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L'algèbre de Hall des courbes et des carquois : fonctions cuspidales, faisceaux pervers et polynômes de Kac

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Chapter 1

Introduction (en français)

Cette thèse est consacrée à l'étude de plusieurs aspects des algèbres de Hall associées à des carquois et à des courbes en lien avec la notion de *fonctions cuspidales* dans ces algèbres. La notion d'algèbre de Hall est apparue dans les travaux de Steinitz au début du vingtième siècle[1] ([Ste01]) lorsqu'il s'est intéressé à la catégorie des groupes abéliens finis de p-torsions, pour un nombre premier p. Plus précisément, il a construit une structure d'algèbre sur l'espace vectoriel de base l'ensemble des classes d'isomorphisme de tels groupes abéliens. Les constantes de structures comptent les extensions entre deux groupes abéliens de p-torsion donnés lorsque la classe d'isomorphisme de l'objet central est fixée. Les résultats de Steinitz furent retrouvés plus d'un demi-siècle plus tard par Hall [Hal]. Le résultat principal est le comportement polynomial en p de ces constantes de structure. L'algèbre obtenue de cette façon est isomorphe à l'algèbre des fonctions symétriques [Mac15]. Ringel [Rin90b, Rin90a] et Green [Gre95] ont élargi l'étude des algèbres de Hall en associant une telle algèbre à la catégorie des modules *finis* (c'est-à-dire des modules dont l'ensemble sous-jacent est un ensemble fini) sur une algèbre *héréditaire* R. Lorsque l'algèbre héréditaire en considération est l'algèbre des chemins d'un carquois sur un corps fini $\mathbf{F}_q$, ils en déterminent partiellement la structure : si le carquois est de type fini, l'algèbre de Hall coïncide avec la partie positive du groupe quantique associé au carquois ; en général, il faut se restreindre à la *sous-algèbre sphérique* de l'algèbre de Hall ([Sch12a, Theorem 3.16]). Un apport crucial à la théorie est la construction du coproduit par Green ([Gre95]), puis la détermination de l'antipode par Xiao [Xia97] ce qui munit l'algèbre de Hall d'une structure d'algèbre de Hopf qui coïncide sur la sous-algèbre sphérique avec celle de la partie positive du groupe quantique. Deux questions naturelles apparaissent. La première est de trouver une réalisation de tout le groupe quantique, et pas seulement de sa partie positive, en termes d'algèbres de Hall. Cette question a été abordée dans divers travaux, en particulier dans [Bri13, Toë06]. Dans [Bri13], Bridgeland réalise le groupe quantique dans l'algèbre de Hall des complexes 2-périodiques (localisée relativement aux complexes acycliques) tandis que dans [Toë06], Toën définit *l'algèbre de Hall dérivée* d'une catégorie différentielle graduée (satisfaisant certaines conditions de finitude), où les triangles distingués jouent le rôle des suites exactes. À la connaissance de l'auteur, les contreparties géométriques[2] des constructions de Bridgeland et Toën

[1] Mais l'appellation *algèbre de Hall* est due à Ringel

[2] C'est-à-dire la construction du groupe quantique à partir du groupe de Grothendieck de la catégorie des faisceaux constructibles sur le champ des objets de la catégorie des complexes 2-périodiques ou de la catégorie triangulée.

n'ont pas encore fait l'objet de publications. La deuxième question concerne la compréhension de *toute* l'algèbre de Hall et pas seulement de sa sous-algèbre sphérique. C'est cette question qui est le point de départ de cette thèse. Un sous-espace générateur minimal de l'algèbre de Hall est formé par les *fonctions cuspidales* ([SVDB01]) (c'est le nom donné aux éléments primitifs de l'algèbre de Hall). On appelle *fonctions cuspidales triviales* les (combinaisons linéaires) des fonctions caractéristiques des représentations simples du carquois concentrées en un sommet. Elles engendrent par définition la sous-algèbre sphérique. Les fonctions cuspidales non triviales sont donc cruciales dans la compréhension de l'algèbre de Hall d'un carquois. On peut reformuler la seconde question de la façon suivante :

Quelles sont les fonctions cuspidales d'un carquois ?

Il faut comprendre cette question en un sens large : elle inclut les questions suivantes. Quelle est la dimension des espaces de fonctions cuspidales ? Peut-on paramétriser explicitement une base de l'espace des fonctions cuspidales, éventuellement à travers la cohomologie d'une variété algébrique ? Comment construire des fonctions cuspidales ? La question de la dimension est étudiée par Bozec et Schiffmann ([BS18]). Le résultat frappant est le comportement polynômial en la taille du corps fini. Les autres questions restent largement ouvertes pour les carquois généraux, même si la question est résolue pour les carquois affines ([Hen19] et des approches géométriques existent ([Dav20]).

On a dit que les fonctions cuspidales forment un sous-espace générateur minimal de l'algèbre de Hall. Le caractère gradué de l'algèbre de Hall est donné par l'exponentielle pléthystique de la série génératrice des polynômes de Kac du carquois (Section 1.4.2, [BS18, Lemma 3.1],[Hua00]), et s'obtient aussi à partir de la dimension graduée de l'espace des fonctions cuspidales grâce à la formule du caractère de Borcherds ([BS18, Section 2.3]). On peut chercher à comprendre l'évolution des polynômes de Kac lorsqu'on donne à chaque flèche du carquois une multiplicité (ce qui produit un nouveau carquois, dans lequel chaque flèche est démultipliée en accord avec la multiplicité) et que l'on fait tendre le vecteur de multiplicités vers une limite. C'est à cette question que l'on s'est intéressés dans [Hen20a], Chapitre 4 de cette thèse. De façon surprenante, la suite de polynômes obtenue converge vers une série formelle (Théorème 1.7.2) et la distribution des coefficients des polynômes de Kac converge conjecturalement vers une distribution continue (voir le Chapitre 4, Conjecture 4.5.1). Ces propriétés laissent penser que l'algèbre de Hall d'un carquois puisse avoir une version aymptotique obtenue en laissant tendre le nombre de flèches vers l'infini, mais nous n'avons pas encore approfondi cette question.

La recherche d'une compréhension géométrique de l'algèbre de Hall d'un carquois mène à la construction de Lusztig [Lus91, Lus98]. Lusztig considère la catégorie des faisceaux pervers sur le champ de représentations du carquois qui sont obtenus comme facteurs directs simples d'inductions de faisceaux constants. Le groupe de Grothendieck additif de la catégorie triangulée associée est alors isomorphe à la partie positive du groupe quantique correspondant au carquois et les faisceaux pervers simples donnent la base canonique, définie et étudiée d'un point de vue purement combinatoire par Kashiwara ([Kas91]). Cette construction en termes de faisceaux pervers sur les champs de représentations a un analogue symplectique en termes de composantes irréductibles de la variété nilpotente de Lusztig, Λ, qui est un sous-champ Lagrangien du champ cotangent du champ de représentations du carquois. Cette deuxième construction donne lieu à la base *semicanonique* de l'algèbre enveloppante de la partie positive de l'algèbre de Kac-Moody associée au carquois ([Lus00]), et le lien entre les deux constructions est fait, de façon imprécise, par l'application cycle caractéristique qui à un faisceau pervers associe une combinaison $\mathbf{Z}$-linéaire des composantes irréductibles de Λ. Mus par notre objectif d'une compréhension géométrique de l'algèbre de Hall, il est naturel de se demander quels sont les faisceaux pervers sur le champ des représentations d'un carquois dont le support singulier (le support du cycle caractéristique) est inclus dans la variété nilpotente de Lusztig. On est capable de répondre à cette question pour les carquois de type fini, les carquois affines et pour les carquois ayant un seul sommet et plusieurs boucles : c'est l'objet du travail [Hen20b]. Dans tous ces cas, il n'y a pas d'autres faisceaux pervers simples à support singulier nilpotent que les faisceaux considérés par Lusztig. La question reste ouverte pour les carquois sauvages généraux bien qu'on dispose de résultats allant dans la même direction pour des carquois sauvages très particuliers (les carquois de Kronecker généralisés en dimension $(1, d)$, $d \geq 0$ ou un carquois arbitraire pour la dimension 1 en chaque sommet). Les méthodes pour traiter ces cas particuliers ne se généralisant pas et étant assez standards, nous ne les avons pas incluses dans cette thèse. Il paraît donc raisonable de conjecturer, à la suite de Lusztig, que cette *caractérisation microlocale* des faisceaux de Lusztig par la nilpotence du support singulier reste valable en toute généralité.

Cette question de la relation entre une catégorie de faisceaux pervers sur un champ et une sous-variété du champ cotangent de cet espace a un analogue pour les courbes projectives lisses définies sur un corps fini. Dans ce nouveau cadre, on remplace le carquois par une courbe, les représentations du carquois par la catégorie des faisceaux cohérents sur la courbe, les faisceaux de Lusztig par les *faisceaux d'Eisenstein sphériques* (définis par Schiffmann, [Sch04a, Sch11]) et la variété nilpotente de Lusztig par le cône global nilpotent (les faisceaux de Higgs nilpotents, [Lau88]). On peut montrer, dans le cas d'une courbe elliptique, que l'application cycle caractéristique induit un isomorphisme entre une complétion du groupe de Grothendieck de la catégorie des faisceaux d'Eisenstein sphériques d'une part et une complétion du $\mathbf{Z}$-module dont une base est formée par les composantes irréductibles du cône global nilpotent d'autre part. Par contre, la caractérisation microlocale ne tient plus. Cependant, il est possible de décrire explicitement (toujours dans le cas d'une courbe elliptique) les faisceaux pervers simples sur le champ des faisceaux cohérents dont le support singulier est nilpotent [Hen21]. Le cas d'une courbe de genre $g \geq 2$ paraît beaucoup plus difficile. Néanmoins, répondre à ces questions fournirait des outils supplémentaires pour étudier l'algèbre de Hall cohomologique associée à une courbe ([SS18]).

Dans la suite de cette introduction, nous allons donner plus de détails sur les ressorts et les motivations ayant abouti aux travaux constituant cette thèse.

1.1 L'analogie entre les courbes et les carquois

Les catégories de faisceaux cohérents sur les courbes projectives lisses et les catégories de représentations de carquois sur un corps sont liées par de nombreuses analogies. Le lecteur peut se référer à l'article de survol [Hos18] pour plus de détails. Le premier objet, la catégorie des faisceaux cohérents sur une courbe projective lisse, est l'objet central en considération dans le programme de Langlands pour les corps de fonctions dans sa version géométrique ([Lau87]), qui vise à établir par des méthodes géométriques la correspondance de Langlands globale. Rappelons que la correspondance de Langlands pour une courbe projective lisse X sur le corps fini $\mathbf{F}_q$ à q éléments pour le groupe GL_r établit une bijection entre les représentations ℓ-adiques irréductibles de rang r du groupe de Galois de X de déterminant d'ordre fini et les représentations automorphes cuspidales irréductibles du groupe des adèles $\mathrm{GL}_r(\mathbb{A})$. Pour les groupes linéaires GL_r ($r \geq 1$), cette correspondance est connue grâce aux travaux de Laurent Lafforgue ([Laf02]). Pour des groupes réductifs généraux, il existe un ensemble de conjectures qui restent largement ouvertes à ce jour, mais font l'objet d'une recherche active. Les carquois, quant-à-eux, sont un objet central de la théorie des représentations des algèbres de dimension finie sur un corps. Leur algèbre de chemins en sont en effet l'exemple archétypal, dont l'étude a commencé avec Gabriel ([Gab72]). Ils ont pénétré de nombreux domaines des mathématiques et permettent souvent de comprendre des objets plus compliqués. À ce titre, on peut mentionner que la structure locale de l'espace de modules de faisceaux sur une surface $K3$ est localement isomorphe à une variété de carquois de Nakajima ([AS18]) ou que les carquois avec potentiel fournissent des modèles intéressants pour les catégories de Calabi-Yau de dimension trois ([Gin06]) et donc pour les champs de faisceaux cohérents sur les variétés de Calabi-Yau de dimension trois. À une courbe projective lisse X sur un corps k, nous associons la catégorie des faisceaux cohérents sur celle-ci, $\mathrm{Coh}(X)$; à un carquois Q, nous associons la catégorie des représentations de celui-ci sur le corps k, $\mathrm{Rep}_Q(k)$. Le premier résultat témoignant de la qualité de ces catégories et qui rapproche leur étude géométrique est le suivant. Nous appelons *dimension homologique* d'une catégorie abélienne $\mathcal{A}$ le plus petit entier d tel que pour tout $i > d$, les bifoncteurs $\mathrm{Ext}^i(-,-)$ soient nuls.

Proposition 1.1.1. *Supposons que le carquois Q contienne au moins une flèche*[3]*. Les catégories* $\mathrm{Coh}(X)$ *et* $\mathrm{Rep}_Q(k)$ *sont des catégories abéliennes de dimension homologique un.*

L'autre propriété importante de ces catégories est l'existence d'une dualité de Serre. Afin d'unifier les notations, nous utilisons la lettre $\mathcal{A}$ pour désigner l'une des catégories $\mathrm{Coh}(X)$ ou $\mathrm{Rep}_Q(k)$. Si ω_X désigne le fibré canonique de X, on a un foncteur

$$\begin{array}{rcccc} F & : & \mathrm{Coh}(X) & \to & \mathrm{Coh}(X) \\ & & \mathcal{F} & \mapsto & \mathcal{F} \otimes \omega_X \end{array}$$

qui s'étend en un foncteur $F : D^b(X) \to D^b(X)$, $\mathcal{F} \mapsto \mathcal{F} \otimes \omega_X$ entre catégories dérivées. Pour le carquois Q (que l'on suppose acyclique), on note $F = \tau$ la transformée d'Auslander-Reiten associée. C'est un endofoncteur de la catégorie $\mathrm{Rep}_Q(k)$, qui s'étend à la catégorie dérivée $D^b(Q,k)$. La dualité de Serre s'exprime par le résultat suivant (cf. [Sch12a, Theorem 3.30, Section 4.1]).

Proposition 1.1.2. *Pour tous objets $\mathcal{F}, \mathcal{G}$ de $D^b(\mathcal{A})$, on a un isomorphisme naturel en les deux variables*

$$\mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{G}, F(\mathcal{F})[1])^*.$$

[3] La catégorie des représentations d'un carquois sans flèches est équivalente à la catégorie des espaces vectoriels gradués par son ensemble de sommets ; elle est de dimension homologique zéro.

Le décalage [1] est un témoin de la dimension homologique un.

La classification des catégories abéliennes sujettes aux énoncés des propositions 1.1.1 et 1.1.2 est un objet d'étude et la littérature regorge de résultats dans cette direction. Nous référons le lecteur au bel article [RVdB02].

Les catégories de faisceaux cohérents sur une courbe, ou de représentations d'un carquois étant de dimension homologique un, les champs classifiant leurs objets sont lisses. C'est une propriété importante pour les considérations géométriques, utilisée dans les chapitres 5 et 6.

Parfois, les catégories de représentations de carquois et de faisceaux cohérents sur une courbe sont plus intimement reliées, comme le montre le théorème suivant où la courbe projective $\mathbf{P}^1_k$ et le carquois de Kronecker K_2 sont en considération (cf. [Sch12a, Section 5.1]).

Théorème 1.1.1. *Les catégories triangulées $D^b(\mathbf{P}^1_k)$ et $D^b(K_2, k)$ sont équivalentes.*

Remarquons toutefois que la catégorie dérivée d'une catégorie abélienne ne garde pas en mémoire la dimension homologique de la catégorie abélienne de départ. Par exemple, les algèbres canoniques sont de dimension homologique 2 mais leurs catégories dérivées sont isomorphes à des catégories dérivées de courbes projectives à poids, et ces dernières sont de dimension homologique 1 (voir l'article [PS19] et les références qu'il contient).

Le théorème 1.1.1 peut s'étendre à certaines droites projectives à poids, pour lesquelles on doit plutôt considérer des carquois de type affine (cf. [Cra10, Section 9]).

1.2 L'algèbre de Hall d'une catégorie finitaire

Nous faisons quelques rappels sur l'algèbre de Hall constructible d'une catégorie abélienne finitaire. Pour de plus amples détails, le lecteur est invité à se référer aux notes [Sch12a].

Soit $\mathcal{A}$ une catégorie abélienne. Supposons qu'elle est petite, au sens où l'ensemble $\mathcal{A}/\sim$ des classes d'isomorphismes de ses objets est bien défini, et finitaire, c'est-à-dire que pour toute paire d'objets $\mathcal{F}$ et $\mathcal{G}$ de $\mathcal{A}$, un nombre fini des ensembles $\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ sont non-triviaux et leurs cardinaux sont finis. Bien souvent, les catégories en jeu seront linéaires sur un corps fini. Nous conseillons au lecteur de garder en mémoire les catégories de représentations de dimension finie d'un carquois sur un corps fini ainsi que celle des faisceaux cohérents sur une courbe projective lisse définie sur un corps fini. La forme d'Euler (dite *multiplicative*) d'une telle catégorie est définie par la formule

$$\langle \mathcal{F}, \mathcal{G} \rangle_m = \prod_{i \in \mathbf{Z}} |\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})|^{(-1)^i}.$$

où pour un ensemble fini E, $|E|$ désigne son cardinal. Un argument standard faisant intervenir une suite exacte longue permet de montrer que la forme d'Euler se factorise par le groupe de Grothendieck de la catégorie $\mathcal{A}$, que nous notons $K_0(\mathcal{A})$, et induit un morphisme de groupes abéliens noté de la même façon:

$$\begin{array}{rccc} \langle -, - \rangle_m & : & K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) & \to & \mathbf{Q}^* \\ & & [\mathcal{F}] \otimes [\mathcal{G}] & \mapsto & \langle \mathcal{F}, \mathcal{G} \rangle_m \end{array}$$

où par le symbole $[\mathcal{F}]$ nous désignons la classe de l'objet $\mathcal{F}$ de la catégorie $\mathcal{A}$ dans son groupe de Grothendieck. Si de plus la catégorie $\mathcal{A}$ est $\mathbf{F}_q$-linéaire, on définit sa *forme d'Euler additive* $\langle -,-\rangle_a : K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) \to \mathbf{Z}$ comme la somme alternée des dimensions des groupes $\mathrm{Ext}^i_{\mathcal{A}}(-,-)$. Elle vérifie $q^{\langle \mathcal{F},\mathcal{G}\rangle_a} = \langle \mathcal{F},\mathcal{G}\rangle_m$ pour tous objets $\mathcal{F},\mathcal{G}$ de $\mathcal{A}$. Souvent, on peut remplacer le groupe abélien $K_0(\mathcal{A})$ par un groupe abélien M muni d'une forme bilinéaire multiplicative $M \otimes M \to \mathbf{Q}^*$ et d'un morphisme $K_0(\mathcal{A}) \to M$ par lequel $\langle -,-\rangle_m$ se factorise. Cette situation se produit pour les carquois $Q = (I,\Omega)$ avec I pour ensemble de sommets pour lesquels la forme d'Euler se factorise par le morphisme de groupes abéliens $\dim : K_0(\mathcal{A}) \to \mathbf{Z}^I$ donné par le vecteur dimension[4] et aussi pour les courbes, où cette fois la forme d'Euler se factorise par le morphisme $K_0(\mathrm{Coh}(X)) \to \mathbf{Z}^2$ associant à $[\mathcal{F}] \in K_0(\mathrm{Coh}(X))$ son rang et son degré.

Si $\mathcal{X}$ est un groupoïde dont l'ensemble des classes d'isomorphismes d'objets est noté $\mathcal{X}/{\sim}$, nous notons $\mathrm{Fun}_c(\mathcal{X}/{\sim},\mathbf{Q})$ l'ensemble des fonctions sur $\mathcal{X}/{\sim}$ à support fini. On notera $[x]$ la classe d'isomorphisme d'un objet x de $\mathcal{X}$. Étant donné un morphisme entre groupoïdes $F : \mathcal{X} \to \mathcal{Y}$, nous pouvons définir des opérations sur les ensembles de fonctions associés :

$$F^* : \mathrm{Fun}_c(\mathcal{Y}/{\sim},\mathbf{Q}) \to \mathrm{Fun}_c(\mathcal{X}/{\sim},\mathbf{Q})$$

et

$$F_! : \mathrm{Fun}_c(\mathcal{X}/{\sim},\mathbf{Q}) \to \mathrm{Fun}_c(\mathcal{Y}/{\sim},\mathbf{Q}).$$

Pour définir $F_!$, il est nécessaire de supposer que les fibres de F sont finies (plus précisément que les ensembles de classes d'isomorphismes des fibres de F sont finis et que les groupes d'automorphismes des objets de $\mathfrak{X}$ et $\mathfrak{Y}$ sont finis). Pour $f \in \mathrm{Fun}_c(\mathcal{Y},\mathbf{Q})$ et un objet x de $\mathcal{X}$, nous définissons

$$F^*(f)([x]) = f([F(x)])$$

et, sous l'hypothèses où les fibres sont finies et les groupes d'automorphismes des objets de $\mathcal{X}$ et de $\mathcal{Y}$ sont des groupes finis, pour y un objet de $\mathcal{Y}$,

$$F_!(f)([y]) = |\mathrm{Aut}(y)| \sum_{[x]\in\mathcal{X}/\sim,[F(x)]=[y]} \frac{f([x])}{|\mathrm{Aut}(x)|}.$$

Le volume $\mathrm{Vol}(\mathcal{X})$ du groupoïde $\mathcal{X}$ est défini comme le pousser-en-avant de la fonction caractéristique $1_{\mathcal{X}}$ prenant 1 pour seule valeur par le morphisme $F : \mathcal{X} \to \mathrm{pt}$ (où pt est le groupoïde ayant un seul et un seul morphisme). En formule,

$$\mathrm{Vol}(\mathcal{X}) = \sum_{[x]\in\mathcal{X}/\sim} \frac{1}{|\mathrm{Aut}(x)|}.$$

Désormais, $\mathcal{X}$ désigne le groupoïde associé à la catégorie $\mathcal{A}$. On définit également $\mathcal{A}^{(1)}$ la catégorie des suites exactes de $\mathcal{A}$ et nous notons $\mathcal{X}^{(1)}$ le groupoïde associé. Nous disposons outre

[4] Si le carquois est acyclique, dim est un isomorphisme.

$\mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q})$ de l'ensemble $\mathrm{Fun}_c(\mathcal{X}^{(1)}/\sim, \mathbf{Q})$. On considère le diagramme de convolution

$$\begin{tikzcd} & \mathcal{X}^{(1)} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{X} \times \mathcal{X} & & \mathcal{X} \end{tikzcd}$$

où p est le foncteur envoyant une suite exacte courte sur son terme central et q sur la paire formée de son troisième et premier termes. De la sorte, nous obtenons les opérateurs d'induction et de restriction

$$\mathrm{Ind} = p_! q^* : \mathrm{Fun}_c(\mathcal{X}/\sim \times \mathcal{X}/\sim, \mathbf{Q}) \to \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q})$$

et

$$\mathrm{Res} = q_! p^* : \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q}) \to \mathrm{Fun}_c(\mathcal{X}/\sim \times \mathcal{X}/\sim, \mathbf{Q}).$$

Remarquons que Res n'est bien défini que lorsqu'une condition de finitude supplémentaire est vérifiée par la catégorie $\mathcal{A}$. En effet, il n'est pas vrai que $\mathrm{Res}(f)$ est à support compact si f l'est. Une condition nécessaire et suffisante pour que ce soit le cas est que tout objet de $\mathcal{A}$ ait un nombre fini de sous-objets. Cette condition est vérifiée pour la catégorie des représentations d'un carquois sur un corps fini, mais pas pour les catégories de faisceaux cohérents sur une courbe projective lisse sur un corps fini. Il est possible de s'en tirer en considérant des complétions mais pour préserver la lisibilité de cette introduction, nous ignorons cette question. Comme on a une identification

$$\mathrm{Fun}_c(\mathcal{X}/\sim \times \mathcal{X}/\sim, \mathbf{Q}) \simeq \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q}),$$

ces opérateurs munissent $\mathrm{Fun}_c(\mathcal{X}, \mathbf{Q})$ de structures d'algèbre et de coalgèbre. La démonstration de l'associativité et de la coassociativité de ces opérations fait intervenir la catégorie $\mathcal{A}^{(2)}$ des filtrations à deux pas $0 \subset M_1 \subset M_2 \subset M$ des objets de $\mathcal{A}$. On peut expliciter les formules pour l'induction et la restriction de fonctions indicatrices de classes d'isomorphismes. Si x est un objet de $\mathcal{A}$, on note $1_{[x]} \in \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q})$ la fonction prenant la valeur 1 sur $[x]$ et la valeur 0 sur $[y] \in \mathcal{X}/\sim$ tel que $[x] \neq [y]$.

Pour trois objets x, y, z de $\mathcal{A}$,

$$\begin{aligned} \mathrm{Ind}(1_{[x]}, 1_{[y]})([z]) &= \sum_{\substack{[e] \in \mathcal{X}^{(1)}/\sim \\ [q(e)] = ([x],[y]) \\ [p(e)] = [z]}} \frac{|\mathrm{Aut}(z)|}{|\mathrm{Aut}(e)|} \\ &= |\mathrm{Aut}(z)| \operatorname{Vol}(\mathcal{X}^{(1)}_{y,z,x}) \end{aligned}$$

où $\mathcal{X}^{(1)}_{y,z,x}$ est le groupoïde des suites exactes courtes $0 \to y' \to z' \to x' \to 0$ avec $x' \simeq x$, $y' \simeq y$, $z' \simeq z$. Le groupoïde $\mathcal{X}^{(1)}_{y,z,x}$ est equivalent au groupoïde quotient

$$\tilde{\mathcal{X}}^{(1)}_{y,z,x} / (\mathrm{Aut}(y) \times \mathrm{Aut}(z) \times \mathrm{Aut}(x))$$

où $\tilde{\mathcal{X}}^{(1)}_{y,z,x}$ est l'ensemble

$$\{(\alpha,\beta)\in \mathrm{Hom}(y,z)\times \mathrm{Hom}(z,x)\mid 0\to y\xrightarrow{\alpha} z\xrightarrow{\beta} x\to 0 \text{ est exacte}\}$$

et l'action de $\mathrm{Aut}(y)\times \mathrm{Aut}(z)\times \mathrm{Aut}(x)$ sur cet ensemble est donnée par

$$(a,b,c)\cdot(\alpha,\beta)=(b\alpha a^{-1},c\beta b^{-1}).$$

On trouve donc

$$\mathrm{Vol}(\mathcal{X}^{(1)}_{y,z,x})=\frac{F^z_{x,y}}{|\mathrm{Aut}(z)|}$$

où $F^z_{x,y}=\{y'\subset z\mid y'\simeq y, z/y'\simeq x\}$ est l'ensemble des sous-objets y' de z isomorphes à y et dont le quotient z/y' est isomorphe à x et donc

$$\mathrm{Ind}(1_{[x]},1_{[y]})([z])=F^z_{x,y}.$$

Pour la restriction, on trouve

$$\begin{aligned}\mathrm{Res}(1_{[z]})([x],[y]) &= \sum_{\substack{[e]\in\mathcal{X}^{(1)}/\sim\\ [p(e)]=[z]\\ [q(e)]=([x],[y])}}\frac{|\mathrm{Aut}(x)|\cdot|\mathrm{Aut}(y)|}{|\mathrm{Aut}(e)|}\\ &= |\mathrm{Aut}(x)|\cdot|\mathrm{Aut}(y)|\,\mathrm{Vol}(\mathcal{X}^{(1)}_{y,z,x})\\ &= \frac{|\mathrm{Aut}(x)|\cdot|\mathrm{Aut}(y)|}{|\mathrm{Aut}(z)|}F^z_{x,y}.\end{aligned}$$

L'unité de la multiplication est la fonction $1\in\mathrm{Fun}_c(\mathcal{X}/\sim,\mathbf{Q})$ définie par

$$\begin{array}{rccc} 1 & : & \mathcal{X}/\sim & \to & \mathbf{Q}\\ & & [x] & \mapsto & \begin{cases}1 \text{ si } x\simeq 0\\ 0 \text{ sinon}\end{cases}\end{array}$$

La counité est l'application $\mathbf{Q}$-linéaire

$$\begin{array}{rccc} \varepsilon & : & \mathrm{Fun}_c(\mathcal{X}/\sim,\mathbf{Q}) & \to & \mathbf{Q}\\ & & f & \mapsto & f([0])\end{array}$$

où 0 est l'objet terminal de la catégorie $\mathcal{A}$. Si $\alpha,\beta: K_0(\mathcal{A})\otimes K_0(\mathcal{A})\to \mathbf{C}^*$ sont deux formes bilinéaires multiplicatives, on peut tordre l'induction et la restriction de la façon suivante. Pour $[M],[N]\in\mathcal{X}/\sim$,

$$\mathrm{Ind}_\alpha(1_{[M]},1_{[N]}):=\alpha([M],[N])\,\mathrm{Ind}(1_{[M]},1_{[N]})$$

et

$$\mathrm{Res}_\beta=\beta\cdot\mathrm{Res}$$

où β est vu comme un élément de $\mathrm{Fun}(\mathcal{X}/\sim\times\mathcal{X}/\sim,\mathbf{C})$ (une fonction à support pas forcément fini) de la façon suivante.

$$\begin{array}{rcccc} \beta & : & \mathcal{X}/\sim \times \mathcal{X}/\sim & \to & \mathbf{C} \\ & & ([x],[y]) & \mapsto & \beta([x],[y]). \end{array}$$

et le point $\cdot$ indique la multiplication ponctuelle.

À l'aide d'une troisième forme bilinéaire multiplicative $\gamma : K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) \to \mathbf{C}^*$, on peut définir une multiplication sur le produit tensoriel $\mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{C}) \otimes \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{C})$:

$$(1_{[x]} \otimes 1_{[y]}) \cdot (1_{[z]} \otimes 1_{[t]}) = \gamma([y],[z])((1_{[x]}1_{[z]}) \otimes (1_{[y]}1_{[t]})).$$

La forme bilinéaire γ est appelée *tressage*. Si α, β sont choisies constantes égales à 1, on prend $\gamma = \langle -,- \rangle_m^{-1}$ l'inverse de la forme d'Euler.

Dans le cas où la catégorie $\mathcal{A}$ est héréditaire (les bifoncteurs Ext^i sont identiquement nuls pour $i \geq 2$), on a le résultat suivant qui donne la compatibilité entre la multiplication et la comultiplication.

Théorème 1.2.1 (Green)**.** *Soit $\mathcal{A}$ une catégorie abélienne héréditaire. Alors*

$$(\mathrm{Fun}_c(\mathcal{X}, \mathbf{Q}), \mathrm{Ind}, \mathrm{Res})$$

est une bialgèbre pour le tressage $\gamma = \langle -,- \rangle_m^{-1}$

D'ordinaire, on tord la multiplication et la comultiplication de l'algèbre de Hall grâce à des formes bilinéaires α, γ non triviales. Le choix usuel, celui que nous considèrerons par défaut, est

$$\alpha = \beta = \langle -,- \rangle_m$$

Dans ce cas, le choix du tressage est

$$\gamma = (-,-)_m,$$

la forme d'Euler symétrisée.

Il n'est pas raisonnable d'espérer décrire la structure de $(\mathrm{Fun}_c(\mathcal{X}, \mathbf{Q}), \mathrm{Ind})$ pour une catégorie abélienne arbitraire. Comme nous l'avons déjà mentionné plusieurs fois, les cas qui nous intéressent sont $\mathcal{A} = \mathrm{Rep}_Q(\mathbf{F}_q)$ et $\mathcal{A} = \mathrm{Coh}(X)$.

La bialgèbre $(\mathrm{Fun}_c(\mathcal{X}, \mathbf{Q}), \mathrm{Ind}_\alpha, \mathrm{Res}_\beta)$ est appelée *algèbre de Hall* de la catégorie $\mathcal{A}$. On la note $\mathbf{H}_\mathcal{A}$. Si $\mathcal{A} = \mathrm{Rep}_Q(\mathbf{F}_q)$, on note $\mathbf{H}_\mathcal{A} = \mathbf{H}_{Q,\mathbf{F}_q}$. C'est une bialgèbre dans la catégorie monoïdale tressée des espaces vectoriels $\mathbf{K}_0(\mathcal{A})$-gradués, où le tressage est donné par γ : si V, W sont des espaces vectoriels $K_0(\mathcal{A})$-gradués, l'isomorphisme de tressage est

$$\begin{array}{rcc} V \otimes W & \to & W \otimes V \\ v \otimes w & \mapsto & \gamma(\deg(w), \deg(v)) w \otimes v \end{array}$$

si v, w sont homogènes de degrés respectifs $\deg(v), \deg(w) \in K_0(\mathcal{A})$.

Il est possible d'obtenir à partir de $(\mathrm{Fun}_c(\mathcal{X}, \mathbf{Q}), \mathrm{Ind}_\alpha, \mathrm{Res}_\beta)$ une bialgèbre au sens classique (une bialgèbre dans la catégorie monoïdale symétrique des espaces vectoriels, c'est-à-dire avec un tressage trivial) en lui adjoignant l'algèbre de groupe $\mathbf{C}[K_0(\mathcal{A})]$. De façon précise, on définit un espace vectoriel

$$\tilde{\mathbf{H}}_{\mathcal{A}} = \mathbf{H}_{\mathcal{A}} \otimes_{\mathbf{C}} \mathbf{C}[K_0(\mathcal{A})]$$

et on le munit d'un produit et d'un coproduit Δ de la façon suivante. Pour $[x],[y] \in \mathcal{X}/\sim$, et $\alpha, \beta \in K_0(\mathcal{A})$,

$$([x] \otimes \alpha)([y] \otimes \beta) = (\alpha, [y])_m (1_{[x]} 1_{[y]}) \otimes (\alpha\beta)$$

et pour un objet z de $\mathcal{A}$,

$$\Delta([z] \otimes \alpha) = \sum_{[x],[y] \in \mathcal{X}/\sim} \langle x, y\rangle_m \frac{|\mathrm{Aut}(x)| \cdot |\mathrm{Aut}(y)|}{|\mathrm{Aut}(z)|} F^z_{x,y}([x] \otimes ([y] + \alpha)) \otimes ([y] \otimes \alpha).$$

Si on a un morphisme de groupes abéliens $K_0(\mathcal{A}) \to M$ à travers lequel la forme d'Euler se factorise, on peut remplacer $K_0(\mathcal{A})$ par M.

Remarque 1.2.1. On peut se demander ce qu'il se passe pour une catégorie non héréditaire $\mathcal{A}$. On ne peut plus espérer avoir la compatibilité du théorème de Green 1.2.1 en utilisant la multiplication et la comultiplication telles qu'on les a définies. Peut-être qu'il existe une formule faisant intervenir les groupes d'extensions Ext^i pour $i \geq 2$ permettant d'exprimer la différence

$$\Delta(\mathcal{F}\mathcal{G}) - \Delta(\mathcal{F})\Delta(\mathcal{G}).$$

Nous laissons cette question en suspend. Notons cependant que la multiplication de l'algèbre de Hall dérivée de Toën ([Toë06]) utilise les Ext^i pour $i \geq 0$ et peut être utile pour aborder cette question.

1.3 Carquois, algèbres de Lie, algèbres enveloppantes et leurs déformations

1.3.1 Carquois et algèbres de Lie

La théorie des algèbres de Lie semisimples sur un corps algébriquement clos de caractéristique nulle est bien connue ([Ser87]). Si $\mathfrak{g}$ est une telle algèbre, le choix d'une sous-algèbre de Cartan $\mathfrak{h}$ et d'une sous-algèbre de Borel $\mathfrak{b}$ qui la contient donne un système de racines et, via une forme bilinéaire invariante, une matrice de Cartan (une matrice inversible à coefficients entiers, ayant des 2 sur la diagonale et telle qu'un coefficient en dehors de la diagonale est nul si et seulement si son symétrique l'est aussi). La donnée d'une telle matrice peut être encodée sous la forme d'un graphe, qui ne dépend d'aucun des choix effectués précédemment. Inversement, à partir de la donné du graphe, on peut définir une algèbre de Lie par générateurs et relations et retrouver ainsi l'algèbre de Lie à laquelle on a fait correspondre le graphe.

Ce procédé inverse peut être appliqué à n'importe quel graphe sans boucles, pas seulement aux graphes de type Dynkin. Les algèbres de Lie obtenues dans ce cadre plus général sont appelées *algèbres de Kac-Moody* ([Kac90]). Si on autorise maintenant des graphes avec des boucles, on trouve les algèbres de Borcherds (ou algèbres de Kac-Moody généralisées), [Bor88]. Malgré la généralité croissante dans la définition de ces algèbres et le fait que les algèbres de Kac-Moody et de Borcherds soient de dimension infinie (à moins d'être des algèbres de Lie semisimples), celles-ci partagent de nombreuses propriétés. Les premières d'entre elles étant la *formule du caractère* et la similarité de leur théorie des représentations (modules de plus haut poids, modules de Verma).

On rappelle brièvement la définition par générateurs et relations des algèbres de Borcherds symétrisables (suivant Borcherds [Bor88, Bor95]). La donnée de départ est la suivante : un espace vectoriel réel H muni d'une forme bilinéaire symétrique $(-,-)$ ainsi qu'un ensemble dénombrable de vecteurs de H, $(h_i)_{i\in I}$, vérifiant les propriétés suivantes :

1. $(h_i, h_j) \leq 0$ si $i \neq j$,
2. $\frac{2(h_i,h_j)}{(h_i,h_i)}$ est un entier si $(h_i, h_i) > 0$.

Afin de simplifier l'exposition, et comme c'est le seul cas qui nous intéressera, on suppose que pour tout $i \in I$, $(h_i, h_i) > 0$ implique $(h_i, h_i) = 2$ (c'est-à-dire que l'on se place dans le cas symétrique[5]). La matrice de Cartan associée à cette donnée est la matrice $A = (a_{i,j})_{i,j\in I} = ((h_i, h_j))_{i,j\in I}$ et l'algèbre de Borcherds (algèbre de Kac-Moody généralisée) associée à cette donnée est l'algèbre de Lie $\mathfrak{g}(A)$ engendrée par les éléments H, e_i, f_i $(i \in I)$ sujets aux relations suivantes.

1. Les éléments de H commutent deux à deux,
2. Pour $h \in H$ et $i \in I$, $[h, e_i] = (h, h_i)e_i$ et $[h, f_i] = -(h, h_i)f_i$,
3. $[e_i, f_j] = h_i$ si $i = j$, $[e_i, f_j] = 0$ si $i \neq j$,
4. Si $a_{i,i} > 0$, $\mathrm{ad}^{1-a_{i,j}}(e_i)(e_j) = \mathrm{ad}^{1-a_{i,j}}(f_i)(f_j) = 0$ (relations de Serre)
5. Si $a_{i,j} = 0$, $[e_i, e_j] = [f_i, f_j] = 0$.

On a une décomposition triangulaire

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus H \oplus \mathfrak{n}_+$$

où $\mathfrak{n}_-$ (resp. $\mathfrak{n}_+$) est la sous-algèbre de Lie de $\mathfrak{g}(A)$ engendrée par les éléments $f_i, i \in I$ (resp. $e_i, i \in I$).

Les algèbres de Kac-Moody symétriques, construites dans [Kac90], peuvent être obtenues comme un cas particulier de cette construction. Soit A une matrice de Cartan symétrique, $A = (a_{i,j})_{1\leq i,j\leq n}$ ($a_{i,i} = 2$, $a_{i,j} \leq 0$, $a_{i,j} = a_{j,i}$ pour tout $1 \leq i,j \leq n$). Une *réalisation* de A est un espace vectoriel $\mathfrak{h}$ muni d'une forme bilinéaire symétrique non dégénérée $(-,-)$, d'un ensemble de vecteurs $\Pi \subset \mathfrak{h} = \{\alpha_i : 1 \leq i \leq n\}$ linéairement indépendants et tel que $(\alpha_i, \alpha_j) = a_{i,j}$. On demande en plus que $\mathfrak{h}$ soit de dimension minimale, ce qui veut ici dire que $\dim \mathfrak{h} = 2n - \operatorname{rank} A$. La donnée de $(\mathfrak{h}, (-,-), (\alpha_i)_{1\leq i\leq n})$ permet de construire une algèbre de Borcherds, c'est l'algèbre de Kac-Moody associée à la matrice de Cartan symétrique A.

[5] plutôt que seulement dans le cas symétrisable

À un carquois $Q = (I, \Omega)$ sans boucles, dont I est l'ensemble de sommets et Ω l'ensemble de flèches, on associe une matrice de Cartan $A = (a_{i,j})_{i,j \in I}$ en posant $a_{i,j} = 2\delta_{i,j} - |\{\alpha \in \Omega \mid \{s(\alpha), t(\alpha)\} = \{i, j\}\}|$. Cette matrice ne dépend pas de l'orientation du carquois. L'algèbre de Kac-Moody associée au carquois Q est notée $\mathfrak{g}_Q$. Si Q possède des boucles, l'algèbre associée est une algèbre de Borcherds.

1.3.2 Algèbres enveloppantes et groupes quantiques

Classiquement, on associe à toute algèbre de Lie $\mathfrak{g}$ une algèbre associative notée $\mathbf{U}(\mathfrak{g})$, appelée algèbre enveloppante de $\mathfrak{g}$, et dont la théorie des représentations coïncide avec celle de $\mathfrak{g}$. Elle est construite de façon effective comme le quotient de l'algèbre tensorielle $T\mathfrak{g}$ par l'idéal bilatère engendré par les relations $[x, y] = x \otimes y - y \otimes x$. Ces algèbres sont étudiées dans [Dix96]. Si $\mathfrak{g}$ est l'algèbre de Kac-Moody symétrique associée à une matrice de Cartan symétrique A ou plus généralement une algèbre de Borcherds associée à la donnée $(H, (-,-), (h_i)_{i \in I})$, son algèbre enveloppante peut être décrite par générateurs et relations de la façon suivante. L'algèbre $\mathbf{U}(\mathfrak{g}(A))$ est l'algèbre associative engendrée par H, e_i, f_i, $i \in I$, avec les relations suivantes.

1. Les éléments de H commutent deux à deux,
2. Pour $h \in H$ et $i \in I$, $he_i - e_ih = (h, h_i)e_i$ et $hf_i - f_ih = -(h, h_i)f_i$,
3. $e_if_j - f_je_i = \delta_{i,j}h_i$,
4. Si $a_{i,i} > 0$, $\sum_{k+l=1-a_{i,j}} (-1)^k e_i^{(k)} e_j e_i^{(l)} = 0$, $\sum_{k+l=1-a_{i,j}} (-1)^k f_i^{(k)} f_j f_i^{(l)} = 0$, où $e_i^{(l)} := \frac{e_i^l}{l!}$ pour $l \in \mathbf{N}$ (relations de Serre),
5. Si $a_{i,j} = 0$, $e_ie_j - e_je_i = f_if_j - f_jf_i = 0$.

L'algèbre $\mathbf{U}(\mathfrak{g}(A))$ admet une décomposition triangulaire

$$\mathbf{U}(\mathfrak{g}(A)) = \mathbf{U}(\mathfrak{n}_-) \otimes \mathbf{U}(H) \otimes \mathbf{U}(\mathfrak{n}_+).$$

L'algèbre enveloppante a une structure plus riche encore. C'est une algèbre de Hopf, lorsqu'on la munit de la comultiplication Δ, de l'antipode S et de la counité ϵ. La comultiplication est un morphisme d'algèbres $\Delta : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g}) \otimes \mathbf{U}(\mathfrak{g})$ uniquement déterminé par la formule

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

pour $x \in \mathfrak{g} \subset \mathbf{U}(\mathfrak{g})$; la counité est un morphisme d'algèbres $\varepsilon : \mathbf{U}(\mathfrak{g}) \to \mathbf{C}$ donné par $\varepsilon(x) = 0$ pour $x \in \mathfrak{g}$ et enfin l'antipode est un antihomomorphisme d'algèbres $S : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g})$ déterminé par $S(x) = -x$ pour $x \in \mathfrak{g}$. Ces nouvelles opérations permettent de définir le produit tensoriel de $\mathfrak{g}$-représentations et de munir le dual vectoriel d'une structure de $\mathfrak{g}$-représentation. En d'autres termes, la catégorie des $\mathbf{U}(\mathfrak{g})$-modules est une catégorie tensorielle (pour une étude générale des catégories tensorielles, on peut se référer à [EGNO15]). L'algèbre de Hopf $\mathbf{U}(\mathfrak{g})$ est cocommutative, mais en général elle n'est pas commutative. Elle l'est si et seulement si $\mathfrak{g}$ est abélienne.

Drinfeld ([Dri87b]) et Jimbo ([Jim85]), motivés par des questions de mécanique statistique, ont introduit indépendemment une déformation à un paramètre des algèbres de Hopf $\mathbf{U}(\mathfrak{g})$ lorsque $\mathfrak{g}$ est une algèbre de Lie semi-simple. La même définition fonctionne lorsque $\mathfrak{g}$ est une algèbre de Kac-Moody, et a été étendue par Kang ([Kan95]) aux algèbres de Borcherds. Cette déformation est notée $\mathbf{U}_q(\mathfrak{g})$; c'est une algèbre sur le corps des fractions rationnelles en une variable, $\mathbf{C}(q)$, engendrée par les éléments q^h pour $h \in H$, e_i, f_i, $i \in I$, vérifiant les relations

1. $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ pour tous $h, h' \in H$,
2. $q^h e_i = q^{(h,h_i)} e_i q^h$,
3. $q^h f_i = q^{-(h,h_i)} f_i q^h$,
4. $e_i f_j - f_j e_i = 0$ si $i \neq j$,
5. $e_i f_i - f_i e_i = \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$,
6. Si $a_{i,i} > 0$, $\sum_{k+l=1-a_{i,j}} (-1)^k e_i^{(k)} e_j e_i^{(l)} = 0$, $\sum_{k+l=1-a_{i,j}} (-1)^k f_i^{(k)} f_j f_i^{(l)} = 0$ où pour $k \in \mathbf{N}$, $e_i^{(k)} = \frac{e_i^k}{[k]!}$, $f_i^{(k)} = \frac{f_i^k}{[k]!}$, $[k]! = \prod_{m=1}^k [m]$,$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$.

À nouveau, l'algèbre $\mathbf{U}_q(\mathfrak{g})$ admet une décomposition triangulaire

$$\mathbf{U}_q(\mathfrak{g}) = \mathbf{U}_q(\mathfrak{n}_-) \otimes \mathbf{U}_q(H) \otimes \mathbf{U}_q(\mathfrak{n}_+).$$

Comme $\mathbf{U}(\mathfrak{g})$, $\mathbf{U}_q(\mathfrak{g})$ a une structure d'algèbre de Hopf. On doit pour cela préciser la comultiplication Δ, l'antipode S et la counité ε. Sur les générateurs, on a

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes q^{-h_i} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + q^{h_i} \otimes f_i,$$

$$S(q^h) = q^{-h}, \quad S(e_i) = -e_i q^{h_i}, \quad S(f_i) = -q^{h_i} f_i,$$

$$\varepsilon(q^h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0.$$

Il y a une bialgèbre munie d'un accouplement de Hopf définie par Lusztig ([Lus11]) à partir d'une matrice de Cartan de type Kac-Moody, que l'on notera $\boldsymbol{f}$, qui est plus adaptée que $\mathbf{U}_q(\mathfrak{n}_+)$ dans les constructions géométriques. Les algèbres de Hopf $\mathbf{U}_q(\mathfrak{n}_\pm)$ et $\boldsymbol{f}$ sont isomorphes seulement en tant qu'algèbres. Leurs structures de coalgèbres diffèrent mais sont néanmoins très proches. Nous rappelons la définition par générateurs et relations de l'algèbre $\boldsymbol{f}$ lorsque la matrice de Cartan est symétrique. Soit $(H, (-,-), (h_i)_{1 \leq i \leq n})$ une donnée de Cartan de type Kac-Moody. En particulier, $(-,-)$ est une forme bilinéaire symétrique sur H, $(h_i, h_i) = 2$. L'algèbre $\boldsymbol{f}$ est une algèbre associative sur le corps $\mathbf{Q}(q)$ engendrée par les éléments θ_i, $1 \leq i \leq n$, avec les relations de Serre quantiques

$$\sum_{k+l=1-a_{i,j}} (-1)^k \theta_i^{(k)} \theta_j \theta_i^{(l)}$$

pour $i \neq j$ et pour $1 \leq i \leq n$, $k \geq 0$, $\theta_i^{(k)} = \frac{\theta_i^k}{[k]!}$. L'algèbre $\boldsymbol{f}$ est $\mathbf{Z}^I$ graduée si l'on pose $\deg \theta_i = e_i$, le i-ème vecteur de la base canonique de $\mathbf{Z}^I$. De plus, $\mathbf{Z}^I$ est muni d'une forme bilinéaire $(-,-)$

vérifiant $(e_i, e_j) = (h_i, h_j)$. La structure de coalgèbre est donnée par la comultiplication Δ qui s'exprime sur les générateurs par la formule

$$\Delta(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i.$$

Pour avoir la compatibilité entre la multiplication et la comultiplication, on doit considérer sur le produit tensoriel $\boldsymbol{f} \otimes \boldsymbol{f}$ le produit

$$(x \otimes y)(z \otimes w) = q^{(\deg y, \deg z)}(xz) \otimes (yw).$$

Il y a une seconde comultiplication sur l'algèbre $\boldsymbol{f}$ obtenue en conjuguant Δ par l'automorphisme de $\mathbf{Q}$-algèbres $\boldsymbol{f} \to \boldsymbol{f}$ agissant sur les coefficients par $f(q) \mapsto f(q^{-1})$. On note $\bar{\Delta}$ cette deuxième comultiplication.

On définit maintenant deux formes bilinéaires sur $\boldsymbol{f}$, $(-,-)$ et $\{-,-\}$. Elles sont caractérisées par le fait d'être des accouplements de Hopf pour Δ (resp. $\bar{\Delta}$) vérifiant les égalités $(\theta_i, \theta_j) = \delta_{i,j}\frac{1}{1-q_i^{-2}}$ (resp. $\{\theta_i, \theta_j\} = \delta_{i,j}\frac{1}{1-q_i^2}$) pour tous i, j.

Il est possible de définir des *formes entières* de $\mathbf{U}_q(\mathfrak{g})$, c'est-à-dire des algèbres sur un sous-anneau A de $\mathbf{C}(q)$ telles que la $\mathbf{C}(q)$-algèbre obtenue par extension des scalaires soit isomorphe à $\mathbf{U}_q(\mathfrak{g})$. Très souvent, on prend $A = \mathbf{C}[q, q^{-1}]$. Il y a deux formes entières qui se sont révélées intéressantes dans l'étude des groupes quantiques : la forme entière de Lusztig, $\mathbf{U}_q^L(\mathfrak{g})$, et la forme entière de De Concini-Kac, $U_q^{DK}(\mathfrak{g})$. Un de leurs intérêts est de pouvoir spécialiser le paramètre formel q à un nombre complexe.

La forme entière de Lusztig, $\mathbf{U}_q^L(\mathfrak{g})$ est la $\mathbf{Z}[q, q^{-1}]$-sous-algèbre de $\mathbf{U}_q(\mathfrak{g})$ engendrée par les puissances divisées des générateurs de Chevalley alors que la forme entière de De Concini-Kac est engendrée par les puissances des générateurs de Chevalley. Seule la forme entière de Lusztig est intéressante dans notre situation. Pour plus de détails sur sa construction, on se réfèrera à [Lus11].

1.3.3 Affinisation du groupe quantique

Une référence intéressante pour cette partie est [CP94]. Étant donnée une algèbre de Lie $\mathfrak{g}$, on appelle affinisation du groupe quantique $\mathbf{U}_q(\mathfrak{g})$ une algèbre de Hopf qui est une déformation à un paramètre de l'algèbre enveloppante de l'algèbre des lacets à valeurs dans $\mathfrak{g}$, $\mathbf{U}(\mathfrak{g}[t, t^{-1}])$. Lorsque $\mathfrak{g}$ est une algèbre de Lie semisimple, Drinfeld a défini une telle déformation. Cette déformation a deux présentations. Il y a d'abord la présentation de type Kac-Moody obtenue en considérant le carquois affine associé à l'algèbre de Lie semisimple ([Dri87b]), puis il y a la nouvelle présentation de Drinfeld ([Dri87a]). Cette dernière peut être formulée pour n'importe quelle algèbre de Kac-Moody (et possiblement même pour n'importe quelle algèbre de Borcherds). Nakajima ([Nak01]) ou Grojnowski pour la partie positive ([Gro]) ont utilisé les algèbres quantiques affines associées à une algèbre de Kac-Moody arbitraire.

Pour préserver la lisibilité de cette introduction, on donne seulement la présentation de la partie positive $\mathbf{U}_q(\hat{\mathfrak{n}}_+)$ de l'algèbre $\mathbf{U}_q(\hat{\mathfrak{g}})$. Celle-ci est engendrée par les éléments $e_{k,r}, k \in I, r \in \mathbf{Z}$ et les relations s'expriment de façon compacte grâce à la série

$$x_k^+(z) = \sum_{r=-\infty}^{+\infty} e_{k,r} z^{-r}$$

et prennent alors la forme

1. $(z - q^2 w) x_k^+(z) x_k^+(w) = (q^2 z - w) x_k^+(w) x_k^+(w)$,
2. $\prod_{p=1}^{-a_{k,l}} (z - q^{-a_{k,l}-2p} w) x_k^+(z) x_l^+(w) = \prod_{p=1}^{-a_{k,l}} (q^{-a_{k,l}-2p} z - w) x_l^+(w) x_k^+(z)$ si $k \neq l$,
3. $\sum_{\sigma \in \mathfrak{S}_b} \sum_{p=0}^{b} \binom{b}{p} x_k^+(z_{\sigma(1)}) \dots x_k^+(z_{\sigma(p)}) x_l^+(w) x_k^+(z_{\sigma(p+1)}) \dots x_k^+(z_{\sigma(b)}) = 0$ si $k \neq l$,

où $b = 1 - a_{k,l}$

Conjecture 1.3.1. *Soit Q un carquois qu'on suppose sans boucles*[6]. *Alors il existe une unique algèbre de Lie $\mathfrak{n}_B$ telle que $\mathbf{U}_q(\hat{\mathfrak{n}}_+)$ soit une déformation de $\mathbf{U}(\mathfrak{n}_B[t, t^{-1}])$. En outre, $\mathfrak{n}_B$ est la partie positive d'une algèbre de Borcherds $\mathfrak{g}_B$.*

On peut formuler la même conjecture pour toute l'algèbre quantique affine $\mathbf{U}_q(\hat{\mathfrak{g}})$. Cette conjecture vient du fait que pour une algèbre de Lie semisimple $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, l'algèbre quantique affine telle que nous l'avons définie est bien une déformation de $\mathbf{U}(\mathfrak{n}_+[t, t^{-1}])$, c'est-à-dire $\mathfrak{n}_B = \mathfrak{n}_+$, mais ce n'est plus vrai pour una algèbre de Kac-Moody générale.

1.3.4 Le Yangien d'une algèbre de Lie

Drinfeld a défini par générateurs et relations le *Yangien* d'une algèbre de Lie semisimple $\mathfrak{g}$, $Y(\mathfrak{g})$. C'est une déformation à un paramètre de l'algèbre enveloppante de l'algèbre de Lie des polynômes à coefficients dans $\mathfrak{g}$, $\mathbf{U}(\mathfrak{g}[z])$. La deuxième présentation du Yangien ([Dri87a]) donne des formules qui s'adaptent à n'importe quelle matrice de Cartan de type Kac-Moody. Comme pour les algèbres quantiques affines, on doit pouvoir aussi adapter cette présentation pour les algèbres de Borcherds. Soit $A = (a_{i,j})_{1 \leq i,j \leq n}$ une matrice de Cartan symétrique. On appellera[7] *Yangien* de l'algèbre de Kac-Moody $\mathfrak{g}_Q$ l'algèbre associative $Y_\hbar(\mathfrak{g}_Q)$ ayant pour générateurs $X_{i,r}^\pm$, $H_{i,r}$, $1 \leq i \leq n$, $r \in \mathbf{N}$, satisfaisant les relations

1. $[H_{i,r}, H_{j,s}] = 0$,
2. $[H_{i,0}, X_{j,s}^\pm] = \pm a_{i,j} X_{j,s}^\pm$,
3. $[H_{i,r+1}, X_{j,s}^\pm] - [H_{i,r}, X_{j,s+1}^\pm] = \pm \frac{\hbar}{2} a_{i,j} (H_{i,r} X_{j,s}^\pm + X_{j,s}^\pm H_{i,r})$,
4. $[X_{i,r}^+, X_{j,s}^-] = \delta_{i,j} H_{i,r+s}$,
5. $[X_{i,r+1}^\pm, X_{j,s}^\pm] - [X_{i,r}^\pm, X_{j,s+1}^\pm] = \pm \frac{\hbar}{2} a_{i,j} (X_{i,r}^\pm X_{j,s}^\pm + X_{j,s}^\pm X_{i,r}^\pm)$,

[6] pour simplifier

[7] L'algèbre que nous allons définir n'est pas une déformation de $\mathbf{U}(\mathfrak{g}[z])$ en général. Si $\mathfrak{g}$ est une algèbre de Lie semisimple, c'est bien le cas.

6. $\sum_{\pi \in \mathfrak{S}_m} [X^{\pm}_{i,r_{\pi(1)}}, [X^{\pm}_{i,r_{\pi(2)}}, \ldots, [X^{\pm}_{i,r_{\pi(m)}}, X^{\pm}_{j,s}] \ldots]] = 0$ pour $m = 1 - a_{i,j}$ et $r_1, \ldots, r_m \geq 0$.

Le Yangien $Y_\hbar(\mathfrak{g}_Q)$ contient une copie de l'algèbre enveloppante $\mathbf{U}(\mathfrak{g}_Q)$, la sous-algèbre engendrée par $H_{i,0}$ et $X^{\pm}_{i,0}$ pour $1 \leq i \leq n$ et il admet une décomposition triangulaire

$$Y_\hbar(\mathfrak{g}_Q) = Y_\hbar(\mathfrak{n}_{Q,-}) \otimes Y_\hbar(\mathfrak{h}) \otimes Y_\hbar(\mathfrak{n}_{Q,+}).$$

Lorsque Q est un carquois de type fini, le Yangien $Y_\hbar(\mathfrak{g}_Q)$ tel que nous l'avons défini est une déformation de $\mathbf{U}(\mathfrak{g}_Q[u])$. Dans les autres cas, le Yangien est beaucoup plus gros qu'une simple déformation de $\mathbf{U}(\mathfrak{g}_Q[u])$. Dans cette perspective, on peut formuler la conjecture suivante.

Conjecture 1.3.2. *Soit Q un carquois qu'on suppose sans boucles*[8]*. Alors il existe une unique algèbre de Lie $\mathfrak{g}_B$ de partie positive $\mathfrak{n}_B$ telle que $Y_\hbar(\mathfrak{n}_{Q,+})$ soit une déformation de $\mathbf{U}(\mathfrak{n}_B)$. En outre, $\mathfrak{g}_B$ est une algèbre de Borcherds.*

De plus, les algèbres de Borcherds $\mathfrak{g}_B$ des conjectures 1.3.1 et 1.3.2 doivent coïncider et on peut naturellement étendre la Conjecture 1.3.2 à toute l'algèbre de Lie.

1.4 L'algèbre de Hall d'un carquois

Dans cette section, on se concentre sur l'algèbre de Hall d'un carquois. Soit $Q = (I, \Omega)$ un carquois, c'est-à-dire un graphe orienté ayant un nombre fini de sommets et d'arêtes, et possiblement des boucles. On a noté I l'ensemble des sommets de Q et Ω l'ensemble de ses arêtes. On fixe un corps fini $\mathbf{F}_q$. La catégorie finitaire héréditaire que nous considérons est $\mathrm{Rep}_Q(\mathbf{F}_q)$, la catégorie des représentations de dimension finie de Q sur le corps $\mathbf{F}_q$. Nous notons $\mathbf{H}_{Q,\mathbf{F}_q}$ l'algèbre de Hall de $\mathrm{Rep}_Q(\mathbf{F}_q)$. Rappelons que c'est une algèbre de Hopf tordue. C'est Ringel, dans l'article fondateur [Rin90b] qui a initié leur étude. Le premier résultat majeur, du à Ringel et à Green, est le suivant (cf. [Sch12a, Theorem 3.16]).

Théorème 1.4.1 (Ringel, Green)**.** *Il existe un unique morphisme de bialgèbres*

$$\begin{array}{ccc} \boldsymbol{f}_{\sqrt{q}} & \to & \mathbf{H}_{Q,\mathbf{F}_q} \\ \theta & \mapsto & [S_i] \end{array}$$

Ce morphisme est injectif. Il est surjectif si et seulement si Q est de type fini.

L'image de ce morphisme est notée $\mathbf{H}^{\mathrm{sph}}_{Q,\mathbf{F}_q}$ et appelée *algèbre de Hall sphérique* du carquois Q. La notation $\boldsymbol{f}_{\sqrt{q}}$ indique la spécialisation en $\sqrt{q}$ de l'algèbre $\boldsymbol{f}$. Ce morphisme est bijectif si et seulement si le carquois est de type fini. On peut alors naturellement se demander quelle est la structure de $\mathbf{H}_{Q,\mathbf{F}_q}$ lorque le carquois Q est arbitraire. Sevenhant et Van den Bergh on répondu à cette question en démontrant qu'elle est isomorphe à la partie positive de la spécialisation en $\sqrt{q}$ de l'algèbre enveloppante quantifiée d'une algèbre de Borcherds pour laquelle le rôle des générateurs de Chevalley est joué par les fonctions cuspidales. Nous reviendrons sur ce résultat dans la Section 1.4.3.

[8] pour simplifier

1.4.1 Caractère des algèbres de Lie et de leur algèbre enveloppante

Exponentielle pléthystique

L'exponentielle pléthystique est la décatégorification dimensionnelle de la puissance symétrique. Ce que nous entendons par là deviendra plus clair dans la Section 1.4.1. Nous adoptons ici un point de vue plus terre-à-terre.

Soit M un monoïde dont on note 0 l'élément neutre. On suppose pour simplifier que pour chaque $m \in M$, l'ensemble $\{(m_1, m_2) \in M \times M \mid m_1 + m_2 = m\}$ est fini. On note $\mathbf{Q}[[M]]$ l'ensemble des séries formelles à coefficients dans $\mathbf{Q}$ de la forme

$$f = \sum_{m \in M} f_m z^m,$$

$f_m \in \mathbf{Q}$. L'ensemble $\mathrm{supp}(f) = \{m \in M \mid f_m \neq 0\}$ est appelé le support de f. L'ensemble $\mathbf{Q}[[M]]$ est naturellement un anneau. On note $\mathbf{Q}[[M]]_+$ *l'idéal d'augmentation* de $\mathbf{Q}[[M]]$, c'est-à-dire l'ensemble des séries formelles dont le terme constant (le coefficient de $z^0 = 1$) est nul. L'*exponentielle pléthystique* est alors l'application

$$\begin{array}{rccc} \mathrm{Exp}_z & : & \mathbf{Q}[[M]]_+ & \to & 1 + \mathbf{Q}[[M]]_+ \\ & & f & \mapsto & \exp\left(\sum_{l \geq 1} \frac{1}{l} f(z^l)\right) \end{array}.$$

On aura parfois besoin de l'exponentielle pléthystique pour des éléments de $\mathbf{Q}[q][[M]]$. Cela rentre dans notre cadre si l'on remplace M par $M \times \mathbf{N}$, étant donnée l'inclusion $\mathbf{Q}[q][[M]] \to \mathbf{Q}[[M \times \mathbf{N}]]$. Dans ce cas, l'exponentielle pléthystique associée sera notée $\mathrm{Exp}_{z,q}$ pour faire la distinction. L'exponentielle pléthystique est un isomorphisme de groupes ($\mathbf{Q}[[M]]_+$ est naturellement un groupe additif ; $1+\mathbf{Q}[[M]]_+$ est un groupe multiplicatif). Son inverse est donné par le *logarithme pléthystique* Log_z (ou bien $\mathrm{Log}_{z,q}$ dans le cas de $\mathbf{Q}[q][[M]]$), défini par la formule

$$\begin{array}{rcccc} \mathrm{Log}_z & : & 1 + \mathbf{Q}[[M]]_+ & \to & \mathbf{Q}[[M]]_+ \\ & & f(z) & \mapsto & \sum_{l \geq 1} \frac{\mu(l)}{l} \log(f)(z^l) \end{array}$$

où μ est la fonction de Moebius.

Caractère de l'algèbre enveloppante d'une algèbre de Lie

Soit M un monoïde et V un espace vectoriel M-gradué. On peut décomposer M en ses composantes graduées : $V = \bigoplus_{m \in N} V_m$. On suppose que chaque composante graduée V_m pour $m \in M$ est de dimension finie. On a alors le caractère de V:

$$\mathrm{ch}(V) = \sum_{m \in N} (\dim V_m) z^m \in \mathbf{Z}[[M]]$$

est dans l'algèbre de groupe de M et où l'élément $m \in M$ induit l'élément $z^m \in \mathbf{Z}[[M]]$. On note $\mathrm{supp}(V) = \{m \in M \mid V_m \neq 0\}$ le support de V. On suppose que $0 \notin \mathrm{supp}(V)$. Alors la puissance symétrique de V a pour caractère

$$S(V) = \sum_{m \in M} (\dim S(V)_m) z^m = \mathrm{Exp}_z\left(\mathrm{ch}(V)\right). \tag{1.4.1}$$

La catégorie des espaces vectoriels M-gradués dont chaque composante homogène est de dimension finie, $M - vect$, a pour groupe de Grothendieck $\mathbf{Z}[[M]]$. Il existe un foncteur de la sous-catégorie $M - vect_+$ des espaces vectoriels M-gradués dont la composante de degré 0 est nulle vers la sous-catégorie $M - vect_1$ des espaces vectoriels gradués dont la composantes de degré 0 est de dimension 1, donné par la puissance symétrique. Sur les sous-ensembles correspondants du groupe de Grothendieck, l'application induite est précisément l'exponentielle pléthystique.

Soit maintenant $\mathfrak{g}$ une algèbre de Lie. On suppose que $\mathfrak{g}$ est graduée par un monoïde M satisfaisant à la condition donnée au début de cette section. Le théorème de Poincaré-Birkhoff-Witt assure que le caractère M-gradué de $\mathbf{U}(\mathfrak{g})$ est le même que celui de la puissance symétrique $S\mathfrak{g}$. Il est donc donné par la formule (1.4.1) appliquée à $V = \mathfrak{g}$.

1.4.2 Caractère de l'algèbre de Hall d'un carquois

Il suit immédiatement de sa définition que le caractère de l'algèbre de Hall du carquois Q sur le corps fini $\mathbf{F}_q$ est donné par la famille de fonctions comptant en chaque dimension le nombre de classes d'isomorphisme de représentations de Q de cette dimension. Plus précisément, pour $\mathbf{d} \in \mathbf{N}^I$, on note $M_{Q,\mathbf{d}}(q)$ le nombre de classes d'isomorphisme de représentations de Q sur $\mathbf{F}_q$ de dimension $\mathbf{d}$, $I_{Q,\mathbf{d}}(q)$ le nombre de classes d'isomorphisme de représentations *indécomposables* de Q sur $\mathbf{F}_q$ de dimension $\mathbf{d}$ et $A_{Q,\mathbf{d}}(q)$ le nombre de classes d'isomorphisme de représentations *absolument indécomposables* de Q sur $\mathbf{F}_q$. Alors Kac [Kac80a, Kac83, Kac82, Kac80b] a démontré que les fonctions $M_{Q,\mathbf{d}}$, $I_{Q,\mathbf{d}}$ et $A_{Q,\mathbf{d}}$ de q sont des polynômes en q, Hausel, Letellier et Rodriguez-Villegas ([HLRV13]) ont démontré que le polynôme $A_{Q,\mathbf{d}}$ est à coefficients entiers positifs et Hausel ([Hau10]) a démontré la conjecture de Kac interprétant le coefficient constant du polynôme $A_{Q,\mathbf{d}}$ comme la multiplicité de $\mathbf{d}$ en tant que racine de l'algèbre de Kac-Moody associé au carquois Q : $A_{Q,\mathbf{d}}(0) = \dim \mathfrak{g}[\mathbf{d}]$. Il ressort de tout celà que la famille de polynômes $A_{Q,\mathbf{d}}$ est celle qui possède les meilleures propriétés. En outre, si on note

$$\mathrm{ch}\, \mathbf{H}_{Q,\mathbf{F}_q} = \sum_{\mathbf{d} \in \mathbf{N}^I} \dim_{\mathbf{C}} \mathbf{H}_{Q,\mathbf{F}_q}[\mathbf{d}] z^{\mathbf{d}} \in \mathbf{C}[[z_i, i \in I]],$$

on a les égalités

$$\mathrm{ch}\, \mathbf{H}_{Q,\mathbf{F}_q} = \sum_{\mathbf{d} \in \mathbf{N}^I} M_{Q,\mathbf{d}}(q) z^{\mathbf{d}} = \mathrm{Exp}_z\left(\sum_{\mathbf{d} \in \mathbf{N}^I} I_{Q,\mathbf{d}}(q) z^{\mathbf{d}}\right) = \mathrm{Exp}_{z,q}\left(\sum_{\mathbf{d} \in \mathbf{N}^I} A_{Q,\mathbf{d}}(q) z^{\mathbf{d}}\right).$$

La première égalité a lieu par définition, la deuxième par la propriété de Krull-Schmidt de la catégorie des représentations d'un carquois et la troisième par un argument de descente galoisienne pour les représentations de carquois (cf. [BS18, Lemma 3.1]).

L'exponentielle pléthystique faisant intervenir les polynômes $I_{Q,\mathbf{d}}$ indique que le caractère de $\mathbf{H}_{Q,\mathbf{F}_q}$ est exactement le caractère de l'algèbre enveloppante d'une algèbre de Lie $\mathfrak{n}$ dont le caractère serait donné par $\operatorname{ch}\mathfrak{n} = \sum_{\mathbf{d}\in\mathbf{N}^I} I_{Q,\mathbf{d}}(q)z^{\mathbf{d}}$. L'exponentielle pléthystique faisant intervenir les polynômes de Kac $A_{Q,\mathbf{d}}$ interprète $\operatorname{ch}\mathbf{H}_{Q,\mathbf{F}_q}$ comme la spécialisation en q du caractère de l'algèbre enveloppante d'une algèbre de Lie $\mathbf{N}$-graduée $\mathfrak{n}_{\mathbf{N}}$ dont le caractère $\mathbf{N}$-gradué serait $\operatorname{ch}\mathfrak{n}_{\mathbf{N}} = \sum_{\mathbf{d}\in\mathbf{N}^I} A_{Q,\mathbf{d}}(q)z^{\mathbf{d}} \in \mathbf{Q}[q][[z_i : i \in I]]$, où q est compris comme une variable formelle.

1.4.3 Générateurs et relations pour l'algèbre de Hall d'un carquois

L'algèbre de Hall d'un carquois est une algèbre de Hopf tordue $\mathbf{N}^I$-graduée, $\mathbf{H}_{Q,\mathbf{F}_q}$, dont on note Δ la comultiplication. On définit l'ensemble des fonctions cuspidales de $\mathbf{H}_{Q,\mathbf{F}_q}$:

$$\mathbf{H}_{Q,\mathbf{F}_q}^{\mathrm{cusp}} = \{f \in \mathbf{H}_{Q,\mathbf{F}_q} \mid \Delta(f) = f \otimes 1 + 1 \otimes f\}.$$

C'est un sous-espace vectoriel $\mathbf{N}^I$-gradué de $\mathbf{H}_{Q,\mathbf{F}_q}$, dont les composantes graduées ont une dimension polynômiale en q ([BS18, Theorem 1.1]). C'est un sous-espace générateur minimal de $\mathbf{H}_{Q,\mathbf{F}_q}$ ([SVDB01]) et si on choisit une base orthonormale graduée quelconque $(f_j)_{j\in J}$ de cet espace, on obtient une présentation de $\mathbf{H}_{Q,\mathbf{F}_q}$ par générateurs et relations. On note d'abord $\nu = \sqrt{q}$; $A = (a_{i,j})_{i,j\in J}$ où $a_{i,j} = (\dim f_i, \dim f_j)$ la matrice de Cartan, où $(-,-)$ est la forme d'Euler symétrisée. Alors l'algèbre de Hall $\mathbf{H}_{Q,\mathbf{F}_q}$ est isomorphe à la $\mathbf{C}$-algèbre associative engendrée par $(f_j)_{j\in J}$ avec les relations suivantes.

1. Pour tous $i, j \in J$, si $a_{i,j} = 0$, alors $f_i f_j = f_j f_i$,

2. Pour tous $i, j \in J$, si $a_{i,i} = 2$, alors

$$\sum_{i=0}^{1-a_{i,j}} (-1)^l \begin{Bmatrix} 1-a_{i,j} \\ l \end{Bmatrix} f_i^l f_j f_i^{1-a_{i,j}+l} = 0,$$

où, pour tous entiers r et s, $\begin{Bmatrix} s \\ r \end{Bmatrix} = \prod_{u=1}^{r} \frac{\nu^{u+s-r} - \nu^{-(u+s-r)}}{\nu^u - \nu^{-u}}$.

Cette présentation de l'algèbre de Hall $\mathbf{H}_{Q,\mathbf{F}_q}$ indique que $\mathbf{H}_{Q,\mathbf{F}_q}$ est la spécialisation en $\nu = \sqrt{q}$ de l'algèbre enveloppante quantifiée d'une algèbre de Borcherds associée à la donnée de Cartan $(\mathbf{Z}^J, (-,-))$.

On comprend mal les fonctions cuspidales d'un carquois arbitraire. Les fonctions cuspidales des carquois de type fini sont facilement identifiables : ce sont les fonctions inducatrices des représentations simples. C'est une conséquence du fait qu'elles forment un sous-ensemble générateur minimal de l'algèbre de Hall et du théorème 1.4.1. Pour le carquois de Jordan, les fonction cuspidales peuvent facilement être décrites grâce à l'isomorphisme explicite avec l'algèbre des fonctions symétriques. Pour les carquois affines, la théorie d'Auslander-Reiten permet une description des fonctions cuspidales en fonction des fonctions cuspidales du carquois de Jordan et des carquois cycliques. C'est l'objet du Chapitre 3 (see also Section 1.7.1, Théorème 1.7.1). En dehors de ces cas là, on n'a pas de classification générale des fonctions cuspidales, seulement des résultats épars.

1.5 Des modèles géométriques pour le Yangien et l'algèbre quantique affine associés à un carquois

Afin de mieux comprendre les groupes quantiques (algèbres enveloppantes quantiques, algèbres quantiques affines, Yangiens), il s'est révélé puissant d'en obtenir des constructions géométriques. Ces constructions géométriques fournissent naturellement des actions sur d'autres objets de nature géométrique, permettant aussi d'étudier leurs représentations. Le formalisme sous-jacent associé est celui des correspondances, développé par Chriss et Ginzburg dans [CG97]. Les travaux allant dans cette direction sont nombreux : [Var00, KS11, SV13a, SV20, SV17, Soi16, YZ18, YZ20, MO19].

1.5.1 L'algèbre de Hall cohomologique associée à un carquois

Les constructions de cette section sont dues à Schiffmann-Vasserot, Kontsevich-Soibelman, Davison, Yang-Zhao... À un carquois $Q = (I, \Omega)$, on associe le carquois dédoublé $\overline{Q}$: chaque arête a de Q est remplacée par une paire d'arêtes (a, a^*) de sens opposés. L'algèbre préprojective de Q est le quotient de l'algèbre des chemins de $\overline{Q}$ par l'idéal bilatère engendré par l'élément $\sum_{a\in\Omega}[a, a^*]$. En chaque dimension $\mathbf{d} \in \mathbf{N}^I$, cette relation définit un sous-schéma fermé $G_{\mathbf{d}}$-invariant de l'espace de représentations :

$$E_{\Pi_Q,\mathbf{d}} := \left\{(x_a, x_{a^*}) \in E_{\overline{Q},\mathbf{d}} \mid \sum_{a\in\Omega}[x_a, x_{a^*}] = 0\right\}.$$

Le quotient par l'action de $G_{\mathbf{d}}$ donne le champ de représentations de l'algèbre préprojective, $\mathcal{M}_{\Pi_Q} := E_{\Pi_Q,\mathbf{d}}/G_{\mathbf{d}}$. L'algèbre de Hall cohomologique associée est une structure de $H_T^*(\mathrm{pt})$-algèbre sur la somme directe

$$\mathrm{CoHA}(Q) := \bigoplus_{\mathbf{d}\in\mathbf{N}^I} H_*^T(\mathcal{M}_{\Pi_Q,\mathbf{d}})$$

des groupes d'homologie de Borel-Moore T-équivariants (pour l'action naturelle d'un tore T, qui encode les paramètres de déformation), [SV20]. Il existe une version de l'algèbre de Hall cohomologique où on considère plutôt le champ des représentations *nilpotentes* (ou semi-nilpotentes) de l'algèbre préprojective. Après localisation (tensorisation par $\mathrm{Frac}(H_T^*(\mathrm{pt}))$, ces algèbres deviennent isomorphes. Cette algèbre admet une construction alternative en termes de carquois avec potentiel (due à Kontsevich-Soibelman, [KS11]). Le lien entre les deux constructions se fait par la *réduction dimensionnelle*, qui est un moyen de comparer l'algèbre de Hall cohomologique d'une catégorie de Calabi-Yau de dimension 3 à l'algèbre de Hall cohomologique d'une catégorie de Calabi-Yau de dimension 2 sous certaines conditions ([RS17, YZ20]). Lorsque $\mathrm{CoHA}(Q)$ est réalisée comme l'algèbre de Hall cohomologique d'une catégorie de Calabi-Yau de dimension 3, on dispose de la *filtration perverse* ([DM20]) qui a permis à Ben Davison et Sven Meinhardt d'extraire l'algèbre de Lie BPS $\mathfrak{g}_{BPS}$ du Yangien. La réalisation de $\mathrm{CoHA}(Q)$ comme algèbre de Hall cohomologique d'une catégorie de Calabi-Yau de dimension 2 donne une autre filtration perverse, qui permet de retrouver l'algèbre enveloppante $\mathbf{U}(\mathfrak{g}_{BPS})$ de l'algèbre de Lie BPS dans l'algèbre de Hall cohomologique ([Dav20]).

1.5.2 L'algèbre de Hall K-théorique associée à un carquois

L'idée de considérer l'algèbre de Hall K-théorique de l'algèbre préprojective d'un carquois est due à Grojnowski [Gro]. Dans cette note non publiée, il définit une structure d'algèbre sur la somme directe des groupes de K-théorie

$$\mathrm{KHA}(Q) = \bigoplus_{\mathbf{d}\in\mathbf{N}^I} K^{\mathbf{C}^*}(\mathcal{M}_{\Pi_Q,\mathbf{d}})$$

où $\mathbf{C}^*$ agit avec poids 1 sur sur les fibres cotangentes, et il montre qu'il y a une structure d'algèbre induite sur la K-théorie de la variété nilpotente

$$\mathrm{KHA}_\Lambda(Q) := \bigoplus_{\mathbf{d}\in\mathbf{N}^I} K^{G_{\mathbf{d}}\times\mathbf{C}^*}(\Lambda_{\mathbf{d}}).$$

Il esquisse des arguments vagues pour montrer que la partie positive de l'algèbre quantique affine associée à Q (définie grâce à la nouvelle présentation de Drinfeld, Section 1.3.3) se surjecte sur $\mathrm{KHA}_\Lambda(Q)$ et pour montrer l'injectivité grâce à une sorte de coproduit et à une forme bilinéaire symétrique non dégénérée. On s'attend à l'existence d'une algèbre de Borcherds $\mathfrak{g}_B$ de partie positive $\mathfrak{n}_B$ telle que $\mathrm{KHA}_\Lambda(Q)$ soit une déformation de l'algèbre enveloppante des lacets à valeurs dans $\mathfrak{n}_B$, $\mathbf{U}(\mathfrak{n}_B[t^{\pm1}])$. L'algèbre de Hall K-théorique du carquois à g boucles intervient dans le papier de Schiffmann et Vasserot [SV10]. Elle permet une compréhension de la correspondance de Langlands géométrique pour une courbe de genre g dans le voisinage du système local trivial. Dans l'article [VV20], Varagnolo et Vasserot démontrent de façon précise l'isomorphisme entre la partie positive de Drinfeld du groupe quantique affine associé à l'algèbre de Kac-Moody d'un carquois Q de type fini ou affine. En général, ils démontrent la surjectivité du morphisme entre le groupe quantique affine et l'algèbre de Hall K-théorique. Le cas de $\mathrm{SL}(2)$ a été considéré en détails dans [SVV19] où en outre le lien avec l'algèbre de Hecke-carquois de type $A_1^{(1)}$ est fait. Récemment, Padurariu a développé la théorie des algèbres de Hall K-théoriques pour les carquois avec potentiel dans [Pad19]. Dans [VV20], Varagnolo et Vasserot s'appuient sur l'approche de Padurariu pour construire des super groupes quantiques en termes d'algèbre de Hall K-théorique. Mentionnons aussi le travail de Okounkov et Smirnov [OS16] qui propose une construction de l'algèbre quantique affine associée à un carquois en termes d'enveloppes stables K-théoriques.

1.5.3 Les enveloppes stables

Okounkov, Maulik, Smirnov et Aganagic ont développé une approche des algèbres que l'on peut associer à un carquois en utilisant le formalisme des R-matrices construites à partir des *enveloppes stables*, [MO19, OS16, AO21]. Leur travail trouve sa source dans la réalisation des représentations des algèbres de Kac-Moody dans la cohomologie des variétés carquois due à Nakajima [Nak98]. De cette façon, ils obtiennent un Yangien $\mathbb{Y}$ et une algèbre de Lie $\mathfrak{g}_{MO}$ telle que $\mathbb{Y}$ est une déformation de $\mathbf{U}(\mathfrak{g}_{MO}[u])$.Schiffmann and Vasserot conjecture que le Yangien obtenu dans [MO19] est isomorphe au Yangien de [SV20].

1.5.4 Caractère del'algèbre de Lie BPS

L'algèbre de Lie BPS de la Section 1.5.1 est $\mathbf{N}^I \times \mathbf{N}$-graduée et son caractère est donné par la série génératrice des polynômes de Kac:

$$\operatorname{ch} \mathfrak{g}_{BPS} = \sum_{\mathbf{d} \in \mathbf{N}^I} A_{Q,\mathbf{d}}(q) z^{\mathbf{d}}. \tag{1.5.1}$$

Dans l'exposé [Oko], Okounkov conjecture que le caractère de la partie positive de l'algèbre de Lie $\mathfrak{g}_{MO}$ est aussi donné par la même série génératrice. De la même façon, on peut s'attendre à ce que le caractères des algèbres de Lie $\mathfrak{n}_B$ des Conjectures 1.3.1 et 1.3.2 soit donné par la même série génératrice. Ainsi, on dispose de nombreux points de vue différents sur la série (1.5.1) (Sections 1.4.2, 1.5.1, 1.3.3, 2.3.4) et de plusieurs moyens de construire des algèbres de Lie ayant (conjecturalement) le même caractère. Pour les carquois de type fini et affines, tous ces résultats sont connus. C'est un problème intéressant de comparer toutes ces constructions. Une question laissée à une réflexion future est de comprendre les implications des propriétés asymptotiques des polynômes de Kac (Theorem 1.7.2) dans ces constructions. Par exemple, on peut étudier la convergencede la R-matrice de Maulik-Okounkov lorsque le nombre de flèches du carquois augmente.

1.6 L'algèbre de Hall d'une courbe algébrique

1.6.1 L'algèbre de Hall des fonctions constructibles

On a vu au début de cette introduction que les catégories de faisceaux cohérents sur une courbe algébrique projective lisse définie sur un corps fini satisfont aux conditions requises pour définir leur algèbre de Hall. Leur étude dans cette nouvelle situation a été initiée par Kapranov [Kap97]. C'est un objet qui a fait l'objet de nombreux travaux : [BK01, Sch06, BS12, KSV17, Sch12b, SV12]. En particulier, Schiffmann a donné une description explicite de la sous-algèbre sphérique pour les droites projectives à poids ainsi que pour les courbes elliptiques. Dans ce dernier cas, on obtient ce qu'on appelle maintenant *l'algèbre de Hall elliptique*. C'est une algèbre d'un intérêt considérable en théorie des représentations, et sa sous-algèbre sphérique est une déformation de l'algèbre de Hopf des fonctions diagonalement symétriques

$$\mathbf{\Lambda}^+ = \mathbf{C}[x_1^{\pm 1}, \dots, y_1, \dots]^{\mathfrak{S}_\infty}.$$

1.6.2 L'algèbre de Hall elliptique

Soit X une courbe elliptique définie sur un corps fini $\mathbf{F}_q$. La catégorie héréditaire que nous considérons ici est $\operatorname{Coh}(X)$, la catégorie des faisceaux cohérents sur X. Pour la construction détaillée de l'algèbre de Hall $\mathbf{H}_X$ de $\operatorname{Coh}(X)$, on se réfèrera à [BS12, Section 2]. Nous donnons ici seulement quelques unes de ses caractéristiques principales. L'algèbre $\mathbf{H}_X$ est $\mathbf{Z}^+$-graduée, où

$$\mathbf{Z}^+ = \{(r,d) \in \mathbf{Z}^2 \mid r \geq 1 \text{ ou } r = 0, d \geq 0\}.$$

Si $(r,d) \in \mathbf{Z}^+$, $\mathrm{Coh}(X)_{(r,d)}/\sim$ désigne l'ensemble des classes d'isomorphisme de faisceaux cohérents sur X de rang r et degré d. Les éléments de $\mathbf{H}_X[(r,d)]$ sont les fonctions $\mathrm{Coh}(X)_{(r,d)}/\sim \to \mathbf{C}$. En considérant seulement les faisceaux cohérents semi-stables, on obtient un sous-ensemble $\mathrm{Coh}(X)^{ss}_{(r,d)}/\sim$. La fonction indicatrice de ce sous-ensemble est notée $1^{ss}_{(r,d)}$. La sous-algèbre sphérique de $\mathbf{H}_X$ est la sous-algèbre de $\mathbf{H}_X$ engendrée par $1^{ss}_{(r,d)}$ pour $(r,d) \in \mathbf{Z}^+$, $r \leq 1$. C'est une sous-bialgèbre topologique de $\mathbf{H}_X$ et (comme on l'a déjà remarqué dans la Section 1.2), pour définir le coproduit, il est nécessaire de considérer une complétion. Il est possible de définir une *algèbre de Hall elliptique générique* $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ sur l'anneau $\mathbf{R} = \mathbf{C}[\sigma^{\pm 1/2}, \bar{\sigma}^{\pm 1/2}]$, où $\sigma, \bar{\sigma}$ sont des paramètres formels jouant le rôle des valeurs propre de l'action du Frobenius sur $H^1(X, \overline{\mathbf{Q}}_\ell)$. La construction de l'algèbre de Hall elliptique en termes de faisceaux pervers sur le champ de représentations est donnée dans [Sch12b]. Cette construction est inspirée de la construction analogue de Lusztig pour les carquois [Lus91]. Voir aussi [SV11] ainsi que les références de cet article pour le lien avec des objets importants de la théorie des représentations, tels les polynômes de MacDonald et les algèbres de Cherednik.

1.7 Résultats

Dans cette section, nous présentons les résultats principaux de cette thèse qui seront détaillés dans les chapitres suivants. Cette thèse est constituée des quatre articles suivants.

1. Isotropic cuspidal functions in the Hall algebra of a quiver, à paraître dans Int. Math. Res. Not., [Hen19],

2. Asymptotic behaviour of Kac polynomials, accepté par Experimental Mathematics, [Hen20a],

3. Microlocal characterization of Lusztig sheaves for affine and negative quivers, [Hen20b],

4. Perverse sheaves with nilpotent singular support on the stack of coherent sheaves on an elliptic curve, [Hen21].

1.7.1 Les fonctions cuspidales des carquois affines

Nous décrivons brièvement le contenu de l'article [Hen19], c'est-à-dire du Chapitre 3 de cette thèse. Nous avons vu dans la Section 1.4.3 que les fonctions cuspidales d'un carquois forment un sous-espace générateur minimal. Dans ce premier travail, on détermine les fonctions cuspidales d'un carquois affine. Ce calcul se déroule en deux étapes. D'abord, on définit *l'algèbre de Hall régulière.* C'est l'algèbre de Hall de la catégorie $\mathrm{Rep}_Q^{\mathcal{R}}(\mathbf{F}_q)$ des représentations régulières du carquois Q (définie grâce à la théorie d'Auslander-Reiten). Cette catégorie a l'avantage d'être un produit indexé par les points fermés de la droite projective $\mathbf{P}^1_{\mathbf{F}_q}$ de catégories équivalentes soit à la catégorie des représentations nilpotentes du carquois de Jordan sur le corps résiduel du point choisi, soit pour un nombre fini de points de degré 1, à la catégorie des représentations d'un carquois cyclique sur $\mathbf{F}_q$. La structure des algèbres de Hall de ces catégories est connue : l'algèbre de Hall de la catégorie des représentations du carquois de Jordan est isomorphe à l'algèbre des fonctions symétriques de MacDonald et est souvent appelée *algèbre de Hall classique* ([Mac15]) et l'algèbre de Hall d'un carquois cyclique a une structure décrite dans [Sch00]. On en déduit la structure de l'algèbre de Hall $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ de la catégorie $\mathrm{Rep}_Q^{\mathcal{R}}(\mathbf{F}_q)$. En particulier, on obtient une description relativement explicite de l'espace des fonctions cuspidales de $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$, que l'on note $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}} = \bigoplus_{\mathbf{d}\in\mathbf{N}^I}\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[\mathbf{d}]$. Des propriétés de base des fonctions cuspidales impliquent que $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}]$ est non nul si et seulement si $\mathbf{d}\in\{e_i : i\in I\}\cup\mathbf{N}_{\geq 1}\delta$, où δ est la racine imaginaire indivisible du carquois Q. De plus, si $r\geq 1$ est un entier, on peut montrer que le support d'une fonction cuspidale $f\in\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]$ est régulier (au sens où f prend pour valeur 0 sur les représentations dont un des facteurs directs n'est pas régulier), de sorte que l'on a l'inclusion

$$\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]\subset\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[r\delta].$$

En outre, on dispose de formules explicites pour les dimensions de ces deux espaces. Ces formules indiquent que $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]$ est un hyperplan de $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[r\delta]$. On dispose également d'une forme linéaire naturelle sur $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[r\delta]$ qui est l'intégration des fonctions sur l'espace de modules :

$$\begin{array}{cccc}\int & : & \mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[r\delta] & \to & \mathbf{C} \\ & & f & \mapsto & \sum_{[M]\in\mathrm{Rep}_Q^{\mathcal{R}}(\mathbf{F}_q)}\frac{f([M])}{|\mathrm{Aut}(M)|}.\end{array}$$

On a alors le théorème principal du Chapitre 3, Théorème 3.1.1.

Théorème 1.7.1. *Le noyau de la forme linéaire $\int$ coïncide avec l'espace des fonctions cuspidales du carquois Q en dimension $r\delta$, $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]$.*

La sratégie de la démonstation est relativement simple. Avec le travail préparatoire exposé avant l'énoncé du théorème, il s'agit de montrer que $\int$ est une forme linéaire non triviale nulle sur $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]$. Cela est fait en montrant que $\int$ est représentée pour le produit scalaire de Green par la fonction constante $1_{r\delta}$ et en utilisant le fait que $1_{r\delta}$ est dans l'algèbre de composition du carquois Q (ce qui est vrai pour n'importe quel carquois sans cycles).

1.7.2 Le comportement asymptotique des polynômes de Kac

Nous décrivons dans cette section quelques aspects de l'article [Hen20a] (Chapitre 4 de cette thèse). Les polynômes de Kac jouent un rôle considérable dans l'étude des algèbres associées à un carquois, en particulier lorsqu'il s'agit de déterminer leur caractère (Section 1.4.2). Le désir de comprendre comment évolue l'algèbre de Hall d'un carquois lorqu'on modifie le carquois nous a amenés à considérer le problème suivant : Que peut-on dire de la suite de polynômes $(A_{Q_{\underline{n}},\mathbf{d}}(q))$ lorsque $Q=(I,\Omega)$ est un carquois, $\underline{n}\in\mathbf{N}^{\Omega}$ est un vecteur de multiplicités et $Q_{\underline{n}}$ désigne le carquois ayant le même ensemble de sommets que Q mais pour lequel chaque arête $\alpha\in\Omega$ est remplacée par n_α arêtes, reliant les mêmes sommets que α? Dans le Chapitre 4, nous apportons quelques réponses et formulons des conjectures supportées informatiquement. Le théorème principal de ce chapitre est le suivant (Theorem 4.3.2).

Théorème 1.7.2. *Soit $\underline{m}=(m_\alpha)_{\alpha\in\Omega}\in(\mathbf{N}_{\geq 1}\cup\{+\infty\})^{\Omega}$.*

1. *Soit Q un carquois sans boucles et $\mathbf{d} \in \mathbf{N}^I$. Alors, pour tout $\underline{n} \in \mathbf{N}^{\Omega}$ la valuation du polynôme $A_{Q_{\underline{n}},\mathbf{d}}$ est $v_{\underline{n}} = 0$ si $A_{Q_{\underline{n}},\mathbf{d}} \neq 0$. La suite de polynômes*
$$A_{Q_{\underline{n}},\mathbf{d}}(q) \in \mathbf{N}[q]$$
converge dans $\mathbf{N}[[q]]$ lorsque $\underline{n} \to \underline{m}$. De plus, la limite est le développement en série entière en $q = 0$ d'une fraction rationnelle.

2. *Soit Q un carquois arbitraire et $\underline{r} \in \mathbf{N}^{\Omega}_{\geq 1} \setminus \{0\}$. Pour $s \in \mathbf{N}$, on définit $v_{s\underline{r}}$ la valuation du polynôme $A_{Q_{s\underline{r}},\mathbf{d}}(q)$. Alors la suite de polynômes*
$$\frac{A_{Q_{s\underline{r}},\mathbf{d}}(q)}{q^{v_{s\underline{r}}}} \in \mathbf{N}[q]$$
converge dans $\mathbf{N}[[q]]$ lorsque $s \to +\infty$ et la limite est le développement en série entière en $q = 0$ d'une fraction rationnelle.

Pour démontrer ce résultat, on utilise la formule explicite de la série génératrice des polynômes de Kac de Hua ([Hua00], voir aussi [HLRV13]). Une de nos conjectures concerne la valuation des polynômes de Kac en présence de boucles (Conjecture 4.3.1).

1.7.3 Étude microlocale des faisceaux pervers sur les champs de représentations de carquois

Nous donnons ici les résultats principaux de [Hen20b]. Motivés par les aspects géométriques des algèbres de Hall, et dans l'espoir de trouver une interprétation en termes de faisceaux pervers des fonctions cuspidales, nous avons étudié les faisceaux de Lusztig et leur interaction avec la variété nilpotente de Lusztig. Soit $Q = (I, \Omega)$ un carquois. Inspiré par la construction de Ringel de la partie positive du groupe quantique associé à un carquois ([Rin90b]), Lusztig a défini une catégorie de faisceaux pervers équivariants $\mathcal{P}$ sur les espaces de représentation $E_{\mathbf{d}}$, $\mathbf{d} \in \mathbf{N}^I$. On a noté
$$E_{\mathbf{d}} = \bigoplus_{\alpha \in \Omega} \mathrm{Hom}(\mathbf{C}^{\mathbf{d}_{s(\alpha)}}, \mathbf{C}^{\mathbf{d}_{t(\alpha)}})$$
l'espace de représentation de Q en dimension $\mathbf{d}$. C'est un espace affine sur lequel agit le groupe réductif
$$G_{\mathbf{d}} = \prod_{i \in I} \mathrm{GL}_{\mathbf{d}_i}.$$
La catégorie $\mathcal{P}$ est semisimple, et les faisceaux pervers simples de $\mathcal{P}$ sont les faisceaux pervers apparaissant comme constituants simples des inductions de faisceaux constants sur les espaces de représentations triviaux (en dimension $e_i, i \in I$), voir par exemple [Lus91, Lus11, Sch09]. Le groupe de Grothendieck additif de la catégorie triangulée obtenue à partir de cette catégorie abélienne a une structure de $\mathbf{Z}[\nu, \nu^{-1}]$-algèbre (où ν est une indéterminée) induite par l'induction de faisceaux constructibles, et est isomorphe à la forme entière de la partie positive du groupe quantique. Les classes des faisceaux pervers simples donnent la base canonique et lorsque cette construction

est réalisée sur un corps fini, la trace de Frobenius des faisceaux constructibles fait le lien avec l'algèbre de Hall constructible ([Lus98]). Il y a maintenant des objets puissants qui permettent de comprendre la base canonique en utilisant les algèbres KLR (ou Hecke-carquois), voir par exemple [KL09, KL11, VV11, VV03] et les références de ces papiers. De plus, Lusztig a défini la variété nilpotente $\Lambda = \bigcup_{\mathbf{d}\in\mathbf{N}^I} \Lambda_{\mathbf{d}}$, une sous-variété fermée du fibré cotangent de l'espace de représentations $T^*E_{\mathbf{d}}$ du carquois ([Lus91]). Les supports singuliers des faisceaux pervers simples considérés par Lusztig sont des unions de certaines des composantes irréductibles de la variété nilpotente Λ. Lusztig a utilisé le $\mathbf{Z}$-module libre engendré par les composantes irréductibles de Λ pour définir la *base semicanonique* de l'algèbre enveloppante de la partie positive de l'algèbre de Kac-Moody associée à Q ([Lus00]). De plus, Kashiwara et Saito ont démontré dans [KS97] que l'application cycle caractéristique induit une bijection entre l'ensemble des classes d'isomorphisme de faisceaux pervers simples de $\mathcal{P}$ et l'ensemble des composantes irréductibles de Λ. Il est alors naturel de se demander quelle est la structure de la catégorie des faisceaux pervers dont le support singulier est contenu dans la variété nilpotente. Dans le Chapitre 5, nous répondons à cette question pour les carquois affines. Nous étendons aussi cette question au cas des carquois avec des boucles, pour lesquels on doit considérer une variété nilpotente adaptée (la variété semi-nilpotente). Nous démontrons un résultat analogue pour les carquois ayant un sommet et au moins deux boucles. Les deux théorèmes principaux de ce chapitre sont les suivants (Theorem 5.1.1 et Theorem 5.1.3).

Théorème 1.7.3. *Soit Q un carquois de type fini ou affine. Soit $\mathscr{F} \in \mathrm{Perv}_{G_{\mathbf{d}}}(E_{\mathbf{d}})$ un faisceau pervers $G_{\mathbf{d}}$-équivariant tel que $SS(\mathscr{F}) \subset \Lambda_{\mathbf{d}}$. Alors $\mathscr{F}$ est un faisceau pervers de Lusztig.*

Ce théorème démontre une conjecture formulée par Lusztig dans [Lus90b].

Théorème 1.7.4. *Soit $g \geq 2$ et $Q = S_g$ le carquois à g boucles. Soit $\mathbf{d} \in \mathbf{N}$ un vecteur dimension et $\mathscr{F}$ un faisceau pervers simple sur $E_{Q,\mathbf{d}}$ tel que $SS(\mathscr{F}) \subset \Lambda^{\flat}$ où $\flat = (\mathrm{nil}, 1)$ ou $\flat = 1$. Alors $\mathscr{F}$ est dans la catégorie $\mathcal{P}^{\flat}$.*

Pour démontrer ce théorème, nous nous appuyons sur les résultats de Lusztig [Lus93] concernant la petitesse des morphismes du champ des paires formées d'une représentation de Q et d'un drapeau d'un type fixé stabilisé par cette représentation vers le champ des représentations de Q.

Nous conjecturons, à la suite de Lusztig, que ces caractérisations microlocales restent vraies en toute généralité. On renvoie le lecteur au Chapitre 5 pour des énoncés précis des conjectures.

1.7.4 Faisceaux d'Eisenstein sphériques sur le champ des faisceaux cohérents d'une courbe elliptique

Nous expliquons dans cette section les résultats principaux de l'article [Hen21]. Soit X une courbe complexe projective lisse. On dispose de la situation analogue à celle des carquois. La catégorie des représentations du carquois est remplacée par la catégorie des faisceaux cohérents sur la courbe X, la catégorie des faisceaux pervers considérée par Lusztig devient la catégorie des faisceaux d'Eisenstein sphériques définie par Schiffmann ([Sch06]), et la variété nilpotente de Lusztig est remplacée par le cône global nilpotent ([Lau88]). On peut alors montrer que les supports singuliers des faisceaux d'Eisenstein sphériques sont des unions de composantes irréductibles du cône global nilpotent. En s'inspirant de la situation pour les carquois, on démontre pour les courbes elliptiques que l'application cycle caractéristique induit une bijection entre (les complétions) du groupe de Grothendieck de la catégorie des faisceaux d'Eisenstein sphériques et du groupe abélien libre engendré par les composantes irréductibles du cône global nilpotent (Theorem 6.1.2). Par contre, l'existence de systèmes locaux non triviaux sur X empêche d'avoir une caractérisation microlocale. Cependant, on détermine de façon explicite les faisceaux pervers simples sur le champ des faisceaux cohérents sur une courbe elliptique dont le support singulier est contenu dans le cône global nilpotent (Theorem 6.1.3). Les résultats principaux du Chapitre 6 sont alors les suivants. On rappelle seulement les notations nécessaires à l'énoncé des Théorèmes 1.7.5 et 1.7.6. Le champ des faisceaux cohérents sur X est noté $\mathfrak{Coh}(X)$. La composante connexe correspondant à $\alpha = (r, d) \in \mathbf{Z}^+$ est notée $\mathfrak{Coh}(X)_\alpha$. Le champ des fibrés de Higgs est $\mathfrak{Higgs}(X) = \bigsqcup_{\alpha \in \mathbf{Z}^+} \mathfrak{Higgs}(X)_\alpha$. Il contient un sous-champ fermé, le cône global nilpotent : pour $\alpha \in \mathbf{Z}^+$, on note $\mathscr{N}_\alpha \subset \mathfrak{Higgs}(X)_\alpha$. Son ensemble de composantes irréductibles est noté $\mathrm{Irr}(\mathscr{N}_\alpha)$. Les chapeaux indiquent des complétions des groupes abéliens en question. On renvoit au Chapitre 6 pour la définition de celles-ci.

Théorème 1.7.5. *Pour toute classe $\alpha \in \mathbf{Z}^+$ L'application cycle caractéristique*

$$CC : \widehat{K_0(\mathcal{P}^\alpha)} \to \mathbf{Z}[\widehat{\mathrm{Irr}(\mathscr{N}_\alpha)}]$$

est un isomorphisme de $\mathbf{Z}$-modules.

Afin d'énoncer le deuxième résultat principal, on a besoin d'introduire quelques notations supplémentaires. Pour $\alpha \in \mathbf{Z}^+$, on note $\mathfrak{Coh}(X)_{(\alpha)}$ le champ des faisceaux cohérents *semistables* de type α. Comme X est une courbe elliptique, on a un isomorphisme $\epsilon_\alpha : \mathfrak{Coh}(X)_{(\alpha)} \simeq \mathfrak{Coh}(X)_{(0, \gcd \alpha)}$ où $\gcd \alpha = \gcd(r, d)$. Le groupe fondamental topologique de X est $\mathbf{Z}^2$. Ainsi, les systèmes locaux irréductibles sur $\mathfrak{Coh}_{(0,1)}$ sont indexés par $z \in (\mathbf{C}^*)^2$. Le système local correspondant est noté $\mathscr{L}_z$. Si $\alpha \in \mathbf{Z}^+$ est tel que $\gcd \alpha = 1$, par l'isomorphisme ϵ_α, on obtient un système local encore noté $\mathscr{L}_z$ sur $\mathfrak{Coh}(X)_{(\alpha)}$.

Théorème 1.7.6. *Les faisceaux pervers sur $\mathfrak{Coh}_\alpha(X)$ ayant un support singulier nilpotent sont précisément les faisceaux pervers qui sont les constituants simples d'inductions de faisceaux pervers de la forme $\mathcal{IC}(\mathscr{L}_z)$ sur $\mathfrak{Coh}_\beta(X)$, $\beta \in \mathbf{Z}^+$ indivisible.*

Chapter 2

Introduction

This thesis is devoted to the study of some aspects of Hall algebras one can associate to quivers and curves, related to the notion of *cuspidal functions* in these algebras. Hall algebras appeared in the work of Steinitz at the beginning of the twentieth century[1] ([Ste01]) when he studied the category of finite abelian p-groups for some prime number p. More precisely, he constructed a multiplication on the complex vector space having as basis the set of isomorphism classes of such groups. Structure constants count the number of extensions between two abelian p-groups when the isomorphism class of the middle object is fixed. More than a half century later, Hall published a paper with similar results [Hal]. The main fact is the polynomial behaviour in p of these structure constants. The algebra obtained in this way is isomorphic to the algebra of symmetric functions [Mac15]. Ringel [Rin90b, Rin90a] and Green [Gre95] enlarged the study of these algebras by associating Hall algebras to the category of *finite* modules (that is, modules whose underlying set is finite) over a *hereditary* algebra R. When the algebra under consideration is the path algebra of a quiver over a finite field $\mathbf{F}_q$, they partially determine the structure of the Hall algebra: when Q is a finite type quiver, it coincides with the positive part of the quantum group associated to the quiver; in general, we have to restrict ourselves to the *spherical subalgebra* of the Hall algebra ([Sch12a, Theorem 3.16]). Important contributions to the theory are the construction of the coproduct by Green ([Gre95]), and an explicit formula for the antipode [Xia97], which endows the Hall algebra with a Hopf algebra structure, of which the spherical subalgebra is a Hopf subalgebra isomorphic to the positive part of the quantum group. There are two natural questions arising now. The first one is to find a realization of the whole quantum group, and not only of its positive part, in terms of Hall algebras. This question has been addressed in various works, in particular in [Bri13, Toë06]. In [Bri13], Bridgeland realizes the whole quantum group in the Hall algebra of 2-periodic complexes (localized with respect to acyclic complexes) while in [Toë06], Toën defines *the derived Hall algebra* of a differential graded category (satisfying some finiteness conditions), for which distinguished triangles play the role of exact sequences. To the author's knowledge, the geometric counterparts[2] of the constructions of Bridgeland and Toën have not yet been published. The second question concerns the understanding of the *whole* Hall algebra, and not only of its spherical subalgebra.

[1] But this terminology is due to Ringel

[2] That is, the construction of the quantum group from the Grothendieck group of the category of constructible sheaves on the stack classifying objects of the category of 2-periodic complexes or of the differential category

This question is the starting point of this thesis. A minimal generating subspace of the Hall algebra is given by the so-called *cuspidal functions* ([SVDB01]) (this is the name given to primitive elements of the Hall algebra). We call *trivial cuspidal functions* the (linear combinations) of the characteristic functions of the simple representations of the quiver concentrated at one vertex. They generate (by definition) the spherical subalgebra. Non-trivial cuspidal functions are thus crucial in the understanding of the Hall algebra of a quiver. One can reformulated our second question in the following manner:

What are the cuspidal functions of a quiver?

This question has to be understood in a broad sense: it includes the following ones. What are the dimensions of the spaces of cuspidal functions? Can one parametrize explicitly a basis of the space of cuspidal functions, possibly through the cohomology of an algebraic variety? Is there a way to construct cuspidal functions? The question concerning the dimension has been studied by Bozec and Schiffmann ([BS18]). The other questions remain largely open for general quivers, although the problem is solved for affine quivers ([Hen19], or Chapter 3 if this thesis) and geometric approaches do exist ([Dav20]).

We saw that cuspidal functions form a minimal generating subspace of the Hall algebra. The graded character of the Hall algebra is given by the plethystic exponential of the generating series of Kac polynomials of the quiver ([BS18, Lemma 3.1], [Hua00], see also Section 2.4.2), and is also obtained from the graded dimension of the space of cuspidal functions thanks to Borcherds character formula ([BS18, Section 2.3]). We can try to understand the evolution of Kac polynomials when we give each arrow of the quiver a multiplicity (which produces a new quiver, in which each arrow is replaced by a certain number of parallel arrows in accordance with the multiplicity) and make the vector of multiplicities tend to a limit. This is the question that is adressed in [Hen20a], which is Chapter 4 of this thesis. Surprisingly, the sequence of polynomials obtained converges to a power series (Theorem 2.7.2) and the distribution of the coefficients of Kac polynomials conjecturally converges to a continuous distribution (see Conjecture 4.5.1 of Chapter 4). These properties suggest that the Hall algebra of a quiver has an aymptotic version obtained by letting the number of arrows tend to infinity, but we have not yet explored this question.

The search for a geometric understanding of the Hall algebra of a quiver leads to the construction of Lusztig [Lus91, Lus98]. Lusztig considers the category of perverse sheaves on the stack of quiver representations which are obtained as simple constituents of the induction of the constant sheaves. The split Grothendieck group of the associated triangulated category is then isomorphic to the positive part of the quantum group corresponding to the quiver and the simple perverse sheaves give the canonical basis, which was defined and studied from a purely combinatorial point of view by Kashiwara ([Kas91]). This construction in terms of perverse sheaves on the stacks of representations has a symplectic analogue in terms of irreducible components of Lusztig's nilpotent variety, Λ, which is a conical Lagrangian substack of the cotangent stack of the representation stack of the quiver. This second construction gives rise to the *semicanonical* basis of the enveloping algebra of the positive part of the Kac-Moody algebra associated with the quiver ([Lus00]), and the link between the two constructions is given (in an unprecise way) thanks to the characteristic cycle map, which to a perverse sheaf associates its characteristic cycle, a $\mathbf{Z}$-linear combination of the irreducible components of Λ. Driven by our objective of a geometric understanding of the Hall algebra, it is natural to ask what are the perverse sheaves on the stack of representations of a quiver whose singular support (the support of the characteristic cycle) is a union of irreducible components of the nilpotent variety. We are able to answer this question for finite, affine quivers and for quivers with a single vertex and several loops: this is the object of the work [Hen20b], which is Chapter 5 of this thesis. In all these cases, there are no other simple perverse sheaves with nilpotent singular support than the sheaves considered by Lusztig. The question remains open for general wild quivers although we have results going in the same direction in some very particular cases (generalized Kronecker quivers in dimension $(1, d)$, $d \geq 0$ or arbitrary quivers for the dimension 1 at each vertex). It therefore seems reasonable to conjecture, following Lusztig, that this *microlocal characterization* of Lusztig sheaves by the nilpotency of the singular support remains valid in general (Conjecture 5.10.17 of Chapter 5).

The question of the relation between a category of perverse sheaves on a stack and a Lagrangian substack of the cotangent stack of this stack has an analog for smooth projective curves defined over the complex numbers. In this new framework, we replace the quiver by a curve, the representations of the quiver by the category of coherent sheaves on the curve, Lusztig sheaves by *spherical Eisenstein sheaves* (defined by Schiffmann, [Sch04a, Sch11]) and the nilpotent variety by the global nilpotent cone (the nilpotent Higgs sheaves, [Lau88]). We can show, in the case of an elliptic curve, that the characteristic cycle map induces an isomorphism between a completion of the Grothendieck group of the category of spherical Eisenstein sheaves on the one hand and a completion of the $\mathbf{Z}$-module whose basis is formed by the irreducible components of the global nilpotent cone on the other hand. However, the microlocal characterization no longer holds. It is nevertheless possible to describe explicitly (still in the case of an elliptic curve) the simple perverse sheaves on the stack of coherent sheaves whose singular support is nilpotent ([Hen21] or Chapter 6). The case of a curve of genus $g \geq 2$ seems much more difficult, but we conjecture the same kinds of results. These results would be helpful for the study of the cohomological Hall algebra of nilpotent Higgs sheaves on the curve.

In the rest of this introduction, we will give more details on the roots and motivations that led to the work constituting this thesis.

2.1 The analogy between curves and quivers

The categories of coherent sheaves on smooth projective curves and of quiver representations over a field share many properties. The reader may refer to the overview article [Hos18] for more details. The first object, the category of coherent sheaves on a smooth projective curve, is the central object considered in the Langlands program for function fields in its geometric version ([Lau87]), whose ultimate goal is understand geometrically the global Langlands correspondence. Recall that the Langlands correspondence for a smooth projective curve X over a finite field $\mathbf{F}_q$ with q elements for the group GL_r establishes a bijection between the irreducible ℓ-adic representations of rank r of the Galois group of X of finite order determinant and the irreducible cuspidal automorphic representations of the group $\mathrm{GL}_r(\mathbb{A})$ where $\mathbb{A}$ is the ring of adèles. For linear groups GL_r $(r \geq 1)$, this correspondence is known thanks to the work of Laurent Lafforgue ([Laf02]). For general reductive groups, there is a similar set of conjectures which remain open. Quivers, on the other hand, are a central object of the representation theory of finite dimensional algebras over a field. Path algebras of acyclic quivers are indeed the archetypal example of finite dimensional hereditary algebras, the study of which began with Gabriel ([Gab72]). Quivers have penetrated many areas of mathematics and often provide an understanding of more complicated objects. We can mention for example that the local structure of moduli spaces of sheaves on a $K3$ surface are locally isomorphic to Nakajima quiver varieties ([AS18]) or that quivers with potential provide interesting models for three-dimensional Calabi-Yau categories ([Gin06]) and thus for moduli spaces of coherent sheaves on three-dimensional Calabi-Yau manifolds. To a smooth projective curve X over a field k, we associate the category of coherent sheaves over it, $\mathrm{Coh}(X)$; to a quiver Q, we associate the category of its representations over the field k, $\mathrm{Rep}_Q(k)$. The first result which brings their geometric study closer is the following. We call *homological dimension* of an abelian category $\mathcal{A}$ the smallest integer d such that $\mathrm{Ext}^d(-,-)$ is nonzero.

Proposition 2.1.1. *If X is a smooth projective curve and Q a quiver with at least one arrow*[3]*, the categories $\mathrm{Coh}(X)$ and $\mathrm{Rep}_Q(k)$ are abelian categories of homological dimension one.*

The other important property of these categories is the existence Serre functors. In order to unify the notations, we use the letter $\mathcal{A}$ to denote one of the categories $\mathrm{Coh}(X)$ or $\mathrm{Rep}_Q(k)$. If ω_X denotes the canonical bundle of X, we have a functor

$$\begin{array}{cccc} F & : & \mathrm{Coh}(X) & \to & \mathrm{Coh}(X) \\ & & \mathcal{F} & \mapsto & \mathcal{F} \otimes \omega_X \end{array}$$

which extends into a functor $F : D^b(X) \to D^b(X)$, $\mathcal{F} \mapsto \mathcal{F} \otimes \omega_X$ between derived categories. For the quiver Q (which is assumed to be acyclic), we denote by $F = \tau$ the associated Auslander-Reiten transform. It is an endofunctor of the category $\mathrm{Rep}_Q(k)$, which extends to the derived category $D^b(Q,k)$. Serre duality is expressed by the following result (cf. [Sch12a, Theorem 3.30, Section 4.1]).

Proposition 2.1.2. *For any two objects $\mathcal{F}, \mathcal{G}$ of $D^b(\mathcal{A})$, we have a canonical isomorphism*

$$\mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(\mathcal{G}, F(\mathcal{F})[1])^*.$$

The shift [1] is a marker of the homological dimension one.

The classification of the abelian categories subject to the properties of Propositions 2.1.1 and 2.1.2 is an object of study and the literature abounds in results in this direction. We refer the reader to the beautiful article [RVdB02].

The categories of coherent sheaves on a curve or of representations of a quiver being of homological dimension one, the stacks classifying their objects are smooth. This property is important

[3] The category of representations of a quiver without arrows is equivalent to the category of vector spaces graded by the set of vertices; it is of homological dimension zero.

for geometric considerations.

Sometimes the categories of coherent sheaves on a curve and of quiver representations are more closely related, as shown by the following theorem where the projective curve $\mathbf{P}^1_k$ and the Kronecker quiver K_2 are considered (cf. [Sch12a, Section 5.1]).

Theorem 2.1.3. *The triangulated categories $D^b(\mathbf{P}^1_k)$ and $D^b(K_2, k)$ are equivalent.*

Note however that the derived category of an abelian category does not determine the homological dimension of the initial abelian category. For example, canonical algebras are of homological dimension 2 but their derived categories are isomorphic to derived categories of weighted projective curves, and the latter are of homological dimension 1 (cf. [PS19] and references therein).

Theorem 2.1.3 can be extended to certain weighted projective lines, for which we rather need to consider quivers of affine type (cf. [Cra10, Section 9]).

2.2 The Hall algebra of a finitary category

We recall some facts on the constructible Hall algebra of a finitary abelian category. For further details, the reader may refer to the notes [Sch12a].

Let $\mathcal{A}$ be an abelian category. We assume that it is small, in the sense that the set $\mathrm{Isom}(\mathcal{A})$ of the isomorphism classes of its objects is well defined, and finitary, that is that for any pair of objects $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$, a finite number of the sets $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ are non-trivial and their numbers of elements are finite. Often, the categories in play will be linear over a finite field. We advise the reader to keep in mind the categories of finite dimensional representations of a quiver over a finite field as well as that of coherent sheaves on a smooth projective curve defined over a finite field. The Euler form (called *multiplicative*) of such a category is defined by the formula

$$\langle \mathcal{F}, \mathcal{G} \rangle_m = \prod_{i \in \mathbf{Z}} |\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})|^{(-1)^i}.$$

where for a finite set E, $|E|$ denotes its cardinality. A standard argument involving a long exact sequence allows to show that the Euler form only depends on the class in the Grothendieck group of the category $\mathcal{A}$, which we denote by $K_0(\mathcal{A})$, and induces a morphism of abelian groups:

$$\begin{array}{ccccc} \langle -, - \rangle_m & : & K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) & \to & \mathbf{Q}^* \\ & & [\mathcal{F}] \otimes [\mathcal{G}] & \mapsto & \langle \mathcal{F}, \mathcal{G} \rangle_m \end{array}$$

where by the symbol $[\mathcal{F}]$ we denote the class of the object $\mathcal{F}$ of the category $\mathcal{A}$ in its Grothendieck group. If moreover the category $\mathcal{A}$ is $\mathbf{F}_q$-linear, we define its *additive Euler form* $\langle -, - \rangle_a : K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) \to \mathbf{Z}$ as the alternated sum of the dimensions of the groups $\mathrm{Ext}^i(-, -)$. We have the equality $q^{\langle \mathcal{F}, \mathcal{G} \rangle_a} = \langle \mathcal{F}, \mathcal{G} \rangle_m$ for any objects $\mathcal{F}, \mathcal{G}$ of $\mathcal{A}$. Often, we can replace the abelian group $K_0(\mathcal{A})$ by an abelian group M endowed with a multiplicative bilinear form $M \otimes M \to \mathbf{Q}^*$ and a morphism $K_0(\mathcal{A}) \to M$ through which $\langle -, - \rangle_m$ factorizes. This situation occurs for quivers $Q = (I, \Omega)$ with set of vertices I, for which the Euler form factorizes through the morphism of abelian groups $\dim : K_0(\mathcal{A}) \to \mathbf{Z}^I$ given by the dimension vector and also for curves, where this time the Euler form factorizes through $K_0(\mathrm{Coh}(X)) \to \mathbf{Z}^2$ which associates to $[\mathcal{F}] \in K_0(\mathrm{Coh}(X))$ its rank and its degree.

If $\mathcal{X}$ is a groupoid whose set of isomorphism classes of objects is denoted $\mathcal{X}/\sim$, we denote by $\mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q})$ the set of functions on $\mathcal{X}/\sim$ having finite support. We denote by $[x]$ the isomorphism class of an object x of $\mathcal{X}$. Given a morphism between groupoids $F : \mathcal{X} \to \mathcal{Y}$, we can define operations on the sets of functions:

$$F^* : \mathrm{Fun}_c(\mathcal{Y}/\sim, \mathbf{Q}) \to \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q})$$

and

$$F_! : \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q}) \to \mathrm{Fun}_c(\mathcal{Y}/\sim, \mathbf{Q}).$$

To define $F_!$, it is necessary to assume that the fibers of F are finite (more precisely that the sets of isomorphism classes of the fibers of F are finite) and that the automorphism groups of objets of $\mathfrak{X}, \mathfrak{Y}$ are also finite. For $f \in \mathrm{Fun}_c(\mathcal{Y}, \mathbf{Q})$ and an object x of $\mathcal{X}$, we define

$$F^*(f)([x]) = f([F(x)])$$

and, under the assumption that the fibers are finite and the automorphism groups of the objects of $\mathcal{X}$ and of $\mathcal{Y}$ are finite, for y an object of $\mathcal{Y}$,

$$F_!(f)([y]) = |\mathrm{Aut}(y)| \sum_{[x]\in\mathcal{X}/\sim, [F(x)]=[y]} \frac{f([x])}{|\mathrm{Aut}(x)|}.$$

The volume $\mathrm{Vol}(\mathcal{X})$ of the groupoid $\mathcal{X}$ is defined as the push-forward of the characteristic function $1_{\mathcal{X}}$ whose single value is 1 by the morphism $F : \mathcal{X} \to \mathrm{pt}$ (where pt is the groupoid having a single object and a single morphism). We have

$$\mathrm{Vol}(\mathcal{X}) = \sum_{[x]\in\mathcal{X}/\sim} \frac{1}{|\mathrm{Aut}(x)|}.$$

From now on, $\mathcal{X}$ denotes the groupoid associated with the category $\mathcal{A}$. We also define $\mathcal{A}^{(1)}$ the category of exact sequences of $\mathcal{A}$ and we denote by $\mathcal{X}^{(1)}$ the associated groupoid. We consider the convolution diagram

$$\begin{tikzcd} & \mathcal{X}^{(1)} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{X} \times \mathcal{X} & & \mathcal{X} \end{tikzcd}$$

where p is the functor sending a short exact sequence to its middle term and q to the pair of its third and first terms. In this way, we obtain the induction and restriction operations

$$\mathrm{Ind} = p_! q^* : \mathrm{Fun}_c(\mathcal{X}/\sim \times \mathcal{X}/\sim, \mathbf{Q}) \to \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q})$$

and

$$\mathrm{Res} = q_! p^* : \mathrm{Fun}_c(\mathcal{X}/\!\sim, \mathbf{Q}) \to \mathrm{Fun}_c(\mathcal{X}/\!\sim \times \mathcal{X}/\!\sim, \mathbf{Q}).$$

Note that Res is only well defined when an additional finiteness condition is satisfied by the category $\mathcal{A}$. Indeed, it is not true that $\mathrm{Res}(f)$ has a finite support if f has. A necessary and sufficient condition for this to be the case is that any object of $\mathcal{A}$ has a finite number of subobjects. This condition is satisfied for the category of representations of a quiver over a finite field, but not for the category of coherent sheaves on a smooth projective curve over a finite field. It is possible to get around this problem by considering completions, but to preserve the readability of this introduction, we ignore this question. As we have an identification

$$\mathrm{Fun}_c(\mathcal{X}/\!\sim \times \mathcal{X}/\!\sim, \mathbf{Q}) \simeq \mathrm{Fun}_c(\mathcal{X}/\!\sim, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathrm{Fun}_c(\mathcal{X}/\!\sim, \mathbf{Q}),$$

these operations endow $\mathrm{Fun}_c(\mathcal{X}, \mathbf{Q})$ with algebra and coalgebra structures. The proofs of the associativity and co-associativity of these operations involve the category $\mathcal{A}^{(2)}$ of two-step filtrations $0 \subset M_1 \subset M_2 \subset M$ of objects of $\mathcal{A}$. We can give explicit formulas for the induction and restriction of characteristic functions of isomorphism classes of objects. If x is an object of $\mathcal{A}$, we denote by $1_{[x]} \in \mathrm{Fun}_c(\mathcal{X}/\!\sim, \mathbf{Q})$ the function taking the value 1 on $[x]$ and the value 0 on $[y] \in \mathcal{X}/\!\sim$ such that $[x] \neq [y]$. For three objects x, y, z of $\mathcal{A}$,

$$\begin{aligned}\mathrm{Ind}(1_{[x]}, 1_{[y]})([z]) &= \sum_{\substack{[e]\in\mathcal{X}^{(1)}/\sim \\ [q(e)]=([x],[y]) \\ [p(e)]=[z]}} \frac{|\mathrm{Aut}(z)|}{|\mathrm{Aut}(e)|} \\ &= |\mathrm{Aut}(z)| \,\mathrm{Vol}(\mathcal{X}^{(1)}_{y,z,x})\end{aligned}$$

where $\mathcal{X}^{(1)}_{y,z,x}$ is the groupoid of short exact sequences $0 \to y' \to z' \to x' \to 0$ with $x' \simeq x$, $y' \simeq y$, $z' \simeq z$. The groupoid $\mathcal{X}^{(1)}_{y,z,x}$ is equivalent to the quotient groupoid

$$\tilde{\mathcal{X}}^{(1)}_{y,z,x}/(\mathrm{Aut}(y) \times \mathrm{Aut}(z) \times \mathrm{Aut}(x))$$

where $\tilde{\mathcal{X}}^{(1)}_{y,z,x}$ is the set

$$\{(\alpha, \beta) \in \mathrm{Hom}(y, z) \times \mathrm{Hom}(z, x) \mid 0 \to y \xrightarrow{\alpha} z \xrightarrow{\beta} x \to 0 \text{ is exact}\}$$

and the action of $\mathrm{Aut}(y) \times \mathrm{Aut}(z) \times \mathrm{Aut}(x)$ on this set is given by

$$(a, b, c) \cdot (\alpha, \beta) = (b\alpha a^{-1}, c\beta b^{-1}).$$

Put together, we find

$$\mathrm{Vol}(\mathcal{X}^{(1)}_{y,z,x}) = \frac{F^z_{x,y}}{|\mathrm{Aut}(z)|}$$

where $F^z_{x,y} = \{y' \subset z \mid y' \simeq y, z/y' \simeq x\}$ is the set of subobjects y' of z isomorphic to y and such that the quotient z/y' is isomorphic to x, and so

$$\mathrm{Ind}(1_{[x]}, 1_{[y]})([z]) = F^z_{x,y}.$$

For the restriction, one finds

$$\begin{aligned}\mathrm{Res}(1_{[z]})([x],[y]) &= \sum_{\substack{[e]\in\mathcal{X}^{(1)}/\sim \\ [p(e)]=[z] \\ [q(e)]=([x],[y])}} \frac{|\mathrm{Aut}(x)|\cdot|\mathrm{Aut}(y)|}{|\mathrm{Aut}(e)|} \\ &= |\mathrm{Aut}(x)|\cdot|\mathrm{Aut}(y)|\,\mathrm{Vol}(\mathcal{X}^{(1)}_{y,z,x}) \\ &= \frac{|\mathrm{Aut}(x)|\cdot|\mathrm{Aut}(y)|}{|\mathrm{Aut}(z)|} F^z_{x,y}.\end{aligned}$$

The unit of the multiplication is the function $1 \in \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q})$ defined by

$$\begin{array}{rcccc} 1 & : & \mathcal{X}/\sim & \to & \mathbf{Q} \\ & & [x] & \mapsto & \begin{cases} 1 \text{ if } x \simeq 0 \\ 0 \text{ else} \end{cases} \end{array}$$

The counit is the $\mathbf{Q}$-linear map

$$\begin{array}{rcccc} \varepsilon & : & \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{Q}) & \to & \mathbf{Q} \\ & & f & \mapsto & f([0]), \end{array}$$

where 0 denotes the terminal object of the category $\mathcal{A}$. If $\alpha, \beta : K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) \to \mathbf{C}^*$ are two multiplicative bilinear forms, one can twist the induction and the restriction in the following way. For $[M], [N] \in \mathcal{X}/\sim$,

$$\mathrm{Ind}_\alpha(1_{[M]}, 1_{[N]}) := \alpha([M],[N])\,\mathrm{Ind}(1_{[M]}, 1_{[N]})$$

and

$$\mathrm{Res}_\beta = \beta \cdot \mathrm{Res}$$

where the dot $\cdot$ indicates the pointwise multiplication and β is seen as an element of $\mathrm{Fun}(\mathcal{X}/\sim \times\mathcal{X}/\sim, \mathbf{C})$ (a function with a not necessarily finite support) in the following way:

$$\begin{array}{rcccc} \beta & : & \mathcal{X}/\sim \times \mathcal{X}/\sim & \to & \mathbf{C} \\ & & ([x],[y]) & \mapsto & \beta([x],[y]). \end{array}$$

With the help of a third multiplicative bilinear form $\gamma : K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) \to \mathbf{C}^*$, one can define a multiplication on the tensor product $\mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{C}) \otimes \mathrm{Fun}_c(\mathcal{X}/\sim, \mathbf{C})$:

$$(1_{[x]} \otimes 1_{[y]}) \cdot (1_{[z]} \otimes 1_{[t]}) = \gamma([y],[z])((1_{[x]}1_{[z]}) \otimes (1_{[y]}1_{[t]})).$$

The bilinear form γ is called *braiding*. If α, β are constant equal to 1, we take $\gamma = \langle -,- \rangle_m^{-1}$, the inverse of the Euler form.

In the case where the category $\mathcal{A}$ is hereditary (the bifunctors $\mathrm{Ext}^i(-,-)$ are identically zero for $i \geq 2$), we have the following result which gives the compatibility between the multiplication and the comultiplication.

Theorem 2.2.1 (Green)**.** *Let $\mathcal{A}$ be an abelian hereditary category. Then,*

$$(\mathrm{Fun}_c(\mathcal{X}, \mathbf{Q}), \mathrm{Ind}, \mathrm{Res})$$

is a bialgebra for the braiding $\gamma = \langle -, - \rangle_m^{-1}$

Usually, we twist the multiplication and comultiplication of the Hall algebra using nontrivial bilinear forms α, γ. The usual choice, the one we will consider by default, is

$$\alpha = \beta = \langle -, - \rangle_m$$

In this case, the choice for the braiding is

$$\gamma = (-, -)_m,$$

the symmetrized Euler form.

It is not reasonable to hope to describe the structure of $(\mathrm{Fun}_c(\mathcal{X}, \mathbf{Q}), \mathrm{Ind})$ for an arbitrary abelian category. As we have already mentioned several times, the cases that interest us are $\mathcal{A} = \mathrm{Rep}_Q(\mathbf{F}_q)$ and $\mathcal{A} = \mathrm{Coh}(X)$.

The bialgebra $(\mathrm{Fun}_c(\mathcal{X}, \mathbf{Q}), \mathrm{Ind}_\alpha, \mathrm{Res}_\beta)$ is called *Hall algebra* of the category $\mathcal{A}$. We write it $\mathbf{H}_{\mathcal{A}}$. If $\mathcal{A} = \mathrm{Rep}_Q(\mathbf{F}_q)$, we write $\mathbf{H}_{\mathcal{A}} = \mathbf{H}_{Q,\mathbf{F}_q}$. It is a bialgebra in the braided monoidal category of $\mathbf{K}_0(\mathcal{A})$-graded vector spaces, where the braiding is given by γ: if V, W are $K_0(\mathcal{A})$-graded vector spaces, the braiding is

$$\begin{array}{ccc} V \otimes W & \to & W \otimes V \\ v \otimes w & \mapsto & \gamma(\deg(w), \deg(v)) w \otimes v \end{array}$$

if v, w are homogeneous of respective degrees $\deg(v), \deg(w) \in K_0(\mathcal{A})$.

It is possible to obtain from $(\mathrm{Fun}_c(\mathcal{X}, \mathbf{Q}), \mathrm{Ind}_\alpha, \mathrm{Res}_\beta)$ a bialgebra in the classical sense (a bialgebra in the symmetric monoidal category of vector spaces, i.e. with trivial braiding) by adding the group algebra $\mathbf{C}[K_0(\mathcal{A})]$ to it. More precisely, we define a vector space

$$\tilde{\mathbf{H}}_{\mathcal{A}} = \mathbf{H}_{\mathcal{A}} \otimes_{\mathbf{C}} \mathbf{C}[K_0(\mathcal{A})]$$

together with a product and a co-product Δ as follows. For $[x], [y] \in \mathcal{X}/\sim$, and $\alpha, \beta \in K_0(\mathcal{A})$,

$$([x] \otimes \alpha)([y] \otimes \beta) = (\alpha, [y])_m (1_{[x]} 1_{[y]}) \otimes (\alpha\beta)$$

and for an object z of $\mathcal{A}$ and $\alpha \in K_0(\mathcal{A})$,

$$\Delta([z] \otimes \alpha) = \sum_{[x],[y] \in \mathcal{X}/\sim} \langle x, y \rangle_m \frac{|\mathrm{Aut}(x)| \cdot |\mathrm{Aut}(y)|}{|\mathrm{Aut}(z)|} F^z_{x,y} ([x] \otimes ([y] + \alpha)) \otimes ([y] \otimes \alpha).$$

Remark 2.2.2. One might wonder what happens for a non-hereditary category $\mathcal{A}$. We can no longer hope to have the compatibility of Green's theorem 2.2.1 between the multiplication and the comultiplication. There may be a formula involving the groups of extensions Ext^i for $i \geq 2$ expressing the difference

$$\Delta(\mathcal{F}\mathcal{G}) - \Delta(\mathcal{F})\Delta(\mathcal{G}).$$

We leave this question open. This is certainely linked with the work of Toën [Toë06], since the multiplication of the derived Hall algebra involves the higher extension groups.

2.3 Quivers, Lie algebras, enveloping algebras and their deformations

2.3.1 Quivers and Lie algebras

The theory of semisimple Lie algebras over an algebraically closed field of characteristic zero is well known ([Ser87]). If $\mathfrak{g}$ is such an algebra, the choice of a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$ which contains it gives a root system and, via an invariant bilinear form, a Cartan matrix (an invertible matrix with integer coefficients, having 2 on the diagonal and such that a coefficient outside the diagonal is zero if and only if the symmetric coefficient is zero too). The data of such a matrix can be encoded in the form of a graph, which does not depend on any of the choices made previously. Conversely, from the data of the graph, we can define a Lie algebra by generators and relations and thus find the Lie algebra to which we made the graph correspond. This reverse process can be applied to any graph without loops, not just Dynkin type graphs. The Lie algebras obtained in this more general framework are called *Kac-Moody algebras* ([Kac90]). If we now allow graphs with loops, we find Borcherds algebras (or generalized Kac-Moody algebras), [Bor88]. Despite the increasing generality in the definition of these algebras and the fact that Kac-Moody and Borcherds algebras are of infinite dimension (unless they are semisimple Lie algebras), they share many properties. The first of these being the *character formula* and the similarity of their representation theory (highest weight modules, Verma modules).

We briefly recall the definition by generators and relations of symmetrizable Borcherds algebras (according to Borcherds [Bor88, Bor95]). The starting data is the following: a real vector space H endowed with a symmetric bilinear form $(-,-)$ as well as a countable set of vectors of H, $(h_i)_{i\in I}$, satisfying the following properties:

1. $(h_i, h_j) \leq 0$ if $i \neq j$,
2. $\frac{2(h_i,h_j)}{(h_i,h_i)}$ is an integer $(h_i, h_i) > 0$.

In order to simplify the exposition, and as this is the only case that will interest us, we suppose that for any $i \in I$, $(h_i, h_i) > 0$ implies $(h_i, h_i) = 2$ (that is we only consider the symmetric case). The Cartan matrix associated with this data is the matrix $A = (a_{i,j})_{i,j\in I} = ((h_i, h_j))_{i,j\in I}$. The Borcherds algebra (generalized Kac-Moody algebra) associated with this data is the Lie algebra $\mathfrak{g}(A)$ generated by the elements H, e_i, f_i $(i \in I)$ subject to the following relations.

1. The elements of H pairwise commute,
2. For $h \in H$ and $i \in I$, $[h, e_i] = (h, h_i)e_i$ and $[h, f_i] = -(h, h_i)f_i$,
3. $[e_i, f_j] = h_i$ si $i = j$, $[e_i, f_j] = 0$ si $i \neq j$,
4. If $a_{i,i} > 0$, $\mathrm{ad}^{1-a_{i,j}}(e_i)(e_j) = \mathrm{ad}^{1-a_{i,j}}(f_i)(f_j) = 0$ (Serre relations)
5. If $a_{i,j} = 0$, $[e_i, e_j] = [f_i, f_j] = 0$.

We have a triangular decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus H \oplus \mathfrak{n}_+$$

where $\mathfrak{n}_-$ (resp. $\mathfrak{n}_+$) is the Lie subalgebra of $\mathfrak{g}(A)$ generated by the elements $f_i, i \in I$ (resp. $e_i, i \in I$).

Symmetric Kac-Moody algebras, constructed in [Kac90], can be obtained as a special case of this construction. Let A be a symmetric Cartan matrix, $A = (a_{i,j})_{1 \leq i,j \leq n}$ ($a_{i,i} = 2$, $a_{i,j} \leq 0$, $a_{i,j} = a_{j,i}$ for all $1 \leq i, j \leq n$). A *realization* of A is a vector space $\mathfrak{h}$ endowed with a non-degenerate symmetric bilinear form $(-, -)$, a set of vectors $\Pi \subset \mathfrak{h} = \{\alpha_i : 1 \leq i \leq n\}$ linearly independent and such that $(\alpha_i, \alpha_j) = a_{i,j}$. We also ask that $\mathfrak{h}$ is of minimal dimension, which means here that $\dim \mathfrak{h} = 2n - \mathrm{rank}\, A$. The data of $(\mathfrak{h}, (-, -), (\alpha_i)_{1 \leq i \leq n})$ allows us to build a Borcherds algebra, it is the Kac-Moody associated with the symmetric Cartan matrix A.

To a quiver $Q = (I, \Omega)$ without loops, of which I is the set of vertices and Ω the set of arrows, we associate a Cartan matrix $A = (a_{i,j})_{i,j \in I}$ by setting $a_{i,j} = 2\delta_{i,j} - |\{\alpha \in \Omega \mid \{s(\alpha), t(\alpha)\} = \{i, j\}\}|$. This matrix does not depend on the orientation of the quiver. The Kac-Moody algebra associated with the quiver Q is denoted $\mathfrak{g}_Q$. If Q has loops, the associated algebra is a Borcherds algebra.

2.3.2 Enveloping algebras and quantum groups

Classically, we associate with any Lie algebra $\mathfrak{g}$ an associative algebra denoted $\mathbf{U}(\mathfrak{g})$ and called the enveloping algebra of $\mathfrak{g}$, whose representation theory coincides with that of $\mathfrak{g}$. It is effectively constructed as the quotient of the tensor algebra $T\mathfrak{g}$ by the two-sided ideal generated by the relations $[x, y] = x \otimes y - y \otimes x$. These algebras are studied in [Dix96]. If $\mathfrak{g}$ is the symmetric Kac-Moody algebra associated with a symmetric Cartan matrix A or more generally a Borcherds algebra associated with the data $(H, (-, -), (h_i)_{i \in I})$, its enveloping algebra can be described by generators and relations as follows. The algebra $\mathbf{U}(\mathfrak{g}(A))$ is the associative algebra generated by H, e_i, f_i, $i \in I$, with the relations

1. The elements of H pairwise commute,
2. For $h \in H$ and $i \in I$, $he_i - e_ih = (h, h_i)e_i$ and $hf_i - f_ih = -(h, h_i)f_i$,
3. $e_if_j - f_je_i = \delta_{i,j}h_i$,
4. If $a_{i,i} > 0$, $\sum_{k+l=1-a_{i,j}} (-1)^k e_i^{(k)} e_j e_i^{(l)} = 0$, $\sum_{k+l=1-a_{i,j}} (-1)^k f_i^{(k)} f_j f_i^{(l)} = 0$, where $e_i^{(l)} := \frac{e_i^l}{l!}$ for $l \in \mathbf{N}$ (Serre relations),

5. If $a_{i,j} = 0$, $e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0$.

The algebra $\mathbf{U}(\mathfrak{g}(A))$ admits a triangular decomposition

$$\mathbf{U}(\mathfrak{g}(A)) = \mathbf{U}(\mathfrak{n}_-) \otimes \mathbf{U}(H) \otimes \mathbf{U}(\mathfrak{n}_+).$$

The enveloping algebra has an even richer structure. It is a Hopf algebra, when we endow it with the comultiplication Δ, the antipode S and the counit ϵ. The comultiplication is an algebra morphism $\Delta : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g}) \otimes \mathbf{U}(\mathfrak{g})$ uniquely determined by the formula

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

for $x \in \mathfrak{g} \subset \mathbf{U}(\mathfrak{g})$; the counit is an algebra morphism $\varepsilon : \mathbf{U}(\mathfrak{g}) \to \mathbf{C}$ given by $\varepsilon(x) = 0$ for $x \in \mathfrak{g}$ and finally the antipode is an antihomomorphism of algebras $S : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g})$ determined by $S(x) = -x$ for $x \in \mathfrak{g}$. These new operations make it possible to define the tensor product of $\mathfrak{g}$-representations and to dualize representations. In other words, the category of $\mathbf{U}(\mathfrak{g})$-modules is a tensor category (for a general study of tensor categories, one can refer to [EGNO15]). The Hopf algebra $U(\mathfrak{g})$ is cocommutative, but in general it is not commutative. It is if and only if $\mathfrak{g}$ is abelian. Drinfeld ([Dri87b]) and Jimbo ([Jim85]), motivated by questions of statistical mechanics, independently introduced a one-parameter deformation of the Hopf algebras $\mathbf{U}(\mathfrak{g})$ when $\mathfrak{g}$ is a semi-simple Lie algebra. The same definition works when $\mathfrak{g}$ is a Kac-Moody algebra, and was extended by Kang ([Kan95]) to Borcherds algebras. This deformation is denoted by $\mathbf{U}_q(\mathfrak{g})$; it is an algebra over the field of rational fractions in one variable, $\mathbf{C}(q)$, generated by the elements q^h for $h \in H$, e_i, f_i, $i \in I$, satisfying the relations

1. $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ for all $h, h' \in H$,
2. $q^h e_i = q^{(h,h_i)} e_i q^h$,
3. $q^h f_i = q^{-(h,h_i)} f_i q^h$,
4. $e_i f_j - f_j e_i = 0$ if $i \neq j$,
5. $e_i f_i - f_i e_i = \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$,
6. If $a_{i,i} > 0$, $\sum_{k+l=1-a_{i,j}} (-1)^k e_i^{(k)} e_j e_i^{(l)} = 0$, $\sum_{k+l=1-a_{i,j}} (-1)^k f_i^{(k)} f_j f_i^{(l)} = 0$ where for $k \in \mathbf{N}$, $e_i^{(k)} = \frac{e_i^k}{[k]!}$, $f_i^{(k)} = \frac{f_i^k}{[k]!}$, $[k]! = \prod_{m=1}^k [m]$, $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$.

Again, the algebra $\mathbf{U}_q(\mathfrak{g})$ admits a triangular decomposition

$$\mathbf{U}_q(\mathfrak{g}) = \mathbf{U}_q(\mathfrak{n}_-) \otimes \mathbf{U}_q(H) \otimes \mathbf{U}_q(\mathfrak{n}_+).$$

Like $\mathbf{U}(\mathfrak{g})$, $\mathbf{U}_q(\mathfrak{g})$ has a Hopf algebra structure. We must specify the comultiplication Δ, the antipode S and the counit ε. On the generators, we have

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes q^{-h_i} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + q^{h_i} \otimes f_i,$$

$$S(q^h) = q^{-h}, \quad S(e_i) = -e_i q^{h_i}, \quad S(f_i) = -q^{h_i} f_i,$$

$$\varepsilon(q^h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0.$$

There is a bialgebra together with a Hopf pairing defined by Lusztig ([Lus11]) from a Cartan matrix of Kac-Moody type, which we will denote by $\boldsymbol{f}$, which is more suitable than $\mathbf{U}_q(\mathfrak{n}_+)$ in geometric constructions. The Hopf algebras $\mathbf{U}_q(\mathfrak{n}_\pm)$ and $\boldsymbol{f}$ are isomorphic only as algebras. Their coalgebra structures differ but are nevertheless very similar. We recall the definition by generators and relations of the algebra $\boldsymbol{f}$ when the Cartan matrix is symmetric. Let $(H, (-,-), (h_i)_{1\leq i\leq n})$ be a Cartan data of Kac-Moody type. In particular, $(-,-)$ is a symmetric bilinear form on H, $(h_i, h_i) = 2$. The algebra $\boldsymbol{f}$ is an associative algebra over the field $\mathbf{Q}(q)$ generated by the elements θ_i, $1 \leq i \leq n$, with the quantum Serre relations

$$\sum_{k+l=1-a_{i,j}} (-1)^k \theta_i^{(k)} \theta_j \theta_i^{(l)}$$

for $i \neq j$ where for $1 \leq i \leq n$, $k \geq 0$, $\theta_i^{(k)} = \frac{\theta_i^k}{[k]!}$. The algebra $\boldsymbol{f}$ is $\mathbf{Z}^I$ graded if we set $\deg \theta_i = e_i$, the i-th vector of the canonical basis of $\mathbf{Z}^I$. Moreover, $\mathbf{Z}^I$ has a bilinear form $(-,-)$ satisfying $(e_i, e_j) = (h_i, h_j)$. The coalgebra structure is given by the comultiplication Δ, which can be expressed on the generators by the formula

$$\Delta(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i.$$

To have the compatibility between the multiplication and the comultiplication, we must consider on the tensor product $\boldsymbol{f} \otimes \boldsymbol{f}$ the product

$$(x \otimes y)(z \otimes w) = q^{(\deg y, \deg z)} (xz) \otimes (yw)$$

if x, y, z, w are homogenous elements. There is a second comultiplication on the algebra $\boldsymbol{f}$ obtained by conjugating Δ by the automorphism of $\mathbf{Q}$-algebras $\boldsymbol{f} \to \boldsymbol{f}$ acting on the coefficients by $f(q) \mapsto f(q^{-1})$. We denote by $\bar{\Delta}$ this second comultiplication. Geometrically, the existence of these two comultiplications are reflected in the fact that the restriction functor of Lusztig acting on constructible complexes is not Verdier self-dual. We now define two bilinear forms on $\boldsymbol{f}$, $(-,-)$ and $\{-,-\}$. They are characterized by the fact that they are Hopf pairings for Δ (resp. $\bar{\Delta}$) satisfying the equalities $(\theta_i, \theta_j) = \delta_{i,j} \frac{1}{1-q_i^{-2}}$ (resp. $\{\theta_i, \theta_j\} = \delta_{i,j} \frac{1}{1-q_i^2}$) for all i, j.

It is possible to define *integral forms* of $\mathbf{U}_q(\mathfrak{g})$, i.e. algebras over a subring A of $\mathbf{C}(q)$ such that the $\mathbf{C}(q)$-algebra obtained by extending the scalars is isomorphic to $\mathbf{U}_q(\mathfrak{g})$. Very often, we take $A = \mathbf{C}[q, q^{-1}]$. There are two integral forms which have proved to be of interest in the study of quantum groups: the integral form of Lusztig, $\mathbf{U}_q^L(\mathfrak{g})$, and the integral form of De Concini-Kac, $U_q^{DK}(\mathfrak{g})$. One of their interests is to allow to specialize the formal parameter q to a complex number.

The integral form of Lusztig, $\mathbf{U}_q^L(\mathfrak{g})$ is the $\mathbf{Z}[q, q^{-1}]$-subalgebra of $\mathbf{U}_q(\mathfrak{g})$ generated by the divided powers of the Chevalley generators while the integral form of De Concini-Kac is generated by the powers of the Chevalley generators. Only the integral form of Lusztig is interesting in our situation. For more details on its construction, the reader may refer to [Lus11].

2.3.3 Affinization of the quantum group

An interesting reference for this part is [CP94]. Given a Lie algebra $\mathfrak{g}$, we call affinization of the quantum group $\mathbf{U}_q(\mathfrak{g})$ a Hopf algebra which is a one-parameter deformation of the enveloping algebra of the loop algebra with values in $\mathfrak{g}$, $\mathbf{U}(\mathfrak{g}[t,t^{-1}])$. When $\mathfrak{g}$ is a semisimple Lie algebra, Drinfeld defined such a deformation. This deformation has two presentations. There is first the Kac-Moody type presentation obtained by considering the *affine* quiver associated with the semisimple Lie algebra ([Dri87b]), then there is the new presentation of Drinfeld ([Dri87a]). The latter can be formulated for any Kac-Moody algebra (and possibly for any Borcherds algebra). Nakajima ([Nak01]) or Grojnowski for the positive part ([Gro]) used quantum affine algebras associated with an arbitrary Kac-Moody algebra. To preserve the readability of this introduction, we only give the presentation of the positive part $\mathbf{U}_q(\hat{\mathfrak{n}}_+)$ of the algebra $\mathbf{U}_q(\hat{\mathfrak{g}})$. It is generated by the elements $e_{k,r}$, $k \in I, r \in \mathbf{Z}$ and the relations are expressed in a compact way thanks to the series

$$x_k^+(z) = \sum_{r=-\infty}^{+\infty} e_{k,r} z^{-r}$$

and have the following form

1. $(z - q^2 w) x_k^+(z) x_k^+(w) = (q^2 z - w) x_k^+(w) x_k^+(w)$,
2. $\prod_{p=1}^{-a_{k,l}} (z - q^{-a_{k,l}-2p} w) x_k^+(z) x_l^+(w) = \prod_{p=1}^{-a_{k,l}} (q^{-a_{k,l}-2p} z - w) x_l^+(w) x_k^+(z)$ if $k \neq l$,
3. $\sum_{\sigma \in \mathfrak{S}_b} \sum_{p=0}^{b} \binom{b}{p} x_k^+(z_{\sigma(1)}) \dots x_k^+(z_{\sigma(p)}) x_l^+(w) x_k^+(z_{\sigma(p+1)}) \dots z_k^+(z_{\sigma(b)}) = 0$ if $k \neq l$,

where $b = 1 - a_{k,l}$

Conjecture 2.3.1. *Let Q be a quiver that we assume without loops*[4]*. Then there exists a unique Lie algebra $\mathfrak{n}_B$ such that $\mathbf{U}_q(\hat{\mathfrak{n}}_Q)$ is a deformation of $\mathbf{U}(\mathfrak{n}_B[t,t^{-1}])$. Furthermore, $\mathfrak{n}_B$ is the positive part of a Borcherds algebra $\mathfrak{g}_B$.*

We can formulate the same conjecture for the whole quantum affine algebra $\mathbf{U}_q(\hat{\mathfrak{g}})$. This conjecture arises from the fact that for a semisimple Lie algebra $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, the quantum affine algebra defined as above is indeed a deformation of $\mathbf{U}(\mathfrak{n}_+)$, but this is no longer true for a general Kac-Moody algebra.

2.3.4 The Yangian of a Lie algebra

Drinfeld defined by generators and relations the *Yangian* of a semisimple Lie algebra $\mathfrak{g}$, $Y(\mathfrak{g})$. It is a one-parameter deformation of the enveloping algebra of the Lie algebra of polynomials with coefficients in $\mathfrak{g}$, $\mathbf{U}(\mathfrak{g}[t])$. The second presentation of the Yangian ([Dri87a]) gives formulas which can be adapted to any Cartan matrix of Kac-Moody type. As for quantum affine algebras, we might

[4] for simplicity

be able to adapt this presentation for Borcherds algebras. Let $A = (a_{i,j})_{1\leq i,j\leq n}$ be a symmetric Cartan matrix. We will call[5] *Yangian* of the Kac-Moody algebra $\mathfrak{g}_Q$ the associative algebra $Y_\hbar(\mathfrak{g}_Q)$ having as generators $X^\pm_{i,r}$, $H_{i,r}$, $1 \leq i \leq n$, $r \in \mathbf{N}$, satisfying relations

1. $[H_{i,r}, H_{j,s}] = 0$,
2. $[H_{i,0}, X^\pm_{j,s}] = \pm a_{i,j} X^\pm_{j,s}$,
3. $[H_{i,r+1}, X^\pm_{j,s}] - [H_{i,r}, X^\pm_{j,s+1}] = \pm\frac{\hbar}{2} a_{i,j}(H_{i,r}X^\pm_{j,s} + X^\pm_{j,s}H_{i,r})$,
4. $[X^+_{i,r}, X^-_{j,s}] = \delta_{i,j} H_{i,r+s}$,
5. $[X^\pm_{i,r+1}, X^\pm_{j,s}] - [X^\pm_{i,r}, X^\pm_{j,s+1}] = \pm\frac{\hbar}{2} a_{i,j}(X^\pm_{i,r}X^\pm_{j,s} + X^\pm_{j,s}X^\pm_{i,r})$,
6. $\sum_{\pi\in\mathfrak{S}_m}[X^\pm_{i,r_{\pi(1)}}, [X^\pm_{i,r_{\pi(2)}}, \dots, [X^\pm_{i,r_{\pi(m)}}, X^\pm_{j,s}]\dots]] = 0$ for $m = 1 - a_{i,j}$ and $r_1, \dots, r_m \geq 0$.

The Yangian $Y_\hbar(\mathfrak{g}_Q)$ contains a copy of the enveloping algebra $\mathbf{U}(\mathfrak{g}_Q)$, the subalgebra generated by $H_{i,0}$ and $X^\pm_{i,0}$ for $1 \leq i \leq n$. We also have a triangular decomposition

$$Y_\hbar(\mathfrak{g}_Q) = Y_\hbar(\mathfrak{n}_{Q,-}) \otimes Y_\hbar(\mathfrak{h}_Q) \otimes Y_\hbar(\mathfrak{n}_{Q,+}).$$

When Q is a quiver of finite type, the Yangian $Y_\hbar(\mathfrak{g}_Q)$ is a deformation of $\mathbf{U}(\mathfrak{g}_Q[u])$. In the other cases, the Yangian is much bigger than just a deformation of $\mathbf{U}(\mathfrak{g}_Q[u])$. In this perspective, we can formulate the following conjecture.

Conjecture 2.3.2. *Let Q be a quiver that we assume without loops*[6]*. Then there exists a unique Lie algebra $\mathfrak{g}_B$ with positive part $\mathfrak{n}_B$ such that $Y_\hbar(\mathfrak{n}_{Q,+})$ is a deformation of $\mathbf{U}(\mathfrak{n}_B[t])$. Furthermore, $\mathfrak{g}_B$ is a Borcherds algebra.*

Conjecture 2.3.2 can of course be formulated for the whole Yangian. Moreover, the Borcherds algebras $\mathfrak{g}_B$ of the conjectures 2.3.1 and 2.3.2 must match.

2.4 The Hall algebra of a quiver

In this section, we focus on the Hall algebra of a quiver. Let $Q = (I, \Omega)$ be a quiver, that is a directed graph having a finite number of vertices and edges, and possibly loops. We denote by I the set of vertices of Q and by Ω the set of its edges. We fix a finite field $\mathbf{F}_q$. The hereditary finitary category that we consider is $\mathrm{Rep}_Q(\mathbf{F}_q)$, the category of finite dimensional representations of Q over the field $\mathbf{F}_q$. We denote by $\mathbf{H}_{Q,\mathbf{F}_q}$ the Hall algebra of $\mathrm{Rep}_Q(\mathbf{F}_q)$. Recall that this is a twisted Hopf algebra (Section 2.2). It was Ringel who initiated their study in the article [Rin90b]. The first major result, due to Ringel and Green, is as follows (cf. [Sch12a, Theorem 3.16]).

[5] The algebra that we are going to define is not a deformation of $\mathbf{U}(\mathfrak{g}[z])$ in general. If $\mathfrak{g}$ is a semisimple Lie algebra, it is.

[6] for simplicity

Theorem 2.4.1 (Ringel, Green)**.** *There exists a unique morphism of bialgebras*

$$\begin{array}{ccc} \boldsymbol{f}_{\sqrt{q}} & \to & \mathbf{H}_{Q,\mathbf{F}_q} \\ \theta & \mapsto & [S_i] \end{array}$$

This morphism is injective. It is surjective if and only if Q is of finite type.

The image of this morphism is denoted $\mathbf{H}^{\mathrm{sph}}_{Q,\mathbf{F}_q}$ and called *spherical Hall algebra* of the quiver Q. The notation $\boldsymbol{f}_{\sqrt{q}}$ indicates the specialization at $\sqrt{q}$ of the algebra $\boldsymbol{f}$. This morphism is bijective if and only if the quiver is of finite type. We can then ask what is the structure of $\mathbf{H}_{Q,\mathbf{F}_q}$. Sevenhant and Van den Bergh answered this question by showing that it is isomorphic to the positive part of the specialization at $\sqrt{q}$ of the quantized enveloping algebra of a Borcherds algebra for which the role of the Chevalley generators is played by the cuspidal functions. We will come back to this result in Section 2.4.3.

2.4.1 Characters of Lie algebras and of their enveloping algebras

Plethystic exponential

We can say that *the plethystic exponential is the dimensional decategorification of the symmetric power*. What we mean by this will become clearer in Section 2.4.1. We take a more down-to-earth point of view here.

Let M be a monoid with neutral element 0. Assume for simplicity that for each $m \in M$, the set $\{(m_1, m_2) \in M \times M \mid m_1 + m_2 = m\}$ is finite. We denote by $\mathbf{Q}[[M]]$ the set of formal power series with coefficients in $\mathbf{Q}$ of the form

$$f = \sum_{m \in M} f_m z^m,$$

$f_m \in \mathbf{Q}$. The set $\mathrm{supp}(f) = \{m \in M \mid f_m \neq 0\}$ is called the support of f. The set $\mathbf{Q}[[M]]$ is naturally a ring. We denote by $\mathbf{Q}[[M]]_+$ *the augmentation ideal* of $\mathbf{Q}[[M]]$, that is the set of formal power series whose constant term (the coefficient of $z^0 = 1$) is zero. The *plethystic exponential* is then the map

$$\begin{array}{cccc} \mathrm{Exp}_z & : & \mathbf{Q}[[M]]_+ & \to & 1 + \mathbf{Q}[[M]]_+ \\ & & f & \mapsto & \exp\left(\sum_{l \geq 1} \frac{1}{l} f(z^l)\right) \end{array}.$$

We will sometimes need the plethystic exponential for elements of $\mathbf{Q}[q][[M]]$. This is within our framework if we replace M by $M \times \mathbf{N}$, given the inclusion $\mathbf{Q}[q][[M]] \to \mathbf{Q}[[M \times \mathbf{N}]]$. In this case, the associated plethystic exponential will be denoted $\mathrm{Exp}_{z,q}$ to make the distinction. The plethystic exponential is an isomorphism of groups ($\mathbf{Q}[[M]]_+$ is naturally an additive group; $1 + \mathbf{Q}[[M]]_+$ is a multiplicative group). Its inverse is given by the *plethystic logarithm* Log_z (or $\mathrm{Log}_{z,q}$ in the case of $\mathbf{Q}[q][[M]]$), given by the formula

$$\begin{array}{cccc} \mathrm{Log}_z & : & 1 + \mathbf{Q}[[M]]_+ & \to & \mathbf{Q}[[M]]_+ \\ & & f(z) & \mapsto & \sum_{l \geq 1} \frac{\mu(l)}{l} \log(f)(z^l) \end{array}$$

where μ is the Moebius function.

Character of the enveloping algebra of a Lie algebra

Let M be a monoid and V a M-graded vector space. We can decompose M into its graded components: $V = \bigoplus_{m \in N} V_m$. We assume that each graded component V_m for $m \in M$ is of finite dimension. We then have the character of V:

$$\mathrm{ch}(V) = \sum_{m \in N} (\dim V_m) z^m \in \mathbf{Z}[[M]]$$

is in the group algebra of M and where the element $m \in M$ induces the element $z^m \in \mathbf{Z}[[M]]$. We denote by $\mathrm{supp}(V) = \{m \in M \mid V_m \neq 0\}$ the support of V. Then the symmetric power of V has character

$$\mathrm{ch}(S(V)) = \sum_{m \in M} (\dim S(V)_m) z^m = \mathrm{Exp}_z\left(\mathrm{ch}(V)\right). \tag{2.4.1}$$

The category of M-graded vector spaces M whose homogeneous components are of finite dimension, $M - vect$, has Grothendieck group $\mathbf{Z}[[M]]$. There exists a functor from the subcategory $M - vect_+$ from M-graded vector spaces whose component of degree 0 is zero to the subcategory $M - vect_1$ of graded vector spaces whose component of degree 0 is of dimension 1, given by the symmetric power. On the corresponding subsets of the Grothendieck group, the induced map is precisely the plethystic exponential. This explains the first sentence of Section 2.4.1.

Now let $\mathfrak{g}$ be a Lie algebra. We suppose that $\mathfrak{g}$ is graded by a monoid M satisfying the condition given in Section 2.4.1. The Poincaré-Birkhoff-Witt theorem ensures that the graded character of $\mathbf{U}(\mathfrak{g})$ is the same as that of the symmetric power $S\mathfrak{g}$. It is therefore given by the formula (1.4.1) applied to $V = \mathfrak{g}$.

2.4.2 Character of the Hall algebra of a quiver

It follows immediately from its definition that the character of the Hall algebra of the quiver Q over the finite field $\mathbf{F}_q$ is given by the family of functions counting in each dimension the number of isomorphism classes of representations of Q of this dimension. More precisely, for $\mathbf{d} \in \mathbf{N}^I$, we denote by $M_{Q,\mathbf{d}}(q)$ the number of isomorphism classes of representations of Q over $\mathbf{F}_q$ of dimension $\mathbf{d}$, $I_{Q,\mathbf{d}}(q)$ the number of isomorphism classes of *indecomposable* representations of Q over $\mathbf{F}_q$ of dimension $\mathbf{d}$ and $A_{Q,\mathbf{d}}(q)$ the number of isomorphism classes of *absolutely indecomposable* representations of Q over $\mathbf{F}_q$ of dimension $\mathbf{d}$. Then Kac [Kac80a, Kac83, Kac82, Kac80b] showed that the functions $M_{Q,\mathbf{d}}$, $I_{Q,\mathbf{d}}$ and $A_{Q,\mathbf{d}}$ of q are polynomials in q, Hausel, Letellier and Rodriguez-Villegas ([HLRV13]) showed that the polynomial $A_{Q,\mathbf{d}}$ has nonnegative integer coefficients and Hausel ([Hau10]) proved Kac's conjecture interpreting the constant coefficient of the polynomial $A_{Q,\mathbf{d}}$ as the multiplicity of $\mathbf{d}$ as the root of the Kac-Moody algebra associated with the quiver Q: $A_{Q,\mathbf{d}}(0) = \dim \mathfrak{g}[\mathbf{d}]$. It emerges from all this that the family of polynomials $A_{Q,\mathbf{d}}$ is the one which has the best properties. Moreover, if we write

$$\operatorname{ch}\mathbf{H}_{Q,\mathbf{F}_q}=\sum_{\mathbf{d}\in\mathbf{N}^I}\dim_{\mathbf{C}}\mathbf{H}_{Q,\mathbf{F}_q}[\mathbf{d}]z^{\mathbf{d}}\in\mathbf{C}[[z_i,i\in I]],$$

we have the equalities

$$\operatorname{ch}\mathbf{H}_{Q,\mathbf{F}_q}=\sum_{\mathbf{d}\in\mathbf{N}^I}M_{Q,\mathbf{d}}(q)z^{\mathbf{d}}=\operatorname{Exp}_z\left(\sum_{\mathbf{d}\in\mathbf{N}^I}I_{Q,\mathbf{d}}(q)z^{\mathbf{d}}\right)=\operatorname{Exp}_{z,q}\left(\sum_{\mathbf{d}\in\mathbf{N}^I}A_{Q,\mathbf{d}}(q)z^{\mathbf{d}}\right).$$

The first equality occurs by definition, the second by the Krull-Schmidt property of the category of quiver representations and the third by a Galois descent argument for quiver representations. (cf. [BS18, Lemma 3.1]).

The plethystic exponential involving the polynomials $I_{Q,\mathbf{d}}$ indicates that the character of $\mathbf{H}_{Q,\mathbf{F}_q}$ is exactly the character of the enveloping algebra of a Lie algebra $\mathfrak{n}$ whose character would be given by $\operatorname{ch}\mathfrak{n}=\sum_{\mathbf{d}\in\mathbf{N}^I}I_{Q,\mathbf{d}}(q)z^{\mathbf{d}}$. The plethystic exponential involving the Kac polynomials $A_{Q,\mathbf{d}}$ interprets $\operatorname{ch}\mathbf{H}_{Q,\mathbf{F}_q}$ as the specialization at q of the character of the enveloping algebra of an $\mathbf{N}$-graded Lie algebra $\mathfrak{n}_{\mathbf{N}}$ whose $\mathbf{N}$-graded character would be $\operatorname{ch}\mathfrak{n}_{\mathbf{N}}=\sum_{\mathbf{d}\in\mathbf{N}^I}A_{Q,\mathbf{d}}(q)z^{\mathbf{d}}\in\mathbf{Q}[q][[z_i: i\in I]]$, where q is understood as a formal variable.

2.4.3 Generators and relations for the Hall algebra of a quiver

The Hall algebra of a quiver is a twisted $\mathbf{N}^I$-graded Hopf algebra, $\mathbf{H}_{Q,\mathbf{F}_q}$, with comultiplication Δ. We define the set of cuspidal functions of $\mathbf{H}_{Q,\mathbf{F}_q}$:

$$\mathbf{H}_{Q,\mathbf{F}_q}^{\text{cusp}}=\{f\in\mathbf{H}_{Q,\mathbf{F}_q}\mid\Delta(f)=f\otimes 1+1\otimes f\}.$$

It is an $\mathbf{N}^I$-graded vector subspace of $\mathbf{H}_{Q,\mathbf{F}_q}$, whose graded components have a polynomial dimension in q ([BS18, Theorem 1.1]). It is a minimal generating subspace of $\mathbf{H}_{Q,\mathbf{F}_q}$ ([SVDB01]) and if we choose an arbitrary graded orthonormal basis $(f_j)_{j\in J}$ of this space, we obtain a presentation of $\mathbf{H}_{Q,\mathbf{F}_q}$ by generators and relations. We first write $\nu=\sqrt{q}$; $A=(a_{i,j})_{i,j\in J}$ where $a_{i,j}=(\dim f_i,\dim f_j)$ the Cartan matrix, where $(-,-)$ is the symmetrized Euler form. Then the Hall algebra $\mathbf{H}_{Q,\mathbf{F}_q}$ is isomorphic to the associative algebra generated by $(f_j)_{j\in J}$ with the relations

1. For any $i,j\in J$, if $a_{i,j}=0$, then $f_if_j=f_jf_i$,
2. For any $i,j\in J$, if $a_{i,i}=2$, then

$$\sum_{i=0}^{1-a_{i,j}}(-1)^l\begin{Bmatrix}1-a_{i,j}\\ l\end{Bmatrix}f_i^lf_jf_i^{1-a_{i,j}+l}=0,$$

where, for two integers r and s, $\begin{Bmatrix}s\\ r\end{Bmatrix}=\prod_{u=1}^r\frac{\nu^{u+s-r}-\nu^{-(u+s-r)}}{\nu^u-\nu^{-u}}$.

This presentation of the Hall algebra $\mathbf{H}_{Q,\mathbf{F}_q}$ indicates that $\mathbf{H}_{Q,\mathbf{F}_q}$ is the specialization at $\nu=$

$\sqrt{q}$ of the quantized enveloping algebra of a Borcherds algebra associated with the Cartan data $(\mathbf{Z}^J, (-,-))$.

It is hard to understand the cuspidal functions of an arbitrary quiver. The cuspidal functions of finite type quivers are easily identified: they are the characteristic functions of isomorphism classes of simple representations. This is a consequence of the fact that they are a minimal generating subspace of the Hall algebra and of Theorem 2.4.1. For the Jordan quiver, cuspidal functions can be easily described thanks to the explicit isomorphism with the algebra of symmetric functions. For affine quivers, Auslander-Reiten theory allows a description of cuspidal functions. This is the subject of Chapter 3. Apart from these cases, we have no general classification of cuspidal functions, only partial results.

2.5 Geometric models for the Yangian and the quantum affine algebra associated with a quiver

Geometric constructions such as cohomological or K-theoretic Hall algebras are not considered in this thesis. However, since they constitute part of the motivations and may provide geometric counterparts of combinatorial properties of generating series, we very briefly summarize the constructions and give some references.

In order to better understand quantum groups (quantum groups, quantum affine algebras, Yangians), it has proved powerful to obtain geometric constructions of them. These geometric constructions also naturally provide actions on other objects of geometric nature, making it possible to study their representations. The underlying formalism is that of correspondences, developed by Chriss and Ginzburg in [CG97]. There are many works going in this direction: [Var00, KS11, SV13a, SV20, SV17, Soi16, YZ18, YZ20, MO19].

2.5.1 The cohomological Hall algebra associated with a quiver

The constructions of this section are due to Schiffmann-Vasserot, Kontsevich-Soibelman, Davison, Yang-Zhao... To a quiver $Q=(I,\Omega)$, we associate the double quiver $\overline{Q}$: each edge a of Q is replaced by a pair of edges a, a^* of opposite directions. The preprojective algebra of Q is the quotient of the path algebra of $\overline{Q}$ by the two-sided ideal generated by the element $\sum_{a\in\Omega}[a,a^*]$. In each dimension $\mathbf{d}\in\mathbf{N}^I$, this relation defines a closed $G_{\mathbf{d}}$-invariant subscheme of the representation space:

$$E_{\Pi_Q,\mathbf{d}} := \left\{(x_a, x_{a^*}) \in E_{\overline{Q},\mathbf{d}} \mid \sum_{a\in\Omega}[x_a, x_{a^*}] = 0\right\}.$$

The quotient by the action of $G_{\mathbf{d}}$ gives the stack of representations of the preprojective algebra, $\mathcal{M}_{\Pi_Q} := E_{\Pi_Q,\mathbf{d}}/G_{\mathbf{d}}$. The associated cohomological Hall algebra is a structure of $H_T^*(\mathrm{pt})$-algebra on the direct sum

$$\mathrm{CoHA}(Q) := \bigoplus_{\mathbf{d}\in\mathbf{N}^I} H_*^T(\mathcal{M}_{\Pi_Q,\mathbf{d}})$$

of T-equivariant Borel-Moore homology groups (for the natural action of a torus T, which encodes the deformation parameters), [SV20]. There is a version of cohomological Hall algebra where we consider rather the stack of *nilpotent* (or semi-nilpotent) representations of the preprojective algebra. After localization (extension of scalars to $\mathrm{Frac}(H_T^*(\mathrm{pt}))$, these algebras become isomorphic. This algebra admits an alternative construction in terms of quiver with potential (due to Kontsevich-Soibelman, [KS11]). The link between the two constructions is made through *dimensional reduction*, which is a way to compare the cohomological Hall algebra of a Calabi-Yau category of dimension 3 to the cohomological Hall algebra of a Calabi-Yau category of dimension 2 under certain conditions ([RS17, YZ20]). When $\mathrm{CoHA}(Q)$ is realized as the cohomological Hall algebra of a Calabi-Yau category of dimension 3, we have the *perverse filtration* ([DM20]) which allowed Ben Davison and Sven Meinhardt to extract the BPS Lie algebra $\mathfrak{g}_{BPS}$. The realization of $\mathrm{CoHA}(Q)$ as a cohomological algebra of a Calabi-Yau category of dimension 2 gives another perverse filtration, which allows to find the enveloping algebra $\mathbf{U}(\mathfrak{g}_{BPS})$ of the BPS Lie algebra in the cohomological Hall algebra ([Dav20]).

2.5.2 The K-theoretic Hall algebra associated to a quiver

The idea of considering the K-theoretic Hall algebra of the preprojective algebra of a quiver is due to Grojnowski [Gro]. In this unpublished note, he defines an algebra structure on the direct sum of K-theory groups

$$\mathrm{KHA}(Q)=\bigoplus_{\mathbf{d}\in\mathbf{N}^I}K^{\mathbf{C}^*}(\mathcal{M}_{\Pi_Q,\mathbf{d}})$$

where $\mathbf{C}^*$ acts with weight 1 on the cotangent fibers, and he shows that there is an induced algebra structure on the K-theory of the nilpotent variety

$$\mathrm{KHA}_{\Lambda}(Q):=\bigoplus_{\mathbf{d}\in\mathbf{N}^I}K^{G_{\mathbf{d}}\times\mathbf{C}^*}(\Lambda_{\mathbf{d}}).$$

He sketches some vague arguments to show that the positive part of the quantum affine algebra associated with Q (defined thanks to the new presentation of Drinfeld, Section 2.3.3) is isomorphic to $\mathrm{KHA}_{\Lambda}(Q)$ and to show the injectivity, which uses a kind of co-product and a non-degenerate symmetric bilinear form. The existence of a Borcherds algebra $\mathfrak{g}_B$ with positive part $\mathfrak{n}_B$ such that $\mathrm{KHA}_{\Lambda}(Q)$ is a deformation of the enveloping algebra of loops with values in $\mathfrak{n}_B$, $\mathbf{U}(\mathfrak{n}_B[t^{\pm1}])$, is expected. The K-theoretic Hall algebra of the quiver with g loops appears in the paper of Schiffmann and Vasserot [SV10]. It allows an understanding of the geometric Langlands correspondence for genus g curves in the neighbourhood of the trivial local system. In the paper [VV20], Varagnolo and Vasserot prove in a precise way the isomorphism with Drinfeld's positive part of the affine quantum group associated with the Kac-Moody algebra of the quiver of finite or affine type Q. In general, they prove the surjectivity of the morphism from the affine quantum group to the K-theoretic Hall algebra. The case of SL(2) was considered in details in [SVV19] where moreover the link with the quiver Hecker algebra of type $A_1^{(1)}$ is made. Recently, Padurariu has developed the theory of K-theoretic Hall algebras for quivers with potential in [Pad19]. In [VV20], Varagnolo and Vasserot use Padurariu's approach to construct super quantum groups in terms of K-theoretic Hall algebras. Let us also mention the work of Okounkov and Smirnov [OS16] who propose a construction of the quantum affine algebra associated with a quiver in terms of stable K-theoretic envelopes.

2.5.3 Stable envelopes

Okounkov, Maulik, Smirnov and Aganagic have developed an approach to algebras that can be associated with a quiver using the formalism of R-matrices built from *stable envelopes*, [MO19, OS16, AO21]. Their works find their source in the realization of representations of Kac-Moody algebras in the cohomology of quiver varieties due to Nakajima [Nak98]. In this way, they obtain a Yangian $\mathbb{Y}$ and a Lie algebra $\mathfrak{g}_{MO}$ so that $\mathbb{Y}$ is a deformation of $\mathbf{U}(\mathfrak{g}_{MO}[u])$. Schiffmann and Vasserot conjecture that the Yangian obtained in [MO19] is isomorphic to the Yangian of [SV20].

2.5.4 Character of the BPS Lie algebra

The BPS Lie algebra of Section 2.5.1 is $\mathbf{N}^I \times \mathbf{N}$-graded and its character is given by the generating series of Kac poynomials:

$$\operatorname{ch} \mathfrak{g}_{BPS} = \sum_{\mathbf{d} \in \mathbf{N}^I} A_{Q,\mathbf{d}}(q) z^{\mathbf{d}}. \tag{2.5.1}$$

In [Oko], Okounkov conjectures that the character of the positive part of the Lie algebra $\mathfrak{g}_{MO}$ of Section 2.5.3 is also given by the same generating series. In the same way, we can expect that the characters of the Lie algebras $\mathfrak{n}_B$ of Conjectures 2.3.1 and 2.3.2 are given by the same generating series. Therefore, we have several points of view on the generating series (2.5.1) (Sections 2.4.2, 2.5.1, 2.3.3, 2.3.4) and many ways to construct Lie algebras with the (conjecturally) same character. For finite type and affine quivers, all of this is known (all conjectures are actually theorems). In general, this is a challenging question to compare these constructions. A question for future work is also to understand the implications of the asymptotic properties of Kac polynomials (Theorem 2.7.2) in the geometric constructions. For example, one may study the convergence of the R-matrix of Maulik-Okounkov when the number of arrows of the quiver grows.

2.6 The Hall algebra of an elliptic curve

2.6.1 The Hall algebra of constructible functions

We saw at the beginning of this introduction that the categories of coherent sheaves on a smooth projective algebraic curve defined over a finite field satisfy the conditions required to define their Hall algebra. The study in this new situation was initiated by Kapranov [Kap97]. It is an object which has been the subject of numerous works: [BK01, Sch06, BS12, KSV17, Sch12b, SV12]. In particular, Schiffmann gave an explicit description of the spherical subalgebra for weighted projective projective lines as well as for elliptic curves. In the latter case, we get what is now called the *elliptic Hall algebra*. It is an algebra of considerable interest in representation theory, and its spherical subalgebra is a deformation of the Hopf algebra of diagonally symmetric functions.

$$\mathbf{\Lambda}^+ = \mathbf{C}[x_1^{\pm 1}, \ldots, y_1, \ldots]^{\mathfrak{S}_\infty}.$$

2.6.2 The elliptic Hall algebra

Let X be an elliptic curve defined over a finite field $\mathbf{F}_q$. The hereditary category we are considering here is $\mathrm{Coh}(X)$, the category of coherent sheaves on X. For the detailed construction of the Hall algebra $\mathbf{H}_X$ of $\mathrm{Coh}(X)$, we refer to [BS12, Section 2]. We give here only some of its main characteristics. The algebra $\mathbf{H}_X$ is $\mathbf{Z}^+$-graded, where

$$\mathbf{Z}^+ = \{(r, d) \in \mathbf{Z}^2 \mid r \geq 1 \text{ or } r = 0, d \geq 0\}.$$

If $(r, d) \in \mathbf{Z}^+$, $\mathrm{Coh}(X)_{(r,d)}/\sim$ denotes the set of isomorphism classes of coherent sheaves on X of rank r and degree d. The elements of $\mathbf{H}_X[(r, d)]$ are the functions $\mathrm{Coh}(X)_{(r,d)}/\sim \to \mathbf{C}$. Considering only semi-stable coherent sheaves, we obtain a subset $\mathrm{Coh}(X)^{ss}_{(r,d)}/\sim$. The characteristic function of this subset is denoted by $1^{ss}_{(r,d)}$. The spherical subalgebra of $\mathbf{H}_X$ is the subalgebra generated by $1^{ss}_{(r,d)}$ for $(r, d) \in \mathbf{Z}^+$, $r \leq 1$. It is a topological sub-bialgebra of $\mathbf{H}_X$ and (as we have already noticed in Section 2.2), to define the co-product, it is necessary to consider a completion). It is possible to define a *generic elliptic Hall algebra* $\boldsymbol{\mathcal{E}}_{\mathbf{R}}$ over the ring $\mathbf{R} = \mathbf{C}[\sigma^{\pm 1/2}, \bar{\sigma}^{\pm 1/2}]$, where $\sigma, \bar{\sigma}$ are formal parameters playing the role of the eigenvalues of the Frobenius action on $H^1(X, \overline{\mathbf{Q}}_\ell)$. The construction of the elliptic Hall algebra in terms of perverse sheaves on the stack of coherent sheaves is given in [Sch12b]. This construction is inspired by the similar construction of Lusztig for the quivers [Lus91]. See also [SV11] for the link with objects of deep representation theoretic interest like MacDonald polynomials and Cherednik algebras.

2.7 Results

In this section, we present the main results of this thesis, which will be detailed in the following chapters. This thesis is made up of the following four articles.

1. Isotropic cuspidal functions in the Hall algebra of a quiver, to appear in Int. Math. Res. Not., [Hen19],

2. Asymptotic behaviour of Kac polynomials, accepted by Experimental Mathematics, [Hen20a],

3. Microlocal characterization of Lusztig sheaves for affine and negative quivers, [Hen20b],

4. Perverse sheaves with nilpotent singular support on the stack of coherent sheaves on an elliptic curve, [Hen21].

2.7.1 Cuspidal functions of affine quivers

We briefly describe the content of the article [Hen19], that is of Chapter 3 of this thesis. We saw in Section 2.4.3 that the cuspidal functions of a quiver form a minimal generating subspace. In this first work, we determine the cuspidal functions of an affine quiver. This calculation is done in two steps. First, we define *the regular Hall algebra*. It is the Hall algebra of the category $\mathrm{Rep}_Q^{\mathcal{R}}(\mathbf{F}_q)$ of regular representations of the quiver Q (defined thanks to Auslander-Reiten theory). This category has the advantage of being a product indexed by the closed points of the projective line $\mathbf{P}^1_{\mathbf{F}_q}$ of categories equivalent to either the category of nilpotent representations of the Jordan quiver on the residue field of the point, or, for a finite number of points of degree 1, to the category of representations of a cyclic quiver on $\mathbf{F}_q$. The structure of the Hall algebras of these categories is known: the Hall algebra of the category of nilpotent representations of the Jordan quiver is isomorphic to MacDonald algebra of symmetric functions and is often called *classical Hall algebra* ([Mac15]) and the Hall algebra of a cyclic quiver has a structure described in [Sch00]. We deduce from this the structure of the Hall algebra $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ of the category $\mathrm{Rep}_Q^{\mathcal{R}}(\mathbf{F}_q)$. In particular, we obtain an explicit description of the space of cuspidal functions of $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$, which we denote by $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}} = \bigoplus_{\mathbf{d}\in\mathbf{N}^I}\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[\mathbf{d}]$. Basic properties of cuspidal functions imply that $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}]$ is non-zero if and only if $\mathbf{d}\in\{e_i : i\in I\}\cup\mathbf{N}_{\geq 1}\delta$, where δ is the indivisible imaginary root of the quiver Q. Moreover, if $r\geq 1$ is an integer, we can show that the support of a cuspidal function $f\in\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]$ is regular (in the sense that f takes 0 as value on representations for which one of the direct factors is not regular), so that we have the inclusion

$$\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]\subset\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[r\delta].$$

In addition, we have explicit formulas for the dimensions of these two spaces. These formulas indicate that $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]$ is a hyperplane of $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[r\delta]$. We also have a natural linear form on $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[r\delta]$ which is the integration of functions on the moduli space:

$$\begin{array}{cccc}\int & : & \mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[r\delta] & \to & \mathbf{C}\\ & & f & \mapsto & \sum_{[M]\in\mathrm{Rep}_Q^{\mathcal{R}}(\mathbf{F}_q)}\frac{f([M])}{|\mathrm{Aut}(M)|}.\end{array}$$

Then, we have the main theorem of 3.

Theorem 2.7.1. *The kernel of the linear form $\int$ coincides with the space of cuspidal functions of the quiver Q in dimension $r\delta$, $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]$.*

The strategy of proof is relatively simple. With the preparatory work exposed before the statement of the theorem, we have to show that $\int$ is non-trivial linear form on $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[r\delta]$. This is done by showing that $\int$ is represented for Green's scalar product by the constant function $1_{r\delta}$ and by using the fact that $1_{r\delta}$ is in the composition algebra of the quiver Q (which is true for any cyclic quiver).

2.7.2 Asymptotic behaviour of Kac polynomials

In this section, we describe some aspects of the article [Hen20a], which is Chapter 4 of this thesis. Kac polynomials play a considerable role in the study of algebras associated with a quiver, in particular to determine their character (Section 2.4.2). The wish to understand how the Hall algebra of a quiver evolves when we modify the quiver led us to consider the following problem: What can we say about the sequence of polynomials $(A_{Q_{\underline{n}},\mathbf{d}}(q))$ when $Q = (I, \Omega)$ is a quiver, $\underline{n} \in \mathbf{N}^{\Omega}$ is a vector of multiplicities and $Q_{\underline{n}}$ denotes the quiver having the same set of vertices as Q but for which each edge $\alpha \in \Omega$ is replaced by n_α edges, connecting the same vertices as α? In Chapter 4, we provide some answers and formulate computer-supported conjectures. The main theorem of that chapter is the following (Theorem 4.3.2).

Theorem 2.7.2. *Soit $\underline{m} = (m_\alpha)_{\alpha\in\Omega} \in (\mathbf{N}_{\geq 1} \cup \{+\infty\})^{\Omega}$.*

1. *Let Q be a quiver without loops and $\mathbf{d} \in \mathbf{N}^I$. Then, for all $\underline{n} \in \mathbf{N}^{\Omega}$ the valuation of the polynomial $A_{Q_{\underline{n}},\mathbf{d}}$ is $v_{\underline{n}} = 0$ if $A_{Q_{\underline{n}},\mathbf{d}} \neq 0$. The sequence of polynomials*
$$A_{Q_{\underline{n}},\mathbf{d}}(q) \in \mathbf{N}[q]$$
converges in $\mathbf{N}[[q]]$ when $\underline{n} \to \underline{m}$. Moreover, the limit is the power series expansion at $q = 0$ of a rational fraction.

2. *Let Q be an arbitrary quiver and $\underline{r} \in \mathbf{N}^{\Omega}_{\geq 1} \setminus \{0\}$. For $s \in \mathbf{N}$, we define $v_{s\underline{r}}$ the valuation of the polynomial $A_{Q_{s\underline{r}},\mathbf{d}}(q)$. Then the sequence of polynomials*
$$\frac{A_{Q_{s\underline{r}},\mathbf{d}}(q)}{q^{v_{s\underline{r}}}} \in \mathbf{N}[q]$$
converges in $\mathbf{N}[[q]]$ when $s \to +\infty$ and the limit is the power series expansion at $q = 0$ of a rational fraction.

To prove this result, we use the explicit formula of the generating series of Kac polynomials of Hua ([Hua00], see also [HLRV13]). One of our conjectures concerns the valuation of Kac polynomials in the presence of loops.

2.7.3 Microlocal study of perverse sheaves on the representation stacks of quivers

Here we give the main results of [Hen20b] (Chapter 5). Motivated by the geometric aspects of Hall algebras, and in the hope of finding an interpretation of cuspidal functions in terms of perverse sheaves, we studied Lusztig sheaves and their interaction with the Lusztig nilpotent variety. Let $Q = (I, \Omega)$ be a quiver. Inspired by Ringel's construction of the positive part of the quantum group associated with a quiver ([Rin90b]), Lusztig defined a category of perverse sheaves $\mathcal{P}$ on the representation spaces $E_{\mathbf{d}}$, $\mathbf{d} \in \mathbf{N}^I$. We let

$$E_{\mathbf{d}} = \bigoplus_{\alpha\in\Omega} \mathrm{Hom}(\mathbf{C}^{\mathbf{d}_{s(\alpha)}}, \mathbf{C}^{\mathbf{d}_{t(\alpha)}})$$

be the representation space of Q in dimension $\mathbf{d}$. This is an affine space on which the reductive group

$$G_{\mathbf{d}} = \prod_{i\in I} \mathrm{GL}_{\mathbf{d}_i}$$

acts. The category $\mathcal{P}$ is semisimple, and the simple perverse sheaves of $\mathcal{P}$ are the perverse sheaves appearing as simple constituents of the inductions of constant sheaves on the trivial representation spaces (in dimension $e_i, i \in I$), see for example [Lus91, Lus11, Sch09]. The additive Grothendieck group of the triangulated category obtained from this abelian category has the structure of a $\mathbf{Z}[\nu, \nu^{-1}]$-algebra (where ν is an indeterminate) induced by the induction of constructible sheaves, and is isomorphic to the integral form of the positive part of the quantum group. The classes of simple perverse sheaves give the canonical basis and when this construction is carried out over a finite field, the Frobenius traces of constructible sheaves make the link with the constructible Hall algebra ([Lus98]). There are now powerful ways to understand the canonical basis using KLR (or quiver-Hecke)-algebras and quiver varieties, see for example [KL09, KL11, VV11, VV03] and references therein. Moreover, Lusztig defined the nilpotent variety $\Lambda = \bigcup_{\mathbf{d} \in \mathbf{N}^I} \Lambda_{\mathbf{d}}$, a closed subvariety of the cotangent bundle of the representation space $T^*E_{\mathbf{d}}$ from the quiver ([Lus91]). The singular supports of the simple perverse sheaves considered by Lusztig are unions of some of the irreducible components of the nilpotent variety Λ. Lusztig used the free $\mathbf{Z}$-module generated by the irreducible components of Λ to define the *semicanonical basis* of the enveloping algebra of the positive part of the Kac-Moody algebra associated with Q ([Lus00]). Moreover, Kashiwara and Saito proved in [KS97] that the characteristic cycle map induces a bijection between the set of isomorphism classes of simple perverse sheaf of $\mathcal{P}$ and the set of irreducible components of Λ. It is natural to ask what is the structure of the category of perverse sheaves whose singular support is contained in the nilpotent variety. In Chapter 5, we answer this question for affine quivers. We also extend this question to the case of quivers with loops, for which we must consider a suitable nilpotent variety (the semi-nilpotent variety). We prove a similar result for quivers having a vertex and at least two loops. The two main theorems of this chapter are as follows (Theorems 5.1.1 and 5.1.3).

Theorem 2.7.3. *Let Q be a finite or affine type quiver. Let $\mathscr{F} \in \mathrm{Perv}_{G_{\mathbf{d}}}(E_{\mathbf{d}})$ be a simple $G_{\mathbf{d}}$-equivariant perverse sheaf such that $SS(\mathscr{F}) \subset \Lambda_{\mathbf{d}}$. Then $\mathscr{F}$ is a Lusztig perverse sheaf.*

This theorem proves a particular case of a conjecture formulated by Lusztig in [Lus90b].

Theorem 2.7.4. *Let $g \geq 2$ and $Q = S_g$ be the quiver with g loops. Let $\mathbf{d} \in \mathbf{N}$ be a dimension vector and $\mathscr{F}$ a simple perverse sheaf on $E_{Q,\mathbf{d}}$ such that $SS(\mathscr{F}) \subset \Lambda^{\flat}$ where $\flat = (\mathrm{nil}, 1)$ or $\flat = 1$. Then $\mathscr{F}$ is in the category $\mathcal{P}^{\flat}$.*

To prove this theorem, we rely on the results of Lusztig [Lus93] concerning the smallness of the morphisms from the stack of pairs of a representation with a flag of a certain type preserved by the representation to the stack of representations.

We conjecture, following Lusztig, that these microlocal characterizations remain true in general. The reader is referred to Chapter 5 for precise statements of the conjectures.

2.7.4 Spherical Eisenstein sheaves on the stack of coherent sheaves on an elliptic curve

In this section we explain the main results of the article [Hen21] (Chapter 6). Let X be a smooth complex projective curve. We are in a situation analogous to that of quivers. The category of representations of the quiver is replaced by the category of coherent sheaves on the curve X, the category of perverse sheaves considered by Lusztig becomes the category of spherical Eisenstein sheaves defined by Schiffmann ([Sch06]), and Lusztig's nilpotent variety is replaced by the global nilpotent cone ([Lau88]). We can then show that the singular supports of the spherical Eisenstein sheaves are unions of irreducible components of the global nilpotent cone. Inspired by the situation for quivers, we prove that for elliptic curves, the characteristic cycle map induces a bijection between (the completions) of the Grothendieck group of the category of spherical Eisenstein sheaves and of the free abelian group generated by the irreducible components of the global nilpotent cone. On the other hand, the existence of non-trivial local systems on X makes it impossible to have a microlocal characterization. However, one can determine explicitly the simple perverse sheaves on the stack of coherent sheaves on an elliptic curve whose singular support is contained in the global nilpotent cone. The main results of Chapter 6 are then as follows. We only recall the notations necessary for the statement of Theorem 1.7.5 and 2.7.6. The stack of coherent sheaves on X is denoted by $\mathfrak{Coh}(X)$. The connected component corresponding to $\alpha = (r,d) \in \mathbf{Z}^+$ is denoted $\mathfrak{Coh}(X)_\alpha$. The stack of Higgs bundles is $\mathfrak{Higgs}(X) = \bigsqcup_{\alpha\in\mathbf{Z}^+}\mathfrak{Higgs}(X)_\alpha$. It contains a closed substack, the global nilpotent cone: for $\alpha \in \mathbf{Z}^+$, we write $\mathscr{N}_\alpha \subset \mathfrak{Higgs}(X)_\alpha$ the substack of nilpotent Higgs bundles. Its set of irreducible components is denoted $\mathrm{Irr}(\mathscr{N}_\alpha)$. The hats indicate completions of the abelian groups in question. We refer to Chapter 6 for the definition of these. The following is Theorem 6.1.1 of Chapter 6.

Theorem 2.7.5. *For any class $\alpha \in \mathbf{Z}^+$, the characteristic cycle map*

$$CC : \widehat{K_0(\mathcal{P}^\alpha)} \to \mathbf{Z}[\widehat{\mathrm{Irr}(\mathscr{N}_\alpha)}]$$

is an isomorphism of $\mathbf{Z}$-modules.

In order to state the second main result, we need to introduce some additional notations. For $\alpha \in \mathbf{Z}^+$, we denote by $\mathfrak{Coh}(X)_{(\alpha)}$ the stack of *semistable* coherent sheaves of type α. As X is an elliptic curve, we have an isomorphism $\epsilon_\alpha : \mathfrak{Coh}(X)_{(\alpha)} \simeq \mathfrak{Coh}(X)_{(0,\gcd\alpha)}$ where $\gcd\alpha = \gcd(r,d)$. The topological fundamental group of X is $\mathbf{Z}^2$. Thus, the irreducible local systems on $\mathfrak{Coh}_{(0,1)}$ are indexed by $z \in (\mathbf{C}^*)^2$. The corresponding local system is denoted by $\mathscr{L}_z$. If $\alpha \in \mathbf{Z}^+$ is such that $\gcd\alpha = 1$, by the isomorphism ϵ_α, we obtain a local system still denoted $\mathscr{L}_z$ on $\mathfrak{Coh}(X)_{(\alpha)}$. The following theorem is Theorem 6.1.3 of Chapter 6.

Theorem 2.7.6. *The perverse sheaves on $\mathfrak{Coh}_\alpha(X)$ having a nilpotent singular support are precisely the perverse sheaves which are the simple constituents of inductions of perverse sheaves of the form $\mathcal{IC}(\mathscr{L}_z)$ on $\mathfrak{Coh}_\beta(X)$, $\beta \in \mathbf{Z}^+$ indivisible.*

Chapter 3

Isotropic cuspidal functions in the Hall algebra of a quiver

From the structure of the category of representations of an affine cycle-free quiver, we determine an explicit linear form on the space of regular cuspidal functions over a finite field: its kernel is exactly the space of cuspidal functions. Moreover, we show that any isotropic cuspidal dimension has an affine support. Brought together, this two results give an explicit description of isotropic cuspidal functions of any quiver. The main theorem together with an appropriate action of some permutation group on the Hall algebra provides a new elementary proof of two conjectures of Berenstein and Greenstein previously proved by Deng and Ruan. We also prove a statement giving non-obvious constraints on the support of the comultiplication of a cuspidal regular function allowing us to connect both mentioned conjectures of Berenstein and Greenstein. Our results imply the positivity conjecture of Bozec and Schiffmann concerning absolutely cuspidal polynomials in isotropic dimensions.

Contents

3.1 Introduction

Primitive elements of Hopf algebras or more generally of bialgebras are of primary importance in their study. A striking result is the Milnor-Moore theorem ([MM65]) asserting that a graded connected cocommutative Hopf algebra with finite dimensional graded parts is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements. Primitive elements of quantum groups ([Lus11]) have no mystery: these are the Chevalley generators. The situation for generalized Borcherds-Kac-Moody algebras ([Bor88, Kan95]) is analogous: these behave in fact very much like quantum groups associated to Kac-Moody algebras, although an infinite number of generators and imaginary simple roots are allowed. The Hall algebra of a quiver gives a way to construct quantum groups. Namely, given a quiver Q, one can consider the category $\mathrm{Rep}_Q(\mathbf{F}_q)$ of finite dimensional representations of Q over some finite field $\mathbf{F}_q$. It can be used to built the so-called Hall algebra of Q over $\mathbf{F}_q$, which is a Hopf algebra object in some braided monoidal category[1]. There is a natural subalgebra of the Hall algebra. It is the subalgebra generated by the classes of simple representations $[S_i]$ at each vertex i of Q. This is the so-called composition algebra. By a theorem

[1]More precisely, in the category of $\mathbf{Z}^I$-graded $\mathbf{C}$-vector spaces with finite dimensional graded components with braiding $X \otimes Y \to Y \otimes X$, $x \otimes y \mapsto \nu^{(x,y)} y \otimes x$ for any objects X and Y and homogeneous x, y, where I is the set of vertices of Q and $(-,-)$ is the symmetrized Euler form defined in Section 3.2.

of Ringel ([Rin90b]), it is isomorphic to the positive part of the quantum group $\mathbf{U}_\nu(\mathfrak{g}_A)$ specialized at $\nu = q^{1/2}$. The work of Sevenhant and Van den Bergh ([SVDB01]) identifies the whole Hall algebra $\mathbf{H}_{Q,\mathbf{F}_q}$ of Q with the quantization of the enveloping algebra of a generalized Kac-Moody algebra. The isomorphism is constructed using primitive elements of $\mathbf{H}_{Q,\mathbf{F}_q}$ and depends on such a choice. This is not completely satisfactory since we would like to determine natural generators of the Hall algebra. This work is the beginning of this project as we provide a way to compute explicitly primitive elements of the Hall algebra in isotropic dimensions. This solves the problem of the calculation of primitive elements of the Hall algebra for affine quivers, but the ambiguity still remains.

Here is a brief overview of what is done in this chapter. In Section 3.2, we introduce notations and known facts of the representation theory of quivers. We focuse in particular on the category of representations of affine quivers and recall their decomposition in blocks. In Section 3.3, we recall the definition of the constructible Hall algebra of a quiver. We provide several formulas for the comultiplication and recall the theorem of Sevenhant and Van den Bergh, which will only be used in Section 3.7 to prove Conjecture 3.7.3. A major role is played by the Kronecker quiver for which the classification of representations is explicit. In Section 3.4, we write the formulas for the number of indecomposable and absolutely indecomposable representations, and for the dimensions of cuspidal functions for affine quivers. In Section 3.5, we calculate explicitly all cuspidal functions of the Jordan quiver. We do not know any formula for nilpotent cuspidal functions of cyclic quivers but we provide sufficient informations on the value they take on indecomposable representations to deal with them. In Section 3.6, we determine regular cuspidal functions of affine quivers. Regular cuspidality is a weaker condition than cuspidality. As a numerical coincidence, in imaginary dimensions, cuspidal functions form a codimension one subspace of regular cuspidal functions. We determine an explicit linear form defining this hyperplane. In Section 3.7, we use our results to prove two conjectures made by Berenstein and Greenstein in [BG16] concerning the symmetry of the Hall algebra. The last Section 3.8 is devoted to show that an isotropic cuspidal dimension of any quiver has affine support. This immediately implies a conjecture of Bozec and Schiffmann in isotropic dimensions. The letter I is used for both the set of vertices of a quiver and an indecomposable representation of a quiver. It should be clear from the context how to distinguish them.

3.1.1 The main results

We state here our main contributions. Let Q be an affine quiver and $\mathbf{F}_q$ a finite field. As in Section 3.3, let $\mathbf{H}_{Q,\mathbf{F}_q}$ be the Hall algebra of Q over $\mathbf{F}_q$.

Cuspidal functions as the kernel of a linear form

In Section 3.6, we consider the subalgebra $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ of $\mathbf{H}_{Q,\mathbf{F}_q}$ generated by classes $[M]$ for M a regular representation. It is a Hopf algebra for a corestriction of the comultiplication of $\mathbf{H}_{Q,\mathbf{F}_q}$[2] endowed with a nondegenerate hermitian product $(-,-)$. The algebra $\mathbf{H}_{Q,\mathbf{F}_q}$ has a well-understood

[2] see the introduction for precisions on the bialgebra structure.

structure. Indeed, we have a bialgebra graded isomorphism

$$\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}} \simeq \bigotimes_{a\in|\mathbf{P}^1_{\mathbf{F}_q}|}{}' \ \mathbf{H}_a$$

where $\mathbf{H}_a$ is isomorphic to Macdonald's ring of symmetric function or to the nilpotent Hall algebra of a cyclic quiver for some finite number of a, and the degree of elements of $\mathbf{H}_a$ for $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$ is multiplied by $\deg(a)$ to obtain the degree in $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$. This decomposition allows us to give an expression of primitive elements of this algebra, called regular cuspidal functions. For $r \geq 1$ and δ the indecomposable imaginary root of Q, let $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[r\delta]$ be the subspace of regular cuspidal functions of dimension $r\delta$. We also let

$$\chi_{r\delta} = \sum_{\substack{[M] \text{ regular} \\ \dim M = r\delta}} [M].$$

Theorem 3.1.1. *The kernel of the linear form*

$$\begin{array}{cccc} L: & \mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}} & \to & \mathbf{C} \\ & f & \mapsto & (f, \chi_{r\delta}) \end{array}$$

is $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]$.

An action of a permutation group on the Hall algebra

To prove Conjecture 3.7.3, we use the following action of a product of permutation groups on the Hall algebra which deserves to be considered separately.

As in Section 3.2, D denote the set of closed points of $\mathbf{P}^1_{\mathbf{F}_q}$ parametrizing non-homogeneous tubes. For $e \geq 2$, we let $N(e)$ be the number of closed points of $\mathbf{P}^1_{\mathbf{F}_q}$ of degree e and $N(1) = q+1-|D|$. Let $\mathfrak{S}$ be the group of degree preserving permutations of $|\mathbf{P}^1_{\mathbf{F}_q}| \setminus D$. The group $\mathfrak{S}$ is isomorphic to

$$\prod_{e\geq 1} \mathfrak{S}_{N(e)}$$

where for a positive integer N, $\mathfrak{S}_N$ is the symmetric group on N letters. We define an action

$$\mathfrak{S} \to \mathrm{Aut}(\mathbf{H}_{Q,\mathbf{F}_q})$$

as follows. For M, N two representations, λ a partition, $x \in |\mathbf{P}^1_{\mathbf{F}_q}| \setminus D$ and $\sigma \in \mathfrak{S}$,

$$\sigma \cdot [M] = [M] \quad \text{if } [M] \text{ is preprojective, preinjective or in a non-homogeneous tube}$$

$$\sigma \cdot [I_\lambda(x)] = [I_\lambda(\sigma(x))]$$

$$\sigma \cdot ([M] \oplus [N]) = \sigma \cdot [M] \oplus \sigma \cdot [N]$$

where for notational reasons, we define here $[M] \oplus [N] = [M \oplus N]$. It is easily seen that σ acts as a graded linear isomorphism on $\mathbf{H}_{Q,\mathbf{F}_q}$. We prove in Section 3.7 the following facts.

1. σ acts as an isometry of $\mathbf{H}_{Q,\mathbf{F}_q}$,
2. The action of σ leaves $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ and $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}$ stable,
3. σ commutes with the linear form L. In particular, it preserves $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}]$ for any dimension $\mathbf{d}$.

This yields the following result.

Theorem 3.1.2. *The group $\mathfrak{S}$ acts on $\mathbf{H}_{Q,\mathbf{F}_q}$ by degree preserving Hopf algebra automorphisms.*

Usually, quantum groups have very few degree preserving Hopf algebra automorphisms. The only ones are obtained by rescaling the Chevalley generators. Here, there is multiplicities for imaginary roots, from which we obtain non-trivial automorphisms.

3.2 Structure of the category of representations of affine quivers over a finite field

3.2.1 Notation and recollections on quiver representations

In this section, let $Q = (I, \Omega)$ be an arbitrary quiver with set of vertices I and set of arrows Ω and k a field. We denote by $\mathrm{Rep}_Q(k)$ the category of finite dimensional representations of Q over k. Equivalently, this is the category of finite dimensional modules over the path algebra kQ of Q. This category is known to be a k-linear abelian category of homological dimension one.

The Euler form

For $\mathcal{A}$ an abelian category, $K_0(\mathcal{A})$ is its Grothendieck group. For M an object of $\mathcal{A}$, $\overline{M} \in K_0(\mathcal{A})$ is its class in the Grothendieck group. The bilinear (usually non-symmetric) form

$$\langle -, - \rangle : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbf{Z}, \qquad \langle \overline{M}, \overline{N} \rangle = \hom(M, N) - \mathrm{ext}^1(M, N)$$

where by definition $\hom(M,N) = \dim_k \mathrm{Hom}_{kQ}(M,N)$ and $\mathrm{ext}^1(M,N) = \dim_k \mathrm{Ext}^1_{kQ}(M,N)$ is called the Euler form of the quiver. It factorizes through the (surjective) morphism of abelian groups $\dim : K_0(\mathcal{A}) \to \mathbf{Z}^I$ given by the dimension $\overline{M} \mapsto \dim M$. In fact, if $\mathbf{d} = \dim M \in \mathbf{N}^I$ and $\mathbf{d}' = \dim N \in \mathbf{N}^I$, we have the explicit formula:

$$\langle \overline{M}, \overline{N} \rangle = \sum_{i \in I} \mathbf{d}_i \, \mathbf{d}'_i - \sum_{\alpha : i \to j} \mathbf{d}_i \, \mathbf{d}'_j \, .$$

We use the same notation

$$\langle -, - \rangle : \mathbf{Z}^I \times \mathbf{Z}^I \to \mathbf{Z}$$

for the induced bilinear form. We will also consider its symmetrized version:

$$(-,-) : \mathbf{Z}^I \times \mathbf{Z}^I \to \mathbf{Z}, \qquad (\mathbf{d}, \mathbf{d}') = \langle \mathbf{d}, \mathbf{d}' \rangle + \langle \mathbf{d}', \mathbf{d} \rangle \quad \text{for any } \mathbf{d}, \mathbf{d}' \in \mathbf{Z}^I.$$

In case of an affine quiver, the symmetrized Euler form is nonnegative with one dimensional kernel generated by an indecomposable integer valued positive vector, called the indecomposable imaginary root, denoted by the letter δ (see [Sch14, Theorem 8.6]).

Dualization

Let Q be a quiver and Q^* the quiver in which we change the orientation of all arrows, which deserves the name of dual quiver. A representation V of Q gives a representation V^* of Q^* obtained by dualizing the vector spaces at the vertices of Q and replacing the linear maps between them by their transpose. Explicitly, if $V = ((V_i)_{i\in I}, (f_\alpha)_{\alpha\in\Omega})$ is a representation of Q, its dual is $V^* = ((V_i^*)_{i\in I}, ({}^t f_\alpha)_{\alpha\in\Omega})$. We obtain in this way an equivalence of categories

$$\tilde{D} : \mathrm{Rep}_Q(k)^{\mathrm{op}} \to \mathrm{Rep}_{Q^*}(k)$$

which associate to a representation its dual. In particular, there is a canonical identification $\mathrm{Hom}_{kQ^*}(M^*, N^*) \simeq \mathrm{Hom}_{kQ}(N, M)$ and $\mathrm{Ext}^1_{kQ^*}(M^*, N^*) \simeq \mathrm{Ext}^1_{kQ}(N, M)$.

3.2.2 The structure of the category of representations of an affine quiver

Cyclic and Jordan quivers

Let k be an arbitrary field. Cyclic and Jordan quivers are the non-acyclic quivers of affine type. This section introduces notations which will be later used. Let J be the Jordan quiver. We let J_n be the $n \times n$ indecomposable nilpotent matrix with ones on the superdiagonal and zeros everywhere else:

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}.$$

We also see J_n as a n-dimensional representation of the Jordan quiver. For a partition $\lambda = (\lambda_1, \dots)$, we write $J_\lambda = \bigoplus_{j\geq 1} J_{\lambda_j}$. Any nilpotent representation of J over k is isomorphic to exactly one representation of the following set:

$$\{J_\lambda : \lambda \text{ partition}\}.$$

All other isomorphism classes of representations of the Jordan quiver also have an explicit representative which will not be used. See [Hua00, Section 3.].

Let $n \geq 1$ be a integer and C_n be the cyclic quiver of length n. What follows also applies to C_1 which is the Jordan quiver. We distinguish two types of representations for cyclic quivers: invertible representations, for which every arrow is invertible, and nilpotent representations, for which the composition of arrows along any sufficiently long path is zero. Before describing them, we introduce some notations. We suppose that the vertices are indexed by $\mathbf{Z}/n\mathbf{Z}$ with exactly one arrow $i \to i+1$ for $i \in \mathbf{Z}/n\mathbf{Z}$. A representation of C_n is a n-tuple $(a_0, \dots, a_{n-1})$ where $a_i : V_i \to V_{i+1}$ for $i \in \mathbf{Z}/n\mathbf{Z}$

and some k-vector spaces V_i. We let $\mathrm{Rep}^{\mathrm{inv}}_{C_n}(k)$ be the full subcategory of invertible representations of C_n over k and $\mathrm{Rep}^{\mathrm{nil}}_{C_n}(k)$ be the full subcategory of nilpotent representations.

1. $\mathrm{Rep}_{C_n}(k) = \mathrm{Rep}^{\mathrm{nil}}_{C_n}(k) \sqcup \mathrm{Rep}^{\mathrm{inv}}_{C_n}(k)$ is a decomposition in blocks of $\mathrm{Rep}_{C_n}(k)$.

2. A full set of representatives of nilpotent indecomposable representations is built as follows. Let l be a nonnegative integer and for $0 \leq m \leq l$, $V_m = ke_m$ a one-dimensional vector space generated by e_m. The $\mathbf{Z}/n\mathbf{Z}$-graded vector space $V_{i,l} = \bigoplus_{\bar{x}\in\mathbf{Z}/n\mathbf{Z}} \bigoplus_{m\equiv x-i \pmod n} V_m$ with the endomorphism sending e_m to e_{m+1} if $0 \leq m < l$ and e_l to zero defines an indecomposable representation of C_n again denoted by $V_{i,l}$. The set
$$\{V_{i,l} : i \in \mathbf{Z}/n\mathbf{Z}, l \geq 0\}$$
contains exactly one representative of each isomorphism class of indecomposable nilpotent representations. Define also $S_i = V_{i,0}$ for $i \in \mathbf{Z}/n\mathbf{Z}$.

3. The following functor is an equivalence of categories:
$$\begin{array}{rccc} G_n : & \mathrm{Rep}^{\mathrm{inv}}_J(k) & \to & \mathrm{Rep}^{\mathrm{inv}}_{C_n}(k) \\ & (V,a) & \mapsto & (\mathrm{id},\dots,\mathrm{id},a). \end{array} \tag{3.2.1}$$

Proof. The statement (3) is straightforward and (2) is proved in [Sch12a, Section 3.5]. We prove (1). Since $\mathrm{Rep}^{\mathrm{nil}}_{C_n}(k)$ and $\mathrm{Rep}^{\mathrm{inv}}_{C_n}(k)$ are stable under taking subobjects and quotients in $\mathrm{Rep}_{C_n}(k)$, any morphism from a nilpotent representation M to an invertible representation N is zero, and conversely. The extension spaces $\mathrm{Ext}^1(M,N)$ and $\mathrm{Ext}^1(N,M)$ also vanish, since using the Euler form and that $\dim N$ is some multiple $r\delta = (r,\dots,r)$ of the indecomposable imaginary root $\delta = (1,\dots,1)$, we have:
$$0 = (\overline{M},\overline{N}) = -\operatorname{ext}^1(M,N) - \operatorname{ext}^1(N,M) = 0.$$
□

Decomposition in blocks of $\mathbf{Rep}_{C_n}(k)$

Let k be a field. We give here a decomposition of $\mathrm{Rep}_{C_n}(k)$ in blocks. Let V be a k-vector space of dimension d. The group $\mathrm{GL}(V)$ acts algebraically on $\mathrm{Hom}(V,V)$ by conjugation with quotient $\mathrm{Hom}(V,V)/\!/\mathrm{GL}(V) \simeq S^n\,\mathbf{A}^1_k$. Denote by π_V the projection $\pi_V : \mathrm{Hom}(V,V) \to S^n\,\mathbf{A}^1_k$.

Theorem 3.2.1. *We have a decomposition in blocks*
$$\mathrm{Rep}_J(k) \simeq \bigsqcup_{a\in|\mathbf{A}^1_k|} \mathrm{Rep}^a_J(k),$$
where $(V,x) \in \mathrm{Rep}^a_J(k)$ if and only if $\pi_V(x) = (a,\dots,a)$.

Theorem 3.2.2. *We have a decomposition in blocks*

$$\mathrm{Rep}_{C_n}(k) \simeq \bigsqcup_{a\in|\mathbf{A}^1_k|} \mathrm{Rep}^a_{C_n}(k)$$

where $\mathrm{Rep}^0_{C_n}(k) = \mathrm{Rep}^{\mathrm{nil}}_{C_n}(k)$ *and* $\mathrm{Rep}^a_{C_n}(k)$ *is the full subcategory* $G_n(\mathrm{Rep}^a_J(k))$ *of* $\mathrm{Rep}^{\mathrm{inv}}_{C_n}(k)$*.*

Acyclic affine quivers

We recollect known facts on the representation theory of acyclic affine quivers (see [Rin84]) over a finite field (some results may hold in greater generality). The exposition here follows and can be completed by [Sch12a, Section 3.6] and [CB, §8]. Troughout this section, $Q = (I, \Omega)$ is an acyclic affine quiver. This condition is equivalent to the finite dimensionality of the path algebra of Q over any field and excludes the Jordan quiver and cyclic quivers which have been studied above.

Theorem 3.2.3. *Let k be an arbitrary field. Then, there exists an adjunction*

$$\tau^- : \mathrm{Rep}_Q(k) \rightleftarrows \mathrm{Rep}_Q(k) : \tau$$

with bi-natural isomorphisms[3] *(the star means the dual with respect to the k-vector space structure):*

$$\mathrm{Ext}^1(M,N)^* \simeq \mathrm{Hom}(N, \tau M), \qquad \mathrm{Ext}^1(M,N)^* \simeq \mathrm{Hom}(\tau^- N, M).$$

The functors τ and τ^- are known as *Auslander-Reiten translates*. From the properties of τ^- and τ, it is immediate that a representation M of Q over k is projective if and only if $\tau(M) = 0$ and injective if and only if $\tau^-(M) = 0$. We call an indecomposable representation M of Q over k

1. preprojective if $\tau^n M = 0$ for $n \gg 0$,
2. preinjective if $\tau^{-n} M = 0$ for $n \gg 0$,
3. regular if $\tau^n M \neq 0$ for all $n \in \mathbf{Z}$.

Furthermore, we call a representation M of Q over k preprojective if all its indecomposable direct summands are preprojective, and we adopt similar terminology for preinjective and regular representations. The full subcategory of $\mathrm{Rep}_Q(k)$ of preprojective (resp. preinjective, resp. regular) representations is denoted by $\mathrm{Rep}^{\mathcal{P}}_Q(k)$ (resp. $\mathrm{Rep}^{\mathcal{I}}_Q(k)$, resp. $\mathrm{Rep}^{\mathcal{R}}_Q$). These are extension closed subcategories of $\mathrm{Rep}_Q(k)$, hence exact categories. Moreover, $\mathrm{Rep}^{\mathcal{R}}_Q(k)$ is an abelian category (though not stable under taking subobjects in the bigger category $\mathrm{Rep}_k Q$). The three categories $\mathrm{Rep}^{\mathcal{R}}_Q(k), \mathrm{Rep}^{\mathcal{P}}_Q(k)$ and $\mathrm{Rep}^{\mathcal{I}}_Q(k)$ are disjoint and the category to which an indecomposable M belongs is given by the sign of its defect defined by $\partial M = \langle \delta, \dim M\rangle$. An indecomposable representation M is preprojective if and only if $\partial M < 0$, preinjective if and only of $\partial M > 0$ and regular if and only if $\partial M = 0$. The following proposition gives the interactions between these three subcategories.

[3] We say that (τ^-, τ) is a Serre adjunction.

Proposition 3.2.4. *For $M \in \mathrm{Rep}_Q^{\mathcal{P}}(k)$, $N \in \mathrm{Rep}_Q^{\mathcal{I}}(k)$, $L \in \mathrm{Rep}_Q^{\mathcal{R}}(k)$, we have*

$$\mathrm{Hom}(N, M) = \mathrm{Hom}(N, L) = \mathrm{Hom}(L, M) = 0,$$

$$\mathrm{Ext}^1(M, N) = \mathrm{Ext}^1(L, N) = \mathrm{Ext}^1(M, L) = 0.$$

The simple objects of the abelian category $\mathrm{Rep}_Q^{\mathcal{R}}(k)$ are called simple regular. A simple regular representation M is called homogeneous if $\tau M \simeq M$.

Theorem 3.2.5 (Ringel, [Rin84])**.** *Let Q be an affine acyclic quiver and k an arbitrary field. Let d and $p_1, \dots, p_d$ be attached to Q as in the table below. Then*

1. *There is a degree preserving bijection $M_a \leftrightarrow a$ between the set of homogeneous regular simple modules and $|\mathbf{P}_k^1| \setminus D$ where D consists of d closed points of degree one*[4]*,*
2. *There are d τ-orbits $\mathcal{O}_1, \dots, \mathcal{O}_d$ of non-homogeneous regular simple modules of size given by $p_1, \dots, p_d$*[5]*,*
3. *The category $\mathrm{Rep}_Q^{\mathcal{R}}(k)$ decomposes as a direct sum of blocks*[6]*:*

$$\mathrm{Rep}_Q^{\mathcal{R}}(k) = \bigsqcup_{a \in |\mathbf{P}^1| \setminus D} \mathcal{C}_{M_a} \sqcup \bigsqcup_{l=1}^{d} \mathcal{C}_{\mathcal{O}_l}$$

where $\mathcal{C}_{M_a}$ is the full subcategory of objects which are extensions of M_a and $\mathcal{C}_{\mathcal{O}}$ is the full subcategory of $\mathrm{Rep}_Q^{\mathcal{R}}(k)$ of objects whose regular simple factors lie in $\mathcal{O}$.

type of Q	d	$p_1, \dots, p_d$
$A_1^{(1)}$	0	
$A_n^{(1)}, n > 1$	2	p_1 =number of arrows going clockwise p_2 =number of arrows going counterclockwise
$D_n^{(1)}$	3	$2, 2, n-2$
$E_n^{(1)}, n = 6, 7, 8$	3	$2, 3, n-3$

Figure 3.1: Non-homogeneous tubes of affine quivers and their period [Sch12a, (3.18)]

In Theorem 5.2.5, the subcategories $\mathcal{C}_{\mathcal{O}_l}$ are called the non-homogeneous tubes while the subcategories $\mathcal{C}_{M_a}$ are the homogeneous tubes. The number of non-homogeneous tubes is d (see however Remark 5.2.6) and the integers $p_1, \dots, p_d$ are the periods. They do not depend on the chosen field. For $a \in \mid \mathbf{P}_Q^1 \mid$, $\mathcal{C}_Q^a$ also denotes the corresponding tube.

Remark 3.2.6. In type $A_n^{(1)}$, in the case where all arrows except one go in the same direction, we have in fact $d = 1$, *i.e.* there is only one non-homogeneous tube.

[4] in the sequel for X a scheme, we denote by $\mid X \mid$ the set of its closed points.

[5] *i.e.* the set of isomorphism classes of simple objects in $\mathcal{O}_j$, $1 \leq j \leq d$ is of cardinality d and the Auslander-Reiten translates τ and τ^- act as inverse cycles on it.

[6] There are no morphisms or extensions between the objects of different categories

We can furthermore precisely identify the tubes $\mathcal{C}_{M_a}$ and $\mathcal{C}_{\mathcal{O}}$ with the help of the Jordan quiver and of cyclic quivers respectively.

Theorem 3.2.7. *Let $a \in |\mathbf{P}^1_k| \setminus D$ a closed point of degree d. Let $K = \mathrm{End}(M_a)$ the k-algebra of endomorphisms of the simple regular M_a of the tube $\mathcal{C}_{M_a}$. This is a finite field extension of k of degree d. There exists a unique equivalence of categories*

$$\begin{array}{rccc} F_a : & \mathrm{Rep}_J(K) & \to & \mathcal{C}_{M_a} \\ & I_{(1)} & \to & M_a. \end{array}$$

We set $I^Q_\lambda(a) := F_a(J_\lambda)$.

Let $a \in D$ be a closed point corresponding to a non-homogeneous tube. Let p be the corresponding period of the non-homogeneous tube $\mathcal{O}_a$ and S a simple regular of $\mathcal{O}_a$. There is a unique equivalence of categories

$$\begin{array}{rccc} F_a : & \mathrm{Rep}_{C_p}(k) & \to & \mathcal{C}_{\mathcal{O}_a} \\ & S_i & \to & \tau^i S. \end{array}$$

In the case of a non-homogeneous tube, there is a reasonable way to choose the simple representation S. For this, we may first choose an extending vertex i_0 of Q. Then, since $\sum_{s=0}^{p} \dim \tau^s S = \delta$ and $\delta_{i_0} = 1$, there is a unique simple representation S in the non-homogeneous tube which is nonzero at vertex i_0. The isomorphism class of the representation S may however depend on the extending vertex we choose.

3.2.3 Identification of the tubes with the help of the Kronecker quiver

Representations of the Kronecker quiver

We recall here the classification of indecomposables of the Kronecker quiver over a finite field $\mathbf{F}_q$ for the sake of completeness. It can be obtained from the classification over the algebraic closure $\overline{\mathbf{F}}_q$ using Galois arguments. For complements, see [Kir16, Section 7.7].

The Kronecker quiver has two vertices connected by two arrows going in the same direction:

$$K_2 : \qquad 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2,$$

We fix a finite field with q elements $\mathbf{F}_q$ and an algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$.

Theorem 3.2.8 ([Kir16, Theorem 7.30])**.**

1. *The set of real roots is $\{(n, n+1) : n \in \mathbf{N}\} \cup \{(n+1, n) : n \in \mathbf{N}\}$. For $n \in \mathbf{N}$, the indecomposable representation of K_2 over $\mathbf{F}_q$ of dimension $(n, n+1)$ is preprojective whereas the indecomposable representation of K_2 over $\mathbf{F}_q$ of dimension $(n+1, n)$ is preinjective. They*

have the following form:

$$\mathbf{F}_q^n \overset{\begin{pmatrix} I_n \\ 0 \end{pmatrix}}{\underset{\begin{pmatrix} 0 \\ I_n \end{pmatrix}}{\rightrightarrows}} \mathbf{F}_q^{n+1},$$

for the preprojective representations and

$$\mathbf{F}_q^{n+1} \overset{\begin{pmatrix} I_n & 0 \end{pmatrix}}{\underset{\begin{pmatrix} 0 & I_n \end{pmatrix}}{\rightrightarrows}} \mathbf{F}_q^{n},$$

for the preinjective representations, where I_n is the $n \times n$ identity matrix.

2. *The regular indecomposable representations of K_2 over $\mathbf{F}_q$ are parametrized by the set $|\mathbf{P}^1_{\mathbf{F}_q}| \times \mathbf{Z}_{\geq 1}$. For $([x : y], n) \in \mathbf{P}^1_{\mathbf{F}_q}(\overline{\mathbf{F}}_q) \times \mathbf{Z}_{\geq 1}$, the corresponding representation of K_2 is*

$$\mathbf{F}^n_{q^{\deg(x)}} \overset{xI_n + J_n}{\underset{yI_n + J_n}{\rightrightarrows}} \mathbf{F}^n_{q^{\deg(x)}},$$

where we consider $\mathbf{F}_{q^{\deg(x)}}$ as a $\mathbf{F}_q$-vector space of dimension $\deg(x)$. The isomorphism class of this representation only depends on the Galois orbit of $[x : y]$. We denote by $I_{t,n}$ where $t = [x : y]$ this representation and when $n = 1$, we use the notation $S_t = I_{t,1}$ for $t \in \mathbf{P}^1_{\mathbf{F}_q}(\overline{\mathbf{F}}_q)$.

For $t \in \mathbf{P}^1_{\mathbf{F}_q}(\overline{\mathbf{F}}_q)$ and $\lambda = (\lambda_1, \lambda_2, \ldots)$ a partition, define $I_{t,\lambda} = \oplus_{i=1}^{l(\lambda)} I_{t,\lambda_i}$. The isomorphism class $[I_{t,\lambda}]$ only depends on the image a of $t : \operatorname{Spec} \overline{\mathbf{F}}_q \to \mathbf{P}^1_{\mathbf{F}_q}$. For any $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$, we fix $t_a \in \mathbf{P}^1_{\mathbf{F}_q}(\overline{\mathbf{F}}_q)$ a geometric point such that the image of $t : \operatorname{Spec} \overline{\mathbf{F}}_q \to \mathbf{P}^1_{\mathbf{F}_q}$ is a. The set of isomorphism classes of regular representations of K_2 over $\mathbf{F}_q$ is

$$\mathcal{M}^{K_2}_{\mathbf{F}_q} = \{[I_{t_a,\lambda}] : a \in |\mathbf{P}^1_{\mathbf{F}_q}|, \lambda \text{ a partition}\}.$$

For $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$, write $I_{a,\lambda} = I_{t_a,\lambda}$ and $S_a = I_{a,(1)}$. For any $r \geq 1$, the number $I_{K_2,(r,r)}(q)$ of indecomposable representations of the Kronecker quiver of dimension (r, r) can be determined from this:

$$I_{K_2,(r,r)}(q) = \text{number of closed points of } \mathbf{P}^1_{\mathbf{F}_q} \text{ of degree dividing } r.$$

This description also gives a natural way to identify the tubes of the Kronecker quiver:

$$\begin{array}{cccc} C: & |\mathbf{P}^1_{\mathbf{F}_q}| & \to & \{\text{tubes of } K_2 \text{ over } \mathbf{F}_q\} \\ & a & \mapsto & C_a = \{I_{t_a,\lambda} : \lambda \text{ partition}\} \end{array}. \tag{3.2.2}$$

The number of automorphisms of regular representations of the Kronecker quiver

For q a power of a prime and $\lambda = (1^{l_1}, 2^{l_2}, \dots)$ a partition, define:

$$a_\lambda(q) = q^{|\lambda|+2n(\lambda)} \prod_i \prod_{j=1}^{l_i} (1-q^{-j}).$$

This is the number of automorphism of the nilpotent representation of type λ J_λ of the Jordan quiver over $\mathbf{F}_q$ (see [Sch12a, Lemma 2.8] or [Mac15, Chapter II, (1.6)]). The identification of the tubes of the Kronecker quiver with nilpotent representations of the Jordan quiver gives the following.

Proposition 3.2.9. *Let $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$ and $n \geq 1$. Then the number of automorphism of $I^{K_2}_{t_a,\lambda}$ is $a_\lambda(q^{\deg(a)})$.*

Regular representations of an affine quiver

Let Q be an arbitrary acyclic affine quiver. Ler i_0 be an extending vertex of Q and δ the indivisible imaginary root of Q. We can restrict ourselves to the case where i_0 is a sink, since it is always possible to choose the extending vertex to be a sink or a source, and the dualization is a way to pass from the second case to the first. Without referring to the dualization process, it is possible to adapt this section in the case where the extending vertex i_0 is a source. By the classification above, all tubes are homogeneous for the Kronecker quiver K_2, *i.e.* in type $A_1^{(1)}$ with the noncyclic orientation.

Let $Q' = (I', \Omega')$ be the finite type quiver associated to Q by erasing the vertex i_0 and all arrows adjacent to it. The element $\theta = \delta - e_{i_0}$ is an element of $\mathbf{N}^{I'}$. We denote by I_θ an indecomposable representation of Q of dimension θ. Thanks to the classification of affine quivers there is either one vertex i_1 adjacent to i_0 (in types $D_n^{(1)}$ $(n \geq 4)$ and $E_n^{(1)}, n = 6,7,8$), in which case $\delta_{i_1} = 2$, or two vertices i_1, i_2 adjacent to i_0 (in types $A_n^{(1)}$, $n > 1$), in which case, $\delta_{i_1} = \delta_{i_2} = 1$. In the first case, we choose an arbitrary identification $I_{i_1} \simeq k^2$. We have a functor

$$\begin{array}{rccc} F: & \mathrm{Rep}_{K_2}(k) & \to & \mathrm{Rep}_Q(k) \\ & (V_0, V_1, \alpha, \beta) & \mapsto & V \end{array} \tag{3.2.3}$$

where V is defined as follows. The restriction to the subquiver Q' is $V_{Q'} = I_\theta \otimes V_0$, $V_{i_0} = V_1$, and if i_0 is connected by two arrows $i_1 \to i_0$ and $i_2 \to i_0$ to Q', then we choose α for the map $V_{i_1} = V_0 \to V_{i_0} = V_1$ and β for the map $V_{i_2} = V_0 \to V_{i_0} = V_1$. If i_0 is connected to Q' by a single arrow $i_1 \to i_0$, then the map $V_{i_1} \simeq V_0 \oplus V_0 \to V_{i_0}$ is choosen to be $\alpha \oplus \beta$. Of course this functor depends in the first case on the choosen order i_1, i_2 of the vertices connected to i_0 and in the second case on the identification $V_{i_1} \simeq k^2$. We implicitly fix such a choice. The action of F on the morphisms is as follows. Let $V = (V_0, V_1, \alpha, \beta)$ and $V' = (V_0', V_1', \alpha', \beta')$ be two representations of K_2 and $f : V \to V'$ a morphism between them. The linear map $f_0 : V_0 \to V_0'$ induces a morphism of representations of Q', $\mathrm{id} \otimes f_0 : I_\theta \otimes V_0 \to I_\theta \otimes V_0'$ and considering $f_1 : V_1 = F(V)_{i_0} \to V_1' = F(V')_{i_0}$

at the vertex i_0, we obtain a morphism of representations $F(f) : F(V) \to F(V')$. Note that $F(S_1) = I_\theta$ and $F(S_2) = S_{i_0}$.

Proposition 3.2.10 ([Kir16, Theorem 7.34])**.** *The functor F is exact and fully faithful.*

Proof. For quiver representations, exactness of a sequence can be checked pointwise (*i.e.* at each vertex). Here, tensor products are over a field so exactness of F is immediate from its definition.

For the full faithfulness, we use that I_θ, being an indecomposable of Q' which is a finite type quiver, is a brick, *i.e.* $\mathrm{End}_{kQ'}(I_\theta, I_\theta) \simeq k$. Let $V = (V_0, V_1, \alpha, \beta)$ and $V' = (V_0', V_1', \alpha', \beta')$ be two representations of K_2 and $g : F(V) \to F(V')$ a morphism of representations. We obtain for free a linear map $f_1 = g_{i_0} : V_1 \to V_1'$. Because I_θ is a brick, the restricted morphism of representations $g_{Q'} : I_\theta \otimes V_0 \to I_\theta \otimes V_0'$ is induced by a unique linear map $f_1 : V_0 \to V_0'$. The datum $(f_0, f_1) : V \to V'$ is a morphism of representations such that $F(f_0, f_1) = g$. Faithfulness is immediate. This concludes the proof. □

Proposition 3.2.11. *The image of the functor F is the full subcategory of $\mathrm{Rep}_Q(\mathbf{F}_q)$ whose objects are extensions of $I_\theta^{\oplus d_1}$ by $S_{i_0}^{\oplus d_2}$ for some $d_1, d_2 \geq 0$.*

Proof. This is a straightforward consequence of the definition of F. By definition, if $V = (V_0, V_1, \alpha, \beta)$ is a representation of K_2, then $F(V)$ is an extension of $I_\theta \otimes V_0$ by $S_{i_0} \otimes V_1$. Conversely let $(V', (f_\alpha)_{\alpha \in \Omega})$ be a representation of Q, extension of $I_\theta^{\oplus d_1}$ by $S_{i_0}^{\oplus d_2}$ for some $d_1, d_2 \geq 0$. We argue separately according to whether $Q = A_n^{(1)}$ or $Q \in \{D_n^{(1)}, E_n^{(1)}\}$. In the first case, we define the representation $V = (V_0, V_1, \alpha, \beta)$ of K_2 by setting $V_1 = V'_{i_0}$, $V_0 = V'_{i_1}$. An isomorphism of the restriction of V' to Q' with $I_\theta^{\oplus d_1}$, $\psi : V'_{Q'} \to I_\theta^{\oplus d_1}$ induces an isomorphism $\psi : V'_{i_1} \to V'_{i_2}$. We define $\alpha = f_{i_1 \to i_0}$ and $\beta = f_{i_2 \to i_0} \circ \psi$. It is easily checked that the isomorphism class of the so defined representation V does not depend on the various choices made and that $F(V) \simeq V'$.

In the second case, an isomorphism $V'/S_{i_0}^{d_2} \simeq I_\theta^{\oplus d_1}$ gives an identification $\varphi : V'_{i_1} \to k^2 \otimes k^{d_1}$. We then define V by setting $V_0 = k^{d_1}$, $V_1 = V_{i_0}$ and we define $\alpha = f_{i_0 \to i_2} \circ \varphi^{-1}((1, 0) \otimes -)$ and $\beta = f_{i_0 \to i_2} \circ \varphi^{-1}((0, 1) \otimes -)$. This defines a representation $V = (V_0, V_1, \alpha, \beta)$ of K_2 such that $F(V) \simeq V'$. □

Following the notations of Kirillov in [Kir16], we define $S_a^Q = F(S_a)$ for $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$.

Theorem 3.2.12 (Identification of the simple regular representations, [Kir16, Theorem 7.37])**.**

1. *Let X be a simple regular representation of Q over $\mathbf{F}_q$ of period 1. Then $\dim X = d\delta$ where d is the degree of the tube containing X and $X \simeq S_a^Q$ for some closed point $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$.*

2. *Let $\mathcal{O}$ be the τ-orbit of a simple regular representation of period $l > 1$. Then*

$$\sum_{X \in \mathcal{O}} \dim X = \delta.$$

Since at the extending vertex i_0, $\delta_{i_0} = 1$, $\mathcal{O}$ contains a unique simple regular representation X such that $X_{i_0} \neq 0$. Moreover, if $X^{(l)} \in \mathcal{C}_{\mathcal{O}}$ is the unique indecomposable representation of dimension δ having X as quotient, then $X^{(l)} \simeq S_a^Q$ for some $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$.

3.2.4 Some examples

We treat here the examples of types $D_4^{(1)}$ and $A_4^{(1)}$ with some particular orientations in dimension δ to illustrate the previous procedure in the two different cases for which (1) the extending vertex is connected to one vertex and (2) the extending vertex is connected to two vertices.

The type $D_4^{(1)}$

Let Q be the quiver

$$\begin{tikzcd} 1 & & 1 \\ & 2 \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\ 1 & & 1 \end{tikzcd} .$$

The labels of the vertices are given by the indecomposable imaginary root δ of Q. The extending vertex can be one of the four vertices with a 1. To fix the notations, we choose the bottom right vertex and call it i_0 as before. An indecomposable representation of Q of dimension $\theta = \delta - e_{i_0}$ is

$$\begin{tikzcd} \mathbf{F}_q & & \mathbf{F}_q \\ & \mathbf{F}_q^2 \arrow[ul, "(1,0)"'] \arrow[ur, "(0,1)"'] \arrow[dl, "(1,1)"'] \arrow[dr] & \\ \mathbf{F}_q & & 0 \end{tikzcd} .$$

Up to isomorphism, it is unique. Then, isomorphism classes of $(1,1)$-dimensional representations of K_2 are parametrized by $\mathbf{P}^1_{\mathbf{F}_q}(\mathbf{F}_q)$. The functor F induces on isomorphism classes of regular representations of dimension δ the following map:

$$\begin{array}{rccl} F: & \mathcal{M}^{K_2}_{\mathbf{F}_q}[\delta] \simeq \mathbf{P}^1_{\mathbf{F}_q}(\mathbf{F}_q) & \to & \mathcal{M}^{Q}_{\mathbf{F}_q}[\delta] \\ & [\lambda : \mu] & \mapsto & [S^Q_{(\lambda,\mu)}] \end{array}$$

where $S^Q_{(\lambda,\mu)}$ is the following representation of Q:

$$\begin{tikzcd} \mathbf{F}_q & & \mathbf{F}_q \\ & \mathbf{F}_q^2 \arrow[ul, "(1,0)"'] \arrow[ur, "(0,1)"'] \arrow[dl, "(1,1)"'] \arrow[dr, "(\lambda,\mu)"] & \\ \mathbf{F}_q & & \mathbf{F}_q \end{tikzcd} .$$

There are three non-homogeneous tubes associated to the parameters $[1:0], [0:1], [1:1] \in \mathbf{P}^1_{\mathbf{F}_q}(\mathbf{F}_q)$. The other parameters give simple regular representations of dimension δ. We now study one non-homogeneous tube. By symmetry of the quiver, this suffices to determine the three non-homogeneous tubes. Let us choose the non-homogeneous tube associated to $[1:0]$. Thanks to Theorem 3.2.12, the regular simple representations of the tube are the indecomposables

$$Y = \begin{tikzcd} 0 & & \mathbf{F}_q \\ & \mathbf{F}_q \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\ \mathbf{F}_q & & 0 \end{tikzcd} \quad \text{and} \quad X = \begin{tikzcd} \mathbf{F}_q & & 0 \\ & \mathbf{F}_q \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\ 0 & & \mathbf{F}_q \end{tikzcd}.$$

In this non-homogeneous tube, there are two indecomposables of dimension δ up to isomorphism: the representation $S^Q_{(1,0)}$ and the unique (up to isomorphism) extension of Y by X:

$$\begin{tikzcd} \mathbf{F}_q & & \mathbf{F}_q \\ & \mathbf{F}_q^2 \arrow[ul, "{(1,1)}"] \arrow[ur, "{(0,1)}"'] \arrow[dl, "{(0,1)}"'] \arrow[dr, "{(1,0)}"] & \\ \mathbf{F}_q & & \mathbf{F}_q \end{tikzcd}.$$

The type $A_4^{(1)}$

Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}.$$

The extending vertex is the bottom right vertex, we call it i_0 and the indivisible imaginary root δ is given in the graph above. The indecomposable representation of dimension $\theta = \delta - e_{i_0}$ is

$$\begin{tikzcd} \mathbf{F}_q \arrow[r, "1"] \arrow[d, "1"'] & \mathbf{F}_q \arrow[d] \\ \mathbf{F}_q \arrow[r] & 0 \end{tikzcd}.$$

The functor F induces on isomorphism classes the following map

$$\begin{array}{rccc} F: & \mathcal{M}^{K_2}_{\mathbf{F}_q}[(1,1)] \simeq \mathbf{P}^1_{\mathbf{F}_q}(\mathbf{F}_q) & \to & \mathcal{M}^Q_{\mathbf{F}_q}[\delta] \\ & [\lambda : \mu] & \mapsto & [S^Q_{(\lambda,\mu)}] \end{array}$$

where $S^Q_{(\lambda,\mu)}$ is the following representation of Q:

$$\begin{tikzcd} \mathbf{F}_q \arrow[r, "1"] \arrow[d, "1"'] & \mathbf{F}_q \arrow[d, "\mu"] \\ \mathbf{F}_q \arrow[r, "\lambda"'] & \mathbf{F}_q \end{tikzcd} .$$

The two non-homogeneous tubes correspond to the parameters $[\lambda : \mu] = [0 : 1]$ and $[\lambda : \mu] = [1 : 0]$ and both are of period two. Because of the symmetry of the quiver, we study only the case $[\lambda : \mu] = [1 : 0]$. The two regular simple representations of this tube are

$$Y = \begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbf{F}_q \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd} \quad \text{and} \quad X = \begin{tikzcd} \mathbf{F}_q \arrow[r] \arrow[d, "1"'] & 0 \arrow[d] \\ \mathbf{F}_q \arrow[r, "1"'] & \mathbf{F}_q \end{tikzcd} .$$

The two indecomposable of dimension δ in this tube are

$$\begin{tikzcd} \mathbf{F}_q \arrow[r, "1"] \arrow[d, "1"'] & \mathbf{F}_q \arrow[d, "0"] \\ \mathbf{F}_q \arrow[r, "1"'] & \mathbf{F}_q \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbf{F}_q \arrow[r, "1"] \arrow[d, "1"'] & \mathbf{F}_q \arrow[d, "1"] \\ \mathbf{F}_q \arrow[r, "0"'] & \mathbf{F}_q \end{tikzcd} .$$

Convention

In the sequel, we assume that a functor as in (3.2.3):

$$F : \mathrm{Rep}_{K_2}(\mathbf{F}_q) \to \mathrm{Rep}_Q(\mathbf{F}_q)$$

is fixed. The bijection (3.2.2) gives an explicit bijection

$$\begin{array}{rccc} C_Q : & |\mathbf{P}^1_{\mathbf{F}_q}| & \to & \{\text{tubes of } Q\} \\ & a & \mapsto & \text{tube of Q containing } F(C_a). \end{array}$$

where $F(C_a)$ is the essential image of the tube C_a of the Kronecker quiver by the functor F. We will sometimes also write C^a_Q the a-tube of Q.

The regular indecomposable representations of an affine quiver

In the previous theorem, we identified the simple regular representations in the homogeneous tubes. We give now an easy consequence concerning indecomposables in the tubes. We keep the notations of the previous sections.

Proposition 3.2.13. *Let $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$ be a closed point. If the a-tube of Q is homogeneous, then for $n \geq 1$, $F(I_{a,n})$ is the unique $n\deg(x)\delta$-dimensional indecomposable representation of this tube.*

Suppose the a-tube is non-homogeneous. Let X_a be the simple regular representation of Q in this tube which is nonzero at the extending vertex. Then $F(I_{a,n})$ is the indecomposable representation *of this tube of dimension $n\delta$ having X_a as quotient.*

Proof. This result is a consequence of the full faithfulness of F and the properties of the representations of a tube given by the identification of a tube with the category of nilpotent representations of the Jordan or cyclic quiver. $\square$

Let $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$ corresponding to a non-homogeneous tube $\mathcal{O}_a$ and $S_a = F(I_{a,1})$ as in the Proposition 3.2.13. Then the subcategory of $\mathcal{O}_a$ generated by S_a is isomorphic to the category of nilpotent representations of the Jordan quiver. When a is associated to a homogeneous tube, the restriction of F induces an equivalence of categories between the corresponding tubes of K_2 and Q.

3.3 The Hall algebra of a quiver

The Hall algebra of a quiver now has a long history beginning with the paper of Ringel, [Rin90b].

3.3.1 Definition of the Hall algebra of a quiver

We refer the reader to [Sch12a] for the proofs and the general theory of constructible Hall algebras. These can be defined for any abelian finitary category. We will consider it for the category $\mathrm{Rep}_Q(k)$ with $k = \mathbf{F}_q$. Let Q be a finite quiver and $k = \mathbf{F}_q$ a finite field. As a vector space, the Hall algebra of Q over k is defined as

$$\mathbf{H}_{Q,k} = \bigoplus_{[M]\in \mathrm{Ob}(\mathrm{Rep}_Q(k))/\sim} \mathbf{C}[M]$$

where $\mathrm{Ob}(\mathrm{Rep}_Q(k))/\sim$ is the set of representations of Q over k up to isomorphism.

It is convenient to see the Hall algebra as an algebra of functions endowed with some sort of convolution product. For this, let $\mathcal{M}_k = \mathrm{Rep}_Q(k)/\sim$ be the set of isomorphism classes of representations of Q over k and

$$\mathcal{M}_k = \bigsqcup_{\mathbf{d}\in\mathbf{N}^I} \mathcal{M}_k[\mathbf{d}]$$

its decomposition with respect to the dimension. Thus,

$$\mathbf{H}_{Q,k} = \bigoplus_{\mathbf{d}\in\mathbf{N}^I} \mathbf{H}_{Q,k}[\mathbf{d}], \qquad \mathbf{H}_{Q,k}[\mathbf{d}] = \mathrm{Fun}(\mathcal{M}_k[\mathbf{d}], \mathbf{C}) = \text{functions } \mathcal{M}_k[\mathbf{d}] \to \mathbf{C}.$$

We now define the operations. Let $\nu = |k|^{1/2}$ and for M a representation of Q, $a_M = |\mathrm{Aut}(M)|$.

1. (Multiplication) For $f, g \in \mathbf{H}_{Q,k}$,

$$f \star g = m(f,g) : [M] \mapsto \sum_{N\subseteq M} \nu^{\langle M/N, N\rangle} f([M/N])g([N])$$

where the sum is over subrepresentations N of M.

2. (Comultiplication) For $f \in \mathbf{H}_{Q,k}$ and $[M], [N] \in \mathcal{M}_k$,

$$\Delta(f)([M],[N]) = \frac{\nu^{-\langle \dim(M), \dim(N) \rangle}}{|\operatorname{Ext}^1(M,N)|} \sum_{\xi \in \operatorname{Ext}^1(M,N)} f([X_\xi]),$$

   where X_ξ is the middle term of an exact sequence representing the extension of M by N given by ξ.

3. (Green's scalar product) For $[M], [N] \in \mathbf{H}_{Q,k}$, define

$$([M],[N]) = \frac{\delta_{[M],[N]}}{a_M}.$$

These operations endow $\mathbf{H}_{Q,\mathbf{F}_q}$ with a twisted bialgebra structure (see the Introduction). The multiplication on $\mathbf{H}_{Q,\mathbf{F}_q} \otimes \mathbf{H}_{Q,\mathbf{F}_q}$ is defined for homogeneous x, y, z, w by $(x \otimes y)(z \otimes w) = \nu^{(y,z)}(xz \otimes yw)$. The comultiplication lies at the center of this chapter and it is in duality with the comultilication by (3.3.1): it is thus necessary to give other formulas for both the multiplication and the comultiplication, more adapted for explicit computations. For this purpose, we introduce the following notations.

For $M, N, R \in \operatorname{Rep}_Q \mathbf{F}_q$ three quiver representations, let us define

$$F'^R_{M,N} = |\{(\alpha,\beta) \in \operatorname{Hom}(N,R) \times \operatorname{Hom}(R,M) \mid 0 \to N \xrightarrow{\alpha} R \xrightarrow{\beta} M \to 0 \text{ is exact}\}|,$$

$$F^R_{M,N} = |\{X \subset R \mid X \simeq N \text{ and } R/X \simeq M\}|,$$
$$F^{M,N}_R = F^R_{M,N} \frac{a_M a_N}{a_R}.$$

The free action of $\operatorname{Aut}(N) \times \operatorname{Aut}(M)$ on $F'^R_{M,N}$ given by $(a,b) \cdot (\alpha,\beta) = (\alpha a^{-1}, b\beta)$ gives the equality

$$F'^R_{M,N} = a_M a_N F^R_{M,N}.$$

We have also Riedtmann's formula:

$$F^{M,N}_R = \frac{|\operatorname{Ext}^1(M,N)_R|}{|\operatorname{Hom}(M,N)|}$$

where $\operatorname{Ext}^1(M,N)_R$ is the subset of $\operatorname{Ext}^1(M,N)$ of extensions represented by an exact sequence with middle term isomorphic to R. We now have the following formulas for the multiplication and comultiplication: for $M, N, R \in \operatorname{Rep}_Q(k)$ three representations of Q over k,

$$[M] \star [N] = \nu^{\langle M,N \rangle} \sum_{[S] \in \mathcal{M}_k} F^S_{M,N}[S],$$

$$\Delta([R]) = \sum_{[U],[V] \in \mathcal{M}_k} \nu^{\langle U,V \rangle} F^{U,V}_R [U] \otimes [V].$$

The Green scalar product is a Hopf pairing, meaning that for any $f, g, h \in \mathbf{H}_{Q,k}$,

$$(fg, h) = (f \otimes g, \Delta(h)). \tag{3.3.1}$$

In this formula, we have implicitly naturally defined the scalar product on $\mathbf{H}_{Q,k} \otimes \mathbf{H}_{Q,k}$ by the formula:

$$([M] \otimes [N], [R] \otimes [S]) = ([M], [R])([N], [S])$$

for any $[M], [N], [R], [S] \in \mathbf{H}_{Q,k}$.

3.3.2 The dualization process and the Hall algebra

We saw in the subsection 3.2.1 how to to dualize representations of quivers to obtain from a representation M of Q a representation M^* of the dual quiver Q^*. This process induces a linear map between the Hall algebras

$$\begin{array}{rccc} D: & \mathbf{H}_{Q,k} & \to & \mathbf{H}_{Q^*,k} \\ & [M] & \mapsto & [M^*] \end{array}$$

which is a Hopf algebra graded anti-isomorphism. In particular, D induces a linear isomorphism between the spaces of primitive elements of a quiver and its dual.

3.3.3 A PBW basis for the Hall algebra

The Hall algebra construction can be extended to any finitary exact category, see [Sch12a] and references therein. Let $\mathbf{H}_{\mathcal{A}}$[7] be the Hall algebra of the finitary exact category $\mathcal{A}$.

Theorem 3.3.1 (Guo-Peng, Berenstein-Greenstein). *Let $\mathcal{A}$ be a finitary exact category. Then for any order on the set* $\operatorname{ind}\mathcal{A}$ *of isomorphism classes of indecomposable objects in $\mathcal{A}$, $\mathbf{H}_{\mathcal{A}}$ is spanned, as a $\mathbf{C}$-vector space, by ordered monomials on* $\operatorname{ind}\mathcal{A}$*. Moreover, if $\mathcal{A}$ is Krull-Schmidt, then such monomials form a basis of $\mathbf{H}_{\mathcal{A}}$.*

Proof. See [BG16, Theorem 2.4]. See [GP97, Theorem 3.1] for a quiver version. □

3.3.4 Cuspidal functions and the theorem of Sevenhant and Van den Bergh

In this section, we let Q be an arbitrary quiver (we allow multiple arrows, oriented cycles and edge loops) and $\mathbf{F}_q$ a finite field. Let $\mathbf{H}_{Q,\mathbf{F}_q}$ be the Hall algebra of Q over $\mathbf{F}_q$.

Cuspidal functions

We define here the objects of main interest in this chapter.

Definition 3.3.2. An element $f \in \mathbf{H}_{Q,\mathbf{F}_q}$ is called a *cuspidal function* if it is primitive *i.e.* if

$$\Delta f = f \otimes 1 + 1 \otimes f.$$

[7] It is an algebra and a coalgebra. This is a bialgebra if $\mathcal{A}$ is hereditary.

We let $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}$ be the space of cuspidal functions. It decomposes as a direct sum

$$\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q} = \bigoplus_{\mathbf{d}\in\mathbf{N}^I} \mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}].$$

A key fact in the proof of Theorem 3.3.4 below is the following.

Lemma 3.3.3. *Let $\mathbf{d} \in \mathbf{N}^I$. Then $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}]$ is the $\mathbf{d}$-graded component of the orthogonal with respect to Green's scalar product of the subspace*

$$\sum_{\mathbf{d}',\mathbf{d}''>0} \mathbf{H}_{Q,\mathbf{F}_q}[\mathbf{d}']\, \mathbf{H}_{Q,\mathbf{F}_q}[\mathbf{d}''].$$

Proof. See [SVDB01, 3.1]. □

The theorem of Sevenhant and Van den Bergh

This theorem motivates the study of cuspidal functions, as they were used by Sevenhant and Van den Bergh in their article [SVDB01] to identify the whole Hall algebra of a quiver with the specialization at $\nu = \sqrt{q}$ of the positive part of the quantized enveloping algebra of a generalized Kac-Moody algebra associated to that quiver.

Let $(f_j)_{j\in J}$ be a graded orthonormal basis of $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}$ with respect to Green's scalar product, so that in particular, for $j \in J$, $\dim f_j \in \mathbf{N}^I$ is well-defined. We define

$$a_{i,j} = (\dim f_i, \dim f_j).$$

for $i, j \in J$. By [SVDB01, Proposition 3.2], $a_{i,j} \leq 0$ for $i \neq j$ and if $a_{i,i} > 0$, then $a_{i,i} = 2$. The infinite matrix $(a_{i,j})$ is a generalized Cartan matrix in the sense of [Bor88].

Theorem 3.3.4 (Sevenhant-Van den Bergh, [SVDB01])**.** *The Hall algebra $\mathbf{H}_{Q,\mathbf{F}_q}$ is the bialgebra generated by the primitive elements $(f_j)_{j\in J}$ subject to the following relations:*

1. *For all $i, j \in J$, if $a_{i,j} = 0$, then $f_i f_j = f_j f_i$,*
2. *For all $i, j \in J$, if $a_{i,i} = 2$, then*

$$\sum_{l=0}^{1-a_{i,j}} (-1)^l \begin{Bmatrix} 1-a_{i,j} \\ l \end{Bmatrix} f_i^l f_j f_i^{1-a_{i,j}+l} = 0$$

where, for any integers r and s, $\begin{Bmatrix} s \\ r \end{Bmatrix}$ is the ν-binomial coefficient defined by:

$$\begin{Bmatrix} s \\ r \end{Bmatrix} = \prod_{u=1}^{r} \frac{\nu^{u+s-r} - \nu^{-(u+s-r)}}{\nu^u - \nu^{-u}}$$

and $\nu = \sqrt{q}$.

Remark 3.3.5. From what we know about the Hall algebra, there is no natural choice for the basis $(f_j)_{j\in J}$. We hope to tackle this question in the future.

3.4 Kac and cuspidal polynomials of an affine quiver over a finite field

3.4.1 Indecomposable and absolutely indecomposable representations count

For a fundamental contribution on the count of representations of quivers over finite fields, see [Hua00]. Let Q be an arbitrary quiver. We denote by $A_{Q,\mathbf{d}}(q)$, $\mathbf{d}\in\mathbf{N}^I$ the Kac polynomial of Q counting absolutely indecomposables representations of dimension $\mathbf{d}$ over $\mathbf{F}_q$ and $I_{Q,\mathbf{d}}(q)$ the polynomial counting indecomposable representations of Q of dimension $\mathbf{d}$ over $\mathbf{F}_q$. The following formula is well-known and is a consequence of Galois descent for quiver representations: for $\mathbf{d}\in\mathbf{N}^I$ indivisible and $r\geq 1$,

$$I_{Q,r\mathbf{d}}(q)=\sum_{l\mid r}\frac{1}{l}\sum_{m\mid l}\mu(m)A_{Q,\frac{r}{l}\mathbf{d}}(q^{\frac{l}{m}}),$$

where μ is the Möbius function. See also [Hua00, Theorem 4.1]. Sometimes, this formula is presented using plethystic operations (see [BS18] for basics on plethystic notation):

$$\mathrm{Exp}_z\left(\sum_{\mathbf{d}>0}I_{Q,\mathbf{d}}(q)z^{\mathbf{d}}\right)=\mathrm{Exp}_{t,z}\left(\sum_{\mathbf{d}>0}A_{Q,\mathbf{d}}(t)z^{\mathbf{d}}\right).$$

These polynomials do not depend on the orientation of the graph Q.

3.4.2 Dimension count of cuspidal functions

Let Q be an arbitrary quiver. For complements on this section, see [BS18]. For $\mathbf{d}\in\mathbf{N}^I$, we let

$$C_{Q,\mathbf{d}}(q)=\dim_{\mathbf{C}}\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}].$$

We will not use the following in the sequel.

Theorem 3.4.1 (Bozec-Schiffmann, [BS18])**.** *The function $C_{Q,\mathbf{d}}(q)$ is a polynomial in $\mathbf{Q}[q]$.*

In *loc. cit.*, Bozec and Schiffmann defined the absolutely cuspidal polynomials of a quiver Q. They are characterized as follows.

Proposition 3.4.2 (Bozec-Schiffmann, [BS18])**.** *The absolutely cuspidal polynomials of Q form the unique family of polynomials $(C^{abs}_{Q,\mathbf{d}}(t))_{\mathbf{d}\in\mathbf{N}^I}$ satisfying the following conditions.*

1. *If $\mathbf{d}\in\mathbf{N}^I$ is hyperbolic, $C^{abs}_{Q,\mathbf{d}}(t)=C_{Q,\mathbf{d}}(t)$,*
2. *If $\mathbf{d}\in\mathbf{N}^I$ is isotropic and indivisible, then*

$$\mathrm{Exp}_z\left(\sum_{r>0}C_{Q,\mathbf{d}}(t)z^{\mathbf{d}}\right)=\mathrm{Exp}_{t,z}\left(\sum_{r>0}C^{abs}_{Q,\mathbf{d}}(t)z^{\mathbf{d}}\right).$$

Conjecture 3.4.3 (Bozec-Schiffmann, [BS18, Conjecture 1.3]). *For any Q and $\mathbf{d} \in \mathbf{N}^I$, we have $C^{abs}_{Q,\mathbf{d}}(t) \in \mathbf{N}[t]$.*

We prove this conjecture for $\mathbf{d}$ isotropic, see Corollary 3.8.3.

3.4.3 Kac and cuspidal polynomials of affine quivers

For affine quivers, explicit expressions of the considered polynomials are known.

Kac polynomials of affine quivers

Theorem 3.4.4. *Let Q be an affine quiver, $\delta \in \mathbf{N}^I$ its indecomposable imaginary root, and $k = \mathbf{F}_q$ a finite field with q elements.*

1. *If $\mathbf{d} \in \mathbf{N}^I$ is a real root (*i.e. *if $\langle \mathbf{d}, \mathbf{d} \rangle = 1$), there is a unique indecomposable representation over $\mathbf{F}_q$, and it is absolutely indecomposable: $A_{Q,\mathbf{d}}(q) = 1 = I_{Q,\mathbf{d}}(q)$,*

2. *If $\mathbf{d} \in \mathbf{N}^I$ is an imaginary root, then it is a positive multiple $r\delta$ of the indecomposable imaginary root and $A_{Q,\mathbf{d}}(t) = t + n_0$,* i.e. *there are $q + n_0$ absolutely indecomposable representations over $\mathbf{F}_q$ where n_0 is the number of vertices of Q minus one (the number of vertices of the finite type quiver associated to Q).*

Proof. See [HX02, 5.1]. □

Cuspidal polynomials of affine quivers

For acyclic affine quivers, [HX02] countains all the formulas we need. Let $Q = (I, \Omega)$ be an affine quiver.

Proposition 3.4.5 (Hua-Xiao, [HX02, 5.2]). *For $r \geq 1$, we have:*

$$C_{Q,r\delta}(t) = I_{Q,r\delta}(t) - n_0 \tag{3.4.1}$$

and

$$C^{abs}_{Q,r\delta}(t) = t.$$

For $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbf{N}^I$, we have

$$C_{Q,e_i} = C^{abs}_{Q,e_i} = 1.$$

For $\mathbf{d} \notin \{e_i : i \in I\} \cup \{r\delta : r \geq 1\}$,

$$C_{Q,\mathbf{d}} = 0.$$

In fact, this formulas remains true for any affine quiver, *i.e.* for the Jordan and cyclic quivers, as it is clear from the formulas of [BS18, 1.2].

3.5 Cuspidal functions of the Jordan and cyclic quivers

Let Q be a quiver. The category of nilpotent representations of Q, $\mathrm{Rep}_Q^{\mathrm{nil}}(\mathbf{F}_q)$ is a sub-category of $\mathrm{Rep}_Q(\mathbf{F}_q)$ stable under extensions and taking subobjects. Therefore, denoting by $\mathcal{M}_{\mathbf{F}_q}^{Q,\mathrm{nil}}$ the set of isomorphism classes of nilpotent representations of Q, the subspace of nilpotently supported functions $\mathbf{H}_{Q,\mathbf{F}_q}^{\mathrm{nil}} = \bigoplus_{[M]\in\mathcal{M}_{\mathbf{F}_q}^{Q,\mathrm{nil}}} \mathbf{C}[M]$ is in fact a sub-Hopf-algebra. Its primitive elements are called nilpotent cuspidal functions. In this section, we put the emphasis on nilpotent cuspidal functions of the Jordan and cyclic quivers. We will also see that arbitrary cuspidal functions can be expressed in terms of nilpotent cuspidal functions.

3.5.1 Nilpotent cuspidal functions of the Jordan quiver

All in this section has been known for almost a century, as Steinitz introduced an algebra structure on the complex vector space generated by isomorphism classes of finite length modules over a discrete valuation ring with finite residue field. The link with Macdonald's ring of symmetric functions is explained in [Mac15]. We recall here what we need for the convenience of the reader – who is asked to look at the references for the proofs.

Let Λ be Macdonald's ring of symmetric functions. It is constructed as the direct limit in the category of graded rings of the system

$$(\mathbf{Z}[x_1,\dots,x_n]^{\mathfrak{S}_n}, f(x_1\dots,x_{n+1})\in\mathbf{Z}[x_1,\dots,x_{n+1}]^{\mathfrak{S}_{n+1}} \to f(x_1,\dots,x_n,0)\in\mathbf{Z}[x_1,\dots,x_n]^{\mathfrak{S}_n}).$$

The $\mathbf{Z}$-algebra Λ is endowed with a structure of Hopf algebra with comultiplication Δ obtained by considering the direct limit of the system of applications

$$\Delta_n : \mathbf{Z}[x_1,\dots,x_{2n}]^{\mathfrak{S}_{2n}} \longrightarrow \mathbf{Z}[x_1,\dots,x_n]^{\mathfrak{S}_n}\otimes\mathbf{Z}[x_1,\dots,x_n]^{\mathfrak{S}_n} \tag{3.5.1}$$

where $\Delta_n(x_i) = x_{i/2}\otimes 1$ if i is even, and $\Delta_n(x_i) = 1\otimes x_{\frac{i+1}{2}}$ if i is odd. As usual, we let

$$e_r = \sum_{i_1<\dots<i_r} x_{i_1}\dots x_{i_r},$$

$$p_r = \sum_i x_i^r$$

for $r\geq 1$. We do not recall the definition of Hall-Littlewood symmetric functions $P_\lambda(x;t)$ for a partition λ: see [Mac15, Chapter III]. The elements p_r are the normalized primitive elements of Λ. We now state the main theorem of this section.

Theorem 3.5.1. *We have an isomorphism of Hopf algebras:*

$$\begin{array}{cccc} \varphi_q: & \mathbf{H}_{Q_0}^{\mathrm{nil}} & \longrightarrow & \Lambda\otimes\mathbf{C} \\ & [I_{(1^r)}] & \longmapsto & q^{\frac{-r(r-1)}{2}}e_r. \end{array} \tag{3.5.2}$$

Moreover, for any partition λ:

$$\varphi_q([I_\lambda]) = q^{-n(\lambda)} P_\lambda(x; q^{-1}) \tag{3.5.3}$$

where P_λ is the Hall-Littlewood symmetric function associated to the partition λ and $n(\lambda) = \sum_i (i-1)\lambda_i$.

Proof. See [Mac15], 3.4 page 217. □

Define

$$\widetilde{p}_r = \varphi_q^{-1}(p_r)$$

the cuspidal function of dimension r of the Jordan quiver. For $m \geq 0$, define $\phi_m(t) = \prod_{i=1}^{m}(1-t^i)$ and let $l(\lambda)$ be the length of the partition λ. We have the following closed formula for the nilpotent cuspidal functions of the Jordan quiver.

Theorem 3.5.2. *The r-dimensional cuspidal function over $\mathbf{F}_q$ of Q_0 is:*

$$\widetilde{p}_r = \sum_{|\lambda|=r} \phi_{l(\lambda)-1}(q)[I_\lambda]. \tag{3.5.4}$$

Proof. This formula already appears in [Sch04a], formula 4.1 or in [BG16, 3.2]. See references therein. □

3.5.2 Cuspidal functions for the Jordan quiver

The following theorem provides a basis for cuspidal functions for the Jordan quiver J.

Theorem 3.5.3. *Let $a \in |\mathbf{A}^1_{\mathbf{F}_q}|$ a point of degree d. Fix $t_a \in \mathbf{A}^1_{\mathbf{F}_q}(\mathbf{F}_q)$ such that the image of $t_a : \operatorname{Spec} \mathbf{F}_{q^d} \to \mathbf{A}^1_{\mathbf{F}_{q^d}}$ is a. We define $\widetilde{p}_{r,t_a} \in \mathbf{H}_{J,\mathbf{F}_{q^d}}$ associated to t_a as*

$$\widetilde{p}_{r,t_a} = \sum_{|\lambda|=r} \phi_{l(\lambda)-1}(q^d)[t_a I_d + I_\lambda]. \tag{3.5.5}$$

This is a cuspidal function of Q over $\mathbf{F}_{q^d}$. We have a linear map:

$$F_d : \mathbf{H}_{J,\mathbf{F}_{q^d}} \to \mathbf{H}_{J,\mathbf{F}_q}$$

induced by the forgetfull functor $F'_d : \operatorname{Rep}_J(\mathbf{F}_{q^d}) \to \operatorname{Rep}_J(\mathbf{F}_q)$. The set

$$\{F_d(\widetilde{p}_{r,t_a}) \mid r \geq 1, a \in |\mathbf{A}^1_{\mathbf{F}_q}|\}$$

is a basis of cuspidal functions of J over $\mathbf{F}_q$.

Proof. The functions $F_d(\widetilde{p}_{r,t_a})$ for $d \geq 1$, $r \geq 1$ and $a \in |\mathbf{A}^1_{\mathbf{F}_q}|$ of degree d are clearly linearly independent, since they have disjoint support. For $e \geq 1$, the number of such functions of dimension e is the number of closed points of $\mathbf{A}^1_{\mathbf{F}_q}$ of degree less or equal to e. This is also the number of irreducible monic polynomials in $\mathbf{F}_q[T]$ with degree less than or equal to e. This number is the dimension of $\mathbf{H}^{\mathrm{cusp}}_{J,\mathbf{F}_q}[e]$, see [BS18, Examples 1.2 and Proposition 4.1]. Therefore, it remains to show that for $d \geq 1$, $r \geq 1$ and $a \in \mathbf{A}^1_{\mathbf{F}_q}$ a closed point of degree d, $F_d(\widetilde{p}_{r,t_a})$ is indeed a cuspidal function in $\mathbf{H}_{J,\mathbf{F}_q}$.

Let $t \in |\mathbf{A}^1_{\mathbf{F}_q}|$ be a closed point of degree d, r an integer and λ a partition of r. It suffices to show that the number of automorphisms of $F'_d(tI_r + J_\lambda)$ and that of $tI_r + J_\lambda$ coincide, and that all subrepresentations of $F'_d(tI_r + J_\lambda)$ are of the form $F'_d(N)$ for N a subrepresentation of $tI_r + J_\lambda$.

Step 1: The number of automorphisms. To prove that the number of automorphisms of $F'_d(tI_r + J_\lambda)$ and that of $tI_r + J_\lambda$ coincide, it suffices to show that an automorphism ψ of $F'_d(tI_r + J_\lambda)$ is $\mathbf{F}_{q^d}$-linear. But – by definition of morphisms of quiver representations – ψ has to commute with $tI_r + J_\lambda$ hence with its semisimple part, which is tI_r. Therefore, ψ commutes with the multiplication by t. Since t is of degree d over $\mathbf{F}_q$, $\mathbf{F}_q[t] = \mathbf{F}_{q^d}$ and ψ is $\mathbf{F}_{q^d}$-linear, that is exactly what we wanted to prove.

Step 2: The subrepresentations. Let $N \subset F'_d(tI_r + J_\lambda)$ be a subrepresentation, *i.e.* N is a $\mathbf{F}_q$-subspace of $\mathbf{F}^r_{q^d}$ stable under the linear map $tI_r + J_\lambda$. Since the semisimple part tI_r of $tI_r + J_\lambda$ is a polynomial with coefficients in $\mathbf{F}_q$ without constant term of $tI_r + J_\lambda$, N is stable under tI_r. Therefore, the multiplication by t leaves N invariant: N is a $\mathbf{F}_{q^d}$-subspace of $\mathbf{F}^r_{q^d}$. □

3.5.3 Nilpotent cuspidal functions of cyclic quivers

Let $n \geq 2$ and C_n be the cyclic quiver of length n.

Theorem 3.5.4. *For any $r \geq 1$, $\dim_{\mathbf{C}} \mathbf{H}^{\mathrm{nil\,cusp}}_{C_n,\mathbf{F}_q}[r\delta] = 1$.*

Proof. In fact, Schiffmann proved the following result, see [Sch12a, Proposition 3.25]. Let U be the two sided ideal of $\mathbf{H}^{\mathrm{nil}}_{C_n,\mathbf{F}_q}$ generated by the classes $[S_i]$ of simple representations of dimension e_i for $0 \leq i \leq n$ and let $\mathbf{R} = U^\perp$ be its orthogonal with respect to Green's scalar product. Then $\mathbf{R}$ is a sub-Hopf algebra of $\mathbf{H}^{\mathrm{nil}}_{C_n,\mathbf{F}_q}$ isomorphic to a graded polynomial ring with countably many variables

$$\mathbf{R} \simeq \mathbf{C}[x_j : j \geq 1] \tag{3.5.6}$$

with $\deg(x_j) = j\delta$ and $\Delta(x_j) = x_j \otimes 1 + 1 \otimes x_j$. This immediately implies our result. Indeed, if f is a cuspidal function of dimension $r\delta$, then it is orthogonal to $[S_i]$ for $i \in I$ and to any nontrivial products. Therefore, $f \in \mathbf{R}$. By (3.5.6), a cuspidal function in $\mathbf{R}$ is a linear combination of x_j, $j \geq 1$. This proves Theorem 3.5.4. □

Lemma 3.5.5. *Let $d \geq 1$. Let $f \in \mathbf{H}^{\mathrm{nil\,cusp}}_{C_n,\mathbf{F}_q}[d\delta]$ be non-zero. Then for I a nilpotent indecomposable representation of C_n of dimension $d\delta$, $f(I) \neq 0$. Furthermore, $f(I)$ does not depend on the chosen indecomposable I.*

Proof. By Theorem 3.3.1, $\{[I] :\ \text{I nilpotent indecomposable}\}$ generates $\mathbf{H}^{\mathrm{nil}}_{Q,\mathbf{F}_q}$ as an algebra. For $d \geq 1$, consider the orthogonal projection with respect to Green's scalar product:

$$\pi : \mathbf{H}^{\mathrm{nil}}_{C_n,\mathbf{F}_q}[d\delta] \to \mathbf{H}^{\mathrm{nil\,cusp}}_{C_n,\mathbf{F}_q}[d\delta].$$

By Lemma 3.3.3, π restricts to a surjective linear map:

$$\pi : \mathbf{H}^{\text{nil,indec}}_{C_n,\mathbf{F}_q}[d\delta] \to \mathbf{H}^{\text{nil cusp}}_{C_n,\mathbf{F}_q}[d\delta]$$

where $\mathbf{H}^{\text{nil indec}}_{C_n,\mathbf{F}_q}$ is the subspace of $\mathbf{H}_{C_n,\mathbf{F}_q}$ generated by the basis elements $[M]$ for M indecomposable. Being surjective, this morphism is nonzero and there exists an indecomposable nilpotent representation M of dimension $d\delta$ of C_n such that $\pi([M]) \neq 0$. This precisely means that $(f, [M]) \neq 0$ since $\mathbf{H}^{\text{nil cusp}}_{Q,\mathbf{F}_q}[d\delta]$ is one-dimensional, and therefore $f([M]) \neq 0$. The group $\mathbf{Z}/n\mathbf{Z}$ acts by rotations on the quiver C_n, inducing an action of $\mathbf{Z}/n\mathbf{Z}$ by Hopf algebra automorphisms on $\mathbf{H}^{\text{nil}}_{C_n,\mathbf{F}_q}$ preserving $\mathbf{H}^{\text{nil cusp}}_{C_n}[r\delta]$. Since the latter space is one-dimensional, $\mathbf{Z}/n\mathbf{Z}$ acts on it through a character $\mathbf{Z}/n\mathbf{Z} \to \mathbf{C}^*$, $i \mapsto \zeta^i$ for some n-th root of unity ζ.

We show that $\zeta = 1$. For M a representation of C_n of dimension $d\delta$, $\Delta([M])(V_{0,dn-1}, S_{-1}) \neq 0$ if and only if $M \simeq V_{0,dn}$ or $M \simeq V_{0,dn-1} \oplus S_{-1}$ and $\Delta([M])(S_{-1}, V_{0,dn-1}) \neq 0$ if and only if $M \simeq V_{-1,dn}$ or $M \simeq V_{0,dn-1} \oplus S_{-1}$. Thus, we may write $f = [V_{0,nd}] + \zeta[V_{-1,nd}] + c[V_{0,nd-1} \oplus S_{-1}] + g$ where $\Delta(g)(S_{-1}, V_{0,dn-1}) = \Delta(g)(V_{0,dn-1}, S_{-1}) = 0$ and $c \in \mathbf{C}$ is some complex number. Computing the comultiplications, we obtain:

$$\begin{aligned} 0 =& \Delta(f)(V_{0,nd-1}, S_{-1}) \\ =& \nu^{\langle V_{0,nd-1}, S_{-1}\rangle} \left(\frac{|\operatorname{Aut}(S_{-1})||\operatorname{Aut}(V_{0,nd-1})|}{|\operatorname{Aut}(V_{0,nd})|} + c\frac{|\operatorname{Aut}(S_{-1})||\operatorname{Aut}(V_{0,nd-1})|}{|\operatorname{Aut}(V_{0,nd-1} \oplus S_{-1})|} \right) \end{aligned}$$

and

$$\begin{aligned} 0 =& \Delta(f)(S_{-1}, V_{0,nd-1}) \\ =& \nu^{\langle S_{-1}, V_{0,nd-1}\rangle} \left(\zeta\frac{|\operatorname{Aut}(S_{-1})||\operatorname{Aut}(V_{0,nd-1})|}{|\operatorname{Aut}(V_{-1,nd})|} + c\frac{|\operatorname{Aut}(S_{-1})||\operatorname{Aut}(V_{0,nd-1})|}{|\operatorname{Aut}(I_{-1,nd-1} \oplus S_{-1})|} \right). \end{aligned}$$

From the first equation, it follows that $c \in \mathbf{Q}_{<0}$ and from the second equation, $\zeta \in \mathbf{Q}\,c$. So $\zeta \in \mathbf{Q}$ and $\zeta \in \mathbf{R}_{>0}$. This concludes the proof.

□

Remark 3.5.6. There is no known closed formula for nilpotent cuspidal functions of the Hall algebra of cyclic quivers.

3.5.4 Cuspidal functions of cyclic quivers

Let C_n be the cyclic quiver of length n. As the functor G_n (see 3.2.1) is an equivalence of categories, we can give an explicit formula for invertible cuspidal functions of cyclic quivers. For $a \in |\mathbf{A}^1_{\mathbf{F}_q}|$ of degree d, let

$$I_{a,\lambda} = G_n(F'_d(t_a I + J_\lambda)),$$

where $t_a \in \mathbf{A}^1_{\mathbf{F}_q}(\mathbf{F}_q)$ represents a (see Theorem 3.5.3).

Proposition 3.5.7. *A basis of* $\mathbf{H}^{\mathrm{cusp\,inv}}_{C_n,\mathbf{F}_q}$ *is given by the functions*

$$f_{a,s} = \sum_{|\lambda|=s} \left(\prod_{j=1}^{l(\lambda)-1} (1-q^j) \right) [I_{a,\lambda}]$$

for $a \in |\mathbf{A}^1_{\mathbf{F}_q}| \setminus \{0\}$ *and* $s \geq 1$.

Proof. This is a consequence of Theorem 3.5.3. □

3.6 Cuspidal functions of affine quivers

3.6.1 Decomposition of cuspidal functions

Proposition 3.6.1. *A cuspidal function* $f \in \mathbf{H}_{Q,k}$ *in the Hall algebra of an affine quiver over a finite field decomposes as* $f = f_{\mathcal{P}} + f_{\mathcal{I}} + f_{\mathcal{R}}$ *where* $f_{\mathcal{R}}$ *(resp.* $f_{\mathcal{I}}$*, resp.* $f_{\mathcal{P}}$*) is a cuspidal function whose support consists of regular (resp. preinjective, resp. preprojective) representations.*

Proof. Let $f \in \mathbf{H}^{\mathrm{cusp}}_{Q,k}$ be a cuspidal function. Let $M = P \oplus R \oplus I$ be a representation of Q over k written as a direct sum of its preprojective, preinjective and regular summands, that is P is the direct sum of the indecomposable preprojective direct summands of M, I is the direct sum of the indecomposable preinjective direct summands of M and R is the direct sum of the indecomposable regular direct summands of M. Suppose moreover that at least two of the representations P, R, I are nonzero. Then the proposition is equivalent to $f([M]) = 0$ for all such M. But, because of the properties of extensions and morphisms (see Proposition 5.2.3), in $\mathbf{H}_{Q,k}$, we have $[M] = [P][R][I]$. This is a non trivial product, and therefore, f is orthogonal to $[M]$ with respect to Green's scalar product. But this precisely means that $f([M]) = 0$ and implies the decomposition $f = f_{\mathcal{P}} + f_{\mathcal{I}} + f_{\mathcal{R}}$. It remains to show that $f_{\mathcal{P}}, f_{\mathcal{R}}$ and $f_{\mathcal{I}}$ are cuspidal. This comes from the fact that f is cuspidal and $\Delta(f_{\mathcal{P}})$ is supported on $\{(M,N) : \partial M + \partial N < 0\}$ while $\Delta(f_{\mathcal{R}})$ is supported on $\{(M,N) : \partial M + \partial N = 0\}$ and $\Delta(f_{\mathcal{I}})$ is supported on $\{(M,N) : \partial M + \partial N > 0\}$. □

Proposition 3.6.2. *Let* $r \geq 1$ *and* $f \in \mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]$*. Then* f *is supported on regular representations.*

Proof. Let $f = f_{\mathcal{P}} + f_{\mathcal{I}} + f_{\mathcal{R}}$ be the decomposition given by Proposition 3.6.1. By Theorem 3.3.1, $\{[I] :\ \text{I indecomposable}\}$ generates $\mathbf{H}_{Q,\mathbf{F}_q}$ as an algebra. For $d \geq 1$, consider the orthogonal projection with respect to Green's scalar product:

$$\pi : \mathbf{H}_{C_n,\mathbf{F}_q}[r\delta] \to \mathbf{H}^{\mathrm{cusp}}_{C_n,\mathbf{F}_q}[r\delta].$$

By Lemma 3.3.3, π restricts to a surjective linear map:

$$\pi : \mathbf{H}^{\mathrm{indec}}_{C_n,\mathbf{F}_q}[r\delta] \to \mathbf{H}^{\mathrm{cusp}}_{C_n,\mathbf{F}_q}[r\delta].$$

Let M be an indecomposable representation of dimension $r\delta$. It is regular. Therefore, $(f_{\mathcal{P}}, [M]) = (f_{\mathcal{I}}, [M]) = 0$ and $f_{\mathcal{I}} = f_{\mathcal{P}} = 0$. □

3.6.2 Regular Hall algebra of affine quivers

In all this section, Q is an acyclic affine quiver and $\mathbf{F}_q$ a finite field.

We denote by $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ the subspace of $\mathbf{H}_{Q,\mathbf{F}_q}$ generated by classes of regular representations $[M]$, for $M \in \mathrm{Rep}_Q^{\mathcal{R}}(\mathbf{F}_q)$. Since $\mathrm{Rep}_Q^{\mathcal{R}}(\mathbf{F}_q)$ is stable under extensions, this is in fact a subalgebra. We denote by m the induced multiplication. The comultiplication $\Delta : \mathbf{H}_{Q,k} \to \mathbf{H}_{Q,k} \otimes \mathbf{H}_{Q,k}$ induces a linear map

$$\Delta_{\mathcal{R}} : \mathbf{H}_{Q,k,\mathcal{R}} \to \mathbf{H}_{Q,k,\mathcal{R}} \otimes \mathbf{H}_{Q,k,\mathcal{R}}$$

which is the composition

$$\mathbf{H}_{Q,k,\mathcal{R}} \to \mathbf{H}_{Q,k} \to \mathbf{H}_{Q,k} \otimes \mathbf{H}_{Q,k} \to \mathbf{H}_{Q,k\mathcal{R}} \otimes \mathbf{H}_{Q,k,\mathcal{R}}$$

where the first arrow is the inclusion, the second is the comultiplication Δ and the last the orthogonal projection with respect to Green's scalar product. When elements of $\mathbf{H}_{Q,\mathbf{F}_q}$ are seen as functions on isoclasses of representations of Q over $\mathbf{F}_q$, the last arrow is the restriction of functions to isoclasses of regular representations.

The restriction of Green's scalar product to $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ induces a hermitian scalar product $(-,-)$.

Proposition 3.6.3. *The two operations m and $\Delta_{\mathcal{R}}$ endow $\mathbf{H}_{Q,k,\mathcal{R}}$ with a bialgebra structure and the multiplication is adjoint to the comultiplication for the restriction of Green's scalar product.*

Proof. We already noticed that m is an associative bilinear map. We have to prove that $\Delta_{\mathcal{R}}$ is coassociative, compatible with the multiplication m and that m and $\Delta_{\mathcal{R}}$ are adjoint for Green's scalar product. The coassociativity will follow from the associativity and the adjunction property.

Adjunction property. Let $M, N, R \in \mathrm{Rep}_Q^{\mathcal{R}}(\mathbf{F}_q)$ be three regular representations of Q. Then,

$$(\Delta([R]), [M] \otimes [N]) = (\Delta_{\mathcal{R}}([R]), [M] \otimes [N]).$$

But by adjunction for m and Δ in $\mathbf{H}_{Q,\mathbf{F}_q}$,

$$(\Delta([R]), [M] \otimes [N]) = ([R], [M][N]).$$

Therefore,

$$(\Delta_{\mathcal{R}}([R], [M] \otimes [N]) = ([R], [M][N]).$$

This proves the adjunction for m and $\Delta_{\mathcal{R}}$.

Compatibility of m and $\Delta_{\mathcal{R}}$. Let $M, N \in \mathrm{Rep}_Q^{\mathcal{R}}(\mathbf{F}_q)$. In $\mathbf{H}_{Q,\mathbf{F}_q}$, we have the equation

$$\Delta([M][N]) = \Delta([M])\Delta([N]).$$

Thanks to the properties of morphisms between preprojective, regular and preinjective representions (see Proposition 5.2.3), if $S \subset M$ is a subrepresentation, then its indecomposable summands can only be preprojective or regular. Moreover, S is regular if and only if its defect ∂S is zero, since preprojective representations have negative defect. Suppose S is not regular. If S' is an arbitrary subrepresentation of N, then for any regular representations A and B, $([S]\otimes[M/S])([S']\otimes[N/S'])(A,B) = 0$ since the support of $[S][S']$ contains only representations of defect $\partial S + \partial S' < 0$. By reversing the role of S and S', we obtain the formula

$$\Delta_{\mathcal{R}}([M][N]) = \Delta_{\mathcal{R}}([M])\Delta_{\mathcal{R}}([N]).$$

This is precisely what we wanted. □

3.6.3 Regular cuspidal functions of affine quivers

A function $f \in \mathbf{H}_{Q,k,\mathcal{R}}$ is said to be *cuspidal regular* if $\Delta_{\mathcal{R}}(f) = f \otimes 1 + 1 \otimes f$. From Proposition 3.6.2, we have the inclusion

$$\mathbf{H}^{\mathrm{cusp}}_{Q,k}[r\delta] \subset \mathbf{H}^{\mathrm{cusp}}_{Q,k,\mathcal{R}}[r\delta]$$

for any $r \geq 1$.

Recall the decomposition of $\mathrm{Rep}^{\mathcal{R}}_{Q}(\mathbf{F}_q)$ into blocks C^Q_a for $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$. As a consequence, we have the following proposition.

Proposition 3.6.4. *There is an isomorphism of Hopf algebras:*

$$\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}} \simeq \bigotimes_{a \in |\mathbf{P}^1_{\mathbf{F}_q}|} \mathbf{H}_{C^Q_a}$$

between the regular Hall algebra and the restricted tensor product of Hall algebras of tubes.

Theorem 3.6.5. *For any $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$ and any $n > 0$, $\dim(\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}} \cap \mathbf{H}_{C^Q_a}[n\deg(a)]) = 1$.*

Proof. This is a consequence of Proposition 3.6.4. Indeed, if C^Q_a is a homogeneous tube, $\mathbf{H}_{C^Q_a} \simeq \mathbf{H}_{J,\mathbf{F}_q^{\deg(a)}}$ and if C^Q_a is a non-homogeneous tube of period p, $\mathbf{H}_{C^Q_a} \simeq \mathbf{H}_{C_p,\mathbf{F}_q}$. □

Normalization of cuspidal functions

The space of cuspidal functions whose support is contained in a given tube is one dimensional. We give here a natural way to normalize them. Take $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$ a closed point and $n \geq 1$. If the corresponding tube is homogeneous, it contains exactly one indecomposable representation $I = I_{a,n}$ of dimension $n\deg(a)\delta$ and if f is a cuspidal function whose support is contained in this tube, thanks to formula (3.5.4), $f(I) \neq 0$. We may normalize f such that $f(I) = 1$.

If a corresponds to a non-homogeneous tube of period p, then for $n \geq 1$, it contains p indecomposables (up to isomorphism) $I_1, \ldots, I_p$ of dimension $n\delta$. By Proposition 3.5.5, if f is a nonzero cuspidal function whose support is contained in this tube, then $f(I_1) = \ldots = f(I_p) \neq 0$. We may therefore normalize f by fixing the value $f(I_1) = 1$.

Definition 3.6.6. A dimension vector $\mathbf{d} \in \mathbf{N}^I$ is said to be cuspidal if there exists a nonzero cuspidal function of dimension $\mathbf{d}$. We say sometimes that a regular cuspidal function is in a given tube when its support is contained in this tube.

We define an homogeneous basis $\mathcal{B}$ of $\mathbf{H}^{\text{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}$. From Theorem 3.6.5, for each tube indexed by an element $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$, the space of cuspidal functions whose support is contained in this tube is one dimensional. We use the previous normalization and for $d \geq 1$, we denote by $f_{a,d}$ the unique corresponding normalized cuspidal function. We define now

$$\mathcal{B} = \{f_{a,d} : a \in |\mathbf{P}^1_{\mathbf{F}_q}|, d \geq 1\}.$$

This is a homogeneous basis of regular cuspidal functions of Q.

3.6.4 Comparison of dimensions

For any $\mathbf{d} = s\delta \in \mathbf{N}^I$, the previous theorem gives a basis of $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ containing

$$I_{Q,\mathbf{d}}(q) - \sum_{i=1}^{d}(p_i - 1) = \text{ number of tubes of degree} \leq s$$

elements, where we recall that d is the number of non-homogeneous tubes and p_i, $1 \leq i \leq d$ denote their respective periods. Moreover, the table in Theorem 5.2.5 gives

$$\sum_{i=1}^{d} p - i - d = n_0 - 1$$

where n_0 is the number of vertices of Q minus one. Therefore,

$$\dim_{\mathbf{C}} \mathbf{H}^{\text{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[s\delta] = I_{Q,s\delta}(q) - n_0 + 1$$

and, by Equation (3.4.1),

$$\dim_{\mathbf{C}} \mathbf{H}^{\mathcal{R}}_{Q,\mathbf{F}_q}[s\delta] = C_{Q,s\delta}(q) + 1$$

which means that the subspace $\mathbf{H}^{\text{cusp}}_{Q,\mathbf{F}_q}[s\delta] \subset \mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}[s\delta]$ is of codimension 1.

In Theorem 3.6.9, we will construct an explicit linear form on $\mathbf{H}^{\text{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[s\delta]$ whose kernel is $\mathbf{H}^{\text{cusp}}_{Q,\mathbf{F}_q}[s\delta]$ for any $s \geq 1$.

3.6.5 Link between regular cuspidal functions of the Kronecker quiver and affine quivers

Let Q be an acyclic affine quiver. In Section 3.2.3, we defined an exact fully faithful functor

$$F : \operatorname{Rep}_{K_2}(k) \to \operatorname{Rep}_Q(k).$$

This functor induces a linear map

$$\tilde{F} : \mathbf{H}_{K_2,\mathbf{F}_q} \to \mathbf{H}_{Q,\mathbf{F}_q}$$

defined by $\tilde{F}[M] = [F(M)]$ for a representation M of K_2 over $\mathbf{F}_q$. This map is an injective algebra morphism (since F is fully faithful and since the essential image of F is closed under extensions). It is usually not a coalgebra homomorphism, but it verifies the following property. For $f \in \mathbf{H}_{Q,\mathbf{F}_q}$, denote by $f^{\perp}$ its orthogonal projection on $\mathrm{im}(\tilde{F})$ with respect to Green's scalar product. When viewing elements of $\mathbf{H}_{Q,\mathbf{F}_q}$ as functions, $f^{\perp}$ is simply the restriction of f to $\{[F(M)] : M \in \mathrm{Rep}_{K_2}(\mathbf{F}_q)\}$. Then, denoting by $\tilde{F}^{-1}$ the inverse of $\tilde{F} : \mathbf{H}_{K_2,\mathbf{F}_q} \to \mathrm{im}(\tilde{F})$, we have the formula

$$\Delta(F^{-1}(f^{\perp})) = \tilde{F}^{-1} \otimes \tilde{F}^{-1}(\Delta(f)^{\perp}). \tag{3.6.1}$$

By construction the functor F preserves indecomposables. Regularity of an indecomposable representation is a property of its dimension. It follows that F restricts to a functor between abelian categories

$$F_{\mathcal{R}} : \mathrm{Rep}^{\mathcal{R}}_{K_2}(\mathbf{F}_q) \to \mathrm{Rep}^{\mathcal{R}}_{Q}(\mathbf{F}_q)$$

which now induces an algebra morphism between regular Hall algebras:

$$\tilde{F}_{\mathcal{R}} : \mathbf{H}_{K_2,\mathbf{F}_q,\mathcal{R}} \to \mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}\,.$$

For $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$ and $n \geq 1$,

$$f^{K_2}_{a,n} = \sum_{|\lambda|=n} \prod_{j=1}^{l(\lambda)-1} (1 - q^{j\deg(a)})[I_{a,\lambda}]$$

are the regular cuspidal functions of the Kronecker quiver.

We first prove the following result giving the restriction of a regular cuspidal function of $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$.

Proposition 3.6.7. *Let $r \geq 1$ and f be the normalized regular cuspidal function of Q in a given tube. Then $\tilde{F}^{-1}(f^{\perp})$ is the normalized cuspidal function of K_2 of a (homogeneous) tube.*

Proof. In case f is in a homogeneous tube, F induces an equivalence of abelian categories with the corresponding tube of $\mathrm{Rep}^{\mathcal{R}}_{K_2}(\mathbf{F}_q)$. Therefore, $f^{\perp} = f$, $\Delta(f)^{\perp} = \Delta(f)$ and the results follows from Equation (3.6.1).

In case f is in a non-homogeneous tube, let C be the corresponding tube of K_2. Let $[M]$ be the isomorphism class of indecomposable representations of dimension (s, s) contained in the tube C. By Lemma 3.5.5, $f([F(M)]) \neq 0$ (the full faithfulness of F implies that $F(M)$ is indecomposable in the same tube as f) and moreover by normalization, $f([F(M)]) = 1$. By formula (3.6.1), $\tilde{F}^{-1}(f^{\perp})$ is cuspidal, and moreover $\tilde{F}^{-1}(f^{\perp}([M]) = 1$. Therefore, $\tilde{F}^{-1}(f^{\perp})$ is the normalized cuspidal function of a homogeneous tube of K_2. □

Corollary 3.6.8. *Let $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$. If F sends the a-tube of K_2 on a homogeneous tube of Q, then*

$$\tilde{F}(f^{K_2}_{a,n}) = f_{a,n}$$

is the normalized cuspidal regular function of Q of in the t-tube of dimension $n\deg(a)$.

If F sends the a-tube of K_2 on a non-homogeneous tube of Q, then

$$\tilde{F}(f_{a,n}^{K_2}) = f_{a,n}^{\perp}.$$

Proof. This is an immediate consequence of the Proposition 3.6.7 since in particular, if $a \in |\mathbf{P}^1_{\mathbf{F}_q}|$ corresponds to an homogeneous tube, for any $n \geq 1$, $f_{a,n}^{\perp} = f_{a,n}$. $\square$

3.6.6 Cuspidal functions of affine quivers

Cuspidal functions of an affine quiver in terms of regular cuspidal functions

Let us introduce some notations concerning partitions. For $(\lambda_1, \dots, \lambda_l)$ a partition, we define the following quantities:

$$|\lambda| = \sum_{i=1}^{l} \lambda_i,$$

$$l(\lambda) = l$$

and

$$n(\lambda) = \sum_{i=1}^{l} (i-1)\lambda_i.$$

For $d \geq 1$ and $q \neq 0$, set

$$\xi(d,q) = \sum_{|\lambda|=d} \frac{\prod_{j=1}^{l(\lambda)-1}(1-q^j)}{a_\lambda(q)},$$

and $\mathbf{d} \in \mathbf{N}^I$, define also

$$\chi_{\mathbf{d}} = \sum_{[M]\in\mathcal{M}_Q(\mathbf{F}_q)[\mathbf{d}]} [M].$$

The main theorem of this chapter is the following.

Theorem 3.6.9. *Let $\mathcal{B} = \{f_{x,n} : x \in |\mathbf{P}^1_{\mathbf{F}_q}|, n \geq 1\}$ be the basis of normalized regular cuspidal functions of Q. Then the kernel of the linear form*

$$\begin{array}{rccc} L: & \mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}} & \to & \mathbf{C} \\ & f_{x,n} & \mapsto & \xi(n, q^{\deg(x)}) \quad = (f_{x,n}, \chi_{n\deg(x)}) \end{array}$$

is the space of cuspidal functions of Q of imaginary dimension, $\bigcup_{r\geq 1} \mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[r\delta]$.

Proof. Let $r \geq 1$ be an integer and $\mathbf{d} = r\delta$ a multiple of the indivisible imaginary root. We denote by $L[\mathbf{d}] : \mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[\mathbf{d}] \to \mathbf{C}$ the restriction of L. We already know that $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}] \subset \mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}[\mathbf{d}]$ is an hyperplane. To prove the theorem, it suffices to prove that $L[\mathbf{d}]$ is a nonzero linear form whose kernel contains $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}]$.

Step 1: We prove that $\mathbf{d}$-dimensional cuspidal functions of Q are in the kernel of $L[\mathbf{d}]$. We can suppose – using the dualization process – that the extending vertex i_0 is a sink. We use previously defined notations, in particular I_θ is an indecomposable of Q of dimension $\theta = \delta - e_{i_0}$. We will show that for $x \in |\mathbf{P}^1_{\mathbf{F}_q}|$,

$$L(f_{x,n}) = \frac{1}{\nu^{\langle I_\theta^{\oplus r}, S_{i_0}^{\oplus r}\rangle} |\operatorname{GL}_n(\mathbf{F}_{q^{\deg(x)}})|^2} \Delta(f_{x,n})(I_\theta^{\oplus r}, S_{i_0}^{\oplus r}). \tag{3.6.2}$$

This formula will imply our claim, since if f is cuspidal of dimension $\mathbf{d}$, then $\Delta(f) = f \otimes 1 + 1 \otimes f$.

Let us now prove Formula (3.6.2). For a representation M of Q, $\Delta([M])(I_\theta^{\oplus r}, S_{i_0}^{\oplus r}) = 0$ if M is not in the essential image of F by Proposition 3.2.11. Thus, since F induces an equivalence of categories between $\operatorname{Rep}_{K_2}(\mathbf{F}_q)$ and the full subcategory of $\operatorname{Rep}_Q(\mathbf{F}_q)$ whose objects are extension of $I_\theta^{\oplus d_1}$ by $S_{i_0}^{\oplus d_2}$ for some nonnegative integers d_1 and d_2. It suffices therefore to prove (3.6.2) for $Q = K_2$ the Kronecker quiver, for which $I = S_1$ is the simple representation at the first vertex. For the Kronecker quiver,

$$f_{x,n} = \sum_{|\lambda|=n} \prod_{j=1}^{l(\lambda)-1} (1 - q^{j\deg(x)})[I_{x,\lambda}^{K_2}].$$

Thus,

$$\Delta(f_{x,n})(S_1^{\oplus r}, S_2^{\oplus r}) = \nu^{\langle S_1^{\oplus r}, S_2^{\oplus r}\rangle} \sum_{|\lambda|=n} \left(\prod_{j=1}^{l(\lambda)-1} (1 - q^{j\deg(x)}) \right) \frac{|\operatorname{Aut}(S_2^{\oplus r})||\operatorname{Aut}(S_1^{\oplus r})|}{a_\lambda(q^{\deg(x)})}.$$

Now, we have $|\operatorname{Aut}(S_1^{\oplus r})| = |\operatorname{Aut}(S_2^{\oplus r})| = |\operatorname{GL}_r(\mathbf{F}_{q^{\deg(x)}})|$, giving (3.6.2).

Step 2: We prove that $L[\mathbf{d}]$ is a nonzero linear form. If $r > 1$ or $r = 1$ and $q \neq 2$, there is at least one homogeneous tube of degree r. Let us choose $f = [S]$ where S is a regular simple in such a tube. By definition, it is an element of the basis $\mathcal{B}$ of dimension $r\delta$ and $L[\mathbf{d}](f) = \xi(1, q^r) = \frac{1}{q^r - 1} \neq 0$, so $L[\mathbf{d}] \neq 0$ in these cases. If $d = 1$ and $q = 2$, then in types D or E, there are only non-homogeneous tubes in dimension δ, since $|\ \mathbf{P}^1_{\mathbf{F}_2}(\mathbf{F}_2)\ |$ has three elements. Let f be a regular cuspidal function of dimension δ in a non-homogeneous tube. By 3.6.1 and because the essential image in dimension δ of the functor F defined above is precisely the full subcategory of objects which are nontrivial extension of I by S_{i_0}, we have

$$\Delta(f)(I, S_{i_0}) = \Delta(f^\perp)(I, S_{i_0}) = \Delta(\tilde{F}^{-1}(f^\perp))(S_1, S_2) \neq 0$$

since $\tilde{F}^{-1}(f^\perp) = [N]$ where $[N]$ is a regular cuspidal function of K_2 of dimension $(1,1)$, therefore N is one of the following representation of K_2 over $\mathbf{F}_2$:

$$\mathbf{F}_2 \overset{1}{\underset{0}{\rightrightarrows}} \mathbf{F}_2\ , \qquad \mathbf{F}_2 \overset{0}{\underset{1}{\rightrightarrows}} \mathbf{F}_2 \quad \text{or} \quad \mathbf{F}_2 \overset{1}{\underset{1}{\rightrightarrows}} \mathbf{F}_2\,.$$

$\square$

Corollary 3.6.10. *The difference of two normalized regular cuspidal functions of the same dimen-*

sion of two tubes of the same degree is a cuspidal function.

Proof. This is clear since for $x \in |\mathbf{P}^1_{\mathbf{F}_q}|$ and $s \geq 1$, $L(f_{x,s})$ depends only on s and $\deg(x)$. $\square$

3.7 Two conjectures of Berenstein and Greenstein

In all this section, Q is an acyclic affine quiver and $\mathbf{F}_q$ a finite field.

3.7.1 Fortuitous cancellation theorem

We prove in this section a result we use to relate Conjecture 3.7.2 and Conjecture 3.7.3 of Berenstein and Greenstein. We prove that for a cuspidal regular function $f \in \mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$, (*i.e.* such that $\Delta_{\mathcal{R}}(f) = f \otimes 1 + 1 \otimes f$), the function $\Delta(f) - (f \otimes 1 + 1 \otimes f)$ is supported on the subset of $\mathcal{I} \times \mathcal{P} = \{([M],[N]) : M \text{ preinjective and } P \text{ preprojective}\}$.

Let $f \in \mathbf{H}_{Q,\mathbf{F}_q}$ a regular cuspidal function. *A priori*, $\Delta(f)$ is a sum of the primitive part and terms of the form: $[I] \otimes [P], [I] \otimes [R \oplus P], [R \oplus I] \otimes [P]$ and $[R \oplus I] \otimes [R' \oplus P]$ where R (resp. P, resp. I) is any regular (resp. preprojective, resp. preinjective) representation. Indeed, if a term of the form $[M] \otimes [N]$ appears with nonzero coefficient in the comultiplication of $[R] \in \mathbf{H}_{Q,\mathbf{F}_q}$, this means that N is a sub-representation of R and M the quotient R/N. But a subrepresentation of a regular representation has only regular and preprojective indecomposable direct summand and the quotient of a regular representation by a subobject has only indecomposable regular and preinjective summands. We will show the following cancellation theorem.

Theorem 3.7.1. *Let $f \in \mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ be a regular cuspidal function. Then the comultiplication $\Delta(f) \in \mathbf{H}_{Q,\mathbf{F}_q} \otimes \mathbf{H}_{Q,\mathbf{F}_q}$ is the sum of its primitive part and terms of the form $[I] \otimes [P]$.*

Proof. Let $f \in \mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ be a regular cuspidal function. The regular cuspidality precisely means that $\Delta_{\mathcal{R}}(f)([M],[N]) = 0$ for any nonzero regular representations M and N of Q. We will use the coassociativity to prove the theorem. Suppose $\Delta(f)(R \oplus I, P) \neq 0$ for some representations P, R and I of Q with R regular, I preinjective and P preprojective. We first prove that for any $g \in \mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$,

$$(\Delta \otimes 1) \circ \Delta(g)(R, I, P) = (1 \otimes \Delta)(\Delta_{\mathcal{R}}(g))(R, I, P) \tag{3.7.1}$$

which will imply our assertion for the terms of the form $(R \oplus I) \otimes P$.

Let M be a regular representation. Then consider a filtration

$$0 \subset A \subset M$$

of M with successive quotients $P, R \oplus I$. Because $\mathrm{Ext}^1(R, I) = 0$ and $\mathrm{Hom}(I, R) = 0$, the datum of such a filtration is equivalent to the datum of a filtration:

$$0 \subset A \subset B \subset M$$

with successive quotients P, I, R (the two sorts of filtrations of M with the given quotients are in one-to-one correspondence). Since M and M/B are by assumption regular, so is B. This proves (3.7.1) for $g = [M]$ and then for any function g by linearity. Therefore,

$$(\Delta \otimes 1) \circ \Delta(f)(R, I, P) = (1 \otimes \Delta)(f \otimes 1 + 1 \otimes f)(R, I, P) = (f \otimes 1 \otimes 1 + 1 \otimes \Delta(f))(R, I, P) = 0.$$

But a term of the form $[R] \otimes [I] \otimes [P]$ in the decomposition of $(\Delta \otimes 1) \circ \Delta(f)$ can only come from the term $[R \oplus I] \otimes [P]$ of $\Delta(f)$, because $\mathrm{Ext}^1(R, I) = 0$, yielding a contradiction if this one appears with nonzero coefficient in $\Delta(f)$.

The case of $I \otimes R \oplus P$ is dual: the ingredients to handle this case are the formula

$$(1 \otimes \Delta) \circ \Delta(f)(I, P, R) = (\Delta \otimes 1)(\Delta_{\mathcal{R}}(f))(I, P, R)$$

and the fact that a term of the form $[I] \otimes [P] \otimes [R]$ in $(1 \otimes \Delta) \circ \Delta(f)$ can only come from the term $[I] \otimes [R \oplus P]$ of $\Delta(f)$.

The case of $[R \oplus I] \otimes [R' \oplus P]$ is more subtle but is a consequence of the formula:

$$(\Delta \otimes 1 \otimes 1) \circ (1 \otimes \Delta) \circ \Delta(f)(R, I, P, R') = (1 \otimes \Delta \otimes 1)(((1 \otimes \Delta) \circ \Delta)_{\mathcal{R}}(f))(R, I, P, R'),$$

where for $f \in \mathbf{H}_{Q,\mathbf{F}_q}$, $((1 \otimes \Delta) \circ \Delta)_{\mathcal{R}}(f)$ is the projection on $\mathbf{H}^3_{Q,\mathbf{F}_q,\mathcal{R}}$ of $((1 \otimes \Delta) \circ \Delta)(f)$. This formula is proved using the fact that filtrations of a regular module M

$$0 \subset A \subset M$$

with successive quotients $P \oplus R'$ and $I \oplus R$ are in one-to-one correspondence with filtrations of M of the form

$$0 \subset B \subset A \subset C \subset M$$

with successive quotients R', P, I, R. As before, we also need the fact that a term of the form $[R] \otimes [I] \otimes [P] \otimes [R']$ in $(\Delta \otimes 1 \otimes 1) \circ (1 \otimes \Delta) \circ \Delta(f)$ can only come from the term $[R \oplus I] \otimes [R' \oplus P]$ of $\Delta(f)$. $\square$

3.7.2 Two conjectures of Berenstein and Greenstein

In their paper [BG16], Berenstein and Greenstein gave the following conjectures proved by Deng and Ruan in [DR17] using weighted projective lines and Hall polynomials.

For $n > 1$, let $N(n) =$ number of closed points of $\mathbf{P}^1_{\mathbf{F}_q}$ of degree n and $N(1) = q + 1 - d$. This is the number of closed points of $\mathbf{P}^1_{\mathbf{F}_q}$ of degree n not in D.

Conjecture 3.7.2. *For $s \geq 1$, $d \geq 1$, and $x \in \mathbf{P}^1_{\mathbf{F}_q}$ a closed point of degree d not in D,*

$$f_{x,s} - \frac{1}{N(d)} \sum_{\substack{y \in |\mathbf{P}^1_{\mathbf{F}_q}| \setminus D \\ \deg(y) = d}} f_{y,s}$$

is a cuspidal function.

Conjecture 3.7.3. *Let P be a preprojective representation and I a preinjective representation. Then, for any partition λ and closed points $x, y \in |\mathbf{P}^1_{\mathbf{F}_q}| \setminus D$,*

$$F^{P,I}_{I_\lambda(x)} = F^{P,I}_{I_\lambda(y)}.$$

Thanks to the fortuitous cancellation Theorem, Conjecture 3.7.3 implies Conjecture 3.7.2. We will prove both conjectures using Theorem 3.6.9. We provide a direct proof of Conjecture 3.7.2

Theorem 3.7.4. *Conjecture 3.7.2 holds.*

Proof. Let $x \in |\mathbf{P}^1_{\mathbf{F}_q}| \setminus D$ a closed point of degree d and

$$f = f_{x,s} - \frac{1}{N(d)} \sum_{\substack{y \in |\mathbf{P}^1_{\mathbf{F}_q}| \setminus D \\ \deg(y) = d}} f_{y,s}.$$

Then f is a function of dimension $s \deg(x)$. From Theorem 3.6.9, f is cuspidal if and only if $L(f) = 0$. But

$$L(f) = \xi(s, q^{deg(x)}) - \frac{1}{N(d)} \sum_{\substack{y \in |\mathbf{P}^1_{\mathbf{F}_q}| \setminus D \\ \deg(y) = d}} \xi(s, q^{\deg(y)}) = 0.$$

□

To prove Conjecture 3.7.3, we first define an action of an infinite permutation group on the Hall algebra $\mathbf{H}_{Q,\mathbf{F}_q}$.

Let $\mathfrak{S}$ be the group of degree preserving permutations of $|\mathbf{P}^1_{\mathbf{F}_q}| \setminus D$. The group $\mathfrak{S}$ is isomorphic to

$$\prod_{e \geq 1} \mathfrak{S}_{N(e)}$$

where for a positive integer N, $\mathfrak{S}_N$ is the symmetric group on N letters. The action

$$\mathfrak{S} \to \mathrm{Aut}(\mathbf{H}_{Q,\mathbf{F}_q})$$

is defined as follows. For M, N two representations, λ a partition, $x \in |\mathbf{P}^1_{\mathbf{F}_q}| \setminus D$ and $\sigma \in \mathfrak{S}$,

$$\sigma \cdot [M] = [M] \quad \text{if } [M] \text{ is preprojective, preinjective or in a non-homogeneous tube}$$
$$\sigma \cdot [I_\lambda(x)] = [I_\lambda(\sigma(x))]$$
$$\sigma \cdot ([M] \oplus [N]) = (\sigma \cdot [M]) \oplus (\sigma \cdot [N])$$

where for notational reasons, we define here $[M] \oplus [N] = [M \oplus N]$. It is easily seen that σ acts as a graded linear isomorphism on $\mathbf{H}_{Q,\mathbf{F}_q}$.

The following is the second important result of this chapter.

Theorem 3.7.5. *The group $\mathfrak{S}$ acts by Hopf-algebra automorphisms on $\mathbf{H}_{Q,\mathbf{F}_q}$.*

Proof. It is a consequence of the following facts.

1. σ acts as an isometry of $\mathbf{H}_{Q,\mathbf{F}_q}$,
2. The action of σ leaves $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ and $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}$ stable,
3. σ commutes with the linear form L. In particular, it preserves $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q}[\mathbf{d}]$ for any dimension $\mathbf{d}$.

Indeed, for (1), we have to prove that for M a representation of Q over $\mathbf{F}_q$, the number of automorphisms of M and σM are the same. We write $M \simeq I \oplus R \oplus P$ with I preinjective, R regular and P preprojective. There is no morphisms from I to R or P and no morphisms from R to P. Therefore, an endomorphism of M is upper triangular in this direct sum decomposition. Since for a partition λ and a closed point $x \in |\mathbf{P}^1_{\mathbf{F}_q}| \setminus D$ the number of automorphisms of $I_\lambda(x)$, $a_\lambda(q^{\deg(x)})$, only depends on the degree of x, we only have to prove that for a partition λ, $x \in |\mathbf{P}^1_{\mathbf{F}_q}| \setminus D$, a preprojective representation P and a preinjective representation I, both numbers

$$\dim \mathrm{Hom}(P, I_\lambda(x)),$$
$$\dim \mathrm{Hom}(I_\lambda(x), I)$$

only depend on the degree of x. This is straightforward since using the Euler form, the first equals $\langle \dim P, |\lambda| \deg(x)\delta\rangle$ and the second equals $\langle |\lambda| \deg(x)\delta, \dim I\rangle$.

For (2), by definition σ let $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ stable and the element $f_{x,n}$ of $\mathbf{H}_{Q,\mathbf{F}_q,\mathcal{R}}$ verifies $\sigma \cdot f_{x,n} = f_{\sigma(x),n}$, so $\mathbf{H}^{\mathrm{cusp}}_{Q,\mathbf{F}_q,\mathcal{R}}$ is stable under σ. For (3), we notice that $L(f_{x,n}) = \xi(n, q^{\deg(x)}) = L(f_{\sigma(x),n})$.

Therefore, σ sends a system of orthogonal cuspidal generators of $\mathbf{H}_{Q,\mathbf{F}_q}$ to an other such system. Theorem 3.3.4 tells us that these systems of generators satisfy the same relations: σ is an algebra morphism. Since σ also preserves cuspidal functions, it is a Hopf algebra automorphism. □

Corollary 3.7.6. *Conjecture 3.7.3 holds.*

Proof. As usual, $\Delta_{I,P}$ denotes the truncation of the comultiplication: for $f \in \mathbf{H}_{Q,\mathbf{F}_q}$, we keep only the terms of the form $[I'] \otimes [P']$ in $\Delta(f)$. A reformulation of conjecture 3.7.3 is

$$\Delta_{I,P}[I_\lambda(x)] = \Delta_{I,P}[I_\lambda(y)]$$

for any partition λ and closed points $x, y \in |\mathbf{P}^1_{\mathbf{F}_q}| \setminus D$. For any $x, y \in |\mathbf{P}^1_{\mathbf{F}_q}| \setminus D$, we choose $\sigma \in \mathfrak{S}$ such that $\sigma(x) = y$. We have

$$\Delta_{I,P}([I_\lambda(y)]) = \Delta_{I,P}(\sigma \cdot [I_\lambda(x)]) = (\sigma \otimes \sigma) \cdot \Delta_{I,P}([I_\lambda(x)]) = \Delta_{I,P}([I_\lambda(x)])$$

as σ acts trivially on preinjective and preprojective representations.

□

3.8 Isotropic cuspidal dimensions of quivers

Let $Q = (I, \Omega)$ be an arbitrary quiver and $\mathbf{F}_q$ a finite field.

3.8.1 The support of an isotropic cuspidal dimension of a quiver

Write $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where all coordinates except the i-th are 0. The following seems to be known and is not difficult to prove but we reproduce the proof for the sake of completeness.

Proposition 3.8.1. *Let $\mathbf{d}$ be a cuspidal dimension of Q over $\mathbf{F}_q$. Then $\mathbf{d}$ has a connected support.*

Proof. Let $\mathbf{d} \in \mathbf{N}^I$ be a cuspidal dimension and f a nonzero cuspidal function of dimension $\mathbf{d}$. Suppose $\operatorname{supp}\mathbf{d}$ is not connected, therefore $P = \operatorname{supp}\mathbf{d} = P_1 \sqcup P_2$ such that there is no arrow between P_1 and P_2. For a representation M of Q over $\mathbf{F}_q$, we write M_{P_j} for $j = 1, 2$ the representation of Q which coincides with M on P_j and is zero at the other vertices. Since P_1 and P_2 are not connected,

$$\operatorname{Ext}^1(M_{P_1}, M_{P_2}) = \operatorname{Ext}^1(M_{P_2}, M_{P_1}) = 0$$

and

$$\operatorname{Hom}(M_{P_1}, M_{P_2}) = \operatorname{Hom}(M_{P_2}, M_{P_1}) = 0.$$

Therefore, in $\mathbf{H}_{Q,\mathbf{F}_q}$,

$$[M_{P_1}][M_{P_2}] = [M_{P_1} \oplus M_{P_2}] = [M_{P_2}][M_{P_1}].$$

Now write

$$f = \sum_{[M], \dim M = \mathbf{d}} c_{[M]}[M] = \sum_{[M], \dim M = \mathbf{d}} c_{[M]}[M_{P_1} \oplus M_{P_2}].$$

Since from the characterization of cuspidal functions, f is orthogonal to any nontrivial products,

$$c_{[M]} = |\operatorname{Aut}(M)|(f, [M]) = |\operatorname{Aut}(M)|(f, [M_1][M_2]) = 0,$$

giving $f = 0$, contradiction.

□

The following Theorem 3.8.2 is a direct consequence of [Kac90, Proposition 5.7] and of the inequalities verified by cuspidal dimensions in case of quivers without loops. We reproduce the proof here for arbitrary quivers for the convenience of the reader.

Theorem 3.8.2. *Let $\mathbf{d}$ be an isotropic cuspidal dimension of Q over $\mathbf{F}_q$ (*i.e.* $(\mathbf{d}, \mathbf{d}) = 0$ where $(-, -)$ is the symmetrized Euler form of the quiver). Then the support $\operatorname{supp}\mathbf{d}$ of $\mathbf{d}$ is an affine quiver.*

Proof. From [SVDB01, Proposition 3.2 1.(a)], which can be extended to an arbitrary quiver with few modifications[8], we have

$$(\mathbf{d}, e_i) \leq 0$$

for any $i \in I$ and moreover $\operatorname{supp} \mathbf{d}$ is connected by Proposition 3.8.1. We set $P = \operatorname{supp} \mathbf{d}$ and see P as a full subquiver of Q. The condition $(\mathbf{d}, \mathbf{d}) = 0$ then implies

$$(\mathbf{d}, e_i) = 0 \tag{3.8.1}$$

for $i \in \operatorname{supp} \mathbf{d}$: in fact, if $\mathbf{d} = \sum_{i \in \operatorname{supp} \mathbf{d}} d_i e_i$ with d_i a positive integer,

$$(d, d) = \sum_{i \in \operatorname{supp} \mathbf{d}} d_i (\mathbf{d}, e_i) = 0$$

and each term of this sum is negative or zero, which implies the result.

We show that if P contains edge loops, P is the Jordan quiver. Let $i \in \operatorname{supp} \mathbf{d}$ be a vertex with g loops. Then

$$(\mathbf{d}, e_i) = 2(1-g)d_i - \sum_{\alpha: i \to j} d_j - \sum_{\alpha: j \to i} d_j.$$

This quantity must be zero. The only possibility is $g = 1$ and P contains no vertices adjacent to i. Thanks to the connectedness of P, we deduce that P is the Jordan quiver.

Suppose now P has no edge loop. Then the Cartan matrix A_P of the subquiver P is a generalized Cartan matrix as defined in [Kac90, Chapter 4], that is for any i, j vertices of P,

$$a_{i,i} = 2 \quad \text{and} \quad a_{i,j} = 0 \implies a_{j,i} = 0.$$

The matrix A_P is indecomposable because P is connected. Moreover, the vector $\mathbf{d}$ has positive nonzero integer coefficients and verifies $A_P \mathbf{d} = 0$ because of equation (3.8.1), since by definition of the symmetrized Euler form of a quiver, $(\mathbf{d}, e_i) = {}^t e_i A \, \mathbf{d}$. The classification of indecomposable generalized Cartan matrices given in [Kac90, Theorem 4.3] implies that A_P is of affine type. From [Kac90, Lemma 4.5], since A_P is by definition symmetric, A_P is positive semidefinite. Thanks to [Sch14, Theorem 8.6 2.], $\operatorname{supp} \mathbf{d}$ is an affine quiver: this concludes the proof. □

3.8.2 Consequences

Theorem 3.8.2 together with Theorem 3.6.9 gives an explicit description of cuspidal isotropic functions of any quiver. The cuspidal functions of a quiver are of three different types. We have the real cuspidal functions: the $[S_i]$ for i a real vertex of Q and S_i the simple representation of dimension e_i, the isotropic cuspidal functions, which are the homogeneous cuspidal functions of dimension $\mathbf{d}$ with $(\mathbf{d}, \mathbf{d}) = 0$ and finally the hyperbolic cuspidal functions, that is cuspidal functions of dimension $\mathbf{d}$ such that $(\mathbf{d}, \mathbf{d}) < 0$.

[8] Sevenhant and Van den Bergh restrict themselves to edge loop free quivers, but their proof can be slightly modified for any quiver.

A conjecture of Bozec and Schiffmann for isotropic dimensions

In the paper, [BS18], Bozec and Schiffmann give the formula:

$$C^{abs}_{Q,\mathbf{d}}(t) = t \tag{3.8.2}$$

if $\mathbf{d}$ is an isotropic dimension of an affine quiver. By Theorem 3.8.2, we can replace the condition (2) of Proposition 3.4.2 by the condition (2)′: if $\mathbf{d}$ is isotropic, then $C^{abs}_{Q,\mathbf{d}}(t) = t$. We can also partially obtain Conjecture 3.4.3.

Corollary 3.8.3. *Conjecture 3.4.3 holds for isotropic dimensions* $\mathbf{d} \in \mathbf{N}^I$.

Chapter 4

Asymptotic behaviour of Kac polynomials

We conjecture a formula supported by computations for the valuation of Kac polynomials of a quiver, which only depends on the number of loops at each vertex. We prove a convergence property of renormalized Kac polynomials of quivers when increasing the number of arrows: they converge in the ring of power series, with a linear rate of convergence. Then, we propose a conjecture concerning the global behaviour of the coefficients of Kac polynomials. All computations were made using SageMath.

Contents

4.1 Introduction

Kac polynomials are a family of polynomials associated to a quiver, one for each positive root. They are the counting polynomials of absolutely indecomposable representations of fixed dimension vector of the given quiver introduced by Kac in the early eighties ([Kac80b, Kac82, Kac83]). These polynomials are of considerable current interest due to their appearance in the different constructions of Yangians associated to a general Kac-Moody algebra and when counting the points of meaningful algebraic stacks, in particular the points of the stack of (nilpotent) representations of the preprojective algebra of the quiver ([MO19, Oko, Dav13, DM20, Dav20, BSV17, SV20]). Lots of conjectures have been proposed by Kac concerning the integrality and positivity of their coefficients and a Lie theoretic interpretation of the constant coefficient ([Kac83]). The integrality and the degree is known since the beginning and the positivity was proved in the last decade by Hausel, Letellier and Rodriguez-Villegas using the geometry of Nakajima quiver varieties ([HLRV13]). Then, Davison interpreted Kac polynomials as Donaldson-Thomas invariants of the tripled quiver associated to Q, giving a new proof of the positivity of the coefficients by a purity argument ([Dav18]). Kac polynomials are now conjectured to give the graded character of the Maulik-Okounkov Lie algebra associated to Q ([Oko]).

In this chapter, we study asymptotic properties of Kac polynomials when making the number of arrows vary and go to infinity. First, we conjecture an explicit formula for the valuation of the polynomials depending only on the number of loops at each vertex. This is supported by computations. Then, we prove that as the number of arrows goes to infinity, renormalized Kac polynomials converge to well-defined power series in $\mathbf{N}[[q]]$ (when the quiver has loops, we need to choose a direction, but we conjecture that the convergence holds without this choice). In fact, we prove a stronger theorem giving a canonical way of writing Kac polynomials as a quotient of two polynomials. The denominator is independent of the number of arrows and its roots are roots of unity. This theorem allows explicit computations of Kac polynomials in a fixed dimension vector as a function of the multiplicity of the arrows. We give several examples of the formulas produced. The code used for the computations is given in the appendix of this chapter and is also available on the author's webpage.

4.2 Kac polynomials

4.2.1 Quiver representations

Let $Q = (I, \Omega)$ be a quiver with set of vertices I and set of arrows Ω, both finite. We work over a finite field $\mathbf{F}_q$. A *representation* of Q is the data of a finite dimensional vector space for any vertex and a linear map for any arrow. Finite dimensional representations of Q form an abelian category which has been studied for some time now, beginning with Gabriel ([Gab72]). To Q, we associate a bilinear form (its *Euler form*):

$$\begin{array}{rclcc} \langle -,- \rangle & : & \mathbf{Z}^I \times \mathbf{Z}^I & \to & \mathbf{Z} \\ & & (\mathbf{d},\mathbf{e}) & \mapsto & \sum_{i\in I} \mathbf{d}_i\,\mathbf{e}_i - \sum_{\alpha:i\to j} \mathbf{d}_i\,\mathbf{e}_j\,. \end{array}$$

It coincides with the Euler form of the abelian category of finite dimensional representations of Q (which is of finite global dimension), which is by definition the alternate sum of the dimensions of the vector spaces over $\mathbf{F}_q$ given by $\mathrm{Ext}^i_Q(M,N)$ for M of dimension vector $\mathbf{d}$ and N of dimension vector $\mathbf{e}$ (and does not depend on the choice of M,N).

4.2.2 Kac-Moody Lie algebras and Weyl group

To an arbitrary loop-free quiver $Q=(I,\Omega)$, we can associate a Kac-Moody Lie algebra, given by generators and relations, generalizing the relation between simple Lie algebras and Dynkin graphs. We denote by $\mathfrak{g}_Q$ the Kac-Moody algebra corresponding to Q. It has a $\mathbf{Z}^I$-grading. We write $\mathfrak{g}_Q=\bigoplus_{\mathbf{d}\in\mathbf{Z}^I}\mathfrak{g}_Q[\mathbf{d}]$. We also associate to Q a Coxeter group W, called the Weyl group of the quiver. We refer to [Kac90] for details on these constructions. A slightly larger class of Lie algebras has been defined by Borcherds using quivers with possible loops in [Bor88]. Although they are very closely related to this work, they won't appear directly here.

4.2.3 Counting representations

Recall that representations of Q over $\mathbf{F}_q$ together with morphisms between them form an abelian category. A representation of Q is called *indecomposable* provided it cannot be non-trivially written as a direct sum of two representations of Q. A representation of Q is called *absolutely indecomposable* when it remains indecomposable when tensored over $\mathbf{F}_q$ with the algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$. At the begining of the 80's, Kac introduced many families of counting functions which happen to be polynomials in q ([Kac82, Kac80a, Kac80b, Kac83]). They are the number of isomorphism classes of representations of Q over $\mathbf{F}_q$ of dimension vector $\mathbf{d}$, $M_{Q,\mathbf{d}}(q)$; the number of isomorphism classes of indecomposable representations of Q over $\mathbf{F}_q$ of dimension vector $\mathbf{d}$, $I_{Q,\mathbf{d}}(q)$; and the number of isomorphism classes of absolutely indecomposable representations of Q over $\mathbf{F}_q$ of dimension vector $\mathbf{d}$, $A_{Q,\mathbf{d}}(q)$. The family of polynomials $(A_{Q,\mathbf{d}}(q))_{\mathbf{d}\in\mathbf{N}^I}$ is the better behaved one, as illustrated by the Kac conjectures proved by Hausel [Hau10] (Theorem 4.2.2) and Hausel–Letellier–Rodriguez-Villegas [HLRV13] (Theorem 4.2.3). The three families of functions $M_{Q,\mathbf{d}}(q), I_{Q,\mathbf{d}}(q)$ and $A_{Q,\mathbf{d}}(q)$ are related as follows ([BS18, Lemma 3.1][Hua00]):

$$\sum_{\mathbf{d}\geq 0} M_{Q,\mathbf{d}}(q)z^{\mathbf{d}} = \mathrm{Exp}_z\left(\sum_{\mathbf{d}>0} I_{Q,\mathbf{d}}(q)z^{\mathbf{d}}\right) = \mathrm{Exp}_{q,z}\left(\sum_{\mathbf{d}>0} A_{Q,\mathbf{d}}(q)z^{\mathbf{d}}\right), \tag{4.2.1}$$

where Exp_z and $\mathrm{Exp}_{z,t}$ denote the plethystic exponentials, see [BS18, Section 1.5]. The first equality follows from the Krull-Schmidt property of the category of representations of the quiver and the second from Galois descent for quiver representations.

The results are as follows.

Theorem 4.2.1 (Kac,[Kac83, §§1.10, 1.15]). *The function $A_{Q,\mathbf{d}}(q)$ is a polynomial in q with coefficients in $\mathbf{Z}$ of degree $1-\langle \mathbf{d},\mathbf{d}\rangle$. It is nonzero if and only if $\mathbf{d}$ is a positive root of $\mathfrak{g}_Q$. It does not depend on the orientation of Q. If $w \in W$ is an element of the Weyl group such that $\mathbf{d}$ and $w\,\mathbf{d}$ are both positive, then $A_{Q,\mathbf{d}}(q) = A_{Q,w\,\mathbf{d}}(q)$.*

A word on the proof. The argument of [Kac83, §1.15] does not suffice to prove that the polynomials $A_{Q,\mathbf{d}}$ have integer coefficients. Instead one can use the argument of [BSV17, §2.3.1] which rests on the deep result of Katz in the appendix of [HRV08], at the end of page 616. The same argument allows Bozec, Schiffmann and Vasserot to prove that the nilpotent Kac polynomials have integer coefficients ([BSV17, §2.3.1]). $\square$

Theorem 4.2.2 (Hausel,[Hau10]). *If Q is loop-free and $\mathbf{d} \in \mathbf{N}^I$ is a positive root, then $A_{Q,\mathbf{d}}(0) = \dim_{\mathbf{C}} \mathfrak{g}_Q[\mathbf{d}]$.*

Theorem 4.2.3 (Hausel–Letellier–Rodriguez-Villegas,[HLRV13]). *The polynomial $A_{Q,\mathbf{d}}(q)$ has nonnegative coefficients.*

Remark 4.2.4.

1. Theorems 4.2.2 and 4.2.3 were first proved by Crawley-Boevey and Van den Bergh for indivisible dimension vectors $\mathbf{d} \in \mathbf{N}^I$ in [CBVdB04],
2. The coefficients of Kac polynomials can be interpreted as the dimensions of certain isotypical components for a Weyl group action on the compactly supported cohomology of a Nakajima quiver variety ([HLRV13]).

We have a closed formula for the generating function of Kac polynomials due to Hua:

Theorem 4.2.5 (Hua, [Hua00]).

$$\sum_{\mathbf{d}\in\mathbf{N}^I\setminus\{0\}} A_{Q,\mathbf{d}}(q) z^{\mathbf{d}} = (q-1)\,\mathrm{Log}_{z,q}\left(\sum_{\pi=(\pi^i)_{i\in I}\in\mathscr{P}^I} \frac{\prod_{i\to j\in\Omega} q^{\langle \pi^i,\pi^j\rangle}}{\prod_{i\in I} q^{\langle \pi^i,\pi^i\rangle}\prod_k \prod_{j=1}^{m_k(\pi^i)}(1-q^{-j})} z^{|\pi|}\right),$$

where $\mathrm{Log}_{z,q}$ is the plethystic logarithm ([BS18, Section 1.5]), $\mathscr{P}$ is the set of partitions (including the unique partition of 0), $m_j(\lambda)$ is the multiplicity of the part j in the partition λ, $|\pi| = (|\pi^i|)_{i\in I}$ and $\langle -,-\rangle$ is the pairing on the set of partitions given by

$$\langle \lambda,\mu\rangle = \sum_{i,j} \min(i,j) m_i(\lambda) m_j(\mu).$$

4.2.4 Nilpotent versions

The category of representations of Q has two natural Serre subcategories (stable under extensions, subobjects and quotients): the category of nilpotent representations (the composition of maps along any sufficiently long path is zero), with associated families of counting functions $M^0_{Q,\mathbf{d}}(q), I^0_{Q,\mathbf{d}}(q)$, $A^0_{Q,\mathbf{d}}(q)$ and the category of 1-nilpotent representations, that is the category of representations of Q so that the restriction of the representation to a vertex $i \in I$ with g_i loops (we only keep the vector space at vertex i and loops at i) gives a nilpotent representation of S_{g_i}, the quiver with g_i loops, with associated families of counting functions $M^1_{Q,\mathbf{d}}(q), I^1_{Q,\mathbf{d}}(q), A^1_{Q,\mathbf{d}}(q)$. The equalities (4.2.1) are still true for these new families of functions, whose definitions first appear in [BSV17]. In *loc. cit.*, Proposition 1.2, it is proved that these functions are polynomials in q, and moreover that $A^{\flat}_{Q,\mathbf{d}}(q)$ has integer coefficients for $\flat = 0, 1$. It was conjectured there (Remark 1.3 of *op. cit.*) and proved by Ben Davison in [Dav16, Theorem 7.8] that these polynomials have nonnegative coefficients. It is important to note that $A^0_{Q,\mathbf{d}}(q)$ depends on the orientation of Q while $A^1_{Q,\mathbf{d}}(q)$ does not ([BSV17, Remark 2.8]).

4.3 Stabilization property of Kac polynomials

4.3.1 Multi-arrows quivers

Let $Q = (I, \Omega)$ be a quiver. We write $I = I^{\mathrm{im}} \sqcup I^{\mathrm{re}}$, where I^{im} is the set of vertices of Q having at least one loop (the imaginary vertices) and I^{re} its complement (the real vertices). For what follows, we may assume that it has no multiple arrows. Let $\underline{n} = (n_\alpha)_{\alpha\in\Omega} \in \mathbf{N}^{\Omega}$. The *multi-arrowed quiver* associated to Q and $\underline{n}$ is the quiver $Q_{\underline{n}} = (I, \Omega_{\underline{n}})$ having the same set of vertices as Q but each arrow $\alpha : i \to j \in \Omega$ of Q is replaced by n_α arrows $\alpha_l : i \to j$, $1 \leq l \leq n_\alpha$. We give two examples in Section 4.3.3. For $\mathbf{d} \in \mathbf{N}^I$, we consider the Kac polynomials $A_{Q_{\underline{n}},\mathbf{d}}(q) \in \mathbf{N}[q]$. We are in the situation where $\mathbf{d}$ is fixed but $\underline{n}$ will vary, and possibly diverge to infinity.

4.3.2 Conjectures

We make the following conjecture, supported by computations for multiloop quivers S_g and tennis-racket quivers (Section 4.3.3). Given a nonzero polynomial $P \in Q[q]$, we call *valuation* of P the smallest integer v such that the coefficient of q^v in P is nonzero.

Conjecture 4.3.1. *For $\underline{n} \in (\mathbf{N}_{\geq 1})^{\Omega}$, the valuation of $A_{Q_{\underline{n}},\mathbf{d}}(q)$ is*

$$v_{\underline{n}} := \sum_{i\in I^{\mathrm{im}}} (1 + d_i(\sum_{\alpha:i\to i} n_\alpha - 1))$$

provided the polynomial is nonzero.

This conjecture is clearly true for loop-free quivers as a consequence of Theorem 4.2.2: if $A_{Q_{\underline{n}},\mathbf{d}}(q)$ is non-zero, then by Theorem 4.2.1, $\mathbf{d}$ is a positive root of $\mathfrak{g}_{Q_{\underline{n}}}$ hence $A_{Q_{\underline{n}},\mathbf{d}}(0) = \dim_{\mathbf{C}}(\mathfrak{g}_{Q_{\underline{n}}}[\mathbf{d}]) \neq 0$.

Our interest in this conjecture is the following theorem.

Theorem 4.3.2. *Let $\underline{m} = (m_\alpha)_{\alpha\in\Omega} \in (\mathbf{N}_{\geq 1} \cup \{+\infty\})^{\Omega}$.*

1. *Let Q be a loop-free quiver and $\mathbf{d} \in \mathbf{N}^I$. Then, for any $\underline{n} \in \mathbf{N}^{\Omega}$ the valuation of $A_{Q_{\underline{n}},\mathbf{d}}$ is $v_{\underline{n}} = 0$ if $A_{Q_{\underline{n}},\mathbf{d}} \neq 0$. The sequence of polynomials*

$$A_{Q_{\underline{n}},\mathbf{d}}(q) \in \mathbf{N}[q]$$

converges in $\mathbf{N}[[q]]$ *as* $\underline{n} \to \underline{m}$. *Moreover, the limit is the power series expansion at* $q = 0$ *of a rational fraction.*

2. *Let* Q *be an arbitrary quiver and* $\underline{r} \in \mathbf{N}_{\geq 1}^{\Omega} \setminus \{0\}$. *For* $s \in \mathbf{N}$, *we let* $v_{s\underline{r}}$ *be the valuation of the polynomial* $A_{Q_{s\underline{r}},\mathbf{d}}(q)$. *Then, the sequence of polynomials*

$$\frac{A_{Q_{s\underline{r}},\mathbf{d}}(q)}{q^{v_{s\underline{r}}}} \in \mathbf{N}[q]$$

converges in $\mathbf{N}[[q]]$ *as* $s \to +\infty$ *and the limit is the power series expansion at* $q = 0$ *of a rational fraction.*

In fact, we can prove the following stronger result concerning the structure of Kac polynomials.

Theorem 4.3.3. *Let* $\mathbf{d} \in \mathbf{N}^I$. *Then there exists* $r \geq 1$, *polynomials with integer coefficients* $Q, P_0, \ldots, P_r$ *such that the complex roots of* Q *are roots of unity and affine linear functions* $l_0, l_1, \ldots, l_r : \mathbf{Z}^{\Omega} \to \mathbf{Z}$ *whose linear parts* $\tilde{l}_j$ *have nonnegative coefficients when written in the canonical basis of* $\mathbf{Z}^{\Omega}$ *(that is,* $\tilde{l}_j(e_\alpha) \geq 0$ *for any* $0 \leq j \leq r$ *and* $\alpha \in \Omega$*) and are pairwise distinct, such that* $A_{Q_{\underline{n}},\mathbf{d}}(q)$ *can be written as the rational fraction*

$$A_{Q_{\underline{n}},\mathbf{d}}(q) = \frac{\sum_{j=0}^{r} q^{l_j(\underline{n})} P_j(q)}{Q(q)}$$

for any $\underline{n} \in \mathbf{N}^{\Omega}$. *Moreover, if we impose* $\gcd(Q, P_0, \ldots, P_r) = 1$, Q *monic and* $P_j(0) \neq 0$ *for any* $0 \leq j \leq r$, *the polynomials* Q, P_j *and the* l_j *are unique up to reordering.*

Remark 4.3.4.
1. The reader should be tempted to speculate on the expression given by Theorem 4.3.3: is there any (maybe geometric, see Section 4.4) interpretation of the affine linear functions l_j or of the polynomials P_j? The appearance of $q^{l_j(\underline{n})}$ could possibly be a clue for a hidden affine fibration of rank $l_j(\underline{n})$. At the moment, the author does not have any reasonable conjecture.

2. Knowing that Kac polynomials can be written under the form given by Theorem 4.3.3 is very powerful to give closed formulas for them using computations, see Section 4.3.3.

3. The decomposition of Kac polynomials given by this theorem seems related to the refined Kac polynomials defined by Rodriguez-Villegas in [Rod11]. However, the formula of Theorem 4.3.3 is really a sum of rational fractions and not of polynomials.

Proof of Theorem 4.3.2. We assume Theorem 4.3.3, whose proof will be given below. We prove the first point. Let $\mathbf{d} \in \mathbf{N}^I$. For $\underline{n} \in \mathbf{N}^{\Omega}$, we write $A_{Q_{\underline{n}},\mathbf{d}}(q)$ as in Theorem 4.3.3. By the remark preceding Theorem 4.3.2, since Q has no loops, $v_{\underline{n}} = 0$ for any $\underline{n} \in \mathbf{N}_{\geq 1}^{\Omega}$ (if $A_{Q_{\underline{n}},\mathbf{d}}(q) \neq 0$). Therefore, one of the l_j's is the zero affine function $\mathbf{Z}^{\Omega} \to \mathbf{Z}$. Assume that $l_0 = 0$. Let $\underline{m} \in (\mathbf{N}_{\geq 1} \cup \{+\infty\})^{\Omega}$. Let $J' = \{1 \leq j \leq r \mid \lim_{\underline{n} \to \underline{m}} l_j(\underline{n}) = +\infty\}$ and $J = \{1, \ldots, r\} \setminus J'$. For $j \in J$, $l_j(\underline{n})$ is a sequence of integers which stabilizes as $\underline{n} \to \underline{m}$ (by the positivity assumption on the coefficients of the linear forms l_j, $0 \leq j \leq r$). We let $l_j(\underline{m})$ be its limit. In particular, $0 \in J$ and $l_0(\underline{m}) = 0$. Then,

$$\lim_{\underline{n} \to \underline{m}} \frac{A_{Q_{\underline{n}},\mathbf{d}}(q)}{q^{v_{\underline{n}}}} = \frac{\sum_{j \in J} q^{l_j(\underline{m})} P_j(q)}{Q(q)}.$$

In particular, for $\underline{m} = (+\infty, \ldots, +\infty)$, $J = \{0\}$ and so

$$\lim_{\underline{n} \to \underline{m}} \frac{A_{Q_{\underline{n}},\mathbf{d}}(q)}{q^{v_{\underline{n}}}} = \frac{P_0(q)}{Q(q)}.$$

For the second point, let Q be an arbitrary quiver and $\underline{r} \in \mathbf{N}_{\geq 1}^{\Omega}$. Write $A_{Q_{\underline{n}},\mathbf{d}}(q)$ as in Theorem 4.3.3. We obtain affine functions

$$\begin{array}{rcccc} l'_j & : & \mathbf{Z} & \to & \mathbf{Z} \\ & & s & \mapsto & l_j(s\underline{r}) \end{array}$$

for $0 \leq j \leq r$. Let $J = \{1 \leq j \leq r \mid l'_j \text{ is constant}\}$. For $j \in J$, we let $l'_j(\infty)$ be the unique value taken by l'_j. Then,

$$\lim_{s \to +\infty} \frac{A_{Q_{s\underline{r}},\mathbf{d}}}{q^{v_{s\underline{r}}}} = \frac{\sum_{j \in J} q^{l'_j(\infty)} P_j(q)}{Q(q)}.$$

This proves the second point of the theorem. $\square$

In the second point of Theorem 4.3.2, we need to choose a direction $\underline{r}$ since it is not clear that the set J in the proof is independent of the direction (and the limit clearly depends on J). Nethertheless, we conjecture it is the case. If this conjecture is true, the limit is independent of the given direction $\underline{r}$. This conjecture is therefore equivalent to the following one.

Conjecture 4.3.5. *Let Q be a quiver, $\mathbf{d} \in \mathbf{N}^I$ and $v_{\underline{n}}$ be the valuation of $A_{Q_{\underline{n}},\mathbf{d}}(q)$. Then, the sequence of polynomials*

$$\frac{A_{Q_{\underline{n}},\mathbf{d}}(q)}{q^{v_{\underline{n}}}} \in \mathbf{N}[q]$$

converges in $\mathbf{N}[[q]]$ as $\underline{n} \to (+\infty, \ldots, +\infty)$ and the limit is the power series expansion at 0 of a rational fraction.

Proof of Theorem 4.3.3. We take advantage of the explicit formula of Theorem 4.2.5. First recall the formula for the plethystic logarithm ([BS18, Section 1.5]):

$$\mathrm{Log}_{q,z} : 1 + \mathbf{Q}[[q, z_i : i \in I]]^+ \to \mathbf{Q}[[q, z_i : i \in I]]$$

where the $+$ upperscript denotes the augmentation ideal, the ideal of $\mathbf{Q}[[q, z_i : i \in I]]$ generated by $q, z_i, i \in I$. For $f \in \mathbf{Q}[[q, z_i : i \in I]]^+$,

$$\mathrm{Log}_{q,z}(1+f) = \sum_{l \geq 1} \frac{\mu(l)}{l} \psi_l \left(\sum_{k \geq 1} \frac{(-1)^{k+1} f^k}{k} \right),$$

where μ is Moebius function, and for $g = g(q, z_i : i \in I) \in \mathbf{Q}[[q, z_i, i \in I]]$, $\psi_l(g) = g(q^l, z_i^l : i \in I)$.

For $\underline{n} \in \mathbf{N}^{\Omega}$, we let

$$f_{\underline{n}} = \sum_{\pi=(\pi^i)_{i\in I}\in\mathscr{P}'^I} \frac{\prod_{i\xrightarrow{\alpha}j\in\Omega} q^{n_\alpha\langle\pi^i,\pi^j\rangle}}{\prod_{i\in I} q^{\langle\pi^i,\pi^i\rangle}\prod_k\prod_{j=1}^{m_k(\pi^i)}(1-q^{-j})} z^{|\pi|}$$

where $\mathscr{P}'$ is the set of partitions of integers not including the trivial partition $\emptyset$ of 0. Then, by Theorem 4.2.5, we have

$$\sum_{\mathbf{d}\in\mathbf{N}^I} A_{Q_{\underline{n}},\mathbf{d}}(q) z^{\mathbf{d}} = (q-1)\operatorname{Log}(1+f_{\underline{n}}).$$

From this, we see that for fixed dimension vector $\mathbf{d} \in \mathbf{N}^I$, the Kac polynomial $A_{Q_{\underline{n}},\mathbf{d}}(q)$ can be written as a finite sum of rational fractions whose denominators are products of terms of the form $q^a(1-q^{-b})$ and whose numerators are finite linear combinations of powers of q where the exponents are degree one polynomials in the n_α, $\alpha \in \Omega$. This proves that one can write Kac polynomials as in Theorem 4.3.3. Since for any partitions π, π', the pairing $\langle\pi,\pi'\rangle$ is a nonnegative integer, this implies the nonnegativity of the coefficients of the linear part of the affine linear functions l_j, $0 \leq j \leq r$.

For the unicity, assume that for any $\underline{n} \in \mathbf{N}^I$ one has

$$\frac{\sum_{j=0}^r q^{l_j(\underline{n})}P_j(q)}{Q(q)} = \frac{\sum_{j=0}^s q^{l'_j(\underline{n})}P'_j(q)}{Q'(q)}$$

for some affine linear functions $l_0,\dots,l_r,l'_0,\dots,l'_s$ and polynomials $P_0,\dots,P_r,P'_0,\dots,P'_s$ verifying the assumptions given in Theorem 4.3.3. Then,

$$\sum_{j=0}^r q^{l_j(\underline{n})}P_j(q)Q'(q) = \sum_{j=0}^s q^{l'_j(\underline{n})}P'_j(q)Q(q).$$

By comparing the behaviour of the degrees as $\underline{n}$ varies, we see that $r = s$, the sets $\{l_0,\dots,l_r\}$ and $\{l'_0,\dots,l'_s\}$ coincide and for j, j' such that $l_j = l_{j'}$

$$P_j(q)Q'(q) = P'_{j'}(q)Q(q).$$

Up to reordering, we can assume that $l_j = l'_j$ for $0 \leq j \leq r$. Since $\gcd(P_0,\dots,P_r,Q) = 1$, this implies that Q divides Q' and conversely that Q' divides Q. Therefore, $Q = Q'$ and $P_j = P'_j$. □

By Theorem 4.3.2, to a set of vertices I we can associate a family of power series $A_{I,\mathbf{d}}(q) \in \mathbf{N}[[q]]$ as follows. Let $Q = (I,\Omega)$ be a quiver given by an arbitrary orientation of the complete graph on the set of vertices I. We let $\underline{\infty} = (+\infty,\dots,+\infty) \in (\mathbf{N}\cup\{+\infty\})^{\Omega}$ and for $\mathbf{d} \in \mathbf{N}^I$,

$$A_{I,\mathbf{d}}(q) = \lim_{\underline{n}\to\underline{\infty}} A_{Q_{\underline{n}},\mathbf{d}}(q).$$

As Kac polynomials do not depend on the orientation of Q, this family of power series only depends on the cardinality of I. For such quivers, the support of any dimension vector is connected (since any two vertices are connected). Therefore, an easy computation (one checks that $\langle \mathbf{d}, \mathbf{d}\rangle \leq 0$ for $\underline{n}$ big enough where $\langle -, -\rangle$ denotes the Euler form defined in Section 4.2.1) shows that any $\mathbf{d} \in \mathbf{N}^I \setminus \{0, e_i, i \in I\}$ (where e_i, $i \in I$ denotes the canonical basis of $\mathbf{N}^I$ as a monoid) is an imaginary root of $Q_{\underline{n}}$ for $\underline{n}$ big enough (see [Kac90, Lemma 5.3]). So $A_{Q_{\underline{n}},\mathbf{d}}(q) \neq 0$ for $\underline{n} \gg 0$. By letting $\underline{n} \to \underline{\infty}$, we consequently get a non-zero power series $A_{Q,\mathbf{d}}(q)$ for any $\mathbf{d} \in \mathbf{N}^I$. It is also possible to imagine lots of variants using for example the same process for a quiver having exactly one arrow between any two vertices and one loop at each vertex.

Remark 4.3.6. Conjecture 4.3.1 can also be formulated for the polynomials $A^{\flat}_{Q_{\underline{n}},\mathbf{d}}(q)$ for $\flat = 0, 1$. We conjecture that the valuation of $A^{\flat}_{Q_{\underline{n}},\mathbf{d}}(q)$ is 0.

Theorem 4.3.7. *Let $\underline{m} = (m_\alpha)_{\alpha\in\Omega} \in (\mathbf{N}_{\geq 1} \cup \{+\infty\})^{\Omega}$. The sequence of polynomials*

$$q^{\deg A_{Q_{\underline{n}},\mathbf{d}}(q)} A_{Q_{\underline{n}},\mathbf{d}}(q^{-1})$$

converges in $\mathbf{N}[[q]]$ as $\underline{n} \to \underline{m}$.

By a result of Kac ([Kac83, §1.15]), the polynomials $A_{Q_{\underline{n}},\mathbf{d}}$ are all monic. Hence, the limit of Theorem 4.3.7 is an element of $1 + q\,\mathbf{N}[[q]]$.

Proof. For $\underline{n} \in \mathbf{N}^{\Omega}$, we write $A_{Q_{\underline{n}},\mathbf{d}}(q) = \frac{\sum_{j=0}^{r} q^{l_j(\underline{n})} P_j(q)}{Q(q)}$ as in Theorem 4.3.3, where the conditions on the polynomials and affine functions appearing in this decomposition are satisfied, which ensures the unicity of the decomposition. Since we are interested in the limit $\underline{n} \to \underline{m}$, we can restrict ourselves to consider $\underline{n}$ in the subset

$$\mathbf{N}^{\Omega}_{\underline{m}} = \{\underline{n} \in \mathbf{N}^{\Omega} \mid \text{for any } \alpha \in \Omega,\ m_\alpha < +\infty \implies n_\alpha = m_\alpha\}.$$

We let $\Omega' = \{\alpha \in \Omega \mid m_\alpha = +\infty\}$. We identify $\mathbf{N}^{\Omega'}$ and $\mathbf{N}^{\Omega}_{\underline{m}}$ by sending $(n'_\alpha)_{\alpha\in\Omega'}$ to $(n_\alpha)_{\alpha\in\Omega}$, where $n_\alpha = n'_\alpha$ if $\alpha \in \Omega'$, $n_\alpha = m_\alpha$ else. For $\alpha \in \Omega'$, we denote by $e_\alpha \in \mathbf{N}^{\Omega}_{\underline{n}}$ the vector such that $(e_\alpha)_\beta = m_\beta$ if $\beta \notin \Omega'$, $(e_\alpha)_\alpha = 1$ and $(e_\alpha)_\beta = 0$ if $\beta \in \Omega' \setminus \{\alpha\}$. The previous identification endows $\mathbf{N}^{\Omega}_{\underline{m}}$ with a monoid structure: for $\underline{n}, \underline{n}' \in \mathbf{N}^{\Omega}_{\underline{m}}$,

$$(\underline{n} + \underline{n}')_\alpha = \begin{cases} n_\alpha + n'_\alpha & \text{if } \alpha \in \Omega' \\ m_\alpha & \text{else} \end{cases}$$

We define in a similar way $\mathbf{Z}^{\Omega}_{\underline{m}}$ together with a $\mathbf{Z}$-module structure on it. We consider the restrictions l'_j $(0 \leq j \leq r)$ of the affine functions l_j to $\mathbf{Z}^{\Omega}_{\underline{m}}$. By sorting the l'_j's according to their linear parts, we obtain the existence of an integer $s \geq 0$, affine functions $\tilde{l}_j : \mathbf{Z}^{\Omega}_{\underline{m}} \to \mathbf{Z}$, $0 \leq j \leq s$, whose linear parts are pairwise distinct and have nonnegative coefficients when written in the canonical basis $(e_\alpha)_{\alpha\in\Omega'}$ of $\mathbf{Z}^{\Omega}_{\underline{m}}$, polynomials with integer coefficients $\tilde{Q}, \tilde{P}_0, \ldots, \tilde{P}_s$ such that the complex roots of $\tilde{Q}$ are roots of unity, $\tilde{Q}$ is monic, $\tilde{P}_j(0) \neq 0$ for any $0 \leq j \leq s$ and $\gcd(\tilde{Q}, \tilde{P}_1, \ldots, \tilde{P}_s) = 1$, such that

$$A_{Q_{\underline{n}},\mathbf{d}}(q) = \frac{\sum_{j=0}^{s} q^{\tilde{l}_j(\underline{n})} \tilde{P}_j(q)}{\tilde{Q}(q)} \tag{4.3.1}$$

for any $\underline{n} \in \mathbf{N}^{\Omega}_{\underline{m}}$. Moreover, this decomposition is unique by the arguments of the proof of Theorem 4.3.3. By Theorem 4.2.1, for $\underline{n} \in \mathbf{Z}^{\Omega}$, the degree of the polynomial $A_{Q_{\underline{n}},\mathbf{d}}(q)$ is given by the affine function

$$\begin{array}{rcccc} \deg & : & \mathbf{Z}^{\Omega} & \to & \mathbf{Z} \\ & & (\underline{n}_\alpha)_{\alpha\in\Omega} & \mapsto & 1-\langle \mathbf{d},\mathbf{d}\rangle_{\underline{n}} \end{array}$$

where $\langle -,-\rangle_{\underline{n}}$ denotes the Euler form of the quiver $Q_{\underline{n}}$ (Section 4.2.1). More explicitly,

$$\deg \underline{n} = 1-\sum_{i\in I}\mathbf{d}_i^2+\sum_{i\xrightarrow{\alpha}j\in\Omega} n_\alpha\,\mathbf{d}_i\,\mathbf{d}_j$$

for any $\underline{n}=(n_\alpha)_{\alpha\in\Omega}\in\mathbf{Z}^{\Omega}$. We consider the restriction of deg to $\mathbf{Z}^{\Omega}_{\underline{m}}$ and denote it $\widetilde{\deg}$. For any affine function l on some space, we let $L(l)$ denote its linear part. As a consequence of the degree formula and the unicity of the decomposition (4.3.1), there is a unique $0\leq j\leq s$ such that $\tilde{l}_j$ has the same linear part as $\widetilde{\deg}$. We assume this is l_0. Then, we claim that for any $1\leq j\leq s$ and for any $\underline{n}\in\mathbf{N}^{\Omega}_{\underline{m}}$,

$$L(\tilde{l}_j)(\underline{n})\leq L(\tilde{l}_0)(\underline{n})$$

and there exists $\alpha\in\Omega'$ such that for $\underline{n}=e_\alpha\in\mathbf{N}^{\Omega}_{\underline{m}}$, the inequality is strict. We first note that by linearity, the claim is equivalent to the same inequality for any $1\leq j\leq s$ and any $\underline{n}\in(\mathbf{Q}_+)^{\Omega}_{\underline{m}}$ and by density, we can replace $\mathbf{Q}_+$ by $\mathbf{R}_+$. Assume it is false. Then, there exists $1\leq j\leq s$ and $\underline{n}\in(\mathbf{Q}_+)^{\Omega}_{\underline{m}}$ such that $L(\tilde{l}_j)(\underline{n})>L(\tilde{l}_0)(\underline{n})$. The same inequality remains true in a small neighbourhood U of $\underline{n}$ in $(\mathbf{R}_+)^{\Omega}_{\underline{m}}$. Since the $\tilde{l}_j$'s have pairwise distinct linear parts,

$$V:=U\setminus\bigcup_{u\neq v}\{\underline{n}\in(\mathbf{R}_+)^{\Omega}_{\underline{m}}\mid L(\tilde{l}_u)(\underline{n})=L(\tilde{l}_v)(\underline{n})\}$$

is a non-empty open subset of U. For each connected component V_0 of V, there exists $1\leq j'\leq s$ such that for any $0\leq u\leq s$, $u\neq j'$ and $\underline{n}\in V_0$,

$$L(\tilde{l}_{j'})(\underline{n})>L(\tilde{l}_u)(\underline{n}).$$

Fix a connected component V_0 of V and j' as above. Let $\underline{n}\in V_0\cap(\mathbf{Q}_+)^{\Omega}_{\underline{m}}$ and $a\in\mathbf{N}_{\geq 1}$ be such that $a\underline{n}\in\mathbf{N}^{\Omega}_{\underline{m}}$. Then, for any $b\in\mathbf{N}$, $ba\underline{n}\in\mathbf{N}^{\Omega}_{\underline{m}}$ and

$$L(\tilde{l}_{j'})(ab\underline{n})-L(\tilde{l}_u)(ab\underline{n})\to+\infty$$

as $b\to+\infty$ for any $0\leq u\leq s$, $u\neq j'$. Therefore, for $b\gg 0$,

$$\deg A_{Q_{ab\underline{n}},\mathbf{d}}(q)=\tilde{l}_{j'}(ab\underline{n})+\deg\tilde{P}_{j'}-\deg\tilde{Q}>\tilde{l}_0(ab\underline{n})+\deg\tilde{P}_0-\deg\tilde{Q}$$

and this is a contradiction with the degree formula for $\deg A_{Q_{ab\underline{n}},\mathbf{d}}(q)$. Now, since $L(\tilde{l}_j)$ and $L(\tilde{l}_0)$ are distinct linear functions on $\mathbf{Z}^{\Omega}_{\underline{m}}$, there exists $\alpha\in\Omega'$ such that $L(\tilde{l}_j)(e_\alpha)<L(\tilde{l}_0)(e_\alpha)$. Consequently, as $\underline{n}\to\underline{m}$, $\underline{n}\in\mathbf{N}^{\Omega}_{\underline{m}}$, $L(\tilde{l}_0)(\underline{n})-L(\tilde{l}_j)(\underline{n})\to+\infty$ for $1\leq j\leq s$. Since $\widetilde{\deg}$ and $\tilde{l}_0$ have the same

linear part, as $\underline{n} \to \underline{m}$, $\underline{n} \in \mathbf{N}^{\Omega}_{\underline{m}}$,

$$\widetilde{\deg}(\underline{n}) - \tilde{l}_j(\underline{n}) \to +\infty$$

and $\widetilde{\deg} - \tilde{l}_0 := c$ is constant. Then, for $\underline{n} \in \mathbf{N}^{\Omega}_{\underline{m}}$, we have

$$q^{\deg A_{Q_{\underline{n}},\mathbf{d}}(q)} A_{Q_{\underline{n}}\,\mathbf{d}}(q^{-1}) = \frac{q^{\widetilde{\deg}(\underline{n}) - \tilde{l}_0(\underline{n})} \tilde{P}_0(q^{-1}) + \sum_{j=1}^{s} q^{\widetilde{\deg}(\underline{n}) - \tilde{l}_j(\underline{n})} \tilde{P}_j(q^{-1})}{\tilde{Q}(q^{-1})}.$$

The limit as $\underline{n} \to \underline{m}$ is then

$$\lim_{\underline{n} \to \underline{m}} q^{\deg A_{Q_{\underline{n}},\mathbf{d}}(q)} A_{Q_{\underline{n}}\,\mathbf{d}}(q^{-1}) = \frac{q^c \tilde{P}_0(q^{-1})}{\tilde{Q}(q^{-1})}.$$

This proves the theorem.

□

4.3.3 Particular cases

To support Conjecture 4.3.1, we give some examples for the quiver S_g with one vertex and g-loops (multi-loop quivers) and for the tennis-racket quiver (Section 4.3.3):

$$S_g = \; 1 \circlearrowleft (g)$$

and the generalized Kronecker quivers K_r with two vertices and r arrows from the first to the second:

$$K_r = \; 1 \xrightarrow{(r)} 2$$

If $Q = S_g$ and $\underline{n} = (n_1, \ldots, n_g)$ (we label the arrows of Q by the integers $1, \ldots, g$), then $Q_{\underline{n}}$ is the one-vertex quiver with $g \cdot \sum_{i=1}^{g} n_i$ loops:

$$Q_{\underline{n}} = \; 1 \circlearrowleft (g \cdot \textstyle\sum_{i=1}^{g} n_i)$$

and if $Q = K_r$, $\underline{n} = (n_1, \ldots, n_r)$, $Q_{\underline{n}}$ is the quiver having two vertices and $r \cdot \sum_{i=1}^{r} n_i$ arrows from the first to the second:

$$Q_{\underline{n}} = \; 1 \xrightarrow{\left(r \cdot \sum_{i=1}^{r} n_i\right)} 2$$

Example 4.3.8. By Theorem 4.3.2, we let $A_{K_\infty,\mathbf{d}} = \lim_{r \to \infty} A_{K_r,\mathbf{d}}$. This Section illustrates Theorem

4.3.2 and its proof. We have the following equalities:

$$
\begin{aligned}
A_{K_r,(2,2)} &= \frac{q^{4r-2}-q^{2(r-1)}(1+2q+3q^2)+2q^{r-1}(q+1)^2-(2q+1)}{(q-1)^3(q+1)^2}\\
A_{K_\infty,(2,2)} &= \frac{1+2q}{(1-q)^3(q+1)^2}\\
&= 1+3q+5q^2+9q^3+12q^4+18q^5+22q^6+30q^7+35q^8+45q^9+51q^{10}+O(q^{11})\\
A_{K_r,(2,3)} &= \frac{1}{(1-q)^3(1+q)^2(1-q^3)}\big((q^{6r-4}-q^{4r-4}(1+q+q^2)-q^{3r-2}(1+3q+3q^2+q^3)+\\
&q^{2r-3}(1+3q+7q^2+8q^3+6q^4+2q^5)-q^{r-2}(1+4q+7q^7+7q^3+4q^4+q^5)+\\
&(2+2q+2q^2+q^3)\big)\\
A_{K_\infty,(2,3)} &= \frac{2+2q+2q^2+q^3}{(1-q)^3(1+q)^2(1-q^3)}\\
&= 2+4q+10q^2+17q^3+29q^4+43q^5+64q^6+87q^7+119q^8+154q^9+199q^{10}+O(q^{11})\\
A_{K_r,(2,4)} &= \frac{1}{(1-q)^4(1+q)^3(1-q^3)(1+q^2)}\big(-q^{8r-7}+q^{6r-7}(1+q+q^2+q^3)\\
&-q^{4r-5}(1+2q-q^2-q^3-5q^4-4q^5-3q^6)-q^{3r-6}(1+q+q^2+4q^3+8q^4+13q^5+15q^6+\\
&12q^7+7q^8+2q^9)+q^{2r-5}(1+2q+4q^2+10q^3+15q^4+22q^5+19q^6+17q^7+7q^8+3q^9)\\
&-q^{r-3}(1+3q+7q^2+12q^3+14q^4+13q^5+9q^6+4q^7+q^8)+(2+4q+3q^2+4q^3+q^4+q^5)\big)\\
A_{K_\infty,(2,4)} &= \frac{2+4q+3q^2+4q^3+q^4+q^5}{(1-q)^4(1+q)^3(1-q^3)(1+q^2)}\\
&= 2+6q+13q^2+27q^3+46q^4+78q^5+118q^6+179q^7+251q^8+355q^9+473q^{10}+O(q^{11})\\
A_{K_\infty,(2,5)} &= 3+7q+18q^2+35q^3+67q^4+113q^5+186q^6+286q^7+431q^8+622q^9+882q^{10}+O(q^{11})\\
A_{K_\infty,(3,3)} &= 3+9q+24q^2+48q^3+92q^4+154q^5+248q^6+376q^7+551q^8+775q^9+1070q^{10}+O(q^{11})\\
A_{K_\infty,(3,4)} &= 5+15q+44q^2+98q^3+200q^4+364q^5+631q^6+1021q^7+1596q^8+2390q^9+O(q^{10})\\
A_{K_\infty,(3,5)} &= 7+23q+70q^2+165q^3+355q^4+685q^5+1247q^6+2129q^7+3491q^8+5488q^9+O(q^{10})\\
A_{K_\infty,(4,4)} &= 8+32q+98q^2+250q^3+547q^4+1101q^5+2036q^6+3574q^7+5933q^8+9513q^9+O(q^{10})
\end{aligned}
$$

The multi-loops quivers

Let S_g be the quiver with one vertex and g loops and $A_{S_g,d,0}=\frac{A_{S_g,d}}{q^{1+d(g-1)}}$. We let

$$A_{S_\infty,d,0}=\lim_{g\to\infty}A_{S_g,d,0}.$$

We have the following:

$$
\begin{aligned}
A_{S_\infty,1,0} &= 1\\
A_{S_g,2,0} &= \frac{1-q^{2g}}{1-q^2}\\
A_{S_\infty,2,0} &= \frac{1}{1-q^2}\\
&= 1+q^2+q^4+q^6+q^8+q^{10}+O(q^{11})\\
A_{S_g,3,0} &= \frac{(1+q)-q^{2g-1}(1+q+q^2)+q^{6g-1}}{(1-q^2)(1-q^3)}\\
A_{S_\infty,3,0} &= \frac{1}{(1-q)(1-q^3)}\\
&= 1+q+q^2+2q^3+2q^4+2q^5+3q^6+3q^7+3q^8+4q^9+4q^{10}+O(q^{11})\\
A_{S_\infty,4,0} &= \frac{1}{(1-q^2)^2(1-q^3)(1+q^2)}\big((1+q+2q^2+q^3+q^4)-q^{2g-1}(1+2q+3q^2+3q^3+2q^4+q^5)+\\
&\quad q^{4g-2}(-1+2q^3+q^4+q^5)+q^{6g-3}(1+q+q^2+q^3)-q^{12g-3}\big)\\
A_{S_\infty,4,0} &= \frac{1+q+2q^2+q^3+q^4}{(1-q^2)^2(1-q^3)(1+q^2)} = \frac{1}{(1-q)(1-q^2)^2}\\
&= 1+q+3q^2+3q^3+6q^4+6q^5+10q^6+10q^7+15q^8+15q^9+21q^{10}+O(q^{11})\\
A_{S_\infty,5,0} &= \frac{1+2q+3q^2+4q^3+4q^4+4q^5+3q^6+2q^7+q^8}{(1-q)^3(1+q)^2(1-q^3)(1+q^2)(1-q^5)} = \frac{1-q+q^2}{(1-q)^3(1-q^5)}\\
&= 1+2q+4q^2+7q^3+11q^4+17q^5+24q^6+33q^7+44q^8+57q^9+73q^{10}+O(q^{11})\\
A_{S_\infty,6,0} &= \frac{1+q+2q^2+2q^3+3q^4+2q^6+q^7}{(1-q)^5(1+q)^3(1-q+q^2)(1+q+q^2)}\\
&= 1+2q+6q^2+11q^3+22q^4+33q^5+57q^6+80q^7+121q^8+164q^9+231q^{10}+O(q^{11})\\
A_{S_\infty,7,0} &= \frac{(1-q+q^2)^2}{(q-1)^6(q^6+q^5+q^4+q^3+q^2+q+1)}\\
&= 1+3q+8q^2+18q^3+36q^4+66q^5+113q^6+184q^7+286q^8+429q^9+624q^{10}+O(q^{11})\\
A_{S_\infty,8,0} &= 1+3q+11q^2+25q^3+59q^4+113q^5+217q^6+371q^7+630q^8+994q^9+O(q^{10})\\
A_{S_\infty,9,0} &= 1+4q+13q^2+37q^3+88q^4+190q^5+382q^6+715q^7+1270q^8+2162q^9+O(q^{10})\\
A_{S_\infty,10,0} &= 1+4q+17q^2+48q^3+130q^4+297q^5+647q^6+1280q^7+2438q^8+4363q^9+O(q^{10})
\end{aligned}
$$

The tennis-racket quiver

We consider the quiver

$$
Q = \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[loop right, "\beta"] \end{tikzcd} .
$$

We let $A_{Q_{\underline{n}},\mathbf{d},0} = \frac{A_{Q_{\underline{n}},\mathbf{d}}}{q^{1+d_2(n_\beta-1)}}$ and

$$
A_{Q_{\underline{\infty}},\mathbf{d},0} = \lim_{\underline{n}\to\underline{\infty}} A_{Q_{\underline{n}},\mathbf{d}}.
$$

We have the following formulas, obtained combining Theorem 4.3.3 and computations. We let $\underline{n} = (n_\alpha, n_\beta)$.

$$
\begin{aligned}
A_{Q_{\underline{n}},(1,1),0} &= \frac{1-q^{n_\alpha}}{1-q}\\
A_{Q_{\underline{\infty}},(1,1),0} &= \frac{1}{1-q}\\
&= 1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10}+O(q^{11})\\
A_{Q_{\underline{n}},(1,2),0} &= \frac{1+q-q^{n_\alpha}(1+q)-q^{2n_\beta}+q^{2(n_\alpha+n_\beta)}}{(1-q)(1-q^2)}\\
A_{Q_{\underline{\infty}},(1,2),0} &= \frac{1+q}{(1-q)(1-q^2)}\\
&= 1+2q+3q^2+4q^3+5q^4+6q^5+7q^6+8q^7+9q^8+10q^9+11q^{10}+O(q^{11})\\
A_{Q_{\underline{\infty}},(1,3),0} &= \frac{1}{(1-q)^3}\\
&= 1+3q+6q^2+10q^3+15q^4+21q^5+28q^6+36q^7+45q^8+55q^9+66q^{10}+O(q^{11})\\
A_{Q_{\underline{\infty}},(1,4),0} &= 1+4q+10q^2+20q^3+35q^4+56q^5+84q^6+120q^7+165q^8+220q^9+286q^{10}+O(q^{11})\\
A_{Q_{\underline{n}},(2,1),0} &= \frac{1-q^{n_\alpha-1}(1+q)+q^{2n_\alpha}}{(1-q)(1-q^2)}\\
A_{Q_{\underline{\infty}},(2,1),0} &= \frac{1}{(1-q)(1-q^2)}\\
&= 1+q+2q^2+2q^3+3q^4+3q^5+4q^6+4q^7+5q^8+5q^9+6q^{10}+0(q^{11})\\
A_{Q_{\underline{n}},(2,2),0} &= \frac{1+q+2q^2-q^{n_\alpha}(2+4q+2q^2)-q^{2n_\beta}+q^{2n_\alpha}(1+3q)+q^{2(n_\alpha+n_\beta)}(1+q)-q^{4n_\alpha+2n_\beta-1}}{(1-q^2)^2(1-q)}\\
A_{Q_{\underline{\infty}},(2,2),0} &= \frac{1+q+2q^2}{(1-q^2)^2(1-q)}\\
&= 1+2q+6q^2+8q^3+15q^4+18q^5+28q^6+32q^7+45q^8+50q^9+66q^{10}+O(q^{11})\\
A_{Q_{\underline{\infty}},(2,3),0} &= \frac{1+q+q^2}{(1-q^2)(1-q)^3}\\
&= 1+4q+11q^2+23q^3+42q^4+69q^5+106q^6+154q^7+215q^8+290q^9+381q^{10}+O(q^{11})\\
A_{Q_{\underline{\infty}},(2,4),0} &= 1+5q+19q^2+45q^3+100q^4+182q^5+322q^6+510q^7+795q^8+1155q^9+1661q^{10}+O(q^{11})\\
A_{Q_{\underline{\infty}},(3,1),0} &= 1+q+2q^2+3q^3+4q^4+5q^5+7q^6+8q^7+10q^8+12q^9+14q^{10}+O(q^{11})\\
A_{Q_{\underline{\infty}},(3,2),0} &= 1+3q+7q^2+14q^3+24q^4+38q^5+57q^6+81q^7+111q^8+148q^9+192q^{10}+O(q^{11})\\
A_{Q_{\underline{\infty}},(3,3),0} &= \frac{(2q+1)(q^5+q^4+2q^3+3q^2+q+1)}{(1-q^2)(1-q)^2(1-q^3)^2}\\
&= 1+5q+15q^2+38q^3+78q^4+144q^5+248q^6+397q^7+605q^8+890q^9+1261q^{10}+O(q^{11})\\
A_{Q_{\underline{\infty}},(3,4),0} &= 1+7q+27q^2+79q^3+191q^4+405q^5+779q^6+1390q^7+2336q^8+3740q^9+5751q^{10}+O(q^{11})\\
A_{Q_{\underline{\infty}},(4,1),0} &= 1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8+18q^9+23q^{10}+O(q^{11})\\
A_{Q_{\underline{\infty}},(4,2),0} &= 1+3q+9q^2+17q^3+35q^4+56q^5+95q^6+139q^7+211q^8+290q^9+410q^{10}=O(q^{11})
\end{aligned}
$$

4.3.4 The constant coefficient of Kac polynomials

When Q has no edge-loops, the constant coefficient of $A_{Q_{\underline{n}},\mathbf{d}}$ can easily be seen to converge when $\underline{n} \to \underline{m}$. Indeed, for a connected dimension vector and $\underline{n}$ big enough (that is, n_α big enough for any $\alpha \in \Omega$), $\mathbf{d}$ is a positive root and therefore, by Theorem 4.2.1,

$$A_{Q_{\underline{n}},\mathbf{d}}(0) = \dim \mathfrak{g}_{Q_{\underline{n}}}[\mathbf{d}] = \dim \mathfrak{n}^+_{Q_{\underline{n}}}[\mathbf{d}].$$

where $\mathfrak{g}_{Q_{\underline{n}}}$ is the Kac-Moody algebra asociated to $Q_{\underline{n}}$ and $\mathfrak{g}_{Q_{\underline{n}}} = \mathfrak{n}^-_{Q_{\underline{n}}} \oplus \mathfrak{h}_{Q_{\underline{n}}} \oplus \mathfrak{n}^+_{Q_{\underline{n}}}$ its triangular decomposition. By [Kac90, Theorem 9.11], the $\mathbf{N}^I$-graded Lie algebra $\mathfrak{n}^+_{Q_{\underline{n}}}$ has (Chevalley) generators E_i, $i \in I$ verifying Serre's relations: if $i \neq j$, and $a_{ij} = -\sum_{\alpha:i\to j} n_\alpha - \sum_{\alpha:j\to i} n_\alpha$

$$\mathrm{ad}(E_i)^{1-a_{ij}}(E_j) = 0$$

This relation is homogeneous of degree $e_j + (1-a_{ij})e_i$. Therefore, when $\underline{n} \to \underline{m}$, $\dim \mathfrak{g}_{Q_{\underline{n}}}[\mathbf{d}]$ stabilizes and converges to $\dim \mathfrak{g}_{Q_{\underline{m}}}$, where $\mathfrak{g}_{Q_{\underline{m}}}$ is defined the same way as $\mathfrak{g}_{Q_{\underline{n}}}$, except that if $m_\alpha = \infty$, $\alpha : i \to j$, there is no relation between E_i and E_j. If Q is complete, that is any two vertices are connected by an arrow, and $\underline{m} = \underline{\infty}$, then $\mathfrak{g}_{Q_{\underline{\infty}}}$ is the free Lie algebra on generators E_i, $i \in I$. Its character is given by Witt's dimension formula. For $\mathbf{d} = \sum_{i\in I} d_i e_i \in \mathbf{N}^I$ and $|\mathbf{d}| = \sum_{i\in I} d_i$,

$$\dim \mathfrak{g}_{Q_{\underline{\infty}}}[\mathbf{d}] = \frac{1}{|\mathbf{d}|} \sum_{d|d_i, i\in I} \mu(d) \frac{\frac{|\mathbf{d}|}{d}!}{\prod_{i\in I} \frac{d_i}{d}!}$$

(see *e.g.* [Hal86, (2.1.21)]). It is easy to check on the examples that for the generalized Kronecker quivers, see Example 4.3.8, this formula indeed gives the constant coefficient of $A_{K_\infty,\mathbf{d}}$.

4.3.5 Rate of convergence

Theorem 4.3.3 also gives the rate of convergence of $A_{Q_{\underline{n}},\mathbf{d}}$ to $A_{Q_{\underline{m}},\mathbf{d}}$. To achieve this, let

$$\alpha_{\underline{n}} = val(A_{Q_{\underline{n}},\mathbf{d}} - A_{Q_{\underline{m}},\mathbf{d}})$$

be the highest power of q dividing $A_{Q_{\underline{n}},\mathbf{d}}(q) - A_{Q_{\underline{m}},\mathbf{d}}(q)$.

Proposition 4.3.9. *The rate of convergence is linear in $\underline{n}$, that is $\alpha_{\underline{n}}(q)$ is bigger than a linear function of $\underline{n}$ whose limit in any infinite direction is infinite.*

In fact, the optimal choice can be determined using the affine linear functions l_j given by Theorem 4.3.3. For example, we expect to have $\alpha_r = r - \max(d_1, d_2) + 1$ for the generalized Kronecker quiver and $\alpha_g = 2g$ for $d = 1$, $\alpha_g = 2g - 1$ for $d \geq 2$ for the multi-loops quivers.

4.3.6 Dimension count of cuspidal functions

In [BS18], Bozec and Schiffmann introduced two families of polynomials, $C_{Q,\mathbf{d}}(q)$ and $C^{abs}_{Q,\mathbf{d}}(q)$. When evaluated at $q = |\mathbf{F}_q|$,

$$C_{Q,\mathbf{d}}(q) = \dim \mathbf{H}^{cusp}_{Q,\mathbf{F}_q}[\mathbf{d}]$$

where $\mathbf{H}^{cusp}_{Q,\mathbf{F}_q}[\mathbf{d}]$ is the vector space of cuspidal functions of degree $\mathbf{d}$ in the Hall algebra of Q over $\mathbf{F}_q$ (see *op. cit.* for more details). The polynomials $C^{abs}_{Q,\mathbf{d}}(q)$ are obtained by inductively inverting Borcherds character formula for Borcherds Lie algebras starting from the formula [BS18, (4.3)]:

$$\pi(\mathrm{ch}(U(\tilde{\mathfrak{n}}_Q^{\mathbf{N}}))) = \mathrm{Exp}_{q,z}\left(\sum_{\mathbf{d}>0} A_{Q,\mathbf{d}}(q) z^{\mathbf{d}}\right),$$

where $\tilde{\mathfrak{n}}_Q^{\mathbf{N}}$ is the positive part of a conjectural Borcherds Lie algebra whose character is given by the right hand side of the above formula. Moreover, these two families of polynomials are related:

$$C^{abs}_{Q,\mathbf{d}}(q) = C_{Q,\mathbf{d}}(q) \text{ if } \mathbf{d} \in \mathbf{N}^I \text{ , } \langle \mathbf{d}, \mathbf{d}\rangle < 0$$

$$\mathrm{Exp}_z\left(\sum_{l\geq 1} C_{Q,l\,\mathbf{d}}(q) z^l\right) = \mathrm{Exp}_{q,z}\left(\sum_{l\geq 1} C^{abs}_{Q,l\,\mathbf{d}}(q) z^l\right) \text{ if } \mathbf{d} \in \mathbf{N}^I \text{ is indivisible and } \langle \mathbf{d}, \mathbf{d}\rangle \leq 0.$$

Combining all of this, Theorem 4.3.2 has as a consequence the stabilization of the family of polynomials $C_{Q_{\underline{n}},\mathbf{d}}(q)$ and $C^{abs}_{Q_{\underline{n}},\mathbf{d}}(q)$ as we increase the number of arrows, if we assume that Q has no loops. It seems more difficult to conclude if Q has loops although it is reasonable to expect they also verify a stabilization property of the same kind. Computing the valuation of cuspidal polynomials is also a problem for which no answer is known.

4.4 Geometric interpretation

It would be nice to be able to understand geometrically these stabilization properties of Kac polynomials. In this Section, we give some ideas of where to look for such an interpretation.

4.4.1 Nakajima quiver varieties

Kac polynomials have an interpretation related to the cohomology of Nakajima quiver varieties. These are symplectic resolutions defined in terms of GIT quotients of moduli spaces of framed representations of a quiver whose link with the representation theory of Kac-Moody algebras has been studied by Nakajima from the 1990s ([Nak94, Nak98]). For indivisible dimension vectors, the link between the Kac polynomial and Nakajima quiver varieties is given in [CBVdB04] and for any dimension vector in [HLRV13]. Following King ([Kin94]), for $\lambda \in \mathbf{Z}^I$ and $\mathbf{d} \in \mathbf{N}^I$ such that $\lambda \cdot \mathbf{d} := \sum_{i\in I} \lambda_i \mathbf{d}_i = 0$, a representation M of Q of dimension vector $\mathbf{d}$ is said to be stable if $\lambda \cdot \dim N < 0$ for any subrepresentation $0 \neq N \subsetneq M$.

Theorem 4.4.1 ([CBVdB04])**.** *Let $\mathbf{d}$ be an indivisible dimension vector, $\lambda \in \mathbf{Z}^I$ such that $\lambda\cdot\mathbf{d} = 0$ and $\lambda \cdot \mathbf{e} \neq 0$ for any $0 < \mathbf{e} < \mathbf{d}$. Then*

$$A_{Q,\mathbf{d}}(q) = \sum_{i=0}^{d} \dim H^{2d-2i}(X_s, \mathbf{C}) q^i$$

where X_s is the moduli space of λ-stable representations of dimension vector $\mathbf{d}$ of the preprojective *algebra* Π_Q *and* $\dim X_s = 2d$. *(The variety* X_s *is an example of a Nakajima quiver variety when the framing is trivial).*

For clarity, we briefly explain the construction of X_s. First recall that the preprojective algebra Π_Q is the quotient of the path algebra of the doubled quiver $\overline{Q}$ (for each arrow a of Q, we add an arrow a^* in the opposite direction) by the two-sided ideal generated by

$$\sum_{a\in\Omega}(aa^* - a^*a).$$

The vector space

$$E_{\overline{Q},\mathbf{d}} = \bigoplus_{a:i\to j\in\Omega} \mathrm{Hom}(\mathbf{C}^{\mathbf{d}_i}, \mathbf{C}^{\mathbf{d}_j}) \oplus \bigoplus_{a:i\to j\in\Omega} \mathrm{Hom}(\mathbf{C}^{\mathbf{d}_j}, \mathbf{C}^{\mathbf{d}_i})$$

is naturally acted on by the product of linear groups

$$\mathrm{GL}_{\mathbf{d}} = \prod_{i\in I} \mathrm{GL}_{\mathbf{d}_i}.$$

Let $E_{\Pi_Q,\mathbf{d}}$ be the closed subvariety of $E_{\overline{Q},\mathbf{d}}$ of elements $(x_a, x_{a^*})_{a\in\Omega}$ such that $\sum_{a\in\Omega}(x_a x_{a^*} - x_{a^*}x_a) = 0$. The set-theoretic quotient $E_{\Pi_Q,\mathbf{d}}/\mathrm{GL}_{\mathbf{d}}$ parametrizes the isomorphism classes of representations of Π_Q of dimension vector $\mathbf{d}$. The action of $\mathrm{GL}_{\mathbf{d}}$ factorizes through $PGL_{\mathbf{d}} = \mathrm{GL}_{\mathbf{d}}/\mathbf{C}^*$ and an element $\lambda \in \mathbf{Z}^I$ such that $\lambda\cdot\mathbf{d} = 0$ gives a character of $PGL_{\mathbf{d}}$:

$$\begin{array}{cccc} \chi & : & PGL_{\mathbf{d}} & \to & \mathbf{C}^* \\ & & (\overline{g_i}))_{i\in I} & \mapsto & \prod_{i\in I}\det(g_i)^{-\lambda_i} \end{array}.$$

This is a linearization of the trivial vector bundle over $E_{\Pi_Q,\mathbf{d}}$ for the $PGL_{\mathbf{d}}$-action and it can be checked that the stable points of $E_{\Pi_Q,\mathbf{d}}$ are precisely whose $x \in E_{\Pi_Q,\mathbf{d}}$ giving a stable representation of Π_Q. We let $E^{\lambda-st}_{\Pi_Q,\mathbf{d}} \subset E_{\Pi_Q,\mathbf{d}}$ be the open subset of λ-stable elements. The variety X_s is obtained as the GIT quotient $E_{\Pi_Q,\mathbf{d}}/\!/_{\chi}PGL_{\mathbf{d}}$.

Example 4.4.2. For $Q = K_r$, $\mathbf{d} = (1,1)$, $\lambda = (1,-1)$, a representation $(a_i, b_i)_{1\leq i\leq r}$ of Π_Q is stable if and only if $(a_1,\dots,a_r) \neq 0$. In this case, X_s is the total space of the tautological bundle over the grassmannian of hyperplanes in $\mathbf{C}^r$. Therefore, the cohomology of X_s is in this case very simple ($H^i(X_s,\mathbf{C})$ is one dimensional for i even and $0\leq i\leq 2(r-1)$ and zero else). Combining this with Theorem 4.4.1, we obtain an explicit formula for the Kac polynomial $A_{K_r,(1,1)}(q)$ from which we obtain a proof of Theorem 4.3.2 for the Kronecker quiver and the dimension vector $(1,1)$ (although this case is trivial).

4.4.2 Lusztig nilpotent variety

Kac polynomials appear in the counting of points of Lusztig nilpotent varieties, as shown in [BSV17].

Theorem 4.4.3. *For any quiver Q,*

$$\mathrm{Exp}_{q,z}\left(\frac{1}{1-q^{-1}}\sum_{\mathbf{d}} A_{\mathbf{d}}(q^{-1})z^{\mathbf{d}}\right) = \sum_{\mathbf{d}} \frac{|\Lambda_{\mathbf{d}}(\mathbf{F}_q)|}{|G_{\mathbf{d}}(\mathbf{F}_q)|} q^{\langle \mathbf{d},\mathbf{d}\rangle} z^{\mathbf{d}}.$$

The interesting point in this formula is that only non-positive powers of q appear in the left hand side. It implies that $q \mapsto \frac{|\Lambda_{\mathbf{d}}(\mathbf{F}_q)|}{|G_{\mathbf{d}}(\mathbf{F}_q)|} q^{\langle \mathbf{d},\mathbf{d}\rangle}$ is a polynomial in q^{-1}. The appearance of $A_{Q,\mathbf{d}}(q^{-1})$ instead of $A_{Q,\mathbf{d}}(q)$ in Theorem 4.4.3 suggests that Theorem 4.3.7 concerning the reciprocal of Kac polynomials $q^{\deg A_{Q,\mathbf{d}}(q)} A_{Q,\mathbf{d}}(q^{-1})$ should have a geometric interpretation.

The point count of the zero-level of the moment map also has something to do with Kac polynomials:

Theorem 4.4.4 ([BSV17, Formula (1.4)])**.** *For any quiver Q, we have*

$$\sum_{\mathbf{d}} \frac{|\mu_{\mathbf{d}}^{-1}(0)(\mathbf{F}_q)|}{|G_{\mathbf{d}}(\mathbf{F}_q)|} = \mathrm{Exp}_{q,z}\left(\frac{q}{q-1}\sum_{\mathbf{d}} A_{\mathbf{d}}(q) z^{\mathbf{d}}\right).$$

Because of this formula, we can expect some interpretation of Conjecture 4.3.1 and Theorem 4.3.2 in terms of some geometric properties of $\mu_{\mathbf{d}}^{-1}(0)$ when we increase the number of arrows.

4.5 Distribution of the coefficients of Kac polynomials

In this Section, we explore global properties of Kac polynomials of quivers. Motivated by the Hausel–Rodriguez-Villegas paper [HRV15] and the interpretation of Kac polynomials in terms of Poincaré polynomials of some quiver varieties (at least for indivisible dimension vectors), Theorem 4.4.1, we were led to study the distribution of the coefficients of Kac polynomials when we increase the number of arrows. More precisely, to a quiver $Q = (I, \Omega)$ and a dimension vector $\mathbf{d} \in \mathbf{N}^I$, we plot on a graph $G_{Q,\mathbf{d},0}$ the renormalized even coefficients of $A_{Q,\mathbf{d}}(q)$, that is the points $\left(\frac{2j}{\deg A_{Q,\mathbf{d}}(q)}, \frac{a_{Q,\mathbf{d},2j}}{a_{Q,\mathbf{d}}}\right)$ where $A_{Q,\mathbf{d}}(q) = q^{N_{\mathbf{d}}} \sum_{j=0}^{\deg A_{Q,\mathbf{d}}(q)} a_{Q,\mathbf{d},j} q^j$ ($N_{\mathbf{d}}$ is some nonnegative integer, the valuation of $A_{Q,\mathbf{d}}$, and $a_{Q,\mathbf{d},0} \neq 0$) and $a_{Q,\mathbf{d}} = \max_{0 \leq j \leq \deg A_{Q,\mathbf{d}}(q)} a_{Q,\mathbf{d},j}$ and on $G_{Q,\mathbf{d},1}$ the renormalized odd coefficients of the polynomial $A_{Q,\mathbf{d}}(q)$, which are the points $\left(\frac{2j+1}{\deg A_{Q,\mathbf{d}}(q)}, \frac{a_{Q,\mathbf{d},2j+1}}{a_{Q,\mathbf{d}}}\right)$. We make the following conjecture. By abuse, we also let $G_{Q,\mathbf{d},a}$ be the piecewise affine curve obtained by joining the dots of $G_{Q,\mathbf{d},a}$ for $a = 0, 1$. Let $\underline{m} \in \mathbf{N}^I_{\geq 1}$. For $u \in \mathbf{N}$, we let $u\underline{m} = (um_\alpha)_{\alpha \in \Omega}$.

Conjecture 4.5.1. *The curves $G_{Q_{u\underline{m}},\mathbf{d}}$ converge to a continuous curve when $u \to \infty$.*

Remark 4.5.2.

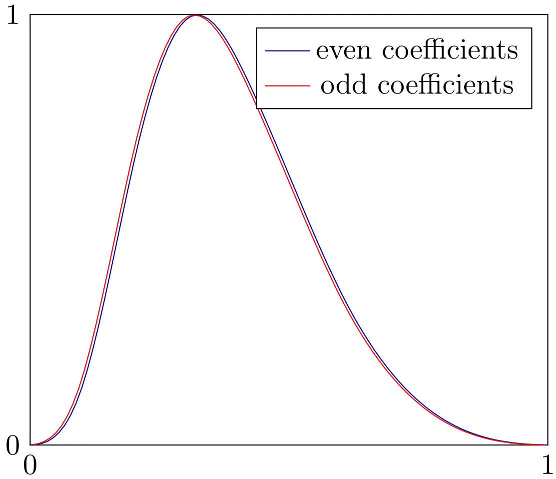
1. The difference with Theorem 4.3.2 is that here, we need to fix a direction $\underline{m}$ to have the convergence.

2. In [HRV15], the authors studied the distribution of the Betti numbers of semiprojective hyperkähler varieties by taking a limit involving a different operation on the quiver than ours: we increase the number of arrows without changing the set of vertices while the authors of *op. cit.* consider growing complete graphs with dimension vector one at each vertex, or the

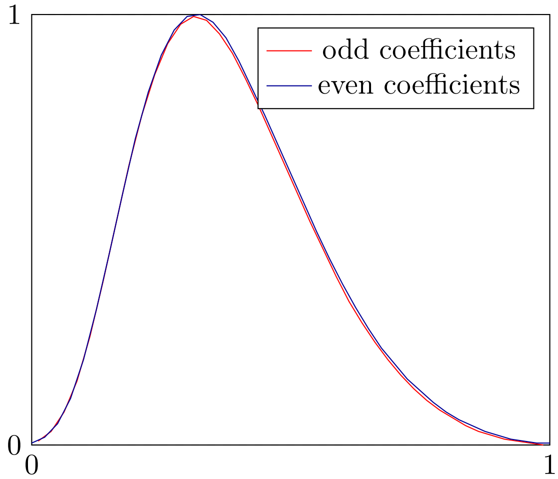
tennis-racket quiver with dimension vector $(1, n)$, n being the dimension at the loop-vertex, and let n goes to infinity.

To illustrate Conjecture 4.5.1, we give the graphs obtained for some specific quivers.

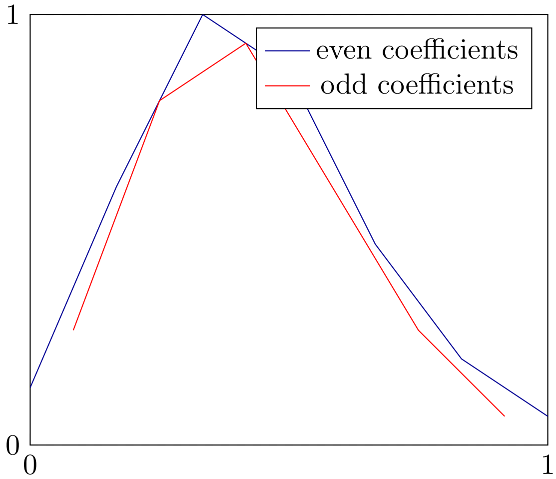
Quiver S_g with $d = 5$, $g = 10$



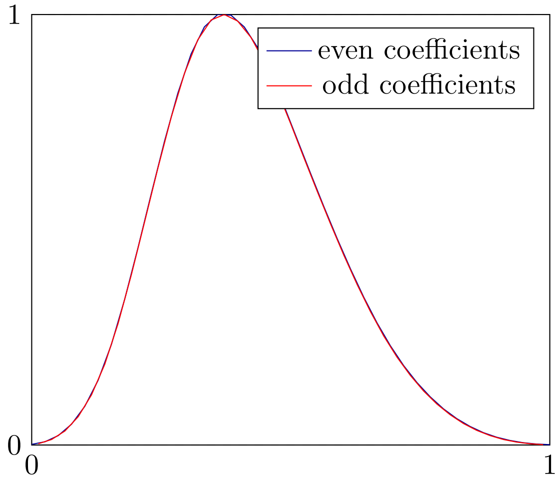
Quiver S_g with $d = 5$, $g = 5$



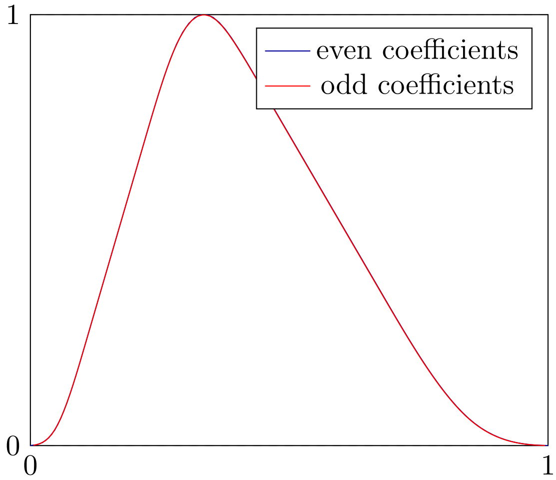
Quiver K_r with $\mathbf{d} = (2, 3)$, $r = 4$



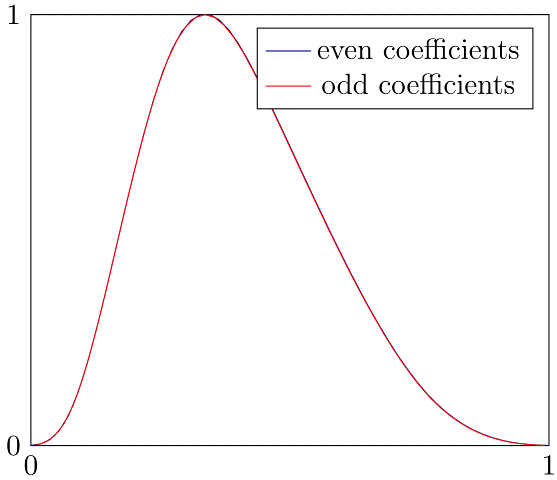
Quiver K_r with $\mathbf{d} = (2, 3)$, $r = 15$



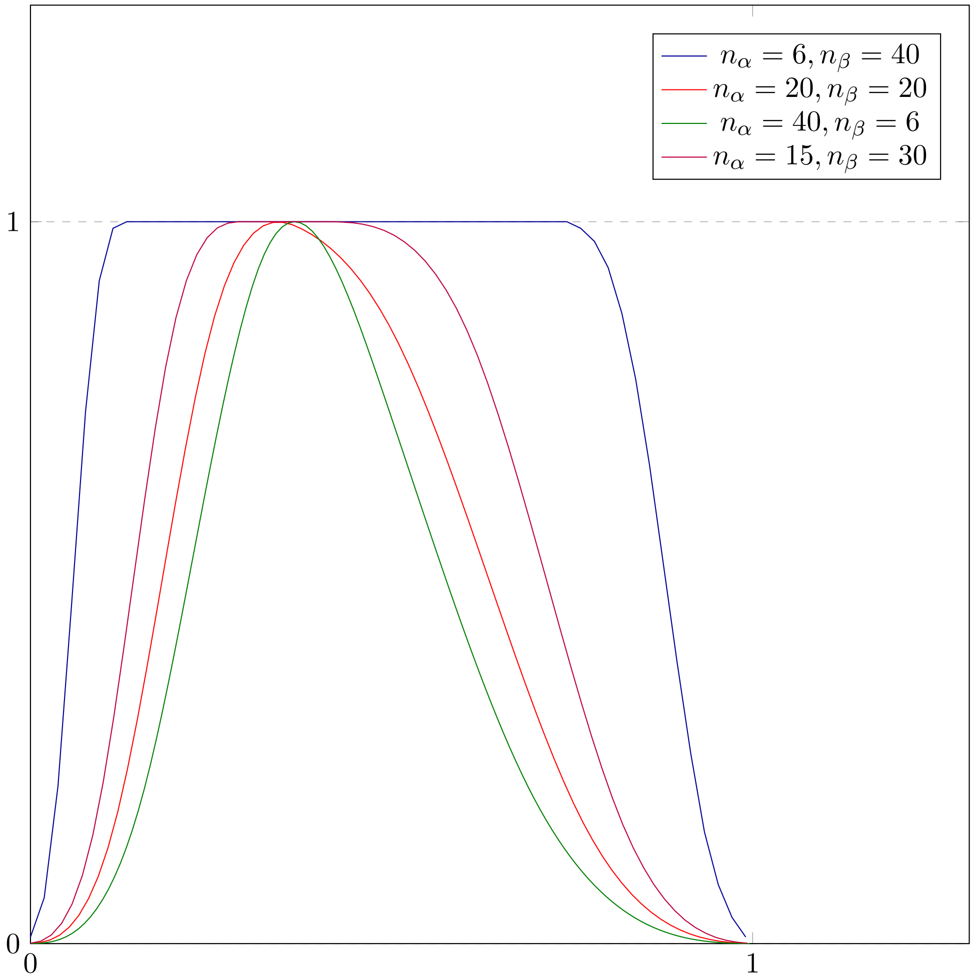
Tennis-racket quiver, $\mathbf{d} = (2, 3)$, $n_\beta = 2n_\alpha = 20$



Tennis-racket quiver, $\mathbf{d} = (2, 3)$, $n_\alpha, n_\beta = 10$



Tennis-racket quiver with $\mathbf{d} = (3, 2)$, even coefficients



The last conjecture follows from the observation of the various graphs above.

Conjecture 4.5.3. *The sequences of even and odd Betti numbers are unimodal (that is, increasing and then decreasing)*

Remark 4.5.4. This is a frequent phenomenon having a strong relationship with the Hard Lefschetz Theorem (at least when we consider the Betti numbers of some varieties), see [HRV15].

4.A Code for the computations of Kac polynomials

In this section, we give the entire code used to obtained the results of Example 4.3.8, Section 4.3.3 and Section 4.3.3 using SageMath. For the sake of practicality, the code is available on the author's webpage https://www.imo.universite-paris-saclay.fr/~hennecart/.

4.A.1 Explanation for the code

We use the formulas in [Hua00].

If $r \geq 0$, we let $\phi_r(q) = \prod_{j=1}^{r}(1-q^j)$ and for a partition λ, $b_\lambda(q) = \prod_{i\geq 1}\phi_{n_i}(q)$ if $\lambda = (1^{n_1}2^{n_2}\ldots)$. Note that if λ' is the partition conjugate to λ, one has $n_i = \lambda'_i - \lambda'_{i+1}$.

If λ, μ are two partitions, we defined in Theorem 4.2.5 the bilinear product $\langle\lambda,\mu\rangle$. We use another expression for it. If λ' (resp. μ') is the partition conjugate to λ (resp. μ), we have

$$\langle\lambda,\mu\rangle = \sum_{i\geq 1}\lambda'_i\mu'_i.$$

We define

$$P((x_i)_{i\in I},q) = \sum_{\pi=(\pi^i)\in\mathscr{P}^I}\frac{\prod_{i\to j\in\Omega}q^{\langle\pi^i,\pi^j\rangle}}{\prod_{i\in I}q^{\langle\pi^i,\pi^i\rangle}b_{\pi^i}(q^{-1})}x^{|\pi|}.$$

The rational functions of q, $H(\mathbf{d},q)$, $\mathbf{d}\in\mathbf{N}^I\setminus\{0\}$ are defined by the equality:

$$\log\left(P((x_i)_{i\in I},q)\right) = \sum_{\mathbf{d}\in\mathbf{N}^I\setminus\{0\}}\frac{H(\mathbf{d},q)x^{\mathbf{d}}}{\overline{\mathbf{d}}}$$

where log is defined by $\log(1-X) = -\sum_{i\geq 1}\frac{X^i}{i}$ and for $\mathbf{d}\in\mathbf{N}^I$, we let $\overline{\mathbf{d}} = \gcd(d_i, i\in I)$.

Theorem 4.A.1 ([Hua00, Theorem 4.6])**.** *For any $\mathbf{d}\in\mathbf{N}^I\setminus\{0\}$, we have*

$$A_{\mathbf{d}}(q) = \frac{q-1}{\overline{\mathbf{d}}}\sum_{d|\alpha}\mu(d)H\left(\frac{\mathbf{d}}{d},q^d\right).$$

4.A.2 The Kronecker quivers

We give here the code for Example 4.3.8.

```
R.<q>=PolynomialRing(QQ)
K=FractionField(R)
#We work with the Kronecker quiver with r arrows from the vertex 1
to the vertex 2.
S.<x1,x2>=PowerSeriesRing(K)

def produitbilin(l1,l2): #returns the bilinear products of the partitions l1,l2
    r=0
    l1c=Partition(l1).conjugate()
    l2c=Partition(l2).conjugate()
    l=min(len(l2c),len(l1c))
    for i in range(l):
        r=r+l1c[i]*l2c[i]
    return(r)

def phi(r): #returns \phi_r(q^{-1})
    p=1
```

```
    for i in range(1,r+1):
        p=p*(1-q^(-i))
    return(p)

def b(l): #l is a partition
    lc=Partition(l).conjugate()+[0]
    r=1
    for i in range(len(lc)-1):
        r=r*phi(lc[i]-lc[i+1])
    return(r)

def polynomeP(d1,d2,r): #returns P(x_1,x_2,q) for the quiver K_r
    P=S(0)
    for i in range(d1+1):
        for j in range(d2+1):
            for p1 in Partitions(i).list():
                for p2 in Partitions(j).list():
                    P=P+((q^(r*produitbilin(p1,p2)))/
                         (q^(produitbilin(p1,p1
                         +produitbilin(p2,p2))*b(p1)*b(p2)))*(x1^i)*(x2^j)
    return(P)

def serieH(d1,d2,N,r): #returns the generating series of the polynomials
                       #H(\dd,q) given by log(P(x_1,x_2,q))
    s=0
    p=polynomeP(d1,d2,r)
    for i in range(1,N+1):
        s=s-(1-p)^i/i
    return(s)

def polyH(N,r,d1,d2): #returns the polynomial H((d_1,d_2),q)
    ser=serieH(d1,d2,N,r)
    d=gcd(d1,d2)
    pol=d*(1/(ZZ(d1).factorial()))*(1/(ZZ(d2).factorial()))*(
        (S((S(ser).derivative(x1,ZZ(d1))).derivative(x2,ZZ(d2)))).
        constant_coefficient())
    return(pol)

def KacPolA(d1,d2,r): #returns the Kac polynomial of K_r for
the dimension vector (d_1,d_2).
    N=d1+d2
```

```
    d=gcd(d1,d2)
    A=0
    div=d.divisors()
    for i in div:
        A=A+moebius(i)*(polyH(N,r,ZZ(d1/i),ZZ(d2/i))(q^i))
    A=A*((q-1)/d)
    return(A)

for i in range(1,5):print(R(KacPolA(2,2,i))) #an example
```

4.A.3 The g-loop quivers

We give here the code for Section 4.3.3. Some of it is identical to that in the previous Section, but we reproduce it for the convenience of the reader.

```
R.<q>=PolynomialRing(QQ)
K=FractionField(R)
S.<x1,x2>=PowerSeriesRing(K)
#We work with the quiver S_g

def polynomePS(d1,g):
    P=S(0)
    for i in range(d1+1):
        for p1 in Partitions(i).list():
            P=P+((q^(g*produitbilin(p1,p1)))/(q^(produitbilin(p1,p1))*b(p1)))*
            (x1^i)
    return(P)

def serieHS(d1,N,g):
    s=0
    p=polynomePS(d1,g)
    for i in range(1,N+1):
        s=s-(1-p)^i/i
    return(s)

def polyHS(N,g,d1):
    ser=serieHS(d1,N,g)
    d=d1
    pol=d*(1/(ZZ(d1).factorial()))*((S((S(ser).derivative(x
    ,ZZ(d1))))).constant_coefficient())
    return(pol)
```

```
def KacPolAS(d1,g):
    N=d1
    d=d1
    A=0
    div=d.divisors()
    for i in div:
        A=A+moebius(i)*(polyHS(N,g,ZZ(d1/i))(q^i))
    A=A*((q-1)/d)
    return(A)

for r in range(1,6): print(KacPolAS(2,g))
```

4.A.4 The Tennis Racket quiver

We give the code for Section 4.3.3.

```
R.<q>=PolynomialRing(QQ)
K=FractionField(R)
S.<x1,x2>=PowerSeriesRing(K)

def produitbilin(l1,l2):
   r=0
   l1c=Partition(l1).conjugate()
   l2c=Partition(l2).conjugate()
   l=min(len(l2c),len(l1c))
   for i in range(l):
       r=r+l1c[i]*l2c[i]
   return(r)
def phi(r):
    p=1
    for i in range(1,r+1):
        p=p*(1-q^(-i))
    return(p)

def b(l):
    lc=Partition(l).conjugate()+[0]
    r=1
    for i in range(len(lc)-1):
        r=r*phi(lc[i]-lc[i+1])
    return(r)

def polynomeP(d1,d2,r,g):
```

```
    P=S(0)
    for i in range(d1+1):
        for j in range(d2+1):
            for p1 in Partitions(i).list():
                for p2 in Partitions(j).list():
                    P=P+((q^(r*produitbilin(p1,p2
                    +g*produitbilin(p2,p2)))/(q^(produitbilin(p1,p1
                    +produitbilin(p2,p2))*b(p1)*b(
                    2)))*(x1^i)*(x2^j)
    return(P)

def serieH(d1,d2,N,r,g):
    s=0
    p=polynomeP(d1,d2,r,g)
    for i in range(1,N+1):
        s=s-(1-p)^i/i
    return(s)

def polyH(N,r,g,d1,d2):
    ser=serieH(d1,d2,N,r,g)
    d=gcd(d1,d2)
    pol=d*(1/(ZZ(d1).factorial()))*(1/(ZZ(d2).factorial()))*(
        (S((S(ser).derivative(x1,ZZ(d1))).derivative(x2,ZZ(d2)))).
        constant_coefficient())
    return(pol)

def KacPolA(d1,d2,r,g):
    N=d1+d2
    d=gcd(d1,d2)
    A=0
    div=d.divisors()
    for i in div:
        A=A+moebius(i)*(polyH(N,r,g,ZZ(d1/i),ZZ(d2/i))(q^i))
    A=A*((q-1)/d)
    return(A)
```

Chapter 5

Microlocal characterization of Lusztig sheaves for affine quivers and g-loops quivers

We prove that for extended Dynkin quivers, simple perverse sheaves in Lusztig category are characterized by the nilpotency of their singular support. This proves a conjecture of Lusztig in the case of affine quivers. For cyclic quivers, we prove a similar result for a larger nilpotent variety and a larger class of perverse sheaves. We formulate conjectures concerning similar results for quivers with loops, for which we have to use the appropriate notion of nilpotent variety, due to Bozec, Schiffmann and Vasserot. We prove our conjecture for g-loops quivers ($g \geq 2$).

Contents

5.1 Introduction

5.1.1 The main results

Motivated by his theory of Character Sheaves ([Lus85]), Lusztig defined in the early 90's a category $\mathscr{Q}$ of equivariant semisimple constructible complexes on the representation spaces of a given acyclic quiver $Q = (I, \Omega)$ ([Lus91]). His construction is detailed in Section 5.3. Here is a brief overview. For a dimension vector $\mathbf{d} \in \mathbf{N}^I$, we let $E_{\mathbf{d}}$ be the representation space of Q of dimension $\mathbf{d}$. It is acted on by the product of linear groups $G_{\mathbf{d}}$. We also let $\mathscr{P}_{\mathbf{d}}$ be the category of $G_{\mathbf{d}}$-equivariant perverse sheaves on $E_{\mathbf{d}}$ which belong to the category $\mathscr{Q}$. By definition, it is the semisimple category whose simple objects are the perverse sheaves appearing as a direct summand of the push-forward of the constant sheaf $\underline{\mathbf{C}}$ by the morphism $\pi : Y \to E_{\mathbf{d}}$, where Y is the variety of pairs $(x, \underline{F})$, $x \in E_{\mathbf{d}}$, $\underline{F}$ is a x-stable I-graded flag of $\mathbf{C}^{\mathbf{d}}$ and π is the natural projection. In *op. cit.*, Lusztig studied the singular support of the perverse sheaves which belong to $\mathscr{P}_{\mathbf{d}}$. This led him to define the so-called nilpotent variety $\Lambda_{\mathbf{d}} \subset T^*E_{\mathbf{d}}$ (Section 5.4). This is a closed, conical and Lagrangian subvariety of $T^*E_{\mathbf{d}}$ and for any $\mathscr{F} \in \mathscr{P}_{\mathbf{d}}$, $SS(\mathscr{F}) \subset \Lambda_{\mathbf{d}}$. The first main result of the present chapter is the proof of the converse under some restrictions on the quiver.

Theorem 5.1.1. *Let Q be a finite type or affine quiver*[1]*. Let $\mathscr{F} \in \mathrm{Perv}_{G_{\mathbf{d}}}(E_{\mathbf{d}})$ be a $G_{\mathbf{d}}$-equivariant simple perverse sheaf such that $SS(\mathscr{F}) \subset \Lambda_{\mathbf{d}}$. Then $\mathscr{F}$ is a Lusztig perverse sheaf.*

The method of proof is briefly described in Section 5.1.3.

Lusztig conjectured more generally in [Lus90b, §10.3], without restriction on the quiver, that perverse sheaves whose singular support is contained in the nilpotent variety should provide the canonical basis of the positive part of the quantum group. When combined with Lusztig's subsequent paper [Lus91], this may be formulated as the following conjecture which was also put forward independently by Webster (see [Web]).

Conjecture 5.1.2 (Lusztig)**.** *Let Q be a loop-free quiver and $\mathbf{d} \in \mathbf{N}^I$ a dimension vector. Then $G_{\mathbf{d}}$-equivariant irreducible perverse sheaves on $E_{\mathbf{d}}$ whose singular support is included in $\Lambda_{\mathbf{d}}$ are exactly Lusztig sheaves.*

[1] In this chapter, we include under the terminology *affine quiver* extended Dynkin quivers, Jordan and cyclic quivers.

See Section 5.1.6 for relevant considerations. In this chapter, we prove this result for (the easy and already known cases of) finite type and cyclic quivers and for the more subtle case of affine quivers, for which the representation theory is heavily used (Auslander-Reiten theory for affine quivers).

We also conjecture (Conjecture 5.10.17) a modification of the previous conjecture for arbitrary quivers (possibly carrying loops) using the appropriate notion of nilpotent variety defined in [BSV17] (Section 5.10). This leads to four different categories of perverse sheaves on the representation spaces of a quiver, $\mathcal{P}^\flat$ for $\flat \in \{\mathrm{nil}, \emptyset, (\mathrm{nil}, 1), 1\}$ (Section 5.10.2), together with four corresponding *nilpotent varieties*, $\Lambda^\flat$, $\flat \in \{\mathrm{nil}, \emptyset, (\mathrm{nil}, 1), 1\}$ (Section 5.10.3). These four situations are paired using the Fourier-Sato transform (nil and $\emptyset$ are paired, $(\mathrm{nil}, 1)$ and 1 also). It is easily shown that the singular support of a perverse sheaf in the category $\mathcal{P}^\flat$ is contained in the nilpotent variety $\Lambda^\flat$ by using favourable functorial properties of the singular support with respect to the pushforward by a proper morphism. In Section 5.10, we prove the converse for g-loops quivers ($g \geq 2$), which constitutes the second main result of this chapter:

Theorem 5.1.3. *Let $g \geq 2$ and $Q = S_g$ be the g-loop quiver. Let $\mathbf{d} \in \mathbf{N}$ be a dimension vector and $\mathscr{F}$ an irreducible perverse sheaf on $E_{Q,\mathbf{d}}$ such that $SS(\mathscr{F}) \subset \Lambda^\flat$ where $\flat = (\mathrm{nil}, 1)$ or $\flat = 1$. Then $\mathscr{F}$ is in the category $\mathcal{P}^\flat$.*

We give the rough idea of the proof in Section 5.1.4. We would like to emphasize the fact that no equivariance assumption is made on $\mathscr{F}$: this is not necessary in the proof and happens to be a consequence of the nilpotency of the singular support.

5.1.2 Analogy with character sheaves on Lie groups and Lie algebras

In [Lus91], motivated by his theory of character sheaves on reductive groups, Lusztig defined a class of perverse sheaves on the representation spaces of an arbitrary loop-free quiver. In both cases, the perverse sheaves under consideration are defined as appropriate shifts of direct summands of the pushforward by a proper morphism of a local system. Character sheaves for a complex reductive group G are obtained in this way using the morphism

$$\pi : \tilde{G} = \{(g, B) \in G \times \mathscr{B} \mid g \in B\} \to G$$

where $\mathscr{B} = G/B$ denotes the flag variety of G.

For quivers, Lusztig considers a family of morphisms

$$\pi_{(i,a)} : \tilde{\mathcal{F}}_{(i,a)} \to E_{\mathbf{d}}$$

where $\tilde{\mathcal{F}}_{(i,a)}$ is a smooth variety and the morphism $\pi_{(i,a)}$ is proper (Section 5.3). By functorial properties of the singular support, it is proved that for groups, the singular support of character sheaves is contained in

$$\Lambda_G = \{(g, \xi^*) \mid g \in Z_G(\xi^*) \text{ and } \xi^* \in \mathscr{N}^*\}$$

where $\mathscr{N}^* \subset \mathfrak{g}^*$ denotes the nilcone of $\mathfrak{g}^*$ and $Z_G(\xi^*)$ is the centralizer of ξ^* in G for the coadjoint action of G on $\mathfrak{g}^*$. For loop-free quivers, the result is the following. Lusztig perverse sheaves on $E_{\mathbf{d}}$ have a singular support which is a subvariety of

$$\Lambda_{\mathbf{d}} = \{(x, x^*) \in E_{\bar{Q},\mathbf{d}} \mid \mu_{\mathbf{d}}(x, x^*) = 0 \text{ and } (x, x^*) \text{ is nilpotent}\}.$$

See Section 5.2.4 for the definition of the moment map $\mu_{\mathbf{d}}$ (the condition $\mu_{\mathbf{d}}(x, x^*) = 0$ is a generalization of the commuting relation we had for Λ_G). In [MV88], Mirković and Vilonen give a proof of a conjecture of Laumon and Lusztig asserting that character sheaves on a complex connected reductive group G are exactly those with singular support included in $G \times \mathscr{N}^*$. Analogously, one can try to characterize $G_{\mathbf{d}}$-equivariant perverse sheaves on $E_{\mathbf{d}}$ with singular support included in $\Lambda_{\mathbf{d}}$. To make the analogy even clearer, note that a G-equivariant perverse sheaf on G has singular support included in $\mu^{-1}(0)$ where

$$\begin{array}{rcrcl} \mu & : & T^*G = G \times \mathfrak{g}^* & \to & \mathfrak{g}^* \\ & & (g, \xi^*) & \mapsto & ad_g^*(\xi^*) \end{array}$$

is a moment map of the hamiltonian action of G on itself by conjugation. Therefore, any G-equivariant perverse sheaf on G whose singular support is contained in $G \times \mathscr{N}^*$ has singular support in $\Lambda_G = \mu^{-1}(0) \cap (G \times \mathscr{N}^*)$ (see also [MV88, 1.4]). It is also possible to prove similar results for complex reductive Lie algebras, as in [Mir04], which is close to the quiver situation. In fact, the case of the Jordan quiver in dimension d coincides by definition with the case of $\mathfrak{gl}_d$.

5.1.3 Steps of the proof for affine quivers

The proof proceeds in the following steps. For finite type (resp. cyclic quivers), we use that the number of orbits (resp. nilpotent orbits) is finite in each dimension and appropriate resolution of their closure. A Fourier-Sato transform allows us to conclude for type A affine quivers. The cases of affine types D and E need a more subtle work. Using an appropriate stratification of the representation spaces (given by Auslander-Reiten theory), we reduce the problem to sheaves on the regular locus. Cyclic quivers allow us to describe a neighbourhood of a non-homogeneous tube in the representation space of Q. The question can now be answered by studying a class of perverse sheaves on the representation spaces of cyclic quivers. This class is slightly larger than the class of Lusztig sheaves. Therefore, we call it *extended Lusztig category*. It turns out that this class of perverse sheaves contains exactly the Fourier-Sato transforms of the intersection cohomology complexes on nilpotent orbits for the opposite orientation. We describe explicitly this class of perverse sheaves together with a microlocal characterization of the simple perverse sheaves it contains, analogously to the the main theorem of this chapter. Transfering the question from an affine quiver to cyclic quivers gives a proof of the theorem.

5.1.4 Steps of the proof for g-loops quivers

In Section 5.10, we consider general quivers, possibly carrying loops. For them, four different *nilpotent varieties* are available: see [BSV17]. Accordingly, there are four different categories of perverse sheaves on the representation spaces of the quiver. We conjecture a relationship between equivariant simple perverse sheaves whose singular support is a union of irreducible components of one of the nilpotent varieties and the corresponding category of perverse sheaves. The conjecture is proved for g-loops quivers with $g \geq 2$. The loops at the vertices ensure the smallness of some proper morphisms and this allows us to describe precisely the categories of perverse sheaves under consideration. We use also that the different closed subvarieties of the representation space contributing to the singular support are of codimension at least two in the support of our perverse sheaf. Then the proof rests essentially on the fact that the proper morphisms we use to define the perverse sheaves on the representation spaces are small and the existence of a bijection between isomorphism classes of simple objects in the category under consideration and irreducible components of the corresponding nilpotent variety. This bijection for quivers with loops and discrete flag-types[2] is due to Bozec. Such a bijection is not known for flag-types which are not discrete (note that for quivers with one vertex, all flag-types are discrete).

5.1.5 Contents of the chapter

We shortly describe the contents of the different sections of this chapter. Section 5.2 contains basic results on the representation theory of finite type quivers and acyclic quivers (Auslander-Reiten theory). We put the emphasis on affine quivers (decomposition of the category of representations in three parts, preprojective, regular and preinjective). We define stratifications of the representation spaces induced by this decomposition of the category (Auslander-Reiten stratification). In the case of affine quivers, we give a refinement of this stratification. In Section 5.3, we recall the definition of Lusztig category for a general quiver. We give an explicit description of this category for finite type quivers and cyclic quivers, together with proofs. We recall the description of Lusztig perverse sheaves for affine quivers (without proof). We define the induction and restriction functors used by Lusztig to categorify the operations of the quantum group. In Section 5.4, we study the singular support of Lusztig sheaves. We define Lusztig nilpotent variety and describe it explicitly for finite type and affine quivers. This uses the stratifications previously defined. We prove two technical lemmata, the first of which allows us to consider only perverse sheaves on the regular locus. In Section 5.5, we prove the microlocal characterization for finite type quivers. In Section 5.6, we prove it for type A affine quivers. In Section 5.7, we define a class of perverse sheaves on the representation spaces of cyclic quivers and give its basic properties: singular support, explicit description and microlocal characterization. In Section 5.8, we explain how to describe a neighbourhood of a non-homogeneous tube in the representation space of an affine quiver using cyclic quivers. Section 5.9 contains the proof of Theorem 5.1.1 for affine quivers. Last, Section 5.10 describes a general conjecture for quivers possibly carrying loops and cycles in the cases of general or discrete flag-types. We prove the conjecture in the case of g-loops quivers, $g \geq 2$. The method is completely different from that for affine quivers. In the appendices, we collect useful facts on local systems, equivariant perverse sheaves, singular supports and Fourier-Sato transform.

[2] A flag-type is called discrete if it is an uplet of dimension vectors, each of them being supported at one vertex.

5.1.6 Remark

After the paper [Hen20b] was written, the author learned from Ben Webster that a proof of Conjecture 5.1.2 based on an unpublished work of Baranovsky and Ginzburg is written in [Web17], see Proposition 4.8 of *op. cit.* and the comment following it. This result of Baranovsky and Ginzburg claims injectivity of the characteristic cycle map for quantized conical resolution (Theorem 4.9 of *op. cit.*). It is proved in the special case of quantized quiver varieties *for finite type quivers* and *for affine quivers with the framing ϵ_0 (one dimensional framing at the extending vertex only)* (not for wild quivers) in the paper of Bezrukavnikov and Losev, [BL]. To the knowledge of the author, no proof of this theorem in full generality has appeared yet. Note that in order to run Webster's argument one needs the injectivity of the characteristic cycle for all framings. Our methods are completely different and rest on a detailed understanding of the geometry of the representation spaces of affine quivers. Note also that a priori the Baranovsky-Ginzburg's theorem does not apply for quivers with loops.

5.1.7 Notations

We denote by $\mathscr{P}$ the set of partitions. For G a linear algebraic group and X a complex G-variety, $D^b_G(X)$ denotes the constructible derived category of X with complex coefficients. We denote by $\mathrm{Perv}_G(X)$ the category of G-equivariant perverse sheaves on X. The underlying formalism is developed in [BL94]. If $\mathscr{F}$ is a simple perverse sheaf on a variety X and $i : Y \to X$ a locally closed immersion such that $\overline{Y} = \operatorname{supp} \mathscr{F}$, we let $\mathscr{F}_Y := i^*\mathscr{F}$ be the restriction of $\mathscr{F}$ to Y. It is still a simple perverse sheaf and $\mathscr{F} = i_{!*}\mathscr{F}_Y$, the intermediate extension of $\mathscr{F}_Y$ to X. A quiver is a pair $Q = (I, \Omega)$ where I is the set of vertices and Ω the set of arrows[3]. Both are assumed to be finite sets. For $\mathbf{d} \in \mathbf{N}^I$ a dimension vector for Q, $\mathbf{C}^{\mathbf{d}}$ is a $\mathbf{N}^I$-graded vector space of dimension $\mathbf{d}$. For $\mathbf{d}' \in \mathbf{N}^I$, will consider the grassmannian $\mathrm{Gr}(\mathbf{d}', \mathbf{d})$ of $\mathbf{d}'$-dimensional graded subspaces of $\mathbf{C}^{\mathbf{d}}$. It is of course smooth, projective, and nonempty if and only if $\mathbf{d}' \leq \mathbf{d}$. For us, cyclic quivers are by definition quivers of type $A_n^{(1)}$ for some $n \geq 0$ with a possibly non-cyclic orientation. We will explicitly write when cyclic quivers are considered with a cyclic orientation. The opposite quiver of Q is the quiver $Q^{\mathrm{op}} = (I, \Omega^{\mathrm{op}})$ having the same set of vertices as Q but all arrows have the reverse direction. When X is an algebraic (or analytic) variety and $d \in \mathbf{N}$ an integer, we let $\Delta \subset X^d$ be the big diagonal, that is the closed subvariety of d-uplets $(x_1, \dots, x_d)$ with two or more of the x_i's equal. We let $S^d X$ be the d-th symmetric power of X and again the symbol Δ denotes its diagonal. If X is a G-variety and $\Lambda \subset T^*X$ a subset, we let $\mathrm{Perv}_G(X, \Lambda)$ be the full, abelian subcategory of $\mathrm{Perv}_G(X)$ of perverse sheaves whose singular support is contained in Λ. The letter k denotes a field. We implicitly specialize to $k = \mathbf{C}$ whenever we use analytic notions. Let $H \subset G$ be a closed subgroup and X an H-variety. The group H acts freely on $X \times G$ by $h \cdot (x, g) = (hx, gh^{-1})$. We let $X \times^H G$ be the quotient variety. It is a G-variety under the G-action $g' \cdot (x, g) = (x, g'g)$. If X is a finite set, $\sharp X$ denotes its number of elements. The symbols $\subset$ and $\subseteq$ are used for inclusions of sets which can be an equality. When the inclusion is strict, we use $\subsetneq$.

[3] The letter I will sometimes be used to denote a preinjective representation of the quiver Q (provided it is acyclic), see Section 5.2. This should not cause any confusion.

5.1.8 Terminology

If X is a G-variety, we say that a G-orbit $\mathcal{O} \subset X$ is *equivariantly simply connected* if its stabilizer is connected. It is the case for all orbits for quiver representations. If $(f, \phi) : (X, G) \to (Y, H)$ is an equivariant morphism between a G-variety (X, G) and an H-variety (Y, H) (for any $(x, g) \in X \times G$, $f(g \cdot x) = \phi(g) \cdot f(x)$), we say that f is an equivariant π_1-equivalence if $f^* : \mathrm{LocSys}_H(Y) \to \mathrm{LocSys}_G(X)$ is an equivalence of categories between the categories of equivariant local systems. If $\phi : H \to G$ is the inclusion of a closed subgroup and X a H-variety, $(f, \phi) : (X, H) \to (X \times^H G, G)$, $x \in X \mapsto (x, e) \in X \times^H G$ is the prototypical example of an equivariant π_1-equivalence.

5.2 Some representation theory of quivers and stratification of the representation spaces

We use in a fundamental manner the very explicit representation theory of finite type and affine quivers to prove the main theorem. For the sake of completeness and to fix notations, we briefly recall the basic facts we will use.

5.2.1 Representation theory of finite type quivers

The theorem ruling the representation theory of finite type quivers is the following. It does not depend on the base field.

Theorem 5.2.1 (Gabriel, [Gab72])**.** *Let Q be a quiver. Then Q has a finite number of indecomposable representations if and only if Q is of type ADE. Moreover, taking the dimension induces a one-to-one correspondence between indecomposable representations of Q and dimension vectors $\mathbf{d} \in \mathbf{N}^I$ such that $\langle \mathbf{d}, \mathbf{d} \rangle = 1$*[4]*.*

5.2.2 Nilpotent representations of cyclic quivers

Let C_n be the cyclic quiver with n vertices (of type $A^{(1)}_{n-1}$) and cyclic orientation. For convenience, we label the vertices by the set $\mathbf{Z}/n\mathbf{Z}$. We assume that for $i \in \mathbf{Z}/n\mathbf{Z}$, we have an arrow $i \to i+1$. For any $i \in \mathbf{Z}/n\mathbf{Z}$ and $l \in \mathbf{N}_{\geq 1}$, there is exactly one indecomposable nilpotent representation of C_n with top S_i and length l. We denote it $I_{i,l}$. Let M be a nilpotent representation of C_n. Then M is called *aperiodic* if for any $l \geq 1$, not all the representations $I_{i,l}$ for $i \in \mathbf{Z}/n\mathbf{Z}$ are direct summands of M. Nilpotent representations of C_n are parametrized by multipartitions, that is functions

$$\mathbf{m} : \mathbf{Z}/n\mathbf{Z} \to \mathscr{P}.$$

The nilpotent representation corresponding to $\mathbf{m}$ is

$$N_{\mathbf{m}} = \bigoplus_{\substack{i \in I \\ l \geq 1}} I_{i, \mathbf{m}(i)^l}$$

[4] $\langle \mathbf{d}, \mathbf{d} \rangle = \sum_{i \in I} \mathbf{d}_i^2 - \sum_{\alpha : i \to j \in \Omega} \mathbf{d}_i \mathbf{d}_j$ is the Euler form of the quiver.

where $\mathbf{m}(i) = (\mathbf{m}(i)^1, \mathbf{m}(i)^2, \ldots)$. Accordingly, the (total) dimension of the multipartition $\mathbf{m}$ is

$$\dim \mathbf{m} = \dim N_{\mathbf{m}} = \sum_i |\mathbf{m}(i)|.$$

5.2.3 Some Auslander-Reiten theory

Let Q be an acyclic quiver. We recall here the needed facts about the representation theory of Q with a particular emphasis on affine quivers. Such theory has been known for some time now. See [Rin84] for a useful account.

Theorem 5.2.2. *Let k be a field. Then, there exists an adjunction*

$$\tau^- : \mathrm{Rep}_Q(k) \rightleftarrows \mathrm{Rep}_Q(k) : \tau$$

with bi-natural isomorphisms[5] *(the star means the dual with respect to the k-vector space structure):*

$$\mathrm{Ext}^1(M,N)^* \simeq \mathrm{Hom}(N, \tau M), \qquad \mathrm{Ext}^1(M,N)^* \simeq \mathrm{Hom}(\tau^- N, M).$$

The functors τ and τ^- are known as *Auslander-Reiten translates.* From the above properties of τ^- and τ, it is immediate that a representation M of Q over k is projective if and only if $\tau(M) = 0$ and injective if and only if $\tau^-(M) = 0$. We call an indecomposable representation M of Q over k

1. preprojective if $\tau^n M = 0$ for $n \gg 0$,
2. preinjective if $\tau^{-n} M = 0$ for $n \gg 0$,
3. regular if $\tau^n M \neq 0$ for all $n \in \mathbf{Z}$.

Furthermore, we call a representation M of Q over k preprojective if all its indecomposable direct summands are preprojective, and we adopt similar terminology for preinjective and regular representations. By abuse, the zero representation is preprojective, regular and preinjective. The full subcategory of $\mathrm{Rep}_Q(k)$ of preprojective (resp. preinjective, resp. regular) representations is denoted by $\mathrm{Rep}_Q^{\mathcal{P}}(k)$ (resp. $\mathrm{Rep}_Q^{\mathcal{I}}(k)$, resp. $\mathrm{Rep}_Q^{\mathcal{R}}(k)$). These are extension closed subcategories of $\mathrm{Rep}_Q(k)$, hence exact categories. Moreover, for Q an affine quiver, $\mathrm{Rep}_Q^{\mathcal{R}}(k)$ is an abelian category (though not stable under taking subobjects in the bigger category $\mathrm{Rep}_Q(k)$). The three categories $\mathrm{Rep}_Q^{\mathcal{R}}(k), \mathrm{Rep}_Q^{\mathcal{P}}(k)$ and $\mathrm{Rep}_Q^{\mathcal{I}}(k)$ are disjoint and a crucial fact for affine quivers is the following: The category to which an indecomposable M belongs is given by the sign of its defect defined by

$$\partial M = \langle \delta, \dim M \rangle, \tag{5.2.1}$$

where

$$\begin{array}{rcccc} \langle -, - \rangle & : & \mathbf{Z}^I \times \mathbf{Z}^I & \to & \mathbf{Z} \\ & & (\mathbf{d}, \mathbf{e}) & \mapsto & \sum_{i \in I} \mathbf{d}_i \, \mathbf{e}_i - \sum_{\alpha: i \to j \in \Omega} \mathbf{d}_i \, \mathbf{e}_j \end{array}$$

[5]We say that (τ^-, τ) is a Serre adjunction.

is the (non-symmetrized) Euler form of Q and δ is the indivisible positive imaginary root of Q (see [Sch14, Fig 8.3]). An indecomposable representation M is preprojective if and only if $\partial M < 0$, preinjective if and only of $\partial M > 0$ and regular if and only if $\partial M = 0$: it only depends on the dimension $\dim M$ of M. The following proposition gives the interactions between these three subcategories.

Proposition 5.2.3. *For $M \in \mathrm{Rep}_Q^{\mathcal{P}}(k)$, $N \in \mathrm{Rep}_Q^{\mathcal{I}}(k)$, $L \in \mathrm{Rep}_Q^{\mathcal{R}}(k)$, we have*

$$\mathrm{Hom}(N, M) = \mathrm{Hom}(N, L) = \mathrm{Hom}(L, M) = 0,$$

$$\mathrm{Ext}^1(M, N) = \mathrm{Ext}^1(L, N) = \mathrm{Ext}^1(M, L) = 0.$$

Corollary 5.2.4. *Let $M = P \oplus R \oplus I$ be a representation of an affine quiver Q with P preprojective, R regular and I preinjective. Let $\mathbf{d}_P = \dim P, \mathbf{d}_R = \dim R$ and $\mathbf{d}_I = \dim I$. Then M has a unique subrepresentation of dimension $\mathbf{d}_I$ and a unique subrepresentation of dimension $\mathbf{d}_I + \mathbf{d}_R$. In particular, it has a unique filtration $(0 \subset M_1 \subset M_2 \subset M_2 = M)$ with subquotients of dimensions $\mathbf{d}_I, \mathbf{d}_R, \mathbf{d}_P$.*

Proof. Let $N = P' \oplus R' \oplus I'$ be a subrepresentation of M with $\dim N = \mathbf{d}_I$ (P' preprojective, R' regular and I' preinjective). Its defect is $\langle \delta, \dim N \rangle = \partial I = \partial P' + \partial I' \leq \partial I'$. By Proposition 5.2.3, $I' \subset I$, and $\partial(I/I') = \partial I - \partial I' \leq 0$. Therefore, if I/I' is nonzero, it has preprojective or regular direct summands. By Proposition 5.2.3 again, I has regular or preprojective direct summands: contradiction. Therefore, $I' = I$ and by considering the dimensions, $N = I$. The same argument works for subrepresentations of dimension $\mathbf{d}_R + \mathbf{d}_I$. □

The simple objects of the abelian category $\mathrm{Rep}_Q^{\mathcal{R}}(k)$ are called simple regular. A simple regular representation M is called homogeneous if $\tau M \simeq M$. It is called non-homogeneous otherwise.

Theorem 5.2.5 (Ringel, [Rin84])**.** *Let Q be an affine acyclic quiver and k an arbitrary field. Let d and $p_1, \ldots, p_d$ be attached to Q as in the table below. Then*

1. *There is a degree preserving bijection $M_a \leftrightarrow a$ between the set of homogeneous regular simple modules and $|\mathbf{P}_k^1| \setminus D$ where D consists of d closed points of degree one*[6]*,*

2. *There are d τ-orbits $\mathcal{O}_1, \ldots, \mathcal{O}_d$ of non-homogeneous regular simple modules of size given by $p_1, \ldots, p_d$*[7]*,*

3. *The category $\mathrm{Rep}_Q^{\mathcal{R}}(k)$ decomposes as a product sum of blocks*[8]*:*

$$\mathrm{Rep}_Q^{\mathcal{R}}(k) = \prod_{a \in |\mathbf{P}_k^1| \setminus D} \mathcal{C}_{M_a} \times \prod_{l=1}^{d} \mathcal{C}_{\mathcal{O}_l}$$

[6] For X a scheme, $|X|$ denotes the set of closed points of X.

[7] *i.e.* the set of isomorphism classes of simple objects in $\mathcal{O}_j$, $1 \leq j \leq d$ is of cardinality p_j and the Auslander-Reiten translates τ and τ^- act as inverse cycles on it.

[8] There are no morphisms or extensions between the objects of different categories in the product.

where $\mathcal{C}_{M_a}$ is the full subcategory of objects which are extensions of M_a and $\mathcal{C}_{\mathcal{O}}$ is the full subcategory of $\mathrm{Rep}_Q^{\mathcal{R}}(k)$ *of objects whose regular simple factors lie in $\mathcal{O}$.*

type of Q	d	$p_1, \dots, p_d$
$A_1^{(1)}$	0	
$A_n^{(1)}, n > 1$	2	p_1 =number of arrows going clockwise p_2 =number of arrows going counterclockwise
$D_n^{(1)}$	3	$2, 2, n-2$
$E_n^{(1)}, n = 6, 7, 8$	3	$2, 3, n-3$

Figure 5.1: Non-homogeneous tubes of affine quivers and their period [Sch12a, (3.18)]

In Theorem 5.2.5, the subcategories $\mathcal{C}_{\mathcal{O}_l}$ are called the non-homogeneous tubes while the subcategories $\mathcal{C}_{M_a}$ are the homogeneous tubes. These are finite-length categories. For any object in one of them, its quasi-length is its length in the given tube. A representation all of those indecomposable direct summands are contained in (non-)homogeneous tubes is called *(non-)homogeneous.* A representation all of those indecomposable direct summands are in homogeneous tubes is called regular homogeneous. The number of non-homogeneous tubes is d (see however Remark 5.2.6) and the integers $p_1, \dots, p_d$ are the periods. They do not depend on the chosen field. For $a \in |\mathbf{P}_k^1| \setminus D$ and $n \geq 1$, we let $S_a[n]$ be an indecomposable representation of Q in the tube a of quasi-length n. For λ a partition, we let $S_a[\lambda] = \bigoplus_{i \geq 1} S_a[\lambda_i]$. We obtain all representations in C_{M_a} in this way.

Remark 5.2.6. In type $A_n^{(1)}$, $n \geq 2$ in the case where all arrows except one go in the same direction, we have in fact $d = 1$, *i.e.* there is only one non-homogeneous tube. The Kronecker quiver $K2$ (type $A_1^{(1)}$) has only homogeneous tubes.

Define $\mathbf{P}_1^{\mathrm{hom}} = |\mathbf{P}_k^1| \setminus D$ the set indexing homogeneous tubes.

5.2.4 Quiver representation varieties

Let $Q = (I, \Omega)$ be a finite quiver with set of vertices I and set of arrows Ω. For a field k and an I-graded vector space V over k, the associated representation variety of Q is

$$E_Q(V) = \bigoplus_{i \xrightarrow{\alpha} j \in \Omega} \mathrm{Hom}(V_i, V_j).$$

If $\mathbf{d} \in \mathbf{N}^I$ is a dimension vector, then the representation variety of Q in dimension $\mathbf{d}$ is

$$E_{Q,\mathbf{d}} := E_Q(V)$$

for $V = (k^{\mathbf{d}_i})_{i \in I}$. When the context is clear, we write $E(V) = E_Q(V)$ and $E_{\mathbf{d}} = E_{Q,\mathbf{d}}$. Elements of $E_{\mathbf{d}}$ can be seen as representations of Q. If $x \in E_{\mathbf{d}}$, we let $(k^{\mathbf{d}}, x)$ be the associated representation. To shorten the notation, we sometimes write only x. Also, we will sometimes consider quivers $Q' = (I, \Omega')$ with the same underlying graph as Q but a possibly different orientation Ω'. In this case, we write $E_{\mathbf{d}}^{\Omega'} = E_{Q',\mathbf{d}}$.

The variety $E_Q(V)$ is acted on by the product of linear groups

$$G(V) = \prod_{i \in I} \mathrm{GL}(V_i).$$

If $V = (k^{\mathbf{d}_i})_{i \in I}$, we write $G_{\mathbf{d}} = G(V) = \prod_{i \in I} \mathrm{GL}_{\mathbf{d}_i}(k)$. The orbits of $E_{\mathbf{d}}$ under $G_{\mathbf{d}}$ are in one-to-one correspondence with isomorphism classes of representations of Q of dimension $\mathbf{d}$ over k. Note that the diagonal embedded copy of $\mathbf{C}^*$ inside $G_{\mathbf{d}}$ acts trivially on $E_{\mathbf{d}}$. If M is a representation of Q of dimension $\mathbf{d}$, we let $\mathcal{O}_M \subset E_{\mathbf{d}}$ be its orbit.

We denote by $\overline{Q} = (I, \Omega \sqcup \overline{\Omega})$ the doubled quiver having the same set of vertices as Q and for each $\alpha \in \Omega$, an additional arrow $\bar{\alpha} \in \overline{\Omega}$ going in the opposite direction. The symplectic action of $G_{\mathbf{d}}$ on $T^* E_{\mathbf{d}}$ identified with $E_{\overline{Q},\mathbf{d}}$ via the trace pairing has (quadratic) moment map

$$\begin{array}{cccc} \mu_{\mathbf{d}} & : & E_{\overline{Q},\mathbf{d}} & \to & \prod_{i \in I} \mathfrak{gl}_{\mathbf{d}_i} \\ & & x & \mapsto & \sum_{\alpha \in \Omega} [x_\alpha, x_{\overline{\alpha}}] \end{array}.$$

Its zero-level $\mu_{\mathbf{d}}^{-1}(0)$ is of fundamental importance in the geometry of quiver representations, as shown in [CB01]. This can be explained by the fact that $[\mu_{\mathbf{d}}^{-1}(0)/G_{\mathbf{d}}]$ has a strong relation to the cotangent stack to $[E_{\mathbf{d}}/G_{\mathbf{d}}]$.

5.2.5 Stratification of the representation spaces of acyclic quivers

We stratify the representation spaces of acyclic quivers using Auslander-Reiten theory. For finite type quivers, the given stratification is trivial, but can be refined by the stratification by orbits. For affine quivers, we give an other refinement which is given by Ringel in [Rin98b]. In both cases, these stratifications have the nice property that Lusztig nilpotent variety is the union of the conormal bundles to some of the strata, as we will see in Section 5.4.2.

Auslander-Reiten stratification of the representation spaces

Let $Q = (I, \Omega)$ be an acyclic quiver and $\mathbf{d}$ a dimension vector. For $x \in E_{Q,\mathbf{d}}$, we let $\mathbf{d}_P(x), \mathbf{d}_R(x)$ and $\mathbf{d}_I(x)$ be the dimension vector of the preprojective, regular and preinjective direct summands of x seen as a representation of Q. For $\mathbf{d}_P, \mathbf{d}_R, \mathbf{d}_I \in \mathbf{N}^I$ such that $\mathbf{d}_P + \mathbf{d}_R + \mathbf{d}_I = \mathbf{d}$, we define

$$E_{\mathbf{d}_P,\mathbf{d}_R,\mathbf{d}_I} = \{x \in E_{Q,\mathbf{d}} \mid \mathbf{d}_P(x) = \mathbf{d}_P, \mathbf{d}_R(x) = \mathbf{d}_R, \mathbf{d}_I(x) = \mathbf{d}_I\}.$$

It is a smooth locally closed subvariety of $E_{Q,\mathbf{d}}$ and we have

$$E_{Q,\mathbf{d}} = \bigsqcup_{\mathbf{d}_P + \mathbf{d}_R + \mathbf{d}_I = \mathbf{d}} E_{\mathbf{d}_P,\mathbf{d}_R,\mathbf{d}_I}.$$

We can refine this stratification by fixing the isoclass of the preprojective and preinjective summands in the following way. Choose a decomposition $\mathbf{d} = \mathbf{d}_P + \mathbf{d}_R + \mathbf{d}_I$, a preprojective representation P

of dimension $\mathbf{d}_P$ and a preinjective representation I of dimension $\mathbf{d}_I$. Define

$$E_{[P],\mathbf{d}_R,[I]} = \{x \in E_{\mathbf{d}} \mid (\mathbf{C}^{\mathbf{d}}, x) \simeq P \oplus R \oplus I,\ R \text{ regular of dimension } \mathbf{d}_R\}.$$

Lemma 5.2.7. *We have the decomposition into smooth locally closed subvarieties*

$$E_{Q,\mathbf{d}} = \bigsqcup_{[P],\mathbf{d}_R,[I]} E_{[P],\mathbf{d}_R,[I]}$$

where the sum runs over the triples $([P],\mathbf{d}_R,[I])$ *where* $[P]$ *is a preprojective isoclass,* $[I]$ *a preinjective isoclass and* $\mathbf{d}_R \in \mathbf{N}^I$ *a dimension vector subject to the condition* $\dim P + \mathbf{d}_R + \dim I = \mathbf{d}$.

Ringel stratification of the representation spaces of acyclic affine quivers

To describe the irreducible components of Lusztig nilpotent variety for affine quivers, Ringel introduced in [Rin98b] a stratification of the representation spaces. Let P an isomorphism class of a preprojective representation, I of a preinjective and N of a non-homogeneous regular representation. Let

$$\mu : \mathscr{P} \to \mathbf{N}$$

be a function with finite support. Then (P, I, N, μ) is called a *type*. Its dimension is given by $\dim(P, I, N, \mu) = \dim P + \dim I + \dim N + \sum_{\lambda\in\mathscr{P}} \mu(\lambda)|\lambda|\delta$. We also let $\dim \mu = \sum_{\lambda\in\mathscr{P}} \mu(\lambda)|\lambda|$. If $\mathbf{d} = \dim(P, I, N, \mu)$, the type (P, I, N, μ) defines a stratum $\Xi(P, I, N, \mu) \subset E_{\mathbf{d}}$. Let first $\lambda_1, \dots, \lambda_r$ be partitions such that any partition λ appears $\mu(\lambda)$ times in this list. The stratum consists of $x \in E_{\mathbf{d}}$ which are isomorphic to a direct sum

$$P \oplus I \oplus N \oplus \bigoplus_{i=1}^{r} S_{a_i}[\lambda_i]$$

where $a_1, \dots, a_r \in |\mathbf{P}^1_{\mathbf{C}}| \setminus D$ are pairwise distinct. It is clear (by Theorem 5.2.5) that $E_{\mathbf{d}}$ is the disjoint union of the strata $\Xi(P, I, N, \mu)$ where (P, I, N, μ) is a type of dimension $\mathbf{d}$. To ease the notation, we will write $\Xi(P, I, N, \mu) = \Xi(P \oplus I \oplus N, \mu)$. Also when $\mu = 0$, $\Xi(P, I, N, 0)$ is the orbit of $P \oplus I \oplus N$ and will usually be denoted by $\mathcal{O}_{P\oplus I\oplus N}$. We also define $E^{reg}_{\mathbf{d}} = \{x \in E_{\mathbf{d}} \mid (\mathbf{C}^{\mathbf{d}}, x) \text{ is regular}\}$ and $E^{\text{reghom}}_{\mathbf{d}} = \{x \in E_{\mathbf{d}} \mid (\mathbf{C}^{\mathbf{d}}, x) \text{ is regular homogeneous}\}$. By Remark 5.2.6, both coincide for the Kronecker quiver. Recall from Theorem 5.2.5 that $D \subset |\mathbf{P}^1|$ is the set of non-homogeneous tubes. It consists of $0, 1, 2$ or 3 points. It $T \subset D$ is a subset, we let $E^T_{\mathbf{d}}$ be the set of $x \in E^{reg}_{\mathbf{d}}$ such that all indecomposable inhomogeneous direct summands of x are contained in the tubes indexed by T. In particular, $E^{\emptyset}_{\mathbf{d}} = E^{\text{reghom}}_{\mathbf{d}}$. It is an open subset of $E^{reg}_{\mathbf{d}}$ and thus of $E_{\mathbf{d}}$ since it is the set of $x \in E^{reg}_{\mathbf{d}}$ such that $\mathrm{Hom}(N, x) = 0$ for any inhomogeneous indecomposable representation of Q in the tubes indexed by $D \setminus T$ of dimension $\leq \mathbf{d}$, and the isoclasses of such N are in finite number. If $T \subset D$ and N is a regular inhomogeneous representation of Q whose indecomposable summands are in the tubes indexed by T, for any $\mathbf{d} \in \mathbf{N}^I$, we let $E^{D\setminus T}_{[N],\mathbf{d}}$ be the set of $x \in E^{reg}_{\mathbf{d}+\dim N}$ such that x is isomorphic to $N \oplus R$ for a regular representation R all of those regular non-homogeneous direct summands belong to the tubes indexed by $D \setminus T$.

The map $\mu : \mathscr{P} \to \mathscr{P}$ is called *regular* if it takes nonzero values only on partitions of length one, and regular semisimple if it is nonzero only on the partition (1). A type $\Xi(P, I, N, \mu)$ is called *regular* (resp. *regular semisimple*) if μ is regular (resp. regular semisimple).

Lemma 5.2.8. *The stratum $\Xi(N, \mu) \subset E_{\mathbf{d}}^{reg}$ where N is regular non-homogeneous and μ is regular semisimple are irreducible, locally closed and smooth.*

We will need such a result to use Lemma 5.C.5 in the Section 5.C. It is more generally also true that the stratum $\Xi(P, I, N, \mu)$ is smooth for any quadruple (P, I, N, μ). The above result is sufficient for our purposes.

Let N be a non-homogeneous regular representation and $\mu : \mathscr{P} \to \mathbf{N}$. Let $d = \dim \mu$.

Lemma 5.2.9. *The natural map induced by the direct sum*

$$(\mathcal{O}_N \times \Xi(\mu)) \times^{G_{\mathbf{d}_N} \times G_{d\delta}} G_{\mathbf{d}} \to \Xi(N, \mu)$$

is an isomorphism.

Proof. It follows directly from the fact that any regular representation of Q can be uniquely decomposed as a direct sum of a regular non-homogeneous representation and a regular homogeneous one. $\square$

Quotient of the regular homogeneous locus

Consider $\theta := \langle \delta, - \rangle : \mathbf{Z}^I \to \mathbf{Z}$. By [HdlPn02], the open subset of E_δ of θ-stable representations coincide with E_δ^{reghom} and $E_\delta^{\text{reghom}}/(G_\delta/\mathbf{C}^*) \simeq \mathbf{P}_1^{\text{hom}}$. As a consequence, we have a map

$$E_\delta^{\text{reghom}} \to \mathbf{P}_1^{\text{hom}}$$

whose fibers are orbits of simple regular representations of Q. More generally, if $d \in \mathbf{N}$, $E_{d\delta}^{\text{reghom}}$ is an open subset of the θ-semistable locus and the quotient map is now

$$E_{d\delta}^{\text{reghom}} \to S^d\,\mathbf{P}_1^{\text{hom}}.$$

If μ is regular semisimple with $\dim \mu = d$, $\Xi(\mu) \subset E_{d\delta}^{\text{reghom}}$. Moreover, the morphism above gives a morphism

$$\chi_\mu : \Xi(\mu) \to S^d\,\mathbf{P}_1^{\text{hom}} \setminus \Delta.$$

If N is a non-homogeneous regular representation, then we obtain a morphism

$$(\mathcal{O}_N \times \Xi(\mu)) \times GL_{\mathbf{d}} \to S^d\,\mathbf{P}_1^{\text{hom}} \setminus \Delta$$

which factorizes through the action of $\mathrm{GL}_{\mathbf{d}_N} \times \mathrm{GL}_{\dim \mu}$ and by Lemma 5.2.9 gives rise to a $G_{\mathbf{d}}$-equivariant morphism:

$$\chi_{N,\mu} : \Xi(N, \mu) \to S^d\,\mathbf{P}_1^{\text{hom}} \setminus \Delta.$$

5.2.6 Stratification of the representation spaces of the Jordan and cyclic quivers

For the Jordan and cyclic quivers, we obtain a stratification similar to that in Section 5.2.5.

Stratification of the representation spaces of the Jordan quiver

Let J be the Jordan quiver (one vertex and one loop). In dimension $d \in \mathbf{N}$, the representation space $E_{J,d}$ is the Lie algebra $\mathfrak{gl}_d$ endowed with the adjoint action of GL_d. We let $G = \mathrm{GL}_d$ and $\mathfrak{g} = \mathfrak{gl}_d$. We describe Lusztig stratification of $\mathfrak{gl}_d$ (the same kind of stratification exists for any reductive Lie algebra and also on any reductive group, [Lus84], [Mir04, 5.5]). For $x \in \mathfrak{g}$, we let $Z_{\mathfrak{g}}(x)$ be the centralizer of x in $\mathfrak{g}$. For a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$, let $Z_r(x) = \{x \in \mathfrak{l} \mid Z_{\mathfrak{g}}(x) = \mathfrak{l}\}$. Let $\mathcal{O} \subset \mathfrak{l}$ be a nilpotent orbit. Then, we obtain a stratum

$$\Xi(\mathfrak{l}, \mathcal{O}) = G \cdot (Z_r(\mathfrak{l}) + \mathcal{O}).$$

Proposition 5.2.10. *The partition*

$$\mathfrak{g} = \bigsqcup_{(\mathfrak{l},\mathcal{O})} \Xi(\mathfrak{l}, \mathcal{O})$$

where the sum is indexed by pairs $(\mathfrak{l}, \mathcal{O})$ up to conjugation is a stratification by smooth locally closed subvarieties.

We will give a very explicit description of the strata. We let $\mathcal{S}$ be the set of pairs $(\mathfrak{l}, \mathcal{O})$ as above up to conjugation. We let $\mathrm{Fun}_d(\mathscr{P}, \mathbf{N})$ be the set of functions $\mu : \mathscr{P} \to \mathbf{N}$ of weight d, *i.e.* such that $\sum_{\lambda \in \mathscr{P}} \mu(\lambda)|\lambda| = d$. We will construct a bijection $\mathrm{Fun}_d(\mathscr{P}, \mathbf{N}) \to \mathcal{S}$. Let $\mu : \mathscr{P} \to \mathbf{N}$ be a function of weight d. Let $\lambda_1, \dots, \lambda_r$ be the collection of partitions ordered by decreasing lengths such that any partition $\lambda \in \mathscr{P}$ appears exactly $\mu(\lambda)$ times. It defines the diagonal Levi subalgebra

$$\mathfrak{l}_\mu = \prod_{i=1}^{r} \mathfrak{gl}_{|\lambda_i|}$$

and the nilpotent orbit

$$\mathcal{O}_\mu = \prod_{i=1}^{r} \mathcal{O}_{\lambda_i}$$

of $\mathfrak{l}$, where for any partition λ of an integer n, $\mathcal{O}_\lambda$ is the corresponding nilpotent orbit of $\mathfrak{gl}_n$ (the partition (n) corresponds to the regular nilpotent orbit).

The map

$$\begin{array}{ccc} \mathrm{Fun}_d(\mathscr{P}, \mathbf{C}) & \to & \mathcal{S} \\ \mu & \mapsto & (\mathfrak{l}_\mu, \mathcal{O}_\mu) \end{array}$$

is a bijection. We write $\Xi(\mu) = \Xi(\mathfrak{l}_\mu, \mathcal{O}_\mu)$. In explicit terms, $\Xi(\mu)$ is the smallest G-invariant subset of $\mathfrak{g}$ containing the matrices

$$J_\mu(x_1,\dots,x_r)=\begin{pmatrix} J_{\lambda_1}(x_1) & & & \\ & J_{\lambda_2}(x_2) & & \\ & & \ddots & \\ & & & J_{\lambda_r}(x_r)\end{pmatrix}.$$

where $x_1,\dots,x_r$ are pairwise distinct complex numbers, and for $x\in\mathbf{C}$ and a partition λ, $J_\lambda(x)$ is the standard Jordan matrix with eigenvalue x. That is, $J_\lambda(x)=xI_{|\lambda|}+J_\lambda(0)$,

$$J_\lambda(0)=\begin{pmatrix} J_{\lambda^1} & & & \\ & J_{\lambda^2} & & \\ & & \ddots & \\ & & & J_{\lambda^s}\end{pmatrix}$$

if $\lambda=(\lambda^1,\dots,\lambda^s)$ and for $\ell\in\mathbf{N}$,

$$J_\ell=(\delta_{i+1,j})_{1\leq i,j\leq \ell}$$

where $\delta_{k,l}$ is the Kronecker symbol.

Stratification of the representation spaces of cyclic quivers

Let C_n be the cyclic quiver with n vertices indexed by $\mathbf{N}/n\mathbf{Z}$ and having arrows $i\to i+1$ for $i\in\mathbf{Z}/n\mathbf{Z}$. Let $d\in\mathbf{N}$ and $\delta=(1,\dots,1)\in\mathbf{N}^{\mathbf{Z}/n\mathbf{Z}}$. A representation of C_n is a n-tuple $(x_i)_{i\in\mathbf{Z}/n\mathbf{Z}}$ of linear maps $x_i:V_i\to V_{i+1}$ for $i\in\mathbf{Z}/n\mathbf{Z}$. We have a closed immersion

$$\begin{array}{cccc} i_d & : & E_{J,d} & \to & E_{Q,d\delta} \\ & & x & \mapsto & (\mathrm{id},\dots,\mathrm{id},x). \end{array}$$

Let N be a nilpotent representation of C_n and $\mathbf{d}_N$ its dimension. Let $\mu:\mathscr{P}\to\mathbf{N}$ be a finitely supported function. We let $\dim\mu=\sum_{\lambda\in\mathscr{P}}\mu(\lambda)|\lambda|$ and $\mathbf{d}=\dim N+\delta\dim\mu$. We define the subset $\Xi(N,\mu)$ of elements x of $E_{Q,\mathbf{d}}$ such that $(\mathbf{C}^{\mathbf{d}},x)$ is isomorphic to $N\oplus R$ as a representation of C_n, where R is a representation of C_n such that the orbit $\mathscr{O}_R\subset E_{Q,\dim\mu}$ intersects $i_d(\Xi(\mu)\cap\mathrm{GL}_{\dim\mu})$, where $\Xi(\mu)$ is the corresponding stratum in $E_{J,\dim\mu}$ defined in Section 5.2.6 and $\mathrm{GL}_{\dim\mu}\subset E_{J,\dim\mu}$ is the set of invertible elements.

Let $\Xi(N,\mu)\subset E_{\mathbf{d}}$, $\mathbf{d}=\dim N+\delta\dim\mu$, be a stratum. Since any representation of C_n can be uniquely decomposed as a direct sum of a nilpotent representation and an invertible one, we obtain the following lemma.

Lemma 5.2.11. *The natural map induced by the direct sum*

$$(\mathscr{O}_N\times\Xi(\mu))\times^{G_{\mathbf{d}_N}\times G_{\delta\dim\mu}}G_{\mathbf{d}}\to\Xi(N,\mu)$$

is an isomorphism.

Proposition 5.2.12. *The partition*

$$E_{C_n,\mathbf{d}} = \bigsqcup_{(N,\mu)} \Xi(N,\mu)$$

where the sum runs over pairs (N,μ), N is a nilpotent representation (taken up to isomorphism) and μ are such that $\dim N + \dim \mu\delta = \mathbf{d}$ *is a locally closed stratification of $E_{C_n,\mathbf{d}}$. Moreover, if $\mathbf{C}^*$ is the action by multiplication on $E_{C_n,\mathbf{d}}$, this stratification is $\mathbf{C}^*$-stable.*

Isomorphism classes of representations of the cyclic quiver

For $\mu : \mathscr{P} \to \mathbf{N}$ such that $\dim \mu = d\delta$ and $(x_1,\dots,x_r) \in (\mathbf{C}^*)^r$, $r = \sum_{\lambda\in\mathscr{P}} \mu(\lambda)$, we let $J_\mu^{C_n}(x_1,\dots,x_r) = i_d(J_\mu(x_1,\dots,x_r))$. For any partition λ of d and $x \in \mathbf{C}^*$, we let $J_\lambda^{C_n}(x) = i_d(J_\lambda(x))$. When the context is clear, we drop the exponent. In particular, we write

$$J_\mu(\underline{x}) = J_\mu^{C_n}(x_1,\dots,x_r).$$

Then, any representation of C_n is isomorphic to a representation of the form

$$N_{\mathbf{m}} \oplus J_\mu(\underline{x})$$

for a unique pair $(\mathbf{m},\mu)$, where $\mathbf{m}$ is a multipartition, $\mu : \mathscr{P} \to \mathbf{N}$ and $\underline{x} \in (\mathbf{C}^*)^{\sum_\lambda \mu(\lambda)}$. When μ is regular semisimple (it means by definition that for any $\lambda \in \mathscr{P}$, $\mu(\lambda) \neq 0 \implies \lambda = (1)$),

$$J_\mu(x) \simeq \bigoplus_{j=1}^r J_1(x_j).$$

5.2.7 Open subsets of the representation spaces of cyclic quivers

Let $\mathbf{d} \in \mathbf{N}^{\mathbf{Z}/n\mathbf{Z}}$. We have a morphism of algebraic varieties

$$\begin{array}{cccc} \varphi : & E_{C_n,\mathbf{d}} & \to & E_{J,\mathbf{d}_0} \\ & (x_0,\dots,x_{n-1}) & \mapsto & x_{n-1}x_{n-2}\dots x_0 \end{array}.$$

Let

$$\chi_J : E_{J,\mathbf{d}_0} \to S^{\mathbf{d}_0}\mathbf{C} = E_{J,\mathbf{d}_0} /\!\!/ \mathrm{GL}_{\mathbf{d}_0}$$

be the quotient map and $\chi = \chi_J \circ \varphi$. Let $S^{\mathbf{d}_0}D(0,1) \subset S^{\mathbf{d}_0}\mathbf{C}$ where $D(0,1) \subset \mathbf{C}$ is the open unit disk. We let $E_{C_n,\mathbf{d}}^{<1} = \chi^{-1}(S^{\mathbf{d}_0}D(0,1))$. It is an open analytic subset of $E_{C_n,\mathbf{d}}$. The stratification of $E_{C_n,\mathbf{d}}$ induces a stratification of $E_{C_n,\mathbf{d}}^{<1}$ (it just constrains the eigenvalues to have absolute value < 1). For a stratum $\Xi(N,\mu) \subset E_{\mathbf{d}}$, we let $\Xi^{<1}(N,\mu) = \Xi(N,\mu) \cap E_{\mathbf{d}}^{<1}$ be the corresponding stratum. To any subset of strata $\mathcal{S}$ of $E_{C_n,\mathbf{d}}$, we define $\mathcal{S}^{<1} = \{S \cap E_{C_n,\mathbf{d}}^{<1} : S \in \mathcal{S}\}$ the corresponding set of strata of $E_{C_n,\mathbf{d}}^{<1}$. For any stratum $\Xi(N,\mu)$, we let

$$j : \Xi(N,\mu) \to E_{C_n,\mathbf{d}},$$

$$j^{<1} : \Xi^{<1}(N,\mu) \to E^{<1}_{C_n,\mathbf{d}}$$

and

$$j_{N,\mu} : \Xi^{<1}(N,\mu) \to \Xi(N,\mu)$$

be the natural inclusions. The group $G_{\mathbf{d}}$ acts on $\Xi(N,\mu)$, $\Xi^1(N,\mu)$, $E^{<1}_{C_n,\mathbf{d}}$ and all the above maps are $G_{\mathbf{d}}$ equivariant. For a subset $D \subset \mathbf{C}$, we let

$$E^{\mathbf{C}\setminus D}_{C_n,\mathbf{d}} = \begin{cases} \chi^{-1}(S^{\mathbf{d}_0}(\mathbf{C}\setminus D)) \text{ if } D \text{ does not contain } 0, \\ \{x \in \chi^{-1}(S^{\mathbf{d}_0}(\mathbf{C}\setminus D)) \mid x \text{ has no nilpotent direct summands}\} \text{ otherwise} \end{cases}$$

By Proposition 5.2.12, if $\mathbf{C}^*$ acts on $E_{C_n,\mathbf{d}}$ with weight one, Lusztig strata $\Xi(N,\mu)$ are $\mathbf{C}^*$-invariant. This gives the following result.

Proposition 5.2.13. *The inclusion $j_{N,\mu}$ induces an isomorphism at the level of fundamental groups.*

Recall the isomorphism

$$(\mathcal{O}_N \times \Xi(\mu)) \times^{G_{\mathbf{d}_N} \times G_{\dim \mu\delta}} G_{\mathbf{d}} \to \Xi(N,\mu)$$

of Lemma 5.2.11. Let $d = \dim\mu$. The map $\chi : E_{C_n d\delta} \to S^d(\mathbf{C})$ restricts to

$$\chi_\mu : \Xi(\mu) \to S^d(\mathbf{C}^*).$$

By the previous isomorphism, it is easily seen to give a map

$$\chi_{N,\mu} : \Xi(N,\mu) \to S^d(\mathbf{C}^*).$$

If moreover μ is regular semisimple, it takes value in the complement of the diagonal:

$$\chi_{N,\mu} : \Xi(N,\mu) \to S^d(\mathbf{C}^*) \setminus \Delta.$$

5.3 Lusztig perverse sheaves, Induction and Restriction functors

In this Section, we briefly recall how Lusztig sheaves are built and the operations of induction and restriction. We leave the reader consult [Lus11], [Lus91], [Sch09] for more details on the link with quantum groups.

5.3.1 Lusztig perverse sheaves

In his foundational paper [Lus91], Lusztig introduced a semisimple category of constructible complexes on the representation varieties of a quiver giving a categorification of one half of the quantum group and providing the so-called canonical basis. We briefly recall here how Lusztig sheaves are obtained.

Let $\mathbf{d} \in \mathbf{N}^I$ be a dimension vector. A flag-type of dimension $\mathbf{d}$ is an uplet $\underline{\mathbf{d}} = (\mathbf{d}_1, \dots, \mathbf{d}_l) \in (\mathbf{N}^I)^l$ for some $l \geq 1$ such that $\sum_{j=1}^{l} \mathbf{d}_i = \mathbf{d}$. Given a flag-type $\underline{\mathbf{d}}$ as above, define the partial flag variety

$$\mathcal{F}_{\underline{\mathbf{d}}} = \{(0 = F_0 \subset \dots \subset F_l = k^{\mathbf{d}}) \mid \dim(F_j/F_{j-1}) = \mathbf{d}_j \text{ for } 1 \leq j \leq l\}.$$

Define also

$$\tilde{\mathcal{F}}_{\underline{\mathbf{d}}} = \{(x, \underline{F}) \in E_{\mathbf{d}} \times \mathcal{F}_{\underline{\mathbf{d}}} \mid x(F_j) \subset F_j \text{ for } 1 \leq j \leq l\}.$$

For quivers with cycles, it is also useful to consider the nilpotent version

$$\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}^{\mathrm{nil}} = \{(x, \underline{F}) \in E_{\mathbf{d}} \times \mathcal{F}_{\underline{\mathbf{d}}} \mid x(F_j) \subset F_{j-1} \text{ for } 1 \leq j \leq l\}.$$

We have natural projections $\pi_{\underline{\mathbf{d}}} : \tilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to E_{\mathbf{d}}$, $\pi_{\underline{\mathbf{d}}}^{\mathrm{nil}} : \tilde{\mathcal{F}}_{\underline{\mathbf{d}}}^{\mathrm{nil}} \to E_{\mathbf{d}}$ which are projective and $\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}, \tilde{\mathcal{F}}_{\underline{\mathbf{d}}}^{\mathrm{nil}}$ are smooth, being affine fibrations over the flag manifold $\mathcal{F}_{\underline{\mathbf{d}}}$.

For quivers without oriented cycles, there is no need to consider both $\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ and $\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}^{\mathrm{nil}}$. However, for quivers with loops or more generally oriented cycles, this leads to different (although related) stories, as is already seen for the Jordan quiver (on the one-hand we have the Grothendieck-Springer resolution and on the other hand the Springer resolution). See for example [Lus93, KKS09, Boz15, Boz16] for some perspective.

For a dimension vector $\mathbf{d} \in \mathbf{N}^I$ and a flag-type $\underline{\mathbf{d}}$, $\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ is smooth and $\pi_{\underline{\mathbf{d}}}$ is proper and therefore by the decomposition theorem ([BBD82, Théorème 6.2.5]) $(\pi_{\underline{\mathbf{d}}})_*\underline{\mathbf{C}}$ is a semisimple constructible complex on $E_{\mathbf{d}}$. In his paper [Lus91], for loop-free quivers, Lusztig considers the category $\mathscr{Q}_{\mathbf{d}}$ of semisimple constructible complexes on $E_{\mathbf{d}}$ whose direct summands are shifts of some of the direct summands of the complexes $(\pi_{\underline{\mathbf{d}}})_*\underline{\mathbf{C}}$ for various *discrete* flag-types $\underline{\mathbf{d}}$ of dimension $\mathbf{d}$. We call $\mathscr{Q} = \prod_{\mathbf{d} \in \mathbf{N}^I} \mathscr{Q}_{\mathbf{d}}$ the Hall category. He also considers the category of perverse sheaves $\mathscr{P}_{\mathbf{d}}$ which are in $\mathscr{Q}_{\mathbf{d}}$. When $i \in I$ and $n \geq 0$, we let $L_{ne_i} = (\pi_{(ne_i)})_*\underline{\mathbf{C}}$. It is a constructible complex on E_{ne_i} (e_i is the i-th vector of the canonical basis of $\mathbf{Z}^I$). Observe that for $n = 0$, it does not depend on i.

5.3.2 Lusztig perverse sheaves for finite type quivers

It is possible to give a complete description of Lusztig sheaves for finite type quivers. This task is easy since for any dimension vector $\mathbf{d}$, $E_{\mathbf{d}}$ has a finite number of $G_{\mathbf{d}}$-orbits. This is the content of Theorem 5.3.2, of which we provide a geometric proof. This theorem can be proved differently using that for finite type quivers, the representation varieties are union of finite number of orbits in any dimension and combining results of Ringel ([Sch09, Theorem 3.16]) and Lusztig (categorification of the quantum group, [Lus91]).

Description of Lusztig perverse sheaves for finite type quivers

This Section relies on the desingularization of finite type orbits given in [Rei03]. The main theorem of *loc. cit.* can be formulated as follows.

Theorem 5.3.1 (Reineke, [Rei03, Theorem 2.2]). *Let $Q = (\Omega, I)$ be a finite type quiver, $\mathbf{d} \in \mathbf{N}^I$ a dimension vector and $\mathcal{O} \subset E_{\mathbf{d}}$ a $G_{\mathbf{d}}$-orbit. Then there exists a flag-type $\underline{\mathbf{d}} = (\mathbf{d}_1, \dots, \mathbf{d}_l)$ with*

$\sum_{i=1}^{l} \mathbf{d}_i = \mathbf{d}$ *such that the projective morphism*

$$\pi_{\underline{\mathbf{d}}} : \mathcal{F}_{\underline{\mathbf{d}}} \to E_{\mathbf{d}}$$

factorizes through $\overline{\mathcal{O}}$ *and induces a desingularization of* $\overline{\mathcal{O}}$.

We obtain immediately the following description (obtained by Lusztig with a different approach) of Lusztig sheaves for finite type quivers.

Theorem 5.3.2 ([Lus90a])**.** *For a dimension vector* $\mathbf{d}$*,* $\mathscr{P}_{\mathbf{d}}$ *is the semisimple category generated by the simple objects* $\mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ *for all* $G_{\mathbf{d}}$*-orbits* $\mathcal{O} \subset E_{\mathbf{d}}$*.*

Proof. It is clear that $\mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ appears in $(\pi_{\underline{\mathbf{d}}})_*\underline{\mathbf{C}}$ where $\underline{\mathbf{d}}$ is given by Theorem 5.3.1. Since $E_{\mathbf{d}}$ has a finite number of $G_{\mathbf{d}}$-orbits and each of them has a connected stabilizer in $G_{\mathbf{d}}$, all $G_{\mathbf{d}}$-equivariant perverse sheaves on $E_{\mathbf{d}}$ are of this form. This concludes the proof. □

5.3.3 Lusztig perverse sheaves for cyclic quivers with cyclic orientation

For cyclic quivers with cyclic orientation, the situation is very close to that of finite type quivers since Lusztig sheaves are supported on the nilpotent locus which has only a finite number of $G_{\mathbf{d}}$-orbits.

Partial resolutions of aperiodic orbits

We give a resolution of aperiodic nilpotent orbits of cyclic quivers in the spirit of [Sch04b, Proposition 1.1]. The idea is to construct a flag-type associated to any nilpotent orbit giving a resolution of its closure and then to refine it in order to consider a discrete flag, which is only possible for aperiodic orbits.

Let $n \geq 2$ and C_n be the cyclic quiver with n vertices and cyclic orientation. For $\mathbf{d} \in \mathbf{N}^{\mathbf{Z}/n\mathbf{Z}}$, define the counterclockwise rotation of $\mathbf{d}$ by

$$\mathbf{d}_{+1} = (\mathbf{d}_{i+1}\, e_i)_{i \in \mathbf{Z}/n\mathbf{Z}}.$$

We first give a lemma.

Recall the parametrization of nilpotent orbits of C_n by multipartitions (Section 5.2.2). Let $\mathbf{m} = (\lambda^{(i)})_{i \in \mathbf{Z}/n\mathbf{Z}}$ be a multipartition of dimension $\mathbf{d} \in \mathbf{N}^{\mathbf{Z}/n\mathbf{Z}}$. Let $x \in \mathcal{O}_{\mathbf{m}}$ and $N = \max\{s \geq 0 \mid x^{s+1} = 0 \text{ and } x^s \neq 0\}$. Let

$$\underline{\mathbf{d}'} = (\mathbf{d}'_0, \dots, \mathbf{d}'_N)$$

where

$$\mathbf{d}'_j = \dim \operatorname{im}(x^{N-j}) / \operatorname{im}(x^{N+1-j}).$$

Then we have the following lemma whose proof is an easy consequence of the description of nilpotent orbits by multisegments.

Lemma 5.3.3. *For any $0 \leq j \leq N$,*

$$\mathbf{d}'_j - (\mathbf{d}'_{j-1})_{+1} = (\sharp\{t : \lambda_t^{(i-(N-j))} = N - j\})_{i \in \mathbf{Z}/n\mathbf{Z}},$$

where it is understood that $\mathbf{d}'_{-1} = 0$. In particular, if $\mathbf{m}$ is aperiodic, $\mathbf{d}'_j - (\mathbf{d}'_{j-1})_{+1}$ has some zero coordinate.

Proof. It suffices to note that, by the very definition of multisegments, for any $0 \leq j \leq N$,

$$(\mathbf{d}'_j)_i = \sharp\{t : \lambda_t^{i-(N-j)} \geq N - j\}.$$

$\square$

Theorem 5.3.4. *Let $\mathbf{d} \in \mathbf{N}^{\mathbf{Z}/n\mathbf{Z}}$ be a dimension vector Let $\mathcal{O} \subset E_{C_n,\mathbf{d}}$ a nilpotent aperiodic orbit. Then there exists a discrete flag-type $\underline{\mathbf{d}}$ such that the proper morphism $\pi_{\underline{\mathbf{d}}} : \mathcal{F}_{\underline{\mathbf{d}}} \to E_{C_n,\mathbf{d}}$ has image $\overline{\mathcal{O}}$ and induces a resolution of singularities of $\overline{\mathcal{O}}$.*

Proof. Assume $\mathcal{O} = \mathcal{O}_{\mathbf{m}}$ for some aperiodic multipartition $\mathbf{m}$. First define a (usually non-discrete) flag-type of dimension $\mathbf{d}$ as follows. Let $x \in \mathcal{O}$ and $N = \max\{s \geq 0 \mid x^s \neq 0 \text{ and } x^{s+1} = 0\}$. Define

$$\underline{\mathbf{d}'} = (\mathbf{d}'_0, \ldots, \mathbf{d}'_N)$$

where

$$\mathbf{d}'_j = \dim \operatorname{im}(x^{N-j}) / \operatorname{im}(x^{N+1-j})$$

as before. Then, the dual of the proof of [Sch04b, Proposition 1.1] shows that $\pi_{\underline{\mathbf{d}'}} : \mathcal{F}^{\mathrm{nil}}_{\underline{\mathbf{d}'}} \to E_{C_n,\mathbf{d}}$ induces by corestriction to $\overline{\mathcal{O}}$ a resolution of singularities of $\overline{\mathcal{O}}$.

The next step if to refine the flag-type into another one $\underline{\mathbf{d}}$, using that $\mathcal{O}$ is aperiodic, such that the forgetful morphism (forgetting the additional steps of the flags) $\mathcal{F}_{\underline{\mathbf{d}}} \to \mathcal{F}^{\mathrm{nil}}_{\underline{\mathbf{d}'}}$ is an isomorphism over $\mathcal{O}$ and the projection $\pi_{\mathbf{d}}$ to $E_{\mathbf{d}}$ has image $\overline{\mathcal{O}}$.

Since $\mathbf{m}$ is aperiodic, $(\mathbf{d}'_0)_i = 0$ for some $i \in \mathbf{Z}/n\mathbf{Z}$. We replace $\mathbf{d}'_0$ by the sequence of discrete dimension vectors

$$((\mathbf{d}'_0)_{i-1}e_{i-1}, (\mathbf{d}'_0)_{i-2}e_{i-2}, \ldots, (\mathbf{d}'_0)_{i-(n-1)}e_{i-(n-1)}).$$

Suppose next by induction that for some $1 \leq j \leq N$, $\mathbf{d}'_0, \ldots, \mathbf{d}'_{j-1}$ have been replaced by sequences of discrete dimension vectors. In particular, $\mathbf{d}'_{j-1}$ has been replaced by

$$(\alpha_1 e_{i_1}, \ldots, \alpha_r e_{i_r})$$

for some nonnegative integers α_j and $i_j \in \mathbf{Z}/n\mathbf{Z}$ for $1 \leq j \leq r$. Let $\tilde{\mathbf{d}} = \mathbf{d}'_j - (\mathbf{d}'_{j-1})_{+1}$. By Lemma 5.3.3, there exists $i \in \mathbf{Z}/n\mathbf{Z}$ such that $\tilde{\mathbf{d}}_i = 0$. We replace now $\mathbf{d}'_j$ by

$$(\alpha_1 e_{i_1-1}, \ldots, \alpha_r e_{i_r-1}, \tilde{\mathbf{d}}_{i-1}e_{i-1}, \ldots, \tilde{\mathbf{d}}_{i-(n-1)}e_{i-(n-1)}).$$

The flag-type $\underline{\mathbf{d}}$ obtained fulfills the conditions of the theorem. Indeed, it suffices to note that by construction, any representation of $\mathcal{O}$ admits a filtration whose subquotients are of the dimensions prescribed by $\underline{\mathbf{d}}$ and that the image of $\pi_{\underline{\mathbf{d}}}$ is included in the image of $\pi_{\underline{\mathbf{d}'}}$. ☐

Description of Lusztig perverse sheaves for cyclic quivers with cyclic orientation

Proposition 5.3.5 ([Lus91, §15])**.** *For a dimension vector $\mathbf{d} \in \mathbf{N}^{\mathbf{Z}/n\mathbf{Z}}$, $\mathscr{P}_{\mathbf{d}}$ is the semisimple category generated by the simple objects $\mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ by varying nilpotent aperiodic $G_{\mathbf{d}}$-orbits $\mathcal{O} \subset E_{\mathbf{d}}$.*

The following proof uses the description of the singular support of Lusztig sheaves for cyclic quivers obtained in Section 5.4.2 and Section 5.4.1 to prove that non-aperiodic orbits do not give rise to Lusztig sheaves.

Proof. Let $\mathcal{O} \subset E_{\mathbf{d}}$ be a nilpotent aperiodic orbit. If $\pi_{\underline{\mathbf{d}}}$ is given by Proposition 5.3.4, then $\mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ appears as a direct summand of $(\pi_{\underline{\mathbf{d}}})_*\underline{\mathbf{C}}$. Conversely, take $\mathscr{F}$ a simple Lusztig sheaf of $E_{\mathbf{d}}$. It is supported on the nilpotent locus, which consists of a finite number of orbits, each of which having a connected stabilizer in $G_{\mathbf{d}}$. Therefore, $\mathscr{F} = \mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ for some nilpotent orbit $\mathcal{O} \subset E_{\mathbf{d}}$. By Section 5.4.2, the singular support of $\mathscr{F}$ is contained in the union of $\overline{T^*_{\mathcal{O}}E_{\mathbf{d}}}$ for aperiodic nilpotent orbits $\mathcal{O} \subset E_{\mathbf{d}}$. Since $\overline{T^*_{\mathcal{O}}E_{\mathbf{d}}}$ is an irreducible component of $SS(\mathscr{F})$, it must be an aperiodic orbit. ☐

5.3.4 Lusztig perverse sheaves for affine quivers

In this Section, we recall the description of Lusztig simple perverse sheaves for affine acyclic quivers. See [Lus92, LL07] for proofs.

Proposition 5.3.6 ([Lus92, Proposition 6.7], [LL07, Proposition 5.10])**.** *Let $\mathbf{d} \in \mathbf{N}^I$ be a dimension vector. If $\mathcal{O}_M \subset E_{\mathbf{d}}$ is the orbit of the representation M where M does not have regular homogeneous direct summands or regular inhomogeneous non-aperiodic summands, then $\mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ is a Lusztig sheaf.*

Let $\Xi(P, I, N, \mu)$ a stratum as in Section 5.2.5, where μ is regular semisimple and N is aperiodic. Let $d \in \mathbf{N}$ such that $\dim \mu = d\delta$. Let

$$\tilde{\Xi}(P, I, N, \mu) = \{(x, \underline{x}) \in \Xi(P, I, N, \mu) \times (\mathbf{P}_1^{\mathrm{hom}})^d \mid x \simeq P \oplus I \oplus N \oplus \bigoplus_{j=1}^{d} S_{x_j}[1]\}.$$

The map $\pi := \pi_{P,I,N,\mu} : \tilde{\Xi}(P, I, N, \mu) \to \Xi(P, I, N, \mu)$ is a $\mathfrak{S}_d$-covering. Therefore, we have a decomposition

$$\pi_*(\underline{\mathbf{C}}) \simeq \bigoplus_{\lambda \in \mathscr{P}_d} \mathscr{L}_\lambda.$$

Theorem 5.3.7 ([LL07])**.** *The simple perverse sheaves in the category $\mathcal{P}_{\mathbf{d}}$ are exactly the intersection cohomology complexes $\mathcal{IC}(\Xi(P, I, N, \mu), \mathscr{L}_\lambda)$ for (P, I, N, μ) and λ as above, with $\dim P + \dim I + \dim N + \dim \mu = \mathbf{d}$.*

We call the local systems $\mathscr{L}_\lambda$ which appear *Lusztig local systems*. A consequence of Theorem 5.3.7 is that a local system $\mathscr{L}$ on $\Xi(P, I, N, \mu)$ with μ regular semisimple is a Lusztig local system if and only if $\pi^*_{P,I,N,\mu}\mathscr{L}$ is the trivial local system on $\tilde{\Xi}(P, I, N, \mu)$.

5.3.5 Local systems on the regular part

Let μ be the regular semisimple type of dimension d and N a regular non-homogeneous representation. We let $\mathbf{d} = \dim N + \delta \dim \mu$. We have a cartesian square (see Section 5.3.4 and 5.2.5 for the notations):

$$\begin{tikzcd} \tilde{\Xi}(N,\mu) \arrow[r, "\pi_{N,\mu}"] \arrow[d, "\tilde{\chi}_{N,\mu}"'] & \Xi(N,\mu) \arrow[d, "\chi_{N,\mu}"] \\ (\mathbf{P}_1^{\mathrm{hom}})^d \setminus \Delta \arrow[r, "\pi_d"] & (S^d\,\mathbf{P}_1^{\mathrm{hom}}) \setminus \Delta. \end{tikzcd}$$

Lemma 5.3.8. *Let $\mathscr{L}$ be a $G_{\mathbf{d}}$-equivariant local system on $\Xi(N,\mu) \subset E_{\mathbf{d}}$ for the regular semisimple type μ of dimension $d\delta$. Then it is the pull-back by $\chi_{N,\mu}$ of a local system $\mathscr{L}'$ on $S^d\,\mathbf{P}_1^{\mathrm{hom}} \setminus \Delta$. Moreover, the intersection complex $\mathcal{IC}(\mathscr{L})$ on $E_{d\delta}$ is a Lusztig sheaf if and only if $\pi_d^*\mathscr{L}'$ is the trivial local system.*

Proof. We postpone the proof to Section 5.B.3 since it is analogous to that of Lemma 5.B.3. □

5.3.6 Induction and restriction of constructible complexes

Induction and restriction are the operations on constructible complexes categorifying respectively the multiplication and the comultiplication of the quantum group. We invite the reader to consult [Lus91, Lus11, Rin90b] for more on this construction and the Ringel-Hall algebra construction of the quantum group of Ringel.

We closely follow [Sch09] and refer to it for properties of the induction and restriction functors.

The induction functor

Let $\mathbf{d}', \mathbf{d}'' \in \mathbf{N}^I$ and $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$. We have the induction diagram:

$$\begin{tikzcd} & E^{(1)}_{\mathbf{d}',\mathbf{d}''} \arrow[r, "r"] \arrow[dl, "p"'] & E_{\mathbf{d}',\mathbf{d}''} \arrow[dr, "q"] & \\ E_{\mathbf{d}'} \times E_{\mathbf{d}''} & & & E_{\mathbf{d}'+\mathbf{d}''}. \end{tikzcd}$$

where

$$E_{\mathbf{d}',\mathbf{d}''} = \{(x,W) \in E_{\mathbf{d}} \times \mathrm{Gr}(\mathbf{d}'',\mathbf{d}) \mid \text{ and } xW \subset W\}$$

and

$$E^{(1)}_{\mathbf{d}',\mathbf{d}''} = \{(x,W,g',g'') \mid (x,W) \in E_{\mathbf{d}',\mathbf{d}''}, g'' : W \xrightarrow{\sim} \mathbf{C}^{\mathbf{d}''}, g' : \mathbf{C}^{\mathbf{d}}/W \xrightarrow{\sim} \mathbf{C}^{\mathbf{d}'}\}.$$

The morphisms r and q are the natural projections while

$$p(x,W,g',g'') = (g'x_{|\mathbf{C}^{\mathbf{d}}/W}(g')^{-1}, g''x_W(g'')^{-1}).$$

The map r is a $GL_{\mathbf{d}'} \times \mathrm{GL}_{\mathbf{d}''}$-torsor, hence induces a triangulated equivalence

$$r^* : D^b_{GL_{\mathbf{d}} \times \mathrm{GL}_{\mathbf{d}'} \times \mathrm{GL}_{\mathbf{d}''}}(E^{(1)}_{\mathbf{d}',\mathbf{d}''}) \to D^b_{\mathrm{GL}_{\mathbf{d}}}(E_{\mathbf{d}',\mathbf{d}''}).$$

A quasi-inverse is denoted by $r_\flat$. The induction functor is

$$m := q_! r_\flat p^*[\dim p - \dim r].$$

For $\mathscr{F} \in D^b_{\mathrm{GL}_{\mathbf{d}'}}(E_{\mathbf{d}'})$ and $\mathscr{G} \in D^b_{\mathrm{GL}_{\mathbf{d}''}}(E_{\mathbf{d}''})$, we define

$$\mathscr{F} \star \mathscr{G} = m(\mathscr{F} \boxtimes \mathscr{G}).$$

It is possible to show the associativity of m (up to a natural transformation)([Sch09, Proposition 1.9]). This allows us to define the induction $\mathscr{F}_1 \star \ldots \star \mathscr{F}_r$ for equivariant constructible complexes $\mathscr{F}_1, \ldots, \mathscr{F}_r$ on $E_{\mathbf{d}_1}, \ldots, E_{\mathbf{d}_r}$ respectively. We write

$$\mathscr{F}_1 \star \ldots \star \mathscr{F}_r = m(\mathscr{F}_1 \boxtimes \ldots \boxtimes \mathscr{F}_r) = m_{\mathbf{d}_1,\ldots,\mathbf{d}_r}(\mathscr{F}_1 \boxtimes \ldots \boxtimes \mathscr{F}_r)$$

depending whether or not we want to put the emphasis on the dimensions.

The restriction functor

The restriction diagram is:

$$\begin{tikzcd} & F_{\mathbf{d}',\mathbf{d}''} \arrow[dl, "\kappa"'] \arrow[dr, "\iota"] & \\ E_{\mathbf{d}'} \times E_{\mathbf{d}''} & & E_{\mathbf{d}} \end{tikzcd}$$

where $F_{\mathbf{d}',\mathbf{d}''} = \{x \in E_{\mathbf{d}} \mid x\,\mathbf{C}^{\mathbf{d}''} \subset \mathbf{C}^{\mathbf{d}''}\}$, $\mathbf{C}^{\mathbf{d}''} \subset \mathbf{C}^{\mathbf{d}}$ being the natural inclusion. The restriction functor is

$$\Delta := \kappa_! \iota^*[-\langle \mathbf{d}', \mathbf{d}'' \rangle]$$

where $\langle -, - \rangle$ is the Euler form of the quiver. As for the induction functor, it is possible to prove the coassociativity of the restriction. It allows us to define

$$\Delta_{\mathbf{d}_1,\ldots,\mathbf{d}_r}(\mathscr{F})$$

for $\mathbf{d}_1, \ldots, \mathbf{d}_r \in \mathbf{N}^I$ and $\mathscr{F}$ a $G_{\mathbf{d}}$-equivariant constructible complex on $E_{\mathbf{d}}$, $\mathbf{d} = \mathbf{d}_1 + \ldots + \mathbf{d}_r$.

5.4 Singular support of Lusztig perverse sheaves

5.4.1 Lusztig nilpotent variety and singular support of Lusztig's sheaves

Lusztig nilpotent variety

We use the notation of Section 5.2.4. In particular, recall the moment map $\mu_{\mathbf{d}}$. For any $\mathbf{d} \in \mathbf{N}^I$, we define Lusztig nilpotent variety as follows:

$$\Lambda_{\mathbf{d}} = \{x \in E_{\overline{Q},\mathbf{d}} \mid \mu_{\mathbf{d}}(x) = 0 \text{ and } x \text{ is nilpotent}\}.$$

The stacky quotient $\Lambda_{\mathbf{d}}/G_{\mathbf{d}}$ parametrizes nilpotent representations of the preprojective algebra of Q. It will be convenient for us to consider restrictions of the nilpotent variety. First let $\pi_{\mathbf{d}}: T^*E_{\mathbf{d}} \to E_{\mathbf{d}}$ be the cotangent bundle of $E_{\mathbf{d}}$. Let $\pi_{\mathbf{d}}^{\Lambda}: \Lambda_{\mathbf{d}} \to E_{\mathbf{d}}$ be its restriction at the source. For an open subset $U \subset E_{\mathbf{d}}$, we let $\Lambda_{\mathbf{d}}^U = (\pi_{\mathbf{d}}^{\Lambda})^{-1}(U)$. When $U = E_{\mathbf{d}}^{reg}$ (resp. $E_{\mathbf{d}}^{\mathrm{reghom}}$), see end of Section 5.2.5, we write $\Lambda_{\mathbf{d}}^{reg}$ (resp. $\Lambda_{\mathbf{d}}^{\mathrm{reghom}}$).

Proposition 5.4.1 (Lusztig)**.** *The variety $\Lambda_{\mathbf{d}}$ is a closed, conical (*i.e. *$\mathbf{C}^{\times}$-invariant), Lagrangian subvariety of $T^*E_{\mathbf{d}}$.*

Such a variety can be written as

$$\Lambda_{\mathbf{d}} = \bigcup_S \overline{T_S^* E_{\mathbf{d}}}$$

for some locally closed subvarieties $S \subset E_{\mathbf{d}}$. The goal for Section 5.4.2 is to identify such strata for finite type and affine quivers. This task is known to be much more difficult for wild quivers. In particular, the description we give rests on the representation theory of finite type and affine quivers. It seems therefore hopeless to give a uniform description for all quivers.

Singular support of Lusztig sheaves

Theorem 5.4.2 (Lusztig, [Lus91, Corollary 13.6])**.** *Let $\mathscr{F}$ be a Lusztig perverse sheaf on $E_{\mathbf{d}}$. Then its singular support is a union of irreducible components of $\Lambda_{\mathbf{d}}$.*

Sketch of the proof. We only deal with constructible sheaves, therefore singular supports are Lagrangian subvarieties of $T^*E_{\mathbf{d}}$ and it suffices to prove that $SS(\mathscr{F}) \subset \Lambda_{\mathbf{d}}$. As a shift of $\mathscr{F}$ appears as a direct summand of $(\pi_{\underline{\mathbf{d}}})_*\underline{\mathbf{C}}$ for some discrete flag-type $\underline{\mathbf{d}}$, it suffices to prove that $SS((\pi_{\underline{\mathbf{d}}})_*\underline{\mathbf{C}}) \subset \Lambda_{\mathbf{d}}$. It follows from the fact that $\pi_{\underline{\mathbf{d}}}$ is proper and Proposition 5.C.3. □

5.4.2 Irreducible components of Lusztig's Lagrangian for finite type and affine quivers

Lusztig nilpotent variety for finite type quivers

For finite type quivers, one can show that the nilpotency hypothesis in the definition of $\Lambda_{\mathbf{d}}$ is redundant, therefore that $\Lambda_{\mathbf{d}} = \mu_{\mathbf{d}}^{-1}(0)$. Moreover we have the following description of its irreducible components.

Proposition 5.4.3 (Lusztig, [Lus91, Proposition 14.2])**.** *We have*

$$\Lambda_{\mathbf{d}} = \bigcup_{\mathcal{O} \subset E_{\mathbf{d}}} \overline{T_{\mathcal{O}}^* E_{\mathbf{d}}}.$$

where the sum is indexed by $G_{\mathbf{d}}$-orbits in $E_{\mathbf{d}}$.

Lusztig nilpotent variety for cyclic quivers

Recall from Section 5.2.2 the two types of nilpotent orbits: aperiodic and not aperiodic orbits. We have the following result due to Lusztig.

Proposition 5.4.4 (Lusztig, [Lus91, Proposition 15.5]). *The nilpotent variety is*

$$\Lambda_{\mathbf{d}} = \bigcup_{\mathcal{O}\subset E_{\mathbf{d}}} \overline{T^*_{\mathcal{O}}E_{\mathbf{d}}}$$

where the union is indexed by aperiodic *nilpotent orbits.*

Lusztig nilpotent variety for acyclic affine quivers

In our terminology, all results of this Section also concern cyclic quivers with acyclic orientation. Recall the Ringel stratification of the representation spaces of an affine quiver defined in Section 5.2.5. We have the following result.

Proposition 5.4.5 (Ringel, [Rin98b, Corollary 5.3]). *We have*

$$\Lambda_{\mathbf{d}} = \bigcup_{P,I,H,\mu} \overline{T^*_{\Xi(P,I,H,\mu)}E_{\mathbf{d}}}$$

where the sum is indexed by the quadruples (P,I,H,μ) as in Section 5.2.5, where μ is regular (if $\mu(\lambda)\neq 0$ for a partition λ, then λ has length one) and $\dim(P,I,H,\mu)=\mathbf{d}$.

This result is analogous to the one in Section 5.10.4 giving a stratification of the Lie algebras of linear groups.

5.4.3 Two technical lemmas

First Lemma

Let $\mathscr{F}\in D^b_{G_{\mathbf{d}}}(E_{\mathbf{d}})$ be a simple $G_{\mathbf{d}}$-equivariant perverse sheaf. Assume that $SS(\mathscr{F})\subset\Lambda_{\mathbf{d}}$. Then by Proposition 5.4.5, there exists $\Xi:=\Xi(P,I,N,\mu)$ such that $\overline{\Xi}=\operatorname{supp}\mathscr{F}$. Let $\mathbf{d}_P=\dim P$, $\mathbf{d}_I=\dim I$ and $\mathbf{d}_R=\dim N+\dim\mu$. We let

$$i:\mathcal{O}_P\times E^{reg}_{\mathbf{d}_R}\times\mathcal{O}_I\to E_{\mathbf{d}}$$

be the direct sum and

$$j:\mathcal{O}_P\times E^{reg}_{\mathbf{d}_R}\times\mathcal{O}_I\to E_{\mathbf{d}_P}\times E_{\mathbf{d}_R}\times E_{\mathbf{d}_I}$$

be the inclusion. We fix the standard flag $\underline{F}=(\mathbf{C}^{\mathbf{d}_I},\mathbf{C}^{\mathbf{d}_I+\mathbf{d}_R},\mathbf{C}^{\mathbf{d}_P+\mathbf{d}_r+\mathbf{d}_I})$ in $\mathbf{C}^{\mathbf{d}}$ with subquotients of dimensions $\mathbf{d}_I,\mathbf{d}_R,\mathbf{d}_P$. Associated to it there is a parabolic subgroup $P_{\underline{F}}$ of $G_{\mathbf{d}}$ and a unipotent subgroup $U_{\underline{F}}\subset P_{\underline{F}}$ of elements respecting this flag.

Lemma 5.4.6. *With the above notations,*

$$\mathscr{G}:=i^*\mathscr{F}[r]\in D^b_{G_{\mathbf{d}_P}\times G_{\mathbf{d}_R}\times G_{\mathbf{d}_I}}(\mathcal{O}_P\times E^{reg}_{\mathbf{d}_R}\times\mathcal{O}_I)$$

is a perverse sheaf for $r=-\dim U_{\underline{F}}-(\dim G_{\mathbf{d}}-\dim P_{\underline{F}})$. Moreover,

$$\mathscr{G}=\underline{\mathbf{C}}[\dim\mathcal{O}_P]\boxtimes\mathscr{G}_R\boxtimes\underline{\mathbf{C}}[\dim\mathcal{O}_I]$$

where $\mathscr{G}_R$ is a $G_{\mathbf{d}}$-equivariant simple perverse sheaf on $E^{reg}_{\mathbf{d}_R}$ such that $SS(\mathscr{G}_R) \subset \Lambda^{reg}_{\mathbf{d}_R}$ and $\mathscr{F}$ appears as a direct summand of $m_{\mathbf{d}_P,\mathbf{d}_R,\mathbf{d}_I}(\mathscr{G})$.

Proof. Recall the notation $E_{[P],\mathbf{d}_R,[I]}$ from Section 5.2.5. To ease the notation, we define E' to be this set. Let $E'_{\underline{F}}$ be the closed subset of E' of elements x preserving the standard flag $\underline{F}$. By Corollary 5.2.4, $E' \simeq E'_{\underline{F}} \times^{P_{\underline{F}}} G_{\mathbf{d}}$. Let $i_1 : E'_{\underline{F}} \to E_{\mathbf{d}}$ be the natural inclusion. E' and $E'_{\underline{F}}$ possess the stratifications induced by Ringel's one on $E_{\mathbf{d}}$. The stratification on $E'_{\underline{F}}$ is

$$E'_{\underline{F}} = \bigsqcup_{P',I',N',\mu'} \Xi(P',I',N',\mu') \cap E'_{\underline{F}}$$

where (P',I',N',μ') is defined by the same conditions as in Section 5.2.5. The notion of *regular* strata is the same as in Proposition 5.4.5. The pull-back $\mathscr{F}_1 := i_1^*\mathscr{F}[\dim P_{\underline{F}} - \dim G_{\mathbf{d}}]$ is perverse on $E'_{\underline{F}}$. By Lemma 5.C.4, its singular support is included in the union of the closures of the conormal bundles to the regular strata of the stratification of $E'_{\underline{F}}$. Let $i_2 : \mathcal{O}_P \times E^{reg}_{\mathbf{d}_R} \times \mathcal{O}_I \to E'_{\underline{F}}$ be the inclusion induced by the direct sum. We also have a projection $\pi : E'_{\underline{F}} \to \mathcal{O}_P \times E^{reg}_{\mathbf{d}_R} \times \mathcal{O}_I$ which is a trivial fiber bundle of rank $r_{\underline{F}} := \sum_{\alpha:i\to j}((\mathbf{d}_R)_i(\mathbf{d}_I)_j + (\mathbf{d}_P)_I(\mathbf{d}_R + \mathbf{d}_I)_j)$. The stratification on $E'_{\underline{F}}$ coincides with the pull-back by π of Ringel stratification on $\mathcal{O}_P \times E^{reg}_{\mathbf{d}_R} \times \mathcal{O}_I$, where the strata are of the form $\mathcal{O}_P \times \Xi(N',\mu) \times \mathcal{O}_I$ for $\mu : \mathscr{P} \to \mathbf{N}$, N' regular non-homogeneous, $\dim N' + \dim \mu = \mathbf{d}_R$. Therefore, there exists an open stratum of the support of $\mathscr{F}_1$ of the form $\pi^{-1}(S)$ for $S \subset \mathcal{O}_P \times E^{reg}_{\mathbf{d}_R} \times \mathcal{O}_I$ a Ringel stratum. By Lemmas 5.C.5 and 5.2.8, the restriction of $\mathscr{F}_1$ to $\pi^{-1}(S)$ is a local system. Therefore, it is of the form $\pi^*\mathscr{L}[\dim S + r_{\underline{F}}]$ for a local system $\mathscr{L}$ on S. Consequently, $\mathscr{F}_1 = \mathcal{IC}(\pi^*\mathscr{L})$, and $i_2^*\mathscr{F}_1[-r_{\underline{F}}] = \mathcal{IC}(\mathscr{L})$. From the definitions of i, i_1 and i_2, this last sheaf is $\mathscr{G}$. Since $\mathscr{G}$ is $G_{\mathbf{d}_P} \times G_{\mathbf{d}_R} \times G_{\mathbf{d}_I}$-equivariant, $\mathscr{G} = \underline{\mathbf{C}}[\dim \mathcal{O}_P] \boxtimes \mathscr{G}_R \boxtimes \underline{\mathbf{C}}[\dim \mathcal{O}_I]$ where $\mathscr{G}_R$ is a simple $G_{\mathbf{d}_R}$-equivariant perverse sheaf on $E^{reg}_{\mathbf{d}_R}$. If x_P (resp. x_I) is any element in the orbit of P (resp. I),

$$\begin{array}{cccc} i_3 & : & E^{reg}_{\mathbf{d}_R} & \to & \mathcal{O}_P \times E^{reg}_{\mathbf{d}_R} \times \mathcal{O}_I \\ & & x & \mapsto & x_P \oplus x \oplus x_I \end{array}$$

then $\mathscr{G}_R = i_3^*\mathscr{G}[-\dim \mathcal{O}_P - \dim \mathcal{O}_I]$. Using Lemma 5.C.4 again, $\mathscr{G}_R$ has its singular support in $\Lambda^{reg}_{\mathbf{d}_R}$. This proves the first part of the lemma.

To prove the second part, we use the following diagram whose lower row is the induction diagram with three terms:

$$\begin{tikzcd} \mathcal{O}_P \times \Xi(N,\mu) \times \mathcal{O}_I \arrow[d] & A \arrow[l, "p'"'] \arrow[r, "r'"] \arrow[d] & B \arrow[r, "q'"] \arrow[d] & \Xi(P,N,I,\mu) \arrow[d] \\ E_{\mathbf{d}_P} \times E_{\mathbf{d}_R} \times E_{\mathbf{d}_I} & E^{(2)}_{\mathbf{d}_P,\mathbf{d}_R,\mathbf{d}_I} \arrow[l, "p"'] \arrow[r, "r"] & E_{\mathbf{d}_P,\mathbf{d}_R,\mathbf{d}_I} \arrow[r, "q"] & E_{\mathbf{d}} \end{tikzcd}$$

where $A = p^{-1}(\mathcal{O}_P \times \Xi(N,\mu) \times \mathcal{O}_I)$, $B = r(A)$. By Corollary 5.2.4, q' is an isomorphism and $\Xi \simeq \Xi_{\underline{F}} \times^{P_{\underline{F}}} G_{\mathbf{d}}$. Moreover, $A \simeq \Xi_{\underline{F}} \times^{U_{\underline{F}}} G_{\mathbf{d}}$ and all squares are cartesian. Therefore, to prove that $\mathscr{F}$ is a direct summand of $m_{\mathbf{d}_P,\mathbf{d}_R,\mathbf{d}_I}(j_{!*}\mathscr{G})$, it suffices to prove that the restriction $\mathscr{F}_\Xi$ of $\mathscr{F}$ to Ξ

is equal to $q'_*(r')_\flat p'^*\mathscr{G}[\dim p - \dim r]$. We now have the following diagram:

$$\begin{tikzcd}
 & \Xi_{\underline{F}} \arrow[dl, "\pi"'] \arrow[d, "\iota_U"'] \arrow[dr, "\iota_P"] & \\
\mathcal{O}_P \times \Xi(N,\mu) \times \mathcal{O}_I & \Xi_{\underline{F}} \times^{U_{\underline{F}}} G_{\mathbf{d}} \arrow[l, "p'"'] \arrow[r, "r'"] & \Xi_{\underline{F}} \times^{P_{\underline{F}}} G_{\mathbf{d}}.
\end{tikzcd}$$

By the same arguments as in the first part of the proof (recall that π is a trivial fiber bundle of rank $r_{\underline{F}}$ and $\mathscr{G}_{\mathcal{O}_P \times \Xi(N,\mu)\times\mathcal{O}_I}$ and $\mathscr{F}_\Xi$ are local systems shifted by the dimension to make them perverse), $\pi^*\mathscr{G}[r_{\underline{F}}] \simeq \iota_P^*\mathscr{F}_\Xi[\dim G_{\mathbf{d}} - \dim P_{\underline{F}}]$. Since the left triangle commutes, $\iota_U^* p'^*\mathscr{G}[\dim p - (\dim G_{\mathbf{d}} - \dim U_{\underline{F}})] \simeq \iota_P^*\mathscr{F}_\Xi[\dim G_{\mathbf{d}} - \dim P_{\underline{F}}]$. Since the right-hand side triangle also commutes, we get

$$\begin{aligned}
\iota_P^*(r')_\flat p'^*\mathscr{G}[\dim p' - \dim r' - (\dim G_{\mathbf{d}} - \dim P_{\underline{F}})] &\simeq \iota_U^* p'^*\mathscr{G}[\dim p' - (\dim G_{\mathbf{d}} - \dim U_{\underline{F}})] \\
&\simeq \iota_P^*\mathscr{F}_\Xi[\dim G_{\mathbf{d}} - \dim P_{\underline{F}}]
\end{aligned}$$

By $G_{\mathbf{d}}$-equivariance of both $\mathscr{F}_\Xi$ and $q'_*(r')_\flat p'^*\mathscr{G}[\dim p' - \dim r']$, this proves that they are isomorphic. □

The following corollary follows immediately from Lemma 5.4.6.

Corollary 5.4.7. *Theorem 5.1.1 is true if and only if for any $\mathbf{d} \in \mathbf{N}^I$ and any $G_{\mathbf{d}}$-equivariant simple perverse sheaf $\mathscr{F}$ on $E_{\mathbf{d}}^{reg}$ such that $SS(\mathscr{F}) \subset \Lambda_{\mathbf{d}}^{reg}$, $\mathscr{F}$ is the restriction to $E_{\mathbf{d}}^{reg}$ of a Lusztig perverse sheaf on $E_{\mathbf{d}}$.*

Second Lemma

Let N be an inhomogeneous regular representation. Let $T \subset D$ be a set of inhomogeneous tubes such that any indecomposable direct summand of N belongs to one of the tubes of T. Let $\mathbf{d}_N = \dim N$ and $\mathbf{d} = \mathbf{d}_N + \mathbf{d}_R$, $\mathbf{d}_R$. We consider the map

$$i : \mathcal{O}_N \times E_{\mathbf{d}_R}^{D\setminus T} \to E_{\mathbf{d}}.$$

induced by the direct sum. We recall (see Section 5.2.5) that $E_{[N],\mathbf{d}_R}^{D\setminus T}$ is the locally closed subset of $E_{\mathbf{d}_N+\mathbf{d}_R}$ parametrizing representations of Q isomorphic to $N \oplus R$ where R is some regular representation of Q, none of whose indecomposable direct summands belongs to a tube indexed by T.

Lemma 5.4.8. *Let $\mathscr{F}$ be a simple $G_{\mathbf{d}}$-equivariant perverse sheaf on $E_{\mathbf{d}}$ such that* $\operatorname{supp}\mathscr{F} \subset \overline{E_{[N]\,\mathbf{d}_R}^{D\setminus T}}$. *Then*

$$\mathscr{G} := i^*\mathscr{F}[r] \in D^b_{G_{\mathbf{d}_N}\times G_{\mathbf{d}_R}}(\mathcal{O}_N \times E_{\mathbf{d}_R}^{D\setminus T})$$

is a perverse sheaf for $r = \dim G_{\mathbf{d}_R} + \dim G_{\mathbf{d}_N} - \dim G_{\mathbf{d}}$. Moreover, $\mathscr{G} = \underline{\mathbf{C}}[\dim \mathcal{O}_N] \boxtimes \mathscr{G}_{reg}$, where $\mathscr{G}_{reg}$ is a $G_{\mathbf{d}_R}$ equivariant perverse sheaf on $E_{\mathbf{d}_R}^{D\setminus T}$. If in addition $SS(\mathscr{F}) \subset \Lambda_{\mathbf{d}}$, then $SS(\mathscr{G}_{reg}) \subset \Lambda_{E_{\mathbf{d}_R}^{D\setminus T}}$.

Proof. It is a consequence of the isomorphism given by Lemma 5.2.9 and Lemma 5.C.4. □

5.5 Proof of the main theorem in the finite type case

Let $\mathscr{F}$ be a $G_{\mathbf{d}}$-equivariant simple perverse sheaf on $E_{\mathbf{d}}$. By Theorem 5.2.1, the action of $G_{\mathbf{d}}$ on $E_{\mathbf{d}}$ has a finite number of orbits, and any orbit $\mathcal{O}_M$ for M a $\mathbf{d}$-dimensional representation of Q has a connected stabilizer (even irreducible as open subset of $\mathrm{End}(M)$). Therefore, any orbit is equivariantly simply-connected and $\mathscr{F} = \mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ for some $G_{\mathbf{d}}$-orbit $\mathcal{O} \subset E_{\mathbf{d}}$. By the explicit description of Lusztig sheaves for finite type quivers (Section 5.3.2), $\mathscr{F}$ is a Lusztig sheaf.

Remark 5.5.1. We did not use explicitly the hypothesis $(\mathscr{F}) \subset \Lambda_{\mathbf{d}}$. In fact, it is a consequence of $G_{\mathbf{d}}$-equivariance for finite type quivers. Indeed, since $\mathscr{F}$ is $G_{\mathbf{d}}$-equivariant, its singular support is a subset of $\mu_{\mathbf{d}}^{-1}(0)$ and $\mu_{\mathbf{d}}^{-1}(0)$ coincides with $\Lambda_{\mathbf{d}}$ for finite type quivers ([Sch09, Proposition 4.14]).

5.6 Proof of the result for cyclic quivers

We now prove the main theorem of this chapter (Theorem 5.1.1) for cyclic quivers. The main tool is the resolution of singularities of nilpotent orbits closure.

5.6.1 Proof for cyclic quivers with cyclic orientation

Let $\mathscr{F}$ be a $G_{\mathbf{d}}$-equivariant perverse sheaf on $E_{\mathbf{d}}$ such that $SS(\mathscr{F}) \subset \Lambda_{\mathbf{d}}$. Then, $\mathrm{supp}(\mathscr{F}) = \pi_{\mathbf{d}}(SS(\mathscr{F})) \subset \pi_{\mathbf{d}}(\Lambda_{\mathbf{d}}) \subset E_{\mathbf{d}}^{\mathrm{nil}}$ where $\pi_{\mathbf{d}} : T^*E_{\mathbf{d}} \to E_{\mathbf{d}}$ denotes the cotangent bundle of $E_{\mathbf{d}}$. Since the nilpotent locus $E_{\mathbf{d}}^{\mathrm{nil}} \subset E_{\mathbf{d}}$ has a finite number of orbits, each of which is equivariantly simply-connected, $\mathscr{F} = \mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ for some nilpotent orbit $\mathcal{O} \subset E_{\mathbf{d}}^{\mathrm{nil}}$. Then, $SS(\mathscr{F})$ contains $\overline{T^*_{\mathcal{O}}E_{\mathbf{d}}}$. But by Section 5.4.2, $\overline{T^*_{\mathcal{O}}E_{\mathbf{d}}}$ is contained in $\Lambda_{\mathbf{d}}$ if and only if $\mathcal{O}$ is a nilpotent aperiodic orbit. By Theorem 5.3.5, $\mathscr{F}$ is a Lusztig sheaf.

Remark 5.6.1. As pointed out to us by Éric Vasserot, we can prove that if $\mathscr{F}$ is a $G_{\mathbf{d}}$-equivariant perverse sheaf on $E_{\mathbf{d}}$ such that $\mathrm{supp}\,\mathscr{F}$ and $\mathrm{supp}\,\Phi\mathscr{F}$ are nilpotent, for Φ the Fourier-Sato transform reversing the orientation, then $\mathscr{F}$ is a Lusztig sheaf for the cyclic quiver. The hypothesis is weaker that the one on the singular support. This result is due to Lusztig.

5.6.2 Proof for cyclic quivers with arbitrary orientation

A $G_{\mathbf{d}}$-equivariant perverse sheaf on $E_{\mathbf{d}}$ whose singular support is in $\Lambda_{\mathbf{d}}$ is monodromic with respect to the actions of $\mathbf{C}^*$ by dilatation on any of the arrows. Indeed, noting that Ringel strata are $\mathbf{C}^*$-invariant with respect to any of these actions, this is a consequence of Corollary 5.D.2. Using a Fourier-Sato transform $\varPhi$ making the orientation cyclic, we obtain the sheaf $\varPhi\mathscr{F}$. By Theorem 5.D.3, this sheaf is accountable to Theorem 5.1.1 for a cyclic quiver with cyclic orientation. It is therefore a Lusztig sheaf and so is $\mathscr{F}$.

Theorem 5.1.1 is now proved for affine quivers of type A. It remains the case of affine quivers of type D and E. It is more subtle since these quivers have three non-homogeneous tubes, and we have to see what happens around each of them. It is possible thanks to cyclic quivers.

5.7 A larger class of perverse sheaves for cyclic quivers

5.7.1 Extension of the Hall category

In this Section, we let C_n be the cyclic quiver, with labeling of vertices and arrows as in Section 5.2.2. Let $\mathbf{d} \in \mathbf{N}^{\mathbf{Z}/n\mathbf{Z}}$. We define $\tilde{\mathscr{Q}}_{\mathbf{d}}$ as the full additive subcategory of $D^b_{c,G_{\mathbf{d}}}(E_{\mathbf{d}})$ generated by direct summands of the constructible sheaves $(\pi_{\underline{\mathbf{d}}})_*\underline{\mathbf{C}}_{\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}}$ for *all* (*i.e.* not necessarily discrete) flag-types $\underline{\mathbf{d}}$ of dimension $\mathbf{d}$. We call $\tilde{\mathscr{Q}}_{\mathbf{d}}$ the *extended Hall category* and we denote by $\tilde{\mathscr{P}}_{\mathbf{d}}$ the full subcategory of perverse sheaves which are in $\tilde{\mathscr{Q}}_{\mathbf{d}}$.

We will describe the simple perverse sheaves in $\tilde{\mathscr{P}}_{\mathbf{d}}$ explicitly by exhibiting for each of them a local system on a smooth open subset of its support. We call such local systems *extended Lusztig local systems*. It is very analogous to Theorem 5.3.7. We will also be able to describe their Fourier transform when reversing all the arrows of C_n and their singular support.

5.7.2 Singular support: the extended nilpotent variety

Recall the moment maps

$$\mu_{\mathbf{d}} : E_{\overline{C}_n,\mathbf{d}} = T^*E_{C_n,\mathbf{d}} \to \mathfrak{gl}_{\mathbf{d}}$$

from Section 5.2.4. An element of $T^*E_{\mathbf{C}_n,\mathbf{d}} = E_{C_n,\mathbf{d}} \oplus E^*_{\mathbf{C}_n,\mathbf{d}}$ is denoted (x,x^*). We first recall the definition of $*$-semi-nilpotent elements from [BSV17, Section 1.1] in the particular case of cyclic quivers (although the general definition is exactly the same). An element $(x,x^*) \in E_{\overline{C}_n,\mathbf{d}}$ is called $*$-semi-nilpotent if there exists a $\mathbf{Z}/n\mathbf{Z}$-graded flag $(0 = F_0 \subset F_1 \subset \ldots \subset F_r = \mathbf{C}^{\mathbf{d}})$ of $\mathbf{C}^{\mathbf{d}}$ such that for any $1 \leq j \leq r$,

$$x^*(F_j) \subseteq F_{j-1} \quad \text{and} \quad x(F_j) \subseteq F_j.$$

We let

$$\tilde{\Lambda}_{\mathbf{d}} = \{(x,x^*) \in E_{\overline{C_n},\mathbf{d}} \mid \mu_{\mathbf{d}}(x,x^*) = 0 \text{ and } (x,x^*) \text{ is } *\text{-semi-nilpotent}\}.$$

Recall the stratification of the representation spaces of cyclic quiver from Section 5.2.6.

Proposition 5.7.1. *The subvariety $\tilde{\Lambda}_{\mathbf{d}}$ of $T^*E_{C_n,\mathbf{d}}$ is closed, conical and Lagrangian. We have*

$$\tilde{\Lambda}_{\mathbf{d}} = \bigcup_{(N,\mu)} \overline{T^*_{\Xi(N,\mu)}E_{C_n,\mathbf{d}}},$$

where the union runs over pairs (N,μ) with N a nilpotent aperiodic representation of C_n and $\mu : \mathscr{P} \to \mathbf{N}$ is regular such that $\dim N + \delta \dim \mu = \mathbf{d}$

Proof. The first two properties (closed and conical) are fairly obvious. It is Lagrangian by [Boz16, Theorem 1.4]. The proof of this decomposition in irreducible components is completely analogous to the one of [Rin98b]. Everything can be adapted by replacing the Auslander translation τ by the rotation of representations: if $x = (x_i)_{i\in\mathbf{Z}/n\mathbf{Z}}$ is a representation of C_n, $\tau(x) = (x_{i-1})_{i\in\mathbf{Z}/n\mathbf{Z}}$. □

Proposition 5.7.2. *From the point of view of the projection* $\tilde{\Lambda}_{\mathbf{d}} \to E^{\mathrm{op}}_{C_n,\mathbf{d}}$, $(x, x^*) \mapsto x^*$,

$$\tilde{\Lambda}_{\mathbf{d}} = \bigcup_{\mathcal{O} \subset E^{\mathrm{op}}_{C_n,\mathbf{d}}} T^*_{\mathcal{O}} E^{\mathrm{op}}_{C_n,\mathbf{d}}$$

where the union runs over all nilpotent orbits $\mathcal{O} \subset E^{\mathrm{op}}_{C_n,\mathbf{d}}$.

Proof. If $(x, x^*) \in T^*_{\mathcal{O}} E^{\mathrm{op}}_{C_n,\mathbf{d}}$ for some nilpotent orbit $\mathcal{O} \subset E^{\mathrm{op}}_{C_n,\mathbf{d}}$, then $x^* \in \mathcal{O}$, therefore x^* is nilpotent. Moreover, for any $a \in \mathfrak{g}_{\mathbf{d}}$, $\mathrm{Tr}([a, x]x^*) = 0$, since $T_x\mathcal{O} = \{[a, x] : a \in \mathfrak{g}_{\mathbf{d}}\}$ is the tangent space of $\mathcal{O}$ at x. Then, for any $a \in \mathfrak{g}_{\mathbf{d}}$, $\mathrm{Tr}(a[x, x^*]) = 0$, and as a consequence, $\mu_{\mathbf{d}}(x, x^*) = [x, x^*] = 0$. We now see x and x^* as endomorphisms of $\mathbf{C}^{\mathbf{d}}$. Let $r \geq 0$ such that $(x^*)^r = 0$. We choose

$$\underline{F} = (0 \subset \ker x^* \subset \ldots \subset \ker(x^*)^r = \mathbf{C}^{\mathbf{d}})$$

and let $F_j = \ker(x^*)^j$ for $0 \leq j \leq r$. Obviously $x^*F_j \subset F_{j-1}$ for $1 \leq j \leq r$ and since x and x^* commute, $xF_j \subset F_j$ for $1 \leq j \leq r$. This shows that (x, x^*) is $*$-semi-nilpotent. We proved that the left-hand-side contains the right-hand-side. Now, if $(x, x^*) \in \tilde{\Lambda}_{\mathbf{d}}$, x^* is nilpotent by definition. Let $\mathcal{O} \subset E^{\mathrm{op}}_{C_n,\mathbf{d}}$ be its orbit. The condition $\mu_{\mathbf{d}}(x, x^*) = 0$ implies that $(x, x^*) \in T^*_{\mathcal{O}} E^{\mathrm{op}}_{C_n,\mathbf{d}}$, proving the reverse inclusion. □

Remark 5.7.3. Proposition 5.7.2 provides an alternative proof that $\tilde{\Lambda}_{\mathbf{d}}$ is Lagrangian.

Theorem 5.7.4. *The singular supports of the sheaves of the extended Hall category* $\tilde{\mathscr{Q}}_{\mathbf{d}}$ *are unions of irreducible components of* $\tilde{\Lambda}_{\mathbf{d}}$.

Proof. The proof is an easy adaptation of [Lus91, Section 13]. □

5.7.3 Explicit description

We give a description of the simple perverse sheaves in the category $\tilde{\mathcal{P}}_{\mathbf{d}}$ in the spirit of [Lus92] and [LL07]. Let $\mathbf{d} \in \mathbf{N}^{\mathbf{Z}/n\mathbf{Z}}$. We consider the Fourier-Sato transform Φ which reverse all the arrows of C_n. The Fourier-Sato transforms of perverse sheaves on $E_{C_n,\mathbf{d}}$ are perverse sheaves on $E^{\mathrm{op}}_{C_n,\mathbf{d}}$. Moreover, the Fourier-Sato transform of a $G_{\mathbf{d}}$-equivariant perverse sheaf is again $G_{\mathbf{d}}$-equivariant.

Lemma 5.7.5. *The Fourier transforms of the simple perverse sheaves in* $\tilde{\mathcal{P}}_{\mathbf{d}}$ *are intersection cohomology complexes* $\mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ *of nilpotents orbits* $\mathcal{O} \subset E^{\mathrm{nil}}_{C_n,\mathbf{d}}$.

Proof. Let $\mathscr{F}$ be a simple perverse sheaf in $\tilde{\mathcal{P}}_{\mathbf{d}}$. Its singular support is contained in $\tilde{\Lambda}_{\mathbf{d}}$ by Theorem 5.7.4. The support of $\Phi\mathscr{F}$ is the projection of $SS(\mathscr{F})$ to $E^{\mathrm{op}}_{C_n,\mathbf{d}}$ by Theorem 5.D.3. It is contained in $E^{\mathrm{nil,op}}_{C_n,\mathbf{d}}$. This proves that $\Phi\mathscr{F} = \mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ for some nilpotent orbit $\mathcal{O} \subset E^{\mathrm{op}}_{C_n,\mathbf{d}}$. □

We now give a combinatorial parametrization of nilpotent orbits of cyclic quiver.

Lemma 5.7.6. *Nilpotent orbits in* $E_{C_n,\mathbf{d}}$ *are parametrized by pairs* (N, λ) *where* N *is aperiodic and* λ *is a partition such that* $\dim N + |\lambda|\delta = \mathbf{d}$. *The nilpotent orbit corresponding to* (N, λ) *is the* *orbit of the representation*

$$N \oplus \bigoplus_{\substack{r\geq 1\\ i\in \mathbf{Z}/n\,\mathbf{Z}}} I_{i,\lambda_r}.$$

Proof. Let M be a nilpotent representation of C_n of dimension $\mathbf{d}$. We can write it $M = N \oplus P$ where N is aperiodic and P is completely periodic, meaning that $P = N_{\mathbf{m}}$ for a multipartition $\mathbf{m} : \mathbf{Z}/n\,\mathbf{Z} \to \mathscr{P}$ such that $\mathbf{m}(i) = \mathbf{m}(j)$ for any $i, j \in \mathbf{Z}/n\,\mathbf{Z}$. Therefore, the data of $\mathbf{m}$ is equivalent to the data of a partition $\lambda = \mathbf{m}(0)$. Moreover, the total dimension of P is $\sum_{i\in\mathbf{Z}/n\,\mathbf{Z}}(\dim P)_i = n|\lambda|$ and by periodicity, $(\dim P)_i = (\dim P)_j$ for any $i, j \in \mathbf{Z}/n\,\mathbf{Z}$. As a consequence, $|\lambda| = (\dim P)_0$ and $\dim N + |\lambda|\delta = \mathbf{d}$. Putting these facts together, we obtain the lemma. □

We are ready to describe extended Lusztig perverse sheaves. Let (N, μ) be a pair with N the isoclass of a nilpotent aperiodic representation and μ regular semisimple such that $\dim N + \delta \dim \mu = \mathbf{d}$. Let $d \in \mathbf{N}$ such that $\dim \mu = d\delta$. We can describe te stratum $\Xi(N, \mu)$:

$$\Xi(N,\mu) = \{x \in E_{C_n,\mathbf{d}} \mid x \simeq N \oplus \bigoplus_{j=1}^{d} J_1(x_j) \text{ for some } (x_1,\dots,x_d) \in (\mathbf{C}^*)^d \setminus \Delta\}.$$

Let $\tilde{\Xi}(N,\mu) = \{(x, x_1, \dots, x_r) \in E_{C_n,\mathbf{d}} \times ((\mathbf{C}^*)^d \setminus \Delta) \mid x \simeq N \oplus \bigoplus_{j=1}^{d} J_1(x_j)\}$. The map

$$\begin{array}{cccc} \pi_{N,\mu} & : & \tilde{\Xi}(N,\mu) & \to & \Xi(N,\mu) \\ & & (x, \underline{x}) & \mapsto & x \end{array}$$

is a $\mathfrak{S}_d$ cover. Therefore, we obtain a family $(\mathscr{L}_{N,\lambda})_{\lambda\in\mathscr{P}_d}$ of local systems on $\Xi(N, \mu)$ indexed by partition of d. The main theorem of this section is the following.

Theorem 5.7.7. *The simple perverse sheaves in $\tilde{\mathcal{P}}_{\mathbf{d}}$ are the intersection cohomology complexes $\mathcal{IC}(\Xi(N,\mu), \mathscr{L}_{N,\lambda})$ for (N, μ) and λ such that N is nilpotent aperiodic, μ is regular semisimple,* $\dim \mu = d$, $\dim N + \dim \mu = \mathbf{d}$ *and λ is a partition of d.*

Proof. Let $\mathbf{d}_N = \dim N$. Let $\underline{\mathbf{d}}_N$ be a discrete flag-type given by Theorem 5.3.4 for the orbit $\mathcal{O}_N$. We define the flag-type $\underline{\mathbf{d}} = (\underline{\mathbf{d}}_N, \delta, \dots, \delta)$ of $\mathbf{d}$ (there is d copies of δ). Then we obtain the projective morphism $\pi_{\underline{\mathbf{d}}} : \tilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to E_{\mathbf{d}}$ whose image is $\overline{\Xi(N,\mu)}$. Moreover, the restriction of $\pi_{\underline{\mathbf{d}}}$ to

$$\pi_{\underline{\mathbf{d}}}^{-1}(\Xi(N,\mu)) \to \Xi(N,\mu)$$

is a $\mathfrak{S}_d$-covering. Hence,

$$\bigoplus_{\lambda\in\mathscr{P}_d} \mathcal{IC}(\mathscr{L}_{N,\lambda})$$

is a direct factor of $(\pi_{\underline{\mathbf{d}}})_*\underline{\mathbf{C}}$. This proves that the perverse sheaves defined in the theorem are extended Lusztig sheaves.

Now, all the perverse sheaves defined are pairwise non-isomorphic. Combining Lemma 5.7.8 and the fact that this sheaves are parametrized by the same set as nilpotent orbits (Lemma 5.7.6), it follows that the theorem gives a complete description of the extended Lusztig category.

□

We will call the local systems $\mathscr{L}_{N,\lambda}$ which appear during this process *extended Lusztig local systems.*

A a corollary of the proof of Theorem 5.7.7, we obtain the following result.

Corollary 5.7.8. *The Fourier-Sato transforms of the simple perverse sheaves in $\tilde{\mathcal{P}}_{\mathbf{d}}$ are exactly the intersection cohomology complexes $\mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ for nilpotents orbits $\mathcal{O} \subset E^{\mathrm{nil}}_{C_n,\mathbf{d}}$.*

Consider the map

$$\chi_{N,\mu} : \Xi(\mu) \to S^d(\mathbf{C}^*) \setminus \Delta$$

defined in Section 5.2.7. We have a cartesian diagram

$$\begin{tikzcd} \tilde{\Xi}(N,\mu) \arrow[r, "\pi_{N,\mu}"] \arrow[d, "\tilde{\chi}_{N,\mu}"'] & \Xi(N,\mu) \arrow[d, "\chi_{N,\mu}"] \\ (\mathbf{C}^*)^d \setminus \Delta \arrow[r, "\pi_d"] & S^d(\mathbf{C}^*) \setminus \Delta \end{tikzcd} \tag{5.7.1}$$

Lemma 5.7.9. *A $G_{\mathbf{d}}$-equivariant local system $\mathscr{L}$ on $\Xi(N,\mu)$ is the pull-back of a local system $\mathscr{L}'$ on $S^d(\mathbf{C}^*) \setminus \Delta$. The local system $\mathscr{L}'$ is unique up to isomorphism. Moreover, $\mathscr{L}$ is an extended Lusztig local system if and only if $\pi_d^*\mathscr{L}'$ is trivial.*

Proof. We postpone the proof to Section 5.B.3. □

5.7.4 Microlocal characterization of sheaves in the extended Hall category

The following result is a statement of the main theorem of this chapter, Theorem 5.1.1, for cyclic quivers and the extended Hall category.

Theorem 5.7.10. *Let $\mathscr{F} \in \mathrm{Perv}_{G_{\mathbf{d}}}(E_{\mathbf{d}})$ be a simple $G_{\mathbf{d}}$-equivariant perverse sheaf on $E_{\mathbf{d}}$ such that $SS(\mathscr{F}) \subset \tilde{\Lambda}_{\mathbf{d}}$. Then, $\mathscr{F}$ is in $\tilde{\mathcal{P}}_{\mathbf{d}}$.*

Proof. Since the strata $\Xi(N,\mu)$ defined is Section 5.2.6 are $\mathbf{C}^*$-invariant, by Corollary 5.D.2, $\mathscr{F}$ is monodromic. By Theorem 5.D.3 and the definition of $\tilde{\Lambda}_{\mathbf{d}}$, its Fourier-Sato transform $\Phi\mathscr{F}$ is a simple perverse sheaf on $E^{\mathrm{op}}_{C_n,\mathbf{d}}$ with nilpotent support. Therefore, $\mathscr{F} = \mathcal{IC}(\mathcal{O}, \underline{\mathbf{C}})$ for some nilpotent orbit $\mathcal{O} \subset E^{\mathrm{op}}_{C_n,\mathbf{d}}$. By Theorem 5.7.8, $\mathscr{F}$ is in $\tilde{\mathcal{P}}_{\mathbf{d}}$. □

5.7.5 Restriction of perverse sheaves to a neighbourhood of nilpotent representations

Define the restriction of the extended nilpotent variety

$$\tilde{\Lambda}^{<1}_{\mathbf{d}} = \bigcup_{(N,\mu)} \overline{T^*_{\Xi^{<1}(N,\mu)} E^{<1}_{\mathbf{d}}}.$$

Let

$$j^{<1}_{\mathbf{d}} : E^{<1}_{\mathbf{d}} \to E_{\mathbf{d}}$$

be the inclusion.

Lemma 5.7.11. *The restriction of perverse sheaves $(j_{\mathbf{d}}^{<1})^*$ induces an equivalence of categories*

$$(j_{\mathbf{d}}^{<1})^* : \mathrm{Perv}_{G_{\mathbf{d}}}(E_{\mathbf{d}}, \tilde{\Lambda}_{\mathbf{d}}) \to \mathrm{Perv}_{G_{\mathbf{d}}}(E_{\mathbf{d}}^{<1}, \tilde{\Lambda}_{\mathbf{d}}^{<1}).$$

Proof. We construct a quasi inverse to $(j_{\mathbf{d}}^{<1})^*$. Let $\mathscr{F}^{<1} \in \mathrm{Perv}_{G_{\mathbf{d}}}(E_{\mathbf{d}}^{<1}, \tilde{\Lambda}_{\mathbf{d}}^{<1})$. There exists a stratum $\Xi^{<1}(N,\mu)$ and a local system $\mathscr{L}^{<1}$ on it such that $\mathscr{F} = \mathcal{IC}(\mathscr{L})$. But since $\Xi(N,\mu)$ and $\Xi^{<1}(N,\mu)$ have isomorphic fundamental groups, $\mathscr{L}^{<1}$ can be extended to a local system $\mathscr{L}$ on $\Xi(N,\mu)$. We set $\mathscr{F} = \mathcal{IC}(\mathscr{L})$. Moreover, since $\mathscr{L}$ is $\mathbf{C}^*$-equivariant, $\mathscr{F}$ is also $\mathbf{C}^*$-equivariant, which implies that $SS(\mathscr{F}) \subset \tilde{\Lambda}_{\mathbf{d}}$. □

5.8 Neighbourhood of non-homogeneous tubes in the representation spaces of affine quivers

We recall the construction of the Hall functor and the Hall morphism appearing in [LL07, Section 3]. Let Q be an affine quiver. Let $D \subset \mathbf{P}^1(\mathbf{C})$ be the subset indexing the non-homogeneous tubes of Q. In the sequel, we assume that D is non-empty. Let $\mathcal{C}$ be a non-homogeneous tube of Q of period $p > 1$ (Theorem 5.2.5) corresponding to $t \in D$. Let $N_0, \dots, N_{p-1}$ be a full set of representatives of simple regular representations in the tube $\mathcal{C}$ ordered such that for any $s \in \mathbf{Z}/p\mathbf{Z}$, $\mathrm{Ext}^1(N_s, N_{s+1}) \simeq \mathbf{C}$. Let us fix for each $s \in \mathbf{Z}/p\mathbf{Z}$ a nontrivial extension:

$$0 \to N_{s+1} \to E_s \to N_s \to 0.$$

As vector spaces, $E_s = N_{s+1} \oplus N_s$. The extension E_s gives linear maps

$$z_{\alpha,s} : (N_s)_i \to (N_{s+1})_j$$

for any $\alpha : i \to j \in \Omega$. The representation N_s of Q is given by the linear maps $a_{s,\alpha} : (N_s)_i \to (N_s)_j$ for $\alpha : i \to j \in \Omega$. Write $\mathbf{d}_s = \dim N_s$ for $s \in \mathbf{Z}/p\mathbf{Z}$. We have the map between dimension lattices

$$\begin{array}{rcccc} \dim & : & \mathbf{Z}^{\mathbf{Z}/p\mathbf{Z}} & \to & \mathbf{Z}^I \\ & & \mathbf{e} = (e_s) & \mapsto & \tilde{\mathbf{e}} = \sum_{s \in \mathbf{Z}/p\mathbf{Z}} e_s \mathbf{d}_s \,. \end{array}$$

Let $\mathbf{e} = (e_s)_{s \in \mathbf{Z}/p\mathbf{Z}} \in \mathbf{Z}^{\mathbf{Z}/p\mathbf{Z}}$. Let $V = (V_s)_{s \in \mathbf{Z}/p\mathbf{Z}}$ be a fixed $\mathbf{Z}/p\mathbf{Z}$-graded vector space. Let W be the underlying I-graded vector space of $\bigoplus_{s \in \mathbf{Z}/p\mathbf{Z}} V_s \otimes N_s$. We construct a fully faithful functor

$$\begin{array}{rcccc} F & : & \mathrm{Rep}_{C_p}(k) & \to & \mathrm{Rep}_Q(k) \\ & & (V_s, v_s)_{s \in \mathbf{Z}/p\mathbf{Z}} & \mapsto & (W, x) \end{array}$$

as follows: for $\alpha : i \to j \in \Omega$,

$$x_\alpha \quad : \quad W_i \quad \to \quad W_j$$

is defined for any $v \otimes n \in V_s \otimes (N_s)_i$ by

$$x_\alpha(v \otimes n) = v \otimes a_{s,\alpha}(n) + v_s(v) \otimes z_{\alpha,s}(n).$$

At the level of representation varieties, the same formulas give a morphism:

$$\iota_E : E_{C_p}(V) \to E_Q(W).$$

Define now the closed embedding of algebraic groups:

$$\begin{array}{cccc} \iota_G & : & GL(V) & \to & GL(W) \\ & & (g_s)_{s\in \mathbf{Z}/p\mathbf{Z}} & \mapsto & (\sum_{s\in\mathbf{Z}/p\mathbf{Z}} g_s \otimes \mathrm{id}_{(N_s)_i})_{i\in I}. \end{array}$$

We obtain a locally closed immersion

$$j_V : E_{C_p}(V) \times^{GL(V)} GL(W) \to E_Q(W).$$

We let $\mathbf{d} = \dim V$ so that $\tilde{\mathbf{d}} = \dim W$ and we see ι_E, ι_G and $j_{\mathbf{d}} := j_V$ as morphism between the corresponding representation varieties:

$$\iota_E : E_{C_p,\mathbf{d}} \to E_{Q,\tilde{\mathbf{d}}},$$

$$\iota_G : G_{C_p,\mathbf{d}} \to G_{W,\tilde{\mathbf{d}}},$$

and

$$j_{\mathbf{d}} : E_{C_p,\mathbf{d}} \times^{G_{C_p,\mathbf{d}}} G_{Q,\tilde{\mathbf{d}}} \to E_{Q,\tilde{\mathbf{d}}}.$$

We will also need restrictions of ι_E and $j_{\mathbf{d}}$:

$$\iota_E^{<1} : E_{C_p,\mathbf{d}}^{<1} \to E_{Q,\tilde{\mathbf{d}}}$$

and

$$j_{\mathbf{d}}^{<1} : E_{C_p,\mathbf{d}}^{<1} \times^{G_{C_p,\mathbf{d}}} G_{Q,\tilde{\mathbf{d}}} \to E_{Q,\tilde{\mathbf{d}}}.$$

From [LL07, Lemma 3.2 and Lemma 3.3], we have the following proposition.

Proposition 5.8.1. *F is an exact fully faithful functor. It takes value in the full subcategory of regular representations of Q. By restriction, it induces a equivalence of categories*

$$\mathrm{Rep}_{C_p}^{\mathrm{nil}}(k) \to \mathcal{C}\,.$$

(Recall that $\mathcal{C}$ is the chosen non-homogeneous tube of Q). Let $\mathrm{Rep}_Q^{reg\,\mathcal{C}}(k)$ be the full subcategory of objects of $\mathrm{Rep}_Q(k)$ isomorphic to a direct sum $N \oplus R$ where N is an object of $\mathcal{C}$ and R is regular homogeneous. Let $\mathrm{Rep}_{C_p}^{\mathcal{C}}(k)$ be the full subcategory of objects of $\mathrm{Rep}_{C_p}(k)$ whose image by F belongs to the subcategory $\mathrm{Rep}_Q^{reg\,\mathcal{C}}(k)$ of regular representations of Q whose non-homogeneous

direct summands lie in $\mathcal{C}$. Then F restricts to an equivalence of categories

$$\mathrm{Rep}^{\mathcal{C}}_{C_p}(k) \to \mathrm{Rep}^{reg\,\mathcal{C}}_{Q}(k).$$

Remark 5.8.2. The image of F contains all homogeneous regular tubes and misses exactly one inhomogeneous tube (if Q has three inhomogeneous tubes, that is Q is of type D or E).

At the level of representation varieties, we consider the subvariety $E^{\{t\}}_{Q,\tilde{\mathbf{d}}}$ of $E_{Q,\tilde{\mathbf{d}}}$ defined in Section 5.2.5. It is an open subvariety contained in the image of $j_{\mathbf{d}}$. The subvariety $j_{\mathbf{d}}^{-1}(E^{\{t\}}_{Q,\tilde{\mathbf{d}}}) = \iota_E^{-1}(E^{\{t\}}_{Q\tilde{\mathbf{d}}}) \times^{G_{\mathbf{d}}} G_{\tilde{\mathbf{d}}}$ is open. The open subset $\iota_E^{-1}(E^{\{t\}}_{Q,\tilde{\mathbf{d}}})$ consists of the elements $x \in E_{C_p,\mathbf{d}}$ which avoid exactly one nonzero eigenvalue if Q has three non-homogeneous tubes, and coincide with $E_{C_p,\mathbf{d}}$ otherwise. In this case, by choosing properly the extensions E_s, we can assume that the avoided eigenvalue is 1. We let

$$D' = \begin{cases} \{0\} \text{ if } Q \text{ has one or two non-homogeneous tubes} \\ \{0,1\} \text{ else.} \end{cases}$$

In dimension δ, we obtain a cartesian diagram

$$\begin{tikzcd} E^{\mathbf{C}\setminus D'}_{C_p,\delta} \times^{G_{C_p,\delta}} G_{Q,\delta} \arrow[r] \arrow[d] & E^{\mathrm{reghom}}_{Q,\delta} \arrow[d] \\ \mathbb{A}^1 \setminus \{0,1\} \arrow[r] & \mathbf{P}^1 \setminus D \end{tikzcd}$$

whose horizontal arrows are isomorphisms.

We also have the cartesian diagram

$$\begin{tikzcd} E^{\mathbf{C}\setminus\{1\}}_{C_p,\delta} \times^{G_{C_p,\delta}} G_{Q,\delta} \arrow[r] \arrow[d] & E^{\{t\}}_{Q,\delta} \arrow[d] \\ (\mathbb{A}^1 \setminus \{1\}) \arrow[r] & (\mathbf{P}^1 \setminus D) \cup \{t\} \end{tikzcd}$$

whose horizontal rows are isomorphisms. If $N = F(N')$ is a non-homogeneous regular representation of Q for some nilpotent representation N' of C_p, and μ is regular semisimple such that $\mathbf{d} = \dim N + (\dim \mu)\delta$, we obtain a cartesian square:

$$\begin{tikzcd} \Xi(N',\mu) \arrow[r] \arrow[d] & \Xi(N,\mu) \arrow[d] \\ S^d(\mathbb{A}^1 \setminus \{1\}) \setminus \Delta \arrow[r] & S^d((\mathbf{P}^1 \setminus D) \cup \{t\}) \setminus \Delta \end{tikzcd}$$

where $d = \dim \mu$.

We will also need the open immersion

$$j^{<1}_{\mathbf{d}} : E^{<1}_{C_p,\mathbf{d}} \times^{G_{C_p,\mathbf{d}}} G_{Q,\tilde{\mathbf{d}}} \to E^{reg}_{Q,\tilde{\mathbf{d}}}.$$

A crucial property is given by the following proposition whose proof is obvious from Proposition 5.4.5.

Proposition 5.8.3. *The restriction of the Lusztig nilpotent variety $\Lambda_{\mathbf{d}}$ by $j_{\mathbf{d}}^{<1}$ is*

$$(j_{\mathbf{d}}^{<1})^*\Lambda_{\mathbf{d}} = \bigcup_{(N,\mu)} \overline{T^*_{\Xi^{<1}(N,\mu)\times^{G_{C_p,\mathbf{d}}}G_{Q,\tilde{\mathbf{d}}}}(E^{<1}_{C_p,\mathbf{d}} \times^{G_{C_p,\mathbf{d}}} G_{Q,\tilde{\mathbf{d}}})}.$$

Consequently, if $\mathscr{F}$ is a perverse sheaf on $E_{Q,\tilde{\mathbf{d}}}$ such that $SS(\mathscr{F}) \subset \Lambda_{\tilde{\mathbf{d}}}$, then

$$\mathscr{G} := (\iota_E^{<1})^*\mathscr{F}[\dim G_{C_p,\mathbf{d}} - \dim G_{Q,\tilde{\mathbf{d}}}]$$

is a perverse sheaf on $E^{<1}_{C_p,\mathbf{d}}$ such that $SS(\mathscr{G}) \subset \tilde{\Lambda}^{<1}_{\mathbf{d}}$.

5.9 Proof of the main theorem in the affine case

5.9.1 Local systems on a stratum of the cyclic quiver

We record the following results, which are an easy consequence of the action of $\mathbf{C}^*$ on all the representation varieties and stratifications considered.

Lemma 5.9.1. *Let $\Xi(N,\mu) \subset E_{C_p,\mathbf{d}}$ be a stratum. Then the inclusion $\Xi^{<1}(N,\mu) \to \Xi(N,\mu)$ induces an isomorphism between the fundamental groups $\pi_1(\Xi(N,\mu))$ and $\pi_1(\Xi^{<1}(N,\mu))$.*

Therefore, there is a canonical bijective correspondance between isomorphism classes of local systems $\mathscr{L}$ on $\Xi(N,\mu)$ and $\mathscr{L}^{<1}$ on $\Xi^{<1}(N,\mu)$.

Lemma 5.9.2. *Let $\mathscr{L}^{<1}$ be a local system on $\Xi^{<1}(N,\mu)$ such that $SS(j_{!*}^{<1}\mathscr{L}^{<1}) = \bigcup_{S\in\mathcal{S}^{<1}} \overline{T^*_S E^{<1}_{C_p,\mathbf{d}}}$ where $\mathcal{S}^{<1}$ is a set of strata of the form $\Xi^{<1}(N,\mu)$. Then, if $\mathscr{L}$ denotes the local system corresponding to $\mathscr{L}^{<1}$ and $\mathcal{S}$ the set of strata corresponding to $\mathcal{S}^{<1}$,*

$$SS(j_{!*}\mathscr{L}) = \bigcup_{S\in\mathcal{S}} \overline{T^*_S E_{C_p,\mathbf{d}}}.$$

5.9.2 Proof of Theorem 5.1.1 for affine quivers

Let $\mathscr{F}$ be a $G_{\mathbf{d}}$-equivariant perverse sheaf on $E^{reg}_{\mathbf{d}}$ whose singular support is contained in $\Lambda^{reg}_{\mathbf{d}}$. Then, by Corollary 5.4.7, it is supported on $\overline{\Xi(N,\mu)}$ for some non-homogeneous regular representation N and μ regular. Let $d = \dim\mu$. Our argument proceeds in two steps: first we prove that μ is regular semisimple; then, since $\Xi(N,\mu)$ is smooth and $(\mathscr{F}_{\Xi(N,\mu)}) = T^*_{\Xi(N,\mu)}E_{\mathbf{d}}$, by Lemma 5.C.5, $\mathscr{F} = \mathcal{IC}(\mathscr{L})$ for a local system $\mathscr{L}$ on $\Xi(N,\mu)$; we then prove that $\mathscr{L}$ is a Lusztig local system on $\Xi(N,\mu)$.

Let $t \in D$ be a non-homogeneous tube. of period $p > 1$. Write $N = N_t \oplus N'$ where N_t is in the tube indexed by t and none of the direct summands of N' belongs to the tube indexed by t. Write $\mathbf{d}' = \dim N_t + \dim\mu\delta$. By the isomorphism

$$(\mathcal{O}_{N'} \times E^{\{t\}}_{\mathbf{d}'}) \times^{G_{\mathbf{d}_{N'}}\times G_{\mathbf{d}'}} G_{\mathbf{d}} \to E^{\{t\}}_{[N'],\mathbf{d}'},$$

we obtain a perverse sheaf $\mathscr{G}$ on $E_{\mathbf{d}'}^{\{t\}}$ such that $SS(\mathscr{G}) \subset (\Lambda_{\mathbf{d}'})_{E_{\mathbf{d}'}^{\{t\}}}$ (see Lemma 5.4.8). By Proposition 5.8.1, there exists $\mathbf{e} \in \mathbf{N}^{\mathbf{Z}/p\mathbf{Z}}$ such that $\tilde{\mathbf{e}} = \mathbf{d}'$.

Let $i_{\mathbf{e}}^{<1} : E_{C_p,\mathbf{e}}^{<1} \to E_{Q,\tilde{\mathbf{e}}}$ be the composition of $j_{\mathbf{e}}^{<1}$ with the closed immersion $E_{C_p\mathbf{e}}^{<1} \to E_{C_p,\mathbf{e}}^{<1} \times^{G_{C_p,\mathbf{e}}} G_{Q,\tilde{\mathbf{e}}}$. Then

$$(i_{\mathbf{e}}^{<1})^*\mathscr{G}[\dim G_{C_p,\mathbf{d}} - \dim G_{Q,\tilde{\mathbf{e}}}]$$

is a perverse sheaf on $E_{C_p,\mathbf{e}}^{<1}$ and by Proposition 5.8.3 and Lemma 5.C.4, $SS(\mathscr{G}) \subset \tilde{\Lambda}_{\mathbf{e}}^{<1}$. By Lemma 5.7.11, it can be extended to a perverse sheaf $\mathscr{G}'$ on $E_{C_p,\mathbf{e}}$ such that $SS(\mathscr{G}') \subset \tilde{\Lambda}_{\mathbf{e}}$. By Theorem 5.7.10, it is a Lusztig sheaf in the extended category. Therefore, by Theorem 5.7.7, μ has to be regular semisimple. Now, as stated above, $\mathscr{F} = \mathcal{IC}(\mathscr{L})$ for some local system $\mathscr{L}$ on $\Xi(N,\mu)$. By Lemma 5.3.8, there exists a local system $\tilde{\mathscr{L}}$ on $S^d\,\mathbf{P}_1^{\mathrm{hom}} \setminus \Delta$ such that $\mathscr{L} \simeq \chi_{N,\mu}^*\tilde{\mathscr{L}}$, where $\chi_{N,\mu}$ is the support map whose construction is sketched in Section 5.2.5. We have to prove that $\pi_d^*\tilde{\mathscr{L}}$ is trivial (recall that $\pi_d : (\mathbf{P}_1^{\mathrm{hom}})^d \setminus \Delta \to S^d\,\mathbf{P}_1^{\mathrm{hom}} \setminus \Delta$ is the $\mathfrak{S}_d$-covering).

Again by Lemma 5.C.5, the perverse sheaf $\mathscr{G}'$ is given by a local system $\mathscr{L}'$ on $\Xi(N_t',\mu)$, where N_t' is a representation of C_p such that $F(N_t') \simeq N$ and F is the functor of Section 5.8. Moreover, by Lemma 5.7.9, $\pi_{N_t',\mu}^*\mathscr{L}'$ is the trivial local system. By Lemma 5.7.9, $\mathscr{L}' = \chi_{N_t',\mu}\tilde{\mathscr{L}}'$ for some local system $\tilde{\mathscr{L}}'$ on $S^d(\mathbf{C}^*) \setminus \Delta$. By the cartesian diagram

$$\begin{tikzcd}
\tilde{\Xi}(N_t',\mu) \arrow[r, "\pi_{N_t',\mu}"] \arrow[d, "\tilde{\chi}_{N_t',\mu}"'] & \Xi(N_t',\mu) \arrow[d, "\chi_{N_t',\mu}"] \\
(\mathbf{C}^*)^d \setminus \Delta \arrow[r, "\pi_d"] & S^d(\mathbf{C}^*) \setminus \Delta
\end{tikzcd},$$

see also Lemma 5.7.9, $\pi_d^*\tilde{\mathscr{L}}'$ is the trivial local system.

By the cartesian diagram

$$\begin{tikzcd}
\Xi^{<1}(N_t',\mu) \arrow[r, "h"] \arrow[d, "\chi_{N_t',\mu}"'] & \Xi(N,\mu) \arrow[d, "\chi_{N,\mu}"] \\
S^dD^*(0,1) \arrow[r, "i_d"] & S^d\,\mathbf{P}_1^{\mathrm{hom}} \setminus \Delta
\end{tikzcd}$$

where i_d is the natural injection, and the upper horizontal arrow h is the composition of the injection $\Xi^{<1}(N_t',\mu) \to \Xi(N_t',\mu)$ and the injection given by ι_E, we have that $h^*\mathscr{L} = \mathscr{L}'$, and therefore, the hypothesis of Lemma 5.A.4 is verified for $\tilde{\mathscr{L}}$ and $t \in D$. By repeating this for all $t \in D$, it shows that $\mathscr{L}$ is trivialized by π_d at all points of D. By Lemma 5.A.4, $(\pi_d)^*\mathscr{L}$ is trivial. By Lemma 5.3.8, $\mathscr{L}$ is a Lusztig local system.

5.10 The case of quivers with loops

In this section, we show that the class of simple perverse sheaves defined by Lusztig on representation spaces of quivers is determined by a nilpotent condition on the singular support for the g-loop quiver. The method is completely different than that for finite type or affine quivers.

5.10.1 Smallness of some morphisms for negative quivers

Let Q be quiver. For a flag-type $\underline{\mathbf{d}}$, we let $E_{\underline{\mathbf{d}}}$ be the image of the proper morphism $\pi_{\underline{\mathbf{d}}}$ and $E_{\underline{\mathbf{d}}}^{\mathrm{nil}}$ the image of $\pi_{\underline{\mathbf{d}}}^{\mathrm{nil}}$. For convenience, we let $E_{\underline{\mathbf{d}}}^{\flat}=E_{\underline{\mathbf{d}}}$ if $\flat=1$ or $\flat=\emptyset$ and $E_{\underline{\mathbf{d}}}^{\flat}=E_{\underline{\mathbf{d}}}^{\mathrm{nil}}$ if $\flat=\mathrm{nil}$ or $\flat=(\mathrm{nil},1)$. Technically speaking, if $\flat=1$ or $\flat=(\mathrm{nil},1)$, $E_{\underline{\mathbf{d}}}^{\flat}$ exists only if $\underline{\mathbf{d}}$ is a discrete flag-type. These are closed irreducible subvarieties of $E_{Q,\mathbf{d}}$. Define $\mathscr{C}_{\mathbf{d}}$ as the set of flag-types of dimension $\mathbf{d}$ and $\mathscr{C}_{\mathbf{d}}^1$ as the set of discrete flag-types of dimension $\mathbf{d}$. To ease the notation, for $\flat\in\{\mathrm{nil},\emptyset,(\mathrm{nil},1),1\}$, we let $\mathscr{C}_{\mathbf{d}}^{\flat}=\mathscr{C}_{\mathbf{d}}$ if $\flat=\mathrm{nil}$ or $\flat=\emptyset$ and $\mathscr{C}_{\mathbf{d}}^{\flat}=\mathscr{C}_{\mathbf{d}}^1$ else and the same remark as before applies. Similarly, $\pi_{\underline{\mathbf{d}}}^{\flat}=\pi_{\underline{\mathbf{d}}}$ if $\flat=\emptyset$ or $\flat=1$ and $\pi_{\underline{\mathbf{d}}}^{\flat}=\pi_{\underline{\mathbf{d}}}^{\mathrm{nil}}$ if $\flat=\mathrm{nil}$ or $\flat=(\mathrm{nil},1)$.

A quadratic form

Let $\mathbf{d}\in\mathbf{N}^I$ be a dimension vector and $\underline{\mathbf{d}}=(\mathbf{d}_1,\dots,\mathbf{d}_r)$ a flag-type of dimension $\mathbf{d}$ (that is, $\sum_{j=1}^r\mathbf{d}_j=\mathbf{d}$). In this section, we reformulate the main result of [Lus93] in the case where the flag type $\underline{\mathbf{d}}$ is not necessarily discrete and for both morphisms $\pi_{\underline{\mathbf{d}}}$ and $\pi_{\underline{\mathbf{d}}}^{\mathrm{nil}}$ (Lusztig considered the case of $\pi_{\underline{\mathbf{d}}}$). Recall that $\mathcal{F}_{\underline{\mathbf{d}}}$ denotes the flag variety of flags of type $\underline{\mathbf{d}}$ inside $\mathbf{C}^{\mathbf{d}}$. As in [BLM90] in the non-graded case or in [Lus93] for quivers and discrete flag-types, the relative position of two flags $\underline{F}$ and $\underline{F'}$ of type $\underline{\mathbf{d}}$ is encoded in an array $z=(z^{pq})_{\substack{1\leq p\leq r\\1\leq q\leq r}}$, such that for any $1\leq s\leq r$,

$$\sum_{1\leq p\leq r}z^{ps}=\sum_{1\leq q\leq r}z^{sq}=\mathbf{d}_s \tag{5.10.1}$$

For any $1\leq p,q\leq r$, $z^{pq}=(z_i^{pq})_{i\in I}\in\mathbf{N}^I$ is defined by

$$z^{pq}=\dim\left(\frac{F_{p-1}+F_p\cap F'_q}{F_{p-1}+F_p\cap F'_{q-1}}\right).$$

By splitting the refined flag

$$(F_{p-1}+F_p\cap F'_q)_{1\leq p,q\leq r}$$

(where tuples (p,q) are lexicographically ordered: $(p,q)\leq(p',q')\iff p<p'$ or ($p=p'$ and $q\leq q'$)), we see that $\underline{F}$ and $\underline{F'}$ are in relative position z for some z as above if and only there exists vector spaces V^{pq} such that $\dim V^{pq}=z^{pq}$, $\mathbf{C}^{\mathbf{d}}=\bigoplus_{1\leq p,q\leq r}V^{pq}$ and for any $1\leq s\leq r$,

$$F_s=\bigoplus_{1\leq q\leq r}V^{sq},\quad F'_s=\bigoplus_{1\leq p\leq r}V^{ps}.$$

Of course, all this applies for I-graded vector spaces (where I will denote the vertices of the quiver Q). We let $\Theta(\underline{\mathbf{d}})$ be the set of all possible relative positions between two flags of type $\underline{\mathbf{d}}$, that is the set of arrays $z=(z^{pq})_{1\leq p,q\leq r}$ satisfying the equalities (5.10.1). The variety $\mathcal{F}_{\underline{\mathbf{d}}}\times\mathcal{F}_{\underline{\mathbf{d}}}$ is then stratified by the relative position:

$$\mathcal{F}_{\underline{\mathbf{d}}}\times\mathcal{F}_{\underline{\mathbf{d}}}=\bigsqcup_{z\in\Theta(\underline{\mathbf{d}})}(\mathcal{F}_{\underline{\mathbf{d}}}\times\mathcal{F}_{\underline{\mathbf{d}}})_z$$

where $(\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}})_z$ denotes the locally closed subvariety of pairs $(\underline{F}, \underline{F'})$ in relative position z. Recall the proper morphism $\pi^\sharp_{\underline{\mathbf{d}}} : \tilde{\mathcal{F}}_{\underline{\mathbf{d}}} \to E_{\mathbf{d}}$ for $\sharp \in \{\emptyset, \mathrm{nil}\}$. We have a morphism $\varphi_{\underline{\mathbf{d}}} : \tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}} \to \mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$. Therefore, we can stratify $\tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}}$ by the pull-back of the stratification of $\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}}$:

$$\tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}} = \bigsqcup_{z \in \Theta(\underline{\mathbf{d}})} (\tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}})_z.$$

In first lecture, the reader can directly jump to the statement of Corollary 5.10.4 and then to Section 5.10.1.

Lemma 5.10.1. *The stratum $(\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}})_z$ is a connected $G_{\mathbf{d}}$-homogeneous space of dimension*

$$\dim G_{\mathbf{d}} - \sum_{\substack{i \in I \\ 1 \le t \le p \le r \\ 1 \le s \le q \le r}} z_i^{pq} z_i^{ts} = \sum_{\substack{i \in I \\ 1 \le t, p \le r}} (\mathbf{d}_p)_i (\mathbf{d}_t)_i - \sum_{\substack{i \in I \\ 1 \le t \le p \le r \\ 1 \le s \le q \le r}} z_i^{pq} z_i^{ts}.$$

Proof. See [Lus93, §17.]. □

Lemma 5.10.2. *The morphism $\varphi_{\underline{\mathbf{d}}}$ restricts to a fiber bundle $(\tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}})_z \to (\mathcal{F}_{\underline{\mathbf{d}}} \times \mathcal{F}_{\underline{\mathbf{d}}})_z$ with fibers of dimension*

$$\sum_{\alpha : i \to j \in \Omega} \sum_{\substack{1 \le t \le p \le r \\ 1 \le s \le q \le r}} z_i^{pq} z_j^{ts}$$

if $\sharp = \emptyset$ and

$$\sum_{\alpha : i \to j \in \Omega} \sum_{\substack{1 \le t < p \le r \\ 1 \le s < q \le r}} z_i^{pq} z_j^{ts}$$

if $\sharp = \mathrm{nil}$.

Note that for $z_{\underline{\mathbf{d}}} = (\mathbf{d}_p \, \delta_{pq})_{1 \le p, q \le r}$, $(\tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}})_{z_{\underline{\mathbf{d}}}} \simeq \tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}}$ is the diagonal copy of $\tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}}$ inside $\tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^\sharp_{\underline{\mathbf{d}}}$.

Proof. See [Lus93, §17.]. Note that for $\sharp = \mathrm{nil}$, the inequalities in the sum are strict as a consequence of the nilpotency condition. □

Lemma 5.10.3. *Let $z = (z^{pq})_{1 \le p, q \le r}$ be a matrix such that for any $1 \le s \le r$,*

$$\sum_{1 \le p \le r} z^{ps} = \sum_{1 \le q \le r} z^{sq}.$$

Then,

$$\sum_{\substack{1 \le t < p \le r \\ 1 \le q \le s \le r}} z^{pq} z^{ts} = \sum_{\substack{1 \le t \le p \le r \\ 1 \le q < s \le r}} z^{pq} z^{ts} \tag{5.10.2}$$

or, equivalently,

$$\sum_{\substack{1 \le t < p \le r \\ 1 \le q \le r}} z^{pq} z^{tq} = \sum_{\substack{1 \le p \le r \\ 1 \le q < s \le r}} z^{pq} z^{ps}. \tag{5.10.3}$$

Proof. The first equality (5.10.2) can be deduced from the second one (5.10.3) by adding on both side $\sum_{\substack{1\leq t<p\leq r\\1\leq q<s\leq r}} z^{pq}z^{ts}$. To prove (5.10.3), note that $z^{pq}z^{tq}$ is symmetric in p,t. Let $L=\sum_{\substack{1\leq t<p\leq r\\1\leq q\leq r}} z^{pq}z^{tq}$ be the left-hand side of (5.10.3) and $R=\sum_{\substack{1\leq p\leq r\\1\leq q<s\leq r}} z^{pq}z^{ps}$ be its right-hand side. Then

$$\begin{aligned}
2L+\sum_{\substack{1\leq p\leq r\\1\leq q\leq r}}(z^{pq})^2 &= \sum_{1\leq p,q,t\leq r} z^{pq}z^{tq}\\
&=\sum_{1\leq q\leq r}\left(\sum_{1\leq p\leq r} z^{pq}\right)\left(\sum_{1\leq t\leq r} z^{tq}\right)\\
&=\sum_{1\leq q\leq r}\left(\sum_{1\leq p\leq r} z^{qp}\right)\left(\sum_{1\leq t\leq r} z^{qt}\right)\\
&=\sum_{1\leq p,q,t\leq r} z^{qp}z^{qt}
\end{aligned}$$

and,

$$2R+\sum_{\substack{1\leq p\leq r\\1\leq q\leq r}}(z^{pq})^2=\sum_{1\leq p,q,s\leq r} z^{pq}z^{ps}$$

so that $L=R$.

$\square$

Corollary 5.10.4. *We have the following formulas:*

$$\dim\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}-\dim(\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}\times_{E_{\mathbf{d}}}\tilde{\mathcal{F}}_{\underline{\mathbf{d}}})_z=\sum_{\alpha:i\to j\in\Omega}\sum_{\substack{1\leq t\leq p\leq r\\1\leq q<s\leq r}} z_i^{pq}z_j^{ts}-\sum_{i\in I}\sum_{\substack{1\leq t\leq p\leq r\\1\leq q<s\leq r}} z_i^{pq}z_i^{ts}$$

and

$$\dim\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}-\dim(\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}\times_{E_{\mathbf{d}}}\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z=\sum_{\alpha:i\to j\in\Omega}\sum_{\substack{1\leq t<p\leq r\\1\leq q\leq s\leq r}} z_i^{pq}z_j^{ts}-\sum_{i\in I}\sum_{\substack{1\leq t<p\leq r\\1\leq q\leq s\leq r}} z_i^{pq}z_i^{ts}$$

Proof. By Lemmas 5.10.1 and 5.10.2,

$$\dim\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}=\sum_{\alpha:i\to j\in\Omega}\sum_{1\leq t\leq p\leq r}(\mathbf{d}_p)_i(\mathbf{d}_t)_j+\sum_{\substack{i\in I\\1\leq t,p\leq r}}(\mathbf{d}_p)_i(\mathbf{d}_t)_i-\sum_{\substack{i\in I\\1\leq t\leq p\leq r}}(\mathbf{d}_p)_i(\mathbf{d}_t)_i \tag{5.10.4}$$

and

$$\dim(\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}\times_{E_{\mathbf{d}}}\tilde{\mathcal{F}}_{\underline{\mathbf{d}}})_z=\sum_{\alpha:i\to j\in\Omega}\sum_{\substack{1\leq t\leq p\leq r\\1\leq s\leq q\leq r}} z_i^{pq}z_j^{ts}+\sum_{\substack{i\in I\\1\leq t,p\leq r}}(\mathbf{d}_p)_i(\mathbf{d}_t)_i-\sum_{\substack{i\in I\\1\leq t\leq p\leq r\\1\leq s\leq q\leq r}} z_i^{pq}z_i^{ts}. \tag{5.10.5}$$

Note that

$$\sum_{\substack{1\leq t\leq p\leq r\\1\leq s\leq q\leq r}} z_i^{pq}z_j^{ts}+\sum_{\substack{1\leq t\leq p\leq r\\1\leq q<s\leq r}} z_i^{pq}z_j^{ts}=\sum_{\substack{1\leq t\leq p\leq r\\1\leq q,s\leq r}} z_i^{pq}z_j^{ts}=\sum_{1\leq t\leq p\leq r}(\mathbf{d}_p)_i(\mathbf{d}_t)_j. \tag{5.10.6}$$

Consequently, by substracting (5.10.4) and (5.10.5), using (5.10.6), we obtain the first formula of the corollary.

For the second formula, again by Lemmas 5.10.1 and 5.10.2,

$$\dim \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} = \sum_{\alpha:i\to j\in\Omega}\sum_{1\leq t<p\leq r}(\mathbf{d}_p)_i(\mathbf{d}_t)_j + \sum_{\substack{i\in I\\1\leq t,p\leq r}}(\mathbf{d}_p)_i(\mathbf{d}_t)_i - \sum_{\substack{i\in I\\1\leq t\leq p\leq r}}(\mathbf{d}_p)_i(\mathbf{d}_t)_i \tag{5.10.7}$$

and

$$\dim(\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}\times_{E_{\mathbf{d}}}\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z = \sum_{\alpha:i\to j\in\Omega}\sum_{\substack{1\leq t<p\leq r\\1\leq s<q\leq r}}z_i^{pq}z_j^{ts} + \sum_{\substack{i\in I\\1\leq t,p\leq r}}(\mathbf{d}_p)_i(\mathbf{d}_t)_i - \sum_{\substack{i\in I\\1\leq t\leq p\leq r\\1\leq s\leq q\leq r}}z_i^{pq}z_i^{ts}. \tag{5.10.8}$$

the only difference with (5.10.4) and (5.10.5) is the strict inequalities in the range of summation of the first sums of (5.10.7) and (5.10.8). Therefore, by substrating (5.10.7) and (5.10.8),

$$\begin{aligned}
\dim \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} - \dim(\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}\times_{E_{\mathbf{d}}}\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z &= \sum_{\alpha:i\to j\in\Omega}\sum_{\substack{1\leq t<p\leq r\\1\leq q\leq s\leq r}}z_i^{pq}z_j^{ts} - \sum_{\substack{i\in I\\1\leq t\leq p\leq r\\1\leq q<s\leq r}}z_i^{pq}z_i^{ts}\\
&= \sum_{\alpha:i\to j\in\Omega}\sum_{\substack{1\leq t<p\leq r\\1\leq q\leq s\leq r}}z_i^{pq}z_j^{ts} - \sum_{\substack{i\in I\\1\leq t<p\leq r\\1\leq q\leq s\leq r}}z_i^{pq}z_i^{ts}+\\
&\quad\underbrace{\sum_{\substack{i\in I\\1\leq t<p\leq r\\1\leq q\leq s\leq r}}z_i^{pq}z_i^{ts} - \sum_{\substack{i\in I\\1\leq t\leq p\leq r\\1\leq q<s\leq r}}z_i^{pq}z_i^{ts}}_{=0\text{ by Lemma 5.10.3}}\\
&= \sum_{\alpha:i\to j\in\Omega}\sum_{\substack{1\leq t<p\leq r\\1\leq q\leq s\leq r}}z_i^{pq}z_j^{ts} - \sum_{\substack{i\in I\\1\leq t<p\leq r\\1\leq q\leq s\leq r}}z_i^{ps}z_i^{ts}
\end{aligned}$$

where the last equality follows from Lemma 5.10.3. We are done. □

Smallness for negative quivers

A quiver is called *negative* provided it carries at least two loops at each vertex.

Proposition 5.10.5. *Let Q be a negative quiver. For any flag-type $\underline{\mathbf{d}}$ of dimension $\mathbf{d}$, the morphisms $\pi_{\underline{\mathbf{d}}}$ and $\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ are small and are an isomorphism over a non-empty open subset of their image.*

Proof. Write $\underline{\mathbf{d}} = (\mathbf{d}_1,\ldots,\mathbf{d}_r)$. For the smallness of $\pi_{\underline{\mathbf{d}}}$, we use Corollary 5.10.4. We have to prove that for any $z\in\Theta(\underline{\mathbf{d}})\setminus\{z_{\underline{\mathbf{d}}}\}$,

$$\dim\tilde{\mathcal{F}}_{\underline{\mathbf{d}}} - \dim(\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}\times_{E_{\mathbf{d}}}\tilde{\mathcal{F}}_{\underline{\mathbf{d}}})_z > 0.$$

For any vertex $i\in I$, we let $g_i\geq 2$ be the number of loops of Q at i. Let $z\in\Theta(\underline{\mathbf{d}})\setminus\{z_{\underline{\mathbf{d}}}\}$. Then there exists $p'\neq q'$ and $i\in I$ such that $z_i^{p'q'}\neq 0$. Choose p' minimal, so that for any $1\leq p<p'$

and $q \neq p$, $z^{pq} = 0$. We therefore have $z^{pp} = \mathbf{d}_p$ for any $1 \leq p < p'$. Since we have $\sum_{p=1}^{r} z^{pq} = \mathbf{d}_q$, for $1 \leq q < p'$, $z^{qq} + \sum_{1\leq p\leq r, p\neq q} z^{pq} = \mathbf{d}_q$. Consequently, $z^{pq} = 0$ for any $1 \leq q < p'$, $1 \leq p \leq r$ and $p \neq q$. We have therefore $q' > p'$. We have

$$\begin{aligned}
\dim \tilde{\mathcal{F}}_{\underline{\mathbf{d}}} - \dim(\tilde{\mathcal{F}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}_{\underline{\mathbf{d}}})_z &\geq \sum_{i\in I}(g_i - 1) \sum_{\substack{t,p,q,s \\ 1\leq t\leq p\leq r \\ 1\leq q<s\leq r}} z_i^{pq} z_i^{ts} \text{ (forgetting arrows which are not loops)} \\
&\geq (g_i - 1) \sum_{\substack{p,s \\ p'\leq p\leq r \\ p'<s\leq r}} z_i^{pp'} z_i^{p's} \\
&\geq (g_i - 1) \sum_{\substack{p, \\ p'\leq p\leq r}} z_i^{pp'} z_i^{p'q'} \text{ (specializing to } t = q = p') \\
&= (g_i - 1)(\mathbf{d}_{p'})_i z_i^{p'q'} \text{ (summing over } p) \\
&> 0.
\end{aligned}$$

The same reasoning shows that

$$\dim \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} - \dim(\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z > 0$$

for any $z \in \Theta(\underline{\mathbf{d}}) \setminus \{z_{\underline{\mathbf{d}}}\}$ and hence that $\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ is small.

To prove that $\pi_{\underline{\mathbf{d}}}$ (resp. $\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}}$) is an isomorphism over an open subset of its image $E_{\underline{\mathbf{d}}}$ (resp. $E^{\mathrm{nil}}_{\underline{\mathbf{d}}}$), it suffices to show that a sufficiently general element $x \in E_{\underline{\mathbf{d}}}$ (resp. $x \in E^{\mathrm{nil}}_{\underline{\mathbf{d}}}$) admits a unique filtration of type $\underline{\mathbf{d}}$. It suffices to do this when $Q = S_g$ is a quiver with one vertex and $g \geq 2$ loops. Since $\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ is small, its fiber over a general element of $E^{\mathrm{nil}}_{S_g,\underline{\mathbf{d}}}$ is finite, but this is not sufficient. However, in type A, the generalized Springer resolutions $T^*(G/P) \to \mathfrak{g}$ are always birational onto their image ([BB82, §2.7]). It shows that $\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ is also birational onto its image. For $\pi_{\underline{\mathbf{d}}}$, a general element of $E_{S_g,\underline{\mathbf{d}}}$ is of the form $(x_1, \ldots, x_g)$ with x_1 regular semisimple. Hence, x_1 stabilizes a finite number of flags of type $\underline{\mathbf{d}}$ and choosing x_2 general enough, there will be a unique flag of type $\underline{\mathbf{d}}$ stabilized by $(x_1, \ldots, x_g)$.

□

5.10.2 Lusztig sheaves

Let $\mathbf{d} \in \mathbf{N}^I$. We define *four* classes of $G_{\mathbf{d}}$-equivariant perverse sheaves on the representation spaces $E_{Q,\mathbf{d}}$ of an arbitrary quiver.

1. $\mathcal{P}^{\mathrm{nil}}_{Q,\mathbf{d}}$ is the semisimple category of $\mathrm{Perv}_{G_{\mathbf{d}}}(E_{Q,\mathbf{d}})$ generated by perverse sheaves appearing (with a possible shift) as a direct summand of $(\pi^{\mathrm{nil}}_{Q,\underline{\mathbf{d}}})_*\underline{\mathbf{C}}$ where $\underline{\mathbf{d}}$ is some flag type of dimension $\mathbf{d}$,

2. $\mathcal{P}_{Q,\mathbf{d}}$ is the semisimple category of $\mathrm{Perv}_{G_{\mathbf{d}}}(E_{Q,\mathbf{d}})$ generated by perverse sheaves appearing (with a possible shift) as a direct summand of $(\pi_{Q,\underline{\mathbf{d}}})_*\underline{\mathbf{C}}$ where $\underline{\mathbf{d}}$ is some flag type of dimension $\mathbf{d}$,

3. $\mathcal{P}^{\mathrm{nil},1}_{Q,\mathbf{d}}$ is the semisimple category of $\mathrm{Perv}_{G_{\mathbf{d}}}(E_{Q,\mathbf{d}})$ generated by perverse sheaves appearing (with a possible shift) as a direct summand of $(\pi^{\mathrm{nil}}_{Q,\underline{\mathbf{d}}})_*\underline{\mathbf{C}}$ where $\underline{\mathbf{d}}$ is some *discrete* flag type of dimension $\mathbf{d}$,

4. $\mathcal{P}^{1}_{Q,\mathbf{d}}$ is the semisimple category of $\mathrm{Perv}_{G_{\mathbf{d}}}(E_{Q,\mathbf{d}})$ generated by perverse sheaves appearing (with a possible shift) as a direct summand of $(\pi_{Q,\underline{\mathbf{d}}})_*\underline{\mathbf{C}}$ where $\underline{\mathbf{d}}$ is some *discrete* flag type of dimension $\mathbf{d}$.

All sheaves in this categories will be called *Lusztig sheaves*, as perverse sheaves defined in this way were first considered by Lusztig.

Remark 5.10.6. We have the following inclusions between these categories:

$$\mathcal{P}^{1}_{Q,\mathbf{d}} \subset \mathcal{P}_{Q,\mathbf{d}}, \quad \mathcal{P}^{\mathrm{nil},1}_{Q,\mathbf{d}} \subset \mathcal{P}^{\mathrm{nil}}_{Q,\mathbf{d}}.$$

Moreover, if Q is acyclic, all these categories coincide:

$$\mathcal{P}^{\mathrm{nil}}_{Q,\mathbf{d}} = \mathcal{P}_{Q,\mathbf{d}} = \mathcal{P}^{\mathrm{nil},1}_{Q,\mathbf{d}} = \mathcal{P}^{1}_{Q,\mathbf{d}},$$

and if Q has no cycles apart from loops, the categories in (1) and (3) one the one side and (2) and (4) on the other side coincide:

$$\mathcal{P}^{\mathrm{nil}}_{Q,\mathbf{d}} = \mathcal{P}^{\mathrm{nil},1}_{Q,\mathbf{d}}, \quad \mathcal{P}_{Q,\mathbf{d}} = \mathcal{P}^{1}_{Q,\mathbf{d}}.$$

If Q has no loops, the categories (3) and (4) coincide: $\mathcal{P}^{\mathrm{nil},1}_{Q,\mathbf{d}} = \mathcal{P}^{1}_{Q,\mathbf{d}}$.

Let $\underline{\mathbf{d}}$ be a flag-type of dimension $\mathbf{d}$. Define

$$L^{\mathrm{nil}}_{Q,\underline{\mathbf{d}}} = (\pi^{\mathrm{nil}}_{Q,\underline{\mathbf{d}}})_*\underline{\mathbf{C}}, \quad L_{Q,\underline{\mathbf{d}}} = (\pi_{Q,\underline{\mathbf{d}}})_*\underline{\mathbf{C}}.$$

These sheaves are related thanks to the Fourier-Sato transform with the corresponding sheaves on the representation space of the opposite quiver.

Lemma 5.10.7. *Let $\Phi : D^b_{G_{\mathbf{d}}}(E_{Q,\mathbf{d}}) \to D^b_{G_{\mathbf{d}}}(E_{Q^{\mathrm{op}},\mathbf{d}})$ be the Fourier-Sato transform reversing all arrows of Q. Then,*

$$\Phi(L_{Q,\underline{\mathbf{d}}}) = L^{\mathrm{nil}}_{Q^{\mathrm{op}},\underline{\mathbf{d}}}, \quad \Phi(L^{\mathrm{nil}}_{Q,\underline{\mathbf{d}}}) = L_{Q^{\mathrm{op}},\underline{\mathbf{d}}}.$$

Proof. It is completely similar to the proof of Lemma 2.2 of [AHJR14]. □

Explicit description of Lusztig sheaves for negative quivers

Proposition 5.10.8. *Let Q be a negative quiver. Then for $\flat \in \{\mathrm{nil}, \emptyset, (1,\mathrm{nil}), 1\}$, the simple objects of the category $\mathcal{P}^{\flat}_{\mathbf{d}}$ are the perverse sheaves $\mathcal{IC}(E^{\flat}_{\underline{\mathbf{d}}})$ for $\underline{\mathbf{d}} \in \mathscr{C}^{\flat}_{\mathbf{d}}$.*

Proof. This is an immediate consequence of Proposition 5.10.5. □

5.10.3 Lusztig nilpotent varieties and singular support of Lusztig sheaves

The notions of nilpotency for representations of the double quiver

Let Q be a quiver and $\overline{Q}$ the doubled quiver (that is, for any $\alpha : i \to j \in \Omega$, add an arrow $\alpha^* : j \to i$). Recall that a representation of $\overline{Q}$ is denoted by $\bar{x} = (x, x^*)$ where $x = (x_\alpha)_{\alpha\in\Omega}$ and $x^* = (x^*_\alpha)_{\alpha\in\Omega}$. Following [BSV17], a representation (x, x^*) is called

1. semi-nilpotent if there exists a flag of I-graded vector spaces $(0 \subset F_1 \subset \ldots \subset F_r = \mathbf{C}^{\mathbf{d}})$ such that for any $\alpha \in \Omega$, $1 \leq j \leq r$,
$$x_\alpha F_j \subset F_{j-1}, \quad x^*_\alpha F_j \subset F_j,$$
2. $*$-semi-nilpotent if there exists a flag of I-graded vector spaces $(0 \subset F_1 \subset \ldots \subset F_r = \mathbf{C}^{\mathbf{d}})$ such that for any $\alpha \in \Omega$, $1 \leq j \leq r$,
$$x_\alpha F_j \subset F_j, \quad x^*_\alpha F_j \subset F_{j-1},$$
3. strongly semi-nilpotent if there exists a *discrete* flag of I-graded vector spaces $(0 \subset F_1 \subset \ldots \subset F_r = \mathbf{C}^{\mathbf{d}})$ such that for any $\alpha \in \Omega$, $1 \leq j \leq r$,
$$x_\alpha F_j \subset F_{j-1}, \quad x^*_\alpha F_j \subset F_j,$$
4. $*$-strongly semi-nilpotent if there exists a *discrete* flag of I-graded vector spaces $(0 \subset F_1 \subset \ldots \subset F_r = \mathbf{C}^{\mathbf{d}})$ such that for any $\alpha \in \Omega$, $1 \leq j \leq r$,
$$x_\alpha F_j \subset F_j, \quad x^*_\alpha F_j \subset F_{j-1}.$$

Remark 5.10.9. These notions of nilpotency interact as follows. For a representation $\bar{x} = (x, x^*)$ of $\overline{Q}$,
$$(x, x^*) \text{ strongly semi-nilpotent } \implies (x, x^*) \text{ semi-nilpotent}$$
and
$$(x, x^*) \text{ is } *\text{-strongly semi-nilpotent } \implies (x, x^*) \text{ is } *\text{-semi-nilpotent.}$$
Moreover, if Q is acyclic, all these notions of nilpotency coincide, and if Q has no cycles apart form loops, being strongly semi-nilpotent is the same as being semi-nilpotent and being $*$-strongly semi-nilpotent is the same as being $*$-semi-nilpotent. Lastly, if Q has no loops, being strongly semi-nilpotent is the same as being $*$-strongly semi-nilpotent.

The nilpotent varieties

Related to the four categories of perverse sheaves defined in Section 5.10.2 and to the four notions of nilpotency defined in Section 5.10.3, we define four different *nilpotent varieties* in the representation space of the doubled quiver $\overline{Q}$. For this purpose, recall the moment map

$$\begin{array}{cccc} \mu_{\mathbf{d}} & : & E_{\overline{Q},\mathbf{d}} & \to & \mathfrak{gl}_{\mathbf{d}} \\ & & (x,x^*) & \mapsto & \sum_{\alpha\in\Omega}[x_\alpha, x^*_\alpha]. \end{array}$$

The nilpotent varieties are defined as follows.

1. $\Lambda^{\mathrm{nil}}_{Q,\mathbf{d}} = \{(x,x^*)\in\mu_{\mathbf{d}}^{-1}(0) \mid (x,x^*) \text{ is semi-nilpotent}\}$,
2. $\Lambda_{Q,\mathbf{d}} = \{(x,x^*)\in\mu_{\mathbf{d}}^{-1}(0) \mid (x,x^*) \text{ is } *\text{-semi-nilpotent}\}$,
3. $\Lambda^{\mathrm{nil},1}_{Q,\mathbf{d}} = \{(x,x^*)\in\mu_{\mathbf{d}}^{-1}(0) \mid (x,x^*) \text{ is strongly semi-nilpotent}\}$,
4. $\Lambda^{1}_{Q,\mathbf{d}} = \{(x,x^*)\in\mu_{\mathbf{d}}^{-1}(0) \mid (x,x^*) \text{ is } *\text{-strongly semi-nilpotent}\}$.

Remark 5.10.10. Using Remark 5.10.9, we have the following inclusions between the nilpotent varieties. For a general quiver Q,

$$\Lambda^{1}_{Q,\mathbf{d}} \subset \Lambda_{Q,\mathbf{d}}, \quad \Lambda^{\mathrm{nil},1}_{Q,\mathbf{d}} \subset \Lambda^{\mathrm{nil}}_{Q,\mathbf{d}},$$

if Q is acyclic, all the nilpotent varieties coincide and if Q has no cycles apart from loops,

$$\Lambda^{1}_{Q,\mathbf{d}} = \Lambda_{Q,\mathbf{d}}, \quad \Lambda^{\mathrm{nil},1}_{Q,\mathbf{d}} = \Lambda^{\mathrm{nil}}_{Q,\mathbf{d}}.$$

Lastly, if Q has no loops,

$$\Lambda^{\mathrm{nil},1}_{Q,\mathbf{d}} = \Lambda^{1}_{Q,\mathbf{d}}.$$

Proposition 5.10.11. *The varieties $\Lambda^{\flat}_{\mathbf{d}}$, $\flat\in\{\mathrm{nil},\emptyset,(\mathrm{nil},1),1\}$, are closed, Lagrangian, conical subvarieties of $E_{\overline{Q},\mathbf{d}}\simeq T^*E_{Q,\mathbf{d}}$.*

Proof. That these varieties are closed and conical is immediate from their definitions. That they are Lagrangian is already mentioned in [BSV17, §1.1]. □

Remark 5.10.12. We have directly from the definitions and the natural identifications $T^*E_{Q,\mathbf{d}} \simeq E_{\overline{Q},\mathbf{d}} = E_{\overline{Q^{\mathrm{op}}},\mathbf{d}} \simeq T^*E_{Q^{\mathrm{op}},\mathbf{d}}$ the equalities

$$\Lambda^{\mathrm{nil}}_{Q^{\mathrm{op}},\mathbf{d}} = \Lambda_{Q,\mathbf{d}} \quad \text{and} \quad \Lambda^{\mathrm{nil},1}_{Q^{\mathrm{op}},\mathbf{d}} = \Lambda^{1}_{Q,\mathbf{d}}.$$

The singular support of Lusztig sheaves

Proposition 5.10.13. *The singular support of sheaves of the category $\mathcal{P}^{\flat}$ ($\flat\in\{\mathrm{nil},\emptyset,(\mathrm{nil},1),1\}$) is a union of irreducible components of $\Lambda^{\flat}_{\mathbf{d}}$.*

Proof. It suffices to show that the singular support of $(\pi^{\flat}_{\underline{\mathbf{d}}})_*\underline{\mathbf{C}}$ is a subvariety of $\Lambda^{\flat}_{\mathbf{d}}$. This is a standard argument similar to the one of [Lus91, Corollary 13.6]. □

Explicit description of the nilpotent varieties

Recall that for a closed subvariety $Z \subset X$ inside a smooth variety X, we let $T_Z^*X = \overline{T_U^*X}$ where U is some smooth and dense open subset of Z.

Proposition 5.10.14. *For any* $\mathbf{d} \in \mathbf{N}^I$ *and* $\flat \in \{\mathrm{nil}, \emptyset, (\mathrm{nil}, 1), 1\}$*, we have the inclusion*

$$\bigcup_{\underline{\mathbf{d}} \in \mathscr{C}_{\mathbf{d}}^{\flat}} T^*_{E^{\flat}_{\underline{\mathbf{d}}}} E_{\mathbf{d}} \subset \Lambda_{\mathbf{d}}^{\flat}.$$

Proof. By Proposition 5.10.8, for any $\underline{\mathbf{d}} \in \mathscr{C}_{\mathbf{d}}^{\flat}$, $\mathcal{IC}(E^{\flat}_{\underline{\mathbf{d}}}) \in \mathcal{P}_{\mathbf{d}}^{\flat}$; by Proposition 5.10.13, $SS(\mathcal{IC}(E^{\flat}_{\underline{\mathbf{d}}})) \subset \Lambda_{\mathbf{d}}^{\flat}$; and $T^*_{E^{\flat}_{\underline{\mathbf{d}}}} E_{\mathbf{d}} \subset SS(\mathcal{IC}(E^{\flat}_{\underline{\mathbf{d}}}))$. This proves the inclusion of the lemma. □

Conjecture 5.10.15. *If Q is a negative quiver, then the inclusion of Proposition 5.10.14 is an equality.*

Proposition 5.10.16. *Conjecture 5.10.15 is true if Q is a negative quiver and $\flat = 1$ or $\flat = (\mathrm{nil}, 1)$.*

Proof. By Remark 5.10.12, it suffices to prove Conjecture 5.10.15 for $\flat = (\mathrm{nil}, 1)$. In this case, Bozec showed that the set of isomorphism classes of simple perverse sheaves in the category $\mathcal{P}_{\mathbf{d}}^{\mathrm{nil},1}$ is in bijection with the irreducible components of $\Lambda_{\mathbf{d}}^{\mathrm{nil},1}$ (see for example [Boz16, Theorem 3.13]). Since by Proposition 5.10.14 each of the $T^*_{E^{\mathrm{nil},1}_{\underline{\mathbf{d}}}} E_{\mathbf{d}}$ is an irreducible component of $\Lambda_{\mathbf{d}}^{\mathrm{nil},1}$, this proves Proposition 5.10.16. □

5.10.4 Microlocal characterization of Lusztig sheaves

Conjectures

Conjecture 5.10.17. *Let Q be a quiver. Any irreducible $G_{\mathbf{d}}$-equivariant perverse sheaf on $E_{\mathbf{d}}$ whose singular support is contained in $\Lambda_{\mathbf{d}}^{\flat}$ ($\flat \in \{\mathrm{nil}, \emptyset, (\mathrm{nil}, 1), 1\}$ is in the category $\mathcal{P}_{\mathbf{d}}^{\flat}$.*

Remark 5.10.18. By Lemma 5.10.7, Remark 5.10.12 and Lemma 5.D.3, this conjecture is true for $\flat = \mathrm{nil}$ if and only if it is true for $\flat = \emptyset$ and similarly, it is true for $\flat = (\mathrm{nil}, 1)$ if and only if $\flat = 1$.

Remark 5.10.19 (State of Conjecture 5.10.17).

1. By Theorem 5.1.1, Remark 5.10.6 and Remark 5.10.10, Conjecture 5.10.17 is true for any finite type or affine acyclic quiver.
2. By Theorem 5.1.1, it is true for cyclic quivers for $\flat = (\mathrm{nil}, 1)$ and $\flat = 1$ (these two cases coincide). By Theorem 5.7.10, Conjecture 5.10.17 is true for cyclic quivers and $\flat = \emptyset$. By Remark 5.10.18, it is also true for cyclic quivers and $\flat = \mathrm{nil}$.
3. For the Jordan quiver, Conjecture 5.10.17 is true, as a consequence of Springer theory for $\mathfrak{gl}_n$ (for any n). We briefly explain it in Section 5.10.4 below.
4. In Section 5.10.4, we prove Conjecture 5.10.17 for g-loops quivers with $g \geq 2$.
5. The conjecture is still open for wild quivers which are not a g-loop quiver.

The situation for the Jordan quiver

For the Jordan quiver, Conjecture 5.10.18 is true. Let $d \in \mathbf{N}$. We have

$$\Lambda_d = \{(x, x^*) \in \mathfrak{gl}_d^2 \mid [x, x^*] = 0 \text{ and } x^* \text{ is nilpotent}\}.$$

There are two projections $\pi_j : \Lambda_d \to \mathfrak{gl}_d$, $j = 1, 2$. We identify $T^*\mathfrak{gl}_d$ with $E_{\overline{Q},d}$ using the trace pairing. From the point view of π_2,

$$\Lambda_d = \bigsqcup_{\mathcal{O} \subset \mathfrak{gl}_d} T^*_{\mathcal{O}}\mathfrak{gl}_d$$

where the sum runs over nilpotent orbits $\mathcal{O} \subset \mathfrak{gl}_d$.

From the point of view of π_1,

$$\Lambda_d = \bigsqcup_{\mu} \overline{T^*_{\Xi(\mu)}\mathfrak{gl}_d}$$

where μ is regular (see Section 5.2.6 for the definition of the strata $\Xi(\mu)$). Moreover, the Fourier transform gives a bijection between intersection cohomology complexes of nilpotent orbits and perverse sheaves appearing as direct summands of the Springer sheaf, that is the pushforward of the constant sheaf by the Grothendieck-Springer resolution (which is $\pi_{\underline{\mathbf{d}}}$ for $\underline{\mathbf{d}} = (1, \dots, 1) \in \mathbf{N}^d$). Therefore, if $\mathscr{F}$ is a GL_d-equivariant perverse sheaf on $\mathfrak{gl}_d$ with singular support in Λ_d^*, then its Fourier transform is the intersection cohomology of a nilpotent orbit, which means that $\mathscr{F}$ is a direct summand of the Springer sheaf.

The case g-loops quivers

In this section, we assume that Q is a g-loops quivers with $g \geq 2$.

Theorem 5.10.20. *Let Q be a g-loop quiver. Let $\mathscr{F}$ be an irreducible perverse sheaf on $E_{Q,\mathbf{d}}$ such that $SS(\mathscr{F}) \subset \Lambda_{\mathbf{d}}^{\mathrm{nil},1}$. Then, $\mathscr{F} \in \mathcal{P}_{\mathbf{d}}^{\mathrm{nil},1}$.*

Remark 5.10.21. Recall that since Q has only one vertex, $\Lambda_{\mathbf{d}}^{\mathrm{nil},1} = \Lambda_{\mathbf{d}}^{\mathrm{nil}}$ and $\mathcal{P}_{\mathbf{d}}^{\mathrm{nil},1} = \mathcal{P}_{\mathbf{d}}^{\mathrm{nil}}$.

Remark 5.10.22. To prove this theorem, we surprisingly do not need to assume that $\mathscr{F}$ is $G_{\mathbf{d}}$-equivariant. It happens to be a consequence of the property on the singular support.

To prove this theorem, we need several lemmas.

Lemma 5.10.23. *Let $Q = S_g$ be the g-loop quiver. Let $\underline{\mathbf{d}}$ and $\underline{\mathbf{d}}'$ be two flag-types of dimension $\mathbf{d}$ such that $E_{\underline{\mathbf{d}}'}^{\mathrm{nil}} \subsetneq E_{\underline{\mathbf{d}}}^{\mathrm{nil}}$. Then the codimension of $E_{\underline{\mathbf{d}}'}^{\mathrm{nil}}$ in $E_{\underline{\mathbf{d}}}^{\mathrm{nil}}$ is at least two.*

Proof. Since $\pi_{\underline{\mathbf{d}}}^{\mathrm{nil}}$ is small (Proposition 5.10.5), it suffices to show that for $\underline{\mathbf{d}}$ and $\underline{\mathbf{d}}'$ as in the lemma, $\dim \tilde{\mathcal{F}}_{\underline{\mathbf{d}}}^{\mathrm{nil}} - \dim \tilde{\mathcal{F}}_{\underline{\mathbf{d}}'}^{\mathrm{nil}} \geq 2$. However, in this case, (5.10.7) reads

$$\dim \tilde{\mathcal{F}}_{\underline{\mathbf{d}}}^{\mathrm{nil}} = (g+1) \sum_{1 \leq t < p \leq r} (\mathbf{d}_p)_i (\mathbf{d}_t)_j$$

and is therefore a multiple of $g + 1 \geq 3$. Therefore the codimension is at least 3. □

We let $Z'' = \bigcup_{\substack{\underline{\mathbf{d}}' \in \mathscr{C}^1_{\underline{\mathbf{d}}} \\ E_{\underline{\mathbf{d}}'} \subsetneq E_{\underline{\mathbf{d}}}}} E^{\mathrm{nil}}_{\underline{\mathbf{d}}'}$. By the previous lemma, it is of codimension at least two (in fact three) in $E^{\mathrm{nil}}_{\underline{\mathbf{d}}}$.

Lemma 5.10.24. *Let $\pi : X \to Y$ be a small map. Assume that Y is irreducible. If $Z \subset Y$ is a closed subvariety of codimension at least two, then $\pi^{-1}(Z) \subset X$ is also a closed subvariety of codimension at least two.*

Proof. Since π is small, one can find a stratification by locally closed subvarieties $Y = \bigsqcup_{S \in \mathcal{S}} S$ such that for any $S \in \mathcal{S}$, the restriction $\pi_S : \pi^{-1}(S) \to S$ is a topological fibration and for any $x \in S$,

$$\dim S + 2 \dim \pi^{-1}(x) < \dim X$$

unless S is the dense stratum, for which we have the equality of the dimensions. We have $Z = \bigsqcup_{S \in \mathcal{S}} Z \cap S$ and $\pi^{-1}(Z) = \bigsqcup_{S \in \mathcal{S}} \pi^{-1}(Z \cap S)$. If S is the dense stratum, $\dim \pi^{-1}(Z \cap S) = \dim Z \cap S$, so $\overline{\pi^{-1}(Z \cap S)}$ is of codimension at least two in X. If S is not the dense stratum, but $\pi : \pi^{-1}(S) \to S$ has finite fibers, $\dim(\pi^{-1}(Z \cap S)) = \dim Z \cap S \leq \dim Z$. Therefore, $\overline{\pi^{-1}(Z \cap S)}$ is of codimension at least two in X. Lastly, if S is a stratum over which the fibers of π have dimension at least one, then by the smallness of π, for any $x \in Z \cap S$,

$$\begin{aligned}
\dim \pi^{-1}(Z \cap S) &= \dim Z \cap S + \dim \pi^{-1}(x) \\
&\leq \dim S + \dim \pi^{-1}(x) \\
&< \dim X - \dim \pi^{-1}(x) \\
&< \dim X - 1.
\end{aligned}$$

This last equality allows us to conclude.

□

Lemma 5.10.25. *Let Z' be the set of $x \in E^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ such that $(\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}})^{-1}(x)$ contains at least two points and $\overline{Z'}$ its Zariski closure (so that $\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}} : \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \setminus (\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}})^{-1}(\overline{Z'}) \to E^{\mathrm{nil}}_{\underline{\mathbf{d}}} \setminus \overline{Z'}$ is an isomorphism). Then $(\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}})^{-1}(\overline{Z'})$ is of codimension at least two in $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$.*

Proof. Consider the diagram

$$\begin{tikzcd}
\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \arrow[r, "pr_1"] \arrow[dr, "p"'] & \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \arrow[d, "\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}}"] \\
 & E^{\mathrm{nil}}_{\underline{\mathbf{d}}}
\end{tikzcd}$$

of proper morphisms. Then, $Z' = p(\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \setminus \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})$ (recall that $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ is identified with its diagonal embedding in $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$) and $(\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}})^{-1}(Z') = pr_1(\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \setminus \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})$. Moreover, since a general element of $E^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ admits a unique filtration of type $\underline{\mathbf{d}}$ (by Proposition 5.10.5), then $\overline{Z'}$ is a proper closed subset of $E^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ and hence is of codimension at least 1 in $E^{\mathrm{nil}}_{\underline{\mathbf{d}}}$. Now, $\overline{Z'} = Z' \sqcup (\overline{Z'} \setminus Z')$ and $\overline{Z'} \setminus Z'$ is of codimension at least two in $E^{\mathrm{nil}}_{\underline{\mathbf{d}}}$. By Lemma 5.10.24, $(\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}})^{-1}(\overline{Z'} \setminus Z')$ is of codimension at least two in $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$. It remains therefore to show that $(\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}})^{-1}(Z') = pr_1(\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}}$

$\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \setminus \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$) is of codimension at least two in $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$. We write

$$\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \setminus \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} = \bigsqcup_{z \in \Theta(\underline{\mathbf{d}}) \setminus \{z_{\underline{\mathbf{d}}}\}} (\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z$$

If $g \geq 3$, by the second formula of Corollary 5.10.4 and the same argument as in the proof of Proposition 5.10.5, we have that $\dim \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} - (\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z$ is a non-zero multiple of $g-1$ and hence is ≥ 2. Consequently, $\dim pr_1((\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z) \leq \dim \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} - 2$.

If $g = 2$, we have to be more careful. If

$$\dim \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} - (\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z \geq 2$$

then the argument above applies. However, it is possible to have

$$\dim \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} - (\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z = 1$$

and by examining the formula

$$\dim \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} - (\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z = (g-1) \sum_{\substack{1 \leq t < p \leq r \\ 1 \leq q \leq s \leq r}} z^{pq} z^{ts}$$

we see that this happens if and only if there exists $1 \leq i \leq r-1$ such that

$$z^{pq} = \begin{cases} 1 \text{ if } p = i,\ q = i+1 \\ 1 \text{ if } p = i+1,\ q = i \\ \mathbf{d}_p \ \text{ if } p = q \\ 0 \text{ else} \end{cases}$$

(In particular, this imposes $\mathbf{d}_i = \mathbf{d}_{i+1} = 1$). If now $x \in p((\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z)$ for such a z, there exists two flags $\underline{F}$ and $\underline{F'}$ of type $\underline{\mathbf{d}}$ and in relative position z such that for any $1 \leq j \leq r$, $xF_j \subset F_{j-1}$ and $xF'_j \subset F'_{j-1}$. Using the particular form of z described above, we have $F_{i+2} = F_{i+1} + F'_{i+1}$ and therefore $xF_j \subset F_{j-1}$ for $1 \leq j \leq i+1$, $xF_{i+2} \subset F_i$ and $xF_j \subset F_{j-1}$ for $i+3 \leq j \leq r$. We consider the new flag-type $\underline{\mathbf{d}}' = (\mathbf{d}'_1, \ldots, \mathbf{d}'_{r-1})$ such that $\mathbf{d}'_j = \mathbf{d}_j$ if $1 \leq j \leq i$, $\mathbf{d}'_{i+1} = 2$, $\mathbf{d}'_j = \mathbf{d}_{j+1}$ if $i+2 \leq j \leq r-1$. Then, $E^{\mathrm{nil}}_{\underline{\mathbf{d}}'} \subset E^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ and $p((\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \times_{E_{\mathbf{d}}} \tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}})_z) \subset E^{\mathrm{nil}}_{\underline{\mathbf{d}}'}$. By Lemma 5.10.23 and Lemma 5.10.24, we are done. □

We let now $Z = \overline{Z'} \cup Z''$.

Corollary 5.10.26. *The restriction $\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}} : (\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}})^{-1}(E^{\mathrm{nil}}_{\underline{\mathbf{d}}} \setminus Z) \to E^{\mathrm{nil}}_{\underline{\mathbf{d}}} \setminus Z$ is an isomorphism and $E_{\underline{\mathbf{d}}} \setminus Z$ is open is $E^{\mathrm{nil}}_{\underline{\mathbf{d}}}$.*

Proof. It is clear form what preceeds. □

Lemma 5.10.27. *For any $\underline{\mathbf{d}} \in \mathscr{C}_{\mathbf{d}}$, the partial flag variety $\mathcal{F}_{\underline{\mathbf{d}}}$ is simply-connected, and hence $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ is also simply-connected. Similarly, $\tilde{\mathcal{F}}_{\underline{\mathbf{d}}}$ is simply-connected.*

Proof. The partial flag variety $\mathcal{F}_{\mathbf{d}}$ is a partial flag variety of a reductive algebraic group and hence admits a cell decomposition where the cells are affine spaces. Therefore, it is simply connected. Now, the second projection $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \to \mathcal{F}_{\underline{\mathbf{d}}}, (x, \underline{F}) \mapsto \underline{F}$ is a fiber bundle. Hence, $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ is also simply connected. □

Proof of Theorem 5.10.20. Let $\mathscr{F}$ be an irreducible perverse sheaf on $E_{Q,\mathbf{d}}$ such that $SS(\mathscr{F}) \subset \Lambda^{\mathrm{nil},1}_{Q,\mathbf{d}}$. Let $p_{\mathbf{d}} : E_{\overline{Q},\mathbf{d}} \to E_{Q,\mathbf{d}}$ be the cotangent bundle map. Since $p_{\mathbf{d}}(SS(\mathscr{F})) = \mathrm{supp}(\mathscr{F})$, by Proposition 5.10.16, there exists a flag-type $\underline{\mathbf{d}} \in \mathscr{C}^1_{\mathbf{d}}$ such that $\mathrm{supp}(\mathscr{F}) = E^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ and moreover, any other irreducible component of $SS(\mathscr{F})$ is of the form $T^*_{E^{\mathrm{nil}}_{Q,\underline{\mathbf{d}}'}} E_{Q,\mathbf{d}}$ for some flag-type $\underline{\mathbf{d}}' \in \mathscr{C}^{\mathrm{nil}}_{\mathbf{d}}$ such that $E^{\mathrm{nil}}_{Q,\underline{\mathbf{d}}'} \subset E_{Q,\underline{\mathbf{d}}}$. Let Z be as before Corollary 5.10.26. By Corollary 5.10.26, the open subset $E^{\mathrm{nil}}_{\underline{\mathbf{d}}} \setminus Z$ is smooth (being isomorphic to an open subset of $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$, which is smooth) and the restriction $\mathscr{F}_U$ of $\mathscr{F}$ to $U := E^{\mathrm{nil}}_{\underline{\mathbf{d}}} \setminus Z$ verifies $SS(\mathscr{F}_U) = T^*_U U$. By Lemma 5.C.5, $\mathscr{F}_U = \mathscr{L}[s]$ for some local system $\mathscr{L}$ on U and $s = \dim U$. Therefore, $(\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}})^*\mathscr{L}$ is a local system on $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}} \setminus (\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}})^{-1}(Z)$. By Lemmas 5.10.23, 5.10.24 and 5.10.25, $(\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}})^{-1}(Z)$ is of codimension at least two in $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$. Since the latter is smooth, $(\pi^{\mathrm{nil}}_{\underline{\mathbf{d}}})^*\mathscr{L}$ can be extended to a local system on $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$. Since $\tilde{\mathcal{F}}^{\mathrm{nil}}_{\underline{\mathbf{d}}}$ is simply-connected (Lemma 5.10.27), this extension is the trivial local system. Hence, by Lemma 5.10.26, $\mathscr{L}$ is the trivial local system on U. Therefore, $\mathscr{F} = \mathcal{IC}(E^{\mathrm{nil}}_{\underline{\mathbf{d}}})$. By Proposition 5.10.8, $\mathscr{F}$ is indeed in $\mathcal{P}^{\mathrm{nil},1}_{\mathbf{d}}$. □

5.A Local systems on the complement of a normal crossing divisor

Lemma 5.A.1. *Let X be a smooth irreducible curve and $d \geq 1$ an integer. Let $D \subset X$ be a finite set. Then $U = (X \setminus D)^d$ is the complement of a simple normal crossing divisor in X^d.*

Proof. Let $x = (x_1, \ldots, x_d) \in X^d$. By symmetry of the question, we can assume that there exists $r \geq 1$ such that $x_1, \ldots, x_r \in D$ and $x_{r+1}, \ldots, x_d \in X \setminus D$. If y_i $(1 \leq i \leq d)$ is a local coordinate of X around x_i, such that x corresponds to $y_1 = \ldots = y_d = 0$, a local equation for $X \setminus U$ in a neighbourhood of x is $\prod_{i=1}^r y_i$. □

Lemma 5.A.2. *Let $D \subset \mathbf{P}^1(\mathbf{C})$ be a finite subset and $d \geq 1$. Let $\mathscr{L}$ be a local system on $U = (\mathbf{P}^1(\mathbf{C}) \setminus D)^d \setminus \Delta$. Then $\mathscr{L}$ extends to a local system on $Y = \mathbf{P}^1(\mathbf{C})^d \setminus \Delta$ if and only if for any $x \in D$, there exists an analytic neighbourhood V of $(x, \ldots, x) \in \mathbf{P}^1(\mathbf{C})^d$ such that $\mathscr{L}_{V\cap U}$ extends to $V \cap Y$.*

Proof. The direct implication is trivial. For the other implication, for any $x \in D$, let V_x a neighbourhood of $(x, \ldots, x) \in \mathbf{P}^1(\mathbf{C})^d$ such that $\mathscr{L}_{V_x\cap U}$ extends to $V_x \cap Y$. Then any branch of the simple normal crossing divisor $Y \setminus U$ of Y intersects at least one of the V_x. Therefore, the monodromies around all branches of $Y \setminus U$ are trivial and $\mathscr{L}$ extends to Y. □

We recall that $D(0,1) \subset \mathbf{C}$ denotes the open unit disk and $D^*(0,1)$ the punctured unit disk. Let $\pi : D^*(0,1)^d \setminus \Delta \to S^d D^*(0,1) \setminus \Delta$. It is a $\mathfrak{S}_d$-covering.

Lemma 5.A.3. *Let $\mathscr{L}$ be a local system on $S^d D^*(0,1) \setminus \Delta$. Then $\mathscr{L}$ can be extended to $S^d D(0,1) \setminus \Delta$ if and only if $\pi^*\mathscr{L}$ can be extended to $D(0,1)^d \setminus \Delta$.*

Proof. The direct implication is obvious. Let $U = S^dD^*(0,1) \setminus \Delta$, $X = S^dD(0,1) \setminus \Delta$, $\tilde{U} = D^*(0,1)^d \setminus \Delta$, $\tilde{X} = D(0,1)^d \setminus \Delta$. Let $\mathscr{L}'$ the extension of $\pi^*\mathscr{L}$ to $\tilde{X}$. Since $\tilde{\pi} : \tilde{X} \to X$ is a $\mathfrak{S}_d$-covering, $\pi_*\mathscr{L}'$ is a local system on X. Moreover, $\pi_*\mathscr{L}'_U = \pi_*\pi^*\mathscr{L}$ and therefore, $\mathscr{L}$ is a direct summand of $\pi_*\mathscr{L}'_U$. The corresponding direct summand of $\pi_*\mathscr{L}'$ gives the extension of $\mathscr{L}$ to X. □

Let $D \subset \mathbf{P}^1(\mathbf{C})$ be a nonempty finite subset. Let $\mathscr{L}$ be a local system on $S^d(\mathbf{P}^1(\mathbf{C}) \setminus D) \setminus \Delta$. We let $U = S^d(\mathbf{P}^1(\mathbf{C}) \setminus D) \setminus \Delta$, $X = S^d\,\mathbf{P}^1(\mathbf{C}) \setminus \Delta$, $\tilde{U} = (\mathbf{P}^1(\mathbf{C}) \setminus D)^d \setminus \Delta$ and $\tilde{X} = \mathbf{P}^1(\mathbf{C})^d \setminus \Delta$. We have the diagram

$$\begin{tikzcd} \tilde{U} \arrow[r] \arrow[d, "\pi"'] & \tilde{X} \arrow[d, "\pi'"] \\ U \arrow[r] & X \end{tikzcd}$$

Lemma 5.A.4. *The local system $\pi^*\mathscr{L}$ is trivial if and only if for any $x \in D$, there exists a neighbourhood $V \subset \mathbf{P}^1(\mathbf{C})$ of x such that, with $V_d = S^d(V \setminus \{x\}) \setminus \Delta$ and $\tilde{V}_d = (V \setminus \{x\})^d \setminus \Delta$, $\pi_V : \tilde{V}_d \to V_d$,*

$$(\pi_V)^*\mathscr{L}_{V_d}$$

is trivial on $\tilde{V}_d$.

Proof. By Lemma 5.A.2 and Lemma 5.A.3, $\mathscr{L}$ extends to $S^d\,\mathbf{P}^1(\mathbf{C}) \setminus \Delta$. Let $y \in \mathbf{P}^1(C) \setminus D$. Then, $\mathbf{P}^1(\mathbf{C}) \setminus \{y\} \simeq \mathbb{A}^1(\mathbf{C})$. The inclusion $\mathbb{A}^d(\mathbf{C}) \setminus \Delta \subset S^d\,\mathbf{P}^1(\mathbf{C}) \setminus \Delta$ is surjective on the fundamental groups, so that it suffices to check that the restriction of $\pi^*\mathscr{L}$ to $\mathbb{A}^d(\mathbf{C}) \setminus \Delta$ is trivial. Now, for $x \in D$, let $D(x,r) \subset V$ be a small neighbourhood of x. Then the inclusion $D(x,r)^d \setminus \Delta \subset \mathbb{A}^d(\mathbf{C}) \setminus \Delta$ induces an isomorphism at the level of fundamental groups. Therefore, it suffices to prove that the restriction of $\pi^*\mathscr{L}$ to $D(x,r)^d \setminus \Delta$ is trivial. But this is clearly implied by the hypotheses of the lemma. □

5.B Equivariant perverse sheaves and local systems

5.B.1 Equivariant perverse sheaves

We will quite often face the situation described in the following lemma.

Lemma 5.B.1. *Let G be a connected algebraic group and $H \subset G$ a normal closed subgroup. Let X be a G-variety on which H acts trivially (*i.e. *a G/H-variety). Then* $\mathrm{Perv}_G(X) \simeq \mathrm{Perv}_{G/H}(X)$.

Proof. By hypothesis, G and G/H are connected. We have the action map

$$a : G \times X \to X$$

and its factorization

$$a' : G/H \times X \to X$$

together with the projections

$$p : G \times X \to X$$

and

$$p' : G/H \times X \to X.$$

Let $\pi : G \to G/H$ be the projection. Then $a = a' \circ (\pi \times \mathrm{id}_X)$ and $p = p' \circ (\pi \times \mathrm{id}_X)$. The forgetful functor $\mathrm{Perv}_G(X) \to \mathrm{Perv}(X)$ identifies $\mathrm{Perv}_G(X)$ with the full subcategory of $\mathrm{Perv}(X)$ of perverse sheaves $\mathscr{F}$ such that $p^*\mathscr{F}$ and $a^*\mathscr{F}$ are isomorphic. Similarly, $\mathrm{Perv}_{G/H}(X)$ is identified with the subcategory of $\mathrm{Perv}(X)$ of perverse sheaves $\mathscr{F}$ such that $p'^*\mathscr{F}$ and $a'^*\mathscr{F}$ are isomorphic. Since $\pi \times \mathrm{id}_X$ is smooth, given $\mathscr{F} \in \mathrm{Perv}(X)$, $p'^*\mathscr{F}$ and $a'^*\mathscr{F}$ are isomorphic if and only if $(\pi \times \mathrm{id}_X)^*p'^*\mathscr{F} = p^*\mathscr{F}$ and $(\pi \times \mathrm{id}_X)^*a'^*\mathscr{F} = a^*\mathscr{F}$ are isomorphic. Hence, $\mathrm{Perv}_G(X)$ and $\mathrm{Perv}_{G/H}(X)$ are both identified with the same full subcategory of $\mathrm{Perv}(X)$ and hence are equivalent. □

Let $H \subset G$ be a closed subgroup of an algebraic group G and X a H-variety. Let

$$\begin{array}{cccc} i & : & X & \to & X \times^H G \\ & & x & \mapsto & (x, e). \end{array}$$

The following lemma is used many times in this chapter without mention.

Lemma 5.B.2 (Induction equivalence, [BL94]). *The functor*

$$i^*[\dim H - \dim G] : D^b_G(X \times^H G) \to D^b_H(X)$$

is a perverse equivalence of categories.

5.B.2 Local systems on the regular semisimple locus of a reductive Lie algebra

We let $G = \mathrm{GL}_d$, $\mathfrak{g} = \mathfrak{gl}_d$, $T \subset G$ is the maximal torus of diagonal matrices and $\mathfrak{t}$ its Lie algebra. We let $W = \mathfrak{S}_d$ be the Weyl group. We let $\mathfrak{g}^{rss} \subset \mathfrak{g}$ be the open subvariety of regular semisimple elements, $f' : \tilde{\mathfrak{g}}^{rss} \to \mathfrak{g}^{rss}$ the W-covering induced by the Grothendieck-Springer simultaneous resolution, and g the semisimplification map. The map g' lifts g in the sense that the following square is cartesian.

$$\begin{tikzcd} \tilde{\mathfrak{g}}^{rss} \arrow[r, "f'"] \arrow[d, "g'"'] & \mathfrak{g}^{rss} \arrow[d, "g"] \\ \mathfrak{t}^{rss} \arrow[r, "f"] & \mathfrak{t}^{rss}/W \end{tikzcd} .$$

Lemma 5.B.3. *A G-equivariant local system $\mathscr{L}$ on $\mathfrak{g}^{rss}$ is the pull-back of a local system $\mathscr{L}'$ on $\mathfrak{t}^{rss}/W$. The local system $\mathscr{L}'$ is unique up to isomorphism. In other words, g is an equivariant π_1-equivalence. Moreover, $(f')^*\mathscr{L}$ is trivial if and only if $f^*\mathscr{L}'$ is trivial.*

Proof. Let $\mathscr{L}$ be a G-equivariant local system on $\mathfrak{g}^{rss}$. We can assume that $\mathscr{L}$ is indecomposable. Since f' is a W covering, $\mathscr{L}$ is a direct summand of $f'_*(f')^*\mathscr{L}$. Assume that $(f')^*\mathscr{L} = (g')^*\mathscr{L}''$ for some local system $\mathscr{L}''$ on $\mathfrak{t}^{rss}$. Then by smooth base-change, $f'_*(g')^*\mathscr{L}'' \simeq g^*f_*\mathscr{L}''$. Therefore, $\mathscr{L}$ is a direct summand of $g^*f_*\mathscr{L}''$. Since $\mathscr{L}$ is indecomposable, there exists an indecomposable summand $\mathscr{L}'$ of $f_*\mathscr{L}''$ such that $\mathscr{L}$ is a direct summand of $g^*\mathscr{L}'$. Since g is smooth, $g^*\mathscr{L}'$ is indecomposable. Therefore, $\mathscr{L} = g^*\mathscr{L}'$. Therefore, it suffices to prove the existence of $\mathscr{L}''$.

But $\tilde{\mathfrak{g}}^{rss} \simeq \mathfrak{t}^{rss} \times^T G$, the action of T on $\mathfrak{t}^{rss}$ being trivial. Therefore, a G-invariant local system on $\tilde{\mathfrak{g}}^{rss}$ is the same thing as a T-equivariant local system on $\mathfrak{t}^{rss}$, that is (since T is connected) a local system on $\mathfrak{t}^{rss}$. Let

$$\mathfrak{t}^{rss} \xrightarrow{i} \tilde{\mathfrak{g}}^{rss} \xrightarrow{g'} \mathfrak{t}^{rss}$$

where the first row is the closed immersion and the second is g'. Let $\mathscr{L}$ be a local system on $\tilde{\mathfrak{g}}^{rss}$. Let $\mathscr{L}'' = i^*\mathscr{L}$. Then, $\mathscr{L} = (g')^*\mathscr{L}''$ since $i^*\mathscr{L} = i^*(g')^*\mathscr{L}''$, because $g' \circ i = \mathrm{id}$. This ends the proof. □

Remark 5.B.4. This lemma can also be deduced from the fact that the algebraic stacks $\mathfrak{g}^{rss}/G$ and $\mathfrak{t}^{rss}/W$ are isomorphic for any reductive group G.

5.B.3 Equivariant local systems on the regular semisimple strata of affine quivers

Let Q be an affine acyclic quiver (resp. a cyclic quiver). Let $\Xi(N,\mu) \subset E_{\mathbf{d}}$ be a regular semisimple stratum, that is N is regular non-homogeneous (resp. nilpotent), μ is regular semisimple and $\mathbf{d} = \dim N + \delta \dim \mu$. Let $d = \dim \mu$. Recall the map $\chi_{N,\mu} : \Xi(N,\mu) \to S^d\,\mathbf{P}_1^{\mathrm{hom}} \setminus \Delta$ from Section 5.2.5 (resp. $\chi_{N,\mu} : \Xi(N,\mu) \to S^d(\mathbf{C}^*) \setminus \Delta$ from Section 5.2.7). Recall also the $\mathfrak{S}_d$-covering $\pi_d : (\mathbf{P}_1^{\mathrm{hom}})^d \setminus \Delta \to S^d\,\mathbf{P}_1^{\mathrm{hom}} \setminus \Delta$ (resp. $\pi_d : (\mathbf{C}^*)^d \setminus \Delta \to S^d(\mathbf{C}^*) \setminus \Delta$). We have the following analog of Lemma 5.B.3.

Lemma 5.B.5. *Any $G_{\mathbf{d}}$-equivariant local system $\mathscr{L}$ on $\Xi(N,\mu)$ is the pull-back by $\chi_{N,\mu}$ of a local system $\mathscr{L}'$ on $S^d(\mathbf{P}_1^{\mathrm{hom}}) \setminus \Delta$ (resp. on $S^d(\mathbf{C}^*) \setminus \Delta$). Moreover, $\mathscr{L}'$ is determined up to isomorphism by $\mathscr{L}$ and $\mathscr{L}$ is a Lusztig local system (resp. an extended Lusztig local system) if and only of $\pi_d^*\mathscr{L}'$ is the trivial local system.*

Proof. The proof is completely similar to that of Lemma 5.B.3 using the following facts (stated for acyclic affine quivers; the analogous facts for cyclic quivers are also true).

We have a cartesian diagram

$$\begin{tikzcd} \tilde{\Xi}(N,\mu) \arrow[r, "\pi_{N,\mu}"] \arrow[d, "\tilde{\chi}_{N,\mu}"'] & \Xi(N,\mu) \arrow[d, "\chi_{N,\mu}"] \\ (\mathbf{P}_1^{\mathrm{hom}})^d \setminus \Delta \arrow[r, "\pi_d"] & S^d\,\mathbf{P}_1^{\mathrm{hom}} \setminus \Delta. \end{tikzcd}$$

where $\pi_{N,\mu}$ and π_d are $\mathfrak{S}_d$-coverings. Moreover, $\tilde{\chi}_{N,\mu}$ (and thus $\chi_{N,\mu}$) is smooth. This can be proved as follows. By the definition of $\chi_{N,\mu}$ in Section 5.2.5, it suffices to prove that $\tilde{\chi}_\mu : \tilde{\Xi}(\mu) \to (\mathbf{P}_1^{\mathrm{hom}})^d \setminus \Delta$ is smooth. In dimension δ, the morphism $E_\delta^{\mathrm{reghom}} \to \mathbf{P}_1^{\mathrm{hom}}$ is a $G_\delta/\,\mathbf{C}^*$-principal bundle. Therefore it is smooth. Consider the following cartesian diagram (which defines Z):

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "p"'] & (E_\delta^{\mathrm{reghom}})^d \arrow[d] \\ (\mathbf{P}_1^{\mathrm{reghom}})^d \setminus \Delta \arrow[r] & (\mathbf{P}_1^{\mathrm{reghom}})^d. \end{tikzcd}$$

By base-change, the vertical left-most map p is smooth. Then, $\tilde{\Xi}(\mu) \simeq Z \times^{(G_\delta)^d} G_{d\delta}$. In the commutative triangle

$$\begin{tikzcd} Z \times G_\delta \arrow[r] \arrow[dr, "p\circ \mathrm{pr}_1"'] & \tilde{\Xi}(\mu) \arrow[d, "\tilde{\chi}_{N,\mu}"] \\ & (\mathbf{P}_1^{\mathrm{reghom}})^d \end{tikzcd} ,$$

the horizontal arrow is a $(G_\delta)^d$-principal bundle, hence is smooth and $p \circ pr_1$ is also smooth. As a consequence, $\tilde{\chi}_{N,\mu}$ is smooth.

□

5.C Singular support of constructible complexes

Let $f : X \to Y$ be a morphism between smooth complex algebraic varieties. Then, we have the classical correspondence between cotangent bundles:

$$\begin{tikzcd} & X \times_Y T^*Y \arrow[dl, "pr_2"'] \arrow[dr, "(df)^*"] & \\ T^*Y & & T^*X \end{tikzcd}$$

There are properties on the morphism f that we recall here ensuring that the singular support satisfies natural functorialities with respect to these morphisms. We refer to the original references for proofs.

5.C.1 Singular support of the pull-back by a smooth morphism

For a proof of the following proposition, we refer to [KS85].

Proposition 5.C.1 ([KS85, Proposition 4.1.2])**.** *Suppose that $f : X \to Y$ is smooth. Let $\mathscr{F}$ be a constructible sheaf on Y. Then*

$$SS(f^*\mathscr{F}) = (df)^*(pr_2^{-1}(SS(\mathscr{F}))).$$

Corollary 5.C.2. *Let $f : X \to Y$ be a smooth morphism and $\mathscr{F} \in D^b(Y)$ a constructible complex. Write*

$$SS(\mathscr{F}) = \bigcup_{S\in\mathcal{S}} \overline{T_S^*Y}$$

for some set $\mathcal{S}$ of locally closed subvariety of Y. Then

$$SS(f^*\mathscr{F}) = \bigcup_{S\in\mathcal{S}} \overline{T_{f^{-1}(S)}^*X}.$$

Proof. In this case, $(df)^*$ is a closed immersion. Moreover, $pr_2^{-1}(T_S^*Y) = X \times_Y T_S^*Y$ is isomorphic to $T_{f^{-1}(S)}^*X$ via $(df)^*$. □

5.C.2 Singular support and the pushforward by a proper morphism

The following proposition is taken from [KS90, Proposition 5.4.4].

Proposition 5.C.3. *Let $X \to Y$ be a proper morphism of manifolds, $\mathscr{F} \in D^b(Y)$. Then*

$$SS(f_*\mathscr{F}) \subset pr_2((df)^{*-1}(SS(\mathscr{F})))$$

and this inclusion is an equality if f is a closed immersion.

5.C.3 Induction of singular supports

Let $H \subset G$ be connected algebraic groups. Let X be a H-variety. Let i be the closed immersion

$$\begin{array}{cccccc} i & : & X & \to & X \times^H G \\ & & x & \mapsto & (x, e) \end{array}.$$

Then, we have a triangulated equivalence of categories preserving the categories of perverse sheaves ([BL94, 2.6]):

$$i^0 := i^*[\dim H - \dim G] : D^b_G(X \times^H G) \to D^b_H(X).$$

The inverse equivalence is described by the induction γ_H^G as follows ([MV88, 1.4]). Consider the diagram:

$$\begin{tikzcd} & X \times G \arrow[dl, "pr_1"'] \arrow[dr, "\pi"] & \\ X & & X \times^H G \end{tikzcd} \tag{5.C.1}$$

Then for $\mathscr{F}$ a constructible H-equivariant complex on X, $pr_1^*\mathscr{F}$ is H-equivariant on $X \times G$, so there is a unique constructible complex (up to isomorphism) on $X \times^H G$, $\mathscr{G}$, such that $\pi^*\mathscr{G} \simeq pr_1^*\mathscr{F}$. We let $\gamma_H^G = \mathscr{G}[\dim G - \dim H]$.

Lemma 5.C.4. *Let X be an H-variety and $H \to G$ an injective group homomorphism. Let $\mathcal{F} \in D^b_H(X)$ and $\mathcal{G} \in D^b_G(X \times^H G)$ be two perverse sheaves corresponding to each other via the equivalence $D^b_H(X) \simeq D^b_G(X \times^H G)$. Then*

$$SS(\mathcal{F}) = \bigsqcup_{S \in \mathcal{S}} T^*_S X$$

is and only if

$$SS(\mathcal{G}) = \bigsqcup_{S \in \mathcal{S}} T^*_{S \times^H G}(X \times^H G).$$

Proof. In the induction diagram (5.C.1), both pr_1 and π are smooth. We only prove the direct implication, the proof of the converse being similar. Write

$$SS(\mathscr{G}) = \bigcup_{S' \in \mathcal{S}'} \overline{T^*_{S'}(X \times^H G)}$$

for some G-invariant locally closed strata $S' \in \mathcal{S}'$ of $X \times^H G$. It is easily seen that by G-invariance, any $S' \in \mathcal{S}'$ can be written $S' = S'_Y \times^H G$ where $S'_Y \subset Y = i^{-1}(S')$ is H-invariant and locally closed in Y. Then, by Corollary 5.C.2,

$$SS(\pi^*(\mathscr{G})) = \bigcup_{S' \in \mathcal{S}'} \overline{T^*_{S'_Y \times G}(X \times G)}.$$

Moreover, $\pi^*\mathscr{G} \simeq pr_1^*\mathscr{F}$ and again by Corollary 5.C.2,

$$SS(pr_1^*(\mathscr{F})) = \bigcup_{S \in \mathcal{S}} \overline{T^*_{S \times G}(X \times G)}.$$

This ends the proof. □

Lemma 5.C.5. *Let $\mathscr{F}$ be a perverse sheaf on a smooth irreducible variety X such that $SS(\mathscr{F}) = T^*_X X$. Then $\mathscr{F} = \mathscr{L}[\dim X]$ for some local system $\mathscr{L}$ on X.*

Proof. This is a particular case of [Dim04, Theorem 4.3.15 iv)] in the case where X is trivially stratified. □

5.D Fourier-Sato transform

5.D.1 Monodromic sheaves

Let $E \to X$ be a complex fiber bundle over an algebraic variety X. We consider the $\mathbf{C}^*$-action of weight 1 contracting the fibers. We call a sheaf on E *monodromic* if it is locally constant on $\mathbf{C}^*$-orbits of E.

Lemma 5.D.1. *Let $i : Y \to E$ be a $\mathbf{C}^*$-invariant locally closed subvariety. Let $\mathscr{F} \in D^b_{mon}(Y, \mathbf{C})$ be a constructible monodromic complex. Then $i_{!*}\mathscr{F}$ is a constructible monodromic complex on E.*

Corollary 5.D.2. *Let $\mathscr{F} \in D^b(E, \mathbf{C})$ be an irreducible perverse sheaf whose singular support is the union of conormal bundles to $\mathbf{C}^*$-invariant subvarieties. Then $\mathscr{F}$ is monodromic.*

Proof. The support of $\mathscr{F}$, $SS(\mathscr{F}) \cap T^*_E E$ is a $\mathbf{C}^*$-invariant closed subvariety of E. There exists a smooth $\mathbf{C}^*$-invariant open subset $j : U \to \operatorname{supp} \mathscr{F}$ such that $SS(j^*\mathscr{F}) = T^*_U U$. Then, $j^*\mathscr{F}$ is a local system on U. Then, by Lemma 5.D.1, $\mathscr{F} = j_{!*}(j^*\mathscr{F})$ is monodromic on E. □

5.D.2 The Fourier-Sato transform

The definition and basic facts for the Fourier-Sato transform in the complex-analytic setting are given in [AHJR14, 2.7].

Let X be a complex algebraic variety and $p : E \to X$ a complex vector bundle. Let $\check{p} : E^* \to X$ be its dual.

Consider the correspondence

$$\begin{tikzcd} & Q \arrow[dl, "q"'] \arrow[dr, "\check{q}"] & \\ E & & E^* \end{tikzcd}$$

where

$$Q = \{(x, y) \in E \times_X E^* \mid \Re(\langle x, y \rangle) \leq 0\} \subset E \times_X E^*.$$

Then the Fourier-Sato transform is the the functor

$$\begin{array}{cccc} \varPhi_E & : & D^b_{mon}(E, \mathbf{C}) & \to & D^b_{mon}(E^*, \mathbf{C}) \\ & & \mathscr{F} & \mapsto & \check{q}_! q^* \mathscr{F}[\operatorname{rank} E]. \end{array}$$

It is an equivalence of categories preserving perverse sheaves. Useful compatibilities between Fourier transform and other functors are stated in [AHJR14, 2.7 and Appendix A]. The Fourier-Sato transform exists also in the equivariant setting. If X is a G-variety for some algebraic group G and $E \to X$ is a G-equivariant vector bundle, its dual is also an equivariant vector bundle and the Fourier-Sato transform gives a perverse equivalence of categories

$$\Phi_E : D^b_{G,mon}(E) \to D^b_{G,mon}(E^*).$$

5.D.3 Action on the singular support

The main result of this section is that the Fourier-Sato transform preserves the singular support.

Let $p : E \to X$ be a complex vector bundle and $\check{p} : E^* \to X$ its dual. We can identify T^*E and T^*E^* as complex fiber bundles (and also symplectic varieties) over E^* as follows ([KS90, V.5.5]). First consider the relative cotangent bundle of p, $T^*(E/X)$. It is the cokernel of the monomorphism of vector bundles $E \times_X T^*X \to T^*E$. Then, we have a morphism $T^*(E/X) \to E^*$ given on fibers over X by

$$T^*(E/X)_x \simeq E_x \times (E_x)^* \to (E_x)^*.$$

Composing with $T^*E \to T^*(E/X)$, we obtain the map $p' : T^*E \to E^*$. Proposition 5.5.1 in [KS90] gives the existence of an isomorphism of vector bundles over E^*

$$\begin{tikzcd} T^*E \arrow[rr, "\Xi"] \arrow[dr, "p'"'] & & T^*E^* \arrow[dl, "\check{p}"] \\ & E^* & \end{tikzcd} .$$

It is given in local coordinates by $(x, y, \zeta, \xi) \mapsto (x, \xi, \zeta, -y)$, where x is a local coordinate on X and y a local coordinate on the fiber E_x. In our context, Theorem 5.5.5 of *loc. cit.* can be formulated as follows.

Theorem 5.D.3. *Let $\mathscr{F} \in D^b_{mon}(E, \mathbf{C})$. Then,*

$$\Xi(SS(\mathscr{F})) = SS(\varPhi(\mathscr{F})).$$

5.D.4 The Fourier-Sato transform for quivers

We briefly describe how we will use the Fourier-Sato transform for quivers. Let $Q = (I, \Omega)$ be a quiver. Let $\Omega = \Omega_1 \sqcup \Omega_2$ be a partition of the set of arrows. We obtain new quivers $Q_1 = (I, \Omega_1)$ and $Q_2 = (I, \Omega_2)$. Let $\overline{\Omega}_2$ be the opposite set of arrows (we reverse the direction of arrows in Ω_2) and $\Omega' = \Omega_1 \sqcup \overline{\Omega}_2$. Let $Q' = (I, \Omega')$. For $\mathbf{d} \in \mathbf{N}^I$, we have two vector bundles over $E_{Q_1,\mathbf{d}}$:

$$\begin{tikzcd} E_{Q,\mathbf{d}} \arrow[dr, "\pi"'] & & E_{Q',\mathbf{d}} \arrow[dl, "\check{\pi}"] \\ & E_{Q_1,\mathbf{d}} & \end{tikzcd}$$

given by the projection on the corresponding direct factor of $E_{Q,\mathbf{d}}$ (resp. $E_{Q',\mathbf{d}}$). The trace maps identifies $\check{\pi}$ with the dual of π. The Fourier-Sato transform then gives a perverse equivalence of categories

$$\Phi : D^b_{G_{\mathbf{d}},mon}(E_{Q,\mathbf{d}}) \to D^b_{G_{\mathbf{d}},mon}(E_{Q',\mathbf{d}}).$$

Chapter 6

Perverse sheaves with nilpotent singular support on the stack of coherent sheaves on an elliptic curve

We define a stratification of the moduli stack of coherent sheaves on an elliptic curve which allows us (1) to give an explicit description of the irreducible components of the global nilpotent cone of elliptic curves, (2) to establish an explicit bijection between the simple objects of the category of perverse sheaves defined by Schiffmann to categorify the elliptic Hall algebra (the so-called spherical Eisenstein sheaves) and the irreducible components of the global nilpotent cone and (3) to give an explicit description and parametrization of the perverse sheaves on the moduli stack of coherent sheaves on an elliptic curve having nilpotent singular support. Along the way, we find a combinatorial parametrization of the irreducible components of the semistable locus of the elliptic global nilpotent cone.

Contents

6.1 Introduction

In this chapter, we study the relationship between the irreducible components of the global nilpotent cone of an elliptic curve and the simple objects of some categories of perverse sheaves on the stack of coherent sheaves (Eisenstein perverse sheaves) relevant in the geometric Langlands program. The global nilpotent cone is a closed substack of the stack of Higgs bundle whose geometry has been studied in depth and is also an essential object in the geometric Langlands program ([Lau88, Lau87]). The stack of Higgs bundles is the cotangent stack to the stack of coherent sheaves and the global nilpotent cone is a Lagrangian substack ([Lau88, Gin01]). When the underlying curve is the affine line, the global nilpotent cone for torsion sheaves of length d is the stack quotient given by pairs of $d \times d$ matrices, the second one being nilpotent, modulo the simultaneous conjugation by the general linear group GL_d. This stack plays a important role in Springer theory and in the microlocal study of character sheaves on reductive Lie algebras ([Mir04]). Spherical Eisenstein sheaves form a category of perverse sheaves on the stack of coherent sheaves and are the perverse sheaves appearing as shifted direct summands of the induction of trivial local systems ([Sch12b]). In this chapter, we define *twisted spherical Eisenstein sheaves*. They form a bigger category: we take the simple constituents of inductions of arbitrary local systems. They can be defined for an arbitrary smooth, projective curve (over a finite field and in the ℓ-adic setting or over a complex curve, working with $\mathbf{Q}$ or $\mathbf{C}$ coefficients). Twisted spherical Eisenstein perverse sheaves supported on some connected component of the stack of torsion sheaves have an explicit description in terms of local systems associated to representations of the symmetric groups and local systems on the curve. This is one of our main results. The particular structure of the stack of coherent sheaves on an elliptic curve which rests upon the description by Atiyah of the category of such objects ([Ati57]) allowed Schiffmann to describe in explicit terms the whole category of spherical Eisenstein sheaves ([Sch12b]). For curves of genus bigger that two, the description remains mysterious. The interest in this category lies in the fact that it gives a geometric categorification of the elliptic Hall algebra defined in [BS12]. The elliptic Hall algebra is a deformation of the Hopf algebra of diagonally symmetric polynomials

$$\mathbf{\Lambda}^+ = \mathbf{C}[x_1^{\pm 1}, \ldots, y_1, \ldots]^{\mathfrak{S}_\infty}.$$

The combinatorial study of such a ring is part of the theory of *multisymmetric* functions, which attempts to generalize to an arbitrary number of sets of variables the classical theory of Macdonald of symmetric functions ([Mac15]). The elliptic Hall algebra has now appeared in a great diversity of problems: in the study of the K-theory of the Hilbert scheme of the affine plane ([SV13b]), in skein theory of tori ([MS17]), in diagrammatic categorification ([CLL$^+$18]),...

Motivated by a putative Lagrangian construction of a specialization of the elliptic Hall algebra in the spirit of Lusztig's semicanonical basis of quantum groups ([Lus00]), by the fact that in the context of quivers, Lusztig sheaves ([Lus91]) are in canonical one-to-one correspondence with irreducible components of Lusztig nilpotent variety ([KS97]) and by a possible geometric interpretation of elliptic Kostka polynomials, which we leave for future investigations, we were led to the study of the characteristic cycle map from the category of spherical Eisenstein sheaves to Lagrangian cycles in the stack of Higgs bundles. Our main results are the description of the irreducible components of the global nilpotent cone of an elliptic curve, the unitriangularity of the characteristic cycle map and the description of simple perverse sheaves on the stack of coherent sheaves on an elliptic curve having nilpotent singular support. As a corollary, we deduce that the characteristic cycle map induces a canonical bijection between isomorphism classes of simple spherical Eisenstein sheaves and irreducible components of the global nilpotent cone.

We would like to mention that Bozec has an other approach to the description of the irreducible components of the global nilpotent cone which works for any genus ([Boz17]). His approach uses the Jordan type of the Higgs field while our is very specific to elliptic curves.

6.1.1 The main results

We give a brief overview of the main results of this chapter. We refer to relevant sections for more details on the notation.

The irreducible components of the elliptic global nilpotent cone

We consider a complex elliptic curve X. The category of coherent sheaves on X is denoted by $\mathrm{Coh}(X)$, the set of isomorphism classes of coherent sheaves by $|\mathrm{Coh}(X)|$ and the moduli stack of objects of $\mathrm{Coh}(X)$ by $\mathfrak{Coh}(X)$. We adopt similar notations for subcategories of $\mathrm{Coh}(X)$ when this makes sense. The class of a coherent sheaf $\mathcal{F}$ on X is the pair $\alpha = (r, d)$ of its rank and its degree, $r = \operatorname{rank} \mathcal{F}$, $d = \deg \mathcal{F}$. It is an element of the monoid $\mathbf{Z}^+ = \{(r, d) \in \mathbf{Z}^2 \mid r > 0 \text{ or } (r = 0 \text{ and } d > 0)\}$. A coherent sheaf $\mathcal{F}$ on X is said *semistable* provided that for any proper nonzero subsheaf $\mathcal{G}$, the inequality $\frac{\deg \mathcal{G}}{\operatorname{rank} \mathcal{G}} \leq \frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}}$ holds. The quantity $\frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}}$ is called the slope of $\mathcal{F}$ and is denoted by $\mu(\mathcal{F})$. If the inequalities are always strict, the sheaf is said stable. Let $\mathscr{P}$ stand for the set of all partitions. In Section 6.2.3, we will define a partition of the set of isomorphism classes of rank r, degree d semistable coherent sheaves on X,

$$|\mathrm{Coh}^{ss}_{\alpha}(X)| = \bigsqcup_{\xi \in (\mathbf{N}^{\mathscr{P}})_{\delta}} |\mathrm{Coh}^{ss}_{\alpha,\xi}(X)|$$

where $\alpha = (r,d)$, $\delta = \gcd(r,d)$ and $(\mathbf{N}^{\mathscr{P}})_{\delta}$ denotes the set of functions $\mathscr{P} \to \mathbf{N}$ such that $\sum_{\lambda \in \mathscr{P}} \xi(\lambda)|\lambda| = \delta$ (in particular, ξ has finite support).

The Harder-Narasinhan filtration of a coherent sheaf $\mathcal{F}$ on X is the unique filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_s = \mathcal{F}$ such that any successive subquotient $\mathcal{F}_j/\mathcal{F}_{j-1}$ is semistable and the sequence $(\mu(\mathcal{F}_j/\mathcal{F}_{j-1}))_{1 \leq j \leq s}$ is strictly decreasing. This sequence is called the Harder-Narasimhan type of $\mathcal{F}$. More generally, for any $\alpha \in \mathbf{Z}^+$, we will define a partition of the set of isomorphism classes of coherent sheaves of Harder-Narasimhan type (abbreviated *HN-type*) $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s)$, where $\alpha_i = (r_i, d_i) \in \mathbf{Z}^+$:

$$|\mathrm{Coh}_{\boldsymbol{\alpha}}(X)| = \bigsqcup_{\boldsymbol{\xi} \in (\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}}} |\mathrm{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)|$$

where $(\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}} = \{\boldsymbol{\xi} = (\xi_1, \ldots, \xi_s) \in (\mathbf{N}^{\mathscr{P}})^s \mid \xi_i \in (\mathbf{N}^{\mathscr{P}})_{\delta_i} \text{ for } 1 \leq i \leq s\}$, $\delta_i = \gcd(\alpha_i)$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_s)$. An element $\boldsymbol{\xi}$ is said *regular* provided that for any $1 \leq i \leq s$ and any partition $\lambda \in \mathscr{P}$, $\xi_i(\lambda) \neq 0$ only if the length of λ is 1. The datum of a regular $\boldsymbol{\xi}$ is equivalent to the datum of the s-tuple of partitions $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s)$ where $\lambda_i = (j^{\xi_i((j))}, j \geq 1)$, and (j) denotes the partition of j of length one. In particular, $\boldsymbol{\lambda} \in \mathscr{P}_{\boldsymbol{\delta}} = \{(\lambda_1, \ldots, \lambda_s) \in \mathscr{P}^s \mid |\lambda_i| = \delta_i\}$. In this case, we denote $\xi_i = \xi_{\lambda_i}$ and $\boldsymbol{\xi} = \boldsymbol{\xi}_{\boldsymbol{\lambda}}$.

This partition induces a locally closed stratification of the moduli stack of coherent sheaves of HN-type $\boldsymbol{\alpha}$:

$$\mathfrak{Coh}_{\boldsymbol{\alpha}}(X) = \bigsqcup_{\boldsymbol{\xi} \in (\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}}} \mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)$$

and a stratification of the moduli stack of coherent sheaves of class $\alpha = (r,d) \in \mathbf{Z}^+$ in the numerical Grothendieck group:

$$\mathfrak{Coh}_{\alpha}(X) = \bigsqcup_{\boldsymbol{\alpha} \in HN(\alpha)} \bigsqcup_{\boldsymbol{\xi} \in (\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}}} \mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)$$

where the first union runs over HN-types of rank r and degree d, $\boldsymbol{\delta} = (\gcd \alpha_i)_{1 \leq i \leq s}$ if $\boldsymbol{\alpha} = (\alpha_i)_{1 \leq i \leq s}$, $(\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}} = \{\boldsymbol{\xi} = (\xi_i)_{1 \leq i \leq s} \in (\mathbf{N}^{\mathscr{P}})^s \mid \xi_i \in (\mathbf{N}^{\mathscr{P}})_{\delta_i}\}$. The moduli stack of Higgs sheaves of type $\alpha = (r,d)$ is denoted $\mathfrak{Higgs}_{\alpha}(X)$ (Section 6.3). It contains the closed substack of nilpotent Higgs sheaves $\mathscr{N}_{\alpha}$ and there is a natural projection $\pi_{\alpha} : \mathfrak{Higgs}_{\alpha}(X) \to \mathfrak{Coh}_{\alpha}(X)$ forgetting the Higgs field. The restriction of this projection to the global nilpotent cone is denoted

$$\pi_{\alpha,\mathscr{N}} : \mathscr{N}_{\alpha} \to \mathfrak{Coh}_{\alpha}.$$

For $\boldsymbol{\alpha} \in HN(\alpha)$, we let $\mathscr{N}_{\boldsymbol{\alpha}}$ be the union of the irreducible components of dimension $\dim \mathscr{N}$ of

$\pi^{-1}_{\alpha,\mathscr{N}}(\mathfrak{Coh}_{\boldsymbol{\alpha}}(X))$. There is an induced projection

$$\pi_{\boldsymbol{\alpha},\mathscr{N}} : \mathscr{N}_{\boldsymbol{\alpha}} \to \mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$$

If furthermore $\boldsymbol{\xi} \in (\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}}$, we let $\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}} = \pi^{-1}_{\boldsymbol{\alpha},\mathscr{N}}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X))$, $\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}}}$ denote the closure of $\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}}$ in $\mathscr{N}_{\boldsymbol{\alpha}}$ and $\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}}}$ denote the closure of $\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}}$ in $\mathscr{N}_{\alpha}$. If $\boldsymbol{\xi} = \boldsymbol{\xi}_{\boldsymbol{\lambda}}$ for some uplet of partitions $\boldsymbol{\lambda} \in \mathscr{P}_{\boldsymbol{\delta}}$, that is $\boldsymbol{\xi}$ is regular, we let $\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}} = \mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}_{\boldsymbol{\lambda}}}$.

The following theorem gives a description of the irreducible components of $\mathscr{N}_{\alpha}$ and $\mathscr{N}_{\boldsymbol{\alpha}}$ different from that of [Boz17] and more convenient for the computation of the characteristic cycle of spherical Eisenstein sheaves (Theorem 6.1.2).

Theorem 6.1.1. *Let $\alpha \in \mathbf{Z}^+$. The irreducible components of $\mathscr{N}_{\alpha}$ are the $\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}$ for $\boldsymbol{\alpha} \in HN(\alpha)$ and $\boldsymbol{\lambda} \in \mathscr{P}_{\boldsymbol{\delta}}$.*

Let $\boldsymbol{\alpha} \in HN(\alpha)$. The irreducible components of $\mathscr{N}_{\boldsymbol{\alpha}}$ are the $\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}$ for $\boldsymbol{\lambda} \in \mathscr{P}_{\boldsymbol{\delta}}$.

We order the set of irreducible components of $\mathscr{N}_{\alpha}$ as follows:

$$\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}} \leq \overline{\mathscr{N}_{\boldsymbol{\beta},\boldsymbol{\nu}}} \iff \begin{cases} \mathfrak{Coh}_{\boldsymbol{\beta}}(X) \subset \overline{\mathfrak{Coh}_{\boldsymbol{\alpha}}}(X) \text{ is a strict inclusion} \\ \text{or} \\ \boldsymbol{\beta} = \boldsymbol{\alpha} \text{ and for any } 1 \leq i \leq s,\ \nu_i \leq \lambda_i \end{cases}$$

The first condition on the containment of Harder-Narasimhan strata can be reformulated in purely combinatorial terms using Harder-Narasimhan polytopes (see [Sch12b, §1.1 e)]). We order similarly the set of irreducible components $\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}$ of $\mathscr{N}_{\boldsymbol{\alpha}}$ (it reduces to the anti-dominant order on the uplet of partitions $\boldsymbol{\lambda}$).

Bijectivity of the characteristic cycle map

Let $\alpha \in \mathbf{Z}^+$. We complete the space $\mathbf{Z}[\mathrm{Irr}(\mathscr{N}_{\alpha})]$ of functions $\mathrm{Irr}(\mathscr{N}_{\alpha}) \to \mathbf{Z}$ with respect to the dimension of the support and let $\mathbf{Z}[\widehat{\mathrm{Irr}(\mathscr{N}_{\alpha})}]$ be the completed space. More precisely, $\mathbf{Z}[\widehat{\mathrm{Irr}(\mathscr{N}_{\alpha})}]$ consists of all sums $\sum_{i\geq 0} a_i[\Lambda_i]$ where $a_i \in \mathbf{Z}$ and Λ_i is an irreducible component of $\mathscr{N}_{\alpha}$. The category of spherical Eisenstein perverse sheaves on $\mathfrak{Coh}_{\alpha}(X)$ is denoted by $\mathcal{P}^{\alpha}$ (Section 6.4). Isomorphism classes of simple spherical Eisenstein perverse sheaves on $\mathfrak{Coh}_{\alpha}(X)$ are parametrized by pairs $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in HN(\alpha)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s)$ is a s-tuple of partitions such that $|\lambda_i| = \delta_i := \gcd \alpha_i$ (Theorem 6.4.1). We let $\mathscr{F}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}$ be the corresponding simple perverse sheaf. Isomorphism classes of simple perverse sheaves in $\mathcal{P}^{\alpha}$ are ordered in the same way as irreducible components of $\mathscr{N}_{\alpha}$:

$$[\mathscr{F}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}] \leq [\mathscr{F}_{\boldsymbol{\beta},\boldsymbol{\nu}}] \iff \begin{cases} \mathfrak{Coh}_{\boldsymbol{\beta}}(X) \subset \overline{\mathfrak{Coh}_{\boldsymbol{\alpha}}}(X) \text{ is a strict inclusion} \\ \text{or} \\ \boldsymbol{\beta} = \boldsymbol{\alpha} \text{ and for any } 1 \leq i \leq s,\ \nu_i \leq \lambda_i \end{cases}$$

We complete the Grothendieck group of $\mathcal{P}^{\alpha}$ in a similar manner, in terms of the dimension of the support. The completed Grothendieck group is then denoted $\widehat{K_0(\mathcal{P}^{\alpha})}$. This completion was defined in [BS12, Sch12b] by defining an adic valuation on $K_0(\mathcal{P}^\alpha)$.

Theorem 6.1.2. *The characteristic cycle map*

$$CC : \widehat{K_0(\mathcal{P}^\alpha)} \to \mathbf{Z}[\widehat{\mathrm{Irr}(\mathscr{N}_\alpha)}]$$

is an isomorphism of $\mathbf{Z}$*-modules. This isomorphism is lower unitriangular with respect to the basis of simple perverse sheaves on the left and the basis of irreducible components of* $\mathscr{N}_\alpha$ *on the right, when ordered as above.*

The isomorphism of Theorem 6.1.2 induces a canonical bijection between the set of isomorphism classes of simple sheaves in $\mathcal{P}^\alpha$ and $\mathrm{Irr}(\mathscr{N}_\alpha)$. This bijection is described by $\mathscr{F}_{\boldsymbol{\alpha},\boldsymbol{\lambda}} \leftrightarrow \overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}$.

Perverse sheaves with nilpotent singular support on the stack of coherent sheaves on an elliptic curve

In our last main result, which we prove in Section 6.6.4, we describe explicitly the simple perverse sheaves on the stack $\mathfrak{Coh}_\alpha(X)$, $\alpha \in \mathbf{Z}^+$, whose singular supports are nilpotent (that is, a union of some of the irreducible components of the global nilpotent cone $\mathscr{N}_\alpha$). We introduce some piece of notation. Let X be an elliptic curve. If $\alpha = (r, d) \in \mathbf{Z}^+$ is coprime, then the semistable locus $\mathfrak{Coh}_{(\alpha)}(X)$ is isomorphic to the stack quotient $X/\mathbf{G}_m$, where the action of $\mathbf{G}_m$ is trivial (Section 6.2.3). Therefore, we have a family of local systems $\mathscr{L}_z$ on $\mathfrak{Coh}_{(\alpha)}(X)$ indexed by characters of $\pi_1(X) \simeq \mathbf{Z}^2$ if we work with the topological fundamental group, or $\pi_1^{ét}(X) \simeq \widehat{\mathbf{Z}^2}$ when considering the étale fundamental group. We prove in Proposition 6.6.5 that such a local system extends to the whole of $\mathfrak{Coh}_\alpha(X)$. Theorem 6.6.10 can be formulated as follows. The second statement is a consequence of Lemma 6.6.8.

Theorem 6.1.3. *The perverse sheaves on* $\mathfrak{Coh}_\alpha(X)$ *having a nilpotent singular support are precisely the perverse sheaves which are the simple constituents of the inductions of perverse sheaves of the form* $\mathcal{IC}(\mathscr{L}_z)$ *on various* $\mathfrak{Coh}_\alpha(X)$*,* $\alpha \in \mathbf{Z}^+$ *coprime as defined above. Moreover, it suffices to take* $\alpha = (0, 1)$ *and* $\alpha = (1, d)$ *for* d *in a subset of the integers not bounded below.*

Such perverse sheaves are called in this chapter *twisted spherical Eisenstein perverse sheaves* as when all the local systems $\mathscr{L}_z$ are trivial (that is $z = 1$), then we obtain the perverse sheaves considered by Schiffmann in [Sch12b] and called there *spherical Eisenstein perverse sheaves.*

The proofs of Theorems 6.1.1 and 6.1.2 will be given in Section 6.5. The proof of Theorem 6.1.3 will be given in Section 6.6.4.

6.1.2 Strategy of proof

Theorem 6.1.1

We compute the dimension of the locally closed substacks $\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}}$ of $\mathscr{N}_{\boldsymbol{\alpha}}$ in two steps. First, we assume that $\boldsymbol{\alpha} = (\alpha)$, so that we only deal with semistable sheaves of fixed rank and degree. By the classification due Atiyah of vector bundles over an elliptic curve, such coherent sheaves form an algebraic stack isomorphic to the stack of torsion sheaves of degree $\gcd(\alpha)$. The case of torsion sheaves corresponds to the Springer situation for which the description of the irreducible components is well-known. Then, we use the morphism

$$p_{\boldsymbol{\alpha}} : \mathfrak{Coh}_{\boldsymbol{\alpha}}(X) \to \prod_{i=1}^{s} \mathfrak{Coh}_{(\alpha_i)}(X)$$

which sends a coherent sheaf of HN-type $\boldsymbol{\alpha}$ to the collection of semistable factors of the Harder-Narasimhan filtration. This morphism has very favourable properties (it is a vector bundle stack ([GPHS14, Hei04])– or more precisely an iteration of vector bundle stacks), and the dimension of the fiber is easy to compute. This allows us to perform efficiently the computation of the dimension of $\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}}$. Combined with the irreducibility of $\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)$, this will allow us to prove Theorem 6.1.1. Our strategy is inspired by the work of Ringel who determined the irreducible components of Lusztig nilpotent variety for affine quivers ([Rin98a, Rin98b]).

Theorem 6.1.2

Let $\alpha \in \mathbf{Z}^+$ and $\boldsymbol{\alpha} \in HN(\alpha)$. We let $\mathfrak{Coh}_{\geq \boldsymbol{\alpha}}(X)$ be the open substack of $\mathfrak{Coh}_{\alpha}(X)$ which is the union of the $\mathfrak{Coh}_{\boldsymbol{\beta}}(X)$ such that $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X) \subset \overline{\mathfrak{Coh}_{\boldsymbol{\beta}}}(X)$. We let $j_{\boldsymbol{\alpha}} : \mathfrak{Coh}_{\geq \boldsymbol{\alpha}}(X) \to \mathfrak{Coh}_{\alpha}(X)$ be the (open) inclusion. Since we work with completions, using the poset structure on the set of Harder-Narasimhan strata (endowed with the containment order) and letting

$$\mathcal{P}^{\boldsymbol{\alpha}} = \{(j_{\boldsymbol{\alpha}})^* \mathscr{F} : \mathscr{F} \in \mathrm{Ob}(\mathcal{P}^{\alpha}), \operatorname{supp} \mathscr{F} = \overline{\operatorname{supp} \mathscr{F} \cap \mathfrak{Coh}_{\boldsymbol{\alpha}}(X)}\},$$

it suffices to show that the characteristic cycle map

$$CC : K_0(\mathcal{P}^{\boldsymbol{\alpha}}) \to \mathbf{Z}[\mathrm{Irr}(\mathscr{N}_{\boldsymbol{\alpha}})]$$

is a unitriangular isomorphism. We proceed in two steps by first assuming $\boldsymbol{\alpha} = (\alpha)$ so that we work on the semistable locus. There we use again the structure of coherent sheaves on an elliptic curve which allows us to consider only torsion sheaves. Last, using the explicit description of the simple ojects of $\mathcal{P}^{\boldsymbol{\alpha}}$, and the morphism $p_{\boldsymbol{\alpha}}$, we are able to give a rather concrete description of the characteristic cycle of any perverse sheaf of the category $\mathcal{P}^{\boldsymbol{\alpha}}$. Proving the isomorphism is now easy.

Theorem 6.1.3

The key fact, which is true not only for elliptic curves but also for curves of genus $g \geq 2$, is that any local system on some connected component of the semisimple locus extends to the whole connected component of $\mathfrak{Coh}(X)$. This can be proved by codimension considerations and by using the determinant morphism to the Picard stack, which provides a retraction to the inclusion of the Picard stack in the stack of rank one coherent sheaves. This allows us to prove that any induction of perverse sheaves as in Theorem 6.1.3 has nilpotent singular support. The other implication of Theorem 6.1.3 does not present difficulties and follows from the consideration of the restriction of the induction diagram to the sheaves of HN-type $\boldsymbol{\alpha} \in HN(\alpha)$ (6.4.3).

6.1.3 Contents of the chapter

In Section 6.2, we recall some standard facts concerning the stack of coherent sheaves on curves with an emphasis on the case of elliptic curves. We follow the presentation of Schiffmann ([Sch12b]). We define stratifications of the semistable loci and glue them using the Harder-Narasimhan stratification to obtain a refined stratification of any Harder-Narasimhan stratum. In Section 6.3, we introduce the stack of Higgs sheaves and the global nilpotent cone. We give a description of the irreducible components of the global nilpotent cone and of its semistable locus for an elliptic curve. A partial order on the set of its irreducible components is defined combining the natural order on Harder-Narasimhan strata given by the dominance order and the antidominance order on partitions. In Section 6.4, we recall the induction and restriction functors on the derived category of constructible sheaves and the definition of spherical Eisenstein sheaves. In this generality, these functors appear in the work of Schiffmann ([Sch12b]). They are defined in analogy with the quiver induction and restriction functors considered by Lusztig ([Lus91]) and with the induction functor studied by Laumon ([Lau87, Lau90]). We recall the explicit description of simple Eisenstein perverse sheaves in terms of a local system on a smooth part of their support due to Schiffmann. This description allows us to describe the multiplicities of the irreducible components of the global nilpotent cone in the singular support of the restriction of a simple Eisenstein sheaf to its supporting Harder-Narasimhan stratum. Perverse sheaves on the nilpotent cone of $\mathfrak{gl}_n$ and the Kostka numbers appear in this description. In Section 6.5, we detail the proof of the first two main theorems, namely the description of the irreducible components of the global nilpotent cone (Theorem 6.1.1) and the lower unitriangularity of the characteristic cycle map from the category of spherical Eisenstein sheaves (Theorem 6.1.2). In Section 6.6, we introduce the material needed to prove the third main result, Theorem 6.1.3, which gives an explicit description of the simple perverse sheaves on the stack of coherent sheaves on an elliptic curve having singular support in the global nilpotent cone. Along the way, we determine codimension one Harder-Narasimhan strata and show that local systems on the semi-stable locus always extend to the whole stack of coherent sheaves, for any curve.

6.1.4 Notations and conventions

We let $\mathscr{P}$ be the set of partitions of positive integers. If $j \geq 1$, the unique partition of j of length one is (j). The set $\mathscr{P}$ is naturally ordered by the dominance order. If $d \geq 1$, $\mathscr{P}_d$ is the subset of partitions of d. If $\lambda \in \mathscr{P}_d$, $|\lambda| = d$ and $l(\lambda)$ is the length of λ (the number of nonzero parts). If $\boldsymbol{d} = (d_1, \dots, d_s)$ is a s-uplet, $\mathscr{P}_{\boldsymbol{d}} = \{(\lambda_1, \dots, \lambda_s) \in \mathscr{P}^s \mid \lambda_i \in \mathscr{P}_{d_i}\}$. We let $\mathbf{N}^{\mathscr{P}}$ be the set of functions $\mathscr{P} \to \mathbf{N}$. For $d \in \mathbf{N}$, $(\mathbf{N}^{\mathscr{P}})_d$ is the set of functions $\xi \in \mathbf{N}^{\mathscr{P}}$ such that $\sum_{\lambda \in \mathscr{P}} \xi(\lambda)|\lambda| = d$. If $\xi \in (\mathbf{N}^{\mathscr{P}})_d$ is such that for any $\lambda \in \mathscr{P}$, $\xi(\lambda) \neq 0 \implies l(\lambda) = 1$, the datum of ξ is equivalent to the datum of the partition $\lambda = (j^{\xi((j))} : j \geq 1) \in \mathscr{P}_d$. In this case, we write $\xi = \xi_\lambda$. If $\boldsymbol{d} = (d_1, \dots, d_s)$ is a s-uplet, $(\mathbf{N}^{\mathscr{P}})_{\boldsymbol{d}}$ is the set of s-uplets of functions $(\xi_1, \dots, \xi_s)$ such that for $1 \leq i \leq s$, $\xi_i \in (\mathbf{N}^{\mathscr{P}})_{d_i}$. If for any $1 \leq i \leq s$ and any $\lambda \in \mathscr{P}$, $\xi_i(\lambda) \neq 0 \implies l(\lambda) = 1$, the datum of $\boldsymbol{\xi}$ is equivalent to the datum of the s-uplet of partitions $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s) \in \mathscr{P}_{\boldsymbol{d}}$, where $\lambda_i = (j^{\xi_i((j))} : j \geq 1)$. In this case, we write $\boldsymbol{\xi} = \boldsymbol{\xi}_{\boldsymbol{\lambda}}$. We denote $\mathfrak{Coh}(X)$ the stack of coherent sheaves on a given fixed curve X. We let $\mathbf{Z}^+ = \{(r,d) \in \mathbf{Z}^2 \mid r > 0 \text{ or } r = 0 \text{ and } d > 0\}$. For $\alpha \in \mathbf{Z}^+$, $\mathfrak{Coh}_\alpha(X)$ is the substack of coherent sheaves of class α and $\mu(\alpha) = \frac{d}{r}$ if $\alpha = (r,d)$. The set of Harder-Narasimhan types of class α is $HN(\alpha)$. Its elements are uplets $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s)$ for some $s \geq 1$, $\alpha_i \in \mathbf{Z}^+$, $\mu(\alpha_1) > \ldots > \mu(\alpha_s)$, $\sum_{i=1}^s \alpha_i = \alpha$. For $\alpha = (r,d) \in \mathbf{Z}^2$, $\delta = \gcd(\alpha) = \gcd(r,d)$. For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in (\mathbf{Z}^2)^s$, $\boldsymbol{\delta} = \gcd(\boldsymbol{\alpha}) = (\gcd(\alpha_1), \ldots, \gcd(\alpha_s))$. For $\alpha \in \mathbf{Z}^+$ and $\boldsymbol{\alpha} \in HN(\alpha)$, $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ denotes the corresponding Harder-Narasimhan stratum. We will consider the derived category of constructible sheaves on the stack of coherent sheaves on a smooth projective curve. Each connected component $\mathfrak{Coh}_\alpha(X)$ of this stack has a presentation as an increasing union of open substacks which are quotient stacks, $\mathfrak{Coh}_\alpha(X) = \varinjlim_n X_n/G_n$ where X_n is an open subvariety of some Quot scheme and G_n some (reductive) algebraic group. The derived category of the Artin stack $\mathfrak{Coh}_\alpha(X)$ can be dealt with using the formalism of equivariant derived categories ([BL94]): $D_c(\mathfrak{Coh}_\alpha(X)) = \varprojlim_n D_{c,G_n}(X_n)$. If X is any algebraic variety and $n \geq 1$ an integer, we let $\Delta \subset X^n$ and $\Delta \subset S^n X$ be the big diagonals (the closed subvarieties of n-uplets $(x_1, \ldots, x_n)$ such that two or more coordinates are equal).

6.2 The moduli stack of coherent sheaves on elliptic curves

6.2.1 Coherent sheaves on a curve

We recall here the fundamental properties of the stack of coherent sheaves on a curve that we will need. We consider a smooth projective curve X over the field of complex numbers $\mathbf{C}$. We let $\mathrm{Coh}(X)$ be the category of coherent sheaves on X. This is an abelian category of homological dimension one. To a coherent sheaf $\mathcal{F}$, we can assign its rank r and its degree d. The pair $\alpha = (r,d)$ belongs to

$$\mathbf{Z}^+ = \{(r,d) \in \mathbf{Z}^2 \mid r > 0 \text{ or } r = 0, d > 0\}.$$

and is called the *type* of $\mathcal{F}$. This assignment yields a group homomorphism

$$K_0(\mathrm{Coh}(X)) \to \mathbf{Z}^2 .$$

For a coherent sheaf $\mathcal{F}$ on X, we let $[\mathcal{F}] \in \mathbf{Z}^+$ be the corresponding pair. The Euler form of the category $\mathrm{Coh}(X)$ factors through this morphism: for any coherent sheaves $\mathcal{F}, \mathcal{G}$ on X such that $[\mathcal{F}] = (r_1, d_1)$ and $[\mathcal{G}] = (r_2, d_2)$,

$$\langle \mathcal{F}, \mathcal{G} \rangle = \dim \mathrm{Hom}(\mathcal{F}, \mathcal{G}) - \dim \mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) = (1-g) r_1 r_2 + (r_1 d_2 - r_2 d_1) \tag{6.2.1}$$

if g is the genus of X, thanks to Serre duality and the Riemann-Roch formula. When X is an elliptic curve, the first term disappears. There is a smooth stack $\mathfrak{Coh}(X)$ parametrizing the objects of $\mathrm{Coh}(X)$. It has an infinite number of connected components indexed by $\mathbf{Z}^+$:

$$\mathfrak{Coh}(X) = \bigsqcup_{\alpha \in \mathbf{Z}^+} \mathfrak{Coh}_\alpha(X)$$

where the connected component $\mathfrak{Coh}_\alpha(X)$ is an Artin stack locally of finite type which parametrizes coherent sheaves on X of class α. The dimension of $\mathfrak{Coh}_\alpha(X)$ is $r^2(g-1)$ if $\alpha = (r, d)$. In particular, it is of dimension 0 when X is an elliptic curve.

6.2.2 The category and moduli stack of torsion sheaves

Torsion sheaves on a curve

The moduli stack of torsion sheaves is the union $\mathfrak{Tor}(X)$ of the connected components $\mathfrak{Coh}_{(0,d)}(X)$, $d > 0$ of $\mathfrak{Coh}(X)$. Fix $d > 0$. There is a support map:

$$\chi : \mathfrak{Coh}_{(0,d)}(X) \to S^d X$$

to the d-th symmetric power of X sending a coherent sheaf to its support (with multiplicities).

The category of torsion sheaves on X is denoted $\mathrm{Tor}(X)$. For any closed point $x \in X$, we let $\mathrm{Tor}_x(X)$ be the subcategory of torsion sheaves supported at x. It is equivalent to the category of nilpotent representations of the Jordan quiver (the quiver with one vertex and one loop). In other words, the datum of a torsion sheaf of degree d supported at a given point is the same thing as the datum of a nilpotent $d \times d$ matrix. In particular, isomorphism classes of torsion sheaves supported at x are indexed by partitions according to the sizes of the Jordan blocks. We let $T_{x,\lambda}$ be the torsion sheaf associated to the partition $\lambda \in \mathscr{P}$.

Stratification of the stack of torsion sheaves on a curve

Let $d \geq 0$. The stack $\mathfrak{Coh}_{(0,d)}(X)$ admits a stratification indexed by the set $(\mathbf{N}^{\mathscr{P}})_d$ of functions $\xi : \mathscr{P} \to \mathbf{N}$ satisfying $\sum_{\lambda \in \mathscr{P}} \xi(\lambda)|\lambda| = d$. Namely, we write

$$\mathfrak{Coh}_{(0,d)}(X) = \bigsqcup_{\xi \in (\mathbf{N}^{\mathscr{P}})_d} \mathfrak{Coh}_{(0,d),\xi}(X)$$

where $\mathfrak{Coh}_{(0,d),\xi}(X)$ is defined as follows. We pick partitions $\lambda_1, \ldots, \lambda_s \in \mathscr{P}$ so that in this collection, any partition $\lambda \in \mathscr{P}$ appears precisely $\xi(\lambda)$ times. Then, geometric points of $\mathfrak{Coh}_{(0,d),\xi}(X)$ parametrize by definition isomorphism classes of torsion sheaves T on X isomorphic to a direct sum

$$\bigoplus_{i=1}^{s} T_{x_i,\lambda_i}$$

for pairwise distinct points $x_i \in X$. This stratification is analogous to the Jordan stratification of $\mathfrak{gl}_d$ (the case when $X = \mathbb{A}^1$) and also to the stratification of the regular locus of the representation stack of affine quivers ([Rin98b]) which has been of great help to the author in the understanding of perverse sheaves with nilpotent singular support (see [Hen20b]).

6.2.3 Category and stack of semistable sheaves of fixed rank and degree

Let $\alpha = (r, d) \in \mathbf{Z}^+$. For a nonzero coherent sheaf $\mathcal{F}$ on X of rank r and degree d, we define its slope $\mu(\mathcal{F}) = \frac{d}{r}$ with the convention that the slope of a torsion sheaf is infinite. We say that $\mathcal{F}$ is semistable if for any nonzero proper subsheaf $\mathcal{G} \subset \mathcal{F}$, $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$. Stable sheaves are those $\mathcal{F}$ for which the inequality is always strict. Given a fixed slope $\mu \in \mathbf{Q}$, the category of semistable coherent sheaves on X of slope μ is abelian, noetherian and artinian. There is a moduli stack of semistable coherent sheaves of slope μ, $\mathfrak{Coh}^{\mu}(X)$. It is an open substack of $\mathfrak{Coh}(X)$. If $\alpha = (r, d) \in \mathbf{Z}^+$ is such that $\mu = \frac{d}{r}$ and $\gcd(r, d) = 1$, we have a decomposition in connected components of $\mathfrak{Coh}^{\mu}(X)$:

$$\mathfrak{Coh}^{\mu}(X) = \bigsqcup_{l \geq 1} \mathfrak{Coh}_{(l\alpha)}(X),$$

where for any $\beta \in \mathbf{Z}^+$, $\mathfrak{Coh}_{(\beta)}(X)$ is the open subtack of $\mathfrak{Coh}_{\beta}(X)$ classifying semistable coherent sheaves of class β. If X is an elliptic curve, there is an equivalence of categories between $\mathrm{Tor}(X) = \mathrm{Coh}^{\infty}(X)$ and $\mathrm{Coh}^{\mu}(X)$, which can be constructed using mutations or equivalently, Fourier-Mukai transforms (see [Sch12b, §1.1 d)], [Kul90]). We let $\mathfrak{Tor}(X)$ be the stack of torsion sheaves on X. It coincides with $\mathfrak{Coh}^{\infty}(X)$. The equivalence sends semistable sheaves of rank r and degree d with $\mu = \frac{d}{r}$ to torsion sheaves of degree $\delta = \gcd(r, d)$. It also induces isomorphisms at the level of stacks

$$\epsilon_{\mu} : \mathfrak{Tor}(X) \to \mathfrak{Coh}^{\mu}(X)$$

and

$$\epsilon_{\alpha} : \mathfrak{Coh}_{(0,\delta)}(X) \to \mathfrak{Coh}_{(\alpha)}(X).$$

Thanks to the isomorphism ϵ_{α}, we transport the stratification of $\mathfrak{Coh}_{(0,\delta)}(X)$ to $\mathfrak{Coh}_{(\alpha)}(X)$:

$$\mathfrak{Coh}_{(\alpha)}(X) = \bigsqcup_{\xi \in (\mathbf{N}^{\mathscr{P}})_{\delta}} \mathfrak{Coh}_{(\alpha),\xi}(X)$$

where $\mathfrak{Coh}_{(\alpha),\xi}(X) = \epsilon_{\alpha}(\mathfrak{Coh}_{(0,\delta),\xi}(X))$.

We have the following properties for semistable sheaves of different slopes which will be useful for understanding the induction of perverse sheaves. If $\nu > \mu \in \mathbf{Q}$ are two slopes, $\mathcal{F} \in \mathrm{Coh}^{\mu}(X)$ and $\mathcal{G} \in \mathrm{Coh}^{\nu}(X)$, then

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) = 0.$$

Since the canonical bundle of an elliptic curve is trivial, Serre duality implies that

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0. \tag{6.2.2}$$

6.2.4 A refinement of the Harder-Narasimhan stratification

The Harder-Narasimhan stratification

Let $\mathcal{F}$ be a coherent sheaf on X. It admits a unique filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_s = \mathcal{F}$ such that for any $1 \leq i \leq s$, the quotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable and if μ_i denotes its slope, $\mu_1 > \mu_2 > \ldots > \mu_s$.

It is called the Harder-Narasimhan stratification of $\mathcal{F}$. The Harder-Narasimhan type of $\mathcal{F}$ is then the collection $([\mathcal{F}_i/\mathcal{F}_{i-1}])_{1\leq i\leq s}$. We let $HN(\alpha)$ be the set of all possible Harder-Narasimhan types of coherent sheaves of type α. For $\alpha \in \mathbf{Z}^+$, the Harder-Narasimhan stratification of $\mathfrak{Coh}_\alpha(X)$ is then

$$\mathfrak{Coh}_\alpha(X) = \bigsqcup_{\boldsymbol{\alpha}\in HN(\alpha)} \mathfrak{Coh}_{\boldsymbol{\alpha}}(X),$$

where $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ is the locally closed substack parametrizing coherent sheaves on X of HN-type $\boldsymbol{\alpha}$. In particular, if $\boldsymbol{\alpha} = (\alpha)$ for some $\alpha \in \mathbf{Z}^+$, then the notation $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X) = \mathfrak{Coh}_{(\alpha)}(X)$ is the one introduced in Section 6.2.3.

Refinement of the Harder-Narasimhan stratification for elliptic curves

Let $\alpha = (r, d) \in \mathbf{Z}^+$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s) \in HN(\alpha)$. There is a morphism of stacks

$$p_{\boldsymbol{\alpha}} : \mathfrak{Coh}_{\boldsymbol{\alpha}}(X) \to \prod_{i=1}^{s} \mathfrak{Coh}_{(\alpha_i)}(X)$$

which sends a coherent sheaf of HN-type $\boldsymbol{\alpha}$ to the collection of the subquotients of the Harder-Narasimhan filtration. If X is an elliptic curve, the Harder-Narasimhan filtration splits (noncanonically): if $\mathcal{F}$ is a coherent sheaf on X and $(\mathcal{F}_i)_{1\leq i\leq s}$ is its Harder-Narasimhan filtration,

$$\mathcal{F} \simeq \bigoplus_{i=1}^{s} \mathcal{F}_i/\mathcal{F}_{i-1}.$$

This is a consequence of the Ext-vanishing property (6.2.2). Therefore, the fiber of $p_{\boldsymbol{\alpha}}$ over a $\mathbf{C}$-point $(\mathcal{F}_1, \dots, \mathcal{F}_s)$ can be identified with the stack quotient

$$\mathrm{pt} / \bigoplus_{j<i} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_i/\mathcal{F}_{i-1}, \mathcal{F}_j/\mathcal{F}_{j-1}).$$

In particular, its dimension is

$$d_{\boldsymbol{\alpha}} = -\sum_{j<i}(r_i d_j - r_j d_i), \tag{6.2.3}$$

where $\alpha_i = (r_i, d_i) = [\mathcal{F}_i/\mathcal{F}_{i-1}]$.

We define a stratification of $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$:

$$\mathfrak{Coh}_{\boldsymbol{\alpha}}(X) = \bigsqcup_{\boldsymbol{\xi}\in(\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}}} \mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)$$

indexed by $(\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}} = \{\boldsymbol{\xi} = (\xi_1, \dots, \xi_s) \in (\mathbf{N}^{\mathscr{P}})^s \mid \sum_{\lambda\in\mathscr{P}} \xi_i(\lambda)|\lambda| = \gcd(\alpha_i) \text{ for any } 1 \leq i \leq s\}$, where

$$(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)) = p_{\boldsymbol{\alpha}}^{-1}\left(\prod_{i=1}^{s} \mathfrak{Coh}_{(\alpha_i),\xi_i}(X)\right).$$

In concrete terms, the $\mathbf{C}$-points of $\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)$ are the isomorphism classes of coherent sheaves on X

having HN-type $\boldsymbol{\alpha}$ and such that the i-th subquotient of the HN-filtration belongs to $\mathfrak{Coh}_{(\alpha_i),\xi_i}(X)$.

6.2.5 $\mathrm{SL}_2(\mathbf{Z})$-action on the stack of coherent sheaves

It is a well-known fact that Harder-Narasimhan strata of $\mathfrak{Coh}(X)$ can be indexed by convex path in $\mathbf{Z}^+$ starting from the origin. Namely, for $\alpha \in \mathbf{Z}^+$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s) \in HN(\alpha)$, we take the piecewise affine path $\bigsqcup_{i=0}^{s-1} [\sum_{j=0}^{i-1} \alpha_{s-j}, \sum_{j=0}^{i} \alpha_{s-j}]$ with the convention that $\alpha_0 = 0$. We denote $\boldsymbol{p_\alpha}$ this path. The convexity comes from the condition on the successive slopes of a Harder-Narasimhan type. The inclusion of Harder-Narasimhan strata can be described in terms of convex paths: for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in HN(\alpha)$, $\mathfrak{Coh}_{\boldsymbol{\beta}}(X) \subset \overline{\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)}$ if and only if $\boldsymbol{p_\beta}$ lies below $\boldsymbol{p_\alpha}$. This defines the order relation $\mathfrak{Coh}_{\boldsymbol{\beta}}(X) \subset \overline{\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)} \iff \boldsymbol{\beta} \geq \boldsymbol{\alpha}$ on $HN(\alpha)$ (the dense stratum is the smallest element of $HN(\alpha)$). The group $\mathrm{SL}_2(\mathbf{Z})$ acts naturally on $\mathbf{Z}^2$ by $\mathbf{Z}$-linear automorphisms. Therefore, it acts on tuples $(\alpha_1, \dots, \alpha_s)$ by $\gamma \cdot (\alpha_1, \dots, \alpha_s) = (\gamma \cdot \alpha_1, \dots, \gamma \cdot \alpha_s)$. If $A \subset HN(\alpha)$ is some set of Harder-Narasimhan types such that for any $\boldsymbol{\beta} \in HN(\alpha)$ and $\boldsymbol{\alpha} \in A$, $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ implies $\boldsymbol{\beta} \in A$ and $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ is such that $\gamma \cdot A = \{\gamma \cdot \boldsymbol{\alpha} : \boldsymbol{\alpha} \in A\} \subset HN(\gamma \cdot \alpha)$ (this is the condition that γ sends $\boldsymbol{p_\alpha}$ to a path contained in $\mathbf{Z}^+$), then γ induces an isomorphism between the open substacks $\mathfrak{Coh}_A = \bigsqcup_{\boldsymbol{\alpha} \in A} \mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ of $\mathfrak{Coh}_\alpha(X)$ and $\mathfrak{Coh}_{\gamma \cdot A} = \bigsqcup_{\boldsymbol{\alpha} \in A} \mathfrak{Coh}_{\gamma \cdot \boldsymbol{\alpha}}(X)$ of $\mathfrak{Coh}_{\gamma \cdot \alpha}(X)$,

$$i_\gamma : \mathfrak{Coh}_A \to \mathfrak{Coh}_{\gamma \cdot A} \tag{6.2.4}$$

already used in the proof of [Sch12b, Proposition 6.7].

6.3 The global nilpotent cone

In this section, we describe the irreducible components of the global nilpotent cone in terms of the stratification of Section 6.2.4.

6.3.1 Higgs sheaves and the global nilpotent cone

We refer to [SS18, §2.3] for more details and general properties of the stack of Higgs sheaves. A Higgs sheaf on X is a pair $(\mathcal{F}, \theta)$ of a coherent sheaf $\mathcal{F}$ and a morphism $\theta : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} K$ where K is the canonical bundle of X. The moduli stack of Higgs sheaves on X is denoted $\mathfrak{Higgs}(X)$. It has an infinite number of connected components indexed by $\mathbf{Z}^+$:

$$\mathfrak{Higgs}(X) = \bigsqcup_{\alpha \in \mathbf{Z}^+} \mathfrak{Higgs}_\alpha(X)$$

where $\mathfrak{Higgs}_\alpha(X)$ parametrizes those Higgs sheaves on X whose underlying coherent sheaf is of class α. There is a projection

$$\pi : \mathfrak{Higgs}(X) \to \mathfrak{Coh}(X)$$

forgetting the Higgs field which is compatible with the decompositions into connected components. For $\alpha \in \mathbf{Z}^+$, $\pi_\alpha : \mathfrak{Higgs}_\alpha(X) \to \mathfrak{Coh}_\alpha(X)$ denotes the restriction of π. It is the cotangent stack of $\mathfrak{Coh}_\alpha(X)$. Moreover, $\mathfrak{Higgs}_\alpha(X)$ is locally of finite type, of dimension $-2\langle \alpha, \alpha \rangle$. In particular, it is of dimension 0 for an elliptic curve. The global nilpotent cone is the closed substack $\mathscr{N}$ parametrizing nilpotent Higgs sheaves, that is pairs $(\mathcal{F},\theta)$ such that the composition

$$\mathcal{F}\xrightarrow{\theta}\mathcal{F}\otimes K\xrightarrow{\theta\otimes\mathrm{id}_K}\mathcal{F}\otimes K^{\otimes 2}\xrightarrow{\theta\otimes\mathrm{id}_{K^{\otimes 2}}}\dots\xrightarrow{\theta\otimes\mathrm{id}_{K^{\otimes(n-1)}}}\mathcal{F}\otimes K^{\otimes n}$$

vanishes for n sufficiently large. We write it $\mathscr{N}=\bigsqcup_{\alpha\in\mathbf{Z}^+}\mathscr{N}_\alpha$. For any $\alpha\in\mathbf{Z}^+$, $\mathscr{N}_\alpha$ is a Lagrangian substack of $\mathfrak{Higgs}_\alpha(X)$ ([Lau88, Gin01]). The restriction of π_α to $\mathscr{N}_\alpha$ is denoted $\pi_{\alpha,\mathscr{N}}$.

6.3.2 Irreducible components of the semistable nilpotent cone

We introduce the notion of stability for Higgs sheaves and show that for elliptic curve it coincides with the stability of the underlying coherent sheaf. Then, we give a parametrization of the irreducible components of the semistable locus of the elliptic global nilpotent cone (Corollary 6.3.2). For higher genus, see [Boz17].

Let $(\mathcal{F},\theta)$ be a Higgs sheaf on X. We say that it is semistable if for any $0\subsetneq\mathcal{G}\subsetneq\mathcal{F}$ such that $\theta(\mathcal{G})\subset\mathcal{G}\otimes K$, $\mu(\mathcal{G})\leq\mu(\mathcal{F})$, and stable if these inequalities are always strict.

Lemma 6.3.1. *Assume X is an elliptic curve. A Higgs sheaf on X is semistable (resp. stable) if and only if the underlying coherent sheaf is semistable (resp. stable).*

Proof. Let $(\mathcal{F},\theta)$ be a Higgs sheaf on X. The canonical bundle of an elliptic curve is trivial, so $\theta\in\operatorname{End}\mathcal{F}$. Moreover, any endomorphism respects the Harder-Narasimhan of $\mathcal{F}$ so the condition of being a semistable or stable Higgs sheaf is the same as the condition on the underlying coherent sheaf of being semistable or stable. □

The same argument shows that the Harder-Narasimhan stratification of $\mathfrak{Higgs}_\alpha(X)$ is induced by the Harder-Narasimhan stratification of $\mathfrak{Coh}_\alpha(X)$ when X is an elliptic curve: the former is the pull-back by π_α of the latter. For a HN-type $\boldsymbol{\alpha}$, we let $\mathfrak{Higgs}_{\boldsymbol{\alpha}}(X)$ be the locally closed substack of Higgs bundles of HN-type $\boldsymbol{\alpha}$. By Lemma 6.3.1, $\mathfrak{Higgs}_{\boldsymbol{\alpha}}(X)=\pi_\alpha^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha}}(X))$. We let $\mathscr{N}_{\boldsymbol{\alpha}}$ be the union of the irreducible components of dimension $\dim\mathscr{N}$ of $\mathfrak{Higgs}_{\boldsymbol{\alpha}}(X)\cap\mathscr{N}_\alpha$. In particular, $\mathscr{N}_{(\alpha)}$ denotes the stack of nilpotent semistable Higgs bundles of class α on an elliptic curve X. By Lemma 6.3.1, we have

$$\mathscr{N}_{(\alpha)}=\pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{(\alpha)}(X)).$$

Therefore, by Theorem 6.1.1, the irreducible components of $\mathscr{N}_{(\alpha)}$ can be described as in Corollary 6.3.2 below.

The semistable nilpotent cone is the union

$$\mathscr{N}^{ss}=\bigsqcup_{\alpha\in\mathbf{Z}^+}\mathscr{N}_{(\alpha)}$$

Since $\mathfrak{Coh}_{(\alpha)}(X)$ is open in $\mathfrak{Coh}_\alpha(X)$, $\mathscr{N}_{(\alpha)}$ is a Lagrangian substack of $\mathscr{N}_{(\alpha)}=\pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{(\alpha)}(X))$. For $\xi\in(\mathbf{N}^{\mathscr{P}})_\delta$ ($\delta=\gcd(\alpha)$), we let $\mathscr{N}_{(\alpha),\xi}=\pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{(\alpha),\xi}(X))$. We let $\overline{\mathscr{N}_{(\alpha),\xi}}$ be the closure of $\mathscr{N}_{(\alpha),\xi}$ in $\mathscr{N}_{(\alpha)}$ and $\overline{\overline{\mathscr{N}_{(\alpha),\xi}}}$ be the closure of $\mathscr{N}_{(\alpha),\xi}$ in $\mathscr{N}_\alpha$. If $\xi=\xi_\lambda$ for some partition $\lambda\in\mathscr{P}_\delta$, we write $\mathscr{N}_{\alpha,\lambda}=\mathscr{N}_{\alpha,\xi}$. We can now formulate a corollary of Theorem 6.1.1.

Corollary 6.3.2. *The irreducible components of $\mathscr{N}_{(\alpha)}$ are the closed substacks $\overline{\mathscr{N}_{(\alpha),\lambda}}$ for $\lambda \in (\mathbf{N}^{\mathscr{P}})_{\delta}$, $\delta = \gcd \alpha$.*

Proof. This corollary is a consequence of Lemma 6.3.1 and Theorem 6.1.1 for $\boldsymbol{\alpha} = (\alpha)$ which we prove in Section 6.5.1 □

6.3.3 Irreducible components of the global nilpotent cone

Let $\alpha \in \mathbf{Z}^+$ and $\boldsymbol{\alpha} \in HN(\alpha)$. We let $\mathscr{N}_{\boldsymbol{\alpha}}$ be the union of the irreducible components of dimension $\dim \mathscr{N}$ of $\pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha}}(X))$. If $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s)$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_s) \in (\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}}$, we let $\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}} = \pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X))$. In particular, for $\boldsymbol{\alpha} = (\alpha)$ and $\boldsymbol{\xi} = \xi \in \mathscr{P}_{\delta}$, we obtain the locally closed substacks $\mathscr{N}_{(\alpha),\xi}$ of $\mathscr{N}_{(\alpha)}$ defined in Section 6.3.2. Also, $\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}}}$ is the closure of $\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}}$ in $\mathscr{N}_{\boldsymbol{\alpha}}$ and $\overline{\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}}}}$ is the closure of $\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}}$ in $\mathscr{N}_{\alpha}$. If $\boldsymbol{\xi} = \xi_{\boldsymbol{\lambda}}$ for some uplet of partitions $\boldsymbol{\lambda} \in \mathscr{P}_{\boldsymbol{\delta}}$, we let $\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}} = \mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\xi}}$.

Let $\Lambda \subset \mathscr{N}_{\alpha}$ be an irreducible component. We call *supporting stratum* of Λ the unique HN-stratum S of $\mathfrak{Coh}_{\alpha}(X)$ such that $\overline{S \cap \pi_{\alpha}(\Lambda)} = \pi_{\alpha}(\Lambda)$. We reformulate here Theorem 6.1.1 whose proof will be given in Section 6.5.1.

Theorem 6.3.3. *Let $\alpha \in \mathbf{Z}^+$ and $\boldsymbol{\alpha} \in HN(\alpha)$. Irreducible components of $\mathscr{N}_{\alpha}$ whose supporting stratum is $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ are the irreducible components of $\overline{\mathscr{N}_{\boldsymbol{\alpha}}} \subset \mathscr{N}_{\alpha}$. They are the $\overline{\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}}$ for $\boldsymbol{\lambda} \in \mathscr{P}_{\boldsymbol{\delta}}$.*

6.3.4 Partial order on the set of irreducible components of the global nilpotent cone

We define a partial order on the set of irreducible components of the global nilpotent cone as follows. If $\overline{\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}}$ and $\overline{\overline{\mathscr{N}_{\boldsymbol{\beta},\boldsymbol{\nu}}}}$ are two irreducible components of $\mathscr{N}_{\alpha}$. We say that $\overline{\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}} \leq \overline{\overline{\mathscr{N}_{\boldsymbol{\beta},\boldsymbol{\nu}}}}$ if $\mathfrak{Coh}_{\boldsymbol{\beta}}(X)$ is strictly contained in $\overline{\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)}$ or $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and $\lambda_i \geq \nu_i$ for any $1 \leq i \leq s$ when we write $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_s)$. We order similarly the set of irreducible components $\mathrm{Irr}(\mathscr{N}_{\boldsymbol{\alpha}}) = \{\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}} : \boldsymbol{\lambda} \in (\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}}\}$ of $\mathscr{N}_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in HN(\alpha)$ (this order reduces to the antidominant order on the uplet of partitions $\boldsymbol{\lambda}$). We define a completion of the $\mathbf{Z}$-modules $\mathbf{Z}[\mathrm{Irr}(\mathscr{N}_{\alpha})]$ of functions $\mathrm{Irr}(\mathscr{N}_{\alpha}) \to \mathbf{Z}$ with finite support. Namely, we let $\mathbf{Z}[\widehat{\mathrm{Irr}(\mathscr{N}_{\alpha})}]$ be the $\mathbf{Z}$-module of all functions $\mathrm{Irr}(\mathscr{N}_{\alpha}) \to \mathbf{Z}$. We define in a similar way the completion $\mathbf{Z}[\widehat{\mathrm{Irr}(\mathscr{N}_{\boldsymbol{\alpha}})}]$ of $\mathbf{Z}[\mathrm{Irr}(\mathscr{N}_{\boldsymbol{\alpha}})]$.

6.4 Spherical Eisenstein perverse sheaves on the moduli stack of coherent sheaves

6.4.1 Induction and restriction functors

We recall the definitions of the induction and restriction functors appearing in the work of Schiffmann [Sch12b]. Let $\alpha, \beta \in \mathbf{Z}^+$. We let $\mathfrak{Exact}_{\alpha,\beta}$ be the Artin stack parametrizing exact sequences of coherent sheaves $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H} \to 0$ such that $[\mathcal{G}] = \alpha$ and $[\mathcal{H}] = \beta$. We consider the correspondence:

$$\begin{tikzcd}
& \mathfrak{Exact}_{\alpha,\beta} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathfrak{Coh}_\beta(X) \times \mathfrak{Coh}_\alpha(X) & & \mathfrak{Coh}_{\alpha+\beta}(X)
\end{tikzcd} \tag{6.4.1}$$

where the morphisms p and q are described on geometric points as follows. If $\mathcal{E} = (0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H} \to 0)$ is an exact sequence with $[\mathcal{G}] = \alpha$ and $[\mathcal{H}] = \beta$, $p(\mathcal{E}) = \mathcal{F}$ and $q(\mathcal{E}) = (\mathcal{H}, \mathcal{G})$. The morphism q is smooth with connected fibers (and in fact is a vector bundle stack, see [GPHS14, Hei04]) while p is proper. We let

$$\begin{array}{rccc} \mathrm{Ind}_{\beta,\alpha} & : & D^b(\mathfrak{Coh}_\beta(X)) \times D^b(\mathfrak{Coh}_\alpha(X)) & \to & D^b(\mathfrak{Coh}_{\alpha+\beta}(X)) \\ & & (\mathscr{F}, \mathscr{G}) & \mapsto & p_! q^*(\mathscr{F} \boxtimes \mathscr{G})[-\langle \beta, \alpha \rangle] \end{array}$$

be the induction functor and

$$\begin{array}{rccc} \mathrm{Res}_{\beta,\alpha} & : & D^b(\mathfrak{Coh}_{\alpha+\beta}(X)) & \to & D^b(\mathfrak{Coh}_\beta(X) \times \mathfrak{Coh}_\alpha(X)) \\ & & \mathscr{F} & \mapsto & q_! p^* \mathscr{F}[-\langle \beta, \alpha \rangle] \end{array}$$

be the restriction functor.

6.4.2 A category of perverse sheaves on the stack of coherent sheaves

In the paper [Sch12b], Schiffmann considers the semisimple category of perverse sheaves on $\mathfrak{Coh}(X)$ whose simple objects are the simple perverse sheaves on $\mathfrak{Coh}(X)$ which appear with a possible shift as a direct summand of an iterated induction of the constant sheaf on $\mathfrak{Coh}_\alpha(X)$ for various $\alpha = (r, d)$ with $r \leq 1$. We let $\mathcal{P}$ be the corresponding category of perverse sheaves on $\mathfrak{Coh}(X)$. These are called *spherical Eisenstein perverse sheaves*. It decomposes as a direct sum $\mathcal{P} = \bigoplus_{\alpha \in \mathbf{Z}^+} \mathcal{P}^\alpha$ according to the decomposition of $\mathfrak{Coh}(X)$ into connected components. In his paper, Schiffmann proves that the induction and restriction functors preserve $\mathcal{P}$. Moreover, he gives an explicit description of the simple objects of $\mathcal{P}$ which we recall now. Recall the support map:

$$\chi : \mathfrak{Tor}_d(X) \to S^d X.$$

It induces an isomorphism

$$\mathfrak{Tor}_d^{rss}(X) := \chi^{-1}(S^d X \setminus \Delta) \to (S^d X \setminus \Delta)/(\mathbf{G}_m)^d$$

where $(\mathbf{G}_m)^d$ acts trivially on $S^d X \setminus \Delta$. In particular, we have a $\mathfrak{S}_d$-cover $X^d \setminus \Delta \to S^d X \setminus \Delta$ which gives a family $(\mathscr{L}_\lambda)_{\lambda \in \mathscr{P}_d}$ of irreducible local systems on $\mathfrak{Tor}_d^{rss}(X)$ indexed by irreducible representations of $\mathfrak{S}_d$, and hence partitions of d. This bijection between partitions and irreducible representations of $\mathfrak{S}_d$ is induced by the Springer correspondence, so that the partition $\lambda = (1^d)$ corresponds to the trivial character of $\mathfrak{S}_d$ and $\lambda = (d)$ corresponds to the sign character. Therefore, $\mathscr{L}_{(1^d)}$ is the trivial local system of rank 1.

Let $\alpha \in \mathbf{Z}^+$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in HN(\alpha)$. We consider the iterated induction diagram:

$$\prod_{i=1}^s \mathfrak{Coh}_{\alpha_{s+1-i}}(X) \xleftarrow{q} \mathfrak{Exact}_{\alpha_1,\ldots,\alpha_s} \xrightarrow{p} \mathfrak{Coh}_{\alpha}(X) \tag{6.4.2}$$

where $\mathfrak{Exact}_{\alpha_1,\ldots,\alpha_s}$ parametrizes filtrations $(0 = \mathcal{F}_0 \subset \ldots \subset \mathcal{F}_s)$ of coherent sheaves such that $[\mathcal{F}_i/\mathcal{F}_{i-1}] = \alpha_i$ for any $1 \leq i \leq s$. In the diagram 6.4.2, whe have the equality $p^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)) = q^{-1}\left(\prod_{i=1}^s \mathfrak{Coh}_{(\alpha_{s+1-i})}(X)\right)$. We let $\mathfrak{Exact}_{\boldsymbol{\alpha}}$ be this locally closed substack of $\mathfrak{Exact}_{\alpha_1,\ldots,\alpha_s}$. The iterated induction diagram restricted to sheaves of HN-type $\boldsymbol{\alpha}$ is:

$$\begin{tikzcd} & \mathfrak{Exact}_{\boldsymbol{\alpha}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \prod_{i=1}^s \mathfrak{Coh}_{(\alpha_{s+1-i})}(X) & & \mathfrak{Coh}_{\boldsymbol{\alpha}}(X) \end{tikzcd} \tag{6.4.3}$$

By uniqueness of the Harder-Narasimhan filtration, the map p is an isomorphism. Moreover, q is smooth with connected fibers of dimension $\dim q = d_\alpha$ given by Formula (6.2.3). If $\mathscr{F}$ is a perverse sheaf on $\mathfrak{Coh}_\alpha(X)$, its supporting stratum is the Harder-Narasimhan stratum S of $\mathfrak{Coh}_\alpha(X)$ such that $\operatorname{supp}\mathscr{F} = \overline{\operatorname{supp}\mathscr{F} \cap S}$. We let $\mathfrak{Coh}_{\geq\boldsymbol{\alpha}}(X)$ be the union of the Harder-Narasimhan strata $\mathfrak{Coh}_{\boldsymbol{\beta}}(X)$ ($\boldsymbol{\beta} \in HN(\alpha)$) such that $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X) \subset \overline{\mathfrak{Coh}_{\boldsymbol{\beta}}(X)}$. It is an open substack of $\mathfrak{Coh}_\alpha(X)$. We denote by $j_{\boldsymbol{\alpha}}$ the open inclusion $\mathfrak{Coh}_{\geq\boldsymbol{\alpha}}(X) \to \mathfrak{Coh}_\alpha(X)$. We let $\mathcal{P}^{\boldsymbol{\alpha}}$ denote the restriction to $\mathfrak{Coh}_{\geq\boldsymbol{\alpha}}(X)$ of perverse sheaves in $\mathcal{P}^\alpha$ whose supporting stratum is $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$:

$$\mathcal{P}^{\boldsymbol{\alpha}} = \{j_{\boldsymbol{\alpha}}^*\mathscr{F} : \mathscr{F} \in \operatorname{Ob}(\mathcal{P}^\alpha), \operatorname{supp}\mathscr{F} = \overline{\operatorname{supp}\mathscr{F} \cap \mathfrak{Coh}_{\boldsymbol{\alpha}}(X)}\}.$$

Theorem 6.4.1 ([Sch12b, Proposition 3.4])**.** *1. Let $d \in \mathbf{N}$. The simple objects of the category $\mathcal{P}^{(0,d)}$ are the perverse sheaves on $\mathfrak{Tor}_d(X)$ isomorphic to one of the intersection complexes $\mathcal{IC}(\mathscr{L}_\lambda)$ for $\lambda \in \mathscr{P}_d$.*

2. Let $\alpha \in \mathbf{Z}^+$. The simple objects of the category $\mathcal{P}^\alpha$ whose supports intersect the semistable locus $\mathfrak{Coh}_{(\alpha)}(X)$ are the perverse sheaves isomorphic to one of the intermediate extensions

$$(j_\alpha)_{!*}(\epsilon_\alpha)_*\mathcal{IC}(\mathscr{L}_\lambda)$$

for $\lambda \in \mathscr{P}_d$, where $j_\alpha : \mathfrak{Coh}_{(\alpha)}(X) \to \mathfrak{Coh}_\alpha(X)$ is the inclusion of the semistable locus. For $\alpha \in \mathbf{Z}^+$, we let $\mathcal{IC}(\mathscr{L}_\lambda) = (\epsilon_\alpha)_*\mathcal{IC}(\mathscr{L}_\lambda)$. The context makes clear on which space we consider the perverse sheaves.

3. Let $\boldsymbol{\alpha} \in HN(\alpha)$. Then, simple perverse sheaves on $\mathfrak{Coh}_\alpha(X)$ whose supporting stratum is $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ are the perverse sheaves isomorphic to one of the following intermediate extensions

$$(j_{\boldsymbol{\alpha}})_{!*}p_{\boldsymbol{\alpha}}^*(\mathcal{IC}(\mathscr{L}_{\lambda_1}) \boxtimes \ldots \boxtimes \mathcal{IC}(\mathscr{L}_{\lambda_s}))[d_{\boldsymbol{\alpha}}]$$

for multipartitions $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s) \in \mathscr{P}_{\boldsymbol{\delta}}$, where $d_{\boldsymbol{\alpha}}$ is the relative dimension of $p_{\boldsymbol{\alpha}}$ given by Formula (6.2.3). We let $\mathscr{F}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}$ denote this perverse sheaf.

We order the isomorphism classes of simple spherical Eisenstein perverse sheaves as follows:

$$[\mathscr{F}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}] \leq [\mathscr{F}_{\boldsymbol{\beta},\boldsymbol{\nu}}] \iff \begin{cases} \mathfrak{Coh}_{\boldsymbol{\beta}}(X) \subset \overline{\mathfrak{Coh}_{\boldsymbol{\alpha}}}(X) \text{ is a strict inclusion} \\ \text{or} \\ \boldsymbol{\beta} = \boldsymbol{\alpha} \text{ and for any } 1 \leq i \leq s,\ \nu_i \leq \lambda_i. \end{cases}$$

We define a completion of the Grothendieck group $K_0(\mathcal{P}^\alpha)$, $\widehat{K_0(\mathcal{P}^\alpha)}$. By definition, $\widehat{K_0(\mathcal{P}^\alpha)}$ consists of all formal sums $\sum_{\substack{\boldsymbol{\alpha}\in HN(\alpha)\\ \boldsymbol{\lambda}\in\mathscr{P}_\delta}} a_{\boldsymbol{\alpha},\boldsymbol{\lambda}}[\mathscr{F}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}]$ with $a_{\boldsymbol{\alpha},\boldsymbol{\lambda}} \in \mathbf{Z}$.

6.4.3 Singular support of spherical Eisenstein perverse sheaves

Nilpotency

Consider $\alpha \in \mathbf{Z}^+$, $s \geq 0$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in (\mathbf{Z}^+)^s$ such that $\sum_{i=1}^s \alpha_i = \alpha$. We let $\mathfrak{Y}_\alpha = \mathfrak{Coh}_\alpha(X)$, $\mathfrak{Y}_{\boldsymbol{\alpha}} = \prod_{i=1}^s \mathfrak{Coh}_{\alpha_i}(X)$ and $\mathfrak{X}_{\boldsymbol{\alpha}}$ be the stack parametrizing filtrations $(0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_s = \mathcal{F})$ such that for $1 \leq i \leq s$, $[\mathcal{F}_i/\mathcal{F}_{i-1}] = \alpha_i$. The iterated induction diagram is

$$\mathfrak{Y}_{\boldsymbol{\alpha}} \xleftarrow{q} \mathfrak{X}_{\boldsymbol{\alpha}} \xrightarrow{p} \mathfrak{Y}_\alpha$$

Proposition 6.4.2. *Spherical Eisenstein perverse sheaves have nilpotent singular support.*

Proof. It suffices to show that $p_! q^* \underline{\mathbf{C}}_{\mathfrak{Y}_{\boldsymbol{\alpha}}}$ has a nilpotent singular support for any $\boldsymbol{\alpha} \in (\mathbf{Z}^+)^s$. Since $q^*\underline{\mathbf{C}}_{\mathfrak{Y}_{\boldsymbol{\alpha}}}$ is the constant local system on $\mathfrak{X}_{\boldsymbol{\alpha}}$, it suffices to understand how the zero section of $T^*\mathfrak{X}_{\boldsymbol{\alpha}}$ transforms under the cotangent correspondence:

$$\begin{tikzcd} & T^*\mathfrak{Y}_\alpha \times_{\mathfrak{Y}_\alpha} \mathfrak{X}_{\boldsymbol{\alpha}} \arrow[dl, "dp^*"'] \arrow[dr, "pr_1"] & \\ T^*\mathfrak{X}_{\boldsymbol{\alpha}} & & T^*\mathfrak{Y}_\alpha \end{tikzcd}$$

We can describe $T^*\mathfrak{Y}_\alpha$ as the stack of Higgs bundles, which parametrizes pairs $(\mathcal{F}, \theta)$, where $\mathcal{F}$ is a coherent sheaf on X verifying $[\mathcal{F}] = \alpha$ and $\theta \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F} \otimes K_X)$ (see Section 6.3.1); analogously, $T^*\mathcal{Y}_{\boldsymbol{\alpha}} = \prod_{i=1}^s T^*\mathfrak{Y}_{\alpha_i}$ and $T^*\mathfrak{Y}_\alpha \times_{\mathfrak{Y}_\alpha} \mathfrak{X}_{\boldsymbol{\alpha}}$ is the stack parametrizing pairs $(\mathcal{F}_\bullet, \theta)$ where $\mathcal{F}_\bullet = (0 = \mathcal{F}_0 \subset \ldots \subset \mathcal{F}_s = \mathcal{F})$ and $\theta \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F} \otimes K_X)$. The map pr_1 sends $(\mathcal{F}_\bullet, \theta)$ to $(\mathcal{F}, \theta)$. If we write $\mathfrak{X}_{\boldsymbol{\alpha}}$ for the zero-section of $T^*\mathfrak{X}_{\boldsymbol{\alpha}}$, the geometric points of $(dp^*)^{-1}(\mathfrak{X}_{\boldsymbol{\alpha}})$ parametrize pairs $(\mathcal{F}_\bullet, \theta)$ such that $\theta(\mathcal{F}_i) \subset \mathcal{F}_{i-1} \otimes K_X$ for any $1 \leq i \leq s$. Consequently, $pr_1(dp^*)^{-1}(\mathfrak{X}_{\boldsymbol{\alpha}}) \subset \mathscr{N}_\alpha$ and since the properness of p implies $SS(p_! q^* \underline{\mathbf{C}}_{\mathfrak{Y}_{\boldsymbol{\alpha}}}) \subset pr_1(dp^*)^{-1}(\mathfrak{X}_{\boldsymbol{\alpha}})$, this proves the nilpotency of the singular support. $\square$

Remark 6.4.3. The same proof shows that for any local system $\mathscr{L}$ on $\mathfrak{X}_{\boldsymbol{\alpha}}$, the singular support of $p_!\mathscr{L}$ is nilpotent.

Remark 6.4.4. The situation of Proposition 6.4.2 is analogous to the situation of the induction of perverse sheaves for reductive Lie algebras. Let $P \subset G$ be a parabolic subgroup of a reductive algebraic group with Lévi quotient L and $\mathfrak{p} \subset \mathfrak{g}$ and $\mathfrak{l}$ be the corresponding Lie algebras. The induction diagram reads

$$\mathfrak{l}/L \xleftarrow{q} \mathfrak{p}/P \xrightarrow{p} \mathfrak{g}/G$$

(see for example [Gun18, §1.1]). The groups act on their Lie algebra by means of the adjoint action. The problem is to show that $p_! q^* \underline{\mathbf{C}}_{\mathfrak{l}/L} = p_! \underline{\mathbf{C}}_{\mathfrak{p}/P}$ has a nilpotent singular support. For stack quotient, the cotangent bundle has a nice description using Hamiltonian reduction. We first fix a nondegerate invariant bilinear form on $\mathfrak{g}$. This allows us to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ and $\mathfrak{p}^*$ with $\mathfrak{g}/\mathfrak{n}$ where $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}$. The moment maps are

$$\begin{array}{rccc} \mu_G & : & T^*\mathfrak{g} \simeq \mathfrak{g} \times \mathfrak{g} & \to & \mathfrak{g} \\ & & (x, y) & \mapsto & [x, y] \end{array}$$

and

$$\begin{array}{rccc} \mu_P & : & T^*\mathfrak{p} \simeq \mathfrak{p} \times \mathfrak{g}/\mathfrak{n} & \to & \mathfrak{g}/\mathfrak{n} \\ & & (x, y) & \mapsto & [x, y] \end{array}$$

so that

$$T^*(\mathfrak{g}/G) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\}/G$$

and

$$T^*(\mathfrak{p}/P) = \{(x, y) \in \mathfrak{p} \times \mathfrak{g}/\mathfrak{n} \mid [x, y] = 0\}/P.$$

Moreover,

$$T^*(\mathfrak{g}/G) \times_{\mathfrak{g}/G} (\mathfrak{p}/P) = \{(x, y, g) \in \mathfrak{g} \times \mathfrak{g} \times G \mid [x, y] = 0, \mathrm{Ad}(g) \cdot x \in \mathfrak{p}\}/(G \times P)$$

where the action of $G \times P$ is as follows. For any $(h, p) \in G \times P$ and $(x, y, g) \in \mathfrak{g} \times \mathfrak{g} \times G$,

$$(h, p) \cdot (x, y, g) = (\mathrm{Ad}(h) \cdot x, \mathrm{Ad}(h) \cdot y, pgh^{-1}).$$

Last, the cotangent map dp^* is given by

$$\begin{array}{rcccc} dp^* & : & T^*(\mathfrak{g}/G) \times_{\mathfrak{g}/G} (\mathfrak{p}/P) & \to & T^*(\mathfrak{p}/P) \\ & & (x, y, g) & \mapsto & (\mathrm{Ad}(g) \cdot x, \mathrm{Ad}(g) \cdot y) \end{array}$$

and the corresponding morphism $G \times P \to P$ is the projection on the second factor. Let $pr_1 : T^*(\mathfrak{g}/G) \times_{\mathfrak{g}/G} (\mathfrak{p}/P) \to T^*(\mathfrak{g}/G)$ be the projection on the first factor. If $\mathfrak{p}/P$ denotes the zero-section of $T^*(\mathfrak{p}/P)$, then

$$pr_1((dp^*)^{-1}(\mathfrak{p}/P)) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0 \text{ and } \exists g \in G \text{ s.t. } \mathrm{Ad}(g) \cdot x \in \mathfrak{p},\ \mathrm{Ad}(g) \cdot y \in \mathfrak{n}\}/G.$$

Consequently, the singular support of $p_! \underline{\mathbf{C}}_{\mathfrak{p}/P}$ is nilpotent.

The singular support over the supporting HN-stratum

In this section, we compute the singular support of the restriction of spherical Eisenstein sheaves to their supporting stratum. Let $\mathscr{F}$ be a simple spherical Eisenstein sheaf on $\mathfrak{Coh}_{\alpha}(X)$ supported on the stratum $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ (that is, $\operatorname{supp}\mathscr{F} = \overline{\operatorname{supp}\mathscr{F}\cap\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)}$). Recall the open immersion $j_{\boldsymbol{\alpha}} : \mathfrak{Coh}_{\geq\boldsymbol{\alpha}}(X) \to \mathfrak{Coh}_{\alpha}(X)$ from Section 6.4.2. The characteristic cycle of $(j_{\boldsymbol{\alpha}})^*\mathscr{F}$ is a $\mathbf{Z}$-linear combination with nonnegative coefficients of the irreducible components of $\overline{\mathscr{N}_{\boldsymbol{\alpha}}}$. By Theorem 6.1.1, we can write

$$CC((j_{\boldsymbol{\alpha}})^*\mathscr{F}) = \sum_{\boldsymbol{\lambda}\in\mathscr{P}_{\boldsymbol{\delta}}} m_{\mathscr{F},\boldsymbol{\lambda}}[\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}]. \tag{6.4.4}$$

See Section 6.3.3 for details on the notation $\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}$. Write

$$\mathscr{F} = (j_{\boldsymbol{\alpha}})_{!*}p_{\boldsymbol{\alpha}}^*(\mathcal{IC}(\mathscr{L}_{\lambda_1})\boxtimes\ldots\boxtimes\mathcal{IC}(\mathscr{L}_{\lambda_s}))[d_{\boldsymbol{\alpha}}]$$

(using Theorem 6.4.1). Then, the restriction of $\mathscr{F}$ to $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ is $p_{\boldsymbol{\alpha}}^*(\mathcal{IC}(\mathscr{L}_{\lambda_1})\boxtimes\ldots\boxtimes\mathcal{IC}(\mathscr{L}_{\lambda_s}))[d_{\boldsymbol{\alpha}}]$. We let $\mathscr{F}_i = \mathcal{IC}(\mathscr{L}_{\lambda_i})$ for $1\leq i\leq s$. This is a perverse sheaf on $\mathfrak{Coh}_{\alpha_i}(X)$. With these notations, we have in particular

$$CC((j_{(\alpha_i)})^*\mathscr{F}_i) = \sum_{\lambda\in\mathscr{P}_{\delta_i}} m_{\mathscr{F}_i,\lambda}[\overline{\mathscr{N}_{(\alpha_i),\lambda}}].$$

This is Formula (6.4.4) applied to $\alpha=\alpha_i$ and $\boldsymbol{\alpha}=(\alpha_i)$, using that $d_{(\alpha_i)}=0$.

Lemma 6.4.5. *The multiplicies $m_{\mathscr{F},\boldsymbol{\lambda}}$ are given by the formula*

$$m_{\mathscr{F},\boldsymbol{\lambda}} = \prod_{i=1}^{s} m_{\mathscr{F}_i,\lambda_i}$$

for any $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_s)\in\mathscr{P}_{\boldsymbol{\delta}}$.

Proof. This formula follows from the fact that the characteristic cycle of an exterior product is the product of the characteristic cycles, from the smoothness of q and its compatibility with the stratifications defined in Section 6.2.4 (see [KS90]). More precisely, Proposition 9.4.5 of *op. cit.* asserts that a shift by k transforms the characteristic cycle by $(-1)^k$, Proposition 9.4.3 that the characteristic cycle map is compatible with pull-back by non-characteristic morphisms and the remark following Definition 5.4.12 that a smooth morphism in non-characteristic. □

Before giving the microlocal multiplicities of spherical Eisenstein sheaves whose supporting stratum is the semistable one, that is $\boldsymbol{\alpha}=(\alpha)$, we need a digression on perverse sheaves on general linear Lie algebras $\mathfrak{gl}_d$ for $d\geq 0$. Let $d\in\mathbf{N}$. Nilpotent orbits of $\mathfrak{gl}_d$ are in bijection with partitions of d in such a way that for any two partitions λ,ν, $\mathcal{O}_{\nu}\subset\overline{\mathcal{O}_{\lambda}}$ if and only if $\nu\leq\lambda$ for the dominance order $\leq$ on partitions. In particular, the orbit $\{0\}\subset\mathfrak{gl}_d$ corresponds to the partition (1^d) and the regular nilpotent orbit to $\lambda=(d)$.

Let $\Lambda_{\nu}=[\overline{T^*_{\mathcal{O}_{\nu}}\mathfrak{gl}_d}]$ as a Lagrangian cycle of $T^*\mathfrak{gl}_d$. Then, we have the following easy lemma.

Lemma 6.4.6. *The characteristic cycle map*

$$CC : K_0(\mathrm{Perv}_{\mathrm{GL}_d}(\mathscr{N})) \to \mathbf{Z}[\Lambda_\nu : \nu \in \mathscr{P}_d]$$

is an isomorphism of $\mathbf{Z}$*-modules. Moreover, it is lower unitriangular with respect to the basis of simple perverse sheaves on the left and the basis given by the* Λ_ν *on the right, both ordered using the antidominance order on partitions.*

Consider the Grothendieck-Springer resolution for $\mathfrak{g} = \mathfrak{gl}_d$:

$$\pi_{\mathfrak{g}} : \tilde{\mathfrak{g}} \to \mathfrak{g}.$$

The decomposition theorem gives the decomposition

$$(\pi_{\mathfrak{g}})_* \underline{\mathbf{C}} = \bigoplus_{\lambda \in \mathscr{P}_d} \mathcal{IC}(\mathscr{L}_\lambda) \otimes V_\lambda$$

where $\mathcal{IC}(\mathscr{L}_\lambda) = \mathfrak{F}\mathcal{IC}(\mathcal{O}_\lambda)$ and V_λ is a (non-zero) multiplicity complex. Here, $\mathfrak{F}$ is the Fourier-Sato transform of perverse sheaves on $\mathfrak{g}$ and $\mathscr{L}_\lambda$ is the local system on $\mathfrak{g}$ associated to the partition λ (the map $\pi_{\mathfrak{g}}$ is small and is a $\mathfrak{S}_d$-cover over the regular semisimple locus of $\mathfrak{g}$). In particular, $\mathscr{L}_{(1^d)} = \underline{\mathbf{C}}_{\mathfrak{g}}$ is the trivial local system while $\mathscr{L}_{(d)}$ is associated to the sign character of $\mathfrak{S}_d$. Precomposing the isomorphism of Lemma 6.4.6 with the Fourier-Sato transform gives a $\mathbf{Z}$-module isomorphism

$$CC : K_0(\mathcal{P}_{\mathfrak{g}}) \to \mathbf{Z}[\Lambda_\nu : \nu \in \mathscr{P}_d], \tag{6.4.5}$$

where $\mathcal{P}_{\mathfrak{g}}$ is the semisimple category of perverse sheaves on $\mathfrak{g}$ generated by the $\mathcal{IC}(\mathscr{L}_\lambda)$, $\lambda \in \mathscr{P}_d$. Moreover, by ordering the basis given by classes of simple perverse sheaves on the left and the basis $(\Lambda_\nu)_{\nu \in \mathscr{P}_d}$ on the right by the antidominance order, this isomorphism is lower unitriangular. Note that identifying $T^*\mathfrak{g}$, $T^*\mathfrak{g}^*$ and $\mathfrak{g} \times \mathfrak{g}$ in the natural way using the trace pairing, the Fourier transform and the characteristic cycle map are compatible with these isomorphisms, that is the microlocal multiplicities are preserved (see [KS90, Exercise IX.7]). It is possible to give explicitly the microlocal multiplicities thanks to the following result of Evens and Mirković.

Theorem 6.4.7 ([EM99, Theorem 0.2 (b)])**.** *Let* $d \in \mathbf{N}$*. Let* λ *and* ν *be two partitions of* d *corresponding to nilpotent orbits* $\mathcal{O}_\lambda$ *and* $\mathcal{O}_\nu$*. The multiplicity* $\alpha_{\nu,\lambda}$ *of* $[\overline{T^*_{\mathcal{O}_\nu}\mathfrak{gl}_d}]$ *in* $CC(\mathcal{IC}(\mathcal{O}_\lambda))$ *is given by the multiplicity of the Springer representation* V_λ *of* $\mathfrak{S}_d$ *in the cohomology of the Springer fiber* $H^*(\mathcal{B}_e)$ *at* $e \in \mathcal{O}_\nu$*, which is the Kostka number* $K_{\lambda\nu}$*.*

Let $\mathscr{F}$ be a simple spherical Eisenstein sheaf on $\mathfrak{Coh}_\alpha(X)$ whose support intersects the semistable stratum $\mathfrak{Coh}_{(\alpha)}(X)$ (that is, the restriction $(j_{(\alpha)})^*\mathscr{F}$ is a nonzero perverse sheaf of the category $\mathcal{P}^{(\alpha)}$). By Theorem 6.4.1, there exists a partition $\lambda \in \mathscr{P}$ such that $\mathscr{F} = \mathcal{IC}(\mathscr{L}_\lambda)$. Then, the characteristic cycle of $(j_{(\alpha)})^*\mathscr{F}$ is given by the following lemma.

Lemma 6.4.8. *We have*

$$CC((j_{(\alpha)})^*\mathscr{F}) = \sum_{\nu \leq \lambda} m_{\nu,\lambda}[\overline{\mathscr{N}_{(\alpha),\nu}}],$$

where $m_{\nu,\lambda} = \alpha_{\nu,\lambda}$. In particular, the map

$$\begin{array}{rcccc} CC & : & K_0(\mathcal{P}^{(\alpha)}) & \to & \mathbf{Z}[\overline{\mathscr{N}_{(\alpha),\nu}} : \nu \in \mathscr{P}_\delta] \\ & & \mathscr{F} & \mapsto & CC(\mathscr{F}) \end{array}$$

is a lower unitriangular isomorphism of $\mathbf{Z}$-modules when the basis of simple perverse sheaves on the left and irreducible components of $\mathscr{N}_{(\alpha)}$ on the right are ordered using the antidominance order on partitions.

Proof. We first reduce to the case $\alpha = (0, d)$ for some $d \geq 0$. Recall the isomorphism of stacks

$$\epsilon_\alpha : \mathfrak{Coh}_{(0,\delta)}(X) \to \mathfrak{Coh}_{(\alpha)}(X)$$

where $\delta = \gcd(\alpha)$. It is induced by an equivalence of categories so it also gives an isomorphism at the level of the stacks of Higgs bundles:

$$\epsilon_\alpha : \mathfrak{Higgs}_{(0,\delta)}(X) \to \mathfrak{Higgs}_{(\alpha)}(X)$$

making the following natural diagram commute:

$$\begin{tikzcd} \mathfrak{Higgs}_{(0,\delta)}(X) \arrow[r, "\epsilon_\alpha"] \arrow[d, "\pi_{(0,\delta)}"'] & \mathfrak{Higgs}_{(\alpha)}(X) \arrow[d, "\pi_\alpha"] \\ \mathfrak{Coh}_{(0,\delta)}(X) \arrow[r, "\epsilon_\alpha"] & \mathfrak{Coh}_{(\alpha)}(X) \end{tikzcd}$$

and ϵ_α also induces an isomorphism of the semistable parts of the global nilpotent cones:

$$\epsilon_\alpha : \mathscr{N}_{(0,\delta)} \to \mathscr{N}_{(\alpha)}.$$

Therefore, we can assume $\alpha = (0, d)$ for some $d \geq 0$. The problem is then local (and does not depend anymore on X being an elliptic curve), so that we can assume $X = \mathbb{A}^1$ (for the same reason as in the proof of [Lau87, Théorème (3.3.13)]). We are now in the situation of the classical Springer correspondence for $\mathfrak{gl}_d$ and the Theorem is a consequence of the properties of the map (6.4.5). □

Lemma 6.4.9. *Let $\alpha \in \mathbf{Z}^+$ and $\boldsymbol{\alpha} \in HN(\alpha)$. The characteristic cycle map*

$$\begin{array}{rcccc} CC & : & K_0(\mathcal{P}^{\boldsymbol{\alpha}}) & \mapsto & \mathbf{Z}[\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}} : \boldsymbol{\lambda} \in \mathscr{P}_{\boldsymbol{\delta}}] \\ & & \mathscr{F} & \mapsto & CC(\mathscr{F}) \end{array}$$

is a lower unitriangular isomorphism of $\mathbf{Z}$-modules when the set of s-uplets of partitions $\mathscr{P}_{\boldsymbol{\delta}}$ is ordered by the antidominance order, that is

$$\boldsymbol{\lambda} \leq \boldsymbol{\nu} \iff \textit{for } 1 \leq i \leq s,\ \lambda_i \geq \nu_i.$$

Proof. It is an immediate consequence of Lemmas 6.4.8 and 6.4.5. □

6.5 Proofs of the main theorems 6.1.1 and 6.1.2

6.5.1 Some Lemmas

Lemma 6.5.1. *Let $d \geq 1$, $\xi \in (\mathbf{N}^{\mathscr{P}})_d$ and $T = \bigoplus_{i=1}^{s} T_{x_i,\lambda_i}$ be a torsion sheaf in the stratum $\mathfrak{Coh}_{(0,d),\xi}(X)$. Then, the closed subset $\mathscr{N}_T$ of* $\mathrm{End}(T)$ *of nilpotent endomorphisms is irreducible of codimension*

$$\sum_{\lambda\in\mathscr{P}} \xi(\lambda)l(\lambda).$$

Proof. Note that $\sum_{\lambda\in\mathscr{P}} \xi(\lambda)l(\lambda)$ is the number of indecomposable summands of T (see Section 6.2.2). Write

$$T = \bigoplus_{j=1}^{t} T_j^{\oplus m_j}$$

where the T_j are the pairwise distinct indecomposable summands of T and m_j are their multiplicities. We have to show that the codimension of $\mathscr{N}_T$ in $\mathrm{End}(T)$ is $\sum_{j=1}^{t} m_j$. Let J be the radical of $\mathrm{End}(T)$. The quotient $\mathrm{End}(T)/J$ is isomorphic to

$$\prod_{j=1}^{t} \mathscr{M}_{m_i}(\mathbf{C})$$

where for any $n \in \mathbf{N}$, $\mathscr{M}_n(\mathbf{C})$ is the ring of $n\times n$ matrices with complex coefficients. The projection

$$p : \mathrm{End}(T) \to \mathrm{End}(T)/J$$

is a fiber bundle and $f \in \mathrm{End}(T)$ is nilpotent if and only if $p(f)$ is nilpotent. Therefore,

$$\mathscr{N}_T = p^{-1}\left(\prod_{j=1}^{t} \mathscr{N}_{m_j}\right)$$

where $\mathscr{N}_{m_j}$ denotes the nilpotent cone of $\mathscr{M}_{m_j}(\mathbf{C})$, and for any $n \in \mathbf{N}$, the nilpotent cone of $\mathscr{M}_n(\mathbf{C})$ is irreducible of codimension n (it is the vanishing locus of the n symmetric polynomials in the n eigenvalues). The result for $\mathscr{N}_T$ follows. □

For $\alpha \in \mathbf{Z}^+$ and $\xi \in (\mathbf{N}^{\mathscr{P}})_\delta$, the dimension of the endomorphism ring of a semistable coherent sheaf whose isomorphism class belongs to $|\mathrm{Coh}_{(\alpha),\xi}(X)|$ is constant and only depends on ξ. We denote it $e(\xi)$.

Next, we need the dimension of the stratum $\mathfrak{Coh}_{(\alpha),\xi}(X)$. Thanks to the isomorphism ϵ_α between $\mathfrak{Coh}_{(0,\delta)}(X)$ and $\mathfrak{Coh}_{(\alpha)}(X)$ ($\delta = \gcd\alpha$), we can assume that $\alpha = (0,\delta)$. Then, $\sum_{\lambda\in\mathscr{P}} \xi(\lambda)$ is the number of parameters of $\mathfrak{Coh}_{(0,\delta),\xi}(X)$ (see Section 6.2.2) and $e(\xi)$ is the dimension of the automorphism group. We easily deduce the following lemma.

Lemma 6.5.2. *Let $\alpha \in \mathbf{Z}^+$ and $\xi \in (\mathbf{N}^{\mathscr{P}})_\delta$. Then, $\mathfrak{Coh}_{(\alpha),\xi}(X)$ is irreducible of dimension*

$$\sum_{\lambda \in \mathscr{P}} \xi(\lambda) - e(\xi).$$

Lemma 6.5.3. *For any $\boldsymbol{\xi} \in (\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}}$, the stratum $\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)$ is irreducible of dimension*

$$\sum_{i=1}^{s} \left(\sum_{\lambda \in \mathscr{P}} \xi_i(\lambda) - e(\xi_i) \right) - \sum_{i<j} (r_j d_i - r_i d_j)$$

Proof. The morphism $p_{\boldsymbol{\alpha}} : \mathfrak{Coh}_{\boldsymbol{\alpha}}(X) \to \prod_{i=1}^{s} \mathfrak{Coh}_{(\alpha_i)}(X)$ is a stack bundle with fibers of dimension $d_{\boldsymbol{\alpha}}$ given by Formula (6.2.3). Together with Lemma 6.5.2, this gives Lemma 6.5.3. □

Lemma 6.5.4. *For any $\boldsymbol{\xi} \in (\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}}$, and any geometric point $\mathcal{F} \in \mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)$, $\pi_{\alpha,\mathscr{N}}^{-1}(\mathcal{F})$ is irreducible of dimension*

$$\sum_{i<j} (r_j d_i - r_i d_j) + \sum_{i=1}^{s} \left(e(\xi_i) - \sum_{\lambda \in \mathscr{P}} \xi_i(\lambda) l(\lambda) \right).$$

Proof. The geometric points of the fiber of $\pi_{\alpha,\mathscr{N}}$ over $\mathcal{F} \in \mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)$ are nilpotent endomorphisms of $\mathcal{F}$. Let $\mathcal{F} = \bigoplus_{i=1}^{s} \mathcal{F}_i$ be its Harder-Narasimhan decomposition (since the Harder-Narasimhan filtration splits), where $[\mathcal{F}_i] = \alpha_i = (r_i, d_i)$. An endomorphism of $\mathcal{F}$ is the datum of $f_{ij} \in \operatorname{Hom}(\mathcal{F}_i, \mathcal{F}_j)$ for $1 \leq j \leq i \leq s$ and it is nilpotent if and only if $f_{i,i}$ is nilpotent for any $1 \leq i \leq s$. From the equality

$$\dim \operatorname{Hom}(\mathcal{F}_i, \mathcal{F}_j) = r_i d_j - r_j d_i$$

and Lemma 6.5.1, we get the desired formula.

□

Lemma 6.5.5. *For $\boldsymbol{\xi} \in (\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}}$, $\pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X))$ is irreducible of dimension*

$$\sum_{i=1}^{s} \sum_{\lambda \in \mathscr{P}} \xi_i(\lambda)(1 - l(\lambda)).$$

In particular, this dimension is nonpositive and equals $0 = \dim \mathscr{N}$ if and only if for any $1 \leq i \leq s$ and $\lambda \in \mathscr{P}$, $\xi_i(\lambda) \neq 0$ implies $l(\lambda) = 1$, which is the definition of $\boldsymbol{\xi}$ being regular.

Proof. The restriction of $\pi_{\alpha,\mathscr{N}}$

$$\pi_{\alpha,\mathscr{N}} : \pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)) \to \mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)$$

is surjective with irreducible target (whose dimension is given by Lemma 6.5.3) and irreducible fibers (of dimension given by Lemma 6.5.4). Hence, $\pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X))$ is irreducible, of dimension

$$\sum_{i=1}^{s} \left(\sum_{\lambda \in \mathscr{P}} \xi_i(\lambda) - e(\xi_i) \right) - \sum_{i<j} (r_j d_i - r_i d_j) + \sum_{i<j} (r_j d_i - r_i d_j) + \sum_{i=1}^{s} \left(e(\xi_i) - \sum_{\lambda \in \mathscr{P}} \xi_i(\lambda) l(\lambda) \right)$$

which yields the formula of the Lemma. $\square$

6.5.2 Proof of Theorem 6.1.1

Let Λ be an irreducible component of $\mathscr{N}_{\boldsymbol{\alpha}}$. It is of dimension 0. Since

$$\mathscr{N}_{\boldsymbol{\alpha}} = \bigcup_{\boldsymbol{\xi} \in (\mathbf{N}^{\mathscr{P}})_{\boldsymbol{\delta}}} \pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X))$$

and this is a locally closed stratification of $\mathscr{N}_{\boldsymbol{\alpha}}$, there exists $\boldsymbol{\xi}$ such that $\pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X)) \cap \Lambda$ is open and dense in Λ. Therefore, $\pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X))$ is of dimension $0 = \dim \mathscr{N}$ hence $\boldsymbol{\xi}$ is regular by Lemma 6.5.5. Write $\boldsymbol{\xi} = \boldsymbol{\xi}_{\boldsymbol{\lambda}}$ for some $\boldsymbol{\lambda} \in \mathscr{P}_{\boldsymbol{\delta}}$. Then, $\Lambda = \overline{\pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X))} = \overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}$.

For the converse, if $\boldsymbol{\xi}$ is regular, by Lemma 6.5.5, $\pi_{\alpha,\mathscr{N}}^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\xi}}(X))$ is an irreducible substack of $\mathscr{N}_{\boldsymbol{\alpha}}$ of dimension 0 and hence its closure is an irreducible component of $\mathscr{N}_{\boldsymbol{\alpha}}$.

This proves the description of the irreducible components of $\mathscr{N}_{\boldsymbol{\alpha}}$. The description of the irreducible components of $\mathscr{N}_{\alpha}$ follows immediately. Indeed, if Λ is such an irreducible component, we let $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ be its supporting stratum. Then, $\Lambda \cap \pi_{\alpha}^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha}}(X))$ is an irreducible component of $\mathscr{N}_{\boldsymbol{\alpha}}$. Therefore, it is of the form $\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}$ for some $\boldsymbol{\lambda} \in \mathscr{P}_{\boldsymbol{\delta}}$. Hence, $\Lambda = \overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}$.

6.5.3 Proof of Theorem 6.1.2

Let $\mathscr{F} \in \mathcal{P}^{\alpha}$ be a simple Eisenstein perverse sheaf. Let $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ be its supporting stratum. Then, its characteristic cycle can be written

$$CC(\mathscr{F}) = \sum_{(\boldsymbol{\beta},\boldsymbol{\lambda})} m_{\boldsymbol{\beta},\boldsymbol{\lambda}} [\overline{\mathscr{N}_{\boldsymbol{\beta},\boldsymbol{\lambda}}}]$$

where the sum runs over pairs $(\boldsymbol{\beta}, \boldsymbol{\lambda})$ of a Harder-Narasimhan type $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s) \in HN(\alpha)$ and a multipartition $\boldsymbol{\lambda} \in \mathscr{P}_{\boldsymbol{\delta}}$, $\boldsymbol{\delta} = \gcd(\boldsymbol{\beta})$. Moreover, $m_{\boldsymbol{\beta},\boldsymbol{\lambda}} = 0$ unless $\mathfrak{Coh}_{\boldsymbol{\beta}}(X) \subset \overline{\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)}$ and

$$CC(j_{\boldsymbol{\alpha}}^* \mathscr{F}) = \sum_{\boldsymbol{\lambda} \in \mathscr{P}_{\boldsymbol{\delta}}} m_{\boldsymbol{\alpha},\boldsymbol{\lambda}} [\overline{\mathscr{N}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}}]$$

is given by Lemma 6.4.5. By Lemma 6.4.9, we obtain the lower unitriangularity of the characteristic cycle map

$$CC : \widehat{K_0(\mathcal{P}^{\alpha})} \to \mathbf{Z}[\widehat{\mathrm{Irr}(\mathscr{N}_{\alpha})}].$$

6.6 Perverse sheaves with nilpotent singular support on the stack of coherent sheaves

In this section, we will describe explicitly the simple objects of the category $\mathrm{Perv}(\mathfrak{Coh}_{\alpha}(X), \mathscr{N}_{\alpha})$ of perverse sheaves on the stack of coherent sheaves on an elliptic curve whose singular support is nilpotent (that is, a union of some of the irreducible components of $\mathscr{N}_{\alpha}$).

6.6.1 Local systems on the semistable locus of the stack of coherent sheaves

Local systems on the Picard stack

Proposition 6.6.1. *Let $\alpha = (1, d) \in \mathbf{Z}^+$ and $\mathscr{L}$ be a local system on $\mathfrak{Pic}_d = \mathfrak{Coh}_{(\alpha)}(X) \subset \mathfrak{Coh}_\alpha(X)$. Then, $\mathscr{L}$ extends to a local system on $\mathfrak{Coh}_\alpha(X)$.*

Proof. Consider the determinant morphism

$$\det : \mathfrak{Coh}_\alpha(X) \to \mathfrak{Pic}_d.$$

It restricts to the identity on $\mathfrak{Coh}_{(\alpha)}(X) = \mathfrak{Pic}_d$. Let $\mathscr{L}$ be a local system on $\mathfrak{Coh}_{(\alpha)}(X)$. Then, $\mathscr{L} = j_{(\alpha)}^* \det^*(\mathscr{L})$. Consequently, $\det^*(\mathscr{L})$ is a local system on $\mathfrak{Coh}_\alpha(X)$ which extends $\mathscr{L}$. □

Codimension one Harder-Narasimhan strata of $\mathfrak{Coh}_\alpha(X)$

Proposition 6.6.2. *Let $\alpha = (r, d) \in \mathbf{Z}^+$. Then, $\mathfrak{Coh}_\alpha(X)$ has a codimension one Harder-Narasimhan stratum if and only if r and d are coprime. In this case, such a stratum is unique.*

Proof. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s) \in HN(\alpha)$ be a Harder-Narasimhan type. By Formula (6.2.3), $\dim \mathfrak{Coh}_{\boldsymbol{\alpha}}(X) = -\sum_{j<i}(r_i d_j - r_j d_i)$ when we write $\alpha_i = (r_i, d_i)$. Consequently, the codimension of $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ in $\mathfrak{Coh}_\alpha(X)$ is the opposite, since $\dim \mathfrak{Coh}_\alpha(X) = 0$. Since the slopes strictly decrease, for any $j < i$, $\frac{d_j}{r_j} > \frac{d_i}{r_i}$. Each term of the sum is therefore positive.

Assume that $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ is of codimension one. Then necessarily $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ has length two and $r_2 d_1 - r_1 d_2 = 1$. Using that $r_2 = r - r_1$ and $d_2 = d - d_1$, we get $r d_1 - r_1 d = 1$, that is a Bézout relation between r and d. This proves that r and d have to be coprime. If $r d_1' - r_1' d = 1$ is an other Bézout relation between r and d, there exists $k \in \mathbf{Z}$ such that

$$\begin{cases} r_1' = r_1 + kr \\ d_1' = d_1 + kd \end{cases}$$

If this new Bézout relation comes from a codimension one stratum given by $\boldsymbol{\alpha}' = (\alpha_1', \alpha_2') \in HN(\alpha)$, we have furthermore $0 \leq r_1' \leq r$. If $0 < r_1 < r$, then necessarily $r_1' = r_1$ and $d_1' = d_1$ so that $\boldsymbol{\alpha}' = \boldsymbol{\alpha}$. The case $r_1 = r$ is excluded since in this case, $\boldsymbol{\alpha} = ((r, d_1), (0, d - d_1))$ and the slopes do not decrease. If $r_1 = 0$, $r_1' = 0$ or $r_1' = r$ but the second case is not allowed for the same reason. Hence $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$ in any case. This proves one implication and the last statement of the Proposition. For the converse, assume (r, d) is coprime. Let $r d_1 - r_1 d = 1$ be the Bézout relation with $0 \leq r_1 < r$. We let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) = ((r_1, d_1), (r - r_1, d - d_1))$. Let $r_2 = r - r_1$ and $d_2 = d - d_1$. The Bézout relation can be rewritten $r_2 d_1 - r_1 d_2 = 1$. Therefore, $\frac{d_1}{r_1} - \frac{d_2}{r_2} = \frac{1}{d_1 d_2} > 0$. This proves that $\boldsymbol{\alpha} \in HN(\alpha)$. □

Remark 6.6.3. When X is a smooth projective curve of genus $g \geq 2$, we can prove a similar result which we mention here, but we will not need it in this chapter.

Proposition 6.6.4. *Let X be a smooth projective curve of genus $g \geq 2$ and $\alpha \in \mathbf{Z}^+$. Then, the stack $\mathfrak{Coh}_\alpha(X)$ has a codimension one Harder-Narasimhan stratum if and only if $\alpha = (1,d)$. In this case, it is unique and corresponds to the Harder-Narasimhan type* $((0,1),(1,d-1))$.

Proof. Write $\alpha = (r,d)$. Let $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_s) \in HN(\alpha)$. Consider the projection $p_{\boldsymbol{\alpha}} : \mathfrak{Coh}_{\boldsymbol{\alpha}}(X) \to \prod_{i=1}^s \mathfrak{Coh}_{(\alpha_i)}(X)$. It is a smooth morphism of relative dimension $-\sum_{j<i}\langle\alpha_i,\alpha_j\rangle$ where $\langle -,-\rangle$ is the Euler form (see Formula (6.2.1)). Therefore,

$$\dim \mathfrak{Coh}_{\boldsymbol{\alpha}}(X) = \sum_{i=1}^s (g-1)r_i^2 + (g-1)\sum_{j<i} r_ir_j - \sum_{j<i}(r_id_j - r_jd_i).$$

Since $\dim \mathfrak{Coh}_\alpha(X) = (g-1)\left(\sum_{i=1}^s r_i\right)^2$, we have,

$$\dim \mathfrak{Coh}_{\boldsymbol{\alpha}}(X) - \dim \mathfrak{Coh}_\alpha(X) = -(g-1)\sum_{j<i} r_ir_j - \sum_{j<i}(r_id_j - r_jd_i).$$

By the condition on the slopes, for any $j<i$, $r_id_j - r_jd_i > 0$. If $s \geq 3$, then the codimension of $\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ in $\mathfrak{Coh}_\alpha(X)$ is at least two (even three). If $\boldsymbol{\alpha} = (\alpha_1,\alpha_2)$,

$$\dim \mathfrak{Coh}_{\boldsymbol{\alpha}}(X) - \dim \mathfrak{Coh}_\alpha(X) = -(g-1)r_1r_2 - (r_2d_1 - r_1d_2)$$

so if $r_1r_2 \neq 0$, then the codimension is at least two. If $r_1r_2 = 0$, we have $r_1 = 0$ for slope reasons. Then, the codimension is one if and only if $r_2 = 1$ and $d_1 = 1$. So $\boldsymbol{\alpha} = ((0,1),(1,d_2))$ This proves the proposition. ☐

Local systems on the semistable locus

Proposition 6.6.5. *Let X be a smooth projective curve, $\alpha \in \mathbf{Z}^+$ and let $\mathscr{L}$ be a local system on the semistable Harder-Narasimhan stratum $\mathfrak{Coh}_{(\alpha)} \subset \mathfrak{Coh}_\alpha$. Then, $\mathscr{L}$ extends to a local system on $\mathfrak{Coh}_\alpha(X)$.*

Proof. We assume X is an elliptic curve. The same arguments combined with Proposition 6.6.4 give a proof for curves of genus $g \geq 2$. If $\gcd(\alpha) > 1$, then the closed complement $\mathfrak{Coh}_\alpha(X) \setminus \mathfrak{Coh}_{(\alpha)}(X)$ of the open substack $\mathfrak{Coh}_{(\alpha)}(X)$ of $\mathfrak{Coh}_\alpha(X)$ is of codimension at least two by Proposition 6.6.2. Since any local system extends over closed substacks of codimension at least two, $\mathscr{L}$ extends to a local system on $\mathfrak{Coh}_\alpha(X)$.

If $\gcd(\alpha) = 1$, we let $\boldsymbol{\alpha} = (\alpha_1,\alpha_2)$ be the Harder-Narasimhan type of the codimension one stratum of $\mathfrak{Coh}_\alpha(X)$ (see Proposition 6.6.2). The open substack $\mathfrak{Coh}_\alpha(X) \cup \mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$ is of codimension at least two. It suffices to show that $\mathscr{L}$ extends over this open substack. Since $\begin{pmatrix} r & d \\ r_1 & d_1 \end{pmatrix}$ has determinant one (see the proof of Proposition 6.6.2), there exists $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ such that $\gamma\cdot\alpha = (1,1)$ and $\gamma\cdot\alpha_1 = (0,1)$. By the isomorphism (6.2.4) of Section 6.2.5, it suffices to consider the case when $\alpha = (1,1)$. In this case, the result is implied by Proposition 6.6.1 ☐

6.6.2 Twisted spherical Eisenstein perverse sheaves on the stack of coherent sheaves on an elliptic curve

The surface braid group

Let X be a connected topological surface. We will mainly be interested in the case when X is an elliptic curve. Let $n \in \mathbf{N}$. The pure braid group $P_n(X)$ is by definition the fundamental group of $X^n \setminus \Delta$ while the braid group $B_n(X)$ is the fundamental group of $S^nX \setminus \Delta$. The $\mathfrak{S}_n$-covering $p_n : X^n \setminus \Delta \to S^nX \setminus \Delta$ induces an exact sequence of groups:

$$1 \to P_n(X) \to B_n(X) \to \mathfrak{S}_n \to 1.$$

Proposition 6.6.6. *There is a canonical quotient $B_n(X) \to K$ making the following diagram commute:*

$$\begin{tikzcd} 1 \arrow[r] & P_n(X) \arrow[r] \arrow[d] & B_n(X) \arrow[r] \arrow[d] & \mathfrak{S}_n \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \pi_1(X)^n \arrow[r] & K \arrow[r] & \mathfrak{S}_n \arrow[r] & 1 \end{tikzcd}$$

Proof. The surjective morphism $P_n(X) \to \pi_1(X^n) \simeq \pi_1(X)^n$ is induced by the inclusion $X^n \setminus \Delta \to X^n$ and the group K is constructed by push-out of the upper exact sequence. $\square$

Remark 6.6.7. It is possible to show that K is isomorphic to the wreath product $\pi_1(X)^n \rtimes \mathfrak{S}_n$. Indeed, we can construct a section to the projection $K \to \mathfrak{S}_n$ as follows. We fix a set $Q = (q_1, \dots, q_n)$ of n distinct points on X and interpret $B_n(X)$ as the group of braids on X, that is of isotopy classes of collections of n paths $p = (p_1, \dots, p_n)$ parametrized by $[0, 1]$ on X starting and ending at Q such that for any $t \in [0, 1]$ and any $i \neq j$, $p_i(t) \neq p_j(t)$. The left-most quotient is the quotient by the (normal) subgroup generated by braids whose strands are trivial in $\pi_1(X)$. For clarity, we assume that Q is contained in an open subset D of X homeomorphic to a disk. For $\sigma \in \mathfrak{S}_n$, we choose a braid $p_\sigma \in B_n(X)$ such that each strand $p_{\sigma,i}$ is contained in D (this is possible since $S^nD \setminus \Delta$ is path-connected) and such that p_σ induces the permutation σ. The section of $K \to \mathfrak{S}_n$ is then the composition of the map $\mathfrak{S}_n \to \mathcal{B}_n(X)$, $\sigma \mapsto p_\sigma$, with the projection $B_n(X) \to K$. It is easily seen that two different choices for p_σ give the same element in K so that the composition is a group homomorphism.

When X is a projective algebraic curve over $\mathbf{C}$, we can consider all local systems, that is representations of $\pi_1(X)$ and we can also work with local systems coming from finite coverings of X, that is with representation of the étale fundamental group $\pi_1^{\acute{e}t}(X)$ which is the profinite completion $\widehat{\pi_1(X)}$ of $\pi_1(X)$. In the case when X is an elliptic curve, $\pi_1^{\acute{e}t}(X) = \hat{\mathbf{Z}} \times \hat{\mathbf{Z}}$ and its representations correspond to local systems on X having finite monodromy. Representations of K are those local systems on $S^n(X) \setminus \Delta$ whose pullback by the $\mathfrak{S}_n$-covering $p_n : X^n \setminus \Delta \to S^nX \setminus \Delta$ extends to X^n.

The following remark will be useful in Section 6.6.3. Let $\rho : P_n(X) \to \mathrm{GL}(V)$ be an irreducible representation of $P_n(X)$ and $\mathscr{L}$ be the corresponding local system on $X^n \setminus \Delta$. The local system $(p_n)_*\mathscr{L}$ on $S^nX \setminus \Delta$ is associated to the induced representation $\mathrm{Ind}_{P_n(X)}^{B_n(X)}(\rho)$. The pull-back

$p_n^*(p_n)_*\mathscr{L}$ is the restriction to $P_n(X)$ of $\mathrm{Ind}_{P_n(X)}^{B_n(X)}(\rho)$. Its decomposition into irreducible representations of $P_n(X)$ is

$$\bigoplus_{\omega P_n(X) \in B_n(X)/P_n(X)=\mathfrak{S}_n} \omega \cdot \rho$$

where $\omega \cdot \rho : P_n(X) = \pi_1(X^n \setminus \Delta) \to \mathrm{GL}(V)$, $g \mapsto \rho(\omega g \omega^{-1})$. This is the representation obtained from ρ by permuting according to ω the factors of $X^n \setminus \Delta$. Therefore, when ρ factors through $\pi_1(X)^n$ and hence corresponds to a local system of the form $\mathscr{L} = \mathscr{L}_1 \boxtimes \ldots\ldots \boxtimes \mathscr{L}_n$ on X^n, $p_n^*(p_n)_*\mathscr{L}_{X^n\setminus\Delta}$ is a local system on $X^n \setminus \Delta$ which extends to X^n and this extension is the local system

$$\bigoplus_{\sigma \in \mathfrak{S}_n} (\mathscr{L}_{\sigma(1)} \boxtimes \ldots \boxtimes \mathscr{L}_{\sigma(n)}). \tag{6.6.1}$$

Consequently, if $\mathscr{L}'$ is a simple direct summand of $(p_n)_*\mathscr{L}_{X^n\setminus\Delta}$, $p_n^*\mathscr{L}'$ is a local system on $X^n \setminus \Delta$ which extends to X^n and the extension is a direct sum of some of the local systems appearing in the direct sum (6.6.1). Consequently, if $\mathscr{L}_i, \mathscr{L}'_i$, $1 \leq i \leq n$ are two collections of simple local systems on X such that that the second is not a permutation of the first, then, letting $\mathscr{L} = \mathscr{L}_1 \boxtimes \ldots \boxtimes \mathscr{L}_n$ and $\mathscr{L}' = \mathscr{L}'_1 \boxtimes \ldots \boxtimes \mathscr{L}'_n$, the simple direct summands of $(p_n)_*\mathscr{L}$ and $(p_n)_*\mathscr{L}'$ are pairwise non-isomorphic.

A class of perverse sheaves on the moduli stack of coherent sheaves on an elliptic curve

In this Section, we define a category of perverse sheaves on the moduli stack of coherent sheaves on an elliptic curve. We will call these sheaves *twisted spherical Eisenstein sheaves*. Let $\alpha \in \mathbf{Z}^+$. We let $\mathcal{P}_{tw}^\alpha$ be the semisimple subcategory of $\mathrm{Perv}(\mathfrak{Coh}_\alpha(X))$ whose simple objects are the simple perverse sheaves appearing with a possible shift as a direct summand of the induction of the perverse sheaves $\mathcal{IC}(\mathscr{L}_i)$, $1 \leq i \leq t$ for some $t \geq 1$, $\mathscr{L}_i$ local systems on $\mathfrak{Coh}_{(\alpha_i)}(X)$ with coprime $\alpha_i \in \mathbf{Z}^+$ such that $\sum_{i=1}^t \alpha_i = \alpha$. We denote $\mathcal{P}_{tw,f}^\alpha$ the category obtained in a similar way when allowing only local systems with finite monodromy. We will respectively denote $\mathcal{Q}_{tw}^\alpha$ and $\mathcal{Q}_{tw,f}^\alpha$ the full semisimple triangulated subcategories of $D(\mathfrak{Coh}_\alpha(X))$ generated by $\mathcal{P}_{tw}^\alpha$ (resp. $\mathcal{P}_{tw,f}^\alpha$). Since the map q of the induction diagram (Section 6.4) is smooth and the map p is proper, by the argument of [Sch12b, §3.3] involving the decomposition theorem, the induction functor induces a functor

$$\mathrm{Ind}_{\beta,\alpha} : \mathcal{Q}_\sharp^\beta \boxtimes \mathcal{Q}_\sharp^\alpha \to \mathcal{Q}_\sharp^{\alpha+\beta}$$

for $\sharp = tw$ and $\sharp = tw, f$. For $\sharp = tw$, we need the generalization of the decomposition theorem which applies to any semisimple perverse sheaves and not only to whose of geometric origin (see for example [dC16]). The next proposition shows that in the definition of $\mathcal{P}_\sharp^\alpha$ ($\sharp = tw$ or $\sharp = tw, f$), it suffices to consider inductions for $\mathrm{rk}(\alpha_i) \leq 1$.

Lemma 6.6.8. *Let $\alpha \in \mathbf{Z}^+$ and $\mathscr{L}$ be a local system on $\mathfrak{Coh}_\alpha(X)$. If $\mathrm{rk}(\alpha) > 1$, there exists $d' \in \mathbf{Z}$ such that $\gcd(r-1, d-d') = 1$, a local system $\mathscr{L}'$ on $\mathfrak{Coh}_{(1,d')}(X)$ and a local system $\mathscr{L}''$ on $\mathfrak{Coh}_{(r-1,d-d')}(X)$ such that $\mathscr{L}$ is a direct summand (possibly shifted) of the induction $\mathrm{Ind}_{(r-1,d-d'),(1,d')}(\mathscr{L}'' \boxtimes \mathscr{L}')$.*

Proof. Let $d' \in \mathbf{Z}$, $d' \ll 0$. Then, any nonzero morphism from a line bundle of degree d' to a semistable coherent sheaf of class α is injective. We choose d' so that $(r-1, d-d')$ is coprime. We let $\alpha' = (1, d)$ and $\alpha'' = (r-1, d-d')$. The restriction diagram is

$$\mathfrak{Coh}_{\alpha''}(X) \times \mathfrak{Coh}_{\alpha'}(X) \xleftarrow{q} \mathfrak{Exact}_{\alpha',\alpha''} \xrightarrow{p} \mathfrak{Coh}_{\alpha}(X)\,.$$

The map p is proper, and surjective by our condition on d'. Therefore, $\mathscr{L}$ appears up to a shift in $p_*p^*\mathscr{L}$. Since $p^*\mathscr{L}$ is a local system on $\mathfrak{Exact}_{\alpha',\alpha''}$ and q is a vector bundle stack, there exists a local system $\tilde{\mathscr{L}}$ on $\mathfrak{Coh}_{\alpha''}(X) \times \mathfrak{Coh}_{\alpha'}(X)$ such that $q^*\tilde{\mathscr{L}} = p^*\mathscr{L}$. We write $\tilde{\mathscr{L}} = \mathscr{L}'' \boxtimes \mathscr{L}'$ for local systems $\mathscr{L}'$ (resp. $\mathscr{L}''$) on $\mathfrak{Coh}_{\alpha'}(X)$ (resp. $\mathfrak{Coh}_{\alpha''}(X)$). Then, $\mathscr{L}$ appears up to a shift in $\mathrm{Ind}_{\alpha'',\alpha'}(\mathscr{L}'' \boxtimes \mathscr{L}')$. $\square$

As in Section 6.4.2, for $\alpha \in \mathbf{Z}^+$ and $\boldsymbol{\alpha} \in HN(\alpha)$, we let $\mathcal{P}_{tw}^{\boldsymbol{\alpha}}$ (resp. $\mathcal{P}_{tw,f}^{\boldsymbol{\alpha}}$) denote the additive subcategory of perverse sheaves $(j_{(\alpha)}^*)\mathscr{F} = \mathscr{F}_{\mathfrak{Coh}_{\geq \boldsymbol{\alpha}}(X)}$ where $\mathscr{F} \in \mathcal{P}_{tw}^{\alpha}$ (resp. $\mathcal{P}_{tw,f}^{\alpha}$) satisfies $\operatorname{supp}\mathscr{F} = \overline{\operatorname{supp}\mathscr{F} \cap \mathfrak{Coh}_{\boldsymbol{\alpha}}(X)}$.

6.6.3 Perverse sheaves with nilpotent singular support on the semistable locus

In this section, we describe the simple perverse sheaf on the semistable locus of the stack of coherent sheaves on an elliptic curve whose singular support is nilpotent. We first define a family of local systems on the open substack of torsion sheaves of degree n supported at n pairwise distinct points of X. Consider the stack of torsion sheaves of degree n on X, $\mathfrak{Tor}_n$. The open substack $\mathfrak{Tor}_n^{rss}$ of torsion sheaves of degree n supported at n distinct points is isomorphic to $(S^nX \setminus \Delta)/\,\mathbf{G}_m^n$ where the action of $(\mathbf{G}_m)^n$ is trivial. It therefore admits a $\mathfrak{S}_n$-cover $p_n : (X^n \setminus \Delta)/\,\mathbf{G}_m^n \to \mathfrak{Tor}_n^{rss}$. Let $\mathscr{L}_i$ be simple local systems on X for $1 \leq i \leq n$. When a basis of $\pi_1(X) \simeq \mathbf{Z}^2$ is fixed, this datum is equivalent to a n-uplet $\boldsymbol{z} \in ((\mathbf{C}^*)^2)^n$ describing the monodromy. The exterior product $\mathscr{L}_1 \boxtimes \ldots \boxtimes \mathscr{L}_n$ is a local system on X^n. We consider its restriction to $X^n \setminus \Delta$ and let $\mathscr{L}$ be the induced local system on $(X^n \setminus \Delta)/\,\mathbf{G}_m^n$. Then, the (underived) pushforward $(p_n)_*\mathscr{L}$ decomposes as a direct sum of local systems on $\mathfrak{Tor}_n^{rss}$ indexed by representations of the symmetric group:

$$\pi_*\mathscr{L} \simeq \bigoplus_{\lambda \in \mathscr{P}_n} \mathscr{L}_{\boldsymbol{z},\lambda} \otimes V_\lambda$$

where V_λ is the multiplicity vector space ($\mathfrak{S}_n$ acts on $\pi_*\mathscr{L}$ and this is its decomposition in isotypical components, where each $\mathscr{L}_{\boldsymbol{z},\lambda}$ is assumed to be irreducible). The multiplicity complexes V_λ are nonzero. One can see this as follows. Since $\mathfrak{S}_n$ acts on $\pi_*\mathscr{L}$, we have an algebra morphism $\mathbf{C}[\mathfrak{S}_n] \to \mathrm{End}(\pi_*\mathscr{L})$. This algebra morphism is injective since the fiber of π over any point $x \in S^nX \setminus \Delta$ has $n!$ points and $(\pi_*\mathscr{L})_x \simeq \mathbf{C}^{n!}$ is acted on by $\mathfrak{S}_n$ by the regular representation. Hence, the composition $\mathbf{C}[\mathfrak{S}_n] \to \mathrm{End}(\pi_*\mathscr{L}) \to \mathrm{End}((\pi_*\mathscr{L})_x)$ is an isomorphism. In particular, this gives non-trivial orthogonal idempotents 1_λ in $\mathrm{End}(\pi_*\mathscr{L})$, one for each partition λ of n. On the other hand, $\mathrm{End}(\pi_*\mathscr{L}) = \bigoplus_{\lambda \in \mathscr{P}_n} \mathrm{Hom}_{\mathbf{C}}(V_\lambda, V_\lambda)$. The non-trivial idempotents of this algebra are the identity morphisms $\mathrm{id}_{V_\lambda} \in \mathrm{Hom}(V_\lambda, V_\lambda)$. Consequently, none of the multiplicity vector spaces V_λ is trivial. Therefore, all the local systems $\mathscr{L}_{\boldsymbol{z},\lambda}$ occur in the direct sum decomposition.

Moreover, by the discussion following Proposition 6.6.6, if $\boldsymbol{z}$ and $\boldsymbol{z}'$ cannot be obtained from each other by permuting the factors of $((\mathbf{C}^*)^2)^n$, then for any partitions λ and ν, $\mathscr{L}_{\boldsymbol{z},\lambda}$ and $\mathscr{L}_{\boldsymbol{z}',\nu}$ are not isomorphic. Of course, $\pi_*\mathscr{L}$ only depends on the local systems $\mathscr{L}_i$ up to permutation.

Let $\alpha \in \mathbf{Z}^+$. By the isomorphism $\epsilon_\alpha : \mathfrak{Tor}_\delta = \mathfrak{Coh}_{(0,\delta)} \to \mathfrak{Coh}_{(\alpha)}(X)$ (Section 6.2.3), we can transport the local systems on $\mathfrak{Tor}_\delta^{rss}$ on local systems on an open substack of $\mathfrak{Coh}_\alpha(X)$. These are still denoted $\mathscr{L}_{\boldsymbol{z},\lambda}$ for $\boldsymbol{z} \in ((\mathbf{C}^*)^2)^\delta$ and $\lambda \in \mathscr{P}_\delta$.

Proposition 6.6.9. *Let $\alpha \in \mathbf{Z}^+$ and $\mathscr{F}$ be a simple perverse sheaf on $\mathfrak{Coh}_{(\alpha)}(X)$ having a nilpotent singular support. Then, $\mathscr{F}$ is the intersection cohomology sheaf $\mathcal{IC}(\mathscr{L}_{\boldsymbol{z},\boldsymbol{\lambda}})$ of one of the local systems defined above.*

Proof. By the isomorphisms ϵ_α between $\mathfrak{Tor}_\delta(X)$ and $\mathfrak{Coh}_{(\alpha)}(X)$, it suffices to consider the case when $\alpha = (0,d)$ for some $d \geq 1$. The singular support only depends on the local behaviour of the perverse sheaf, so we can assume that $X = \mathbb{A}^1$ is the affine line (as in the proof of [Lau87, Théorème (3.3.13)]). In this case, $\mathfrak{Coh}_{(0,d)}(X) \simeq \mathfrak{gl}_d/\operatorname{GL}_d$ and the nilpotent cone is $\mathscr{N}_{\mathfrak{gl}_d} = \{(x,\xi) \in \mathfrak{gl}_d \times \mathscr{N} \mid [x,\xi] = 0\}/\operatorname{GL}_d$. By Springer theory, perverse sheaves on $\mathfrak{gl}_d/\operatorname{GL}_d$ with singular support in $\mathscr{N}_{\mathfrak{gl}_d}$ are the Fourier-Sato transforms of the intersection cohomology sheaves of nilpotent orbits of $\mathfrak{gl}_d$. These are given by the local systems $\mathscr{L}_\lambda$ on the open substack $\mathfrak{gl}_d^{rss}/\operatorname{GL}_d \simeq (\mathbb{A}^n\setminus\Delta)/\mathfrak{S}_n$ of regular semisimple elements which appears in $(\pi_{\mathfrak{g}})_!\underline{\mathbf{C}}_{\tilde{\mathfrak{g}}^{rss}}$ (see Section 6.4.3). This proves the proposition. $\square$

6.6.4 Proof of Theorem 6.1.3

For the convenience of the reader, we recall Theorem 6.1.3.

Theorem 6.6.10. *The simple objects of the category $\operatorname{Perv}(\mathfrak{Coh}_\alpha(X), \mathscr{N}_\alpha)$ of perverse sheaves having a nilpotent singular support are precisely the simple twisted spherical Eisenstein perverse sheaves.*

Proof. We first show that simple twisted spherical Eisenstein perverse sheaves have nilpotent singular support. This follows from the definition of these perverse sheaves as direct summands of induction of perverse sheaves of the form $\mathcal{IC}(\mathscr{L})$ where $\mathscr{L}$ is a shifted local system on $\mathfrak{Coh}_{(\alpha)}(X)$, $\gcd(\alpha) = 1$ by the same argument as in the proof of Proposition 6.4.2 and the fact that $\mathcal{IC}(\mathscr{L})$ is by Proposition 6.6.5 a (shifted) local system on $\mathfrak{Coh}_\alpha(X)$, so its singular support is the zero section of $T^*\mathfrak{Coh}_\alpha(X)$.

Conversely, we need to show that a simple perverse sheaf on $\mathfrak{Coh}_\alpha(X)$ having nilpotent singular support belongs to $\mathcal{P}_{tw}^\alpha$. Let $\mathscr{F}$ be such a perverse sheaf. Let $\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}(X)$ be the stratum of $\mathfrak{Coh}_\alpha(X)$ such that $\operatorname{supp}\mathscr{F} = \overline{\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}(X)}$ (which exists since $\operatorname{supp}\mathscr{F} = \pi_\alpha(SS(\mathscr{F}))$). We consider the iterated induction diagram restricted to the HN-stratum of type $\boldsymbol{\alpha} = (\alpha_s, \ldots, \alpha_1)$:

$$\prod_{i=1}^s \mathfrak{Coh}_{(\alpha_i)}(X) \xleftarrow{q} \mathfrak{Exact}_{\boldsymbol{\alpha}} \xrightarrow{p} \mathfrak{Coh}_{\boldsymbol{\alpha}}(X)$$

The map p is an isomorphism and q a vector bundle stack. Let $\mathscr{H} = p^*\mathscr{F}_{\mathfrak{Coh}_{\boldsymbol{\alpha}}(X)}$. Write $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s)$. The singular support of the restriction of $\mathscr{H}$ to $q^{-1}\left(\prod_{i=1}^s \mathfrak{Coh}_{(\alpha_i),\lambda_i}\right) =$

$p^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}(X))$ is the conormal bundle to $p^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}(X))$. Therefore, $\mathscr{H} = \mathcal{IC}(\mathscr{L})$ for a local system $\mathscr{L}$ on $p^{-1}(\mathfrak{Coh}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}(X))$. Hence, there exists a local system $\mathscr{L}'$ on $\prod_{i=1}^{s}\mathfrak{Coh}_{(\alpha_i),\lambda_i}(X)$ such that $\mathscr{L} = q^*\mathscr{L}'$. We can write $\mathscr{L}' = \mathscr{L}_1\boxtimes\ldots\boxtimes\mathscr{L}_s$ for local systems $\mathscr{L}_i$ on $\mathfrak{Coh}_{(\alpha),\lambda_i}$, $1\leq i\leq s$. Then, $\mathscr{H} = q^*(\mathscr{G}_1\boxtimes\ldots\boxtimes\mathscr{G}_s)[\dim q]$ and $\mathscr{F}$ is a simple constituent (up to some shift) of the induction $\mathrm{Ind}_{\alpha_1,\ldots,\alpha_s}(\mathscr{G}_1\boxtimes\ldots\boxtimes\mathscr{G}_s)$ for $\mathscr{G}_i = \mathcal{IC}(\mathscr{L}_i)$. Since the singular support of a product is the product of the singular supports and q is compatible with the stratifications (by definition), for any $1\leq i\leq s$, $\mathscr{G}_i$ is a perverse sheaf on $\mathfrak{Coh}_{(\alpha_i)}(X)$ with nilpotent singular support. By Proposition 6.6.9, for $1\leq i\leq s$, $\mathscr{G}_i$ is a twisted spherical Eisenstein perverse sheaf, so the same is true for $\mathscr{F}$. $\square$

6.6.5 The simple twisted spherical Eisenstein perverse sheaves

In this section, we describe explicitly the simple objects of the categories $\mathcal{P}_{tw}^{\alpha}$ and $\mathcal{P}_{tw,f}^{\alpha}$ in the spirit of [Sch12b, Proposition 3.4]. We let $\boldsymbol{\mu}\subset\mathbf{C}^*$ be the subgroup of roots of unity.

Proposition 6.6.11. *The simple objects of $\mathcal{P}_{tw}^{(\alpha)}$ (resp. $\mathcal{P}_{tw,f}^{(\alpha)}$) are the intermediate extensions $\mathcal{IC}(\mathscr{L}_{\boldsymbol{z},\lambda})$ of the local systems described above for $\boldsymbol{z}\in((\mathbf{C}^*)^2)^{\delta}$ (resp. $\boldsymbol{z}\in(\boldsymbol{\mu}^2)^{\delta}$). Moreover, the local systems $\mathscr{L}_{\boldsymbol{z},\lambda}$ and $\mathscr{L}_{\boldsymbol{z}',\lambda'}$ on $\mathfrak{Coh}_{\alpha}(X)$, for some $\boldsymbol{z},\boldsymbol{z}'\in((\mathbf{C}^*)^2))^{\delta}$ are not isomorphic if $\boldsymbol{z}$ and $\boldsymbol{z}'$ cannot be deduced from each other by a permutation.*

Furthermore, for each $\alpha\in\mathbf{Z}^+$, there is a canonical bijection

$$\theta_{\alpha}:\mathcal{P}_{\sharp}^{\alpha}\to\bigsqcup_{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_s)\in HN(\alpha)}\prod_{i=1}^{s}\mathcal{P}_{\sharp}^{(\alpha_i)}$$

($\sharp = tw$ or $\sharp = tw,f$) such that if $\theta(\mathscr{F}) = (\mathscr{F}_1,\ldots,\mathscr{F}_s)$, then $\mathrm{Ind}_{\boldsymbol{\alpha}}(\mathscr{F}_s\boxtimes\ldots\boxtimes\mathscr{F}_1) = \mathscr{F}\oplus\mathscr{G}$ where $\operatorname{supp}\mathscr{G}\subsetneq\operatorname{supp}\mathscr{F}$. Last, $\mathbb{D}(\mathcal{IC}(\mathscr{L}_{\boldsymbol{z},\lambda})) = \mathcal{IC}(\mathscr{L}_{\boldsymbol{z}^{-1},\lambda})$ where $\mathbb{D}$ denotes the Verdier duality.

Proof. The statement involving the Verdier duality follows from the fact that the dual of a local system on an elliptic curve is the local system with the inverse monodromy (equivalently, the dual of a representation $\rho:\mathbf{Z}^2\to\mathbf{C}^*$ is the representation $\rho^{-1}:\mathbf{Z}^2\to\mathbf{C}^*$, $z\mapsto\rho(z)^{-1}$). The existence of the bijection θ_{α} is analogous to that of the similar map of [Sch12b, Proposition 3.4] and follows from the fact that the map p in the iterated induction diagram restricted to sheaves of HN-type $\boldsymbol{\alpha}$ (6.4.3) is an isomorphism.

For the first statement, the intermediate extensions $\mathcal{IC}(\mathscr{L}_{\boldsymbol{z},\boldsymbol{\lambda}})$ for $\boldsymbol{z},\boldsymbol{\lambda}$ as in the theorem are simple objects of $\mathcal{P}_{tw}^{\geq(\alpha)}$ or $\mathcal{P}_{tw,f}^{\geq(\alpha)}$. Indeed, if $\alpha=\delta\alpha'$ with α' coprime and $\boldsymbol{z}=(z_1,\ldots,z_{\delta})$, this intersection cohomology sheaf appears as a simple constituent of the induction $\mathrm{Ind}_{\alpha',\ldots,\alpha'}(\mathcal{IC}(\mathscr{L}_{z_1}\boxtimes\ldots\boxtimes\mathscr{L}_{z_{\delta}}))$. To conclude, we need to show that when performing inductions, no more simple perverse sheaves on $\mathfrak{Coh}_{\alpha}(X)$ whose support intersects the semistable stratum appear. This is a consequence of Theorem 6.6.10 above and of its proof since all sheaves appearing in the inductions have nilpotent singular support, which is also proved in the proof of Theorem 6.6.10. $\square$

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Titre : L'algèbre de Hall des courbes et des carquois : fonctions cuspidales, faisceaux pervers et polynômes de Kac

Mots clés : algèbre de Hall, carquois, courbes, fonctions cuspidales, algèbres de Kac-Moody, Polynômes de Kac, faisceaux pervers, support singulier, cône global nilpotent, faisceaux d'Eisenstein

Résumé: Cette thèse a pour objet l'étude de plusieurs aspects des algèbres de Hall des courbes et des carquois. Ces algèbres encodent les extensions entre les objets de la catégorie des représentations d'un carquois ou des des faisceaux cohérents sur une courbe projective lisse. Pour les carquois, elles sont apparues dans les travaux de Ringel dans sa construction géométrique des groupes quantiques. L'algèbre de Hall des courbes a été en premier considérée par Kapranov pour comprendre les équations fonctionnelles entre les séries d'Eisenstein. Dans un premier chapitre, on décrit explicitement les fonctions cuspidales des carquois affines. Ce travail repose la théorie d'Auslander-Reiten. Dans un deuxième chapitre, on étudie le comportement des polynômes de Kac lorsqu'on augmente le nombre de flèches du carquois. Des propriétés asymptotiques et de convergence émergent. Notre troisième chapitre aborde des aspects plus géométriques : nous démontrons que les faisceaux pervers simples de la catégorie de Lusztig (une catégorie de faisceaux pervers sur le champ de représentations du carquois) sont caractérisés par la nilpotence de leur support singulier. Enfin, le dernier chapitre étudie l'analogue de cette question pour les courbes. Nous démontrons que le cycle caractéristique donne un isomorphisme entre une complétion du groupe de Grothendieck de la catégorie des faisceaux d'Eisenstein sphériques sur le champ des faisceaux cohérents sur une courbe elliptique et le groupes des fonctions à valeurs entières sur l'ensemble des composantes irréductibles du cône global nilpotent elliptique.

Title: The Hall algebra of curves and quivers: cuspidal functions, perverse sheaves and Kac polynomials

Keywords: Hall algebra, quiver, curves, cuspidale functions, Kac-Moody algebras, Kac polynomials, perverse sheaves, singular support, global nilpotent cone, Eisenstein sheaves

Abstract: This thesis aims to study several aspects of Hall algebras associated with curves and quivers. These algebras encode the extensions between the objects of the category of representations of a quiver or of coherent sheaves on a smooth projective curve. For quivers, they appeared in Ringel's work in his geometric construction of quantum groups. The Hall algebra of curves was first considered by Kapranov to understand the functional equations between Eisenstein series. In the first chapter, we explicitly describe the cuspidal functions of affine quivers. This work is based on Auslander-Reiten. In a second chapter, we are interested in the behavior of Kac polynomials when we increase the number of arrows. Asymptotic and convergence properties emerge. The third chapter deals with more geometric aspects: we prove that the simple perverse sheaves of the Lusztig category (a subcategory of the category of perverse sheaves on the stack of quiver representations) are characterized by the nilpotency of their singular support. Finally, the last chapter studies the analog of this question for elliptic curves. We show that the characteristic cycle gives an isomorphism between a completion of the Grothendieck group of the category of spherical Eisenstein sheaves on the stack of coherent sheaves on an elliptic curve and the group of integer-valued functions on the set irreducible components of the elliptic global nilpotent cone.

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