Entropy of Random Matchcircuits

We consider a chain of L qubits and associate with each qubit a Jordan-Wigner fermionic mode

$|0\rangle \to |\text{vac}\rangle \; \sigma_j^+ \to (-1)^{n_{<j}} a_j^\dagger \; \sigma_j^- \to (-1)^{n_{<j}} a_j.$

Define Majorana modes

$\gamma_{2n-1} = a_n + a_n^\dagger \; \gamma_{2n} = i(a_n - a_n^\dagger),$

the $2L \times 2L$ correlation matrix of the initial state $|00\ldots0\rangle$ is then given by

$$M_{mn} = \langle \gamma_m \gamma_n \rangle = \begin{pmatrix} 1 & -i & & & \\ +i & 1 & & & \\ & & 1 & -i & \\ & & +i & 1 & \\ & & & & \ddots \end{pmatrix} = \delta_{mn} + i\Gamma_{mn}$$

The goal is to propagate M through a series of p Matchgate layers, i.e. $M[0] \to M[1] \to \cdots \to M[p]$. We will show later how to get the entanglement entropy and spectrum from M in time polynomial in L. The entanglement of the spin state is identical (Peschel and Eisler 2009).

Each layer of gates $r = 1 \ldots p$ can be associated to a fermionic Hamiltonian $H[r] = \sum_{mn} h[r]_{mn} \gamma_m \gamma_n$ with real antisymmetric coefficients $h[r]_{mn} = -h[r]_{nm}$, such that $U[r] = e^{-iH[r]}$, where $U[r]$ contains all gates in layer r and where the Jordan-Wigner transformation is assumed.

Using the Heisenberg picture, one can show that $M[r] = e^{4h[r]} M[r-1] e^{-4h[r]}$ (note that there is a factor of i missing in eq.(4.29) in . If we are interested in random circuits, we need to choose the ensemble to sample from and just picking $h[r]$ real, antisymmetric, uniformly random probably leads to some sort of Haar measure in the gates.

Peschel, Ingo, and Viktor Eisler. 2009. “Reduced Density Matrices and Entanglement Entropy in Free Lattice Models.” *Journal of Physics A: Mathematical and Theoretical* 42 (50): 504003. https://doi.org/10.1088/1751-8113/42/50/504003.