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──────────────────── MODULE RelationUtils ────────────────────
Relation related operators.

LOCAL INSTANCE Naturals
LOCAL INSTANCE FiniteSets
LOCAL INSTANCE Sequences
LOCAL INSTANCE SequencesExt
LOCAL INSTANCE Functions
──────────────────────────────────────────────────────────────
Basic definitions.

Dom(R)     ≜ {a : ⟨a, b⟩ ∈ R}   Domain of R
Ran(R)     ≜ {b : ⟨a, b⟩ ∈ R}   Range of R
Support(R) ≜ Dom(R) ∪ Ran(R)    Support of R
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Basic operations.

Image(R, a) ≜ {b ∈ Ran(R) : ⟨a, b⟩ ∈ R}
LeftRestriction(R, a) ≜ {⟨a, b⟩ : b ∈ Image(R, a)}

InverseRelation(R) ≜ {⟨b, a⟩ : ⟨a, b⟩ ∈ R}
InverseImage(R, b) ≜ {a ∈ Dom(R) : ⟨a, b⟩ ∈ R}

R | S ≜ R ∩ (S × S)   Restriction of R on S

R ** T ≜   Composition of R and T
   LET SR ≜ Support(R)
       ST ≜ Support(T)
   IN  {⟨r, t⟩ ∈ SR × ST : ∃ s ∈ SR ∩ ST : (⟨r, s⟩ ∈ R) ∧ (⟨s, t⟩ ∈ T)}

GT(R, a) ≜ {b ∈ Ran(R) : ⟨a, b⟩ ∈ R}    ≜ Image(R, a)
LT(R, b) ≜ {a ∈ Dom(R) : ⟨a, b⟩ ∈ R}    ≜ InverseImage(R, b)
The following definition is from https://github.com/jameshfisher/tlaplus/blob/master/examples/TransitiveClosure/TransitiveClos
It also contains several other methods for computing TC.
TC(R) ≜   Transitive closure of R
     LET S ≜ Support(R)
         RECURSIVE TCR(_)
          TCR(T) ≜ IF T = {}
                     THEN R
                     ELSE LET r ≜ CHOOSE s ∈ T : TRUE
                              RR ≜ TCR(T \ {r})
                          IN  RR ∪ {⟨s, t⟩ ∈ S × S :
                                      ⟨s, r⟩ ∈ RR ∧ ⟨r, t⟩ ∈ RR}
     IN  TCR(S)
Example: SeqToRel(⟨1, 2, 3⟩) = {⟨1, 2⟩, ⟨1, 3⟩, ⟨2, 3⟩}
RECURSIVE Seq2Rel(_)
Seq2Rel(s) ≜   Transform a sequence s into a strict total order relation
   IF s = ⟨⟩ THEN {}
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$\quad\quad \text{ELSE} \ \text{LET} \ h \triangleq Head(s)$
$\quad\quad\quad\quad\quad\quad\quad\ t \triangleq Tail(s)$
$\quad\quad\quad\quad\ \text{IN} \quad \{\langle h, r\rangle : r \in Range(t)\} \cup Seq2Rel(t)$

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Basic properties.

$IsReflexive(R, S) \triangleq \forall a \in S : \langle a, a\rangle \in R$
$IsIrreflexive(R, S) \triangleq \forall a \in S : \langle a, a\rangle \notin R$

$IsSymmetric(R, S) \triangleq \forall a, b \in S : \langle a, b\rangle \in R \equiv \langle b, a\rangle \in R$
$IsAntisymmetric(R, S) \triangleq \forall a, b \in S : \langle a, b\rangle \in R \land \langle b, a\rangle \in R \Rightarrow a = b$

$IsTransitive(R, S) \triangleq$
$\quad \forall a, b, c \in S : (\langle a, b\rangle \in R \land \langle b, c\rangle \in R) \Rightarrow \langle a, c\rangle \in R$

$IsTotal(R, S) \triangleq$
$\quad \forall a, b \in S : \langle a, b\rangle \in R \lor \langle b, a\rangle \in R$

$IsPartialOrder(R, S) \triangleq$
$\quad \land IsReflexive(R, S)$
$\quad \land IsAntisymmetric(R, S)$
$\quad \land IsTransitive(R, S)$

$IsTotalOrder(R, S) \triangleq$
$\quad \land IsPartialOrder(R, S)$
$\quad \land IsTotal(R, S)$

$IsStrictPartialOrder(R, S) \triangleq$
$\quad \land IsIrreflexive(R, S)$
$\quad \land IsTransitive(R, S)$

$IsStrictTotalOrder(R, S) \triangleq$
$\quad \land IsStrictPartialOrder(R, S)$
$\quad \land IsTotal(R, S)$

$Respect(R, T) \triangleq T \subseteq R$ Does $R$ respect $T$?

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Special elements in a relation

$Minimal(R, S) \triangleq$ the set of minimal elements in relation $R$ on the set $S$
$\quad \{m \in S : \neg \exists a \in Dom(R) : \langle a, m\rangle \in R\}$
$Maximal(R, S) \triangleq$ the set of maximal elements in relation $R$ on the set $S$
$\quad \{m \in S : \neg \exists b \in Ran(R) : \langle m, b\rangle \in R\}$

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A variant of *Kahn*'s algorithm for topological sorting

See https://*en.wikipedia.org*/wiki/*Topological_sorting* $\neq$ *Kahn*'s_algorithm

$Cyclic(R) \triangleq$ Is $R$ cyclic?
$\quad \text{LET RECURSIVE} \ CyclicUtil(\_, \_)$
$\quad\quad\quad CyclicUtil(rel, set) \triangleq$ remaining relation; set: remaining set

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                IF set = {} THEN FALSE
                 ELSE LET mins ≜ Minimal(rel, set)
                      IN  IF mins = {} THEN TRUE
                           ELSE LET m ≜ CHOOSE x ∈ mins : TRUE
                                IN  CyclicUtil(rel \ LeftRestriction(R, m), set \ {m})
    IN  CyclicUtil(R, Support(R))
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  Kahn's algorithm for topological sorting.

  See https://en.wikipedia.org/wiki/Topological_sorting ≠ Kahn's_algorithm
AnyLinearExtension(R, S) ≜  \* return an arbitrary linear extension of R on the set S
    LET RECURSIVE LinearExtensionUtil(_, _)
        LinearExtensionUtil(rel, set) ≜  \* rel: remaining relation; set: remaining set
            IF set = {} THEN ⟨⟩
             ELSE LET m ≜ CHOOSE x ∈ Minimal(rel, set) : TRUE
                  IN  ⟨m⟩ ∘ LinearExtensionUtil(rel \ LeftRestriction(R, m), set \ {m})
    IN  LinearExtensionUtil(R, S)
  A variant of Kahn's algorithm for topological sorting

  See https://en.wikipedia.org/wiki/Topological_sorting ≠ Kahn's_algorithm

  For some TLA+ issue, see https://groups.google.com/g/tlaplus/c/mtyEmqhlRVg
AllLinearExtensions(R, S) ≜  \* return all possible linear extensions of R on the set S
    LET RECURSIVE LinearExtensionsUtil(_, _)
        LinearExtensionsUtil(rel, set) ≜
            IF set = {} THEN {⟨⟩}
             ELSE LET Extend(m) ≜ {⟨m⟩ ∘ l :  \* extend recursively by the minimal element m
                              l ∈ LinearExtensionsUtil(rel \ LeftRestriction(R, m), set \ {m})}
                  IN  UNION {Extend(m) : m ∈ Minimal(rel, set)}  \* for each minimal element
    IN  LinearExtensionsUtil(R, S)

LinearExtensions(R, S) ≜  \* return the set of all possible linear extensions of R on the set S
    {l ∈ TupleOf(S, Cardinality(S)) : Respect(Seq2Rel(l), R)}
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