MATH-6800, Problem Set 3, 10-1-15

Name: Michael Hennessey

1. Let $P \in \mathbb{C}^{m\times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with equality if and only if P is an orthogonal projector.

Proof. $\Rightarrow$ If P is orthogonal, we can write it in the form $P = Q^*\Sigma Q$ where Q is unitary and Σ is diagonal with entries $\sigma_{ii} = 1$ or 0. Then, by the properties of the 2-norm:

$$\|P\|_2 = \|Q^*\Sigma Q\|_2 = \|\Sigma\|_2 = 1.$$

$\Leftarrow$ We begin by showing that $\|P\|_2 > 1$ for oblique P.
Let $v = x + \lambda a$ with $x = Pv, \lambda a = (I - P)v$. We choose x and a such that $\|x\|_2 = \|a\|_2 = 1$. Therefore, we are choosing $\|Pv\|_2 = 1$. Now, we attempt to minimize $\|v\|_2$ in order to maximize the 2-norm of P.

$$\|v\|_2^2 = \|x + \lambda a\|_2^2 = (x + \lambda a)^*(x + \lambda a) = x^*x + 2\lambda x^*a + \lambda^2 a^*a$$

$$= \|x\|_2^2 + \lambda^2 \|a\|_2^2 + 2\lambda x^*a = 1 + \lambda^2 + 2\lambda x^*a$$

If we choose $\lambda = -x^*a$, we get $\|v\|_2 < 1$:

$$\|v\|_2^2 = 1 + |x^*a|^2 - 2|x^*a|^2 = 1 - |x^*a|^2.$$

Since P is oblique, $x^*a \neq 0$, and

$$\|v\|_2^2 = 1 - |x^*a|^2 < 1 \implies \|v\|_2 < 1.$$

Thus

$$\|P\|_2 = max_{\|y\|\neq 0}\frac{\|Py\|_2}{\|y\|_2} \geq \frac{\|Pv\|_2}{\|v\|_2} > 1.$$

However, if P is orthogonal, we know $x^*a = 0$, then $\|v\|_2 = 1$. Then,

$$\|P\|_2 = max_{\|y\|_2\neq 0}\frac{\|Py\|_2}{\|y\|_2} \geq \frac{\|Pv\|_2}{\|v\|_2} = 1.$$

Therefore, if $\|P\|_2 = 1$, P must be orthogonal. ■

2. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) Using any method, determine a reduced QR factorization $A = \hat{Q}\hat{R}$ and a full QR factorization $A = QR$.

$$r_{11} = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2} \implies q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$r_{12} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$r_{22} = \|a_2 - r_{12}q_1\|_2 = \|a_2\|_2 = 1 \implies q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A = \hat{Q}\hat{R} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$$

To find $A = QR$ we must find $q_3 = [q_x, q_y, q_z]^T$ orthogonal to q_1, q_2:

$$q_1^* q_3 = 0 \implies q_x = -q_z$$

$$q_2^* q_3 = 0 \implies q_y = 0$$

Then,

$$q_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$A = QR = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) Determine reduced and full QR factorizations $B = \hat{Q}\hat{R}$ and $B = QR$.

$$r_{11} = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2} \implies q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$r_{12} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \sqrt{2}$$

$$r_{22} = \left\| \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\|_2 = \sqrt{3}$$

$$\implies q_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

$$B = \hat{Q}\hat{R} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

To find $B = QR$ we must find $q_3 = [q_x, q_y q_z]^T$ orthogonal to q_1, q_2. We solve the matrix equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = 0$$

We then get

$$\tilde{q}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \implies q_3 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Then

$$B = QR = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

3. Let A be an $m \times n$ matrix ($m \geq n$), and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization.

   (a) Show that A has rank n if and only if all the diagonal entries of $\hat{R}$ are nonzero.

   Proof. $\Rightarrow$ If A has rank n, $r_{jj} > 0$ by Theorem 7.2.
   $\Leftarrow$ We begin by looking at the matrix equation $Ax = 0$ to determine the dimension of the null space of A. We then factor A to get

   $$QRx = 0.$$

   We then multiply both sides of the equation by Q^* to get

   $$Rx = 0.$$

   We can then solve this equation by back substitution. Since all diagonal elements of R are nonzero, we know that the dimension of the null space of A is zero. Therefore, the rank of A is n. ∎

   (b) Suppose $\hat{R}$ has k nonzero diagonal entries for some k with $0 \leq k < n$. What does this imply about the rank of A?

   $rank(A) \geq k$
   Proof. Following the same logic as the proof above, the number of independent columns of A can be determined by the null space of A.

   $$Ax = 0 \implies QRx = 0 \implies Rx = 0$$

   Since R has k nonzero diagonal entries, we know that the dimension of the null space of A can be equal to $n-k$. Thus A would have $n-(n-k) = k$ independent columns. We look at the matrix

   $$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   . We know $rank(A) = 1$, but $r_{11} = r_{22} = 0$. Therefore, $rank(A) \geq k$. ∎

4. Write matlab functions $[Qc, Rc] = clgs(A)$ and $[Qm, Rm] = mgs(A)$ that implement the reduced QR factorization using the classical Gram-Schmidt and modified Gram-Schmidt algorithms, respectively. Test the implementations by computing the QR factorization for the $m \times m$ Vandermonde matrix for points $x_i \equiv (i-1)/(m-1)$, and compare the results from the built-in Matlab function $[Q, R] = qr(A)$.

   (a) For $m = 5$, compute $\|A - QR\|_2$ for each approximation. Also compute the 2-norm differences $\|Q_i - Q\|_2, \|R_i - R\|_2$ for $i = c$ and $i = m$, and also compute the error $\|Q^*Q - I\|_2$ for each approximation to Q.

   (b) Repeat (a), but with $m = 100$.

In order to compute the norms we use the algorithms below:

```
    %Vandermonde Matrix creator
function [Vm]=Vandermonde(m)
for i=1:m
    x(i)=(i-1)/(m-1);
    for j=1:m
        Vm(i,j)=x(i)^(j-1);
    end
end
```

```
function [Qm,Rm]=mgs(A)
%Modified Gram-Schmidt
%Compute the reduced QR factorization

[m,n]=size(A);
Rm=zeros(n,n);
Qm=zeros(m,n);
I=1:m;
for i=1:n
    v(I,i)=A(I,i); %column i
end
for i=1:n
    Rm(i,i)=norm(v(I,i),2);
    Qm(I,i)=v(I,i)/Rm(i,i);
    for j=(i+1):n
        Rm(i,j)=dot(Qm(I,i),v(I,j));
        v(I,j)=v(I,j)-Rm(i,j)*Qm(I,i);
    end
end
```

```
function [Qc,Rc]=clgs(A)
%classical G-S
% Compute the reduced QR factorization

[m,n]=size(A); %dimensions of A
Rc=zeros(n,n);
Qc=zeros(m,n);
I=1:m; %q0 index range 1,2,...,m
for j=1:n
    vj=A(I,j); %column j
    for i=1;(j-1);
        Rc(i,j)=dot(Qc(I,i),A(I,j));
        vj=vj-Rc(i,j)*Qc(I,i);
    end
    Rc(j,j)=norm(vj,2); %2-norm
    Qc(I,j)=vj/Rc(j,j);
end
```

```
function [Qp,Rp]=qr_plus(Q,R)
%make all diagonal elements of R and corresponding columns of Q positive

[m,m]=size(R);
I=1:m;
for i=1:m
        if R(i,i)<0
            Rp(i,I)=-1*R(i,I);
            Qp(I,i)=-1*Q(I,i);
        else
            Rp(i,I)=R(i,I);
            Qp(I,i)=Q(I,i);
        end
end
```

See attached printouts for computations.