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Posets on up to 16 Points

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Abstract. In this article we describe a very efficient method to construct pairwise non-isomorphic posets (equivalently, T_0 topologies). We also give the results obtained by a computer program based on this algorithm, in particular the numbers of non-isomorphic posets on 15 and 16 points and the numbers of labelled posets and topologies on 17 and 18 points.

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1. Introduction

Various algorithms have been proposed and applied for the constructive enumeration of posets (see [16, 5, 14, 4, 2, 3]). The fastest algorithm so far was developed by Heitzig and Reinhold ([8]) who were able to determine the number of posets on 14 points. The generation rate of the computer program based on their algorithm is about 90 000 posets per second (scaling approximately from the computer they used to a 1 GHz Pentium III). The number of posets on 14 points was later independently confirmed by Lygerōs and Zimmermann [10]. The algorithm we will describe in this article is able to list more than 4 million non-isomorphic posets per second (scaled to the same machine), with the rate increasing with the number of points for these small sizes.

We will describe posets $(V, \leq)$ by their corresponding Hasse diagrams as directed graphs, that is we have the directed graph $P = (V, E)$ with $(a, b) \in E$ iff $a, b \in V$, $a < b$ and there exists no $c \in V$ with $a < c < b$. Points with indegree 0 are called the *minimal* points while the points with outdegree 0 are called the maximal points. The *level* $l(v)$ of a point v is the length of a longest directed path from a minimal point to v. So the minimal points are on level 0 and for $l \geq 1$ for every point b on level l there is a point a on level $l - 1$ such that $(a, b) \in E$.

2. The Construction of Posets

For every $n \in \mathbb{N}$ there is exactly one poset with n points and only one level, so we list that first. To construct posets on n points with at least two levels from posets on $n-1$ points, we start with the unique posets with 1 to $n-1$ isolated points on level 0 and recursively add new points either on the highest level m (if $m \geq 1$), or on a new highest level $m+1$, and connect them to the smaller poset in every possible way to form a larger poset.

Isomorphism rejection will be done by the canonical construction path method described in [13]. In this method, a family of labelled posets, called *candidates*, is defined that has the property that each isomorphism class of posets is represented in the family at least once. Then, without any direct comparison of candidates, each candidate is subject to an *acceptance test* which accepts exactly one from each isomorphism class. The basis of the acceptance test is a function f from the set of all candidates (which are labelled posets) on the points $0, \ldots, n-1$ to the set $2^{\{0,\ldots,n-1\}}$ of subsets of $\{0, \ldots, n-1\}$, such that, given such a poset P, f chooses an orbit of points under the automorphism group of P. Precisely:

(a) For each candidate P, $f(P)$ is an orbit of the automorphism group of P consisting of points on the highest level. (Note that the automorphism group preserves the levels, so such an orbit exists.)
(b) For each pair of candidates P, P', any isomorphism $P \to P'$ maps $f(P)$ onto $f(P')$.

A function f with both these properties will be called a *canonical choice function* and $f(P)$ will be called the *canonical orbit* of P. If only (b) is satisfied, we just call f a *choice function*.

Our implementation defines a canonical choice function f via a sequence of choice functions $f_1, \ldots, f_6 = f$ where, for each candidate P, $f_1(P) \supseteq f_2(P) \supseteq \cdots \supseteq f_6(P)$, $f_1(P)$ is the set of all points of P on the highest level and $f_6(P)$ is a single orbit. The exact definition of $f_2, \ldots, f_6$ will be given later, as the correctness of the algorithm depends only on f being a canonical choice function.

We define the *parent* of a candidate to be the poset obtained when any vertex v of the canonical orbit together with all edges ending in v is removed. Since this point is on the highest level, the remaining part is again a (Hasse diagram of a) poset. Note that the parent of a candidate is uniquely defined up to isomorphism. We accept a newly constructed candidate iff it is constructed from its parent (that is: the last vertex added belongs to the canonical orbit) and reject it otherwise. So – provided we have only one member of every isomorphism class of smaller posets – if two isomorphic posets are accepted, they were generated from the same parent.

In order to extend a poset $P = (V, E)$ with maximum level m by adding a point on level $l \in \{m, m+1\}$, we compute the set A of all antichains S (that is, the set of all $S \subseteq V$ so that for all $a, b \in S$, $a \neq b$, neither $a \leq b$ nor $b \leq a$), that contain at least one point on level $l-1$. We call these antichains l-antichains. These l-antichains are exactly the subsets S of V, so that after connecting the new vertex v with all elements of S the resulting graph represents a poset on $|V| + 1$ points with v on the highest level. If the automorphism group G of P maps two such sets S, S' onto each other, the posets obtained by connecting the new point to all elements of S, resp. S' will be isomorphic, so we compute the orbits of G on A and choose one representative of every orbit to connect the new point with.

Summarizing, the extension of a poset with maximum level m by adding a point on level $l \in \{m, m+1\}$ can be described in three steps:

Step 1: Compute the orbits of l-antichains.

Step 2: Connect the new point with one representative of each orbit in turn.

Step 3: Compute the canonical choice function for the extended poset and accept it if and only if the new point is in the canonical orbit.

Formally, we can define the *candidates* as those posets formed at Step 2, and the *acceptance test* as that performed at Step 3.

THEOREM 1. *For any canonical choice function f, the algorithm described above accepts exactly one element of every isomorphism class of posets.*

Proof. The theorem is obviously true for posets on 1 point or with just one level, so assume it has been proven for posets on up to $p - 1$ points.

For a given poset P on p points with at least two levels, let P_v denote the poset obtained by removing a point v from the last level. By induction, a poset isomorphic to P_v has been accepted and extended in all ways (up to isomorphism) by adding a point at the highest level. Therefore, there are candidates isomorphic to P and we might as well assume that P is a candidate. Choosing $w \in f(P)$, we now find that a poset isomorphic to P_w is extended to a candidate P' isomorphic to P with the isomorphism mapping w onto the last point of P'. This candidate P' is accepted, so we have that at least one member of each isomorphism class is accepted.

If there are two isomorphic candidates P, P' which are both accepted, there is an isomorphism mapping the last points of P, P' onto each other. This induces an isomorphism of the parents, so by induction these parents are the same and the isomorphism is an automorphism which maps the neighbourhoods of the last points onto each other. So the parent was extended using two antichains from the same orbit, contradicting Step 2. □

Using, e.g., `nauty` (see [11, 12]), a computer program which is able to compute the automorphism group of a graph as well as a canonical numbering of the vertices one can easily define a canonical choice function and hence a program for constructing posets by this method. However, such a program would be much slower than can be obtained by defining the canonical choice function more carefully, as we next show.

3. Improvements to the Basic Algorithm

Although our canonical choice function f ultimately relies on `nauty`, the nested sequence $f_1, \ldots, f_6 = f$ of choice functions in our implementation is designed so that it is usually possible to accept or reject candidates without calling `nauty`. Often we will also be able to infer the orbits of antichains without calling `nauty`.

DEFINITION 1. Let P be a poset with maximum level m and automorphism group G. Then, for $l \leq m$, P_l denotes the poset induced by the points on levels $0, \ldots, l$ and G_l denotes its automorphism group. By $G|_l$ we denote the group G restricted to the points on levels $0, \ldots, l$, so $G|_l \subseteq G_l$.

If we have a coloured poset, that is a poset with coloured vertices, and we only allow automorphisms that do not map points of different colours onto each other (this group is denoted by G^c, and the similar subgroup of G_l by G_l^c), then $G = G^c$ if each orbit of G consists of points of the same colour. Nevertheless the colouring may help to compute G, since it is easier to detect non-equivalent points.

So we need some simple criteria to determine which points are in different orbits of G and therefore can be coloured differently:

LEMMA 1. *Let P be a poset with maximum level m and let G denote its automorphism group.*

(a) *Points on different levels are in different orbits of G.*
(b) *If a, b are on level $l < m$, $l \geq 0$, and they are in different orbits of G_l, then they are in different orbits of G.*
(c) *If a, b are on level $l \leq m$, $l \geq 1$, and their neighbourhoods are in different orbits of G_{l-1} on the set of l-antichains, then a, b are in different orbits of G.*

Though the information needed for (b) and (c) can not – unlike that for (a) – be trivially deduced from the poset, it is information that the described algorithm has to compute anyway: the information on the orbits of points needed for (b) is computed with the canonical choice function for the last point in level l (Step 3) and the information on the orbits of antichains is computed in Step 1.

Of course small groups are easier to handle, e.g., the number of differently coloured points is larger and the operation of the groups on the set of points and antichains can be computed faster. But, as known from [8, 15], the ratio of posets with a trivial group converges to 1 fairly slowly compared to some other classes of combinatorial structures. E.g., only 71% of the posets on 14 points have a trivial group and from the number of labelled posets on 16 points which was given in [8], we can deduce from our computations that the harmonic mean of the automorphism group size for unlabelled posets on 16 points is 1.1236, which is quite large. Therefore it is helpful to use a subgroup of the automorphism group instead:

LEMMA 2. *Let P be a poset with maximum level m and automorphism group G, such that for some $l < m$ the group G_l operates like the identity on the set of*

$(l+1)$-antichains. Let G^{co} denote the automorphism group of the coloured poset with the vertices on levels $l+1, \ldots, m$ coloured according to the rules in Lemma 1 *and each vertex on levels* $0, \ldots, l$ *having a unique colour.*

Then two $(m+1)$-antichains are equivalent under G if and only if they are equivalent under G^{co}.

Proof. Since $G^{co} \subseteq G$, it is clear that if they are equivalent under G^{co}, they are equivalent under G, so we only have to show the other direction.

First note that G_l fixes points that are contained in $(l+1)$-antichains: For every point x on level l the set $\{x\}$ is an antichain itself, so this is true by definition. Now let y be a point on level $l' < l$ that occurs in an $(l+1)$-antichain, say, together with x on level l. Then $\{x, y\}$ is an antichain which must be fixed as a set by every automorphism of G_l, but since x and y are on different levels and therefore cannot be mapped onto each other by an automorphism, it must also be fixed pointwise, so y must be fixed too.

Now assume two $(m+1)$-antichains s, s' to be equivalent by an automorphism $\gamma \in G$, that is $s = \gamma(s')$. We have to show that there is an automorphism $\gamma' \in G^{co}$, so that $s = \gamma'(s')$. Define

$$\gamma'(x) := \begin{cases} x & \text{if } x \text{ is on level } 0, \ldots, l; \\ \gamma(x) & \text{if } x \text{ is on level } l+1, \ldots, n. \end{cases}$$

We show that $\gamma(x) = \gamma'(x)$ for all $x \in s'$ and therefore $s = \gamma'(s')$: This is just by definition if $l(x) \geq l+1$, so assume $l(x) \leq l$. Let $y \in s'$ be on level m and z a point on level l on the path from a minimal point to y. Then $\{x, z\}$ is an $(l+1)$-antichain and therefore fixed pointwise by γ, so $\gamma(x) = x = \gamma'(x)$.

It remains to be shown that $\gamma' \in G^{co}$. Since γ as well as the identity respect the colouring, all that is left to be shown is the fact that if (x, y) is an edge in G, then $(\gamma'(x), \gamma'(y))$ is an edge in G. For $l(x), l(y) \leq l$ it follows from the identity being an automorphism of P_l and for $l(x), l(y) > l$ it follows from γ being an automorphism, so let $l(x) \leq l, l(y) = l' > l$. Again let z be a point on level l on the path from a minimal point to y. Then $\{z, x\}$ is an $(l+1)$-antichain. In fact $z = x$ is possible, but in any case x occurs in an $(l+1)$-antichain and therefore $\gamma(x) = x$. So we have $(\gamma'(x), \gamma'(y)) = (\gamma(x), \gamma(y))$ which is an edge of G since γ is an automorphism. □

In Figure 1, we give an example of a poset with nontrivial group acting like the identity on the set of all $(m+1)$-antichains.

The other time-critical part is the computation of the canonical choice function. We have to choose the functions $f_1, \ldots, f_6 = f$ in such a way that in many cases we can determine canonicity very fast and – if possible – even avoid adding points that would not be canonical.

As the statistics we present at the end of this paper show, a very large fraction of the posets generated contain just one point on the last level. In these cases our function f_1 has a single point as the image, so all candidates with one point on

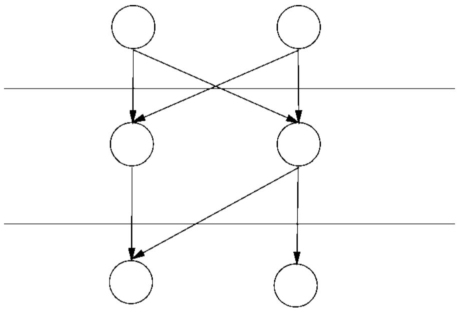

Figure 1. Example for Lemma 2.

the highest level are accepted without having to compute the group or any more expensive canonical choice functions.

Our function f_2 is chosen as follows: Consider a poset with maximum level m. Number the orbits of m-antichains of G_{m-1} in an arbitrary (but fixed) way, e.g., in the order they are found when doing Step 1 for the *first* point on level m (this way the numbering causes very few additional costs). Denote the orbit number of an antichain s by $o(s)$ and also write $o(x)$ for the orbit number of the neighbourhood of the point x. The image of our function f_2 is the set of all points x on level m for which $o(x)$ is greatest.

We have to show that f_2 is a choice function. Assume P, P' are two isomorphic candidates (not necessarily distinct) with maximum level m, say $P = \gamma(P')$. Then $P|_{m-1} = P'|_{m-1}$ since for each smaller poset there is only one isomorphic copy. So if $x \in P$ and $y \in P'$ are on level m, $x = \gamma(y)$, then the set s_y of neighbours of y is mapped onto the set s_x of neighbours of x. So there is an automorphism of $P|_{m-1}$ mapping s_y onto s_x, implying that they are in the same orbit and therefore $o(x) = o(y)$. Thus γ preserves the values of $o()$ and in particular maps $f_2(P')$ onto $f_2(P)$.

This canonicity criterion allows us to avoid making a lot of candidates that would not be accepted anyway. We know that points must be added in a way that the values of $o()$ are monotonically increasing on each level. In fact we have:

LEMMA 3. *Let P be a poset that was accepted by the algorithm with maximum level m and x the last point added. Let y be a new point added to level m to form P'.*

(a) *If $o(y) < o(x)$, P' is rejected.*
(b) *If $o(y) > o(x)$, P' is accepted.*
(c) *If all the points on level m with the maximum value of $o()$ have the same neighbourhood (which is always the case if, e.g., G_{m-1} acts like the identity on the set of m-antichains), P' is accepted if and only if $o(y) \geq o(x)$.*

While (a) and (b) are obvious from the definition, (c) follows since in this case the set of all points with maximal value of $o()$ forms an orbit (so $f(P')$ cannot be a proper subset of $f_2(P')$ and therefore y is in the canonical orbit).

Of course points that would be rejected due to criterion (a) will not be added by the algorithm at all.

Summarizing, the only cases where nontrivial canonicity tests have to be applied are when there are at least two points on the last level with different neighbourhoods but the same, largest, orbit number and one of them is the last point added.

In these cases we apply several further criteria before applying `nauty`. We will only sketch them and leave it to the reader to prove that they are choice functions.

For an m-antichain s, let $n(s)$ denote the number of points on level m with neighbourhood s. Among all points in $f_2(P)$ we choose those with the property that their neighbourhood s has the maximum value of $n(s)$. This set is $f_3(P)$. If all elements of $f_3(P)$ have the same neighbourhood, they form an orbit, so $f(P)$ is determined.

If $f(P)$ is not determined yet and the last point is still contained in $f_3(P)$ we compute $f_4(P)$. To this end let S denote the union of all points neighbouring a point on the maximum level m that is not contained in $f_3(P)$. For a point $x \in f_3(P)$ let S_x denote the intersection of S with the neigbours of x and for $0 \leq j < m$ let $S_{x,j}$ denote the set of all points in S_x that are on level j. Note that every $S_{x,j}$ is a $(j+1)$-antichain, so that $o(S_{x,j})$ is well defined except for $S_{x,j} = \emptyset$ for which we define it as 0.

For all points $x \in f_3(P)$ we define

$$p(x) := \sum_{j=0}^{l-1} 2^j o(S_{x,j})$$

and define $f_4(P)$ to be the set of points in $f_3(P)$ with the maximum value of $p()$.

In unusual cases, there are points with different neighbourhoods even in $f_4(P)$, so we have an additional step. Let H be the underlying undirected graph of the Hasse diagram of P, and let π be the colouring defined in Lemmas 1 and 2. Apply the partition refinement algorithm of [11] to find the coursest equitable colouring π' finer than π. (A colouring is equitable if the number of neighbours of colour c of each vertex v depends only on c and the colour of v.) We now define $f_5(P)$ to be the last cell of π' which includes points of $f_4(P)$. The isomorphism invariance nature of the refinement procedure (Lemma 2.8(b) of [11]) implies that $f_5(P)$ is a choice function.

Lemma 2.25 of [11] gives some sufficient conditions on π' that imply that its cells are orbits. These apply in the majority of cases at this point, enabling us to say that $f(P) = f_5(P)$ is a canonical choice function. In the simplest cases (π' having at least $n-2$ cells, which it usually does) generators for the automorphism group can be seen immediately as well. In the few remaining instances we can apply `nauty` to H with colouring π' and define $f(P) = f_6(P)$ to be the orbit of the automorphism group which contains the vertex of $f_5(P)$ which has the highest number in the canonical labelling given by `nauty`. It follows from Lemmas 1 and 2 that the orbits of points on the last level of the coloured poset and of the uncoloured poset coincide, so that the coloured poset can be used for the computations performed by `nauty`.

When computing the posets on 16 points, `nauty` had to be called for less than 1% of the posets generated.

A last improvement of the algorithm comes from the following observation: Let P be a poset on n points with exactly one minimal point (or equivalently: one point on level 0). Then all points on level 1 are connected to it. Removing this point, a poset on $n-1$ points is obtained and two such posets on n points are isomorphic if an only if the corresponding posets on $n-1$ points are isomorphic. So the algorithm never starts with posets with just one point on the first level, but generates these by adding a point to every poset on $n-1$ points and connecting all points on level 0 to a new vertex. No additional group or canonicity computations must be done for these posets.

4. Labelled and Derived Counts

In order to obtain the number $T_0(n)$ of labelled posets with n points, equal to the number of T_0 topologies, we used the following version of a formula given in [7]:

Let $A(P)$ denote the number of antichains of a poset P, $\mathrm{Aut}(P)$ its automorphism group, $\mathcal{P}_n$ the set of all non-isomorphic posets on n points and

$$s(n,k) = \sum_{m=0}^{n} \binom{n+k-1-m}{k-1} \binom{n+k}{m} \sum_{P \in \mathcal{P}_m} \frac{m!}{|\mathrm{Aut}(P)|} (-A(P))^{n+k-m}$$

then

$$\begin{aligned} T_0(n+1) &= -s(n,1), \\ T_0(n+2) &= (n+2)T_0(n+1) + s(n,2), \\ T_0(n+3) &= \binom{n+4}{2} T_0(n+2) - s(n,3). \end{aligned}$$

These formulas were also used in [8] (where unfortunately some typos occurred in print) in order to determine the labelled counts for up to 16 points.

To determine $T_0(n+3)$, one has to know the size of the automorphism group as well as the number of antichains for all posets on up to n points. We modified our program to compute these numbers too, which slowed down the generation by a factor of approximately 18.5 for $n = 14$, increasing with the number of points. So it was impossible to run the modified program for as many points as the original one. We restricted ourselves to $n = 15$.

From the numbers discussed so far, others can be computed. The counts of connected posets, labelled and unlabelled, can be computed from the counts of all of them by standard means (quoted incorrectly in [8]).

Define a poset P to be *vertically indecomposable* if there is no proper subset A of the points such that $x \leq y$ for all $x \in A, y \notin A$. The numbers of such posets, labelled and unlabelled, can be computed using the recurrences given in [8]. In addition, the numbers of labelled quasiorders (equivalently, topologies) can be computed by the same method as in [8]. Unfortunately, there seems to be no mechanical way to deduce the numbers of unlabelled quasiorders (topologies), but these can be counted with additional computations which we will describe in a future paper [1].

It was proved by Kleitman and Rothschild [9] that a large random poset typically has three levels with approximately $n/4$, $n/2$ and $n/4$ vertices, respectively. Our results do not show this behaviour, emphasizing again that the convergence of posets to their asymptotic behaviour is quite slow. A survey of other enumerative results for posets is given by El Zahar [6].

5. Correctness

Some considerable effort was taken to ensure that the results were correct. The numbers of unlabelled posets up to 14 points as well as labelled posets up to 16 points were checked against previously known data and the labelled counts obtained in various ways were compared against each other.

The correctness of the program is fairly confidently established by the computations up to 14 points and the labelled counts, but we encountered another problem. A few of the more than 200 computers employed in the task gave wrong answers during some runs, without machine exceptions and without any obvious sign of error in the output. In each case the answer was only slightly wrong and running the same task again gave the right answer. A very small rate of random error due to cosmic rays is expected [17] but the clustering of our errors in a few computers suggests more prosaic hardware causes. In fact it turned out that all the computers that gave wrong answers sometimes crashed with other programs. In some cases this seems to depend on the circumstances (load), but some of them continuously failed memory test programs.

To deal with these problems as well as we could, the computation for 15 points was performed more than once.

The case of 16 points is more of a problem since the computation is so long that independent confirmation was not practical. We excluded all results by machines of the type that had failed (not only the specific machines) and reran them on other machines. We also ran about 10% of all cases twice to look for other failures, without finding any.

In addition, we collected some statistics as a check. Reversing the order in a poset defines a bijection between the set of posets with fixed number of points and itself mapping a poset with n_1 maximal and n_2 minimal points onto one with

n_1 minimal and n_2 maximal points, so the matrix of counts indexed by n_1 and n_2 must be symmetric. Moreover, the posets counted by symmetrically opposite entries are usually distributed over different subcases and so executed on different machines. The matrix accumulated by our many runs was indeed symmetric, which we believe to be quite a good check (good enough to have detected any of the known incorrect runs).

As noted by Heitzig and Reinhold [8], a lemma of Borevič shows that $T_0(17) \equiv 4557871 \pmod{12252240}$, and by the same method we find that $T_0(18) \equiv 8352783 \pmod{12252240}$. Both congruences are satisfied by our results.

6. Results

In order to be able to run the program in parts on a large cluster of machines, we added an option that for arbitrary $r, m \in \mathbb{N}$, $r < m$ when computing posets on n points generates all posets on $p - 4$ points and numbers them in the order they occur, but only generates successors of those posets whose number is equivalent to $r \bmod m$. This allowed us to split the generation of posets on 16 points into 30 000 parts. The total cpu time for 16 points, scaled to a 1 GHz Pentium III, was about 30 years.

Table I gives the number of isomorphism classes of posets up to 16 points. In Tables V–LX much more information about these posets is given.

Tables II–IV give the numbers of various types of labelled posets and topologies.

Table I. Numbers of unlabelled posets (equivalently, T_0 topologies).

Points	All	Connected	Vertically indecomposable
1	1	1	1
2	2	1	1
3	5	3	2
4	16	10	7
5	63	44	31
6	318	238	184
7	2045	1650	1351
8	16999	14512	12524
9	183231	163341	146468
10	2567284	2360719	2177570
11	46749427	43944974	41374407
12	1104891746	1055019099	1008220289
13	33823827452	32664984238	31559446774
14	1338193159771	1303143553205	1269310589336
15	68275077901156	66900392672168	65562045668340
16	4483130665195087	4413439778321689	4345161435996517

Table II. Numbers of labelled posets (equivalently, T_0 topologies).

Points	All	Connected
1	1	1
2	3	2
3	19	12
4	219	146
5	4231	3060
6	130023	101642
7	6129859	5106612
8	431723379	377403266
9	44511042511	40299722580
10	6611065248783	6138497261882
11	1396281677105899	1320327172853172
12	414864951055853499	397571105288091506
13	171850728381587059351	166330355795371103700
14	98484324257128207032183	96036130723851671469482
15	77567171020440688353049939	76070282980382554147600692
16	83480529785490157813844256579	82226869197428315925408327266
17	122152541250295322862941281269151	120722306604121583767045993825620
18	241939392597201176602897820148085023	239727397782668638856762574296226842

Table III. Numbers of vertically indecomposable labelled posets.

Points	All
1	1
2	1
3	7
4	97
5	2251
6	80821
7	4305127
8	332273257
9	36630174931
10	5711638291981
11	1249898984911567
12	381230073532620577
13	161042140788424003291
14	93667063572594041040421
15	74610767840852891620692727
16	80997478506602342803118178457
17	119313601058907927882431190269731
18	237541348427311374857037021264415741

Table IV. Numbers of labelled topologies.

Points	All	Connected
1	1	1
2	4	3
3	29	19
4	355	233
5	6942	4851
6	209527	158175
7	9535241	7724333
8	642779354	550898367
9	63260289423	56536880923
10	8977053873043	8267519506789
11	1816846038736192	1709320029453719
12	519355571065774021	496139872875425839
13	207881393656668953041	200807248677750187825
14	115617051977054267807460	112602879608997769049739
15	88736269118586244492485121	86955243134629606109442219
16	93411113411710039565210494095	91962123875462441868790125305
17	134137950093337880672321868725846	132524871920295877337189590203
18	261492535743634374805066126901117203	259048612476248175744581063815546423

Table V. Numbers of posets on 3 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	3
2	1
3	1

Table VI. Numbers of posets on 3 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	3
3	1

Table VII. Numbers of posets on 3 points grouped with respect to the number of nontrivial relations.

Relations	Posets
0	1
1	1
2	2
3	1

Table VIII. Numbers of posets on 3 points with x minimal and y maximal points (symmetric upper half omitted)

$x \setminus y$	1	2	3
1	1		
2	1	1	
3	0	0	1

Table IX. Numbers of posets on 4 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	9
2	5
3	1
4	1

Table X. Numbers of posets on 4 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	8
3	6
4	1

Table XI. Numbers of posets on 4 points grouped with respect to the number of nontrivial relations.

Relations	Posets
0	1
1	1
2	3
3	4
4	3
5	3
6	1

Table XII. Numbers of posets on 4 points with x minimal and y maximal points (symmetric upper half omitted).

x\y	1	2	3	4
1	2			
2	2	4		
3	1	1	1	
4	0	0	0	1

Table XIII. Numbers of posets on 5 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	36
2	18
3	7
4	1
5	1

Table XIV. Numbers of posets on 5 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	20
3	31
4	10
5	1

Table XV. Numbers of posets on 5 points grouped with respect to the number of nontrivial relations.

Relations	Posets
0	1
1	1
2	3
3	6
4	10
5	10
6	12
7	9
8	6
9	4
10	1

Table XVI. Numbers of posets on 5 points with x minimal and y maximal points (symmetric upper half omitted).

$x\backslash y$	1	2	3	4	5
1	5				
2	7	12			
3	3	7	4		
4	1	1	1	1	
5	0	0	0	0	1

Table XVII. Numbers of posets on 6 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	184
2	87
3	35
4	10
5	1
6	1

Table XVIII. Numbers of posets on 6 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	55
3	162
4	84
5	15
6	1

Table XIX. Numbers of posets on 6 points grouped with respect to the number of nontrivial relations.

Relations	Posets
0	1
1	1
2	3
3	7
4	16
5	25
6	36
7	43
8	46
9	44
10	35
11	28
12	17
13	10
14	5
15	1

Table XX. Numbers of posets on 6 points with x minimal and y maximal points (symmetric upper half omitted).

$x \setminus y$	1	2	3	4	5	6
1	16					
2	27	59				
3	15	36	29			
4	4	11	7	4		
5	1	1	1	1	1	
6	0	0	0	0	0	1

Table XXI. Numbers of posets on 7 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	1187
2	575
3	201
4	67
5	13
6	1
7	1

Table XXII. Numbers of posets on 7 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	163
3	940
4	734
5	185
6	21
7	1

Table XXIII. Numbers of posets on 7 points grouped with respect to the number of nontrivial relations.

Relations	Posets
0	1
1	1
2	3
3	7
4	18
5	38
6	74
7	113
8	167
9	209
10	243
11	249
12	239
13	204
14	168
15	123
16	83
17	54
18	29
19	15
20	6
21	1

Table XXIV. Numbers of posets on 7 points with x minimal and y maximal points (symmetric upper half omitted).

$x \backslash y$	1	2	3	4	5	6	7
1	63						
2	134	334					
3	88	251	213				
4	27	79	78	29			
5	5	15	11	7	4		
6	1	1	1	1	1	1	
7	0	0	0	0	0	0	1

Table XXV. Numbers of posets on 8 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	9752
2	4891
3	1730
4	484
5	123
6	17
7	1
8	1

Table XXVI. Numbers of posets on 8 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	556
3	6372
4	7305
5	2380
6	356
7	28
8	1

Table XXVII. Numbers of posets on 8 points grouped with respect to the number of nontrivial relations.

Relations	Posets
0	1
1	1
2	3
3	7
4	19
5	44
6	107
7	208
8	381
9	619
10	915
11	1219
12	1506
13	1705
14	1792
15	1767
16	1621
17	1402
18	1136
19	874
20	629
21	434
22	274
23	166
24	94
25	46
26	21
27	7
28	1

Table XXVIII. Numbers of posets on 8 points with x minimal and y maximal points (symmetric upper half omitted).

$x \backslash y$	1	2	3	4	5	6	7	8
1	318							
2	814	2404						
3	642	2093	2068					
4	221	777	839	392				
5	43	149	170	78	29			
6	6	20	15	11	7	4		
7	1	1	1	1	1	1	1	
8	0	0	0	0	0	0	0	1

Table XXIX. Numbers of posets on 9 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	103926
2	52185
3	20487
4	5211
5	1178
6	221
7	21
8	1
9	1

Table XXX. Numbers of posets on 9 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	2222
3	52336
4	86683
5	35070
6	6259
7	623
8	36
9	1

Table XXXI. Numbers of posets on 9 points grouped with respect to the number of nontrivial relations.

Relations	Posets	Relations	Posets	Relations	Posets
0	1	13	8294	26	3551
1	1	14	10921	27	2386
2	3	15	13363	28	1528
3	7	16	15419	29	939
4	19	17	16687	30	541
5	46	18	17119	31	300
6	124	19	16578	32	153
7	287	20	15309	33	69
8	636	21	13421	34	28
9	1257	22	11253	35	8
10	2311	23	8999	36	1
11	3830	24	6897		
12	5891	25	5054		

Table XXXII. Numbers of posets on 9 points with x minimal and y maximal points (symmetric upper half omitted).

$x \setminus y$	1	2	3	4	5	6	7	8	9
1	2045								
2	6258	21360							
3	5828	21775	24535						
4	2319	9227	11518	5976					
5	477	1978	2606	1472	392				
6	64	253	329	170	78	29			
7	7	25	20	15	11	7	4		
8	1	1	1	1	1	1	1	1	
9	0	0	0	0	0	0	0	0	1

Table XXXIII. Numbers of posets on 10 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	1441424
2	718566
3	301817
4	86274
5	15856
6	2941
7	378
8	26
9	1
10	1

Table XXXIV. Numbers of posets on 10 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	10765
3	534741
4	1261371
5	619489
6	125597
7	14258
8	1016
9	45
10	1

Table XXXV. Numbers of posets on 10 points grouped with respect to the number of nontrivial relations.

Relations	Posets	Relations	Posets	Relations	Posets
0	1	16	90614	32	31401
1	1	17	119179	33	21414
2	3	18	148255	34	14096
3	7	19	174838	35	8974
4	19	20	196135	36	5492
5	47	21	209729	37	3240
6	130	22	214283	38	1836
7	329	23	209692	39	986
8	841	24	196824	40	506
9	1946	25	177576	41	237
10	4251	26	154148	42	99
11	8526	27	128998	43	36
12	15891	28	104101	44	9
13	27259	29	81200	45	1
14	43572	30	61145		
15	64851	31	44566		

Table XXXVI. Numbers of posets on 10 points with x minimal and y maximal points (symmetric upper half omitted).

$x \setminus y$	1	2	3	4	5	6	7	8	9	10
1	16999									
2	60877	238134								
3	66612	281051	361231							
4	30698	137620	193546	115336						
5	7015	32777	49981	32368	9811					
6	931	4431	6913	4607	1472	392				
7	90	401	581	329	170	78	29			
8	8	31	25	20	15	11	7	4		
9	1	1	1	1	1	1	1	1	1	
10	0	0	0	0	0	0	0	0	0	1

Table XXXVII. Numbers of posets on 11 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	25927626
2	13005670
3	5561265
4	1838428
5	357724
6	50769
7	7286
8	626
9	31
10	1
11	1

Table XXXVIII. Numbers of posets on 11 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	64955
3	6915309
4	22902794
5	13452868
6	3012577
7	370057
8	29241
9	1569
10	55
11	1

Table XXXIX. Numbers of posets on 11 points grouped with respect to the number of nontrivial relations.

Relations	Posets	Relations	Posets	Relations	Posets
0	1	19	1190889	38	430451
1	1	20	1587016	39	300981
2	3	21	2015412	40	204974
3	7	22	2446957	41	135976
4	19	23	2844542	42	87786
5	47	24	3174558	43	55127
6	132	25	3405232	44	33614
7	346	26	3518608	45	19897
8	950	27	3505930	46	11385
9	2468	28	3374784	47	6306
10	6171	29	3141073	48	3351
11	14411	30	2831400	49	1694
12	31724	31	2473385	50	811
13	64772	32	2096755	51	353
14	123620	33	1725908	52	137
15	219868	34	1380922	53	45
16	366672	35	1074413	54	10
17	574347	36	813564	55	1
18	849968	37	599653		

Table XL. Numbers of posets on 11 points with x minimal and y maximal points (symmetric upper half omitted).

$x \backslash y$	1	2	3	4	5	6	7	8	9	10	11
1	183231										
2	755323	3338843									
3	959941	4529404	6595603								
4	514525	2561974	4033606	2722606							
5	133610	693156	1182440	881608	310416						
6	18836	100414	182730	144667	53123	9811					
7	1686	9037	16416	12658	4607	1472	392				
8	122	602	966	581	329	170	78	29			
9	9	37	31	25	20	15	11	7	4		
10	1	1	1	1	1	1	1	1	1	1	
11	0	0	0	0	0	0	0	0	0	0	1

Table XLI. Numbers of posets on 12 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	602712285
2	308959917
3	132633575
4	47610599
5	11295712
6	1487660
7	173235
8	17722
9	1002
10	37
11	1
12	1

Table XLII. Numbers of posets on 12 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	501695
3	114388439
4	524615468
5	364407028
6	88556125
7	11417478
8	947791
9	55334
10	2320
11	66
12	1

Table XLIII. Numbers of posets on 12 points grouped with respect to the number of nontrivial relations.

Relations	Posets	Relations	Posets	Relations	Posets
0	1	23	25468042	46	4491015
1	1	24	33157695	47	3100063
2	3	25	41495336	48	2094942
3	7	26	50008606	49	1386092
4	19	27	58130096	50	897535
5	47	28	65270723	51	568627
6	133	29	70888253	52	352196
7	352	30	74562234	53	213115
8	997	31	76042383	54	125818
9	2753	32	75275671	55	72382
10	7558	33	72402491	56	40515
11	19801	34	67726046	57	21985
12	49795	35	61666534	58	11545
13	117875	36	54699028	59	5808
14	263019	37	47302979	60	2779
15	550013	38	39908316	61	1249
16	1080422	39	32869931	62	509
17	1993865	40	26443898	63	184
18	3469819	41	20792175	64	55
19	5707944	42	15984309	65	11
20	8909624	43	12020498	66	1
21	13234277	44	8844848		
22	18766663	45	6370240		

Table XLIV. Numbers of posets on 12 points with x minimal and y maximal points (symmetric upper half omitted).

$x \backslash y$	1	2	3	4	5	6	7	8	9	10	11	12
1	2567284											
2	11988791	59171127										
3	17501138	91355719	149736952									
4	10872251	59594908	104246102	79146756								
5	3259310	18541183	34850173	29246211	11991803							
6	511239	2987994	6041859	5552773	2443800	510498						
7	46358	276164	589070	562409	246756	53123	9811					
8	2885	17170	35837	31690	12658	4607	1472	392				
9	160	868	1532	966	581	329	170	78	29			
10	10	44	37	31	25	20	15	11	7	4		
11	1	1	1	1	1	1	1	1	1	1	1	
12	0	0	0	0	0	0	0	0	0	0	0	1

Table XLV. Numbers of posets on 13 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	18079670883
2	9528171074
3	4174056737
4	1533639687
5	432497911
6	68658720
7	6466574
8	622505
9	41761
10	1555
11	43
12	1
13	1

Table XLVI. Numbers of posets on 13 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	5067145
3	2433814790
4	15262584657
5	12418379471
6	3235850114
7	429543249
8	36303228
9	2183161
10	98246
11	3311
12	78
13	1

Table XLVII. Numbers of posets on 13 points grouped with respect to the number of nontrivial relations.

Relations	Posets	Relations	Posets	Relations	Posets
0	1	27	651206672	54	72028601
1	1	28	840404152	55	49849120
2	3	29	1048785819	56	33906587
3	7	30	1267416540	57	22666616
4	19	31	1484925018	58	14891283
5	47	32	1688672630	59	9613263
6	133	33	1865878896	60	6096379
7	354	34	2005172954	61	3796733
8	1014	35	2097659160	62	2320757
9	2874	36	2138021170	63	1391478
10	8305	37	2124818344	64	817624
11	23513	38	2060635454	65	470396
12	65215	39	1951423800	66	264558
13	173481	40	1805816407	67	145258
14	441249	41	1633935577	68	77647
15	1062532	42	1446444433	69	40260
16	2419194	43	1253457366	70	20165
17	5194267	44	1063880995	71	9660
18	10529510	45	884825225	72	4391
19	20169973	46	721452090	73	1862
20	36606102	47	576924933	74	714
21	63090851	48	452654555	75	241
22	103573457	49	348576046	76	66
23	162384152	50	263545083	77	12
24	243809985	51	195684307	78	1
25	351390204	52	142728742		
26	487237576	53	102283393		

Table XLVIII. Numbers of posets on 13 points with x minimal and y maximal points (symmetric upper half omitted).

$x \setminus y$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	46749427												
2	243933969	1329542918											
3	404358420	2312717460	4234403325										
4	289254098	1730450968	3344379685	2838730365									
5	100592501	623089510	1278525949	1188998944	556049771								
6	18106243	114748355	251883423	258911191	134193604	34606532							
7	1785349	11506533	27062777	30261059	16566570	4225741	510498						
8	106801	699238	1723805	1965720	1023681	246756	53123	9811					
9	4721	30824	73235	73837	31690	12658	4607	1472	392				
10	205	1210	2338	1532	966	581	329	170	78	29			
11	11	51	44	37	31	25	20	15	11	7	4		
12	1	1	1	1	1	1	1	1	1	1	1	1	
13	0	0	0	0	0	0	0	0	0	0	0	0	1

Table XLIX. Numbers of posets on 14 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	699530381610
2	377985520266
3	172661066675
4	63789361847
5	19769331324
6	4008790354
7	415426688
8	30902911
9	2280740
10	94958
11	2346
12	50
13	1
14	1

Table L. Numbers of posets on 14 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	67997749
3	66782709433
4	566229792680
5	535118128301
6	148207017576
7	19994749859
8	1686752496
9	101215375
10	4625995
11	165626
12	4588
13	91
14	1

Table LI. Numbers of posets on 14 points grouped with respect to the number of nontrivial relations.

Relations	Posets	Relations	Posets	Relations	Posets
0	1	31	20075844431	62	1770675277
1	1	32	25908418088	63	1239039207
2	3	33	32478923331	64	854569812
3	7	34	39592931224	65	580970065
4	19	35	46979502004	66	389321463
5	47	36	54307727007	67	257162269
6	133	37	61211924728	68	167425100
7	355	38	67323642504	69	107425128
8	1020	39	72305442194	70	67918114
9	2921	40	75882802702	71	42302121
10	8632	41	77869195951	72	25947267
11	25486	42	78181684187	73	15667333
12	75366	43	76844939859	74	9307819
13	217990	44	73984081629	75	5436986
14	613701	45	69807866652	76	3120222
15	1659796	46	64585100397	77	1757459
16	4290662	47	58617924003	78	970342
17	10541968	48	52215061893	79	524242
18	24574810	49	45668403312	80	276596
19	54282974	50	39234427091	81	142020
20	113695015	51	33122134980	82	70684
21	226043606	52	27486947185	83	33907
22	427434085	53	22430638497	84	15527
23	770364708	54	18005406984	85	6724
24	1326574992	55	14221396017	86	2701
25	2187816296	56	11055548408	87	978
26	3463953205	57	8461181255	88	309
27	5276878418	58	6376703312	89	78
28	7750426625	59	4733381232	90	13
29	10995675430	60	3461298437	91	1
30	15093556839	61	2493875402		

Table LII. Numbers of posets on 14 points with x minimal and y maximal points (symmetric upper half omitted).

$x\setminus y$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1104891746													
2	6366721037	37963016310												
3	11855130462	73662811179	149465890442											
4	9683027437	62734061887	133247611504	125880710785										
5	3899073218	26140403808	58283416727	59464742336	31178543111									
6	816935110	5593236350	13182659058	14786975028	8709276857	2708547371								
7	91976607	635987406	1599205404	1984290362	1292927965	425418235	64217336							
8	5830593	40709212	109572564	147387203	99998975	31204532	4225741	510498						
9	233526	1658023	4668582	6313845	3896030	1023681	246756	53123	9811					
10	7446	52855	141798	162566	73837	31690	12658	4607	1472	392				
11	257	1642	3458	2338	1532	966	581	329	170	78	29			
12	12	59	51	44	37	31	25	20	15	11	7	4		
13	1	1	1	1	1	1	1	1	1	1	1	1	1	
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table LIII. Numbers of posets on 15 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	34893120378937
2	19204949481269
3	9252156316741
4	3496443516959
5	1115013740650
6	273523272107
7	37096135865
8	2601515187
9	165063601
10	8268404
11	207920
12	3457
13	57
14	1
15	1

Table LIV. Numbers of posets on 15 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	1224275497
3	2365611469711
4	26848965254249
5	29244031053807
6	8551130657200
7	1161662376301
8	96523871906
9	5664856954
10	254650419
11	9161348
12	267456
13	6201
14	105
15	1

Table LV. Numbers of posets on 15 points grouped with respect to the number of nontrivial relations.

Relations	Posets	Relations	Posets	Relations	Posets
0	1	36	968292512943	72	32718529348
1	1	37	1224860865896	73	22689363476
2	3	38	1511663110403	74	15543903659
3	7	39	1821590242855	75	10520260606
4	19	40	2144804502345	76	7034463466
5	47	41	2469204340113	77	4647004975
6	133	42	2781195285381	78	3032775681
7	355	43	3066670032486	79	1955278120
8	1022	44	3312121994235	80	1245196819
9	2938	45	3505735430841	81	783203830
10	8759	46	3638355413459	82	486455510
11	26352	47	3704194540655	83	298296646
12	80667	48	3701232395160	84	180536275
13	246029	49	3631243264510	85	107805875
14	743726	50	3499498739062	86	63487758
15	2196960	51	3314170353212	87	36853838
16	6293046	52	3085547401437	88	21073614
17	17331079	53	2825141576761	89	11861055
18	45677285	54	2544803900420	90	6564648
19	114772936	55	2255915243141	91	3568480
20	274516141	56	1968736493888	92	1902199
21	624577059	57	1691935910960	93	992262
22	1352399184	58	1432323100830	94	505213
23	2789759597	59	1194762477976	95	250105
24	5491144190	60	982251643492	96	119854
25	10332084297	61	796114254015	97	55229
26	18622307970	62	636275137247	98	24242
27	32218765020	63	501569106336	99	10025
28	53620482381	64	390055996001	100	3827
29	86015906434	65	299310153475	101	1312
30	133256946201	66	226671448238	102	389
31	199726182784	67	169445230529	103	91
32	290081890065	68	125051716992	104	14
33	408865932583	69	91125892656	105	1
34	559997138829	70	65575816940		
35	746178128487	71	46606485237		

Table LVI. Numbers of posets on 15 points with x minimal and y maximal points (symmetric upper half omitted).

$x \backslash y$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	33823827452														
2	213138659769	1379395251754													
3	441411111230	2958162317549	6599173835521												
4	407935285336	2842002522615	6598278270417	6910409836249											
5	189072354434	1363910599153	3295053573741	3668501150154	2131419142311										
6	46255308910	341232392927	860699630256	1039056210280	676615932644	243878724103									
7	6098509077	45284694670	120298824744	160202474475	117793069283	47383620103	9694440854								
8	439497398	3256278783	9217469242	13612010889	11073974378	4668660480	884807585	64217336							
9	18105035	134888703	409543545	655880725	549912393	211010642	31204532	4225741	510498						
10	489411	3726292	11862922	18917065	13848587	3896030	1023681	246756	53123	9811					
11	11388	87200	262459	341513	162566	73837	31690	12658	4607	1472	392				
12	317	2177	4986	3458	2338	1532	966	581	329	170	78	29			
13	13	67	59	51	44	37	31	25	20	15	11	7	4		
14	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table LVII. Numbers of posets on 16 points grouped with respect to the number of points on the highest level.

Points on highest level	Posets
1	2241667678990734
2	1248591486850771
3	634172561373830
4	251723045196005
5	80707015937796
6	21992038897136
7	3914575788228
8	343227318249
9	18053106884
10	952204516
11	29086987
12	438907
13	4977
14	65
15	1
16	1

Table LVIII. Numbers of posets on 16 points grouped with respect to the number of levels.

Levels	Posets
1	1
2	29733449509
3	108202550561783
4	1629158711278940
5	2030394404859309
6	623385524584982
7	84687296694936
8	6866110095603
9	388876221560
10	16850129605
11	589739135
12	17154919
13	416480
14	8204
15	120
16	1

Table LIX. Numbers of posets on 16 points grouped with respect to the number of nontrivial relations.

Relations	Posets	Relations	Posets	Relations	Posets
0	1	41	55836016178502	82	920803479394
1	1	42	69912711566335	83	638663233321
2	3	43	85687690625967	84	438383565830
3	7	44	102863454246219	85	297806418975
4	19	45	121011175452828	86	200228176126
5	47	46	139584511784057	87	133240295007
6	133	47	157945973168869	88	87753807189
7	355	48	175404254305947	89	57201841370
8	1023	49	191259320332441	90	36902250854
9	2944	50	204850995061806	91	23559777068
10	8806	51	215606091115193	92	14884395519
11	26694	52	223079280730975	93	9304399446
12	82995	53	226983540149572	94	5754198387
13	260378	54	227207348359286	95	3520045581
14	822366	55	223817358905730	96	2129555791
15	2580062	56	217046992906415	97	1273789835
16	7975769	57	207272866974585	98	753082629
17	24043721	58	194982128680769	99	439910631
18	70185724	59	180734419028653	100	253790516
19	197154614	60	165122329113286	101	144526273
20	530859423	61	148733927337996	102	81191237
21	1366423376	62	132120240174366	103	44960494
22	3357919175	63	115769717435331	104	24519498
23	7875025845	64	100090696305987	105	13153589
24	17631961542	65	85402009661905	106	6931176
25	37722769048	66	71931057198939	107	3580785
26	77220026691	67	59818161377709	108	1809403
27	151480418620	68	49125643296315	109	891588
28	285258548460	69	39850014233762	110	426658
29	516616044791	70	31935695429053	111	197308
30	901468446817	71	25288952861798	112	87538
31	1518385135369	72	19790967333965	113	36906
32	2473045797606	73	15309336855707	114	14604
33	3901491671598	74	11707566302920	115	5312
34	5971161534217	75	8852415906032	116	1728
35	8878571583162	76	6619148512049	117	482
36	12842633026541	77	4894902114503	118	105
37	18092835277333	78	3580468527005	119	15
38	24852135710100	79	2590825272279	120	1
39	33315075467438	80	1854739482749		
40	43622621481836	81	1313754264463		

Table LX. Numbers of posets on 16 points with x minimal and y maximal points (symmetric upper half omitted).

$x \setminus y$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1338193159771															
2	9146521421660	63827832364661														
3	20882716706672	150031400188209	365205996831511													
4	21640672903228	161115708920775	406569659513010	470098000227809												
5	11454003722027	88373060271526	230738148047255	279586261548307	178772616534766											
6	3260005461782	25820662747738	69697079557248	90023141662542	63691426906292	25804518119613										
7	507672701692	4055581632806	11368030861207	15977859262185	12773765357901	5926433367339	1527479040250									
8	43221401853	342165092735	1006867109447	1566627161192	1422328779334	738854762915	199503215876	23397419986								
9	2015339894	15744151300	49366535022	85563616233	86237937130	46672289943	11264027805	884807585	64217336							
10	54075336	424363585	1434411081	2708549807	2798390306	1329607443	211010642	31204532	4225741	510498						
11	989872	8007478	28564679	53434156	46520033	13848587	3896030	1023681	246756	53123	9811					
12	16969	139241	467444	689643	341513	162566	73837	31690	12658	4607	1472	392				
13	385	2831	7026	4986	3458	2338	1532	966	581	329	170	78	29			
14	14	76	67	59	51	44	37	31	25	20	15	11	7	4		
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

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