The Dual Simplex Method

Combinatorial Problem Solving (CPS)

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April 26, 2021

Basic Idea

- **Abuse of terminology:**

  Henceforth sometimes by "optimal" we will mean
  "satisfying the optimality conditions"

  If not explicit, the context will disambiguate

- The algorithm as explained so far is known as primal simplex:

  starting with feasible basis,

  find optimal basis (= satisfying optimality conds.) while keeping feasibility

- There is an alternative algorithm known as dual simplex:

  starting with optimal basis (= satisfying optimality conds.),

  find feasible basis while keeping optimality

Basic Idea

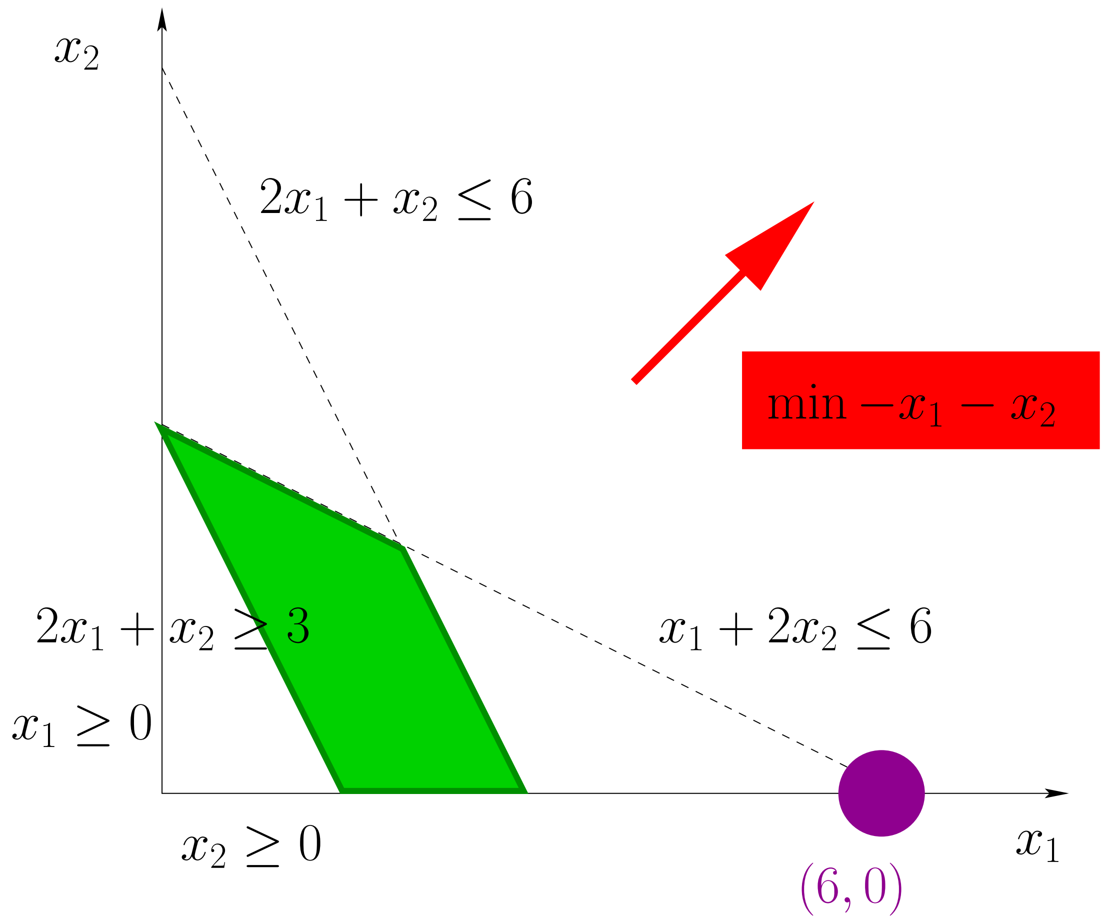
$$\begin{cases} \min\ -x_1 - x_2 \\ 2x_1 + x_2 \geq 3 \\ 2x_1 + x_2 \leq 6 \\ x_1 + 2x_2 \leq 6 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases} \Longrightarrow \begin{cases} \min\ -x_1 - x_2 \\ 2x_1 + x_2 \geq 3 \\ -2x_1 - x_2 \geq -6 \\ -x_1 - 2x_2 \geq -6 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases} \Longrightarrow \begin{cases} \min\ -x_1 - x_2 \\ 2x_1 + x_2 - x_3 = 3 \\ -2x_1 - x_2 - x_4 = -6 \\ -x_1 - 2x_2 - x_5 = -6 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{cases}$$

$$\begin{cases} \min -6 + x_2 + x_5 \\ x_1 = 6 - 2x_2 - x_5 \\ x_3 = 9 - 3x_2 - 2x_5 \\ x_4 = -6 + 3x_2 + 2x_5 \end{cases}$$

Basis (x_1, x_3, x_4) is optimal
($=$ satisfies optimality conditions)

but is not feasible!

Basic Idea



x_2
$2x_1 + x_2 \leq 6$
$\min -x_1 - x_2$
$2x_1 + x_2 \geq 3$
$x_1 + 2x_2 \leq 6$
$x_1 \geq 0$
$x_2 \geq 0$
x_1
$(6, 0)$

Basic Idea

- Let us make a violating basic variable non-negative ...
  - Increase x_4 by making it non-basic: then it will be 0
- ... while preserving optimality ($=$ optimality conditions are satisfied)
  - If x_5 replaces x_4 in the basis,
    then $x_5 = 3 + \frac{1}{2}(x_4 - 3x_2), -x_1 - x_2 = -3 + \frac{1}{2}(x_4 - x_2)$
  - If x_2 replaces x_4 in the basis,
    then $x_2 = 2 + \frac{1}{3}(x_4 - 2x_5), -x_1 - x_2 = -4 + \frac{1}{3}(x_4 + x_5)$

Basic Idea

- Let us make a violating basic variable non-negative ...
  - Increase x_4 by making it non-basic: then it will be 0
- ... while preserving optimality ($=$ optimality conditions are satisfied)
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  - If x_2 replaces x_4 in the basis,
    then $x_2 = 2 + \frac{1}{3}(x_4 - 2x_5), -x_1 - x_2 = -4 + \frac{1}{3}(x_4 + x_5)$
  - To preserve optimality, we must swap x_2 and x_4

Basic Idea

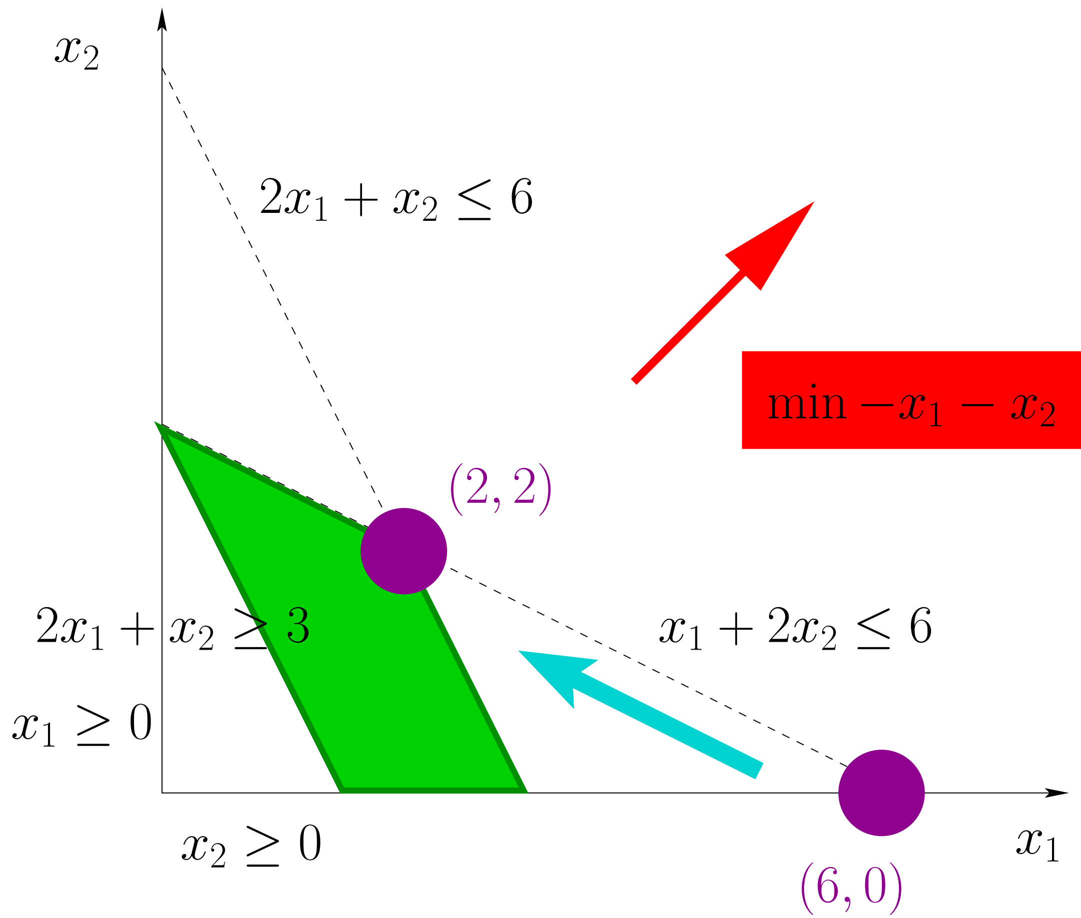
$$\begin{cases} \min -6 + x_2 + x_5 \\ x_1 = 6 - 2x_2 - x_5 \\ x_3 = 9 - 3x_2 - 2x_5 \\ x_4 = -6 + 3x_2 + 2x_5 \end{cases} \Longrightarrow \begin{cases} \min \ -4 + \frac{1}{3}x_4 + \frac{1}{3}x_5 \\ x_1 = 2 - \frac{2}{3}x_4 + \frac{1}{3}x_5 \\ x_2 = 2 + \frac{1}{3}x_4 - \frac{2}{3}x_5 \\ x_3 = 3 - x_4 \end{cases}$$

Basic Idea

$$\begin{cases} \min -6 + x_2 + x_5 \\ x_1 = 6 - 2x_2 - x_5 \\ x_3 = 9 - 3x_2 - 2x_5 \\ x_4 = -6 + 3x_2 + 2x_5 \end{cases} \Longrightarrow \begin{cases} \min \ -4 + \frac{1}{3}x_4 + \frac{1}{3}x_5 \\ x_1 = 2 - \frac{2}{3}x_4 + \frac{1}{3}x_5 \\ x_2 = 2 + \frac{1}{3}x_4 - \frac{2}{3}x_5 \\ x_3 = 3 - x_4 \end{cases}$$

- Current basis is feasible and optimal!

Basic Idea



x_2
$2x_1 + x_2 \leq 6$
$\min -x_1 - x_2$
$(2, 2)$
$2x_1 + x_2 \geq 3$
$x_1 + 2x_2 \leq 6$
$x_1 \geq 0$
$x_2 \geq 0$
$(6, 0)$
x_1

Outline of the Dual Simplex

1. `Initialization:` Pick an optimal basis
   (= satisfies optimality conditions).

2. `Dual Pricing:` If all values in basic solution are ≥ 0,
   then return OPTIMAL.
   Else pick a basic variable with value < 0.

3. `Dual Ratio Test:` Find non-basic variable for swapping that preserves optimality, i.e., non-negativity constraints on reduced costs.

   If it does not exist,
   then return INFEASIBLE.
   Else swap chosen non-basic variable with violating basic variable.

4. `Update:` Update the tableau and go to 2.

Duality

- To understand better how the dual simplex works: theory of duality
- We can get lower bounds on LP optimum value
  by adding constraints in a convenient way

$$\begin{cases} \min\ -x_1 - x_2 \\ 2x_1 + x_2 \geq 3 \\ -2x_1 - x_2 \geq -6 \\ -x_1 - 2x_2 \geq -6 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

$$\begin{array}{rcr} -x_1 - 2x_2 & \geq & -6 \\ x_2 & \geq & 0 \\ \hline -x_1 - x_2 & \geq & -6 \end{array}$$

Duality

- In general we can get lower bounds on LP optimum value by linearly combining constraints with convenient multipliers

$$
\begin{cases}
\min\ -x_1 - x_2 \\
2x_1 + x_2 \geq 3 \\
-2x_1 - x_2 \geq -6 \\
-x_1 - 2x_2 \geq -6 \\
x_1 \geq 0 \\
x_2 \geq 0
\end{cases}
$$

$$
\begin{array}{rrcll}
1 \cdot (& 2x_1 + x_2 & \geq & 3 &) \\
2 \cdot (& -2x_1 - x_2 & \geq & -6 &) \\
1 \cdot (& x_1 & \geq & 0 &) \\
\hline
 & 2x_1 + x_2 & \geq & 3 & \\
 & -4x_1 - 2x_2 & \geq & -12 & \\
 & x_1 & \geq & 0 & \\
\hline
 & -x_1 - x_2 & \geq & -9 &
\end{array}
$$

- There may be different choices, each giving a different lower bound

Duality

- Let $\mu_1, \ldots, \mu_5 \geq 0$:

$$\begin{cases} \min \ -x_1 - x_2 \\ 2x_1 + x_2 \geq 3 \\ -2x_1 - x_2 \geq -6 \\ -x_1 - 2x_2 \geq -6 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

$$\begin{array}{rrcll}
\mu_1 \cdot (& 2x_1 + x_2 & \geq & 3 &) \\
\mu_2 \cdot (& -2x_1 - x_2 & \geq & -6 &) \\
\mu_3 \cdot (& -x_1 - 2x_2 & \geq & -6 &) \\
\mu_4 \cdot (& x_1 & \geq & 0 &) \\
\mu_5 \cdot (& x_2 & \geq & 0 &) \\
\hline
 & 2\mu_1 x_1 + \mu_1 x_2 & \geq & 3\mu_1 & \\
 & -2\mu_2 x_1 - \mu_2 x_2 & \geq & -6\mu_2 & \\
 & -\mu_3 x_1 - 2\mu_3 x_2 & \geq & -6\mu_3 & \\
 & \mu_4 x_1 & \geq & 0 & \\
 & \mu_5 x_2 & \geq & 0 &
\end{array}$$

$$(2\mu_1 - 2\mu_2 - \mu_3 + \mu_4)\, x_1 \ + \ (\mu_1 - \mu_2 - 2\mu_3 + \mu_5)\, x_2 \ \geq \ 3\mu_1 - 6\mu_2 - 6\mu_3$$

- If $2\mu_1 - 2\mu_2 - \mu_3 + \mu_4 = -1$, $\mu_1 - \mu_2 - 2\mu_3 + \mu_5 = -1$, $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$, $\mu_4 \geq 0$, $\mu_5 \geq 0$,
  then $3\mu_1 - 6\mu_2 - 6\mu_3$ is a lower bound

Duality

- We can skip the multipliers of the non-negativity constraints

- We have:

$$\begin{cases} \min \ -x_1 - x_2 \\ 2x_1 + x_2 \geq 3 \\ -2x_1 - x_2 \geq -6 \\ -x_1 - 2x_2 \geq -6 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

$$\begin{array}{lrcll} \mu_1 \cdot (& 2x_1 + x_2 & \geq & 3 &) \\ \mu_2 \cdot (& -2x_1 - x_2 & \geq & -6 &) \\ \mu_3 \cdot (& -x_1 - 2x_2 & \geq & -6 &) \\ \hline & 2\mu_1 x_1 + \mu_1 x_2 & \geq & 3\mu_1 & \\ & -2\mu_2 x_1 - \mu_2 x_2 & \geq & -6\mu_2 & \\ & -\mu_3 x_1 - 2\mu_3 x_2 & \geq & -6\mu_3 & \end{array}$$

$$(2\mu_1 - 2\mu_2 - \mu_3)\, x_1 \ + \ (\mu_1 - \mu_2 - 2\mu_3)\, x_2 \ \geq \ 3\mu_1 - 6\mu_2 - 6\mu_3$$

- Imagine $2\mu_1 - 2\mu_2 - \mu_3 \leq -1$.
  In the coefficient of x_1 we can "complete" $2\mu_1 - 2\mu_2 - \mu_3$ to reach -1 by adding a suitable multiple of $x_1 \geq 0$ (the multiplier will be the slack)

- If $2\mu_1 - 2\mu_2 - \mu_3 \leq -1$, $\mu_1 - \mu_2 - 2\mu_3 \leq -1$, $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$, then $3\mu_1 - 6\mu_2 - 6\mu_3$ is a lower bound

Duality

- Best possible lower bound with this "trick" can be found by solving

$$\begin{cases} \max & 3\mu_1 - 6\mu_2 - 6\mu_3 \\ & 2\mu_1 - 2\mu_2 - \mu_3 \leq -1 \\ & \mu_1 - \mu_2 - 2\mu_3 \leq -1 \\ & \mu_1, \mu_2, \mu_3 \geq 0 \\ & \end{cases}$$

- How far will it be from the optimum?

Duality

- Best possible lower bound with this "trick" can be found by solving

$$\begin{cases} \max & 3\mu_1 - 6\mu_2 - 6\mu_3 \\ & 2\mu_1 - 2\mu_2 - \mu_3 \le -1 \\ & \mu_1 - \mu_2 - 2\mu_3 \le -1 \\ & \mu_1, \mu_2, \mu_3 \ge 0 \end{cases}$$

- How far will it be from the optimum?

- A best solution is given by $(\mu_1, \mu_2, \mu_3) = (0, \frac{1}{3}, \frac{1}{3})$

$$\begin{array}{rlrcrl} 0 \cdot (& & 2x_1 + x_2 & \ge & 3 &) \\ \frac{1}{3} \cdot (& & -2x_1 - x_2 & \ge & -6 &) \\ \frac{1}{3} \cdot (& & -x_1 - 2x_2 & \ge & -6 &) \\ \hline & & -x_1 - x_2 & \ge & -4 & \end{array}$$

Matches the optimum!

Dual Problem

- If we multiply $Ax \geq b$ by multipliers $y^T \geq 0$ we get $y^T Ax \geq y^T b$
- If $y^T A \leq c^T$ then we get a lower bound $y^T b$ for the cost function $c^T x$
- Given an LP (called primal problem)

$$\begin{array}{c} \min \ c^T x \\ Ax \geq b \\ x \geq 0 \end{array}$$

its dual problem is the LP

$$\begin{array}{l} \max \ y^T b \\ y^T A \leq c^T \\ y^T \geq 0 \end{array} \qquad \text{or equivalently} \qquad \begin{array}{l} \max \ b^T y \\ A^T y \leq c \\ y \geq 0 \end{array}$$

- Primal variables associated with columns of A
- Dual variables (multipliers) associated with rows of A
- Objective and right-hand side vectors swap their roles

Dual Problem

- **Prop.** The dual of the dual is the primal.

  Proof:

$$\begin{array}{lcl} \max\ b^T y & & -\min\ (-b)^T y \\ A^T y \le c & \Longrightarrow & -A^T y \ge -c \\ y \ge 0 & & y \ge 0 \end{array}$$

$$\begin{array}{lcl} -\max\ -c^T x & & \min\ c^T x \\ (-A^T)^T x \le -b & \Longrightarrow & Ax \ge b \\ x \ge 0 & & x \ge 0 \end{array}$$

- We say the primal and the dual form a primal-dual pair

Dual Problem

- **Prop.** $\begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array}$ and $\begin{array}{l} \max\ b^T y \\ A^T y \leq c \end{array}$ form a primal-dual pair

Proof:

$$\begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \min\ c^T x \\ Ax \geq b \\ -Ax \geq -b \\ x \geq 0 \end{array}$$

$$\begin{array}{l} \max\ b^T y_1 - b^T y_2 \\ A^T y_1 - A^T y_2 \leq c \\ y_1, y_2 \geq 0 \end{array} \quad \overset{y := y_1 - y_2}{\Longrightarrow} \quad \begin{array}{l} \max\ b^T y \\ A^T y \leq c \end{array}$$

Duality Theorems

- **Th. (Weak Duality)** Let (P, D) be a primal-dual pair

$$(P) \quad \begin{array}{l} \min \ c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{l} \max \ b^T y \\ A^T y \leq c \end{array}$$

If x is feasible solution to P and y is feasible solution to D then $b^T y \leq c^T x$

Proof:

$c - A^T y \geq 0$, i.e., $c^T - y^T A \geq 0$, and $x \geq 0$ imply $c^T x - y^T Ax \geq 0$.

So $c^T x \geq y^T Ax$, and

$$c^T x \geq y^T Ax = y^T b = b^T y$$

Duality Theorems

- Feasible solutions to D give lower bounds on P
- Feasible solutions to P give upper bounds on D
- Will the two optimum values be always equal?

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- **Th. (Strong Duality)** Let (P, D) be a primal-dual pair

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If any of P or D has a feasible solution and a finite optimum then the same holds for the other problem and the two optimum values are equal.

Duality Theorems

- Proof (Th. of Strong Duality):

  By duality it is sufficient to prove only one direction.
  Wlog. let us assume P is feasible with finite optimum.

Duality Theorems

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  Wlog. let us assume P is feasible with finite optimum.

  After executing the Simplex algorithm to P we find
  B optimal feasible basis. Then:

  - $c_{\mathcal{B}}^T B^{-1} a_j \leq c_j$ for all $j \in \mathcal{R}$ (optimality conds hold)
  - $c_{\mathcal{B}}^T B^{-1} a_j = c_j$ for all $j \in \mathcal{B}$

  Hence $c_{\mathcal{B}}^T B^{-1} A \leq c^T$.
  So $\pi^T := c_{\mathcal{B}}^T B^{-1}$ is a dual feasible solution: $\pi^T A \leq c^T$, i.e., $A^T \pi \leq c$

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  Moreover, $c_{\mathcal{B}}^T \beta = c_{\mathcal{B}}^T B^{-1} b = \pi^T b = b^T \pi$
  By the theorem of weak duality, π is optimum for D

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  Moreover, $c_{\mathcal{B}}^T \beta = c_{\mathcal{B}}^T B^{-1} b = \pi^T b = b^T \pi$

  By the theorem of weak duality, π is optimum for D

- If B is an optimal feasible basis for P,
  then simplex multipliers $\pi^T := c_{\mathcal{B}}^T B^{-1}$ are optimal feasible solution for D
- We can solve the dual by applying the simplex algorithm on the primal
- We can solve the primal by applying the simplex algorithm on the dual

Duality Theorems

- **Prop.** Let (P, D) be a primal-dual pair

$$(P)\quad \begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D)\quad \begin{array}{l} \max\ b^T y \\ A^T y \leq c \end{array}$$

(1) If P has a feasible solution but is unbounded, then D is infeasible
(2) If D has a feasible solution but is unbounded, then P is infeasible

Proof:

Let us prove (1) by contradiction.

If y were a feasible solution to D,
by the weak duality theorem, objective of P would be lower bounded!

(2) is proved by duality.

Duality Theorems

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- And the converse?
Does infeasibility of one imply unboundedess of the other?

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(1) If P has a feasible solution but is unbounded, then D is infeasible
(2) If D has a feasible solution but is unbounded, then P is infeasible

- And the converse?
  Does infeasibility of one imply unboundedess of the other?

$$\begin{array}{lrl} \min & 3x_1 + 5x_2 & \\ & x_1 + 2x_2 & = 3 \\ & 2x_1 + 4x_2 & = 1 \\ & x_1, x_2 \text{ free} & \end{array} \qquad \begin{array}{lrl} \max & 3y_1 + y_2 & \\ & y_1 + 2y_2 & = 3 \\ & 2y_1 + 4y_2 & = 5 \\ & y_1, y_2 \text{ free} & \end{array}$$

Duality Theorems

$$\boxed{\begin{array}{lcl} \text{Primal unbounded} & \Longrightarrow & \text{Dual infeasible} \\ \text{Dual unbounded} & \Longrightarrow & \text{Primal infeasible} \\ \text{Primal infeasible} & \Longrightarrow & \text{Dual} \begin{cases} \text{infeasible} \\ \text{unbounded} \end{cases} \\ \text{Dual infeasible} & \Longrightarrow & \text{Primal} \begin{cases} \text{infeasible} \\ \text{unbounded} \end{cases} \end{array}}$$

Karush Kuhn Tucker Opt. Conds.

- Consider a primal-dual pair of the form

$$\begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array} \quad \text{and} \quad \begin{array}{l} \max\ b^T y \\ A^T y \leq c \end{array} \iff \begin{array}{l} \max\ b^T y \\ A^T y + w = c \\ w \geq 0 \end{array}$$

- Karush-Kuhn-Tucker (KKT) optimality conditions are
  - $Ax = b$
  - $A^T y + w = c$
  - $x, w \geq 0$
  - $x^T w = 0$ (complementary slackness)

- They are necessary and sufficient conditions for optimality of the pair of primal-dual solutions (x, y, w)

- Used, e.g., as a test of quality in LP solvers

Karush Kuhn Tucker Opt. Conds.

$$(P)\ \begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad (D)\ \begin{array}{l} \max\ b^T y \\ A^T y + w = c \\ w \geq 0 \end{array}$$

(KKT)

- $Ax = b$
- $A^T y + w = c$
- $x, w \geq 0$
- $x^T w = 0$

- **Th.** (x, y, w) is solution to KKT iff
  x optimal solution to P and (y, w) optimal solution to D

  Proof:

  $\Rightarrow$ By $0 = x^T w = x^T(c - A^T y) = c^T x - b^T y$, and Weak Duality

  $\Leftarrow$ x is feasible solution to P, (y, w) is feasible solution to D.
  By Strong Duality $x^T w = x^T(c - A^T y) = c^T x - b^T y = 0$
  as both solutions are optimal

Relating Bases

- Consider a primal-dual pair of the form

$$
(P)\quad \begin{array}{l} \min\ z = c^T x \\ Ax = b \\ x \geq 0 \end{array}
\qquad\qquad
(D)\quad \begin{array}{l} \max\ Z = b^T y \\ A^T y + w = c \\ w \geq 0 \end{array}
$$

- Let us denote by a_1, ..., a_n the columns of A, i.e., $A = (a_1, \dots, a_n)$
- Let B be a basis of P. Let us see how we can get a basis of D.

  Assume that the basic variables are the first m: $B = (a_1, \dots, a_m)$.

  Then $R = (a_{m+1}, \dots, a_n)$.

  If slacks w are split into $w_{\mathcal{B}}^T = (w_1, \dots, w_m)$, $w_{\mathcal{R}}^T = (w_{m+1}, \dots, w_n)$, then

$$
A^T y + w = \begin{pmatrix} a_1^T y \\ \dots \\ a_m^T y \\ \hline a_{m+1}^T y \\ \dots \\ a_n^T y \end{pmatrix} + \begin{pmatrix} w_1 \\ \dots \\ w_m \\ \hline w_{m+1} \\ \dots \\ w_n \end{pmatrix} = \begin{pmatrix} B^T y + w_{\mathcal{B}} \\ \hline R^T y + w_{\mathcal{R}} \end{pmatrix}
$$

Relating Bases

- Hence we have

$$A^T y + w = \left(\begin{array}{c} B^T y + w_{\mathcal{B}} \\ \hline R^T y + w_{\mathcal{R}} \end{array} \right)$$

- Then the matrix of the system in the dual problem D is

$$\left(\begin{array}{c|c|c} B^T & I & 0 \\ \hline R^T & 0 & I \end{array} \right) \left(\begin{array}{c} y \\ w_{\mathcal{B}} \\ w_{\mathcal{R}} \end{array} \right)$$

- Now let us consider the submatrix of vars y and vars $w_{\mathcal{R}}$:

$$\hat{B} = \left(\begin{array}{c|c} B^T & 0 \\ \hline R^T & I \end{array} \right)$$

- Note $\hat{B}$ is a square $n \times n$ matrix

Relating Bases

- Dual variables $\hat{\mathcal{B}} = (y, w_{\mathcal{R}})$ determine a basis of D:

$$\hat{B} = \left(\begin{array}{c|c} B^T & 0 \\ \hline R^T & I \end{array}\right)$$

$$\hat{B}^{-1} = \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right)$$

Relating Bases

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$$\hat{B}^{-1} = \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right)$$

- In the next slides we answer the following questions:

  1. If basis $\hat{B}$ of the dual D is feasible,
     what can we say about basis B of the primal P?

  2. If basis $\hat{B}$ of the dual D is optimal (satisfies the optimality conds.),
     what can we say about basis B of the primal P?

  3. If we apply the simplex algorithm to the dual D using basis $\hat{B}$,
     how does that translate into the primal P and its basis B?

Relating Bases

- Dual variables $\hat{\mathcal{B}} = (y, w_{\mathcal{R}})$ determine a basis of D:

$$\hat{B} = \left(\begin{array}{c|c} B^T & 0 \\ \hline R^T & I \end{array}\right)$$

$$\hat{B}^{-1} = \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right)$$

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     how does that translate into the primal P and its basis B?

- Recall that each variable w_j in D is associated to a variable x_j in P.
- Note that w_j is $\hat{\mathcal{B}}$-basic iff x_j is not $\mathcal{B}$-basic

Dual Feasibility = Primal Optimality

- If $\hat{B}$ is feasible for dual D, what about B in primal P?
- Let us compute the basic solution of basis $\hat{B}$ in the dual problem D

$$\left(\frac{y}{w_{\mathcal{R}}}\right) = \hat{B}^{-1}c = \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right)\left(\frac{c_{\mathcal{B}}}{c_{\mathcal{R}}}\right) = \left(\frac{B^{-T}c_{\mathcal{B}}}{-R^T B^{-T} c_{\mathcal{B}} + c_{\mathcal{R}}}\right)$$

- Recall that there is no restriction on the sign of $y_1, ..., y_m$
- Variables w_j have to be non-negative. But

$$-R^T B^{-T} c_{\mathcal{B}} + c_{\mathcal{R}} \geq 0 \quad \text{iff} \quad c_{\mathcal{R}}^T - c_{\mathcal{B}}^T B^{-1} R \geq 0$$

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- $\hat{B}$ is dual feasible iff $d_j \geq 0$ for all $j \in \mathcal{R}$

- Dual feasibility is primal optimality!

Dual Optimality $=$ Primal Feasibility

- If $\hat{B}$ satisfies the optimality conds. for dual D, what about B in primal P?
- Let us formulate the optimality conds. of basis $\hat{B}$ in the dual problem D
- Non $\hat{\mathcal{B}}$-basic vars: $w_{\mathcal{B}}$ with costs (0)
- $\hat{\mathcal{B}}$-basic vars: $(y \mid w_{\mathcal{R}})$ with costs $(b^T \mid 0)$
- Matrix of non $\hat{\mathcal{B}}$-basic vars: $\left(\begin{array}{c} I \\ \hline 0 \end{array}\right)$
- Optimality condition: $0 \geq$ reduced costs (maximization!)

$$0 \geq (\ 0\ ) - (\ b^T \mid 0\ ) \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right) \left(\begin{array}{c} I \\ \hline 0 \end{array}\right) =$$

$$= (\ 0\ ) - (\ b^T B^{-T} \mid 0\ ) \left(\begin{array}{c} I \\ \hline 0 \end{array}\right) = -b^T B^{-T} = -\beta^T \text{ where } \beta = B^{-1} b$$

Dual Optimality = Primal Feasibility

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- In the dual, for all $1 \leq p \leq m$ var w_{k_p} satisfies optimality cond. iff $\beta_p \geq 0$
- Dual optimality is primal feasibility!

Improving a Non-Optimal Solution

- Next we apply the simplex algorithm to basis $\hat{B}$ in the dual problem D and translate it to the primal problem P

- Let p (where $1 \leq p \leq m$) be such that $\beta_p < 0$.

  I.e., the reduced cost of non-basic dual variable w_{k_p} is positive.

  So by giving w_{k_p} a larger value we can improve the dual objective value.

  If w_{k_p} takes value $t \geq 0$:

$$\begin{pmatrix} y(t) \\ w_{\mathcal{R}}(t) \end{pmatrix} = \hat{B}^{-1}c - \hat{B}^{-1}te_p =$$

$$= \begin{pmatrix} B^{-T}c_{\mathcal{B}} \\ d_{\mathcal{R}} \end{pmatrix} - \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right) \left(\frac{te_p}{0}\right) = \left(\frac{B^{-T}c_{\mathcal{B}} - tB^{-T}e_p}{d_{\mathcal{R}} + tR^T B^{-T} e_p}\right)$$

- Dual objective value improvement is

$$\Delta Z = b^T y(t) - b^T y(0) = -tb^T B^{-T} e_p = -t\beta^T e_p = -t\beta_p$$

Improving a Non-Optimal Solution

- Of all basic dual variables, only $w_{\mathcal{R}}$ variables need to be ≥ 0
- For $j \in \mathcal{R}$

$$w_j(t) = d_j + t a_j^T B^{-T} e_p = d_j + t e_p^T B^{-1} a_j = d_j + t e_p^T \alpha_j = d_j + t\alpha_j^p$$

where α_j^p is the p-th component of $\alpha_j = B^{-1} a_j$. Hence:

$$w_j(t) \geq 0 \iff d_j + t\alpha_j^p \geq 0 \iff d_j \geq t(-\alpha_j^p)$$

  - If $\alpha_j^p \geq 0$ the constraint is satisfied for all $t \geq 0$
  - If $\alpha_j^p < 0$ we need $\frac{d_j}{-\alpha_j^p} \geq t$

- Best improvement achieved with

$$\Theta_D := \min\{\frac{d_j}{-\alpha_j^p} \mid \alpha_j^p < 0\}$$

- Variable w_q is blocking when $\Theta_D = \frac{d_q}{-\alpha_q^p}$

Improving a Non-Optimal Solution

1. If $\Theta_D = +\infty$ (there is no $j \in \mathcal{R}$ such that $\alpha_j^p < 0$):

   Value of dual objective can be increased infinitely.

   Dual LP is unbounded.

   Primal LP is infeasible.

2. If $\Theta_D < +\infty$ and w_q is blocking:

   When setting $w_{k_p} = \Theta_D$,
   non-negativity constraints of basic vars of dual are respected

   We can make a basis change:

   - In dual: w_{k_p} enters $\hat{\mathcal{B}}$ and w_q leaves
   - In primal: x_{k_p} leaves $\mathcal{B}$ and x_q enters

Update

- We do not actually need to form the dual LP:
  it is enough to have a representation of the primal LP

- New basic indices: $\bar{\mathcal{B}} = (k_1, \ldots, k_{p-1}, q, k_{p+1} \ldots, k_m)$
- New dual objective value: $\bar{Z} = Z - \Theta_D \beta_p$
- New dual basic sol: $\bar{y} = y - \Theta_D \rho_p$ where $\rho_p = B^{-T} e_p$
  $\bar{d}_j = d_j + \Theta_D \alpha_j^p$ if $j \in \mathcal{R}$, $\bar{d}_{k_p} = \Theta_D$
- New primal basic sol: $\bar{\beta}_p = \Theta_P$, $\bar{\beta}_i = \beta_i - \Theta_P \alpha_q^i$ if $i \neq p$
  where $\Theta_P = \frac{\beta_p}{\alpha_q^p}$
- New basis inverse: $\bar{B}^{-1} = E B^{-1}$
  where $E = (e_1, \ldots, e_{p-1}, \eta, e_{p+1}, \ldots, e_m)$ and
  $\eta^T = \left(\left(\frac{-\alpha_q^1}{\alpha_q^p} \right), \ldots, \left(\frac{-\alpha_q^{p-1}}{\alpha_q^p} \right), \frac{1}{\alpha_q^p} \left(\frac{-\alpha_q^{p+1}}{\alpha_q^p} \right), \ldots, \left(\frac{-\alpha_q^m}{\alpha_q^p} \right) \right)^T$

Algorithmic Description

1. `Initialization:` Find an initial dual feasible basis $\mathcal{B}$
   Compute B^{-1}, $\beta = B^{-1}b$,
   $y^T = c_{\mathcal{B}}^T B^{-1}$, $d_{\mathcal{R}}^T = c_{\mathcal{R}}^T - y^T R$, $Z = b^T y$

2. `Dual Pricing:`
   If for all $i \in \mathcal{B}$, $\beta_i \geq 0$ then return OPTIMAL
   Else let p be such that $\beta_p < 0$.
   Compute $\rho_p^T = e_p^T B^{-1}$ and $\alpha_j^p = \rho_p^T a_j$ for $j \in \mathcal{R}$

3. `Dual Ratio Test:` Compute $\mathcal{J} = \{j \mid j \in \mathcal{R}, \alpha_j^p < 0\}$.
   If $\mathcal{J} = \emptyset$ then return INFEASIBLE
   Else compute $\Theta_D = \min_{j \in \mathcal{J}} (\frac{d_j}{-\alpha_j^p})$ and q st. $\Theta_D = \frac{d_q}{-\alpha_q^p}$

Algorithmic Description

4. `Update:`

$\bar{\mathcal{B}} = \mathcal{B} - \{k_p\} \cup \{q\}$
$\bar{Z} = Z - \Theta_D \beta_p$

<u>`Dual solution`</u>

$\bar{y} = y - \Theta_D \rho_p$
$\bar{d}_j = d_j + \Theta_D \alpha_j^p$ if $j \in \mathcal{R}$, $\bar{d}_{k_p} = \Theta_D$

<u>`Primal solution`</u>

Compute $\alpha_q = B^{-1} a_q$ and $\Theta_P = \frac{\beta_p}{\alpha_q^p}$
$\bar{\beta}_p = \Theta_P$, $\bar{\beta}_i = \beta_i - \Theta_P \alpha_q^i$ if $i \neq p$

$\bar{B}^{-1} = E B^{-1}$

Go to 2.

Primal vs. Dual Simplex

PRIMAL	DUAL
■ Can handle bounds efficiently	■ Can handle bounds efficiently (not explained here)
■ Many years of research and implementation	■ Developments in the 90's made it an alternative
■ There are classes of LP's for which it is the best	■ Nowadays on average it gives better performance
■ **Not suitable for solving LP's with integer variables**	■ **Suitable for solving LP's with integer variables**