: $1.44 n \ln n$ comparisons, but up to $8.66 n ln(n)$ array accesses (mergesort is not swap based, so we cannot count that).
- INSERTIONSORT: $\frac{1}{4} n^2$ comparisons and $\frac{1}{4} n^2$ swaps on average.



Robert Sedgewick



https://algs4.cs.princeton.edu/home/

Contents

Priority Queue: ADT

Priority queue

A data structure for maintaining a set S of elements, each with an associated value called a key.

Priority Queue: ADT

Priority queue

A data structure for maintaining a set S of elements, each with an associated value called a key.

A **max-priority queue** supports the following operations:

INSERT(S, x) inserts the element x into the set S, which is equivalent to the operation $S = S \cup \{x\}$.

MAXIMUM(S) returns the element of S with the largest key.

EXTRACT-MAX(S) removes and returns the element of S with the largest key.

INCREASE-KEY(S, x, k) increases the value of element x's key to the new value k, which is assumed to be at least as large as x's current key value.

Priority Queue: ADT

Q-8: What is key difference between a Queue and a Priority-queue?

Queue

FIFO: First-In-First-Out

Priority Queue

- Order does not matter
- Priority matters

Priority Queue: Implementation

Heap → Priority Queue

```
HEAP-MAXIMUM(A)
1  return A[1]
```

```
HEAP-EXTRACT-MAX(A)
1  if A.heap-size < 1
2        error "heap underflow"
3  max = A[1]
4  A[1] = A[A.heap-size]
5  A.heap-size = A.heap-size − 1
6  MAX-HEAPIFY(A, 1)
7  return max
```

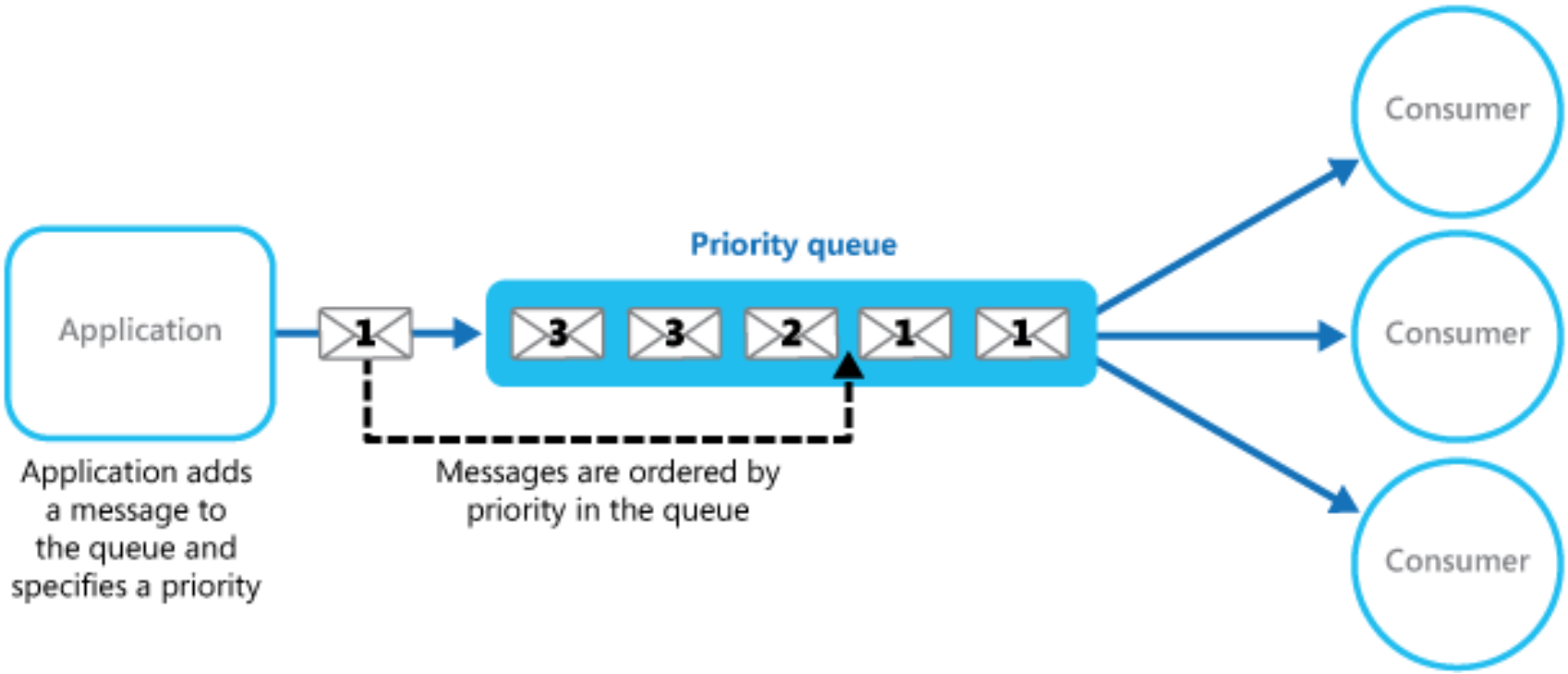
```
HEAP-INCREASE-KEY(A, i, key)
1  if key < A[i]
2        error "new key is smaller than current key"
3  A[i] = key
4  while i > 1 and A[PARENT(i)] < A[i]
5        exchange A[i] with A[PARENT(i)]
6        i = PARENT(i)
```

```
MAX-HEAP-INSERT(A, key)
1  A.heap-size = A.heap-size + 1
2  A[A.heap-size] = −∞
3  HEAP-INCREASE-KEY(A, A.heap-size, key)
```

Priority Queue: Applications

Message Queue

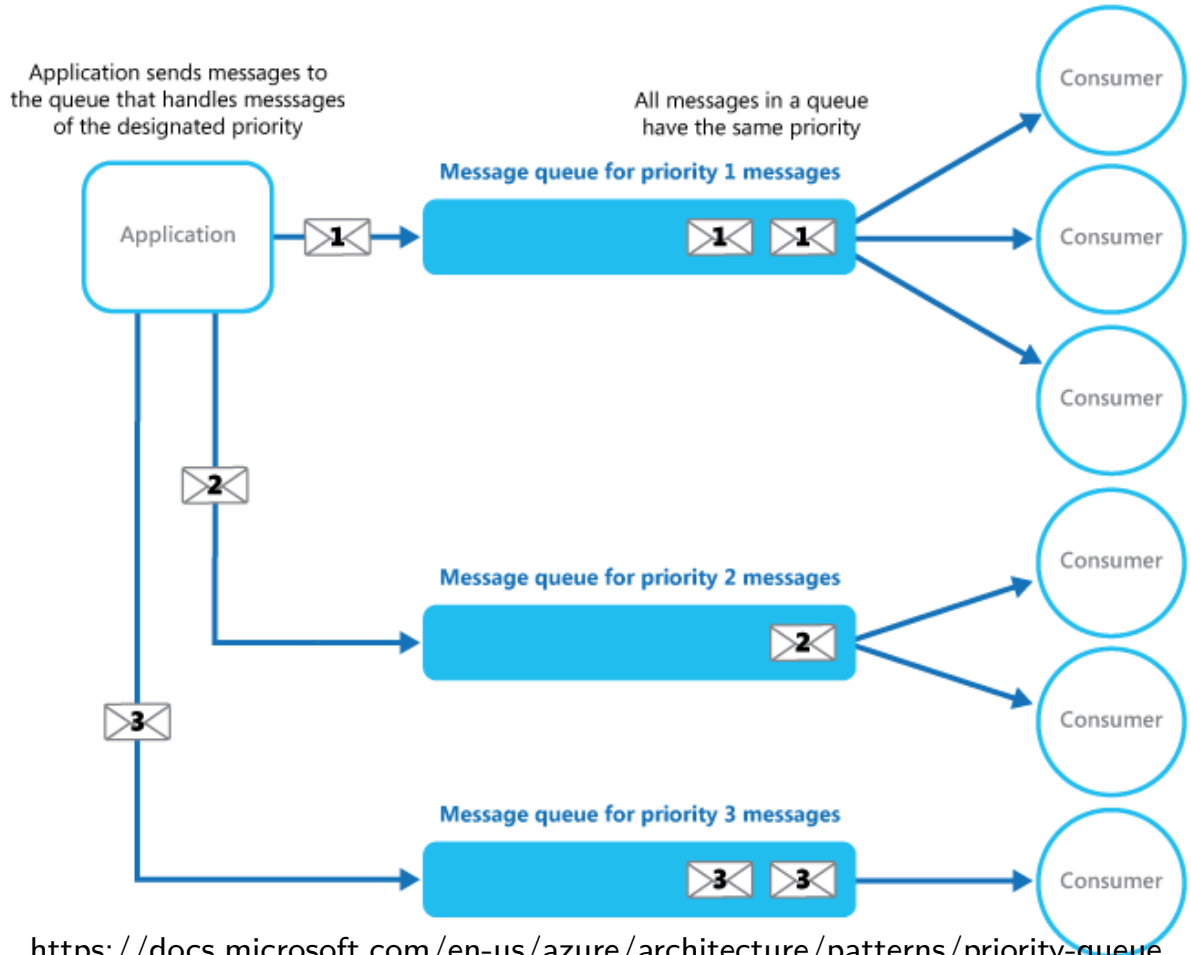
In the cloud, a message queue is typically used to delegate tasks to background processing.



https://docs.microsoft.com/en-us/azure/architecture/patterns/priority-queue

Priority Queue: Applications

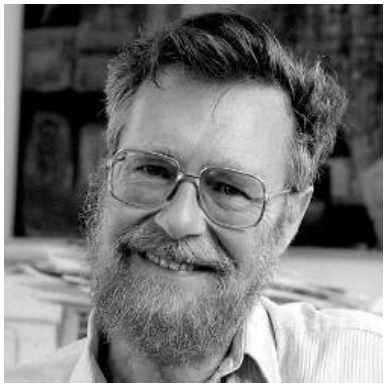
Message Queue (Without Priority Queue)



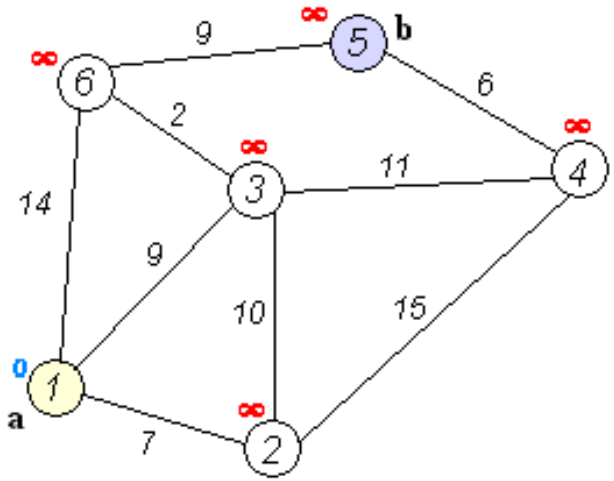
https://docs.microsoft.com/en-us/azure/architecture/patterns/priority-queue

Priority Queue: Applications

Dijkstra's Shortest Path Algorithm



Edsger W. Dijkstra.
1972 ACM A.M. Turing Award winner

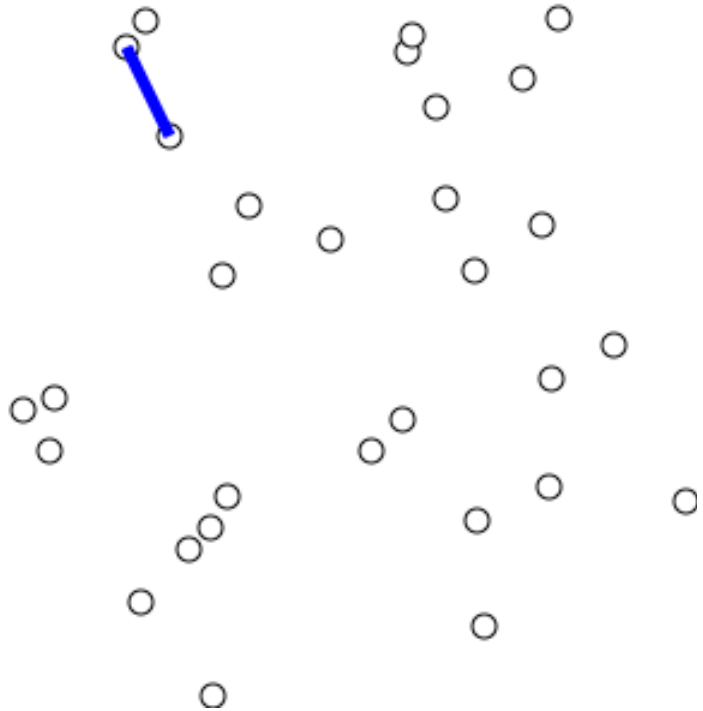


https://en.wikipedia.org/wiki/Dijkstra%27s_algorithm

"I designed in about twenty minutes".
In 1956, "One morning I was shopping in Amsterdam with my young fiancée, and tired, we sat down on the café terrace to drink a cup of coffee and I was just thinking about whether I could do this, and I then designed the algorithm for the shortest path"

Priority Queue: Applications

Prim Algorithm for MST



https://en.wikipedia.org/wiki/Dijkstra%27s_algorithm

Thank You!
Questions?

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majun@nju.edu.cn