BPS SHEAF FOR COMMUTING VARIETIES

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Abstract. In this short note, we give a description of the BPS sheaf of the triple commuting variety of a reductive Lie algebra. It is shown to be isomorphic to a direct sum of constant sheaves on the cube of the center of the Lie algebra. The multiplicity of the constant sheaf is given by the number of distinguished nilpotent orbits in the Lie algebra. This answers a conjecture of the author, motivated by cohomological Donaldson–Thomas theory, in this particular case. This study relies on the virtual smallness of the approximation by schemes of the good moduli space morphism, which we establish in this paper for general (weakly) symmetric representations of reductive groups. The virtual smallness gives a lower bound on the perverse filtration, which also allows us to give a partial identification of the BPS sheaf for symmetric representations of reductive groups.

Contents

1. Introduction

In this paper, we investigate the BPS sheaf of the varieties of commuting triples and of commuting pairs in a reductive Lie algebra, introduced in [Hen24a] for all symmetric representations of reductive groups, extending the definitions for quivers with potential of [DM20]. The commuting variety and the variety of commuting triples are related via dimensional reduction [Dav17]. The BPS sheaf categorifies the *refined Donaldson–Thomas (DT)-invariants*. These newly defined enumerative invariants are expected to carry meaningful geometric and representation-theoretic information, and extend cohomological DT-theory [Sze14] to the study of a large class of algebraic stacks.

While original DT-invariants were first introduced for the moduli spaces of stable sheaves on compact Calabi–Yau threefolds [Tho00], the theory has then been extended to moduli spaces of semistable sheaves, allowing the presence of strictly semistable objects [JS12]. The category of sheaves on a Calabi–Yau threefold forms a 3-Calabi–Yau category (see e.g. [BD19; KW21]), and Kontsevich–Soibelman paved the path to an extension of Donaldson–Thomas theory to a large class of such categories, not necessarily arising from geometry [KS11]. For example, their formalism deals with categories of representations of the Jacobi algebra of a quiver with potential. It also provides a *cohomological refinement* of Donaldson–Thomas theory. In particular, instead of just numbers, it gives cohomologically graded vector spaces, whose Euler characteristics recover classical DT-invariants (when they are defined).

It is now understood that cohomological DT-theory can be extended to more general geometric situations than the ones arising from 3-Calabi–Yau categories. In particular, it admits extensions to all (-1)-shifted symplectic stacks [Ben+15]. However, the study of arbitrary (-1)-shifted symplectic stacks is at the very

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beginning. This work is a sequel to [Hen24b; Hen24a] where we initiated the study of the enumerative geometry of a particular, meaningful class of such stacks. The stacks involved are the quotients of a smooth affine algebraic variety by a reductive group and the critical loci of regular functions in such stacks. By dimensional reduction, this encompasses the study of stacky weak Hamiltonian reductions of smooth, weakly symplectic algebraic varieties.

Given a stack presented as the zero locus of the differential of regular function on a smooth stack (i.e. as a critical locus), one considers the critical cohomology (i.e. the cohomology of the vanishing cycle sheaf), [KS11; Ben+15]. The induction formalism provides an induction structure on this cohomology. One goal of the theory is to determine precisely this induction structure. This work is the first step towards a description of this structure for the triple commuting variety. When $G=\mathrm{GL}_n(\mathbf{C})$, the induction morphisms give the cohomological Hall algebra structure, and this description is known and given in [Dav22]. It involves the $W_{1+\infty}$ Lie algebra.

1.1. **Main results.**

1.1.1. *The BPS sheaf of symmetric representations of reductive groups.* Let V be a symmetric representation of a reductive group G (that is, $V\cong V^*$ as G-representations [Hen24b, Definition 2.3]). For $\lambda\in \mathrm{X}_*(T)$ a cocharacter of a maximal torus T of G, there is a Levi subgroup $G^{\lambda}\subseteq G$, a representation $V^{\lambda}\subseteq V$ of G^{λ} (§2) and a complex of mixed Hodge modules $\underline{\mathcal{BPS}}_{V,\lambda}\in\mathcal{D}^+(\mathrm{MMHM}(V^{\lambda}/\!\!/G^{\lambda}))$ [Hen24a, Theorem 1.5]. By construction, the complex of mixed Hodge modules $\underline{\mathcal{BPS}}_{V,\lambda}$ only depends on the representation V^{λ} of G^{λ} and not V. We let $G_{\lambda}\coloneqq\ker(G^{\lambda}\to\mathrm{GL}(V^{\lambda}))\cap \mathrm{Z}(G^{\lambda})$. We let $\pi_{\lambda}\colon V^{\lambda}/G^{\lambda}\to V^{\lambda}/\!\!/G^{\lambda}$ be the good moduli space morphism.

We let $\mathrm{X}_*(T)^{\mathrm{st}}$ be the subset of *stable* cocharacters, that is $\lambda\in\mathrm{X}_*(T)$ for which a general closed G^{λ}-orbit in V^{λ} has stabilizer whose neutral component concides with G_{λ}° [Hen24a, Definition 8.12]. If X is a smooth G-variety, we let $\underline{\mathbf{Q}}_{X/G}^{\mathrm{vir}}\coloneqq\underline{\mathbf{Q}}_{X/G}\otimes\mathscr{L}^{-\frac{\dim X-\dim G}{2}}$ be the Tate-twisted constant mixed Hodge module on the quotient stack X/G, so that in particular we have an identification of the underlying constructible complex $\mathbf{rat}(\underline{\mathbf{Q}}_{V/G}^{\mathrm{vir}})\cong\mathcal{IC}(V/G)$.

Theorem 1.1 (=Proposition 3.21+Corollary 3.17)**.** *Let V be a symmetric representation of a reductive group G. Then, for any $\lambda\in\mathrm{X}_*(T)^{\mathrm{st}}$, the perverse filtration on $(\pi_{\lambda})_*\underline{\mathbf{Q}}_{V^{\lambda}/G^{\lambda}}^{\mathrm{vir}}$ starts in degree $\dim G_{\lambda}$, that is $\underline{\tau}^{<\dim G_{\lambda}}\pi_*\underline{\mathbf{Q}}_{V^{\lambda}/G^{\lambda}}^{\mathrm{vir}}=0$. Moreover,*

$$\underline{\tau}^{\leqslant\dim G_{\lambda}}(\underline{\mathcal{BPS}}_{V,\lambda})\cong\underline{\mathcal{IC}}(V^{\lambda}/\!\!/G^{\lambda})\otimes\mathscr{L}^{\dim G_{\lambda}/2}.$$

This theorem gives a partial proof of [Hen24a, Conjecture 8.14]. The remaining part of the conjecture is the following.

Conjecture 1.2. *For any $\lambda\in\mathrm{X}_*(T)^{\mathrm{st}}$, we have $\underline{\tau}^{>\dim G_{\lambda}}\underline{\mathcal{BPS}}_{V,\lambda}=0$ and for any $\lambda\in\mathrm{X}_*(T)\smallsetminus\mathrm{X}_*(T)^{\mathrm{st}}$, $\underline{\mathcal{BPS}}_{V,\lambda}=0$.*

While this paper was under corrections, the preprint [Bu+25] appeared. It claims Conjecture 1.2, and even its generalization to smooth stacks with a good moduli space.

1.1.2. *The BPS sheaf of the tripled commuting variety.* The objects of main interest in this paper are the commuting variety of a reductive Lie algebra, the variety of commuting triples in a reductive Lie algebra, the associated moduli stacks and moduli spaces. The commuting variety has been studied thoroughly [Ric79; Pre03] over field of characteristic 0 or good positive characteristics. In arbitrary positive characteristics, it is still an object of study [Rom24]. In this paper, we work in characteristic zero.

We let G be a reductive group and $\mathfrak{g}\coloneqq\mathrm{Lie}(G)$ its Lie algebra. The commuting variety of triples $\mathscr{C}^3(\mathfrak{g})\coloneqq\{(x,y,z)\in\mathfrak{g}\mid[x,y]=[y,z]=[x,z]=0\}$ admits a natural description as the critical locus of a function of $\mathfrak{g}^3$. Namely, we let $\langle -,-\rangle\colon\mathfrak{g}\times\mathfrak{g}\to\mathbf{C}$ be a nondegenerate (up to center) bilinear pairing on $\mathfrak{g}$. It defines a G-invariant function $f\in\mathbf{C}[\mathfrak{g}^3]^G$, $(x,y,z)\mapsto\langle x,[y,z]\rangle$. We have $\mathscr{C}^3(\mathfrak{g})\cong\mathrm{crit}(f)$ (Proposition 4.5).

Consequently, the stack $\mathfrak{C}^3(\mathfrak{g})/G$ has a natural derived enhancement that is a (-1)-shifted symplectic stack [Pan+13, Corollary 2.11]. It is possible to define the corresponding cohomologically refined DT-invariants, as in [Hen24a]. In this paper, we deal with their computation.

Our main result is the following

Theorem 1.3 (=Theorem 5.3)**.** *Let G be a reductive group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{z}\subseteq\mathfrak{g}$ its center. We let r be the number of distinguished nilpotent orbits in $\mathfrak{g}$. Then, $\underline{\mathcal{BPS}}_{\widetilde{\mathfrak{g}},f}\cong\underline{\mathcal{IC}}(\mathfrak{z}^3)^{\oplus r}\otimes\mathscr{L}^{\dim\mathfrak{z}/2}$.*

A consequence of Theorem 1.3, combined with Theorem 1.1, is a computation of the vanishing cycle functor applied to the intersection cohomology complex of the GIT quotient $\mathfrak{g}^3/\!\!/G$ for the function f. We have a natural closed immersion $\mathfrak{z}^3\to\mathfrak{g}^3/\!\!/G$.

Corollary 1.4. *We have $\varphi_f\underline{\mathcal{IC}}(\mathfrak{g}^3/\!\!/G)\cong\underline{\mathbf{Q}}_{\mathfrak{z}^3}^{\oplus r}\otimes\mathscr{L}^{-3\dim\mathfrak{z}/2}\cong\underline{\mathcal{IC}}(\mathfrak{z}^3)^{\oplus r}$.*

It is in general a very difficult task to give an explicit description of the vanishing cycle functor for a regular function applied to some sheaf on some space. In particular, we do not know any direct computation giving the isomorphism of Corollary 1.4.

1.2. **The case of $G=\mathrm{GL}_n(\mathbf{C})$.** When $G=\mathrm{GL}_n(\mathbf{C})$, the results exposed in this paper are already known and encompassed in the Donaldson–Thomas theory of the tripled Jordan quiver $\widetilde{Q}$ with its canonical potential. If the arrows of $\widetilde{Q}$ are x,y,z, the potential is $W=z[x,y]$. This case is developped in [Dav16, §5]. The specialization of Theorem 1.3 or more precisely, Theorem 5.3 to GL_n is [Dav16, Theorem 5.1].

1.3. **Conventions and Notations.**

(1) The letter G denotes a reductive group, and reductive groups are assumed to be connected.
(2) The constructible derived category with rational coefficients of a complex algebraic variety X is denoted by $\mathcal{D}_{\mathrm{c}}(X)$. The category of monodromic mixed Hodge modules on X is denoted by $\mathrm{MMHM}(X)$. The corresponding derived category is $\mathcal{D}(\mathrm{MMHM}(X))$. There is a forgetful functor $\mathbf{rat}\colon\mathcal{D}(\mathrm{MMHM}(X))\to\mathcal{D}_{\mathrm{c}}(X)$ sending a complex of mixed Hodge modules to the corresponding complex of constructible sheaves.
(3) We distinguish complexes of (monodromic) mixed Hodge modules from constructible complexes by underlining the former. For example, $\underline{\mathbf{Q}}_X$ denotes the constant mixed Hodge module on an algebraic variety X while $\mathbf{Q}_X$ denotes the constant constructible complex. We have similarly the intersection complexes $\underline{\mathcal{IC}}(X)$ and $\mathcal{IC}(X)$. Analogously, the cohomology functors for complexes of mixed Hodge modules are denoted by $\underline{\mathcal{H}}$ and the truncation functors by $\underline{\tau}$. After applying the functor $\mathbf{rat}$, they correspond to the perverse cohomology functors ${}^p\mathcal{H}$ and the perverse truncation functors ${}^p\tau$.
(4) The superscript "vir" denotes the virtual dimension, when it is defined. In practice, when $\mathfrak{X}$ is a smooth stack, the virtual dimension is the dimension $\dim\mathfrak{X}$. If $\mathfrak{X}$ is a smooth stack and $\mu\colon\mathfrak{X}\to\mathbf{A}^N$ is a regular function to an affine space, the virtual dimension of $\mu^{-1}(0)$ is defined to be $\dim\mathfrak{X}-N$. It is the rank of the cotangent complex to the derived zero locus $\mu^{-1}(0)$. We denote by $\underline{\mathbf{Q}}_{\mathfrak{X}}^{\mathrm{vir}}:=\underline{\mathbf{Q}}_{\mathfrak{X}}\otimes\mathscr{L}^{-\mathrm{vir}/2}$ the shift of the constant sheaf by the virtual dimension.
(5) If $H\subseteq G$ is a pair of linear algebraic groups and X is a H-variety, we let $X\times^H G:=(X\times G)/H$ be the free quotient, where H acts diagonally on $X\times G$.
(6) If X is a quasi-projective complex algebraic variety with an action of an algebraic group G, we denote by X/G the corresponding quotient stack. If X is affine and G reductive $X/\!\!/G:=\mathrm{Spec}(\mathbf{C}[X]^G)$ denotes the GIT quotient. The natural morphism $\pi\colon X/G\to X/\!\!/G$ is the *good moduli space map* (good moduli spaces may be defined in bigger generality [Alp13]).
(7) Let X be an algebraic variety and $f\colon X\to\mathbf{A}^1$ a regular function. We denote by $\varphi_f\colon\mathcal{D}(\mathrm{MHM}(X))\to\mathcal{D}(\mathrm{MMHM}(X))$ the perverse exact vanishing cycle sheaf functor. It sends mixed Hodge modules to monodromic mixed Hodge modules [DM20, §2]. We denote by $\mathscr{L}^{1/2}\in\mathcal{D}(\mathrm{MMHM}(\mathrm{pt}))$ the square-root of the Tate twist [DM20, §2.1].

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2. BPS sheaf for critical symmetric representations

In [Hen24b], we proved a cohomological integrality isomorphism for any symmetric representation V of a reductive group G [Hen24b, Theorem 1.1], generalizing [Efi12], which deals with symmetric quivers. In [Hen24a], we used this isomorphism to upgrade it at the sheaf level over the GIT quotient, giving the sheafified cohomological integrality isomorphism [Hen24a, Theorem 1.5] (see also Theorem 2.1). For quivers, a motivic version of this upgrade is given in [MR19] while the cohomological upgrade is given in [DM20]. By applying the vanishing cycle sheaf functor with respect to some invariant function to this isomorphism, we obtain a cohomological integrality isomorphism for critical cohomology [Hen24a, Theorem 1.7], recalled as Theorem 2.3. It involves a BPS complex of mixed Hodge modules, whose definition we recall here. For quivers with potentials, this study was carried out in [DM20].

2.1. **BPS sheaf and cohomological integrality for symmetric representations.** Let V be a symmetric representation of a reductive group G [Hen24b, Definition 2.3]. We let T be a maximal torus of G. We denote by $\mathrm{X}_*(T)=\{\mathbf{G}_{\mathrm{m}}\to T\}$ the lattice of cocharacters of T and by $\mathrm{X}^*(T)=\{T\to\mathbf{G}_{\mathrm{m}}\}$ the lattice of characters. Given a cocharacter $\lambda\in\mathrm{X}_*(T)$, we define

(1) a Levi subgroup $G^{\lambda}:=\{g\in G\mid \lambda(t)g\lambda(t)^{-1}=g \text{ for any } t\in\mathbf{C}^*\}\subseteq G$,
(2) $V^{\lambda}:=\{v\in V\mid \lambda(t)\cdot v=v \text{ for any } t\in\mathbf{C}^*\}$, a (symmetric) representation of G^{λ} [Hen24b],
(3) a parabolic subgroup $G^{\lambda\geqslant 0}:=\{g\in G\mid \lim_{t\to 0}\lambda(t)g\lambda(t)^{-1} \text{ exists}\}\subseteq G$,
(4) $V^{\lambda\geqslant 0}:=\{v\in V\mid \lim_{t\to 0}\lambda(t)\cdot v \text{ exists}\}$, a representation of $G^{\lambda\geqslant 0}$,
(5) $V^{\lambda>0}:=\{v\in V\mid \lim_{t\to 0}\lambda(t)\cdot v=0\}$.

We let $V\cong\bigoplus_{\alpha\in\mathrm{X}^*(T)}V_{\alpha}$ be the weight space decomposition of V, where $V_{\alpha}=\{v\in V\mid t\cdot v=\alpha(t)v \text{ for any } t\in T\}$. We let $\mathcal{W}(V)=\{\alpha\in\mathrm{X}^*(T)\mid V_{\alpha}\neq 0\}$ be the set of weights of V. For $\lambda\in\mathrm{X}_*(T)$, we let $\mathcal{W}^{\lambda>0}:=\{\alpha\in\mathcal{W}(V)\mid\langle\lambda,\alpha\rangle>0\}$. We define $\mathcal{W}^{\lambda<0}(V)$, $\mathcal{W}^{\lambda=0}(V)$ similarly.

We let $W:=\mathrm{N}_G(T)/T$ be the Weyl group of G. It is a finite Coxeter group. It acts naturally on $\mathrm{X}_*(T)$ via $(w\cdot\lambda)(t)=\dot{w}\lambda(t)\dot{w}^{-1}$ for any $t\in\mathbf{C}^*$, where $w\in W$ and $\dot{w}\in\mathrm{N}_G(T)$ is any lift. We define an equivalence relation $\sim$ on $\mathrm{X}_*(T)$ via

$$\lambda\sim\mu\iff\begin{cases}V^{\lambda}=V^{\mu}\\ G^{\lambda}=G^{\mu}\,.\end{cases}$$

for any two cocharacters $\lambda,\mu\in\mathrm{X}_*(T)$. We denote by $\mathscr{P}_V:=\mathrm{X}_*(T)/\sim$ the set of equivalence classes of cocharacters of T. This is a finite set. It inherits an action of W. If $\lambda\in\mathrm{X}_*(T)$, we denote by $\overline{\lambda}$ its class in $\mathscr{P}_V$ and by $\widetilde{\lambda}$ its class in $\mathscr{P}_V/W$. Given $\overline{\lambda}\in\mathscr{P}_V$ or $\widetilde{\lambda}\in\mathscr{P}_V/W$, we denote by $\lambda\in\mathrm{X}_*(T)$ an arbitrary lift. For $\lambda\in\mathrm{X}_*(T)$, we let $G_{\lambda}=\ker(G^{\lambda}\to\mathrm{GL}(V^{\lambda}))\cap\mathrm{Z}(G^{\lambda})$. We define $\mathfrak{g}_{\lambda}:=\mathrm{Lie}(G_{\lambda})$. We let $W_{\lambda}:=\{w\in W\mid\overline{w\cdot\lambda}=\overline{\lambda}\}\subseteq W$. It contains W^{λ} as a normal subgroup, and we let $\overline{W}^{\lambda}:=W_{\lambda}/W^{\lambda}$ be the quotient. the group $\overline{W}_{\lambda}$ acts naturally on $\mathrm{H}^*(V^{\lambda}/G^{\lambda})$. Moreover, there is a character $\varepsilon_{V,\lambda}\colon\overline{W}_{\lambda}\to\{\pm1\}$, see [Hen24b, Proposition 4.8] for its precise definition. If $\lambda\in X_*(T)$, we have a closed immersion $V^{\lambda}\to V$ which induces a finite morphism $\imath_{\lambda}\colon V^{\lambda}/\!\!/G^{\lambda}\to V/\!\!/G$ [Hen24b, Lemma 2.2]. We let $\pi\colon V/G\to V/\!\!/G$ be the good moduli space map. When $\lambda=0\in\mathrm{X}_*(T)$ is the trivial cocharacter, we have $V^0=V$, $G^0=G$, $G_0=\ker(G\to\mathrm{GL}(V))\cap Z(G)$, $\overline{W}_0\cong\{1\}$, $\mathfrak{g}_0=\mathrm{Lie}(\mathfrak{g}_0)$, etc.

Theorem 2.1 ([Hen24a, Theorem 1.5])**.** *Let V be a symmetric representation of a reductive group G. Then, there exists cohomologically graded monodromic mixed Hodge modules $\underline{\mathcal{BPS}}_{V,\lambda}\in\mathcal{D}^{+}(\mathrm{MMHM}(V^{\lambda}/\!\!/G^{\lambda}))$ for*

$\lambda \in \mathrm{X}_*(T)$ such that there is an isomorphism

$$\pi_*\underline{\mathbf{Q}}^{\mathrm{vir}}_{V/G} \cong \bigoplus_{\widetilde{\lambda}\in\mathscr{P}_V/W} ((\imath_\lambda)_*\underline{\mathcal{BPS}}_{V,\lambda}\otimes \mathrm{H}^*_{G_\lambda}(\mathrm{pt}))^{\varepsilon_{V,\lambda}},$$

where $(\underline{\mathcal{BPS}}_{V,\lambda}\otimes \mathrm{H}^*_{G_\lambda}(\mathrm{pt}))^{\varepsilon_{V,\lambda}}$ denotes the $\varepsilon_{V,\lambda}$-isotypic component for the natural $\overline{W}_\lambda$-action.

Remark 2.2. Theorem 2.1 can be adapted to any smooth affine G-variety X. We refer to [Hen24a] for more details. In *loc.cit.*, the isomorphism of Theorem 2.1 is induced by parabolic induction [Hen24a, §4].

This theorem admits an enhancement. We consider a G-equivariant LG-model (V,f), where V is a complex representation of G and $f\colon V\to\mathbf{C}$ is a G-invariant regular function. By abuse of notation, we let $\varphi_f^{\mathrm{vir}} := \varphi_f\underline{\mathbf{Q}}^{\mathrm{vir}}_{V/G}\in\mathrm{MMHM}(V/G)$ be the vanishing cycle sheaf. We also let $\underline{\mathcal{BPS}}_{V,f,\lambda} := \varphi_f\underline{\mathcal{BPS}}_{V,\lambda}$, where $f\colon V^\lambda/\!\!/G^\lambda\to\mathbf{C}$ is the function induced by f.

Theorem 2.3 ([Hen24a, Theorem 1.7])**.** *We have an isomorphism*

$$\pi_*\varphi_f^{\mathrm{vir}} \cong \bigoplus_{\widetilde{\lambda}\in\mathscr{P}_V/W} ((\imath_\lambda)_*\underline{\mathcal{BPS}}_{V,f,\lambda}\otimes \mathrm{H}^*_{G_\lambda}(\mathrm{pt}))^{\varepsilon_{V,\lambda}}.$$

In [Hen24a], Theorem 2.3 is deduced from Theorem 2.1. In particular, Theorem 2.1 is Theorem 2.3 for the special case $f=0$, for which φ_f is the identity functor.

When the symmetric representation V and the function f have a particular form, we can use the dimensional reduction isomorphisms (see [KS11, Proposition 4.5], [Dav17, Appendix A]). Namely, we let W be a representation of a reductive group G and $\mathrm{T}^*W = W\oplus W^*$ be the cotangent representation. There is a moment map $\mu\colon \mathrm{T}^*W\to\mathfrak{g}^*$ for the induced Hamiltonian action of G on T^*W. We define $\widetilde{W} := \mathrm{T}^*W\times\mathfrak{g}$. There is a natural function $f\colon\widetilde{W}\to\mathbf{C}$, $(x,\xi)\mapsto\langle\mu(x),\xi\rangle$ for $(x,\xi)\in\mathrm{T}^*W\times\mathfrak{g}$, obtained by contracting the moment map. For any $\lambda\in\mathrm{X}_*(T)$, we have $\mathrm{T}^*(W^\lambda)\cong(\mathrm{T}^*W)^\lambda$ as G^λ-representations, and the (co)-restriction of the moment map $\mu_\lambda\colon\mathrm{T}^*W^\lambda\to\mathfrak{g}_\lambda^*$ is the moment map for T^*W^λ.

There is a natural closed immersion $\mathrm{T}^*W^\lambda\times\mathfrak{g}_\lambda\to\widetilde{W^\lambda}$. It induces the closed immersion $\imath\colon(\mathrm{T}^*W^\lambda/\!\!/G^\lambda)\times\mathfrak{g}_\lambda\to\widetilde{W^\lambda}/\!\!/G$.

Lemma 2.4 (Support lemma [Hen24a, Lemma 1.9])**.** *The support of $\underline{\mathcal{BPS}}_{\widetilde{W},f,\lambda}$ is contained in $(\mathrm{T}^*W^\lambda)/\!\!/G^\lambda\times\mathfrak{g}_\lambda$. Moreover, we have $\underline{\mathcal{BPS}}_{\widetilde{W},f,\lambda}\cong\imath_*(\underline{\mathcal{BPS}}^{\mathrm{red}}_{\widetilde{W},f,\lambda}\boxtimes\underline{\mathbf{Q}}_{\mathfrak{g}_\lambda})$ for some complex of mixed Hodge modules $\underline{\mathcal{BPS}}^{\mathrm{red}}_{\widetilde{W},f,\lambda}$ on $(\mathrm{T}^*W^\lambda)/\!\!/G^\lambda$ (which we call the* dimensionally reduced BPS sheaf*).*

We let $\pi_\lambda\colon\mu_\lambda^{-1}(0)/G^\lambda\to\mu_\lambda^{-1}(0)/\!\!/G^\lambda$ be the good moduli space map. In particular, when λ is the trivial cocharacter, we have the map $\pi\colon\mu^{-1}(0)/G\to\mu^{-1}(0)/\!\!/G$. We let $\imath_\lambda\colon\mu_\lambda^{-1}(0)/\!\!/G^\lambda\to\mu^{-1}(0)/\!\!/G$. By [Hen24b, Lemma 2.2], it is a finite map. The dimensional reduction of Theorem 2.3 gives a cohomological integrality isomorphism for the stacky Hamiltonian reduction $\mu^{-1}(0)/G$.

Theorem 2.5 ([Hen24a, Proposition 12.12])**.** *We have an isomorphism*

$$\pi_*\mathbb{D}\underline{\mathbf{Q}}^{\mathrm{vir}}_{\mu^{-1}(0)/G} \cong \bigoplus_{\widetilde{\lambda}\in\mathscr{P}_V/W} ((\imath_\lambda)_*\underline{\mathcal{BPS}}^{\mathrm{red}}_{\widetilde{W},f,\lambda}\otimes \mathrm{H}^*_{G_\lambda}(\mathrm{pt}))^{\varepsilon_{V,\lambda}}.$$

Theorem 2.5 holds more generally for *stacky weak Hamiltonian reductions* [Hen24a, §10].

2.2. **The case of the Lie algebra of a reductive group.** In this paper, we are particularly interested in the case where $W=\mathfrak{g}$ is the adjoint representation of a reductive group. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using a G-invariant nondegenerate pairing $\langle -,-\rangle$ (one can take the Killing form if $\mathfrak{g}$ is semisimple, and identify the center $\mathfrak{z}$ and its dual $\mathfrak{z}^*$ arbitrarily if $\mathfrak{g}$ is reductive, using the decomposition $\mathfrak{g}\cong\mathfrak{z}\oplus[\mathfrak{g},\mathfrak{g}]$). In this case, $f\colon\mathfrak{g}^3\to\mathbf{C}$, $(x,y,z)\mapsto\langle x,[y,z]\rangle$. Moreover, $\mathscr{P}_{\mathfrak{g}}/W$ is the set of Levi subgroups of G up to G-conjugation or, equivalently, the set of standard Levi subgroups of G up to W-conjugation. Moreover, for any $\lambda\in\mathrm{X}_*(T)$ we have $G_\lambda=\mathrm{Z}(G^\lambda)$ is the center of G^λ and $\mathfrak{g}_\lambda=\mathrm{Z}(\mathfrak{g}^\lambda)$ is the center of $\mathfrak{g}^\lambda$.

We let $\mathscr{C}(\mathfrak{g}) := \{(x,y) \in \mathfrak{g}^2 \mid [x,y] = 0\}$ be the commuting variety, $\mathfrak{C}(\mathfrak{g}) := \mathscr{C}(\mathfrak{g})/G$ be the commuting stack and $\mathcal{C}(\mathfrak{g}) := \mathscr{C}(\mathfrak{g})/\!\!/G$ be the good moduli space (the GIT quotient). We let $\pi\colon \mathfrak{C}(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g})$ be the good moduli space morphism. By specialising Theorem 2.5 to the commuting stack, we obtain the following.

Corollary 2.6. *We have an isomorphism*

$$\pi_*\mathbb{D}\underline{\mathbf{Q}}^{\mathrm{vir}}_{\mathfrak{C}(\mathfrak{g})} \cong \bigoplus_{\widetilde{\lambda} \in \mathscr{P}_V/W} ((\imath_\lambda)_*\underline{\mathcal{BPS}}^{\mathrm{red}}_{\mathfrak{g}^3,f,\lambda} \otimes \mathrm{H}^*_{G_\lambda}(\mathrm{pt}))^{\varepsilon_{V,\lambda}}\,.$$

We note that the virtual dimension of $\mathfrak{C}(\mathfrak{g})$ is 0, since it is the zero locus of the function $(\mathfrak{g}\times\mathfrak{g})/G \to \mathfrak{g}^*$ given by μ. Therefore, one can drop the superscript "vir" in Corollary 2.6.

2.3. **Conjectures.** The complexes of mixed Hodge modules $\underline{\mathcal{BPS}}_{V,f,\lambda}$ are a priori not easy to describe, and we formulated the following conjecture in [Hen24a].

Conjecture 2.7 ([Hen24a, Conjecture 8.14])**.**
(1) The perverse filtration on $\pi_*\underline{\mathbf{Q}}^{\mathrm{vir}}_{V/G}$ starts in degree at least $\dim G_0$, that is $\underline{\mathcal{H}}^j(\pi_*\underline{\mathbf{Q}}^{\mathrm{vir}}_{V/G}) = 0$ for $j < \dim G_0$.
(2) We have $\underline{\mathcal{BPS}}_V \cong \underline{\mathcal{H}}^{\dim G_0}(\pi_*\underline{\mathbf{Q}}^{\mathrm{vir}}_{V/G})$.
(3) We have $\underline{\mathcal{BPS}}_V \cong \begin{cases} \underline{\mathcal{IC}}(V/\!\!/G) & \text{if } (V/\!\!/G)_{(G_0^\circ)} \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$. where $(V/\!\!/G)_{(G_0^\circ)}$ is the set of closed G orbits in V whose stabilizer is conjugated to the neutral component G_0°.

This conjecture is proven for any symmetric representation V of $\mathbf{C}^*$ [Hen24a, §B.5]. For quivers, it is proven in [MR19; DM20]. In general, it implies the following generalised conjecture by applying the vanishing cycle sheaf functor.

Conjecture 2.8. *Let V be a representation of a reductive group G. We let $f\colon V \to \mathbf{C}$ be a regular G invariant function, which may also be seen as a function $f\colon V/\!\!/G \to \mathbf{C}$.*
(1) The perverse filtration on $\pi_*\varphi_f\underline{\mathbf{Q}}^{\mathrm{vir}}_{V/G}$ starts in degree at least $\dim G_0$, that is $\underline{\mathcal{H}}^j(\pi_*\varphi_f\underline{\mathbf{Q}}^{\mathrm{vir}}_{V/G}) = 0$ for $j < \dim G_0$.
(2) We have $\underline{\mathcal{BPS}}_{V,f} \cong \underline{\mathcal{H}}^{\dim G_0}(\pi_*\varphi_f\underline{\mathbf{Q}}^{\mathrm{vir}}_{V/G})$.
(3) We have $\underline{\mathcal{BPS}}_{V,f} \cong \begin{cases} \varphi_f\underline{\mathcal{IC}}(V/\!\!/G) & \text{if } (V/\!\!/G)_{(G_0)} \neq \varnothing \\ 0 & \text{otherwise}. \end{cases}$.

Our goal in this paper is to prove Conjecture 2.8 when $V = \mathfrak{g}^3$ is the cube of the adjoint representation of a reductive group G and f is given by $f(x,y,z) = \langle x,[y,z]\rangle$ where $\langle -,-\rangle$ denotes the Killing form. In this case, we are in addition able to describe $\underline{\mathcal{BPS}}_{V,f}$ explicitly (Theorem 5.3). Our proof involves the study of the approximation by schemes of the good moduli space morphism $\pi\colon V/G \to V/\!\!/G$ given in [Hen24a, §5] for general symmetric representations of reductive groups (§3), in particular of the *virtual smallness* (in the sense of [MR19], but for more flexibility, we use weakened assumptions, see Definition 3.1). This study also allows us to give a partial proof of Conjecture 2.7.

3. Virtual smallness for the approximation of the good moduli space morphism

We let G be a reductive algebraic group and $\mathfrak{g}$ be its Lie algebra. We let V be a representation of G. We let $\pi\colon V/G \to V/\!\!/G$ be the good moduli space map.

3.1. **Approximation of the pushforward.** In this section, we recall how the pushforward $\pi_*\underline{\mathbf{Q}}_{V/G}$ can be computed in terms of approximations by schemes of the quotient stack V/G, following [Hen24a, §5, §A.3]. We choose $k \in \mathbf{N}$ such that there is an embedding $G \to \mathrm{GL}_k(\mathbf{C})$. We let G act naturally on the space of linear maps $V_n := \mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^k)$ via the embedding $G \subseteq \mathrm{GL}_k(\mathbf{C})$. We let $U_n \subseteq V_n$ be the open subset of surjective linear maps $\mathbf{C}^n \to \mathbf{C}^k$. We let $V'_n := V \times U_n$ and $\overline{V'_n} := V'_n/G$. We let $\pi_n \colon \overline{V'_n} \to V/\!\!/G$ Since the codimension of $V_n \smallsetminus U_n$ in V_n tends to $+\infty$ as n tends to $+\infty$, we have

$$\underline{\mathcal{H}}^j(\pi_*\underline{\mathbf{Q}}_{V/G'}) \cong \underline{\mathcal{H}}^j((\pi_n)_*\underline{\mathbf{Q}}_{\overline{V'_n}})$$

for $n \gg 0$.

3.2. **Virtual smallness.**

Definition 3.1. Let Y be an irreducible algebraic variety and $\pi \colon X \to Y$ be a morphism. We say that π is *virtually small* if there exists a finite stratification $Y = \bigsqcup_{i\in I} Y_i$ such that $Y_0 \subseteq Y$ is open for some $0 \in I$, for any $i \in I$, $\pi^{-1}(Y_i) \to Y_i$ has equidimensional fibers, and for any $y_i \in Y_i$, $y_0 \in Y_0$,

$$\dim \pi^{-1}(y_i) - \dim \pi^{-1}(y_0) \leqslant \frac{1}{2}\mathrm{codim} Y_i\,, \tag{3.1}$$

with equality only if $i = 0$.

Remark 3.2. Note that for flexibility, we do not assume that π is proper. Also, in [MR19], the stratification is required to satisfy the property that $\pi^{-1}(Y_i) \to Y_i$ is étale locally trivial. We only require $\pi^{-1}(Y_i) \to Y_i$ to have equidimensional fibers. When the general fiber of π has dimension 0, Definition 3.1 is similar to the definition of small morphisms (with the properness assumption dropped).

Proposition 3.3. *Let $\pi \colon X \to Y$ be a virtually small morphism with X smooth. We let d be the dimension of the generic fiber. Then,*

$$\pi_*\underline{\mathbf{Q}}_X^{\mathrm{vir}} \in \mathcal{D}^{\geqslant -d}(\mathrm{MHM}(Y))\,.$$

In other words, $\underline{\mathcal{H}}^j(\pi_*\underline{\mathbf{Q}}_X^{\mathrm{vir}}) = 0$ for $j < -d$.

Proof. It suffices to prove the statement after applying the functor $\mathbf{rat} \colon \mathcal{D}(\mathrm{MHM}(Y)) \to \mathcal{D}_{\mathrm{c}}(Y)$. We show that $\pi_*\mathbf{Q}_X[\dim X] \in {}^p\mathcal{D}_{\mathrm{c}}^{\geqslant -d}(Y)$. From the definition of the perverse t-structure by supports and cosupports (see e.g. [CM09]), we have

$${}^p\mathcal{D}_{\mathrm{c}}^{\geqslant -d}(Y) = \{\mathcal{F} \in \mathcal{D}_{\mathrm{c}}(Y) \mid \forall i \in \mathbf{Z}, \dim\{y \in Y \mid \mathrm{H}^i((\mathbb{D}\mathcal{F})_y) \neq 0\} \leqslant d - i\}\,.$$

For $y \in Y$, by base-change in the Cartesian diagram

$$\begin{tikzcd} \pi^{-1}(y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "\pi"] \\ y \arrow[r, "\imath_y"] & Y \end{tikzcd},$$

we have $(\mathbb{D}\pi_*(\mathbf{Q}_X[\dim X]))_y \cong \imath_y^*\pi_!\mathbf{Q}_X[\dim X] \cong \mathrm{H}_{\mathrm{c}}^{*+\dim X}(\pi^{-1}(y))$. Therefore, it is concentrated in degrees $[-\dim X, 2\dim\pi^{-1}(y) - \dim X]$.

Let $i \in \mathbf{Z}$. Then, by the remark above,

$$\begin{aligned} \{y \in Y_j \mid \mathrm{H}^i((\mathbb{D}\pi_*(\mathbf{Q}_X[\dim X]))_y) \neq 0\} &= \varnothing \quad \text{if } \dim\pi^{-1}(y) < (\dim X + i)/2 \\ &\subseteq Y_j \quad \text{otherwise.} \end{aligned}$$

If $\dim\pi^{-1}(y) \geqslant (\dim X + i)/2$ for $y \in Y_j$ $(*)$, we have

$$\dim\{y \in Y_j \mid \mathrm{H}^i((\mathbb{D}\pi_*(\mathbf{Q}_X[\dim X]))_y) \neq 0\} \leqslant \dim Y_j \leqslant d - i$$

since, by using (3.1):

$$\begin{aligned} \dim Y_j &= \dim Y - \mathrm{codim}(Y_j) \\ &= \dim X - d - \mathrm{codim}(Y_j) \quad \text{since } \dim Y + d = \dim X \\ &\leqslant \dim X - d + 2d - 2\dim\pi^{-1}(y) \qquad \text{by (3.1)} \\ &\leqslant \dim X + d - \dim X - i \qquad \text{by using } (*) \text{ above} \\ &= d - i\,. \end{aligned}$$

By letting $j\in I$ vary, we can conclude. $\square$

Remark 3.4. If we assume that π is proper in Proposition 3.3, the same kind of argument can be used to show that $\pi_*\underline{\mathbf{Q}}_X^{\mathrm{vir}}\in\mathcal{D}^{[-d,d]}(\mathrm{MHM}(Y))$. In this work, we are only interested in determining lower bounds, which is why we do not need the properness assumption.

Proposition 3.5. *Let $\pi\colon X\to Y$ be a virtually small morphism with X smooth and generic fiber of dimension d. Then, $\underline{\mathcal{H}}^{-d}(\pi_*\underline{\mathbf{Q}}_X[\dim X])\cong(\jmath_{Y_0})_{!*}(\underline{\mathcal{H}}^{-d}(\pi_*\underline{\mathbf{Q}}_X[\dim X]))_{|Y_0}$. That is, simple subquotients of $\underline{\mathcal{H}}^{-d}(\pi_*\underline{\mathbf{Q}}_X^{\mathrm{vir}})$ have full support in Y.*

Proof. The proof is very similar to the proof of Proposition 3.3. It suffices to prove that $\underline{\mathcal{H}}^{-d}(\pi_*\underline{\mathbf{Q}}_X)$ does not have any simple summands supported on $\overline{Y_j}$ for $j\neq 0$. After applying the functor **rat**, it suffices to work in the constructible derived category of Y. It suffices to prove that $\imath_{Y_j}^!\pi_*\mathbf{Q}_X[\dim X]\in{}^p\mathcal{D}_{\mathrm{c}}^{\geqslant -d+1}(Y_j)$ for any $j\neq 0$. We use the description of ${}^p\mathcal{D}_{\mathrm{c}}^{\geqslant -d+1}(Y_j)$ using the support/cosupport condition:

$$ {}^p\mathcal{D}_{\mathrm{c}}^{\geqslant -d+1}(Y_j)=\{\mathcal{F}\in\mathcal{D}_{\mathrm{c}}(Y_j)\mid\forall i\in\mathbf{Z},\dim\{y\in Y_j\mid \mathrm{H}^i((\mathbb{D}\mathcal{F})_y)\neq 0\}\leqslant d-1-i\}\,. $$

For $y\in Y_j$, by base-change, we have $(\mathbb{D}\pi_*(\mathbf{Q}_X[\dim X]))_y\cong \mathrm{H}_{\mathrm{c}}^{*+\dim X}(\pi^{-1}(y))$. It is concentrated in degrees $[-\dim X, 2\dim\pi^{-1}(y)-\dim X]$.

Let $i\in\mathbf{Z}$. Then,

$$ \begin{aligned}\{y\in Y_{j'}\mid \mathrm{H}^i((\mathbb{D}\pi_*(\mathbf{Q}_X[\dim X]))_y)\neq 0\}&=\varnothing\quad \text{if }\dim\pi^{-1}(y)<(\dim X+i)/2\\ &\subseteq Y_{j'}\quad\text{otherwise.}\end{aligned} $$

If $\dim\pi^{-1}(y)\geqslant(\dim X+i)/2$ for $y\in Y_{j'}$ $(*)$, then the inequality is strict since $j'\neq 0$ (by Definition 3.1) and moreover,

$$ \dim\{y\in Y_{j'}\mid \mathrm{H}^i((\mathbb{D}\pi_*(\mathbf{Q}_X[\dim X]))_y)\neq 0\}\leqslant\dim Y_{j'}<d-i $$

since, by using (3.1):

$$ \begin{aligned}\dim Y_{j'}&=\dim Y-\mathrm{codim}(Y_{j'})\\ &=\dim X-d-\mathrm{codim}(Y_{j'})\quad\text{since }\dim Y+d=\dim X\\ &\leqslant\dim X-d+2d-2\dim\pi^{-1}(y)\quad\text{by using (3.1)}\\ &<\dim X+d-\dim X-i\quad\text{by }(*)\text{ above}\\ &=d-i\,.\end{aligned} $$

By letting $j'\in I$, $Y_{j'}\subseteq Y_j$, vary, we can conclude. $\square$

Remark 3.6. When $d=0$, we recover the very-well known property of small morphisms, which is that the pushforward of the contant perverse sheaf by a small morphism with smooth source is a perverse sheaf with full support.

Let V be a representation of a reductive group G. We let $\pi\colon V/G\to V/\!\!/G$ be the good moduli space map, $\widetilde{\pi}\colon V\to V/\!\!/G$ the map induced by the inclusion of the algebra of regular functions $\mathbf{C}[V]^G\to\mathbf{C}[V]$ and $N_V:=\widetilde{\pi}^{-1}(0)$ be the *nilpotent cone* of the representation V.

Lemma 3.7. *For any $\lambda\in\mathrm{X}_*(T)$ and $v\in V$ such that $v\in V^{\lambda>0}$, there exists a generic cocharacter $\mu\in\mathrm{X}_*(T)$ such that $v\in V^{\mu>0}$.*

Proof. It suffices to take a generic perturbation μ of λ such that $\mathcal{W}^{\lambda>0}(V)\subseteq\mathcal{W}^{\mu>0}(V)$. One way to achieve this is to consider the cocharacter associated to Ny for $y\in\mathfrak{t}_{\mathbf{Q}}$ a general rational approximation of $\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t)_{t=0}\in\mathfrak{t}$ for $N\geqslant 1$ an integer such that $Ny\in\mathfrak{t}_{\mathbf{Z}}\cong\mathrm{X}_*(T)$. $\square$

For $\lambda\in\mathrm{X}_*(T)$, we have a natural closed immersion $V^{\lambda>0}\to V$. By using the $G^{\lambda\geqslant 0}$ action of $V^{\lambda>0}$ and the G-action on V, we obtain a morphism $\widetilde{p}_\lambda\colon V^{\lambda>0}\times^{G^{\lambda\geqslant 0}}G\to V$.

Proposition 3.8. *If V is symmetric, then*

$$\dim N_V - \dim G \leqslant \frac{1}{2}\left(\dim(V) - \dim(G) - \dim V^T - \dim T\right)$$

Proof. Let $v \in N_V$. Then, $0 \in \overline{G \cdot v}$, and so there exists a one-parameter subgroup $\lambda\colon \mathbf{G}_{\mathrm{m}} \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot v = 0$. This implies that the nilpotent cone is

$$N_V = \bigcup_{\lambda \in \mathrm{X}_*(T)} \widetilde{p}_\lambda(V^{\lambda>0} \times^{G^{\lambda\geqslant 0}} G)\,.$$

The union is over the lattice of cocharacters of T since $\widetilde{p}_\lambda(V^{>\lambda} \times^{G^{\lambda\geqslant 0}} G)$ is G-invariant for any λ. By Lemma 3.7, we can restrict the union to generic cocharacters. Moreover, we can restrict the union to *finitely many* generic cocharacters. Indeed, given a cocharacter $\lambda \in \mathrm{X}_*(T)$, what imports is $V^{\lambda>0}$. Therefore, $\dim N_V$ is bounded above by the maximum of the quantities $\dim(V^{\lambda>0} \times^{G^{\lambda\geqslant 0}} G)$ for generic cocharacters $\lambda \in \mathrm{X}_*(T)$. Let $\lambda \in \mathrm{X}_*(T)$ be a generic cocharacter. Then, $G^{\lambda\geqslant 0}$ is a Borel subgroup of G. The natural map $V^{\lambda>0} \times^{G^{\lambda\geqslant 0}} G \to G/G^{\lambda\geqslant 0}$ induced by the second projection is a fibration with fibers $V^{\lambda>0}$. Moreover, by symmetry of V and genericity of λ, $V^\lambda = V^T$ and $\dim V^{\lambda>0} = \frac{1}{2}(\dim V - \dim V^T)$.

Therefore,

$$\dim N_V \leqslant \dim G - \dim G^{\lambda\geqslant 0} + \frac{1}{2}(\dim V - \dim V^T)$$

and so

$$\begin{aligned}\dim N_V - \dim G &\leqslant -\frac{1}{2}(\dim G + \dim T) + \frac{1}{2}(\dim V - \dim V^T) \quad \text{since } G^\lambda \text{ is a Borel subgroup of } G\\ &= \frac{1}{2}\left(\dim(V) - \dim(G) - \dim(V^T) - \dim(T)\right)\,.\end{aligned}$$

This concludes. □

We can now optimise Proposition 3.8 by quotienting out by the kernel of the action map $G \to \mathrm{GL}(V)$.

Corollary 3.9. *Let V be a representation of a reductive group G. Let $K = \ker(G \to \mathrm{GL}(V))$. Then,*

$$\dim N_V - \dim(G/K) \leqslant \frac{1}{2}\left(\dim V - \dim(G/K) - \dim V^T - \dim T_{G/K}\right)$$

where $T_{G/K}$ is a maximal torus of G/K (for example, the image of T by the projection $G \to G/K$).

Proof. This is Proposition 3.8 applied to the representation V of G/K. □

We consider the map $\pi_n\colon \overline{V_n'} \to V/\!\!/G$ forgetting the framing datum. We let $\mathcal{L}_G$ be the set of conjugacy classes of reductive subgroups of G. The Luna stratification of $V/\!\!/G$ is given by

$$V/\!\!/G = \bigsqcup_{(H)\in\mathcal{L}_G} (V/\!\!/G)_{(H)}$$

where $(V/\!\!/G)_{(H)}$ is the set of points $x \in V/\!\!/G$ whose stabilizer is conjugated to H (it may be empty for some choice of H). We let $V_{(H)} = \widetilde{\pi}^{-1}((V/\!\!/G)_{(H)})$ where $\widetilde{\pi}\colon V \to V/\!\!/G$ is the projection.

Lemma 3.10. *Let $x \in V$, be a point with reductive stabilizer group G_x. Let $V \cong \mathrm{T}_x V \cong \mathrm{T}_x(G\cdot x) \oplus N_x$ be a decomposition of $\mathrm{T}_x V$ as G_x-representations. Then,*

$$\dim N_x - \dim G_x = \dim V - \dim G\,.$$

Proof. This is an easy calculation:

$$\begin{aligned}\dim N_x - \dim G_x &= \dim V - \dim G\cdot x - \dim G_x\\ &= \dim V - \dim G + \dim G_x - \dim G_x\\ &= \dim V - \dim G\,.\end{aligned}$$

□

Lemma 3.11. *Let $K \coloneqq \ker(G \to \mathrm{GL}(V))$. Let $x \in V/\!\!/G$ and N_x the normal bundle to $G \cdot x$ at x. Then,*

$$\dim \widetilde{\pi}^{-1}(x) - \dim(G/K) \leqslant \frac{1}{2}\left(\dim V - \dim(G/K) - \dim N_x^{T_x} - \dim T_{G_x/K}\right) .$$

where $T_{G_x/K}$ is a maximal torus of G_x/K.

Proof. By Luna slice theorem, the morphism

$$\pi_K \colon V/(G/K) \to V/\!\!/(G/K) \cong V/\!\!/G$$

is locally modelled around $\pi_K^{-1}(x) \to x$ by the map

$$\pi_x \colon N_x/(G_x/K) \to N_x/\!\!/(G_x/K)$$

around $\pi_x^{-1}(0) \to 0$. We let $\widetilde{\pi}_x \colon N_x \to N_x/\!\!/G_x$.

We have

$$\begin{aligned}
\dim \widetilde{\pi}^{-1}(x) - \dim(G/K) &= \widetilde{\pi}_x^{-1}(0) - \dim(G_x/K) && \text{by Luna slice theorem,}\\
&\leqslant \frac{1}{2}(\dim N_x - \dim(G_x/K) - \dim N_x^{T_x} - \dim T_{G_x/K}) && \text{by Lemma 3.8,}\\
&= \frac{1}{2}(\dim V - \dim(G/K) - \dim N_x^{T_x} - \dim T_{G_x/K}) && \text{by Lemma 3.10}.
\end{aligned}$$

□

Lemma 3.12. *For any $(H) \in \mathcal{L}_G$ and $x \in (V/\!\!/G)_{(H)}$, we have*

$$\dim \pi_n^{-1}(x) = \dim \widetilde{\pi}^{-1}(x) + nk - \dim G .$$

In particular, the map π_n is equidimensional over Luna strata.

Proof. This follows immediately from the fact that the G-action on $V \times U_n$ is free and the morphism $V_n' \to V/\!\!/G$ is obtained by first projecting on the first factor $V_n' \to V \to V/\!\!/G$. Therefore, $\dim \pi_n^{-1}(x) = \dim \widetilde{\pi}^{-1}(x) + \dim \mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^k) - \dim G$. Moreover, $\widetilde{\pi}^{-1}(x)$ is constant over Luna strata by the étale slice theorem. □

Proposition 3.13. *Let $K \coloneqq \ker(G \to \mathrm{GL}(V))^\circ$. We assume that the (G/K)-action on V has generically finite stabilizers and a general orbit is closed. Then, the map $\pi_n \colon \overline{V_n'} \to V/\!\!/G$ is virtually small and the dimension of the generic fiber is $d = nk - \dim K$.*

Proof. We have $\dim V/\!\!/G = \dim V - (\dim G - \dim K)$ since a general G-orbit in V is closed and has finite stabilizer.

Let d be the dimension of the generic fiber of π_n. Then,

$$\begin{aligned}
d &= \dim \overline{V_n'} - \dim V/\!\!/G\\
&= \dim V + nk - \dim G - \dim V + \dim G - \dim K\\
&= nk - \dim K.
\end{aligned}$$

We let H be the neutral component of a stabilizer of a closed G-orbit inside V. Let $x \in (V/\!\!/G)_{(H)}$. Then, by Lemmas 3.12 and 3.11, the inequality

$$\dim \pi_n^{-1}(x) - d \leqslant \frac{1}{2}\mathrm{codim}_{V/\!\!/G}((V/\!\!/G)_{(H)})$$

is implied by the inequality

$$\begin{aligned}
(3.2)\quad \frac{1}{2}(\dim V - \dim(G/K) - \dim N_x^{T_x} - \dim T_{G_x/K}) + nk - \dim G + \dim(G/K) - nk + \dim K \leqslant\\
\frac{1}{2}(\dim V - \dim(G/K) - \dim(V/\!\!/G)_{(H)}) .
\end{aligned}$$

Since we have an étale description $V/\!\!/G$ by $N_x/\!\!/G_x^\circ$ where G_x° is the neutral component of the stabilizer of x and N_x is the normal bundle to $G \cdot x$ at x, we have $\dim(V/\!\!/H)_{(H)} = \dim(N_x/\!\!/G_x)_{(G_x^\circ)}$. Therefore (3.2) can be simplified to

$$\dim(N_x/\!\!/G_x)_{(G_x^\circ)} - \dim N_x^{T_x} \leqslant \dim T_{G_x^\circ/K}\,.$$

The l.h.s. is nonpositive, since $(N_x/\!\!/G_x)_{(G_x^\circ)} = (N_x^{G_x^\circ})/\!\!/(G_x/G_x^\circ)$ is a finite quotient of $N_x^{G_x^\circ}$ and $N_x^{G_x^\circ} \subseteq N_x^{T_x}$ while the r.h.s. is nonnegative. Therefore, this inequality is always satisfied. A necessary condition to have equality is $\dim T_{G_x^\circ/K} = 0$ which means that $G_x^\circ/K = 1$. This implies $K = G_x^\circ = H$. This implies in turn that $(V/\!\!/G)_{(H)} = (V/\!\!/H)_{(K)}$ is the open stratum inside $(V/\!\!/G)$. Therefore, the map π_n is indeed virtually small. □

Corollary 3.14. *We let $Y_0 := (V/\!\!/G)_{(G_0^\circ)}$. Then, $\underline{\mathcal{H}}^{\dim G_0}(\pi_*\underline{\mathbf{Q}}_{V/G}^{\mathrm{vir}}) \cong (\jmath_{Y_0})_{!*}(\underline{\mathcal{H}}^{\dim G_0}(\pi_*\underline{\mathbf{Q}}_{V/G}^{\mathrm{vir}}))_{|Y_0}$.*

Proof. We may assume that $Y_0 \neq \varnothing$, otherwise the corollary is obvious. By §3.1, we have $\underline{\mathcal{H}}^{\dim G_0}(\pi_*\underline{\mathbf{Q}}_{V/G'}^{\mathrm{vir}})[-\dim G_0]\otimes \mathscr{L}^{-\dim G_0/2} \cong \underline{\mathcal{H}}^{-d_n}(\pi_n\underline{\mathbf{Q}}_{\overline{V'_n}})[d_n] \otimes \mathscr{L}^{d_n/2}$ for $n \gg 0$, where d_n is the dimension of the general fiber of π_n. Since π_n is virtually small by Proposition 3.13, the corollary follows from Proposition 3.5. □

Lemma 3.15. *Let G be a finite group acting on a smooth quasiprojective variety X and $\pi\colon X/G \to X/\!\!/G$ the good moduli space map. Then, $\pi_*\underline{\mathbf{Q}}_{X/G} \cong \underline{\mathbf{Q}}_{X/\!\!/G}$.*

Proof. The fiber of $\pi_*\underline{\mathbf{Q}}_{X/G}$ over any point $x \in X/\!\!/G$ is isomorphic to $\mathrm{H}^*_{G_{\widetilde{x}}}(\mathrm{pt}) \cong \underline{\mathbf{Q}}$ (since $G_{\widetilde{x}}$ is finite) where $\widetilde{x} \in \pi^{-1}(x)$ is any lift of x. Therefore, the adjunction map $\underline{\mathbf{Q}}_{X/\!\!/G} \to \pi_*\underline{\mathbf{Q}}_{X/G}$ is an isomorphism. □

Lemma 3.16. *Let X be a smooth quasiprojective algebraic variety on which a group G acts with finite stabilizers. Let $\pi\colon X/G \to X/\!\!/G$ be a good moduli space map. Then $\pi_*\underline{\mathbf{Q}}_{X/G} \cong \underline{\mathbf{Q}}_{X/\!\!/G}$.*

Proof. We have a natural adjunction map $\underline{\mathbf{Q}}_{X/\!\!/G} \to \pi_*\underline{\mathbf{Q}}_{X/G}$. We can check that it is an isomorphism locally. Since X/G is locally a finite group quotient [Beh04, Exercise 34], this follows from Lemma 3.15 □

Corollary 3.17. *Let $\lambda \in \mathrm{X}_*(T)^{\mathrm{st}}$. We have an isomorphism $\underline{\mathcal{H}}^{\dim\mathfrak{g}_\lambda}(\pi_*\underline{\mathbf{Q}}_{V^\lambda/G^\lambda}^{\mathrm{vir}})[-\dim\mathfrak{g}_\lambda] \cong \underline{\mathcal{IC}}(V^\lambda/\!\!/G^\lambda)\otimes \mathscr{L}^{\dim\mathfrak{g}_\lambda/2}$.*

Proof. By Corollary 3.14, it suffices to show that $\underline{\mathcal{H}}^{\dim\mathfrak{g}_\lambda}((\pi_\lambda)_*\underline{\mathbf{Q}}_{V^\lambda/G^\lambda})_{|(V^\lambda/\!\!/G^\lambda)_{(G_\lambda^\circ)}}[-\dim\mathfrak{g}_\lambda] \cong \underline{\mathbf{Q}}_{(V^\lambda/\!\!/G^\lambda)_{(G_\lambda^\circ)}} \otimes \mathscr{L}^{\dim\mathfrak{g}_\lambda/2}$. We let $V^\lambda_{(G_\lambda^\circ)} := \widetilde{\pi}_\lambda^{-1}((V^\lambda/\!\!/G^\lambda)_{(G_\lambda^\circ)}) \subseteq V^\lambda$. It is a G^λ-invariant open subvariety. It is acted on by G^λ and every orbit has a stabilizer with neutral component G_λ°, and they are all closed in V^λ. Therefore, G^λ/G_λ acts with finite stabilizers on $V^\lambda_{(G_\lambda^\circ)}$.

The restriction $\pi_\lambda\colon \pi_\lambda^{-1}((V^\lambda/\!\!/G^\lambda)_{(G_\lambda^\circ)}) \cong V^\lambda_{(G_\lambda^\circ)}/G^\lambda \to (V^\lambda/\!\!/G^\lambda)_{(G_\lambda^\circ)}$ can be decomposed as the composition

$$V^\lambda_{(G_\lambda^\circ)}/G^\lambda \xrightarrow{\pi'_\lambda} V^\lambda_{(G_\lambda^\circ)}/(G^\lambda/G_\lambda^\circ) \xrightarrow{\pi''_\lambda} (V^\lambda/\!\!/G^\lambda)_{(G_\lambda^\circ)}\,.$$

The morphism π'_λ is a $\mathrm{pt}/G_\lambda^\circ$-gerbe, and therefore, $(\pi'_\lambda)\underline{\mathbf{Q}}_{V^\lambda_{(G_\lambda^\circ)}/G^\lambda} \cong \underline{\mathbf{Q}}_{V^\lambda_{(G_\lambda^\circ)}/(G^\lambda/G_\lambda^\circ)} \otimes \mathrm{H}^*_{G_\lambda^\circ}(\mathrm{pt})$. The morphism π''_λ is the good moduli space morphism for a smooth quasiprojective variety with an action of $G^\lambda/G_\lambda^\circ$ which has finite stabilizers, and hence we are in the situation of Lemma 3.16. This concludes. □

3.3. **Perverse filtration.**

Lemma 3.18. *Let G be a reductive group and $\mathfrak{g}$ its lie algebra. Then, for any $g \geqslant 2$ and $x \in \mathfrak{g}^g$ such that $G\cdot x$ is a general closed orbit, the stabilizer of x is the center $\mathrm{Z}(G)$ of G.*

Proof. This comes from the fact that the intersection of any two general tori in G is $\mathrm{Z}(G)$. □

Corollary 3.19. *General G-orbits in $\mathfrak{g}^g$ are closed. Moreover, the stabilizer of a general G-orbit is $\mathrm{Z}(G)$.*

Corollary 3.20. *For $V = \mathfrak{g}^g$ with $g \geqslant 2$, we have $\mathrm{X}_*(T)^{\mathrm{st}} = \mathrm{X}_*(T)$.*

Proof. We let $\lambda \in \mathrm{X}_*(T)$. Then, $V^\lambda = (\mathfrak{g}^\lambda)^g$ and $\lambda \in \mathrm{X}_*(T)$ if a generic orbit of $G^\lambda/\mathrm{Z}(G^\lambda)$ in V^λ is closed and has finite stabilizer. This is a consequence of Corollary 3.19 applied to G^λ. □

Proposition 3.21. *Let $\lambda \in \mathrm{X}_*(T)^{\mathrm{st}}$. We let $\mathfrak{g}_\lambda \coloneqq \mathrm{Lie}(G_\lambda)$. Then, we have $\underline{\mathcal{H}}^j(\pi_*\underline{\mathbf{Q}}^{\mathrm{vir}}_{V/G}) = 0$ for $j < \dim \mathfrak{g}_\lambda$.*

Proof. Let $n \in \mathbf{N}$. Since the map π_n is virtually small with generic fiber of dimension $d = nk - \dim \mathfrak{g}_\lambda$, we have $(\pi_n)_*\underline{\mathbf{Q}}_{\overline{V'_n}/G^\lambda} \otimes \mathscr{L}^{-(\dim V'_n - \dim G^\lambda)/2} \in \mathcal{D}^{-(\geqslant nk - \dim \mathfrak{g}_\lambda)}(\mathrm{MHM}(V^\lambda /\!\!/ G^\lambda))$. Therefore, $(\pi_n)_*\underline{\mathbf{Q}}_{V'_n/G'} \otimes \mathscr{L}^{-(\dim V^\lambda - \dim G^\lambda)/2} \in \mathcal{D}^{\geqslant r}(\mathrm{MHM}(V^\lambda /\!\!/ G^\lambda))$ where

$$\begin{aligned} r &= -d - (\dim V^\lambda - \dim G^\lambda - (\dim V'_n - \dim G^\lambda)) \\ &= \dim \mathfrak{g}_\lambda \,. \end{aligned}$$

Since for any $j \in \mathbf{Z}$, $\underline{\mathcal{H}}^j(\pi_*\underline{\mathbf{Q}}_{V/G}[\dim V^\lambda - \dim G^\lambda]) \cong \underline{\mathcal{H}}^j((\pi_n)_*\underline{\mathbf{Q}}_{V'_n/G^\lambda}[\dim V^\lambda - \dim G^\lambda])$ for $n \gg 0$, this proves that $\pi_*\underline{\mathbf{Q}}_{V^\lambda/G^\lambda} \otimes \mathscr{L}^{-(\dim V^\lambda - \dim G^\lambda)/2} \in \mathcal{D}^{\geqslant \dim \mathfrak{g}_\lambda}(V^\lambda /\!\!/ G^\lambda)$. □

4. Geometry of the nilpotent commuting variety

4.1. **The commuting variety and its nilpotent version.** Let G be an affine algebraic groups and $\mathfrak{g} \coloneqq \mathrm{Lie}(G)$ its Lie algebra. We let $T \subseteq G$ be a maximal torus and $\mathfrak{t} \coloneqq \mathrm{Lie}(T)$ its Lie algebra. We denote by W the Weyl group of G. There is a canonical map $\chi \colon \mathfrak{g} \to \mathfrak{g}/\!\!/G$ to the GIT quotient. When G is reductive, there is an isomorphism $\mathfrak{g}/\!\!/G \cong \mathfrak{t}/\!\!/W$.

Definition 4.1. The *commuting variety* of $\mathfrak{g}$ is the closed subset $\mathscr{C}(\mathfrak{g}) \coloneqq \{(x,y) \in \mathfrak{g}^2 \mid [x,y] = 0\}$. The stacky quotient by the diagonal adjoint G-action is $\mathfrak{C}(\mathfrak{g}) \coloneqq \mathscr{C}(\mathfrak{g})/G$. We let $\mathcal{C}(\mathfrak{g}) \coloneqq \mathscr{C}(\mathfrak{g})/\!\!/G$ be the corresponding GIT quotient.

When G is reductive, there is an isomorphism $\mathscr{C}(\mathfrak{g})/\!\!/G \cong \mathfrak{t}^2/\!\!/W$. We let $\mathscr{N} \coloneqq \chi^{-1}(0) \subseteq \mathfrak{g}$ be the nilpotent cone. It is an irreducible Zariski closed subset of $\mathfrak{g}$ of dimension $\dim(G) - \mathrm{rank}(G)$.

Definition 4.2. We let $\mathscr{C}^{\mathrm{nil}}(\mathfrak{g}) \coloneqq \{(x,y) \in \mathscr{N} \mid [x,y] = 0\}$ be the *nilpotent commuting variety.* We define the commuting stack as the quotient stack $\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}) \coloneqq \mathscr{C}^{\mathrm{nil}}(\mathfrak{g})/G$.

4.2. **Irreducibility and dimension of the commuting variety.** The commuting variety $\mathscr{C}(\mathfrak{g})$ is singular. Its geometry is not so well-understood, but we have the following..

Theorem 4.3 ([Ric79])**.** *Let $\mathfrak{g}$ be a reductive Lie algebra. The commuting variety $\mathscr{C}(\mathfrak{g})$ is the Zariski closure of $G \cdot (\mathfrak{t} \times \mathfrak{t}) \subseteq \mathfrak{g} \times \mathfrak{g}$. In particular, it is irreducible and of dimension* $\dim G + \dim T$.

4.3. **Irreducible components of the nilpotent commuting variety and dimension.** The geometry of the nilpotent commuting variety has remained mysterious for a long time. It is now rather well-understood – at least, we know enough for our purposes.

If $H \subseteq G$ and $e \in \mathfrak{g}$, we let $C_H(e) \coloneqq \{h \in H \mid \mathrm{Ad}(h)(e) = 0\}$ be the centralizer of e in H. An element $e \in \mathscr{N}$ is called *distinguished* if the neutral component of $C_{[G,G]}(e)$ is unipotent, that it if $\mathrm{Lie}(C_{[G,G]}(e)) = \{x \in \mathfrak{g} \mid [x,e] = 0\} \subseteq \mathscr{N}$. The orbit $G \cdot e$ of a distinguished nilpotent element $e \in \mathscr{N}$ is called a *distinguished nilpotent orbit.* Let $e \in \mathscr{N}$ be a distinguished nilpotent element. We have $[e, \mathrm{Lie}(C_{(G,G)}(e))] \subseteq \mathscr{C}^{\mathrm{nil}}(\mathfrak{g})$. We denote by $\overline{\mathscr{C}(e)} \coloneqq \overline{G \cdot [e, \mathrm{Lie}(C_{(G,G)}(e))]}$ the Zariski closure of $G \cdot [e, C_{(G,G)}(e)]$. It is a closed subset of $\mathscr{C}^{\mathrm{nil}}(\mathfrak{g})$.

Theorem 4.4 ([Pre03])**.** *Let $e_1, \dots, e_r$ be representatives of the distinguished nilpotent orbits in $\mathfrak{g}$. Then, the closed sets $\overline{\mathscr{C}(e_1)}, \dots, \overline{\mathscr{C}(e_r)}$ are pairwise distinct. They have the same dimension equal to* $\dim[G,G]$ *and $\mathscr{C}^{\mathrm{nil}}(\mathfrak{g}) = \bigsqcup_{i=1}^r \overline{\mathscr{C}(e_i)}$.*

The nilpotent commuting variety of a reductive Lie algebra already appears in the context of the theory of character sheaves for reductive Lie algebras, [Mir04] in characteristic 0 and [Zho24] in positive characteristic. In these papers, it is proven that a simple perverse sheaf on a reductive Lie algebra is a *cuspidal character sheaf* if and only if its characteristic cycle is contained in the nilpotent commuting variety ([Mir04, Theorem 4.7] and [Zho24, Theorem 4.1]). Our study takes place at the level of cohomology, and relates the nilpotent character variety to the BPS sheaf of the commuting varieties (which, in some sense, could be called *cuspidal cohomology* of the commuting stack of triples).

4.4. **Moment map and critical locus.** We let $\langle -,-\rangle\colon \mathfrak{g}\times\mathfrak{g}\to\mathbf{C}$ be the Killing form of $\mathfrak{g}$. It is a symmetric bilinear form whose kernel is the center $\mathfrak{z}$ of $\mathfrak{g}$. We have a canonical decomposition as Lie algebras and G-representations $\mathfrak{g}\cong\mathfrak{z}\oplus[\mathfrak{g},\mathfrak{g}]$ and the Lie algebra $[\mathfrak{g},\mathfrak{g}]$ is semisimple. We may identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by choosing an isomorphism $\mathfrak{z}\cong\mathfrak{z}^*$ and identifying $[\mathfrak{g},\mathfrak{g}]^*$ with $[\mathfrak{g},\mathfrak{g}]$ using the Killing form, which is nondegenerate on $[\mathfrak{g},\mathfrak{g}]$. The adjoint action of G on $\mathfrak{g}$ induces a Hamiltonian action of G on $\mathrm{T}^*\mathfrak{g}\cong\mathfrak{g}\oplus\mathfrak{g}^*$ and the moment map is

$$\mu\colon \mathfrak{g}\oplus\mathfrak{g}^*\to\mathfrak{g}^*,\quad (x,\xi)\mapsto[y\mapsto\langle\xi,[y,x]\rangle]\,.$$

If we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ as above, the moment map can be rewritten

$$\mu\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g},\quad (x,y)\mapsto[x,y]\,.$$

The moment map induces a regular function $f\colon\mathfrak{g}^2\times\mathfrak{g}^*$, $(x,y,\xi)\mapsto\langle\xi,[y,x]\rangle$. By identifying $\mathfrak{g}^*$ with $\mathfrak{g}$, this function becomes

$$f\colon\mathfrak{g}^3\to\mathbf{C},\quad (x,y,z)\mapsto\langle z,[y,x]\rangle\,.$$

Proposition 4.5. *(1) We have* $\mu^{-1}(0)=\mathscr{C}(\mathfrak{g})$.
(2) We have $\mathrm{crit}(f)=\mathscr{C}^3(\mathfrak{g}):=\{(x,y,z)\in\mathfrak{g}^3\mid[x,y]=[y,z]=[x,z]=0\}$.

Proof. The first point is obvious from the description of μ. For the second point, we note that $\mathrm{d}_zf(x,y,z)=\langle[y,z],-\rangle$ and by using the invariance of the Killing form, $\mathrm{d}_xf(x,y,z)=\langle[z,y],-\rangle$ and $\mathrm{d}_yf(x,y,z)=\langle[x,z],-\rangle$. Therefore, the vanishing of $\mathrm{d}_zf(x,y,z)$ is equivalent to the condition $[x,y]\in\mathfrak{z}$. However, $[x,y]\in[\mathfrak{g},\mathfrak{g}]$ and $\mathfrak{z}\cap[\mathfrak{g},\mathfrak{g}]=0$. Therefore, $[x,y]=0$. Similarly the vanishing of $\mathrm{d}_xf(x,y,z)$ is equivalent to $[y,z]=0$ and the vanishing of $\mathrm{d}_yf(x,y,z)$ is equivalent to $[x,z]=0$. These are the equations defining the triple commuting variety $\mathscr{C}^3(\mathfrak{g})$. □

5. BPS sheaf of commuting varieties

5.1. **Support of the BPS sheaf.** We let $\widetilde{\mathfrak{g}}:=\mathfrak{g}^3$ with the diagonal adjoint action, and $f\colon\mathfrak{g}^3\to\mathbf{C}^3$, $(x,y,z)\mapsto\langle z,[y,x]\rangle$ is the G-invariant function descrived in §4.4.

Lemma 5.1. *Let G be a reductive group and $\mathfrak{g}=\mathrm{Lie}(G)$ is Lie algebra. We let $\mathfrak{z}$ be the center of $\mathfrak{g}$. Then, the support of the BPS sheaf $\underline{\mathcal{BPS}}_{\widetilde{\mathfrak{g}},f}$ is included inside $\mathfrak{z}^3\subseteq\mathfrak{g}^3/\!\!/G$.*

Proof. By applying the support lemma (Lemma 2.4) to the three copies of $\mathfrak{g}\subseteq\mathfrak{g}^3$, we obtain

$$\mathrm{supp}(\underline{\mathcal{BPS}}_{\mathfrak{g}^3,f})\subseteq(\mathfrak{z}\times\mathfrak{g}^2)/\!\!/G$$
$$\mathrm{supp}(\underline{\mathcal{BPS}}_{\mathfrak{g}^3,f})\subseteq(\mathfrak{g}\times\mathfrak{z}\times\mathfrak{g})/\!\!/G$$
$$\mathrm{supp}(\underline{\mathcal{BPS}}_{\mathfrak{g}^3,f})\subseteq(\mathfrak{g}^2\times\mathfrak{z})/\!\!/G\,.$$

The intersection of these three subsets is $\mathfrak{z}^3\subseteq\mathfrak{g}^3/\!\!/G$. □

Corollary 5.2. *There exists a sequence of nonnegative integers n_i, $i\in\mathbf{Z}$ such that only finitely many of them are nonzero and $\underline{\mathcal{BPS}}_{\widetilde{\mathfrak{g}},f}\cong\bigoplus_{i\in\mathbf{Z}}\underline{\mathcal{IC}}(\mathfrak{z}^3)^{\oplus n_i}\otimes\mathscr{L}^{i/2}$. Moreover, $n_i=0$ if $i<\dim\mathfrak{z}$.*

Proof. We let $\mathfrak{z}^3$ act on $\mathfrak{g}^3$ by translation, that is $(z_1,z_2,z_3)\cdot(\xi_1,\xi_2,\xi_3)=(\xi_1+z_1,\xi_2+z_2,\xi_3+z_3)$. It induces an action of $\mathfrak{z}^3$ on $\mathfrak{g}^3/\!\!/G$ such that $\pi\colon\mathfrak{C}(\mathfrak{g})\to\mathcal{C}(\mathfrak{g})$ is equivariant. Then, $\pi_*\underline{\mathbf{Q}}^{\mathrm{vir}}_{V/G}$ is $\mathfrak{z}^3$-equivariant. Therefore, $\underline{\mathcal{BPS}}_{\mathfrak{g}^3,f}$ is also $\mathfrak{z}^3$-equivariant. By combining this equivariance with the support property (Lemma 5.1), we obtain the decomposition of the lemma. The virtual smallness of the approximation of π by schemes (more precisely, Proposition 3.21) implies that $n_i=0$ if $i<\dim\mathfrak{z}$. □

5.2. **Description of the BPS sheaf.** The following theorem is our main theorem and gives a precise description of the integers n_i of Corollary 5.2.

Theorem 5.3. *Let G be a reductive group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{z} \subseteq \mathfrak{g}$ its center. We let r be the number of distinguished nilpotent orbits in $\mathfrak{g}$. Then, $\underline{\mathcal{BPS}}_{\widetilde{\mathfrak{g}},\widetilde{W}} \cong \underline{\mathcal{IC}}(\mathfrak{z}^3)^{\oplus r} \otimes \mathscr{L}^{\dim \mathfrak{z}/2}$. In other words, in Corollary 5.2, we have $n_{\dim \mathfrak{z}} = r$ and $n_i = 0$ if $i \neq \dim \mathfrak{z}$.*

Proof. We have a Cartesian diagram

$$\begin{tikzcd} \mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{C}(\mathfrak{g}) \arrow[d] \\ 0 \arrow[r] & \mathcal{C}(\mathfrak{g}) \end{tikzcd}$$

By base-change, Corollary 2.6 gives an isomorphism

$$\mathrm{H}^{\mathrm{BM}}_{-*}(\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}), \mathbf{Q}) \cong \bigoplus_{\widetilde{\lambda} \in \mathscr{P}_V/W} \left((i_0^! \underline{\mathcal{BPS}}_{\mathfrak{g}^3, f, \lambda} \otimes \mathrm{H}^*_{G_\lambda}(\mathrm{pt})\right)^{\overline{W}_\lambda}.$$

and by Corollary 5.2 applied to $\mathfrak{g}^\lambda$, we have $\underline{\mathcal{BPS}}_{\widetilde{\mathfrak{g}}, f, \lambda} \cong \bigoplus_{i \geqslant \dim \mathfrak{z}^\lambda} \underline{\mathcal{IC}}((\mathfrak{z}^\lambda)^3)^{n_i^\lambda} \otimes \mathscr{L}^{i/2}$ where $\mathfrak{z}^\lambda$ denotes the center of $\mathfrak{g}^\lambda$. We have $i_0^! \underline{\mathbf{Q}}_{\mathfrak{z}^\lambda} \cong \underline{\mathbf{Q}}_0 \otimes \mathscr{L}^{\dim \mathfrak{z}^\lambda}$. Therefore,

$$\mathrm{H}^{\mathrm{BM}}_{-*}(\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}), \mathbf{Q}) \cong \bigoplus_{\widetilde{\lambda} \in \mathscr{P}_V/W} \left(\left(\bigoplus_{i \geqslant \dim \mathfrak{z}^\lambda} \underline{\mathbf{Q}}_0^{n_i^\lambda} \otimes \mathscr{L}^{(2\dim \mathfrak{z}^\lambda - i)/2}\right) \otimes \mathrm{H}^*_{G_\lambda}(\mathrm{pt})\right)^{\overline{W}_\lambda}.$$

We have $\dim \mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}) = -\dim \mathfrak{z}$ and therefore, $\mathrm{H}^{\mathrm{BM}}_i(\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}), \mathbf{Q}) = 0$ for $i < 2\dim \mathfrak{z}$. Therefore, we have $n_j = 0$ for $j > \dim \mathfrak{z}$. Moreover, $n_{\dim \mathfrak{z}} = \dim \mathrm{H}^{\mathrm{BM}}_{-2\dim \mathfrak{z}}(\mathfrak{C}^{\mathrm{nil}}(\mathfrak{g}), \mathbf{Q})$, which has a basis indexed by the irreducible components of the nilpotent commuting variety. By Theorem 4.4, the dimension of this space is equal to the number of distinguished nilpotent orbits of $\mathfrak{g}$. This concludes. ☐

We end with a corollary, giving the computation of the vanishing cycle sheaf of some intersection cohomology complex.

Corollary 5.4. *We have $\varphi_f \underline{\mathcal{IC}}(\mathfrak{g}^3 /\!\!/ G) \cong \underline{\mathcal{IC}}(\mathfrak{z}^3)^{\oplus r}$ where r is the number of distinguished nilpotent orbits in $\mathfrak{g}$.*

Proof. It suffices to prove that $\underline{\mathcal{H}}^{\dim \mathfrak{z}}(\pi_* \underline{\mathbf{Q}}_{\mathfrak{g}^3/G})[-\dim \mathfrak{z}] \cong \underline{\mathcal{IC}}(\mathfrak{g}^3 /\!\!/ G) \otimes \mathscr{L}^{\dim \mathfrak{z}/2}$. This is Corollary 3.17 applied to $V = \mathfrak{g}^3$. ☐

This corollary gives an explicit description of the vanishing cycle sheaf functor for the function $f(x, y, z) = \langle x, [y, z] \rangle$ applied to the intersection cohomology complex of the singular affine variety $\mathfrak{g}^3 /\!\!/ G$. It is in general a very challenging problem to compute the vanishing cycle functor for some function on some space and some constructible complex.

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