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BPS algebras and generalised Kac–Moody algebras from 2-Calabi–Yau categories

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Introduction

2-Calabi–Yau (CY) categories feature prominently throughout algebraic geometry and representation theory:

1. **semistable sheaves** on K3 or Abelian surfaces,
2. **semistable Higgs sheaves** on smooth projective curves,
3. representations of **preprojective algebras** of quivers,
4. representations of the **fundamental group of Riemann surfaces**.

We are interested in the **topology and singularities of the moduli stacks** and the good moduli spaces of objects in these categories. Our aim is to understand the **Borel–Moore homologies** of these geometric objects. We achieve this goal in three steps.

1. we define a sheaf-theoretic cohomological Hall algebra for a large class of Abelian categories of dimension at most two,
2. we define the BPS Lie algebra, by generators and relations,
3. we relate the BPS Lie algebra to the BPS algebra of the category and to the whole cohomological Hall algebra

Consequences are multiple. We obtain

1. the **cohomological integrality** of all categories involved,
2. a **stacky nonabelian Hodge isomorphism** for curves,
3. the **positivity of cuspidal polynomials** of quivers (a strengthening of Kac positivity conjecture),
4. a lowest weight vector description for the **cohomology of Nakajima quiver varieties**.

This is an overview of some parts of [1] and [2].

2. 2-dimensional categories

The major examples of 2-CY categories we will be interested in are the following.

1. **Preprojective algebras of quivers**
   - $Q=(Q_0,Q_1)$ quiver, $\overline{Q}=(Q_0,Q_1\sqcup Q_1^*)$ its double,
   - $\rho=\sum_{\alpha\in Q_1}[\alpha,\alpha^*]$ the preprojective relation,
   - $\Pi_Q=\mathbf{C}\overline{Q}/\rho$ the preprojective algebra.
2. **Semistable sheaves on K3 and Abelian surfaces**
   - S symplectic surface, H polarisation, $\boldsymbol{v}\in\mathrm{H}^{\mathrm{even}}(S,\mathbf{Z})$ primitive Mukai vector
   - $\mathrm{Coh}_{\boldsymbol{v}}^{H-\mathrm{ss}}(S)$ category of H-semistable coherent sheaves on S with Mukai vector in $\mathbf{N}\boldsymbol{v}$.
3. **Semistable Higgs sheaves on smooth projective curves**
   - C smooth projective curve, $\mu\in\mathbf{Q}$ slope
   - $\theta\colon\mathcal{F}\to\mathcal{F}\otimes_{\mathcal{O}_C}K_C$ Higgs sheaf
   - $\mathrm{Higgs}^{\mu-\mathrm{ss}}(C)$ category of semistable Higgs sheaves of slope μ.
4. (Twisted) **fundamental group algebras of Riemann surfaces**
   - S (closed) Riemann surface, ξ root of unity
   - $G=\langle\lambda,x_i,y_i:1\leq i\leq g\mid\lambda\prod x_iy_ix_i^{-1}y_i^{-1}=1\rangle=\pi_1(S\setminus\{\mathrm{pt}\})$
   - $A=\mathbf{C}G/(\xi-\lambda)$ twisted fundamental group algebra

3. Cohomological Hall algebras for 2-dimensional categories

Let $\mathcal{A}$ be a d-dimensional Abelian category ($d\leq 2$) and $\mathtt{JH}\colon\mathfrak{M}_{\mathcal{A}}\to\mathcal{M}_{\mathcal{A}}$ the Jordan–Hölder (semisimplification) map from the stack of objects to the good moduli space. The formula $\mathscr{F}\boxdot\mathscr{G}:=\oplus_*(\mathscr{F}\boxtimes\mathscr{G})$ gives $\mathcal{D}^+(\mathrm{MHM}(\mathcal{M}_{\mathcal{A}}))$ a monoidal product, where $\oplus\colon\mathcal{M}_{\mathcal{A}}\times\mathcal{M}_{\mathcal{A}}\to\mathcal{M}_{\mathcal{A}}$ is the direct sum.

Theorem. *The complex of mixed Hodge modules* $\mathscr{A}_{\mathcal{A}}:=\mathtt{JH}_*\mathbb{D}\mathbf{Q}_{\mathfrak{M}_{\mathcal{A}}}^{\mathrm{vir}}\in\mathcal{D}^+(\mathrm{MHM}(\mathcal{M}_{\mathcal{A}}))$ *admits a (relative) cohomological Hall algebra structure.*

This algebra structure is constructed using the commutative diagram

$$\begin{tikzcd}
\mathfrak{M}_{\mathcal{A}}\times\mathfrak{M}_{\mathcal{A}} \arrow[d, "\mathtt{JH}\times\mathtt{JH}"'] & \mathfrak{Exact}_{\mathcal{A}} \arrow[l, "q"'] \arrow[r, "p"] & \mathfrak{M}_{\mathcal{A}} \arrow[d, "\mathtt{JH}"] \\
\mathcal{M}_{\mathcal{A}}\times\mathcal{M}_{\mathcal{A}} \arrow[rr, "\oplus"'] & & \mathcal{M}_{\mathcal{A}}
\end{tikzcd}$$

Key-fact. *1. The RHom complex over $\mathfrak{M}_{\mathcal{A}}\times\mathfrak{M}_{\mathcal{A}}$ can be represented by a 3-term complex of vector bundles. Letting $\mathcal{C}=\mathrm{RHom}[1]$, there is a canonical virtual pullback map $\mathbb{D}\mathbf{Q}_{\mathfrak{M}_{\mathcal{A}}\times\mathfrak{M}_{\mathcal{A}}}\to q_*(\mathbb{D}\mathbf{Q}_{\mathfrak{Exact}_{\mathcal{A}}})[2(-,-)_{\mathcal{A}}]$ for the map $q\colon\mathfrak{Exact}_{\mathcal{A}}=\mathrm{Tot}(\mathcal{C})\to\mathfrak{M}_{\mathcal{A}}\times\mathfrak{M}_{\mathcal{A}}$.*

2. The map p is proper.

5. The PBW isomorphism

Theorem. *We have a* ***PBW isomorphism*** *in* $\mathcal{D}^+(\mathrm{MHM}(\mathcal{M}_{\mathcal{A}}))$:

$$\mathrm{Sym}_{\boxdot}(\mathcal{BPS}_{\mathcal{A},\mathrm{Lie}}\otimes\mathrm{H}^*(\mathrm{B}\mathbf{C}^*))\to\mathscr{A}_{\mathcal{A}}.$$

In particular, we have **cohomological integrality** for the category $\mathcal{A}$.

References

[1] Ben Davison, Lucien Hennecart, and Sebastian Schlegel Mejia. “BPS Lie algebras for totally negative 2-Calabi-Yau categories and nonabelian Hodge theory for stacks”. In: *arXiv preprint arXiv:2212.07668* (2022).

[2] Ben Davison, Lucien Hennecart, and Sebastian Schlegel Mejia. “BPS algebras and generalised Kac-Moody algebras from 2-Calabi-Yau categories”. In: *arXiv preprint arXiv:2303.12592* (2023).

[3] Lucien Hennecart. “On geometric realizations of the unipotent enveloping algebra of a quiver”. In: *arXiv preprint arXiv:2209.06552* (2022).

4. The BPS Lie algebra

Roots. The monoid of connected components of $\mathcal{M}_{\mathcal{A}}$ has the bilinear form induced by the Euler form $(-,-)$. We have the set of primitive positive roots

$$\Sigma_{\mathcal{A}}:=\{a\in\pi_0(\mathcal{M}_{\mathcal{A}})\mid\mathcal{M}_{\mathcal{A},a}\text{ contains simples}\}$$

and the set of positive roots $\Phi_{\mathcal{A}}^+:=\Sigma_{\mathcal{A}}\cup\{la\mid l\in\mathbf{N},a\in\Sigma_{\mathcal{A}},(a,a)=0\}$.

Generators. For $a\in\Sigma_{\mathcal{A}}$, we let $\mathscr{G}_{\mathcal{A},a}:=\mathcal{IC}(\mathcal{M}_{\mathcal{A},a})$. For $a\in\Sigma_{\mathcal{A}}$, $(a,a)=0$ and $l\geq 2$, we let $\mathscr{G}_{\mathcal{A},a}:=(u_m)_*\mathcal{IC}(\mathcal{M}_{\mathcal{A},a})$ where $u_m\colon\mathcal{M}_{\mathcal{A},a}\to\mathcal{M}_{\mathcal{A},la},x\mapsto x^{\oplus l}$.

The BPS Lie algebra. The (relative) BPS Lie algebra is the Lie algebra object $\mathcal{BPS}_{\mathcal{A},\mathrm{Lie}}=\mathfrak{n}^+_{\pi_0(\mathcal{M}_{\mathcal{A}}),\mathscr{G}_{\mathcal{A}}}\in\mathrm{MHM}(\mathcal{M}_{\mathcal{A}})$ generated by $\mathscr{G}_{\mathcal{A},a}$, $a\in\Phi_{\mathcal{A}}^+$, modulo the relations

1. $\mathrm{ad}(\mathscr{G}_{\mathcal{A},a})(\mathscr{G}_{\mathcal{A},b})=0$ if $(a,b)=0$,
2. $\mathrm{ad}(\mathscr{G}_{\mathcal{A},a})^{1-(a,b)}(\mathscr{G}_{\mathcal{A},b})=0$ if $(a,a)=2$.

The BPS algebra The (relative) BPS algebra is defined as $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}}:=\mathcal{H}^0(\mathscr{A}_{\mathcal{A}})\in\mathrm{MHM}(\mathcal{M}_{\mathcal{A}})$.

Theorem. *We have a canonical isomorphism* $\mathcal{BPS}_{\mathcal{A},\mathrm{Alg}}\cong\mathbf{U}(\mathfrak{n}^+_{\pi_0(\mathcal{M}_{\mathcal{A}}),\mathscr{G}_{\mathcal{A}}})$.

Proof. Local neighbourhood theorem for 2-CY categories and identification of the top strictly seminilpotent CoHA of quivers [3]. □

6. Nonabelian Hodge isomorphism for stacks

Let $(r,d)\in\mathbf{Z}_{\geq 1}\times\mathbf{Z}$. Classical NAHT provides us with a diagram in which the middle arrow is an homeomorphism:

$$\mathfrak{M}_{r,d}^{\mathrm{Dol}}(C)\xrightarrow{\mathtt{JH}}\mathcal{M}_{r,d}^{\mathrm{Dol}}\xrightarrow{\Psi}\mathcal{M}_{g,r,d}^{\mathrm{Betti}}\xleftarrow{\mathtt{JH}}\mathfrak{M}_{g,r,d}^{\mathrm{Betti}}.$$

Theorem. *We have a canonical isomorphism of constructible complexes*

$$\Psi_*\mathtt{JH}_*\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{\mathfrak{M}_{r,d}^{\mathrm{Dol}}(C)}\cong\mathtt{JH}_*\mathbb{D}\mathbf{Q}^{\mathrm{vir}}_{\mathfrak{M}_{g,r,d}^{\mathrm{Betti}}}$$

and in particular, a canonical isomorphism in Borel–Moore homology:

$$\mathrm{H}_*^{\mathrm{BM}}(\mathfrak{M}_{r,d}^{\mathrm{Dol}}(C))\cong\mathrm{H}_*^{\mathrm{BM}}(\mathfrak{M}_{g,r,d}^{\mathrm{Betti}})$$

Question. *1. Do we have an isomorphism in cohomology?*

2. Do we have stacky nonabelian isomorphisms for higher dimensional varieties?

7. Positivity of cuspidal polynomials

Let Q be a quiver, $\mathbf{F}_q$ a finite field and $H_{Q,\mathbf{F}_q}$ the Hall algebra of Q over $\mathbf{F}_q$. This is a $\mathbf{N}^{Q_0}$-graded twisted bialgebra. Its primitive elements are called *cuspidal functions*:

$$H_{Q,\mathbf{F}_q}^{\mathrm{cusp}}[\boldsymbol{d}]:=\{f\in H_{Q,\mathbf{F}_q}\mid\Delta(f)=f\otimes 1+1\otimes f\},$$

and $C_{Q,\boldsymbol{d}}(q)$ denote its dimension.

Theorem. *For* $\mathbf{d}\in\mathbf{\Sigma}_{\mathbf{\Pi}_{\mathbf{Q}}}$, *we have* $C_{Q,\mathbf{d}}(q^{-2})=\mathrm{IP}(\mathcal{M}_{\Pi_Q,\mathbf{d}})$, *and so* $C_{Q,\mathbf{d}}(q)\in\mathbf{N}[q]$.

This gives a way to compute the intersection cohomology of all Nakajima quiver varieties, using the Borcherds–Kac–Weyl character formula for generalised Kac–Moody algebras.

8. Decomposition of the cohomology of Nakajima quiver varieties

Let $N_{Q,\mathbf{f},\mathbf{d}}$ be the Nakajima quiver variety for the quiver Q and framing data $\mathbf{f}$. We let $\mathbb{M}_{\mathbf{f}}(Q):=\bigoplus_{\mathbf{d}\in\mathbf{N}^{Q_0}}\mathrm{H}^*(N(Q,\mathbf{f},\mathbf{d}),\mathbf{Q}^{\mathrm{vir}})$. This is a representation of the Lie algebra $\mathfrak{g}_{\Pi_Q}^{\mathrm{BPS}}$ (the double of $\mathfrak{n}_{\Pi_Q}^{\mathrm{BPS},+}:=\mathrm{H}^*(\mathcal{BPS}_{\Pi_Q,\mathrm{Lie}})$.

Theorem. *We have the decomposition*

$$\mathbb{M}_{\mathbf{f}}(Q)=\bigoplus_{(\mathbf{d},1)\in\Sigma_{\Pi_{Q_{\mathbf{f}}}}}\mathrm{IH}^*(N(Q,\mathbf{f},\mathbf{d}))\otimes L_{((\mathbf{d},1),(-,0))_{Q_{\mathbf{f}}}}.$$

$L_{\mathbf{e}}$ ($\mathbf{e}\in\mathrm{Hom}(\mathbf{Z}^{Q_0},\mathbf{Z})$): lowest weight module for the generalised Kac–Moody algebra $\mathfrak{g}_{\Pi_Q}^{\mathrm{BPS}}$.