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How to Count Rabbits:

The evolution of generating functions and their uses

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The idea of generating functions play an important role in the field of Combinatorics. They play a vital role in solving a whole slew of counting problems, and simplify life greatly when solving these problems. The idea of counting is one of the most basic ideas in math and in reality forms the basis for which all math was created. Deservingly so there are quite a few fields devoted to the study of counting and all the interesting properties counting different things can spawn. The field of Combinatorics have a long and colorful history ranging from ancient India, to Euler, to the modern computer era. The field of Combinatorics first found it's creation in ancient India in the text *Bhagabati Sutra* with the following question:

How many ways could one take six tastes one, two, or three tastes at a time?[7]

With that simple sentence written in 300 BC, Combinatorics was born. Throughout the years there were many mathematicians who pioneered the frontier of Combinatorics, but none quite as important as Abraham de Moivre in the 17^{th} century. De Moivre is the man credited with discovering and first using generating functions in combinatorics, more specifically for solving for the solution of the Fibonacci Numbers. He also is credited with first applying the Inclusion-Exclusion Principle which is part of the core for the study of Combinatorics. The generating function is a wonderful tool in Combinatorics and with it all manners of problems can be solved quite easily and in a straight forward manner. As we delve a bit deeper into the ideas of counting and the relations that arise from counting, we shall see the power of these generating functions and all the wonders that they unlock.

1 Counting

One of the most elementary and important ideas of mathematics is counting. Counting is simply stated, the summing up of all similar individuals into a group. The subject of Combinatorics deals heavily with counting, and further expands on this into counting the ways we can take multiple groups and form them into a singular group. Enumerative Combinatorics focuses on the counting of permutations and the counting of combinations. For our purposes the counting of combinations has a heavy hand in the development of generating functions. The idea of counting combinations is the amount of different ways a group of objects can be chosen to form a set. For example if we wanted to count the number of ways to count different colors of m&m's we might run into the following problem:

How many ways can one choose 10 m&m's from a collection of 4 green m&m's, 7 red m&m's, 3 blue m&m's, and 5 yellow m&m's. We have a total $GGGGRRRRRRRBBBYYYYY$ m&m's, where each letter represents the corresponding color. The usual way to do this is using the Inclusion-Exclusion Principle. The Inclusion-Exclusion Principle is defined as follows[6]:

$$|\overline{A_1}\cap\overline{A_2}\cap...\cap\overline{A_n}| = |S|-\sum|A_i|+\sum|A_i\cap A_j|-\sum|A_i\cap A_j\cap A_k|+...+(-1)^n|A_1\cap A_2\cap...\cap A_n|$$

In this equation $|S|$ is the number of possible combinations for choosing S objects. A_i, A_j, and A_k refer to the smaller subsets of S, where i,j,k= $1, 2, ..., n$. $\overline{A_i}$ is the subset of S with certain predetermined constraints upon it relating to how you can choose the objects of set A_i. For our m&m problem we'll set A_1 as the set of green m&m's, A_2 as the red, A_3 as the blue, and A_4 as the yellow. Applying the Inclusion-Exclusion we get the following:

$$|\overline{A_1}\cap\overline{A_2}\cap\overline{A_3}\cap\overline{A_4}| = |S| - (|A_1| + |A_2| + |A_3| + |A_4|)$$

$$+(|A_1\cap A_2| + |A_1\cap A_3| + |A_1\cap A_4| + |A_2\cap A_3| + |A_2\cap A_4| + |A_3\cap A_4|)$$

$$-(|A_1\cap A_2\cap A_3|+|A_1\cap A_3\cap A_4|+|A_1\cap A_2\cap A_4|+|A_2\cap A_3\cap A_4|)+|A_1\cap A_2\cap A_3\cap A_4|$$

$\overline{A_1}$, $\overline{A_2}$, $\overline{A_3}$, and $\overline{A_4}$ are the amount of times each m&m color can appear based on the constraints of each color appearing once, and each color having a finite amount of times it can be chosen. The A_1, A_2, A_3, and A_4 terms refer to the amount of times that each color of m&m might be chosen more than the allotted amount if we were choosing from piles of infinite amounts of m&m's. Continuing on with this method we arrive at the answer of:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = 84 - (10 + 20 + 4 + 0) + 0 - 0 + 0 = 50$$

Where 84 is the number of 10 combinations that can be done using the fact that each color has to appear once, so there are 6 choices up for grabs. $10 + 20 + 4 + 0$ is the sum of the number of single combinations where each color is chosen more than the allowed amount. We see that there are 0 ways to choose multiple colors more than the allotted amount.

Now solving for the amount of combinations using this method is rather clunky and drawn out (remember that non of the actual work using this method was shown here), and leaves one wondering if there are other ways to find the amount of combinations and indeed there are. One method is to assign polynomials to each group of items. Using the Boolean "'or"'[5] we can write each group of m&m's as:

$$G \oplus G^2 \oplus G^3 \oplus G^4$$

$$R \oplus R^2 \oplus R^3 \oplus R^4 \oplus R^5 \oplus R^6 \oplus R^7$$

$$B \oplus B^2 \oplus B^3$$

$$Y \oplus Y^2 \oplus Y^3 \oplus Y^4 \oplus Y^5$$

Where the letter corresponds to the color of m&m and the exponent refers to how many times it was chose. We can link these into a single product to get:

$$(G \oplus G^2 \oplus G^3 \oplus G^4)(R \oplus R^2 \oplus R^3 \oplus R^4 \oplus R^5 \oplus R^6 \oplus R^7)(B \oplus B^2 \oplus B^3)(Y \oplus Y^2 \oplus Y^3 \oplus Y^4 \oplus Y^5)$$

When finding the total amount of combinations we could make by choosing 10 m&m's, the order in which the m&m's are chosen is not all that important. Because of this we can substitute x into the product above, and replace the $\oplus$ with a $+$. This gives us the following expression:

$$(x+x^2+x^3+x^4)(x+x^2+x^3+x^4+x^5+x^6+x^7)(x+x^2+x^3)(x+x^2+x^3+x^4+x^5)$$

When multiplied out we get:

$$x^4+4x^5+10x^6+19x^7+30x^8+41x^9+50x^{10}+55x^{11}+55x^{12}+50x^{13}+41x^{14}+30x^{15}+$$
$$19x^{16} + 10x^{17} + 4x^{18} + x^{19}$$

The theory behind doing all these substitutions is that the coefficients of our x's will be the number of possible ways to choose an amount of m&m's where the amount given by the exponent of x. Indeed if we look at the x^{10} term, the number of ways to choose ten m&m's according to our polynomial expansion is fifty. So the method of finding the coefficients for the desired x gives us the same result as the more traditional method of solving for combinations. This idea can be expanded if we want to choose m&m's from infinite piles and having the ability to not select certain colors. In this case the choices we can make from each pile may be represented as following:

$$G^0 \oplus G^1 \oplus G^2 \oplus G^3 \oplus ... \oplus G^i \oplus ...$$

$$R^0 \oplus R^1 \oplus R^2 \oplus R^3 \oplus ... \oplus R^j \oplus ...$$

$$B^0 \oplus B^1 \oplus B^2 \oplus B^3 \oplus ... \oplus B^k \oplus ...$$

$$Y^0 \oplus Y^1 \oplus Y^2 \oplus Y^3 \oplus ... \oplus Y^l \oplus ...$$

If we want to know how many ways we can choose n mm's we can once again substitute x in for each G, R, B, and Y and change the boolean exclusive "'or"'s to +. This would give us:

$$(1+x+x^2+x^3+...+x^i+...)(1+x+x^2+x^3+...+x^j+...)(1+x+x^2+x^3+...+x^k+...)(1+x+x^2+x^3+...+x^l+...)$$

The number of ways we can choose n m&m's then would be the coefficient of the $x^{i+j+k+l}$ term where $i+j+k+l=n$. The expressions in the parenthesis $(1+x+x^2+...+x^i+...)$ are very important in mathematics and are the most basic form of an infinite series known as the Power Series.

2 Power Series

A power series is an expression represented by the sum $\sum_{i=0}^{\infty} a_i(x-c)^i$, where a_i is a constant and so is c. When expanded this becomes:

$$\sum_{i=0}^{\infty} a_i(x-c)^i = a_0(x-c)^0 + a_1(x-c)^1 + a_2(x-c)^2 + ... + a_i(x-c)^i + ...$$

The series on the right can be thought of as a Taylor series, which can be written as[1]:

$$f(a) + \frac{f'(a)}{1!}(x-c) + \frac{f''(a)}{2!}(x-c)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-c)^n$$

where the function coefficients are already known and are a constant. The first instance of the power series being used was by the Kerala School in India. The founder of this school was Madhavan of Sangamagrama, and it is to him whom the first work with power series is attributed to. The Kerala mathematicians were at the time working on finding ways to calculate values for the trigonometric functions since they were vital in calculations that arose in Astronomy, mostly for the movement of celestial objects. Madhavan was able to construct the taylor series (a specific type of power series that uses the derivatives of a function) for $sin(x)$, $cos(x)$, and $arctan(x)$. Madhavan also discovered the power series expansion for the value of π several centuries before Gottfried Leibniz, one of the fathers of calculus along side Newton. More recently the Leibniz series (the series for π) has been modified in some instances to be named the Madhaven-Leibniz Series in proper recognition of the true discoverer of this series.

For our purposes the most interesting power series is the series known as the generating function which is written as $g(x)$. This series has a value of 0 for c so the form is[6]:

$$g(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 x^0 + a_1 x^1 + a_2 x^2 + ... + a_i x^i + ...$$

The generic power series gives us a specific Taylor series, and similarly the generating function also gives a specific type of Taylor series. The c term in the Taylor series is 0 the series is known as a Maclaurin series, named after Collin Maclaurin. So our generating function then is a Maclaurin series where the functions $f(a)$ and it's derivatives are known. The name generating function comes from the fact that it generates information about an infinite sequence. It might pertain to either the terms of a sequence made by a recurrence relation, or information about a combinatorial problem. It's interesting to note that the value of the x term in our generating functions are really are not all that important, as we are indifferent to whether or not the series converges or not. However the exponent of the x term is very important as it serves as a sort of road sign for our sequence. The tenth term of the sequence will be the coefficient of the $x^1 0$ term. All the information that will be desired is contained within the coefficients of the x's. The x only serves as an indexing tool, a sort of library card for the information we're after. Like power series, generating functions can be operated on like a polynomial. One of the most interesting operations comes from multiplying the generating function for the sequence $1, 1, 1, 1, ...$

by $1-x$.[5] The generating function for the sequence of 1's is $\sum_{i=0}^{\infty} x^i$. When you multiply the two you get the following:

$$(1-x)\sum_{i=0}^{\infty} x^i = (1+x+x^2+...+x^i+...)(1-x)$$

$$(1-x)\sum_{i=0}^{\infty} x^i = (1+x+x^2+...+x^i+...)+(-x-x^2-...-x^i-...)$$

It's fairly clear to see that when those two polynomials are added the answer is equal to 1. Since $(1-x)\sum_{i=0}^{\infty} x^i = 1$ then:

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

In the inversion circle problem Thales This series is the most basic form of the geometric series and it is hugely important when using generating functions to work with recurrence relations. Recurrence relations are a type of infinite sequence where each term is found by performing some operation on the previous terms. The above series is one of the more important geometric series, but there is also another which has great implications when working with generating functions. This geometric series uses Newton's binomial theorem and is the single most useful tool available when solving for the general solution of a recurrence relation using generating functions[6]:

$$\frac{1}{(1-rx)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} r^k x^k$$

One might be wondering what are these recurrence relations, and how do generating functions have anything to do with them? To begin answering this question it's best to start off first with what a recurrence relation is.

3 Recurrence Relations

A recurrence relation is an infinite sequence where each term is found in the following way:

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + ... + a_k h_{n-k} + b_n$$

In this sequence each term h_n is made up by some relation involving the previous terms. In our generic recurrence relation the k term refers to the order of the relation, the a terms are constants which are either given or can be derived from the description of the relation, and the b_n term is either an expression or constant that determines the type of recurrence relation. Recurrence relations come in two forms, linear homogeneous recurrence relations, and non-homogeneous recurrence relations. Linear homogeneous recurrence relations have a b_n term that is equal to zero, while the non-homogeneous recurrence relations have a non-zero b_n term, and this term can range from being a constant to a polynomial with exponential terms. However the b_n term can never involve h_{n-k} term, although it may involve n.

One of the most famous recurrence relations arises from the Fibonacci Numbers. The Fibonacci Numbers are a sequence were first discovered not by Fibonacci but rather by an Indian mathematician named Pingala who was a musical theorist. Pingala noticed a pattern in the meter structure of Indian prose and when describing this repetition with numbers, the first few Fibonacci numbers popped out. However the most famous example of Fibonacci numbers comes from Fibonacci and the problem of reproducing rabbits. Fibonacci proposed this problem in his famous work *Liber Abaci* in 1202 *AD*. The problem presents the following:

A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

This means that at the start of the second month, the pair of rabbits will start producing a new pair every month. At the beginning of two more months the first pair of young will begin to be sexually active as well. We can chart this progression as follows:

Generation	*Mature Pairs*	*Pairs of Young*	*Total*
1	0	1	1
2	1	0	1
3	1	1	2
4	2	1	3
5	3	2	5
6	5	3	8
7	8	5	13
8	13	8	21
9	21	13	34
10	34	21	55

From this table we can create some rules for finding the number of rabbits in a generation. One might notice that the number of mature rabbits in a generation is equal to the sum of mature and young pairs of rabbits from the previous generation. Call this number m_n where m is the amount of pairs of mature rabbits and n is the generation. Let the amount of young pairs be denoted as y_n.

$$m_n = m_{n-1} + y_{n-1}$$

It is also easy to see that the number of young pairs of rabbits in a current generation will be equal to the amount of mature rabbits from the previous generation.

$$y_n = m_{n-1}$$

The total number of rabbits for a generation can be written as:

$$t_n = m_n + y_n$$

where n is the current generation. Then looking again at the number of mature pairs of rabbits one might notice that it is equal to the total of from the previous generation. Then by this same logic the number of young pairs of rabbits in a current generation is equal to the total of rabbits from two generations previous. So we have:

$$t_n = m_n + y_n$$

$$t_n = (m_{n-1} + y_{n-1}) + (m_{n-2} + y_{n-2})$$

$$t_n = t_{n-1} + t_{n-2}$$

This is the relation that describes the Fibonacci Sequence and is usually written as $f_n = f_{n-1} + f_{n-2}$ where f_n is the n^{th} Fibonacci number. From working with the rabbit chart we see that $f_1 = 1$ and $f_2 = 1$, which is important because we have to start at $n > 2$ for the relation to be workable since there is no 0 Fibonacci number nor are there Fibonacci numbers with a negative index. This type of recurrence relation is a linear homogeneous recurrence relation of order 2.

As mentioned earlier there exists another type of recurrence relation besides linear homogeneous recurrence relations, the non-homogeneous recurrence relation. They are similar to linear relations but with one noticeable difference, the b_n term is not equal to zero. This little difference has a noticible impact on the relation especially when trying to solve for the general solution of the relation. One of the more famous recurrence relations of this type is the problem of the Towers of Hanoi.

The Towers of Hanoi tells the story of a group of monks performing a puzzle exercise which the story gets its name from. Before the monks stands three posts, on one post is a stack of sixty four discs stacked from the biggest on bottom to the smallest on top as shown in figure 1, and the goal is to move the entire stack to the third post. The catch is that no disc can have a larger one stacked on top of it. Legend says that when the monks finish moving all sixty four discs to the third post, the world will end.

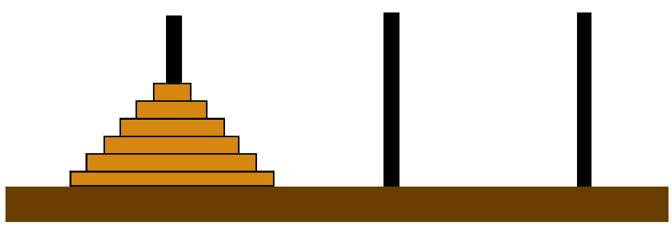

Figure 1

Now one might be confused where recurrence relations play a role in all this, which is quickly cleared up when looking at how many moves it takes to move a certain amount of discs. To move a n amount of discs to the third peg, one first must make the number of moves to get $n-1$ discs to the second peg, then move the largest disc to the third peg, and then move the rest of the discs onto the large one. So we have the number of moves to move $n-1$ discs , then the moving of the large disc, and then a further $n-1$ moves. So the amount of moves total can be represented as:

$$h_n = 2h_{n-1} + 1$$

From what we've learned earlier we can see that the order of the relation is 1, and the b_n term is equal to 1. Also we have a a_1 term that is not equal to 1 something that was not the case in the Fibonacci Sequence. Now that we have a better idea on what recurrence relations are, several questions might be popping up. Is there a way to find the h_n term without having to find all the terms previous to it? If so how would one go about finding it? The answer these questions is yes and yes, with generating functions being the key.

4 Ordinary Generating Functions

"A generating function is a clothesline on which we hang up a sequence of numbers for display."[4] This quote is a nice way to sum up how generating functions are used to work with recurrence relations. We can think of the generating function as a filing cabinet where we store our sequence.[4] To find the n^{th} term in the sequence, one simply finds the n^{th} term in the power series $a_n x^n$, and that coefficient a_n will be the term we are seeking. This means that no member in a sequence is out of our grasp no matter how complex the sequence is, since all we have to do is move down the line on our generating function and look at the coefficient on the corresponding x. To solve recurrence relations with generating functions, we first start by multiplying the entire equation by $\sum_n x^n$.[1] This lets us start manipulating the equation into the form of $g(x) = \sum_n a_n x^n$. Once $g(x)$ is found, we can use geometric series identities to condense the generating function into a sum. The coefficient for the n^{th} term will be the general solution for the recurrence relation. We can revist the Fibonacci Sequence to demonstrate this.

We recall that the Fibonacci Sequence is given as:

$$h_n = h_{n-1} + h_{n-2}$$

With $h_0 = 0$ and $h_1 = 1$. We can rewrite this relation as:

$$h_{n+2} = h_{n+1} + h_n$$

Now to find the generating function for this relation we multiple by x^n and take the sum, as described in the above paragraph.

$$\sum_{n=0}^{\infty} h_{n+2} x^n = \sum_{n=0}^{\infty} h_{n+1} x^n + \sum_{n=0}^{\infty} h_n x^n$$

$$\frac{1}{x^2}\sum_{n=0}^{\infty} h_{n+2}x^{n+2} = \frac{1}{x}\sum_{n=0}^{\infty} h_{n+1}x^{n+1} + \sum_{n=0}^{\infty} h_n x^n$$

The sums of $h_{n+2}x^{n+2}$ and $h_{n+1}x^{n+1}$ are not quite in the form we want for our generating function. In order to fix this we can list these sums out and see what appropriate actions we'll want to take to move forward.

$$\sum_{n=0} h_{n+2}x^{n+2} = h_2x^2 + h_3x^3 + h_4x^4...$$

and

$$\sum_{n=0} h_{n+1}x^{n+1} = h_1x + h_2x^2 + h_3x^3...$$

Recall that $g(x) = h_0 + h_1x + h_2x^2...$ so we can make the following changes:

$$\sum_{n=0} h_{n+2}x^{n+2} = h_0 + h_1x + h_2x^2 + h_3x^3 + h_4x^4... - h_0 - h_1x$$

and

$$\sum_{n=0} h_{n+1}x^{n+1} = h_0 + h_1x + h_2x^2 + h_3x^3... - h_0$$

Since we now have the proper series for a generating function we can move forward and solve for $g(x)$.

$$\frac{1}{x^2}\sum_{n=0}^{\infty} h_{n+2}x^{n+2} = \frac{1}{x}\sum_{n=0}^{\infty} h_{n+1}x^{n+1} + \sum_{n=0}^{\infty} h_n x^n$$

$$\frac{1}{x^2}(g(x) - h_0 - h_1x) = \frac{1}{x}(g(x) - h_0) + g(x)$$

$$g(x) - h_0 - h_1x = x(g(x) - h_0) + x^2g(x)$$

Since we know the values for h_0 and h_1 we can change the equation to:

$$x^2g(x) + xg(x) - g(x) = -x$$

$$(x^2 + x - 1)g(x) = -x$$

$$g(x) = \frac{-x}{x^2 + x - 1}$$

The roots of the denominator are:

$$x = \frac{-1 \pm \sqrt{1^2 + 4}}{2}$$

Using this we can factor the denominator to get:

$$g(x) = \frac{-x}{\left(\frac{-2}{-1+\sqrt{5}}x + 1\right)\left(\frac{2}{-1-\sqrt{5}}x - 1\right)}$$

Now we want to split our fraction into partial fractions in the form of $\frac{c_1}{x+a} + \frac{c_2}{x+b}$:

$$\frac{-x}{\left(\frac{-2}{-1+\sqrt{5}}x + 1\right)\left(\frac{2}{-1-\sqrt{5}}x - 1\right)} = \frac{c_1}{\frac{-2}{-1+\sqrt{5}}x + 1} + \frac{c_2}{\frac{2}{-1-\sqrt{5}}x - 1}$$

$$-x = \left(\frac{2}{-1-\sqrt{5}}x - 1\right)c_1 + \left(\frac{-2}{-1+\sqrt{5}}x - 1\right)c_2$$

$$-x = (c_2 - c_1) + \left(\frac{2c_1}{-1-\sqrt{5}} - \frac{2c_2}{-1+\sqrt{5}}\right)x$$

So we know that:

$$c_2 - c_1 = 0$$

$$\frac{2c_1}{-1-\sqrt{5}} - \frac{2c_2}{-1+\sqrt{5}} = -1$$

It's easy to see that $c_1 = c_2$ so we have:

$$\frac{2c_2}{-1-\sqrt{5}} - \frac{2c_2}{-1+\sqrt{5}} = -1$$

$$\frac{2c_2\left(-1+\sqrt{5}\right) - 2c_2\left(-1-\sqrt{5}\right)}{\left(-1-\sqrt{5}\right)\left(-1+\sqrt{5}\right)} = -1$$

$$\frac{-2c_2 + 2c_2\sqrt{5} + 2c_2 + 2c_2\sqrt{5}}{1+\sqrt{5}-\sqrt{5}-5} = -1$$

$$\frac{4c_2\sqrt{5}}{-4} = -1$$

$$-c_2\sqrt{5} = -1$$

$$c_2 = \frac{1}{\sqrt{5}}$$

So

$$c_1 = \frac{1}{\sqrt{5}}$$

Putting this back into the generating function we get:

$$g(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{\frac{-2}{-1+\sqrt{5}}x+1}\right) + \frac{1}{\sqrt{5}}\left(\frac{1}{\frac{2}{-1-\sqrt{5}}x-1}\right)$$

We want to rearrange the terms into the form of $(1-rx)^{-n}$ to get that special geometric series we discussed earlier:

$$(1-rx)^{-n} = \sum_{k=0}^{\infty}\left(\begin{array}{c} n+k-1 \\ k \end{array}\right) r^k x^k$$

This is done with the following step:

$$g(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{2}{-1+\sqrt{5}}x}\right) + \frac{1}{\sqrt{5}} * \frac{1}{-1}\left(\frac{1}{1-\frac{2}{1-\sqrt{5}}x}\right)$$

$$g(x) = \frac{1}{\sqrt{5}}\sum_{k=0}^{\infty}\left(\begin{array}{c} 1+k-1 \\ k \end{array}\right)\left(\frac{2}{-1+\sqrt{5}}\right)^k x^k - \frac{1}{\sqrt{5}}\sum_{k=0}^{\infty}\left(\begin{array}{c} 1+k-1 \\ k \end{array}\right)\left(\frac{2}{-1-\sqrt{5}}\right)^k x^k$$

$$g(x) = \sum_{n=0}^{\infty} h_n x^n = \frac{1}{\sqrt{5}}\sum_{k=0}^{\infty}\left(\frac{2}{-1+\sqrt{5}}\right)^k x^k - \frac{1}{\sqrt{5}}\sum_{k=0}^{\infty}\left(\frac{2}{-1-\sqrt{5}}\right)^k x^k$$

To find the general solution for the Fibonacci sequence we take the n^{th} coefficient of the sums which are:

$$h_n = \frac{1}{\sqrt{5}}\left(\frac{2}{-1+\sqrt{5}}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{2}{-1-\sqrt{5}}\right)^n$$

We can rearrange the terms taken to n to get a tidier solution of:

$$h_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This general solution is both very powerful and a bit strange at first glance. It is powerful as it lets us find any Fibonacci number without having to know the previous two. It's also a bit strange since the square root terms seem rather out of place in an equation that gives either 0 or a natural number. However the square root terms cancel out when specific values of n are taken.
The true beauty of generating functions though, is that the method never

changes. No matter how many terms or what those terms are in the recurrence relation, solving for the general solution with generating functions will always follow the same predescribed path. Lets look at a non-homogeneous recurrence relation. Earlier we talked about the Towers of Hanoi puzzle where the number of moves needed to move n amount of discs can be written as:

$$h_n = 2h_{n-1} + 1$$

Which has an initial condition of $h_0 = 0$. From here we can find the general solution to the relation:

$$h_{n+1} = 2h_n + 1$$

$$\sum_{n=0}^{\infty} h_{n+1}x^n = 2\sum_{n=0}^{\infty} h_n x^n + \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{x}\sum_{n=0}^{\infty} h_{n+1}x^{n+1} = 2\sum_{n=0}^{\infty} h_n x^n + \sum_{n=0}^{\infty} x^n$$

We can recall from earlier how to change the sum of $h_{n+1}x^{n+1}$ into the appropriate generating function.

$$\frac{1}{x}(g(x) - h_0) = 2g(x) + \frac{1}{1-x}$$

$$g(x) = 2xg(x) + \frac{x}{1-x}$$

$$(1-2x)g(x) = \frac{x}{1-x}$$

$$g(x) = \frac{x}{(1-x)(1-2x)}$$

Once again we will want to split the fraction on the right into partial fractions to get:

$$\frac{x}{(1-x)(1-2x)} = \frac{c_1}{1-x} + \frac{c_2}{1-2x}$$

$$x = (1-2x)c_1 + (1-x)c_2$$

$$x = c_1 + c_2 + (-2c_1 - c_2)x$$

This means that

$$c_1 + c_2 = 0$$

$$-2c_1 - c_2 = 1$$

From above we notice that $c_1 = -c_2$ so we have:

$$-2(-c_2) - c_2 = 1$$

$$c_2 = 1 \quad so \quad c_1 = -1$$

$$g(x) = \frac{-1}{1-x} + \frac{1}{1-2x}$$

$$g(x) = \sum_{n=0}^{\infty} h_n x^n = -1 \sum_{k=0}^{\infty} x^k + \sum_{k=0}^{\infty} 2^k x^k$$

Now we find the coefficients of x when $k = n$ to find that:

$$h_n = (2)^n - 1$$

Now we've seen a little bit of what generating functions can do, we can expand on it. One might still be a little unconvinced on the usefullness of the generating function. Might there be a way to solve for whole groups of recurrence relations instead of solving for each on individually? Let us return then to the original generic recurrence relation and see if we can find a general solution for a whole order class without needing to know any specific values for our a's.

The Fibonacci Sequence is a 2^{nd} order linear homogeneous recurrence relation. The general form of this type of relation is $h_n = a_1 h_{n-1} + a_2 h_{n-2}$. Which has initial conditions of h_0 and h_1. Going through the same motions as before with generating functions lets see if we can find an exact solution to this relation:

$$h_n = a_1 h_{n-1} + a_2 h_{n-2}$$

$$h_{n+2} = a_1 h_{n+1} + a_2 h_n$$

$$\sum_{n=0}^{\infty} h_{n+2} x^n = a_1 \sum_{n=0}^{\infty} h_{n+1} x^n + a_2 \sum_{n=0}^{\infty} h_n x^n$$

$$\frac{1}{x^2} \sum_{n=0}^{\infty} h_{n+2} x^{n+2} = \frac{a_1}{x} \sum_{n=0}^{\infty} h_{n+1} x^{n+1} + a_2 \sum_{n=0}^{\infty} h_n x^n$$

As we've seen earlier, these sums can be changed into $g(x)$.

$$\frac{1}{x^2}(g(x) - h_0 - h_1 x) = \frac{a_1}{x}(g(x) - h_0) + a_2 g(x)$$

$$g(x) - h_0 - h_1 x = a_1 x g(x) - a_1 x h_0 + a_2 x^2 g(x)$$

$$a_2x^2g(x) + a_1xg(x) - g(x) = a_1xh_0 - h_1x - h_0$$
$$(a_2x^2 + a_1x - 1)g(x) = (a_1h_0 - h_1)x - h_0$$
$$g(x) = \frac{(a_1h_0 - h_1)x - h_0}{(a_2x^2 + a_1x - 1)}$$

We can use the quadratic formula to find the roots:

$$x = \frac{-a_1 \pm \sqrt{a_1^2 + 4a_2}}{2a_2}$$

Through a little trial and error, we can find the correct signs in the factorization to give us:

$$g(x) = \frac{(a_1h_0 - h_1)x - h_0}{\left(-1 + \frac{2a_2}{-a_1+\sqrt{a_1^2+4a_2}}x\right)\left(1 - \frac{2a_2}{-a_1-\sqrt{a_1^2+4a_2}}x\right)}$$

From there we change the equation into partial fractions:

$$g(x) = \frac{c_1}{x_1 + a} + \frac{c_2}{x_2 + b}$$

Which gives us:

$$\frac{(a_1h_0 - h_1)x - h_0}{\left(-1 + \frac{2a_2}{-a_1+\sqrt{a_1^2+4a_2}}x\right)\left(1 - \frac{2a_2}{-a_1-\sqrt{a_1^2+4a_2}}x\right)} = \frac{c_1}{\left(-1 + \frac{2a_2}{-a_1+\sqrt{a_1^2+4a_2}}x\right)} + \frac{c_2}{\left(1 - \frac{2a_2}{-a_1-\sqrt{a_1^2+4a_2}}x\right)}$$

We can multiply the denominator of the left side into the whole equation to give us:

$$(a_1h_0 - h_1)x - h_0 = \left(1 - \frac{2a_2}{-a_1 - \sqrt{a_1^2 + 4a_2}}x\right)c_1 + \left(-1 + \frac{2a_2}{-a_1 + \sqrt{a_1^2 + 4a_2}}x\right)c_2$$

Multiplying the right side out and isolating the x results in:

$$(a_1h_0 - h_1)x - h_0 = c_1 - c_2 + \left(\frac{2a_2c_2}{-a_1 + \sqrt{a_1^2 + 4a_2}} - \frac{2a_2c_1}{-a_1 - \sqrt{a_1^2 + 4a_2}}\right)x$$

From this we see that:

$$c_1 - c_2 = -h_0$$
$$c_1 = c_2 - h_0$$

and

$$\frac{2a_2c_2}{-a_1+\sqrt{a_1^2+4a_2}}-\frac{2a_2c_1}{-a_1-\sqrt{a_1^2+4a_2}}=a_1h_0-h_1$$

Using this we can solve for c_2:

$$\frac{2a_2c_2}{-a_1+\sqrt{a_1^2+4a_2}}-\frac{2a_2(c_2-h_0)}{-a_1-\sqrt{a_1^2+4a_2}}=a_1h_0-h_1$$

$$\frac{2a_2c_2\left(-a_1-\sqrt{a_1^2+4a_2}\right)-2a_2\left(c_2-h_0\right)\left(-a_1+\sqrt{a_1^2+4a_2}\right)}{-4a_2}=a_1h_0-h_1$$

$$\frac{-2a_1a_2c_2-2a_2c_2\sqrt{a_1^2+4a_2}+2a_2a_1c_2-2a_2c_2\sqrt{a_1^2+4a_2}-2a_2a_1h_0+2a_2h_0\sqrt{a_1^2+4a_2}}{-4a_2}=a_1h_0-h_1$$

$$\frac{-4a_2c_2\sqrt{a_1^2+4a_2}-2a_2a_1h_0+2a_2h_0\sqrt{a_1^2+4a_2}}{-4a_2}=a_1h_0-h_1$$

$$\frac{2c_2\sqrt{a_1^2+4a_2}+a_1h_0-h_0\sqrt{a_1^2+4a_2}}{2}=a_1h_0-h_1$$

$$2c_2\sqrt{a_1^2+4a_2}+a_1h_0-h_0\sqrt{a_1^2+4a_2=2a_1h_0-2h_1}$$

$$2c_2\sqrt{a_1^2+4a_2}-h_0\sqrt{a_1^2+4a_2=a_1h_0-2h_1}$$

$$(2c_2-h_0)\sqrt{a_1^2+4a_2}=a_1h_0-2h_1$$

$$2c_2-h_0=\frac{a_1h_0-2h_1}{\sqrt{a_1^2+4a_2}}$$

$$c_2=\frac{a_1h_0-2h_1}{2\sqrt{a_1^2+4a_2}}+\frac{h_0}{2}$$

so:

$$c_1=\frac{a_1h_0-2h_1}{2\sqrt{a_1^2+4a_2}}-\frac{h_0}{2}$$

Putting this back into the generating function we get:

$$g(x)=\left(\frac{a_1h_0-2h_1}{2\sqrt{a_1^2+4a_2}}-\frac{h_0}{2}\right)\left(\frac{1}{-1+\frac{2a_2}{-a_1+\sqrt{a_1^2+4a_2}}x}\right)+\left(\frac{a_1h_0-2h_1}{2\sqrt{a_1^2+4a_2}}+\frac{h_0}{2}\right)\left(\frac{1}{1-\frac{2a_2}{-a_1-\sqrt{a_1^2+4a_2}}x}\right)$$

We can pull a -1 out of the denominator of the first term on the right side to get:

$$g(x) = \left(\frac{-a_1h_0 + 2h_1}{2\sqrt{a_1^2 + 4a_2}} + \frac{h_0}{2}\right)\left(\frac{1}{1 - \frac{2a_2}{-a_1 + \sqrt{a_1^2+4a_2}}x}\right) + \left(\frac{a_1h_0 - 2h_1}{2\sqrt{a_1^2 + 4a_2}} + \frac{h_0}{2}\right)\left(\frac{1}{1 - \frac{2a_2}{-a_1 - \sqrt{a_1^2+4a_2}}x}\right)$$

Now we have two terms on the right hand side of the equation that can be simplified into the geometric series:

$$g(x) = \sum_{n=0}^{\infty} h_n x^n = \left(\frac{-a_1h_0 + 2h_1}{2\sqrt{a_1^2 + 4a_2}} + \frac{h_0}{2}\right)\sum_{k=0}^{\infty}\left(\frac{2a_2}{-a_1 + \sqrt{a_1^2 + 4a_2}}\right)^k x^k +$$

$$\left(\frac{a_1h_0 - 2h_1}{2\sqrt{a_1^2 + 4a_2}} + \frac{h_0}{2}\right)\sum_{k=0}^{\infty}\left(\frac{2a_2}{-a_1 - \sqrt{a_1^2 + 4a_2}}\right)^k x^k$$

By taking the n^{th} coefficient of those power series we can find the general solution for h_n:

$$h_n = \left(\frac{-a_1h_0 + 2h_1}{2\sqrt{a_1^2 + 4a_2}} + \frac{h_0}{2}\right)\left(\frac{2a_2}{-a_1 + \sqrt{a_1^2 + 4a_2}}\right)^n + \left(\frac{a_1h_0 - 2h_1}{2\sqrt{a_1^2 + 4a_2}} + \frac{h_0}{2}\right)\left(\frac{2a_2}{-a_1 - \sqrt{a_1^2 + 4a_2}}\right)^n$$

We can make the following changes to the terms being taken to the n^{th} power to tidy them up and get rid of the roots in the denominator:

$$\frac{2a_2}{-a_1 + \sqrt{a_1^2 + 4a_2}} * \frac{-2}{-2} * \frac{-a_1 - \sqrt{a_1^2 + 4a_2}}{-a_1 - \sqrt{a_1^2 + 4a_2}}$$

Which will give us:

$$\frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2}$$

For the other term we do:

$$\frac{2a_2}{-a_1 - \sqrt{a_1^2 + 4a_2}} * \frac{-2}{-2} * \frac{-a_1 + \sqrt{a_1^2 + 4a_2}}{-a_1 + \sqrt{a_1^2 + 4a_2}}$$

$$= \frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2}$$

So a tidier version of the general solution is:

$$h_n = \left(\frac{-a_1 h_0 + 2h_1}{2\sqrt{a_1^2 + 4a_2}} + \frac{h_0}{2} \right) \left(\frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2} \right)^n + \left(\frac{a_1 h_0 - 2h_1}{2\sqrt{a_1^2 + 4a_2}} + \frac{h_0}{2} \right) \left(\frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2} \right)^n$$

The answer we arrived at is pretty incredible when thought about. This means that given any 2nd order linear homogeneous recurrence relation, we already know the general solution to it because of the above relation we found. Let us test this on the Fibonacci Sequence. We'll recall that the Fibonacci Sequence is given as:

$$h_n = h_{n-1} + h_{n-2}$$

With initial conditions $h_0 = 0$. Putting this into our general solution gives us:

$$h_n = \left(\frac{-1 * 0 + 2 * 1}{2\sqrt{1^2 + 4 * 1}} + \frac{0}{2} \right) \left(\frac{1 + \sqrt{1^2 + 4 * 1}}{2} \right)^n + \left(\frac{1 * 0 - 2 * 1}{2\sqrt{1^2 + 4 * 1}} + \frac{0}{2} \right) \left(\frac{1 - \sqrt{1^2 + 4 * 1}}{2} \right)^n$$

$$h_n = \left(\frac{2}{2\sqrt{5}} \right) \left(\frac{1 + \sqrt{5}}{2} \right)^n + \left(\frac{-2}{2\sqrt{5}} \right) \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

$$h_n = \left(\frac{1}{\sqrt{5}} \right) \left(\frac{1 + \sqrt{5}}{2} \right)^n + \left(\frac{-1}{\sqrt{5}} \right) \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

So the general solution does infact work with the Fibonacci Sequence! Let us test it further with a couple more relations:

$$h_n = 2h_{n-1} + 2h_{n-2}$$

Which has the initial conditions of $h_0 = 1$ and $h_1 = 3$. Since we're trying to validate our general solution for all second order relations, the steps involved for solving this particular relation are unimportant, only that the answers are the same. The general solution to this particular relation is:

$$h_n = \frac{2 + \sqrt{3}}{2\sqrt{3}} \left(1 + \sqrt{3} \right)^n + \frac{-2 + \sqrt{3}}{2\sqrt{3}} \left(1 - \sqrt{3} \right)^n$$

Plugging in the values $a_1 = 2$, $a_2 = 2$, $h_0 = 1$, and $h_1 = 3$ into our general solution for all second order linear homogenous relations we get:

$$h_n = \left(\frac{-2*1+2*3}{2\sqrt{2^2+4*2}} + \frac{1}{2}\right)\left(\frac{2+\sqrt{2^2+4*2}}{2}\right)^n + \left(\frac{2*1-2*3}{2\sqrt{2^2+4*2}} + \frac{1}{2}\right)\left(\frac{2-\sqrt{2^2+4*2}}{2}\right)^n$$

$$h_n = \left(\frac{4}{2*2\sqrt{3}} + \frac{1}{2}\right)\left(\frac{2+2\sqrt{3}}{2}\right)^n + \left(\frac{-4}{2*2\sqrt{3}} + \frac{1}{2}\right)\left(\frac{2-2\sqrt{3}}{2}\right)^n$$

$$h_n = \left(\frac{2}{2\sqrt{3}} + \frac{\sqrt{3}}{2\sqrt{3}}\right)\left(1+\sqrt{3}\right)^n + \left(\frac{-2}{2\sqrt{3}} + \frac{\sqrt{3}}{2\sqrt{3}}\right)\left(1-\sqrt{3}\right)^n$$

$$h_n = \left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right)\left(1+\sqrt{3}\right)^n + \left(\frac{-2+\sqrt{3}}{2\sqrt{3}}\right)\left(1-\sqrt{3}\right)^n$$

Indeed the answers are the same, thus meaning that the general solution for all second order linear homogeneous recurrence relations hold up for this relation. Just to hit home that final nail in the coffin where any doubts may lie, lets look at one more relation:

$$h_n = 5h_{n-1} - 6h_{n-2}$$

The initial conditions are given as $h_0 = 1$ and $h_1 = 1$. The general solution to this relation is:

$$h_n = -\left(3^n\right) + 2\left(2^n\right)$$

Using our general solution for all of these relations we have:

$$h_n = \left(\frac{-5*1+2*1}{2\sqrt{5^2-4*6}} + \frac{1}{2}\right)\left(\frac{5+\sqrt{5^2-4*6}}{2}\right)^n + \left(\frac{5*1-2*1}{2\sqrt{5^2-4*6}} + \frac{1}{2}\right)\left(\frac{5-\sqrt{5^2-4*6}}{2}\right)^n$$

$$h_n = \left(\frac{-3}{2\sqrt{1}} + \frac{1}{2}\right)\left(\frac{5+\sqrt{1}}{2}\right)^n + \left(\frac{3}{2\sqrt{1}} + \frac{1}{2}\right)\left(\frac{5-\sqrt{1}}{2}\right)^n$$

$$h_n = \left(\frac{-3+1}{2}\right)\left(\frac{6}{2}\right)^n + \left(\frac{3+1}{2}\right)\left(\frac{4}{2}\right)^n$$

$$h_n = (-1)\,(3)^n + (2)\,(2)^n$$

Thus our general solution for all second order linear homogeneous recurrence relations seems to hold true. It works with a multitude of different relations of this type, all of which have very different solutions and conditions.

The usefulness of the generating function extends beyond solving for combinatorial problems and recurrence relations. One of the most important fields of mathematics in the modern era is the field of statistics, and more specifically the study of chance, or the probability of an action occurring. The field of Probability is hugely important in the world today, most especially in actuarial practices, and various fields within economics. The Probability Generating function is stated as:[1]

$$G(x) = \sum p_k x^k$$

Where p_k is the probability that the k^{th} event will happen. Let us think about rolling a four sided dice. The probability that any one of the four numbers will show face up is $\frac{1}{4}$. So it's probability generating function will be given as:

$$G(x) = \frac{1}{4} + \frac{1}{4}x + \frac{1}{4}x^2 + \frac{1}{4}x^3$$

It's important to remember that p_0 is the first event, which in this case represents 1 being face up. You might notice that all the coefficients add up to 1, which is important because in probability you can never have the chance of any event occurring be more than 100%! We can expand on this by thinking of rolling two four sided dice. The chances of any number i resulting, where i is the sum of the sides of the die that are face up, and $i = 2, 3, ..., 8$ are the coefficients of the corresponding Probability generating function. In this case we will have:

$$G(x) = \frac{1}{16} + \frac{2}{16}x + \frac{3}{16}x^2 + \frac{4}{16}x^3 + \frac{3}{16}x^4 + \frac{2}{16}x^5 + \frac{1}{16}x^6$$

The possibile outcomes for the die roll are $2, 3, 4, 5, 6, 7, 8$ which is represented by x^i with $i = 0, 1, ..., 6$. The probability that any particular sum will result is given by the coefficients of these x terms, and it's easy to see that they all sum to 1. Like how we found particular terms in a sequence of numbers, the probability of any event occurring tucks in nicely into the power series that results from our generating function. This idea of a Probability generating function makes life much easier for statisticians when trying to find specific probabilities for events occurring when the total number of event possibilities is quite large.

Generating functions are an incredible tool when it comes to solving counting problem. Not only do they solve combinatorial problems in a straight forward manner, but the do so in the same few steps every time, no matter what the problem at hand is. The generating function is particularly useful when solving recurrence relations. As we've seen no matter what type the relation is, each is solvable in an easy manner using generating functions. Indeed entire order classes of recurrence relations can be condensed into one equation and solved with generating functions, eliminating the necessity to solve each relation individually. Also we have seen that generating functions while developed for solving counting problems extend much further into other fields of mathematics, and we find them useful for a wide variety of problems.

References

[1] Roberts, Fred S., *Applied Combinatorics*, Prentice-Hall Inc., New Jersey, 1984.

[2] Tucker, Alan, *Applied Combinatorics: Third Edition*, John Wiley & Sons Inc, New York, 1995.

[3] Rosen, Kenneth H., *Discrete Mathematics and Its Applications: Fifth Edition*, McGraw-Hill Higher Education, New York, 2003.

[4] Wilf, Herbert S., *Generatingfunctionology*, Academic Press, Pennsylvania, 1994.

[5] Bogart, Kenneth P., *Introductory Combinatorics: Second Edition*, Harcourt Brace Jovanovich Publishers, 1990.

[6] Brualdi, Richard A., *Introductory Combinatorics: Third Edition*, Prentice-Hall Inc., New Jersey, 1999.

[7] Indian Mathematic, *Wikipedia*, Web. http://en.wikipedia.org/wiki/Indian_mathematics.

[8] Kerala School of Astronomy and Mathematics, *Wikipedia*, Web. http://en.wikipedia.org/wiki/Kerala_School.