11 有限元离散方法

（一维问题）

微分方程解和有限元方法下的数值解

Poisson方程	$-\Delta u = f$	微分方程
	↓ 分部积分	↓
变分问题	$D(u,v) = F(v)$	泛函问题
	↓ Ritz-Galerkin过程	↓
离散变分问题	$D(u_h, v_h) = F(v_h)$	有限元离散
	↓ 有限维基上表示	↓
线性代数问题	$\mathbf{K}\vec{u} = \mathbf{f}$	代数方程
	↓ 快速算法	↓
方程求解	$\vec{u} = \mathbf{K} \backslash \mathbf{f}$	迭代求解

5个部分

8.1 一维问题的有限元法（线性元）

1）讨论对象：常微分方程两点边值问题

$$\begin{cases} -\dfrac{d}{dx}\left(p\dfrac{du}{dx}\right)+qu=f, \quad x\in(a,b) \\ u(a)=0, \\ p(b)u'(b)+\sigma u(b)=\beta, \end{cases}$$

$$p\in C^1[a,b],\ p(x)\geq p_0>0, q\in C[a,b], q(x)\geq 0$$

2）转化为变分问题　考虑空间：

$$V=S_0^1=\left\{v\,\middle|\,\int_a^b\left[v^2+\left(\frac{dv}{dx}\right)^2\right]dx<\infty\ v(a)=0\right\}$$

任取$v\in V$，乘方程两边 $$-\frac{d}{dx}\left(p\frac{du}{dx}\right)+qu=f,$$

$$\left(-\frac{d}{dx}\left(p\frac{du}{dx}\right)+qu\right)v=fv$$

两边积分 $$\underline{\int_a^b\left(-\frac{d}{dx}\left(p\frac{du}{dx}\right)+qu\right)vdx}=\int_a^b fvdx$$

$$\underline{\int_a^b -\frac{d}{dx}\left(p\frac{du}{dx}\right)vdx}+\int_a^b quvdx$$

$$\boxed{\int_a^b u'vdx=uv\Big|_a^b-\int_a^b v'udx}$$

$$-p\frac{du}{dx}v\Big|_a^b+\int_a^b p\frac{du}{dx}\frac{dv}{dx}dx$$

$$-p\frac{du}{dx}v\bigg|_a^b+\int_a^b p\frac{du}{dx}\frac{dv}{dx}dx$$

$$v(a)=0$$

$$p(b)u'(b)+\sigma u(b)=\beta,$$

$$p(a)u'(a)v(a)-p(b)u'(b)v(b)$$

$$=\left(\sigma u(b)-\beta\right)v(b)$$

综上推理，有

$$\int_a^b\left(p\frac{du}{dx}\frac{dv}{dx}+quv\right)dx+\sigma u(b)v(b)-\beta v(b)=\int_a^b fvdx$$

$$\boxed{\int_a^b \left(p\frac{du}{dx}\frac{dv}{dx} + quv \right) dx + \sigma u(b)v(b)} = \boxed{\int_a^b fv\,dx + \beta v(b)}$$

$$D(u,v) \qquad\qquad F(v)$$

原微分方程转化为变分问题：

即找一个函数 $u \in V$

满足 $D(u,v) = F(v) \quad \forall v \in V$

$$D(u,v)=\int_a^b\left(p\frac{du}{dx}\frac{dv}{dx}+quv\right)dx+\sigma u(b)v(b)$$

3）有限元离散

找一个函数$u\in V$, 满足

$$D(u,v)=F(v)\quad \forall v\in V$$

$$\left.\begin{array}{l} D:\ \underline{(u,v)\in V\times V} \to R \\ \text{自变量: 函数} \to R\ \text{因变量: 实数} \end{array}\right\}\ \text{泛函}$$

$$\downarrow$$

函数空间

补充：空间的维数和基　　　基：最大线性无关组

R：1维，基：1，　$\forall a \in R$，$a = a \bullet 1$

子空间

R^2：2维，基: $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$\forall a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in R^2, a = a_1 e_1 + a_2 e_2$$

子空间

R^n：n维，基:$e_1, e_2, \cdots e_n$ $\forall a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in R^n$,

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$$

$\boxed{\text{有限维}}$

补充：空间的维数和基

$C[a,b]$: $[a,b]$所有连续函数

$\forall u \in C[a,b]$

$$u(x) = \sum_{m=-\infty}^{m=+\infty} v(m)\boxed{e^{i2m\pi x}}$$

$\{e^{i2m\pi x}, m = 0, \pm 1, \cdots\}$ 基

$C[a,b]$: 无穷维线性空间

$X = span\{1, x, \cdots x^n\}$
$\subset C[a,b]$

n次多项式函数空间

基为：

$\{1, x, \cdots x^n\}$

X： n维线性空间

子空间

有限元离散

$$V = S_0^1 = \left\{ v \,\middle|\, \int_a^b \left[v^2 + \left(\frac{dv}{dx} \right)^2 \right] dx < \infty \;\; v(a) = 0 \right\}$$

无穷维线性空间

找一个函数$u \in V$, 满足 $D(u,v) = F(v) \quad \forall v \in V$

找一个函数 $u_h \in V_h$, 满足

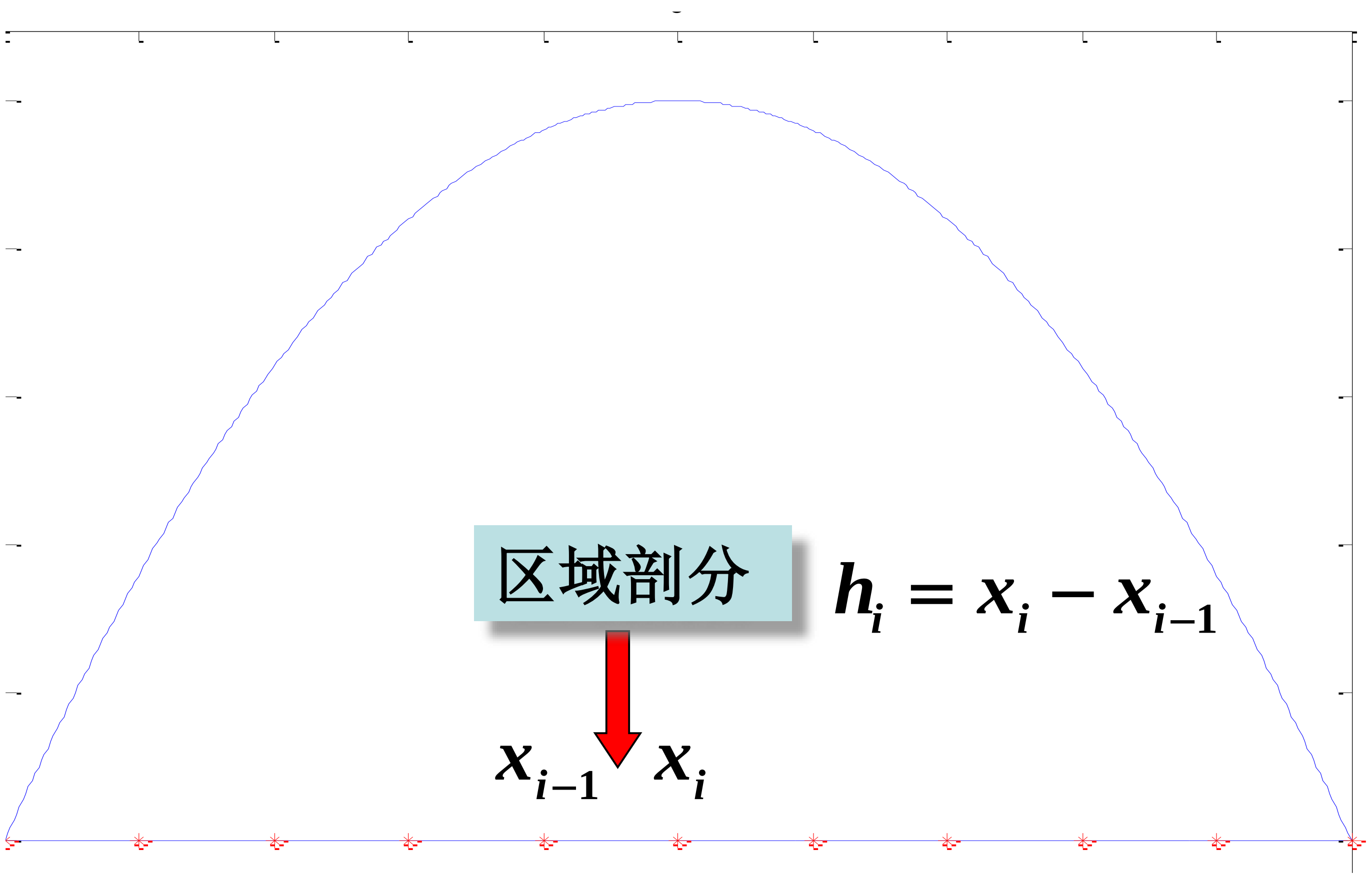
$$D(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

V_h是V 的有限维子空间

找子空间的一个途径：找基函数，从而张成子空间

$$V_h = span\{\varphi_0, \varphi_1, \ldots \varphi_n\}$$

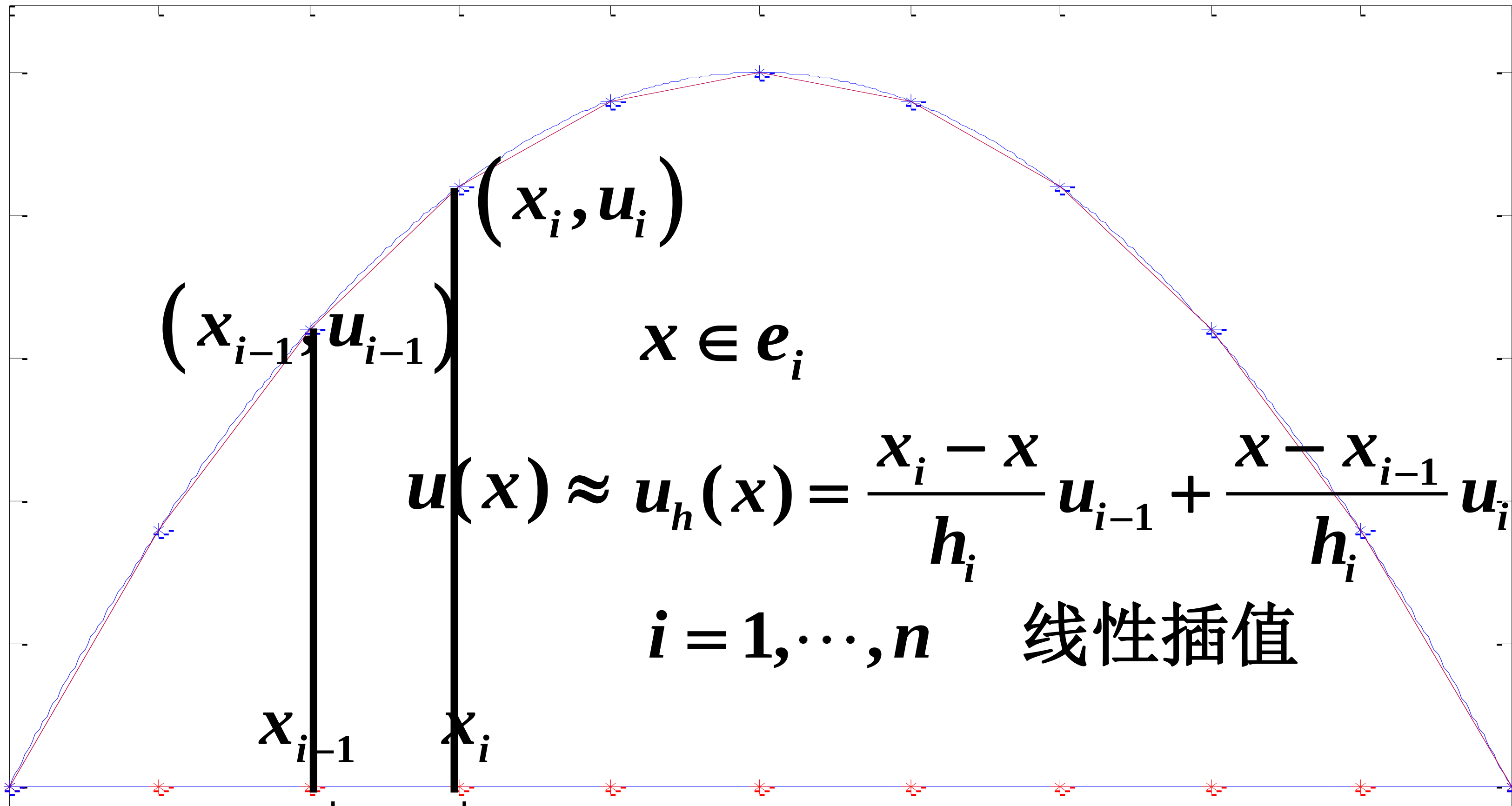
定义离散区间



剖分单元$\mathbf{e_i}$

$$e_i = [x_{i-1}, x_i] \qquad i = 1, \cdots, n$$

u在单元上的取值　　$u_i = u(x_i), i = 1 \cdots n$



$x \in e_i$

$$u(x) \approx u_h(x) = \frac{x_i - x}{h_i} u_{i-1} + \frac{x - x_{i-1}}{h_i} u_i$$

$i = 1, \cdots, n$　　线性插值

$$|u - u_h| \le Ch^2 \left| \frac{d^2 u}{dx^2} \right|$$

当h->0时，插值函数收敛到函数u

基函数

$$u_h(x) = \overset{\varphi_{i-1}}{\boxed{\frac{x_i - x}{h_i}}} u_{i-1} + \overset{\varphi_i}{\boxed{\frac{x - x_{i-1}}{h_i}}} u_i ,$$

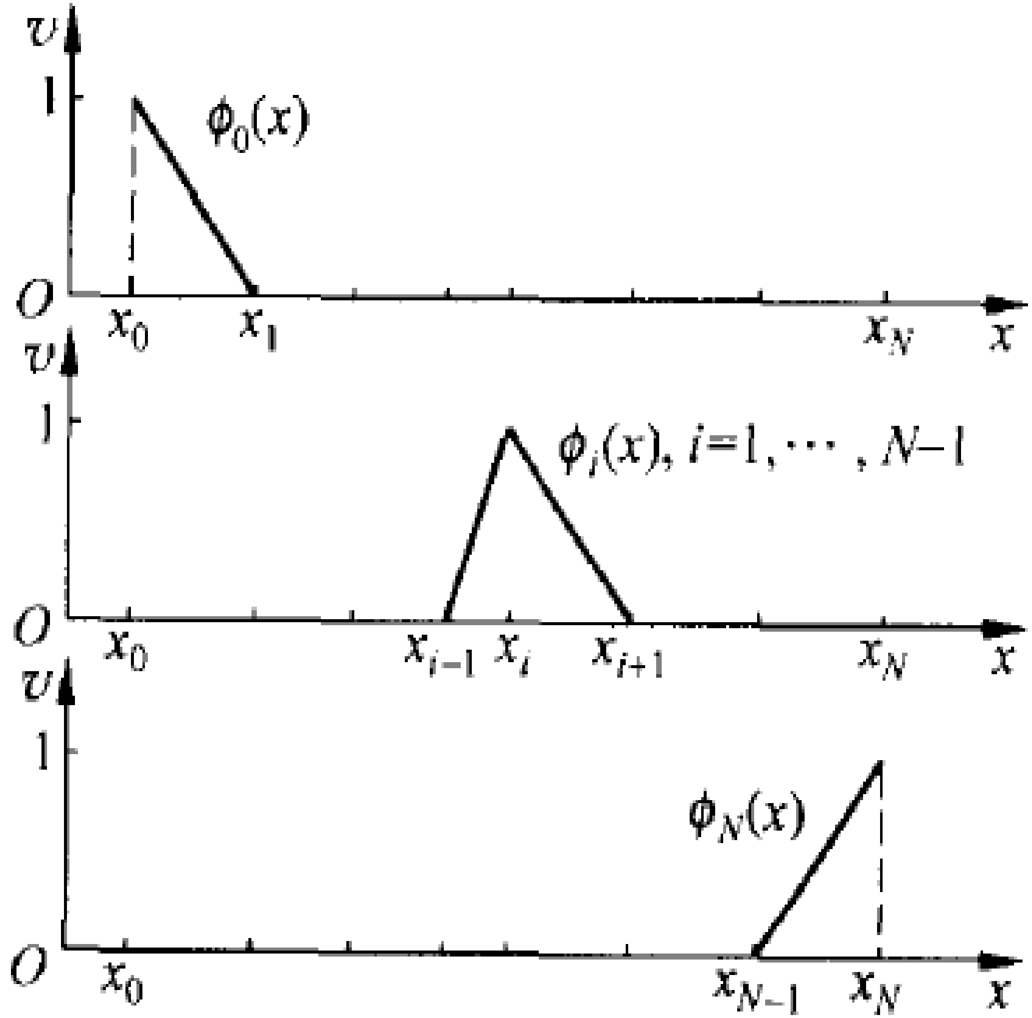
$$x \in e_i = [x_{i-1}, x_i], \quad i = 1, \cdots, n$$

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_i} = 1 + \dfrac{x - x_i}{h_i}, & x \in e_i = [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{h_{i+1}} = 1 - \dfrac{x - x_i}{h_{i+1}}, & x \in e_{i+1} = [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{cases}$$

$$\varphi_0(x) = \begin{cases} \dfrac{x_1 - x}{h_1}, & x \in e_0 = [x_0, x_1] \\ 0, & otherwise \end{cases}$$

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_i} = 1 + \dfrac{x - x_i}{h_i}, & x \in e_i = [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{h_{i+1}} = 1 - \dfrac{x - x_i}{h_{i+1}}, & x \in e_{i+1} = [x_i, x_{i+1}] \\ 0, & \text{otherwise} \\ i = 1, \cdots, n-1 & \end{cases}$$

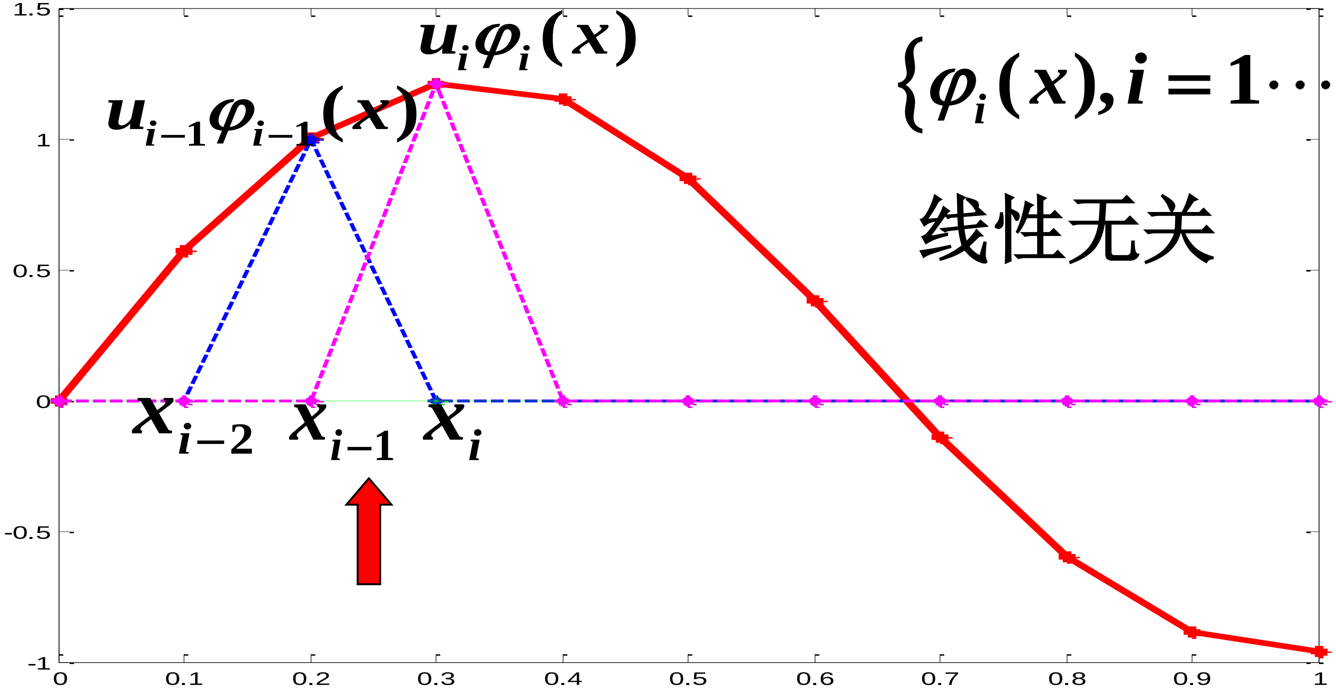
$$\varphi_n(x) = \begin{cases} \dfrac{x - x_{n-1}}{h_n} = 1 + \dfrac{x - x_n}{h_n}, & x \in e_n = [x_{n-1}, x_n] \\ 0, & \text{otherwise} \end{cases}$$



$$\varphi_i(x_k) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

$$u_h(x) = \varphi_{i-1}(x)u_{i-1} + \varphi_i(x)u_i, \qquad x \in e_i,$$

$$= \sum_{i=1}^{n} \varphi_i(x)u_i \qquad x \in [a,b]$$



$$\{\varphi_i(x), i = 1\cdots n\} \subset V$$

线性无关

定义空间： $U_h = span\{\varphi_i(x), i = 0,1\cdots n\}$ n+1维

$$V_h = \{v_h \mid v_h \in U_h, v_h(a) = 0\}$$

试探函数空间

找一个函数$u \in V$, 满足

$$D(u,v) = F(v)$$

$$\forall v \in V$$

找一个函数$u_h \in V_h$, 满足

$$D(u_h, v_h) = F(v_h),$$

$$\forall v_h \in V_h$$

有限元化

4）化为代数方程组--(4.1)单元分析(4.2)总体合成

找一个函数 $u_h \in V_h$

$$\int_a^b \left(p \frac{du_h}{dx} \frac{dv_h}{dx} + q u_h v_h \right) dx + \sigma u_h(b) v_h(b) =$$

$$\int_a^b f v_h dx + \beta v_h(b)$$

$$\forall v_h \in V_h$$

$$[a,b] = \bigcup e_i$$

$$\sum_{e_i} \int_{e_i} \left(p \frac{du_h}{dx} \frac{dv_h}{dx} + q u_h v_h \right) dx + \sigma u_h(b) v_h(b) =$$

$$\sum_{e_i} \int_{e_i} f v_h dx + \beta v_h(b)$$

$$\forall v_h \in V_h$$

(4.1)单元分析

$$\sum_{e_i}\int_{e_i}\left(p\frac{du_h}{dx}\frac{dv_h}{dx}+qu_hv_h\right)dx+\sigma u_h(b)v_h(b)=$$

$$\sum_{e_i}\int_{e_i} fv_h dx+\beta v_h(b)$$

任取一个单元 $e_i=[x_{i-1},x_i]$ $\quad \forall v_h\in V_h$

$$u_h(x)=u_{i-1}\frac{x_i-x}{h_i}+u_i\frac{x-x_{i-1}}{h_i}=Nu_{e_i}$$

$$N=\left[\frac{x_i-x}{h_i},\frac{x-x_{i-1}}{h_i}\right],\qquad u_{e_i}=\begin{pmatrix}u_{i-1}\\ u_i\end{pmatrix}$$

$$u_h(x)=Nu_{e_i},\text{其中}N=\left[\frac{x_i-x}{h_i},\frac{x-x_{i-1}}{h_i}\right],u_{e_i}=\begin{pmatrix}u_{i-1}\\u_i\end{pmatrix}$$

$$\frac{du_h(x)}{dx}=\frac{dN}{dx}u_{e_i}=\left[\frac{-1}{h_i},\frac{1}{h_i}\right]u_{e_i}=Bu_{e_i}$$

$$\text{其中}B=\left[\frac{-1}{h_i},\frac{1}{h_i}\right]$$

同理:$v_h(x)=Nv_{e_i}$, $\quad \dfrac{dv_h}{dx}=Bv_{e_i}$

$$\sum_{e_i}\int_{e_i}\left(p\frac{du_h}{dx}\frac{dv_h}{dx}+qu_hv_h\right)dx+\sigma u_h(b)v_h(b)=\cdots$$

$$u(x)=Nu_{e_i},\qquad \frac{du}{dx}=Bu_{e_i}$$

$$\int_{e_i}\left(pBu_{e_i}Bv_{e_i}+qNu_{e_i}Nv_{e_i}\right)dx$$

$$=\int_{e_i}\left(p\left(Bv_{e_i}\right)^T Bu_{e_i}+q\left(Nv_{e_i}\right)^T Nu_{e_i}\right)dx$$

$$=v_{e_i}^{\ T}\underbrace{\left[\boxed{\int_{e_i}\left(pB^TB+qN^TN\right)dx}\right]}_{K_{e_i}}u_{e_i}=v_{e_i}^{\ T}K_{e_i}u_{e_i}$$

单元刚度矩阵

单元荷载

$$\cdots = \int_{e_i} f v_h dx = \int_{e_i} f N v_{e_i} dx = \int_{e_i} \left(N v_{e_i} \right)^T f dx$$

$$= v_{e_i}{}^T \boxed{\int_{e_i} N^T f dx}$$

令$$F_{e_i} = \int_{e_i} N^T f dx = \begin{bmatrix} F_{i-1}^{e_i} \\ F_i^{e_i} \end{bmatrix}$$

这样，由 $\sum_{i=1}^{n} \int_{e_i} \square + \sigma u_h(b) v_h(b) = \sum_{i=1}^{n} \int_{e_i} \square + \beta v_h(b)$

$$\sum_{i=1}^{n} v_{e_i}{}^T K_{e_i} u_{e_i} + \sigma u_n v_n = \sum_{i=1}^{n} v_{e_i}{}^T F_{e_i} + \beta v_n$$

(4.2)总体分析，边界处理

$$\sum_{i=1}^{n} v_{e_i}{}^{T} K_{e_i} u_{e_i} + \sigma u_n v_n = \sum_{i=1}^{n} v_{e_i}{}^{T} F_{e_i} + \beta v_n$$

$$u = \left(u_0, \cdots, u_n\right)^T, \quad v = \left(v_0, \cdots, v_n\right)^T$$

$$\text{令} C_{e_i} = \begin{array}{c} \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \\ \begin{array}{cccccccc} 1 & & & i-1 & i & & & n+1 \end{array} \end{array}$$

$$\text{则}\ u_{e_i} = C_{e_i} u, \quad v_{e_i} = C_{e_i} v$$

$$\sum_{i=1}^{n} v_{e_i}^{\ T} K_{e_i} u_{e_i} + \sigma u_n v_n = \sum_{i=1}^{n} v_{e_i}^{\ T} F_{e_i} + \beta v_n$$

$$u_{e_i} = C_{e_i} u, \quad v_{e_i} = C_{e_i} v$$

$$v^T \left(\sum_{i=1}^{n} C_{e_i}^{\ T} K_{e_i} C_{e_i} \right) u + \underbrace{\sigma u_n v_n}_{\sigma v^T (0, \cdots 0, u_n)^T}$$

$$= v^T \left(\sum_{i=1}^{n} C_{e_i}^{\ T} F_{e_i} \right) + \underbrace{\beta v_n}_{\beta v^T (0, \cdots 0, 1)^T}$$

$$\cancel{v}^T\left(\sum_{i=1}^{n} C_{e_i}{}^T K_{e_i} C_{e_i}\right)u+\sigma \cancel{v}^T\left(0,\cdots 0,u_n\right)^T$$

$$=\cancel{v}^T\left(\sum_{i=1}^{n} C_{e_i}{}^T F_{e_i}\right)+\beta \cancel{v}^T\left(0,\cdots 0,1\right)^T$$

令$K=\sum_{i=1}^{n} C_{e_i}{}^T K_{e_i} C_{e_i}$ $\quad F=\sum_{i=1}^{n} C_{e_i}{}^T F_{e_i}$

$\Rightarrow$ $Ku+\sigma\left(0,\cdots 0,u_n\right)^T=F+\beta\left(0,\cdots 0,1\right)^T$

$$K_{e_i} = \int_{x_{i-1}}^{x_i} \left(pB^T B + qN^T N \right) dx$$

$$C_{e_i}{}^T K_{e_i} C_{e_i} =$$

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & k_{i-1,i-1}^{e_i} & k_{i-1,i}^{e_i} & 0 & \cdots \\ \cdots & 0 & k_{i,i-1}^{e_i} & k_{i,i}^{e_i} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{(n+1)\times(n+1)}$$

K_{e_i} 单元刚度矩阵

$K=$ 总刚度矩阵

$u_0 = 0$

$D(\varphi_i, \varphi_j)$

$$
K=\begin{bmatrix}
k_{00}^{(1)} & k_{01}^{(1)} & & & & & & \\
k_{10}^{(1)} & k_{11}^{(1)}+k_{11}^{(2)} & k_{12}^{(2)} & & & & & \\
 & k_{21}^{(2)} & k_{22}^{(2)} & \ddots & & & & \\
 & & \ddots & +k_{i-1,i-1}^{(i)} & k_{i-1,i}^{(i)} & & & \\
 & & & k_{i,i-1}^{(i)} & k_{i,i}^{(i)}+k_{i,i}^{(i+1)} & k_{i,i+1}^{(i+1)} & & \\
 & & & & k_{i+1,i}^{(i+1)} & k_{i+1,i+1}^{(i+1)} & \ddots & \\
 & & & & & \ddots & +k_{n-1,n-1}^{(n)} & k_{n-1,n}^{(n)} \\
 & & & & & & k_{n,n-1}^{(n)} & k_{nn}^{(n)}
\end{bmatrix}
$$

K_{e_i}

单元刚度矩阵

同理

$$u_0 = 0$$

$$F = \begin{bmatrix} \cancel{F_0^{(1)}} \\ F_1^{(1)} + F_1^{(2)} \\ F_2^{(2)} + \\ \ddots \\ F_{i-1}^{(i)} \\ F_i^{(i)} + F_i^{(i+1)} \\ F_{i+1}^{(i+1)} + \\ \ddots \\ F_{n-1}^{(n)} \\ F_n^{(n)} \end{bmatrix}$$

总荷载向量

$$F_{e_i}$$

单元荷载矩阵

两点边值问题有限元方法总结

$$\begin{cases} -\dfrac{d}{dx}\left(p\dfrac{du}{dx}\right)+qu=f, \quad x\in(a,b) \\ u(a)=0, \\ p(b)u'(b)+\sigma u(b)=\beta, \end{cases}$$

$$p\in C^1[a,b],\ p(x)\geq p_0>0, q\in C[a,b], q(x)\geq 0$$

1. 区域剖分

$$[a,b]=\bigcup e_i \qquad e_i=[x_{i-1},x_i]$$

准备必要的原始信息

1）单元编号，节点编号，节点坐标

2）节点的局部和整体编码对应关系

3）边界点相关信息

2 计算“单元刚度矩阵”和“单元荷载向量”

$$N=\left[\frac{x_i-x}{h_i},\frac{x-x_{i-1}}{h_i}\right]$$

$$B=\left[\frac{-1}{h_i},\frac{1}{h_i}\right]$$

$$K_{e_i}=\int_{x_{i-1}}^{x_i}\left(pB^TB+qN^TN\right)dx$$

$$F_{e_i}=\int_{x_{i-1}}^{x_i}N^Tf\,dx$$

3 计算“总刚度矩阵”和“总荷载向量”，形成代数方程组

$$K=\sum_{i=1}^{n} C_{e_i}{}^{T} K_{e_i} C_{e_i}$$

$$F=\sum_{i=1}^{n} C_{e_i}{}^{T} F_{e_i}$$

$$Ku+\sigma\left(0,\cdots 0,u_n\right)^{T}=F+\beta\left(0,\cdots 0,1\right)^{T}$$

4 边界条件处理

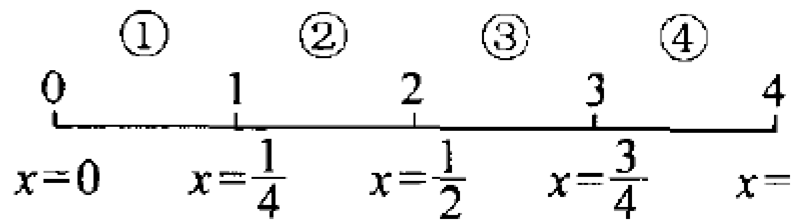
根据边界调整K和F，即掐头去尾

5 解代数方程组, 相关结果输出

例题：

$$\begin{cases} -\dfrac{d^2u}{dx^2} = 2, \quad x \in [0,1] \\ u(0) = 0, \\ u'(1) = 0. \end{cases}$$

解：



1）单元剖分

4等分，即$h_i=h=1/4$，$i=1,2,3,4$

$x_i=ih$，$i=1,2,3,4$; $x_0=0$

2）计算“单元刚度矩阵”和“单元荷载向量”

$$N = \left[\frac{x_i - x}{h}, \frac{x - x_{i-1}}{h} \right] \qquad B = \left[\frac{-1}{h}, \frac{1}{h} \right]$$

$$-\frac{d}{dx}\left(p\frac{du}{dx}\right)+qu=f$$

$$\Rightarrow -\frac{d^2u}{dx^2}=2$$

$$K_{e_i}=\int_{x_{i-1}}^{x_i}\left(pB^TB+qN^TN\right)dx$$

$$=\int_{x_{i-1}}^{x_i}\frac{1}{h^2}\begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}-1 & 1\end{pmatrix}dx$$

$$=\frac{1}{h}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

$$B=\left[\frac{-1}{h},\frac{1}{h}\right]$$

$$F_{e_i}=\int_{x_{i-1}}^{x_i}N^Tfdx \quad =\begin{pmatrix}\displaystyle\int_{x_{i-1}}^{x_i}\frac{x_i-x}{h}fdx\\ \displaystyle\int_{x_{i-1}}^{x_i}\frac{x-x_{i-1}}{h}fdx\end{pmatrix}$$

$$F_{e_i} = \begin{pmatrix} \displaystyle\int_{x_{i-1}}^{x_i} \frac{x_i - x}{h} f dx \\ \displaystyle\int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{h} f dx \end{pmatrix} \qquad f = 2$$

$$= h\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad i = 1, 2, 3, 4$$

3）计算“总刚度矩阵”和“总荷载向量”

$$K=\frac{1}{h}\begin{pmatrix} \boxed{\begin{matrix}1 & -1\\ -1 & 1\end{matrix}} + \boxed{\begin{matrix}1 & -1\\ -1 & 1\end{matrix}} + \boxed{\begin{matrix}1 & -1\\ -1 & 1\end{matrix}} + \boxed{\begin{matrix}1 & -1\\ -1 & 1\end{matrix}} \end{pmatrix}$$

$$=\frac{1}{h}\begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}$$

$$K_{e_i}=\frac{1}{h}\begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix} \quad \text{i=1,2,3,4}$$

3）计算“总刚度矩阵”和“总荷载向量”

$$F_{e_i} = h\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad i = 1,2,3,4$$

$$F = h\begin{pmatrix} 1 \\ 1+1 \\ 1+1 \\ 1+1 \\ 1 \end{pmatrix} = h\begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

4）根据边界调整K和F，得

$$Ku = F \qquad u(0) = 0 \Rightarrow u_0 = 0$$

$$K = \frac{1}{h}\begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix} \qquad F = h\begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

5）求解代数方程组

$$\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\left(u_1, u_2, u_3, u_4\right)^T = \left(\frac{7}{16}, \quad \frac{12}{16}, \quad \frac{15}{16}, 1\right)^T$$

- 有限元法内容总结

1 明确微分方程

2 转化为变分形式；

3 有限元离散√

- 区域单元剖分；确定单元基函数；

4 转化为代数方程 √

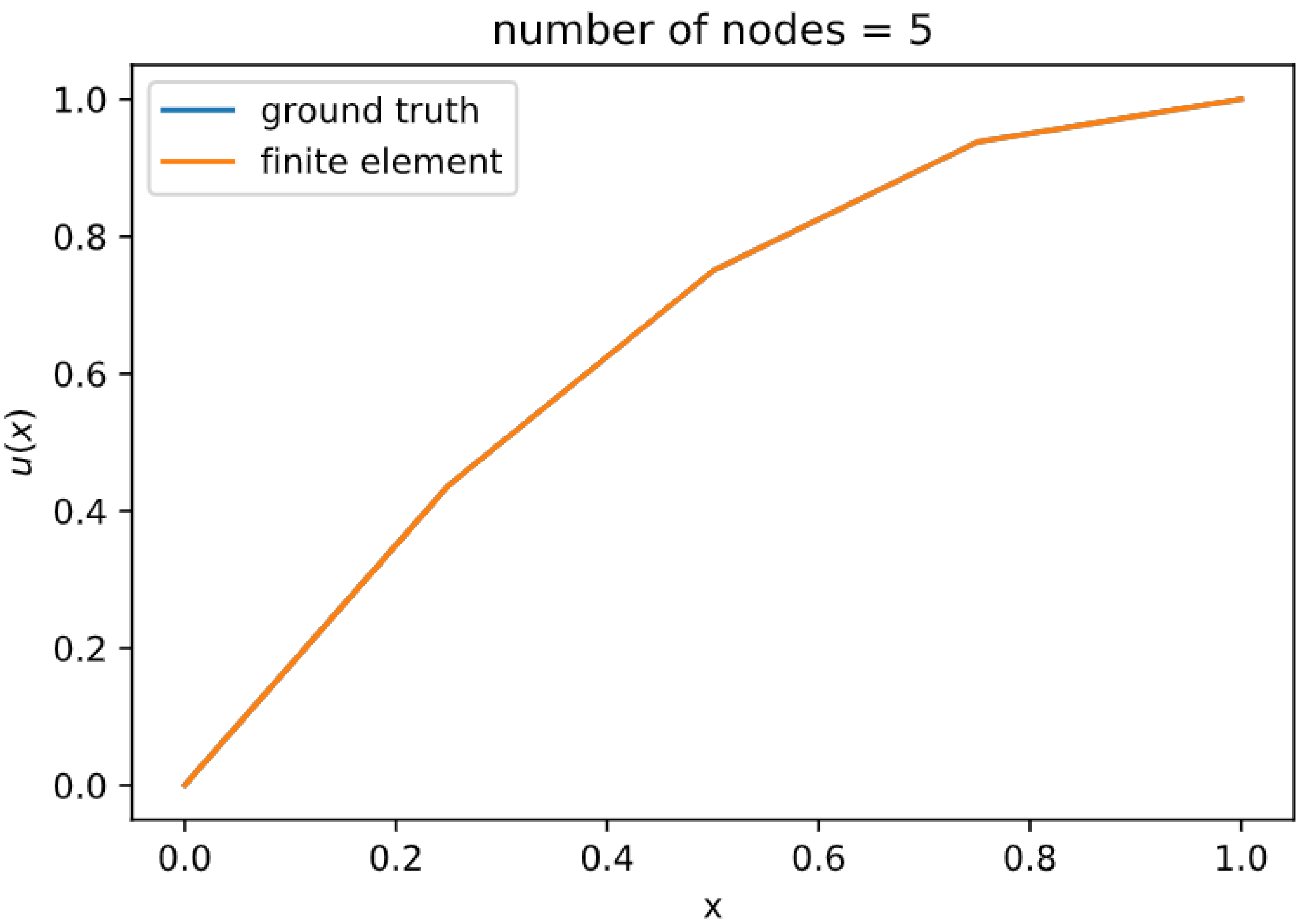
- 单元分析；总体合成；边界条件的处理；

5 解代数方程。 √

```python
from scipy.sparse import dia_matrix
from scipy.sparse.linalg import inv
from numpy import pi

class FEM:
    def __init__(self, nodes, xmin=0, xmax=1):
        self.nodes = nodes
        x = np.linspace(xmin, xmax, nodes)
        self.x = x
        self.h = x[1] - x[0]

    def Kmatrix(self):
        n = self.nodes
        m = 1/self.h * np.ones(n-1)
        m1= 1/self.h * np.ones(n-1)
        m1[n-2] = 0.0
        data = [-m, m+m1, -m]
        offsets = [-1, 0, 1]
        # 使用 scipy.sparse 稀疏矩阵库，构造 3 对角稀疏矩阵 K 压缩.tocsc()
        K = dia_matrix((data, offsets), shape=(n-1, n-1)).tocsc()
        return K
```

```
def bvec(self):
        m2 = np.ones(n-1)
        m3 = np.ones(n-1)
        m3[n-2] = 0.0
        return self.h * (m2+m3)

    def solve(self):
        K = self.Kmatrix()
        b = self.bvec()
        return inv(K) * b

    def compare(self):
        ground_truth = -(self.x)**2 + 2 * self.x
        fem_res = self.solve()
        print(fem_res)
        femsol = np.array([0])
        femsol = np.append(femsol,fem_res)
        plt.plot(self.x, ground_truth, label="ground truth")
        plt.plot(self.x, femsol, label="finite element")
        plt.title("number of nodes = %s"%self.nodes)
        plt.xlabel("x")
        plt.ylabel(r"$u(x)$")
        plt.legend(loc='best')
        plt.show()
```

调用 fem = FEM(1001) fem.compare()



课堂练习

1.请描述有限元方法求解问题的步骤。

2.请采用四个长度相等的单元计算下列一维问题：

$$\begin{cases} -u'' = f = \begin{cases} 1 & x \in [0,1/2) \\ 2 & x \in [1/2,1) \end{cases} \\ u(0) = 0, \quad u(1) = 0 \end{cases}$$