# Langlands duality for character varieties via BPS cohomology

based on joint work w/ Tasuki Kinjo.

0

$M$ smooth 3-manifold, compact, oriented

$G$ reductive group

$G^L$ Langlands dual of $G$

$\mathcal{L}oc_G(M)$ $(-1)$-symplectic stack of $G$-local systems on $M$.

Donaldson-Thomas theory [Brav-Bussi-Dupont-Joyce-Szendrői, Kiem-Li ]

$$\to \varphi_{\mathcal{L}oc_G(M)} \in \mathrm{Perv}(\mathcal{L}oc_G(M)) \quad \text{DT perverse sheaf}$$

**Conjecture A** [Ben Zvi, Jordan, Safronov] (= Conj B of Tasuki's talk)

$$H^*(\mathcal{L}oc_G(M), \varphi_{\mathcal{L}oc_G(M)}) \cong H^*(\mathcal{L}oc_{G^L}(M), \varphi_{\mathcal{L}oc_{G^L}(M)})$$

Recently, coh. int. for these stacks produces a perverse sheaf

$\mathcal{BPS}_{\mathrm{Loc}_G(M)} \in \mathrm{Perv}(\mathrm{Loc}_G(M))$ where

$p : \mathcal{L}oc_G(M) \longrightarrow \mathrm{Loc}_G(M)$ is the good moduli space.

**Conjecture B**:

$$H^*(\mathrm{Loc}_G(M), \mathcal{BPS}_{\mathrm{Loc}_G(M)}) \cong H^*(\mathrm{Loc}_{G^L}(M), \mathcal{BPS}_{\mathrm{Loc}_{G^L}(M)}).$$

Conj A and Conj B are closely related: implication relations.

* $M = \mathbb{T}^3$, $G = SL_p$ $G^L = PGL_p$, $p$ premier (Kaubrys)
* other cases : widely open.

Today : $M = \mathbb{T}^3$ and any semisimple $G \to$ formula for $\dim H^*_{BPS}(\mathrm{Loc}_G(M))$

$\pi_1(\mathbb{T}^3) \cong \mathbb{Z}^3$ and so stacks of local systems are commuting stacks.

$\Rightarrow$ conjecture B for $SL_n / PGL_n$ (Kaubrys for $n$ prime)

$E_6^{sc} / E_6^{ad}$

$SO(2n+1) / Sp(2n)$

$Spin(7) / PSp(6)$

In principle : the formula allows a case by case check of the conjecture.

Thm : (H-Kinjo) For $G$ semisimple, $H^*_{BPS}(\mathrm{Loc}_G(M))$ is concentrated in degree 0 and:

1. a) $$\dim H^0_{BPS}\left(\mathrm{Loc}_{SO(2n+1)}(\mathbb{T}^3)\right) = \frac{1}{8}[x^{2n+1}] \prod_{n \geq 0} (1+x^{2n+1})^8$$

   b) $$\dim H^0_{BPS}\left(\mathrm{Loc}_{Sp(2n)}(\mathbb{T}^3)\right) = [x^n] \prod_{n \geq 1} (1+x^n)^8$$

   These quantities coincide

2. $$\dim H^0_{BPS}\left(\mathrm{Loc}_{E_6^{sc}}(\mathbb{T}^3)\right) = \dim H^0_{BPS}\left(\mathrm{Loc}_{E_6^{ad}}(\mathbb{T}^3)\right) = 416$$

3. $$\dim H^0_{BPS}\left(\mathrm{Loc}_{PSp(6)}(\mathbb{T}^3)\right) = \dim H^0_{BPS}\left(\mathrm{Loc}_{Spin(7)}(\mathbb{T}^3)\right) = 65$$

$$\dim H^0_{BPS}\left(\mathrm{Loc}_{Sp(6)}(T^3)\right) = \dim H^0_{BPS}\left(\mathrm{Loc}_{SO(7)}(T^3)\right) = 128$$

The fact that (a) and (b) coincide is a consequence of a functional equation involving Jacobi $\theta$ function which is expressed in terms of $\prod_{k\geq 1}(1+x^k)$.

## ⊙ Donaldson–Thomas perverse sheaf

gluing of vanishing cycle sheaves on local critical charts.

* $\mathfrak{M}$ $(-1)$-shifted symplectic derived 1-Artin stack.
* $\mathfrak{M}$ is covered by critical charts: $(\mathcal{U}, f)$ where $\mathcal{U}$ is a smooth Artin stack and $f: \mathcal{U} \to \mathbb{C}$ a regular function such that $\mathrm{crit}(f) \longrightarrow \mathfrak{M}$ smooth.
* $\varphi_f \in \mathrm{Perv}(\mathrm{crit}(f))$ glue to $\varphi_{\mathfrak{M}} \in \mathrm{Perv}(\mathfrak{M})$

When $\mathfrak{M}$ has a good moduli space $p: \mathfrak{M} \longrightarrow \mathcal{M}$, one studies $p_* \varphi_{\mathfrak{M}} \in \mathcal{D}_c(\mathcal{M})$ constructible derived category.

* only depends on the d-critical structure induced. For example, given any $(-1)$-shifted sympl. stack w/ smooth classical truncation, $\varphi_{\mathfrak{M}} \cong \mathbb{Q}_{\mathfrak{M}}[\dim \mathfrak{M}]$ is the constant sheaf.

## ① Cohomological integrality for commuting stacks

down to earth viewpoint

$G$ a reductive group and $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra.

$$\mathcal{C}^n(\mathfrak{g}) = \{(x_1, \dots, x_n) \in \mathfrak{g}^n \mid [x_i, x_j] = 0 \ \forall i,j\}$$

$$\mathcal{E}^n(G) = \{(x_1, \dots, x_n) \in G^n \mid \text{pairwise commute}\}$$

$n$-commuting varieties

$$\mathfrak{Comm}^n_{\mathrm{Lie}}(G) = \mathcal{E}^n(\mathfrak{g})/G$$

$$\mathfrak{Comm}^n(G) = \mathcal{E}^n(G)/G \qquad \text{quotient stacks}$$

$$\mathrm{Comm}^n_{\mathrm{Lie}}(G) = \mathcal{E}^n(\mathfrak{g})/\!\!/G$$

$$\mathrm{Comm}^n(G) = \mathcal{E}^n(G)/\!\!/G \qquad \text{good moduli spaces.}$$

$p : \mathfrak{M} \to \mathcal{M}$ good moduli space morphism in each of these situations.

**Remark**: can imagine variants where some $x_i$'s are in $\mathfrak{g}$ and others in $G$.

**Facts** $\mathfrak{Comm}^2_{\mathrm{Lie}}(G)$, $\mathfrak{Comm}^2(G)$ are $0$-shifted symplectic stacks

$\mathfrak{Comm}^3_{\mathrm{Lie}}(G)$, $\mathfrak{Comm}^3(G)$ are $(-1)$-shifted symplectic.

**short explanation**: $\mathfrak{Comm}^2_{\mathrm{Lie}}(G) = \mu^{-1}(0)/G$ for $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$
$(x,y) \mapsto [x,y]$

$$\mathcal{C}omm^3_{\mathrm{Lie}}(G) = \mathrm{crit}(f)/G \quad \text{for}$$

$$f : \mathfrak{g}^{\times 3} \to \mathfrak{g}$$

$$(x,y,z) \mapsto \langle x, [y,z] \rangle,$$

$\langle -,- \rangle$ Killing form.

$L \subset G$ levi subgroup $\rightsquigarrow$ $W_G(L)$ relative Weyl group.

**Theorem**

For every reductive group $G$, $p_* \varphi_{\mathcal{C}omm^3(G)} \in \mathcal{D}^+_c(\mathrm{Comm}^3(G))$ is semisimple and there exists $W_G(L)$-equivariant cohomologically graded complexes of perverse sheaves $\mathcal{BPS}_{\mathrm{Comm}^3(G)} \in \mathcal{D}^+_c(\mathrm{Comm}^3(G))$ and an isomorphism

$$p_* \varphi_{\mathcal{C}omm^3_{\mathrm{Lie}}(G)} \cong \bigoplus_{L \in \mathrm{Levi}(G)/\text{conjugation}} \left( \iota_{L*} \mathcal{BPS}_{\mathrm{Comm}^3_{\mathrm{Lie}}(L)} \otimes H^*(BZ_L) \right)^{W_G(L)}$$

$$\mathcal{BPS}_{\mathrm{Comm}^3(G)} \cong {}^{p}\mathcal{H}^{\dim Z_G}\left( p_* \varphi_{\mathcal{C}omm^3(G)} \right)[-\dim Z_G]$$

$$\boxed{\mathcal{BPS}_{\mathrm{Comm}^3_{\mathrm{Lie}}(G)} \cong \varphi_f \mathcal{IC}(\mathfrak{g}^{\times 3}/\!/G)[-\dim Z_G]}$$

$\iota_L : \mathrm{Comm}^3(L) \to \mathrm{Comm}^3(G)$.

In particular, the perverse filtration on $p_* \varphi_{\mathcal{E}^3_{\mathrm{Comm}}(G)}$ starts in degree $\dim Z_{G_3}$.

For $\mathcal{E}\mathrm{omm}_{\mathrm{Lie}}(G)$, there is dimensional reduction:

$$\left(T^*[-1]\mathfrak{M} \longrightarrow \mathfrak{M}\right)_* \varphi_{T^*[-1]\mathfrak{M}} \cong \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}} \quad \text{(Kinjo)}$$

$$p_* \underbrace{\mathbb{D}\mathbb{Q}_{\mathcal{E}^2_{\mathrm{Comm}}(G)}}_{\text{dualizing sheaf}} \cong \bigoplus_{L \in \mathrm{levi}(G)/\mathrm{conj}} \left(\mathcal{BPS}_{\mathrm{Comm}^2_{\mathrm{Lie}}(G)} \otimes H^*(BZ_L)\right)^{W_G(L)}$$

The power of dim. red. comes from the fact that while most constructions are made at the 3-level [(-1)-shifted symplectic stacks], computations are only possible / easier at the 2d-level [0-shifted symplectic stack]

## ② BPS sheaf for the Lie commuting varieties

I suggest to explain how to compute $\mathcal{BPS}_{\mathrm{Comm}^3_{\mathrm{Lie}}(G)}$.

$$f : \mathfrak{g}^{\times 3}/G \to \mathbb{C}$$
$$(x,y,z) \mapsto \langle x, [y,z] \rangle$$

**Proposition** [support lemma]

* $\mathrm{supp}\, \mathcal{BPS}_{\mathrm{Comm}^3_{\mathrm{Lie}}(G)} \subset \mathfrak{z}_G^3 \subset \mathrm{Comm}^3_{\mathrm{Lie}}(G)$, $\quad \mathfrak{z}_G = \mathrm{Lie}\, Z_G$.

* $\mathcal{BPS}_{\mathrm{Comm}^3_{\mathrm{Lie}}(G)}$ is $\mathfrak{z}_G^3$-equivariant, where $\mathfrak{z}_G^3$ acts on $\mathrm{Comm}^3_{\mathrm{Lie}}(G)$ by translation.

$$\Rightarrow \boxed{\mathcal{BPS}_{\mathrm{Comm}^3_{\mathrm{Lie}}(G)} \cong \mathbb{Q}_{\mathfrak{z}_G^3}^{\oplus r_G}[2 \dim \mathfrak{z}_G]} \text{ for some } r_G \geq 0.$$

Computing $r_G$ can be done by studying the nilpotent commuting stack:

$$\mathrm{Comm}^{2,\mathrm{nil}}(G) = \{(x,y) \in \mathfrak{g} \mid [x,y] = 0 \text{ and } x,y \text{ nilpotent}\}/G$$

$$\begin{tikzcd}
\mathrm{Comm}^{2,\mathrm{nil}}(G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Comm}^2(G) \arrow[d] \\
\{(0,0)\} \arrow[r, "\iota"] & \mathrm{Comm}^2(G)
\end{tikzcd}$$

Applying $\iota^!$ to the dimensionally reduced coh. integrality iso, we obtain

$$H_*^{BM}\left(\mathcal{E}\mathrm{omm}^{2,\mathrm{nil}}_{\mathrm{Lie}}(G)\right) \cong \bigoplus_{L \in \mathrm{levi}(G)/\mathrm{conj.}} \left(\mathbb{Q}^{r_L} \otimes H^*(BZ_L)\right)^{W_G(L)}$$

and also

$$H^{BM}_{-\dim Z_G}\left(\mathcal{E}\mathrm{omm}^{2,\mathrm{nil}}_{\mathrm{Lie}}(G)\right) \cong \mathbb{Q}^{r_G}$$

__fact__: $\mathcal{E}\mathrm{omm}^{2,\mathrm{nil}}_{\mathrm{Lie}}(G)$ is equidimensional of dimension $-\dim Z_G$

and so, $r_L = \# \underbrace{\mathrm{Irr}\left(\mathcal{E}\mathrm{omm}^{2,\mathrm{nil}}_{\mathrm{Lie}}(G)\right)}_{\text{studied by Premet in early 2000's}}$.

Thm (Premet) $r_L = \#$ distinguished nilpotent orbits of $G$.

nilcone

$$\begin{array}{ccc} \mathcal{N}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{g}/\!/G \end{array}$$

* nilpotent orbits are $G$-orbits of $\mathcal{N}_{\mathfrak{g}}$
* a nilpotent orbit $\mathcal{O} \subset \mathcal{N}_{\mathfrak{g}}$ is __distinguished__ if $\mathcal{O}$ does not intersect proper levi subalgebras $\mathfrak{l} \subset \mathfrak{g}$.

e.g.

| Lie algebra | nilp orbits | distinguished nilp. orbits | number |
|---|---|---|---|
| $\mathfrak{sl}_n$ | partitions of $n$ | $(n)$ | $1$ |
| $\mathfrak{so}(2n+1)$ | partitions of $2n+1$ even parts appear w/ even multiplicity | no repeated parts | $[x^{2n+1}]\prod_{k\geq 0}(1+x^{2k+1})$ |
| $\mathfrak{sp}(2n)$ | partitions of $2n$, odd parts appear w/ even multi'ci'l | no repeated parts | $[x^{n}]\prod_{k\geq 1}(1+x^{k})$. |

③ Local structure for $Loc_G(T^3)$ : tool to go from $BPS_{Comm^3_{Lie}(G)}$ to $BPS_{Comm^3(G)}$.

Theorem

Let $u \in Comm^3(G)$ be a closed point. Then étale locally around $p^{-1}(\bar{u}) \longrightarrow \bar{u}$, the morphism $p$ looks like

$$Comm^3_{Lie}(C_G(u)) \xrightarrow{p'} Comm^3_{Lie}(C_G(u)) \text{ around } p'^{-1}(0) \longrightarrow 0.$$

+ compatibility of d-critical structures

Compatibility of DT perverse sheaf and perverse filtration w/ étale morphisms gives the support of the BPS sheaf $BPS_{Comm^3(G)}$ :

$$\operatorname{supp} BPS_{Comm^3(G)} \subset \left\{ \bar{u} \in Comm^3(G) \;\middle|\; \dim Z_{C_G(u)} = \dim Z_G \right\}$$

## ④ Dimension of the BPS cohomology of $Loc_G(T^3)$

Let $G$ be a semisimple group.

$$S_G^3 = \{\overline{(x,y,z)} \in Comm^3(G) \mid \underset{\text{center}}{C_G(x,y,z)} \text{ has } 0\text{-dim.}\}$$

($\Leftrightarrow$ not contained in any proper Levi subgroup)

<u>Fact</u>: $G$ semisimple $\Rightarrow$ $S_G^3 \subset Comm^3(G)$ is discrete subset: finitely many points

**Thm**:

$$BPS_{Comm^3(G)} \cong \bigoplus_{\bar{u} \in S_G^3} \mathbb{Q}_u^{\oplus r_{C_G(u)}}$$

direct sum of skyscraper sheaves.

Recipe for computing $BPS_{Comm^3(G)} / H^0(Comm^3(G), BPS_{Comm^3(G)})$.

① Determine $S_G^3$ : classification of quasi-isolated elements of reductive groups [Bonnafé, Digne-Michel].

② For $\bar{u} \in S_G^3$, determine $C_G(u)$

③ determine $r_{C_G(u)}$ : classification of distinguished nilpotent orbits

$$\dim H^0(Comm^3(G), BPS_{Comm^3(G)}) = \sum_{\bar{u} \in S_G^3} r_{C_G(u)}$$

## ⑤ Langlands duality

We gave a recipe to compute $H^0_{BPS}(\mathrm{Comm}^3(G))$ for any group $G$.

Example: for $Sp(2n)$

classify isolated elements: $g \in Sp(2n)$ s.t. $C_{Sp(2n)}(g)^0$ is semisimple

$$\rightsquigarrow \quad Sp(2n) = \{ M \in GL(2n) \mid M J \, {}^tM = J \}.$$

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

max torus $\quad T = \{ \mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) \}$

$$u_i = \mathrm{diag}(\underbrace{1, \dots, 1}_{i}, \underbrace{-1, \dots, -1}_{2n-2i}, \underbrace{1, \dots, 1}_{i})$$

$$C_{Sp(2n)}(u_i) \cong Sp(2i) \times Sp(2(n-i))$$

From this, it is easy to compute $S^3_{Sp(2n)}$: it is in bijection with $\{ (a_j)_{1 \le j \le 8} \mid \sum_{j=1}^{8} a_j = n \}$ and the stabilizer of the triple in $S^3_{Sp(2n)}$ corresponding to $(a_j)$ is $\prod_{j=1}^{8} Sp(2a_j)$.

$$\Rightarrow \quad \sum_{n \geq 0} \dim H^0_{BPS}(\mathrm{Comm}^3(Sp(2n)))\, x^n = \prod_{k \geq 1} (1+x^k)^8.$$