

The DPLL algorithm

Combinatorial Problem Solving (CPS)

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Overview of the session

- Designing an efficient SAT solver
- DPLL: A Bit of History
- Abstract DPLL:
 - ◆ Rules
 - ◆ Examples
 - ◆ Theoretical Results

Designing an efficient SAT solver

INPUT: formula F in **CNF**

OUTPUT:

- If F is SAT: YES + model
- If F is UNSAT: NO + refutation (proof of unsatisfiability)

Two possible methods:

- resolution-based:
 - not direct to obtain model
 - + straightforward to give refutation
- DPLL-based:
 - + straightforward to obtain model
 - not direct to give refutation

Due to their efficiency, DPLL-based solvers are the method of choice

Overview of the session

- Designing an efficient SAT solver

- DPLL: A Bit of History

- Abstract DPLL:

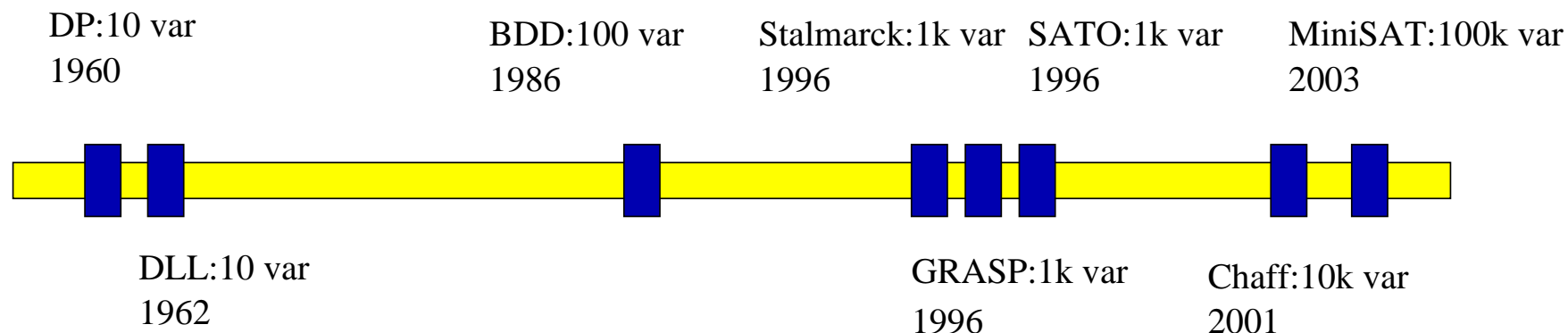
 - ◆ Rules

 - ◆ Examples

 - ◆ Theoretical Results

DPLL - A Bit of History

- Original DPLL was incomplete method for FOL satisfiability
- First paper (Davis and Putnam) in 1960: memory problems
- Second paper (Davis, Logemann and Loveland) in 1962: Depth-first-search with backtracking
- Late 90's and early 00's improvements make DPLL efficient:
 - ◆ Break-through systems: GRASP, SATO, Chaff, MiniSAT



Overview of the session

- Designing an efficient SAT solver
- DPLL: A Bit of History
- **Abstract DPLL:**
 - ◆ Rules
 - ◆ Examples
 - ◆ Theoretical Results

Our Abstraction of DPLL

- Given F in CNF, DPLL tries to build assignment M s.t. $M \models F$
- Assignments M are represented as sequences of literals (those to be true):
EXAMPLE: sequence $p\bar{q}r$ is $M(p) = 1$, $M(q) = 0$, $M(r) = 1$
(overlining bar $\bar{}$ may be used to represent negation, like \neg)
 - ◆ Order in M matters
 - ◆ No literal appears twice in M
 - ◆ No contradictory literals in M
- Sequences may have decision literals, denoted l^d .
- We will introduce a transition system modelling DPLL
- States in the transition system are pairs $M \parallel F$,
where M is a (partial) assignment and F is a CNF
- The algorithm starts with an empty assignment (and the original formula)
- The transition rules in the transition system indicate which steps
 $M \parallel F \Longrightarrow M' \parallel F'$
are allowed.

Abstract DPLL - Rules

Extending the assignment:

Decide

$$M \parallel F \implies M l^d \parallel F \text{ if } \begin{cases} l \text{ or } \bar{l} \text{ occurs in } F \\ l \text{ is undefined in } M \end{cases}$$

UnitProp

$$M \parallel F, C \vee l \implies M l \parallel F, C \vee l \text{ if } \begin{cases} M \models \neg C \\ l \text{ is undefined in } M \end{cases}$$

Abstract DPLL - Rules (2)

Repairing the assignment:

Fail

$$M \parallel F, C \implies \text{fail} \text{ if } \begin{cases} M \models \neg C \\ M \text{ contains no decision literals} \end{cases}$$

Backtrack

$$M l^d N \parallel F, C \implies M \bar{l} \parallel F, C \text{ if } \begin{cases} M l^d N \models \neg C \\ N \text{ contains no decision lits} \end{cases}$$

Abstract DPLL - Example 1

$$\emptyset \parallel \bar{1} \vee 2, \bar{3} \vee 4, \bar{5} \vee \bar{6}, 6 \vee \bar{5} \vee \bar{2} \implies$$

Abstract DPLL - Example 1

$$\emptyset \parallel \bar{1} \vee 2, \bar{3} \vee 4, \bar{5} \vee \bar{6}, 6 \vee \bar{5} \vee \bar{2} \implies (\text{Decide})$$

Abstract DPLL - Example 1

$$\begin{array}{ll} \emptyset & \parallel \quad \bar{1} \vee 2, \quad \bar{3} \vee 4, \quad \bar{5} \vee \bar{6}, \quad 6 \vee \bar{5} \vee \bar{2} \quad \Longrightarrow \quad (\text{Decide}) \\ 1^d & \parallel \quad \bar{1} \vee 2, \quad \bar{3} \vee 4, \quad \bar{5} \vee \bar{6}, \quad 6 \vee \bar{5} \vee \bar{2} \quad \Longrightarrow \end{array}$$

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| $1^d 2 3^d 4 \bar{5} 6^d$ | \parallel | $\bar{1} \vee 2, \bar{3} \vee 4, \bar{5} \vee \bar{6}, 6 \vee \bar{5} \vee \bar{2}$ | | |

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| $1^d 2$ | \parallel | $\bar{1} \vee 2, \bar{3} \vee 4, \bar{5} \vee \bar{6}, 6 \vee \bar{5} \vee \bar{2}$ | \implies | (Decide) |
| $1^d 2 3^d$ | \parallel | $\bar{1} \vee 2, \bar{3} \vee 4, \bar{5} \vee \bar{6}, 6 \vee \bar{5} \vee \bar{2}$ | \implies | (UnitProp) |
| $1^d 2 3^d 4$ | \parallel | $\bar{1} \vee 2, \bar{3} \vee 4, \bar{5} \vee \bar{6}, 6 \vee \bar{5} \vee \bar{2}$ | \implies | (Decide) |
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| $1^d 2 3^d 4 5^d \bar{6}$ | \parallel | $\bar{1} \vee 2, \bar{3} \vee 4, \bar{5} \vee \bar{6}, 6 \vee \bar{5} \vee \bar{2}$ | \implies | (Backtrack) |
| $1^d 2 3^d 4 \bar{5}$ | \parallel | $\bar{1} \vee 2, \bar{3} \vee 4, \bar{5} \vee \bar{6}, 6 \vee \bar{5} \vee \bar{2}$ | \implies | (Decide) |
| $1^d 2 3^d 4 \bar{5} 6^d$ | \parallel | $\bar{1} \vee 2, \bar{3} \vee 4, \bar{5} \vee \bar{6}, 6 \vee \bar{5} \vee \bar{2}$ | | Final state found! |

Abstract DPLL - Example 2

$$\emptyset \parallel \bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3} \implies$$

Abstract DPLL - Example 2

$$\emptyset \parallel \bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3} \implies (\text{UnitProp})$$

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$$\begin{array}{ll} \emptyset \parallel & \bar{1} \vee 2 \vee 3, \quad 1, \quad \bar{2} \vee 3, \quad \bar{2} \vee \bar{3}, \quad 2 \vee 3, \quad 2 \vee \bar{3} \quad \Longrightarrow \quad (\text{UnitProp}) \\ 1 \parallel & \bar{1} \vee 2 \vee 3, \quad 1, \quad \bar{2} \vee 3, \quad \bar{2} \vee \bar{3}, \quad 2 \vee 3, \quad 2 \vee \bar{3} \quad \Longrightarrow \end{array}$$

Abstract DPLL - Example 2

$$\begin{array}{llllllll} \emptyset & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies & (\text{UnitProp}) \\ 1 & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies & (\text{Decide}) \end{array}$$

Abstract DPLL - Example 2

$$\begin{array}{llllllllll} \emptyset & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies & (\text{UnitProp}) \\ 1 & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies & (\text{Decide}) \\ 1 \text{ } 2^d & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies & \end{array}$$

Abstract DPLL - Example 2

$$\begin{array}{llllllllll} \emptyset & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies & (\text{UnitProp}) \\ 1 & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies & (\text{Decide}) \\ 1 \text{ } 2^d & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies & (\text{UnitProp}) \end{array}$$

Abstract DPLL - Example 2

$$\begin{array}{llllllllll}
 \emptyset & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies & (\text{UnitProp}) \\
 1 & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies & (\text{Decide}) \\
 1 \text{ } 2^d & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies & (\text{UnitProp}) \\
 1 \text{ } 2^d \text{ } 3 & \parallel & \bar{1} \vee 2 \vee 3, & 1, & \bar{2} \vee 3, & \bar{2} \vee \bar{3}, & 2 \vee 3, & 2 \vee \bar{3} & \implies &
 \end{array}$$

Abstract DPLL - Example 2

| | | | | |
|-------------|-------------|--|------------|-------------|
| \emptyset | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (UnitProp) |
| 1 | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (Decide) |
| 1 2^d | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (UnitProp) |
| 1 2^d 3 | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (Backtrack) |

Abstract DPLL - Example 2

| | | | | |
|-------------|-------------|--|------------|-------------|
| \emptyset | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (UnitProp) |
| 1 | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (Decide) |
| 1 2^d | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (UnitProp) |
| 1 2^d 3 | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (Backtrack) |
| 1 $\bar{2}$ | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | |

Abstract DPLL - Example 2

| | | | | |
|-------------|-------------|--|------------|-------------|
| \emptyset | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (UnitProp) |
| 1 | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (Decide) |
| 1 2^d | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (UnitProp) |
| 1 2^d 3 | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (Backtrack) |
| 1 $\bar{2}$ | \parallel | $\bar{1} \vee 2 \vee 3, 1, \bar{2} \vee 3, \bar{2} \vee \bar{3}, 2 \vee 3, 2 \vee \bar{3}$ | \implies | (UnitProp) |

Abstract DPLL - Example 2

| | | | | | | | | | |
|---------------------------------|-------------|--------------------------|------|-------------------|-------------------------|-------------|------------------|------------|-------------|
| \emptyset | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (UnitProp) |
| 1 | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (Decide) |
| $1 \text{ } 2^d$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (UnitProp) |
| $1 \text{ } 2^d \text{ } 3$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (Backtrack) |
| $1 \text{ } \bar{2}$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (UnitProp) |
| $1 \text{ } \bar{2} \text{ } 3$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | |

Abstract DPLL - Example 2

| | | | | | | | | | |
|---------------------------------|-------------|--------------------------|------|-------------------|-------------------------|-------------|------------------|------------|-------------|
| \emptyset | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (UnitProp) |
| 1 | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (Decide) |
| $1 \text{ } 2^d$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (UnitProp) |
| $1 \text{ } 2^d \text{ } 3$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (Backtrack) |
| $1 \text{ } \bar{2}$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (UnitProp) |
| $1 \text{ } \bar{2} \text{ } 3$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (Fail) |

Abstract DPLL - Example 2

| | | | | | | | | | |
|---------------------------------|-------------|--------------------------|------|-------------------|-------------------------|-------------|------------------|------------|-------------|
| \emptyset | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (UnitProp) |
| 1 | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (Decide) |
| $1 \text{ } 2^d$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (UnitProp) |
| $1 \text{ } 2^d \text{ } 3$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (Backtrack) |
| $1 \text{ } \bar{2}$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (UnitProp) |
| $1 \text{ } \bar{2} \text{ } 3$ | \parallel | $\bar{1} \vee 2 \vee 3,$ | $1,$ | $\bar{2} \vee 3,$ | $\bar{2} \vee \bar{3},$ | $2 \vee 3,$ | $2 \vee \bar{3}$ | \implies | (Fail) |
| <i>fail</i> | | | | | | | | | |

Abstract DPLL

- There are **no infinite sequences** of the form $\emptyset \parallel F \implies \dots$
- If $\emptyset \parallel F \implies^* M \parallel F$ with state $M \parallel F$ final, then
 - ◆ F is **satisfiable**
 - ◆ M is a model of F
- If $\emptyset \parallel F \implies^* fail$ then F is **unsatisfiable**

Hence the transition system gives a **decision procedure** for SAT

Bibliography - Some further reading

- Martin Davis, Hilary Putnam. *A Computing Procedure for Quantification Theory*. J. ACM 7(3): 201-215 (1960)
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