# Cohomological Mackey formula for representations of reductive groups

IHP
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Algebra seminar

① Context : representations of reductive groups
equivariant cohomology & critical cohomology

red. groups, char, cochar, pairing, weights, symmetric, Weyl group, equivariant cohomology

potential

brief motivation : $IH(V/\!/G)$.

② The Coxeter complex

flats, cells, Tits product, partial order, subrepresentations associated to flats.

③ Induction morphisms fixed locus, attracting loci, with example
induction diagram for $C \leq F$, induction morphism, associativity
explicit formula for a potential

④ Cohomological integrality for (weakly) symmetric representations

equivalence classes of cocharacters, sign, theorem

example

⑤ Restriction morphisms . localized cohomology

- restriction morphism
- coassociativity

⑥ Mackey formula braiding operators, thm.

## 0. Quick introduction

$G$ reductive group

$V$ representation of $G$

$f : V \to \mathbb{C}$ $G$-invariant function

Understand the structure of $H^*_{crit,G}(V, f)$.

→ induction

→ decomposition into smaller pieces : "cohomological integrality"

→ restriction and Mackey formula :

↳ compatibility between induction and restriction.

# ① Context

$G = GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $(\mathbb{C}^*)^N$, $Sp_{2n}(\mathbb{C})$, ...

More generally, $G$: reductive group (unipotent radical is trivial)

$\underset{\text{char } 0}{=}$ linearly reductive

(finite-dimensional representations are semisimple)

<u>non-example</u>: $G = \mathbb{G}_a$ additive group

acts on $V = \mathbb{C}^2$ via $\mathbb{G}_a \hookrightarrow \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$.

$V$ is non-trivial extension of $\mathbb{C}$ by $\mathbb{C}$.

* $T \subset G$ maximal torus. $T \cong (\mathbb{C}^*)^{\mathrm{rank}(G)}$

e.g. diag $\cong (\mathbb{C}^*)^n \subset GL_n(\mathbb{C})$.

* representation: $G \longrightarrow GL(V)$, $V$ $\mathbb{C}$ vector space, finite-dimensional.

$GL_2(\mathbb{C}), SL_2(\mathbb{C}) \curvearrowright \mathbb{C}^2$.

**Characters:** $X^*(T) = \{\alpha : T \to \mathbb{G}_m\} \cong \mathbb{Z}^{\mathrm{rk}\, G}$

**Cocharacters** $X_*(T) = \{\lambda : \mathbb{G}_m \to T\} \cong \mathbb{Z}^{\mathrm{rk}\, G}$

**Pairing**

$$\begin{aligned} \alpha \circ \lambda : \mathbb{G}_m &\longrightarrow \mathbb{G}_m \\ z &\longmapsto z^{\langle \lambda, \alpha \rangle} \end{aligned}$$

$$\langle -,- \rangle : X_*(T) \times X^*(T) \to \mathbb{Z}.$$

**Weights** $T \curvearrowright V$ diagonalizable:

$$V = \bigoplus_{\alpha \in X^*(T)} V_\alpha$$

$$V_\alpha = \{ v \in V \mid t \cdot v = \alpha(t) \cdot v \quad \forall t \in T \}$$

$$\mathcal{W}(V) = \{ \alpha \in X^*(T) \mid V_\alpha \neq 0 \} \text{ weights of } V.$$

In particular, $\mathcal{W}(\mathfrak{g})$ weights of $\mathfrak{g} = \mathrm{Lie}(G)$.

ex. $GL_2(\mathbb{C}) \curvearrowright \mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$

$\quad\cup \qquad\qquad\qquad\qquad (1,0) \qquad (0,1)$

$(\mathbb{C}^*)^2$

$$(t_1, t_2) e_1 = t_1 e_1$$
$$(t_1, t_2) e_2 = t_2 e_2$$

$V$ symmetric: $\dim V_\alpha = \dim V_{-\alpha} \quad \forall \alpha \in X^*(T)$

$\Leftrightarrow$ $V \cong V^*$ (a representation is determined by its character)

sort of weakening of symplecticity, appears sometimes when def Coulomb branches.

ex: $\cdot\, T^*V = V \oplus V^*$, $V$ rep of $G$

- any $V$ rep of $SL_2(\mathbb{C})$
- $\mathfrak{g}$ adjoint of $G$
- any representations in type $B_n, C_n, E_7, E_8, G_2, F_4$

$O(2n+1)$ $Sp(2n)$

<u>More generally</u>: $V$ <u>weakly symmetric</u> if $\mathcal{W}(V) \bmod Q_+^*$

$\subset X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q} / Q_+^*$

is symmetric.

<u>Question</u>: rep. theoretic interpretation?

Weyl group $\quad W = N_G(T)/T$

$N_G(T) = \{ g \in G \mid gTg^{-1} = T \}$

$W_{GL_n} \cong W_{SL_n} \cong \mathfrak{S}_n$ symmetric group.

In general: $W$ is a Coxeter group.

$T$ torus: $W_T = \{e\}$

$W \curvearrowright$ weights of $V$ $= \mathcal{W}(V)$.

# Equivariant cohomology

$H_G^*(V)$ equivariant cohomology

$V$ v.space $\Rightarrow$ contractible

$H^*_G(V) \cong H^*_G(pt) \cong H^*(BG)$

$E_G$ contractible space with free $G$ action.

$BG = EG/G$.

ex: $H^*_{\mathbb{C}^*}(pt)$

$G = \mathbb{C}^* \curvearrowright \mathbb{C}^N \setminus \{0\}$ free

$$\mathbb{C}^\infty \setminus \{0\} = \bigcup_N \mathbb{C}^N \setminus \{0\}$$

$$\mathbb{P}^\infty = \mathbb{C}^\infty \setminus \{0\} / \mathbb{C}^*$$

$$H^*(\mathbb{P}^N) \cong \mathbb{Q}[x]/(x^{N+1}) \qquad \deg(x) = 2$$

$$H^*(\mathbb{P}^\infty) = H^*_{\mathbb{C}^*}(pt) \cong \mathbb{Q}[x]$$

$G = T = (\mathbb{C}^*)^n$ $\quad H^*_T(pt) \cong \mathbb{Q}[x_1, \dots, x_n]$

$G$ general $\quad H^*_G(pt) \cong H^*_T(pt)^W \quad T \subset G$ max torus.

$\rightsquigarrow H^*_G(pt) \cong \mathbb{Q}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ symmetric polynomials

$G = GL_2(\mathbb{C}) \qquad \mathbb{Q}[x_1 + x_2, x_1 x_2]$

**In general** $H^*_G(pt)$ is a polynomial algebra

in particular, $\dim_{\mathbb{Q}} H^*_G(pt) = +\infty$.

**Cohomological integrality**

Extract a finite-dimensional subspace

$$P_0 \subset H^*(V/G)$$

that generates (in the sense of parabolic induction)

$P_0$ = " cuspidal cohomology " of $V/G$.

$\hookrightarrow$ analogy with $\begin{cases} \text{character sheaves (rep of fin grps of Lie type)} \\ \text{Hecke eigensheaves (Langlands)} \end{cases}$

**Motivation: understand $IH^*(V/\!/G)$**

ex: ① $\mathbb{C}^* \curvearrowright \mathbb{C}^N$ poids 1 $\qquad \mathbb{C}^N/\!/\mathbb{C}^* = pt$

② $\mathbb{C}^* \curvearrowright \mathbb{C}^2 \qquad t\cdot(u,v) = (tu, t^{-1}v)$

$\lambda \neq 0$ $\{xy = \lambda\}$ are the closed orbits

$\{0\}$

$\rightsquigarrow \mathbb{C}^2/\!/\mathbb{C}^* \cong \mathbb{C}$

$\mathbb{C}[x,y]^{\mathbb{C}^*} \cong \mathbb{C}[xy] \subset \mathbb{C}[x,y]$.

③ $G \curvearrowright \mathfrak{g}$ adjoint rep.

$\mathfrak{g}/\!/G \cong \mathfrak{t}/\!/W \qquad \mathfrak{t} = \mathrm{Lie}\, T$

$\cong \mathbb{A}^{\mathrm{rk}\, G}$

④ non smooth:

$$\mathbb{C}^* \curvearrowright \mathbb{C}^4 \qquad t\cdot(u,v,w,x) = (tu, tv, t^{-1}w, t^{-1}x)$$

$$\mathbb{C}^4/\!/\mathbb{C}^* \cong \mathrm{Spec}\left(\mathbb{C}[ac, ad, bc, bd]\right)$$

$$\cong \mathrm{Spec}\left(\mathbb{C}[A,B,C,D]/\langle AD = BC\rangle\right)$$

==Cohomological integrality== $\rightsquigarrow$ algorithmic computation of

$\mathrm{IH}^*(V/\!/G)$ [Bu - Davison - Ibanez Nunez - Kinjo - Padurariu]

$$\mathrm{IH}(X) = \begin{cases} \text{intersection cohomology} = \begin{cases} \text{singular cohomology if } X \text{ smooth} \\ \text{encodes information regarding singularities otherwise.} \end{cases} \end{cases}$$

==Critical cohomology==

$V$ representation of a reductive group $G$.

$f : V \to \mathbb{C}$ regular function

$$H^*_{crit,G}(V, f) := H^*_G(V, \varphi_f)$$

vanishing cycle sheaf

encodes the singularities of the function $f$.

All my talk generalizes to this context.

## (2) The Coxeter complex

$V$ representation of $G$

$$V = \bigoplus_{\alpha \in W(V)} V_\alpha$$

$$\mathfrak{g} = \bigoplus_{\alpha \in W(\mathfrak{g})} V_\alpha$$

$\mathfrak{h} := X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ real vector space

For $\alpha \in X^*(T)$, $H_\alpha := \{\alpha = 0\} \subset \mathfrak{h}$ hyperplane

$$\mathcal{H} = \bigcup_{\alpha \in W(V) \cup W(\mathfrak{g})} H_\alpha \quad \subset \mathfrak{h} \quad \text{hyperplane arrangement}$$

<u>flat</u>: intersection of hyperplanes, $\mathcal{F}$ = set of flats

<u>cell</u>: connected component of

$$\bigcap_{H \in \mathcal{H}_1} H \setminus \bigcup_{H \subset \mathcal{H}_2} H .$$

$\mathcal{H} = \mathcal{H}_1 \sqcup \mathcal{H}_2$ , $\mathcal{C}$ = set of cells

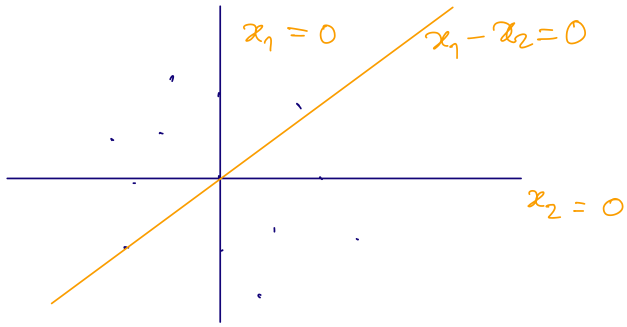
- a cell generates a flat $\langle C \rangle$
- partial order: $F_1 \preceq F_2 \iff F_2 \subseteq F_1$

  $C_1 \preceq C_2 \iff C_2 \subset \overline{C_1}$

  $C \preceq F \iff F \subset \langle C \rangle$

**Example:** $G = GL_2(\mathbb{C})$

$V = \mathbb{C}^2$, $G$ acts via det




$\mathcal{F} = \{\mathbb{Q}^2, \{x_1 = 0\}, \{x_2 = 0\}, \{(0,0)\}, \{x_1 - x_2 = 0\}\}$

$\mathcal{C}$ has 13 elements

W acts naturally on $\mathcal{H}, \mathcal{F}, \mathcal{C}, \ldots$, respecting the partial order.

**Tits product:**

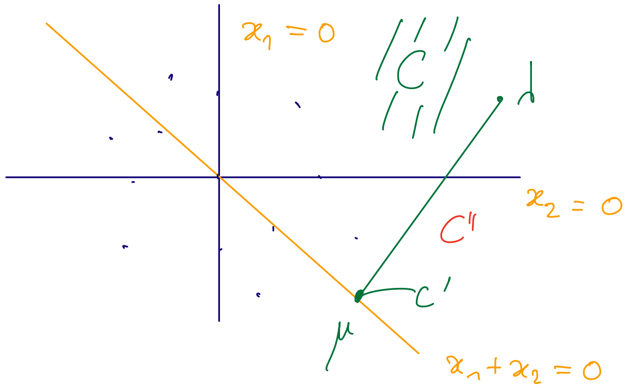
Bilinear operation $\circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

$C, C' \in \mathcal{C}$

$\lambda \in C$, $\mu \in C'$ $\quad$ $L_t \subset \mathfrak{h}$ line parametrized by $[0,1]$ by $(1-t)\lambda + \mu$.

$C \circ C'$ is the first cell in $\mathcal{C}$ met by the line $L_t$ for $t > 0$

example




$$\begin{cases} C \circ C' = C \\ C' \circ C = C'' \end{cases}$$

$\circ$ is $W$-equivariant: $\quad w C \circ w C' = w(C \circ C')$.

## ③ The induction morphisms

### Parabolic induction

$V$ representation of $G$

$\lambda : \mathbb{G}_m \longrightarrow T$ cocharacter

$$G^\lambda = \{ g \in G \mid \lambda(t) g \lambda(t)^{-1} = g \quad \forall t \in \mathbb{C}^* \}$$

$\subset G$ levi subgroup

<u>Note</u> $G^\lambda$ reductive

$T \subset G^\lambda$.

$$V^\lambda = \{ v \in V \mid \lambda(t) v \lambda(t)^{-1} = v \quad \forall t \in \mathbb{C}^* \}$$

$\subset V$ subspace

$$G^{\lambda \geq 0} = \{ g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists} \}$$

$\subset G$ parabolic subgroup

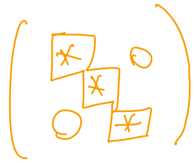
$$V^{\lambda \geq 0} = \{ v \in V \mid \lim_{t \to 0} \lambda(t) \cdot v \text{ exists} \}$$

$\subset V$

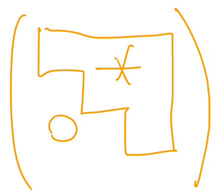
subspace

$G = GL_n \qquad V = T^*\mathbb{C}^n$

$$\begin{aligned} \mathbb{G}_m &\longrightarrow GL_n \\ t &\longmapsto \begin{pmatrix} t^2 & & 0 \\ & t & \\ 0 & & 1 \end{pmatrix} \end{aligned}$$



$$V^\lambda = T^* \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$$



$$V^{\lambda \geq 0} = \mathbb{C}^n \oplus \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$$

* $V^{\lambda \geq 0}$, $G^{\lambda \geq 0}$ only depend on $C \in \mathcal{C}$ containing $\lambda$ so we write $V^{C \geq 0}, G^{C \geq 0}$

* $V^{\lambda}, G^{\lambda}$ only depend on $\underbrace{\quad}_{\text{the minimal}}$ $F \in \mathcal{F}$ containing $\lambda$. We write $V^F, G^F$

For $C \in \mathcal{C}$, $F \in \mathcal{F}$ such that $C \leq F$, we have an induction diagram

$$\begin{tikzcd}
 & V^{C \geq 0, F} / G^{C \geq 0, F} \arrow[dl, "q_{C,F}"'] \arrow[dr, "p_{C,F}"] & \\
V^{\langle C \rangle} / G^{\langle C \rangle} & & V^F / G^F
\end{tikzcd}$$

equivariant for the respective group actions.

$q_{C,F}$ is smooth

$p_{C,F}$ is proper

$$\mathrm{Ind}_C^F := (p_{C,F})_* (q_{C,F})^* : \underbrace{H^*_{G^{\langle C \rangle}}(V^{\langle C \rangle})}_{= \mathcal{H}_{\langle C \rangle}} \longrightarrow \underbrace{H^*_{G^F}(V^F)}_{= \mathcal{H}_F}$$

$$\mathbb{Q}[x_1, \dots, x_n]^{W^{\langle C \rangle}} \longrightarrow \mathbb{Q}[x_1, \dots, x_n]^{W^F}$$

(with $H^*_{G^{\langle C \rangle}}(V^{\langle C \rangle}) \cong \mathbb{Q}[x_1, \dots, x_n]^{W^{\langle C \rangle}}$ and $H^*_{G^F}(V^F) \cong \mathbb{Q}[x_1, \dots, x_n]^{W^F}$)

explicit formula :

$$k_{C,F} := \frac{\displaystyle\prod_{\substack{\alpha \in \mathcal{W}(V) \\ \alpha|_C < 0 \\ \alpha|_F = 0}} \alpha^{\dim V_\alpha}}{\displaystyle\prod_{\substack{\alpha \in \mathcal{W}(V) \\ \alpha|_C < 0 \\ \alpha|_F = 0}} \alpha^{\dim \mathfrak{g}_\alpha}} \in \mathrm{Frac}\, H^*_T(pt)$$

$$\mathrm{Ind}_C^F(f) = \frac{1}{\# W^{\langle C\rangle}} \sum_{w \in W^F} w \cdot (k_{C,F} \cdot f).$$

**Relative Weyl group:** $W_C^F := \{ w \in W^F \mid w\langle C\rangle = \langle C\rangle \}$

**Proposition:** Assume $V$ is weakly symmetric. Then, there exists a character

$$\varepsilon_{V,C,F} : W_C^F \longrightarrow \mathbb{Q}^* \quad \text{s.t.} \quad \forall w \in W_C^F$$

$$w \cdot k_{C,F} = \varepsilon_{V,C,F}(w)\, k_{C,F}.$$

**Proof:** $V$ symmetric $\Rightarrow$ $\mathcal{W}^{C<0,F}(V) = -\mathcal{W}^{C>0,F}(V)$

$\mathfrak{g}$ also symmetric $\Rightarrow$ $\mathcal{W}^{C<0,F}(\mathfrak{g}) = -\mathcal{W}^{C>0,F}(\mathfrak{g})$

$$\pm k_{C,F}^2 = \frac{\prod_{\substack{\alpha \in \mathcal{W}(V) \\ \alpha|_F = 0 \\ \alpha|_C \neq 0}} \alpha^{\dim V_\alpha}}{\prod_{\substack{\alpha \in \mathcal{W}(\mathfrak{g}) \\ \alpha|_F = 0 \\ \alpha|_C \neq 0}} \alpha^{\dim \mathfrak{g}_\alpha}}$$

is preserved by $W_C^F$.

∎

# (4) Cohomological integrality in the weakly symmetric case

For $F \in \mathcal{F}$, we let $G_F = Z(G^F) \cap \ker(G^F \to GL(V^F))$

**Theorem** (H-2024) Let $V$ be a weakly symmetric representation of a reductive group $G$. Then, for $C \in \mathcal{E}$, there exists cohomologically graded vector spaces $P_C \subset H^*_{G^{\langle C\rangle}}(V^{\langle C\rangle})$ which are $W_C$-invariant and such that for any $F \in \mathcal{F}$

$$\bigoplus_{\substack{\widetilde{\langle C\rangle} \in \mathcal{F}/W^F \\ C \leq F}} \left(P_C \otimes H^*_{G_{\langle C\rangle}}(pt)\right)^{W_C^F} \xrightarrow{\oplus \mathrm{Ind}_C^F} \begin{array}{c} H^*(V^F/G^F) \\ \parallel \\ \mathcal{H}_F \end{array}$$

is an isomorphism

Question of interest: identifying $P_{\langle C\rangle}$. Usually $P_C = IH^*(V^{\langle C\rangle}/\!\!/G^{\langle C\rangle})$.

Meinhardt-Reineke: $(V, G)$ comes from a symmetric quiver

H: calculations for $G = \mathbb{C}^*$, $G = SL(2,\mathbb{C})$ & $V = Mat_{2\times n}(\mathbb{C})$.

$G$ arbitrary & $V/\!\!/G = pt$

Bu-Davison-Ibanez Nunez-Kinjo-Padurariu: most precise results

If $V$ is an orthogonal representation of $G$, then

$$P_C = \begin{cases} IH(V^{\langle C\rangle}/\!\!/G^{\langle C\rangle}) & \text{if } V^{\langle C\rangle}/G^{\langle C\rangle}/G_{\langle C\rangle} \to V^{\langle C\rangle}/\!\!/G^{\langle C\rangle} \text{ is generically a finite gerbe.} \\ 0 & \text{else.} \end{cases}$$

Generalization of this in [BDINKP].

## 5 Restriction morphisms

Induction morphisms never come without restrictions!

### a Localized cohomology.

For $C \lesssim F$, define

$$\mathcal{H}_{C,F} = \mathcal{H}_{\langle C\rangle}[k_{C,F}^{-1}]$$ localized cohomology

### b Restriction morphisms

: $C \in \mathcal{C}$, $F \in \mathcal{F}$, $-C$ = opposite chamber

$$\begin{tikzcd}
 & V^{-C\geq 0,F}/G^{-C\geq 0,F} \arrow[dl, "q_{-C,F}"'] \arrow[dr, "p_{-C,F}"] & \\
V^{\langle C\rangle}/G^{\langle C\rangle} = V^{\langle C\rangle}/G^{\langle C\rangle} & & V^F/G^F
\end{tikzcd}$$

$$q_{-C,F}^{*} : H^{*}_{G^{\langle C\rangle}}(V^{\langle C\rangle}) \longrightarrow H^{*}(V^{-C\geq 0,F}/G^{-C\geq 0,F})$$

is an isomorphism.

We define $\mathrm{Res}_F^C : \mathcal{H}_F \longrightarrow \mathcal{H}_{C,F}$

$$f \longmapsto \frac{1}{k_{C,F}} (q^{*}_{-C,F})^{-1} p^{*}_{-C,F} f$$

• **coassociativity**: for any $C \lneq C' \lneq F$,

$$\mathrm{Res}^{C}_{F} = \mathrm{Res}^{C}_{\langle C' \rangle} \circ \mathrm{Res}^{C'}_{F}.$$

## ⑥ Mackey formula

Classically, Mackey formula expresses the commutation between induction and restriction morphisms.

• **Braiding operators**:

For $C, C' \in \mathscr{C}$ such that $\langle C \rangle = \langle C' \rangle$ and $F \in \mathscr{F}$ such that $C, C' \lneq F$, define

$$\tau^{F}_{C,C'} : \mathcal{H}_{C,F} \longrightarrow \mathcal{H}_{C',F} ;$$
$$f \longmapsto \frac{k_{C,F}}{k_{C',F}} f$$

It is an isomorphism.

• Mackey formula : Theorem

For any $C, C' \in \mathcal{C}$ and $F \in \mathcal{F}$ such that $C, C' \not\leq F$,
one has

$$\mathrm{Res}^{C'}_{F} \circ \mathrm{Ind}^{F}_{C} = \sum_{w \in W^{\langle C' \rangle} \backslash W^F / W^{\langle C \rangle}} \underbrace{\mathrm{Ind}^{\langle C' \rangle}_{C' \circ \dot{w} C} \circ \tau^{F}_{\dot{w} C \circ C' ; C' \circ \dot{w} C} \circ \mathrm{Res}^{\dot{w} C \circ C'}_{\langle \dot{w} C \rangle} \circ \dot{w}}_{\text{does not depend on the lift } \dot{w}.}$$

where, for each $w \in {}_{W^{\langle C \rangle}} \backslash W^F / W^{\langle C \rangle}$ , $\dot{w} \in W^F$ is
a lift and $\dot{w} : \mathcal{H}_{\langle C \rangle} \xrightarrow{\sim} \mathcal{H}_{\langle \dot{w} C \rangle}$

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Paves the way to the extension of Mackey formula to general
smooth / 0-shifted symplectic / (-1) shifted symplectic stacks.

## Examples

(1) $\mathbb{C}^* \curvearrowright V = \bigoplus_{k \in \mathbb{Z}} V_k$

For simplicity, we assume $V_0 = 0$.

$$\lambda_0 : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$t \longmapsto 1$$

$$\lambda_1 : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$t \longmapsto t$$

$\mathcal{P}_V = \{\bar{\lambda}_0, \bar{\lambda}_1\}$ ; no Weyl group

$V^{\lambda_0} = V$, $G^{\lambda_0} = G$, $G_{\lambda_0} = \{1\}$, $k_{\lambda_0} = 1$

$V^{\lambda_1} = pt$, $G^{\lambda_1} = G$, $G_{\lambda_1} = G$, $k_{\lambda_1} = \prod_{k>0} (kx)^{\dim V_k} \in \mathbb{Q}[x]$.

$$\mathrm{Ind}_{\lambda_1,\lambda_0} : \mathbb{Q}[x] \longrightarrow \mathbb{Q}[x]$$
$$f(x) \longmapsto \underset{\substack{\| \\ C_V \cdot x^{\sum_{k>0} \dim V_k}}}{k_{\lambda_1}} \cdot f(x)$$

$$P_{\lambda_0} = \mathbb{Q}[x]_{\deg < 2\sum_{k>0} \dim V_k}$$

$$P_{\lambda_1} = \mathbb{Q}.$$

Integrality isomorphism

$$P_{\lambda_0} \oplus (P_{\lambda_1} \otimes \mathbb{Q}[x]) \longrightarrow \mathbb{Q}[x]$$
$$(f, g) \longmapsto f + k_{\lambda_1} \cdot g.$$

clearly an isomorphism

② $\overset{\overset{G}{\shortparallel}}{GL_2(\mathbb{C})} \circlearrowright \overset{\overset{V}{\shortparallel}}{(T^*\mathbb{C}^2)^g}$ $\quad g \geq 0 \qquad T = (\mathbb{C}^*)^2 \subset GL_2(\mathbb{C})$

Integrality isomorphism

$$P_{d_0} \oplus \left(P_{d_1} \otimes \mathbb{Q}[x_1]\right) \oplus \left(P_{d_2} \otimes \mathbb{Q}[x_1, x_2]\right)^{sgn} \longrightarrow \mathbb{Q}[x_1+x_2, x_1x_2]$$

$$(f, h, k) \mapsto f + \frac{x_1^g h(x_1, x_2) - x_2^g h(x_2, x_1)}{x_1 - x_2} + \frac{2(x_1 x_2)^g\, k(x_1, x_2)}{x_1 - x_2}.$$

$$P_{d_0} = \bigoplus_{j=0}^{g-2} \mathbb{Q}(x_1+x_2)^j \subset H^*(V/G) \cong \mathbb{Q}[x_1+x_2, x_1x_2]$$

$$P_{d_1} = \bigoplus_{j=0}^{g-1} \mathbb{Q}x_2^j \subset H^*(V^{d_1}/G^{d_1}) \cong \mathbb{Q}[x_1, x_2]$$

$$P_{d_2} = \mathbb{Q} \subset H^*(V^{d_2}/G^{d_2}) \cong \mathbb{Q}[x_1, x_2]$$

$$P_{d_3} = \{0\} \subset \mathbb{Q}[x_1+x_2, x_1x_2].$$

## 3 $SL_2 \curvearrowright Mat_{2\times n}$

$P_{triv} = \mathbb{Q}[x^2]_{\deg \leq 2(n-2)}$ $\qquad \deg x = 2$

$Mat_{2\times n} /\!/ SL_2$ is the affine cone over $Gr(2,n)$.

$\mathbb{IH}^*(Mat_{2\times n}/\!/SL_2)$ = primitive cohomology of $Gr(2,n)$.

$$\mathbb{IH}^j(Mat_{2\times n}/\!/SL_2) = \begin{cases} \mathbb{Q} & \text{if } j \equiv 0\ (4)\ \&\ j \leq 2(n-2) \\ 0 & \text{otherwise.} \end{cases}$$