7.2 The moment generating function is defined as:

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$f(x)$ as probability density

$$\frac{d^k}{dt^k} M_X(t) = \frac{d^k}{dt^k} E(e^{tX}) = E(X^k e^{tX})$$

$$E(X^k e^{tX})\Big|_{t=0} = E(X^k) \quad \text{k-th moment}$$

moment generating functions:

Laplace: $$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}\,dx = \frac{e^{\mu t}}{1-\sigma^2 t^2}$$

Gauß: $$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Uniform: $$M_X(t) = \int_{a}^{b} e^{tx} \frac{1}{b-a}\,dx = \left[\frac{1}{t(b-a)} e^{tx}\right]_a^b = \frac{e^{bt} - e^{at}}{t(b-a)}$$

The centered moment generating function:

$$M_{X_c}(t) = E(e^{t(X-\mu)}) = E(e^{tX}) \cdot E(e^{-t\mu}) = M_X(t) \cdot e^{-t\mu}$$

The standardised moments are then calculated as described in the exercise.

	Laplace(μ, b)	Gauß(μ, σ)	uniform(a, b)
$\{x\}$	μ	μ	$\frac{1}{2}(a+b)$
$\langle x^2 \rangle_c$	$2\sigma^2$	σ^2	$\frac{1}{12}(b-a)^2$
$\langle x^3 \rangle_S$	0	0	0
$\langle x^4 \rangle_S$	6	0	$-\frac{6}{5}$