K-means Clustering

In this problem set we will implement and apply the standard (batch) K-means algorithm, the online version, and the "soft" clustering procedures. The file `cluster.dat` contains a data set of $p = 500$ (2-dimensional) observations generated from four different Gaussians with four different means.

10.1 K-means Clustering (3 points)

Write a program that implements the *standard* version of K-means clustering and partitions the given data set into K clusters. Repeat the clustering procedure for different initializations of the prototypes and $K = 2, 3, 4, 5, 6, 7, 8$. Include the following steps:

Initialization –

- Set the initial prototypes $\mathbf{w}_q$ randomly around the data set mean
- Set the maximum number of iterations t_{max}, e.g. 5

Optimization –

Implement the k-means update (see lecture notes). Each iteration should contain the following two steps

- assign all datapoints to their closest prototype
- re-compute the new positions of the prototypes for this assignment

Visualization –

(a) Visualize data points and prototypes for each iteration in a sequence of scatter plots.

(b) Plot the error function E against the iteration number t

$$E_{\{m_q^{(\alpha)}\},\{\mathbf{w}_q\}} = \frac{1}{2p} \sum_{q=1}^{K} \sum_{\alpha=1}^{p} m_q^{(\alpha)} \left\| \mathbf{x}^{(\alpha)} - \mathbf{w}_q \right\|^2$$

(c) Create a plot (Voronoi-Tesselation) to show how the resulting solution assigns different regions of input space (e.g. new data points $\mathbf{x} \in \mathbb{R}^2$) to the different clusters.

10.2 Online K-means Clustering (3 points)

Write a program that implements the *online* version of K-means clustering (see lecture notes) and partitions the given data set into $K = 4$ clusters. Include the following steps:

Initialization –

- Set the initial prototypes $\mathbf{w}_q$ randomly around the data set mean
- Select an initial learning step η_0
- Set the maximum number of iterations t_{max}, e.g. equal to the data set size p.

Optimization –

- Choose a suitable $\tau < 1$ and implement online K-means clustering using the following "annealing" schedule for η:

$$\eta_t = \eta_0 \quad \text{for } t = 0, ..., \frac{t_{max}}{4} \qquad \text{and} \qquad \eta_t = \tau\eta_{t-1} \quad \text{for } t = \frac{t_{max}}{4} + 1, ..., t_{max}$$

Visualization –

(a) Visualize data points and the prototypes for each iteration in a sequence of scatter plots, but only show the first, the final, and four intermediate iterations. In the final plot additionally show for each cluster the the sequence of centroid positions $\mathbf{w}_q$ by connecting them with straight lines.

(b) Plot the error function E (as above) against the iteration number t

10.3 Soft K-means Clustering (4 points)

"Soft" clustering is a mean-field approximation of pairwise clustering with squared Euclidean distances. Implement the *soft* K-means algorithm with squared Euclidean distances (cf. lecture notes) and apply it to the same data as before. Proceed as follows:

(a) Set $K = 8$ initial prototypes $\mathbf{w}_q$ randomly around the data set mean and choose a convergence tolerance γ.

(b) For fixed β (no annealing), let the optimization procedure run until convergence, that is $\|\mathbf{w}_q^{new} - \mathbf{w}_q^{old}\| < \gamma \ \forall q$. Repeat this for different $\beta \in [0.2, 20]$ e.g. in steps of $\Delta\beta = 0.2$. Use the same initial prototypes for all runs.

(c) Visualize the data set, initial and final prototypes for each (fixed) β in one scatter plot. Therein show how "soft" each data point $\mathbf{x}^{(\alpha)}$ is assigned to a cluster, e.g. by scaling the brightness of the respective plot symbol with the largest assignment probability, i.e., $\max_q \langle m_q^{(\alpha)} \rangle$.

(d) Plot in two seperate subplots the first and second coordinate of the final prototypes $\mathbf{w}_q$ against the β (i.e. K lines per subplot) and interpret the result.

(e) In additional simulations, run the optimization for $K = 2, 4, 6, 8$ using an annealing schedule: increase β after each iteration. E.g. $\beta_0 = 0.2,\ \tau = 1.1,\ \beta_{t+1} = \tau\beta_t$.

(f) Show the data set, initial and final prototypes of the "annealed" clustering solutions for $K = 2, 4, 6, 8$ in a scatter plot. How "soft" are data points assigned now?

Total points: 10