1. (8 points) In medical imaging numerous measurements are taken and processed by a computer to construct a 3D image of the organ the physician wishes to study. The process is similar to the slicing process we have used to find the volume of a solid.

   Suppose than an MRI scan indicates that the cross-sectional areas of a tumor are given by the values in the following table:

x (cm)	0	0.2	0.4	0.6	0.8	1.0	1.2
area $A(x)$ (cm^2)	0	0.4	0.6	1.2	0.6	0.3	0.1

   Use Simpson's rule with 6 subintervals to estimate the volume of the tumor. Show all main steps. Round your answer to three decimal places and include units.

$$V = \int_0^{1.2} A(x)\,dx$$

$$\cong S_6 = \frac{\Delta x}{3}\left[A(0)+4A(0.2)+2A(0.4)+4A(0.6)+2A(0.8)+4A(1)+A(1.2)\right]$$

$$= \frac{0.2}{3}\left[0+4(0.4)+2(0.6)+4(1.2)+2(0.6)+4(0.3)+0.1\right]$$

$$= \frac{0.2}{3}[10.1] = \frac{2.02}{3} \cong \boxed{0.673\ \text{cm}^3}$$

Note regarding units:

$$\int_0^{1.2} \underbrace{A(x)}_{\text{cm}^2}\cdot\underbrace{dx}_{\text{cm}} \to \text{cm}^3$$

$$\underbrace{\frac{\Delta x}{3}}_{\text{cm}}\ \underbrace{\left[A(0)+4A(0.2)+\dots\right]}_{\text{cm}^2} \to \text{cm}^3$$

2. (10 points) Evaluate the following integral. Show work, and box your final answer.

$$\int \frac{\sin x \cos x}{\sin^2 x + 5\sin x + 6}\,dx$$

$$= \int \frac{u}{u^2+5u+6}\,du$$

(1) $u = \sin x$
$du = \cos x\,dx$

$$= \int \frac{-2}{u+2} + \frac{3}{u+3}\,du$$

(2) P.F.

$$\frac{u}{(u+2)(u+3)} = \frac{A}{u+2} + \frac{B}{u+3}$$

$$u = A(u+3) + B(u+2)$$

$u=-2$: $-2 = A(1) \Rightarrow A = -2$

$u=-3$: $-3 = B(-1) \Rightarrow B = 3$

$$= -2\ln|u+2| + 3\ln|u+3| + C$$

$$= \boxed{-2\ln|\sin x+2| + 3\ln|\sin x+3| + C}$$

3. (10 points) Evaluate the following definite integral. Simplify and leave your answer in exact form.

$$\int_1^{1.5} \sqrt{-x^2+2x}\,dx$$

$$-x^2+2x = -(x^2-2x) = -\left[(x-1)^2-1\right] = 1-(x-1)^2$$

$$\int_1^{1.5} \sqrt{-x^2+2x}\,dx = \int_1^{1.5} \sqrt{1-(x-1)^2}\,dx$$

TRIG SUB.

$x-1 = \sin\theta$

$dx = \cos\theta\,d\theta$

Bounds:

$x=1$: $0 = \sin\theta \Rightarrow \theta = 0$

$x=1.5$ $\frac{1}{2} = \sin\theta \Rightarrow \theta = \frac{\pi}{6}$

$$= \int_0^{\pi/6} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$= \int_0^{\pi/6} \cos^2\theta\,d\theta$$

$$= \int_0^{\pi/6} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]\Big|_0^{\pi/6}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{6} + \frac{\sin \pi/3}{2}\right) - (0)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{6} + \frac{(\sqrt{3}/2)}{2}\right]$$

$$= \boxed{\frac{\pi}{12} + \frac{\sqrt{3}}{8}}$$

4. (a) (8 points) Evaluate the integral $\int \frac{\ln x}{x^2}\,dx$.

IBP: $u = \ln x \quad dv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}dx \quad v = -1/x$

$$= -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx$$

$$= \boxed{-\frac{1}{x}\ln x - \frac{1}{x} + C}$$

(b) (4 points) Does the improper integral $\int_4^\infty \frac{\ln x}{x^2}\,dx$ converge or diverge? If it diverges, show why. If it converges, compute its value. Show your work, including any limits you compute.

$$\int_4^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_4^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\ln x - \frac{1}{x}\right]\Big|_4^t$$

$$= \lim_{t\to\infty}\left[-\frac{1}{t}\ln t - \frac{1}{t}\right] - \left[-\frac{1}{4}\ln 4 - \frac{1}{4}\right]$$

$(0)(\infty)$ ← INDETERMINATE TYPE, MUST COMPUTE

$$= \lim_{t\to\infty}\left(-\frac{\ln t}{t}\right) - \lim_{t\to\infty}\frac{1}{t} + \frac{1}{4}\ln 4 + \frac{1}{4}$$

$\frac{\infty}{\infty}$, L'H

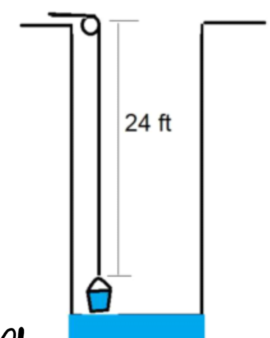
$$= \underbrace{\lim_{t\to\infty}\left(-\frac{1/t}{1}\right)}_{=0.} - 0 + \frac{1}{4}\ln 4 + \frac{1}{4} = \boxed{\frac{1}{4}\ln 4 + \frac{1}{4}}$$

5. (10 points) A bucket that weighs 3 lbs and a rope that weighs 0.6 lbs/ft are used to draw water from a well that is 24 ft deep.

   The bucket initially holds 40 lbs of water, but it leaks at a constant rate of 0.4 lbs of water for each foot that the bucket is lifted.

   Find the work required to pull the bucket half way to the top of the well.

   So we can follow your solution: please indicate what the various expressions, integrals and variables you write down represent.



Let y = how far the bucket is lifted

The force applied when the bucket is y feet from the bottom is:

$$F(y) = F_{bucket} + F_{rope} + F_{water}$$

$$= 3 + 0.6(24-y) + (40-0.4y) \text{ lbs}$$

$$3 + 14.4 - 0.6y + 40 - 0.4y$$

$$= 57.4 - y$$

$$W = \int_0^{12} F(y)\,dy = \int_0^{12} (57.4-y)\,dy = \left(57.4y - \tfrac{1}{2}y^2\right)\Big|_0^{12}$$

$$= \boxed{616.8 \text{ ft-lb}}$$

There are other correct ways to do this. For example, you can separate it into 3 parts:

1) $W_{bucket} = (3)(12) = 36$ ft-lb
2) $W_{rope} = \int_0^{12} 0.6(24-y)\,dy = 129.6$ ft-lb (or $\int_{12}^{24} 0.6y\,dy$ w/a different setup)
3) $W_{water} = \int_0^{12} (40-0.4y)\,dy = 451.2$ ft-lb

Total $W = 36 + 129.6 + 451.2 = \boxed{616.8}$ ft-lb