Project 1 - INF5620

Henrik Andersen Sveinsson

September 4, 2013

1 Setting up the differential equation

Beginning with newtons second law for an object falling with quadratic air resistance and a source term for verification.

$$\sum F = m\ddot{x} \tag{1}$$

$$mg - b\dot{x}|\dot{x}| + F_s = m\ddot{x} \tag{2}$$

Or, written in terms of velocity.

$$mg - bv|v| + F_s = m\dot{v} \tag{3}$$

$$\dot{v} = g - \frac{b}{m}v|v| + F_s = f(t, v) \tag{4}$$

2 Setting up the discrete equation

We choose the Crank-Nicholson scheme for discretizing the equation.

$$[D_t v = f(t, \bar{v}^{t,\frac{1}{2}})]^{n+\frac{1}{2}} \tag{5}$$

To linearize the equation, we decide to take the geometric average of the velocities in $|v|v$, such that:

$$\frac{v^{n+1} - v^n}{\Delta t} = g - \frac{b}{m}|v^n|v^{n+1} + F_s \tag{6}$$

This gives a linear equation in v^{n+1}, which makes it easier to handle with an implicit solver. Solving this equation for v^{n+1} gives:

$$v^{n+1} = \frac{v^n + (g + F_s)\Delta t}{1 + \frac{b}{m}|v^n|\Delta t} \tag{7}$$

Checking whether a linear solution will solve the discrete equation without the source term. I do this by inserting the solution $v(t) = t$, and solving for F_s.

$$\left[D_t t = g - \frac{b}{m}|t_n|t_{n+1} + F_s\right]^{n+\frac{1}{2}} \tag{8}$$

$$F_s = \frac{b}{m}t(t+\Delta t) - g + 1 \tag{9}$$

Since the source term is non-zero, we conclude that a linear function is not a solution to the disctrete equations. With this source term, $v(t) = t$ is the exact soultion to the discrete equation. Inserting this source term into the solver should give a solution that holds up to machine precision.

3 Improved force expression for parachuter

A simple force model for a parachuter at high reynolds numbers is the following.

$$F(t,v) = F_d^{(q)} + F_g + F_b \tag{10}$$

The quadratic drag expression is chosen to be $-\frac{1}{2}C_D\rho A|v|v$, where C_D is a dimensionless drag coefficient and A is the cross sectional area of the falling body. ρ is the air density. The gravitational pull is taken as the constant mg, implying that we are close to the surface of the earth. We assume constant air density, so the bouyancy term is $\rho g V$, where V is the volume of the body.

Applying Newtons 2. law, gives:

$$m\dot{v}(t,v) = -\frac{1}{2}C_D\rho A|v|v + mg - \rho g V \tag{11}$$

The positgive direction of motion is towards ground. Now, a few manipulations to get the equation more suited for numerical treatment: We introduce the density ρ_b of the falling body, so that $m = \rho_b V$. Dividing by the mass, ans inserting $\rho_b V$ gives:

$$\dot{v}(t,v) = -\frac{1}{2}C_D\frac{\rho A}{\rho_b V}|v|v + g\left(1 - \frac{\rho}{\rho_b}\right) \tag{12}$$

We now define $a = \frac{1}{2}C_D\frac{\rho A}{\rho_b V}$ and $b = g\left(1 - \frac{\rho}{\rho_b}\right)$, such that the equation to solve numerically is:

$$\dot{v}(t,v) = -a|v|v + b + f_s(t) \tag{13}$$

Where we have added a source term $f_s(t)$ to be able to manufacture solutions.

Applying the Crank-Nicolson scheme on this equation:

$$\left[D_t v = f(t, \bar{v}^{t,\frac{1}{2}})\right]^{n+\frac{1}{2}} \tag{14}$$

Written out, with $[|v|v]^{n+\frac{1}{2}} \approx |v^n|v^{n+1}$

$$\frac{v^{n+1} - v^n}{\Delta t} = -a|v^n|v^{n+1} + b + f_s(t_{n+\frac{1}{2}}) \tag{15}$$

Where the source term is supposed to be defined later, and we assume it to be evaluable at $t_{n+\frac{1}{2}}$.

$$v^{n+1} = \frac{v^n + (b + f_s(t))\Delta t}{1 + a|v^n|\Delta t}. \tag{16}$$

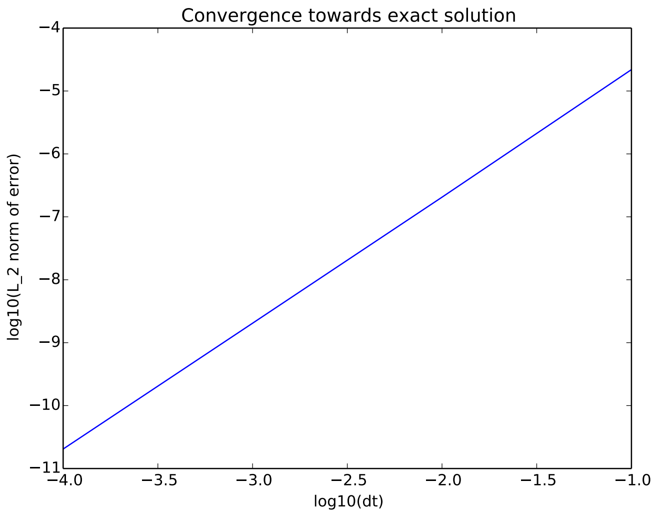

Figure 1: Error estimate shows second order convergence in Δt

We now manufacture a linear solution to the problem, $v(t) = At + B$, Where the initial state $v(0) = B$. Inserted into the CN-scheme, this gives:

$$A = -a|At + B|(A(t + \Delta t) + B) + b + f_s(t_{n+\frac{1}{2}}) \tag{17}$$

Which gives:

$$f_s(t_{n+\frac{1}{2}}) = a|At + B|(A(t + \Delta t) + B) - b + A \tag{18}$$

A test for this manufactured solution is made.

To check the convergence rate, we construct a solution to the continous problem, still on the form $At + B$.

$$D[At + B] = -a|At + B|(At + B) + b + f_s(t) \tag{19}$$

Such that:

$$f_s(t) = a|At + B|(At + B) - b + A \tag{20}$$

Using this source term, the convergence is confirmed to be quadratic.