Basics on Linear Programming

Combinatorial Problem Solving (CPS)

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Linear Programs (LP's)

- A linear program is an optimization problem of the form

$$\begin{aligned} &\min c^T x \\ &A_1 x \le b_1 \\ &A_2 x = b_2 \\ &A_3 x \ge b_3 \\ &x \in \mathbb{R}^n \end{aligned}$$

$$c \in \mathbb{R}^n, b_i \in \mathbb{R}^{m_i}, A_i \in \mathbb{R}^{m_i \times n}, i = 1, 2, 3$$

- x is the vector of variables
- $c^T x$ is the cost or objective function
- $A_1 x \le b_1$, $A_2 x = b_2$ and $A_3 x \ge b_3$ are the constraints

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- Example:

$$\begin{aligned}
&\min \; x + y + z \\
&x + y = 3 \\
&0 \le x \le 2 \\
&0 \le y \le 2
\end{aligned}$$

Notes on the Definition of LP

- Solving minimization or maximization is equivalent:

$$\max\{\ f(x) \mid x \in S\ \} \;\; = \;\; -\min\{\ -f(x) \mid x \in S\ \}$$

- Satisfiability problems are a particular case:
  take arbitrary cost function, e.g., $c = 0$

Equivalent Forms of LP's

- This form is not the most convenient for algorithms
- We will assume problems to be in canonical form:

$$\begin{aligned} &\min c^T x \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

$$c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, n \geq m, \mathrm{rank}(A) = m$$

- Often variables are identified with columns of the matrix, and constraints are identified with rows

Methods for Solving LP's

- Simplex algorithms
- Interior-point algorithms

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Basic Definitions (1)

$$\begin{aligned}&\min c^T x\\&Ax = b\\&x \geq 0\end{aligned}$$

- Any vector x such that $Ax = b$ is called a solution
- A solution x satisfying $x \geq 0$ is called a feasible solution
- An LP with feasible solutions is called feasible;
  otherwise it is called infeasible
- A feasible solution x^* is called optimal
  if $c^T x^* \leq c^T x$ for all feasible solution x
- A feasible LP with no optimal solution is unbounded

Basic Definitions (2)

$$\max x + 2y$$

$$x + y + s_1 = 3$$
$$x + s_2 = 2$$
$$y + s_3 = 2$$
$$x, y, s_1, s_2, s_3 \geq 0$$

- $(x, y, s_1, s_2, s_3) = (-1, -1, 5, 3, 3)$ is a solution but not feasible
- $(x, y, s_1, s_2, s_3) = (1, 1, 1, 1, 1)$ is a feasible solution

Basic Definitions (3)

$$\max x + \beta y$$
$$x + y + s_1 = \alpha$$
$$x + s_2 = 2$$
$$y + s_3 = 2$$
$$x, y, s_1, s_2, s_3 \geq 0$$

- If $\alpha = -1$ the LP is not feasible
- If $\alpha = 3, \beta = 2$ then $(x, y, s_1, s_2, s_3) = (1, 2, 0, 1, 0)$ is the (only) optimal solution

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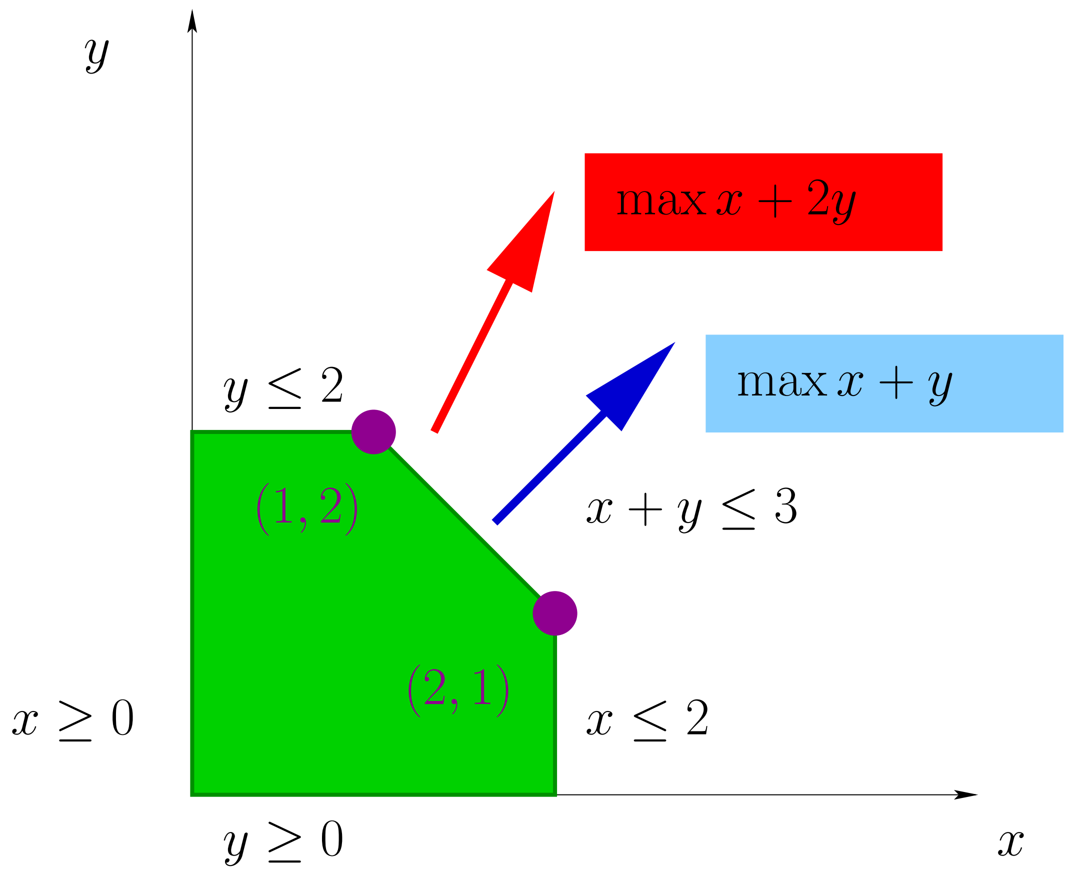
- If $\alpha = -1$ the LP is not feasible
- If $\alpha = 3, \beta = 2$ then
  $(x, y, s_1, s_2, s_3) = (1, 2, 0, 1, 0)$ is the (only) optimal solution
- There may be more than one optimal solution:
  If $\alpha = 3$ and $\beta = 1$ then
  $\{(1, 2, 0, 1, 0), (2, 1, 0, 0, 1), (\frac{3}{2}, \frac{3}{2}, 0, \frac{1}{2}, \frac{1}{2})\}$ are optimal

Basic Definitions (4)

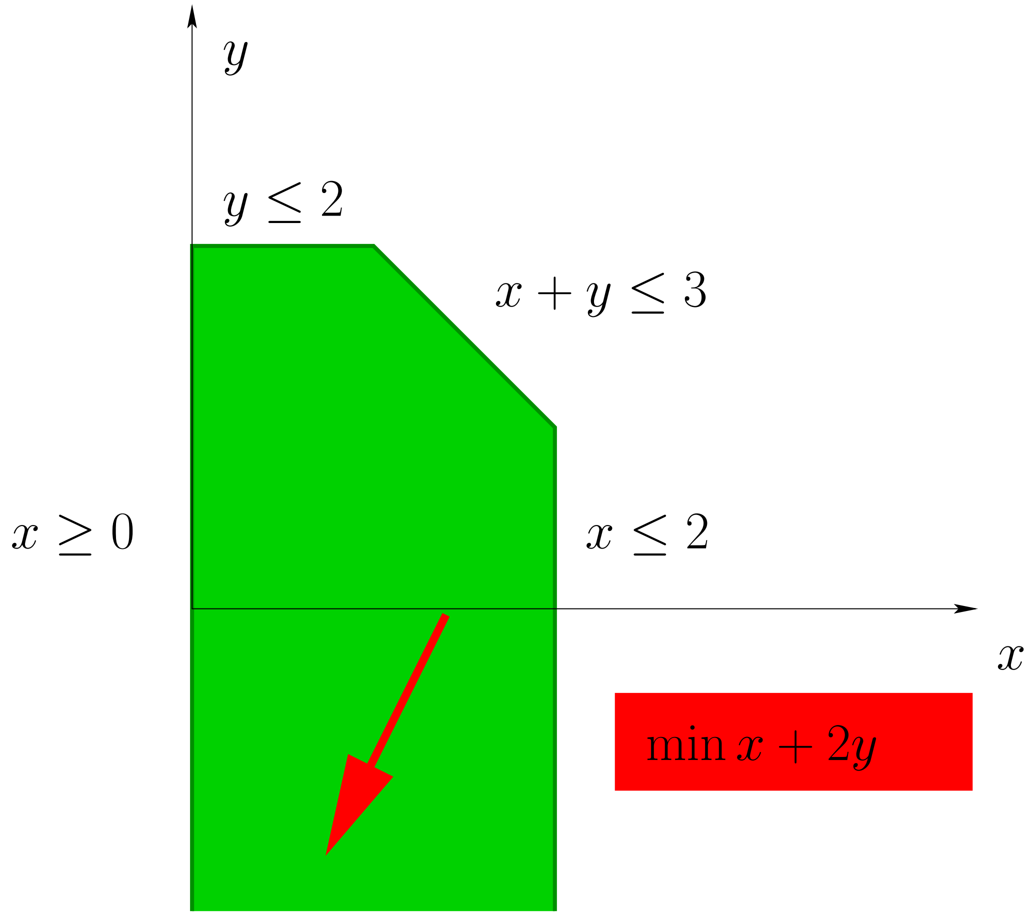


Basic Definitions (5)

$$\min x + 2y$$

$$x + y \leq 3$$
$$0 \leq x \leq 2$$
$$y \leq 2$$

Unbounded LP

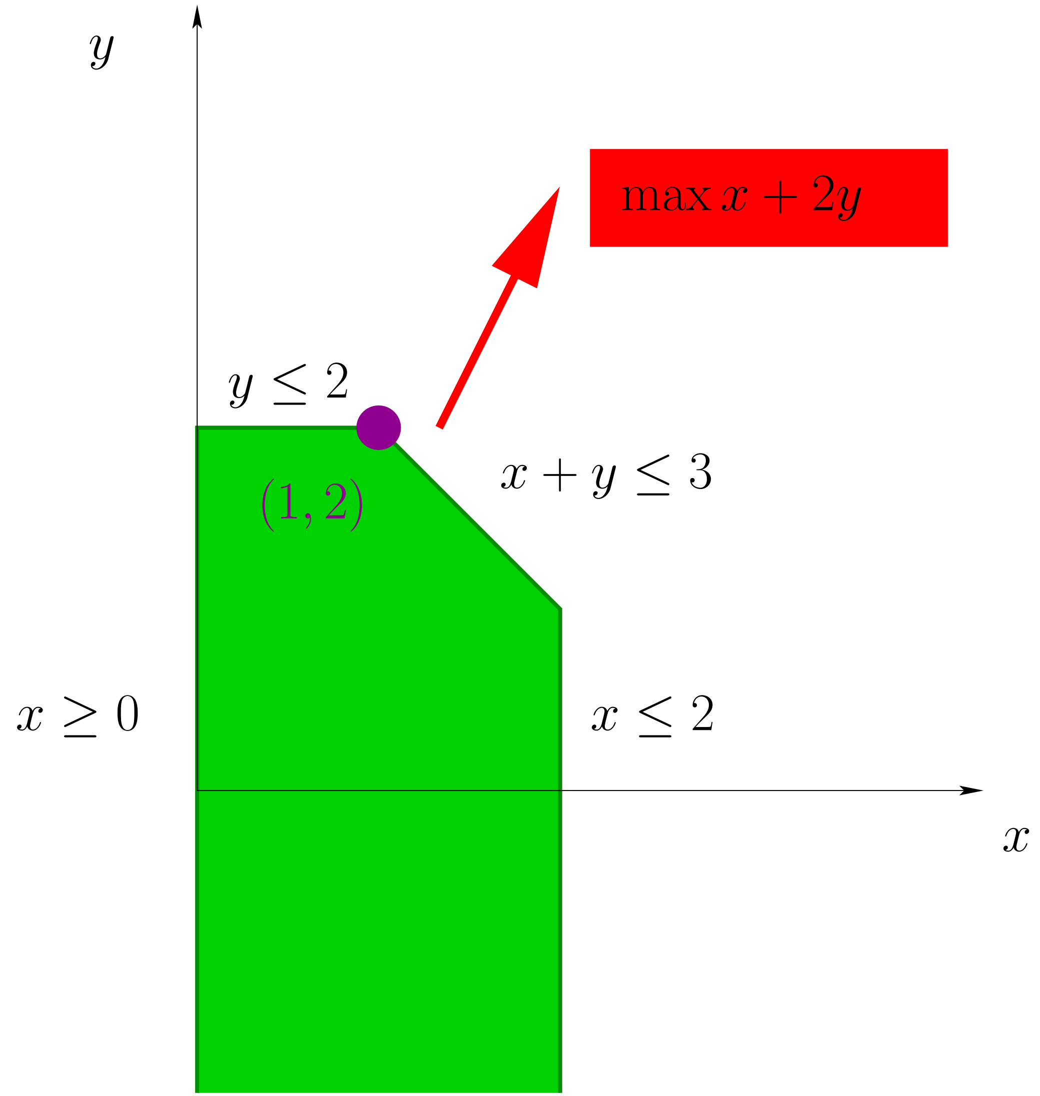


Basic Definitions (6)

$$\max x + 2y$$

$$x + y \leq 3$$
$$0 \leq x \leq 2$$
$$y \leq 2$$

LP is bounded,
but set of feasible
solutions is not



Bases (1)

Let us denote by a_1, ..., a_n the columns of A

Recall that $n \geq m, \; \mathrm{rank}(A) = m$.

- A matrix of m columns $(a_{k_1}, ..., a_{k_m})$ is a basis if the columns are linearly independent
- Note that a basis is a square matrix!
- If $(a_{k_1}, ..., a_{k_m})$ is a basis, then the variables $(x_{k_1}, ..., x_{k_m})$ are called basic
- We usually denote

  by $\mathcal{B}$ the list of indices $(k_1, ..., k_m)$, and
  by $\mathcal{R}$ the list of indices $(1, 2, ..., n) - \mathcal{B}$; and

  by B the matrix $(a_i \mid i \in \mathcal{B})$, and
  by R the matrix $(a_i \mid i \in \mathcal{R})$

  $x_{\mathcal{B}}$ the basic variables, $x_{\mathcal{R}}$ the non-basic ones

Bases (2)

$$\begin{aligned} &\max x + 2y \\ &x + y + s_1 = 3 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$

$$A = \begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

- (x, s_1, s_2) do not form a basis:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ does not have linearly independent columns

- (s_1, s_2, s_3) form a basis, where $x_{\mathcal{B}} = (s_1, s_2, s_3)$, $x_{\mathcal{R}} = (x, y)$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Bases (3)

- If B is a basis, then the following holds

  $$Bx_{\mathcal{B}} + Rx_{\mathcal{R}} = b$$

  Hence:

  $$x_{\mathcal{B}} = B^{-1}b - B^{-1}Rx_{\mathcal{R}}$$

  Non-basic variables determine values of basic ones

- If non-basic variables are set to 0, we get the solution

  $$x_{\mathcal{R}} = 0, x_{\mathcal{B}} = B^{-1}b$$

  Such a solution is called a basic solution

- If a basic solution satisfies $x_{\mathcal{B}} \geq 0$ then it is called a basic feasible solution, and the basis is feasible

Bases (4)

Consider basis (s_1, s_2, s_3)

$$\begin{aligned} &\max x + 2y \\ &x + y + s_1 = 3 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Equations $x_{\mathcal{B}} = B^{-1}b - B^{-1}Rx_{\mathcal{R}}$ are

$$\begin{cases} s_1 = 3 - x - y \\ s_2 = 2 - x \\ s_3 = 2 - y \end{cases}$$

Basic solution is

$$\sigma_{\mathcal{B}} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \qquad \sigma_{\mathcal{R}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So basis (s_1, s_2, s_3) is feasible

Bases (5)

Basis (x, y, s_1) is not feasible

$$\max x + 2y$$
$$x + y + s_1 = 3$$
$$x + s_2 = 2$$
$$y + s_3 = 2$$
$$x, y, s_1, s_2, s_3 \geq 0$$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} x = 2 - s_2 \\ y = 2 - s_3 \\ s_1 = -1 + s_2 + s_3 \end{cases}$$

$$\sigma_{\mathcal{B}} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \qquad \sigma_{\mathcal{R}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Bases (6)

A basis is called degenerate
when at least one component of its basic solution $x_{\mathcal{B}}$ is null

For example:

$$\begin{aligned}
&\max x + 2y \\
&x + y + s_1 = 4 \\
&x + s_2 = 2 \\
&y + s_3 = 2 \\
&x, y, s_1, s_2, s_3 \geq 0
\end{aligned}
\qquad
B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\qquad
R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

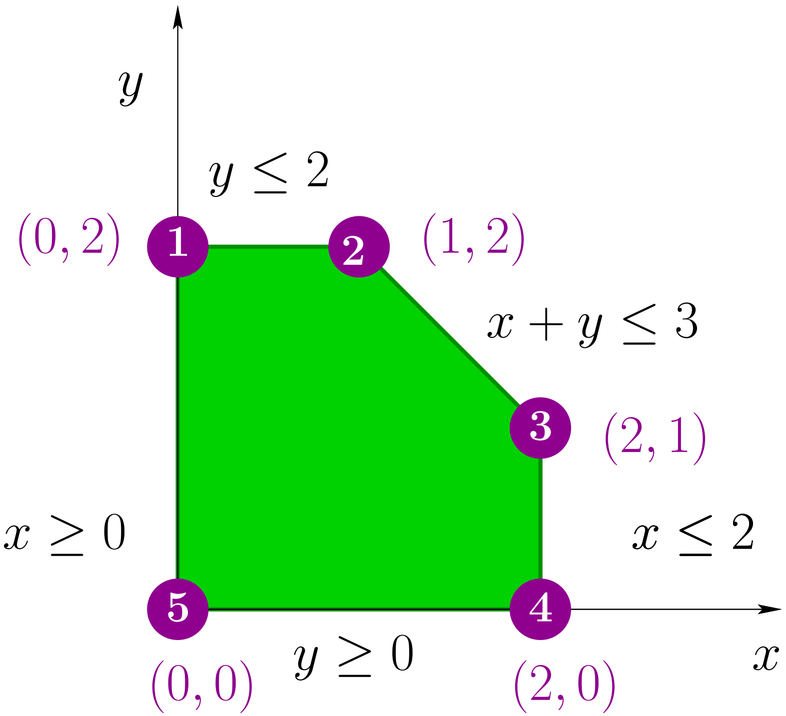
$$\left\{ \begin{array}{l} x = 2 + s_3 - s_1 \\ y = 2 - s_3 \\ s_2 = s_1 - s_3 \end{array} \right.
\qquad
\sigma_{\mathcal{B}} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

Geometry of LP's (1)

- Set of feasible solutions of an LP is a convex polyhedron
- Basic feasible solutions are vertices of the convex polyhedron

Geometry of LP's (2)

- $$\begin{aligned} &\max x + 2y \\ &x + y + s_1 = 3 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$
- $x_{\mathcal{B}_1} = (y, s_1, s_2)$
- $x_{\mathcal{B}_2} = (x, y, s_2)$
- $x_{\mathcal{B}_3} = (x, y, s_3)$
- $x_{\mathcal{B}_4} = (x, s_1, s_3)$
- $x_{\mathcal{B}_5} = (s_1, s_2, s_3)$



Geometry of LP's (3)

- **Theorem (Minkowski-Weyl)**

  Let P be a feasible LP.

  There exist basic feasible solutions $v_1, \ldots, v_r \in \mathbb{R}^n$ and vectors $r_1, \ldots, r_s \in \mathbb{R}^n$ such that a point x is a feasible solution to P iff

$$x = \sum_{i=1}^{r} \lambda_i v_i + \sum_{j=1}^{s} \mu_j r_j$$

  for certain λ_i, μ_j such that $\sum_{i=1}^{r} \lambda_i = 1$ and $\lambda_i, \mu_j \geq 0$.

Possible Outcomes of an LP (1)

- **Theorem (Fundamental Theorem of Linear Programming)**

  Let P be an LP.

  Then exactly one of the following holds:

  1. P is infeasible
  2. P is unbounded
  3. P has an optimal basic feasible solution

  It is sufficient to investigate basic feasible solutions!

Possible Outcomes of an LP (2)

Proof: Assume P feasible and with optimal solution x^*.

Let us see we can find a basic feasible solution as good as x^*.

By Minkowski-Weyl theorem, we can write

$$x^* = \sum_{i=1}^{r} \lambda_i^* v_i + \sum_{j=1}^{s} \mu_j^* r_j$$

where $\sum_{i=1}^{r} \lambda_i^* = 1$ and $\lambda_i^*, \mu_j^* \geq 0$. Then

$$c^T x^* = \sum_{i=1}^{r} \lambda_i^* c^T v_i + \sum_{j=1}^{s} \mu_j^* c^T r_j$$

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- If there is j such that $c^T r_j < 0$ then objective value can be decreased by taking μ_j^* larger. Contradiction!

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- Otherwise $c^T r_j \geq 0$ for all j. Assume $c^T x^* < c^T v_i$ for all i.

$$c^T x^* \geq \sum_{i=1}^{r} \lambda_i^* c^T v_i > \sum_{i=1}^{r} \lambda_i^* c^T x^* = c^T x^* \sum_{i=1}^{r} \lambda_i^* = c^T x^*$$

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$$c^T x^* \geq \sum_{i=1}^r \lambda_i^* c^T v_i > \sum_{i=1}^r \lambda_i^* c^T x^* = c^T x^* \sum_{i=1}^r \lambda_i^* = c^T x^*$$

Contradiction! Thus there is i such that $c^T x^* \geq c^T v_i$; in fact, $c^T x^* = c^T v_i$ by the optimality of x^*.