Z3 Python API 入门

SAT SMT by example (Basics and Equations)

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在线观看幻灯片

本地Jupyter观看幻灯片需要安装RISE

Z3 介绍

Z3是微软开发的一个跨平台的高性能定理证明器, 用于许多应用程序, 例如: 软件/硬件验证和测试, 约束求解, 混合系统分析, 安全性验证, 生物学和几何问题等.

Z3提供多种语言的API, 使得调用Z3变得十分方便. 我们使用Python API.

主页: https://github.com/Z3Prover/z3

SAT和SMT的关系

SMT solver是SAT solver的前端, 它将输入的SMT表达式转换为CNF, 再调用SAT solver.

SMT就像高级编程语言, SAT像汇编语言.

一些SMT solver使用第三方SAT solver作为后端, Z3有自己的SAT solver后端.

常见SMT solver

- MathSAT (http://mathsat.fbk.eu/)
- Z3 (https://github.com/Z3Prover/z3/)
- CVC4 (http://cvc4.cs.nyu.edu/web/)
- Yices 2 (http://yices.csl.sri.com/)
- CUDD (http://vlsi.colorado.edu/~fabio/CUDD/)
- PicoSAT (http://fmv.jku.at/picosat/)
- Boolector (http://fmv.jku.at/boolector/)
- PySMT (https://github.com/pysmt/pysmt)
- JavaSMT (https://github.com/sosy-lab/java-smt)
- 作者自己开发的玩具 MK85 (https://yurichev.com/MK85/)

CNF表达式

举例: $(\neg A \vee B) \wedge (C \vee \neg D)$

SAT solver专用于解CNF形式的布尔表达式

常见SAT solver

- CaDiCaL (https://github.com/arminbiere/cadical)
- Glucose (http://www.labri.fr/perso/lsimon/glucose/)
- Lingeling (http://fmv.jku.at/lingeling/)
- MapleCM (http://sat2018.forsyte.tuwien.ac.at/solvers/main_and_glucose_hack/)
- Maplesat (https://sites.google.com/a/gsd.uwaterloo.ca/maplesat/)
- Minicard (https://github.com/liffiton/minicard)
- Minisat (http://minisat.se/MiniSat.html)
- PySAT (https://github.com/pysathq/pysat)

NP问题的另一种解释

许多算法`快`或`慢`依赖于输入规模.

- 常数时间 $O(1)$
- 线性时间 $O(n)$
- 对数时间 $O(log(n))$
- 指数时间 $O(2^{p(n)})$

NP问题没有大O表示法.

CNF表达式的求解是NP问题, 可以简单使用暴力枚举的方法. SAT solver实际上也是bruteforce, 但它使用很多优化的方法, 然而没有任何的SAT/SMT solver能预测求解需要的时间.

SAT/SMT solver到底是什么?

可以理解为解NP问题的API, 因为所有NP都可以规约到SAT问题.

资源链接

幻灯片中的资料主要来源:

- SAT SMT by example (GIT)
- SAT SMT by example (PDF)
- MK85 toy-level bit-blaster (SRC)
- Z3Py Guide (Jupyter)
- Z3 GitHub
- Z3Examples

环境搭建

最新的发布版: https://github.com/Z3Prover/z3/releases

使用PyCharm的配置步骤: https://github.com/hengxin/sat-smt-satisfying/tree/master/z3-config

或通过Python PIP安装(只包含Python API, 没有Z3可执行程序):

In [ ]:
```
!pip3 install z3-solver
```

第一件事

`import z3` 或 `from z3 import *`

用多了发现还是 `from z3 import *` 方便

In [6]:
```
from z3 import *
```

一个简单的例子

$x > 2$
$y < 10$
$x + 2 * y = 7$

In [7]:

```
x = Int('x')
y = Int('y')
solve(x > 2, y < 10, x + 2*y == 7)
```

```
[y = 0, x = 7]
```

- `Int('x')`在Z3中创建了整型变量x,
- `solve()`函数可以求解一系列规约是否可满足,这里有三个规约,
- `<, <=, >, >=, ==, !=`是用于比较的操作符,类似C++中的重载操作符.

化简函数`simplify()`和网页格式化设置`set_option(html_mode=True)`

In [8]:
```
print(simplify(And(x + 1 >= 3, x**2 + x**2 + y**2 + 2 >= 5)))
```

```
And(x >= 2, 2*x**2 + y**2 >= 3)
```

In [9]:
```
set_option(html_mode=True)
print(simplify(And(x + 1 >= 3, x**2 + x**2 + y**2 + 2 >= 5)))
set_option(html_mode=False)
```

```
x &ge; 2 &and; 2&middot;x<sup>2</sup> + y<sup>2</sup> &ge; 3
```

上面的html代码显示效果: $x \geq 2 \wedge 2 \cdot x^{2} + y^{2} \geq 3$

实数

Z3中能表示任意大的整数,有理数和代数数

In [10]:
```
x = Real('x')
y = Real('y')
solve(x**2 + 0.12 * y**2 > 3, x**3 + y < 5)
```

```
[x = 1/8, y = 639/128]
```

In [11]:
```
print(1/3)
print(RealVal(1)/3)
x = Real('x')
print(x + 1/3)
print(x + "1/3")
print(x + 0.25)
```

```
0.3333333333333333
1/3
x + 3333333333333333/10000000000000000
x + 1/3
x + 1/4
```

有理数也可以用十进制小数表示, 最后的?表示输入被截断到指定的精度:

In [12]:
```
x = Real('x')
solve(3*x*x == 2)
```

[x = -0.8164965809?]

In [13]:
```
set_option(rational_to_decimal=True)
solve(3*x == 1)

set_option(precision=20)
solve(3*x == 1)
set_option(rational_to_decimal=False)
solve(6*x == 37)
```

[x = 0.3333333333?]
[x = 0.33333333333333333333?]
[x = 37/6]

布尔逻辑

Z3支持的布尔逻辑操作: `And()` , `Or()` , `Not()` , `Implies()` (implication), `If()` (if-then-else). `==` (Bi-implications).

举例: $p \Rightarrow q, r \Leftrightarrow \neg q, \neg p \vee r$

In [14]:
```
p = Bool('p')
q = Bool('q')
r = Bool('r')
solve(Implies(p, q), r == Not(q), Or(Not(p), r))
```

[p = False, q = True, r = False]

Solver()

如果规约复杂, `solve()` 难以直接使用.

In [15]:
```
x = Int('x')
y = Int('y')

s = Solver()

print(s)

s.add(x > 10, y == x + 2)
print(s)
print(s.check())
if s.check() == sat:
    print(s.model())
```

```
[]
[x > 10, y == x + 2]
sat
[x = 11, y = 13]
```

Z3无法解非线性多项式时间的规约，如求解 $2^x = 3$ 将会输出 `unknown`：

In [16]:
```
x = Real('x')
s = Solver()
s.add(2**x == 3)
print(s.check())
```

```
unknown
```

约束无法满足时，`solve()` 输出 `no solution`，`s.check()` 输出 `unsat`

In [17]:
```
x = Real('x')
solve(x > 4, x < 0)
```

```
no solution
```

In [18]:
```
x = Real('x')
s = Solver()
s.add(x > 4, x < 0)
print(s.check())
```

```
unsat
```

一次创建多个变量

In [19]:
```
a, b, c = Ints('a b c')
d, e = Reals('d e')
solve(a > b + 2, a == 2*c + 10, c + b <= 1000, d >= e)
```

[c = 0, b = 0, e = 0, d = 0, a = 10]

In [20]:
```
X = [ Int('x%s' % i) for i in range(5) ]
print(X)

X = IntVector('x', 5)
Y = RealVector('y', 5)
P = BoolVector('p', 5)
print(X, Y, P, sep='\n')
```

[x0, x1, x2, x3, x4]

[x__0, x__1, x__2, x__3, x__4]
[y__0, y__1, y__2, y__3, y__4]
[p__0, p__1, p__2, p__3, p__4]

机器数表示

In [21]:

```
x, y = BitVec('x', 16), BitVec('y', 16)
print(x + 2)

# -1 is equal to 65535 for 16-bit integers
print(simplify(x + y - 1))

# Creating bit-vector constants

a, b = BitVecVal(-1, 16), BitVecVal(65535, 16)
print(simplify(a == b))

a, b = BitVecVal(-1, 32), BitVecVal(65535, 32)
# -1 is not equal to 65535 for 32-bit integers
print(simplify(a == b))
```

```
x + 2
65535 + x + y
True
False
```

Z3并不区分有符号或无符号,对于无符号的操作,有特定的操作符:`ULT, ULE, UGT, UGE, UDiv, URem and LShR`对应`<, <=, >, >=, /, % and >>`

In [22]:

```
solve(x & y == ~y)


solve(x < 0)

# using unsigned version of <
solve(ULT(x, 0))
```

```
[y = 65535, x = 0]
[x = 65535]
```

no solution

为什么使用 Z3 Python API?

接下来将对比使用Python和直接使用Z3解下面的三元一次方程组:

$$
\begin{aligned}
&3x + 2y - z = 1 \\
&2x - 2y + 4z = -2 \\
&-x + \tfrac{1}{2}y - z = 0
\end{aligned}
$$

使用Z3 Python API:

In [23]:

```
x, y, z = Reals('x y z')
s = Solver()
s.add(3 * x + 2 * y - z == 1)
s.add(2 * x - 2 * y + 4 * z == -2)
s.add(-x + 0.5 * y - z == 0)
print(s.check())
print(s.model())
```

```
sat
[z = -2, y = -2, x = 1]
```

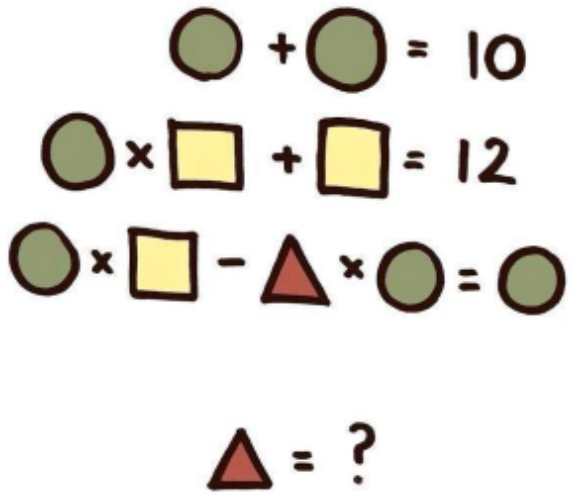
直接使用Z3命令行工具(需要`z3`所在目录在`PATH`中):

In [24]:

```
!echo \
(declare-const x Real) \
(declare-const y Real) \
(declare-const z Real) \
(assert (=(-(+(* 3 x) (* 2 y)) z) 1)) \
(assert (=(+(-(* 2 x) (* 2 y)) (* 4 z)) -2)) \
(assert (=(-(+ (- 0 x) (* 0.5 y)) z) 0)) \

(check-sat) \
(get-model) | z3 -in
```

```
sat
(model
  (define-fun z () Real
    (- 2.0))
  (define-fun y () Real
    (- 2.0))
  (define-fun x () Real

    1.0)
)
```

另一个解方程的例子

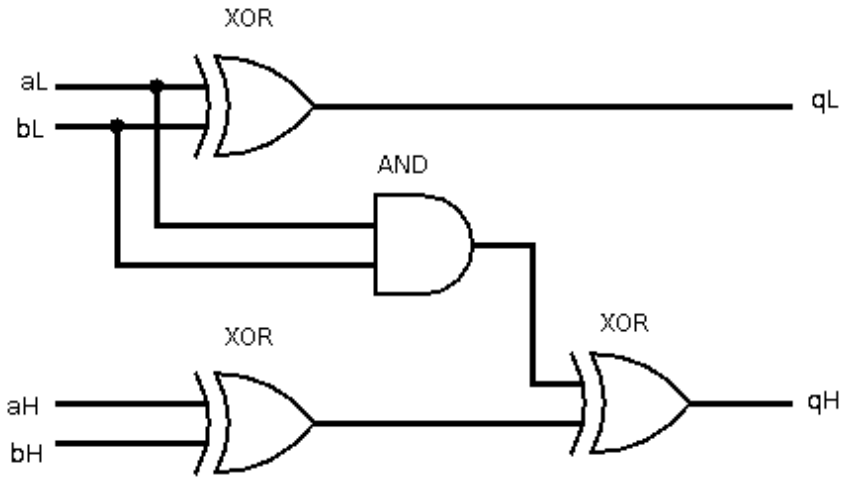


In [25]:

```
circle, square, triangle = Ints('circle square triangle')
s = Solver()
s.add(circle + circle == 10)
s.add(circle * square + square == 12)
s.add(circle * square - triangle * circle == circle)
print(s.check())

if s.check() == sat:
    print(s.model())
```

```
sat
[triangle = 1, square = 2, circle = 5]
```

直接使用SAT的例子



上图中的2-bit 加法器电路没有进位,求使得输出为3(qL和qH均为1)的输入

容易想到有四种可能的输入:$0+3=3, 1+2=3, 2+1=3, 3+0=3$

将该问题改写为CNF: $(\neg a \vee \neg c) \wedge (a \vee c) \wedge (\neg b \vee \neg d) \wedge (b \vee d)$

使用MiniSat解该问题:

```
p cnf 4 4
-1 -3 0
1 3 0
-2 -4 0
2 4 0
```

输出:

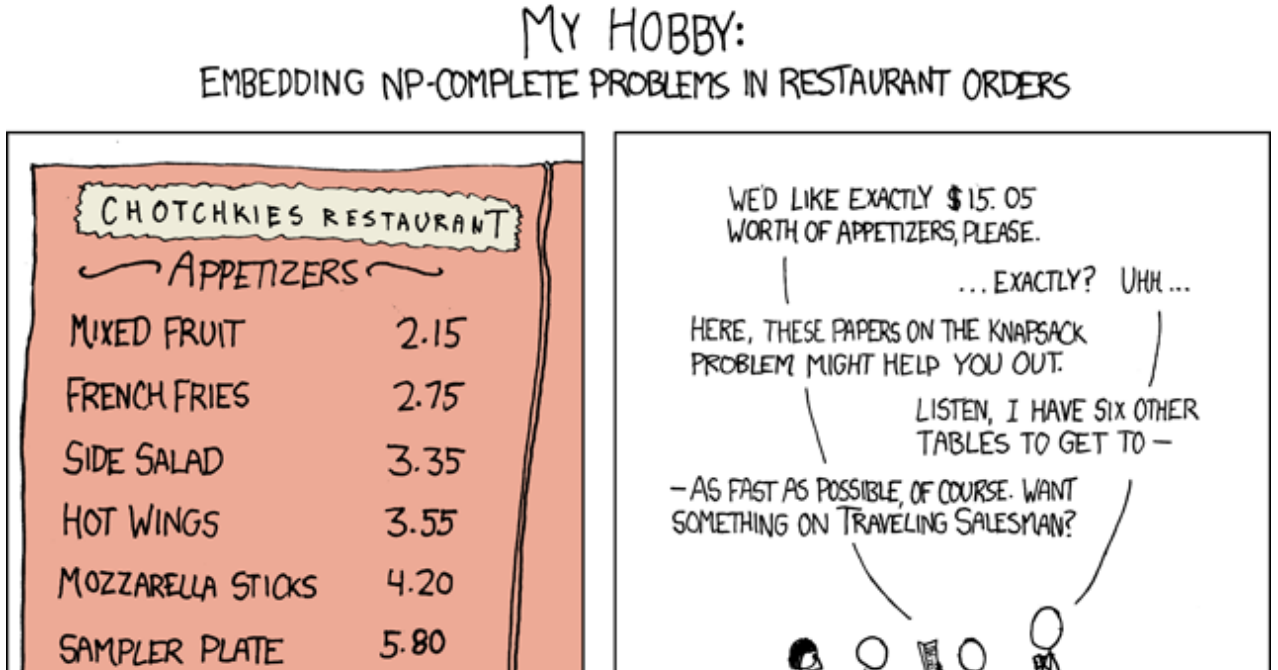
```
SAT
-1 -2 3 4 0
```

SAT与SMT相比, 就像汇编语言与C语言.

更多示例 (第三章 Equations)

解XKCD 287

https://www.xkcd.com/287/



等价于 $2.15a + 2.75b + 3.35c + 3.55d + 4.20e + 5.80f == 15.05$, 其中 $a \ldots f$ 是整数.

这是一个丢番图方程(Diophantine Equation, 又称不定方程, 整系数多项式方程).

下面使用Z3进行求解(原文中使用了他自己开发的MK85):

In [27]:

```
var = [BitVec(chr(i+ord('a')), 16) for i in range(6)]

s = Solver()
for i in var:
    s.add(ULE(i, 10))
s.add(var[0] * 215 + var[1] * 275 + var[2] * 335 + var[3] * 355 + var[4] * 420 + var[5] * 580 == 1505)

while s.check() == sat:
    m = s.model()
    print(m)
    # block current solution and solve again:

    fmt = And(*[i == m.evaluate(i) for i in var])
    s.add(Not(fmt))
```

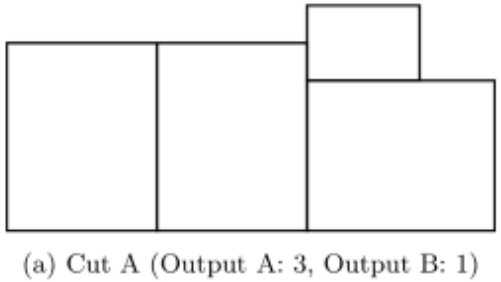
```
[f = 1, b = 0, a = 1, c = 0, d = 2, e = 0]
[f = 0, b = 0, a = 7, c = 0, d = 0, e = 0]
```

XKCD 287的其他解法(各种奇奇怪怪的语言):
https://stackoverflow.com/questions/141779/solving-the-np-complete-problem-in-xkcd

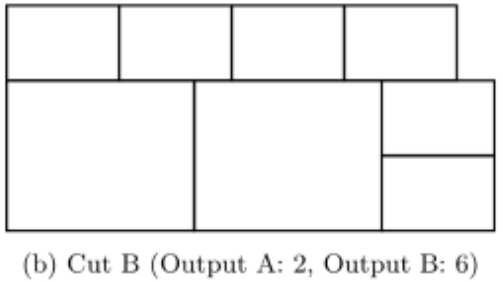
优化问题: Wood workshop

有很多大小为6x13的木块, 需加工为800个4x5的小木块和400个2x3的小木块.

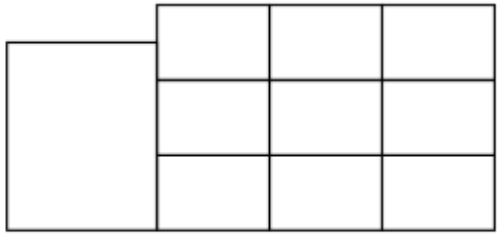
有四种加工方式:



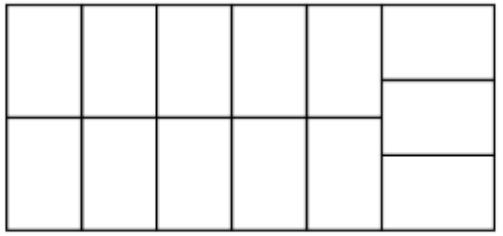
(a) Cut A (Output A: 3, Output B: 1)



(b) Cut B (Output A: 2, Output B: 6)



(c) Cut C (Output A: 1, Output B: 9)



(d) Cut D (Output A: 0, Output B: 13)

怎样切使用最少的木块?

In [28]:

```
s = Optimize()
workpieces_total = Int('workpieces_total')
cut_A, cut_B, cut_C, cut_D = Ints('cut_A cut_B cut_C cut_D')
out_A, out_B = Ints('out_A out_B')
s.add(workpieces_total == cut_A + cut_B + cut_C + cut_D)
s.add(cut_A >= 0)
s.add(cut_B >= 0)
s.add(cut_C >= 0)
s.add(cut_D >= 0)
s.add(out_A == cut_A * 3 + cut_B * 2 + cut_C * 1)
s.add(out_B == cut_A * 1 + cut_B * 6 + cut_C * 9 + cut_D * 13)
s.add(out_A == 800)
s.add(out_B == 400)
s.minimize(workpieces_total)
print(s.check(), s.model())
```

```
sat [cut_B = 25,
 cut_D = 0,
 cut_A = 250,
 out_B = 400,
 out_A = 800,
 workpieces_total = 275,
 cut_C = 0]
```

子集和问题(Subset sum)

给一个整数集合，问是否存在某个非空子集，使得子集内中的数字和为某个特定数值。

例：给定集合{−7, −3, −2, 5, 8}，是否存在子集和为0的集合?答案是YES，因为子集{−3, −2, 5}的数字和是0。

In [29]:

```
set = [-7, -3, -2, 5, 8]
var = [Int('var_%d' % i) for i in range(len(set))]
s = Solver()
rt = []
for i, j in zip(var, set):
    rt.append(i * j)
    s.add(Or(i == 0, i == 1))  # like bools
s.add(sum(rt) == 0)

s.add(sum(var) >= 1)  # subset must not be empty
if s.check() == sat:
    m = s.model()
    print([j for i, j in zip(var, set) if m[i].as_long() == 1])
else:
    print("unsat")
```

```
[-3, -2, 5]
```

2017 AMC 12A Problems/Problem 2

The sum of two nonzero real numbers is 4 times their product. What is the sum of the reciprocals of the two numbers?

$x + y = 4xy$

$\frac{1}{x} + \frac{1}{y} = ?$

In [32]:

```
x, y = Reals('x y')
s = Solver()
s.add(x > 0)
s.add(y > 0)
s.add(x + y == 4 * x * y)
print(s.check())
m = s.model()
print("the model: ", m)
print("the answer:", m.evaluate(1 / x + 1 / y))
```

```
sat
the model:  [x = 1, y = 1/3]
the answer: 4
```

模逆元

给定一个机器数(能被3整除),它乘以什么数的结果就像它除以3一样?

In [33]:
```
m = BitVec('m', 32)
s = Solver()
divisor = 3  # wouldn't work for 10, etc
random_const = 0x1234567  # random constant, must be divisible by divisor
const = (random_const * divisor)
s.add(const * m == const / divisor)
print(s.check())
magic_number = s.model()[m].as_long()
print("0x%x" % magic_number)
```

```
sat
0xaaaaaaab
```

In [34]:
```
print('验证: 0x%x' % (const * magic_number & 0xffffffff))
```

```
验证: 0x1234567
```

扫雷游戏

In [35]:

```
known = [
    "01?10001?",
    "01?100011",
    "011100000",
    "000000000",
    "111110011",
    "????1001?",
    "????3101?",
    "?????211?",
    "?????????"]

WIDTH = len(known[0])
HEIGHT = len(known)

print("WIDTH=", WIDTH, "HEIGHT=", HEIGHT)
```

```
WIDTH= 9 HEIGHT= 9
```

In [36]:

```
def chk_bomb(row, col):
    s = Solver()
    cells = [[Int('r%d_c%d' % (r, c)) for c in range(WIDTH + 2)] for r in range(HEIGHT + 2)]
    for c in range(WIDTH + 2):  # make border
        s.add(cells[0][c] == 0)
        s.add(cells[HEIGHT + 1][c] == 0)
    for r in range(HEIGHT + 2):
        s.add(cells[r][0] == 0)
        s.add(cells[r][WIDTH + 1] == 0)
    for r in range(1, HEIGHT + 1):
        for c in range(1, WIDTH + 1):
            s.add(Or(cells[r][c] == 0, cells[r][c] == 1))  # otherwise -1 is possible, etc
            t = known[r - 1][c - 1]
            if t in "012345678":
                s.add(cells[r][c] == 0)
                # we need empty border so the following expression would be able to work for all possible cases:
                expr = cells[r - 1][c - 1] + cells[r - 1][c] + cells[r - 1][c + 1] + cells[r][c - 1] + \
                       cells[r][c + 1] + cells[r + 1][c - 1] + cells[r + 1][c] + cells[r + 1][c + 1] == int(t)
                s.add(expr)
    s.add(cells[row][col] == 1)  # place bomb

    if s.check() == unsat:
        print("row=%d col=%d, unsat!" % (row, col))
```

In [37]:

```
# enumerate all hidden cells:
for r in range(1, HEIGHT + 1):
    for c in range(1, WIDTH + 1):
        if known[r - 1][c - 1] == "?":
            chk_bomb(r, c)
```

```
row=1 col=3, unsat!
row=6 col=2, unsat!
row=6 col=3, unsat!
row=7 col=4, unsat!
row=7 col=9, unsat!
row=8 col=9, unsat!
```

输出unsat的坐标是没有地雷的, 在游戏中点击后将结果再传给Z3求解

破解线性同余伪随机数生成器(LCG)

LCG非常高效, 使用广泛, 如C语言的`rand()`函数.

LCG定义: $X_{n+1} = (aX_n + c) \mod m$

但它的安全性非常低, 我们使用Z3来破解LCG.

常见参数

Source	modulus *m*	multiplier *a*	increment *c*
Numerical Recipes	2^{32}	1664525	1013904223
Borland C/C++	2^{32}	22695477	1

In [39]:

```
import time

modulus = 2 ** 31
a = 1103515245
c = 12345

def lcg(seed):
    """Linear congruential generator."""
    while True:
        seed = (a * seed + c) % modulus
        yield (seed >> 16) & 0x7fff  # 提高破解效率


generator = lcg(int(time.time()))
rand_var = []
rand_state = []

for g, _ in zip(generator, range(10)):
    rand_var.append(g % 100)
    rand_state.append(g)
print('random var:', rand_var)
print('random state:', rand_state)
```

```
random var: [97, 17, 49, 55, 8, 42, 14, 58, 68, 28]
random state: [19497, 8317, 27149, 20155, 21308, 15142, 29814, 22358, 30868, 15328]
```

In [40]:

```
state = [BitVec('state%d' % i, 32) for i in range(10)]

output_prev = BitVec('output_prev', 32)
output_next = BitVec('output_next', 32)

s = Solver()
for i in range(9):
    s.add(state[i + 1] == ((state[i] * a + c) % modulus))

for i in range(1, 9):
    s.add(URem((state[i] >> 16) & 0x7fff, 100) == rand_var[i])
s.add(output_prev == URem((state[0] >> 16) & 0x7fff, 100))
s.add(output_next == URem((state[9] >> 16) & 0x7fff, 100))

if s.check() == sat:
    m = s.model()
    print('cracked state', [m.evaluate((state[i] >> 16) & 0x7fff) for i in range(10)])
    print('predicate first:', m.evaluate(output_prev), 'last:', m.evaluate(output_next))
else:
    print('unsat')
```

```
cracked state [19497, 8317, 27149, 20155, 21308, 15142, 29814, 22358, 30868, 15328]
predicate first: 97 last: 28
```

这个LCG能产生4个连续的0吗? (运行较慢)

In [41]:

```
s = Solver()
for i in range(4):
    s.add(state[i + 1] == ((state[i] * a + c) % modulus))

for i in range(1, 5):
    s.add(URem((state[i] >> 16) & 0x7fff, 100) == 0)


print(s.check())
if s.check() == sat:
    print(s.model())
```

```
sat
[state3 = 3228838068,
 state1 = 2789793222,
 state2 = 3451711879,
 state0 = 3538020257,
 state4 = 2232704221]
```

因式分解 (运行较慢)

用这种方法可以破解RSA (规模较小, 比如位数少于2^{40})

In [21]:

```
import random
from functools import reduce
from operator import mul


def factor(n):
    print("factoring", n)
    in1, in2, out = Ints('in1 in2 out')
    s = Solver()
    s.add(out == n, in1 * in2 == out, in1 > 1, in2 > 1)
    if s.check() == unsat:
        print(n, "is prime (unsat)")
        return [n]

    if s.check() == unknown:
        print(n, "is probably prime (unknown)")
        return [n]
    m = s.model()
    # get inputs of multiplier:
    in1_n = m[in1].as_long()
    in2_n = m[in2].as_long()
    print("factors of", n, "are", in1_n, "and", in2_n)
    # factor factors recursively:
    rt = sorted(factor(in1_n) + factor(in2_n))
    # self-test:
    assert (reduce(mul, rt, 1) == n)
    return rt
```

In [20]:
```
print(factor(1234567890))
```

```
factoring 1234567890
factors of 1234567890 are 11409 and 108210
factoring 11409
factors of 11409 are 3803 and 3
factoring 3803
3803 is prime (unsat)
factoring 3
3 is prime (unsat)
factoring 108210
factors of 108210 are 15 and 7214
factoring 15
factors of 15 are 5 and 3
factoring 5
5 is prime (unsat)
factoring 3
3 is prime (unsat)
factoring 7214
factors of 7214 are 3607 and 2
factoring 3607
3607 is prime (unsat)
factoring 2
2 is prime (unsat)
[2, 3, 3, 5, 3607, 3803]
```

申子表格公式计算

	⓪	①	②	③
A	1	0	B0+B2	A0*B0*C0
B	123	10	12	11
C	667	A0+B1	C1*A0*122	A3+C2

s.unsat_core()

如果数据出现循环依赖, 则不能求解. 而直接给用户返回unsat非常不友好, 可以通过unsat_core返回更多信息 (退出幻灯片查看下面的例子).

In [15]:

```
import sys, re


# MS Excel or LibreOffice style.
# first top-left cell is A0, not A1
def coord_to_name(R, C):
    return "ABCDEFGHIJKLMNOPQRSTUVWXYZ"[R] + str(C)


ar = [['1', '0', 'B0+B2', 'A0*B0*C0'],
      ['123', '10', '12', '11'],
      ['667', 'A0+B1', 'C1*A0*122', 'A3+C2']]
# ar = [['1', '0', 'B0+B2', 'A0*B0*C0'],
#       ['123', '10', '12', '11'],

#       ['C1+123', 'C0*123', 'C1*A0*122', 'A3+C2']]

WIDTH = len(ar[0])
HEIGHT = len(ar)

s = Solver()
s.set(unsat_core=True)

# cells{} is a dictionary with keys like "A0", "B9", etc:
cells = {}
for R in range(HEIGHT):
    for C in range(WIDTH):
        name = coord_to_name(R, C)
        cells[name] = BitVec(name, 32)
```

In [16]:

```
cur_R = 0
cur_C = 0

for row in ar:
    for c in row:
        # string like "A0+B2" becomes "cells["A0"]+cells["B2"]":
        c = re.sub(r'([A-Z]{1}[0-9]+)', r'cells["\1"]', c)
        st = "cells[\"%s\"]==%s" % (coord_to_name(cur_R, cur_C), c)
        # evaluate string. Z3Py expression is constructed at this step:
        e = eval(st)
        # add constraint:
#         s.add(e)  # 如果没使用unsat_core
        s.assert_and_track(e, coord_to_name(cur_R, cur_C))  # 如果使用了unsat_core
        cur_C = cur_C + 1
    cur_R = cur_R + 1
    cur_C = 0

if s.check() == sat:
    m = s.model()
    for r in range(HEIGHT):
        for c in range(WIDTH):
            print(m.evaluate(cells[coord_to_name(r, c)]), end="\t")
        print()
else:
    print('unsat', s.unsat_core())
```

```
1     0     135   82041
123   10    12    11
667   11    1342        83383
```

物理问题

一辆车的速度为$30m/s$,红灯亮了,刹车加速度为 $-8m/s^2$,求开始刹车到停止的位移.

运动方程:

$d = v_i * t + \frac{1}{2} * a * t^2$
$v_f = v_i + a * t$

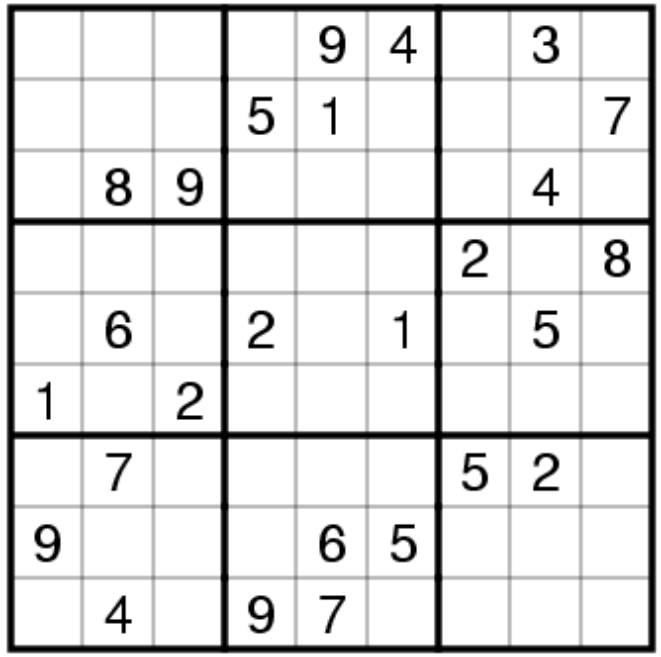
In [22]:

```
d, a, t, v_i, v_f = Reals('d a t v__i v__f')

equations = [ d == v_i * t + (a*t**2)/2, v_f == v_i + a*t ]
print("Kinematic equations:")
print(equations)

# Given v_i, v_f and a, find d
problem = [ v_i == 30, v_f == 0, a  == -8]
print("Problem:")
print(problem)


print("Solution:")
solve(equations + problem)
```

```
Kinematic equations:
[d == v__i*t + (a*t**2)/2, v__f == v__i + a*t]
Problem:
[v__i == 30, v__f == 0, a == -8]
Solution:
[a = -8, v__f = 0, v__i = 30, t = 15/4, d = 225/4]
```

Sudoku

每一行/每一列/小方框 不能出现相同的数字



In [25]:

```
 # 9x9 matrix of integer variables
X = [ [ Int("x_%s_%s" % (i+1, j+1)) for j in range(9) ] for i in range(9) ]
# each cell contains a value in {1, ..., 9}
cells_c  = [ And(1 <= X[i][j], X[i][j] <= 9) for i in range(9) for j in range(9) ]
# each row contains a digit at most once
rows_c   = [ Distinct(X[i]) for i in range(9) ]
# each column contains a digit at most once
cols_c   = [ Distinct([ X[i][j] for i in range(9) ]) for j in range(9) ]
# each 3x3 square contains a digit at most once
sq_c     = [ Distinct([ X[3*i0 + i][3*j0 + j] for i in range(3) for j in range(3) ])
             for i0 in range(3) for j0 in range(3) ]
sudoku_c = cells_c + rows_c + cols_c + sq_c


# sudoku instance, we use '0' for empty cells
instance = ((0,0,0,0,9,4,0,3,0),
            (0,0,0,5,1,0,0,0,7),
            (0,8,9,0,0,0,0,4,0),
            (0,0,0,0,0,0,2,0,8),
            (0,6,0,2,0,1,0,5,0),
            (1,0,2,0,0,0,0,0,0),
            (0,7,0,0,0,0,5,2,0),
            (9,0,0,0,6,5,0,0,0),
            (0,4,0,9,7,0,0,0,0))

instance_c = [ If(instance[i][j] == 0, True, X[i][j] == instance[i][j])
               for i in range(9) for j in range(9) ]
```

In [27]:

```
s = Solver()
s.add(sudoku_c + instance_c)
if s.check() == sat:
    m = s.model()
    r = [ [ m.evaluate(X[i][j]) for j in range(9) ]
          for i in range(9) ]
    print_matrix(r)
else:
    print("failed to solve")
```

```
[[7, 1, 5, 8, 9, 4, 6, 3, 2],
 [2, 3, 4, 5, 1, 6, 8, 9, 7],
 [6, 8, 9, 7, 2, 3, 1, 4, 5],
 [4, 9, 3, 6, 5, 7, 2, 1, 8],
 [8, 6, 7, 2, 3, 1, 9, 5, 4],
 [1, 5, 2, 4, 8, 9, 7, 6, 3],
 [3, 7, 6, 1, 4, 8, 5, 2, 9],
 [9, 2, 8, 3, 6, 5, 4, 7, 1],
 [5, 4, 1, 9, 7, 2, 3, 8, 6]]
```

In [29]:

```
# Let us remove 9 from the first row and see if there is more than one solution
instance = ((0,0,0,0,0,4,0,3,0),
            (0,0,0,5,1,0,0,0,7),
            (0,8,9,0,0,0,0,4,0),
            (0,0,0,0,0,0,2,0,8),
            (0,6,0,2,0,1,0,5,0),
            (1,0,2,0,0,0,0,0,0),
            (0,7,0,0,0,0,5,2,0),
            (9,0,0,0,6,5,0,0,0),
            (0,4,0,9,7,0,0,0,0))

instance_c = [ If(instance[i][j] == 0, True, X[i][j] == instance[i][j])

               for i in range(9) for j in range(9) ]

def n_solutions(n):
    s = Solver()
    s.add(sudoku_c + instance_c)
    i = 0
    while s.check() == sat and i < n:
        m = s.model()
        print([[ m.evaluate(X[i][j]) for j in range(9)] for i in range(9)])
        fml = And([X[i][j] == m.evaluate(X[i][j]) for i in range(9) for j in range(9)])
        s.add(Not(fml))
        i += 1
```

In [30]:
```
n_solutions(10)
```

```
[[7, 1, 6, 8, 2, 4, 9, 3, 5], [2, 3, 4, 5, 1, 9, 6, 8, 7], [5, 8, 9, 7, 3, 6, 1, 4, 2], [4, 9, 3, 6, 5, 7, 2, 1, 8], [8,
6, 7, 2, 9, 1, 4, 5, 3], [1, 5, 2, 4, 8, 3, 7, 9, 6], [6, 7, 1, 3, 4, 8, 5, 2, 9], [9, 2, 8, 1, 6, 5, 3, 7, 4], [3, 4,
5, 9, 7, 2, 8, 6, 1]]
[[7, 1, 5, 8, 9, 4, 6, 3, 2], [2, 3, 4, 5, 1, 6, 8, 9, 7], [6, 8, 9, 7, 2, 3, 1, 4, 5], [4, 9, 3, 6, 5, 7, 2, 1, 8], [8,
6, 7, 2, 3, 1, 9, 5, 4], [1, 5, 2, 4, 8, 9, 7, 6, 3], [3, 7, 6, 1, 4, 8, 5, 2, 9], [9, 2, 8, 3, 6, 5, 4, 7, 1], [5, 4,
1, 9, 7, 2, 3, 8, 6]]
[[5, 1, 7, 6, 8, 4, 9, 3, 2], [2, 3, 4, 5, 1, 9, 6, 8, 7], [6, 8, 9, 7, 2, 3, 1, 4, 5], [7, 9, 3, 4, 5, 6, 2, 1, 8], [4,
6, 8, 2, 9, 1, 7, 5, 3], [1, 5, 2, 8, 3, 7, 4, 9, 6], [3, 7, 6, 1, 4, 8, 5, 2, 9], [9, 2, 1, 3, 6, 5, 8, 7, 4], [8, 4,
5, 9, 7, 2, 3, 6, 1]]
[[5, 1, 6, 7, 8, 4, 9, 3, 2], [2, 3, 4, 5, 1, 9, 6, 8, 7], [7, 8, 9, 6, 2, 3, 1, 4, 5], [3, 9, 7, 4, 5, 6, 2, 1, 8], [4,
6, 8, 2, 9, 1, 7, 5, 3], [1, 5, 2, 8, 3, 7, 4, 9, 6], [6, 7, 3, 1, 4, 8, 5, 2, 9], [9, 2, 1, 3, 6, 5, 8, 7, 4], [8, 4,
5, 9, 7, 2, 3, 6, 1]]
[[7, 1, 6, 8, 2, 4, 9, 3, 5], [2, 3, 4, 5, 1, 9, 6, 8, 7], [5, 8, 9, 6, 3, 7, 1, 4, 2], [3, 9, 7, 4, 5, 6, 2, 1, 8], [4,
6, 8, 2, 9, 1, 7, 5, 3], [1, 5, 2, 7, 8, 3, 4, 9, 6], [6, 7, 3, 1, 4, 8, 5, 2, 9], [9, 2, 1, 3, 6, 5, 8, 7, 4], [8, 4,
5, 9, 7, 2, 3, 6, 1]]
[[6, 1, 7, 8, 2, 4, 9, 3, 5], [2, 3, 4, 5, 1, 9, 6, 8, 7], [5, 8, 9, 6, 3, 7, 1, 4, 2], [7, 9, 3, 4, 5, 6, 2, 1, 8], [4,
6, 8, 2, 9, 1, 7, 5, 3], [1, 5, 2, 7, 8, 3, 4, 9, 6], [3, 7, 6, 1, 4, 8, 5, 2, 9], [9, 2, 1, 3, 6, 5, 8, 7, 4], [8, 4,
5, 9, 7, 2, 3, 6, 1]]
[[6, 1, 7, 8, 2, 4, 9, 3, 5], [2, 3, 4, 5, 1, 9, 6, 8, 7], [5, 8, 9, 6, 3, 7, 1, 4, 2], [4, 9, 3, 7, 5, 6, 2, 1, 8], [7,
6, 8, 2, 9, 1, 4, 5, 3], [1, 5, 2, 4, 8, 3, 7, 9, 6], [3, 7, 6, 1, 4, 8, 5, 2, 9], [9, 2, 1, 3, 6, 5, 8, 7, 4], [8, 4,
5, 9, 7, 2, 3, 6, 1]]
[[6, 1, 7, 8, 2, 4, 9, 3, 5], [2, 3, 4, 5, 1, 9, 6, 8, 7], [5, 8, 9, 7, 3, 6, 1, 4, 2], [4, 9, 3, 6, 5, 7, 2, 1, 8], [7,
6, 8, 2, 9, 1, 4, 5, 3], [1, 5, 2, 4, 8, 3, 7, 9, 6], [3, 7, 6, 1, 4, 8, 5, 2, 9], [9, 2, 1, 3, 6, 5, 8, 7, 4], [8, 4,
5, 9, 7, 2, 3, 6, 1]]
[[5, 1, 7, 6, 8, 4, 9, 3, 2], [2, 3, 4, 5, 1, 9, 6, 8, 7], [6, 8, 9, 3, 2, 7, 1, 4, 5], [4, 9, 3, 7, 5, 6, 2, 1, 8], [7,
6, 8, 2, 9, 1, 3, 5, 4], [1, 5, 2, 4, 3, 8, 7, 9, 6], [8, 7, 6, 1, 4, 3, 5, 2, 9], [9, 2, 1, 8, 6, 5, 4, 7, 3], [3, 4,
5, 9, 7, 2, 8, 6, 1]]
[[7, 1, 5, 6, 8, 4, 9, 3, 2], [2, 3, 4, 5, 1, 9, 6, 8, 7], [6, 8, 9, 3, 2, 7, 1, 4, 5], [3, 9, 7, 4, 5, 6, 2, 1, 8], [4,
6, 8, 2, 9, 1, 7, 5, 3], [1, 5, 2, 7, 3, 8, 4, 9, 6], [8, 7, 6, 1, 4, 3, 5, 2, 9], [9, 2, 1, 8, 6, 5, 3, 7, 4], [5, 4,
3, 9, 7, 2, 8, 6, 1]]
```

The end

感谢聆听!