A 2D Godunov Solver for the Euler Equations on Non-Cartesian Domains

Numerical Solutions of Wave Equation - MATH 6890

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Outline

Motivation

- Gas Dynamics
- Atmospheric Science
- Reactive Euler Equations and Detonations

Governing Equations

$$\mathbf{U}_t + \mathbf{F}_x + \mathbf{G}_y = 0, \tag{1}$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{bmatrix}. \tag{2}$$

Equation of State: $E = \frac{1}{2}\rho(u^2 + v^2) + \rho e$.
Ideal Gas:

$$PV = nRT \quad \Longrightarrow \quad e = \frac{p}{\rho(\gamma - 1)}. \tag{3}$$

Model Problems

- 1D Equations on a Line
- 2D Equations on a Box
    - Closed Box
    - Open Box
    - Free Space (Rectangular Domain)
- 2D Equations on an Annulus
- 2D Equations on a Quarter Circle

Solution Methods for the Riemann Problem

In order to compute fluxes $(\mathbf{F}, \mathbf{G})$ across interfaces, we need to have some solution method to the Riemann problem

$$\begin{aligned} &\mathbf{U}_t + \mathbf{F}_x + \mathbf{G}_y = 0, \\ &\mathbf{U}(x, y, 0) = \begin{cases} \mathbf{U}_L, & x < 0 \\ \mathbf{U}_R, & x > 0 \end{cases}, \text{or} \begin{cases} \mathbf{U}_B, & y < 0 \\ \mathbf{U}_T, & y > 0 \end{cases}. \end{aligned} \tag{4}$$

- Exact Solution
  - Pros: High Accuracy, Easy to Extend to Higher Orders of Accuracy, Preserves Exact Wave-Form
  - Cons: Heavy Computation Cost - Adaptive Iterative Solver
- Approximate Solution (HLLC)
  - Pros: OK Accuracy, Low Computation Cost
  - Cons: Does not Preserve Exact Wave Form

Exact Riemann Solution

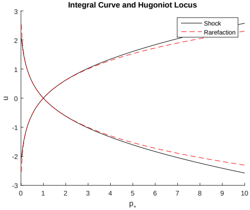
- Shocks and the Hugoniot Locus
  - The Rankine-Hugoniot Conditions

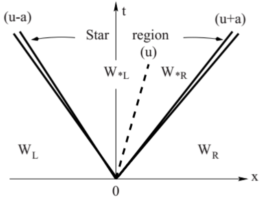
$$\mathbf{F}(\tilde{\mathbf{U}}) - \mathbf{F}(\hat{\mathbf{U}}) = S(\tilde{\mathbf{U}} - \hat{\mathbf{U}}) \tag{5}$$

- Rarefactions and Integral Curves
  - The Generalized Riemann Invariants

$$\frac{dp}{d\rho} = a^2 = \frac{\gamma p}{\rho}, \quad a d\rho + \rho du = 0. \tag{6}$$

- Form of the Exact Solution





The Adaptive Iterative Solver

- Parameterization
  - p_* - Easy, Only for Nice EoS
  - ρ_{*R}, ρ_{*L} - Hard, Works well for Hard EoS
- Initial Guess
  - Linearized Guess
  - Double Rarefaction Guess
  - Double Shock Guess
- The Newton Problem

$$p^{n+1} = p^n - \frac{F_L(p^n) + F_R(p^n) + u_R - u_L}{F_L'(p^n) + F_R'(p^n)} \tag{7}$$

Where F_k is either an integral curve or a Hugoniot locus.
Solution is Adaptive - changes F_k each iteration based on wave-form!

Approximate Riemann Solution

- Many approximate solutions - Roe, HLL, PVRS, TRRS,TSRS, ANRS,...
- HLLC Solver - Harten, Lax, van Leer + Contact
  - Direct Approximation of Intercell Flux
  - Complete Solver - All Characteristic Fields Accounted For
  - Algorithmically Determine Wave Speeds $\rightarrow$ Closed-Form Expression for Flux

A Comparison of the Exact and HLLC Solvers

We will solve the Riemann Problem using each Solver

$$\mathbf{U}_t + \mathbf{F}_x + \mathbf{G}_y = 0$$

$$\mathbf{U}(x, y, 0) = \begin{cases} \mathbf{U}_L, & x < 0 \\ \mathbf{U}_R, & x > 0 \end{cases}$$

- Double Shock

$$\mathbf{U}_L = \begin{bmatrix} 5.99924 \\ 19.5975 \\ 0 \\ 460.894 \end{bmatrix}, \quad \mathbf{U}_R = \begin{bmatrix} 5.99242 \\ -6.19633 \\ 0 \\ 46.0950 \end{bmatrix}. \tag{8}$$

- Double Rarefaction

$$\mathbf{U}_L = \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0.4 \end{bmatrix}, \quad \mathbf{U}_R = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0.4 \end{bmatrix}. \tag{9}$$

A Comparison of the Exact and HLLC Solvers

$$\mathbf{U}_t + \mathbf{F}_x + \mathbf{G}_y = 0$$

$$\mathbf{U}(x, y, 0) = \begin{cases} \mathbf{U}_L, & x < 0 \\ \mathbf{U}_R, & x > 0 \end{cases}$$

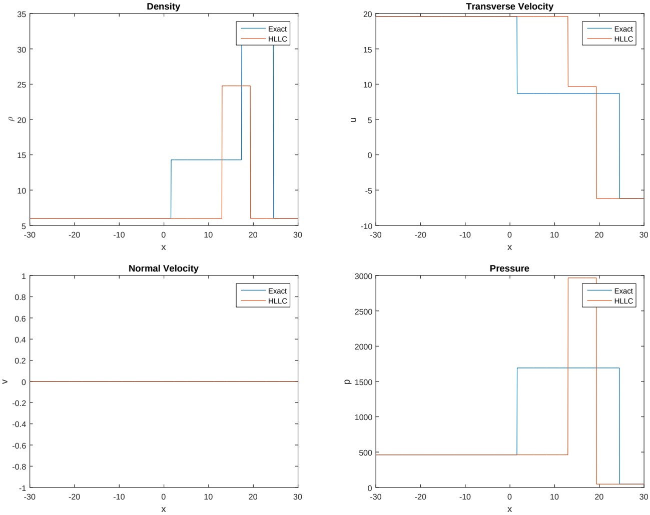
- Left Rarefaction, Right Shock

$$\mathbf{U}_L = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1000 \end{bmatrix}, \quad \mathbf{U}_R = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0.01 \end{bmatrix}. \tag{10}$$

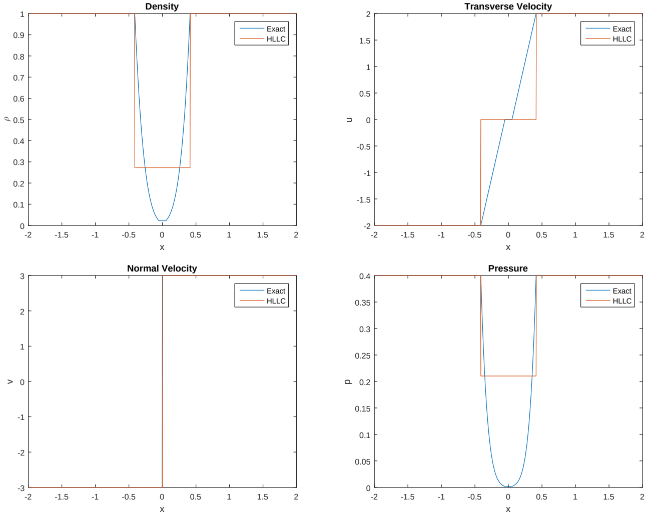
- Left Shock, Right Rarefaction

$$\mathbf{U}_L = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{U}_R = \begin{bmatrix} 1.5 \\ 0 \\ 0 \\ 2 \end{bmatrix}. \tag{11}$$

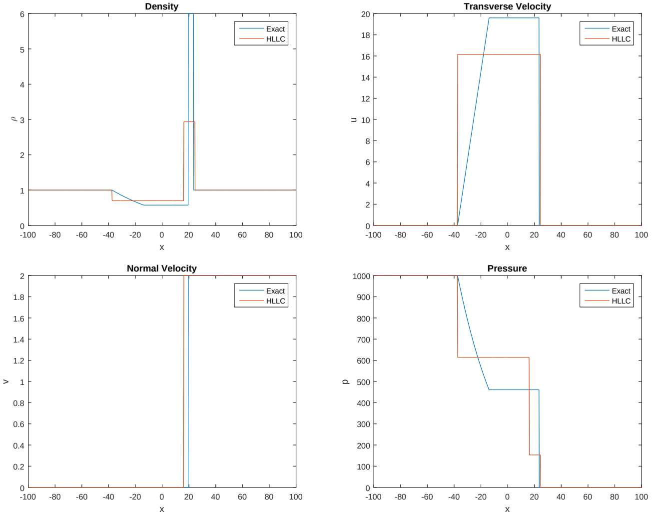
Double Shock



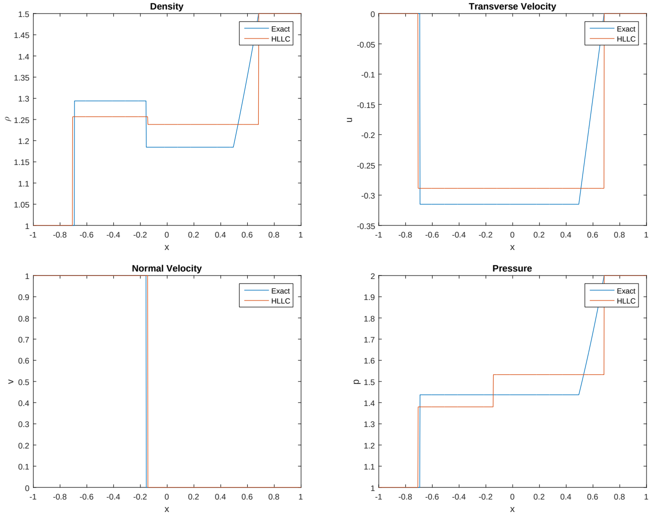
Double Rarefaction



Left Rarefaction, Right Shock



Left Shock, Right Rarefaction



Discretization

Finite Volume Method:
Cell Average: $\mathbf{U}^n_{j,k} = \frac{1}{\Delta x \Delta y} \int_{y_{k-1/2}}^{y_{k+1/2}} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}(\xi, \eta, t_n) d\xi d\eta$
Integrate Conservation Law:

$$\frac{1}{\Delta x \Delta y} \int_{t_n}^{t_{n+1}} \int_{y_{k-1/2}}^{y_{k+1/2}} \int_{x_{j-1/2}}^{x_{j+1/2}} [\mathbf{U}_t(\xi, \eta, \tau) + \mathbf{F}_x(\mathbf{U}(\xi, \eta, \tau)) + \mathbf{G}_y(\mathbf{U}(\xi, \eta, \tau))] d\xi d\eta d\tau = 0 \tag{12}$$

Resultant Discretization:

$$\begin{aligned} \mathbf{U}^{n+1}_{j,k} = \mathbf{U}^n_{j,k} &- \frac{\Delta t}{\Delta x}[\mathbf{F}(\mathbf{U}(x_{j+1/2}, y_k, t_{n+1/2})) - \mathbf{F}(\mathbf{U}(x_{j-1/2}, y_k, t_{n+1/2}))] \\ &- \frac{\Delta t}{\Delta y}[\mathbf{G}(\mathbf{U}(x_j, y_{k+1/2}, t_{n+1/2})) - \mathbf{G}(\mathbf{U}(x_j, y_{k-1/2}, t_{n+1/2}))] \end{aligned} \tag{13}$$

Boundary Conditions

- Reflective Boundary

$$\rho^n_{M+1,k} = \rho^n_{M,k}, \;\; u^n_{M+1,k} = -u^n_{M,k}, \;\; v^n_{M+1,k} = v^n_{M,k}, \;\; p^n_{M+1,k} = p^n_{M,k} \tag{14}$$

- Transmissive Boundary

$$\rho^n_{M+1,k} = \rho^n_{M,k}, \;\; u^n_{M+1,k} = u^n_{M,k}, \;\; v^n_{M+1,k} = v^n_{M,k}, \;\; p^n_{M+1,k} = p^n_{M,k} \tag{15}$$

Time-Step Restriction

Inspect Stability for 2D Advection with Periodic BCs

$$u_t + au_x + bu_y = 0 \tag{16}$$

Assume $a, b > 0$, discretize upwind:

$$\frac{1}{\Delta t}[u^{n+1}_{j,k} - u^n_{j,k}] + aD^x_- u^n_{j,k} + bD^y_- u^n_{j,k} = 0 \tag{17}$$

Seek Normal Mode Solutions $u^n_{j,k} = \rho^n e^{-ij\xi} e^{-ik\eta}$

$$\rho = 1 - \lambda(1 - e^{-i\xi}) - \mu(1 - e^{-i\eta}), \;\; \lambda = \frac{a\Delta t}{\Delta x}, \;\; \mu = \frac{b\Delta t}{\Delta y} \tag{18}$$

Stability Bound:

$$\rho \leq 1 \implies \lambda + \mu \leq 1 \tag{19}$$

Time-Step Restriction

2D Advection Time-Step Restriction

$$\Delta t = \frac{\nu \Delta x \Delta y}{a\Delta y + b\Delta x} \tag{20}$$

Generalize:

$$a = max|\sigma_x(x, y, t_n)|, \quad b = max|\sigma_y(x, y, t_n)| \tag{21}$$

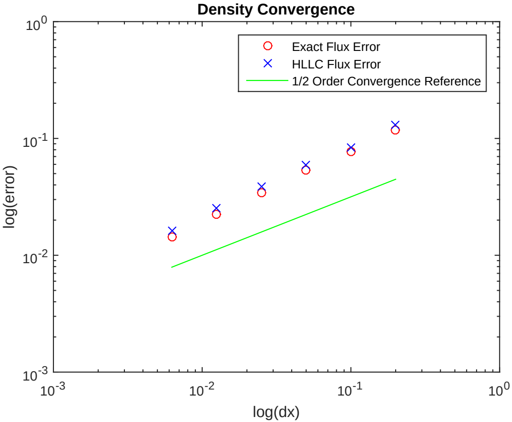
2D Euler Time-Step Restriction

$$\Delta t = \frac{\nu \Delta x \Delta y}{max|\sigma_x(x, y, t_n)|\Delta y + max|\sigma_y(x, y, t_n)|\Delta x} \tag{22}$$

Numerical Convergence Results

Discontinuous Data

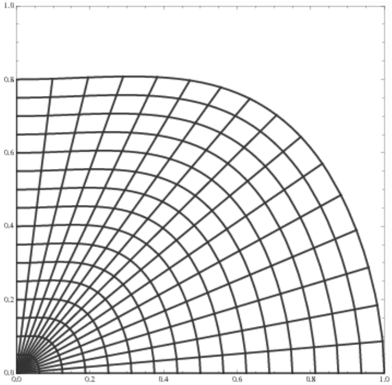
Smooth Data - TBD



Simulation Results

Non-Cartesian Domains

Smooth, invertible mapping $\mathbf{x} = C_g(\mathbf{r})$ from parameter space $\mathbf{r} = (r, s)$ to physical space $\mathbf{x} = (x, y)$



Equations in Parameter Space

The mapping C_g results in the transformed conservation law

$$\phi_t + \mathbf{f}(\phi)_r + \mathbf{g}(\phi) = 0, \tag{23}$$

where

$$\phi = J\mathbf{U}, \quad \mathbf{f} = y_s\mathbf{F} - x_s\mathbf{G}, \quad \mathbf{g} = -y_r\mathbf{F} + x_r\mathbf{G}, \tag{24}$$

and

$$J(r,s) = \left|\frac{\partial(x,y)}{\partial(r,s)}\right|.$$

Discretization

Grid: $(r_i, s_j) = (i\Delta r, j\Delta s)$
Cell Average: $\phi^n_{i,j} = \frac{1}{\Delta r \Delta s} \int_{s_{j-1}}^{s_j} \int_{r_{i-1}}^{r_i} \phi(r, s, t_n) drds$
Conservative Difference Formula:

$$\phi^{n+1}_{i,j} = \phi^n_{i,j} - \frac{\Delta t}{\Delta r}\left(\mathbf{f}^{n+1/2}_{i,j} - \mathbf{f}^{n+1/2}_{i-1,j}\right) - \frac{\Delta t}{\Delta s}\left(\mathbf{g}^{n+1/2}_{i,j} - \mathbf{g}^{n+1/2}_{i,j-1}\right) \tag{25}$$

Time-Step Restriction Same as in Cartesian Case

Inter-Cell Fluxes

Godunov - Follow Similar Procedure for Cartesian Domains (Approximate Solver Only)

$$
\begin{aligned}
&\phi_t + \mathbf{f}(\phi)_r = 0, \quad t > 0, \quad |r| < \infty, \\
&\phi(r,0) = \begin{cases} \phi_L, & r < 0, \\ \phi_R, & r > 0 \end{cases}
\end{aligned} \tag{26}
$$

Eigenvalue Decomposition of Linearized Jacobian of $\mathbf{f}$ Leads to the Numerical Flux

$$
\mathbf{f}_{i,j}^{n+1/2} = \begin{cases} \mathbf{f}(\phi_L), & \lambda_1 > 0, \\ \mathbf{f}(\phi_L) + \alpha_1\lambda_1 v_1, & \lambda_1 < 0 \text{ and } \lambda_2 > 0, \\ \mathbf{f}(\phi_R) - \alpha_4\lambda_4 v_4, & \lambda_4 > 0 \text{ and } \lambda_2 < 0, \\ \mathbf{f}(\phi_R), & \lambda_4 < 0. \end{cases} \tag{27}
$$

with $\phi_R - \phi_L = \sum_p \alpha_p v_p$.

Finite Volume - Transformed Equations

Conservation Equations Satisfied at Discrete Level

Cell Average: $\mathbf{U}^n_{j,k} = \frac{1}{\Delta x \Delta y} \int_{y_{k-1/2}}^{y_{k+1/2}} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}(\xi, \eta, t_n) d\xi d\eta$

Outward unit vector normal to side s of a grid cell $\mathbf{n}_s = (\cos\theta_s, \sin\theta_s)$

Transormed Equation:

$$\mathbf{U}_t = -\frac{1}{|V|} \sum_{s=1}^{N} \int_{A_s}^{A_{s+1}} [\cos\theta_s \mathbf{F}(\mathbf{U}) + \sin\theta_s \mathbf{G}(\mathbf{U})] dA \tag{28}$$

N =number of sides of the cell

Transformed Equations

Rotational Invariance - $\cos\theta_s \mathbf{F}(\mathbf{U}) + \sin\theta_s \mathbf{G}(\mathbf{U}) = \mathbf{T}_s^{-1}\mathbf{F}(\mathbf{T}_s\mathbf{U})$, where T_s is the rotation matrix

$$T_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A Simpler Transformed Equation

$$\mathbf{U}_t = -\frac{1}{|V|}\sum_{s=1}^{N}\int_{A_s}^{A_{s+1}} \mathbf{T}_s^{-1}\mathbf{F}(\mathbf{T}_s\mathbf{U})dA \tag{29}$$

Finite-Volume Numerical Methods

Intercell Fluxes: $\hat{\mathbf{U}} = \mathbf{T}_s\mathbf{U}$, $\hat{\mathbf{F}} = \mathbf{F}(\hat{\mathbf{U}})$

$$\begin{aligned} &\hat{\mathbf{U}}_t + \hat{\mathbf{F}}_{\hat{x}} = 0 \\ &\hat{\mathbf{U}}(\hat{x}, \hat{y}, 0) = \begin{cases} \hat{\mathbf{U}}_L, & \hat{x} < 0 \\ \hat{\mathbf{U}}_R, & hatx > 0 \end{cases} \end{aligned} \tag{30}$$

Approximate Integrals in (29)

$$\int_{A_s}^{A_{s+1}} \mathbf{T}_s^{-1}\mathbf{F}(\mathbf{T}_s\mathbf{U})dA \approx L_s \mathbf{T}_s^{-1}\hat{\mathbf{F}}_s \tag{31}$$

where L_s is the length of the side s.

Discretization

We now present the final discretization obtained by (30) and (29)

$$\mathbf{U}_{j,k}^{n+1} = \mathbf{U}_{j,k}^{n} - \frac{\Delta t}{|I_{j,k}|} \sum_{s=1}^{N} L_s \mathbf{T}_s^{-1} \hat{\mathbf{F}}_s \tag{32}$$

where $|I_{j,k}|$ is the area of cell $I_{j,k}$. Time-Step Restriction is the Same as in the Cartesian Case

Future Steps

- Implement and Compare Non-Cartesian Domain Numerical Methods
- Grid Generation Techniques
- High-Order Methods (WENO, Slope-correcting/limiting,...)
- Generalization to 3D Equations
- ...

Sources

- Toro, E. F. Riemann Solvers and Numerical Methods for Fluid Dynamics: A Practical Introduction. Berlin: Springer-Verlag, 2009.
- Henshaw, William D., and Donald W. Schwendeman. "An Adaptive Numerical Scheme for High-speed Reactive Flow on Overlapping Grids." (2003)

Thanks for your time!