Homework7

June 16, 2016

In [3]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.io import loadmat as loadmat
% matplotlib inline
```

1 7.1 Natural Gradient (3 points)

(a) Extend your code from the previous problem sheet to get an ICA-learning scheme based on the natural gradient with a learning rate η that decays slowly to 0 (e.g. $\eta t+1 = \lambda \eta t$ with $\lambda \approx 1$, $\lambda < 1$). Note that depending on λ you have to iterate over the (shuffled) data more than once for proper convergence.

In [15]:
```
def fhat(y):
    return 1 / (1 + np.exp(-y))

def fhatpp_fhatp(y):
    return 1 - 2*fhat(y)

def dWnatural(eta, W, x):
    #x0 x1
    #x0 x1
    #xv = np.vstack( (x.T, x.T) )
    xv = np.vstack( (x.T, x.T, x.T) )
    #x0 x0
    #x1 x1
    #xh = np.hstack( (x.reshape(-1,1), x.reshape(-1,1)) )
    xh = np.hstack( (x.reshape(-1,1), x.reshape(-1,1), x.reshape(-1,1)) )
    #W-1 + f(W . xh) * xv
    subtotal = (np.eye(W.shape[0]) + np.multiply( fhatpp_fhatp(np.dot(W, xh)), xv))
    #return eta * np.dot(subtotal, np.dot(W.T, W))
    return eta * np.dot(subtotal, W)

def buildcorrelations(N, s, x):
    p = np.zeros((N, N))
    for i in range(N):
        for j in range(N):
            #print("i: "+str(i)+" j: "+str(j)+" corr:\n"+str((np.cov(s[i], x[j]) / (np.std(s[i
            p[i, j] = (np.cov(s[i], x[j]) / (np.std(s[i]) * np.std(x[j])))[0,1]
    return p



def gradientdescent(A, W, s, eta, l):
```

```
print("Mixing Matrix:\n"+str(A)+"\nDet: "+str(np.linalg.det(A)))
x = np.dot(A, s)

#(c) Remove the temporal structure by permuting the columns of the N × p matrix X randomly
xs = np.random.permutation(x)

#(e) Center the data to zero mean.
#print( np.mean(x, axis=1).shape )
x_mean = np.mean(xs, axis=1).reshape(-1,1)
xsc = xs - x_mean

#(f) Initialize the unmixing matrix W with random values.
#s = W.x, s: Nxp, x: Nxp -> W: NxN
#W = np.random.uniform(0, 1, (N, N))

#(c) Choose a suitable learning rate η and apply both versions to the data to unmix the so
epsilon = .000000001
Wb = W.copy()
Wbbs = W.copy()
BConverged = False
BbsConverged = False
etan = eta

print("Initial W:\n"+str(W))
for n in range(10):
    for t in range(0, xsc.shape[1]):
        xa = xsc[:,t]
        #xa = xsc[:,np.random.randint(xsc.shape[1])]
        etan = etan*l

        #Natural
        if not BConverged:
            natural = dWnatural(etan, Wb, xa)
            if np.absolute(natural).sum() > epsilon:
                Wb = Wb + natural
            else:
                BConverged = True
                print("(b) converged at: "+str(n*t))

        #Natural with Bell-Sejnowski regularization
        if not BbsConverged:
            naturalbs = dWnatural(etan, Wbbs, xa)
            for n in range(N):
                naturalbs[n, n] = 0
            if np.absolute(naturalbs).sum() > epsilon:
                Wbbs = Wbbs + naturalbs
                for n in range(N):
                    Wbbs[n, n] = 1
            else:
                BbsConverged = True
                print("(b.bs) Bell-Sejnowski converged at: "+str(n*t))
        else:
            pass
return Wb, Wbbs, x, x_mean
```

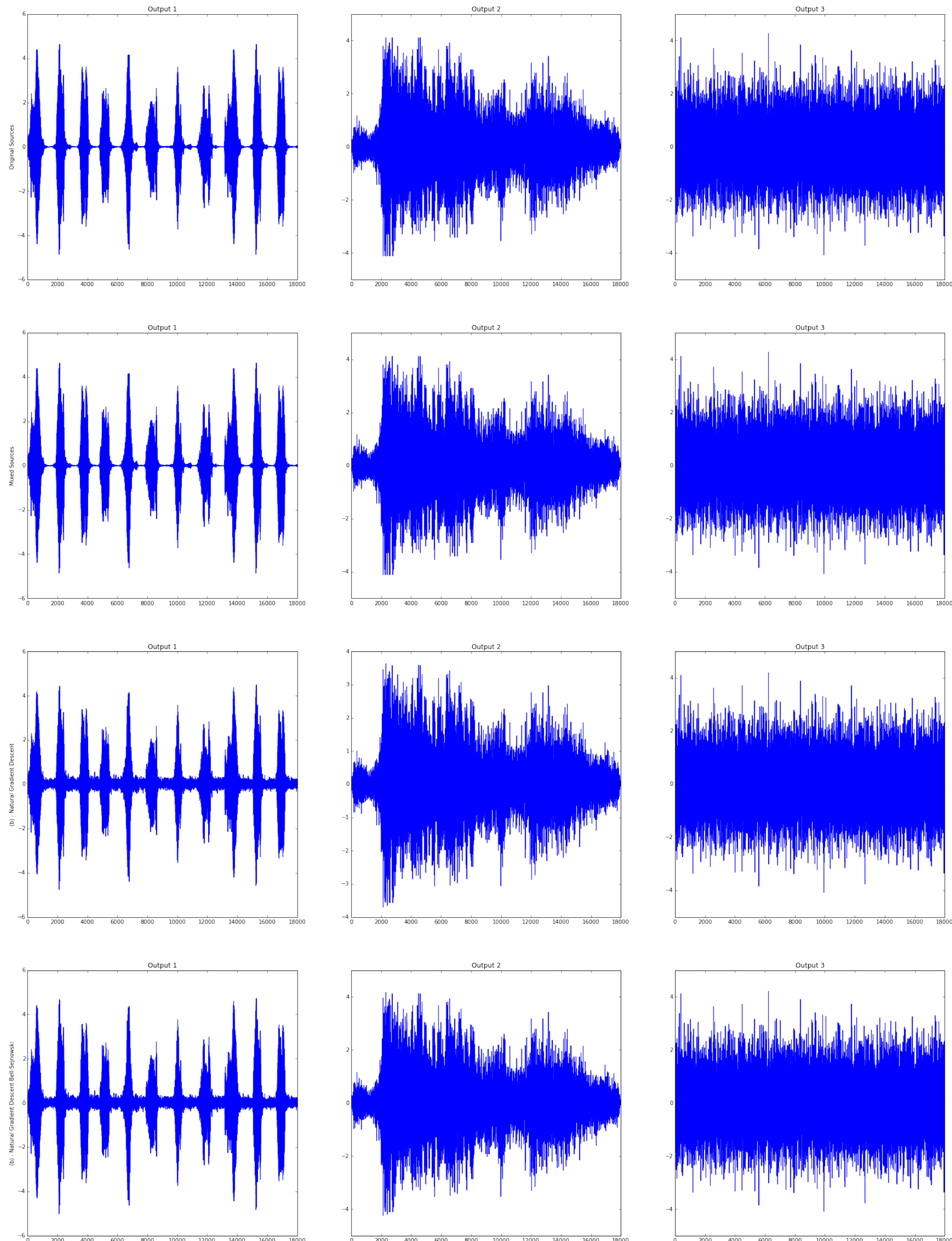
```
def myplot(Wb, Wbbs, x, x_mean, s):
    #retrieve shats - ^s
    shatb = np.dot(Wb, x)
    shatb_decentered = shatb + x_mean
    shatbbs = np.dot(Wbbs, x)
    shatbbs_decentered = shatbbs + x_mean

    #Check correlations, to check whcih source goes to which output channel
    mixcors = buildcorrelations(N, s, x)
    naturalcors = buildcorrelations(N, shatb_decentered, s)
    naturalbscors = buildcorrelations(N, shatbbs_decentered, s)

    isflippedmixes = (mixcors[0,0] < mixcors[0,1]) and (mixcors[1,0] > mixcors[1,1])
    isflippednatural = (naturalcors[0,0] < naturalcors[0,1]) and (naturalcors[1,0] > naturalco
    isflippednaturalbs = (naturalbscors[0,0] < naturalbscors[0,1]) and (naturalbscors[1,0] > na

    ys = [(s, 'Original Sources', False),
          (x, 'Mixed Sources', isflippedmixes),
          (shatb_decentered, '(b) - Natural Gradient Descent', isflippednatural),
          (shatbbs_decentered, '(b) - Natural Gradient Descent Bell-Sejnowski', isflippednatura
    ylimits = [-14,14]
    f, axarr = plt.subplots(len(ys), 3)
    f.set_size_inches(30, len(ys)*10, forward=False)

    axarr[0, 0].set_title("Source 1")
    axarr[0, 1].set_title("Source 2")
    axarr[0, 2].set_title("Source 3")

    for r in range(len(ys)):
        #axarr[r, 0].set_ylim(ylimits)
        #axarr[r, 1].set_ylim(ylimits)
        #axarr[r, 2].set_ylim(ylimits)
        axarr[r, 0].set_ylabel(ys[r][1])

        axarr[r, 0].plot(range(ys[r][0].shape[1]), ys[r][0][0])
        axarr[r, 1].plot(range(ys[r][0].shape[1]), ys[r][0][1])
        axarr[r, 2].plot(range(ys[r][0].shape[1]), ys[r][0][2])
        axarr[r, 0].set_title("Output 1")
        axarr[r, 1].set_title("Output 2")
        axarr[r, 2].set_title("Output 3")

    plt.show()
```

(b) Use the two sound signals from the last problem sheet and add (as third source s3) an additional "noise" source (normally distributed random numbers with a standard deviation similar to the two signals). Mix the signals using a mixing matrix of your choice and apply your ICA-algorithm. Plot the Mixed Sounds and recovered Sources

In [68]:
```
s1 = np.loadtxt("sounds/sound1.dat")
s2 = np.loadtxt("sounds/sound2.dat")
s3 = np.random.normal(0, 1, len(s1))
s = np.array([s1, s2, s3])
print("Standard Deviation s1: "+str(np.std(s[0])))
print("Standard Deviation s2: "+str(np.std(s[1])))
```

```
print("Standard Deviation s3: "+str(np.std(s[2])))
N = s.shape[0]

A = np.eye(N)
W = np.random.uniform(0, .1, (N, N))
W[0,0] = np.random.uniform(.8, 1)
W[1,1] = np.random.uniform(.8, 1)
W[2,2] = np.random.uniform(.8, 1)

Wb, Wbbs, x, x_mean = gradientdescent(A, W, s, .0000001, .9999)
print("Wb:")
print(Wb)
print("Wbbs:")
print(Wbbs)
myplot(Wb, Wbbs, x, x_mean, s)
```

```
Standard Deviation s1: 0.998545937615
Standard Deviation s2: 0.998854345166
Standard Deviation s3: 0.986100852986
Mixing Matrix:
[[ 1.  0.  0.]
 [ 0.  1.  0.]
 [ 0.  0.  1.]]
Det: 1.0
Initial W:
[[ 0.94736096  0.06200152  0.09854062]
 [ 0.04283364  0.86946837  0.00626526]
 [ 0.05738817  0.05842622  0.99659339]]
(b.bs) Bell-Sejnowski converged at: 12422
(b) converged at: 35552
Wb:
[[ 0.94790721  0.06201358  0.09855416]
 [ 0.04284101  0.87002072  0.00626386]
 [ 0.05739633  0.05844015  0.99721645]]
Wbbs:
[[ 1.          0.06200926  0.0985515 ]
 [ 0.04283859  1.          0.00626381]
 [ 0.05739381  0.05843595  1.        ]]
```

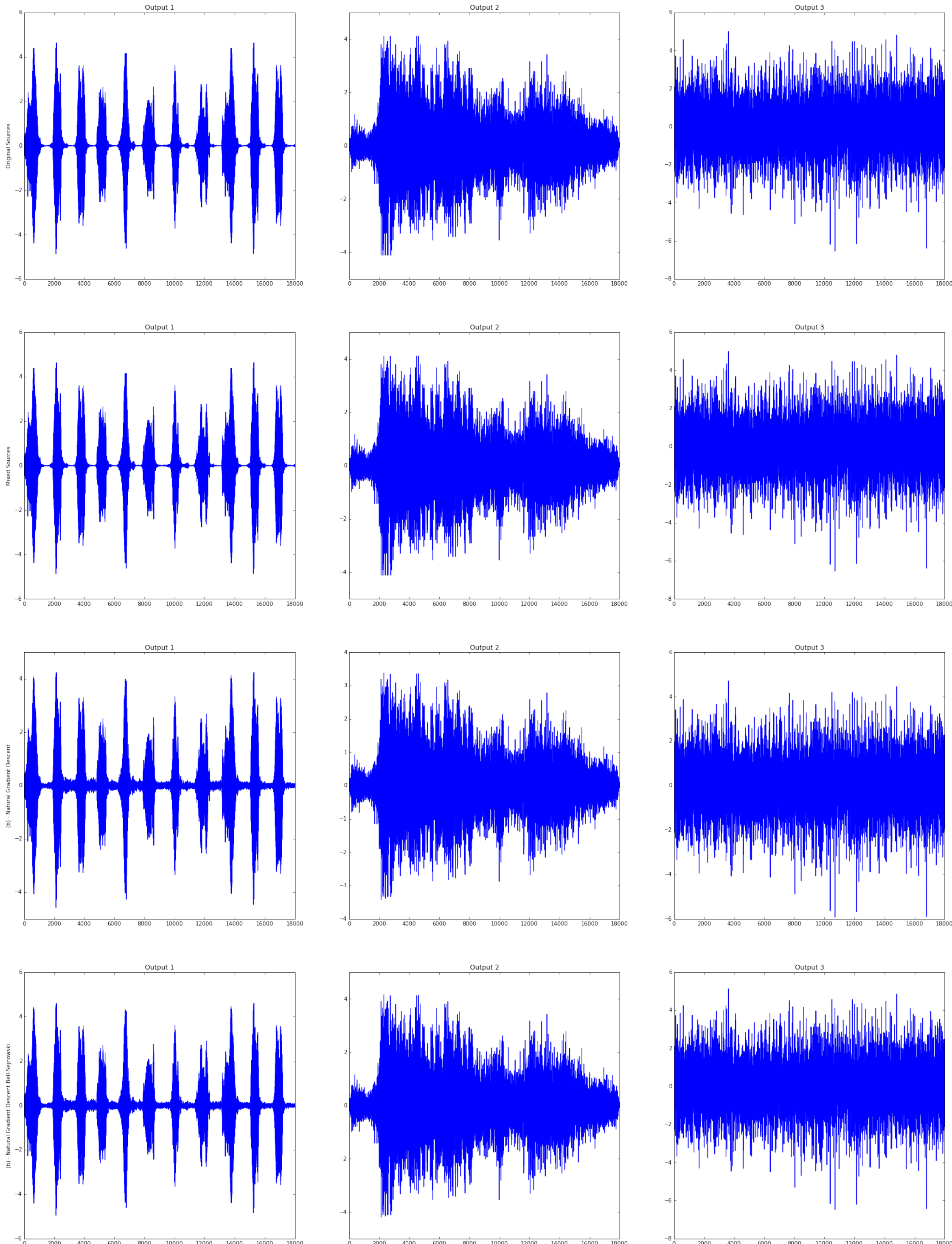


(c) Do the same analysis but adding a different "noise"-source (e.g. Laplace distributed) instead of the normal one.

```
In [69]: s1 = np.loadtxt("sounds/sound1.dat")
         s2 = np.loadtxt("sounds/sound2.dat")
```

```
s3 = np.random.laplace(0.0, 0.7, len(s1))
s = np.array([s1, s2, s3])


print("Standard Deviation s1: "+str(np.std(s[0])))
print("Standard Deviation s2: "+str(np.std(s[1])))
print("Standard Deviation s3: "+str(np.std(s[2])))
N = s.shape[0]
A = np.eye(N)
W = np.random.uniform(0, .1, (N, N))
W[0,0] = np.random.uniform(.8, 1)
W[1,1] = np.random.uniform(.8, 1)
W[2,2] = np.random.uniform(.8, 1)


Wb, Wbbs, x, x_mean = gradientdescent(A, W, s, .0000001, .9999)
print("Wb:")
print(Wb)
print("Wbbs:")
print(Wbbs)
myplot(Wb, Wbbs, x, x_mean, s)
```

```
Standard Deviation s1: 0.998545937615
Standard Deviation s2: 0.998854345166
Standard Deviation s3: 0.980715089168
Mixing Matrix:
[[ 1.  0.  0.]
 [ 0.  1.  0.]
 [ 0.  0.  1.]]
Det: 1.0
Initial W:
[[ 0.92268291  0.06539629  0.02809297]
 [ 0.02891274  0.81202717  0.01726697]
 [ 0.07903204  0.09890617  0.91645418]]
(b.bs) Bell-Sejnowski converged at: 17768
(b) converged at: 32670
Wb:
[[ 0.92328144  0.06540724  0.02809883]
 [ 0.028915    0.81256614  0.01726954]
 [ 0.0790489   0.09892605  0.91699215]]
Wbbs:
[[ 1.          0.06539828  0.02809692]
 [ 0.02891187  1.          0.01726796]
 [ 0.07904427  0.09891487  1.        ]]
```



2 Assignment 7.3

The file distrib.mat contains three toy datasets (uniform, normal, laplacian), each 10000 samples of 2 sources. Do the following for each dataset (which can be read for example using Python with loadmat from scipy.io):

7.2 The moment generating function is defined as:

$$M_x(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$f(x)$ as probability density

$$\frac{d^k}{dt^k} M_x(t) = \frac{d^k}{dt^k} E(e^{tX}) = E(X^k e^{tX})$$

$$E(X^k e^{tX}) \Big|_{t=0} = E(X^k)$$

k-th moment

moment generating functions:

Laplace: $$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}} dx = \frac{e^{\mu t}}{1-\sigma^2 t^2}$$

Gauß: $$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Uniform: $$M_x(t) = \int_a^b e^{tx} \frac{1}{b-a} dx = \left[\frac{1}{t(b-a)} e^{tx}\right]_a^b = \frac{e^{bt} - e^{at}}{t(b-a)}$$

The centered moment generating function:

$$M_{x_c}(t) = E(e^{t(X-\mu)}) = E(e^{tX}) \cdot E(e^{t\mu}) = M_x(t) \cdot e^{t\mu}$$

The standardised moments are then calculated as described in the exercise.

	Laplace(μ, b)	Gauß(μ, σ)	uniform(a, b)
$\langle x \rangle$	μ	μ	$\frac{1}{2}(a+b)$
$\langle x^2 \rangle_c$	$2\sigma^2$	σ^2	$\frac{1}{12}(b-a)^2$
$\langle x^3 \rangle_s$	0	0	0
$\langle x^4 \rangle_s$	6	0	$-\frac{6}{5}$

In [19]:
```
laplacian = loadmat("distrib.mat")["laplacian"]
normal = loadmat("distrib.mat")["normal"]
uniform = loadmat("distrib.mat")["uniform"]
```

(a) Apply the following mixing matrix A to the original data s:

```
4 3
2 1
```

In [20]:
```
A = np.array([[4,3],[2,1]])
xl = np.dot(A, laplacian)
xn = np.dot(A, normal)
xu = np.dot(A, uniform)
```

(b) Center the mixed data to zero mean.

In [21]:
```
#keep as pxd
xlc = (xl - np.mean(xl, axis=1).reshape(-1,1)).T
xnc = (xn - np.mean(xn, axis=1).reshape(-1,1)).T
xuc = (xu - np.mean(xu, axis=1).reshape(-1,1)).T
```

(c) Decorrelate the data by applying principal component analysis (PCA) and project them onto the principal components (PCs).

In [46]:
```
#Scatter the centered data, dxd
Sl = np.dot(xlc.T, xlc)
Sn = np.dot(xnc.T, xnc)
Su = np.dot(xuc.T, xuc)

#SOlve the eigensystem
Dl, Ul = np.linalg.eigh(Sl)
Dn, Un = np.linalg.eigh(Sn)
Du, Uu = np.linalg.eigh(Su)

#reverse the orders
Dl = Dl[::-1]
Dn = Dn[::-1]
Du = Du[::-1]
Ul = Ul.T[::-1]
Un = Un.T[::-1]
Uu = Uu.T[::-1]

#Projections
xlp = (np.dot(Ul, xlc.T)).T
xnp = (np.dot(Un, xnc.T)).T
xup = (np.dot(Uu, xuc.T)).T
```

(d) Scale the data to unit variance in each PC direction (now the data is whitened or sphered).

In [47]:
```
#scale to unit variance
xlpuv = xlp / np.std(xlp, axis=0)
xnpuv = xnp / np.std(xnp, axis=0)
xupuv = xup / np.std(xup, axis=0)
```

(e) Rotate the data by different angles θ and calculate the kurtosis1 empirically for each dimension:

```
xθ = | cos(θ)   −sin(θ) | x
     | sin(θ)   cos(θ)  |
θ = 0,π,...,2π
```

```
In [62]: #compute all rotation amounts
         thetas = [(np.pi*x)/50 for x in range(0, 100)]
         #compute rotation matrices
         rs = [np.array([[np.cos(theta), -np.sin(theta)],[np.sin(theta), np.cos(theta)]]) for theta in t
         #compute kurtosises
         kurtls = np.array([np.mean(np.power(np.dot(r, xlpuv.T),4), axis=1)-3 for r in rs])
         kurtns = np.array([np.mean(np.power(np.dot(r, xnpuv.T),4), axis=1)-3 for r in rs])
         kurtus = np.array([np.mean(np.power(np.dot(r, xupuv.T),4), axis=1)-3 for r in rs])
```

```
(100, 2)
```

(f) Find the minimum and maximum kurtosis value for the first dimension and rotate the data accordingly.

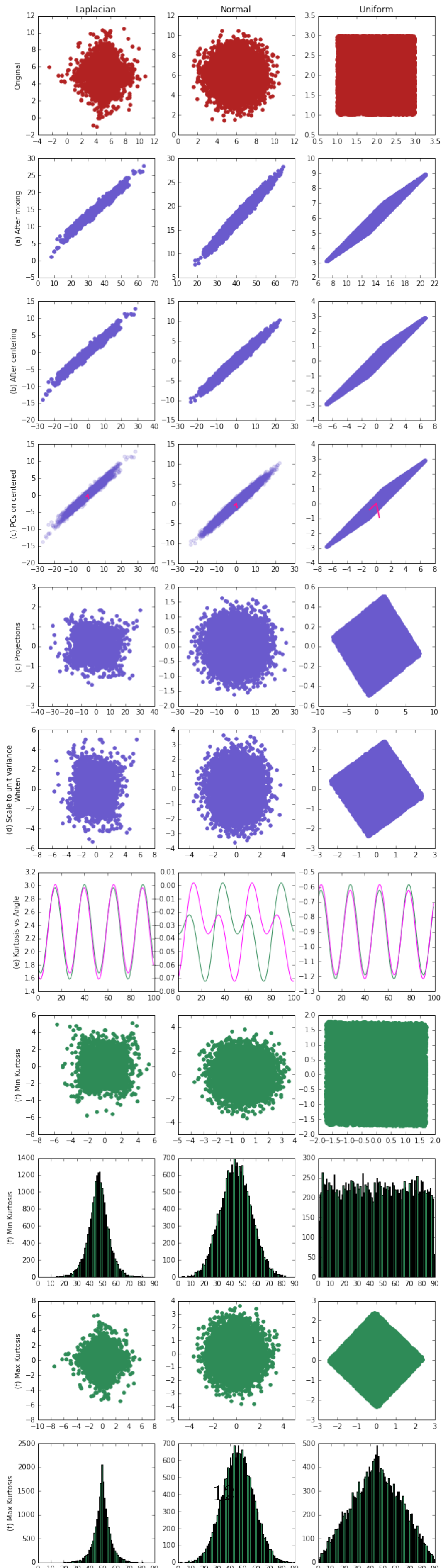
```
In [63]: kurtlminidx = np.argmin(kurtls[:,0])
         kurtlmaxidx = np.argmax(kurtls[:,0])

         kurtnminidx = np.argmin(kurtns[:,0])
         kurtnmaxidx = np.argmax(kurtns[:,0])

         kurtuminidx = np.argmin(kurtus[:,0])
         kurtumaxidx = np.argmax(kurtus[:,0])
```

- Plot the original dataset (sources) and the mixed dataset after the steps (a), (b), (c), (d), and (f) as a scatter plot and display the respective marginal histograms. For step (e) plot the kurtosis value as a function of angle for each dimension.

```
In [65]: bins = 90
         f, axarr = plt.subplots(11, 3)
         f.set_size_inches(10, 39, forward=False)

         r = 0
         axarr[r, 0].set_title("Laplacian")
         axarr[r, 1].set_title("Normal")
         axarr[r, 2].set_title("Uniform")

         axarr[r, 0].set_ylabel("Original")
         axarr[r, 0].scatter(laplacian[0], laplacian[1], color="firebrick")
         axarr[r, 1].scatter(normal[0], normal[1], color="firebrick")
         axarr[r, 2].scatter(uniform[0], uniform[1], color="firebrick")

         r = r+1
         axarr[r, 0].set_ylabel("(a) After mixing")
         axarr[r, 0].scatter(xl[0], xl[1], color="slateblue")
         axarr[r, 1].scatter(xn[0], xn[1], color="slateblue")
         axarr[r, 2].scatter(xu[0], xu[1], color="slateblue")

         r = r+1
         axarr[r, 0].set_ylabel("(b) After centering")
         axarr[r, 0].scatter(xlc.T[0], xlc.T[1], color="slateblue")
         axarr[r, 1].scatter(xnc.T[0], xnc.T[1], color="slateblue")
         axarr[r, 2].scatter(xuc.T[0], xuc.T[1], color="slateblue")
```

```
r = r+1
axarr[r, 0].set_ylabel("(c) PCs on centered")
axarr[r, 0].scatter(xlc.T[0], xlc.T[1], color="slateblue", alpha=.2)
axarr[r, 0].plot([0, Ul[0,0]], [0,Ul[0,1]], color="deeppink", linewidth=2)
axarr[r, 0].plot([0, Ul[1,0]], [0,Ul[1,1]], color="deeppink", linewidth=2)
axarr[r, 1].scatter(xnc.T[0], xnc.T[1], color="slateblue", alpha=.2)
axarr[r, 1].plot([0, Ul[0,0]], [0,Ul[0,1]], color="deeppink", linewidth=2)
axarr[r, 1].plot([0, Ul[1,0]], [0,Ul[1,1]], color="deeppink", linewidth=2)
axarr[r, 2].scatter(xuc.T[0], xuc.T[1], color="slateblue", alpha=.2)
axarr[r, 2].plot([0, Ul[0,0]], [0,Ul[0,1]], color="deeppink", linewidth=2)
axarr[r, 2].plot([0, Ul[1,0]], [0,Ul[1,1]], color="deeppink", linewidth=2)

r = r+1
axarr[r, 0].set_ylabel("(c) Projections")
axarr[r, 0].scatter(xlp.T[0], xlp.T[1], color="slateblue")
axarr[r, 1].scatter(xnp.T[0], xnp.T[1], color="slateblue")
axarr[r, 2].scatter(xup.T[0], xup.T[1], color="slateblue")

r = r+1
axarr[r, 0].set_ylabel("(d) Scale to unit variance\nWhiten")
axarr[r, 0].scatter(xlpuv.T[0], xlpuv.T[1], color="slateblue")
axarr[r, 1].scatter(xnpuv.T[0], xnpuv.T[1], color="slateblue")
axarr[r, 2].scatter(xupuv.T[0], xupuv.T[1], color="slateblue")

r = r+1
axarr[r, 0].set_ylabel("(e) Kurtosis vs Angle")
axarr[r, 0].plot(range(len(kurtls[:,0])), kurtls[:,0], color="seagreen")
axarr[r, 0].plot(range(len(kurtls[:,1])), kurtls[:,1], color="magenta")
axarr[r, 1].plot(range(len(kurtns[:,0])), kurtns[:,0], color="seagreen")
axarr[r, 1].plot(range(len(kurtns[:,1])), kurtns[:,1], color="magenta")
axarr[r, 2].plot(range(len(kurtus[:,0])), kurtus[:,0], color="seagreen")
axarr[r, 2].plot(range(len(kurtus[:,1])), kurtus[:,1], color="magenta")

r = r+1
axarr[r, 0].set_ylabel("(f) Min Kurtosis")
axarr[r, 0].scatter(np.dot(rs[kurtlminidx], xlpuv.T)[0], np.dot(rs[kurtlminidx], xlpuv.T)[1], c
axarr[r, 1].scatter(np.dot(rs[kurtnminidx], xnpuv.T)[0], np.dot(rs[kurtnminidx], xnpuv.T)[1], c
axarr[r, 2].scatter(np.dot(rs[kurtuminidx], xupuv.T)[0], np.dot(rs[kurtuminidx], xupuv.T)[1], c

r = r+1
axarr[r, 0].set_ylabel("(f) Min Kurtosis")
histodata = np.histogram(np.dot(rs[kurtlminidx], xlpuv.T), bins)
axarr[r, 0].bar(np.arange(len(histodata[0])), histodata[0], color="seagreen")
histodata = np.histogram(np.dot(rs[kurtnminidx], xnpuv.T), bins)
axarr[r, 1].bar(np.arange(len(histodata[0])), histodata[0], color="seagreen")
histodata = np.histogram(np.dot(rs[kurtuminidx], xupuv.T), bins)
axarr[r, 2].bar(np.arange(len(histodata[0])), histodata[0], color="seagreen")

r = r+1
axarr[r, 0].set_ylabel("(f) Max Kurtosis")
axarr[r, 0].scatter(np.dot(rs[kurtlmaxidx], xlpuv.T)[0], np.dot(rs[kurtlmaxidx], xlpuv.T)[1], c
axarr[r, 1].scatter(np.dot(rs[kurtnmaxidx], xnpuv.T)[0], np.dot(rs[kurtnmaxidx], xnpuv.T)[1], c
axarr[r, 2].scatter(np.dot(rs[kurtumaxidx], xupuv.T)[0], np.dot(rs[kurtumaxidx], xupuv.T)[1], c
```

```
r = r+1
axarr[r, 0].set_ylabel("(f) Max Kurtosis")
histodata = np.histogram(np.dot(rs[kurtlmaxidx], xlpuv.T), bins)
axarr[r, 0].bar(np.arange(len(histodata[0])), histodata[0], color="seagreen")
histodata = np.histogram(np.dot(rs[kurtnmaxidx], xnpuv.T), bins)
axarr[r, 1].bar(np.arange(len(histodata[0])), histodata[0], color="seagreen")
histodata = np.histogram(np.dot(rs[kurtumaxidx], xupuv.T), bins)
axarr[r, 2].bar(np.arange(len(histodata[0])), histodata[0], color="seagreen")

plt.show()
```


Laplacian
Normal
Uniform
Original
(a) After mixing
(b) After centering
(c) PCs on centered
(c) Projections
(d) Scale to unit variance
Whiten
(e) Kurtosis vs Angle
(f) Min Kurtosis
(f) Min Kurtosis
(f) Max Kurtosis
(f) Max Kurtosis