

Coexistence of Coherence and Incoherence in Nonlocally Coupled Phase Oscillators

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The phase oscillator model with global coupling is extended to the case of finite-range nonlocal coupling. Under suitable conditions, peculiar patterns emerge in which a quasi-continuous array of identical oscillators separates sharply into two domains, one composed of mutually synchronized oscillators with unique frequency and the other composed of desynchronized oscillators with distributed frequencies. We apply a theory similar to the one which successfully explained the onset of collective synchronization in globally coupled phase oscillators with frequency distribution. A space-dependent order parameter is thus introduced, and an exact functional self-consistency equation is derived for this quantity. Its numerical solution is confirmed to reproduce the simulation results accurately.

Key words: nonlocal coupling, phase oscillators, order parameter

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1 Introduction

Large populations and continuous fields of coupled oscillators form a representative class of synergetic systems met in a wide range of scientific disciplines from physics, chemistry, engineering, biology to brain science [1, 2, 3, 4, 5]. Collective dynamics of coupled oscillators depends crucially on the range of their mutual coupling. It has recently been realized that when the coupling is nonlocal, the patterns which emerge could be drastically different from those which we expect for oscillators with local or global coupling [6, 7, 8]. The implication of this fact is relevant even to what we conventionally call locally coupled systems, typically reaction-diffusion systems. This is because it may happen that nonlocality can arise *effectively* as a result of elimination of some variables, e.g., rapidly diffusing components in the case of reaction-diffusion dynamics. Among

the variety of patterns which are characteristic to nonlocally coupled oscillators, we will focus our attention below on a particular class of patterns in which the whole medium is separated into two domains of qualitatively different dynamics. Specifically, the oscillators are mutually synchronized in one domain while they are completely desynchronized in the other domain. A preliminary work on such dynamics was reported recently [8]. We will present below a more thorough investigation of this problem.

The collective dynamics of our concern is similar to the collective synchronization in globally coupled oscillators with distributed natural frequencies [4, 9, 10] where the whole population splits into two subpopulations each composed of synchronized and desynchronized oscillators. There the systems is stationary in a statistical sense within a constant drift of the collective phase corresponding to the os-

cillation of the population as a whole. Similarly to the case of global coupling, the individual oscillators can be regarded as being controlled by a common mean field, although the latter is now space dependent. The oscillators under consideration are identical in nature, still we may develop an exact mean-field theory similar to the one developed for globally coupled oscillators. Commonly to these theories, a suitably defined order parameter representing the collective state is determined from a condition for self-consistency which must exist between the collective dynamics and the dynamics of the individual oscillators.

In section 2, some results of our numerical simulation will be presented. The simulation is carried out first on a nonlocal version of the complex Ginzburg-Landau (CGL) equation in one space dimension with periodic boundary conditions, and then on another equation obtained through the phase reduction of the nonlocal CGL. In section 3, our theory will be developed and a functional self-consistency equation will be derived for the order parameter. Then the numerical solution of the self-consistency equation will be compared with the simulation results presented in Section 2. Concluding remarks will be given in the final section.

2 Coexistence of coherence and incoherence

As a simplest model for densely and uniformly distributed oscillators with nonlocal coupling, let us consider the following equation for a complex amplitude A which we call nonlocally coupled complex Ginzburg-Landau equation[5, 6]. In one spatial dimension our model is expressed as

$$\frac{\partial}{\partial t} A(x, t) = (1 + i\omega_0)A - (1 + ib)|A|^2 A + K(1 + ia) \left(Z(x, t) - A(x, t) \right). \quad (1)$$

The quantity $Z(x, t)$, which we call the mean field, represents the effects of the nonlocal coupling and

is given by

$$Z(x, t) = \int G(x - x') A(x', t) dx'. \quad (2)$$

The coupling function G changes with the distance as

$$G(y) = \frac{\kappa}{2} \exp(-\kappa|y|) \quad (3)$$

and is normalized. The exponential form of G is in fact a natural consequence of the reductive derivation of Eq. (1) from a certain class of reaction-diffusion systems when the latter involve an inactive diffusive component to be eliminated adiabatically [6, 7]. Note that the last term in Eq. (1) representing the coupling is so arranged that it may vanish when A is uniform in space. If A is nonuniform but sufficiently long-waved, this term can be approximated with a diffusion term, and then Eq. (1) is reduced to the ordinary CGL.

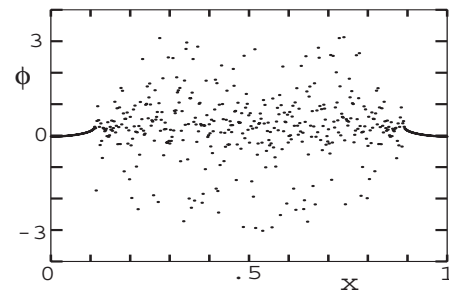


FIG. 1. Instantaneous spatial distribution of the phases obtained from Eq. (1) approximated with an array of 512 oscillators distributed over an interval of length 1 with periodic boundary conditions. Parameter values are $a = -1.0$, $b = 0.88$, $\kappa = 4.0$.

Equation (1) is numerically simulated on a finite interval $[0, 1]$ with periodic boundary conditions. In a suitable range of parameters, in which the uniform oscillation is linearly stable, the system can develop peculiar patterns when the initial perturbation is finite. Figure 1 shows a typical example of such phase patterns. The initial condition is such that the modulus of A equals 1 everywhere and its phase changing randomly still its envelope being nearly symmetric about the midpoint $x = 1/2$. Since the coupling constant K is assumed to be relatively

small, the local oscillations attain almost the full amplitude, so that we will mainly be interested in the phase pattern. The pattern consists of two distinct domains. In one domain which appears near the boundaries, the pattern is spatially continuous and smooth, whereas in the central domain spatial continuity seems completely lost. The pattern as a whole looks steadily advancing upward, implying that the entire population is undergoing regular collective oscillation with a definite frequency Ω .

To investigate the nature and origin of such patterns in further detail, Eq. (1) is reduced to a phase equation which would be much easier to analyze. The phase reduction is valid when the coupling strength K is small, which is actually the case, and the reduced equation takes the form

$$\frac{\partial}{\partial t}\phi(x, t) = \omega - \int G(x - x') \times \sin \left(\phi(x, t) - \phi(x', t) + \alpha \right) dx', \quad (4)$$

where the time scale has been so changed as to normalize the coupling strength, ω is the natural frequency rescaled accordingly, and the phase constant α in the phase-coupling function is related to the original parameters through

$$\tan \alpha = \frac{b - a}{1 + ab}, \quad \alpha(b - a) > 0. \quad (5)$$

Under the parameter condition corresponding to the previous numerical simulation of Eq. (1), a phase pattern similar to FIG. 1 was obtained, which is displayed in FIG. 2.

We now focus on the phase patterns obtained for our phase equation. Let us introduce relative phase ψ by

$$\phi = \Omega t + \psi \quad (6)$$

which describes the dynamics of the phase deviation from the reference motion with some drift velocity Ω whose value is still open. We rewrite Eq. (4) using ψ as

$$\frac{\partial}{\partial t}\psi(x, t) = \omega - \Omega - \int dx' G(x - x') \times \sin \left(\psi(x, t) - \psi(x', t) + \alpha \right). \quad (7)$$

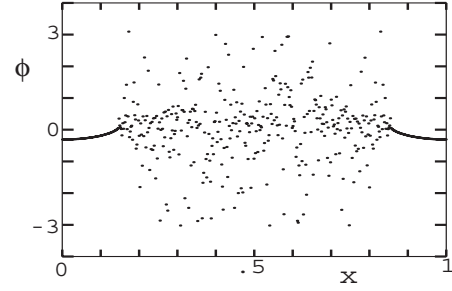


FIG. 2. Instantaneous spatial distribution of the phases obtained from Eq. (4) approximated with an array of 512 oscillators distributed over an interval of length 1 with periodic boundary conditions. Parameter values are $\alpha = 1.457$, $\kappa = 4.0$.

As a generalization of the theory of synchronization transition of globally coupled phase oscillators with frequency distribution, we introduce a complex order parameter with modulus R and phase Θ through

$$\int dx' G(x - x') \exp[i\psi(x', t)] = R(x, t) \exp[i\Theta(x, t)]. \quad (8)$$

Unlike the order parameter defined for globally coupled oscillators, the above quantity is space-dependent. Still a physical picture is valid such that we are practically working with an assembly of *independent* oscillators under the control of a common forcing field represented by R and Θ . This is confirmed by rewriting Eq. (7) in terms of the order parameter into a forced one-oscillator equation as

$$\frac{\partial}{\partial t}\psi(x, t) = \omega - \Omega - R(x, t) \sin(\psi(x, t) + \alpha + \Theta(x, t)). \quad (9)$$

The spatial profiles of $R(x)$ and $\Theta(x)$ after a long-time average are displayed in FIG. 3a and 3b, respectively. We see that the forcing mean-field amplitude is stronger near the boundaries and weaker near the center of the system. If the phase distribution such as shown in FIG. 2 is statistically stationary within a constant overall drift with velocity Ω , then R and Θ are time-independent. Their approximate constancy is confirmed by the fact that the fluctuation of the time sequence of $R(x, t)$ observed at the midpoint of the system $x = 1/2$ gives

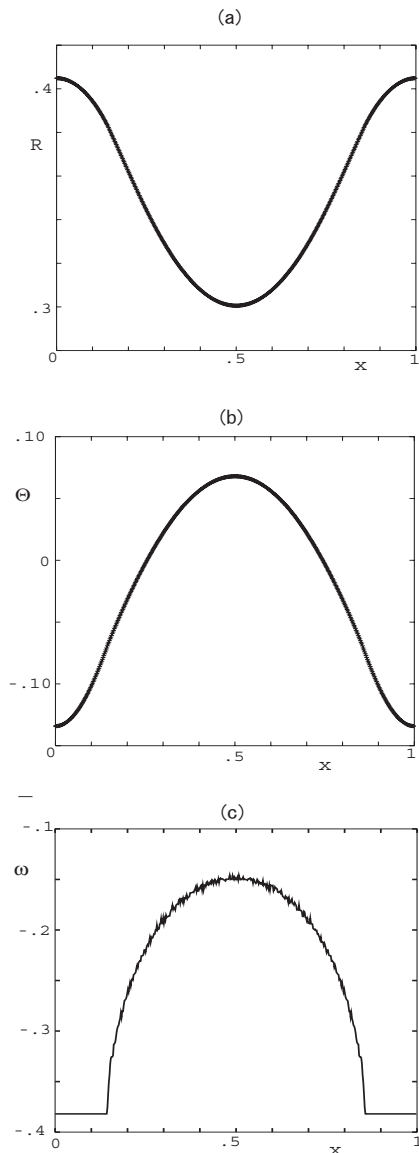


FIG. 3. Spatial profiles of the long-time average of the order parameter amplitude $R(x)$ (a), the order parameter phase $\Theta(x)$ (b) and actual frequencies $\Omega(x)$ (c), each obtained from Eq. (4).

a standard deviation as small as 0.008 which comes possibly from a finite-size effects. Similar property can also be confirmed for $\Theta(x, t)$.

Figure 3c shows a distribution of the actual frequencies $\bar{\omega}(x)$ of the individual oscillators. This was also obtained through a long-time average. In the coherent domain, the oscillation frequencies have an

identical value Ω , while in the incoherent domain they are distributed but give a well-defined continuous curve.

The dynamics which is going on is now clear. The system is divided into two subgroups of oscillators. In the first group, the forcing amplitude is large enough for the oscillators to be entrained, so that they oscillate with an identical frequency Ω . In the second group, in contrast, the forcing amplitude is too weak for entrainment, so that the frequencies of the individual oscillators differ from Ω . Since the actual frequencies of these desynchronized oscillators should depend on the local amplitude of the forcing, the nonuniform but smooth pattern of $R(x)$ as shown in FIG. 3a implies a smooth distribution of the actual frequencies, and this is consistent with FIG. 3c showing $\bar{\omega}(x)$. Our next problem is how to predict stationary patterns of $R(x)$ and $\Theta(x)$ theoretically, and this will be the subject of the next section.

3 Theory

The basic equations to work with below are Eqs. (8) and (9). Assuming R and Θ to be time-independent, we first try to find solutions of Eq. (9) for the phases $\psi(x, t)$ each of which should be a function of $R(x)$ and $\Theta(x)$. These solutions are then substituted into Eq. (8), leading to an equation for determining R and Θ as a *functional* of R and Θ in a self-consistent manner. Since the collective frequency Ω is still to be determined, we will be working with a nonlinear eigenvalue problem.

Here a question may arise as to the seeming contradiction between the assumed time-independence of R and Θ and the apparent time-dependence of the solution of Eq. (9) which is true if $R(x)$ is below a certain critical value. The same problem was encountered in the problem of synchronization transition in globally coupled phase oscillators with frequency distribution. The answer to this question is that the factor $\exp[i\psi(x', t)]$ appearing in the integral of Eq. (8) should be replaced with its statistical

average $\langle \exp[i\psi(x', t)] \rangle$. Such an average can be calculated using a suitable invariant measure which is easy to find.

The successive steps to be taken for reaching the final solution are the following. Note first that Eq. (9) admits two types of solutions depending on the location of the oscillators of concern. They are constant solutions and drifting solutions. If $|R/(\omega - \Omega)| \geq 1$, we have a time-independent solution given by

$$\theta_0(R) = \sin^{-1} \left(\frac{\omega - \Omega}{R(x)} \right) - \alpha. \quad (10)$$

If $|R/(\omega - \Omega)| < 1$, in contrast, we have a drifting solution. The actual frequency $\bar{\omega}$ of the corresponding desynchronized oscillator as a function of R is given by

$$\bar{\omega} = \Omega + (\omega - \Omega) \sqrt{1 - \left(\frac{R}{\omega - \Omega} \right)^2}, \quad (11)$$

which equals the collective frequency Ω plus the mean drift velocity of ψ . Instead of using such drifting solutions in Eq. (9), we use an invariant measure, i.e., the probability density $p(\theta, R)$ such that ψ takes on value θ for given R , which is inversely proportional to the drift velocity of ψ at $\psi = \theta$. Thus,

$$p(\theta, R) = C(\omega - \Omega - R \sin \theta)^{-1}, \quad (12)$$

where C is a normalization constant given by $C = (2\pi)^{-1} \sqrt{(\omega - \Omega)^2 - R^2}$. As noted earlier, this invariant measure is used for replacing the exponential factor appearing on the left-hand side of Eq. (8) with its statistical average. The physical idea underlying this replacement is that an equilibrium statistical ensemble may be applicable to each small local subsystem because infinitely many oscillators which are independently oscillating are already contained in such a subsystem.

The spatial profile of $R(x)$ in FIG. 3 implies that there are spatial points $x = \frac{1}{2} \pm x_c$ at which R takes a critical value separating the two types of solutions

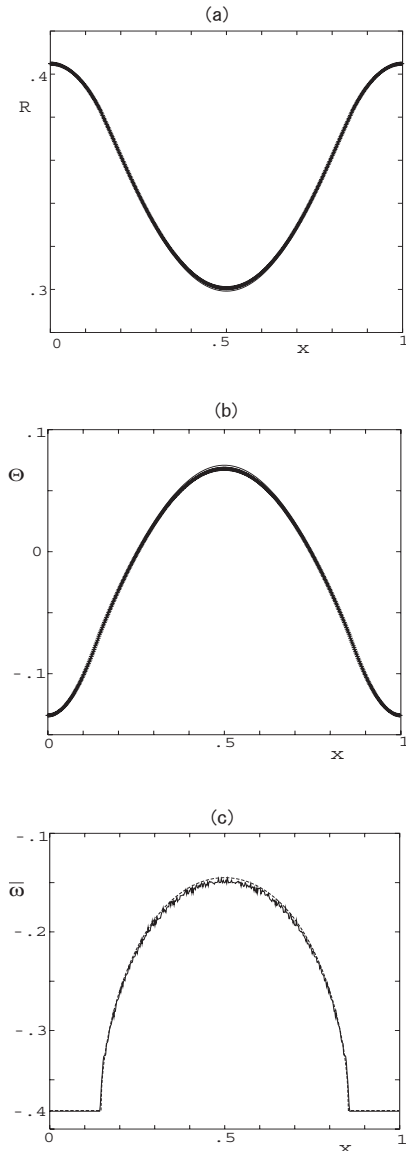


FIG. 4. Theoretical curves for $R(x)$ (solid line in (a)), $\Theta(x)$ (solid line in (b)) and $\Omega(x)$ (broken line in (c)) are compared with the corresponding quantities in FIG. 3 obtained numerically.

described above. Namely, in the inner domain ($|x' - \frac{1}{2}| < x_c$) the solutions are drifting, while in the outer domain ($|x' - \frac{1}{2}| \geq x_c$) they are constant in time. Thus Eq. (8) becomes

$$\int_0^1 dx' G(|x - x'|) \exp[i(\Theta(x') - \alpha)] h(R(x')) = R(x) \exp[i\Theta(x)], \quad (13)$$

where

$$h(R) = \begin{cases} \exp[i\theta_0(R)] & (|x' - \frac{1}{2}| \geq x_c) \\ \int_0^{2\pi} \exp(i\theta)p(\theta, R)d\theta & (|x' - \frac{1}{2}| < x_c) \end{cases} \quad (14)$$

The above expression for $h(R)$ in each domain can further be made explicit as

$$\exp[i\theta_0(R)] = \sqrt{1 - \left(\frac{\omega - \Omega}{R}\right)^2} + i\frac{\omega - \Omega}{R}, \quad (15)$$

$$\int_0^{2\pi} \exp(i\theta)p(\theta, R)d\theta = \frac{i}{R} \left\{ \omega - \Omega - \sqrt{(\omega - \Omega)^2 - R^2} \right\}. \quad (16)$$

Equations (13)~(16) constitute a functional self-consistency condition from which $R(x)$, $\Theta(x)$ and Ω are to be determined. By using $R(x)$ and Ω thus obtained, the actual frequencies of the desynchronized oscillators are determined from Eq. (11).

There is a unique value Ω of the collective frequency for which our functional self-consistency equation admits a solution. Such solution can be obtained numerically, and is compared with our numerical simulation. Some of the results are given in FIG. 4a~4c. We find that the agreement between the theory and numerical experiments is almost perfect for each quantity of $R(x)$, $\Theta(x)$, Ω and $\bar{\omega}(x)$.

4 Concluding remarks

The present theory makes full use of the fact that the number of oscillators contained in the range of coupling is practically infinite. This fact allows us to apply a mean field theory, and our problem is reduced to a one-oscillator problem supplemented with a self-consistency condition. It is well known that the same idea works nicely for globally coupled oscillators. The present theory suggests that, beyond a special problem discussed in the present paper, the mean-field idea may work more generally for nonlocally coupled systems as far as the coupling radius remains much longer than the minimum length scale associated with the discreteness of the spatial distribution of the

constituent oscillators.

Acknowledgement

The present work gives a slight extension of my old work on globally coupled phase oscillators which was developed in the middle of 1970's, shortly after which I spent a very fruitful year at Professor Hermann Haken's Institute in Stuttgart. Therefore, it is my great pleasure to dedicate this small work to Professor Haken.

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