Universal Virasoro Constraints for Additive Categories

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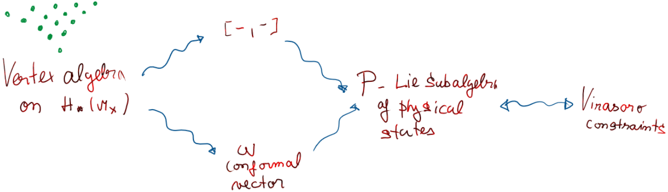
August 24, 2024



Structure of the talk.

1. History and background
2. Geometric formulation of Virasoro constraints and the main claim
3. Reformulation in terms of vertex algebras
   - 3.1 Joyce's construction of VA's
   - 3.2 The conformal element
   - 3.3 Virasoro constraints make virtual fundamental classes into physical states
4. Main results for quivers and varieties

Witten's conjecture

1. The moduli space of algebraic pointed curves $\overline{\mathcal{M}}_{g,n}$ parametrizing semistable[1] $(C, x_1, \ldots, x_n)$:

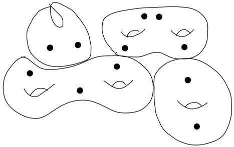


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Figure: $(C, x_1, \ldots, x_n)$ where C has genus 7 here, $n = 11$, and $x_1, \ldots, x_{11}$ are represented by the black dots.

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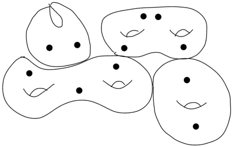

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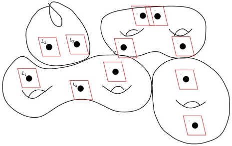

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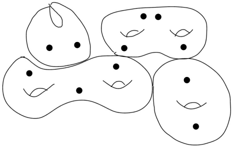

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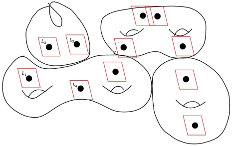

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3. Denote the powers of the first Chern classes by $\tau_d := \psi_i^d := c_1(L_i)^d$

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$$F(\vec{t}) = \sum_{g\geq 0} \Big\langle \exp\Big[\sum_{d\geq 0} \tau_d t_d\Big]\Big\rangle_g \lambda^{2g-2} = \int_{\sum_{g,n\geq 0}\left[\overline{\mathcal{M}}_{g,n}\right]\lambda^{2g-2}} \exp\Big[\sum_{d\geq 0} \tau_d t_d\Big].$$

which collects the invariants

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2. The interpretation of Witten's conjecture by Dijkgraaf, Verlinde, Verlinde (90') states that the τ-function

$$\tau = \exp\left[-\frac{1}{2}F(x,\vec{t})\right]$$

   satisfies Virasoro constraints.

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3. DVV showed that Witten's conjecture is equivalent to the Virasoro constraints

$$L_k \tau = 0 \,, \quad \text{for} \quad k \geq -1$$

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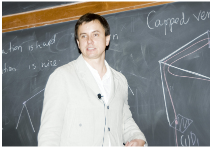

Figure: Kontsevich, Okounkov and Mirzakhani have received Fields medals in part for their work on Witten's conjecture.

Virasoro constraints for $X \neq \mathrm{pt}$

1. Let us now move on to $\overline{\mathcal{M}}_{g,n}(X,\beta)$ where X is a smooth projective variety and $\beta \in H_2(X)$ an algebraic curve class. It parametrizes stable maps $(C, f, x_1, \ldots, x_n)$ with $f : C \to X$ such that $f_*([C]) = \beta \in H_2(X)$.

[2] Here ev_j is the evaluation map for the j'th marked point.

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2. Fix a basis $B = \{v\} \subset H^*(X)$ with $1 \in B$ for the generator of $H^0(X)$ and define the classes [2]

$$\tau_k(v) = (\psi_j)^k \mathrm{ev}_j^*(v)\,, \quad \text{for all} \quad k \geq 0, v \in B$$

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which collects the Gromov–Witten invariants
$$\langle \tau_{k_1}(v_1)^{a_1}\tau_{k_2}(v_2)^{a_2}\cdots\tau_{k_l}(v_l)^{a_l}\rangle_{\beta,g}^X = \int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}} \tau_{k_1}(v_1)^{a_1}\tau_{k_2}(v_2)^{a_2}\cdots\tau_{k_l}(v_l)^{a_l}$$
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5. Virasoro constraints for toric 3-folds X were transported to stationary descendents of PT stable pairs using the GW–PT correspondence by Moreira–Oblomkov–Okounkov–Pandharipande (20'). Dimensional reduction was used to prove these constraints for $\mathrm{Hilb}^n(S)$ – a rather long round-about proof, which hides how natural the Virasoro constraints on the sheaf side are.

Virasoro constraints for sheaves and pairs

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4. Some examples to see how much variety the theory offers:

	Without framing	With framing
Sheaves	Gieseker stable torsion-free sheaves on curves or surface, dimension 1 sheaves on surfaces, Fano 3-folds, CY fourfolds[3]	Bradlow pairs on curves or surfaces, DT/PT pairs on $\leq$ 4-folds, Quot schemes
Quivers with relations (quasi-smooth, CY4)	Bridgeland stable quiver representations	Framed quiver representations: e.g., Grassmanians and Flag varieties

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Quivers with relations and frozen vertices

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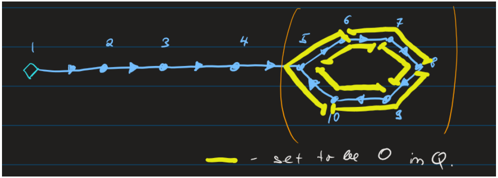

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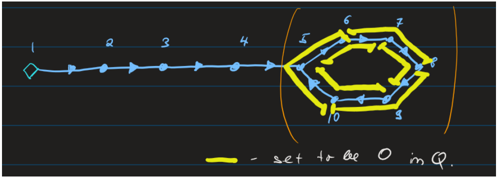

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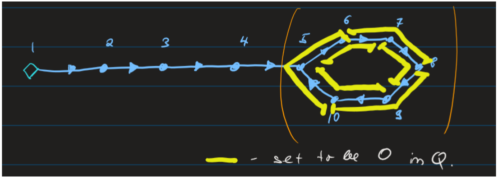

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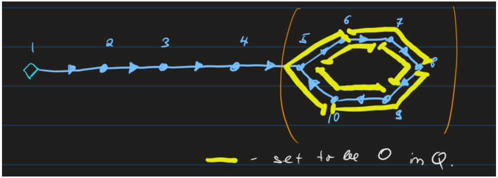

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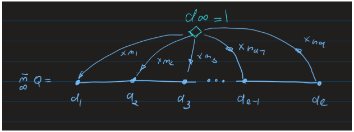
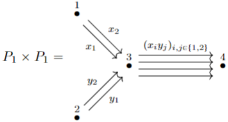


Flag

$(x_i y_k) \circ x_j = (x_j y_k) \circ x_i \,, \qquad (x_k y_i) \circ y_j = (x_k y_j) \circ y_i$



$D^b(\mathbb{P}^1 \times \mathbb{P}^1)$

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$$\mathbb{D}^X = \mathrm{SSym}[\![\tau_i^H(v), i > 0, v \in B]\!]$$

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2. The realization map: depends on a choice of a universal object. We use it when integrating against $[M]^{\mathrm{vir}}$

   $$\tau_i(v) \mapsto \mathrm{ch}_i(\mathbb{U}_v)$$

   $\mathbb{U}_v$ is a universal vector space at v

   $$\tau_i^H(v) = \pi_{2,*}\Big(\pi_1^*(\bar{v})\mathrm{ch}_{i+p}(\mathbb{G})\Big)$$

   $\mathbb{G}$ is a universal sheaf on $X \times M$
   $v \in H^{p,q}(X)$, $\overline{v}$ its Poincaré dual

Virasoro constraints for quivers and sheaves

Next, I will describe the equivalent ingredients for defining Virasoro constraints on quiver representations and sheaves from left to right.

1. The descendent algebras

$$\mathbb{D}^Q = \mathrm{Sym}[\![\tau_i(v), i > 0, v \in V]\!]$$

$$\mathbb{D}^X = \mathrm{SSym}[\![\tau_i^H(v), i > 0, v \in B]\!]$$

$B \subset H^*(X)$ is a basis that respects the Hodge grading

The Virasoro operators $\mathsf{L}_k = \mathsf{T}_k + \mathsf{R}_k$ for $k \geq -1$ will now be differential operators on $\mathbb{D}^{Q/X}$.

2. The realization map: depends on a choice of a universal object. We use it when integrating against $[M]^{\mathrm{vir}}$

$$\tau_i(v) \mapsto \mathrm{ch}_i(\mathbb{U}_v)$$

$\mathbb{U}_v$ is a universal vector space at v

$$\tau_i^H(v) = \pi_{2,*}\left(\pi_1^*(\bar{v})\mathrm{ch}_{i+p}(\mathbb{G})\right)$$

$\mathbb{G}$ is a universal sheaf on $X \times M$

$v \in H^{p,q}(X)$, $\overline{v}$ its Poincaré dual

3. Euler pairing and Todd classes

$$\mathrm{td}(Q) = \mathrm{id} - A^Q + S^Q : \mathbb{Z}^V \to \mathbb{Z}^V = K^0(Q)$$
$$A^Q_{v,w} = \#\{e \in E\colon t(e) = v, h(e) = w\}$$
$$S^Q_{v,w} = \#\{r \in R\colon t(r) = v, h(r) = w\}$$

Euler pairing $\chi(\overline{e}, \overline{d}) = \langle \overline{e}, \mathrm{td}(Q)\overline{d}\rangle$

Todd class $\mathrm{td}(X) : K^0(X, \mathbb{Q}) \cong H^*(X) \to H^*(X)$

Euler pairing $\chi(v, w) = \int_X v^\vee \cdot w \cdot \mathrm{td}(X)$

Virasoro operators

1. Quadratic terms from the diagonal pushforward

$$\begin{aligned}\Delta_* \mathrm{td}(Q) &= \textstyle\sum_{v\in V} \mathrm{td}(Q)\cdot v \boxtimes v \\ &=: \textstyle\sum_v v^L \otimes v^R \in \mathbb{Z}^V \otimes \mathbb{Z}^V\end{aligned}$$

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The operators R_k are the same

$$\mathsf{R}_k\left(\tau_i^{(H)}(v)\right) = i^{(k+1)} \tau_{i+k}^{(H)}(v)$$

where $i^{(k+1)} = i(i+1)\cdots(i+k)$ is the rising factorial.

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3. We can predict the form of T_k from the virtual tangent bundle T^{vir}.

$$\mathrm{ch}\big(T^{\mathrm{vir}}\big) = -\textstyle\sum_{i,j}(-1)^i\tau_i\tau_j\big(\mathrm{td}(Q)\big)(+1) \qquad -\textstyle\sum_{i,j}(-1)^{i+p_L}\tau_i^H\tau_j^H\big(\mathrm{td}(X)\big)(+1)\,,$$

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4. All of this can be done for any nice additive category! This will be clear from the proof.

Defining Virasoro constraints

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$$[\mathsf{L}_m, \mathsf{L}_n] = (n - m)\mathsf{L}_{m+n}\,.$$

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3. Instead, we need to make up for the non-uniqueness of the choice of a universal object. Use another operator S^k compatible with the universal object. It can also absorb fixing determinants of sheaves.

Claim (B.–Lim–Moreira(22'), B.(23'))

Let M be a fine moduli space of stable objects with a virtual fundamental class, then it often satisfies Virasoro constraints
$$\int_{[M]^{\mathrm{vir}}} (\mathsf{L}_k + \mathsf{S}_k)(D) = 0 \qquad \text{for} \quad k \geq 0, D \in \mathbb{D}^{Q/X}\,.$$

Weiht zero Virasoro constraints

1. To avoid talking about S_k, we introduced the weight-zero operator

$$L_{\mathrm{wt}=0} = \sum_{n \geq -1} \frac{(-1)^n}{(n+1)!} \mathsf{L}_n \circ \mathsf{R}_{-1}^{n+1} \,.$$

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The previous claim is equivalent to

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2. This formulation a) is independent of the choice of the universal object when it exists, b) can be defined formally without the universal object.

Flag varieties

1. The partial flag variety $\mathrm{Flag}(d_1, d_2, \ldots, d_l)$ for $d_1 > d_2 \cdots d_{l-1} > d_l$ parametrizes sequences of quotients

$$\mathbb{C}^{d_1} \twoheadrightarrow \mathbb{C}^{d_2} \twoheadrightarrow \ldots \twoheadrightarrow \mathbb{C}^{d_{l-1}} \twoheadrightarrow \mathbb{C}^{d_l}.$$

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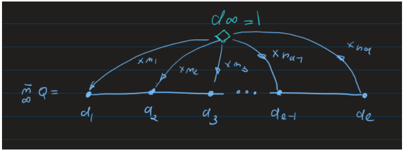

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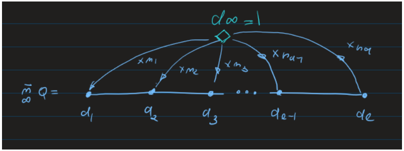

   and choosing the right stability condition, we obtain the flag variety as a moduli space of representations of this quiver. We also identify $\mathcal{Q}_v = \mathbb{U}_v$.

3. The only non-constant descendents are $\tau_i(v) = \mathrm{ch}_i(\mathcal{Q}_v)$ for $v \neq \infty, 1$. We see then that

$$\mathrm{T}_k = \sum_{\substack{i+j=k \\ v \geq 2}} \tau_i(v)\big(\tau_j(v) - \tau_j(v+1)\big) - d_1 \tau_k(2)\,,$$

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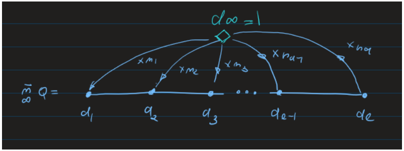

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   and that R_k only acts on these descendents.

4. Choosing a polynomial D in $\mathrm{ch}_i(\mathcal{Q}_v)$ for $v \geq 2$, Virasoro constraints tell us that

$$\int_{\mathrm{Flag}} (\mathrm{T}_k + \mathrm{R}_k)(D) = 0 \qquad \text{for} \quad k \geq 0 .$$

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$$H^*(\mathcal{M}_Q) = \mathrm{Sym}[\![\tau_i(v), i > 0, v \in V]\!] .$$
5. Define homology classes $t_{k,v}$ as duals of $\tau_k(v)$:
$$\tau_k(v) \cap (-) = \frac{\partial}{\partial t_{k,v}} .$$
6. The homology of $\mathcal{M}_{\overline{d}}$ is the polynomial algebra
$$H_*(\mathcal{M}_{\overline{d}}) = e^{\overline{d}} \otimes \mathrm{Sym}[t_{i,v}, i > 0, v \in V] .$$

Recovering a GW-like formulation of Virasoro constraints

1. Let us introduce the notation $\langle \tau_{k_1}(v_1) \cdots \tau_{k_n}(v_n) \rangle_M = \int_{[M]^{\mathrm{vir}}} \tau_{k_1}(v_1) \cdots \tau_{k_n}(v_n)$

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2. It becomes clear that

$$\begin{aligned} \iota_*[M]^{\mathrm{vir}} &= \Big\langle \exp\Big[\sum_{k,v}\tau_k(v)t_{k,v}\Big]\Big\rangle_M \\ &= \int_{[M]^{\mathrm{vir}}} \exp\Big[\sum_{k,v}\tau_k(v)t_{k,v}\Big], \end{aligned}$$

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3. The GW formulation of Virasoro constraints is dual to the one given previously. So we define $T_k = \mathsf{T}_k^*, R_k = \mathsf{R}_k^*, S_k = \mathsf{S}_k^*, L_k = \mathsf{L}_k^*$ on $H_*(\mathcal{M}_Q)$.

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4. As before T_k is a second order differential operator in t's and R_k is a degree changing operator

$$T_k = \sum_{i+j=k} i!j!\tau_i\tau_j\big(\mathrm{td}(Q)\big)\cap,$$

$$R_k = \sum_{\substack{j\geq 1\\ v\in V}} j_{(k+1)}\, t_{j-k,v} \frac{\partial}{\partial t_{j,v}}.$$

   Here $a_{(b)} = a(a-1)\cdots(a-b+1)$ is the falling factorial.

Virasoro constraints for sheaves: the operators

1. The homology version of Virasoro constraints:

$$(L_k + S_k)\, \iota_*[M]^{\mathrm{vir}} = 0 \qquad \text{for} \quad k \geq 0$$

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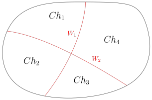
2. The operator on the second line is defined by

$$L_{\mathrm{wt}=0} = \sum_{n \geq -1} \frac{(-1)^n}{(n+1)!} T^{n+1} \circ L_n\,.$$

Wall-crossing and vertex algebras

1. I mainly use Joyce's geometric vertex algebras and wall-crossing for proving Virasoro constraints



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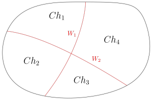

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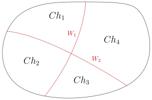

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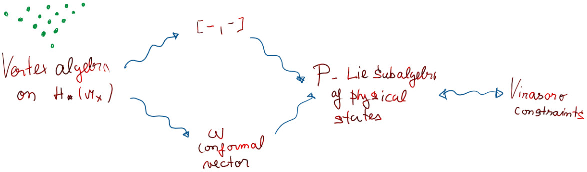

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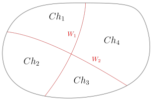

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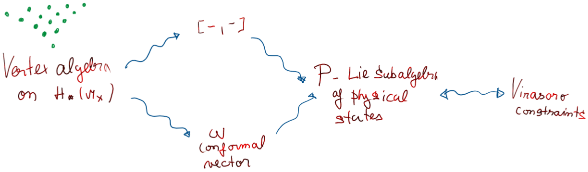

4. The upshot is not only the interpretation that VFC's satisfying Virasoro constraints are physical states, but also that this property is preserved under changing stability conditions.

Example of a wall-crossing formula

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4. This means that all the information about $[M^{\sigma}_{\overline{d}}]^{\mathrm{in}}$ is already contained in the Lie algebra structure.

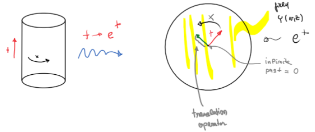
Vertex algebras

1. A vertex algebra is the data of a $\mathbb{Z}$-graded vector space V_* over $\mathbb{C}$ together with
   1.1 a vacuum vector $|0\rangle \in V_0$,
   1.2 a linear operator $T\colon V_* \to V_{*+2}$ called the translation operator,
   1.3 and a state-field correspondence which is a degree 0 linear map

$$Y\colon V_* \longrightarrow \mathrm{End}(V_*)[\![z, z^{-1}]\!]\,,$$

   denoted by $Y(a,z) := \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, where $\deg(z) = -2$.

2. These need to satisfy some axioms that have enlightening interpretations in terms of CFTs



- $\lim_{|z|\to 0} Y(v,z)|0\rangle = v$ – state-field correspondence

- $e^{zT}|0\rangle = |0\rangle$, $Y(|0\rangle, z) = \mathrm{id}$, – vacuum invariance

- $Y(v,z)w = e^{zT} Y(w,-z)v$

– skew-symmetry

- $(z_1-z_2)^N Y(v,z_1) Y(w,z_2) = (z_1-z_2)^N Y(w,z_2) Y(v,z_1)$

– Locality

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$$\mathrm{Ext}_Q = \bigoplus_{v \in V \setminus F} \mathcal{U}_v^\vee \boxtimes \mathcal{U}_v \xrightarrow{\varphi_E} \bigoplus_{e \in E} \mathcal{U}_{t(e)}^\vee \boxtimes \mathcal{U}_{h(e)} \xrightarrow{\varsigma_R} \bigoplus_{r \in R} \mathcal{U}_{t(r)}^\vee \boxtimes \mathcal{U}_{h(r)}$$

   on $\mathcal{M}_Q \times \mathcal{M}_Q$ is the last piece necessary to write down $Y(v, z)$. Satisfies:

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2. The vertex algebra V_* was shown to be a lattice vertex algebra by Joyce (17'), which is the most natural vertex algebra with the underlying graded vector space

$$V_* = \mathbb{Q}[\mathbb{Z}^V] \otimes \mathrm{Sym}[\![t_{i,v}, i > 0, v \in V]\!]\,.$$

The geometric construction of a Lie algebra

1. The stacky quotient of $\mathcal{M}_Q$ by $B\mathbb{G}_m$ is denoted by $\mathcal{M}_Q^{\mathrm{rig}}$. As one would expect, there is roughly the correspondence

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2. Now there are two roles that K_* plays:
   2.1 The classes $[M]^{\mathrm{vir}}$ live in K_* even if M is not fine.
   2.2 K_* has a Lie bracket:

$$[\overline{v}, \overline{w}] = \overline{v_0 w}\,, \quad \forall v, w \in V_*\,.$$

Conformal element

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Remark

When working with varieties X, we need to include $H^{\mathrm{odd}}(X) \cong K^1(X,\mathbb{Q})$ into the lattice. One can still write $\omega = \frac{1}{2}\sum_{v\in B} t^H_{1,v}t^H_{1,\hat{v}}$ where ^{H} denotes some holomorphic grading shift leading to odd degrees. The conformal charge is given by $\chi(X)$.

Physical states and the main claim

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Theorem (BLM(22'), B. (23'))

The condition that $[M_\alpha^\sigma]^{\mathrm{in}}$ satisfies Virasoro constraints is equivalent to it being a physical state with respect to the ω given above. I.e. $[M_\alpha^\sigma]^{\mathrm{in}} \in \check{P}$.
In particular, wall-crossing, stated in terms of iterated Lie brackets, preserves Virasoro constraints from the RHS to the LHS of the formula.

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Remark

In the second point, I used derived equivalences of the surfaces to quivers. This is the first proof of Virasoro constraints for sheaves on surfaces independent of Witten's conjecture. Using a universality of Virasoro constraints for $\mathrm{Hilb}^n(S)$, gives an independent proof of Virasoro constraints for any surface S. In particular, this establishes them as an autonomous phenomenon.