Kac polynomials and generalised Kac–Moody algebras

Ben Davison, Victor Kac's 80th birthday conference, Rome 2023

Quiver representations

Quivers

A *quiver* Q is the data of a set of vertices Q_0, a set of arrows Q_1, and a pair of morphisms $s, t\colon Q_1 \to Q_0$ taking an arrow to its source/target.

Example

Fix a field K. The *path algebra* KQ is a K-algebra with basis given by paths (including lazy paths e_i of length 0 at each vertex i), and multiplication given by concatenation of paths.

Representations

A representation of Q of dimension $\mathrm{d} \in \mathbb{N}^{Q_0}$ is the data of a set of K-vector spaces ρ_i of dimension d_i along with morphisms $\rho(a)\colon \rho_{s(a)} \to \rho_{t(a)}$.
Equivalent to giving a KQ-module N with $\dim(e_i \cdot N) = \mathrm{d}_i$.

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Kronecker quiver: $\bullet \rightrightarrows \bullet$ (multiple edges allowed)

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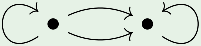

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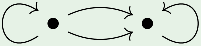

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$$\mathrm{a}_{Q,\mathrm{d}}(q) = \{\text{abs. indecomposable d-dimensional } \mathbb{F}_q Q\text{-modules}\}/\cong$$

Example

Let Q be the Kronecker quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & \bullet \end{tikzcd}$$

Let N be a $(1,1)$-dimensional KQ-module. $N(a)$ and $N(b)$ given by scalars. Iff $N'(a) = \lambda N(a)$ and $N'(b) = \lambda N(b)$ for $\lambda \in K^\times$ then $N \cong N'$.

- $\mathrm{a}_{Q,(1,1)}(q) = \#\mathbb{P}^1_{\mathbb{F}_q} = q + 1$
- $\mathrm{a}_{Q,(n,n)}(q) = \#\mathbb{P}^1_{\mathbb{F}_q} = q + 1$ (e.g. use Beilinson's equivalence)

Theorem (Kac)

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Kac polynomials: some results

To start with, we allow Q to have loops

- Positivity conjecture (Kac), Theorem (Hausel, Letellier, Rodriguez–Villegas): $\mathrm{a}_{Q,\mathrm{d}}(q) \in \mathbb{Z}_{\geq 0}[q]$.

Now we (temporarily) forbid loops:

- Constant term conjecture (Kac), Theorem (Hausel): $\mathrm{a}_{Q,\mathrm{d}}(0) = \dim(\mathfrak{g}_{Q,\mathrm{d}})$, the dth weight space of the Kac–Moody Lie algebra associated to underlying graph of Q.
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What is this hinting at?:

These facts suggest that *all* the coefficients of $\mathrm{a}_{Q,\mathrm{d}}(q)$ are the dimensions of the cohomologically graded pieces of a *generalised* Kac–Moody algebra $\tilde{\mathfrak{g}}_Q$ such that $\tilde{\mathfrak{g}}_Q^0 = \mathfrak{g}_Q$.

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GKM algebras

We define a class of generalised Kac–Moody algebras, following Borcherds.

Ingredients

- Let $\mathfrak{h}$ be a $\mathbb{Q}$-vector space with symmetric bilinear form $(-,-)$.
- Let $\{h_i\}_{i\in\Phi^+} \subset \mathfrak{h}$ be a countable set of *"positive roots"*: $(h_i,h_j) \in \mathbb{Z}_{\leq 0}$ if $i \neq j$ and $(h_i,h_i) \in 2\cdot\mathbb{Z}_{\leq 1}$ for all $i \in \Phi_+$.
- Pick $\mathcal{G} = \bigoplus_{i\in\Phi_+} \mathcal{G}_i$, a Φ_+-graded, cohomologically graded set of *positive Chevalley generators*.

Recipe

$\mathfrak{g}_{\mathcal{G}} :=$ the free Lie algebra generated by $\mathfrak{h} \oplus \mathcal{G} \oplus \mathcal{G}^\vee$ modulo the relations:

- $[h,h'] = 0$ for all $h,h' \in \mathfrak{h}$.
- $[h,\alpha_i^{(\vee)}] = (-)(h,h_i)\alpha_i^{(\vee)} \qquad (\alpha_i^{(\vee)} \in \mathcal{G}_i^{(\vee)})$
- $[\alpha_i,\alpha_j^\vee] = \delta_{ij}\alpha_j^\vee(\alpha_i)h_i$
- (Serre relations): $[\alpha_i^{(\vee)},-]^{1-(h_i,h_j)}(\alpha_j^{(\vee)}) = 0$ if $(h_i,h_j) = 0$ or $(h_i,h_i) = 2$.

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GKM algebras

We define a class of generalised Kac–Moody algebras, following Borcherds.

Ingredients

- Let $\mathfrak{h}$ be a $\mathbb{Q}$-vector space with symmetric bilinear form $(-,-)$.
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Features of GKM algebras

Trichotomy of roots

Roots h_i come in three types:

1. Real: $(h_i, h_i) = 2$
2. Isotropic: $(h_i, h_i) = 0$
3. Hyperbolic: $(h_i, h_i) < 0$

The last two types of roots are called *imaginary*.

Many of your favourite theorems about Kac–Moody Lie algebras generalise to GKM algebras with minimal modifications, for example:

Triangular decomposition

There is a decomposition of vector spaces $\mathfrak{g}_{\mathcal{G}} = \mathfrak{n}^+_{\mathcal{G}} \oplus \mathfrak{h} \oplus \mathfrak{n}^-_{\mathcal{G}}$ into the Lie algebras generated by $\mathcal{G}, \mathfrak{h}, \mathcal{G}^\vee$ respectively. Moreover $\mathfrak{n}^+_{\mathcal{G}}$ is the free Lie algebra generated by $\mathcal{G}$ modulo the Serre relations.

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Examples of GKM algebras

- Fix a quiver Q (maybe with loops). Fix $\mathfrak{h} = \mathbb{Q}^{Q_0}$ and $(\mathrm{d}, \mathrm{e}) = \chi_Q(\mathrm{d}, \mathrm{e}) + \chi_Q(\mathrm{e}, \mathrm{d}) (= 2\sum_{i \in Q_0} \mathrm{d}_i \mathrm{e}_i - \sum_{a \in Q_1} (\mathrm{d}_{s(a)} \mathrm{e}_{t(a)} + \mathrm{d}_{t(a)} \mathrm{e}_{s(a)}))$.
- So 1_i is a real simple root if no loops at i, an isotropic simple root if exactly one loop, and a hyperbolic root otherwise.

Example (KM Lie algebras)

Assume Q has no loops. We set $\Phi_+ = Q_0$ and $h_i = 1_i$. Fix $\mathcal{G}_i = \mathbb{Q}$. Then $\mathfrak{g}_Q := \mathfrak{g}_{\mathcal{G}}$ is the Kac–Moody Lie algebra associated to underlying graph of Q.

Example (Nakajima's Heisenberg algebra)

Let Q have one vertex i, and one loop. Then $(-,-) = 0$. Set $\Phi_+ = \mathbb{Z}_{\geq 1} \subset \mathbb{Q} = \mathfrak{h}$. Set $\mathcal{G}_n = \mathbb{Q}[2]$ for all $n \geq 1$. Then $\mathfrak{g}_{\mathcal{G}} = \mathtt{heis}_\infty$ is Nakajima's Lie algebra of operators on $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathrm{H}(\mathrm{ncHilb}_n(\mathbb{A}^2), \mathbb{Q}^{\mathrm{vir}})$.
N.B. $\chi_{q^{1/2}}(\mathfrak{g}_{\mathcal{G},n}) := \sum_{i \in \mathbb{Z}} \dim(\mathfrak{g}^i_{\mathcal{G},n}) q^{i/2} = \mathrm{a}_{Q,n}(q^{-1})$.

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- Fix a quiver Q (maybe with loops). Fix $\mathfrak{h} = \mathbb{Q}^{Q_0}$ and $(\mathrm{d}, \mathrm{e}) = \chi_Q(\mathrm{d}, \mathrm{e}) + \chi_Q(\mathrm{e}, \mathrm{d}) (= 2\sum_{i \in Q_0} \mathrm{d}_i \mathrm{e}_i - \sum_{a \in Q_1} (\mathrm{d}_{s(a)} \mathrm{e}_{t(a)} + \mathrm{d}_{t(a)} \mathrm{e}_{s(a)}))$.
- So 1_i is a real simple root if no loops at i, an isotropic simple root if exactly one loop, and a hyperbolic root otherwise.

Example (KM Lie algebras)

Assume Q has no loops. We set $\Phi_+ = Q_0$ and $h_i = 1_i$. Fix $\mathcal{G}_i = \mathbb{Q}$. Then $\mathfrak{g}_Q := \mathfrak{g}_{\mathcal{G}}$ is the Kac–Moody Lie algebra associated to underlying graph of Q.

Example (Nakajima's Heisenberg algebra)

Let Q have one vertex i, and one loop. Then $(-, -) = 0$. Set $\Phi_+ = \mathbb{Z}_{\geq 1} \subset \mathbb{Q} = \mathfrak{h}$. Set $\mathcal{G}_n = \mathbb{Q}[2]$ for all $n \geq 1$. Then $\mathfrak{g}_{\mathcal{G}} = \mathtt{heis}_\infty$ is Nakajima's Lie algebra of operators on $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathrm{H}(\mathrm{ncHilb}_n(\mathbb{A}^2), \mathbb{Q}^{\mathrm{vir}})$.

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Okounkov's conjecture

On the previous slide, we began to see how nonconstant terms in Kac polynomials *might* be recording graded dimensions of an associated GKM algebra, helping to motivate

Okounkov's question

This is closely related to *Okounkov's conjecture*:

- Using stable envelopes, Maulik and Okounkov define $\mathrm{Y}_{Q,\mathtt{MO}}$, a subalgebra of the endomorphism algebra of the cohomology of all Nakajima quiver varieties associated to Q.
- They prove that there is a PBW isomorphism $\mathrm{Y}_{Q,\mathtt{MO}} \cong \mathrm{U}\left(\mathfrak{g}_{\mathtt{MO},Q} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q})\right)$, where $\mathfrak{g}_{\mathtt{MO},Q}$ is a Lie algebra containing $\mathfrak{g}_{Q^{\mathrm{real}}}$.
- MO Conjecture: $\mathfrak{g}_{Q,\mathtt{MO}}$ can be defined as coh. graded vector space over $\mathbb{Q}$. Okounkov conjecture: $\chi_{q^{1/2}}(\mathfrak{g}_{\mathtt{MO},Q,\mathrm{d}}) = \mathrm{a}_{Q,\mathrm{d}}(q^{-1})$.

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Given a quiver Q, Is there a (non-cohomologically graded) GKM algebra $\mathfrak{g}^\circ$ with $\dim(\mathfrak{g}^\circ_{\mathrm{d}}) = \mathtt{a}_{Q,\mathrm{d}}(1)$ for all $\mathrm{d} \in \mathbb{N}^{Q_0}$?

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The Kronecker quiver revisited

We plot Kac polynomials $a_{Q,d}(q^{-1})$ against dimension vectors:

3	0	0	$1+$	$1+q^{-1}$
2	0	$1+$	$1+q^{-1}$	$1+$
1	$1+$	$1+q^{-1}$	$1+$	0
0	$*$	$1+$	0	0
	0	1	2	3

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We plot Kac polynomials $\mathsf{a}_{Q,\mathsf{d}}(q^{-1})$ against dimension vectors: Using Kac's vanishing result

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The Kronecker quiver revisited

We plot Kac polynomials $\mathrm{a}_{Q,\mathrm{d}}(q^{-1})$ against dimension vectors: Using constant term theorem

3	0	0	$1+$	$1+q^{-1}$
2	0	$1+$	$1+q^{-1}$	$1+$
1	$1+$	$1+q^{-1}$	$1+$	0
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The Kronecker quiver revisited

We plot Kac polynomials $a_{Q,d}(q^{-1})$ against dimension vectors: By earlier calculations and Weyl-group invariance

3	0	0	1	$1+q^{-1}$
2	0	1	$1+q^{-1}$	1
1	1	$1+q^{-1}$	1	0
0	$*$	1	0	0
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The Kronecker quiver revisited

The following choice of generating $\mathcal{G}$ is then *forced*:

3	0	0	0	$\mathbb{Q}[2]$
2	0	0	$\mathbb{Q}[2]$	0
1	$\mathbb{Q}$	$\mathbb{Q}[2]$	0	0
0	$*$	$\mathbb{Q}$	0	0
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The Kronecker quiver revisited

Yielding the GKM algebra $\mathfrak{g}_{\mathcal{G}}$ with $\mathbb{Z}_{\geq 0}^{Q_0}$-graded pieces:

3	0	0	$\mathbb{Q}$	$\mathbb{Q}\oplus\mathbb{Q}[2]$
2	0	$\mathbb{Q}$	$\mathbb{Q}\oplus\mathbb{Q}[2]$	$\mathbb{Q}$
1	$\mathbb{Q}$	$\mathbb{Q}\oplus\mathbb{Q}[2]$	$\mathbb{Q}$	0
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Preprojective algebras

- We denote by $\overline{Q}$ the *doubled quiver*, obtained by adding an arrow a^* with the opposite orientation to a for all $a \in Q_1$.
- We define the *preprojective algebra* $\Pi_Q := \mathbb{C}\overline{Q}/\langle\sum_{a\in Q_1}[a, a^*]\rangle$

Example

If Q is the one loop quiver then $\mathbb{C}\overline{Q} = \mathbb{C}\langle a, a^*\rangle$, the ring of polynomials in two noncommuting variables, and $\Pi_Q \cong \mathbb{C}[a, a^*]$.

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If Q is the Kronecker quiver then Π_Q is derived equivalent to $\mathrm{QCoh}(\mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-2)))$.

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Moduli spaces

The stack of Π_Q-modules

- Let $\mathbb{A}_{\overline{Q},\mathrm{d}} := \prod_{a\in\overline{Q}_1} \mathrm{Hom}(\mathbb{C}^{\mathrm{d}_{s(a)}}, \mathbb{C}^{\mathrm{d}_{t(a)}}) \cong \mathrm{T}^*\mathbb{A}_{Q,\mathrm{d}}$.
- Moment map

$$\mu_d : \mathbb{A}_{\overline{Q},\mathrm{d}} \to \prod_{i\in Q_0} \mathfrak{gl}_{\mathrm{d}_i}$$

$$(N(a), N(a^*))_{a\in Q_1} \mapsto \sum_{a\in Q_1} [N(a), N(a^*)].$$

- The stack $\mathfrak{M}_{\mathrm{d}}(\Pi_Q)$ of d-dimensional Π_Q-modules is equivalent to the quotient $\mu_{\mathrm{d}}^{-1}(0)/\,\mathrm{GL}_{\mathrm{d}}$ with $\mathrm{GL}_{\mathrm{d}} := \prod_{i\in Q_0} \mathrm{GL}_{\mathrm{d}_i}(\mathbb{C})$.

The coarse moduli space

If we wish to have an algebraic variety instead of a stack, we may consider the coarse quotient: $X_{\mathrm{d}}(\Pi_Q) := \mathrm{Spec}(\Gamma(\mu_{\mathrm{d}}^{-1}(0))^{\mathrm{GL}_{\mathrm{d}}})$
Points of this space are in bijection with d-dimensional semisimple Π_Q-modules.

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From moduli spaces to Kac polynomials

- There is a canonical morphism $\mathtt{JH}\colon \mathfrak{M}(\Pi_Q) \to X(\Pi_Q)$; geometrically this is the affinization map, at the level of points it takes a module to its semisimplification.
- This map is rather stacky, and the spaces very singular. For $\zeta \in \mathbb{Q}^{Q_0}$ a stability condition, the analogous morphism $\mathtt{JH}^{\zeta}_{\mathrm{d}}\colon \mathfrak{M}^{\zeta\text{-sst}}_{\mathrm{d}}(\Pi_Q) \to X^{\zeta\text{-sst}}_{\mathrm{d}}(\Pi_Q)$ to the GIT moduli space can be better behaved:

Theorem (Nakajima, CB+VdB)

Assume that ζ is d*-generic (equivalently: no strictly semistables). Then*

1. *JH^{ζ}_{d} is a $\mathbb{C}^*$-gerbe*
2. *$X^{\zeta\text{-sst}}_{\mathrm{d}}(\Pi_Q)$ is a smooth quasiprojective variety.*
3. *There is an equality $\mathrm{a}_{Q,\mathrm{d}}(q^{-1}) = \chi_{q^{1/2}}(\mathrm{H}(X^{\zeta\text{-sst}}_{\mathrm{d}}(\Pi_Q), \mathbb{Q}^{\mathrm{vir}}))$.*

The catch: if $\mathrm{d} = l\mathrm{d}'$ with $\mathrm{d}' \in \mathbb{N}^{Q_0}$ and $l \geq 2$ then there are *no* d-generic stability conditions!

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By definition

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\mathtt{a}_{Q,\bullet}(q) &= \#^{\text{naive}}\{\text{abs. indec. } \mathbb{F}_q Q \text{ modules}\}/\cong \\
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- Where Q^+ obtained by adding loops ω_i at every $i \in Q_0$ and $R_I = \langle \omega_{t(a)} a - a\omega_{s(a)} \rangle$
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Theorem (D,Meinhardt)

(Proved for arbitrary symmetric quivers with potential). There is a subspace $\mathfrak{g}^{\mathrm{BPS}}_{\tilde{Q},\tilde{W}} \subset \mathcal{H}_{\tilde{Q},\tilde{W}}$, closed under the commutator Lie bracket, such that the natural map

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By dimensional reduction $\mathcal{H}_{\tilde{Q},\tilde{W}} \cong \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q), \mathbb{Q}^{\mathrm{vir}})$.

With respect to the diagram

$$\begin{tikzcd} \mathfrak{M}(\Pi_Q) \arrow[d, "\mathrm{JH}"] \arrow[dr, "p"] & \\ X(\Pi_Q) \arrow[r, "r"] & \mathrm{pt} \end{tikzcd}$$

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$$\begin{aligned} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q), \mathbb{Q}^{\mathrm{vir}}) :=& p_* \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}(\Pi_Q)} \\ =& r_* \, \mathrm{JH}_* \, \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}(\Pi_Q)} \end{aligned}$$

Theorem (Decomposition theorem for 2CY categories (D))

There is an isomorphism

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$$\mathtt{JH}_*\, \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}(\Pi_Q)} \cong \bigoplus_{n \in 2\mathbb{Z}_{\geq 0}} \bigoplus_{s \in S_n} \mathcal{IC}_{\overline{Z_s}}(\mathcal{L}_s)[\dim(Z_s) - n]$$

with $\mathcal{L}_s$ simple local systems on smooth locally closed $Z_s \subset X(\Pi_Q)$.

The decomposition theorem

By dimensional reduction $\mathcal{H}_{\tilde{Q},\tilde{W}} \cong \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q), \mathbb{Q}^{\mathrm{vir}})$.
With respect to the diagram

$$\begin{tikzcd} \mathfrak{M}(\Pi_Q) \arrow[d, "\mathtt{JH}"] \arrow[dr, "p"] & \\ X(\Pi_Q) \arrow[r, "r"] & \mathrm{pt} \end{tikzcd}$$

we write

$$\begin{aligned} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q), \mathbb{Q}^{\mathrm{vir}}) :=& p_* \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}(\Pi_Q)} \\ =& r_* \, \mathtt{JH}_* \, \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}(\Pi_Q)} \end{aligned}$$

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A sheaf-theoretic Hall algebra

Convolution tensor product for sheaves on the coarse moduli space

The coarse moduli space $X(\Pi_Q)$ is a commutative monoid: the morphism $m\colon X(\Pi_Q)^{\times 2} \to X(\Pi_Q)$ sends a pair of points representing modules N, N' to the point representing $N \oplus N'$. For $\mathcal{F}, \mathcal{G} \in \mathcal{D}^+(X(\Pi_Q))$ two complexes of constructible sheaves, we define

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Theorem (D)

- *There is Lie subobject* $\mathcal{L} \subset \mathcal{A}^0_{\Pi_Q}$ *with* $\mathrm{H}(X(\Pi_Q), \mathcal{L}) \cong \mathfrak{g}^{\mathrm{BPS}}_{\tilde{Q},\tilde{W}}$
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The main result

Analyzing the possible summands appearing in the decomposition of $\mathcal{L}$, we prove the following, for general Q (loops allowed):

Theorem (D, Hennecart, Schlegel Mejia)

Let Q be a quiver. Let Φ_+ be the set of positive roots of $\mathfrak{g}_Q$. We define

- *Real roots: $\mathcal{G}_{\mathrm{d}} = \mathbb{Q}$ if $\mathrm{d} = 1_i$, and no loops at i.*
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Let $\mathfrak{g}_{\mathcal{G}}$ be the GKM algebra generated by $\bigoplus_{\mathrm{d}\in\Phi_+} \mathcal{G}_{\mathrm{d}}$. Then $\mathfrak{g}_{\mathcal{G}} \cong \mathfrak{g}_{\tilde{Q},\tilde{W}}$ and so

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