```
1  ---------- MODULE RelationUtils ----------
   Relation related operators.
5  LOCAL INSTANCE Naturals
6  LOCAL INSTANCE FiniteSets
7  LOCAL INSTANCE Sequences
8  LOCAL INSTANCE SequencesExt
9  LOCAL INSTANCE Functions
10 ----------
   Basic definitions.
14 Dom(R) ≜ {a : ⟨a, b⟩ ∈ R}  Domain of R
15 Ran(R) ≜ {b : ⟨a, b⟩ ∈ R}  Range of R
16 Support(R) ≜ Dom(R) ∪ Ran(R)  Support of R
17 ----------
   Basic operations.
21 Image(R, a) ≜ {b ∈ Ran(R) : ⟨a, b⟩ ∈ R}
22 LeftRestriction(R, a) ≜ {⟨a, b⟩ : b ∈ Image(R, a)}

24 InverseRelation(R) ≜ {⟨b, a⟩ : ⟨a, b⟩ ∈ R}
25 InverseImage(R, b) ≜ {a ∈ Dom(R) : ⟨a, b⟩ ∈ R}

27 R | S ≜ R ∩ (S × S)  Restriction of R on S

29 R ∗∗ T ≜  Composition of R and T
30     LET SR ≜ Support(R)
31         ST ≜ Support(T)
32     IN  {⟨r, t⟩ ∈ SR × ST : ∃ s ∈ SR ∩ ST : (⟨r, s⟩ ∈ R) ∧ (⟨s, t⟩ ∈ T)}

34 GT(R, a) ≜ {b ∈ Ran(R) : ⟨a, b⟩ ∈ R}  ≜ Image(R, a)
35 LT(R, b) ≜ {a ∈ Dom(R) : ⟨a, b⟩ ∈ R}  ≜ InverseImage(R, b)
   The following definition is from https://github.com/jameshfisher/tlaplus/blob/master/examples/TransitiveClosure/TransitiveClos
   It also contains several other methods for computing TC.
41 TC(R) ≜  Transitive closure of R
42     LET S ≜ Support(R)
43         RECURSIVE TCR(_)
44         TCR(T) ≜ IF T = {}
45                  THEN R
46                  ELSE LET r ≜ CHOOSE s ∈ T : TRUE
47                           RR ≜ TCR(T \ {r})
48                       IN  RR ∪ {⟨s, t⟩ ∈ S × S :
49                                ⟨s, r⟩ ∈ RR ∧ ⟨r, t⟩ ∈ RR}
50     IN  TCR(S)
   Example: SeqToRel(⟨1, 2, 3⟩) = {⟨1, 2⟩, ⟨1, 3⟩, ⟨2, 3⟩}
54 RECURSIVE Seq2Rel(_)
55 Seq2Rel(s) ≜  Transform a sequence s into a strict total order relation
56     IF s = ⟨⟩ THEN {}
```

```
57      ELSE  LET h  ≜  Head(s)
58                t  ≜  Tail(s)
59            IN    {⟨h, r⟩ : r ∈ Range(t)} ∪ Seq2Rel(t)
60 ----------------------------------------------------------------------------
```

Basic properties.

```
64  IsReflexive(R, S) ≜ ∀ a ∈ S : ⟨a, a⟩ ∈ R
65  IsIrreflexive(R, S) ≜ ∀ a ∈ S : ⟨a, a⟩ ∉ R

67  IsSymmetric(R, S) ≜ ∀ a, b ∈ S : ⟨a, b⟩ ∈ R ≡ ⟨b, a⟩ ∈ R
68  IsAntisymmetric(R, S) ≜ ∀ a, b ∈ S : ⟨a, b⟩ ∈ R ∧ ⟨b, a⟩ ∈ R ⇒ a = b

70  IsTransitive(R, S) ≜
71      ∀ a, b, c ∈ S : (⟨a, b⟩ ∈ R ∧ ⟨b, c⟩ ∈ R) ⇒ ⟨a, c⟩ ∈ R

73   https://en.wikipedia.org/wiki/Connected_relation
74  IsTotal(R, S) ≜
75      ∀ a, b ∈ S : ⟨a, b⟩ ∈ R ∨ ⟨b, a⟩ ∈ R

77   https://en.wikipedia.org/wiki/Connected_relation
78  IsSemiconnex(R, S) ≜
79      ∀ a, b ∈ S : a ≠ b ⇒ (⟨a, b⟩ ∈ R ∨ ⟨b, a⟩ ∈ R)

81   partial order: https://en.wikipedia.org/wiki/Partially_ordered_set ≠ Formal_definition
82  IsPartialOrder(R, S) ≜
83      ∧ IsReflexive(R, S)
84      ∧ IsAntisymmetric(R, S)
85      ∧ IsTransitive(R, S)

87   total order: https://en.wikipedia.org/wiki/ Total_order
88  IsTotalOrder(R, S) ≜
89      ∧ IsPartialOrder(R, S)
90      ∧ IsTotal(R, S)  Atually, IsTotal(R, S) ⇒ IsReflexive(R, S)

92   strict partial order: https://en.wikipedia.org/wiki/Partially_ordered_set ≠ Strict_and_non-strict_partial_orders
93  IsStrictPartialOrder(R, S) ≜
94      ∧ IsIrreflexive(R, S)
95      ∧ IsTransitive(R, S)

97   strict total order: https://en.wikipedia.org/wiki/ Total_order ≠ Strict_total_order
98  IsStrictTotalOrder(R, S) ≜
99      ∧ IsIrreflexive(R, S)
100     ∧ IsTransitive(R, S)
101     ∧ IsSemiconnex(R, S)

103 Respect(R, T) ≜ T ⊆ R  Does R respect T?
104 ----------------------------------------------------------------------------
```

Special elements in a relation

```
Minimal(R, S) ≜ the set of minimal elements in relation R on the set S
    {m ∈ S : ¬∃ a ∈ Dom(R) : ⟨a, m⟩ ∈ R}
Maximal(R, S) ≜ the set of maximal elements in relation R on the set S
    {m ∈ S : ¬∃ b ∈ Ran(R) : ⟨m, b⟩ ∈ R}
```

A variant of *Kahn*'s algorithm for topological sorting

See https://*en.wikipedia.org*/wiki/*Topological_sorting* ≠ *Kahn*'s_algorithm

```
Cyclic(R) ≜ Is R cyclic?
    LET RECURSIVE CyclicUtil(_, _)
        CyclicUtil(rel, set) ≜ remaining relation; set: remaining set
            IF set = {} THEN FALSE
             ELSE LET mins ≜ Minimal(rel, set)
                  IN IF mins = {} THEN TRUE
                     ELSE LET m ≜ CHOOSE x ∈ mins : TRUE
                          IN CyclicUtil(rel \ LeftRestriction(R, m), set \ {m})
    IN  CyclicUtil(R, Support(R))
```

Kahn's algorithm for topological sorting.

See https://*en.wikipedia.org*/wiki/*Topological_sorting* ≠ *Kahn*'s_algorithm

```
AnyLinearExtension(R, S) ≜ return an arbitrary linear extension of R on the set S
    LET RECURSIVE LinearExtensionUtil(_, _)
        LinearExtensionUtil(rel, set) ≜ rel: remaining relation; set: remaining set
            IF set = {} THEN ⟨⟩
             ELSE LET m ≜ CHOOSE x ∈ Minimal(rel, set) : TRUE
                  IN ⟨m⟩ ∘ LinearExtensionUtil(rel \ LeftRestriction(R, m), set \ {m})
    IN  LinearExtensionUtil(R, S)
```

A variant of *Kahn*'s algorithm for topological sorting

See https://*en.wikipedia.org*/wiki/*Topological_sorting* ≠ *Kahn*'s_algorithm

For some TLA+ issue, see https://*groups.google.com*/g/tlaplus/*c*/*mtyEmqhlRVg*

```
AllLinearExtensions(R, S) ≜ return all possible linear extensions of R on the set S
    LET RECURSIVE LinearExtensionsUtil(_, _)
        LinearExtensionsUtil(rel, set) ≜
            IF set = {} THEN {⟨⟩}
             ELSE LET Extend(m) ≜ {⟨m⟩ ∘ l : extend recursively by the minimal element m
                                  l ∈ LinearExtensionsUtil(rel \ LeftRestriction(R, m), set \ {m})}
                  IN UNION {Extend(m) : m ∈ Minimal(rel, set)} for each minimal element
    IN  LinearExtensionsUtil(R, S)

LinearExtensions(R, S) ≜ return the set of all possible linear extensions of R on the set S
    {l ∈ TupleOf(S, Cardinality(S)) : Respect(Seq2Rel(l), R)}
```

Test cases

```
rel0 ≜ {}

set1 ≜ {2, 3, 5, 7, 8, 9, 10, 11}
rel1 ≜  from https://en.wikipedia.org/wiki/Topological_sorting
    {⟨3, 8⟩, ⟨3, 10⟩, ⟨5, 11⟩, ⟨7, 8⟩, ⟨7, 11⟩,
     ⟨8, 9⟩, ⟨11, 2⟩, ⟨11, 9⟩, ⟨11, 10⟩}

set2 ≜ 0 .. 5
rel2 ≜  from https://www.geeksforgeeks.org/topological-sorting/
    {⟨2, 3⟩, ⟨3, 1⟩, ⟨4, 0⟩, ⟨4, 1⟩, ⟨5, 0⟩, ⟨5, 2⟩}

set3 ≜ 1 .. 6
rel3 ≜  from https://leetcode.com/discuss/general-discussion/1078072/introduction-to-topological-sort
    {⟨1, 2⟩, ⟨1, 4⟩, ⟨2, 3⟩, ⟨4, 2⟩, ⟨4, 5⟩, ⟨4, 6⟩, ⟨5, 6⟩}

set4 ≜ {1}
rel4 ≜ {⟨1, 1⟩}

set5 ≜ {1, 2}
rel5 ≜ {⟨1, 2⟩, ⟨2, 1⟩}

set6 ≜ 1 .. 4
rel6 ≜ {⟨1, 2⟩, ⟨2, 3⟩, ⟨3, 4⟩, ⟨4, 1⟩}

set7 ≜ 1 .. 4
rel7 ≜ {⟨1, 2⟩, ⟨2, 3⟩, ⟨3, 4⟩}

set8 ≜ 1 .. 3
rel8 ≜ {⟨1, 2⟩, ⟨1, 3⟩, ⟨2, 3⟩}

all ≜ {rel0, rel1, rel2, rel3, rel4, rel5, rel6, rel7}
----------
LETest ≜  test of linear extensions
    ∧ AllLinearExtensions(rel1, set1) = LinearExtensions(rel1, set1)
    ∧ AllLinearExtensions(rel2, set2) = LinearExtensions(rel2, set2)
    ∧ AllLinearExtensions(rel3, set3) = LinearExtensions(rel3, set3)
----------
CyclicTest ≜  test of Cyclic(R)
    LET cyclic ≜ {rel4, rel5, rel6}
    IN  ∧ ∀ c ∈ cyclic : Cyclic(c)
        ∧ ∀ c ∈ all \ cyclic : ¬Cyclic(c)
----------
IsStrictTotalOrderTest ≜
    ∧ ¬IsStrictTotalOrder(rel7, set7)
    ∧ IsStrictTotalOrder(rel8, set8)
----------
VARIABLES x
==========
```

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