MODULE *RelationUtils*

Relation related operators.

LOCAL INSTANCE *Naturals*
LOCAL INSTANCE *FiniteSets*
LOCAL INSTANCE *Sequences*
LOCAL INSTANCE *SequencesExt*
LOCAL INSTANCE *Functions*

Basic definitions.

$Dom(R) \triangleq \{a : \langle a, b\rangle \in R\}$ Domain of R
$Ran(R) \triangleq \{b : \langle a, b\rangle \in R\}$ Range of R
$Support(R) \triangleq Dom(R) \cup Ran(R)$ Support of R

Basic operations.

$Image(R, a) \triangleq \{b \in Ran(R) : \langle a, b\rangle \in R\}$
$LeftRestriction(R, a) \triangleq \{\langle a, b\rangle : b \in Image(R, a)\}$

$InverseRelation(R) \triangleq \{\langle b, a\rangle : \langle a, b\rangle \in R\}$
$InverseImage(R, b) \triangleq \{a \in Dom(R) : \langle a, b\rangle \in R\}$

$R \mid S \triangleq R \cap (S \times S)$ Restriction of R on S

$R ** T \triangleq$ Composition of R and T
  LET $SR \triangleq Support(R)$
    $ST \triangleq Support(T)$
  IN $\{\langle r, t\rangle \in SR \times ST : \exists s \in SR \cap ST : (\langle r, s\rangle \in R) \land (\langle s, t\rangle \in T)\}$

$GT(R, a) \triangleq \{b \in Ran(R) : \langle a, b\rangle \in R\}$ $\triangleq Image(R, a)$
$LT(R, b) \triangleq \{a \in Dom(R) : \langle a, b\rangle \in R\}$ $\triangleq InverseImage(R, b)$

The following definition is from https://*github.com*/jameshfisher/tlaplus/blob/master/examples/*TransitiveClosure*/*TransitiveClos*
It also contains several other methods for computing TC.

$TC(R) \triangleq$ Transitive closure of R
  LET $S \triangleq Support(R)$
    RECURSIVE $TCR(\_)$
    $TCR(T) \triangleq$ IF $T = \{\}$
        THEN R
        ELSE LET $r \triangleq$ CHOOSE $s \in T$: TRUE
            $RR \triangleq TCR(T \setminus \{r\})$
          IN $RR \cup \{\langle s, t\rangle \in S \times S :$
              $\langle s, r\rangle \in RR \land \langle r, t\rangle \in RR\}$
  IN $TCR(S)$

Example: $SeqToRel(\langle 1, 2, 3\rangle) = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$

RECURSIVE $Seq2Rel(\_)$
$Seq2Rel(s) \triangleq$ Transform a sequence s into a strict total order relation
  IF $s = \langle\rangle$ THEN $\{\}$

$\quad\quad \text{ELSE LET } h \triangleq Head(s)$
$\quad\quad\quad\quad\quad\quad t \triangleq Tail(s)$
$\quad\quad\quad\quad \text{IN } \{\langle h, r\rangle : r \in Range(t)\} \cup Seq2Rel(t)$

Basic properties.

$IsReflexive(R, S) \triangleq \forall a \in S : \langle a, a\rangle \in R$
$IsIrreflexive(R, S) \triangleq \forall a \in S : \langle a, a\rangle \notin R$

$IsSymmetric(R, S) \triangleq \forall a, b \in S : \langle a, b\rangle \in R \equiv \langle b, a\rangle \in R$
$IsAntisymmetric(R, S) \triangleq \forall a, b \in S : \langle a, b\rangle \in R \land \langle b, a\rangle \in R \Rightarrow a = b$

$IsTransitive(R, S) \triangleq$
$\quad \forall a, b, c \in S : (\langle a, b\rangle \in R \land \langle b, c\rangle \in R) \Rightarrow \langle a, c\rangle \in R$

$IsTotal(R, S) \triangleq$
$\quad \forall a, b \in S : \langle a, b\rangle \in R \lor \langle b, a\rangle \in R$

$IsPartialOrder(R, S) \triangleq$
$\quad \land IsReflexive(R, S)$
$\quad \land IsAntisymmetric(R, S)$
$\quad \land IsTransitive(R, S)$

$IsTotalOrder(R, S) \triangleq$
$\quad \land IsPartialOrder(R, S)$
$\quad \land IsTotal(R, S)$

$IsStrictPartialOrder(R, S) \triangleq$
$\quad \land IsIrreflexive(R, S)$
$\quad \land IsTransitive(R, S)$

$IsStrictTotalOrder(R, S) \triangleq$
$\quad \land IsStrictPartialOrder(R, S)$
$\quad \land IsTotal(R, S)$

$Respect(R, T) \triangleq T \subseteq R$ Does R respect T?

Special elements in a relation

$Minimal(R, S) \triangleq$ the set of minimal elements in relation R on the set S
$\quad \{m \in S : \neg\exists a \in Dom(R) : \langle a, m\rangle \in R\}$
$Maximal(R, S) \triangleq$ the set of maximal elements in relation R on the set S
$\quad \{m \in S : \neg\exists b \in Ran(R) : \langle m, b\rangle \in R\}$

A variant of *Kahn*'s algorithm for topological sorting

See https://*en.wikipedia.org*/wiki/*Topological_sorting* $\neq$ *Kahn*'s_algorithm

$Cyclic(R) \triangleq$ Is R cyclic?
$\quad \text{LET RECURSIVE } CyclicUtil(\_, \_)$
$\quad\quad\quad CyclicUtil(rel, set) \triangleq$ remaining relation; set: remaining set

```
                IF set = {} THEN FALSE
                 ELSE LET mins ≜ Minimal(rel, set)
                      IN  IF mins = {} THEN TRUE
                           ELSE LET m ≜ CHOOSE x ∈ mins : TRUE
                                IN   CyclicUtil(rel \ LeftRestriction(R, m), set \ {m})
    IN     CyclicUtil(R, Support(R))
```

Kahn's algorithm for topological sorting.

See https://*en.wikipedia.org*/wiki/*Topological_sorting* ≠ *Kahn*'s_algorithm

```
AnyLinearExtension(R, S) ≜   return an arbitrary linear extension of R on the set S
    LET RECURSIVE LinearExtensionUtil(_, _)
         LinearExtensionUtil(rel, set) ≜   rel: remaining relation; set: remaining set
            IF set = {} THEN ⟨⟩
             ELSE LET m ≜ CHOOSE x ∈ Minimal(rel, set) : TRUE
                  IN  ⟨m⟩ ∘ LinearExtensionUtil(rel \ LeftRestriction(R, m), set \ {m})
    IN     LinearExtensionUtil(R, S)
```

A variant of *Kahn*'s algorithm for topological sorting

See https://*en.wikipedia.org*/wiki/*Topological_sorting* ≠ *Kahn*'s_algorithm

For some TLA+ issue, see https://*groups.google.com*/g/tlaplus/*c*/*mtyEmqhlRVg*

```
AllLinearExtensions(R, S) ≜   return all possible linear extensions of R on the set S
    LET RECURSIVE LinearExtensionsUtil(_, _)
         LinearExtensionsUtil(rel, set) ≜
            IF set = {} THEN {⟨⟩}
             ELSE LET Extend(m) ≜ {⟨m⟩ ∘ l :   extend recursively by the minimal element m
                              l ∈ LinearExtensionsUtil(rel \ LeftRestriction(R, m), set \ {m})}
                  IN  UNION {Extend(m) : m ∈ Minimal(rel, set)}   for each minimal element
    IN     LinearExtensionsUtil(R, S)

LinearExtensions(R, S) ≜   return the set of all possible linear extensions of R on the set S
    {l ∈ TupleOf(S, Cardinality(S)) : Respect(Seq2Rel(l), R)}
```

Test cases

```
rel0 ≜ {}

set1 ≜ {2, 3, 5, 7, 8, 9, 10, 11}
rel1 ≜   from https://en.wikipedia.org/wiki/Topological_sorting
    {⟨3, 8⟩, ⟨3, 10⟩, ⟨5, 11⟩, ⟨7, 8⟩, ⟨7, 11⟩,
     ⟨8, 9⟩, ⟨11, 2⟩, ⟨11, 9⟩, ⟨11, 10⟩}

set2 ≜ 0 . . 5
rel2 ≜   from https://www.geeksforgeeks.org/topological-sorting/
    {⟨2, 3⟩, ⟨3, 1⟩, ⟨4, 0⟩, ⟨4, 1⟩, ⟨5, 0⟩, ⟨5, 2⟩}

set3 ≜ 1 . . 6
```

$rel3 \triangleq$ from https://*leetcode.com*/discuss/general-discussion/1078072/introduction-to-topological-sort

$\quad \{\langle 1, 2\rangle, \langle 1, 4\rangle, \langle 2, 3\rangle, \langle 4, 2\rangle, \langle 4, 5\rangle, \langle 4, 6\rangle, \langle 5, 6\rangle\}$

$rel4 \triangleq \{\langle 1, 1\rangle\}$

$rel5 \triangleq \{\langle 1, 2\rangle, \langle 2, 1\rangle\}$

$rel6 \triangleq \{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 4\rangle, \langle 4, 1\rangle\}$

$rel7 \triangleq \{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 4\rangle\}$

$all \triangleq \{rel0, rel1, rel2, rel3, rel4, rel5, rel6, rel7\}$

THEOREM $LETest \triangleq$ test of linear extensions

$\quad \land AllLinearExtensions(rel1, set1) = LinearExtensions(rel1, set1)$

$\quad \land AllLinearExtensions(rel2, set2) = LinearExtensions(rel2, set2)$

$\quad \land AllLinearExtensions(rel3, set3) = LinearExtensions(rel3, set3)$

THEOREM $CyclicTest \triangleq$ test of $Cyclic(R)$

$\quad$ LET $\quad cyclic \triangleq \{rel4, rel5, rel6\}$

$\quad$ IN $\quad \land \forall c \in cyclic : Cyclic(c)$

$\quad\quad\quad \land \forall c \in all \setminus cyclic : \neg Cyclic(c)$

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* Modification *History*

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