第八章　有限元离散方法
（二维问题）

8.2 二维边值问题有限元方法

1）讨论对象：椭圆型偏微分方程边值问题

$$\begin{cases} -\Delta u = f, & (x,y)\in\Omega \\ \dfrac{\partial u}{\partial n} + \sigma u = g, & (x,y)\in\Gamma\subset\partial\Omega \end{cases}$$

2）转化为变分问题

$$V = H^1(\Omega) = \left\{ v \,\middle|\, \iint_\Omega \left[v^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right] dxdy < \infty \right\}$$

$$\iint_{\Omega} \nabla u \bullet \nabla v dxdy - \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds = \iint_{\Omega} f v dxdy$$

$$\frac{\partial u}{\partial n} = g - \sigma u$$

$$\begin{cases} find \quad u \in V \quad s.t. \\ D(u,v) - F(v) = 0, \ \forall v \in V \end{cases}$$

其中 $D(u,v) = \iint_{\Omega} \nabla u \bullet \nabla v dxdy + \int_{\partial\Omega} \sigma u v ds$

$$F(v) = \iint_{\Omega} f v dxdy + \int_{\partial\Omega} g v ds$$

3）有限元离散

找一个函数$u\in V$, 满足 $D(u,v)=F(v) \quad \forall v\in V$

$$V=H^1(\Omega)=\left\{v \mid \iint_\Omega \left[v^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]dxdy<\infty\right\}$$

$\Rightarrow$ 找一个函数$u_h\in V_h$，满足

$$D(u_h,v_h)=F(v_h) \quad \forall v_h\in V_h$$

V_h是V 的有限维子空间

下面通过找基函数来构造有限维子空间

$u \in V$ 定义区域离散

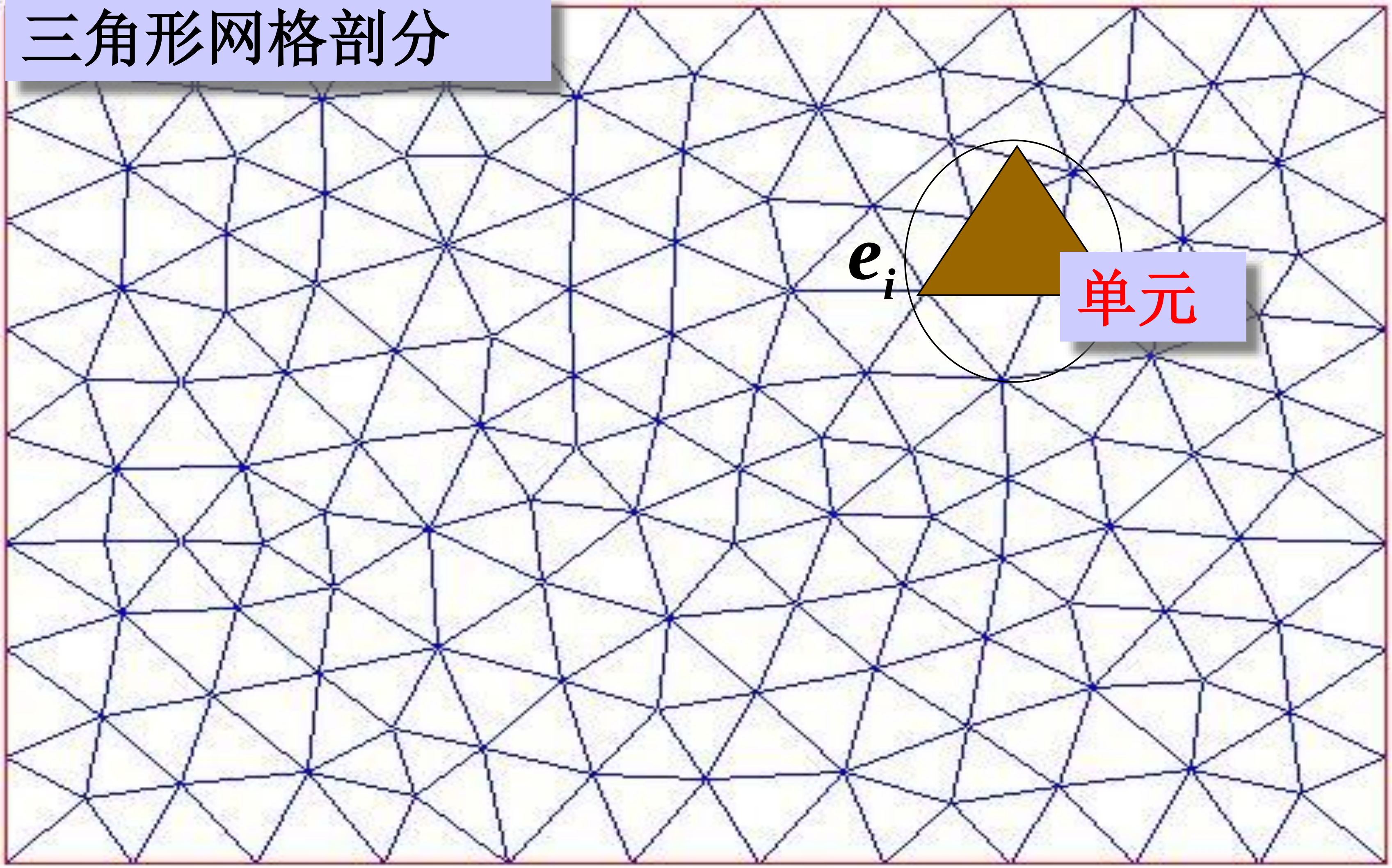
1）单元编号，节点编号，节点坐标

2）节点的局部和整体编码对应关系

3）边界点相关信息

$u \in V$ 定义区域离散 $\quad \Omega \approx \Omega_h = \bigcup e$

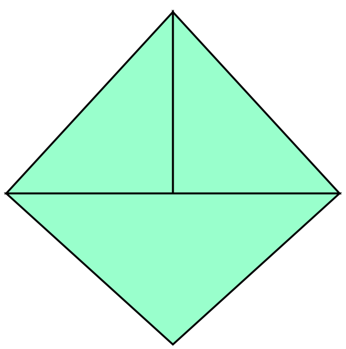
三角形网格剖分

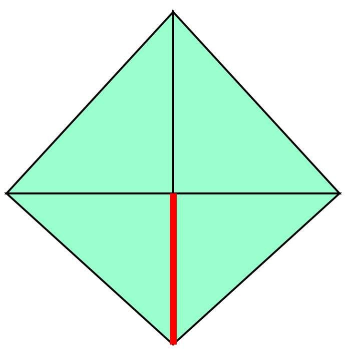


区域三角形网格剖分要求：

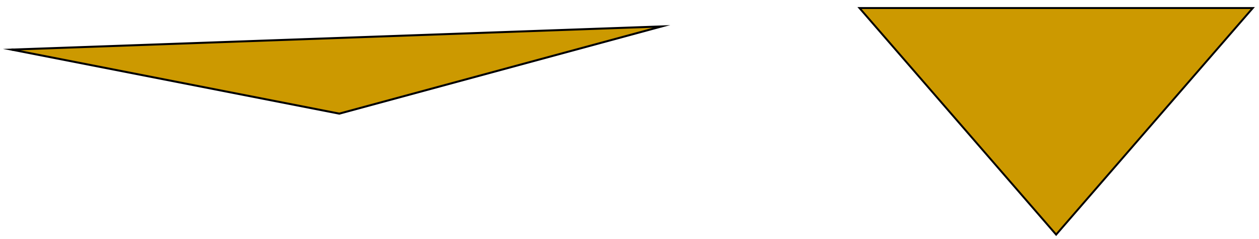
1 区域被分割成有限个互不重叠的三角形单元

2 任一个单元的顶点不能是其他三角形的内点或除顶点之外的边界点

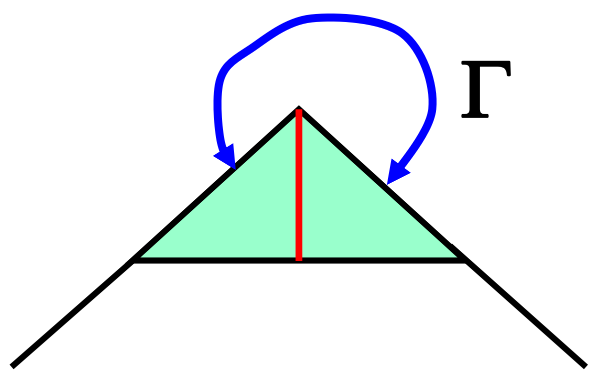




3 单元最小角有下界，尽量大些



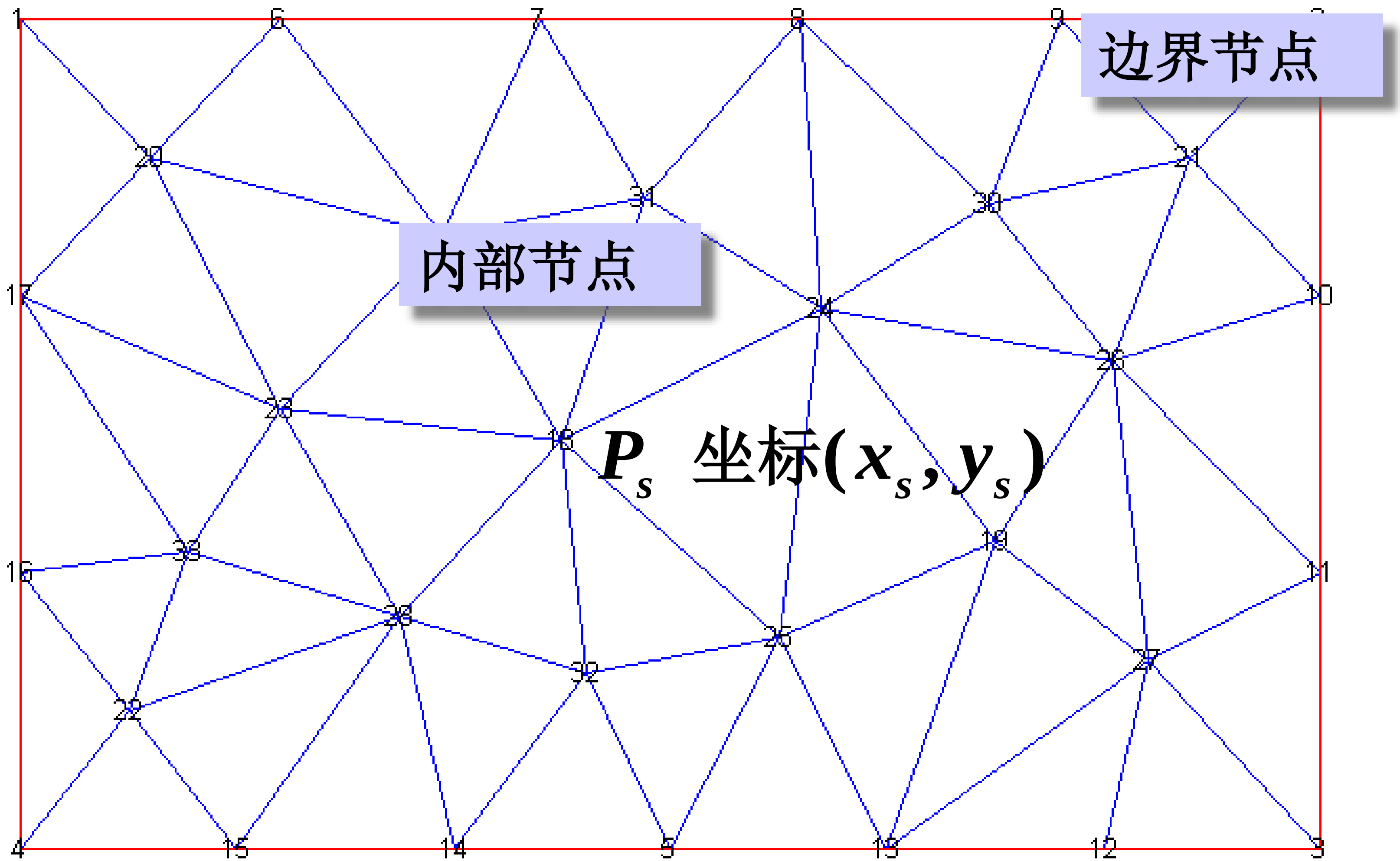
4 任一个单元至多有两个顶点在边界上



5 解变化剧烈处结点密些，变化平缓处结点疏些

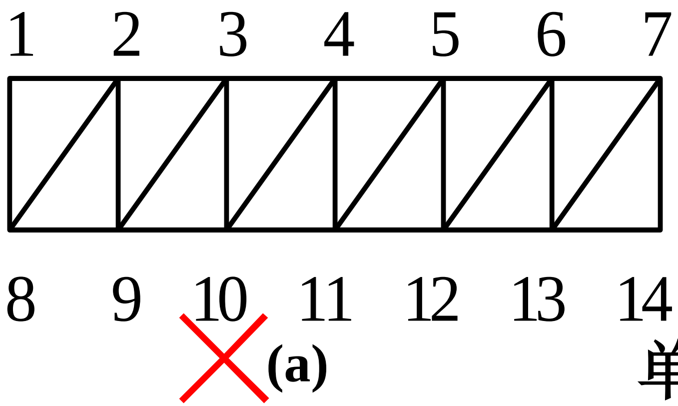
顶点(节点)编号

顶点总数N_p



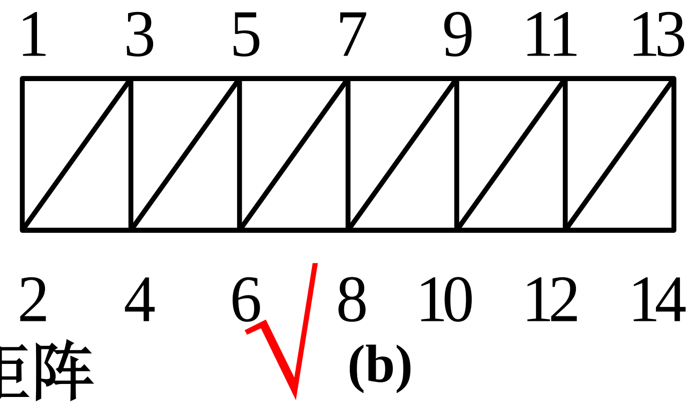
$$u_s = u(x_s, y_s), \quad s = 1 \cdots N_p$$

结点编号原则：（1）单元内3个编号差尽可能的小



(a)

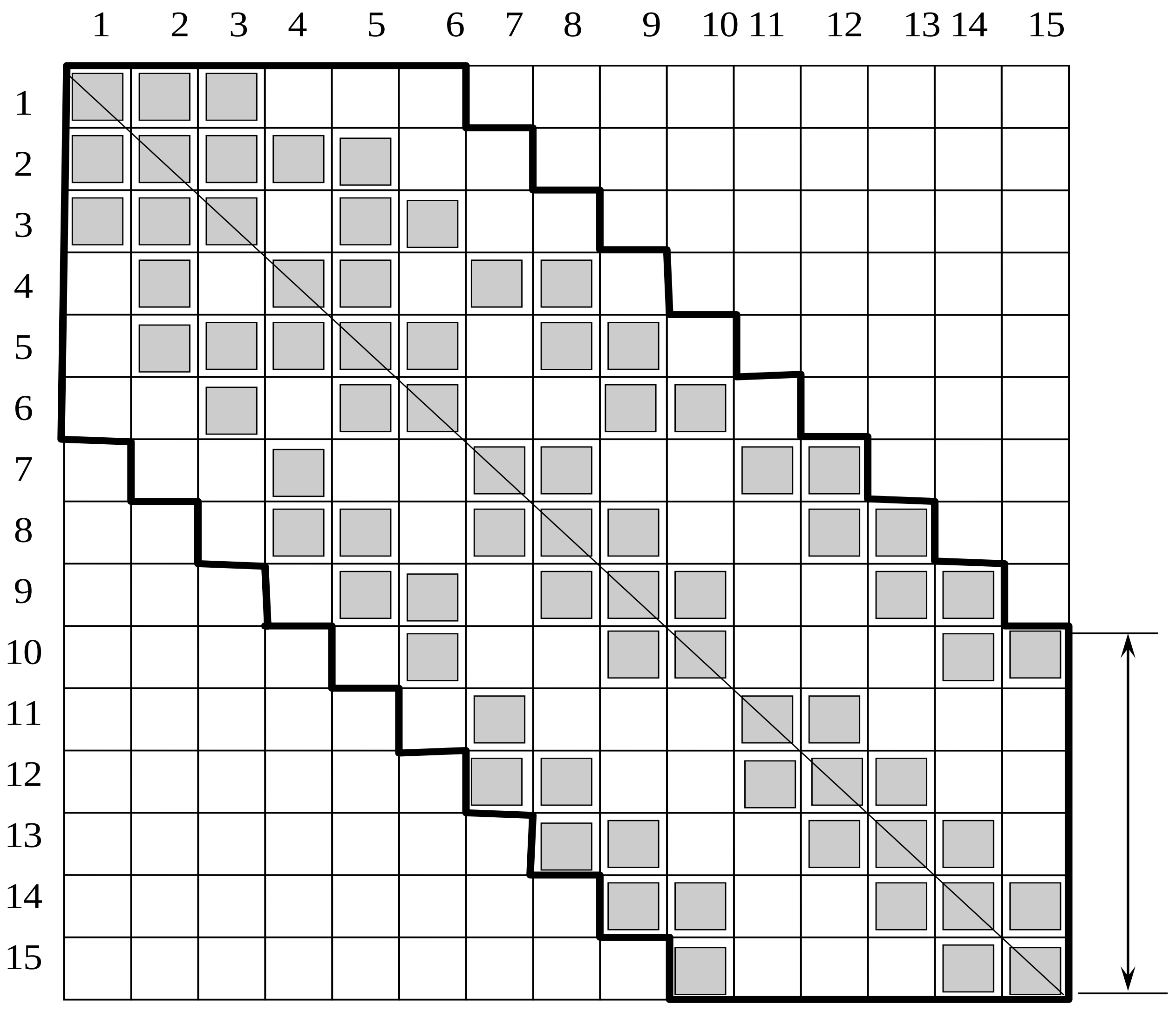
单元刚度矩阵



(b)

例如上例中，两图单元划分相同，且节点总数都等于**14**，但两者的节点编号方式却完全不同。**(a)**图结点编号是按长边进行编号，编号差$d=7$，若采取带宽压缩存储，则刚性矩阵存储量$N=14*8=112$；而**(b)**图是按短边进行编号，$d=2$，$N=42$。显然**(b)**的编号方式可比**(a)**的编号方式节省**70**个存储单元。

（2）先内点后边界点编号

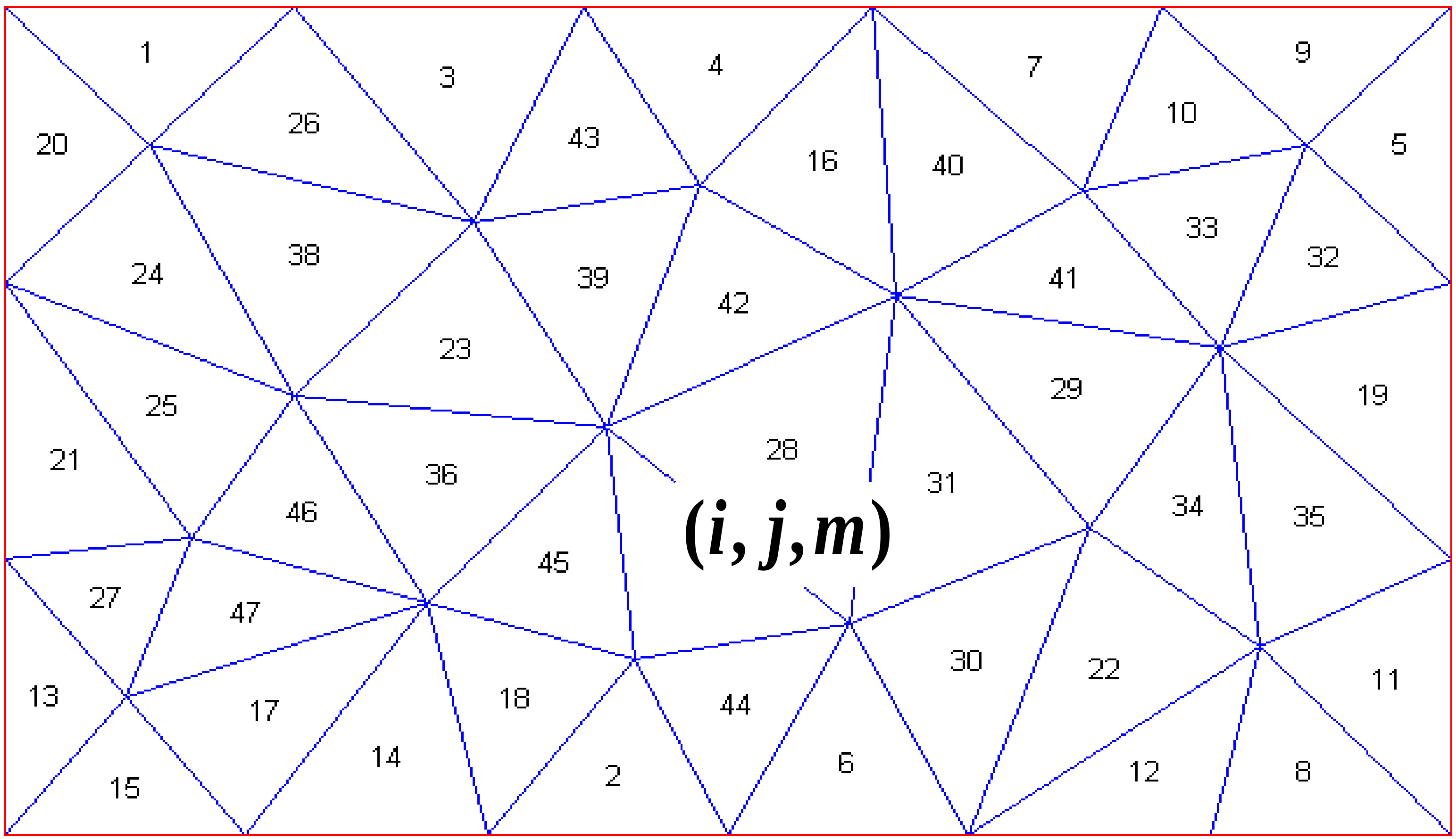


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

单元编号

$$\Omega \approx \Omega_h = \bigcup_i e_i$$

单元总数N_e



结点编号原则： 先内点后边界点，用3个结点号逆时针顺序表示

子空间---线性插值

定义空间：$V_h = \left\{ v_h \in C(\overline{\Omega}) : \left. v_h \right|_{e_i} = ax + by + c \right\}$

$a, b, c = ?$

找一个函数$u \in V$, 满足

$$D(u, v) = F(v)$$

$\forall v \in V$

找一个函数 $u_h \in V_h$ 满足

$$D(u_h, v_h) = F(v_h),$$

$\forall v_h \in V_h$

$\boxed{\text{有限元离散}}$

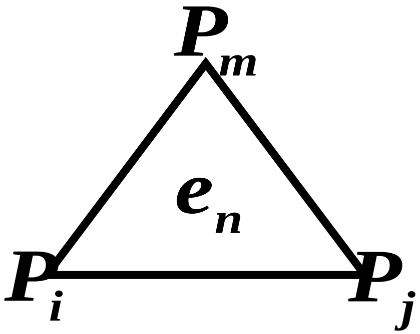
定义空间：$V_h=\left\{v_h\in C(\overline{\Omega}):\ v_h\big|_{e_i}=ax+by+c\right\}$

任取一个单元 $e_n=\Delta P_iP_jP_m$

$$P_i\left(x_i,y_i\right),\quad P_j\left(x_j,y_j\right),\quad P_m\left(x_m,y_m\right)$$

记 $u_i=u(x_i,y_i)$ 那么a,b,c满足

$$\begin{cases} ax_i+by_i+c=u_i \\ ax_j+by_j+c=u_j \\ ax_m+by_m+c=u_m \end{cases}$$



$$u_h(x,y)=ax+by+c \qquad (x,y)\in e_n$$

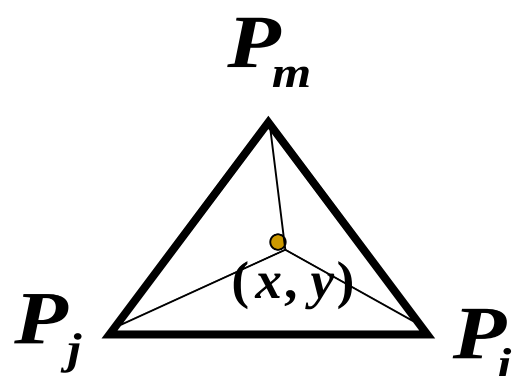
$$=N_i(x,y)u_i+N_j(x,y)u_j+N_m(x,y)u_m$$

$N_i(x, y)$

$$N_i(x, y) = \frac{1}{2\Delta_e}\begin{vmatrix} x & y & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix} \qquad \Delta_e = \frac{1}{2}\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix}$$

单元e的面积

$$N_j(x, y) = \frac{1}{2\Delta_e}\begin{vmatrix} x & y & 1 \\ x_m & y_m & 1 \\ x_i & y_i & 1 \end{vmatrix}$$



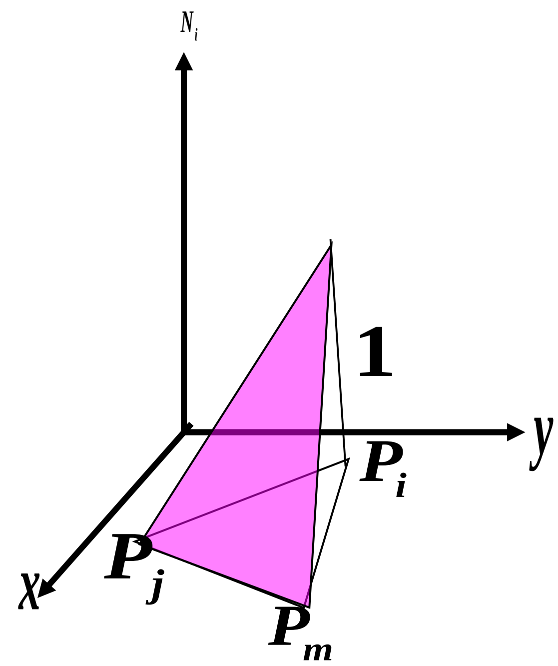
$$N_m(x, y) = \frac{1}{2\Delta_e}\begin{vmatrix} x & y & 1 \\ x_i & y_i & 1 \\ x_j & y_j & 1 \end{vmatrix}$$

e上线性插值的基函数

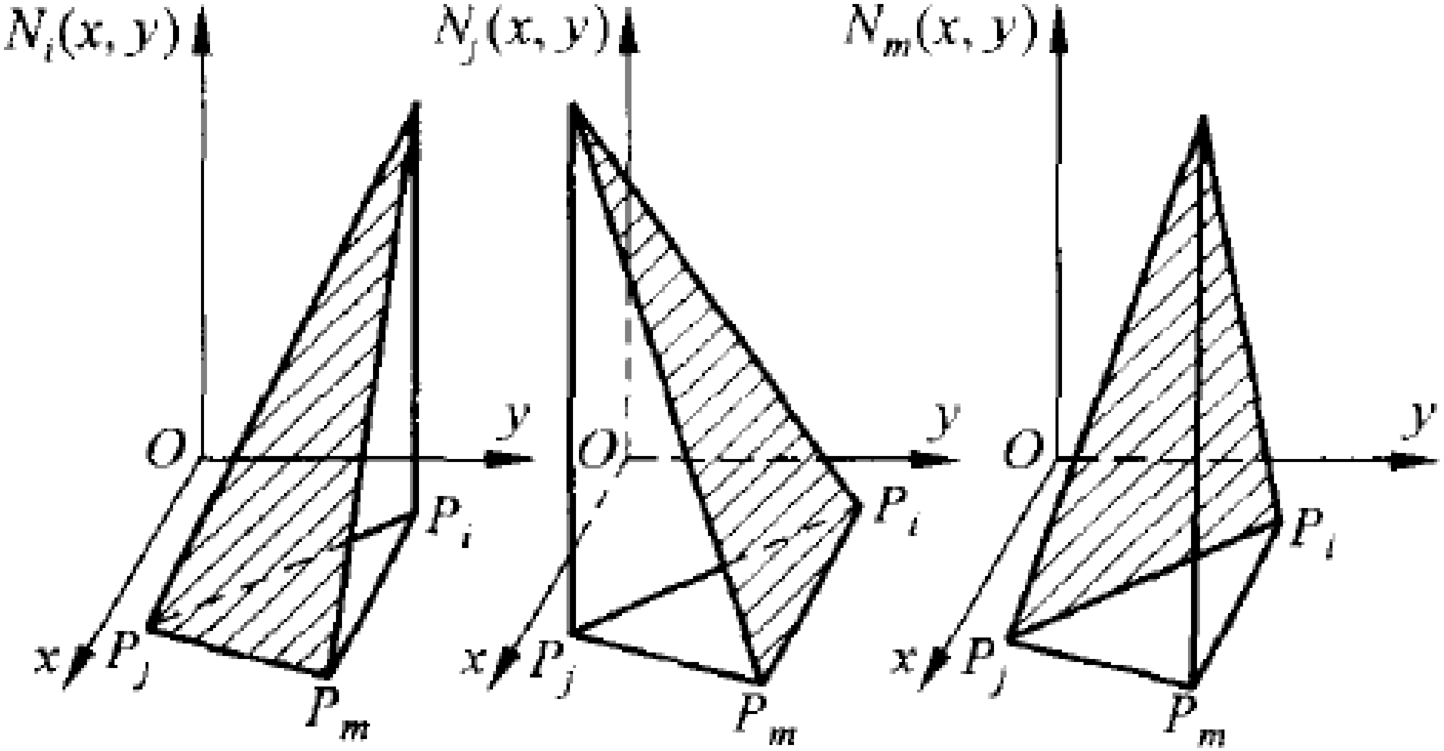
$$N_i(x,y)=\frac{1}{2\Delta_e}\begin{vmatrix} x & y & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix}=\frac{1}{2\Delta_e}\left(a_i x+b_i y+c_i\right)$$

$$a_i=\begin{vmatrix} y_j & 1 \\ y_m & 1 \end{vmatrix},\quad b_i=-\begin{vmatrix} x_j & 1 \\ x_m & 1 \end{vmatrix},$$

$$c_i=\begin{vmatrix} x_j & y_j \\ x_m & y_m \end{vmatrix}$$



$$N_i(x_l,y_l)=\begin{cases} 0, & l=j,m \\ 1, & l=i \end{cases}$$



$$N_i(x_l, y_l) = \begin{cases} 0, & l = j, m \\ 1, & l = i \end{cases}$$

即，子空间V_h定义为

$$V_h = \left\{ u \in C(\overline{\Omega}):\ u\big|_{e_i} = N_i(x,y)u_i + N_j(x,y)u_j + N_m(x,y)u_m \right\}$$

$$\boldsymbol{u}_h = N_i(x,y)\boldsymbol{u}_i + N_j(x,y)\boldsymbol{u}_j + N_m(x,y)\boldsymbol{u}_m$$

$$= N\boldsymbol{u}_e$$

$$N = \left(N_i, N_j, N_m \right) \qquad \boldsymbol{u}_e = \left(\boldsymbol{u}_i, \boldsymbol{u}_j, \boldsymbol{u}_m \right)^T$$

4）化为代数方程组——(4.1)单元分析(4.2)总体合成

找一个函数 $u_h \in V_h$ 满足

$$D(u_h, v_h) = F(v_h), \forall v_h \in V_h$$

$$\iint_{\Omega} \nabla u_h \bullet \nabla v_h dxdy + \int_{\partial\Omega} \sigma u_h v_h ds = \iint_{\Omega} f v_h dxdy + \int_{\partial\Omega} g v_h ds$$

$$\sum_{e_n} \iint_{e_n} \nabla u_h \bullet \nabla v_h dxdy + \sum_{e_n} \int_{\partial e_n \cap \partial\Omega} \sigma u_h v_h ds$$

$$= \sum_{e_n} \iint_{e_n} f v_h dxdy + \sum_{e_n} \int_{\partial e_n \cap \partial\Omega} g v_h ds$$

$$N=\left(N_i,N_j,N_m\right),u_{e_n}=\begin{pmatrix}u_i\\u_j\\u_m\end{pmatrix}$$

单元分析

任取一个单元 $e_n=\Delta P_iP_jP_m$

$$P_i\left(x_i,y_i\right),\quad P_j\left(x_j,y_j\right),\quad P_m\left(x_m,y_m\right)$$

记函数u_h和v_h在结点P_s 的取值分别为 $u_s,v_s,s=i,j,m$

$$u_h=N_i(x,y)u_i+N_j(x,y)u_j+N_m(x,y)u_m$$

$$=Nu_{e_n}$$

$$v_h=N_i(x,y)v_i+N_j(x,y)v_j+N_m(x,y)v_m$$

$$=Nv_{e_n}$$

$$u_h = N_i(x,y)u_i + N_j(x,y)u_j + N_m(x,y)u_m$$

$$\nabla u_h = \begin{pmatrix} \frac{\partial u_h}{\partial x} \\ \frac{\partial u_h}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial N_i(x,y)}{\partial x}u_i + \frac{\partial N_j(x,y)}{\partial x}u_j + \frac{\partial N_m(x,y)}{\partial x}u_m \\ \frac{\partial N_i(x,y)}{\partial y}u_i + \frac{\partial N_j(x,y)}{\partial y}u_j + \frac{\partial N_m(x,y)}{\partial y}u_m \end{pmatrix} = Bu_{e_n}$$

$$\nabla v_h = \begin{pmatrix} \frac{\partial v_h}{\partial x} \\ \frac{\partial v_h}{\partial y} \end{pmatrix} = Bv_{e_n}$$

$$B = \begin{pmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_m}{\partial y} \end{pmatrix}, u_{e_n} = \begin{pmatrix} u_i \\ u_j \\ u_m \end{pmatrix}$$

$$\frac{\partial u_h}{\partial x}\frac{\partial v_h}{\partial x} + \frac{\partial u_h}{\partial y}\frac{\partial v_h}{\partial y} = \nabla u_h \bullet \nabla v_h = Bu_{e_n} \bullet Bv_{e_n}$$

$$\sum_{e_n}\iint_{e_n}\nabla u_h\bullet\nabla v_h dxdy+\sum_{e_n}\int_{\partial e_n\cap\partial\Omega}\sigma u_h v_h ds$$

$$=\sum_{e_n}\iint_{e_n} f v_h dxdy+\sum_{e_n}\int_{\partial e_n\cap\partial\Omega} g v_h ds$$

$$\sum_{e_n}\iint_{e_n}(Bv_{e_n})^T Bu_{e_n} dxdy$$

$$+\sum_{e_n}\int_{\partial e_n\cap\partial\Omega}\sigma(Nv_{e_n})^T Nu_{e_n} ds$$

$$=\sum_{e_n}\iint_{e_n}(Nv_{e_n})^T f dxdy+\sum_{e_n}\int_{\partial e_n\cap\partial\Omega}(Nv_{e_n})^T g ds$$

在e_n上

$$u_h = Nu_{e_n}$$

$$v_h = Nv_{e_n}$$

$$\nabla u_h = Bu_{e_n}$$

$$\nabla v_h = Bv_{e_n}$$

$$u_h v_h$$

$$=(Nv_{e_n})^T Nu_{e_n}$$

$$\nabla u_h\bullet\nabla v_h$$

$$=(Bv_{e_n})^T Bu_{e_n}$$

$$\sum_{e_n}\left(v_{e_n}\right)^T\left(\boxed{\iint_{e_n} B^T B\,dxdy}\right)u_{e_n} \longrightarrow \bar{K}_e$$

面元刚度矩阵

$$+\sum_{e_n}\left(v_{e_n}\right)^T\left(\boxed{\int_{\partial e_n\cap\partial\Omega} \sigma N^T N\,ds}\right)u_{e_n} \longrightarrow \tilde{K}_e$$

线元刚度矩阵

$$=\sum_{e_n}\left(v_{e_n}\right)^T\left(\boxed{\iint_{e_n} N^T f\,dxdy}\right)+\sum_{e_n}\left(v_{e_n}\right)^T\left(\boxed{\int_{\partial e_n\cap\partial\Omega} N^T g\,ds}\right)$$

面元荷载向量 $\bar{F}_e$

$\tilde{F}_e$ 线元荷载向量

$$\sum_{e_n}\left(v_{e_n}\right)^T\left(\boxed{\bar{K}_{e_n}+\tilde{K}_{e_n}}\right)u_e=\sum_{e_n}\left(v_{e_n}\right)^T\left(\boxed{\bar{F}_{e_n}+\tilde{F}_{e_n}}\right)$$

K_{e_n} 单元刚度矩阵

F_{e_n} 单元荷载

面元刚度矩阵的计算

$$\bar{K}_{e_n} = \iint_{e_n} B^T B dxdy$$

$$= \frac{1}{4\Delta_{e_n}} \begin{pmatrix} a_i & b_i \\ a_j & b_j \\ a_m & b_m \end{pmatrix} \begin{pmatrix} a_i & a_j & a_m \\ b_i & b_j & b_m \end{pmatrix}$$

$$= \begin{pmatrix} \bar{k}^e_{ii} & \bar{k}^e_{ij} & \bar{k}^e_{im} \\ \bar{k}^e_{ji} & \bar{k}^e_{jj} & \bar{k}^e_{jm} \\ \bar{k}^e_{mi} & \bar{k}^e_{mj} & \bar{k}^e_{mm} \end{pmatrix}$$

其中$\bar{k}^e_{st} = \dfrac{1}{4\Delta_e}\left(a_s a_t + b_s b_t\right),\ s,t = i,j,m.$

线元刚度矩阵的计算

$$e_n = \Delta P_i P_j P_m \quad \text{设} \quad \partial e \cap \partial\Omega = \overline{P_i P_j} = \gamma_n$$

$$\left|\overline{P_i P_j}\right| = l,$$

$$P_i:\ t = 0,$$

$$P_j: t = l,$$

$$N_i(x_i, y_i) = 1,$$

$$N_i(x_j, y_j) = 0,$$

$$\Rightarrow$$

$$N_i\big|_{\overline{P_i P_j}} = 1 - \frac{t}{l},$$

$$N_j\big|_{\overline{P_i P_j}} = \frac{t}{l},$$

$$N_m\big|_{\overline{P_i P_j}} = 0.$$

$$K_{e_n} = \int_0^l \sigma N^T N dt$$

$$= \begin{pmatrix} k_{ii}^{e_n} & k_{ij}^{e_n} & k_{im}^{e_n} \\ k_{ji}^{e_n} & k_{jj}^{e_n} & k_{jm}^{e_n} \\ k_{mi}^{e_n} & k_{mj}^{e_n} & k_{mm}^{e_n} \end{pmatrix}$$

其中$\begin{cases} k_{ii}^{e_n} = \int_0^l \sigma\left(1 - \frac{t}{l}\right)^2 dt, \\ k_{ij}^{e_n} = k_{ji}^{e_n} = \int_0^l \sigma\left(1 - \frac{t}{l}\right)\frac{t}{l} dt, \\ k_{jj}^{e_n} = \int_0^l \sigma\left(\frac{t}{l}\right)^2 dt, \\ k_{mi}^{e_n} = k_{mj}^{e_n} = k_{mm}^{e_n} = k_{jm}^{e_n} = k_{im}^{e_n} = 0. \end{cases}$

面元荷载的计算

$$\overline{F}_{e_n}=\iint\limits_{e_n} N^T f dxdy=\begin{bmatrix}\overline{F}_i^{e_n}\\ \overline{F}_j^{e_n}\\ \overline{F}_m^{e_n}\end{bmatrix},\text{其中}\overline{F}_i^{e_n}=\iint\limits_{e_n} N_s f dxdy,\ s=i,j,m.$$

线元荷载的计算

$$\overline{F}_{e_n}=\int_{\gamma_n} N^T g ds=\begin{bmatrix}F_i^{e_n}\\ F_j^{e_n}\\ F_m^{e_n}\end{bmatrix}=\begin{bmatrix}\int_0^l\left(1-\frac{t}{l}\right)gdt,\\ \int_0^l\frac{t}{l}gdt,\\ 0\end{bmatrix}$$

可以利用的公式

$$\iint\limits_{e} N_1^{\lambda_1}N_2^{\lambda_2}N_3^{\lambda_3}dxdy=\frac{\lambda_1!\lambda_2!\lambda_3!}{\left(\lambda_1+\lambda_2+\lambda_3+2\right)!}\cdot 2\Delta_e$$

总体合成

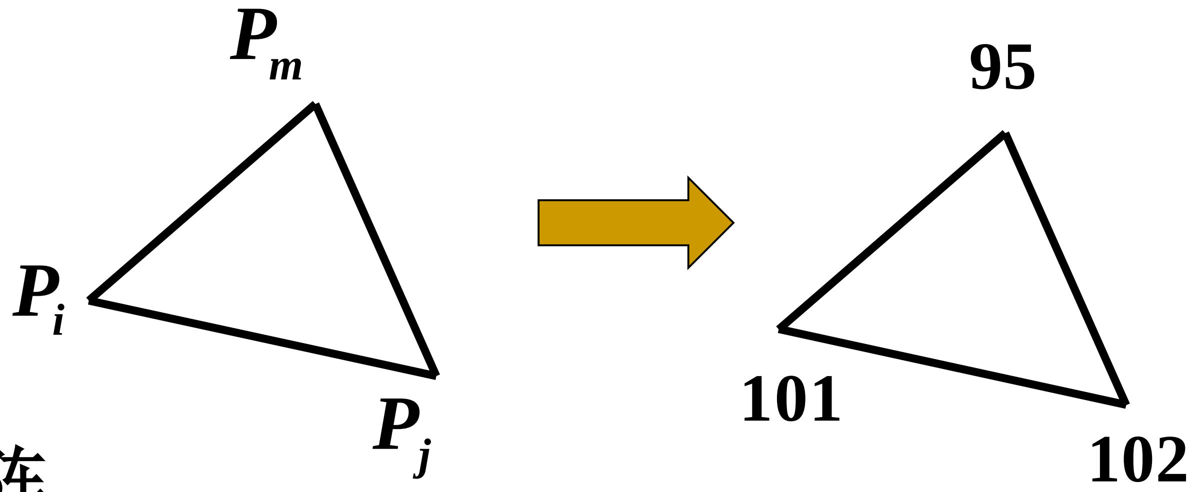
$$\sum_{e_n}\left(v_{e_n}\right)^T\left(\bar{K}_{e_n}+\tilde{K}_{e_n}\right)u_{e_n}=\sum_{e_n}\left(\bar{F}_{e_n}+\tilde{F}_{e_n}\right)v_{e_n}$$

其中 $e_n=\Delta P_i P_j P_m$

$$u_{e_n}=\left(u_i,u_j,u_m\right)^T,\quad v_{e_n}=\left(u_i,u_j,u_m\right)^T$$

记函数u_h和v_h在结点 P_s的取值分别为$u_s,v_s,s=1,\cdots,N_p$

$$u=\left(u_1,u_2,\cdots u_{N_p}\right)^T,\qquad v=\left(v_1,v_2,\cdots v_{N_p}\right)^T$$



记C_{e_n}为$3\times N_p$的矩阵

$$
\begin{array}{c}
C_{e_n} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & 0 & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 & \cdots \\ 0 & \cdots & 0 & 1 & \cdots & \cdots & \cdots & 0 & \cdots \end{pmatrix} \\
\begin{array}{ccccccccc} & & & \begin{matrix} 95 \\ (m) \end{matrix} & & \begin{matrix} 101 \\ (i) \end{matrix} & \begin{matrix} 102 \\ (j) \end{matrix} & & \end{array}
\end{array}
$$

则

$$
v_{e_n} = \begin{pmatrix} v_i \\ v_j \\ v_m \end{pmatrix} = C_{e_n} v
$$

$$\left(v_e\right)^T \bar{K}_{e_n} u_{e_n} = \left(v_i, v_j, v_m\right) \begin{pmatrix} \bar{k}_{ii}^{e_n} & \bar{k}_{ij}^{e_n} & \bar{k}_{im}^{e_n} \\ \bar{k}_{ji}^{e_n} & \bar{k}_{jj}^{e_n} & \bar{k}_{jm}^{e_n} \\ \bar{k}_{mi}^{e_n} & \bar{k}_{mj}^{e_n} & \bar{k}_{mm}^{e_n} \end{pmatrix} \begin{pmatrix} u_i \\ u_j \\ u_m \end{pmatrix}$$

$$= (C_{e_n} v)^T \bar{K}_{e_n} C_{e_n} u = v^T C_{e_n}^T \bar{K}_{e_n} C_{e_n} u$$

$$= \left(\cdots v_i \cdots v_j \cdots v_m \cdots\right) \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 & \vdots & 0 \\ \vdots & \bar{k}_{ii}^{e_n} & \vdots & \bar{k}_{ij}^{e_n} & \vdots & \bar{k}_{im}^{e_n} & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \bar{k}_{ji}^{e_n} & \vdots & \bar{k}_{jj}^{e_n} & \vdots & \bar{k}_{jm}^{e_n} & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \bar{k}_{mi}^{e_n} & \vdots & \bar{k}_{mj}^{e_n} & \vdots & \bar{k}_{mm}^{e_n} & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \vdots \\ u_i \\ \vdots \\ u_j \\ \vdots \\ u_m \\ \vdots \end{pmatrix}$$

$$e_n = \Delta P_i P_j P_m \text{ 设 } \partial e \cap \partial\Omega = \overline{P_i P_j} = \gamma_n$$

$$\int_{\gamma_n} \sigma u_h v_h ds = \int_0^l \sigma (N v_{e_n})^T (N u_{e_n}) ds = v^T C_{e_n}^T K_{e_n} C_{e_n} u$$

$$= \begin{pmatrix} \cdots v_i \cdots v_j \cdots v_m \cdots \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 & \vdots & 0 \\ \vdots & \tilde{k}_{ii}^{e_n} & \vdots & \tilde{k}_{ij}^{e_n} & \vdots & 0 & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \tilde{k}_{ji}^{e_n} & \vdots & \tilde{k}_{jj}^{e_n} & \vdots & 0 & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & 0 & \vdots & 0 & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \vdots \\ u_i \\ \vdots \\ u_j \\ \vdots \\ u_m \\ \vdots \end{pmatrix}$$

$$\bar{F}_{e_n} = \iint_{e_n} N^T f dxdy \quad v_{e_n}^T \bar{F}_{e_n} = v_{e_n}^T \left(\bar{F}_i^e, \bar{F}_j^e, \bar{F}_m^{e_n} \right)^T = (C_{e_n} v)^T \bar{F}_{e_n}$$

$$= v^T C_{e_n}^T \bar{F}_{e_n} = v^T \left(\cdots \bar{F}_i^{e_n} \cdots \bar{F}_j^{e_n} \cdots \bar{F}_m^{e_n} \cdots \right)^T$$

$$\tilde{F}_{e_n} = \int_{\partial e_n \cap \partial \Omega} N^T g ds$$

若 $\partial e_n \cap \partial \Omega = P_i P_j$

$$v_{e_n}^T \tilde{F}_{e_n} = v_{e_n}^T \left(\tilde{F}_i^{e_n}, \tilde{F}_j^{e_n}, 0 \right)^T = v^T C_{e_n}^T \tilde{F}_{e_n} = v^T \left(\cdots \tilde{F}_i^{e_n} \cdots \tilde{F}_j^{e_n} \cdots 0 \cdots \right)^T$$

$$\sum_{e_n} \left(v_{e_n} \right)^T \left(\bar{K}_{e_n} + \tilde{K}_{e_n} \right) u_{e_n} = \sum_{e_n} v_{e_n}^T \left(\bar{F}_{e_n} + \tilde{F}_{e_n} \right)$$

$$\Leftrightarrow v^T \left(\sum_{e_n} \left(\bar{K}_{e_n} + \tilde{K}_{e_n} \right) \right) u = v^T \left(\sum_{e_n} \left(\bar{F}_{e_n} + \tilde{F}_{e_n} \right) \right)$$

$$\Leftrightarrow v^T K u = v^T F$$

5）约束边界处理

讨论对象：椭圆型偏微分方程边值问题

$$\begin{cases} -\Delta u = f, & (x,y)\in\Omega \\ u = 0, & (x,y)\in\Gamma=\partial\Omega \end{cases}$$

转化为变分问题

$$V = H_0^1(\Omega)$$

$$= \left\{ v \,\middle|\, \iint_\Omega \left[v^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right] dxdy < \infty, v\big|_{\partial\Omega} = 0 \right\}$$

$$\begin{cases} find \quad u \in V \quad s.t. \\ D(u,v) - F(v) = 0, \ \forall v \in V \end{cases}$$

其中

$$D(u,v) = \iint_{\Omega} \nabla u \bullet \nabla v dxdy, F(v) = \iint_{\Omega} fvdxdy$$

有限元化

$$V_{h0} = \left\{ v_h \in C(\overline{\Omega}) : \ v_h\big|_{e_i} = ax + by + c, v_h\big|_{P_i} = 0, i = \underbrace{1, 2, \cdots, l}_{} \right\}$$

试探函数空间

边界节点编号

找一个函数 $u_h \in V_{h0}$ 满足

$$D(u_h, v_h) = F(v_h), \ \forall v_h \in V_{h0}$$

$$v=\left\{0,\cdots 0,v_{l+1},\cdots,v_{N_p}\right\},$$

$$v=\begin{pmatrix}0\\ v_{II}\end{pmatrix},u=\begin{pmatrix}u_I\\ u_{II}\end{pmatrix},F=\begin{pmatrix}F_I\\ F_{II}\end{pmatrix}$$

$$\left(0,v_{II}^T\right)\left(\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}u_I\\ u_{II}\end{pmatrix}-\begin{pmatrix}F_I\\ F_{II}\end{pmatrix}\right)=0,$$

$$v_{II}^T\left(K_{21}u_I+K_{22}u_{II}-F_{II}\right)=0, \qquad v^T K u=v^T F$$

$$K_{22}u_{II}=F_{II}-K_{21}u_I$$

若不在开头，总刚矩阵划去边界节点相应的行与列

- 有限元法内容总结

1 明确微分方程

2 转化为变分形式；

3 有限元离散；√

- 区域剖分；确定单元基函数；

4 转化为代数方程 √

- 单元分析；总体合成；边界条件的处理；

5 解代数方程。 √

总结：实际计算步骤

1）单元剖分 $e_n = \Delta P_i P_j P_m$

- 节点编号和坐标
- 单元编号与对应节点(i,j,m)
- 边界条件节点编号

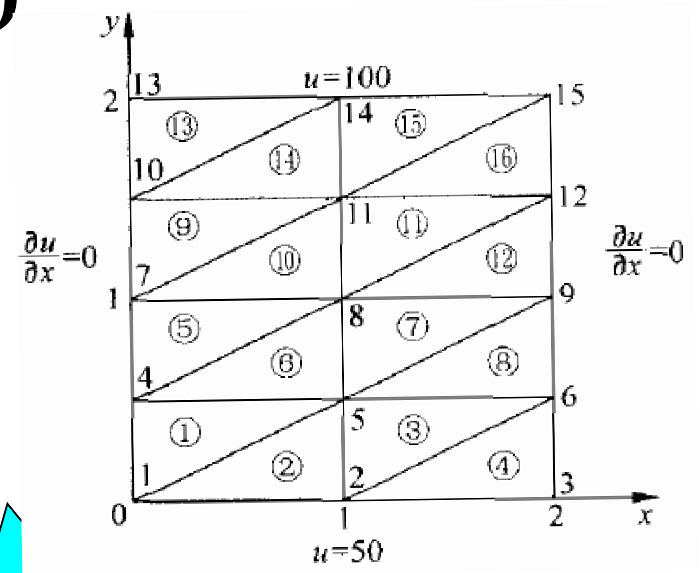
2）有限元离散

- **基函数—单元号和单元顶点i处参数(a_i,b_i,c_i)**
- **计算单元刚度矩阵、单元荷载向量**
- **合成总刚度矩阵与总荷载向量，形成代数方程组**

3）根据边界调整K和F，即“掐头或去尾”

4）求解上代数方程组，得数值解

例题：$\begin{cases} -\Delta u = 0, & (x,y)\in(0,2)^2 \\ u|_{y=0} = 50, & u|_{y=2} = 100 \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, & \left.\dfrac{\partial u}{\partial x}\right|_{x=2} = 0 \end{cases}$



解：分析

平面区域上无热源的定常温度场，两边给出温度值，另两边给出绝热条件

采用三角形剖分，共16个单元，15个节点

$$\begin{cases} -\Delta u = 0, & (x,y)\in(0,2)^2 \\ u\big|_{y=0} = 50, & u\big|_{y=2} = 100 \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, & \left.\dfrac{\partial u}{\partial x}\right|_{x=2} = 0 \end{cases}$$

$$\sum_{e_n}\iint_{e_n}(Bv_{e_n})^T Bu_{e_n}\,dxdy$$

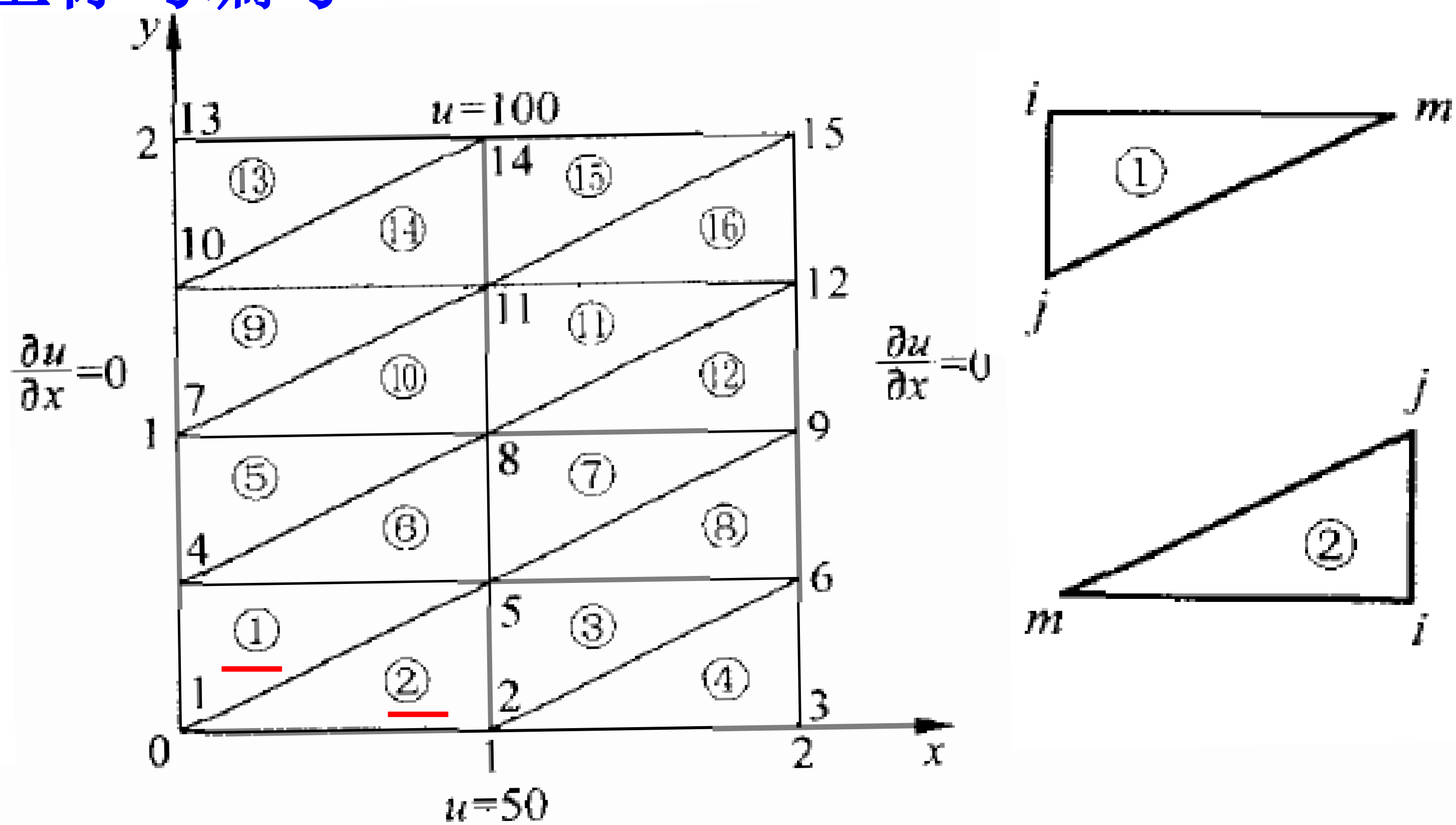
$$+\sum_{e_n}\int_{\partial e_n\cap\Gamma_1}\sigma(Nv_{e_n})^T Nu_{e_n}\,ds$$

$$=\sum_{e_n}\iint_{e_n}(Nv_{e_n})^T f dxdy+\sum_{e_n}\int_{\partial e_n\cap\Gamma_1}(Nv_{e_n})^T g ds$$

其中

$$\Gamma_1=\{(x,y)\,|\,x=\{0\}\cup\{2\},0\le y\le 2\}$$

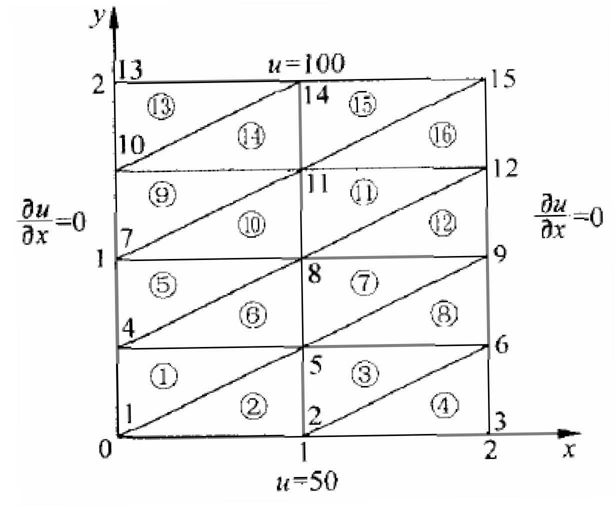
1）节点坐标与编号



节点号	(x,y)	节点号	(x,y)	节点号	(x,y)
1	(0,0)	6	(2,0.5)	11	(1,1.5)
2	(1,0)	7	(0,1)	12	(2,1.5)
3	(2,0)	8	(1,1)	13	(0,2)
4	(0,0.5)	9	(2,1)	14	(1,2)
5	(1,0.5)	10	(0,1.5)	15	(2,2)

2）基函数与相应参数

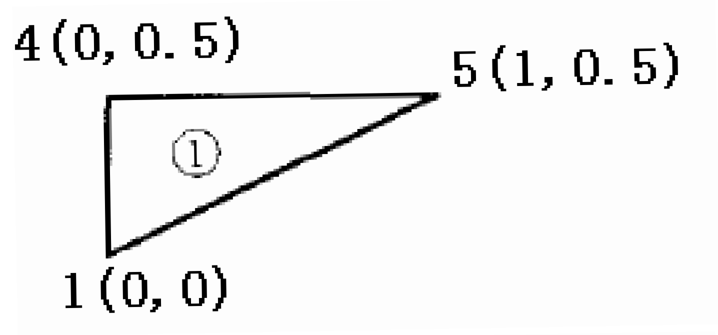
单元 信息	1	2	3	…
结点号 (i,j,k)	(4,1,5)	(2,5,1)	(5,2,6)	…
参数值 (a_s, b_s) $s=i,j,k$	(-0.5,1) (0,-1) (0.5,0)	(0.5,-1) (0,1) (-0.5,0)	…	…

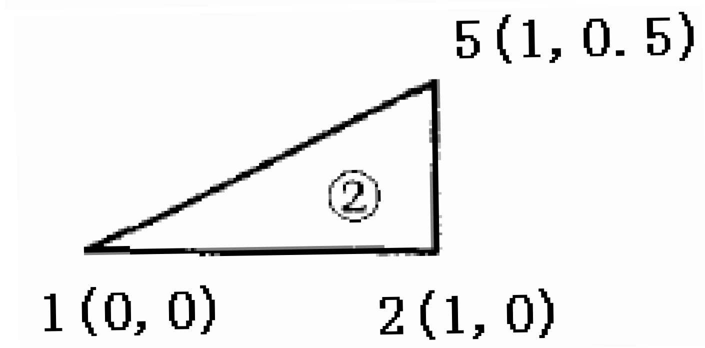


$$N_s(x,y)=\frac{1}{2\Delta_e}\left(a_s x+b_s y+c_s\right)$$

$$s=i,j,k$$

单元 \ 信息	1	2	3	…
结点号 (i,j,k)	(4,1,5)	(2,5,1)	(5,2,6)	…
(a_i, b_i) (a_j, b_j) (a_k, b_k)	(-0.5,1) (0,-1) (0.5,0)	(0.5,-1) (0,1) (-0.5,0)	…	…

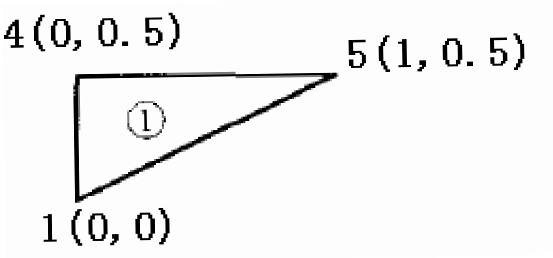




$$a_i = \begin{vmatrix} y_j & 1 \\ y_m & 1 \end{vmatrix}, \quad b_i = -\begin{vmatrix} x_j & 1 \\ x_m & 1 \end{vmatrix}, c_i = \begin{vmatrix} x_j & y_j \\ x_m & y_m \end{vmatrix}$$

通过计算可知，奇数单元上参数值相同, 偶数单元上参数值相同

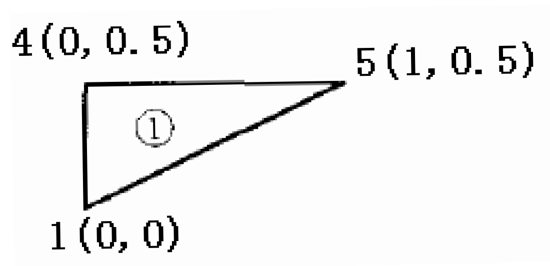
$$2\Delta_e = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix} = 2area(e) = 0.5$$



$$B_{e_1} = \begin{pmatrix} \dfrac{\partial N_4(x,y)}{\partial x} & \dfrac{\partial N_1(x,y)}{\partial x} & \dfrac{\partial N_5(x,y)}{\partial x} \\ \dfrac{\partial N_4(x,y)}{\partial y} & \dfrac{\partial N_1(x,y)}{\partial y} & \dfrac{\partial N_5(x,y)}{\partial y} \end{pmatrix}$$

$$= \frac{1}{2\Delta_e}\begin{pmatrix} a_4 & a_1 & a_5 \\ b_4 & b_1 & b_5 \end{pmatrix} = 2\begin{pmatrix} -0.5 & 0 & 0.5 \\ 1 & -1 & 0 \end{pmatrix}$$

单元1刚度矩阵



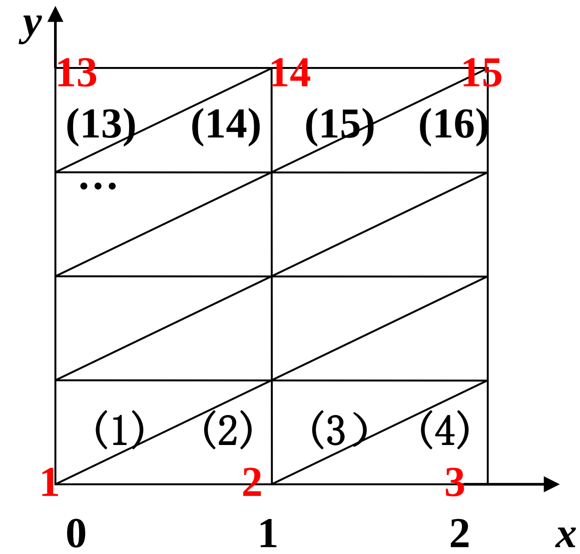
$$K_{e_1} = \iint\limits_{e_1} B_{e_1}^T B_{e_1}\, dx\, dy$$

$$= 4\iint\limits_{e_1} \begin{pmatrix} -0.5 & 1 \\ 0 & -1 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} -0.5 & 0 & 0.5 \\ 1 & -1 & 0 \end{pmatrix} dx\, dy$$

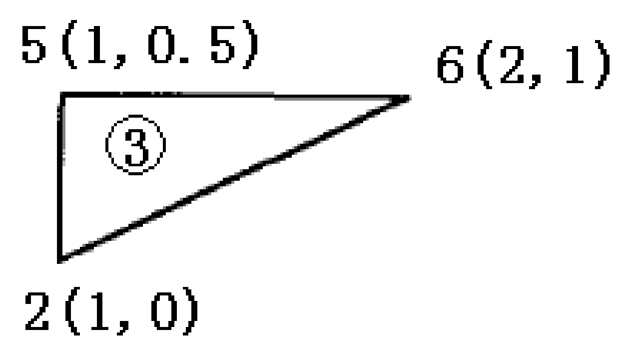
$$= \iint\limits_{e_1} \begin{pmatrix} 5 & -4 & -1 \\ -4 & 4 & 0 \\ -1 & 0 & 1 \end{pmatrix} dxdy = \frac{1}{4} \begin{array}{c} \begin{pmatrix} 5 & -4 & -1 \\ -4 & 4 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{matrix} 4 \\ 1 \\ 5 \end{matrix} \\ \begin{matrix} 4 & \quad 1 & \quad 5 \end{matrix} \end{array}$$

奇数单元刚度矩阵

$$K_{e_1} = \frac{1}{4}\begin{pmatrix} 5 & -4 & -1 \\ -4 & 4 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

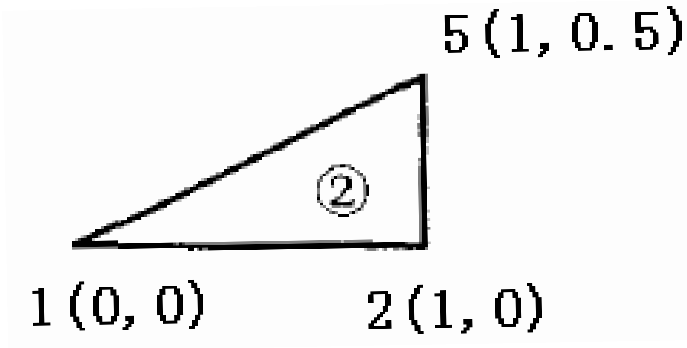


$$K_{e_3} = \frac{1}{4}\begin{array}{cc} \begin{pmatrix} 5 & -4 & -1 \\ -4 & 4 & 0 \\ -1 & 0 & 1 \end{pmatrix} & \begin{array}{c} 5 \\ 2 \\ 6 \end{array} \\ \begin{array}{ccc} 5 & \quad 2 & \quad 6 \end{array} & \end{array}$$



奇数单元上的刚度矩阵等于 K_{e_1}

$$K_{e_1} = K_{e_3} = \cdots = K_{e_{15}},$$



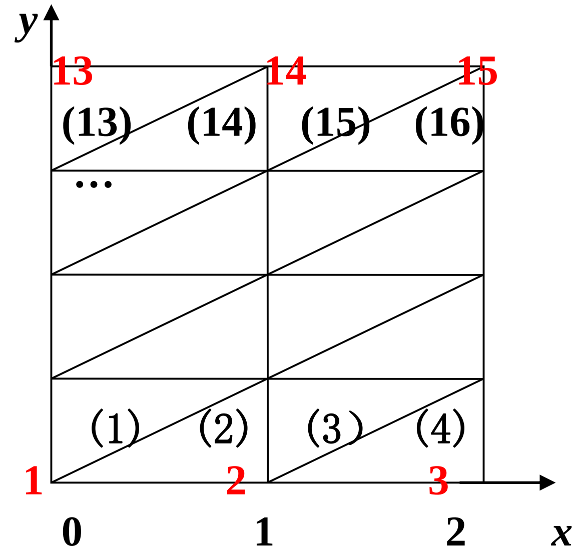
单元2的刚度矩阵

$$B_{e_2} = \frac{1}{2\Delta_e}\begin{pmatrix} a_2 & a_5 & a_1 \\ b_2 & b_5 & b_1 \end{pmatrix} = 2\begin{pmatrix} 0.5 & 0 & -0.5 \\ -1 & 1 & 0 \end{pmatrix}$$

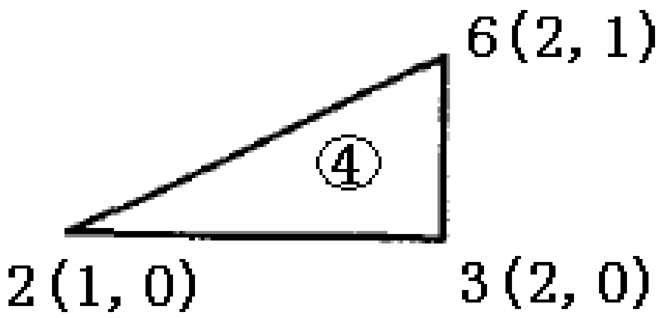
$$K_{e_2} = \iint_{e_2} B_{e_2}^T B_{e_2}\, dxdy = \frac{1}{4}\begin{array}{cc} \begin{pmatrix} 5 & -4 & -1 \\ -4 & 4 & 0 \\ -1 & 0 & 1 \end{pmatrix} & \begin{array}{c} 2 \\ 5 \\ 1 \end{array} \\ \begin{array}{ccc} 2 & \quad 5 & \quad 1 \end{array} & \end{array}$$

偶数单元刚度矩阵

$$K_{e_2}=\frac{1}{4}\begin{pmatrix}5 & -4 & -1\\ -4 & 4 & 0\\ -1 & 0 & 1\end{pmatrix}$$



$$K_{e_4}=\frac{1}{4}\begin{array}{c}\begin{pmatrix}5 & -4 & -1\\ -4 & 4 & 0\\ -1 & 0 & 1\end{pmatrix}\begin{matrix}3\\6\\2\end{matrix}\\ \begin{matrix}3 & \quad 6 & \quad 2\end{matrix}\end{array}$$



偶数单元上的刚度矩阵等于 K_{e_2}

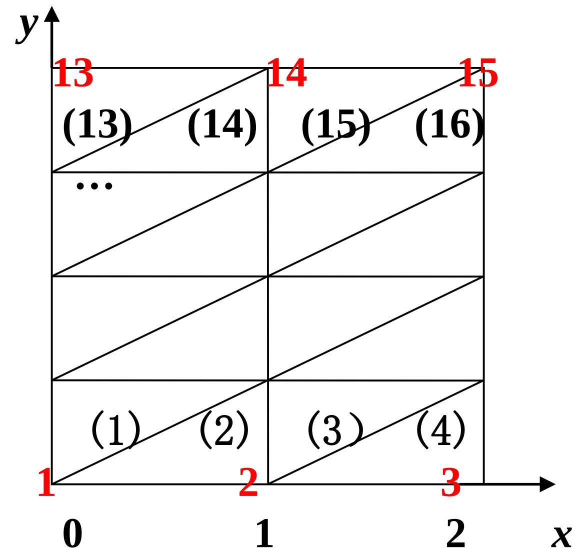
$$K_{e_2}=K_{e_4}=\cdots=K_{e_{16}},$$

扩充单元刚度矩阵

$$K_{e_2}=\frac{1}{4}\begin{array}{cc}\begin{pmatrix}5&-4&-1\\-4&4&0\\-1&0&1\end{pmatrix}&\begin{array}{c}2\\5\\1\end{array}\\\begin{array}{ccc}2&5&1\end{array}&\end{array}$$

$$\Rightarrow\frac{1}{4}\begin{array}{cc}\begin{pmatrix}1&-1&0&0&0&\\-1&5&0&0&-4&\\0&0&0&0&0&\cdots\\0&0&0&0&0&\\0&-4&0&0&4&\cdots\\&\cdots&&\cdots&&\end{pmatrix}&\begin{array}{c}1\\2\\3\\4\\5\\\vdots\end{array}\\\begin{array}{cccccc}1&2&3&4&5&\cdots\end{array}&\end{array}$$

叠加得总体刚度矩阵



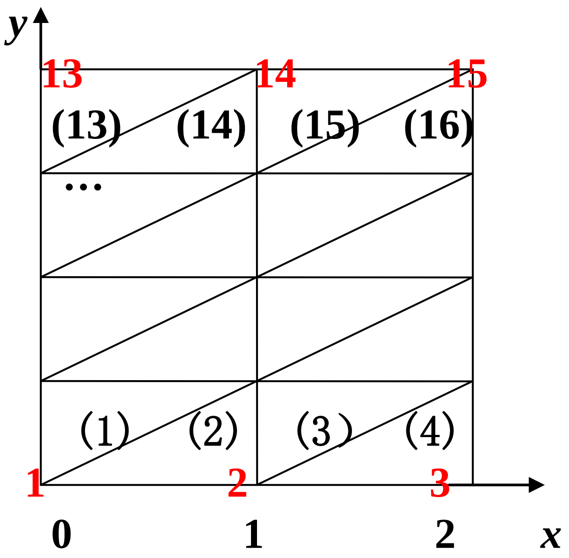
$$K=\frac{1}{4}\begin{array}{cc}\begin{pmatrix} 5 & -1 & 0 & -4 & 0 & \cdots \\ -1 & 10 & -1 & 0 & -8 & 0 \\ 0 & -1 & 5 & 0 & 0 & -4 \\ -4 & 0 & 0 & 10 & -2 & 0 \\ 0 & -8 & 0 & -2 & 20 & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{array} \\ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \end{array} & \end{array}$$

半带宽为4

带宽为7

对称

3）边界处理



$$u\big|_{y=0}=50,\qquad u\big|_{y=2}=100$$

$$u_1=u_2=u_3=50$$

$$u_{13}=u_{14}=u_{15}=100$$

$$u_I=(u_1,u_2,u_3)^T,\ u_{II},\ u_{III}=(u_{13},u_{14},u_{15})^T$$

$$v^T Ku=0$$

$$\Leftrightarrow(0,v_{II}^T,0)\begin{pmatrix}K_{11}&K_{12}&K_{13}\\K_{21}&K_{22}&K_{23}\\K_{31}&K_{32}&K_{33}\end{pmatrix}\begin{pmatrix}u_I\\u_{II}\\u_{III}\end{pmatrix}=0$$

$$\Leftrightarrow \left(0, \mathrm{v}_{II}^{T}, 0\right) \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} u_{I} \\ u_{II} \\ u_{III} \end{pmatrix} = 0$$

$$\Leftrightarrow v_{II}^{T} \left(K_{21} u_{I} + K_{22} u_{II} + K_{23} u_{III} \right) = 0$$

$$\Leftrightarrow K_{22} u_{II} = -K_{21} u_{I} - K_{23} u_{III}$$

其中 $-K_{21} u_{I} = \left(4u_{1}, 8u_{2}, 4u_{3}, 0 \cdots 0\right)^{T}$

$$-K_{23} u_{III} = \left(0 \cdots 0, 4u_{13}, 8u_{14}, 4u_{15}\right)^{T}$$

总体刚度矩阵

$$
K=\frac{1}{4}\left(\begin{array}{cccccccccccc}
5 & -1 & 0 & -4 & & & & & & & & \\
 & 10 & -1 & 0 & -8 & & & & & & & \\
 & -1 & 5 & 0 & 0 & -4 & & & & & & \\
-4 & 0 & 0 & 10 & -2 & 0 & -4 & & & & & \\
 & -8 & 0 & -2 & 20 & -2 & 0 & -8 & & & & \\
 & & -4 & 0 & -2 & 10 & 0 & 0 & -4 & & & \\
 & & & & & & 10 & -2 & 0 & -4 & & \\
 & & & & & & & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & -4 & 0 & 2 & 10 & 0 & 0 & -4 \\
 & & & & & & -4 & 0 & 0 & 5 & -1 & 0 \\
 & & & & & & & -8 & 0 & -1 & 10 & -1 \\
 & & & & & & & & -4 & 0 & -1 & 5
\end{array}\right)
\begin{array}{c}
1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \vdots \\ 12 \\ 13 \\ 14 \\ 15
\end{array}
$$

得代数方程组

$$\frac{1}{4}\begin{pmatrix} 10 & -2 & 0 & -4 & & & & & \\ -2 & 20 & -2 & 0 & -8 & & & & \\ 0 & -2 & 10 & 0 & 0 & -4 & & & \\ -4 & 0 & 0 & 10 & -2 & 0 & -4 & & \\ & -8 & 0 & -2 & 20 & -2 & 0 & -8 & \\ & & -4 & 0 & -2 & 10 & 0 & 0 & -4 \\ & & & -4 & 0 & 0 & 10 & 0 & 0 \\ & & & & -8 & 0 & 0 & 20 & -2 \\ & & & & & -4 & 0 & -2 & 10 \end{pmatrix}\begin{pmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{pmatrix} = 4\begin{pmatrix} 50 \\ 100 \\ 50 \\ 0 \\ 0 \\ 0 \\ 100 \\ 200 \\ 100 \end{pmatrix}$$

利用软件求解即可

课堂练习

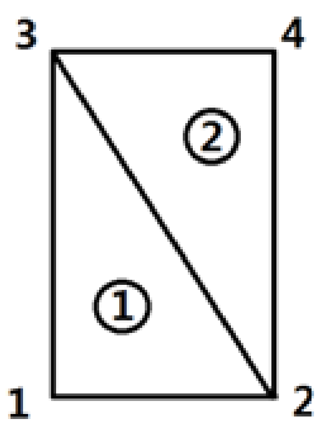
1、请给出有限元法程序设计流程。

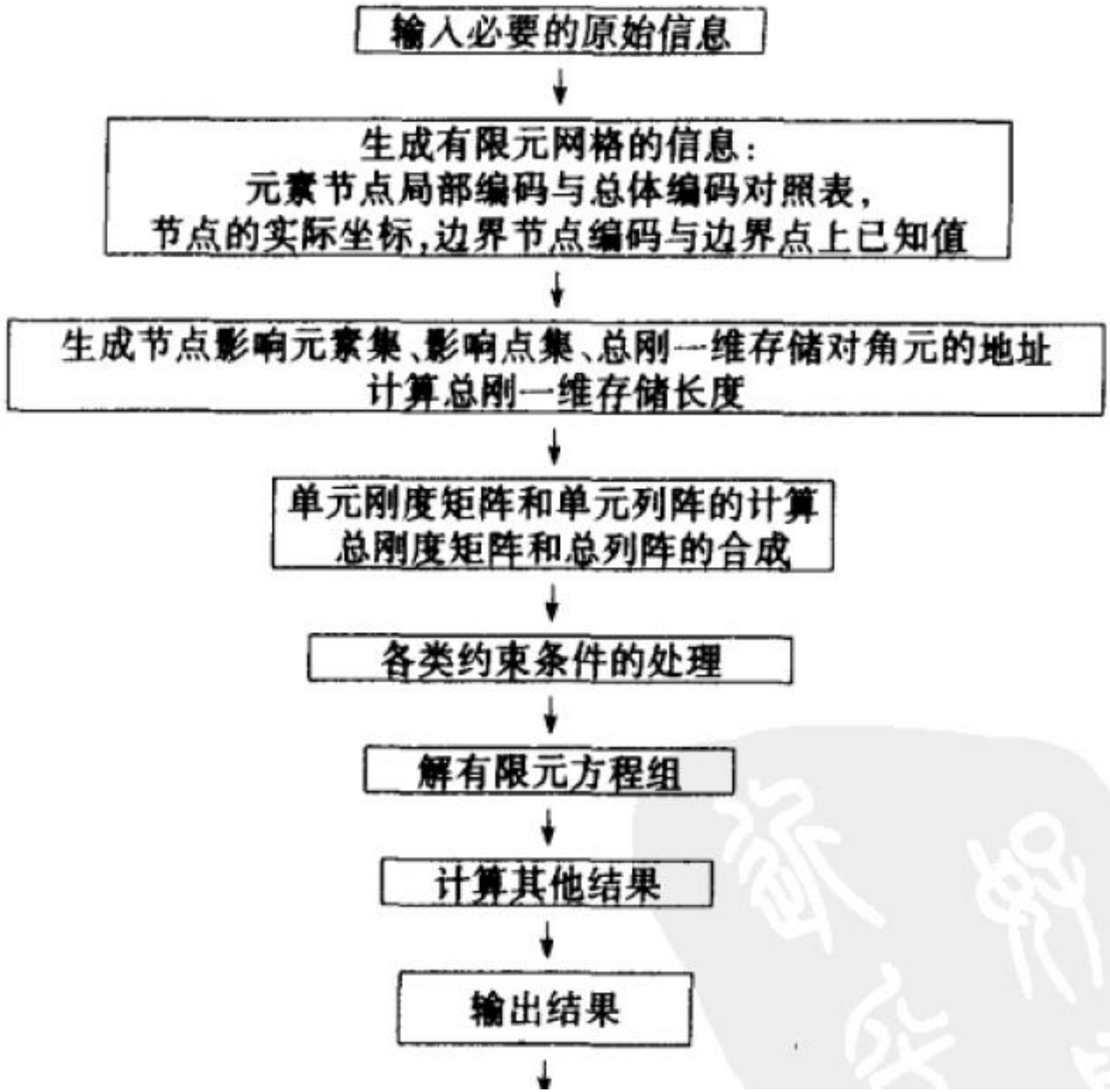
2、

已知椭圆问题为
$$\begin{cases} -\Delta u = 2, & (x,y)\ \text{in}\ \Omega, \\ u\big|_{\Gamma_1} = 0, \dfrac{\partial u}{\partial n}\bigg|_{\partial\Omega/\Gamma_1} = 0. \end{cases}$$

其中 $\Omega = [0,1]\times[0,2]$，$\Gamma_1 = \{(x,y) \mid x = 0, 0 \le y \le 2\}$.

将 Ω 剖分为两个三角形单元，以单元的顶点函数值为自由度，请求出每个单元的单元刚度矩阵，并合成总刚度矩阵。




输入必要的原始信息
生成有限元网格的信息：
元素节点局部编码与总体编码对照表，
节点的实际坐标，边界节点编码与边界点上已知值
生成节点影响元素集、影响点集、总刚一维存储对角元的地址
计算总刚一维存储长度
单元刚度矩阵和单元列阵的计算
总刚度矩阵和总列阵的合成
各类约束条件的处理
解有限元方程组
计算其他结果
输出结果