

# FINE-TUNE YOUR SMILE

## CORRECTION TO HAGAN ET AL

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ABSTRACT. Using results derived in Berestycki *et al.* [1] we correct the celebrated formulae of Hagan *et al.* [5, 4]. We derive explicitly the correct zero order term in the expansion of the implied volatility in time to maturity. The new term is consistent as  $\beta \rightarrow 1$ . Furthermore, numerical simulations show that it reduces or eliminates known pathologies of the earlier formula.

We discuss here<sup>1</sup> the celebrated formulae of Hagan and Woodward [5] and Hagan *et al.* [4], point out some inconsistencies and compare them with the results of Berestycki *et al.* [1]. This leads to a new corrected version of the original formula derived in [4]. This is a short technical paper and the reader is assumed to be familiar with the SABR model and asymptotic expansions of the implied volatility surface. We refer to Gatheral [3, Chp.7] for background.

Consider the following model for the underlying (we take  $r = 0$  which should be interpreted as  $S_t$  representing the forward value process)

$$\begin{cases} dS_t = \sigma_t S_t^\beta dW_t^1, & S_0 = s, \\ d\sigma_t = \nu \sigma_t dW_t^2, & \sigma_0 = \alpha, \end{cases} \quad \text{where } d\langle W^1, W^2 \rangle_t = \rho dt, \quad (1)$$

$\beta \in (0, 1)$ ,  $|\rho| \leq 1$ , which is known as the SABR model. The model (1) is written in the pricing measure under which  $W^1, W^2$  are correlated Brownian motions. We take  $\nu \geq 0$  and note that putting  $\nu = 0$  we obtain a local volatility model. We refer interchangeably to strike  $K$  or to  $x = \ln(s/K)$ . The implied volatility is denoted  $I(x, \tau)$ , i.e. a European call with strike  $K$  and time to maturity  $\tau$  has the same price under the SABR model (1) and under the Black-Scholes model with volatility  $I(x, \tau)$ .

Consider now the Taylor expansion of the implied volatility surface  $I(x, \tau)$  in time to maturity  $\tau$

$$I(x, \tau) = I^0(x) \left( 1 + I^1(x) \tau \right) + O(\tau^2). \quad (2)$$

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	Hagan et al [4]	Berestycki et al [1]	Notation
$I^0(0)$	$\alpha K^{\beta-1}$	$= \alpha K^{\beta-1}$	
$I^0(x) _{\nu=0}$	$\frac{x\alpha(1-\beta)}{s^{1-\beta}-K^{1-\beta}}$	$= \frac{x\alpha(1-\beta)}{s^{1-\beta}-K^{1-\beta}}$	
$I^0(x) _{\beta=1}$	$\nu x / \ln \left( \frac{\sqrt{1-2\rho z+z^2+z-\rho}}{1-\rho} \right)$	$= \nu x / \ln \left( \frac{\sqrt{1-2\rho z+z^2+z-\rho}}{1-\rho} \right)$	$z = \frac{\nu x}{\alpha}$
$I^0(x) _{\beta<1}$	$\nu x \frac{\zeta}{z} / \ln \left( \frac{\sqrt{1-2\rho\zeta+\zeta^2+\zeta-\rho}}{1-\rho} \right)$	$\neq \nu x / \ln \left( \frac{\sqrt{1-2\rho z+z^2+z-\rho}}{1-\rho} \right)$	$z = \frac{\nu}{\alpha} \frac{s^{1-\beta}-K^{1-\beta}}{1-\beta}$ $\zeta = \frac{\nu}{\alpha} \frac{s-K}{(sK)^{\beta/2}}$

TABLE 1. Comparison of  $I^0$  term in [4] and [1].

Hagan *et al.* [4] were first to obtain explicit expressions for  $I^0$  and  $I^1$ . They used perturbation theory (combined with impressive intuition). More recently, Berestycki *et al.* [1] treated the subject in a rigorous analytical manner. They proved in particular that  $I^0(x) = \lim_{\tau \rightarrow 0} I(x, \tau)$  is well defined and characterized it via a solution to eikonal equation (or via Varadhan's signed geodesic distance). Direct, but somewhat tedious, computations allow us to compare their expression for  $I^0$  with that of [4] and of [5] (case of  $\nu = 0$ ). The findings are summarised in Table 1.

The formula for  $I^0(x)$  in the local volatility case  $\nu = 0$  agrees in both works and appears to be correct. We note however that it differs from the formula in Hagan and Woodward [5]. Numerically the difference is small but persistent. Unfortunately, it is the formula in [5] which is often used, as in Bourgade and Croissant [2].

The formula for  $I^0(x)$  when  $\beta = 1$  is the same in [4] and in [1]. However the two papers differ<sup>2</sup> when  $\beta < 1$  and we believe it is the formula of [1] which is correct<sup>3</sup> and should be used. This is natural as authors in [1] present an analytic treatment of the subject and characterize  $I^0$  theoretically. Furthermore, the form of their formula we derived here coincides with the general formula obtained by Labordère [6, Eq. (5.1)] (taken for the case of 0-SABR). We give now two arguments in favor of the formula of [1]: theoretical and numerical.

From the theoretical point of view, the formula of Hagan *et al.* [4] has an internal flaw: it is inconsistent as  $\beta \rightarrow 1$ . Indeed,  $I_\beta^0(x)$  in the formula of [4] does *not* converge to the known value for  $I^0(x)_{\beta=1}$  as  $\beta \rightarrow 1$ .<sup>4</sup> In contrast, as

$$z = z(\beta) = \frac{\nu}{\alpha} \frac{s^{1-\beta} - K^{1-\beta}}{1-\beta} \longrightarrow \frac{\nu x}{\alpha}, \quad \text{when } \beta \rightarrow 1,$$

we have convergence using the formula of [1].

<sup>2</sup>Note there is a typo in (6.5) in [1] which we corrected here.

<sup>3</sup>Strictly speaking Hagan *et al.* [4] do not claim to obtain the Taylor expansion in maturity, however their formula (A.65) is of the form (2).

<sup>4</sup>However, when one uses the simplified formula (2.17a), which is (A.69c), instead of the general formula (A.65), the convergence issue (magically) disappears.

The formula of Hagan *et al.* [4] is known to produce wrong prices in region of small strikes for large maturities. It assigns a negative price to a structure with a positive payoff, or equivalently it implies a negative probability density for the stock price process in some region. If one uses the formula we derive here the problem either appears in yet lower strikes or disappears completely. To see this, we price the following structure in interest rates: long 3% put and short 1% put at three times the notional, whose payoff is a triangle, zero at zero and at 3% and with a peak at 1%. We denote it  $T(1)$ . More generally,  $T(K)$  is the following: long  $(K + 2)\%$  put and short  $K\%$  put at  $(K + 2)/K$  times notional. The payoff of  $T(K)$  is a triangle, zero at zero and at  $(K + 2)\%$ , with a peak at  $K\%$ .

Let  $I_H^0(x)$  and  $I_B^0(x)$  be the formulae for  $I^0$  from [4] and [1] respectively. The implied volatility  $I_H(x)$  in [4, Eq. (A.65)] is given via

$$I_H(x) = I_H^0(1 + I_H^1(x)\tau),$$

$$I_H^1(x) = \frac{(\beta - 1)^2}{24} \frac{\alpha^2}{(sK)^{1-\beta}} + \frac{1}{4} \frac{\rho\nu\alpha\beta}{(sK)^{(1-\beta)/2}} + \frac{2 - 3\rho^2}{24} \nu^2 \quad (3)$$

and we define the implied volatility  $I_B(x) = I_B^0(x)(1 + I_H^1(x)\tau)$ , as we did not derive an explicit formulae for  $I_B^0$  from [1]. An explicit formula for  $I_B^1$ , albeit complicated, is derived in Labordère [6], who also argues that  $I_H^1$  is a valid approximation. Figures 1 – 2 compare the prices of the structure

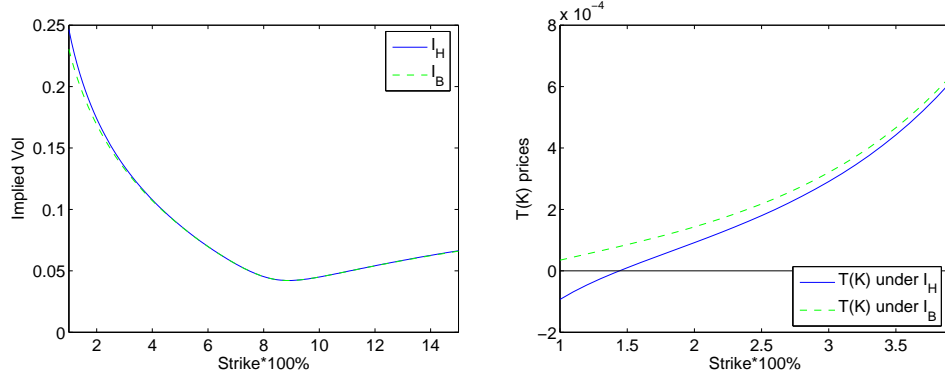


FIGURE 1. Implied volatilities and  $T(K)$  prices for maturity  $\tau = 15$   
 $\rho = -33\%$ ,  $\beta = 0.4$ ,  $ATM = 4.25\%$ ,  $\nu = 25\%$ ,  $s = 8.01\%$

$T(K)$  derived using  $I_H$  and  $I_B$ . The anomalies do not occur, or occur at lower strikes, if we price using the implied volatility  $I_B$  instead of  $I_H$ .

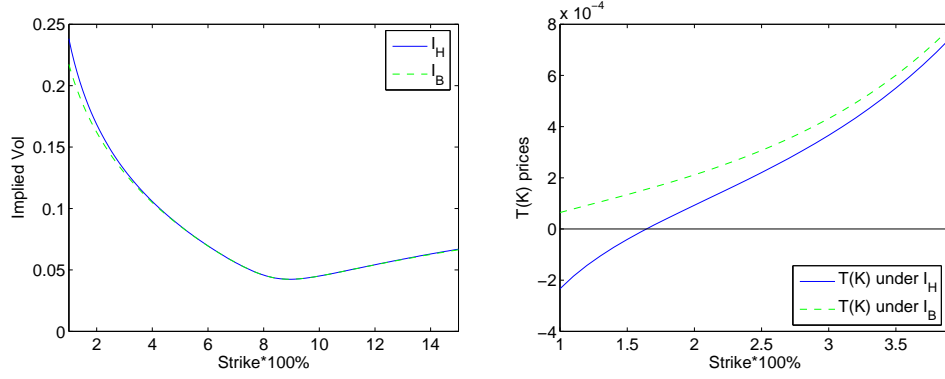


FIGURE 2. Implied volatilities and  $T(K)$  prices for maturity  $\tau = 20$   
 $\rho = -37\%$ ,  $\beta = 0.6$ ,  $ATM = 4.25\%$ ,  $\nu = 24.5\%$ ,  $s = 8.01\%$

### REFERENCES

- [1] H. Berestycki, J. Busca, and I. Florent. Computing the implied volatility in stochastic volatility models. *Comm. Pure Appl. Math.*, 57(10):1352–1373, 2004.
- [2] P. Bourgade and O. Croissant. Heat kernel expansion for a family of stochastic volatility models:  $\delta$  geometry. arXiv:cs/051102, 2006.
- [3] J. Gatheral. *The Volatility Surface*. Wiley, 2006.
- [4] P. S. Hagan, D. Kumar, A. S. Lesniewski, and D. E. Woodward. Managing smile risk. *Wilmott Magazine*, pages 84–108, July 2002.
- [5] P. S. Hagan and D. E. Woodward. Equivalent Black volatilities. *Applied Math Finance*, 6:147–157, 1999.
- [6] P. H. Labordère. A general asymptotic implied volatility for stochastic volatility models. arXiv:cond-mat/0504317, 2005.

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