Fitting the smile
Smart parameters for SABR and Heston

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Abstract

In this paper we revisit the problem of calibrating stochastic volatility models. By finding smart initial parameters, we improve robustness of Levenberg-Marquardt. Applying this technique to the SABR and Heston models reduces calibration time by more than 90% compared to global optimization techniques such as Simplex or Differential Evolution.

Keywords: Stochastic volatility, SABR, Heston, Smile volatility, Calibration, Optimization.

1 Introduction

In order to model the smile efficiently, stochastic volatility models are a popular approach. They enable to have distinct processes for the stock return and its variance. Thus, they may generate volatility smiles. Moreover, if the variance process embeds a mean reversion term, these models can capture the term-structure in the variance dynamics. Popular stochastic volatility models include Heston [4], SABR [3] and square-root models to name but a few.
These advanced models often involve quite a number of parameters, which enable to fit the volatility smile. Calibrating stochastic volatility models to the smile presents some challenging features. A very efficient and robust optimization algorithm should be chosen. Practitioners often have the choice between local optimization algorithms (such as Levenberg-Marquardt) and global optimization algorithms (such as Simplex and Differential Evolution). The former

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are very efficient but may be stuck into local minima if the search space is very large. On the contrary, the latter are quite robust but are much slower to converge.
Through this article, we propose to increase convergence speed and robustness of local optimization algorithms by plugging ad-hoc initial values.

2 Smart parameters

Local optimization algorithms, such as Levenberg-Marquardt, demonstrate great efficiency provided that initial values enable to avoid being trapped in local minima and valleys. Due to its dimensionality, the calibration problem presents these annoying features. As a result, local optimization algorithms only explore a small subset of the search space. Our leading idea is to find initial parameters such that the initial volatility is similar to the target volatility. Thus, a satisfying solution (either a global minimum or a good enough local minimum) may be located in the search space of the local optimization algorithm. Moreover, the algorithm is very likely to converge to this solution in a few iterations.

We will show how to find smart parameters for two popular models and exhibit the benefits of this technique in terms of computational speed.

3 Implied volatility

Calibration of stochastic volatility models aims at reproducing the market implied volatility smile as closely as possible. It may be convenient to have the closest match on ATM volatility for various reasons. First, ATM data is more liquid hence more reliable. Second, pricing of most derivatives depend highly on ATM volatility. Eventually, analytical formulas may have simpler and more accurate expressions at ATM points. In order to match the smile shape, two methods can be thought of, either matching volatility around the money (figure 1) or at two strikes (figure 2). We will demonstrate both methods by studying the SABR and Heston models.

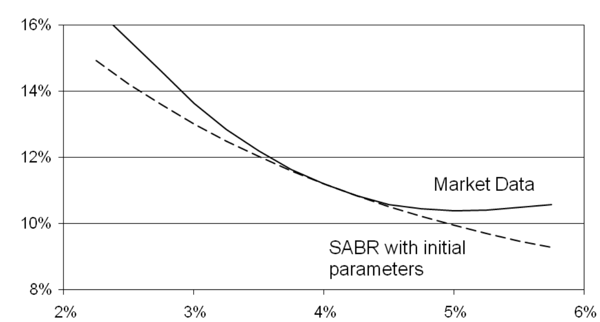


Figure 1: Fitting the volatility around the money – SABR Model

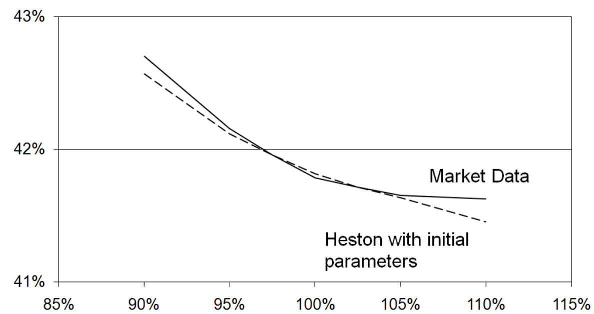


Figure 2: Fitting the volatility at two strikes (97.5% and 102.5%) – Heston Model

4 SABR model: smile fitting around the money

4.1 SABR Model

The SABR model belongs to the large class of stochastic volatility models. It has been originally introduced by Hagan et al. [3] and has become very successful since then, in particular in the interest rates derivatives world. The SABR model assumes CEV dynamics for the forward F and lognormal dynamics for its volatility σ:

$$dF_t = \sigma_t F_t^{\beta} dW_t \tag{1}$$
$$d\sigma_t = \nu \sigma_t dZ_t \tag{2}$$

where W_t and Z_t are two Wiener processes with correlation ρ, $\nu \geq 0$ is the volatility of volatility, $\alpha = \sigma_0$ is the initial volatility and $0 \leq \beta \leq 1$. The special case $\beta = 1$ leads to a lognormal diffusion for the forwards while $\beta = 0$ leads to a normal diffusion.

The SABR model does not have any exact explicit formula for equivalent Black implied volatility but following [3] many approximation formula have been proposed [6, 7, 8]. Actually, as discussed in part 6.1, this choice has little implication on the procedure results, therefore we will use the one given in [6] as it has a analytical tractable form:

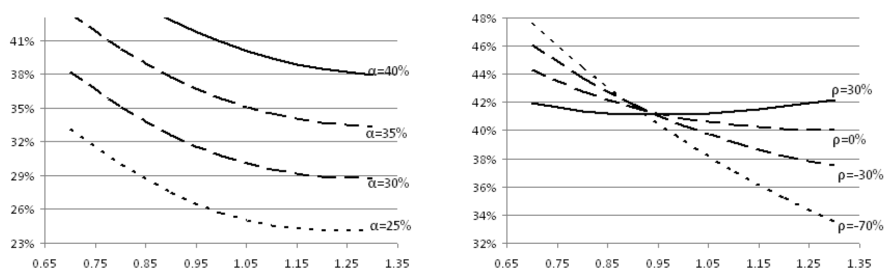
$$I(\alpha, \beta, \nu, \rho, k) = I_0(k)\left(1 + I_1(k)T\right) + \mathcal{O}(T^2) \tag{3}$$

where

$$
\begin{cases}
I_0(k) &= \dfrac{-\nu \log k}{\log \dfrac{z-\rho+\sqrt{1+z^2-2z\rho}}{1-\rho}} \\
I_1(k) &= \frac{1}{24}k^{-1+\beta}\alpha^2(-1+\beta)^2 + \frac{1}{4}k^{\frac{1}{2}(-1+\beta)}\alpha\beta\nu\rho + \frac{1}{24}\nu^2\left(2-3\rho^2\right) \\
z &= \dfrac{\left(1-k^{1-\beta}\right)\nu}{\alpha(1-\beta)}
\end{cases}
\tag{4}
$$

4.2 Influence of parameters

Alpha Increasing the initial volatility level α shifts the implied volatility smile upwards while it has little effects on the shape of the smile (see figure 3(a)).



(a) Initial volatility α

(b) Correlation ρ

Figure 3: Influence of SABR parameters (1/2)

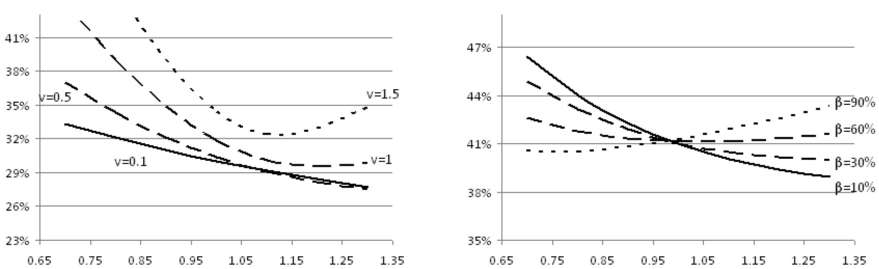
Correlation The correlation parameter, ρ changes the slope of the implied volatility enabling the model to fit skew or smile volatilities (see figure 3(b)).

Volatility of Volatility The main effect of the ν parameter is to increase the curvature of the curve from close to linear to highly convex.

Beta From figure 4(b), beta effect on the smile is similar to the correlation. However, as explained in [3], it plays other roles which overweight the curvature effect (see figure 3(b)).

4.3 Initial values

We are looking for values of α, β, ν and ρ such that the implied smile volatility has the same value and slope at the money as the market volatility.



(a) Volatility of volatility ν

(b) Beta β

Figure 4: Influence of SABR parameters (2/2)

Fitting β As the SABR model is over parametrized, the β is used to fit the backbone of the smile or the practitioner's beliefs on the market (log)normality [3].

Solving for ρ

$$I(\alpha, \beta, \nu, \rho, f) = \sigma_{ATM} \tag{5}$$

This implies that ρ is the root of a second degree polynomial equation. We'll denote the two solutions as $\rho_-(\alpha, \beta, \nu, k, \sigma_{ATM}) \leq \rho_+(\alpha, \beta, \nu, k, \sigma_{ATM})$.

Fitting the slope of the volatility at the money produces the following equation for ρ_+

$$\left.\frac{\partial I\left(\alpha, \beta, \rho_+(\alpha, \beta, \nu, k, \sigma_{ATM}), \nu, k\right)}{\partial k}\right|_{k=f} = \left.\frac{\partial \sigma}{\partial k}\right|_{k=f}, \tag{6}$$

and another similar equation with ρ_- :

$$\left.\frac{\partial I\left(\alpha, \beta, \rho_-(\alpha, \beta, \nu, k, \sigma_{ATM}), \nu, k\right)}{\partial k}\right|_{k=f} = \left.\frac{\partial \sigma}{\partial k}\right|_{k=f}, \tag{7}$$

Those equations have only two unkwowns left: α and ν. Any zero that satisifies the following conditions will produce an acceptable smile (i.e. matching the value and the slope of volatility at the money):

- $\alpha > 0$
- $\nu > 0$
- $-1 < \rho < 1$ (implies than equation 5 has real solutions)

Those zeroes can be found using a two-dimensional solver, however we have devised a simple heuristic to quickly find zeroes of equation 6 that relies on simple observation of the various equations:

Lower bound on α To ensure that equation 5 has at least one real root, we have to satisfy the following constraint on α:

$$s^{3\beta}t\left(1-2\beta+4\beta^2\right)\alpha^3+2s^{2+\beta}\left(12+t\nu^2\right)\alpha-24s^3\sigma_{ATM}\geq 0. \tag{8}$$

From equation 8, we can deduce the minimum value of α, i.e. the first positive root of the third-degree polynomial in α.

More bounds on α As ρ lies within $[-1,+1]$ and $\rho \to I(\alpha,\beta,\nu,\rho,f)$ is a parabola with negative leading coefficient, we deduce two conditions on α (see figure 5):

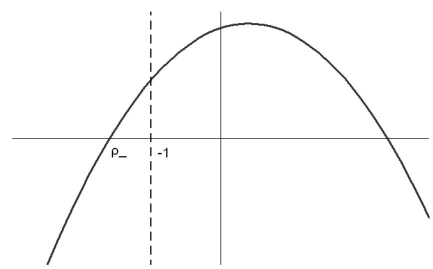


Figure 5: Upper bound on α: $I(\alpha,\beta,\nu,-1,f)>0$ implies $\rho_- < -1$

$$\rho_+ \leq 1 \Longleftrightarrow I(\alpha,\beta,\nu,1,f)-\sigma_{ATM}\leq 0, \tag{9}$$

$$\rho_- \geq -1 \Longleftrightarrow I(\alpha,\beta,\nu,-1,f)-\sigma_{ATM}\leq 0. \tag{10}$$

Any of those two equations yields an upper bound on acceptable value of α.

Finding an appropriate ν To solve equation 6, we first select a value of ν_0 then deduce the bounds in which α lies using equation 9 (or equation 8). Using a root-finding algorithm, we try to find a solution. If we fail, we select a new value of ν based on:

- if $\min_\alpha \left.\frac{\partial I(\alpha,\beta,\rho_+,\nu,k)}{\partial k}\right|_{k=f} > \left.\frac{\partial\sigma}{\partial k}\right|_{k=f}$, then $\nu_0 \leftarrow \nu_0\,/\,1.2$
- if $\max_\alpha \left.\frac{\partial I(\alpha,\beta,\rho_+,\nu,k)}{\partial k}\right|_{k=f} < \left.\frac{\partial\sigma}{\partial k}\right|_{k=f}$, then $\nu_0 \leftarrow \nu_0 \times 1.2$.

If solving equation 7, the selection of updated values for ν is reversed.
By trying several values of ν, we can find at least on solution (α_0,ν_0) from which we deduce ρ_0. $(\alpha_0,\beta_0,\nu_0,\rho_0)$ yields a SABR model which match the volatility and the slope of the volatility at the money.
From our experiments, this heuristic for solving equations 6 and 7 has proven both fast and reliable.

4.4 Numerical results

We calibrated SABR models on a swaption volatility cube (6 expiries, 4 maturities and 9 strikes per volatility smile) using a simplex algoritm, a differential evolution algorithm and Levenberg-Marquardt with our smart parameters methodology. The target precision for the sum of the squared errors on implied volatility was set to $4e-6$ (for each expiry/maturity couple). In table 1, we show the time needed to calibrate the 24 independant SABR models and the maximum number of iterations required.

In this example, which from our experience is fairly typical, the Levenberg-Marquardt with Smart Parameters technique reduce calibration time by more than 90% compared to global optimization techniques. It is worth noticing that the Levenberg-Marquardt requires at most 5 iterations to converge as the initial parameters produce a smile close to the target smile.

Method	**Time (total)**	**Iterations (per smile)**
DE	0.26 s	≤ 120
Simplex	0.15 s	≤ 1300
Smart Parameters	≤ 0.001 s	-
Smart LM	0.01 s	≤ 5

Table 1: Comparison of calibration strategies for SABR

5 Heston model: smile fitting at 2 strikes

Our technique suits SABR model remarkably well since implied volatility can be written explicitly as a function of SABR parameters. However, this is not necessarily the case in other models. For instance, regarding Heston model, the call price is expressed as an integral of some complex function. There is no way to have a nice and robust formula for the implied volatility from this formula. In such a case, we will have to fit the call prices instead of the volatilities.

5.1 Heston model

In his eponym model that he introduced in 1993, Heston assumes lognormal dynamics for the spot and Cox-Ingersoll-Ross dynamics for the variance:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t \tag{11}$$

$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t} dZ_t \tag{12}$$

where the two Wiener processes are correlated with constant correlation ρ, r is the risk-free rate, θ is the long-term variance, κ is the mean reversion, ξ is the volatility of volatility and V_0 the initial variance.

As there is no explicit formula that gives the implicit volatility in Heston model, we have chosen to conduct our analysis on the call price. A widely used formula is taken from Lewis ([5]):

$$C(K,T) = S_0 - \frac{1}{\pi}\sqrt{K}e^{-\frac{rT}{2}} \int_0^{\infty} \mathcal{R}e \left[e^{iuk}\Psi\left(u - \frac{i}{2}\right)\right] \frac{du}{u^2 + \frac{1}{4}}. \qquad (13)$$

where $k = rT - \log K$.

This formula has been derived by means of Fourier techniques for option pricing, introduced by Carr and Madan [2]. Even though this formula is very robust and reliable, it doesn't serve well our purpose. We want indeed to somehow inverse the call price formula: inversing such an integral is surely not an easy task.

Using expansion techniques, Benhamou, Gobet and Miri [1] obtain an accurate approximation for Heston call price that does not involve any integral. Thus, it may better serve our purpose since we want to inverse the formula in terms of Heston parameters. The Heston call price is expressed as a weigthed sum of the Black formula and its derivatives:

$$C_{Hes} = C_{BS}[x, var_T] + \sum_{i=1}^{2} a_{i,T} C_{BS}^{(i,1)}[x, var_T] + \sum_{i=0}^{1} b_{2i,T} C_{BS}^{(2i,2)}[x, var_T]. \quad (14)$$

5.2 Influence of parameters

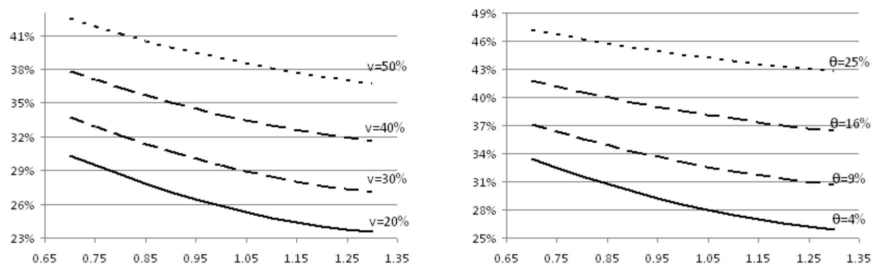
First of all, it is important to understand the meaning of Heston parameters. Thus, we will understand the role played by each parameter when fitting the implied volatility smile. We can already remark that this problem is certainly overparameterized. While the market quotes implied volatility for more or less 7 strikes, some of the 5 parameters may be redundant. Therefore, the objective of this short study will be to identify unnecessary degrees of freedom [1].

Initial Volatility Increasing the initial volatility level $\sqrt{v_0}$ moves the implied volatility smile upwards. This behavior is quite intuitive hence we will not comment it much. However, it is important to note that there is neither a 1-to-1 nor a linear relation (see figure 6(a)).

Long-Term Volatility As a matter of fact, long-term variance θ and initial volatility have a similar influence upon the implied volatility smile. Hence, when performing a global optimization of Heston model, it is certainly useless (and even counter-productive) to calibrate both parameters at a time (see figure 6(b)).

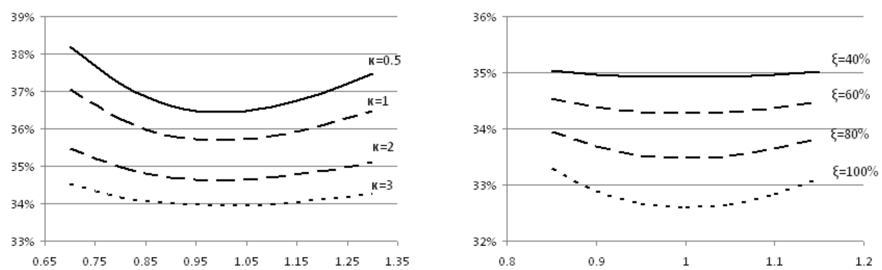
Mean Reversion The mean reversion parameter controls the curvature of the curve. Increasing the mean reversion flattens the implied volatility smile (see figure 7(a)).

[1] Please note that the following remarks are valid only when considering one specific smile curve. Some of the parameters, such as long-term volatility and mean reversion, have a strong impact on the term structure of smile curves. This aspect is not covered in our short analysis.



(a) Initial volatility $v = \sqrt{V_0}$ (b) Long-term volatility θ

Figure 6: Influence of Heston parameters (1/3)

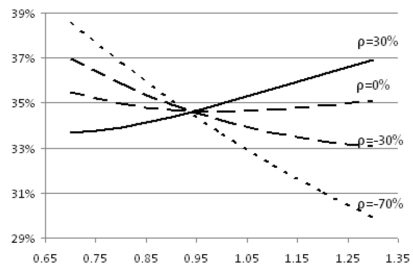


(a) Mean reversion κ (b) Volatility of volatility ξ

Figure 7: Influence of Heston parameters (2/3)

Volatility of Volatility The volatility of volatility ξ and mean reversion both change the curvature. Decreasing the mean reversion has a similar effect as increasing the volatility of volatility in terms of curvature. Once more, we have identified two redundant parameters. Regarding the implied volatility level, the impact of both parameters is clearly different. However, it doesn't matter too much since we have already identified two parameters that control it (see figure 7(b)).

Correlation As in the SABR model, the correlation ρ controls the slope of the volatility smile. This feature is extremely interesting as it is the only parameter to play such a role. Therefore, it is probably easier to isolate this parameter (see figure 8(a)).
This short study shows that calibrating only 3 parameters out of 5 can lead to good results. We have indeed seen that 2 couples of parameters had the same role. Most practitioners choose to calibrate the correlation, the volatility of volatility and either the initial volatility or the long-term volatility. This first remark simplifies greatly the calibration problem.



(a) Correlation ρ

Figure 8: Influence of Heston parameters (3/3)

5.3 Initial parameters

As a matter of fact, even if formula 14 is much simpler, we will have difficulties in inversing the formula in terms of v_0, θ or κ. However, regarding the correlation ρ and the volatility of volatility ξ, the task seems quite feasible. The formula can indeed be rewritten as follows:

$$C_{Hes} = A + B\xi^2 + C\rho\xi + D\rho^2\xi^2 \tag{15}$$

where

$$\begin{aligned}
A &= C_{BS}[x, var_T] \\
B &= (v_0 r_0 + \theta r_1) C_{BS}^{(0,2)}[x, var_T] \\
C &= (v_0 C_0 + \theta C_1) C_{BS}^{(1,1)}[x, var_T] \\
D &= \left((v_0 q_0 + \theta q_1) C_{BS}^{(2,1)}[x, var_T] + \frac{1}{2}(v_0 C_0 + \theta C_1)^2 C_{BS}^{(2,2)}[x, var_T] \right).
\end{aligned}$$

Writing this equation for two strikes yields the following system:

$$\begin{cases} C_{Hes}(K_1) &= A(K_1) + B(K_1)\xi^2 + C(K_1)\rho\xi + D(K_1)\rho^2\xi^2 \\ C_{Hes}(K_2) &= A(K_2) + B(K_2)\xi^2 + C(K_2)\rho\xi + D(K_2)\rho^2\xi^2 \end{cases} \tag{16}$$

This system can be solved analytically and has four solutions, potentially in $\mathbb{C}$. Two of them can be immediatly ignored since they lead to a negative volatility of volatility. By adjusting the long-term volatility (or the initial volatility), both solutions can become real [2].

[2]Empirically, it takes only a few iterations to obtain real solutions if the system admit real solutions. If it requires more time, you had better switch to a global optimization.

5.4 Numerical results

In the following example, we calibrate the Heston model on a market implied volatility smile using four different techniques. Two global optimization methods, Differential Evolution and Simplex, are compared to the Smart Parameters technique, with and without local optimization. In order to be able to compare the results, we have defined a target error level of $5e-7$, the error being defined as the sum of the squared discrepancies between model and market implied volatilities. All methods, except the Smart Parameters, produce a succesful calibration.

Method	**Time**	**Iterations**
DE	10.797 s	738
Simplex	5.031 s	2025
Smart Parameters	$<$0.001 s	-
Smart LM	0.093 s	9

Table 2: Comparison of calibration strategies

For the smart parameters algorithm, arbitrary initial values have to be chosen for the non-calibrated parameters. We have set $v = 42\%$, $\theta = 20\%$ and $\kappa = 2$. Initial volatility and long-term volatility correspond roughly to the ATM volatility level of 42%. κ is fixed at a typical market level.

Calibration times are higher than in the SABR model case since computing the implied volatility takes longer in the Heston model. By reducing considerably the number of iterations, our technique fastens considerably the calibration procedure. In this experiment, initial parameters already produce a decent fit.

6 Practical considerations

6.1 On the choice of the approximation formula

Once initial parameters are found, a local optimization should be run in order to ensure a good fit on the whole implied volatility curve. The formula used for initial parameters need not be used during Levenberg-Marquardt, for instance, in the Heston calibration, we used Miri formula to find smart parameters as it is both fast and tractable, but we switched to Lewis formula during the Levenberg-Marquardt optimization since it is the industry standard. This switch is acceptable provided the approximations are close enough. The same could be done for SABR in order to use other implied volatility formulae.

6.2 Robustness

Although our approach lacks a robustness proof, empirical evidence shows that calibration speed is dramatically increased. In order to ensure robustness, if we cannot find smart initial parameters, we perform a global optimization on all parameters. Thus, the maximum calibration time is slightly higher but average calibration time is much smaller. Besides, calibration error with smart initial parameters is often lower than with a global optimization algorithm.

7 Conclusion

We have exposed a new calibration technique to leverage the speed of Levenberg-Marquardt algorithm while retaining the robustness of global optimization algorithms. The technique of smart initial parameters has been applied to stochastic volatility models such as SABR and Heston model yielding a 10-times faster calibration.

Applying this methodology relies upon the ability to find initial parameters that provide a partial fit to the market volatility smile. This ability requires a careful study of the model parameters and the availability of analytically tractable formulae for implied volatility or option price. Thanks to expansion techniques, several models (Schoebel-Zhu, Wishart) belong to this class and, hence, should be successfully calibrated with our methodology.

Extending our technique to time-dependent parameters requires further research as there is a need to design a procedure that bootstraps an acceptable parameter set for each maturity.

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A Approximated option price in Heston model

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$$C_{Hes} = C_{BS}[x, var_T] + \sum_{i=1}^{2} a_{i,T} C_{BS}^{(i,1)}[x, var_T] + \sum_{i=0}^{1} b_{2i,T} C_{BS}^{(2i,2)}[x, var_T] \quad (17)$$

where the following parameters have been introduced

$$\begin{aligned} var_T &= m_0 v_0 + m_1 \theta, & & \\ a_{1,T} &= \rho\xi(p_0 v_0 + p_1 \theta), & a_{2,T} &= (\rho\xi)^2 (q_0 v_0 + q_1 \theta), \\ b_{0,T} &= \xi^2 (r_0 v_0 + r_1 \theta), & b_{2,T} &= \frac{a_{1,T}^2}{2}, \end{aligned}$$

and

$$\begin{aligned} m_0 &= \frac{e^{-\kappa T}\left(e^{\kappa T}-1\right)}{\kappa}, & m_1 &= T - \frac{e^{-\kappa T}\left(e^{\kappa T}-1\right)}{\kappa}, \\ p_0 &= \frac{e^{-\kappa T}\left(e^{\kappa T}-\kappa T-1\right)}{\kappa^2}, & p_1 &= \frac{e^{-\kappa T}\left(e^{\kappa T}(\kappa T-2)+\kappa T-2\right)}{\kappa^2}, \\ q_0 &= \frac{e^{-\kappa T}\left(2e^{\kappa T}-\kappa T(\kappa T+2)-2\right)}{2\kappa^3}, & q_1 &= \frac{e^{-\kappa T}\left(2e^{\kappa T}(\kappa T-3)+\kappa T(\kappa T+4)+6\right)}{2\kappa^3}, \\ r_0 &= \frac{e^{-2\kappa T}\left(2e^{2\kappa T}-4e^{\kappa T}\kappa T-2\right)}{4\kappa^3}, & r_1 &= \frac{e^{-2\kappa T}\left(e^{2\kappa T}(2\kappa T-5)+4e^{\kappa T}(\kappa T+1)+1\right)}{4\kappa^3}. \end{aligned}$$