Background/Review on Integers and Bases (lecture)

ICS312

Machine-Level and Systems Programming

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Numbers and Computers

- Throughout this course we will
  - use binary and hexadecimal representations of numbers
  - need to be aware of the ways in which the computer stores numbers
- So let us go through a simple review before we start learning how to write assembly code
  - Numbers in different bases (these slides)
  - Number representation in computers and basic arithmetic (next set of slides)
    - More to come later on arithmetic

Numbers and bases

- We are used to thinking of numbers as written in decimal, that is, in base 10

  $25 = 2*10^1 + 5*10^0$

  $136 = 1*10^2 + 3*10^1 + 6*10^0$

- Each number is decomposed into a sum of terms
- Each term is the product of two factors
  - A digit (from 0 to 9)
  - The base (10) raised to a power corresponding to the digit's position in the number

$136 = \ldots + 0*10^4 + 0*10^3 + 1*10^2 + 3*10^1 + 6*10^0$

$= \ldots00000136$

  - We typically don't write (an infinite number of) leading 0's

Numbers and Bases

- Any number can be written in base b, using b digits
  - If b = 10 we have "decimal" with 10 digits [0-9]
  - If b = 2 we have "binary" with 2 digits [0,1], which are also called bits
  - If b = 8 we have "octal" with 8 digits [0-7]
  - If b = 16 we have "hexadecimal" with 16 digits [0-9,A,B,C,D,E,F]
- Computers use binary internally
  - It's easy to associate two states to an electrical current
    - Low voltage = 0, high voltage = 1
    - Associating 16 states to a current is more complicated and error-prone
- However, binary is cumbersome
  - The lower the base the longer the numbers!
  - It's really difficult for a human to remember binary
- Therefore we, as humans, like to use higher bases
- Bases that are powers of 2 make for easy translation to binary, and thus are particularly useful, and in particular hexadecimal

Binary Numbers

- Counting in binary:

0_2	0_{10}
1_2	1_{10}
10_2	2_{10}
11_2	3_{10}
100_2	4_{10}
101_2	5_{10}
110_2	6_{10}
111_2	7_{10}
1000_2	8_{10}
…	

- A binary number with d bits corresponds to integer values between 0 and 2^d-1

$$\sum_{k=0}^{d-1} 2^k = 2^d - 1$$

- Example:
  - An integer stored in 8 bits has values between 0 and 255
  - 128+64+32+16+8+4+2+1 = 255

Converting from Binary to Decimal

- We denote by $XXXX_2$ a binary representation of a number and by $XXXX_{10}$ a decimal representation
- Converting from binary to decimal is straightforward:

$$10010110_2 = 1*2^7 + 1*2^4+1*2^2+1*2^1$$
$$= 1*128 + 1*16 + 1*4 + 1*2$$
$$= 150_{10}$$

- The rightmost bit of a binary number is called the least significant bit (smallest power of 2)
- The leftmost non-zero bit of a binary number is called the most significant bit (largest power of 2)
- **If the least significant bit is 0, then the number is even, otherwise it's odd**

Converting from Decimal to Binary

- The conversion proceeds by a series of integer divisions by 2, and by recording the remainder of the division
  - Integer division a/b: a = b* q + remainder, where all are integers
- Example: converting 37_{10} into binary
  - Divide 37 by 2: 37 = 2*18 + 1
  - Divide 18 by 2: 18 = 2*9 + 0
  - Divide 9 by 2: 9 = 2*4 + 1
  - Divide 4 by 2: 4 = 2*2 + 0
  - Divide 2 by 2: 2 = 2*1 + 0
  - Divide 1 by 2: 1 = 2*0 + 1
  - Result: 100101_2
- The least significant bit is computed first
- The most significant bit is computed last
- Note that if we continue dividing, we get extraneous leading 0s
  - …00000100101_2

Binary Arithmetic

- Appending a 0 to the right of a binary number multiplies it by 2
  - $10101_2 = 16_{10} + 4_{10} + 1_{10} = 21_{10}$
  - $101010_2 = 32_{10} + 8_{10} + 2_{10} = 42_{10}$
- Adding two binary numbers is just like adding decimal numbers: using a carry

With no previous carry				With a previous carry			
0	0	1	1	0	0	1	1
+ 0	+ 1	+ 0	+ 1	+ 0	+ 1	+ 0	+ 1
= 0	= 1	= 1	= 0	= 1	= 0	= 0	= 1
			c		c	c	c

Binary Addition

```
          c  c  c  c
             1  0  0  1          9₁₀
     +       1  1  1  1      + 15₁₀
     =    1  1  0  0  0      = 24₁₀
```

```
       c           c  c
          1  0  1  0  0  1  1  0       166₁₀
     +    1  1  0  0  0  0  1  1   + 195₁₀
     = 1  0  1  1  0  1  0  0  1   =  361₁₀
```

Counting in Hexadecimal

$0_{16}=0_{10}$	$A_{16}=10_{10}$	$14_{16}=20_{10}$	$1E_{16}=30_{10}$
$1_{16}=1_{10}$	$B_{16}=11_{10}$	$15_{16}=21_{10}$	$1F_{16}=31_{10}$
$2_{16}=2_{10}$	$C_{16}=12_{10}$	$16_{16}=22_{10}$	$20_{16}=32_{10}$
$3_{16}=3_{10}$	$D_{16}=13_{10}$	$17_{16}=23_{10}$	$21_{16}=33_{10}$
$4_{16}=4_{10}$	$E_{16}=14_{10}$	$18_{16}=24_{10}$	$22_{16}=34_{10}$
$5_{16}=5_{10}$	$F_{16}=15_{10}$	$19_{16}=25_{10}$	$23_{16}=35_{10}$
$6_{16}=6_{10}$	$10_{16}=16_{10}$	$1A_{16}=26_{10}$	$24_{16}=36_{10}$
$7_{16}=7_{10}$	$11_{16}=17_{10}$	$1B_{16}=27_{10}$	$25_{16}=37_{10}$
$8_{16}=8_{10}$	$12_{16}=18_{10}$	$1C_{16}=28_{10}$	$26_{16}=38_{10}$
$9_{16}=9_{10}$	$13_{16}=19_{10}$	$1D_{16}=29_{10}$	$27_{16}=39_{10}$

Converting from hex to decimal

- This is again straightforward

$$A203DE_{16} = 10*16^5 + 2*16^4 + 3*16^2 + 13*16^1 + 14*16^0 = 10{,}617{,}822_{10}$$

Converting from decimal to hex

- Use the same idea as for binary
- Example: convert 1237_{10}
  - 1237 = 77*16 + 5
  - 77 = 4*16 + 13
  - 4 = 0*16 + 4
  - Result: $4D5_{16}$

Hexadecimal addition

	c	
	A 2 3 F	41535_{10}
+	3 D 1 3	+ 15635_{10}
=	D F 5 2	= 57170_{10}

	c c c	
	D 1 F F	53759_{10}
+	A 4 D F	+ 42207_{10}
=	1 7 6 D E	= 95966_{10}

Why is hexadecimal useful?

- We need to think in binary because computers operate on binary quantities
- But binary is cumbersome
- However, hexadecimal makes it possible to represent binary quantities in a compact form
- Conversions back and forth from binary to hex are straightforward
  - Just convert hex digits into 4-bit numbers
  - Just convert 4-bit binary numbers into hex digits

Converting from hex to binary

- Consider $A43FE2_{16}$
- We convert each hex digit into a 4-bit binary number:
  - A_{16}: 1010_2
  - 4_{16}: 0100_2
  - 3_{16}: 0011_2
  - F_{16}: 1111_2
  - E_{16}: 1110_2
  - 2_{16}: 0010_2
- We "glue" them all together:
  - $A43FE2_{16} = 101001000011111111100010_2$
- Important:
  - You must have the leading 0's for the 4-bit numbers, which is what a computer would store anyway
  - It all works because $F_{16} = 15_{10}$, and a 4-bit number has maximum value of $2^4-1 = 15_{10}$

Converting from binary to hex

- Let's convert 1001010101111_2 into hex
- We split it in 4-bit numbers, which we convert separately
- First **we add leading 0's** to have a number of bits that's a multiple of 4:

  0001 0010 1010 1111
- Then we convert
  - 0001_2 : 1_{16}
  - 0010_2 : 2_{16}
  - 1010_2 : A_{16}
  - 1111_2 : F_{16}
- And the result: $1001010101111_2 = 12AF_{16}$

Conclusion

- Hopefully, what we just went through was already solid knowledge for many of you
- But if it wasn't, no worries, just make sure you practice this until it is solid
  - You'd be surprised how many job interviews have a "convert this from/to hex" questions!
- Onward to how computers store numbers…