

Project 1 - INF5620

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1 Setting up the differential equation

Beginning with newtons second law for an object falling with quadratic air resistance and a source term for verification.

$$\sum F = m\ddot{x} \quad (1)$$

$$mg - b\dot{x}|\dot{x}| + F_s = m\ddot{x} \quad (2)$$

Or, written in terms of velocity.

$$mg - bv|v| + F_s = m\dot{v} \quad (3)$$

$$\dot{v} = g - \frac{b}{m}v|v| + F_s = f(t, v) \quad (4)$$

2 Setting up the discrete equation

We choose the Crank-Nicholson scheme for discretizing the equation.

$$[D_tv = f(t, \bar{v}^{t, \frac{1}{2}})]^{n+\frac{1}{2}} \quad (5)$$

To linearize the equation, we decide to take the geometric average of the velocities in $|v|v$, such that:

$$\frac{v^{n+1} - v^n}{\Delta t} = g - \frac{b}{m}|v^n|v^{n+1} + F_s \quad (6)$$

This gives a linear equation in v^{n+1} , which makes it easier to handle with an implicit solver. Solving this equation for v^{n+1} gives:

$$v^{n+1} = \frac{v^n + (g + F_s)\Delta t}{1 + \frac{b}{m}|v^n|\Delta t} \quad (7)$$

Checking whether a linear solution will solve the discrete equation without the source term. I do this by inserting the solution $v(t) = t$, and solving for F_s .

$$\left[D_tv = g - \frac{b}{m}|t_n|t_{n+1} + F_s \right]^{n+\frac{1}{2}} \quad (8)$$

$$F_s = \frac{b}{m}t(t + \Delta t) - g + 1 \quad (9)$$

With this source term, $v(t) = t$ is the exact solution to the discrete equation. Inserting this source term into the solver should give a solution that holds up to machine precision. This holds, and a nose test with a tolerance of $1e - 13$ is implemented.

3 Improved force expression for parachuter

A simple force model for a parachuter at high Reynolds numbers is the following.

$$F(t, v) = F_d^{(q)} + F_g + F_b \quad (10)$$

The quadratic drag expression is chosen to be $-\frac{1}{2}C_D\rho A|v|v$, where C_D is a dimensionless drag coefficient and A is the cross sectional area of the falling body. ρ is the air density. The gravitational pull is taken as the constant mg , implying that we are close to the surface of the earth. We assume constant air density, so the buoyancy term is ρgV , where V is the volume of the body.

Applying Newton's 2. law, gives:

$$m\dot{v}(t, v) = -\frac{1}{2}C_D\rho A|v|v + mg - \rho gV \quad (11)$$

The positive direction of motion is towards ground. Now, a few manipulations to get the equation more suited for numerical treatment: We introduce the density ρ_b of the falling body, so that $m = \rho_b V$. Dividing by the mass, and inserting $\rho_b V$ gives:

$$\dot{v}(t, v) = -\frac{1}{2}C_D\frac{\rho A}{\rho_b V}|v|v + g\left(1 - \frac{\rho}{\rho_b}\right) \quad (12)$$

We now define $a = \frac{1}{2}C_D\frac{\rho A}{\rho_b V}$ and $b = g\left(1 - \frac{\rho}{\rho_b}\right)$, such that the equation to solve numerically is:

$$\dot{v}(t, v) = -a|v|v + b + f_s(t) \quad (13)$$

Where we have added a source term $f_s(t)$ to be able to manufacture solutions. Applying the Crank-Nicolson scheme on this equation:

$$\left[D_t v = f(t, \bar{v}^{t, \frac{1}{2}})\right]^{n+\frac{1}{2}} \quad (14)$$

Written out, with $[|v|v]^{n+\frac{1}{2}} \approx |v^n|v^{n+1}$

$$\frac{v^{n+1} - v^n}{\Delta t} = -a|v^n|v^{n+1} + b + f_s(t_{n+\frac{1}{2}}) \quad (15)$$

Where the source term is supposed to be defined later, and we assume it to be evaluable at $t_{n+\frac{1}{2}}$.

$$v^{n+1} = \frac{v^n + (b + f_s(t))\Delta t}{1 + a|v^n|\Delta t}. \quad (16)$$

We now manufacture a linear solution to the problem, $v(t) = At + B$, Where the initial state $v(0) = B$. Inserted into the CN-scheme, this gives:

$$A = -a|At + B|(A(t + \Delta t) + B) + b + f_s(t_{n+\frac{1}{2}}) \quad (17)$$

Which gives:

$$f_s(t_{n+\frac{1}{2}}) = a|At + B|(A(t + \Delta t) + B) - b + A \quad (18)$$

A test for this manufactured solution is made.

To check the convergence rate, we construct a solution to the continous problem, still on the form $At + B$.

$$D[At + B] = -a|At + B|(At + B) + b + f_s(t) \quad (19)$$

Such that $f_s(t) = a|At + B|(At + B) - b + A$.