The Simplex Method

Combinatorial Problem Solving (CPS)

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Global Idea

- The Fundamental Theorem of Linear Programming ensures it is sufficient to explore basic feasible solutions to find the optimum of a feasible and bounded LP

- The simplex method moves from one basic feasible solution to another that does not worsen the objective function while
  - optimality or
  - unboundedness

  are not detected

Bases and Tableaux

- Given a basis B, its tableau is the system of equations

$$x_{\mathcal{B}} = B^{-1}b - B^{-1}Rx_{\mathcal{R}}$$

which expresses values of basic variables in terms of non-basic variables

$$\begin{array}{l} \min -x - 2y \\ x + y + s_1 = 3 \\ x + s_2 = 2 \\ y + s_3 = 2 \\ x, y, s_1, s_2, s_3 \geq 0 \end{array}$$

$$\mathcal{B} = \{x, y, s_2\}$$

$$\left\{ \begin{array}{l} x = 1 + s_3 - s_1 \\ y = 2 - s_3 \\ s_2 = 1 - s_3 + s_1 \end{array} \right.$$

Basic Solution in a Tableau

- The basic solution can be easily obtained from the tableau by looking at independent terms

$$\begin{cases} x = 1 + s_3 - s_1 \\ y = 2 - s_3 \\ s_2 = 1 - s_3 + s_1 \end{cases}$$

Note that by definition of basic solution,
the values for non-basic variables are null

Detecting Optimality (1)

- Tableaux can be extended with
  the expression of the cost function in terms of the non-basic variables

$$
\begin{cases}
\min -x - 2y \Longrightarrow \min -5 + s_1 + s_3 \\
x = 1 + s_3 - s_1 \\
y = 2 - s_3 \\
s_2 = 1 - s_3 + s_1
\end{cases}
$$

- Value of objective function at basic solution
  can be easily found by looking at independent term

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- Value of objective function at basic solution
  can be easily found by looking at independent term
- Coefficients of non-basic variables in objective function after substitution
  are called reduced costs
- By convention, reduced costs of basic variables are 0

Detecting Optimality (1)

- Tableaux can be extended with
  the expression of the cost function in terms of the non-basic variables

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- Value of objective function at basic solution
  can be easily found by looking at independent term
- Coefficients of non-basic variables in objective function after substitution are called reduced costs
- By convention, reduced costs of basic variables are 0
- Sufficient condition for optimality: all reduced costs are ≥ 0

  The cost of any other feasible solution can't improve on the basic solution

  So the basic solution is optimal!

Detecting Optimality (2)

- If reduced costs ≥ 0:
  sufficient condition for optimality but not necessary

- In the example, both bases are optimal
  but in one we cannot detect optimality!

$$\begin{array}{l} \min x + 2y \\ x + y = 0 \\ x, y \geq 0 \end{array} \qquad \mathcal{B} = \{x\} \quad \left\{ \begin{array}{l} \min \; y \\ x = -y \end{array} \right. \qquad \mathcal{B} = \{y\} \quad \left\{ \begin{array}{l} \min \; -x \\ y = -x \end{array} \right.$$

Improving the Basic Solution

- What to do when the tableau does not satisfy the optimality condition?

$$\begin{aligned} &\min -x - 2y \\ &x + y + s_1 = 3 \\ &x + s_2 = 2 \\ &y + s_3 = 2 \\ &x, y, s_1, s_2, s_3 \geq 0 \end{aligned} \qquad \mathcal{B} = (s_1, s_2, s_3) \quad \begin{cases} \min \ -x - 2y \\ s_1 = 3 - x - y \\ s_2 = 2 - x \\ s_3 = 2 - y \end{cases}$$

- E.g. variable y has a negative reduced cost
- If we can get a new solution where $y > 0$ and the rest of non-basic variables does not worsen the objective value, we will get a better solution
- In general, to improve the objective value:
  increase the value of a non-basic variable with negative reduced cost
  while the rest of non-basic variables are frozen to 0

  E.g. increase y while keeping $x = 0$

Improving the Basic Solution

- Let us increase value of variable y
  while satisfying non-negativity constraints on basic variables

$$\begin{cases} s_1 = 3 - x - y & \text{Limits new value to} \leq 3 \\ s_2 = 2 - x & \text{Does not limit new value} \\ s_3 = 2 - y & \text{Limits new value to} \leq 2 \end{cases}$$

- Best possible new value for y is $\min(3, 2) = 2$

- The bound due to s_3 is tight, i.e.,
  the constraint $s_3 \geq 0$ limits the new value for y

Improving the Basic Solution

- The new solution does not seem to be basic... but in fact it is. We only need to change the basis.
- When increasing the value of the improving non-basic variable, all basic variables for which the bound is tight become 0

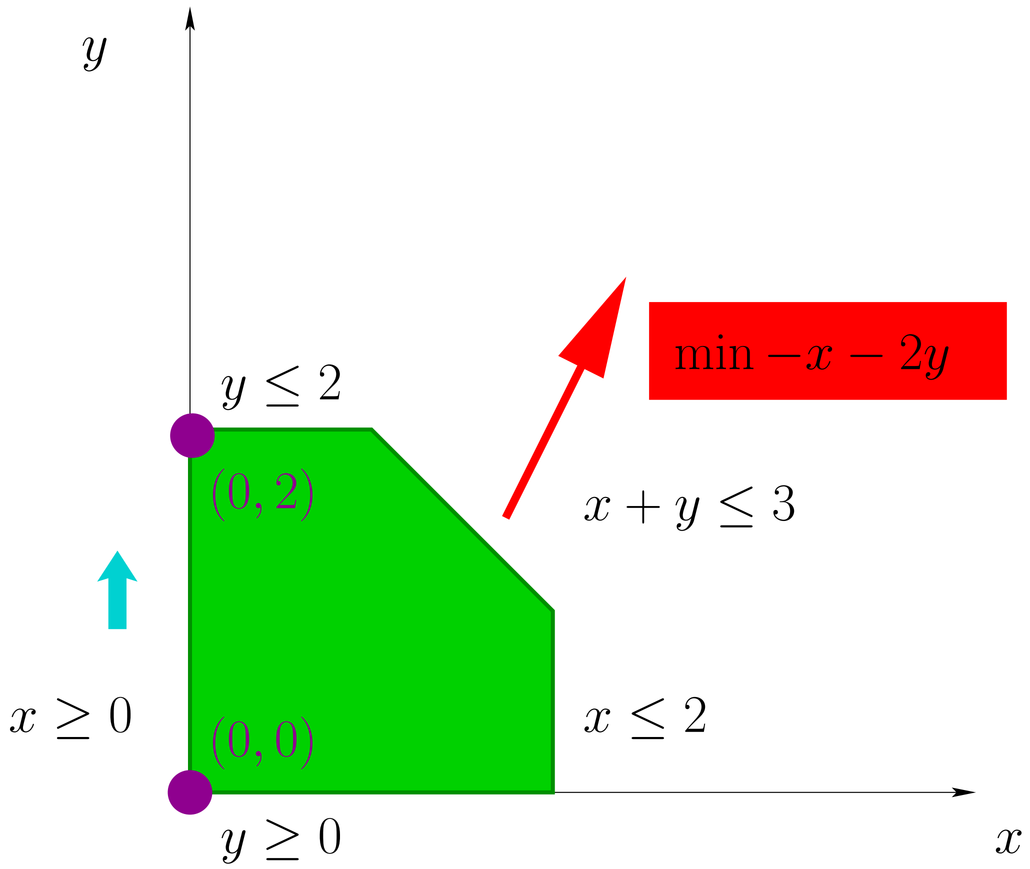
$$y = 2 \rightarrow s_3 = 0$$

- Choose a tight basic variable, here s_3, to be exchanged with the improving non-basic variable, here y
- We can get the tableau of the new basis by solving the non-basic variable in terms of the basic one and substituting:

$$s_3 = 2 - y \quad \Rightarrow \quad y = 2 - s_3$$

$$\begin{cases} \min \ -x - 2y \\ s_1 = 3 - x - y \\ s_2 = 2 - x \\ s_3 = 2 - y \end{cases} \quad \Longrightarrow \quad \begin{cases} \min \ -4 - x + 2s_3 \\ s_1 = 1 + s_3 - x \\ s_2 = 2 - x \\ y = 2 - s_3 \end{cases}$$

Improving the Basic Solution



Improving the Basic Solution

- Let us now increase value of variable x

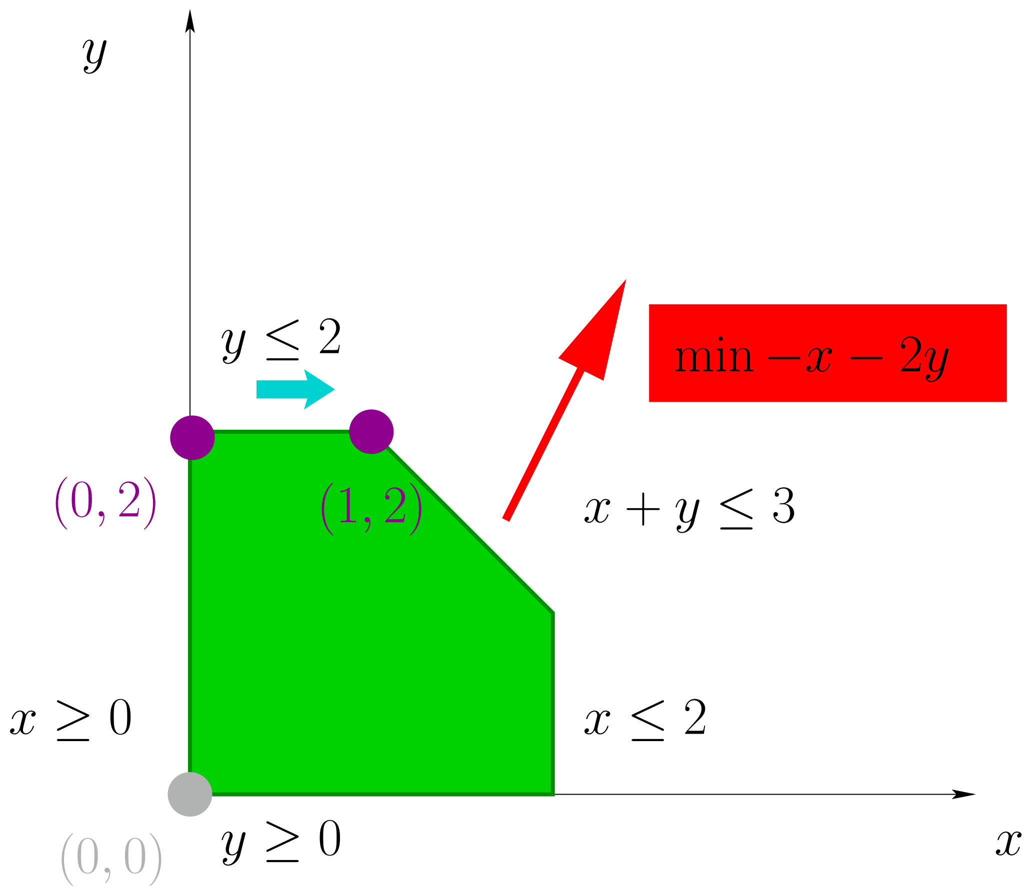
$$\begin{cases} \min\ -4 - x + 2s_3 & \\ s_1 = 1 + s_3 - x & \text{Limits new value to } \leq 1 \\ s_2 = 2 - x & \text{Limits new value to } \leq 2 \\ y = 2 - s_3 & \text{Does not limit new value} \end{cases}$$

- Best possible new value for x is $\min(2, 1) = 1$

- Variable s_1 leaves the basis and variable x enters

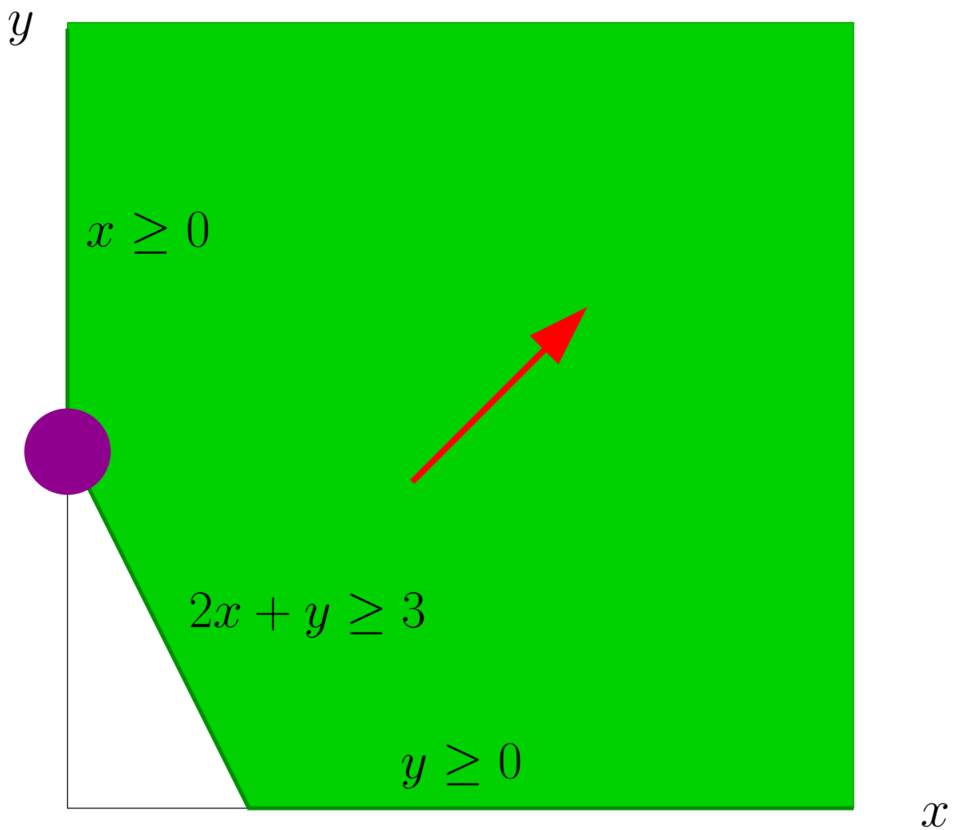
$$\begin{cases} \min\ -4 - x + 2s_3 \\ s_1 = 1 + s_3 - x \\ s_2 = 2 - x \\ y = 2 - s_3 \end{cases} \Longrightarrow \begin{cases} \min -5 + s_1 + s_3 \\ x = 1 + s_3 - s_1 \\ s_2 = 1 - s_3 + s_1 \\ y = 2 - s_3 \end{cases}$$

Improving the Basic Solution



Unboundedness

- Unboundedness is detected when
  the new value for the chosen non-basic variable is not bounded.



$$\max \; x + y$$
$$2x + y \geq 3$$
$$x, y \geq 0$$

$$\Downarrow$$

$$\begin{cases} \min -x - y \\ -2x - y + s = -3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \min -3 + x - s \\ y = 3 - 2x + s \end{cases}$$

Outline of the Simplex Algorithm

1. `Initialization:` Pick a feasible basis.

2. `Pricing:` If all reduced costs are ≥ 0,
   then return OPTIMAL.
   Else pick a non-basic variable with reduced cost < 0.

3. `Ratio Test:` Compute best value for improving non-basic variable respecting non-negativity constraints of basic variables.
   If best value is not bounded,
   then return UNBOUNDED.
   Else select tight basic variable for exchange with non-basic variable.

4. `Update:` Update the tableau and go to 2.

Finding an Initial Basis

- Note that to optimize

$$\begin{aligned} &\min c^T x \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

  initially we need a feasible basis at step 1.

- Step 1 (looking for a feasible basis) is called Phase I of the simplex algorithm

- Steps 2-4 (optimizing) are called phase II

- We will see how to get a feasible basis with the same simplex algorithm by solving another LP for which phase I is trivial

Finding an Initial Basis

For example

$$\begin{cases} \min \ -x - 2y \\ 1 \leq x + y \leq 3 \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \end{cases} \Rightarrow \begin{cases} \min -x - 2y \\ x + y + s_1 = 3 \\ x + y - s_2 = 1 \\ x + s_3 = 2 \\ y + s_4 = 2 \end{cases}$$

An initial basis consisting of slacks is simple as the inverse is the identity:

$$\begin{cases} s_1 = 3 - x - y \\ s_2 = -1 + x + y \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases}$$

But in this example it turns out not to be feasible!

Finding an Initial Basis

Problem: the slack of constraint $x + y \geq 1$ has the "wrong" sign

$$x + y \geq 1 \quad \rightarrow \quad x + y - s_2 = 1 \quad \rightarrow \quad s_2 = -1 + x + y$$

We can add an artificial variable z_1 to the equation with the "right" sign and use it in the basis instead of s_2

$$\begin{cases} x + y + s_1 = 3 \\ x + y - s_2 + z_1 = 1 \\ x + s_3 = 2 \\ y + s_4 = 2 \end{cases} \Rightarrow \begin{cases} s_1 = 3 - x - y \\ z_1 = 1 - x - y + s_2 \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases}$$

Variable z_1 represents how far we are from satisfying constraint $x + y \geq 1$

So we should minimize it
(and forget the original objective function for the time being)

Finding an Initial Basis

So let us solve

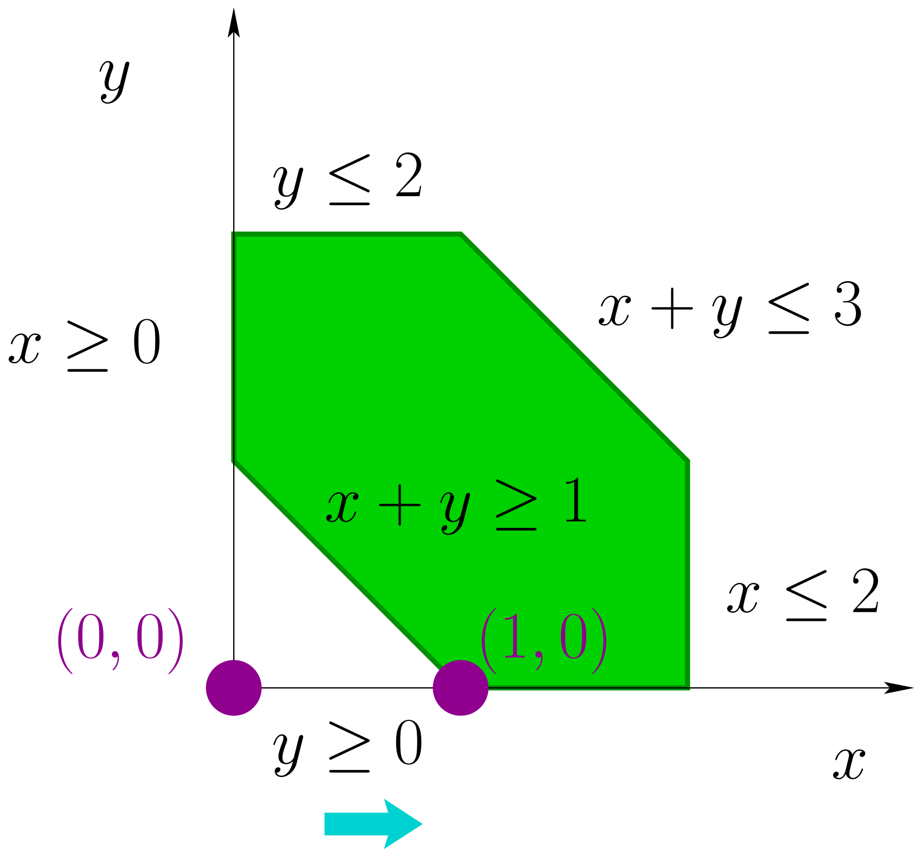
$$\begin{cases} \min z_1 \\ x + y + s_1 = 3 \\ x + y - s_2 + z_1 = 1 \\ x + s_3 = 2 \\ y + s_4 = 2 \end{cases}$$

Applying the simplex algorithm:

$$\begin{cases} \min 1 - x - y + s_2 \\ s_1 = 3 - x - y \\ z_1 = 1 - x - y + s_2 \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases} \Rightarrow \begin{cases} \min z_1 \\ s_1 = 2 + z_1 - s_2 \\ x = 1 - z_1 - y + s_2 \\ s_3 = 1 + z_1 + y - s_2 \\ s_4 = 2 - y \end{cases}$$

Feasible tableau for original LP $\begin{cases} s_1 = 2 - s_2 \\ x = 1 - y + s_2 \\ s_3 = 1 + y - s_2 \\ s_4 = 2 - y \end{cases}$

Finding an Initial Basis



Finding an Initial Basis

- In general, let us imagine we want to get an initial feasible basis for

$$\begin{aligned}&\min c^T x\\&Ax = b\\&x \geq 0\end{aligned}$$

- Let us assume wlog. that $b \geq 0$

- We introduce new vector of artificial variables z and solve

$$\begin{aligned}&\min 1^T z\\&Ax + z = b\\&x, z \geq 0\end{aligned}$$

- We may not need to add an artificial variable for each row if the slack has the right sign

  (but we will do so here, for the sake of presentation)

Finding an Initial Basis

$$[LP] \quad \begin{array}{l} \min c^T x \\ Ax = b \\ x \geq 0 \end{array} \quad \Longrightarrow \quad [LP'] \quad \begin{array}{l} \min 1^T z \\ Ax + z = b \\ x, z \geq 0 \end{array} \qquad \text{where } b \geq 0$$

- LP' is feasible, and a trivial feasible basis is $\mathcal{B} = (z)$
- LP' cannot be unbounded: $z \geq 0$ implies $1^T z \geq 0$

  So LP' has optimal solution with objective value ≥ 0
- If x^* is feasible solution to LP then $(x, z) = (x^*, 0)$ is optimal solution to LP' with objective value 0
- If $(x, z) = (x^*, z^*)$ is optimal solution to LP' with objective value 0 then $z^* = 0$ and so x^* is feasible solution to LP

Finding an Initial Basis

$$[LP] \quad \begin{array}{l} \min c^T x \\ Ax = b \\ x \geq 0 \end{array} \quad \Longrightarrow \quad [LP'] \quad \begin{array}{l} \min 1^T z \\ Ax + z = b \\ x, z \geq 0 \end{array} \quad \text{where } b \geq 0$$

- LP is feasible iff optimum of LP' is 0
- Still: how can we get a feasible basis for LP?
- Assume that optimum of LP' is 0. Then:

  1. If all artificial variables are non-basic,
     then an optimal basis for LP' is a feasible basis for LP
  2. Any basic artificial variable can be made non-basic
     by Gaussian elimination (since A has full rank)

Big M Method

- Alternative to phase I + phase II approach
- LP is changed as follows, where M is a "big number"

$$\begin{array}{lcll} \min c^T x & & \min c^T x + M \cdot 1^T z & \\ Ax = b & \Longrightarrow & Ax + z = b & \text{where } b \geq 0 \\ x \geq 0 & & x, z \geq 0 & \end{array}$$

- Again by taking the artificial variables we get an initial feasible basis
- The search of a feasible basis for the original problem is not blind wrt. cost
- Problems:
  - If M is a fixed big number,
    then the algorithm becomes numerically unstable
  - If M is kept symbolically,
    then handling costs becomes too expensive

Big M Method

$$\begin{cases} \min \ -x-2y \\ 1 \le x+y \le 3 \\ 0 \le x \le 2 \\ 0 \le y \le 2 \end{cases} \Rightarrow \begin{cases} \min -x-2y \\ x+y+s_1=3 \\ x+y-s_2=1 \\ x+s_3=2 \\ y+s_4=2 \end{cases} \Rightarrow \begin{cases} \min \ -x-2y+Mz \\ x+y+s_1=3 \\ x+y-s_2+z=1 \\ x+s_3=2 \\ y+s_4=2 \end{cases}$$

Let us solve

$$\begin{cases} \min \ -x-2y+Mz \\ x+y+s_1=3 \\ x+y-s_2+z=1 \\ x+s_3=2 \\ y+s_4=2 \end{cases}$$

starting with initial feasible basis (s_1, z, s_3, s_4)

Big M Method

$$\begin{cases} \min\ M + (-1-M)x + (-2-M)y + Ms_2 \\ s_1 = 3 - x - y \\ z = 1 - x - y + s_2 \\ s_3 = 2 - x \\ s_4 = 2 - y \end{cases}$$

$$\Longrightarrow$$

$$\begin{cases} \min\ x - 2 - 2s_2 + (M+2)z \\ s_1 = 2 + z - s_2 \\ y = 1 - x - z + s_2 \\ s_3 = 2 - x \\ s_4 = 1 + z + x - s_2 \end{cases}$$

Once z is non-basic we can drop it and continue the optimization:

$$\begin{cases} \min\ x - 2 - 2s_2 \\ s_1 = 2 - s_2 \\ y = 1 - x + s_2 \\ s_3 = 2 - x \\ s_4 = 1 + x - s_2 \end{cases}$$

Termination and Complexity

- A step of the simplex algorithm is degenerate if the increment of the chosen non-basic variable is 0

- At each step of the simplex algorithm:

  cost improvement = reduced cost · increment (of chosen non-basic var)

- There is no cost improvement if and only if the step is degenerate

- But degenerate steps can only happen with degenerate bases

- Assume no degenerate bases occur.

  Then there is a strict improvement from a base to the next one

  So simplex terminates, as bases cannot be repeated

  No. steps is at most exponential: there are $\leq \binom{n}{m}$ bases

  Tight bound for pathological cases (Klee-Minty cube)

  In practice the cost is polynomial

Termination and Complexity

- When there is degeneracy simplex may loop forever

- Termination guaranteed with anticycling rules, e.g. Bland's rule:

  Assume there is a fixed ordering of variables.

  Pricing: among non-basic vars with reduced cost < 0, take the least one

  Ratio Test: among tight basic vars, take the least one

Pricing Strategies

1. Full pricing

   Choose the variable with the most negative reduced cost

2. Partial pricing

   Make a list with the P variables with the most negative reduced costs.

   In next iterations choose variables from list until reduced costs are all ≥ 0

Pricing Strategies

3. Best-improvement pricing

   Let θ_k be the increment for a non-basic variable x_k with reduced cost $d_k < 0$. Choose the variable j such that

   $$|d_j| \cdot \theta_j = \max\{|d_k| \cdot \theta_k \ \text{ such that } \ d_k < 0, k \in \mathcal{R}\}$$

4. Normalized pricing.

   Let $n_k = ||\alpha_k||$ (in practice $n_k = \sqrt{1 + ||\alpha_k||^2}$)

   where α_k is the column in the tableau of variable x_k.

   Take criteria 1. or 2. but using $\frac{d_k}{n_k}$ instead of d_k

5. Other more sophisticate normalized pricing strategies:

   steepest edge, devex

Bounded Variables

- LP solvers implement a variant of the simplex algorithm that handles bounds more efficiently for LP's of the form

$$
\begin{aligned}
&\min c^T x \\
&Ax = b \\
&\ell \le x \le u
\end{aligned}
$$

- These bounded LP's arise when solving combinatorial problems
- Bounds are incorporated into pricing and ratio test
- Now non-basic variables will take values at the lower or the upper bound

Bounded Variables

$$\begin{aligned} &\min -x - 2y \\ &x + y \leq 3 \\ &0 \leq x \leq 2 \\ &0 \leq y \leq 2 \end{aligned} \implies \begin{aligned} &\min -x - 2y \\ &x + y + s = 3 \\ &0 \leq x \leq 2 \\ &0 \leq y \leq 2 \\ &s \geq 0 \end{aligned} \implies \begin{aligned} &\min -x - 2y \\ &s = 3 - x - y \\ &0 \leq x \leq 2 \\ &0 \leq y \leq 2 \\ &s \geq 0 \end{aligned}$$

- Initially non-basic variables x, y are at lower bound
- We choose variable x in pricing

Bounded Variables

$$\begin{cases} \min -x - 2y \\ s = 3 - x - y & \text{Limits new value to } \leq 3 \text{ as } s \geq 0 \\ 0 \leq x \leq 2 & \text{Limits new value to } \leq 2 \text{ as } x \leq 2 \\ 0 \leq y \leq 2 \\ s \geq 0 \end{cases}$$

- Best possible new value for x is $\min(3, 2) = 2$
- Bound flip: x is still non-basic, but is now at upper bound

$$\begin{cases} \min -x - 2y \\ s = 3 - x - y \\ 0 \leq x \leq 2 \\ 0 \leq y \leq 2 \\ s \geq 0 \end{cases}$$

Bounded Variables

- Pricing considers the bound status of non-basic variables
- A non-basic variable x_j with reduced cost d_j can improve the cost function
  - if x_j is at lower bound and $d_j < 0$; or
  - if x_j is at upper bound and $d_j > 0$
- Choose y in pricing:

$$\left\{\begin{array}{ll} \min -x - 2y & \\ s = 3 - x - y & \text{Limits new value to } \leq 1 \text{ as } s \geq 0 \\ 0 \leq x \leq 2 & \\ 0 \leq y \leq 2 & \text{Limits new value to } \leq 2 \text{ as } y \leq 2 \\ s \geq 0 & \end{array}\right.$$

- Best possible new value for y is $\min(1, 2) = 1$

Bounded Variables

- Usual pivoting step now:

$$s = 3 - x - y \quad \Rightarrow \quad y = 3 - x - s$$

$$\left\{\begin{array}{l} \min -x - 2y \\ s = 3 - x - y \\ 0 \le x \le 2 \\ 0 \le y \le 2 \\ s \ge 0 \end{array}\right. \qquad \Longrightarrow \qquad \left\{\begin{array}{l} \min -6 + x + 2s \\ y = 3 - x - s \\ 0 \le x \le 2 \\ 0 \le y \le 2 \\ s \ge 0 \end{array}\right.$$

Bounded Variables

- Choose x in pricing. To respect bounds for y:

$$0 \le y(x) \le 2$$
$$0 \le 3 - x \le 2$$
(since x decreases its value, $0 \le y(x)$ is OK)
$$3 - x \le 2$$
$$1 \le x$$

$$\left\{ \begin{array}{ll} \min -6 + x + 2s & \\ y = 3 - x - s & \text{Limits new value to } \ge 1 \\ 0 \le x \le 2 & \text{Limits new value to } \ge 0 \\ 0 \le y \le 2 & \\ s \ge 0 & \end{array} \right.$$

- Best possible new value for x is $\max(1, 0) = 1$

Bounded Variables

- Usual pivoting step now:

$$y = 3 - x - s \quad \Rightarrow \quad x = 3 - y - s$$

Bounded Variables

- Usual pivoting step now:

$$y = 3 - x - s \quad \Rightarrow \quad x = 3 - y - s$$

$$\left\{ \begin{array}{l} \min -6 + x + 2s \\ y = 3 - x - s \\ 0 \le x \le 2 \\ 0 \le y \le 2 \\ s \ge 0 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} \min -3 + s - y \\ x = 3 - y - s \\ 0 \le x \le 2 \\ 0 \le y \le 2 \\ s \ge 0 \end{array} \right.$$

- Since upper bound of y was tight, now y is set to its upper bound
- Optimal solution: $(x, y, s) = (1, 2, 0)$ with cost -5
- Now reading the basic solution and its cost is more involved!

Bounded Variables