

A tiling algorithm for the aperiodic monotile Tile(1,1)

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Abstract — An algorithm is provided to tile the plane with the aperiodic monotile Tile(1,1) recently discovered by Smith et al. (2023). Their geometric construction guidelines are expanded into a numerical algorithm that does not require any manual intervention. The intention is to remove a main obstacle for researchers interested in applications of this fundamental tiling of the plane.

Key words—Aperiodic tile, tiling of the plane, 2-dimensional nonperiodic tiling, Tile(1,1).

Recently, Smith et al. discovered a family of aperiodic monotiles of the plane, $T(a,b)$, with geometric parameters a and b [1]. In particular, the special case of Tile(1,1) stands out, as it is a chiral monotile; in other words, Tile(1,1) can tile the plane only in a nonperiodic fashion, and mirrored tiles are not needed nor allowed. A slight generalization of Tile(1,1), the so-called “Specter”, consists of a family of Tile(1,1) monotiles with deformed edges.

Smith et al. provide geometric construction guidelines that can be used to assemble the tiling in an iterative way. A published algorithm to automatically generate a tiling with Tile(1,1) seems to be lacking. The provided algorithm solves this problem. It can construct tilings with Tile(1,1) for an in principle arbitrary numbers of tiles, limited only by computing power. Visual verification of the correctness of the algorithm was performed up to the 8th iteration of tilings, consisting of 16,908,641 tiles.

The algorithm translates the geometric construction guidelines of the tiling into a numeric algorithm. Details of the implementation including some non-trivial procedures that were not evident to me from the geometric guidelines can be best understood by consulting the annotated MATLAB script provided in the Appendix. It is described briefly as follows.

The goal is to tile the plane with the aperiodic monotile Tile(1,1). The geometric guidelines achieve this by using “Specter” clusters S and “Mystic” clusters M . S and M consist of one and two tiles Tile(1,1), respectively, to seed the iteration. From one iteration to the next, S and M are mirrored. Refer to Figure A1.2a in Smith et al. (2023). The final tiling of the plane is then given by the S cluster at the final iteration, to be set by the user. In the MATLAB script, first, two functions are defined, which are used to place tiles or clusters of tiles at specified locations and at specified angles, and to mirror the tiling from one iteration to the next. Then, the expected number of tiles for a tiling with the final number of iterations is computed, to provide an estimate of resources needed. The first pattern $S_0 = \text{Tile}(1,1)$ is defined, for initiating the iteration, and labelled with four key points. Then the Mystic pattern M_0 is defined as a two-cluster of S_0 s. For the first two iterations, the key points used to construct the tiling of the next iteration are provided explicitly. They were read off from the beforementioned figure. The first rotation angles for placement of the initial tiles to generate S_1 and S_2 were obtained by trial and error. After construction of S_1 , its key points are cleared and four new key points are being defined to construct S_2 from S_1 in a similar way as S_1 has been constructed from S_0 . The rotation angles for S_2 were obtained by trial and error. Luckily, for subsequent iterations they remained constant, up to iteration number 8. It is conjectured here that they converged at iteration 2 to their asymptotic value. This insight was not expected from the geometric guidelines. After S_2 is complete, S_n ($n > 2$) are assembled by extracting the key points from the previous iteration. This final step was not straightforward and constitutes the most non-trivial part of this contribution.

The provided MATLAB algorithm “TileOneOne” is written in a self-explanatory way and commented. It should be very easy to convert it to any other programming language. It should also be easy to use it with minor modifications to tile the plane with any of the Specters. The script is provided in the Appendix and can also be downloaded at <https://github.com/henningle/TileOneOne.git>.

Figure 1 shows the tiling for iteration 5, consisting of 34649 tiles. Figure 2 show the tiling for iteration 8, which has 16,908,641 tiles.

S5

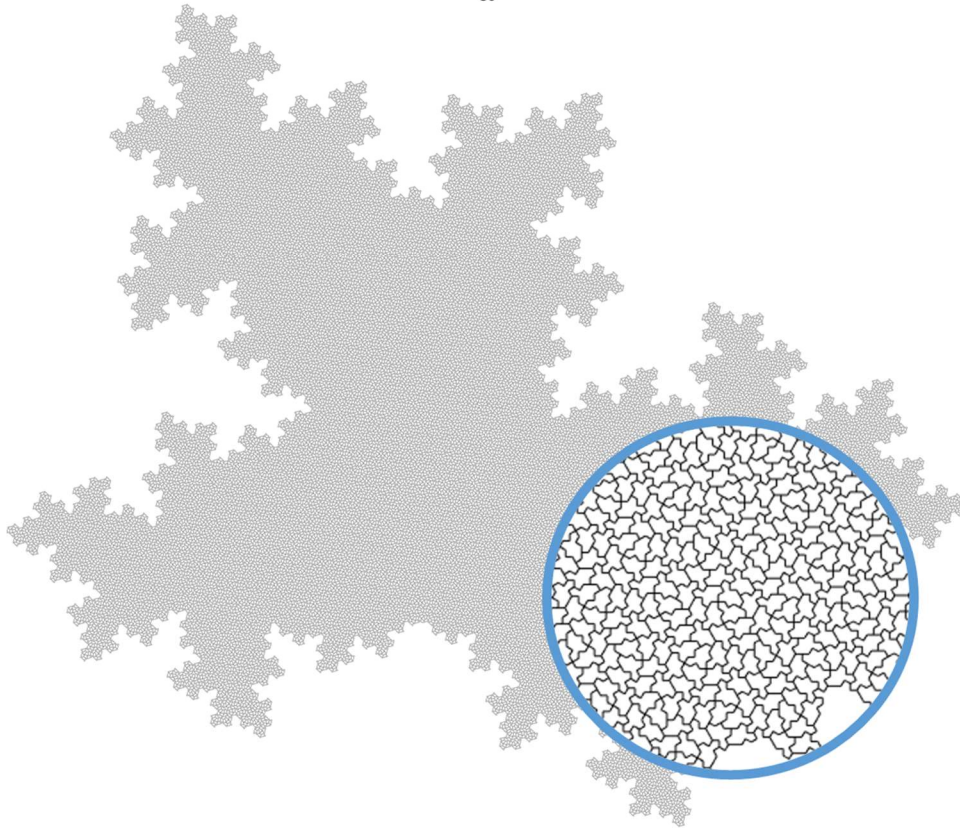


Figure 1: Tiling with the aperiodic monotile $\text{Tile}(1,1)$ for the fifth iteration, shown with a “magnifying glass” to show the monotile components.

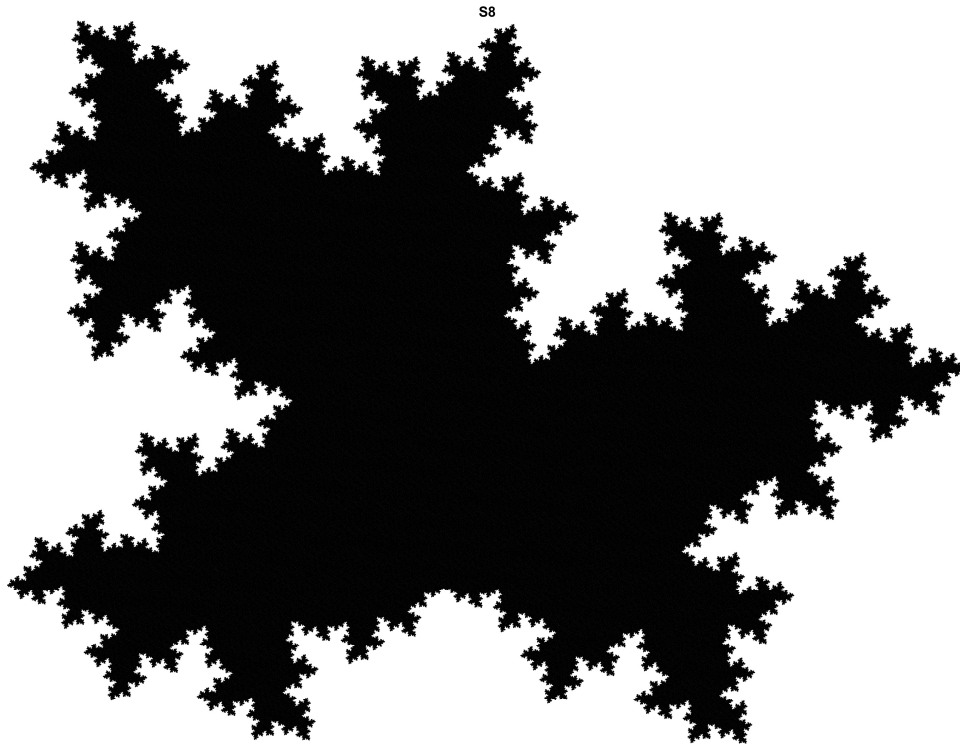


Figure 2: Tiling with the aperiodic monotile Tile(1,1) for the eighth iteration, highlighting the fractal boundary of the tiling set.

Appendix

The MATLAB algorithm TileOneOne.m.

```
%% TileOneOne.m, 6.6.2024
% To construct a tiling with the aperiodic Tile(1,1)
% Please cite as
% "H.U. Voss, A tiling algorithm for the aperiodic monotile Tile(1,1), arXiv (2024)"
% The license attached in GitHub applies, at https://github.com/henninge/TileOneOne.git

%% Parameters

maxN=4; % Maximum iteration. The largest maxN tested was maxN=8 (16,908,641 tiles)

plotkeypoints=true; % Set to false if plotting key points not wanted

%% Main script

% Tile placement function
% Each tile has 3 components: x, y, key point number (0 for no key point, 1 to 4 if key point)
% The placetile function takes in a matrix of size (number of rows, 3)
% The first 2 columns are the coordinates of the tiling, the third is a label of key points
placetile=@(x, angle, coord) ...
    [transpose( [cos(angle*pi/180), -sin(angle*pi/180); ...
    sin(angle*pi/180), cos(angle*pi/180)]*x(:,1:2)') ...
    +transpose(repmat(coord,[size(x,1),1])) , x(:,3)];

mirror=@(x) [transpose([-1, 0; 0, 1]*(x(:,1:2)')), x(:,3)];
```

A tiling algorithm for Tile(1,1)

```
% Number of expected tiles for each iteration
nS=zeros(1,maxN+1);
nM=nS;
nS(1)=1; nM(1)=2; % This is index 0
for n=2:maxN+1
    nS(n)=nM(n-1)+7*nS(n-1);
    nM(n)=nM(n-1)+6*nS(n-1);
end
nS=nS(2:end);
nM=nM(2:end);
disp(['Predicted number of tiles for N = 1, 2, ...: ' num2str(nS)])

%% S0 and M0

% Some definitions
cos30=cos(30*pi/180); sin30=sin(30*pi/180);
cos60=cos(60*pi/180); sin60=sin(60*pi/180);
cos120=cos(120*pi/180); sin120=sin(120*pi/180);
cos150=cos(150*pi/180); sin150=sin(150*pi/180);
cos210=cos(210*pi/180); sin210=sin(210*pi/180);
cos240=cos(240*pi/180); sin240=sin(240*pi/180);
cos300=cos(300*pi/180); sin300=sin(300*pi/180);
cos330=cos(330*pi/180); sin330=sin(330*pi/180);

% One Tile(1,1)
% Each tile has 14 edges and 15 points, the last point for closure
% Plus one NaN for separation of tiles while plotting
% x-components of specter S
Stmp1=[ 0
        0
        cos30
        cos30+cos120
        cos30+cos120+cos60
        cos30+cos120+cos60+cos330
        cos30+cos120+cos60+cos330
        cos30+cos120+cos60+cos330+1
        cos30+cos120+cos60+cos330+1+cos300
        cos30+cos120+cos60+cos330+1+cos300+cos210
        cos30+cos120+cos60+cos330+1+cos300+cos210
        cos30+cos120+cos60+cos330+cos300+cos210
        cos30+cos120+cos60+cos330+cos300+cos210+cos240
        cos30+cos120+cos60+cos330+cos300+cos210+cos240+cos150
        cos30+cos120+cos60+cos330+cos300+cos210+cos240+cos150
        NaN];
% y-components of specter S
Stmp2=[0
        1
        1+sin30
        1+sin30+sin120
        1+sin30+sin120+sin60
        1+sin30+sin120+sin60+sin330
        sin30+sin120+sin60+sin330
        sin30+sin120+sin60+sin330
        sin30+sin120+sin60+sin330+sin300
        sin30+sin120+sin60+sin330+sin300+sin210
        sin30+sin120+sin60+sin330+sin300+sin210-1
        sin30+sin120+sin60+sin330+sin300+sin210-1
        sin30+sin120+sin60+sin330+sin300+sin210-1+sin240
        sin30+sin120+sin60+sin330+sin300+sin210-1+sin240+sin150]
```

A tiling algorithm for Tile(1,1)

```
sin30+sin120+sin60+sin330+sin300+sin210+sin240+sin150
NaN];
% Specter S
S=[Stmp1,Stmp2];

% Augment S with key points to be carried over during iterative construction
keypoints=zeros(16,1); keypoints(4)=1; keypoints(6)=2; keypoints(8)=3; keypoints(14)=4;
S=[S,keypoints];
keyind=find(S(:,3)>0)'; % The indexes that point to key points
% S(keyind,3)' % Should be 1,2,3,4 for iteration 0

% The Mystic
M=[S; placetile(S,-30,[sqrt(3)/2,-1.5])];
M(:,3)=0; % M should not have key points not to contaminate S

%% Iteration loop

for N=1:maxN

    plotting=0; if N==maxN; plotting=1; end % Plot only the last iteration

    S=mirror(S);
    M=mirror(M);

    % The first two iterations require explicit key points for the next iteration
    switch N
        case 1
            Sind=[4,6,8,14]; % Key points on S0, for making S1
            Mind=Sind+16;
            angles=[150,90,90,30,-30,-30,-90];
        case 2
            Sind=[84,118,132,38]; % Key points on S1, for making S2
            % Indexing changes in M2 due to the missing patch
            Mind=[84-16,118-16,132-16,38]; %
            angles=[120,60,60,0,-60,-60,-120]; % Conjecture: Angles converged from here on
    end

    % For iteration 3 and higher, keypoints are computed from the previous two iterations
    if N > 2
        Sind=circshift(keyind,-1);
        Mind(1:3)=Sind(1:3)-16*nS(N-2);
    end

    % Define some tiling cells
    cell0=M;
    cell1=placetile([S(:,1:2)-S(Sind(3),1:2), S(:,3)],angles(1), M(Mind(3),1:2));
    cell2=placetile([S(:,1:2)-S(Sind(2),1:2), S(:,3)],angles(2),cell1(Sind(4),1:2));
    cell3=placetile([S(:,1:2)-S(Sind(3),1:2), S(:,3)],angles(3),cell2(Sind(1),1:2));
    cell4=placetile([S(:,1:2)-S(Sind(2),1:2), S(:,3)],angles(4),cell3(Sind(4),1:2));
    cell5=placetile([S(:,1:2)-S(Sind(2),1:2), S(:,3)],angles(5),cell4(Sind(4),1:2));
    cell6=placetile([S(:,1:2)-S(Sind(3),1:2), S(:,3)],angles(6),cell5(Sind(1),1:2));
    cell7=placetile([S(:,1:2)-S(Sind(2),1:2), S(:,3)],angles(7),cell6(Sind(4),1:2));

    %% Assemble S and M

    S=[cell0 % 2x16 elements because this is the mystic M
        cell1 % 16 elements
        cell2
        cell3
```

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```
cell14
cell15
cell16
cell17
];
M=[cell10 % 2x16 elements
cell11 % 16 elements
cell12
cell14 % Note we skipped cell13
cell15
cell16
cell17
];
M(:,3)=0; % Again, M should not have key points

if plotting
    figure('position',[500.,100.,1000,1000]);
    plot(S(:,1),S(:,2),'k',LineWidth=.1);
    axis image
    title(['S' num2str(N)])
end

%% Key points of this iteration

keyind=find(S(:,3)>0); % Always 7*4=28 elements after pruning previous iteration

if plotting
    if plotkeypoints
        % Plot all keypoints before pruning, in red
        hold on
        for n=1:4:length(keyind)
            plot(S([keyind(n:n+3);keyind(n)],1),S([keyind(n:n+3);keyind(n)],2),'r0-');
        end
    end
end

%% Key points for next iteration

% At the beginning, there are always 9*4=36 key points. At the end, there should be 4
keyind2=keyind([2,13,22,25]); % Read off from Smith23b Fig. A1.2a and previous plot
if N>1; keyind2=keyind([2,13,22,25]+1); end % Don't ask
% S(keyind2,3)' % Should be (2,1,2,1) for iteration 1 and (3,2,3,2) for iteration >= 2

S(:,3)=0; % Clean out this iteration's key points

% Set new keypoints for constructing the next iteration
S(keyind2(1),3)=1;
S(keyind2(2),3)=2;
S(keyind2(3),3)=3;
S(keyind2(4),3)=4;

keyind=find(S(:,3)>0); % Must be 38 84 118 132 for iteration 2

if plotting
    if plotkeypoints
        % Plot only key points taken over to next iteration, in blue
        plot(S([keyind;keyind(1)],1),S([keyind;keyind(1)],2),'b0-');
    end
end
```

A tiling algorithm for Tile(1,1)

```
% Count tiles
nStrue=size(S,1)/16;
disp(['Iteration ' num2str(N) ' has ' num2str(nStrue) ' tiles'])

end

%% Save tiling to file

S=S(:,1:2); % Saving only the lattice proper
save(['TileOneOne_Ni' num2str(N) '.mat'], "S");
```

References

[1] D. Smith, J. S. Myers, C. S. Kaplan, and C. Goodman-Strauss: "A chiral aperiodic monotile," <https://arxiv.org/abs/2305.17743>, 1-23, 2023.