Power of Free QAOA

All code, data, plots and files are available at

https://github.com/henrik-dreyer/GaussianCircuits

01 Preliminaries

We consider a chain of L qubits and associate with each qubit a Jordan-Wigner fermionic mode

$|0\rangle \to |\text{vac}\rangle\ \sigma_j^+ \to (-1)^{n_{<j}} a_j^\dagger\ \sigma_j^- \to (-1)^{n_{<j}} a_j.$

Define Majorana modes

$\gamma_{2n-1} = a_n + a_n^\dagger\ \gamma_{2n} = i(a_n - a_n^\dagger),$

the $2L \times 2L$ correlation matrix of the initial state $|00\ldots 0\rangle$ is then given by

$$M_{mn} = \langle \gamma_m \gamma_n \rangle = \begin{pmatrix} 1 & -i & & & \\ +i & 1 & & & \\ & & 1 & -i & \\ & & +i & 1 & \\ & & & & \ddots \end{pmatrix} =: \delta_{mn} + i\Gamma_{mn}$$

After the Jordan-Wigner mapping and subsequent transformation to Majoranas,

$\sum t_j X_j X_{j+1} \sim \sum h^{XX}(\mathbf{t})_{mn} \gamma_m \gamma_n,$

$$\text{where } h^{XX}(\mathbf{t})_{mn} = \begin{pmatrix} 0 & & & & & & t_L \\ & 0 & t_1 & & & & \\ & -t_1 & 0 & & & & \\ & & & 0 & t_2 & & \\ & & & -t_2 & 0 & & \\ & & & & & \ddots & \\ -t_L & & & & & & \end{pmatrix}$$

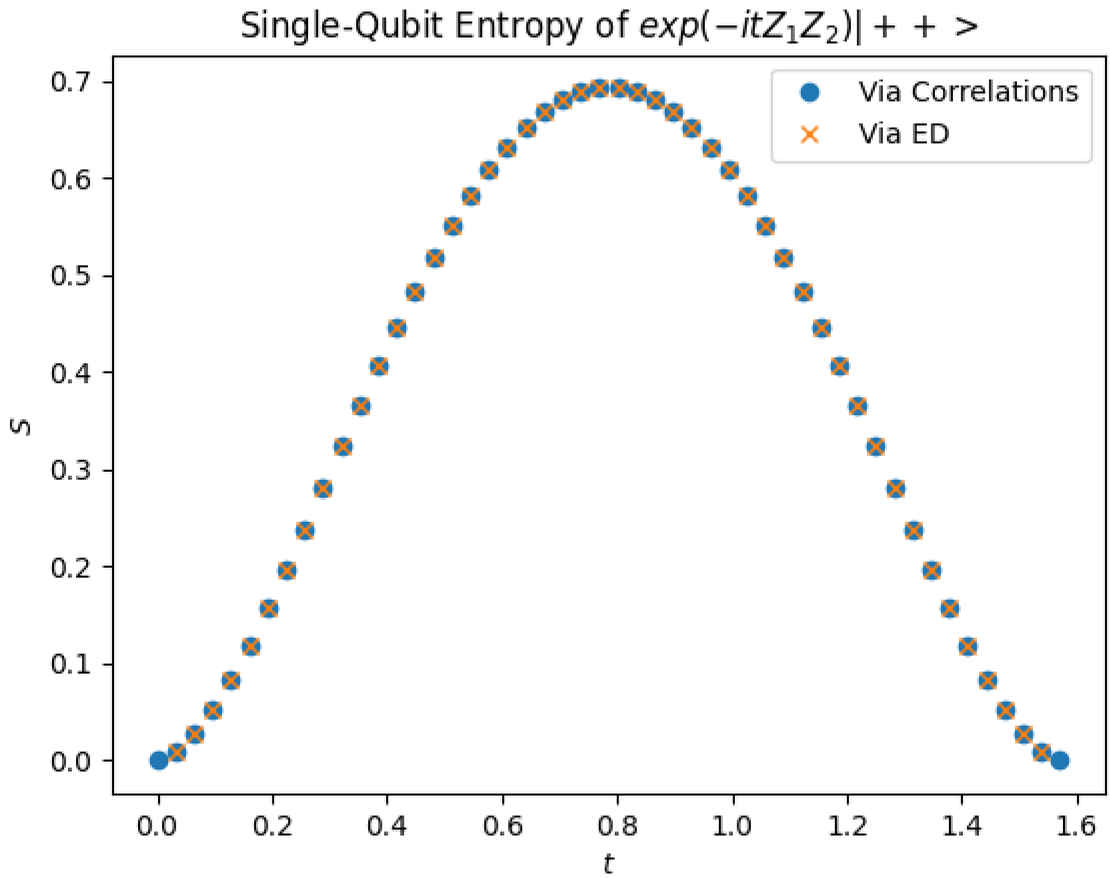
and $t_L = 0$ for open boundaries.

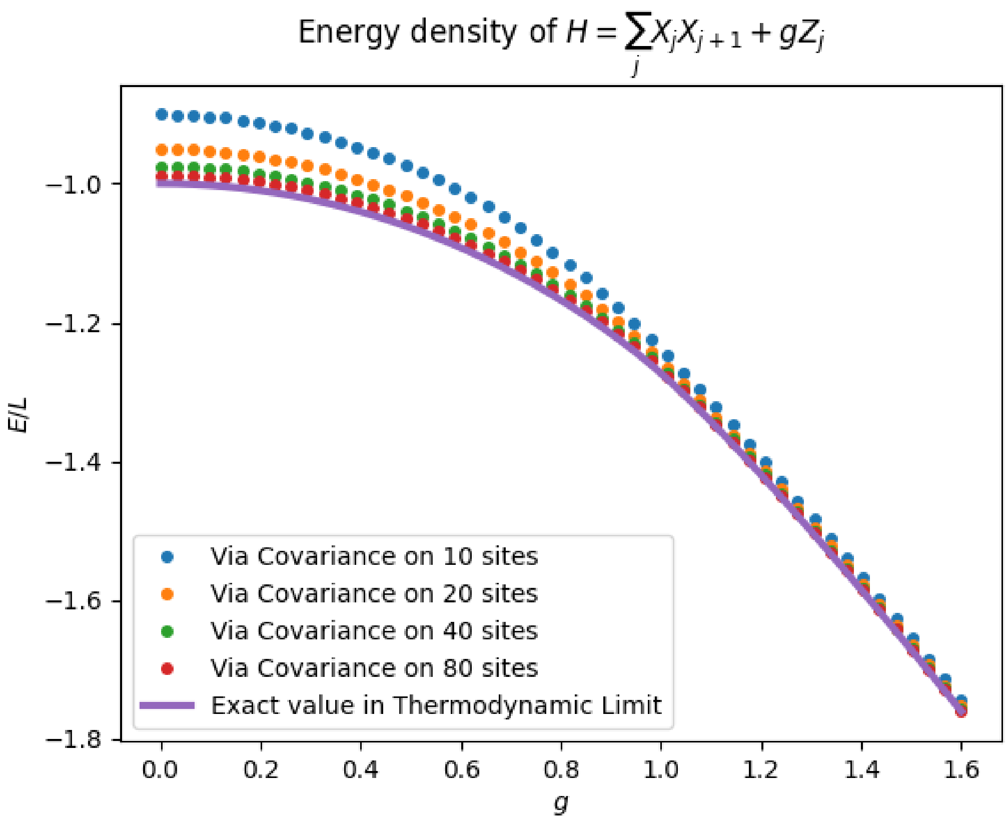
$$\text{Similarly } \sum s_j Z_j \sim \sum h^Z(\mathbf{s})_{mn} \gamma_m \gamma_n \text{ with } h^Z(\mathbf{s})_{mn} = \begin{pmatrix} 0 & s_1 & & & \\ -s_1 & 0 & & & \\ & & 0 & s_2 & \\ & & -s_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Using the Heisenberg picture, one can show that evolution through a layer of gates $U \sim \sum h_{mn}\gamma_m\gamma_n$ keeps the state Gaussian and the covariance matrix evolves as $M' = e^{2h}Me^{-2h}$ (note that there is a factor of $i/2$ missing in eq.(4.29) in .

The energy of the final state with respect to the Ising Hamiltonian $H(\mathbf{t},\mathbf{s}) = \sum_j t_j X_j X_{j+1} + s_j Z_j \sim \sum [h^{XX}(\mathbf{t}) + h^Z(\mathbf{s})]_{mn}\gamma_m\gamma_n$ is given by $\mathrm{Tr}[H(\mathbf{t},\mathbf{s})\Gamma]$ with Γ defined above. That is our cost function.

To check whether implementation is correct, we compute entropy and energy of a propagated state and compare with ED/exact solution:





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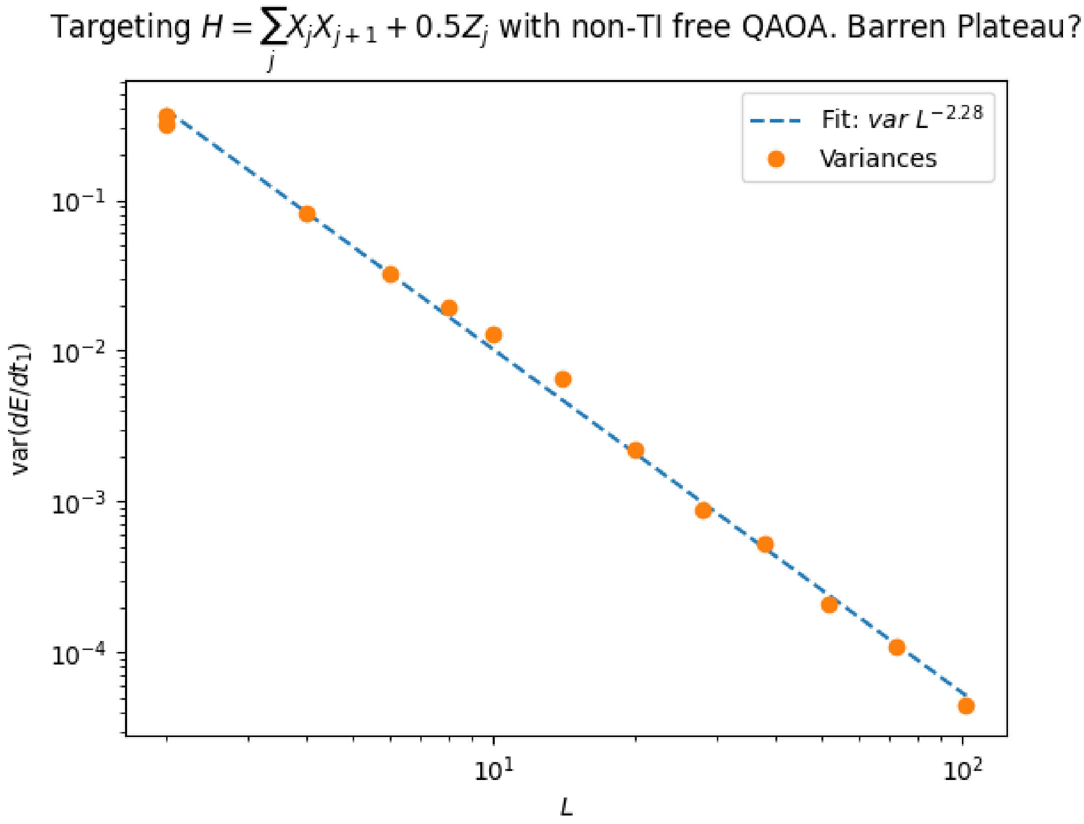
We can now answer the following questions.

02 Trainability

We know translational invariant free QAOA is trainable. Is general free QAOA trainable on the Ising model, or are ther barren plateaus? Taking the gradient (with Jax)

Translational Invariant QAOA



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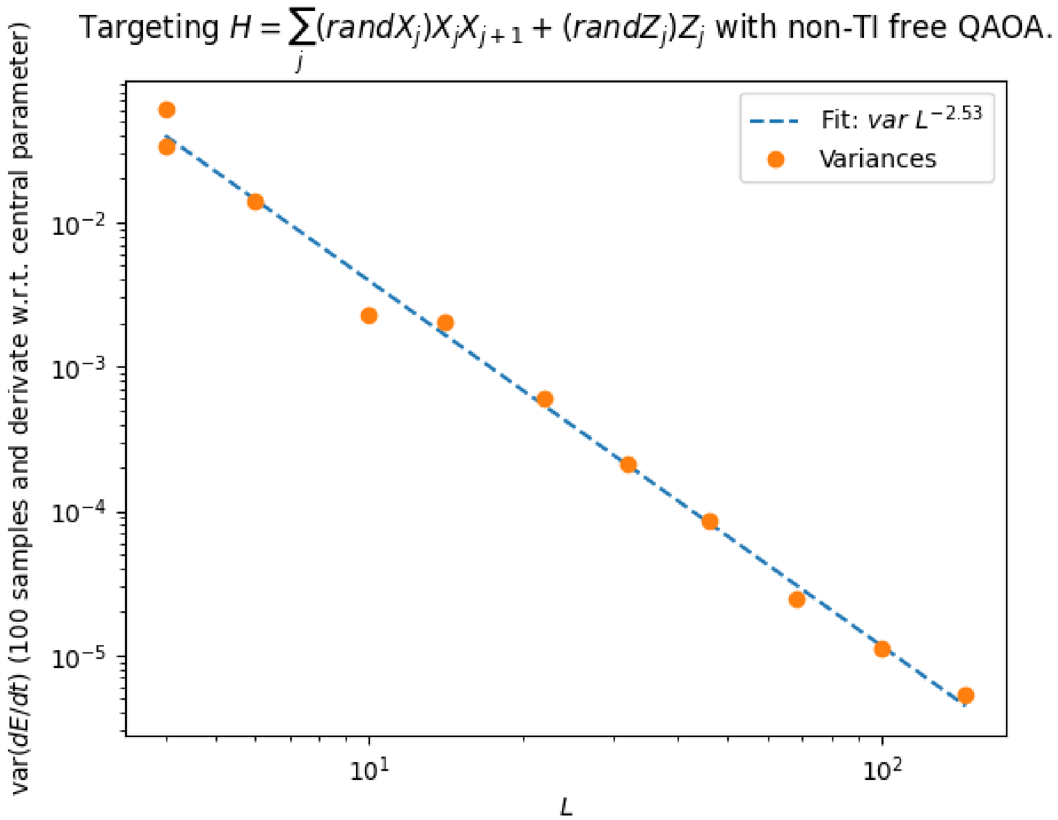
Non-Translational Invariant QAOA



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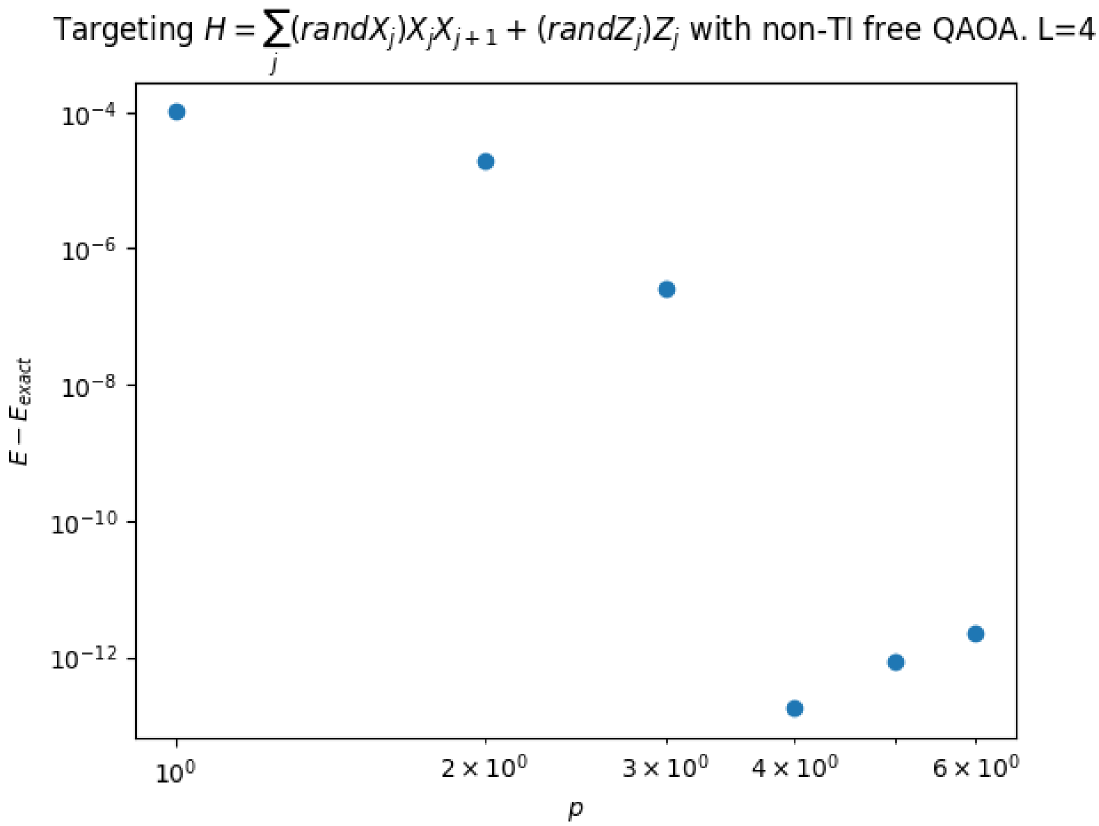
It seems insignificantly harder to train disordered QAOA.

03 Exact Preparability

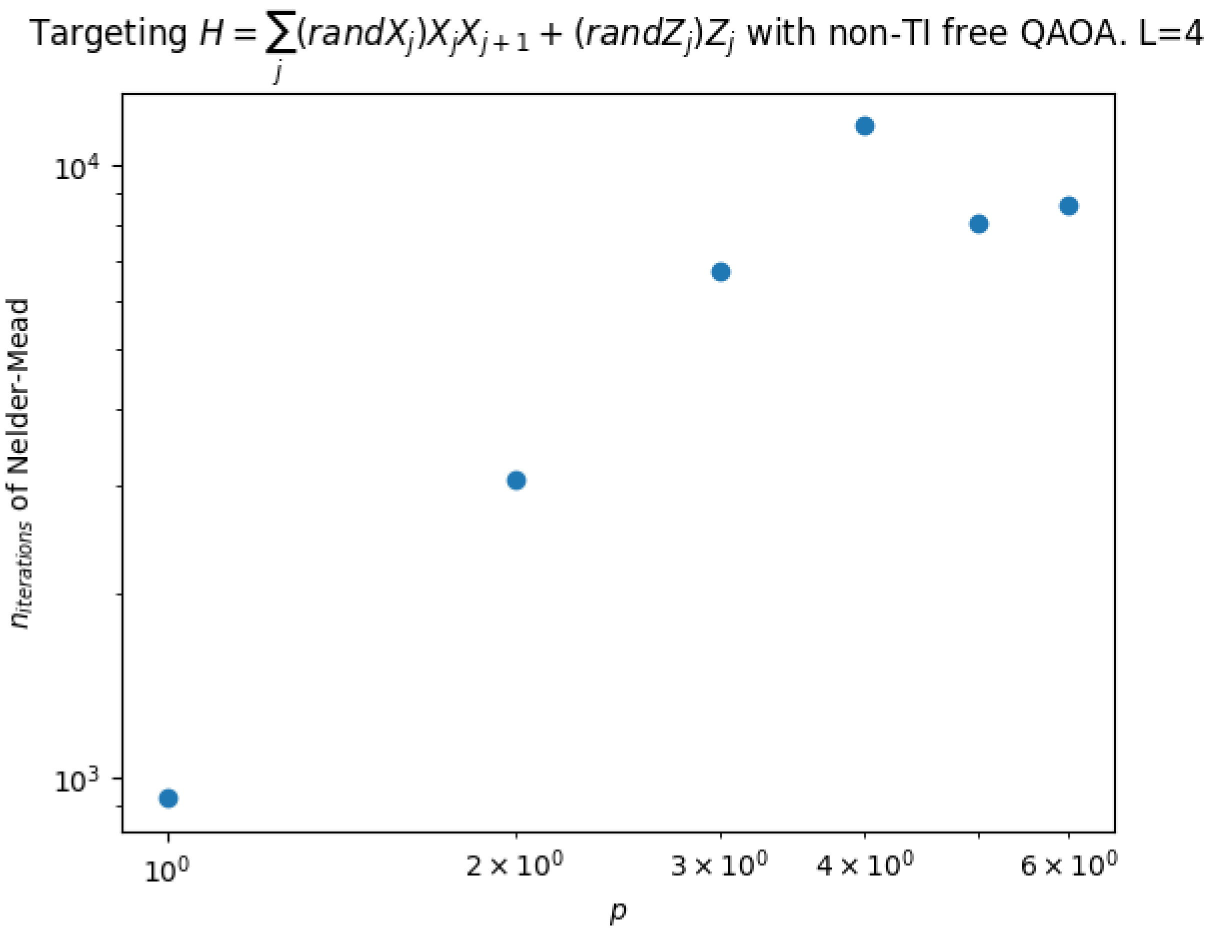
First of all we have to establish what we mean by exact, i.e. what precision we can expect. We know we can find the exact solution with depth $p = L/2$ for translational invariant QAOA on periodic boundary conditions. Let's move to a non-translational invariant random Ising ground state now:

L=4



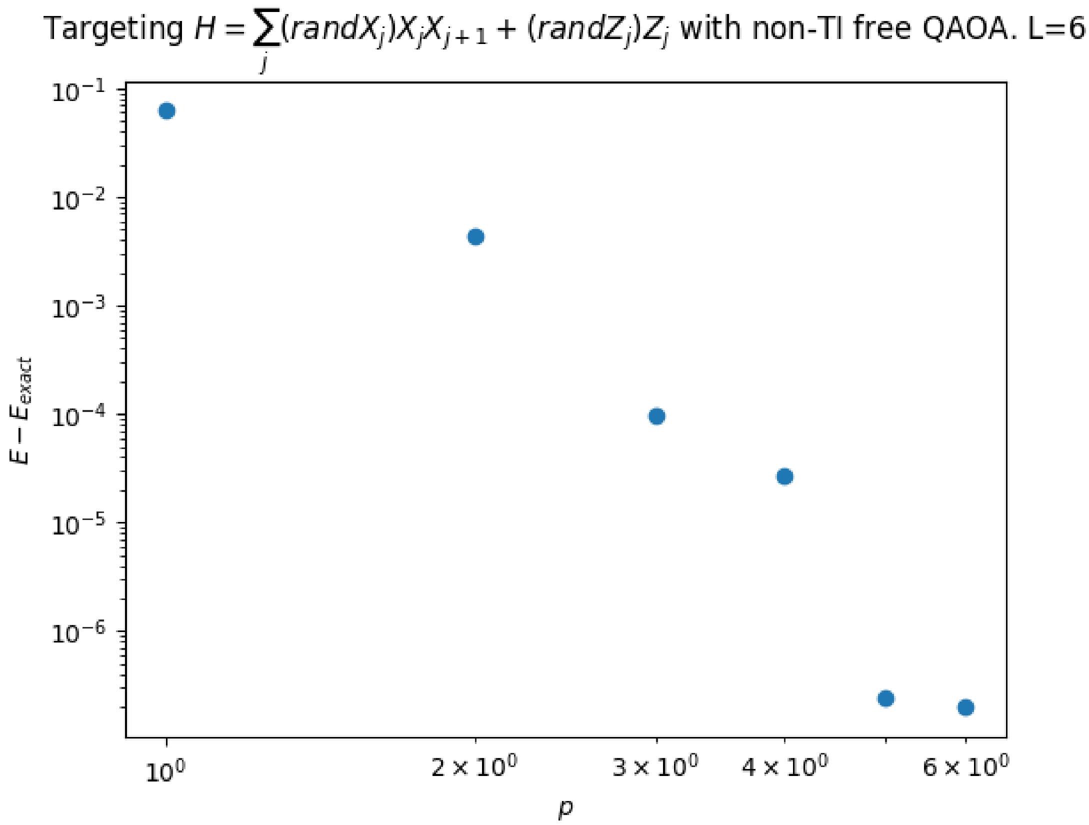
ProducingScript=07_optimization_gradient_free.py_L=4_energy.png

Seems like $p = L$ is enough.



Optimization time is approximately $\mathcal{O}(p^2)$.

L=6



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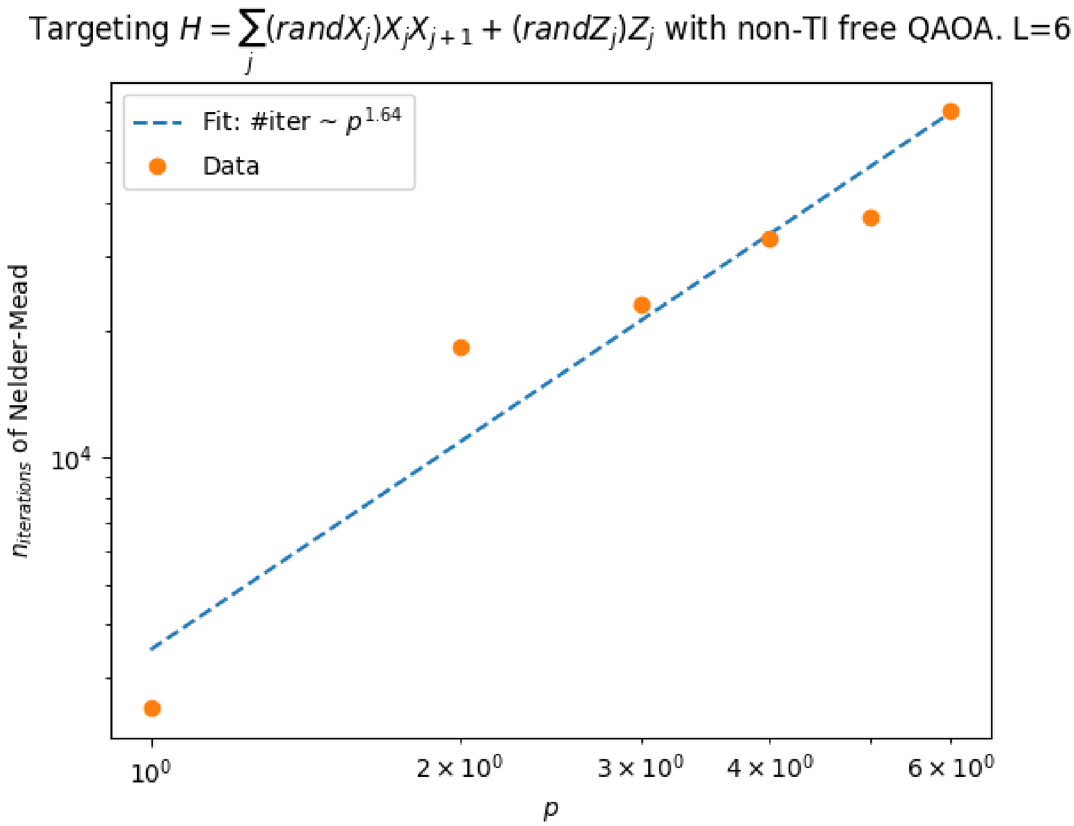
Local minimum?



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04 Open Problems

1. Access larger systems by jiting gradient descent
2. Unlike in my previous work using gradient descent on a TI chain, Nelder-Mead seems to get stuck in local minima ($E(p) - E_{\text{exact}}$ is not smooth).
3. Target excited state
4. Target ground states of next-nearest-neighbour models etc.

References

(Peschel and Eisler 2009)

Peschel, Ingo, and Viktor Eisler. 2009. “Reduced Density Matrices and Entanglement Entropy in Free Lattice Models.” *Journal of Physics A: Mathematical and Theoretical* 42 (50): 504003. https://doi.org/10.1088/1751-8113/42/50/504003.