MATH-6600, CLA Problem Set No. 7, 11-12-15

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1. Derive a block LU decomposition assuming no pivoting of a block tridiagonal matrix D. What are the conditions you need for this LU decomposition to exist?

   Solution:

   We can essentially use the Gaussian Elimination (GE) algorithm without pivoting to compute the block LU factorization. We can begin by eliminating A_2:

$$\begin{bmatrix} I & & & & \\ F & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{bmatrix} \begin{bmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & \\ & A_3 & B_3 & C_3 & \\ & & \ddots & \ddots & \ddots \\ & & & A_n & B_n \end{bmatrix} = \begin{bmatrix} B_1 & C_1 & & & \\ 0 & B_2 + FC_1 & C_2 & & \\ & A_3 & B_3 & C_3 & \\ & & \ddots & \ddots & \ddots \\ & & & A_n & B_n \end{bmatrix}$$

   Then

$$FB_1 + A_2 = 0 \implies F = -A_2 B_1^{-1}.$$

   This gives the first step of the block LU factorization:

$$L_1^{-1} D = U_1 \implies$$

$$\begin{bmatrix} I & & & & \\ -A_2B_1^{-1} & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{bmatrix} \begin{bmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & \\ & A_3 & B_3 & C_3 & \\ & & \ddots & \ddots & \ddots \\ & & & A_n & B_n \end{bmatrix} = \begin{bmatrix} B_1 & C_1 & & & \\ 0 & B_2 - A_2B_1^{-1}C_1 & C_2 & & \\ & A_3 & B_3 & C_3 & \\ & & \ddots & \ddots & \ddots \\ & & & A_n & B_n \end{bmatrix}.$$

   We can multiply both sides by

$$L_1 = \begin{bmatrix} I & & & & \\ A_2B_1^{-1} & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{bmatrix}$$

to get the $D = LU$ form, just as in the non-block case.

$$\begin{bmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & \\ & A_3 & B_3 & C_3 & \\ & & \ddots & \ddots & \ddots \\ & & & A_n & B_n \end{bmatrix} = \begin{bmatrix} I & & & \\ A_2 B_1^{-1} & I & & \\ & & I & \\ & & & \ddots & \\ & & & & I \end{bmatrix} \begin{bmatrix} B_1 & C_1 & & & \\ 0 & B_2 - A_2 B_1^{-1} C_1 & C_2 & & \\ & A_3 & B_3 & C_3 & \\ & & \ddots & \ddots & \ddots \\ & & & A_n & B_n \end{bmatrix}$$

Then continuing on in this fashion will result in the block LU decomposition:

$$\begin{bmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & \\ & A_3 & B_3 & C_3 & \\ & & \ddots & \ddots & \ddots \\ & & & A_n & B_n \end{bmatrix} = \begin{bmatrix} I & & & \\ A_2 B_1^{-1} & I & & \\ & (B_2 - A_2 B_1^{-1} C_1)^{-1} & I & \\ & & \ddots & \ddots & \\ & & & \tilde{B}_n & I \end{bmatrix} \begin{bmatrix} B_1 & C_1 & & & \\ 0 & B_2 - A_2 B_1^{-1} C_1 & C_2 & & \\ & 0 & \tilde{B}_3 & C_3 & \\ & & \ddots & \ddots & \ddots \\ & & & & \tilde{B}_n \end{bmatrix}$$

where $\tilde{B}_j$ is defined thus:

$$\tilde{B}_j = B_j - A_j(B_{j-1} - A_{j-1}(B_{j-2} - ... - A_2 B_1^{-1} C_1)^{-1} C_2)^{-1} ... C_{j-2})^{-1} C_{j-1}.$$

Therefore, for the block LU decomposition to exist, the diagonal elements $\tilde{B}_j$ must be invertible for $j = 1, ..., n-1$.

2. Suppose an $m \times m$ matrix A is written in the block form $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} is $n \times n$ and A_{22} is $(m-n) \times (m-n)$.
Assume that A is nonsingular and has a unique LU factorization.

   (a) Verify the formula

$$\begin{bmatrix} I & \\ -A_{21} A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{bmatrix}$$

   for "elimination" of the block A_{21}. The matrix $A_{22} - A_{21} A_{11}^{-1} A_{12}$ is known as the Schur complement of A_{11} in A.

   Solution:

Multiplying the left hand side of the equation using block-multiplication readily verifies the formula:

$$\begin{bmatrix} I & \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ -A_{21}A_{11}^{-1}A_{11} + A_{21} & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ -A_{21} + A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

(b) Suppose A_{21} is eliminated row by row by means of n steps of Gaussian elimination. Show that the bottom-right $(m-n) \times (m-n)$ block of the result is again the Schur complement.

Solution:

If A_{21} is eliminated by GE, we must have some lower triangular matrix L^{-1} multiplying the original matrix A on the left. As the only rows that were eliminated were the last $m-n$ rows of A, we can express L^{-1} as a block matrix:

$$L^{-1} = \begin{bmatrix} I & \\ C & I \end{bmatrix}.$$

Then if we multiply A by L^{-1} we should get some upper triangular block matrix with the first n rows unchanged:

$$\begin{bmatrix} I & \\ C & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ & S \end{bmatrix}.$$

We then solve for C:

$$CA_{11} + A_{21} = 0 \implies CA_{11} = -A_{21} \implies C = -A_{21}A_{11}^{-1}.$$

We can then define S:

$$CA_{12} + A_{22} = S \implies -A_{21}A_{11}^{-1}A_{12} + A_{22} = S.$$

Thus S is the Schur complement, and we are done.

3. The matrix $A \in \mathbb{C}^{m\times m}$ is diagonally dominant if

$$|a_{ii}| > \sum_{j=1, j\neq i}^{m} |a_{ij}|, \ i = 1, 2, ..., m.$$

(a) Prove that if A is diagonally dominant, then any principle submatrix of A is diagonally dominant.

Solution:

Any principal submatrix of a diagonally dominant matrix A has diagonal composed of only elements from the diagonal of A. Since the principal submatrix has fewer columns than the matrix A, the sum above is always going to be, for a principal submatrix, less than or equal to the sum for A. Thus any principal submatrix of A must be diagonally dominant.

(b) Prove that if A is diagonally dominant, then A is nonsingular.

Solution:

We use Gerschgorin disks to prove this conjecture. For A to be nonsingular, it must have all nonzero eigenvalues. We note that each eigenvalue is located within a Gerschgorin disk defined

$$D_i = \{z| \, |z - a_{ii}| \leq \sum_{j=1, j\neq i}^{m} |a_{ij}|\}.$$

We can expand the expression for the disk as such:

$$- \sum_{j=1,j\neq i}^{m} |a_{ij}| \leq z - a_{ii} \leq \sum_{j=1,j\neq i}^{m} |a_{ij}|$$

$$\implies a_{ii} - \sum_{j=1,j\neq i}^{m} |a_{ij}| \leq z \leq a_{ii} + \sum_{j=1,j\neq i}^{m} |a_{ij}|.$$

Since A is diagonally dominant, $a_{ii} - \sum_{j=1,j\neq i}^{m} |a_{ij}| > 0$ for $a_{ii} > 0$ and $a_{ii} + \sum_{j=1,j\neq i}^{m} |a_{ij}| < 0$ for $a_{ii} < 0$. Therefore, there can be no zero eigenvalues and A is nonsingular.

(c) Prove that if A is diagonally dominant then it will have an LU decomposition.

Solution:

Since each principal submatrix, including the upper left $k \times k$ block for each k, is diagonally dominant, each block is therefore nonsingular. Thus, by NLA 20.1 A has an LU decomposition.

4. lufactor code:

```
% Gaussian Elimination with Partial Pivoting
function [L,U,P]=lufactor(A)
[~,m]=size(A);
U=A;
L=eye(m,m);
P=eye(m,m);
for k=1:(m-1)
    max=abs(U(k,k));
    maxindex=k;
    for i=k+1:m
        if abs(U(i,k))>max
            max=abs(U(i,k));
            maxindex=i;
        end
    end
    b=U(k,k:m);
    U(k,k:m)=U(maxindex,k:m);
    U(maxindex,k:m)=b;
    c=L(k,1:(k-1));
    L(k,1:(k-1))=L(maxindex,1:(k-1));
    L(maxindex,1:(k-1))=c;
    d=P(k,1:m);
    P(k,1:m)=P(maxindex,1:m);
    P(maxindex,1:m)=d;
    for j=k+1:m
        L(j,k)=U(j,k)/U(k,k);
        U(j,k:m)=U(j,k:m)-L(j,k)*U(k,k:m);
    end
end
```

lusolve code:

```
function x=lusolve(b,L,U,P)
[~,m]=size(L);
 z=P*b;


 y=zeros(m,1);
 x=zeros(m,1);

 for i=1:m
     y(i)=(z(i)-dot(y,L(i,1:m)))/L(i,i);
 end

 for k=m:-1:1
     x(k)=(y(k)-dot(x,U(k,1:m)))/U(k,k);
 end
```

(a) LU factorization of A.

```
A =

     2     1     1     0
     4     3     3     1
     8     7     9     5
     6     7     9     8

>> [L,U,P]=lufactor(A)

L =

    1.0000         0         0         0
    0.7500    1.0000         0         0
    0.5000   -0.2857    1.0000         0
    0.2500   -0.4286    0.3333    1.0000


U =

    8.0000    7.0000    9.0000    5.0000
         0    1.7500    2.2500    4.2500
         0         0   -0.8571   -0.2857
         0         0         0    0.6667


P =

     0     0     1     0
     0     0     0     1
     0     1     0     0
     1     0     0     0

>> norm(P*A-L*U,2)

ans =

   1.1102e-16

>> P*A

ans =

     8     7     9     5
     6     7     9     8
     4     3     3     1
     2     1     1     0

>> L*U

ans =

    8.0000    7.0000    9.0000    5.0000
    6.0000    7.0000    9.0000    8.0000
```

```
    4.0000    3.0000    3.0000    1.0000
    2.0000    1.0000    1.0000    0.0000
```

(b) Using LU factorization to solve $Ax = b$.

```
>> A

A =

     2     1     1     0
     4     3     3     1
     8     7     9     5
     6     7     9     8

>> b

b =

     7
    23
    69
    79

>> [L,U,P]=lufactor(A);
>> x=lusolve(b,L,U,P)

x =

    1.0000
    2.0000
    3.0000
    4.0000

>> A*x

ans =

    7.0000
   23.0000
   69.0000
   79.0000

>> norm(A*x-b,2)

ans =

   1.7764e-15
```