Explanation Draft

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We begin the analogy with the Naiver-Stokes equation which governs Incompressible Newtonian fluids. Let v denote the velocity vector field of the fluid, which associates to each position $r = (x, y, z)$ and time t, a vector representative of fluids velocity. Let p be a scalar that denotes pressure. The equation is,

$v_t + v \cdot \nabla v = -\nabla p + \nu \nabla v$, $\nabla \cdot v = 0$. (1)

The designation that $\nabla \cdot v = 0$ is termed divergence free, i.e. v is a divergence free velocity field. A stream function $\Psi(x, y)$ is a scalar function that satisfies the relationship $\nabla^{\perp} = v = (v_x, v_y)$, i.e., $\nabla\Psi_y = v_x, \nabla\Psi_x = -v_y$. The vorticity of flow field is $\omega = \nabla \times v$.

We can take the curl of (1),
$\nabla \times (v_t + v \cdot \nabla v) = \nabla \times (-\nabla p + \nu \nabla v)$

$\nabla \times (v_t + v \cdot \nabla v) = \nabla \times v_t + \nabla \times (v \cdot \nabla v)$ since $\nabla \times (A + B) = \nabla \times A + \nabla \times B$.
$= \dfrac{\delta(\nabla \times v)}{\delta t} + \nabla(v \cdot \nabla v)$ since $\delta \times \dfrac{\delta A}{\delta t} = \dfrac{\delta}{\delta t}(\nabla \times A)$
$= w_t + \nabla \times (v \cdot \nabla v)$ since $w = \nabla \times v$ by definition.
$= w_t + (v \cdot \nabla)(\nabla \times v) - (w \cdot \nabla)(v)$ for some reason. Since w is a scalar value, $\nabla \cdot w = 0$. Hence,
$\nabla \times (v_t + v \cdot \nabla v) = w_t + (v \cdot \nabla)w$.

The right hand side becomes $\nu \nabla w$, hence, $w_t + (v \cdot \nabla)w = \nu \nabla w$.
With the assumption that the fluid has no viscocity, i.e., $\nu = 0$, we have that $\omega_t + (v \cdot \nabla)\omega = 0$. Furthermore, with the assumption of a steady state flow, $w_t = 0$, we have that $(v \cdot \nabla)\omega = 0$. Noticing that $\triangle\Psi = \omega$, $(v \cdot \nabla)\triangle\Psi = 0$.
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$v \cdot \nabla\triangle\Psi = 0$.

By definition, $\dfrac{\delta\Psi}{\delta y} = v_x$ and $\dfrac{\delta\Psi}{\delta x} = -v_y$, hence,

$\nabla^{\perp}\Psi = (v_x, v_y)$ and so,

$\nabla^{\perp}\Psi \cdot \nabla\triangle\Psi = 0$.

"which says that the Laplacian of the stream function, and hence the vorticity, must have the same level curves as the steam function".

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To see this, $\nabla\Psi$ is normal to streamlines of Ψ since streamlines are the level curves of Ψ. $\nabla^{\perp}\Psi$ is a 90 degree rotation of $\nabla\Psi$, hence, tangent to streamlines of Ψ. The equation implies that the tangents of stream lines are orthogonal to gradients of the Laplacian of I. Since the gradient of $\triangle I$ is orthogonal to level curves of $\triangle I$, it follows that the streamlines of I are parallel to the level curves of I.

This derivation justifies the use of the technique. "The concept of smooth continuation of information in the level-lines direction has been addressed in [2]". ... "The proposed algorithm propagates the image Laplacian in the level-lines (isophotes) direction. The algorithm attempts to imitate basic approaches used by professional restorators."

With the justification sorted, we use (1) with the stream function Ψ now being represented by the image intensity I. We begin with equation $\omega_t + v \cdot \nabla w = \nu \triangle w$ which was obtained with the assumption of steady state flow and no viscocity. We add anistrophic diffusion through g.